LINEAR ALGEBRA II
Chapter one
Characteristic Equation

1.1. *Eigen values and Eigen vectors*

Definition1.1.1: Let A be an $n \times n$ matrix. A Number λ is called an Eigen value of A if there exists a non zero vector $v \in F^n$ such that $Av = \lambda v$.

- The vector v is then called an eigenvector of A corresponding to the Eigen value λ.

- **Example:** Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. Then taking $\lambda = 5$ and $v = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$,.

Cont…

- we have $Av = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 10 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \lambda v$ so 5 is an eigenvalue of A and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector of A corresponding to the eigenvalue 5
- **Definition1.1.2**: Let A be an $n \times n$ matrix. The set of all eigenvalues of A is called the spectrum of A.
- **Theorem 1.1.1**: Let A be an $n \times n$ matrix. A Number λ is an Eigen value of A if and only if $det(A - \lambda I) = 0$, where I denotes the $n \times n$ identity matrix.

Properties of Eigen Values

i. The sum of the eigen values of a matrix is the sum of the elements of the principal diagonal.

ii. The product of the eigen values of a matrix A is equal to its determinant.

iii. If λ is an eigen value of a matrix A, then $^1/_\lambda$ is the eigen value of A^{-1}.

iv. If λ is an eigen value of an orthogonal matrix, then $^1/_\lambda$ is also its eigen value.

v. The eigen values of a triangular matrix are precisely the entries of diagonal entries.

Cont…

vi. If $\lambda_1, \lambda_2,\ldots, \lambda_n$ are the eigenvalues of A, then

a. $k\lambda_1, k\lambda_2,\ldots, k\lambda_n$ are the eigenvalues of the matrix kA, where k is a non – zero scalar.

b. $\frac{1}{\lambda_1}, \frac{1}{\lambda_2} \ldots \frac{1}{\lambda_n}$ are the eigenvalues of the inverse matrix A^{-1}.

c. ${\lambda_1}^p, {\lambda_2}^p \ldots {\lambda_n}^p$ are the eigenvalues of A^P, where p is any positive integer.

1.2 Characteristic polynomial

- **Definition 1.2.1:** Polynomial of degree n in x is an expression of the form $a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, where $a_0, a_{1,} \ldots a_n \in F$ and $a_n \neq 0$ $, n\ is\ non\ negative\ integer.$

- We can write $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$, if we replace x every where by a given number λ and we obtain $p(\lambda) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0$

- **Definition 1.2.2**: Let $p(x)$ be a polynomial in x. A number λ is called a root of $p(x)$ if $p(\lambda) = 0$.

Cont.…

- **Definition 1.2.3**: Let $A = (a_{ij})$ be an $n \times n$ matrix. Then the polynomial

$$det(A - \lambda I) = det\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{21} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{bmatrix}$$ is called the characteristic polynomial of A.

- Denoted as $C_A(x)$, the leading coefficient is $(-1)^n$ and the constant term is $detA$.

Cont…

- The equation $det(A - \lambda I) = 0$ or equivalently $det(\lambda I - A) = 0$ is called characteristic equation of A.

- **Definition 1.2.4**: If an eigenvalue λ occur k times as a root of the characteristic polynomial $C_A(x)$, then k is called the multiplicity of the eigenvalue .

- **Example 1**: Let $A = \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix}$ then,

Cont.

i. Find characteristic polynomial and characteristic equation of A.
ii. Eigenvalues and eigenvectors of A.
iii. Eigen values of A^{-1}
iv. Eigen values of A^{10}
v. Eigen values of 10A.

Solution:

Characteristic polynomial is

$$det(A-\lambda I) \quad = \quad \left| \begin{bmatrix} 3 & -1 & -1 \\ -12 & 0 & 5 \\ 4 & -2 & -1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right|$$

Cont.

$$= \left|\begin{bmatrix} 3-\lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1-\lambda \end{bmatrix}\right|$$

$= (3-\lambda)[(\lambda^2+\lambda)+10] + 1[(12\lambda+12)-20] - 1(24+4\lambda)$

characteristic polynomial is $-\lambda^3 + 2\lambda^2 + \lambda - 2$ and characteristic equation of A is $-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$

ii. To find Eigenvalues of A we have to find root of

$-\lambda^3 + 2\lambda^2 + \lambda - 2 = 0$

$\Longrightarrow -(\lambda+1)(\lambda-1)(\lambda-2) = 0$

λ= -1 or λ= 1 or λ=2

Cont.

