Introduction to Topology (math 313)

Chapter -1-

Metric Spaces

1.1 Definition and Examples of a metric space

Def^n Let X be a non-empty set and $d : X \times X \to [0, \infty)$ be a function satisfying the following properties for all x, y, z in X

a) $d(x,y) \geq 0$ (Axiom of non-negativity)

b) $d(x,y) = 0$ iff $x = y$ (Axiom of Coincidence)

c) $d(x,y) = d(y,x)$ (Axiom of Symmetry)

d) $d(x,y) \leq d(x,z) + d(z,y)$ (Axiom of Triangle Inequality)

Then d is called a metric on X and the ordered pair (X, d) is called a metric space.

Theorem (The Cauchy - Schwarz Inequality)

For any points $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$, $|a \cdot b| \leq \|a\| \|b\|$

Proof Ex!

Theorem (The Minkowski Inequality)

For any points $a = (a_1, a_2, \ldots, a_n)$ and $b = (b_1, b_2, \ldots, b_n)$ in $\mathbb{R}^n$, $\|a+b\| \leq \|a\| + \|b\|$

Proof $$\|a+b\|^2 = \sum_{i=1}^{n} (a_i + b_i)^2 = \sum_{i=1}^{n} \left(a_i^2 + 2a_i b_i + b_i^2\right)$$

$$= \sum_{i=1}^{n} a_i^2 + 2\sum_{i=1}^{n} a_i b_i + \sum_{i=1}^{n} b_i^2$$

$= \|a\|^2 + 2\, a \cdot b + \|b\|^2$

$\leq \|a\|^2 + 2|a \cdot b| + \|b\|^2$

$\leq \|a\|^2 + 2\|a\|\|b\| + \|b\|^2$

$= (\|a\| + \|b\|)^2$

Taking square roots, then $\|a+b\| \leq \|a\| + \|b\|$.

Examples

1) The function defined by $d(x,y) = |x-y|$ is a metric on $\mathbb{R}$.

We call this metric the absolute value metric or the usual metric.

To show this: let $x, y, z \in \mathbb{R}$

a) $d(x,y) = |x-y| \geq 0, \ \forall x, y \in \mathbb{R}$

b) $d(x,y) = |x-y| = 0$ iff $x = y$

c) $d(x,y) = |x-y| = |y-x| = d(y,x)$

d) $d(x,y) = |x-y| = |x-z+z-y| \leq |x-z| + |z-y| = d(x,z) + d(z,y)$

$\therefore$ d is a metric on $\mathbb{R}$.

2) For any non empty set X, define $d(p,q) = \begin{cases} 1 & \text{if } p \neq q \\ 0 & \text{if } p = q \end{cases}$

Then, i) $d(p,q) \geq 0$ since $d(p,q) = 1$ or 0

ii) $d(p,q) = 0$ iff $p = q$

iii) $d(p,q) = 1 = d(q,p)$ if $p \neq q$

$d(p,q) = 0 = d(q,p)$ if $p = q$

$\therefore d(p,q) = d(q,p), \forall p, q \in X$

iii) $d(p,q) \le d(p,r) + d(r,q), \ \forall p,q \in X$

if $p = q = r$, then $d(p,q) = 0$, $d(p,r) = 0$, $d(r,q) = 0$

$\Rightarrow d(p,q) = d(p,r) + d(r,q)$

if $p = q \neq r$, then $d(p,q) = 0$, $d(p,r) = d(r,q) = 1$

$\Rightarrow d(p,q) < d(p,r) + d(r,q)$

If $p = r \neq q$, then $d(p,r) = 0$, $d(p,q) = d(r,q) = 1$

$\Rightarrow d(p,q) = d(q,r) + d(r,q)$

If $p \neq q \neq r$, then $d(p,q) = d(q,r) = d(q,r) = 1$

$\Rightarrow d(p,q) < d(p,r) + d(r,q)$

In all cases, $d(p,q) \le d(p,r) + d(r,q)$

$\therefore$ (X,d) is a metric space and "d" is called the discrete metric on X.

③ Let $\mathbb{R}^n$ be the set of all ordered n-tuples of real numbers

Then $(\mathbb{R}^n, d)$ is a metric space, where $d(x,y) = \left(\sum_{i=1}^{n} (x_i - y_i)^2\right)^{\frac{1}{2}}$

$\forall x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$

$\rightarrow$ To show triangle inequality

If $z = (z_1, z_2, \dots, z_n)$, let $a_i = x_i - z_i$ and

$b_i = z_i - y_i$

for $i = 1, 2, \dots, n$.

Then $d(x,z) = \left(\sum_{i=1}^{n} a_i^2\right)^{1/2}$, $d(z,y) = \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}$ and

$$d(x,z) = \left(\sum_{i=1}^{n} (a_i + b_i)^2\right)^{1/2}$$

we must thus show that

$$\left(\sum_{i=1}^{n} (a_i+b_i)^2\right)^{1/2} \leq \left(\sum_{i=1}^{n} a_i^2\right)^{1/2} + \left(\sum_{i=1}^{n} b_i^2\right)^{1/2}$$

But this follows from the Minkowski Inequality

4) In $\mathbb{R}^2$, define $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$

i) $d(x,y) \geq 0 \ \forall x, y \in \mathbb{R}^2$

ii) ~~but~~ $d(x,y) = 0$ iff $|x_1 - y_1| + |x_2 - y_2| = 0$

iff $|x_1 - y_1| = |x_2 - y_2| = 0$

iff $x_1 = y_1$ and $x_2 = y_2$

iff $(x_1, x_2) = (y_1, y_2)$

iff $x = y$

iii) $d(x,y) = |x_1 - y_1| + |x_2 - y_2| = |y_1 - x_1| + |y_2 - x_2| = d(y,x)$

iv) $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$

$= |x_1 - z_1 + z_1 - y_1| + |x_2 - z_2 + z_2 - y_2|$, where $z = (z_1, z_2) \in \mathbb{R}^2$

$\leq |x_1 - z_1| + |z_1 - y_1| + |x_2 - z_2| + |z_2 - y_2|$

$= d(x,z) + d(z,y)$

$\Rightarrow d(x,y) \leq d(x,z) + d(z,y)$

$\therefore d$ is a metric on $\mathbb{R}^2$.

<u>Note</u> In general if $d : \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$ defined by

$d(x,y) = \sum_{i=1}^{n} |x_i - y_i|$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$

is a metric on $\mathbb{R}^n$ which is called the Taxicab metric

4) Let d is defined on $\mathbb{R}^n$ as $d(x,y) = \max\{|x_i - y_i|\}_{i=1}^{n}$

Then $(\mathbb{R}^n, d)$ is a metric space. The metric d is called the maximum metric on $\mathbb{R}^n$.

To show triangle inequality,

Let $x = (x_1, x_2, \dots, x_n)$, $y = (y_1, y_2, \dots, y_n)$ and $z = (z_1, z_2, \dots, z_n) \in \mathbb{R}^n$

Then
$$d(x,z) = \max\{|x_i - z_i|\}_{i=1}^{n}$$
$$= \max\{|x_i - y_i + y_i - z_i|\}_{i=1}^{n}$$
$$\le \max\{|x_i - y_i| + |y_i - z_i|\}_{i=1}^{n}$$
$$\le \max\{|x_i - y_i|\}_{i=1}^{n} + \max\{|y_i - z_i|\}_{i=1}^{n}$$
$$= d(x,y) + d(y,z)$$

5) Let l^∞ denote the set of all bounded sequences of real numbers.

If $x = \{x_n\}_{n=1}^{\infty}$ and $y = \{y_n\}_{n=1}^{\infty}$ are in l^∞, define
$$d(x,y) = \underset{1 \le n < \infty}{\mathrm{l.u.b}} |x_n - y_n|$$

Then d is a metric on $\mathbb{R}^n$.

