Preface

Mathematical modelling is a bridge between the study of mathematics and the application of mathematics to various fields. The module affords the student an early opportunity to see how the pieces of an applied problem fit together. The student investigates meaningful and practical problems chosen from common experiences encompassing many academic disciplines, including the mathematical sciences, operation research, engineering, and the management and life science.

This module provides an introduction to the entire modelling process. The student will have occasions to practice the module facets of modelling and enhance their problem solving capabilities. The aim of this module is to display by examples some of the many facets of mathematical modelling.

This module needs a student with a good knowledge of calculus, ordinary differential equation and a little probability and matrix theory would find all of it accessible.

The module are organized according to a definite point of view, the first chapter focused on introduction of modelling and how to relate mathematical model with other models. Although in this part we try to show steps of modelling and properties of modelling.

The second and third chapter tries to explain about dimensional analysis and modelling using graphical methods respectively.

The last two chapters focused on applications. In this chapter we try to show examples of mathematical modelling related to differential equations and optimization.

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CHAPTER 1

Introduction to Modeling

Learning objectives

After studying this chapter students will be able to:

- Identify properties of Models & its relation with reality.
- Apply the mathematical modeling cycle to simple problems
- Understand some of the assumptions used when modeling.
- Formulate a Mathematical model of real life problems.

Introduction

To help us better understand our world, we often describe a particular phenomenon mathematically (by means of a function or an equation, for instance). Such description is an idealization (model) of the real-world phenomenon and never a completely accurate representation. Although any model has its limitations, a good one can provide valuable results and conclusions.

In modeling our world, we are often interested in predicting the value of a variable at some time in the future. Perhaps it is a population, a real estate value, or the number of people with a communicative disease. Often a mathematical model can help us understand a behavior better or aid us in planning for the future. Let’s think of a mathematical model as a mathematical construct designed to study a particular real-world system or behavior of interest. The model allows us to reach mathematical conclusions about the behavior. These conclusions can be interpreted to help a decision maker plan for the future.

Simplification

Most models simplify reality. Generally, models can only approximate real-world behavior. One very powerful simplifying relationship is proportionality.

DEFINITION:

Two variables y and x are proportional (to each other) if one is always a constant multiple of the other; that is, if $y = kx$ for some non zero constant *k*, we write $y \propto x.$

The definition means that the graph of y versus x lies along a straight line through the origin. This graphical observation is useful in testing whether a given data collection reasonably assumes a proportionality relationship. If proportionality is reasonable, a plot of one variable against the other should approximate a straight line through the origin. Here is an example.

EXAMPLE: ***Testing for Proportionality***

Consider a spring-mass system, such as the one shown in Figure 1.1. We conduct an experiment to measure the stretch of the spring as a function of the mass (measured as weight) placed on the spring. Consider the data collected for this experiment, displayed in Table 1.1. A scatter plot graph of the stretch or elongation of the spring versus the mass or weight placed on it reveals an approximate straight line passing through the origin, (Figure 1.2).

The data appear to follow the proportionality rule that elongation *e* is proportional to the mass *m*, or symbolically, $e \propto m$. The straight line appears to pass through the origin. This geometric understanding allows us to look at the data to determine if proportionality is a reasonable simplifying assumption and, if so, to estimate the slope *k*. In this case, the assumption appears valid, so we estimate the constant of proportionality by picking the two points (200, 3.25) and (300, 4.875) as lying along the straight line. We calculate the slope of the line joining these points as

$$slope = \frac{4.875 - 3.25}{300 - 200} = 0.01625$$

Table 1.1
Spring–mass system

Mass	Elong
50	1.000
100	1.875
150	2.750
200	3.250
250	4.375
300	4.875
350	5.675
400	6.500
450	7.250
500	8.000
550	8.750

Figure 1.1
Spring-mass system

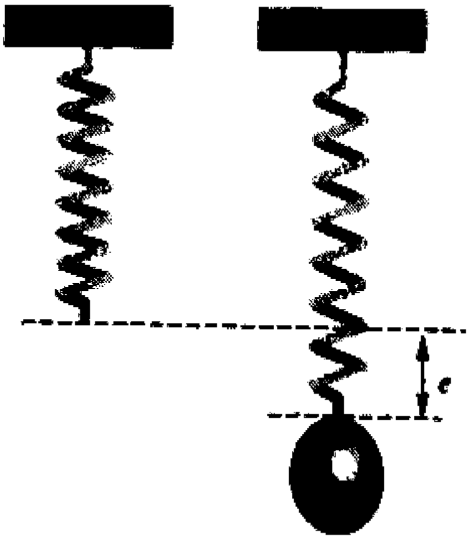
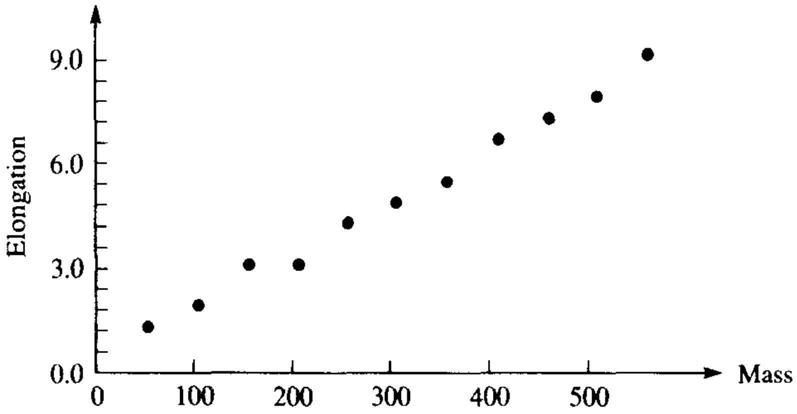


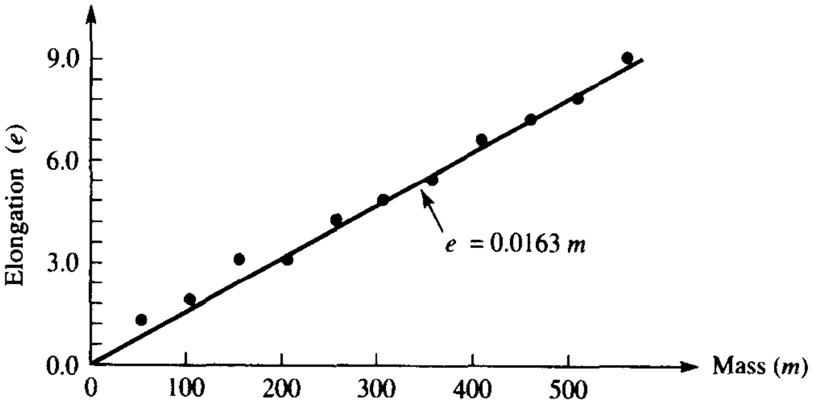
Figure 1.2
Data from spring-mass system



Thus the constant of proportionality is approximately 0.0163 and we estimate our model as:
e=0163m

We then examine how close our model fits the data by plotting the line it represents superimposed on the scatter plot (Figure 1.3). The graph reveals that the simplifying proportionality model is reasonable.

Figure 1.3
Data from spring-mass system with proportionality line



Modeling Change

A powerful paradigm to use in modeling change is

future value = present value + change

Often, we wish to predict the future on what we know now, in the present, and add the change that has been carefully observed. In such cases, we begin by studying the change itself according to the formula

change = future value — present value

By collecting data over a period of time and pot1ing that data, we often can discern patterns to model that capture the trend of the change. If the behavior is taking place over *discrete time periods,* the preceding construct leads to a difference equation, which we study in this chapter. If the behavior is taking place *continuously* with respect to time, then the construct leads to a differential equation. Both are powerful methodologies for studying change to explain and predict behavior.

Modeling Change with Difference Equations

In this section we build mathematical models to describe change in an observed behavior. When we observe change, we are often interested in understanding why the change occurs in the way it does, perhaps to analyze the effects of different conditions on the behavior or to predict what will happen in the future. A mathematical model helps us better understand a behavior while allowing us to experiment mathematically with different conditions affecting it.

DEFINITION: For a sequence of numbers $A = \{a_0, a_1, a_2, a_3, \ldots\}$ the first differences are

$$\Delta a_0 = a_1 - a_0$$

$$\Delta a_1 = a_2 - a_1$$

$$\Delta a_2 = a_3 - a_2$$

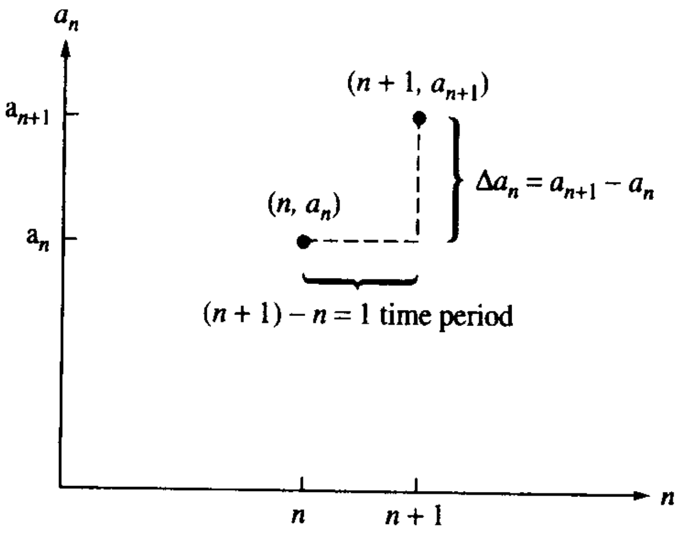
$$\Delta a_3 = a_4 - a_3$$

For each positive integer *n,* the nth first difference is

$$\Delta a_n = a_{n+1} - a_n$$

Note from Figure 1.4 that the first difference represents the rise or fall between consecutive values of the sequence; that is, the vertical *change* in the graph of the sequence during one time period.

Figure 1.4
The first difference of a sequence is the rise in the graph during one time period



Example: *A savings Certificate*

Consider the value of a savings certificate initially worth \$1000 that accumulates interest paid each month at 1% per month. The following sequence of numbers represents the value of the certificate month by month:

$$A = (1000, 1010, 1020.10, 1030.30, \ldots)$$

The first differences are as follows:

$$\Delta a_0 = a_1 - a_0 = 1010 - 1000 = 10$$

$$\Delta a_1 = a_2 - a_1 = 1020.10 - 1010 = 10.10$$

$$\Delta a_2 = a_3 - a_2 = 1030.30 - 1020.10 = 10.20$$

Note that the first differences represent the *change in the sequence* during one time period, or the *interest earned* in the case of the savings certificate example.

The first difference is useful for modeling change taking place in discrete intervals. In this example, the change in the value of the certificate from one month to the next is merely the interest paid during that month. If *n* is the number of months and $\mathbf{a_n}$, the value of the certificate after *n* months, then the change or interest growth in each month is represented by the n^th difference

$$\Delta a_n = a_{n+1} - a_n = 0.01a_n$$

This expression can be rewritten as the difference equation

$$a_{n+1} = a_n + 0.01a_n$$

We also know the initial deposit (initial value) that then gives the dynamical system model

$$\begin{aligned} a_{n+1} &= 1.01a_n, \quad n = 0, 1, 2, 3, .. \\ a_0 &= 1000 \end{aligned} \tag{1.1}$$

where an , represents the amount accrued after *n* months. Because *n* represents the nonnegative integers (0, I, 2, 3 . . .), Equation (1.1) represents an *Infinite set* of algebraic equations, called a dynamical system. Dynamical systems allow us to describe the *change* from one period to the next. The difference equation formula computes the next term knowing the immediately previous term in the sequence, but it does not compute the value of a specific term directly (e.g., the savings after 100 periods).

Because it is change we often observe, we can construct a difference equation by representing or approximating the change from one period to the next. To modify our example, if we were to withdraw $50 from the account each month, the change during a period would be the interest earned during that period minus the monthly withdrawal, or

$$\Delta a_n = a_{n+1} - a_n = 0.01a_n - 50$$

In most examples, mathematically describing the change is no going to be as precise a procedure as illustrated here. Often it is necessary to *plot the change, observe a pattern,* and then *describe the change* in mathematical terms. Thai is, we will be trying to find

$$\textit{change} = \Delta a_n = \textit{some function } f$$

The change may be a function of previous terms in the sequence (as was the case with no monthly withdrawals), or it may also involve some external terms (such as the amount of money withdrawn in the current example or an expression involving the period *n).* Thus, in constructing models representing change in this chapter we will be modeling change in discrete intervals, where

$$\textit{change} = \Delta a_n = a_{n+1} - a_n = f(\textit{terms in the sequence, external terms})$$

Modeling change in this way becomes the art of determining or approximating a function *f* that represents the change.

Consider a second example in which a difference equation exactly models a behavior in the real world.

Examples of problems where modeling could be used:

- to determine the maximum speed of a car round a bend,
- to help define the design requirements of a sports stadium,
- to evaluate new design options for a mountain bike,
- to work out how to send a space station into orbit.

etc.

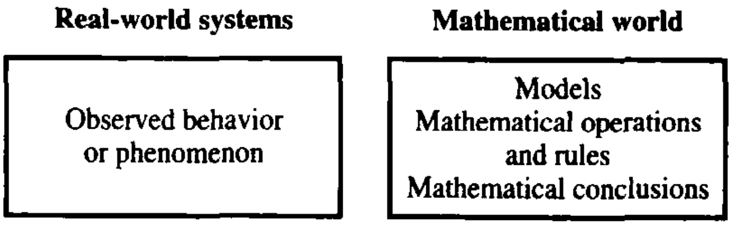
1.1 Models and reality

The theoretical and scientific study of a situation centers around a model, that is, something that mimics (imitate) relevant features of the situation being studied. For example, a road map, a geological map, and a plant collection are all models that mimic different aspects of a portion of the earth's surface.

The ultimate test of a model is how well it performs when it is applied to the problems it was designed to handle.

Now we examine more closely the process of mathematical modeling. To gain an understanding of the processes involved in mathematical modeling, consider the two worlds depicted in Figure 1.5. Suppose we want to understand some behavior or phenomenon in the real world. We may wish to make predictions about that behavior in the future and analyze the effects various situations have on it.

Figure 1.5
The real and mathematical worlds



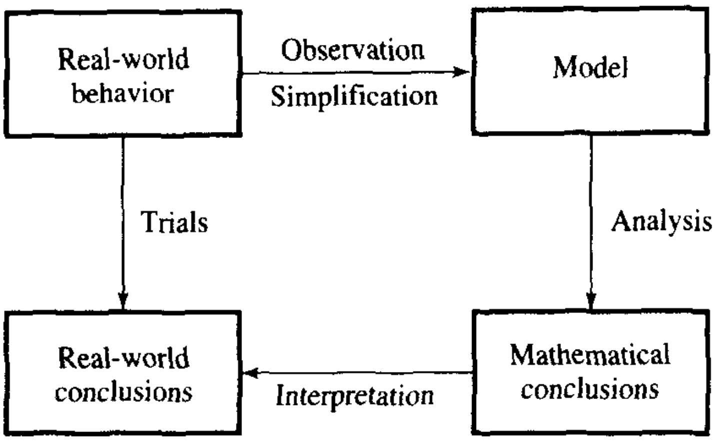
For example, when studying the populations of two interacting species, we may wish to know if the species can coexist within their environment or if one species will eventually dominate and drive the other to extinction. In the case of the administration of a drug to a person, it is important to know the correct dosage and the time between doses to maintain a safe and effective level of the drug in the bloodstream.

How can we construct and use models in the mathematical world to help us better understand real-world systems? Before discussing how we link the two worlds together, let's consider what we mean by a real-world system and why we would be interested in constructing a mathematical model for a system in the first place.

A system is an assemblage of objects joined in some regular interaction or interdependence. The modeler is interested in understanding how a particular system works, what causes changes in the system, and how sensitive the system is to certain changes. He or she is also interested in predicting what changes might occur and when they occur. How might such information be obtained?

For instance, suppose the goal is to draw conclusions about an observed phenomenon in the real world. One procedure would be to conduct some real-world behavior trials or experiments and observe their effect on the real-world behavior.

Figure .1.6
Reaching conclusions about the behavior of real-world systems



This is depicted on the left side of Figure 1.6. Although such a procedure might minimize the loss in fidelity incurred by a less direct approach, there are many situations in which we would not want to follow such a course of action. For instance, there may be prohibitive costs for conducting even a single experiment, such as determining the level of concentration at which a drug proves to be fatal or studying the radiation effects of a failure in a nuclear power plant near a major population area. Or we may not be willing to accept even a single experimental failure, such as when investigating different designs for a heat shield for a manned spacecraft. Moreover, it may not even be possible to produce a trial, as in the case of investigating specific change in the composition of the ionosphere and its corresponding effect on the polar ice cap. Furthermore, we may be interested in generalizing the conclusions beyond the specific conditions set by one trial (such as a cloudy day in Addis Ababa with temperature 82°F, wind 15—20 miles per hour, humidity 42k. and so on). Finally, even though we succeed in predicting the real-world behavior under some very specific conditions, we have not necessarily explained why the particular behavior occurred. (Although the abilities to predict and explain are often closely related, the ability to predict a behavior does not necessarily imply an understanding of it). The preceding discussion underscores the need to develop indirect methods for studying real-world systems.

An examination of Figure 1.6 suggests an alternative way of reaching conclusions about the real world. First, we make specific observations about the behavior being studied and identify the factors that seem to be involved. Usually we cannot consider, or even identify, all the factors involved in the behavior, so we make simplifying assumptions that eliminate some factors. For instance, we may choose to neglect the humidity in Addis Ababa, at least initially, when studying radioactive effects from the failure of a nuclear power plant. Next, we conjecture tentative relationships among the factors we have selected, thereby creating a rough model of the behavior. Having constructed a model, we then apply appropriate mathematical analysis leading to conclusions about the model. Note that these conclusions pertain only to the model, not to the actual real-world system under investigation. Because we made some simplifications in constructing the model and the observations on which the model is based invariably contain errors and limitations, we must carefully account for these anomalies before drawing any inferences about the real-world behavior. In summary, we have the following rough modeling procedure:

1. Through observation, identify the primary factors involved in the real- world behavior, possibly making simplifications.
2. Conjecture tentative relationships among the factors.
3. Apply mathematical analysis to the resultant model.
4. Interpret mathematical conclusions in terms of the real-world problem.

Given some real-world system, we gather sufficient data to formulate a model. Next we analyze the model and reach mathematical conclusions about it. Then we interpret the model and make predictions or offer explanations. Finally, we test our conclusions about the real- world system against new observations and data. We may then find we need to go back and refine the model to improve its predictive or descriptive capabilities. Or perhaps we will discover that the model really does not fit the real world accurately. so we must formulate a new model. We will study the various components of this modeling process in detail throughout the module.

Mathematical Models

Definition.

