NUMERICAL ANALYSIS I (MATH 2061)

For: Second Year Mathematics Student

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Numerical analysis

- is the branch of mathematics that is used to find approximations to difficult problems such as:

- finding the roots of non-linear equations
- integration involving complex expressions
- solving differential equations for which analytical solutions do not exist

- It is applied to a wide variety of disciplines such as :

  -business

  -all fields of engineering

  -computer science

  -education

  -geology

  -meteorology and others.

- It is the area of mathematics and computer science that creates, analyzes, and implements algorithms for solving numerically the problems of continuous mathematics.

Chapter 1

Basic concepts in error estimation

Source of Error

- **Modeling Error**
  - -Blunders
  - -Formulation Error
  - -Data uncertainty
- **Numerical Error**
  - -Rounding Error
  - -Truncation Error

Round off Error

- which result when numbers having limited significant figures are used to represent exact numbers.
- Caused by representing a number approximately.

Example: $\frac{1}{3} \cong 0.333333$ $\sqrt{2} \cong 1.4142...$

TRUNCATION ERROR

- Error caused by truncating or approximating a mathematical procedure.

Example of Truncation Error

1. Taking only a few terms of a Maclaurin series to e^x

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\dots\dots\dots\dots\dots\dots$$

If only 3 terms are used,

$$Truncation\ Error = e^x - \left(1 + x + \frac{x^2}{2!}\right)$$

2. Using a finite Δx to approximate $f'(x)$

$$f'(x) \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

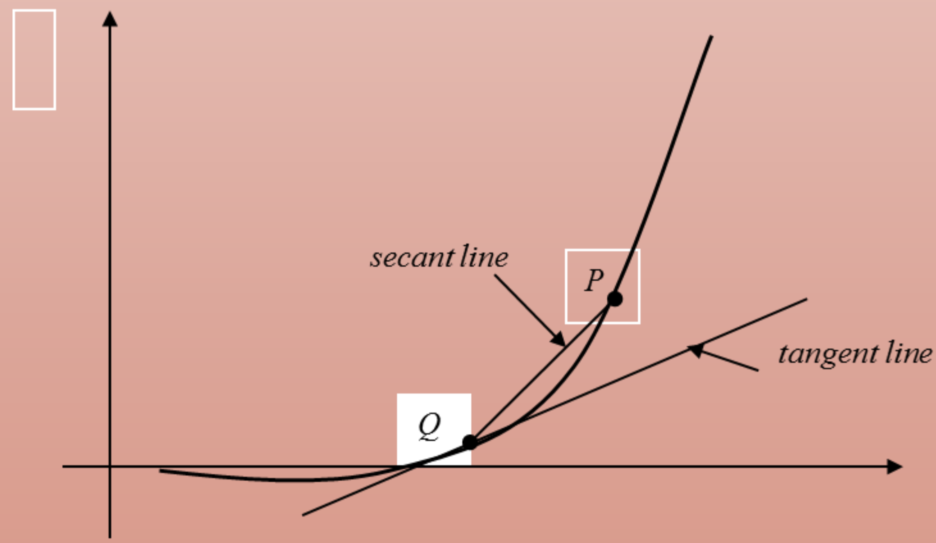
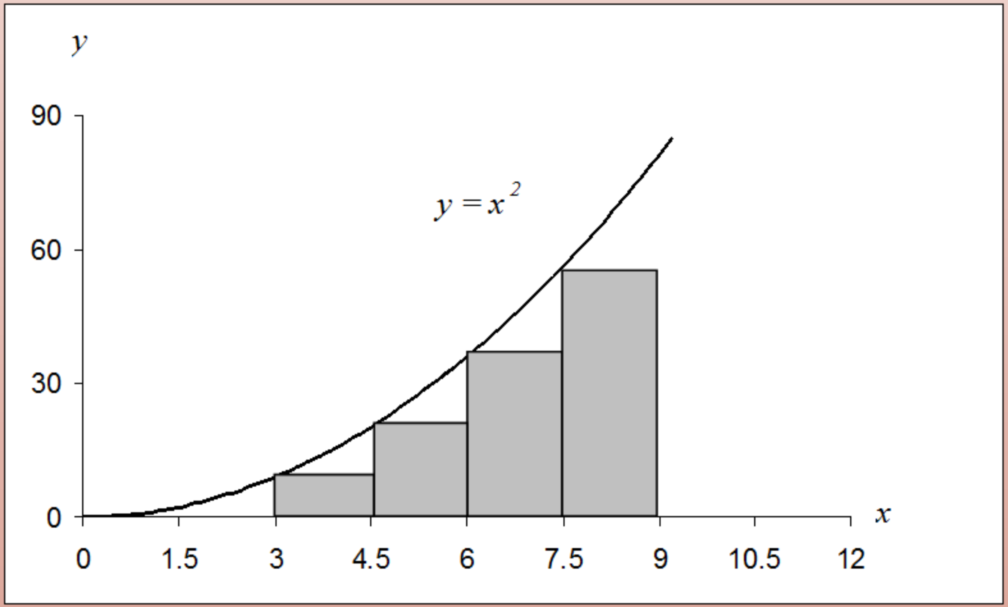