Example if $x_n = \{1 + 1/n\}_{n=1}^{\infty}$, $y = \{2 - 1/n\}_{n=1}^{\infty}$, then
$$d(x,y) = \underset{1 \le n < \infty}{\mathrm{l.u.b}} \left|\left(1+\tfrac{1}{n}\right) - \left(2 - \tfrac{1}{n}\right)\right| = \underset{1 \le n < \infty}{\mathrm{l.u.b}} \left|-1 + \tfrac{2}{n}\right| = 1$$

To show triangle inequality

Let $z = \{z_n\}_{n=1}^{\infty}$ is in l^∞

for any $k \in \mathbb{N}$, $|x_k - y_k| = |x_k - z_k + z_k - y_k| \le |x_k - z_k| + |z_k - y_k|$
$$\le \underset{1 \le n < \infty}{\mathrm{l.u.b}} |x_n - z_n| + \underset{1 \le n < \infty}{\mathrm{l.u.b}} |z_n - y_n|$$

and so, $|x_k - y_k| \le d(x,z) + d(z,y)$, $\forall k \in \mathbb{N}$.

$$\Rightarrow \underset{1 \le k \le n}{\mathrm{l.u.b}} |x_k - y_k| \le d(x,z) + d(z,y)$$

$d(x,y) \le d(x,z) + d(z,y)$

Excercise suppose (X, d) be any metric space

for any $x, y \in X$, define $\mu(x,y) = \dfrac{d(x,y)}{1+d(x,y)}$

Then (X, μ) is also a metric space.

i) $d(x,y) \geq 0 \Rightarrow \mu(x,y) \geq 0$

ii) $\mu(x,y) = 0 \Leftrightarrow \dfrac{d(x,y)}{1+d(x,y)} = 0$

$\Leftrightarrow d(x,y) = 0 \Leftrightarrow x = y$

iii) $\mu(x,y) = \dfrac{d(x,y)}{1+d(x,y)} = \dfrac{d(y,x)}{1+d(y,x)} = \mu(y,x)$

iv) we have to show that $\mu(x,y) \leq \mu(x,z) + \mu(z,y)$

If possible, let $\mu(z,y) + \mu(x,z) < \mu(x,y)$

$$\Rightarrow \frac{d(z,y)}{1+d(z,y)} + \frac{d(x,z)}{1+d(x,z)} < \frac{d(x,y)}{1+d(x,y)}$$

$$\Rightarrow \underbrace{\{d(x,z) + d(z,y) - d(x,y)\}}_{\substack{\geq 0 \\ \text{by the inequality} \\ \text{of } d}} + \underbrace{2\,d(z,y)d(x,z) + d(z,y)d(x,z)\,d(x,y)}_{\geq 0} < 0$$

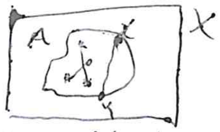
But this is not possible since $d(x,y) \geq 0, \forall x,y \in X$

$\Rightarrow \mu(x,y) \leq \mu(x,z) + \mu(z,y)$

$\therefore (X, \mu)$ is a metric on $\mathbb{R}^n$.

Def'n (diameter of a set)

Let (X,d) be a metric space and A be non-empty subset of X. If $\{d(x,y): x,y \in A\}$ has an upper bound, then A is called a bounded set and $lub\{d(x,y): x,y \in A\}$ is called the diameter $D(A)$ of A.



Remark we define the diameter of the empty set to be zero

Def'n If X is bdd, then (X,d) is called a bounded metric space.

Example

1) Let $A=(0,1)$, then $D(A)=1-0=1$

Let $B=\{3\}$, then $D(B)=3-3=0$

2) Consider the unit square $S=\{x=(x_1,x_2): 0 \le x_i \le 1, i=1,2\}$ in $\mathbb{R}^2$ with the usual metric d, ~~then~~ S has diameter $\sqrt{2}$.

with the taxicab metric d, S has diameter 2

with the max metric d, S has diameter 1

with the discrete metric d, S has diameter 1.

1.2 open sets and closed sets in metric spaces

Def'n Let (X,d) be a metric space, $a \in X$ and $r>0$.

1) The open ball $B_d(a,r)$ with center a and radius r is the set $B_d(a,r)=\{x \in X: d(a,x)<r\}$. And the corresponding closed ball $B_d[a,r]$ is defined by $B_d[a,r]=\{x \in X: d(a,x) \le r\}$

2) The sphere $S_d(a,r)$ with center a and radius r is the set $S_d(a,r)=\{$

Remark 1) $B_d(a,r) \cup S_d(a,r) = B_d[a,r]$

2) $a \in B_d(a,r)$ since $a \in X$ and $d(a,a) = 0 < r$.

Hence $B_d(a,r) \neq \phi$

Examples

(1) Let $X = \mathbb{R}^2$ with the usual metric d.

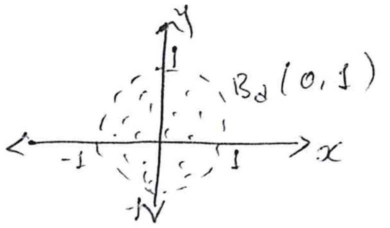
Then, $B_d(0,1) = \{x = (x_1, x_2) \in X : d(0,x) < 1\}$

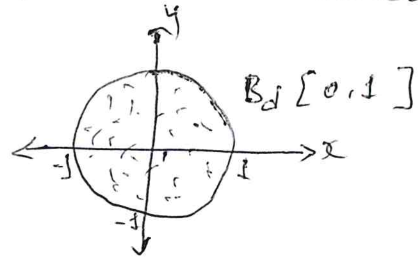
$= \{x : (x_1, x_2) \in X : \sqrt{(0-x_1)^2 + (0-x_2^2)} < 1\}$

$= \{(x_1, x_2) : (0-x_1)^2 + (0-x_2^2) < 1\}$

$= \{(x_1, x_2) : x_1^2 + x_2^2 < 1\}$ which is the region inside the circle with center at the origin and radius 1.

And $B_d[0,1]$ is the union of $B_d(0,1)$ with the boundary circle.





(2) In case of Real line with usual metric d, Every open interval is an open ball.

B/C $B_d(a,r) = \{x \in \mathbb{R} : d(a,x) < r\}$

$= \{x \in \mathbb{R} : |a-x| < r\} = \{x \in \mathbb{R} : |x-a| < r\}$

$= \{x \in \mathbb{R} : -r < x-a < r\} = \{x \in \mathbb{R} : a-r < x < a+r\}$

$= (a-r, a+r)$ which is an open interval.