A mathematical model is a mathematical construct designed to study a particular real-world system or phenomenon. We include graphical, symbolic, simulation, and experimental constructs. Many real problems can be very complex and so the idea of creating a mathematical model is to simplify the real situation, so that it can be described using equations or graphs. These equations or graphs are referred to as a mathematical model. These mathematical models can provide solutions to the original problem. It is often necessary to interpret these answers in the context of the original problem and to check that the answers that you have obtained are reasonable.

Mathematical models can be differentiated further. There are existing mathematical models that can be identified with some particular real-world phenomenon and used to study it. Then there are those mathematical models that we construct specifically to study a special phenomenon. Starting from some real-world phenomenon, we can represent it mathematically by constructing a new model or selecting an existing model. On the other hand, we can replicate the phenomenon experimentally or with some kind of simulation.

The real world refers to

- engineering
- physics
- physiology
- ecology

- wildlife management
- chemistry
- economics
- sports

…

Regarding the question of constructing a mathematical model, a variety of conditions can cause us to abandon hope of achieving any success. The mathematics involved may be so complex and intractable that there is little hope of analyzing or solving the model, thereby defeating its utility.

This complexity can occur, for example, when attempting to use a model given by a system of partial differential equations or a system of nonlinear algebraic equations. Or the problem may be so large (in terms of the number of factors involved) that it is impossible to capture all the necessary information in a single mathematical model. Predicting the global effects of the interactions of a population, the use of resources, and pollution is an example of such an impossible situation. In such cases we may attempt to replicate the behavior directly by conducting various experimental trials. Then we collect data from these trials and analyze the data in some way. Possibly using statistical techniques or curve-fitting procedures. From the analysis, we can reach certain conclusions.

There may be distinction between the various model types. For example, the distinction between experiments and simulations is based on whether the observations are obtained directly (experiments) or indirectly (simulations). In practical models this distinction is not nearly so sharp, one master model may employ several models as sub models, including selections from existing models, simulations, and experiments. Nevertheless, it is informative to contrast these types of models and compare their various capabilities for portraying the real world.

1.2 Properties of models

To that end, consider the following properties of a model:

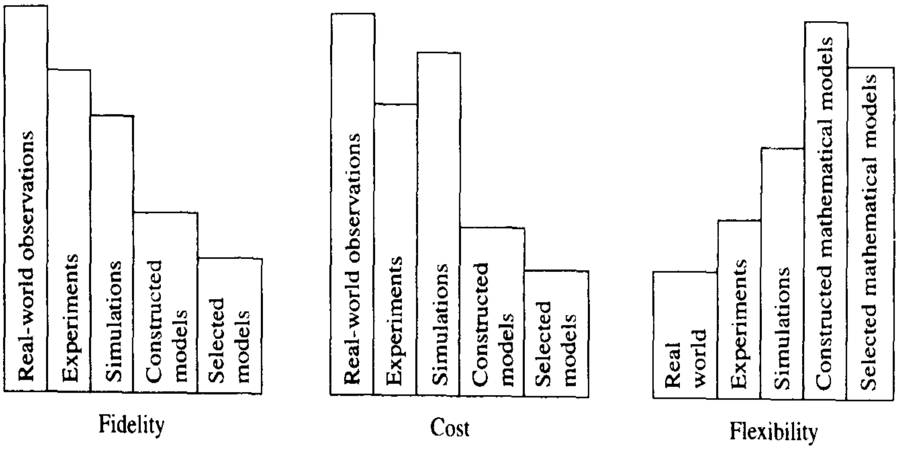
Fidelity: The preciseness of a model's representation of reality

Costs: The total cost of the modeling process

Flexibility: The ability to change and control conditions affecting the model as required data are gathered.

It is useful to know the degree to which a given model possesses each of these characteristics. However, since specific models vary greatly, the best we can hope for is a comparison of the relative performance between the classes of models for each of the characteristics. The comparisons are depicted in Figure 1.7, where the ordinate axis denotes the degree of effectiveness of each class.

Figure 1.7
Comparisons among the model types



Let's summarize the results shown in Figure 1.7. First, consider the characteristic of fidelity. We would expect observations made directly in the real world to demonstrate the greatest fidelity, even though some testing bias and measurement error may be present. We would expect experimental models to show the next greatest fidelity because behavior is observed directly in a more controlled environment such as a laboratory. Because simulations incorporate indirect observations, they suffer a further loss in fidelity. Whenever a mathematical model is constructed, real-world conditions are simplified, resulting in more loss of fidelity. Finally, any selected model is based on additional simplifications that are no even tailored to the specific problem, and these simplifications imply still further loss in fidelity.

Next, consider cost. Generally, we would expect any selected mathematical model to be the least expensive. Constructed mathematical models bear an additional cost of tailoring the simplifications to the phenomenon being studied. Experiments are usually expensive to set up and operate. Likewise, simulations use indirect devices that are often expensive to develop, and simulations commonly involve large amounts of computer space, time, and maintenance.

Finally, consider flexibility. Constructed mathematical models are generally the most flexible because different assumptions and conditions can be chosen relatively easily. Selected models are less flexible because they are developed under specific assumptions; nevertheless, specific conditions can often be varied over wide ranges. Simulations usually entail the development of some other indirect device to alter assumptions and conditions appreciably. Experiments are even less flexible because some factors are very difficult to control beyond specific ranges. Observations of real-word behavior have little flexibility because the observer is limited to the specific conditions that pertain at the time of the observation. Moreover, other conditions might be highly improbable, or impossible, to create. It is important to understand that our discussion is only qualitative in nature, and that there are many exceptions to these generalizations.

1.3 Building a model

Model building involves imagination and skill. It is like listing rules and provides a framework around which to build skill and develop imagination.

1.3.1 Constriction of Models

In the preceding discussion we viewed modeling as a process and considered briefly the form of the model. Now let's focus attention on the construction of mathematical models. We begin by presenting an outline of a procedure that is helpful in constructing models. In the next section, we illustrate the various steps in the procedure by discussing several real-world examples.

STEP 1. **Identify the problem**. What is the problem you would like to explore? Typically this is a difficult step because in real-life situations no one simply hands you a mathematical problem to solve. Usually you have to sort through large amount of data and identify some particular aspect of the situation to study. Moreover, it is imperative to be sufficiently precise (ultimately) in the formulation of the problem to allow for translation of the verbal statements describing the problem into mathematical symbology. This translation is accomplished through the next steps. It is important to realize that the answer to the question posed might not lead directly to a usable problem identification.

STEP 2. **Make assumptions.** Generally, we cannot hope to capture in a usable mathematical model all the factors influencing the identified problem. The task is simplified by reducing the number of factors under consideration. Then, relationships among the remaining variables must be determined. Again, by assuming relatively simple relationships, we can reduce the complexity of the problem. Thus the assumptions fall into two main activities:
a. Classify the variables: What things influence the behavior of the problem identified in Step 1? List these things as variables. The variables the model seeks to explain are the dependent variables and there may be several of these. The remaining variables are the independent variables. Each variable is classified as dependent, independent, or neither.

You may choose to neglect some of the independent variables for either of two reasons. First, the effect of the variable may be relatively small compared to other factors involved in the behavior. Second, a factor that affects the various alternatives in about the same way may be neglected, even though it may have a very important influence on the behavior under investigation. For example, consider the problem of determining the optimal shape for a lecture hall, where readability of a chalkboard or overhead projection is a dominant criterion. Lighting is certainly a crucial factor, but it would affect all possible shapes in about the same way. By neglecting such a variable, possibly incorporating it later in a separate, more refined model, the analysis can be simplified considerably.

b. Determine interrelationships among the variables selected for Study: Before we can hypothesize a relationship between the variables, we generally must make sonic additional simplifications. The problem may be sufficiently complex so that we cannot see a relationship among all the variables initially. In such cases it may be possible to study sub models. That is we study one or more of the independent variables separately. Eventually we will connect the Sub models together. Studying various techniques, such as proportionality, will aid in hypothesizing relationships among the variables.

STEP 3. **Solve or interpret the model**. Now put together all the sub models to see what the model is telling us. In some cases the model may consist of mathematical equations or inequalities that must be solved to find the information we are seeking. Often, a problem statement requires a best or optimal solution to the model. Models of this type are discussed latter.

Often, we will find that we are not quite ready to complete this step or we may end up with a model so unwieldy we cannot solve or interpret it. In such situations we might return to Step 2 and make additional simplifying assumptions. Sometimes we will even want to return to Step 1 to redefine the problem. This point will be amplified in the following discussion.

STEP 4. **Verify the model**. Before we can use the model, we must test it out. There are several questions to ask before designing these tests and collecting data—a process that can be expensive and time-consuming. First, does the model answer the problem identified in Step 1, or did it stray from the key issue as we constructed the model? Second, is the model usable in a practical sense; that is, can we really gather the data necessary to operate the model? Third, does the model make common sense?

Once the commonsense tests are passed, we will want to test many models using actual data obtained from empirical observations. We need to be careful to design the test in such a way as to include observations over the same range of values of the various independent variables we expect to encounter when actually using the model. The assumptions made in Step 2 may be reasonable over a restricted range of the independent variables but very poor outside of those values. For instance, a frequently used interpretation of Newton's second law states that the net force acting on a body is equal to the mass of the body times its acceleration. This law is a reasonable model until the speed of the object approaches the speed of light.

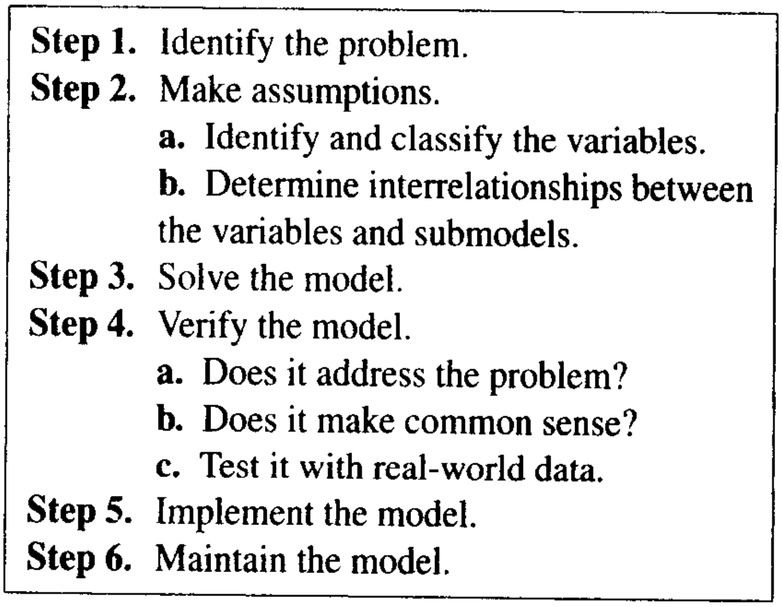
Be careful about the conclusions you draw from any tests. Just as we cannot prove a theorem simply by demonstrating many cases that support the theorem, likewise, we cannot extrapolate broad generalizations from the particular evidence we gather about our model. A model does not become a law just because it is verified repeatedly in some specific instances. Rather, we *corroborate the reasonableness* of our model through the data we collect.

STEP 5. **Implement the model**. Of course, our model is of no use just sitting in a filing cabinet. We will want to explain our model in terms that the decision makers and users can understand if it is ever to be of use to anyone. Furthermore, unless the model is placed in a user-friendly mode, it will quickly fall into disuse. Expensive computer programs sometimes suffer such a demise. Often the inclusion of an additional step to facilitate the collection and input of the data necessary to operate the model determines its success or failure.

STEP 6. **Maintain the model**. Remember that the model is derived from a specific problem identified in Step 1 and from the assumptions made in Step 2. Has the original problem changed in any way, or have some previously neglected factors become important? Does one of the sub models need to be adjusted?

We summarize the steps for constructing mathematical models in Figure 1.8. We should no be too enamored with our work. Like any model, our procedure is an approximation process and therefore has its limitations. For example, the procedure seems to consist of discrete steps leading to a usable result, but that is rarely the case in practice. Before offering an alternative procedure that emphasizes the iterative nature of the modeling process. Let us discuss the advantages of the methodology depicted in Figure 1.8.

Figure 1.8
Construction of a mathematical model



Step 1. Identify the problem.
Step 2. Make assumptions.
  a. Identify and classify the variables.
  b. Determine interrelationships between the variables and submodels.
Step 3. Solve the model.
Step 4. Verify the model.
  a. Does it address the problem?
  b. Does it make common sense?
  c. Test it with real-world data.
Step 5. Implement the model.
Step 6. Maintain the model.

The process shown in Figure 1.8 provides a methodology for progressively focusing on those aspects of the problem we wish to study. Furthermore, it demonstrates a curious blend of creativity with the scientific method used in the modeling process. The first two steps are more artistic or original in nature. They involve abstracting the essential features of the problem under study, neglecting any factor judged to be unimportant and postulating relationships precise enough to help answer the questions posed by the problem. However, these relationships must be simple enough to permit the completion of the remaining steps. Although these steps admittedly involve a degree of craftsmanship, we will learn some scientific techniques we can apply to appraise the importance of a particular variable and the preciseness of an assumed relationship. Nevertheless, when generating numbers in Steps 3 and 4, remember that the process has been largely inexact and intuitive.

EXAMPLE 1. *Vehicular Stopping Distance*

Scenario Consider the following rule often given in driver education classes: Allow one car length for every 10 miles of speed under normal driving conditions, but more distance in adverse weather or road conditions. One way to accomplish this is to use the 2-second rule for measuring the correct following distance no matter what your speed. To obtain that distance, watch the vehicle ahead of you pass some definite point on the highway, like a tar strip or overpass shadow. Then count to yourself “one thousand and one, one thousand and two;” that is 2 seconds. If you reach the mark before you finish saying those words, then you are following too close behind.

The preceding rule is implemented easily enough, but how good is it?

Problem Identification Our ultimate goal is to test this rule and suggest another rule if it fails. However, the statement of the problem, How good is the rule? is vague. We need to be more specific and spell out a problem, or ask a question. Whose solution or answer will help us accomplish our goal while permitting a more exact mathematical analysis? Consider the following problem statement:

Predict the vehicle’s total stopping distance as a function of its speed.

Assumptions We begin our analysis with a rather obvious model for total stopping distance:

total stopping distance = reaction distance + braking distance

By reaction distance, we mean the distance the vehicle travels from the instant the driver perceives a need to stop to the instant when the brakes are actually applied. Braking distance is the distance required for the brakes to bring the vehicle to a complete stop.

First let’s develop a sub model for reaction distance. The reaction distance is a function of many variables, and we start by listing just two of them:

reaction distance = f (response time, speed)

We could continue developing the sub model with as much detail as we like. For instance, response time is influenced by both individual driving factors and the vehicle operating system. System time is the time from which the driver touches the brake pedal until the brakes are mechanically applied. For modern cars we would probably neglect the influence of the system because it is quite small in comparison to the human factors. The portion of the response time determined by the driver depends on many things, such as reflexes, alertness, and visibility. Because we are developing only a general rule, we could just incorporate average values and conditions for these latter variables. Once all the variables deemed important to the sub model have been identified, we can begin to determine interrelationships among them. We suggest a sub model for reaction distance in the next section.

Next consider the braking distance. The weight and speed of the vehicle are certainly important factors to be taken into account. The efficiency of the brakes, type and condition of the tires, road surface, and weather conditions are other legitimate factors. As before, we would must likely assume average values and conditions for these latter factors. Thus, our initial sub models give braking distance as a function of vehicular weight and speed:

braking distance = h(weight, speed)

In the next section we also suggest and analyze a sub model for braking distance.

Finally, let’s discuss briefly the last three steps in the modeling process for this problem. We would want to test our model against real-world data. Do the predictions afforded by the model agree with real driving situations? If not, we would want to assess some of our assumptions and perhaps restructure one (or both) of our sub models. If the model does predict real driving situations accurately, then does the rule stated in the opening discussion agree with the model? The answer gives an objective basis for answering, how good is the rule? Whatever rule we come up with (to implement the model), it must be easy to understand and easy to use if it is going to be effective. In this example, maintenance of the model does not seem to be a particular issue. Nevertheless, we would want to be sensitive to the effects on the model of such changes as power brakes or disc brakes, a fundamental change in tire design, and so on.

Let’s contrast the modeling process presented in Figure 1.8 with the scientific method. One version of the scientific method is as follows:

STEP 1. Make some general observations of a phenomenon.
STEP 2. Formulate a hypothesis about the phenomenon.
STEP 3. Develop a method to test that hypothesis.
STEP 4. Gather data to use in the test.
STEP 5 . Test the hypothesis using the data.
STEP 6. Confirm or deny the hypothesis.

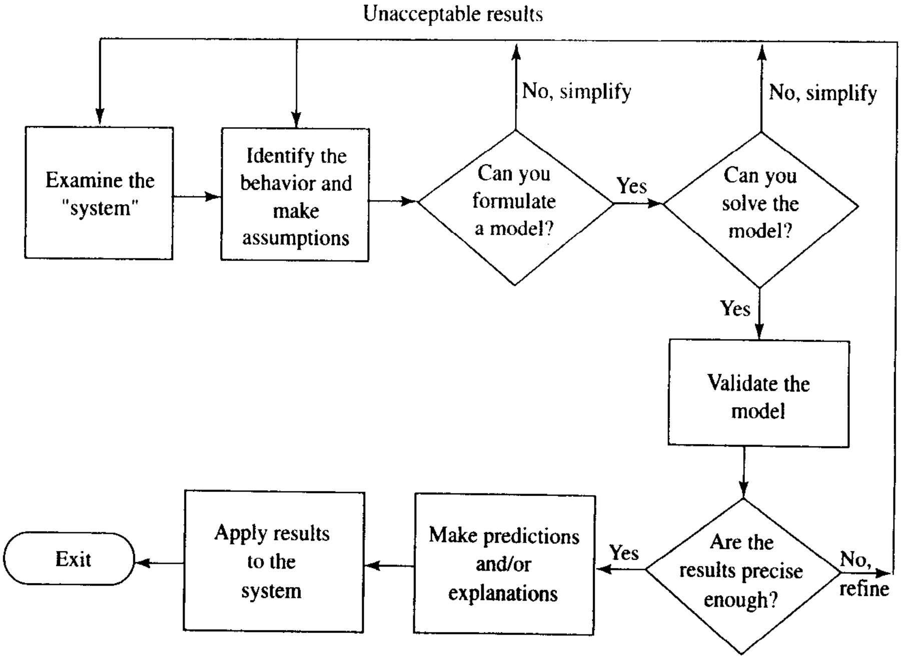
By design, the mathematical modeling process and scientific method have similarities. For instance, both processes involve making assumptions or hypotheses, gathering real-world data, and testing or verification using that data. These similarities should not be surprising; though recognizing that part of the modeling process is an art, we do attempt to be scientific and objective whenever possible.