Figure 1. Approximate derivative using finite Δx

- Using finite rectangles to approximate an integral.



Example 1: Maclaurin series

Calculate the value of $e^{1.2}$ with an absolute relative approximate error of less than 1%.

$$e^{1.2} = 1 + 1.2 + \frac{1.2^2}{2!} + \frac{1.2^3}{3!} + \dots\dots\dots\dots\dots\dots$$

n			
1	1	___	___
2	2.2	1.2	54.545
3	2.92	0.72	24.658
4	3.208	0.288	8.9776
5	3.2944	0.0864	2.6226
6	3.3151	0.020736	0.62550

6 terms are required. How many are required to get at least 1 significant digit correct in your answer?

Example 2: Diffrentiation

Find $f'(3)$ for $f(x) = x^2$ using $f'(x) \approx \dfrac{f(x+\Delta x) - f(x)}{\Delta x}$

and $\Delta x = 0.2$

$$f'(3) = \frac{f(3+0.2) - f(3)}{0.2} = \frac{f(3.2) - f(3)}{0.2} = \frac{3.2^2 - 3^2}{0.2} = \frac{10.24 - 9}{0.2}$$

$$= \frac{1.24}{0.2} = 6.2$$

The actual value is

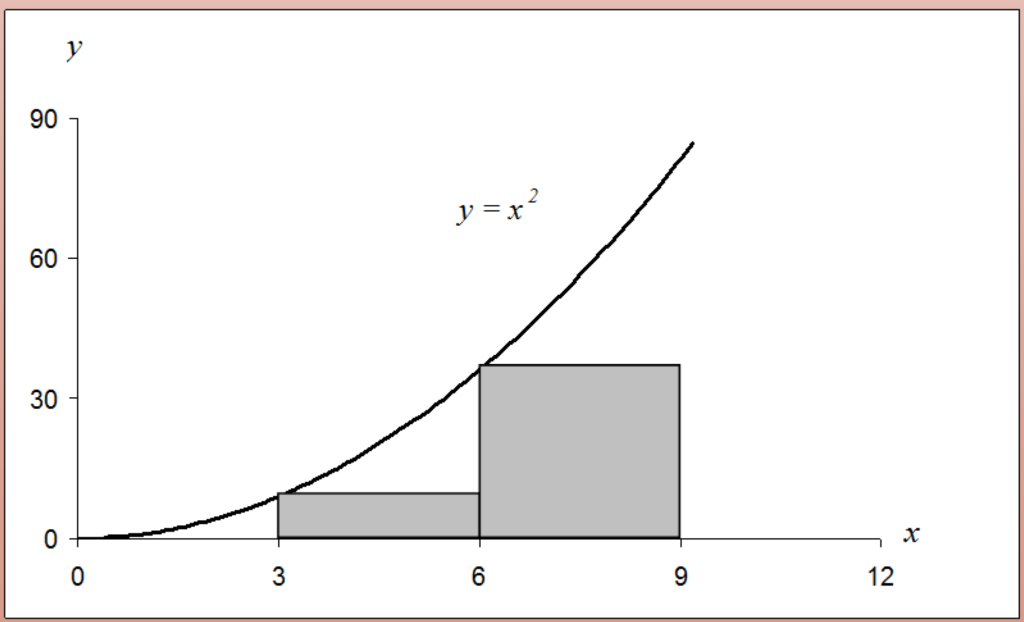
$$f'(x) = 2x, \quad f'(3) = 2 \times 3 = 6$$

Truncation error is then, $6 - 6.2 = -0.2$

Can you find the truncation error with $\Delta x = 0.1$?