- So the eigenvalues are -1,1 and 2 .
- To find eigenvector corresponding to eigenvalues
- write $Av = \lambda v$

$$\Longrightarrow (A - \lambda I)v = 0 \text{ that is}$$

$$\Longrightarrow \begin{bmatrix} 3-\lambda & -1 & -1 \\ -12 & -\lambda & 5 \\ 4 & -2 & -1-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Then we can solve for x_1, x_2 and x_3 by appropriate method.

Let λ= -1 ,then we have

Cont.

$$\begin{bmatrix} 4 & -1 & -1 \\ -12 & 1 & 5 \\ 4 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Solve by using Gaussian elimination .

Since the constant matrix is zero matrix, reduce the coefficient matrix to row echelon form

$$\begin{bmatrix} 4 & -1 & -1 \\ -12 & 1 & 5 \\ 4 & -2 & 0 \end{bmatrix} \xrightarrow[R_3 - R_1]{R_2 + 3R_1} \begin{bmatrix} 4 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & -1 & 1 \end{bmatrix} \xrightarrow{R_3 - {}^1/_2\, R_2} \begin{bmatrix} 4 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Cont.

$$\begin{bmatrix} 4 & -1 & -1 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Thus the linear system becomes

$4x_1 - x_2 - x_3 = 0$

$-2x_2 + 2x_3 = 0$

$$\Rightarrow x_2 = x_3, x_1 = \frac{1}{2} x_3$$

- Let $x_3 = t \ \ then \ \ x_2 = t \ and \ x_1 = {}^1/_2 \, t$ where t is arbitrary

Cont.

Thus v $= \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \begin{bmatrix} 1/2\ t \\ t \\ t \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$

So $\begin{bmatrix} 1/2 \\ 1 \\ 1 \end{bmatrix}$ is eigenvector corresponding to eigenvalue λ = -1

- Similarly, we obtain $\begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}$ as an eigenvector corresponding to eigenvalue λ=1and $\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$ as an eigenvector corresponding to eigenvalue 2.

Cont.

iii. Eigen values of A^{-1} . 1, -1 and $\frac{1}{2}$ are eigenvalues of A^{-1} by the property of eigenvalue.

iv. Eigen values of A^{10} are -1 , 1 and 2^{10}

v. Eigen values of 10A are -10,10 and 20

Example 2: Let $A = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$

Solution:

- The characteristic equation of matrix A is $\lambda^3 - 5\lambda^2 + 8\lambda - 4 = 0$, or, in factored form $(\lambda - 1)(\lambda - 2)^2 = 0$

Cont.

- Thus the eigenvalues of A are $\lambda = 1$and $\lambda_{2,3} = 2$, so there are two eigenvalues of A.

$\mathrm{v} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$,is an eigenvector of A corresponding to λ if and only $\boldsymbol{v}$ is a nontrivial solution of $(A - \lambda I)v = 0$

that is, of the form

$$\begin{bmatrix} -\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

cont.

- If $\lambda = 1$, then $\begin{bmatrix} -1 & 0 & -2 \\ 1 & 1 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

❑Solving this system using Gaussian elimination yields $x_1 = -2s, x_2 = s, x_3 = s$ (verify)

❑Thus the eigenvectors corresponding to $\lambda = 1$are the nonzero vectors of the form

$\begin{bmatrix} -2s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ so that $\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ is the Eigen vector corresponding to eigenvalue 1

Cont.

- If $\lambda = 2$, then it becomes $\begin{bmatrix} -2 & 0 & -2 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Solving this system using Gaussian elimination yields

$x_1 = -s, x_2 = t, x_3 = s$(verify)

- Thus, the eigenvectors of A corresponding to $\lambda = 2$ are the nonzero vectors of the form

❑Thus, the eigenvectors of A corresponding to $\lambda = 2$are the nonzero vectors of the form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -s \\ t \\ s \end{bmatrix} = s\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Cont…

- Since $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$ are linearly independent, these vectors form a basis for the Eigen space corresponding to $\lambda = 2$.
- So the Eigen vectors of A corresponding Eigen values 1and 2 are

$\begin{bmatrix}-2\\1\\1\end{bmatrix}$, $\begin{bmatrix}-1\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\0\end{bmatrix}$

Exercise 1:

1. Find eigenvalues and eigenvectors of

$$A = \begin{bmatrix}1 & 1 & 2 & 3 & 4\\0 & 1 & 1 & 2 & 1\\0 & 0 & 2 & 1 & 1\\0 & 0 & 0 & 3 & 1\\0 & 0 & 0 & 0 & 4\end{bmatrix}$$

Cont.