(3) For $\mathbb{R}^2$ with the Taxicab metric d,

$B_d(y,1) = \{ x = (x_1, x_2) \in \mathbb{R}^2 : |x_1 - y_1| + |x_2 - y_2| < 1 \}$, $y = (y_1, y_2)$

we have 4 cases

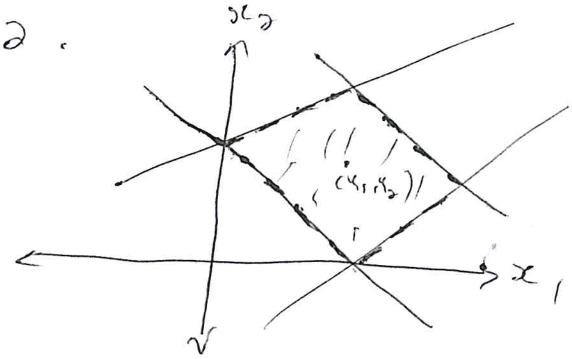
$\{ x = (x_1, x_2) : x_1 - y_1 + x_2 - y_2 < 1 \}$

$\{ x = (x_1, x_2) : x_1 - y_1 - (x_2 - y_2) < 1 \}$

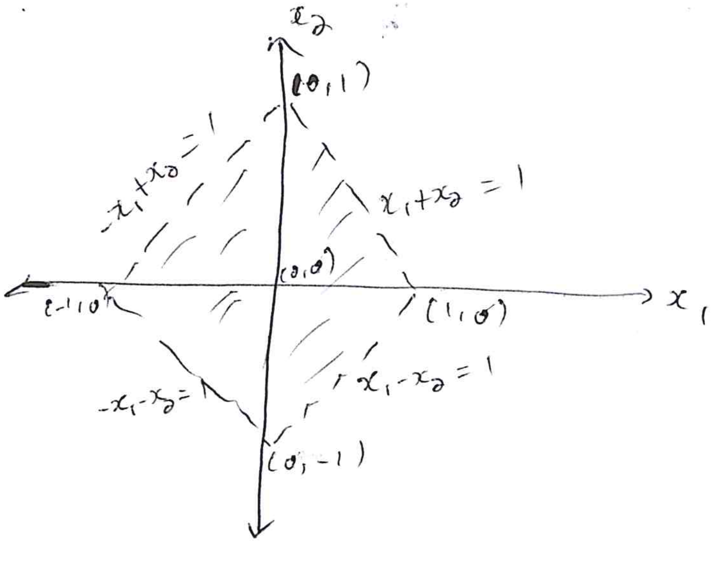
$\{ x = (x_1, x_2) : -(x_1 - y_1) + (x_2 - y_2) < 1 \}$

$\{ x = \{ (x_1, x_2) : -(x_1 - y_1) - (x_2 - y_2) < 1 \}$

$\Rightarrow B_d(y,1)$ is the interior of diamond with center y and height and width 2.



In particular, $B_d(0,1)$ is the interior of the diamond shown in figure below.



(3) For $\mathbb{R}^2$ with the max metric d,

$B_d(y,1) = \{x = (x_1,x_2) : \max\{|x_1-y_1|, |x_2-y_2|\} < 1$.
where $y = (y_1,y_2)$

Case I, let $\max\{|x_1-y_1|, |x_2-y_2|\} = |x_1-y_1|$

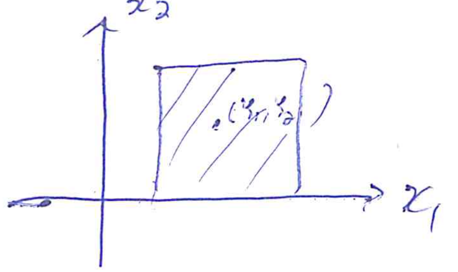
$\Rightarrow |x_1-y_1| < 1$

$\Rightarrow \begin{cases} x_1-y_1 < 1 \\ -(x_1-y_1) < 1 \end{cases}$

Case II: let $\max\{|x_1-y_1|, |x_2-y_2|\} = |x_2-y_2|$

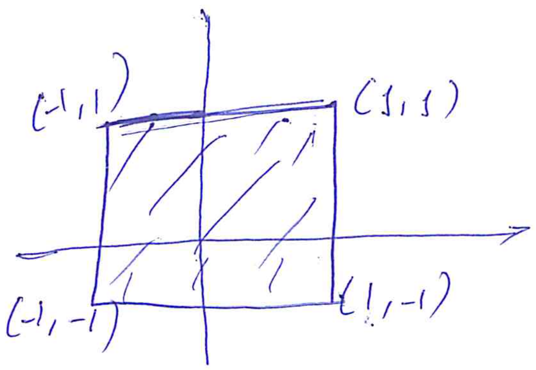
$\Rightarrow |x_2-y_2| < 1$

$\Rightarrow \begin{cases} x_2-y_2 < 1 \\ -(x_2-y_2) < 1 \end{cases}$



i.e., $B_d(y,1)$ = the interior of the square with center y and sides of length 2.

In particular, $B_d(0,1)$ = the interior of the square of side 2 centered at 0 and $B_d[0,1]$ is the union of $B_d(0,1)$ with the four bounding line segments.



(4) For any set X with the discrete metric

$B(a,r) = \{a\}$ if $r \le 1$

? $B[a,r] = \{a\}$ if $r < 1$

$B[a,r] = X$ if $r = 1$

$B(a,r) = B[a,r] = X$ if $r > 1$.

11 -

Def'n : A subset O of a metric space (X, d) is an open set with respect to the metric d provided that O is a union of open balls. The family of open sets is called the topology for X generated by d. A subset G of X is a closed set with respect to d provided that its complement $X \setminus G$ is an open set w.r.t d.

Theorem The following statements are equivalent for a subset of O of a metric space (X, d).

(a) O is an open set

(b) For each $x \in O$, there is an open ball $B(x, \varepsilon_x)$ for some positive radius ε_x, which is contained in O. For $O \neq X$ (a) and (b) are equivalent to

distance b/n x and a point out side X is > 0

(c) For each $x \in O$, $d(x, X \setminus O) > 0$.

Theorem The open subsets of a metric space (X, d) have the following properties.

(a) X and $\emptyset$ are open sets

(b) The union of any family of open sets is open

(c) The intersection of any finite family of open sets is open

Proof (a) The entire space X is open since it is the union of all open balls of all possible centers and radii. The empty set ϕ is open since it is the union of the empty collection of open balls (or we can't find any point that contradicts the condition.

(b) If $\{O_\alpha : \alpha \in A\}$ is a collection of open sets in X, then for each α in the index set A, O_α is a union of open balls. Then $\bigcup_{\alpha \in A} O_\alpha$ is the union of all the open balls of which the open sets O_α are composed and is therefore, an open set.

(c) Let $\{O_i\}_{i=1}^n$ be a finite collection of open sets and let $x \in \bigcap_{i=1}^n O_i$. then by Theorem above (b) there is for each $i = 1, \dots, n$ a positive number ε_i such that $B(x, \varepsilon_i) \subseteq O_i$. Then

$$\bigcap_{i=1}^n B(x, \varepsilon_i) \subseteq \bigcap_{i=1}^n O_i$$

But the intersection of the balls $B(x, \varepsilon_i)$ is simply the ball $B(x, \varepsilon)$ where $\varepsilon = \min\{\varepsilon_i\}_{i=1}^n$.

So, $B(x, \varepsilon)$ is an open ball centered at x and ~~[illegible]~~ contained in $\bigcap_{i=1}^n O_i$. thus $\bigcap_{i=1}^n O_i$ is open

<u>Theorem</u>: The closed subsets of a metric space (X, d) have the following properties.

a) X and ϕ are closed sets

b) The intersection of any family of closed sets is closed

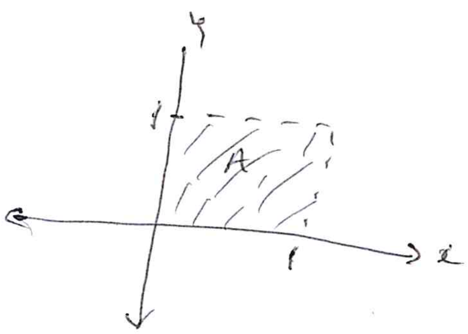
c) The union of any finite family of closed sets is closed

<u>Remark</u> whether a set is open or not depends up on the space in which it is considered.

<u>Example</u> 1) $\mathbb{R}$ contains no open balls in $\mathbb{R}^2$, and hence $\mathbb{R}$ is not open when considered as a subset of $\mathbb{R}^2$ (since open balls in $\mathbb{R}^2$ are circular discs whereas in $\mathbb{R}$ is a line segment)

2) In the plane with the usual metric, the set

$A = \{(x_1, x_2) \in \mathbb{R}^2 : 0 \le x_i < 1 : i = 1, 2\}$ is neither open nor closed.