There are also subtle differences between the two processes. One difference lies in the primary goal of the two processes. In the modeling process, assumptions are made in selecting which variables to include or neglect and postulating the interrelationships among the included variables. The goal in the modeling process is to hypothesize a model, and as with the scientific method, evidence is gathered to corroborate that model. Unlike the scientific method, however, the objective is not to confirm or deny the model (we already know it is not precisely correct because of the simplifying assumptions we have made) but rather to test its reasonableness. We may decide that the model is quite satisfactory and useful, and elect to accept it. Or we may decide that the model needs to be refined or simplified. In extreme cases we may even need to redefine the problem, in a sense rejecting the model altogether. We will see in subsequent chapters that this decision process really constitutes the heart of mathematical modeling.

1.3.2 Iterative Nature of Model Construction

Model construction is an iterative process. We begin by examining some system and identifying the particular behavior we wish to predict or explain. Next we identify the variables and simplifying assumptions, and then we

Figure 1.9
The iterative nature of model construction



generate a mode. We will generally start with a rather simple model, progress through the modeling process, and then refine the model as the results of our validation procedure dictate. If we cannot come up with a model or solve the one we have, we must simplify it (Figure 1.9). This is done by treating some variables as constants, by neglecting or aggregating some variables, by assuming simple relationships (such as linearity) in any sub model, or by further restricting the problem under investigation. On the other hand, if our results are not precise enough, we must refine the model (Figure 1.9).

Refinement is generally achieved in the opposite way to simplification we introduce additional variables, assume more sophisticated relationships among the variables, or expand the scope of the problem. By simplification and refinement, we determine the generality, realism, and precision of our model. This process cannot be overemphasized and constitutes the art of modeling. These ideas are summarized in Table 1.2.

Table 1.2 The art of mathematical modeling: simplifying or refining the model as required

Model simplification	Model refinement
1. Restrict problem identification.	I. Expand the problem.
2. Neglect variables.	2. Consider additional variables.
3. Conglomerate effects of several variables.	3. Consider each variable in detail.
4. Set some variables to be constant.	4. Allow variation in the variables.
5. Assume simple (linear) relationships.	5. Consider nonlinear relationships.
6. Incorporate more assumptions.	6. Reduce the number of assumptions

We complete the section by introducing several terms that are useful in describing models. A model is said to be **robust** when its conclusions do not depend on the precise satisfaction of the assumptions. A model is **fragile** if its conclusions do depend on the precise satisfaction of some sort of conditions. The term **sensitivity** refers to the degree of change in a models conclusions as some condition on which they depend is varied; the greater the change, the more sensitive is the model to that condition.

Problems

In Problems 1 - 4, the scenarios are vaguely stated. From these vague scenarios, identify a problem you would like to study Which variables affect the behavior you have identified in the problem identification? Which variables are the most important? Remember there are really no right answers.

1. The population growth of a single species.
2. A retail store intends to construct a new parking lot. How should the lot be illuminated?
3. How would you design a lecture hail for a large class?
4. How should a manufacturer of some product decide how many units of that product

should be manufactured each year and how much to charge for each Unit?

Projects

1. Consider the taste of brewed coffee. What are some of the variables affecting taste? Which variables might be neglected initially? Suppose you hold all variables fixed except water temperature. Most coffeepots use boiled water in some manner to extract the flavor from the ground coffee. Do you think boiled water is optimal for producing the best flavor? How would you test this sub model? What data would you collect and how would you gather it?
2. A transportation company is considering transporting people between skyscrapers in New York City via helicopter. You are hired as a consultant to determine the number of helicopters needed. Identify an appropriate problem precisely. Use the model-building process to identify the data you would like to have to determine the relationships between the variables you select. You may want to redefine your problem as you proceed.

1.4 Modeling Using Proportionality

We introduced the concept of proportionality in the introduction part . Recall that:

$$y \propto x \text{ if and only if } y = kx \text{ for some constant } k > 0 \tag{1.2}$$

Of course, if $y \propto x$, then $x \propto y$ because the constant k in Equation (1.2) is greater than zero and then $x = \left(\frac{1}{k}\right)y$. The following are other examples of proportionality relationships:

$$y \propto x^2 \text{ if and only if } y = k_1x^2 \text{ for } k_1 \text{ a constant} \tag{1.3}$$

$$y \propto \ln x \text{ if and only if } y = k_2 \ln x \text{ for } k_2 \text{ a constant} \tag{1.4}$$

$$y \propto e^x \text{ if and only if } y = k_3e^x \text{ for } k_3 \text{ a constant} \tag{1.5}$$

In Equation (1.3), $y = kx^2, k > 0$, so we also have $x \propto y^{1/2}$ because $x = \left(\frac{1}{\sqrt{k}}\right) y^{\frac{1}{2}}$. This leads us to consider how to link proportionalities together, a transitive rule for proportionality:

$$y \propto x \quad \text{and} \quad x \propto z, \text{ then} \quad y \propto z$$

Thus, any variables proportional to the same variables are proportional to one another.

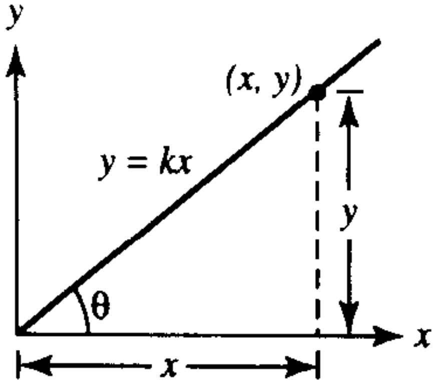


Figure 1.10
Geometrical interpretation of $y \propto x$

Now let's explore a geometric interpretation of proportionality. In Equation (1.2) , $y = kx \text{ yields } k = \frac{y}{x}$. Thus, k may be interpreted as the tangent of the angle θ depicted in Figure 1.10, and the relation $y \propto x$ defines a set of points along a line in the plane with angle of inclination θ.

Comparing the general form of a proportionality relationship $y = kx$ with the equation for a straight line $y = mx + b$; we can see that the graph of a proportionality relationship is a line (possibly extended) passing through the origin. If we plot the proportionality variables for Models (1.3)—(1.5), we obtain the straight-line graphs presented in Figure 1.11.

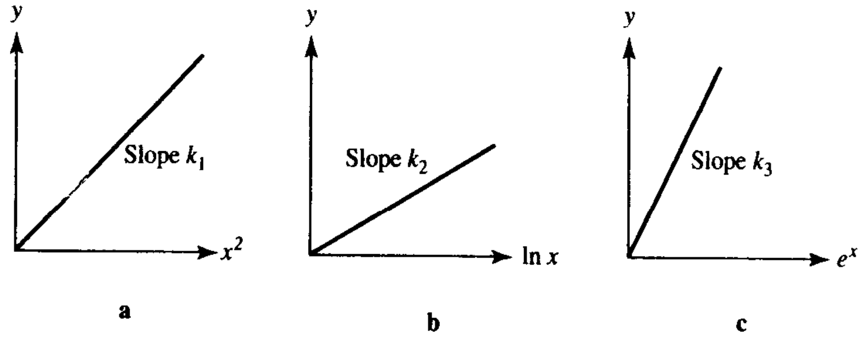


Figure 1.11
Geometrical interpretation of Models (a) (2.2), (b) (2.3), and (c) (2.4)

It is important to note that not just any straight line represents a proportionality relationship: the y-intercept must be zero so that the line passes through the origin. Failure to recognize this point can lead to erroneous results when using our model. For example, suppose we are interested in predicting the volume of water displaced by a boat as i is loaded with cargo. Because a floating object displaces a volume of water equal to its weight, we might be tempted to assume that the total volume y of displaced water is proportional to the weight x of the added cargo. However, there is a flaw with that assumption because the unloaded boat already displaces a volume of water equal to its weight. Although the graph of total volume of displaced water versus weight of added cargo is given by a straight line, it is not given by a line passing through the origin (Figure 1.12). so the proportionality assumption is incorrect.

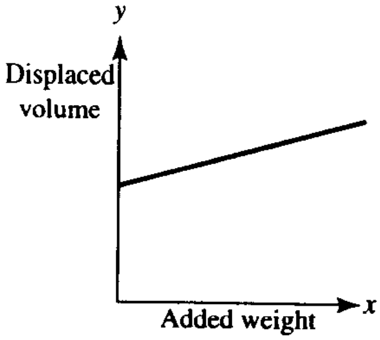
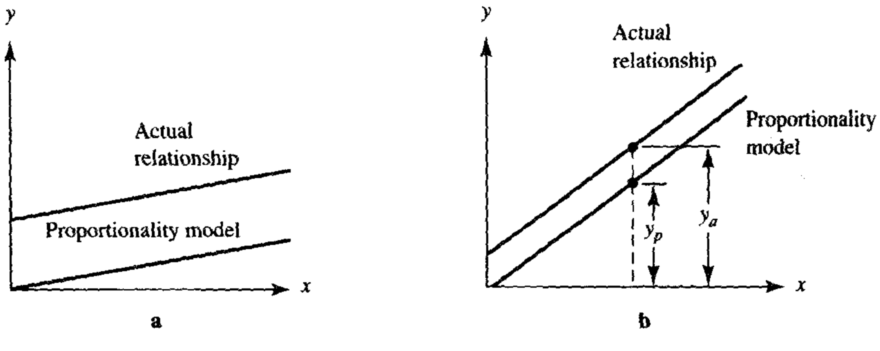


Figure 1.12
A straight-line relationship exists between displaced volume and total weight, but it is not a *proportionality* because the line fails to pass through the origin

A proportionality relationship may, however, be a reasonable simplifying assumption, depending on the Size of the y-intercept and the slope of the line. The domain of the independent variable can also be significant since the relative error

$$\frac{y_a - y_p}{y_a}$$

Figure 1.13
Proportionality as a simplifying assumption



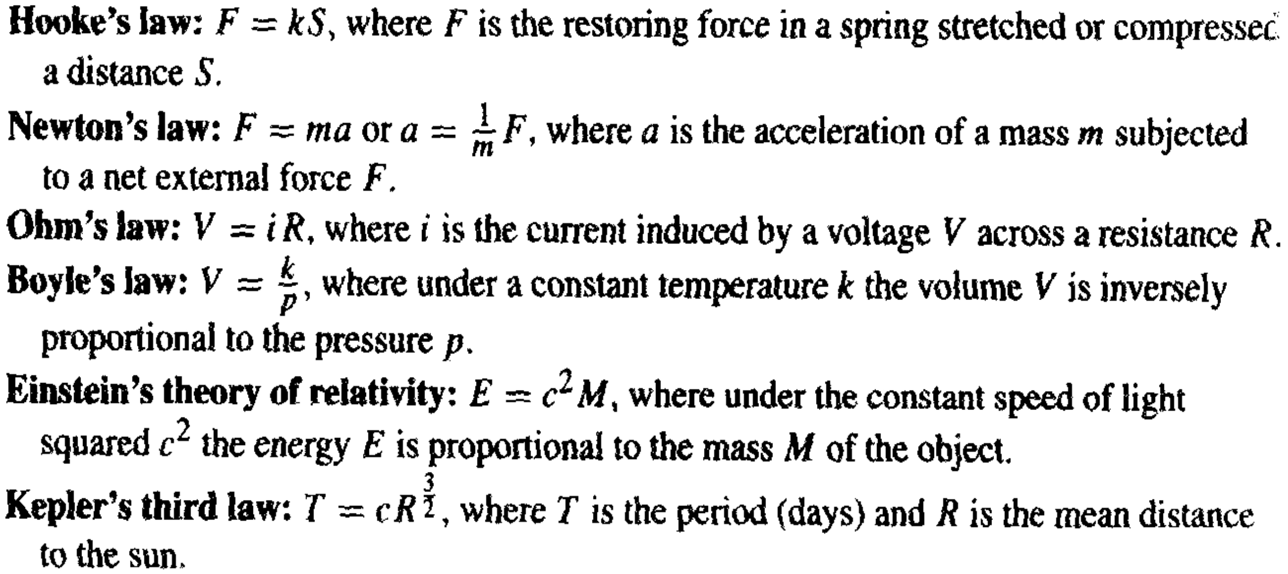
is greater for small values of x. These features are depicted in Figure 2.13. If the slope is nearly zero, proportionality may be a poor assumption because the initial displacement dwarfs the effect of the added weight. For example, there would be virtually no effect in placing 400 lbs. on an aircraft carrier already weighing many tons. On the other hand, if the initial displacement is relatively small and the slope is large, the effect of the initial displacement is dwarfed quickly, and proportionality is a good simplifying assumption.

EXAMPLE 1. Kepler's Third Law

To assist in further understanding the idea of proportionality, let's examine one of the famous proportionalities from Table 1.3, Kepler's third law. In 1601, the German astronomer Johannes Kepler became director of the Prague Observatory. Kepler had been helping Tycho Braise in collecting 13 years of observations on thy relative motion of the planet Mars. By 1609, Kepler had formulated his first two laws:

1. Each planet moves along an ellipse with the sun at one focus.
2. For each planet, the line from the sun to the planet sweeps out equal areas in equal times.

Table 1.3 Famous proportionalities



Hooke's law: $F = kS$, where F is the restoring force in a spring stretched or compressed a distance S.

Newton's law: $F = ma$ or $a = \frac{1}{m}F$, where a is the acceleration of a mass m subjected to a net external force F.

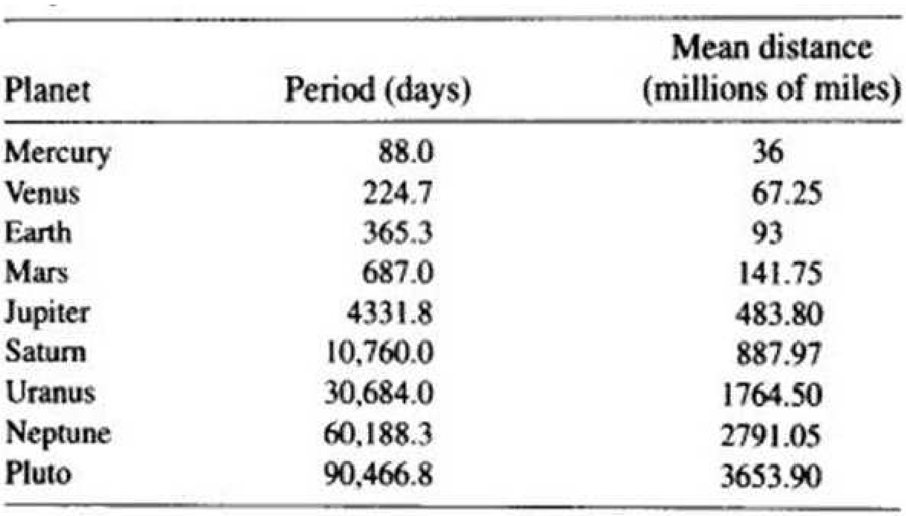
Ohm's law: $V = iR$, where i is the current induced by a voltage V across a resistance R.

Boyle's law: $V = \frac{k}{p}$, where under a constant temperature k the volume V is inversely proportional to the pressure p.

Einstein's theory of relativity: $E = c^2M$, where under the constant speed of light squared c^2 the energy E is proportional to the mass M of the object.

Kepler's third law: $T = cR^{\frac{3}{2}}$, where T is the period (days) and R is the mean distance to the sun.

Table 1.4 Famous proportionalities Orbital periods & mean distances of Planets from the sun.



Planet	Period (days)	Mean distance (millions of miles)
Mercury	88.0	36
Venus	224.7	67.25
Earth	365.3	93
Mars	687.0	141.75
Jupiter	4331.8	483.80
Saturn	10,760.0	887.97
Uranus	30,684.0	1764.50
Neptune	60,188.3	2791.05
Pluto	90,466.8	3653.90

Kepler spent many years verifying these laws and formulating the third law given in Table 1.3, which relates the orbital periods and mean distances of the planets from the sun. The data shown in Table 1.4 are from the 1993 World Almanac.

Modeling Vehicular Stopping Distance

. Consider again the scenario posed in Example 1 of Section 1.3. Recall the general rule that allows one car length for every 10 mph of speed. It was also stated that this rule is the same as allowing for 2 seconds between cars. The rules are in fact different from one another (at least for most cars). For the rules to be the same, at 10 mph both should allow one car length:

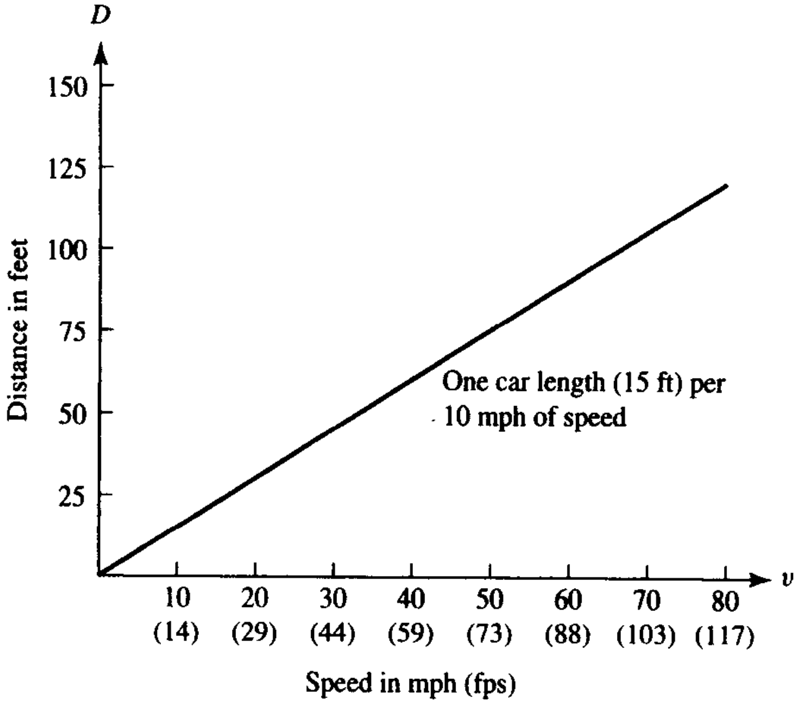
$$\begin{aligned}
1 \text{ car length} = \text{distance} &= \left(\frac{\text{speed in ft}}{\text{sec}}\right)(2 \text{ sec}) \\
&= \left(\frac{10 \text{ miles}}{\text{hr}}\right)\left(\frac{5280 \text{ ft}}{\text{mi}}\right)\left(\frac{1 \text{ hr}}{3600 \text{ sec}}\right)(2 \text{ sec}) \\
&= 29.33 \text{ ft}
\end{aligned}$$

This is an unreasonable result for an average car length of 15 ft, so the rules are not the same.

Let's interpret the one-car-length rule geometrically. If we assume a car length of 15 ft and plot this rule, we obtain the graph shown in Figure 2.14, which shows that the distance allowed by the rule is proportional to the speed. In fact, if we plot the speed in feet per second, the constant of proportionality has the units seconds and represents the total time for the equation D = kv to make sense. Moreover, in the case of a 15-ft car, we obtain a constant of proportionality as follows:

$$k = \frac{15 \text{ ft}}{10 \text{ mph}} = \frac{15 \text{ ft}}{52{,}800 \text{ ft}/3600 \text{ sec}} = \frac{90}{88} \text{ sec}$$

Figure 1.13
Geometrical interpretation of the one-car-length rule



In our previous discussion of this problem, we presented the model

total stopping distance = *reaction distance* + *braking distance.*

Let's consider the sub models for reaction distance and braking distance.