2. Let A= $\begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$,then find

a. Find characteristic polynomial and characteristic equation of A.

b. Eigenvalues and eigenvectors of A.

c. Eigen values of A^{-1}

d. Eigen values of A^{11}

e. Eigen values of $\begin{bmatrix} 30 & -20 & 20 \\ 40 & -40 & 60 \\ 20 & -30 & 50 \end{bmatrix}$.

1.3.Similarity of matrix

- **Definition1.3.1:** If A and B are n xn matrices, then A is similar to B if there is an invertible matrix P such that $P^{-1}AP = B$, or, equivalently

$A = PBP^{-1}$

- Changing A into B is called a similarity transformation.
- **Theorem 1.3.1:** If nxn matrices A and B are similar, then they have the same characteristic polynomial and hence the same eigenvalues (with the same multiplicities).
- **Proof:**
- Suppose A and B are similar matrices of order n.

Cont.

- The characteristic polynomial of B is given by

$$\det(B - \lambda I)$$
$$= \det(P^{-1}AP - \lambda I)$$
$$= \det(p^{-1}(A - \lambda I)p)$$
$$= \det(p^{-1})\det(A - \lambda I)\det(p)$$
$$= \det(p^{-1}p)\det(A - \lambda I)$$
$$= \det(A - \lambda I)$$

- **Definition 1.3.2**: The set of *distinct* eigenvalues, denoted by σ(A) , is called the ***spectrum*** of **A**.
- **Definition 1.3.3:** For square matrices A , the number $\rho(A) = \max|\lambda|, \lambda \in \sigma(\mathbf{A})$ is called the *spectral radius* of A.

1.4. Diagonalization

- ***Definition1.4.1***: A square matrix A is called *diagonalizable* if there exist an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix; the matrix P is said to *diagonalize* A.

❑**THEOREM 1.4.1** : *If* P_1 , P_2, …, P_k are Eigen vectors of A corresponding to distinct eigenvalues

,λ_1, λ_2 , …,λ_k then {P_1 , P_2, …, P_k } is a linearly independent set.

- **Proof**

❑ Let , P_1 , P_2, …, P_k be eigenvectors of A corresponding to distinct eigenvalues λ_1, λ_2 , …,λ_k .

Cont.

- We shall assume that, P_1 , $P_2, \ldots, P_k$ are linearly dependent and obtain a contradiction. We can then conclude that , P_1 , $\mathrm{P}_2, \ldots, \mathrm{P}_\mathrm{k}$ are linearly independent.
- Since an eigenvector is nonzero by definition, $\{\mathrm{p}_1\}$ is linearly independent. Let *r* be the largest integer such that P_1 , $\mathrm{P}_2, \ldots, \mathrm{P}_\mathrm{r}$ is linearly independent. Since we are assuming that P_1 , $\mathrm{P}_2, \ldots,$ P_k is linearly dependent, *r* satisfies $1 \leq \mathrm{r} \leq \mathrm{k}$.

❑ More over, by definition of *r*, P_1 , $\mathrm{P}_2, \ldots, \mathrm{P}_{\mathrm{r}+1}$ is linearly dependent.

Cont.

- Thus there are scalars c_1 , $c_2, \ldots, c_{r+1}$, not all zero, such that

$P_1c_1 + P_2c_2 + \ldots, P_{r+1}c_{r+1} = 0$ (4)

Multiplying both sides of 4 by A and using

$AP_1 = \lambda_1 p_1, AP_2 = \lambda_2 P_2 \ldots, AP_n = \lambda_{r+1}P_{r+1}$

- we obtain $c_1\lambda_1 p_1 + c_2\lambda_2 p_2 + \ldots + c_{r+1}\lambda_{r+1}p_{r+1} = 0 \quad \ldots$ (5)
- Multiplying both sides of 4 by λ_{r+1} and subtracting the resulting equation from 5 yields
- $c_1(\lambda_1 - \lambda_{r+1})p_1 + c_2(\lambda_2 - \lambda_{r+1})p_2 + \ldots + c_r(\lambda_r - \lambda_{r+1})p_r = 0$

Cont.