Example 3: Integrations

Use two rectangles of equal width to approximate the area under the curve for $f(x) = x^2$ over the interval $[3,9]$



$$\int_3^9 x^2 dx$$

INTEGRATION EXAMPLE (CONT'D…)

- Choosing a width of 3, we have

$$\int_3^9 x^2\,dx = (x^2)\Big|_{x=3}(6-3) + (x^2)\Big|_{x=6}(9-6) \quad = (3^2)3 + (6^2)3$$

$$= 27 + 108 = 135$$

- Actual value is given by

$$\int_3^9 x^2\,dx = \left[\frac{x^3}{3}\right]_3^9 = \left[\frac{9^3 - 3^3}{3}\right] = 234$$

Truncation error is then

$$234 - 135 = 99$$

Can you find the truncation error with 4 rectangles?

APPROXIMATIONS AND ROUND-OFF ERRORS

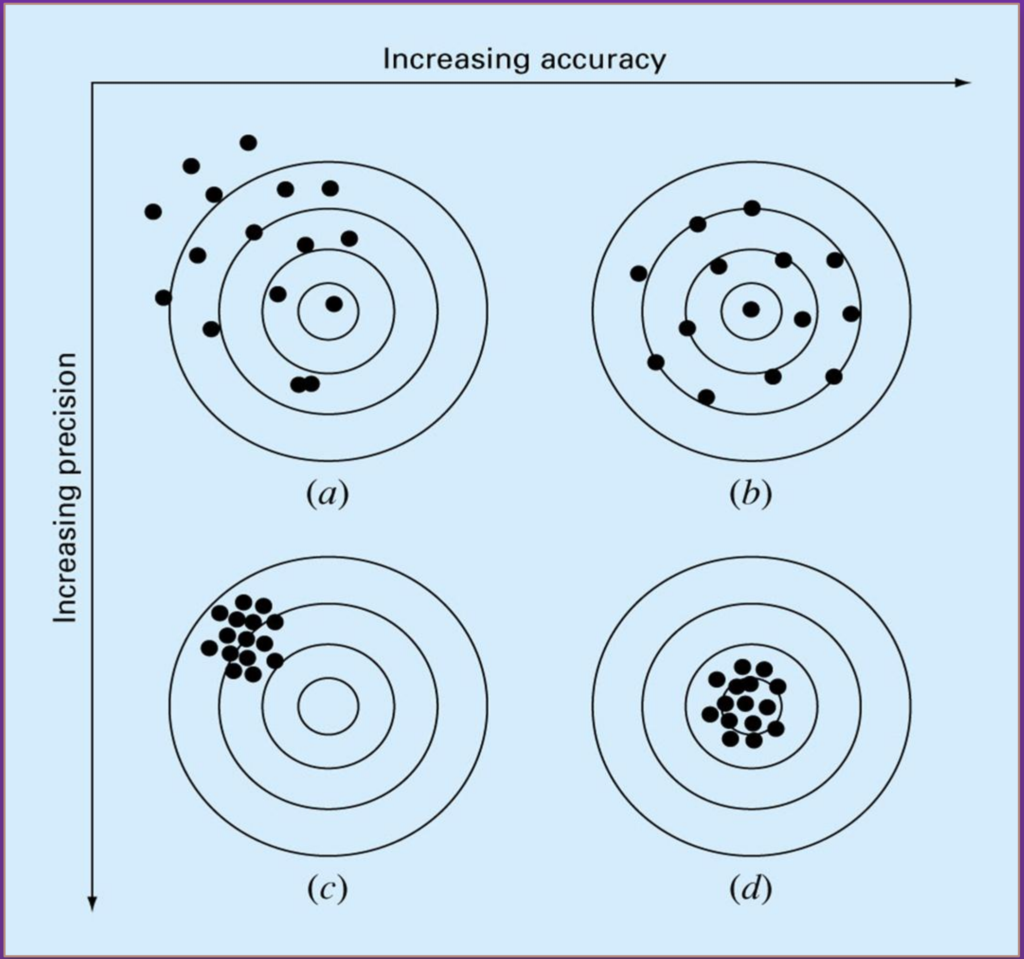
- For many engineering problems, we cannot obtain analytical solutions.
- Numerical methods yield approximate results, results that are close to the exact analytical solution. We cannot exactly compute the errors associated with numerical methods.
  - Only rarely given data are exact, since they originate from measurements. Therefore there is probably error in the input information.

CONT'D

- Algorithm itself usually introduces errors as well, e.g., unavoidable round-offs, etc
- The output information will then contain error from both of these sources.