Recall that

reaction distance = f(response time, speed)

Now assume that the vehicle continues at constant speed from the time the driver determines the need to stop until the brakes are applied. Under this assumption, reaction distance d_r is simply the product of response time t_r and velocity *v*:

$$d_r = t_r v \tag{1.6}$$

To test sub model (1.6), plot measured reaction distance versus velocity, if the resultant graph approximates a straight line through the origin, we could estimate the slope t_r and feel fairly confident in the sub model. Alternatively, we could test a group of drivers representative of the assumptions made in the example in Section 1.3 and estimate t_r directly.

Next, consider the braking distance:

braking distance = h(weight , speed)

Suppose there is a panic stop and that the maximum brake force F is applied throughout the stop. The brakes are basically an energy-dissipating device; that is, the brakes do work on the vehicle producing a change in the velocity that results in a loss of kinetic energy. Now, the work done is the force F times the braking distance d_b. This work must equal the change in kinetic energy, which, in this situation, is simply 0.5 mv^2. Thus, we have

$$\text{work done} = F d_b = 0.5 m v^2 \tag{1.7}$$

Next, we consider how the force F relates to the mass of the car. A reasonable design criterion would be to build cars in such a way that the maximum deceleration is constant when the maximum brake force is applied regardless of the mass of the car. Otherwise, the passengers and driver would experience an unsafe jerk during the braking to a complete stop. This assumption means that the panic deceleration of a larger car, such as a Cadillac, is the same as that of a small car, such as a Honda, owing to the design of the braking system. Moreover, constant deceleration occurs throughout the panic stop. From Newton’s second law, F = ma, it follows that the force F is proportional to the mass. Combining this result with Equation (1.7) gives the proportionality relation

$$d_b \propto v^2$$

At this point we might want to design a test for the two sub models, or we could test the sub models against the data provided by the U.S. Bureau of Public Roads given in Table 1.5.

Figure 1. 15 depicts the plot of driver reaction distance against velocity using the data in Table 1.5. The graph is a straight line of approximate slope 1.1 passing through the origin; our results are too good. Because we always expect some deviation in experimental results, we should be suspicious.

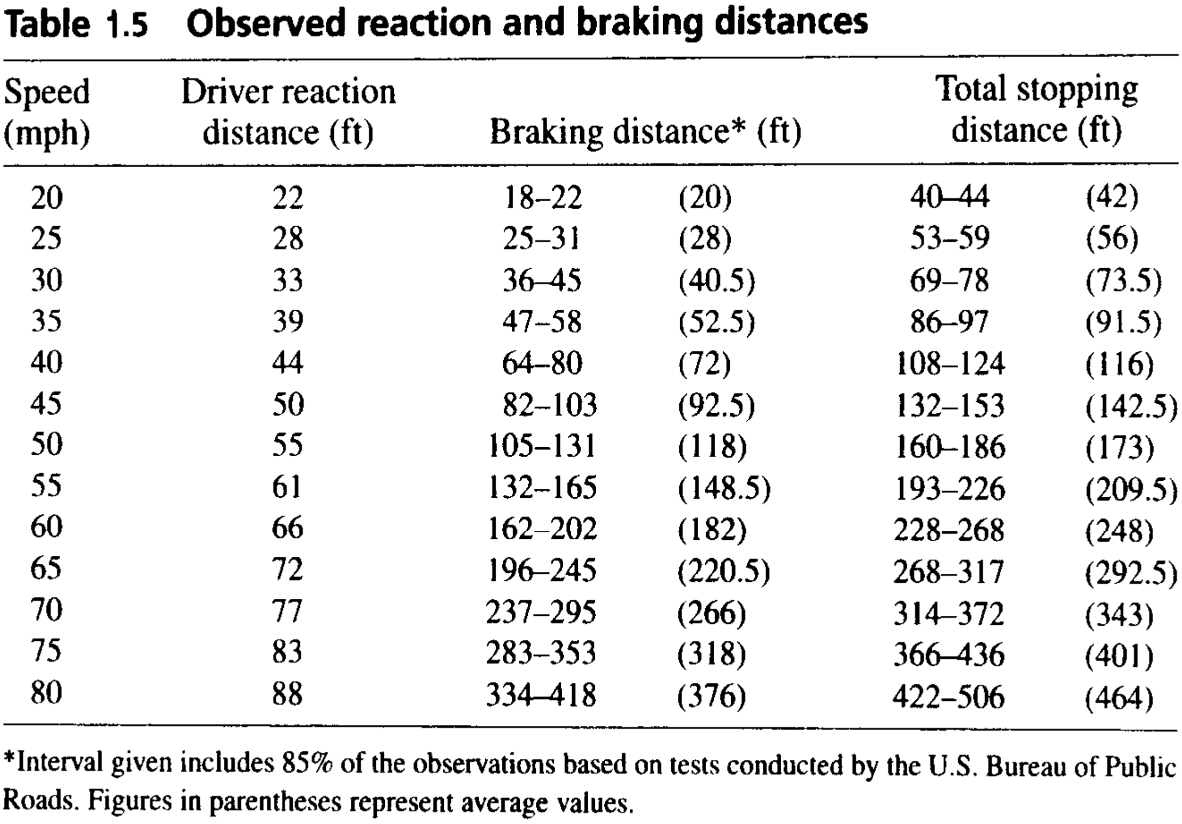
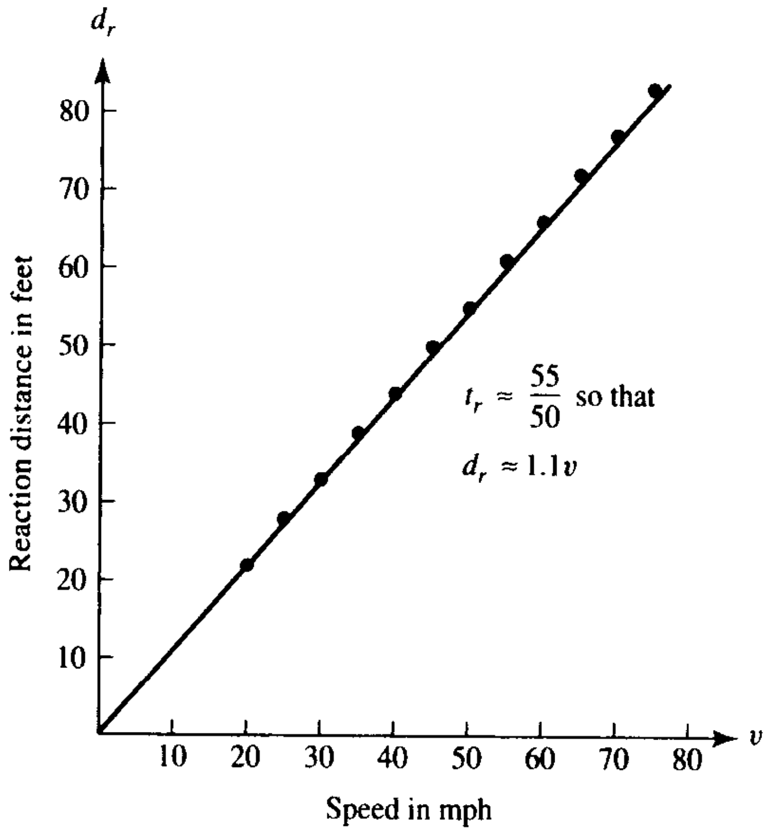


Table 1.5 Observed reaction and braking distances

Speed (mph)	Driver reaction distance (ft)	Braking distance* (ft)		Total stopping distance (ft)	
20	22	18–22	(20)	40–44	(42)
25	28	25–31	(28)	53–59	(56)
30	33	36–45	(40.5)	69–78	(73.5)
35	39	47–58	(52.5)	86–97	(91.5)
40	44	64–80	(72)	108–124	(116)
45	50	82–103	(92.5)	132–153	(142.5)
50	55	105–131	(118)	160–186	(173)
55	61	132–165	(148.5)	193–226	(209.5)
60	66	162–202	(182)	228–268	(248)
65	72	196–245	(220.5)	268–317	(292.5)
70	77	237–295	(266)	314–372	(343)
75	83	283–353	(318)	366–436	(401)
80	88	334–418	(376)	422–506	(464)

*Interval given includes 85% of the observations based on tests conducted by the U.S. Bureau of Public Roads. Figures in parentheses represent average values.

Figure 1.15
Proportionality of reaction distance and speed

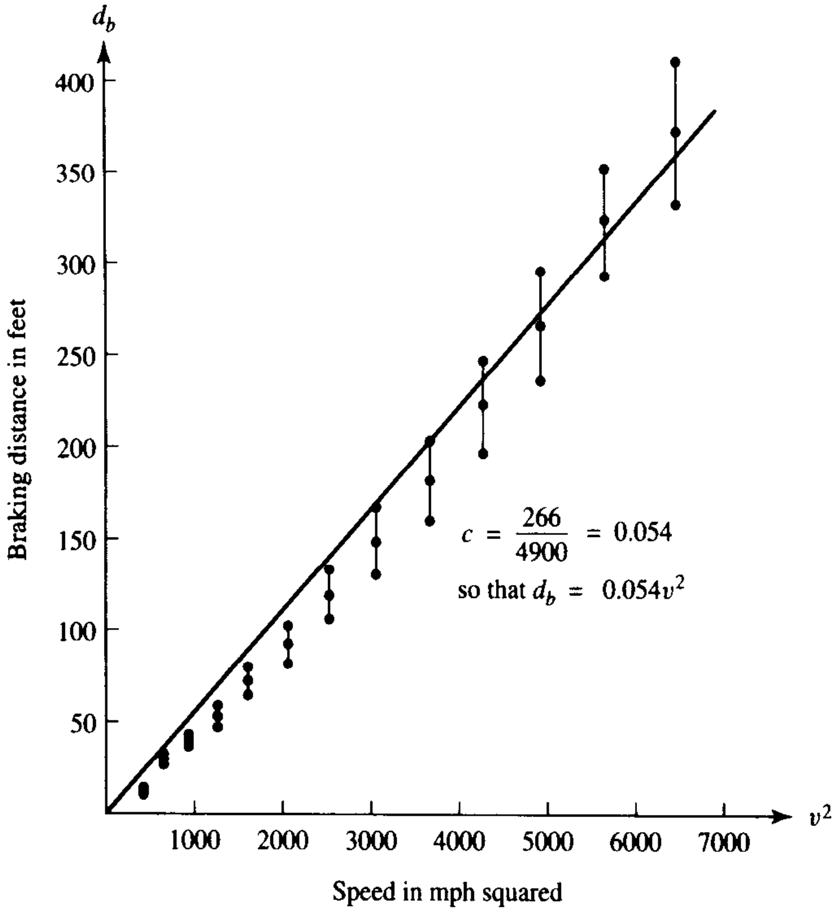


In fact, the results of Table 1.5 are based on Sub model (1.5),where an average response time of 3/4 sec was obtained independently. So we might later decide to design another test for the sub model.

To test the sub model for braking distance, we plot the observed braking distance recorded in Table 1.5 against v^2, as shown in Figure 1.16. Proportionality seems to be a reasonable assumption at the lower speeds, although it does seem to be less convincing at the higher speeds. By graphically tilting a straight line to the data, we estimate the slope and obtain the sub model:

$$d_b = 0.054v^2 \tag{1.8}$$

Figure 1.16
Proportionality of braking distance and the speed squared



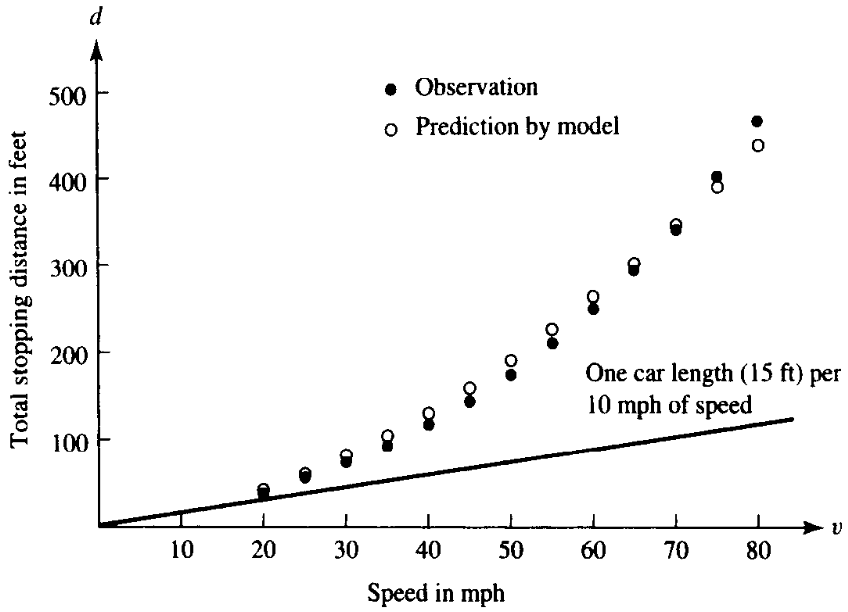
Summing Sub models (1.7) and (1.8), we obtain the following model for the total stopping distance d:

$$d = 1.1v + 0.054v^2 \tag{1.9}$$

The predictions of Model (1.9) and the actual observed stopping distance recorded in Table 1.5 are plotted in Figure 1.17. Considering the grossness of the assumptions and the inaccuracies of the data, the model seems to agree fairly reasonably with the observations up to 70 mph. The rule of thumb of one 15-ft car length for every 10 mph of speed is also plotted in Figure 1.17. We can see that the rule significantly underestimates the total stopping distance at speeds exceeding 40mph.

Let's suggest an alternative rule of thumb that is easy to understand and use. Assume the driver of the trailing vehicle must be fully stopped by the time he or she reaches the point occupied by the lead vehicle at the exact time of the observation. Thus, the driver must trail the lead vehicle by the total slopping distance, based either on Model (1.9) or on the observed data in Table 1.4. The maximum stopping distance can readily be converted to a trailing time. The results of these computations for the observed distances, in which 85% of the drivers were able to stop, are given in Table 1.6. These computations suggest the following general rule:

Figure 1.17
Total stopping distance



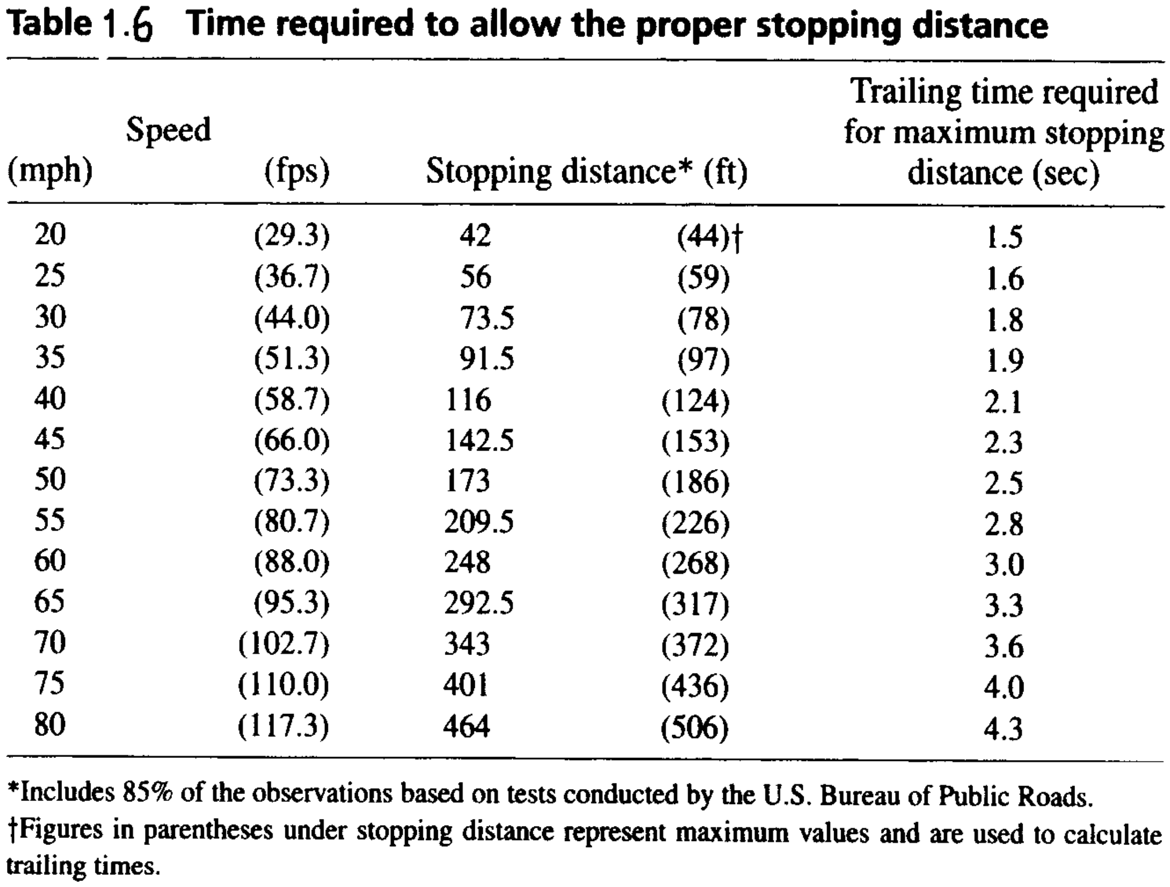
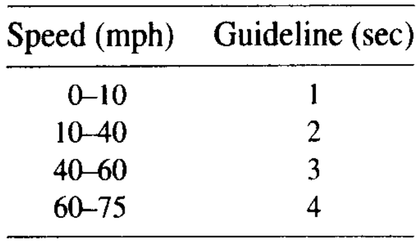


Table 1.6 Time required to allow the proper stopping distance

Speed		Stopping distance* (ft)		Trailing time required for maximum stopping distance (sec)
(mph)	(fps)			
20	(29.3)	42	(44)†	1.5
25	(36.7)	56	(59)	1.6
30	(44.0)	73.5	(78)	1.8
35	(51.3)	91.5	(97)	1.9
40	(58.7)	116	(124)	2.1
45	(66.0)	142.5	(153)	2.3
50	(73.3)	173	(186)	2.5
55	(80.7)	209.5	(226)	2.8
60	(88.0)	248	(268)	3.0
65	(95.3)	292.5	(317)	3.3
70	(102.7)	343	(372)	3.6
75	(110.0)	401	(436)	4.0
80	(117.3)	464	(506)	4.3

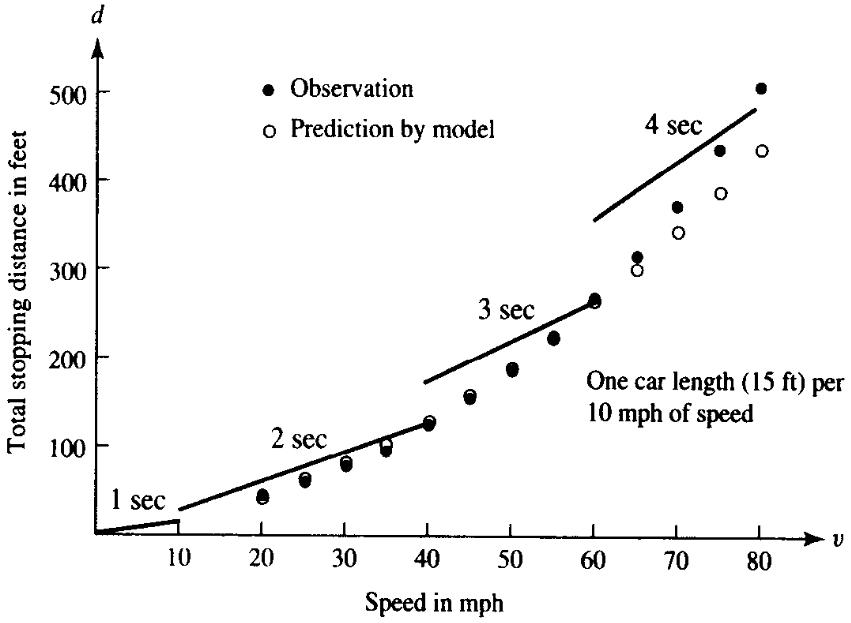
*Includes 85% of the observations based on tests conducted by the U.S. Bureau of Public Roads.
†Figures in parentheses under stopping distance represent maximum values and are used to calculate trailing times.