- Since $\{P_1, P_2, \ldots, P_r\}$ is a linearly independent set, this equation implies that $c_1(\lambda_1 - \lambda_{r+1}) + c_2(\lambda_2 - \lambda_{r+1}) + \ldots + c_r(\lambda_r - \lambda_{r+1}) = 0$ and since , $\lambda_1, \lambda_2, \ldots, \lambda_r$ are distinct by hypothesis, it follows that
- $c_1 = c_2 = \ldots = c_r = 0$ (6)

Substituting these values in 4 yields

- $P_{r+1} c_{n+1} = 0$ (7)

❑ Since the eigenvector P_{r+1} is nonzero, it follows that $c_{r+1} = 0$

❑ Equations 6 and 7 contradict the fact that $c_1, c_2, \ldots, c_{r+1}$ are not all zero; this completes the proof

Cont.

- **Theorem 1.4.2:** If A is an n × n matrix, then the following are equivalent.
- ✓a. A is diagonalizable.
- ✓b. A has n linearly independent eigenvectors.
- **<u>Proof</u>**
- $a \Rightarrow b$ Since A is assumed diagonalizable, there is an invertible matrix

$$\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$$

such that $P^{-1}AP$ is diagonal, say $P^{-1}AP = D$,

Cont.

where D $= \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

- It follows from the formula $P^{-1}AP = D$ that $AP = PD$; that is ,
- AP $= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$

Cont.

$$= \begin{bmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & \cdots & \lambda_n p_{1n} \\ \lambda_1 p_{21} & \lambda_2 p_{22} & \cdots & \lambda_n p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1 p_{n1} & \lambda_2 p_{n2} & \cdots & \lambda_n p_{nn} \end{bmatrix} \qquad 1$$

- If we denote the column vectors of P, by P_1 ,P_2, …, P_n, then from 1, the successive columns of PD are $\lambda_1 P_1$, $\lambda_2 P_2$, …, $\lambda_n P_n$.
- The successive columns of AP are AP_1 , AP_2, …, AP_n .
- Thus we must have

$$AP_1 = \lambda_1 p_1, AP_2 = \lambda_2 P_2 \ldots , AP_n = \lambda_n P_n \qquad 2$$

Cont….

- Since P is invertible, its column vectors are all nonzero; thus, it follows from 2 that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of A, and P_1 , P_2, …, P_n are corresponding eigenvectors.

Since P is invertible, $P_1, P_2, \ldots, P_n$ are linearly independent.

❑ Thus A has n linearly independent eigenvectors.

❑ $b \Rightarrow a$ Assume that A has n linearly independent eigenvectors P_1 , P_2, …, P_n with corresponding eigenvalues , $\lambda_1, \lambda_2, \ldots, \lambda_n$

Cont….

- Let P = $\begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix}$ be the matrix whose column vectors are

P_1 , P_2, …, P_n

- The column vectors of the product AP are AP_1 , AP_2, …, AP_n

❑But , $AP_1 = \lambda_1 p_1$, $AP_2 = \lambda_2 P_2$ … , $AP_n = \lambda_n P_n$ Why?

Cont.

$$AP = \begin{bmatrix} \lambda_1 p_{11} & \lambda_2 p_{12} & \cdots & \lambda_n p_{1n} \\ \lambda_1 p_{21} & \lambda_2 p_{22} & \dots & \lambda_2 p_{2n} \\ \vdots & \vdots & \dots & \vdots \\ \lambda_1 p_{n1} & \lambda_2 p_{n2} & \cdots & \lambda_2 p_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1n} \\ p_{21} & p_{22} & \cdots & p_{2n} \\ \vdots & \vdots & \dots & \vdots \\ p_{n1} & p_{n2} & \cdots & p_{nn} \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} = PD \qquad 3$$

Cont.

- Where D is the diagonal matrix having the eigenvalues, λ_1, λ_2 , …,λ_n on the main diagonal.
- Since the column vectors of P are linearly independent, P is invertible. Thus 3 can be rewritten as $P^{-1}AP = D$; that is, A is diagonalizable
- **THEOREM 1.4.3**: If an $n \times n$ matrix A has n distinct eigenvalues,λ_1, λ_2 , …,λ_n then A is diagonalizable
- **Proof:** If P_1 , P_2, …, P_n are eigenvectors corresponding to the distinct eigenvalues ,λ_1, λ_2 , …,λ_n

then by Theorem , P_1 , P_2, …, P_n are linearly independent.

- Thus A is diagonalizable by Theorem .