- How confident we are in our approximate result?
- The question is "*how much error is present in our calculation and is it tolerable?*"

- Accuracy: How close is a computed or measured value to the true value
- Precision (or *reproducibility*): How close is a computed or measured value to previously computed or measured values.
- Inaccuracy (or *bias*): A systematic deviation from the actual value.
- Imprecision (or *uncertainty*): Magnitude of scatter.


Increasing accuracy
Increasing precision
(a)
(b)
(c)
(d)

SIGNIFICANT FIGURES

- Number of significant figures indicates precision. Significant digits of a number are those that can be *used* with *confidence.* e.g., the number of certain digits plus one estimated digit.
- 53,8<u>00</u> How many significant figures?

5.38×10^4 3

5.380×10^4 4

5.3800×10^4 5

Zeros are sometimes used to locate the decimal point not significant figures.

0.00001753	4
0.0001753	4
0.001753	4

Error Definitions

True Value = Approximation + Error

E_t = True value – Approximation (+/-)

True error

$$\text{True fractional relative error} = \frac{\text{true error}}{\text{true value}}$$

$$\text{True percent relative error}, \varepsilon_t = \frac{\text{true error}}{\text{true value}} \times 100\%$$

- For numerical methods, the true value will be known only when we deal with functions that can be solved analytically (simple systems).
- In real world applications, we usually not know the answer a priori. Then

CON'D

$$\varepsilon_a = \frac{\text{Approximate error}}{\text{Approximation}} \times 100\%$$

Iterative Approach, example Newton's method

$$\varepsilon_a = \frac{\text{Current approximation - Previous approximation}}{\text{Current approximation}} \times 100\%$$

- Use absolute value.
- Computations are repeated until stopping criterion is satisfied

$$|\varepsilon_a| < \varepsilon_s$$

Pre-specified % tolerance based on the knowledge of your solution

CONT'D…

- If the following criterion is met

$$\varepsilon_s = (0.5 \times 10^{(2-n)})\%$$

you can be sure that the result is correct to at least n significant figures.

a) Inaccurate and imprecise; (*b*) accurate and imprecise; (*c*) inaccurate and precise; (*d*) accurate and precise

Absolute and Relative Errors

Absolute Error (E_a)

Absolute error= $\left| Exact\ value - Approximate\ value \right|$

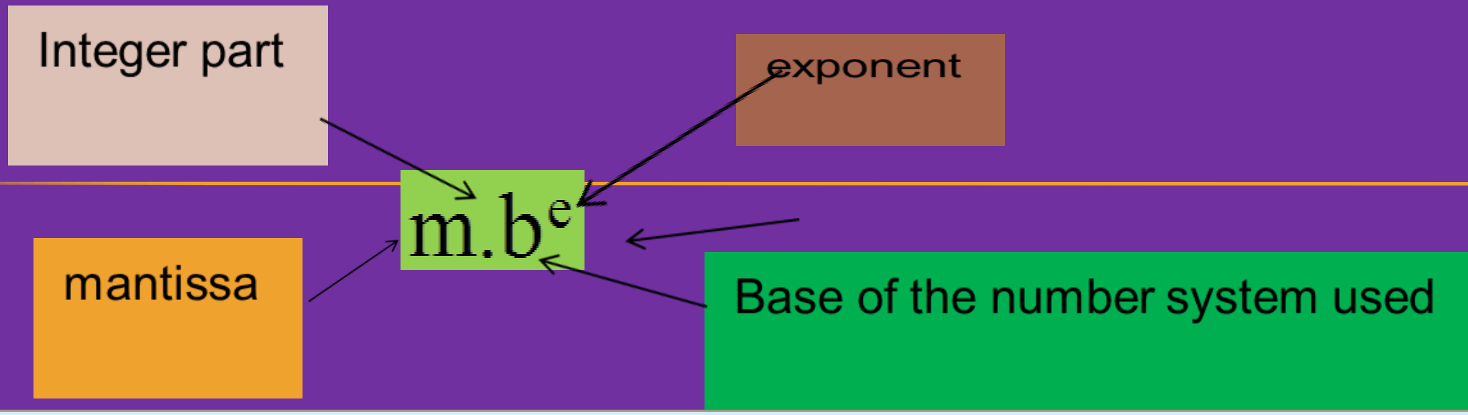
Relative Errors (E_r)