Def'n: Let (X, d) be a metric space and A a subset of X. A point $x \in X$ is a limit point or accumulation point of A provided that every open set containing x contains a point of A distinct from x. The set of limit points of A is called its derived set, denoted A'.

Theorem Let (X, d) be a metric space and A a subset of X. A point $x \in X$ is a limit point of A iff $d(x, A \setminus \{x\}) = 0$

eg for $\mathbb{R}^2$ with the usual metric

a) The origin is the only limit point of the sequence $\{(-1/n, 1/n)\}_{n=1}^{\infty}$

b) The derived set of the closed unit square $S = \{(x_1, x_2) : 0 \le x_i \le 1;\ i = 1, 2\}$ is the set S itself.

c) A finite set has no limit point

Theorem A subset A of a metric space (X, d) is closed iff A contains all its limit points.

Def'n Let (X, d) be a metric space and $\{x_n\}_{n=1}^{\infty}$ a sequence of points of X. Then $\{x_n\}_{n=1}^{\infty}$ converges to the point $x \in X$, or x is the limit of the sequence, provided that given $\varepsilon > 0$ there is a positive integer N such that if $n \ge N$, then $d(x_n, x) < \varepsilon$. A sequence that converges is called a convergent sequence.

Remark since $d(x_n, x) < \varepsilon$ is equivalent to $x_n \in B(x, \varepsilon)$, the definition of convergence can be restated as follows: A sequence in a metric space X converges to $x \in X$ iff for each $\varepsilon > 0$ the open ball $B(x, \varepsilon)$ contains x_n for all but a finite number of positive integers n.

Theorem: A sequence in a metric space cannot converge to more than one limit.

Proof suppose that $\{x_n\}_{n=1}^{\infty}$ converges to two distinct limits a and b in the metric space (X, d). let $\varepsilon = \frac{1}{2} d(a, b)$.
There must exist integers N_a and N_b such that if $n \geq N_a$, then $d(x_n, a) < \varepsilon$ and if $n \geq N_b$, then $d(x_n, b) < \varepsilon$. This means that both $d(x_n, a)$ and $d(x_n, b)$ are less than ε when n is greater than or equal to the larger of N_a and N_b.
Then
$$d(a, b) \leq d(a, x_n) + d(x_n, b)$$
$$< \varepsilon + \varepsilon = 2\varepsilon = d(a, b)$$

So, $d(a, b) < d(a, b)$ $\rightarrow\leftarrow$

Thus, the assumption that $\{x_n\}_{n=1}^{\infty}$ converges to more than one limit must be false.

Theorem let (X, d) be a metric space and A a subset of X.

a) A point x in X is a limit point of A iff there is a sequence of distinct points of A which converges to x.

b) The set A is closed iff each convergent sequence of points of A converges to a point of A.

1.3 Interior, closure and boundary

Def'n: Let A be a subset of a metric space X. A point x in A is an interior point of A, or A is a nbhd of x, provided that there is an open set O which contains x and is contained in A.

The interior of A, denoted $\operatorname{int} A$, is the set of all interior points of A.

Note that If O is an open set contained in A, then every point of O is an interior point of A. Hence the interior of A contains every open set contained in A and is the union of this family of open sets.

If $\operatorname{int} A = \phi$, obvious
or $\operatorname{int} A \neq \phi$, $x \in \operatorname{int} A$
$\Rightarrow \exists$ open set $O \ni$
$x \in O \subseteq A$
$\Rightarrow A$ is nbhd of each elmts of O $\Rightarrow O \subseteq \operatorname{int} A$
$\Rightarrow \operatorname{int} A$ is nbhd of x
$\operatorname{int} A$ is open

Remark (1) The interior of a set A is an open set

(2) $\operatorname{int} A$ is the largest open set contained in A.

eg Consider $\mathbb{R}$ with the usual metric, then

(1) $a, b \in \mathbb{R}$ with $a < b$

$\operatorname{int}(a,b) = \operatorname{int}[a,b) = \operatorname{int}(a,b] = \operatorname{int}[a,b] = (a,b)$

(2) The interior of finite set is empty since such a set contains no open interval.

(3) $\operatorname{int}\phi = \phi$, $\operatorname{int}\mathbb{R} = \mathbb{R}$

(4) Int of irrational no. $= \phi$, since every interval contains some rational nos.

Int rational no. $= \phi$

eg: Consider $\mathbb{R}^2$ with the usual metric

every open ball is an open set

(1) If $a \in \mathbb{R}^2$ and $r > 0$, then

$\text{int } B(a,r) = \text{int } B[a,r] = B(a,r)$

(2) The interior of a finite set is empty since finite set has no open ball (open set)

(3) $\text{Int } \phi = \phi$; $\text{Int } \mathbb{R}^2 = \mathbb{R}^2$

Def'n The closure $\bar{A}$ of a subset A of a metric space X is the union of A with its set of limit points:

$\bar{A} = A \cup A'$, where A' is the derived set

Remark (1) $x \in \bar{A}$ provided that either $x \in A$ or every open set containing x contains a point of A distinct from x.

(2) If $x \in A$, every open set containing x contains a pt of A, namely x itself

$x \in \bar{A}$ iff if every open set containing x contains a point of A.

eg Cosider $\mathbb{R}$ with the usual metric

(a) for $a, b \in \mathbb{R}$, with $a < b$

$\overline{(a,b)} = \overline{[a,b)} = \overline{(a,b]} = \overline{[a,b]} = [a,b]$

(b) If A is a finite set, then $\bar{A} = A$ b/c the derived set A' is empty.

c) closure of $\mathbb{Q}$ $= \mathbb{R}$

closure of $\mathbb{I}\mathbb{Q} = \mathbb{R}$

b/c every open interval contains both rational and irrational no s.

eg Consider $\mathbb{R}^n$ with the usual metric

(a) If $a \in \mathbb{R}^n$ and $r > 0$ then $\overline{B(a,r)} = \overline{B[a,r]} = B[a,r]$

(b) If A is finite set, then $\bar{A} = A$

(c) $\bar{\emptyset} = \phi$ and $\overline{\mathbb{R}^n} = \mathbb{R}^n$.

Theorem: Let A is a subset of a metric space X, then $\bar{A}$ is a closed set and is a subset of every closed set containing A.

Proof NTS $\bar{A}$ contains all its limit points

suppose $x \notin \bar{A}$

$\Rightarrow x \notin A$ and $x \notin A'$

$\Rightarrow$ There is an open set O containing x which contains no point of A.

But if O contains no point of A, then it cannot contain a limit point of A. (b/c if an open set contains limit point of A then it must contain a point of A, by def^n of limit point)

Thus, O contains no point of $\bar{A}$

$\Rightarrow x$ is not a limit point of $\bar{A}$

This means that all limit points of $\bar{A}$ must necessarily be in $\bar{A}$

$\therefore \bar{A}$ is closed.

suppose F is closed subset of X such that $A \subseteq F$

NTS $\bar{A} \subseteq F$

$A \subseteq F \Rightarrow \bar{A} \subseteq \bar{F}$ (EX)

Since F $\Rightarrow$ b/c closed set contains all its limit points $\Rightarrow F' \subseteq F$ $\quad \bar{F} = F \cup F' = F$

Thus, $\bar{A} \subseteq F$

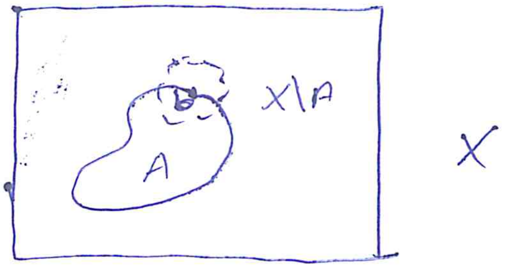
i.e., $\bar{A}$ is the smallest closed set which contains A.