Speed (mph)	Guideline (sec)
0–10	1
10–40	2
40–60	3
60–75	4

This alternative rule is plotted in Figure 1.18. An alternative to using such a rule might be to convince manufactures to modify existing speedometers to compute stopping distance and time for the car’s speed v based on Equation (1.9).

Figure 1.18
Total stopping distance and alternate general rule. The plotted observations are the maximum values from Table 1.4



Problems

1. Show graphically the meaning of the proportionality $y \propto u/v$.
2. If a spring is stretched 0.37 in. by a 14-lb force, what stretch will be produced by a 9-lb force? By a 22-lb force? Assume Hooke's law, which asserts the distance stretched is proportional to the force applied.

1.6 Why Study Modeling?

Mathematical modeling is the art of translating problems from an application area into tractable mathematical formulations whose theoretical and numerical analysis provides insight, answers, and guidance useful for the originating application.

Mathematical modeling

- is indispensable (crucial) in many applications
- is successful in many further applications
- gives precision and direction for problem solution
- enables a thorough(detail) understanding of the system modeled
- prepares the way for better design or control of a system
- allows the efficient use of modern computing capabilities

Learning about mathematical modeling is an important step from a theoretical mathematical training to an application-oriented mathematical expertise, and makes the student fit for mastering the challenges of our modern technological culture.

Mathematical modeling plays a big role in the description of a large part of phenomena in the applied sciences and in several aspects of technical and industrial activity. By a "mathematical model" we mean a set of equations and/or other mathematical relations capable of capturing the essential features of a complex natural or artificial system, in order to describe, forecast and control its evolution. The applied sciences are not confined to the classical ones; in addition to physics and chemistry, the practice of mathematical modeling heavily affects disciplines like finance, biology, ecology, medicine, sociology.

In the industrial activity (e.g. for aerospace or naval projects, nuclear reactors, combustion problems, production and distribution of electricity, traffic control, etc.) the mathematical modeling, involving first the analysis and the numerical simulation and followed by experimental tests, has become a common procedure, necessary for innovation, and also motivated by economic factors. It is clear that all of this is made possible by the enormous computational power now available.

Self Test Exercises.1

I. For the scenarios presented in Problems 1—3, identify a problem worth studying and list the variables that affect the behavior you have identified. Which variables would be neglected completely? Which might be considered as constants initially? Can you identify any sub models you would want to study in detail? Identify any data you would want collected.

1. A botanist is interested in studying the shapes of leaves and the forces that mold them. She clips some leaves from the bottom of a white oak tree and finds the leaves to he rather broad, not very deeply indented. When she goes to the top of the tree, she finds very deeply indented leaves with hardly any broad expanse of blade.
2. Animals of different size work differently. Small ones have squeaky voices, their hearts beat faster, and they breathe more often than larger ones. On the other hand, the

skeleton of a larger animal is more robustly built than that of a small animal. The ratio of the diameter to the length in a larger animal is greater than it is in a smaller one. So there are regular distortions in the proportions of animals as the size increases from small to large.

3. A physicist is interested in studying properties of light. He wants to understand the path of a ray of light as it travels through the air into a smooth lake, particularly at the interface of the two different media.

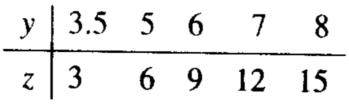
II. ANSWER THE FOLLOWING QUESTIONS.

1. Should a couple buy or rent a home? As the cost of a mortgage rises, intuitively, it would seem that there is a point where it no longer pays to buy a house. What variables determine the total cost of a mortgage?

2. Consider the operation of a medical office. Records have to be kept on individual patients, and accounting procedures are a daily task. Should the office buy or lease a small computer system? Suggest objectives that might be considered. What variables would you consider? How would you relate the variables? What data would you like to have to determine the relationships between the variables you select? Why might solutions to this problem differ from office to office?

3. Determine whether the following data support a proportionality argument for

$y \propto z^{1/2}$.



y	3.5	5	6	7	8
z	3	6	9	12	15

CHAPTER 2

DIMENSIONAL ANALYSIS

Objectives:-

At the end of this chapter you will be able to:

- Define dimensional analysis
- Express dimensions of different physical quantities as products of dimensions in MLT system
- Understand the concept dimensional compatibility
- Describe the process of dimensional analysis
- Apply Buckingham's theorem to produce all possible dimensionally homogeneous equations among the variables under consideration
- Understand the basic procedures in applying dimensional analysis in model building process
- Use dimensional analysis in model building process

2.1 Introduction:

Activity 2.1:-

- Define the term "**dimension**".
- When do you say "a **quantity is dimensionless**".
- Define **dimensional analysis**.

In the process of constructing a mathematical model, we have seen that the variables influencing the behaviour must be identified and classified. We must then determine appropriate relationships among those variables retained for consideration. In the case of a single dependent variable this procedure gives rise to some unknown function: . $y = f(x_1, x_2, \ldots, x_n)$ where the x_i measure the various factors influencing the phenomenon under investigation. In some situations the discovery of the nature of the function *f* for the chosen factors comes about by making some reasonable assumption based on a law of nature or previous experience and construction of a mathematical model. We were able to use this methodology in constructing our model on vehicular stopping distance (see section 2.2). On the other hand, especially for those models designed to predict some physical phenomenon, we may find it difficult or impossible to construct a solvable or tractable explicative model because of the inherent complexity of the problem. In certain instances we might conduct a series of experiments to determine how the dependent variable *y is related to* various values of the independent variable(s). In such cases we usually prepare a figure or table and apply an appropriate curve-fitting or interpolation method that can be used to predict the value of y for suitable ranges of the independent variable(s).

Dimensional analysis is a method for helping determine how the selected variables are related and for reducing significantly the amount of experimental data that must be collected. It is based on the premise that physical quantities have dimensions and that physical laws are not altered by changing the units measuring dimensions. Thus, the phenomenon under investigation can be described by a dimensionally correct equation among the variables. A dimensional analysis provides qualitative information about the model. It is especially important when it is necessary to conduct experiments in the modeling process because the method is helpful in testing the validity of including or neglecting a particular factor, in reducing the number of experiments to be conducted to make predictions, and in improving the usefulness of the results by providing alternatives for the parameters employed to present them. Dimensional analysis has proven useful in physics and engineering for many years and now even plays a role in the study of the life sciences, economics, and operations research. Let's consider an example illustrating how dimensional analysis can be used in the modeling process to increase the efficiency of an experimental design.

Consider the situation of a simple pendulum as suggested in Figure 2.1. Let r denote the length of the pendulum, m its mass, andθ **the initial angle of** displacement from the vertical. One characteristic that is vital in understanding the behaviour of the pendulum is the period, which is the time required for the pendulum bob to swing through one complete cycle and return to its original position (as at the beginning of the cycle) We represent the period of the pendulum by the dependent variable *t*.

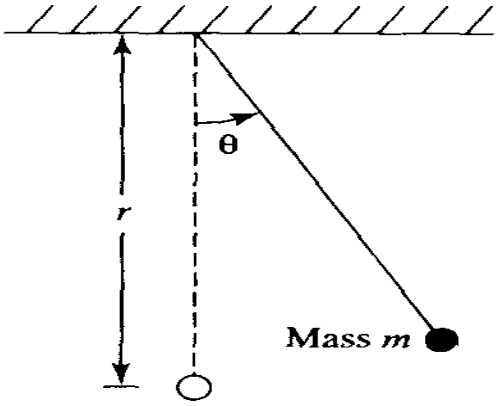


Figure 2.1 A simple pendulum

P**roblem identification** For a given pendulum system determine its speed.

Assumptions First, we list the factors that influence the period. Some of these factors are the length r, the mass m, the initial angle of displacement θ ,the acceleration due to gravity g, and frictional forces such as the friction at the hinge and the drag on the pendulum. Assume initially that the hinge is frictionless, that the mass of the pendulum is concentrated at one end of the pendulum, and that the drag force is negligible. Other assumptions about the frictional forces will be examined in section 2.3. Thus the problem is to determine or approximate the function $t = f(r, m, \theta, g)$ and test its worthiness as a predictor.

Experimental Determination of the Model Because gravity is essentially constant under the assumptions, the period t is a function of the tree variables length r, mass m, and initial angle of displacement θ .At this point we could systematically conduct experiments to determine how t varies with these three variables. We could want to choose enough values of the independent variables to feel confident in predicting the period t over that range. How many experiments will be necessary?

For the sake of illustration, consider a function of one independent variable y=f(x) and assume that four points been deemed necessary to predict y over a suitable domain for x. The situation is depicted in Figure 2.2.

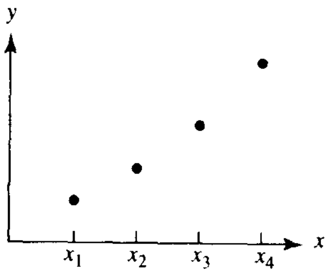


Figure 2.2 Four points have been deemed necessary to predict y for this function of one variable x

An appropriate curve-fitting or interpolation method could be used to predict y within the domain for x.

Next consider what happens when a second independent in variable affects the situation under investigation. We then have a function $y = f(x, z)$.

For each data value of x in figure2.2, experiments must be conducted to obtain y for four values of z. Thus, 16(i.e.,4^2) experiments are required. These observations are illustrated in figure 2.3. Likewise a function of three variables requires 64(i.e.,4^3) experiments. In general, 4^n experiments are required to predict y when n is the number of arguments of the function, assuming four points for the domain of each argument. Thus, a procedure that reduces the number arguments of the function f will dramatically reduce the total number of required experiments. Dimensional analysis is one such procedure.

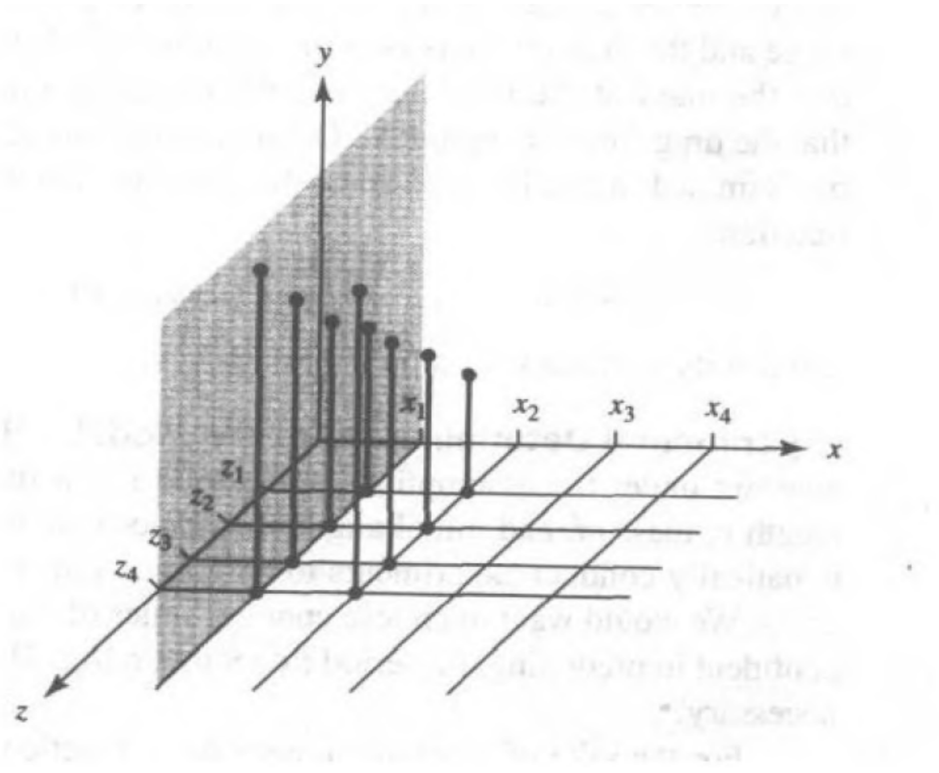


Figure 2.3 Sixteen points are necessary to predict y for this function of the two variables x and z.

The power of dimensional analysis can also be appreciated when we examine the interpolation curves that would be determined after collecting the data represented in figure 2.2and 2.3. Let's assume it is decided to pass a cubic polynomial through the four points shown in figure 2.2. That is, the four points are used to determine the four constants C_1-C_4 in the interpolating curve:

$$y = C_1x^3 + C_2x^2 + C_1x + C_4 .$$

Now consider interpolating from figure 2.3. If for a fixed value of x, say $x=x_1$, we decide to connect our points using a cubic polynomial in z, the equation of the interpolating surface is

$$\begin{aligned} y = D_1x^3 + D_2x^2 + D_3x + D_4 + (D_5x^3 + D_6x^2 + D_7x + D_8)z \\ + (D_9x^3 + D_{10}x^2 + D_{11}x + D_{12})z^2 + (D_{13}x^3 + D_{14}x^2 + D_{15}x + D_{16})z^3 \end{aligned}$$

Note from the equation that there are 16 constants --D_1,D_2,...,D_{16}--to determine rather than 4 as in the two dimensional case. This procedure again illustrates the dramatic reduction in effort required when we reduce the number of arguments of the function we will finally investigate.

At this point we make the important observation that the experimental effort required depends more heavily on the number of arguments of the function to be investigated than on the true number of independent variables the modeler originally selected. For example, consider a function of two arguments, say y=f(x,z).The discussion concerning the number of experiments necessary would not be altered if x were some particular combination of several variables. That is, x could be uv/w, where u, v, and w are the variables originally selected in the model.

Consider now the following preview of dimensional analysis, which describes how it reduces our experimental effort. Beginning with a function of n variables (hence, n arguments), the number of arguments is reduced (ordinarily by three) by combining the original variables into products. These resulting (n-3) products are called **dimensionless products** of the original variables. After applying dimensional analysis, we still need to conduct experiments to make our predictions, but the amount of experimental effort that is required will have been reduced exponentially.

In chapter 1 we discussed the trade-offs of considering additional variables for increased precision versus neglecting variables for simplification. In constructing models based on experimental data, the preceding discussion suggests that the cost of each additional variable is an exponential increase in the number of experimental trials that must be conducted. In the next two sections we present the main ideas underlying the dimensional analysis process. You may find that some of these ideas are slightly more difficult than previous ones we have investigated, but the methodology is powerful when modeling physical behaviour.

2.2 Dimensions as Products

Activity 2.2:-

- Describe basic physical quantities.
- What dimensions are associated with the physical quantities mass, length, and time?
- Can you assign dimensions of other physical quantities in terms of those of mass, length, and time?
- Determine the dimensions of physical quantities force, velocity, density, momentum, power, and energy as products of dimensions of mass, length, and time.
- How can you determine dimensionless products among the variables?

The study of physics is based on abstract concepts such as mass, length, time, velocity, acceleration, force, energy, work, and pressure. To each such concept there is assigned a unit of measurement. A physical law such as F=ma is true, provided that the units of measurement are consistent. Thus, if mass is measured in kilograms and acceleration in meters per second squared, then the force must be in newtons. These units of measurement belong to the MKS (meter-kilogram-second) mass system. It would be inconsistent with the equation F=ma to measure mass in slugs, acceleration in feet per second squared, and force in newtons. In this illustration, force must be measured in pounds, giving the American Engineering System of measurement. There are other systems of measurement, but all are prescribed by international standards so as to be consistent with the laws of physics.

The three primary physical quantities we consider in this chapter are mass, length, and time. We associate with these quantities the dimensions M, L, and T respectively. The dimensions are symbols that reveal how the numerical value of a quantity changes when the units of measurement change in certain ways. The dimensions of other quantities follow from definitions or from physical laws and are expressed in terms of M, L, and T. For example, velocity v is defined as the ratio of distance s (dimension L) travelled to time t (dimension T) of travel--that is, $v=st^{-1}$, so the dimension of velocity is LT^{-1}. Similarly, because area is fundamentally a product of two lengths, its dimension is L^2. These dimension expressions hold true regardless of the particular system of measurement, and they show, for example, that velocity may be expressed in meters per second, feet per second, miles per hour, and so forth. Likewise area can be measured in terms of square meters, square feet, square miles, and so on.

There are still other entities in physics that are more complex in the sense that they are not usually defined directly in terms of mass, length, and time alone: instead, their definitions include other quantities, such as velocity. We associate dimensions with these more complex quantities in accordance with algebraic operations involved in the definitions. For example, because momentum is the product of mass with velocity, its dimension is $M(LT^{-1})$ or simplyMLT^{-1}.

The basic definition of a quantity may also involve dimensionless constants; these are ignored in finding dimensions. Thus the dimension of kinetic energy, which is one-half (a dimensionless constant) the product of mass with velocity squared, is $M(LT^{-1})^2$ or simply ML^2T^{-2}. As you will see in example 2, some constants (dimensional constants), such as gravity g, do have an associated dimension, and these must be considered in a dimensional analysis.

These examples illustrate the following important concepts regarding dimensions of physical quantities.

1. We have based the concept of dimension on three physical quantities: mass m, length s, and time t. These quantities are measured in some appropriate system of units whose choice does not affect the assignment of dimensions. (This underlying system must be linear. A dimensional analysis will not work if the scale is logarithmic, for example.)

2. There are other physical quantities, such as area and velocity that are defined as simple products involving only mass, length, or time. Here we use the term **product** to indicate any quotient because we may indicate division by negative exponents.