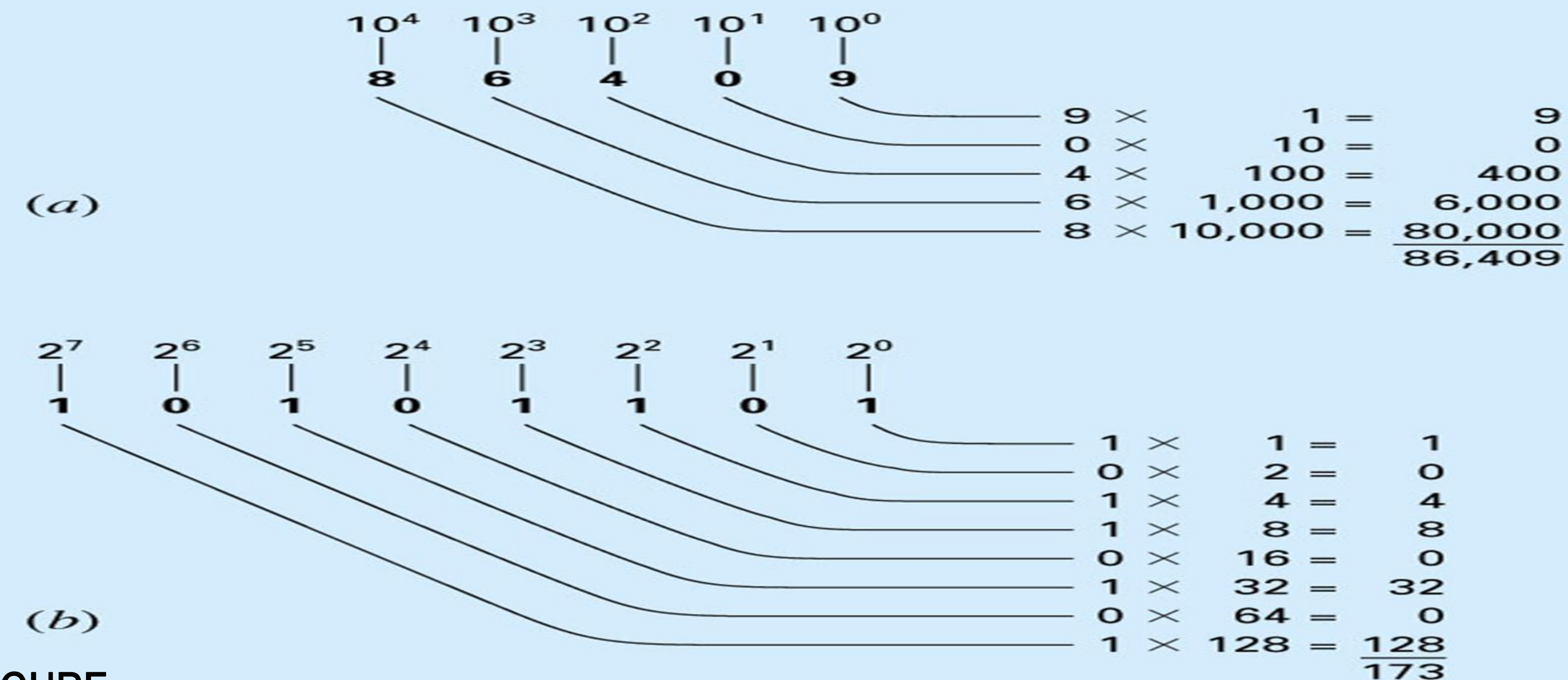
$$E_r = \frac{\left| Exact\ value - Approximate\ value \right|}{\left| Exact\ value \right|}\ x\ 100\%$$

$$\varepsilon_a = \frac{\left| \text{current approximation} - \text{previous approximation} \right|}{\text{current approximation}}\ X\ 100\%$$

Round of Errors

- Round-off errors: originate from the fact that computers retain only a fixed number of significant figures during a calculation.
- Numbers such as π, e, or $\sqrt{7}$ cannot be expressed by a fixed number of significant figures.
- Therefore, they cannot be represented exactly by the computer.





FIGURE

How the (*a*) decimal (base 10) and the (*b*) binary (base 2) systems work. In (*b*), the binary number 10101101 is equivalent to the decimal number 173.