Theorem Let A be a subset of a metric space X.

(a) A is open iff $A = \text{int}\,A$

(b) A is closed iff $A = \bar{A}$

Def'n Let A be a subset of a metric space X. A point $x \in X$ is a boundary point of A provided that x belongs to $\bar{A}$ and to $\overline{(X \setminus A)}$. The set of boundary points of A is called the boundary of A and is denoted by $\text{bdy}\,A$.



The point b is boundary point of A.

Remark From the definition a set and its complement have the same boundary.

eg ① The boundary of any interval in $\mathbb{R}$ with end points a and b is $\{a, b\}$.

② $\text{bdy}\,B(a,r) = \text{bdy}\,B[a,r] = \{x \in \mathbb{R}^n : d(a,x) = r\}$

③ The boundary of the set of all points of $\mathbb{R}^n$ having only rational coordinates is $\mathbb{R}^n$.

④ In any metric space X, $\text{bdy}\,\phi = \text{bdy}\,X = \phi$

Theorem Let X be a metric space, $A \subseteq X$ and $x \in X$. Then the following statements are equivalent.

(1) $x \in \text{bdy}\,A$

(2) $x \in (\bar{A} \setminus \text{Int}\,A)$

(3) Every open set containing x contains a point of A and a point of $X \setminus A$.

(4) Every nbhd of x contains a point of A and a point of $X \setminus A$

(5) $d(x, A) = d(x, X \setminus A) = 0$

(6) $x \in \bar{A} \cap \overline{(X \setminus A)}$

1.4 Continuous functions

Def'n Let $A \subseteq \mathbb{R}$ and $f: A \to \mathbb{R}$. The function f is said to be continuous at the point $a \in A$, if given $\varepsilon > 0$, there is a $\delta > 0$, such that $|f(x) - f(a)| < \varepsilon$, whenever $|x - a| < \delta$. The function f is called continuous if it is continuous at each point of A.

Remark The definition of continuity is applicable for any metric spaces.

Def'n: Let (X, d) and (Y, d') be metric spaces and let $a \in X$. A function $f: X \to Y$ is said to be continuous at the point $a \in X$ if given $\varepsilon > 0$, there is a $\delta > 0$, such that $d'(f(x), f(a)) < \varepsilon$, whenever $x \in X$ and $d(x, a) < \delta$. A function f is called continuous if it is continuous at every point of X.

Note: If p is limit point of X, then f is continuous at p iff $\lim_{x \to p} f(x) = f(p)$.

eg (1) Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x+2 & \text{if } x \neq 2 \\ 0 & \text{if } x = 2 \end{cases}$

Then 2 is limit point of $\mathbb{R}$.

f is cont. at 2 iff $\lim_{x \to 2} f(x) = f(2)$

but $\lim_{x \to 2} f(x) = 4 \neq 0 = f(2)$

$\therefore f$ is not continuous at $x = 2$

(2) Define $f: \mathbb{R} \to \mathbb{R}$ by $f(x) = x + 2$

Then $x = 2$ is limit pt of $\mathbb{R}$ and

$\lim_{x \to 2} f(x) = 4 = f(2)$

$\Rightarrow f$ is cont. at 2.

Example If $f: X \to Y \ni d_Y(f(x), f(y)) \leq d_X(x,y), \forall x,y \in X$ then show that f is continuous on X.

Sol'n Let $\varepsilon > 0$ is given

Then we need to find $\delta > 0$ s.t $\forall x \in X$ $d_X(x,y) < \delta \Rightarrow d_Y(f(x), f(y)) < \varepsilon$

but $d_Y(f(x), f(y)) \leq d_X(x,y) < \delta = \varepsilon$, Choose $\delta = \varepsilon$

$\Rightarrow f$ is continuous.

Example for any $x = (x_1, x_2, \cdots, x_k) \in \mathbb{R}^k$, let us define $\|x\| = \left(\sum_{i=1}^{k} x_i^2\right)^{1/2}$ then the map $f: \mathbb{R}^k \to \mathbb{R}$ defined by $f(x) = \|x\|$ is continuous.

Sol'n $\forall \varepsilon > 0$ we need to find $\delta > 0 \ni \forall x \in \mathbb{R}^k$ and $d_X(x,y) < \delta$

$\Rightarrow d_Y(f(x), f(y)) < \varepsilon$.

$d_X(x,y) = \|x - y\| < \delta$

Take $d_Y(f(x), f(y)) = |f(x) - f(y)| = \big|\|x\| - \|y\|\big|$

$\leq \|x - y\| < \delta = \varepsilon$

$\therefore f$ is continuous on $\mathbb{R}^k$. , Take $\delta = \varepsilon$

Remark The definition of continuity can be restated interms open balls as follows:

Def'n f is continuous at $a \in A$ if for each open ball $B_{d'}(f(a), \varepsilon)$ centered at $f(a)$, there is an open ball $B_d(a, \delta)$ such that the image $f(B_d(a, \delta))$ is a subset of $B_{d'}(f(a), \varepsilon)$

eg ① Let (X, d) be a metric space. Prove that the identity function $i : X \to X$ is continuous.

Sol: Let $a \in X$ and $\varepsilon > 0$ be given.

Choose $\delta = \varepsilon$, then whenever $d(x, a) < \delta$

we have $d(i(x), i(a)) = d(x, a) < \delta = \varepsilon$

② Let $f : (X, d) \to (Y, d')$ be a constant function.

Show that f is cont.

Sol: Let $a \in X$ and $\varepsilon > 0$ be given

Choose any $\delta > 0$, say 1

Then whenever $d(x, a) < \delta$, we've

$d'(f(x), f(a)) = 0 < \varepsilon$.

EX Let $i : \mathbb{R}^n \to \mathbb{R}^n$ be the identity function. Then show that $i : (\mathbb{R}^n, d) \to (\mathbb{R}^n, d')$ is continuous, where d is the max. metric and d' is usual metric.

Sol: Let $a = (a_1, a_2, \dots, a_n) \in \mathbb{R}^n$ and $\varepsilon > 0$ be given

Choose $\delta = \varepsilon / \sqrt{n}$

Suppose $x = (x_1, \dots, x_n)$ is such that $d(x, a) < \delta$

i.e., $\max\limits_{1 \le i \le n} \{ |a_i - x_i| \} < \delta$. Then

$$d'(i(x), i(a)) = d'(x, a) = \sqrt{\sum_{i=1}^{n} (a_i - x_i)^2} < \sqrt{n\delta^2} = \sqrt{\varepsilon^2} = \varepsilon$$

i.e., $\forall \varepsilon > 0 \ \exists \delta > 0 \ni d'(i(x), i(a)) < \varepsilon$ whenever $d(x, a) < \delta$.

Theorem Let $f: X \to Y$ be a function from metric space (X, d) to metric space (Y, d') and let $a \in X$. Then f is continuous at a iff for each sequence $\{x_n\}_{n=1}^{\infty}$ in X converging to a the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$.

Proof ($\Rightarrow$) Let f is cont. at a and $\{x_n\}_{n=1}^{\infty}$ be a sequence in X converging to a.

Let $\varepsilon > 0$ be given

since f is cont. there is $\delta > 0 \ni$ if $x \in X$ and $d(x, a) < \delta$ then $d'(f(x), f(a)) < \varepsilon$. ---(*)

since $\{x_n\}_{n=1}^{\infty}$ converges to a, there is $N \in \mathbb{N}$ such that if $n \geq N$, then $d(x_n, a) < \delta$.