3.There still other, more complex, physical entities, such as momentum and kinetic energy, whose definitions involve quantities other than mass, length, and time. Because the simpler quantities from (1) and (2) are products, these more complex quantities can also be expressed as products involving mass, length, and time by algebraic simplification. We use the term product to refer to any physical quantity from item (1), (2), or (3); a product from (1) is trivial because it has only one factor.

4. To each product, there is assigned a dimension--that is, an expression of the form

$$M^n L^p T^q \qquad (2.1)$$

where n, p, and q are real numbers that may be positive, negative, or zero.

When a basic dimension is missing from a product, the corresponding exponent is understood to be zero. Thus, the dimension $M^2L^0T^{-1}$ may also appear as M^2T^{-1}. When n, and q are all zero in an expression of the form (2.1), so that the dimension reduces to

$$M^0 L^0 T^0 \qquad (2.2)$$

the quantity, or product, is said to be *dimensionless*.

Special care must be taken in forming sums of products because just as we cannot add apples and oranges, in an equation we cannot add products that have unlike dimensions. For example, if F denotes force, m mass, and v velocity, we know immediately that the equation $F = mv + v^2$ cannot be correct because mv has dimension MLT^{-1}, where as v^2 has dimension L^2T^{-2}. These dimensions are unlike; hence, the products mv and v^2 cannot be added. An equation such as this--that is, one that contains among its terms two products having unlike dimensions--is said to be *dimensionally incompatible*. Equations that involve only sums of products having the same dimension are *dimensionally compatible*.

The concept of dimensional compatibility is related to another *important* concept called dimensional homogeneity. In general, an equation that is true regardless of the system of units in which the variables are measured is said to be dimensionally homogeneous. For example, $t = \sqrt{\frac{2s}{g}}$ giving the time a body falls a distance s under gravity (neglecting air resistance) is dimensionally homogeneous (true in all systems), whereas the equation $t = \sqrt{\frac{s}{16.1}}$ is not dimensionally homogeneous (because it depends on a particular system). In particular, if an equation involves only sums of dimensionless products (i.e., it is a dimensionless equation), then the equation is dimensionally homogeneous. Because the products are dimensionless, the factors used for conversion from one system of units to another would simply cancel.

The application of dimensional analysis to a real- world problem is based on the assumption that the solution to the problem is given by a dimensionally homogeneous equation that the problem is given by a dimensionally homogeneous equation in terms of the appropriate variables. Thus, the task is to determine the form of the desired equation by finding an appropriate dimensionless equation and then solving for the dependent variable. To accomplish this task, we must decide which variables enter into the physical problem under investigation and determine all the dimensionless products among them. In general, there may be infinitely many such products, so they will have to be described rather than actually written out. Certain subsets of these dimensionless products are then used to construct dimensionally homogeneous equations. In section 2.2 we investigate how the dimensionless products are used to find all dimensionally homogeneous equations. The following example illustrates how the dimensionless products may be found.

Example 1 A simple Pendulum Revised

Consider again the simple pendulum discussed in the introduction. Analysing the dimensions of the variables for the pendulum problem, we have

Variable	m	g	t	r	θ
Dimension	M	LT^{-2}	T	L	$M^0L^0T^0$

Next we find all the dimensionless products among the variables. Any product of these variables must be of the form

$$m^a g^b t^c r^d \theta^e \qquad (2.3).$$

and hence must have dimension

$$(M)^a (LT^{-2})^b (T)^c (L)^d (M^0 L^0 T^0)^e$$

Therefore, a product of the form (2.3) is dimensionless if and only if

$$M^a L^{b+d} T^{c-2b} = M^0 L^0 T^0 \qquad (2.4)$$

Equating the exponents on both sides of this last equation leads to the system of linear equations

$$\left.\begin{array}{lll} a & +0e & = 0 \\ b & +d+0e & = 0 \\ -2b+c & +0e & = 0 \end{array}\right\} \qquad (2.5)$$

Solution of the system (2.5) gives a=0, c=2b, d=-b, where b is arbitrary. Thus, there are infinitely many solutions. Here are some general rules for selecting arbitrary variables: (1) choose the dependent variable so it will appear only once, (2) select any variable that expedites the solution of the other equations (i.e., a variable that appears in all equations), and (3) choose a variable that always has a zero coefficient, if possible. Notice that the exponent e does not really appear in (2.4) (because it has a zero coefficient in each equation) so that it is also arbitrary. One dimensionless product is obtained by setting b=0 and e=1, yielding a=c=d=0. A second, independent dimensionless product is obtained when b=1 and e=0, yielding a=0, c=2, and d=-1. These solutions give the dimensionless products

$$\pi_1 = m^0 g^0 t^0 r^0 \theta^1 = \theta$$

In section 2.2, we will learn a methodology for relating these products to carry the modeling process to completion. For now, we will develop a relationship in an intuitive manner.

Assuming t=f(r, m, g,θ), to determine more about the function f, we observe that if the units in which we measure mass are made smaller by same factor(e.g., 10), then the measure of the period t will not change because it is measured in units (T) of time. Because m is the only factor whose dimension contains M, it cannot appear in the model. Similarly, if the scale of the units (L) for measuring length is altered, it cannot change the measure of the period. For this to happen, the factors r and g must appear in the model as r/g, g/r, or, more generally, $(g/r)^k$. This ensures that any linear change in the way length is measured will be cancelled. Finally, if we make the units (T) that measure time smaller by a factor of 10, for example, the measure of the period will directly increase by the same factor 10. Thus, to have the dimension of T on the right side of the equation $t=f(r,m,g,\theta)$, g and r must appear as $\sqrt{r/g}$ because T appears to the power -2 in the dimension of g. Note that none of the preceding conditions places any restrictions on the angle θ . Thus, the equation of the period should be of the form

$$t = \sqrt{\frac{r}{g}}h(\theta)$$

where the function h must be determined or approximated by experimentation.

We note two things in this analysis that are characteristic of a dimensional analysis. First, in the MLT system, three conditions are placed on the model, so we should generally expect to reduce the number of arguments of the function present at the end of a dimensional analysis (in this case,θ) are dimensionless products.

In the problem of the undamped pendulum we assumed that friction and drag were negligible. Before proceeding with experiments (which might be costly), we would like to know if that assumption is reasonable. Consider the model obtained so far:

$$t = \sqrt{\frac{r}{g}}h(\theta)$$

Keeping θ constant while allowing r to vary, form the ratio

$$\frac{r_1}{r_2} = \frac{\sqrt{\frac{r_1}{g}}h(\theta_0)}{\sqrt{\frac{r_2}{g}}h(\theta_0)} = \sqrt{\frac{r_1}{r_2}}$$

Hence the model predicts that t will vary as $\sqrt{r}$ for constant θ. Thus, if plot t versus r with fixed θ for some observations, we would expect to get a straight line (figure 2.4). If we do not obtain a reasonable straight line, then we need to re-examine the assumptions. Note that our judgment here is qualitative. The final measure of the adequacy of any model is always how will it predicts or explains the phenomenon under investigation. Nevertheless, this initial test is useful for eliminating obviously bad assumptions and for choosing among competing sets of assumptions.

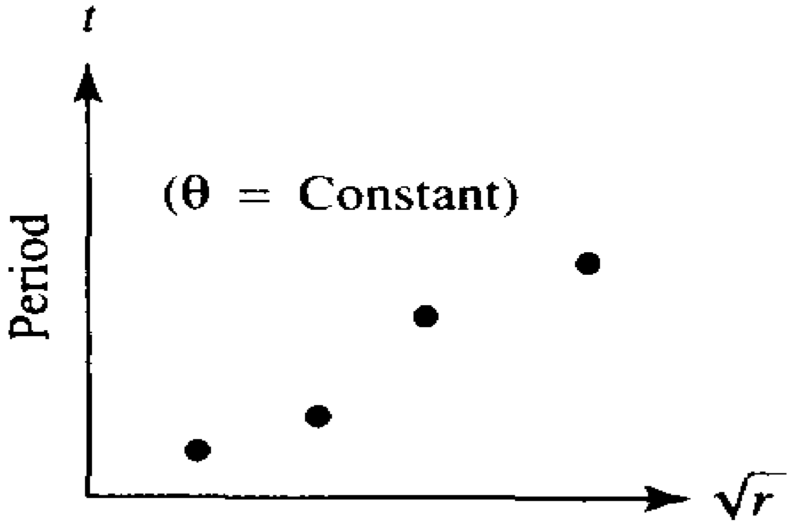
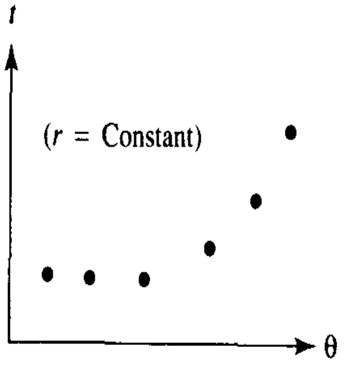


Figure 2.4 Testing the assumptions of the simple pendulum model by plotting the period t versus the square root of the length r for constant displacementθ

Dimensional analysis has helped construct a model t=f(r, m, g,θ) f or the undamped pendulum as $t = \sqrt{\frac{r}{g}}h(\theta)$. If we are interested in predicting the behaviour of the pendulum, we could isolate the effect of h by holding r constant and varyingθ. This provides the ratio

$$\frac{t_1}{t_2} = \frac{\sqrt{\frac{r_0}{g}}h(\theta_1)}{\sqrt{\frac{r_0}{g}}h(\theta_2)} = \frac{h(\theta_1)}{h(\theta_2)}$$

Hence a plot of t versus θ for several observations would reveal the nature of h. This plot is illustrated in figure 2.5. We may never discover the true function h relating the variables. In such cases, an important model might be constructed from the experimental data, as discussed in Chapter 4. When we are interested in using our model to predict t, based on experimental results, it is convenient to use the equation $t\sqrt{g/r} = h(\theta)$ and to plot $t\sqrt{g/r}$ versus θ , as in figure 2.6. Then, for a given value of θ , we would determine $\sqrt{g/r}$, multiply it by $\sqrt{r/g}$ for a specific r, and finally determine t.



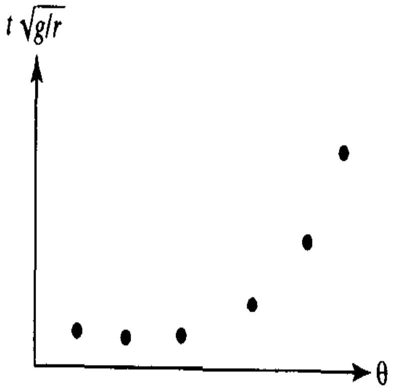


Figure 2.5Determining the unknown

Function h

Figure 2.6 presenting the

results for the

Simple pendulum

Example 2 Wind Force on a Van

Suppose you are driving a van down a highway with gusty winds. How does the speed of your vehicle affect the wind force you are experiencing?

The force F of the wind on the van is certainly affected by the speed v of the van and the surface area A of the van directly exposed to the wind's direction. Thus, we might hypothesize that the force is proportional to some power of the speed times some power of the surface area; that is,

$$F = kv^a A^b \qquad (2.6)$$

for some (dimensionless) constant k. Analyzing the dimensions of the variables gives

variable	F	k	v	A
Dimension	MLT^{-2}	$M^0L^0T^0$	LT^{-1}	L^2

Hence, dimensionally, Equation (2.6) becomes

$$MLT^{-2} = (M^0L^0T^0)(LT^{-1})^a(L^2)^b$$

This last equation cannot be correct because the dimension M for mass does not enter into the right-hand side with nonzero exponent.

So consider again equation (2.6). What is missing in our assumption concerning the wind force? Wouldn't the strength of the wind be affected by its density?

After some reflection we would probably agree that density does have an effect. If we include the density ρ as a factor, then our refined model becomes

$$F = kv^a A^b \rho^c \qquad (2.7)$$

Because density is mass per unit volume, the dimension of density is ML^{-3}. Therefore, dimensionally, equation (2.7) becomes

$$MLT^{-2} = (M^0L^0T^0)(LT^{-1})^a(L^2)^b(ML^{-3})^c$$

Equating the exponents on both sides of this last equation leads to the system of linear equations:

$$\left.\begin{array}{lll} & c & = 1 \\ a+2b & -3c & = 1 \\ -a & & = -2 \end{array}\right\} \qquad (2.8)$$

Solution of the system (2.8) gives a=2, b=1, and c=1. When substituted into equation (2.7) these values give the model

$$F = kv^2 A\rho$$

At this point we make an important observation. When it was assumed that F=kv^{a}A^b, the constant was assumed to be dimensionless. Subsequently, our analysis revealed that for a particular medium (so ρ is constant)

$$F \propto Av^2$$

giving $F = k_1 Av^2$. However,k$_1$ does have a dimension associated with it and is called a **dimensional constant**. In particular, the dimension of k$_1$ is

$$\frac{MLT^{-2}}{L^2\,(L^2T^{-2})} = ML^{-3}$$

Dimensional constants contain important information and must be considered when a dimensional analysis. We consider dimensional constants again in section 2.3 when we investigate a damped pendulum. If we assume the density ρ is constant, our model shows that the force of the wind is proportional to the square of the speed of the van times its surface area directly exposed to the wind. We can test the model by collecting data and plotting the wind force F versus v^2A to determine if the graph approximates a straight line through the origin. This example illustrates one of the ways dimensional analysis can be used to test our assumptions and check whether we have a faulty list of variables identifying the problem. Table 2.1 gives a summary of the dimensions of some common physical entities.

Table 2.1 Dimensions of physical entities in the MLT system

Mass	M	Momentum	MLT^{-1}
Length	L	Work	ML^2T^{-2}
Time	T	Density	ML^{-3}
Velocity	LT^{-1}	Viscosity	$ML^{-1}T^{-1}$
Acceleration	LT^{-2}	Pressure	$ML^{-1}T^{-2}$
Specific weight	$ML^{-2}T^{-2}$	Surface tension	MT^{-2}
Force	MLT^{-2}	Power	ML^2T^{-3}
Frequency	T^{-1}	Rotational inertia	ML^2
Angular velocity	T^{-1}	Torque	ML^2T^{-2}
Angular acceleration	T^{-2}	Entropy	ML^2T^{-2}
Angular momentum	ML^2T^{-1}	Heat	ML^2T^{-2}
Energy	ML^2T^{-2}		

Problems 2.1

1. Determine whether the equation

$$s = s_0 + v_0 t - 0.5gt^2$$

is dimensionally compatible, if s is the position (measured vertically from a fixed reference point) of a body at time t, s_0 is the initial velocity, and g is the acceleration caused by gravity.

.

2. The various constants of physics often have physical dimensions (dimensional constants) because their values depend on the system in which they are expressed. For example, Newton's law of gravitation asserts that the attractive force between two bodies is proportional to the product of their masses divided by the square of the distance between them, or symbolically,

$$F = \frac{Gm_1 m_2}{r^2}$$

where G is the gravitational constant. Find the dimension of G so that Newton's law is dimensionally Acompatible.

3. Certain stars, whose light and radial velocities undergo periodic vibrations, are thought to be pulsating. It is hypothesized that the period t of pulsation depends on the star's radius r, its mass m, and the gravitational constant G, (see problem 3 for the dimension of G.) Express t as a period of m, r, and G, so the equation

$$t = m^a r^b G^c$$

is dimensionally compatible.

4. In checking the dimensions of an equation, you should note that derivatives also p

$$\frac{dE}{dt} = \left[mr^2 \left(\frac{d^2\theta}{dt^2}\right) mgr sin\theta\right] \frac{d\theta}{dt}$$

for the time rate of total energy E in a pendulum system with damping force is dimensionally compatible.

5. For a body moving along a straight-line path, if the mass of the body is changing over time, then an equation governing its motion is given by

$$m\frac{dv}{dt} = F + u\frac{dm}{dt}$$

where m is the mass of the body, v is the velocity of the body, F is the total force acting on the body, dm is the mass joining or leaving the body in the time interval dt, and u is the velocity of dm at the moment it joins or leaves the body (relative to an observer stationed on the body). Show that the preceding equation is dimensionally compatible.

6. In humans, the hydrostatic pressure of blood contributes to the total blood pressure. The hydrostatic pressure P is a product of blood density, height h of the blood column between the heart and some lower point in the body, and gravity g. Determine

$$P = k\rho^{a}h^{b}g^{c}$$

where k is a dimensionless constant.

7. Assume the force F opposing the fall of a raindrop through air is a product of viscosity μ, velocity v, and the diameter r of the drop. Assume that density is neglected. Find

$$F = k\mu^{a}v^{b}r^{c}$$

where k is a dimensionless constant.

2.3 The process of the Dimensional Analysis

Activity 2.3:-

- What are dimensionally homogeneous equations?
- How can you use dimensionless products to determine dimensionally homogeneous equations?
- When is an equation dimensionally homogeneous?
- What are the basic steps in dimensional analysis process?

In the preceding section we learned how to determine all dimensionless products among the variables selected in the problem under investigation. Now we investigate how to use the dimensionless products to find all possible dimensionally homogeneous equations among the variables. The key result is Buckingham's theorem, which summarizes the entire theory of dimensional analysis.

Example 1 in the preceding section shows that in general many dimensionless products may be formed from the variables of a given system. In that example we determined every dimensionless product to be of the form

$$g^{b}t^{2b}r^{-b}\theta^{e} \qquad (2.9)$$

where b and e are arbitrary real numbers. Each one of these products corresponds to a solution of the homogenous system of linear algebraic equations given by Equation (2.5). The two products

$$\pi_1 = \theta \quad and \quad \pi_2 = \frac{gt^2}{r}$$

obtained when b=0, e=1, and b=1, e=0, respectively, are special in the sense that any of the dimensionless products (2.9) can be given as a product of some power of π_1 times some power of π_2 . Thus, for instance,

$$g^3t^6r^{-3}\theta^{1/2} = \pi_1^{1/2}\pi_2^3$$

This observation follows from the fact that b=0, e=1, and b=1, e=0 represent, in some sense, independent solutions of the system (8.5). Let's explore these ideas further.

Consider the following system of m linear algebraic equations in the n unknowns $x_1, x_2, \ldots, x_n$:

$$\begin{array}{llll} a_{11}x_1 + a_{12}x_2 + & \ldots + a_{1n}x_n & = b_1 \\ \vdots & \vdots & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + & \ldots + a_{mn}x_n & = b_m \end{array} \qquad (2.10)$$

The numbers a_{ij} and b_i denote real numbers for each i=1, 2, . . ., m and j=1, 2, . . ., n. The numbers a_{ij} are called the **coefficients** of the system and the b_i are referred to as the **constants**. The subscript i in the symbol a_{ij} refers to the ith equation of the system (2.10) and the subscript j refers to the jth unknown x_j to which a_{ij} belongs. Thus, the subscripts serve to locate aij. It is customary to read a_{13} as "a, one, three" and a_{42} as "a, four, two," for example, rather than "a, thirteen" and "a, fourty-two".