Procedure to diagonalize a matrix

- ✓ Find the linearly independent eigenvectors
- ✓ Construct P from eigenvectors
- ✓ Construct $D = P^{-1}AP$ from eigenvalues

- **Example 1**: Finding a Matrix P that diagonalize a matrix A

$$\mathrm{A} = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

- **Solution**
- The characteristic equation of A is $(\lambda - 1)(\lambda - 2)^2 = 0$ and we found the following bases for the Eigen spaces:

Cont.

- $\lambda = 2$: $p_1 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $p_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\lambda = 1$: $p_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$
- There are three basis vectors in total, so the matrix A is diagonalizable and

$\mathrm{P} = \begin{bmatrix} -1 & 0 & -2 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ diagonalize A.

Cont.

Verify that $P^{-1}AP =$

$$\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

- **EXAMPLE 2:**
- Find a matrix P that diagonalizable $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ -3 & 5 & 2 \end{bmatrix}$

Cont.

Solution

- The characteristic polynomial of A is

❑ $\det(A-\lambda I) = \begin{pmatrix} 1-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ -3 & 5 & 2-\lambda \end{pmatrix}$

$$= (\lambda-1)(\lambda-2)^2$$

❑Thus the eigenvalues of A are $\lambda_1 = 1$ and $\lambda_{2,3} = 2$.

Cont.

- Verify that eigenvalues are corresponding to

$$\lambda_1 = 1 \; is \; p_1 = \begin{bmatrix} -1 \\ 1 \\ -8 \end{bmatrix}$$ and corresponding to $$\lambda_{2,3} = 2 \; is \; p_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

- Since A is a 3×3 matrix and there are only two basis vectors in total, A is not diagonalizable by theorem.

1.5.Cayley-Hamilton Theorem

❑Theorem: If $C_A(x)$ is characteristic polynomial of A, then $C_A(x) = 0$

- Let A = $\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$ then characteristic polynomial of A

$$C_A(x) = \det(A - \lambda I) = \det\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$$

Cont.

- The characteristic equation is given by $C_A(x) = x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_1x^1 + a_0 = 0$
- $C_A(x) = A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1A^1 + Ia_0 = 0$
- **Theorem**: Let A be a non singular $n \times n$ matrix, and let its characteristic polynomial be $C_A(x) = A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1A^1 + Ia_0 = 0$,then $A^{-1} = -\frac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots + Ia_1)$
- Proof :
- By Cayley-Hamilton theorem $A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1A^1 + Ia_0 = 0$

Cont.

- $a_0 = (-1)^n \det A \neq 0$, Since A is non singular .
- So the above equation can be written as

$$I = -\frac{1}{a_0}(A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1 A^1$$

$$= -\frac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots + Ia_1)A$$

- $A^{-1} = -\frac{1}{a_0}(A^{n-1} + a_{n-1}A^{n-2} + \cdots + Ia_1)$

Cont….

Example 1: $Let\ \mathrm{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

Verify Cayley -Hamilton theorem and compute A^{-1}

- <u>Solution</u>
- The characteristic equation of A $is\ |A-\lambda I| = 0$
- $\begin{vmatrix} 2-\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & 2-\lambda \end{vmatrix} = 0$

$\Rightarrow -(\lambda^3 - 6\lambda^2 + 9\lambda - 4) = 0$

Cont.

- To verify Cayley – Hamilton theorem, we have to show that $A^3 - 6A^2 + 9A - 4I = 0$

- $A^2 = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$

- $A^3 = A^2A = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 22 & -21 & 22 \end{bmatrix}$

Cont.

- Therefore $A^3 - 6A^2 + 9A - 4I$

✓$= \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 22 & -21 & 22 \end{bmatrix} - 6\begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} + 9\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} -$

$$4\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Now multiplying both side of (1) by A^{-1} yields
- $A^2 - 6A^1 + 9I - 4A^{-1} = 0$
- $4A^{-1} = (A^2 - 6A^1 + 9I)$

Cont.

$$4A^{-1} = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix} - 6\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix} + 9\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$$\Longrightarrow A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

Cont.

- **Example 2**: given A $= \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$ find AdjA by using Cayley-Hamilton theorem.

  - Solution
- The characteristic equation of a matrix A is
- $|A - \lambda I| = 0$ i.e., $\left|\begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & -1 \\ 3 & -1 & 1-\lambda \end{bmatrix}\right| = 0$
- $\Longrightarrow \lambda^3 - 3\lambda^2 + 5\lambda + 3 = 0$
- by Cayley-Hamilton theorem $A^3 - 3A^2 + 5A + 3I = 0$
- Multiply both side by A^{-1} ,we get $A^2 - 3A + 5I + 3A^{-1} = 0$

Cont.