$\Rightarrow d'(f(x_n), f(a)) < \varepsilon$ for $n \geq N$

so, $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$.

($\Leftarrow$) suppose that $f(x_n) \to f(a)$

Let us assume that f is not continuous at a.

$\exists$ no $\varepsilon > 0$ satisfies (*) above

i.e.,) there is $\varepsilon > 0$ s.t if $\delta > 0$ then there is an $x \in X$ such that $d(x, a) < \delta$ but $d'(f(x), f(a)) \geq \varepsilon$ ---(1)

In particular choose $x_n \in X$ for $\delta_n = 1/n$, $n \in \mathbb{N}$

i.e., $d(x_n, a) < 1/n$ but $d'(f(x_n), f(a)) \geq \varepsilon$ from (1)

$\Rightarrow x_n \to a$ in X but $f(x_n)$ doesnot tends to $f(a)$ which is a contradiction of the supposition $f(x_n) \to f(a)$

Hence f is cont. at a.

Theorem Let f be a function from metric space (X, d) to metric space (Y, d'). Then the following statements are equivalent

1. f is continuous.
2. For each sequence $\{x_n\}_{n=1}^{\infty}$ converging to a point a in X, the sequence $\{f(x_n)\}_{n=1}^{\infty}$ converges to $f(a)$.
3. For each open set O in Y, $f^{-1}(O)$ is open in X
4. For each closed set C in Y, $f^{-1}(C)$ is closed in X.

1.5 Equivalence of metric spaces

Def'n Two metric spaces (X, d) and (Y, d') are metrically equivalent or isometric if there exist a fun $f: X \to Y$ such that

1. f is 1-1
2. f is onto
3. for each $a, b \in X$, $d(a, b) = d'(f(a), f(b))$

The function f is called an isometry

Remark (a) The identity fn on any metric space is an isometry.

Proof Let $i: (X, d) \to (X, d)$ be an identity function

Let $x_1 \neq x_2 \Rightarrow i(x_1) \neq i(x_2) \Rightarrow i$ is 1-1

$\forall y \in X \ \exists x = y \in X \ \ni i(x) = i(y) = y$

$\Rightarrow i$ is onto.

Let $a, b \in X$, then $d(a, b) = d(i(a), i(b))$.

$\Rightarrow (X, d)$ is metrically equivalent to itself.

Thus, metric equivalence is reflexive relation

-23-

$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$

$f^{-1}(x_1) \neq f^{-1}(x_2) \Rightarrow x_1 \neq x_2$

b) If $f: X \to Y$ is an isometry from X onto Y, then $f^{-1}: Y \to X$ is an isometry from Y onto X. ~~Thus~~

Pf Let $f: X \to Y$ is an isometry

$y_1 \neq y_2$

$f(x) \quad f(x')$

$\Rightarrow$ f is 1-1 and on to

$\Leftrightarrow$ f^{-1} is 1-1 and on to

And, let $c, e \in Y$. Then $d(f^{-1}(c), f^{-1}(e)) =$

$d'(f(f^{-1}(c)), f(f^{-1}(e)))$

$d'(c, e)$.

$\Rightarrow (Y, d') \,\&\, (X, d')$ are isometric.

Thus, isometric relation is symmetric.

c) The composition of two isometries is an isometry.

Pf Let $f: (X, d) \to (Y, d')$ ~~and~~ and $g: (Y, d') \to (Z, d'')$ be isometric.

Then, $g \circ f$ is 1-1 and on to

and let $a, b \in X$, $d(a,b) = d'(f(a), f(b))$

$= d''(g(f(a)), g(f(b)))$

i.e., $d(a,b) = d''(g(f(a)), g(f(b)))$

$\Rightarrow$ isometric relation is ~~symmetric~~ transitive

Hence metric equivalence is an equivalence relation.

$\forall x \in X \; \exists y \in Y$

$\ni f(x) = y$

$x_1 \neq x_2$ $\quad$ $y_1 \neq y_2$

$f(x_1) \neq f(x_2)$ $\quad$ $f^{-1}(y_1) \neq f^{-1}(y_2)$

$y_1 \neq y_2$

O open in Y ⇒ $f^{-1}(O)$ open in X
O open in X ⇒ $f(O)$ open in Y
⇒ $f^{-1}(O)$ open in X ⇒ $f(f^{-1}(O)) = O$ open in Y

Def'n: metric spaces (X, d) and (Y, d') are topologically equivalent or homeomorphic if there is a one-to-one function $f: X \to Y$ from X onto Y for which f and f^{-1} both continuous. The function f is called a homeomorphism.

Remark: From the above theorem.

(1) $f: X \to Y$ is continuous provided that $f^{-1}(O)$ is open in X for each open subset O of Y.

(2) $f^{-1}: Y \to X$ is continuous provided that $(f^{-1})^{-1}(U) = f(U)$ is open in Y for each open subset U of X.

Thus, a one to one function f from X onto Y is a homeomorphism provided that a subset O of Y is open iff $f^{-1}(O)$ is open in X.

EX: Show that topological equivalence is an equivalence relation.

Remark: Since each isometry is continuous (by def'n of isometry 3rd condition) map, it follows that, if (X, d) and (Y, d') are metrically equivalent, then they must be topological equivalent.

$f: (X, d) \to (Y, d')$
$d(a, b) = d'(f(a), f(b))$
$f^{-1}: (Y, d') \to (X, d)$

$f^{-1}: (Y, d') \to (X, d)$
$d(a, b) = d'(f(a), f(b))$
$d'(c, d) < \delta \Rightarrow$
$d(f^{-1}(c), f^{-1}(d)) < \varepsilon$

eg Consider the metric spaces $X = (0,1)$ and $Y = (0,2)$ with metrices determined by the usual metric on the real line.
Then, $f : X \to Y$ defined by $f(x) = 2x$, $\forall x \in (0,1)$ is a homeomorphism but not an isometry.

Sol'n i) one to oneness

Let $f(x) = f(y) \Rightarrow 2x = 2y \Rightarrow x = y$

$\therefore f$ is 1-1

ii) on to ness

For any $y \in (0,2)$, let $x = \frac{1}{2}y$
Then $x \in (0,1)$ and $f(x) = f(\frac{1}{2}y) = 2(\frac{1}{2}y) = y$
$\Rightarrow f$ is on to

iii) find $f^{-1}(x)$

let $y = f(x) \Rightarrow y = 2x \Rightarrow x = \frac{1}{2}y \Rightarrow y = \frac{1}{2}x$

$\Rightarrow f^{-1}(x) = \frac{1}{2}x$

and f^{-1} is continuous $\forall x \in \mathbb{R}$ since it is a pln
$\Rightarrow f^{-1}$ is continuous on $(0,2)$
and f is cont. $\forall x \in \mathbb{R}$ since it is a pln
$\Rightarrow f$ is cont. on $(0,1)$
$\therefore f$ is homeomorphism.

but for any $a, b \in (0,1)$, $d(a,b) = |a-b| \neq$
$d(f(a), f(b)) = |f(a) - f(b)|$
$= |2a - 2b| = 2|a-b|$

$\therefore f$ is not isometry.

eg Intervals (a,b), $a<b$, and $(0,1)$ are topologically equivalent with the metrics given by usual method of measuring distance in $\mathbb{R}$. This follows the function $f:(0,1)\to(a,b)$ defined by $f(x)=(b-a)x+a$, $x\in(0,1)$ is a homeomorphism

eg The function $f:(-\pi/2,\pi/2)\to\mathbb{R}$ defined by $f(x)=\tan x$, $x\in(-\pi/2,\pi/2)$ is one to one correspondence, continuous and has inverse function arctangent function, which is also continuous. Thus, $(-\pi/2,\pi/2)$ is topologically equivalent to $\mathbb{R}$.