A solution to the system (8.10) is a sequence of numbers $s_1, s_2, \ldots, s_n$ for which $x_1=s_1$, $x_2=s_2, \ldots, x_n=s_n$ solves each equation in the system. If $b_1=b_2= \ldots =b_m=0$, the system (8.10) is said to be **homogeneous**. The solution $s_1=s_2= \ldots =s_n =0$ always solves the homogeneous system and is called the **trivial solution**. For a homogeneous system there are two solution possibilities: Either the trivial solution is the only solution or there are infinitely many solutions.

Whenever $s_1, s_2, \ldots, s_n$ and $s'_1, s'_2, \ldots, s'_n$ are solutions to the homogeneous system, the sequences $s_1+s'_1, s_2+s'_2, \ldots, s_n+s'_n$, and $cs_1, cs_2, \ldots, cs_n$ are also solutions for any constant c. These solutions are called the **sum** and **scalar multiple** of the original solutions, respectively. If S and S' refer to the original solutions, then we use the notations S+S' to refer to their sum and cS to refer to a scalar multiple of the first solution. If S_1, S_2, . . ., S_k is a collection of k solutions to the homogeneous system, then the solution

$$c_1S_1+c_2S_2+ . . . +c_kS_k$$

is called a **linear combination** of the k solutions, where c_1, c_2, . . . ,c_k are arbitrary real numbers. It is an easy exercise to show that any linear combination of solutions to the homogeneous system is still another solution to the system.

A set of solutions to a homogeneous system is said to be **independent** if no solution in the set is a linear combination of the remaining solutions in the set. A set of solutions is **complete** if it is independent and every solution is expressible as a linear combination of solutions in the set. For a specific homogeneous system, we seek some complete set of solutions because all other solutions are produced from them using linear combinations. For example, the two solutions corresponding to the two choices b=0, e=1 and b=1, e=0 form a complete set of solutions to the homogeneous system (2.5).

It is not our intent to present the theory of linear algebraic equations. Such a study is appropriate for a course in linear algebra. We do point out that there is an elementary algorithm known as Gaussian elimination for producing a complete set of solutions to a given system of linear equations. Moreover, Gaussian elimination is readily implemented on computers and handheld programmable calculators. The system of equations we will encounter in this book are simple enough to be solved by the elimination method learned in intermediate algebra.

How does our discussion relate to dimensional analysis? Our basic goal thus far has been to find all possible dimensionless products among the variables that influence the physical phenomenon under investigation. We developed a homogeneous system of linear algebraic equations to help us determine these dimensionless products. This system of equations usually has infinitely many solutions. Each solution product among the variables. If we sum two solutions, we produce another solution that yields the same dimensionless product as does multiplication of the dimensionless products corresponding to the original two solutions. For example, the sum of the solutions corresponding to b=0, e=1 and b=1, e=0 for equation (2.5) yields the solution corresponding to b=1, e=1 with the corresponding dimensionless product from equation (8.9) given by

$$gt^2r^{-1}\theta = \pi_1\pi_2$$

The reason for this result is that the system of equations is the exponents in the dimensionless products, and addition of exponents algebraically corresponds to multiplication of numbers having the same base: $x^{m+n}=x^mx^n$. Moreover, multiplication of a solution by a constant produces a solution that yields the same dimensionless product as does raising the product corresponding to the original solution to the power of the constant. For example, -1 times the solution corresponding to b=1, e=0 yields the solution corresponding to b=-1, e=0 with the corresponding dimensionless product

$$g^{-1}t^{-2}r = \pi_2^{-1}$$

The reason for this last result is that algebraic multiplication of an exponent by a constant corresponds to raising a power to a power, $x^{mn}=(x^m)^n$.

In summary, addition of solutions to the homogeneous system of equations results in multiplication of their corresponding dimensionless products and multiplication of a solution by a constant results in raising the corresponding product to the power given by that constant. Thus, if S_1 and S_2 are two solutions corresponding to the dimensionless products $\pi_1$and π_2, respectively, then the linear combination aS_1+bS_2 corresponds to the dimensionless product

$$\pi_1^a\pi_2^b$$

It follows from our preceding discussion that a complete set of solutions to the homogeneous system of equations produces all possible solutions through linear combination.The dimensionless products corresponding to a complete set of solutions are therefore called a complete set of dimensionless products. All dimensionless products can be obtained by forming powers and products of the members of a complete set.

Next, let's investigate how these dimensionless products can be used to produce all possible dimensionally homogeneous equations among the variables. In section 2.1 we defined an equation to be dimensionally homogeneous if it remains true regardless of the system of units in which the variables are measured. The fundamental result in dimensional analysis that provides for the construction of all dimensionally homogeneous equations from complete sets of dimensionless products is the following theorem.

Theorem 1

Buckingham's Theorem

An equation is dimensionally homogeneous if and only if it can be put into the form

$$f(\pi_1,\pi_2,\dots,\pi_n) = 0 \qquad (2.11)$$

where f is some function of n arguments and $\{\pi_1,\pi_2,\dots,\pi_n\}$

is a complete set of dimensionless products.

Let's apply Buckingham's theorem to the simple pendulum discussed in the preceding sections. The two dimensionless products

$$\pi_1 = \theta \quad and \quad \pi_2 = \frac{gt^2}{r}$$

form a complete set for the pendulum problem. Thus, according to Buckingham's theorem, there is a function f such that

$$f\left(\theta, \frac{gt^2}{r}\right) = 0$$

Assuming we can solve this equation for $\frac{gt^2}{r}$ as a function of , it follows that

$$t = \sqrt{\frac{r}{g}}h(\theta) \qquad (2.12)$$

where h is some function of the single variableθ. Notice that this last result agrees with our intuitive formulation for the simple pendulum presented in section2.1. Observe that Equation (2.12) represents only a general form for the relationship among the variables m, g, t, r, and θ. However, it can be concluded from this expression that t does not depend on the mass m and is related to $r^{1/2}$ and $g^{-1/2}$ by some function of the initial angle of displacementθ. Knowing this much, we can determine the nature of the function h experimentally or approximate it, as discussed in section 2.1.

Consider equation (2.11) in Buckingham's theorem. For the case in which a complete set consists of a single dimensionless product, for example, π_1, the equation reduces to the form

$$f(\pi_1) = 0$$

In this case we assume that the function f has one real root at k (to assume otherwise has little physical meaning). Hence, the solution $\pi_1 = k$ is obtained.

Using Buckingham's theorem, let's reconsider the example from section 2.1 of the wind force on a van driving down a highway. Because the four variables F, v, A, and were selected and all three equations in (2.8) are independent, a complete set of dimensionless products consists of a single

$$\pi_1 = \frac{F}{v^2 A\mu\rho}$$

Application of Buckingham's theorem gives

$$f(\pi_1) = 0$$

which implies from the preceding discussion that $\pi_1 = k$, or

$$F = kv^2 A\rho$$

where k is a dimensionless constant as before. Thus, when a complete set consists of a *single dimensionless product*, as is generally the case when we begin with four variables, the application of Buckingham's theorem yields the desired relationship up to a *constant of proportionality*. Of course, the predicted proportionality must be tested to determine the adequacy of our list of variables. If the list does prove to be adequate, then the constant of proportionality can be determined by experimentation, thereby completely defining the relationship.

For the case n=2, Equation (2.11) in Buckingham's theorem takes the form

$$f(\pi_1, \pi_2) = 0 \tag{2.13}$$

If we choose the products in the complete set $\{\pi_1, \pi_2\}$ so that the dependent variable appears in only one of them, for example, π_2, we can proceed under the assumption that equation (2.13) can be solved for that chosen product π_2 in terms of the remaining product π_1. Such a solution takes the form

$$\pi_2 = H(\pi_1)$$

and then this latter equation can be solved for the dependent variable. Note that when a complete set consists of more than one dimensionless product, the application of Buckingham's theorem determines the desired relationship up to an arbitrary function. After Verifying the adequacy of the list variables, we may be lucky enough to recognize the underlying functional relationship. However, in general we can expect to construct an empirical model, although the task has been eased considerably.

For the general case of n dimensionless products in the complete set for Buckingham's theorem, we again choose the products in the complete set $\{\pi_1, \pi_2, \dots, \pi_n\}$ so that the dependent variable appears in only one of them, say π_n for definiteness. Assuming we can solve equation (2.11) for that product π_n in terms of the remaining ones, we have the form

$$\pi_n = H(\pi_1, \pi_2, \dots, \pi_{n-1})$$

We then solve this last equation for the dependent variable.

Summary of Dimensional Analysis Methodology

STEP 1 Decide which variables enter the problem under investigation.

STEP 2 Determine a complete set of dimensionless products $\{\pi_1, \pi_2, \ldots, \pi_n\}$ among the variables. Make sure the dependent variable of the problem appears in only one of the dimensionless products.

STEP 3 Check to ensure that the products found in the previous step are dimensionless and independent. Otherwise you have an algebra error.

STEP 4 Apply Buckingham's theorem to produce all possible dimensionally homogeneous equations among the variables. This procedure yields an equation of the form (2.11).

STEP 5 Solve the equation in Step 4 for the dependent variable.

STEP 6 Test to ensure that the assumptions made in Step 1 are reasonable. Otherwise the list of variables is faulty.

STEP 7 Conduct the necessary experiments and present the results in a useful format.

Let's illustrate the first five steps of this procedure.

Example 1 Terminal Velocity of a Raindrop

Consider the problem of determining the terminal velocity v of a raindrop falling from a motionless cloud. We examined this problem from a very simplistic point of view in chapter 2, but let's take another look using dimensional analysis.

What are the variables influencing the behaviour of the raindrop? Certainly the terminal velocity will depend on the size of the raindrop given by, say, its radius r. The density ρ of the air and the viscosity μ of the air will also affect the behaviour. (Viscosity measures resistance to motion---a sort of internal molecular friction. In gases this resistance is caused by collisions between fast-moving molecules.) The acceleration due to gravity g is another variable to consider. Although the surface tension of the raindrop is a factor that does influence the behaviour of the fall, we will ignore this factor. If necessary, surface tension can be taken into account in a later, refined model. These considerations give the following table relating the selected variables to their dimensions:

Variable	v	r	g	ρ	μ
Dimension	LT^{-1}	L	LT^{-2}	ML^{-3}	$ML^{-1}T^{-1}$

Next we find all the dimensionless products among the variables. Any such product must be of the form

$$v^a r^b g^c \rho^d \mu^e \tag{2.14}$$

and hence must have dimension

$$(LT^{-1})^a (L)^b (LT^{-2})^c (ML^{-3})^d (ML^{-1}T^{-1})^e$$

Therefore, a product of the form (2.14) is dimensionless if and only if the following system of equations in the exponents is satisfied:

$$\left.\begin{array}{rrl} & d+e & =0 \\ a+b+c & -3d-e & =0 \\ -a-2c & -e & =0 \end{array}\right\} \tag{2.15}$$

Solution of the system (2.15) gives b=(3/2)d-(1/2)a, c=(1/2)d-(1/2)a, and e=-d, where a and d are arbitrary. One dimensionless product π_1 is obtained by setting a=1, d=0; another, independent dimensionless product $\pi_2$is obtained when a=0, d=1. These solutions give

$$\pi_1 = vr^{-1/2}g^{-1/2} \;\; and \;\; \pi_2 = r^{3/2}g^{1/2}\rho\mu^{-1}$$

Next, we check the results to ensure that the products are indeed dimensionless:

$$\frac{LT^{-1}}{L^{1/2}(LT^{-2})^{1/2}} = M^0L^0T^0 \qquad \text{and} \qquad \frac{L^{3/2}(LT^{-2})^{1/2}(ML^{-3})}{ML^{-1}T^{-1}} = M^0L^0T^0$$

Thus, according to Buckingham's theorem, there is a function f such that

$$f\left(vr^{-1/2}g^{-1/2}, \frac{r^{3/2}g^{1/2}\rho}{\mu}\right) = 0$$

Assuming we can solve this last equation for$vr^{-1/2}g^{-1/2}$ as a function of the second product π_2 , it follows that

$$v = \sqrt{rg}h(\frac{r^{3/2}g^{1/2}\rho}{\mu})$$

where h is some function of the single productπ_2.

The preceding example illustrates a characteristic future of dimensional analysis. Normally the modeler studying a given physical system has an intuitive idea of variables involved and has a working knowledge of general principles and laws (such as Newton's second law) but lacks the precise laws governing the interaction of the variables. Of course, the modeler can always experiment with each independent variable separately, holding the other constant and the effect on the system. Often, however, the efficiency of the experimental work can be improved through an application of dimensional analysis. Although we did not illustrate steps 6 and 7 of the dimensional analysis process for the preceding example, these steps will be illustrated in section 2.3.

We now make some observations concerning the dimensional analysis process. Suppose n variables have been identified in the physical problem under investigation. When determining a complete set of dimensionless products, we form a system of three linear algebraic equations by equating the exponents for M, L, and T to zero. That is, we obtain a system of three equations in n unknowns (the exponents). If the three equations are independent, we can solve the system for three of the unknowns in terms of the remaining n-3 unknowns (declared to be arbitrary). In this case, we find n-3 independent dimensionless products that make up the complete set seek. For instance, in the preceding example, there are five unknowns, a, b, c, d, e, and we determined three of them (b, c, and e) in terms of the remaining (5-3) two arbitrary ones (a and d). Thus, we obtained a complete set of two dimensionless products. When choosing the n-3 dimensionless products, we must be sure that the dependent variable appears in only one of them. We can then solve equation (2.11) guaranteed by Buckingham's theorem for the dependent variable, at least under suitable assumption on the function f in that equation. (The full story telling when such a solution is possible is the content of an important result in advanced calculus known as the implicit function theorem.)

We acknowledge that we have been rather sketchy in our presentation for solving the system of linear algebraic equations that results in the process of determining all dimensionless products. Recall how to solve simple linear systems by the method of elimination of variables. We conclude this section with another example.

Example 2 Automobile Gas Mileage Revisited.

Consider again the automobile gasoline mileage problem presented in Chapter 1. One of our submodels in that problem was for the force of propulsion F_p. The variables we identified that affect the propulsion force are C_r, the amount of fuel burned per unit time, the amount K of energy contained in each gallon of gasoline, and the speed v. Let's perform a dimensional analysis. The following table relates the variables to their dimensions:

Variable	F_p	C_r	K	v
Dimension	MLT^{-2}	L^3T^{-1}	$ML^{-1}T^{-2}$	LT^{-1}

Thus, the product

$$F_p^a C_r^b K^c v^d \tag{2.16}$$

Must have the dimension

$$(MLT^{-2})^a (L^3T^{-1})^b (ML^{-1}T^{-2})^c (LT^{-1})^d$$

The requirement for a dimensionless product leads to the system

$$\left.\begin{array}{lll} a & +c & =0 \\ a+3b & -c+d & =0 \\ -2a-b & -2c & =0 \end{array}\right\} \tag{2.17}$$

Solution of the system (8.17) gives b=-a, c=-a, and d=a, where a is arbitrary. Choosing a=1, we obtain the dimensionless product

$$\pi_1 = F_p C_r^{-1} K^{-1} v$$

From Buckingham's theorem there is a function f with $f(\pi_1) = 0$, so π_1 equals a constant. Therefore,

$$F_p \propto \frac{C_r K}{v}$$

In agreement with the conclusion reached in Chapter 1.

Problems2.2

1.Predict the time of revolution for two bodies of mass m_1 and mass m_2 in empty space revolving about each other under their mutual gravitational attraction.

2. A projectile is fired with initial velocity v at an angle θ with the horizon. Predict the range R.

3. Consider an object falling under the influence of gravity. Assume that air resistance is negligible. Using dimensional analysis, find the speed v of the object after it has fallen a distance s. Let v=f(m,g,s), where m is the mass of the object and g is the acceleration due to gravity. Does yor answer agree with your knowledge of the physical situation?

4. One would like to know the nature of the drg forces experienced by a sphere as it passes through a fluid,. It is assumed that the sphere has a law speed. Therefore, the drag force is highly dependent on the viscosity of the fluid. The fluid density is to be neglected. Use the dimensional anlysis process to develo a model for drag force F as a function of the radius r and velocity m of the sphere and the viscosity $\mu$of the flid.

5. The volume flow rate q for laminar flow in a pipe depends on the pipe radius r, the viscosity μ of the fluid; and the pressure drop per unit length $\frac{dp}{dz}$. develop a model for the flow rate q as a function of r, μ and $\frac{dp}{dz}$.

6. In fluid mechanics, the Reynold number is a dimensionless number involving the fluid velocity v, density ρ, viscosityμ, and a characteristic length r. Use dimensional analysis to find the Reynolds number.

2.4 A Damped Pendulum

Activity 2.4:-

- Can you apply dimensional analysis process on a pendulum problem?

In section 2.1 we investigated the pendulum problem under the assumptions that the hinge is frictionless, the mass is concentrated at one end of the pendulum, and the drag force is neglijible. Suppose we are not satisfied with the results predicted by the concentrated model. Then we can refine the model by incorporating drag forces. If F represnts the total drag force, the problem now is to determine the function

$$t = f(r, m, g, \theta, F)$$

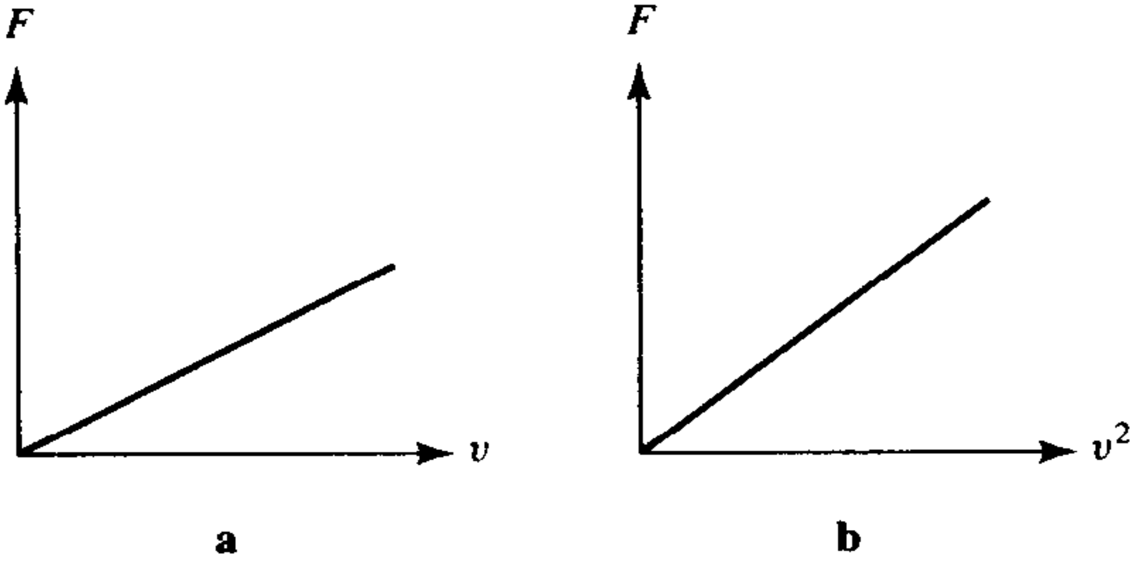


Figure 2.7 Possible submodels for the drag force

Let's consider a submodel for the drag force. As we have seen in previous examples, the modeler is usually faced with a trade-off between simplicity and accuracy. For the pendulum it might seem reasonable to expect the drag force to be proportional to some positive power of the velocity. To keep our model simple, we assume that F is proportional to either v or v^2, as depicted in Figure 2.7.