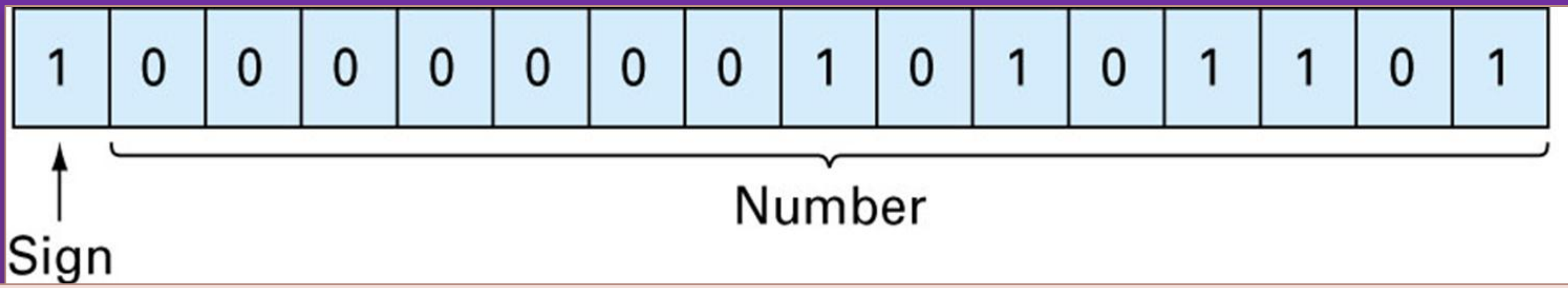


FIGURE
The representation of the decimal integer −173 on a 16-bit computer using the signed magnitude method.

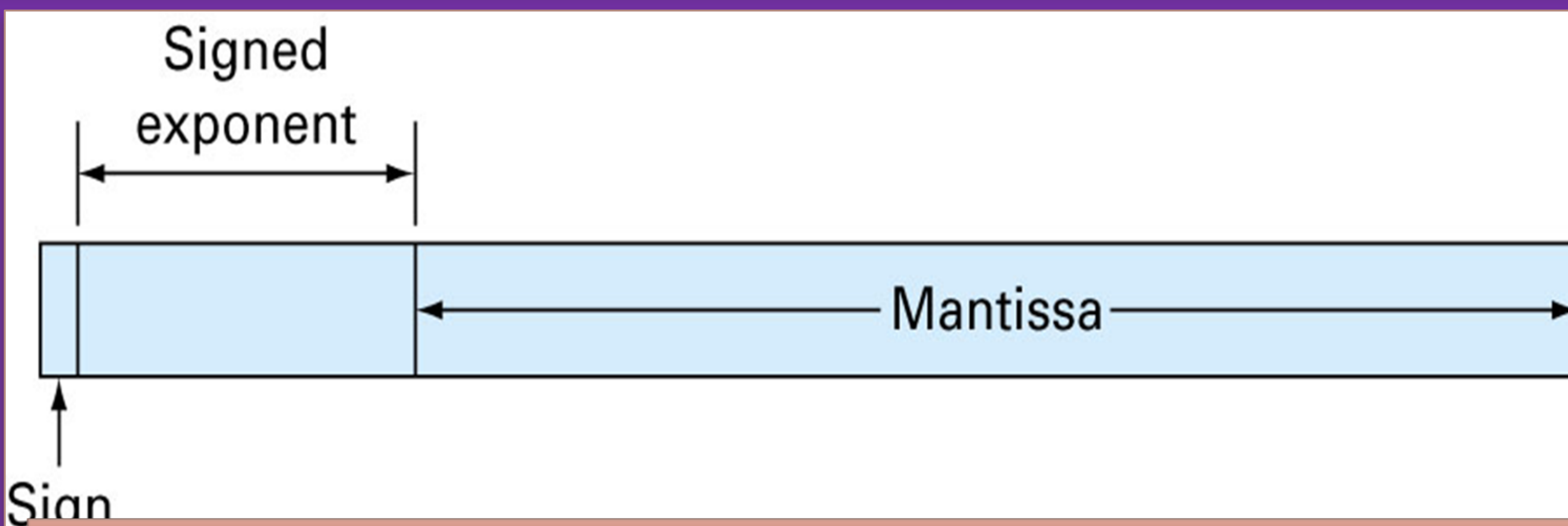


FIGURE
The manner in which a floating-point number is stored in a word

CHOPPING

Example:

π=3.14159265358 to be stored on a base-10 system carrying 7 significant digits.

π=3.141592 chopping error ε_t=0.00000065

If rounded

π=3.141593 ε_t=0.00000035

- Some machines use chopping, because rounding adds to the computational overhead. Since number of significant figures is large enough, resulting chopping error is negligible.