EX Show that unbounded open intervals $(-\infty,a)$ and (a,∞) are topologically equivalent to $\mathbb{R}$.

Remark All open intervals on $\mathbb{R}$ are topologically equivalent to each other and the entire real line since topologically equivalence is an equivalence relation and by the above EX.

EX
Lemma Let d_1 and d_2 be two metrics for the set X and suppose that there is a positive number c such that $d_1(x,y)\le c\,d_2(x,y)$ for all $x,y\in X$. Then the identity function $i:(X,d_2)\to(X,d_1)$ is continuous.

Proof Let $a\in X$ and ε be a positive number.

Then if $\delta=\varepsilon/c$ and $x\in X$ for which $d_2(x,a)<\delta$, then

$$d_1(i(x),i(a))=d_1(x,a)\le c\,d_2(x,a)<c\delta=\varepsilon$$

Thus $d_1(i(x),i(a))<\varepsilon$ whenever $d_2(x,a)<\delta$

$\therefore$ $i:(X,d_2)\to(X,d_1)$ is continuous.

Theorem: Let d_1 and d_2 be two metrics for the set X and suppose there are positive numbers c and c' such that $d_1(x,y) \leq c\, d_2(x,y)$,

$$d_2(x,y) \leq c'\, d_1(x,y)$$

for all $x,y \in X$. Then the identity function on X is a homeomorphism b/n (X, d_1) and (X, d_2)

Proof The identity map is one to one correspondence from X on to itself.

Continuity in both directions is guaranteed by the preceding lemma.

Def^n Metrics d_1 and d_2 for a set X which determine the same topology are called equivalent metrics.

Remark Two metrics d_1 and d_2 on the same set X of points are called equivalent if the identity mapping of (X, d_1) onto (X, d_2) is a homeomorphism.

eg Show that the following set of metrics for the set of n-tuples of real numbers are equivalent.

$$p(x,y) = [(x_1-y_1)^2 + \cdots + (x_n-y_n)^2]^{1/2}$$

$$p^*(x,y) = |x_1-y_1| + \cdots + |x_n-y_n|$$

$$p^+(x,y) = \max\{|x_1-y_1|, \ldots, |x_n-y_n|\}$$

Sol^n

i) Show that $i : (\mathbb{R}^n, p) \to (\mathbb{R}^n, p^*)$ is homeomorphism (i.e., i & i^{-1} are continuous, i is 1-1 and onto)

→ To show i is continuous

$\forall \varepsilon > 0$ we need to find $\delta > 0 \ni p(x,y) < \delta \Rightarrow$

$p^*(i(x), i(y)) < \varepsilon$

$\Rightarrow \rho^*(i(x), i(y)) = \rho^*(x,y)$

$= |x_1-y_1| + \cdots + |x_n-y_n|$

$\le \sqrt{n[(x_1-y_1)^2 + \cdots + (x_n-y_n)^2]}$

$= \sqrt{n}\sqrt{(x_1-y_1)^2 + \cdots + (x_n-y_n)^2}$

$= \sqrt{n}\,\rho(x,y) < \sqrt{n}\,\delta = \varepsilon$, take $\delta = \varepsilon/\sqrt{n}$

Thus i is continuous.

→ To show i^{-1} is continuous

$i^{-1} : (\mathbb{R}^n, \rho^*) \to (\mathbb{R}^n, \rho)$

since i is 1-1 and onto i^{-1} is also 1-1 and onto

$\forall \varepsilon > 0$ we need to find $\delta > 0 \ni \rho^*(x,y) < \delta \Rightarrow \rho(i^{-1}(x), i^{-1}(y)) < \varepsilon$

Take $\rho(i^{-1}(x), i^{-1}(y)) = \rho(x,y) \le \rho^*(x,y) < \delta = \varepsilon$

Thus, i^{-1} is continuous.

$\therefore \rho$ and ρ^* are equivalent.

iii) show that $i : (\mathbb{R}^n, \rho) \to (\mathbb{R}^n, \rho^+)$ is homeomorphism

→ To show $i : (\mathbb{R}^n, \rho) \to (\mathbb{R}^n, \rho^+)$ is continuous

$\forall \varepsilon > 0$ we need to find $\delta > 0 \ni \rho(x,y) < \delta \Rightarrow \rho^+(i(x), i(y)) < \varepsilon$

$\rho^+(i(x), i(y)) = \rho^+(x,y) = \max\{|x_1-y_1|, \ldots, |x_n-y_n|\}$

$\le \rho(x,y) < \delta = \varepsilon$

Thus, i is continuous

To show $i^{-1} : (\mathbb{R}^n, \rho^+) \to (\mathbb{R}^n, \rho)$ is continuous

$\forall \varepsilon > 0$ we need to find $\delta > 0 \ni \rho^+(x,y) < \delta \Rightarrow \rho(i^{-1}(x), i^{-1}(y)) < \varepsilon$

Then, $\rho(i^{-1}(x), i^{-1}(y)) = \rho(x,y) \le \sqrt{n}\,\rho^+(x,y) < \sqrt{n}\,\delta = \varepsilon$

Take $\delta = \varepsilon/\sqrt{n}$

Thus, i & i^{-1} are continuous.

$\therefore \rho$ and ρ^+ are ~~continuous~~ equivalent

Hence, ρ, ρ^* and ρ^+ are equivalent.

1.6 Complete metric spaces

Def. A sequence $\{p_n\}$ in a metric space X is said to converge if there is a point $p \in X$ with the following property: for every $\varepsilon > 0$ there exists a positive integer N such that $n \geq N \Rightarrow d(p_n, p) < \varepsilon$ (where d is the metric in X).
In this case we say that $\{p_n\}$ converges to p or p is the limit of $\{p_n\}$ and write $p_n \to p$ or $\lim_n p_n = p$.

Remark (1) If $\{p_n\}$ does not converge then we say that $\{p_n\}$ diverges.

~~Def.~~ (2) In geometric terms, $\{p_n\}$ converges to $\{p\}$ if every ball about p contains all but a finite number of terms of the sequence.

Def. A sequence $\{p_n\}$ in a metric space (X, d) is called Cauchy sequence if the following condition (called Cauchy condition) is satisfied.
For every $\varepsilon > 0$, there exists a positive integer N such that $n \geq N, m \geq N \Rightarrow d(p_n, p_m) < \varepsilon$.

Theorem In a metric space, every convergent sequence is a Cauchy sequence.

Proof Let $\{p_n\}$ is convergent sequence in a metric space X.
So, there exists $p \in X$ such that $p_n \to p$.
Let $\varepsilon > 0$, so, there exists $N \in \mathbb{N}$ such that
$n \geq N \Rightarrow d(p_n, p) < \varepsilon/2$
Now, $n \geq N, m \geq N \Rightarrow d(p_n, p_m) \leq d(p_n, p) + d(p, p_m)$
$< \varepsilon/2 + \varepsilon/2 = \varepsilon$

Hence $\{p_n\}$ is a Cauchy sequence.

Note: The converse of the above theorem is not true.
i.e., every Cauchy sequence need not be cvt.
For instance, the sequence $\{p_n\}$ where $p_n = \frac{1}{n}$ in $(\mathbb{R},$
converges to 0 (b/c $0 \in \mathbb{R}$).
But every cvt sequence is Cauchy sequence in $\mathbb{R}$], so $\{p_n\}$ is
a Cauchy sequence in $(0,1]$ (b/c $(0,1] \subseteq \mathbb{R}$)
but not cvt in $(0,1]$ (b/c $0 \notin (0,1]$).

Remark: In $\mathbb{R}^n$, every Cauchy sequence is convergent.