- $A^{-1} = -\frac{1}{3}(A^2 - 3A + 5I)$
- $= -\frac{1}{3}\left(\begin{bmatrix} -2 & 5 & -4 \\ -3 & 2 & -2 \\ 6 & 4 & -1 \end{bmatrix} - \begin{bmatrix} 3 & 6 & -3 \\ 0 & 3 & -3 \\ 9 & -3 & 3 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}\right)$
- $A^{-1} = \begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 1 & \frac{-4}{3} & -\frac{1}{3} \\ 1 & -\frac{7}{3} & -\frac{1}{3} \end{bmatrix}$

Cont.

- We know that , $A^{-1} = \frac{adjA}{|A|}$

- Therefore $adjA = |A|A^{-1} = -3\begin{bmatrix} 0 & \frac{1}{3} & \frac{1}{3} \\ 1 & \frac{-4}{3} & -\frac{1}{3} \\ 1 & -\frac{7}{3} & -\frac{1}{3} \end{bmatrix}$

$$= \begin{bmatrix} 0 & -1 & -1 \\ 1 & 4 & 1 \\ 1 & 7 & 1 \end{bmatrix}$$

Cont.

- Another application of Cayley-Hamilton theorem is to compute power of square matrix A
- Example: Let $A = \begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$ and then find A^{735}

- **Solution**

- The characteristic polynomial of A is $x^2 - 9$. Eigen values are -3,3 .
- Division algorithm applied to the polynomial x^{735}, $x^2 - 9$ will give

equation of the form $x^{735} = (x^2 - 9)q(x) + (a_0 + a_1 x)$, (1)

Where $(a_0 + a_1 x)$ is remainder obtained by dividing x^{735} by $x^2 - 9$.

Cont.

- Note that the degree of remainder is less than the degree of the divisor $x^2 - 9$.
- By Cayley-Hamilton theorem $A^2 - 9I = 0$.
- Inserting A for x in equation (1), we get
- $A^{735} = a_0 + a_1 A$ (2)
- Inserting eigen values 3 and -3 for x successively in equation (1), we get
- $3^{735} = a_0 + 3a_1$

Cont….

- $(-3)^{735} = a_0 - 3a_1$

- This gives $a_0 = 0, a_1 = 3^{734}$.

- Then $A^{735} = a_0 + a_1 A$
- gives
- $A^{735} = 3^{735} A.$

1.6.Minimal polynomial

- **Definition 1.6.1**: The minimal polynomial of a matrix A, denoted by $m_A(\lambda)$ is the unique monic polynomial of least degree such that $m_A(\lambda) = 0$.
- Theorem: a scalar λ is Eigen values of a matrix A if and only if is root of minimal polynomial.
- proof(exercise)
- **Example**: find minimal polynomial $m_A(\lambda)$ of $A = \begin{bmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{bmatrix}$
- **Solution**
- First find the characteristic polynomial of A.

Exercise 3

- Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 3 & -1 & 1 \end{bmatrix}$, then

1. Verify Cayley-Hamilton theorem
2. Find A^{-1} and AdjA by using Cayley-Hamilton theorem.

Cont.

- $$p(\lambda) = -\lambda^3 + 5\lambda^2 - 7\lambda + 3 = -(\lambda - 1)^2(\lambda - 3).$$
  The minimal polynomial must divide characteristic polynomial.
  Thus minimal polynomial is exactly one of the following
  $$p(\lambda) = (\lambda - 3)(\lambda - 1)^2 \text{or } g(\lambda) = (\lambda - 3)(\lambda - 1)$$
  $$g(A) = (A - I)(A - 3I) = \begin{pmatrix} 1 & 2 & -5 \\ 3 & 6 & -15 \\ 1 & 2 & -5 \end{pmatrix}\begin{pmatrix} -1 & 2 & -5 \\ 3 & 4 & -15 \\ 1 & 2 & -7 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Cont.

Thus $g(t) = (\lambda - 1)(\lambda - 3) = \lambda^2 - 4\lambda + 3$ is a minimal polynomial of A

Exercise 3 : Find minimal polynomial m (λ) of

$$i. \quad A = \begin{bmatrix} 3 & -2 & 2 \\ 4 & -4 & 6 \\ 2 & -3 & 5 \end{bmatrix}$$

$$i. \quad B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$