Propagation of Error

- The purpose of this section is to study how errors in numbers can propagate through mathematical functions.
- If we multiply two numbers that have errors, we would like to estimate the error in the product.

  * Functions of a Single Variable

  * Functions of More than One Variable

- Suppose that we have a function f (x) that is dependent on a single independent variable x.
- Assume that x is an approximation of x.
- to assess the effect of the discrepancy between x and $\tilde{x}$ on the value of the function.
- We would like to estimate by

$$\triangle f(x) = \left| f(x) - f(x) \right|$$

By expansion of Taylor's series, we obtain:

$$\triangle f(x) = \left| f^{'}(x) \right| \triangle x, \text{ where } \triangle x = \left| x - x \right|$$

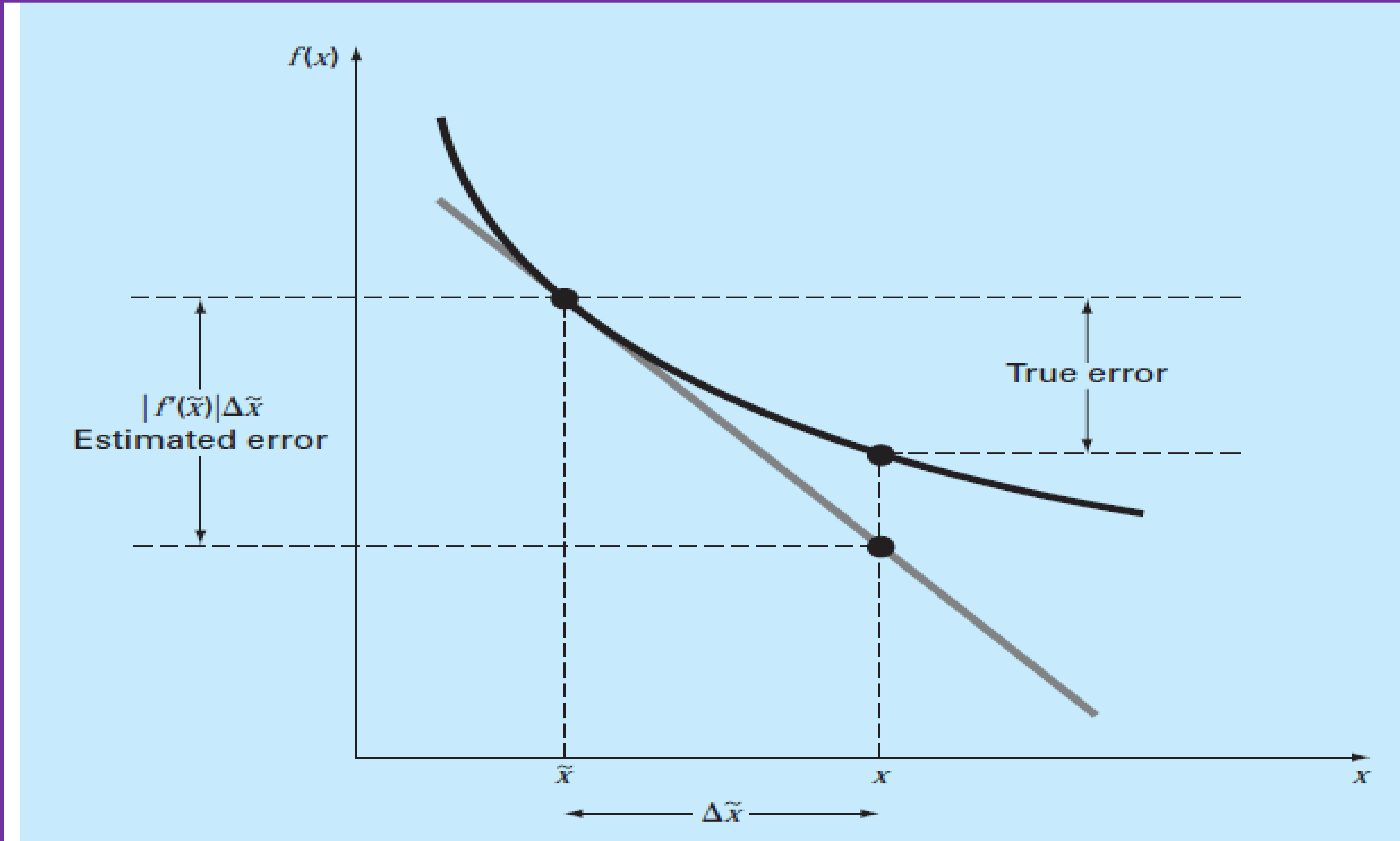


FIGURE 4.7

Graphical depiction of first order error propagation

Example: Given a value of x = 2.5 with an error of Δx= 0.01, estimate the resulting error in the function $f(x) = x^3$.