Def^n: A metric space (X,d) is called complete if every Cauchy sequence in X converges to a point in X.

eg ① $\mathbb{R}^k$ is complete metric space (show!)
In particular if $k=1$, then $\mathbb{R}$ is complete.

② Closed interval $[a,b]$ is complete.
To show this, consider a Cauchy sequence $\{x_k\}_{k=1}^{\infty}$ in $[a,b]$.
Since $\mathbb{R}$ is complete, this sequence converges to a real number x in $\mathbb{R}$. x belongs to $[a,b]$ since $[a,b]$ is closed.

③ Open intervals, half open intervals and half close intervals are not complete.
eg $\{\frac{1}{n}\}_{n=1}^{\infty}$ is a Cauchy sequence in $(0,1]$ which does not converge to a point of $(0,1]$.

Theorem: Let (X,d) be a complete metric space. A subspace A of X is complete iff A is closed.

Proof (⇒) Suppose that A is complete subspace.
If x is a limit point of A, then by previous thm, there is a sequence of distinct points of A which converge to x. Since each cvt sequence is Cauchy and A is complete, the limit of this sequence, namely x, must be in A. Thus A is closed.
(⇐) Suppose that A is a closed subspace of a complete metric space X.
Consider a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ of points of A. Since X is complete, this sequence converges to a point x belonging to X. By the previous thm, A is closed insures that the limit x belongs to A. Thus every Cauchy sequence of points of A converges to a point of A, and we conclude that A is a complete subspace.

Def'n : A subset A of a metric space X is nowhere dense in X if $\bar{A}$ has empty interior.

eg a) As subsets of the real line $\mathbb{R}$, each of the following is nowhere dense.
i) any finite set
ii) The range of the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$
iii) The set of integers

b) As subsets of the plane, each of the following is nowhere dense.
i) any finite set
ii) The points whose coordinates are integers
iii) a circle

Def'n A set A which fails to be nowhere dense, int $\bar{A} \neq \emptyset$, is called somewhere dense.

a set either finite or countably infinite (equivalent to $\mathbb{Z}^+$; $f: A \to \mathbb{Z}^+$ is 1-1 & onto)

Def'n A metric space or subspace that is the union of a countable family of nowhere dense set is said to be of the first category. A metric space which is not the first category is said to be the second category.

eg ① As a subspace of $\mathbb{R}$, the set $\mathbb{Q}$ of rational numbers is of the first category. Since it is the union of a countable collection of nowhere dense singleton set, each containing one rational numbers.

② The set of points in $\mathbb{R}^n$ having all coordinates rational is also of the first category.

③ The next Baire Category Theorem, shows that every complete metric space is of the second category.

Lemma A subset A of a metric space X is nowhere dense in X iff each non empty open set in X contains an open ball whose closure is disjoint from A.

Theorem (The Bair Category Theorem)
Every complete metric space, considered as a subset of itself, is of the second category.

Def'n Let (X, d) be a metric space and $f: X \to X$ a function. Then f is contractive w.r.t the metric d provided that there is a positive number $\alpha < 1$ such that for all $x, y \in X$, $d(f(x), f(y)) \leq \alpha d(x, y)$

Remark A contractive function is continuous.

Theorem (The contraction Lemma)
Let (X, d) be a complete metric space and $f : X \to X$ a contractive function. Then there is exactly one point x in X for which $f(x) = x$.

Def'n Let (X, d) and (Y, d') be metric spaces. A distance preserving function $f : X \to Y$ from X into Y is called an isometric embedding.

Def'n A subset A of a metric space X is dense or everywhere dense in X provided that $\bar{A} = X$. That is A is dense if every point of X is either a point of A or a limit point of A.

eg The set of rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$ with the usual metric.

since b/n any two real numbers, there are infinitely many rational numbers and every number in $\mathbb{R}$ is either a rational number or a limit point of $\mathbb{Q}$.

Def'n Let (X, d) and (Y, d') be metric spaces and $\{f_n : X \to Y\}$, a sequence of functions from X to Y. This sequence converges uniformly to a function $f : X \to Y$ provided that for each positive number ε there is a positive integer N such that if $n \geq N$ and x is a point of X, then $d'(f_n(x), f(x)) <$

Theorem Let (X,d) and (Y,d') be metric spaces and $\{f_n\}_{n=1}^{\infty}$ a sequence of continuous functions from X to Y which converges uniformly to a function $f: X \to Y$. Then f is continuous.

Proof Let $f_n : X \to Y$

$f : X \to Y$

Let $f_n \to f$ uniformly on X

NTS f is cont.

Let $x_0 \in X$, To show that f is cont at x_0,

Let $\varepsilon > 0$, we need to find $\delta > 0 \ni x \in X \wedge d(x, x_0) < \delta$

$\Rightarrow d'(f(x), f(x_0)) < \varepsilon$.

Since $f_n \to f$ uniformly on X $\exists$ a +ve integer $N \ni$

$d'(f_n(x), f(x)) < \varepsilon/3 \quad (n \geq N, x \in X)$

In particular, $d'(f_N(x), f(x)) < \varepsilon/3$, $x \in X$

Since f_N is cont at x_0 $\exists \delta > 0 \ni x \in X$ and

$d(x, x_0) < \delta \Rightarrow d'(f_N(x), f_N(x_0)) < \varepsilon/3$

now, $d'(f(x), f(x_0)) \leq d'(f(x), f_N(x)) + d'(f_N(x), f(x_0))$

$\leq d'(f(x), f_N(x)) + d'(f_N(x), f_N(x_0)) + d'(f_N(x_0), f(x_0))$

$< \varepsilon/3 + d'(f_N(x), f_N(x_0)) + \varepsilon/3$ (by uniform conv

$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3$, provided $d(x, x_0) < \delta$

$= \varepsilon$

$\therefore f$ is cont at x_0

Q) Let $f_n(x) = \frac{x}{n} e^{-\frac{x}{n}}$ $(0 \le x < \infty)$, with the usual metrics

a) Does $\{f_n\}_{n=1}^{\infty}$ converges uniformly to 0 on $[0,\infty)$?

b) Does $\{f_n\}_{n=1}^{\infty}$ converges uniformly to 0 on $[0, 500]$

Sol:

If $x=0$, $f_n(0)=0 \Rightarrow f_n(0) \to 0$ (converges), $\forall n \in \mathbb{N}$

Let $x > 0$

a) Suppose $f_n \to f = 0$ on $[0,\infty)$

Let $\varepsilon = \frac{1}{2} e^{-1}$

then $\exists N = N(\varepsilon) \in \mathbb{Z}^+$ such that $|f_n(x) - f(x)| < \varepsilon$

$\forall n \ge N$, $x \in [0,\infty)$

$\Rightarrow \frac{x}{n} e^{-\frac{x}{n}} < \varepsilon$, $n \ge N$, $x \in [0,\infty)$

In particular, $\frac{x}{N} e^{-\frac{x}{N}} < \frac{1}{2} e^{-1}$, $x \in [0,\infty)$

Set $x = N$, we get $\frac{N}{N} e^{-N/N} < \frac{1}{2} e^{-1}$

$$\Rightarrow e^{-1} < \frac{1}{2} e^{-1} \to \bot$$

$\therefore$ f_n do not converge uniformly to 0 on $[0,\infty)$

b) Let $\varepsilon > 0$ we need to find a +ve integer N

such that $|f_n(x) - f(x)| < \varepsilon$, $\forall n \ge N$, $x \in [0, 500]$

$$|f_n(x) - f(x)| = \frac{x}{n} e^{-\frac{x}{n}} \le \frac{x}{n} \le \frac{500}{n} < \varepsilon$$

If $\frac{500}{n} < \varepsilon$, then $n > \frac{500}{\varepsilon}$

Choose $N \ni N > \frac{500}{\varepsilon}$