n

Ans: f(2.5) = 15.625 ± 0.1875

Functions of More than One Variable

For n independent variables $x_1, x_2, ..., x_n$ having errors $\Delta x_1, \Delta x_2, ..., \Delta x_n$ the following general relationship holds:

$$\Delta f(x_1, x_2, ..., x_n) = \left|\frac{\partial f}{\partial x_1}\right| \Delta x_1 + \left|\frac{\partial f}{\partial x_2}\right| \Delta x_2 + ... + \left|\frac{\partial f}{\partial x_n}\right| \Delta x_n$$

EXAMPLE 1

Find the bounds for the propagation in adding two numbers. For example if one is calculating $X+Y$ where

$$X = 1.5 \pm 0.05$$
$$Y = 3.4 \pm 0.04$$

Solution

Maximum possible value of $X = 1.55$ and $Y = 3.44$

Maximum possible value of $X + Y = 1.55 + 3.44 = 4.99$

Minimum possible value of $X = 1.45$ and $Y = 3.36$.

Minimum possible value of $X + Y = 1.45 + 3.36 = 4.81$

Hence

$$4.81 \le X + Y \le 4.99.$$

EXAMPLE

The strain in an axial member of a square cross-section is given by $\in = \dfrac{F}{h^2 E}$

Given

$$F = 72 \pm 0.9 \text{ N}$$
$$h = 4 \pm 0.1 \text{ mm}$$
$$E = 70 \pm 1.5 \text{ GPa}$$

Find the maximum possible error in the measured strain.

Solution:

$$\in = \frac{72}{(4\times10^{-3})^2(70\times10^9)}$$

$$= 64.286\times10^{-6} \quad = 64.286\mu$$

$$\Delta\in = \left|\frac{\partial\in}{\partial F}\Delta F\right| + \left|\frac{\partial\in}{\partial h}\Delta h\right| + \left|\frac{\partial\in}{\partial E}\Delta E\right|$$

$$\frac{\partial\in}{\partial F} = \frac{1}{h^2E} \qquad \frac{\partial\in}{\partial h} = -\frac{2F}{h^3E} \qquad \frac{\partial\in}{\partial E} = -\frac{F}{h^2E^2}$$

CONT'D…

Thus

$$\Delta E = \left|\frac{1}{h^2 E}\Delta F\right| + \left|\frac{2F}{h^3 E}\Delta h\right| + \left|\frac{F}{h^2 E^2}\Delta E\right|$$

$$= \left|\frac{1}{(4\times 10^{-3})^2(70\times 10^9)}\times 0.9\right| + \left|\frac{2\times 72}{(4\times 10^{-3})^3(70\times 10^9)}\times 0.0001\right|$$

$$+ \left|\frac{72}{(4\times 10^{-3})^2(70\times 10^9)^2}\times 1.5\times 10^9\right|$$

$$= 5.3955\mu$$

Hence

$$\in = (64.286\mu \pm 5.3955\mu)$$

EXAMPLE 3

Subtraction of numbers that are nearly equal can create unwanted inaccuracies. Using the formula for error propagation, show that this is true.

Solution: Let $z = x - y$

Then

$$|\Delta z| = \left|\frac{\partial z}{\partial x}\Delta x\right| + \left|\frac{\partial z}{\partial y}\Delta y\right|$$

$$= |(1)\Delta x| + |(-1)\Delta y|$$

$$= |\Delta x| + |\Delta y|$$

So the relative change is

$$\left|\frac{\Delta z}{z}\right| = \frac{|\Delta x| + |\Delta y|}{|x - y|}$$

CONT'D

For example if $x = 2 \pm 0.001$

$$y = 2.003 \pm 0.001$$

$$\left|\frac{\Delta z}{z}\right| = \frac{|0.001| + |0.001|}{|2 - 2.003|}$$

$$= 0.6667$$

$$= 66.67\%$$

Numerical stability

- refers to the accuracy of an algorithm in the presence of rounding errors

• an algorithm is unstable if rounding errors cause large errors in the result

- rigorous definition depends on what 'accurate' and 'large error' mean

THE END