Now we can experiment to determine directly the nature of the drag force. However, we will first perform a dimensional analysis because we expect it to reduce our experimental effort. Assume F is proportional to v so that $F = kv$. For convenience we choose to work with the dimensional constant $k = \frac{F}{v}$, which has dimension$\frac{MLT^{-2}}{LT^{-1}}$, or simply MT^{-1}. Notice that the dimensional constant captures the assumption about the drag force. Thus, we apply dimensional analysis to the model

$$t = f(r, m, g, \theta, k)$$

An analysis of the dimensions of the variables gives

Variable	t	r	m	g	θ	k
Dimension	T	L	M	LT^{-2}	$M^0L^0T^0$	MT^{-1}

Any product of the variables must be of the form

$$t^a r^b m^c g^d \theta^e k^f \qquad (2.18)$$

And hence must have dimension

$$(T)^a (L)^b (M)^c (LT^{-2})^d (M^0L^0T^0)^e (MT^{-1})^f$$

Therefore, a product of the form (8.18) is dimensionless if and only if

$$\left.\begin{array}{lll} & c+f & =0 \\ b & +d & =0 \\ a & -2d-f & =0 \end{array}\right\} \qquad (2.19)$$

The equations in the system (2.19) are independent, so we know we can solve for three of the variables in terms of the remaining (6-3) three variables. We would like to choose the solutions in such a way that t appears in only one of the dimensionless products. Thus, we choose a, e, and f as the arbitrary variables with

$$c = -f,\ b = -d = \frac{-a}{2} + \frac{f}{2},\ d = \frac{a}{2} - \frac{f}{2}$$

Setting a=1, e=0, and f=0, we obtain c=0, b=-1/2, and d=1/2 with the corresponding dimensionless product $t\sqrt{g/r}$. Similarly, choosing a=0, e=1, and f=0, we get c=0, b=0, and d=0, corresponding to the dimensionless product θ. Finally, choosing a=0, e=0, and f=1, we obtain c=-1, b=1/2, and d=-1/2, corresponding to the dimensionless product $\frac{k\sqrt{r}}{m\sqrt{g}}$. Notice that t appears in only the first of these products. From Buckingham's theorem, there is a function h with

$$h\left(t\sqrt{g/r}, \theta, \frac{k\sqrt{r}}{m\sqrt{g}}\right) = 0$$

Assuming we can solve this last equation for $t\sqrt{g/r}$, we obtain

$$t = \sqrt{r/g}H(\theta, \frac{k\sqrt{r}}{m\sqrt{g}})$$

for some function H of two arguments.

Testing the Model (Step 6)

Given $t = \sqrt{r/g}H(\theta, \frac{k\sqrt{r}}{m\sqrt{g}})$, our model predicts that $\frac{t_1}{t_2} = \sqrt{\frac{r_1}{r_2}}$ if the parameters of the function H (namely, θ and $\frac{k\sqrt{r}}{m\sqrt{g}}$) could be held constant. Now there is no difficulty in keeping θ and k constant. However, varying r while simultaneously keeping $\frac{k\sqrt{r}}{m\sqrt{g}}$ constant is more complicated. Because g is constant, we could try to vary r and m in such a manner that remains constant. This might be done using a pendulum with a hollow mass to vary m without altering the drag characteristics. Under these conditions we would expect the plot in figure 2.8.

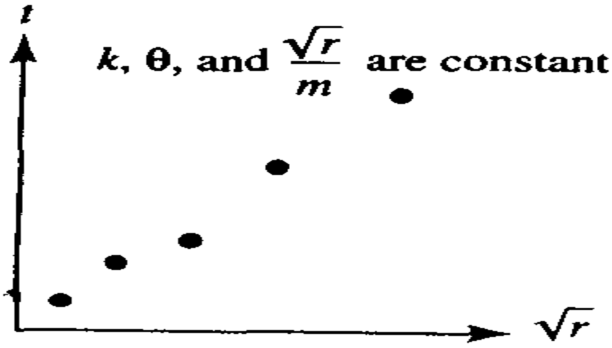


Figure 2.8 A plot of t versus $\sqrt{r}$ keeping the variables k, θ, and $\sqrt{r}$/m constant

Presenting the Results (Step 7)

was suggested in predicting the period of the undamped pendulum, we can plot $t\sqrt{g/r} = H(\theta, \frac{k\sqrt{r}}{m\sqrt{g}})$ However, because H is here a function of two arguments, this would yield a three-dimensional figure that is not easy to use. An alternative technique is to plot $t\sqrt{g/r}$ versus $\frac{k\sqrt{r}}{m\sqrt{g}}$ for various values ofθ. This is illustrated in figure 2.9. To be safe in predicting t over the range of interest for representative values ofθ, it would be necessary to conduct sufficient experiments at various values of $\frac{k\sqrt{r}}{m\sqrt{g}}$. Note that once data are collected, various empirical models could be constructed using an appropriate interpolating scheme for each value of θ.

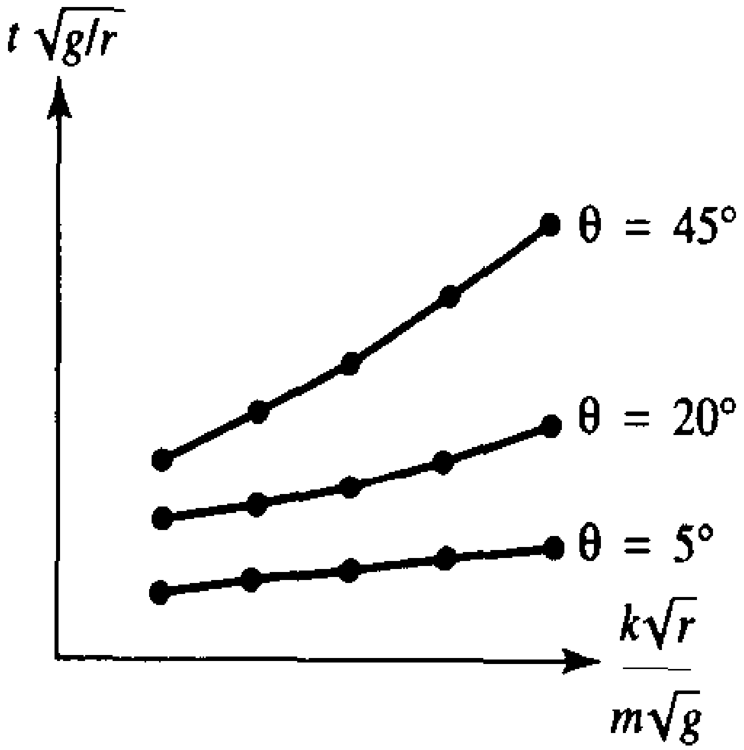


Figure 2.9 presenting the results

Choosing Among Competing Models

Because dimensional analysis involves only algebra, one is tempted to develop several models under different assumptions before proceeding with, perhaps quite costly, experimentation. In the case of the pendulum, under different assumptions, we can develop the following three models (see Problem 1 in the 2.3 problem set):

A: $t = \sqrt{r/g}h(\theta)$ No drag forces

B: $t = \sqrt{r/g}h(\theta, \frac{k\sqrt{r}}{m\sqrt{g}})$ Drag forces proportional to v: F=kv

C: $t = \sqrt{r/g}h(\theta, \frac{k_1 r}{m})$ Drag forces proportional to v^2: $F=k_1v^2$

Because all the preceding models are approximations, it is reasonable to ask which, if any, is suitable in a particular situation. We now describe the experimentation necessary to distinguish among these models, and we present some experimental results.

Model **A** predicts that when the angle of displacement θ is held constant, the period t is proportional to $\sqrt{r}$. Model **B** predicts that when θ and $\frac{r}{m}$ are both held constant, while maintaining the same drag characteristics k, t is proportional to $\sqrt{r}$. Finally, Model **C** predicts that if $\theta, \frac{r}{m}$ and k1 are held constant, then t is proportional to $\sqrt{r}$.

The following discussion describes our experimental results for the pendulum. Various types of balls were suspended from a string in such a manner as to minimize the friction at the hinge. The kinds of balls included tennis balls and various types and sizes of plastic balls. A hole was made in each ball to permit variations in the mass without altering appreciably the aerodynamic characteristics of the ball or the location of the center of mass. The models were then compared with one another. In the case of the tennis ball, Model A proved to be superior. The period was independent of the mass, and a plot of t versus $\sqrt{r}$ for constant θ is shown in figure 2.10.

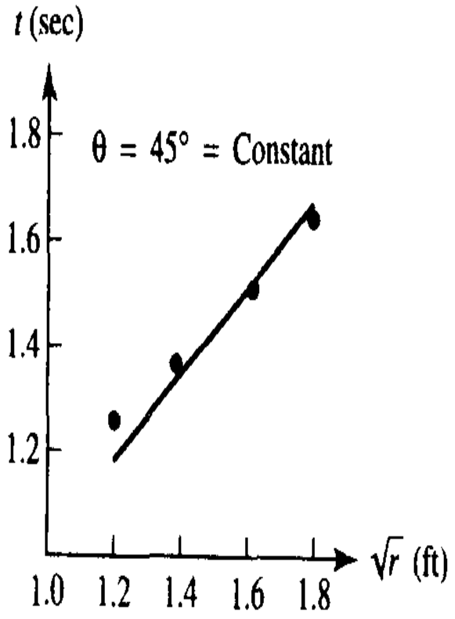


Figure 2.10

Model for a tennis ball

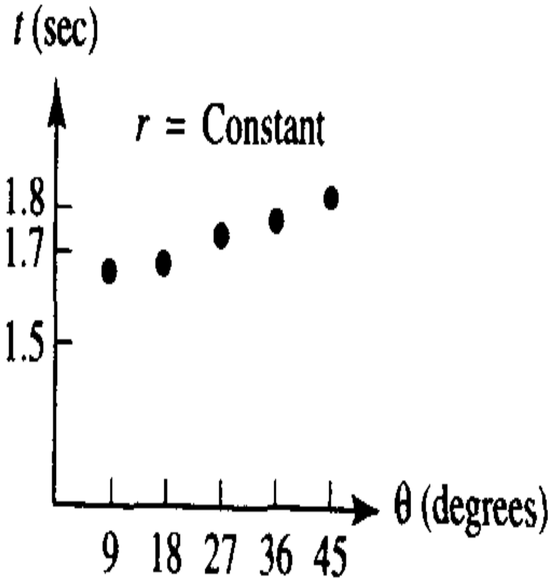


Figure 2.11

Isolating the effect of θ

Having decided that $t = \sqrt{r/g}h(\theta)$ is the best of the models for the tennis ball, we isolated the effect of θ by holding r constant to gain insight into the nature of the function h. A plot of t versus θ for constant r is shown in figure 2.11.

Note from figure 2.11 that for small angles of initial displacement θ, the period is virtually independent of θ. However the displacement effect becomes more noticeable as θ is increased. Thus, for small angles we might hypothesize that $t = c\sqrt{r/g}$ for some constant c. If one plots t versus $\sqrt{r}$ for small angles, the slope of the resulting straight line should be constant.

For larger angles, the experiment demonstrates that the effect of θ needs to be considered. In such cases, one may desire to estimate the period for various angles. For example, if $\theta = 45^0$ and we know a particular value of $\sqrt{r}$, we can estimate t from Figure 2.10. Although not shown, plots for several different angles can be graphed in the same figure.

2.5 Dimensional Analysis in the Model-Building Process

Activity 2.5:-

- In what ways is dimensional analysis useful in the model building process?
- Describe the basic steps of model-building using dimensional analysis.

Let's summarize how dimensional analysis assists in the model-building process. In the determination of a model we must first decide which factors to neglect and which to include. A dimensional analysis provides additional information on how the included factors are related. Moreover, in large problems, we often determine one or more submodels before dealing with the larger problem. For example, in the pendulum problem we had to develop a submodel for drag forces. A dimensional analysis helps us choose among the various submodels.

A dimensional analysis is also useful for obtaining an initial test of the assumptions in the model. For example, suppose we hypothesize that the dependent variable y is some function of five variables:

$y = f(x_1, x_2, x_3, x_4, x_5)$.A dimensional analysis in the MLT system in general yields $\pi_1 = h(\pi_2, \pi_3)$, where each π_i is a dimensionless product. The model predicts that π_1 will remain constant if π_2 and π_3 are held constant, even though the components of π_2 and π_3 may vary. Because there are, in general, an infinite number of ways of choosing π_i, we should choose those that can be controlled in laboratory experiments. Having determined that $\pi_1 = h(\pi_2, \pi_3)$, we can isolate the effect of π_2 by holding π_3 constant and vice versa. This can help explain the functional relationship among the variables. For instance, we say in our example that the period of the pendulum did not depend on the initial displacement for small displacements.

Perhaps the greatest contribution of dimensional analysis is that it reduces the number of experiments required to predict the behavior. If we wanted to conduct experiments required to predict values of y for the assumed relationship $y = f(x_1, x_2, x_3, x_4, x_5)$ and it was decided that 5 data points would be necessary over the range of each variable, 5^5 or 3125 experiments would be necessary. Because a two-dimensional chart is required to interpolate conveniently, y might be plotted against x_1 for five values of x_1, holding x_2, x_3, x_4, x_5 constant. Because x_2, x_3, x_4, and x_5 must vary as well, 5^4 or 625 charts would be necessary. However, after a dimensional analysis yields $\pi_1 = h(\pi_2, \pi_3)$, only 25 data points would be required. Moreover, π_1 can be plotted versus π_2, for various values of π_3 on a single chart. Ultimately, the task is far easier after applying a dimensional analysis. Finally dimensional analysis helps in presenting the results. It is usually best to present experimental results using those π_i that are classical representations within the field of study. For instance, in the field of fluid mechanics there are eight factors that might be significant in a particular situation: velocity v, length r, mass density ρ , viscosityμ , acceleration of gravity g, speed of sound c, surface tension σ , and pressure p. Thus, a dimensional analysis could require as many as five independent dimensionless products. The five generally used are the Reynolds number, Froude number, Mach number, Weber number, and the pressure coefficient. These numbers are defined as follows.

Reynolds number	$\frac{vr\rho}{\mu}$
Froude number	$\frac{v^2}{rg}$

Mach number	$\frac{v}{c}$
Weber number	$\frac{\rho v^2 r}{\sigma}$
Pressure coefficient	$\frac{p}{\rho v^2}$

Thus, the application of dimensional analysis becomes quite easy. Depending on which of the eight variables are considered in a particular problem, the following steps are performed:

1. Choose an appropriate subset from the preceding five dimensionless products.
2. Apply Buckingham's theorem.
3. Test the reasonableness of the choices of variables.
4. Conduct the necessary experiments and present the results in a useful format.

Problems2.3

1. For the damped pendulum,

(a) Assume that F is proportional to v^2 and use dimensional analysis to show that

$$t = \sqrt{r/g}h(\theta, \frac{rk_1}{m}).$$

(b) Assume that F is proportional to v^2 and describe an experiment to test the model

$$t = \sqrt{r/g}h(\theta, \frac{rk_1}{m}).$$

2. Under appropriate conditions, all three models for the pendulum imply that t is proportional to $\sqrt{r}$. Explain how the conditions distinguish between the three models by considering how m must vary in each case.

3. Use a model employing a differential equation to predict the period of a simple frictionless pendulum for small initial angles of displacement. (Hint: Let$\sin\theta = \theta$.) Under these conditions, what should be the constant of proportionality? Compare your results with those predicted by Model **A** in the text.

2.6 Examples Illustrating Dimensional Analysis

Example 1 Explosion Analysis

In excavation and mining operations it is important to be able to predict the size of a crater resulting from a given explosive such as TNT in some particular soil medium. Direct experimentation is often impossible or too costly. Thus, it is desirable to use small laboratory or field tests and then scale this up in some manner to predict the results for explosions far greater in magnitude.

We may wonder how the modeler determines which variables to include in the initial list. Experience is necessary to intelligently determine which variables can be neglected. Even with experience, however, the task is usually difficult in practice, as this example will illustrate. It also illustrates that the modeler must often change the list of variables to get usable results.

Problem Identification Predict the crater volume V produced by a spherical explosive located at some depth d in a particular soil medium.

Assumptions and Model Formulation initially, let's assume that the craters are geometrically similar (see chapter 1), where the crater size depends on three variables: the radius r of the crater, the density ρ of the soil, and the mass W of the explosive. These variables are composed of only two primary dimensions, length L and mass M, and the dimensional analysis results in only one dimensionless product (see problem 1 a in 2.4 problem set):

$$\pi_r = r\left(\frac{\rho}{W}\right)^{1/3}$$

According to Buckingham's theorem, π_r must equal a constant. Thus, the crater dimensions of radius or depth vary with the cube root of the mass of the explosive. Because the crater volume is proportional to r^3, it follows that the volume of the crater is proportional to the mass of the explosive for constant soil density. Symbolically we have

$$V \propto \frac{W}{\rho} \qquad (2.20)$$

Experiments have shown that the proportionality (2.20) is satisfactory for small explosions (less than 300lb of TNT) at zero depth in soils, such as moist alluvium, that have good cohesion. For larger explosions, however, the rule proves unsatisfactory. Other experiments suggest that gravity plays a key role in the explosion process, and because we want to consider extraterrestrial craters as well, we need to incorporate gravity as a variable.

If gravity is taken into account, then we assume crater size to be dependent on four variables: crater radius r, density of soil ρ, gravity g, and charge density E. Here, the charge energy is the mass W of the explosive times its specific energy. Applying a dimensional analysis to these four variables again leads to a single dimensionless product (see problem 1 b in the 2.4 problem set):

$$\pi_{rg} = r(\frac{\rho g}{E})^{1/4}$$

Thus, π_{rg} equals a constant and the linear crater dimensions (radius or depth of the crater) vary with the one-fourth root of the energy (or mass) of the explosive for a constant soil density. This leads to the following proportionality known as the quarter-root scaling and is a special case of gravity scaling:

$$V \propto (\frac{E}{\rho g})^{3/4} \qquad (2.21)$$

Experimental evidence indicates that gravity scaling holds for large explosions (more than 100 tons of TNT) where the stresses in the cratering process are much larger than the material strengths of the soil. The proportionality (2.21) predicts that crater volume decreases with increased gravity. The effect of gravity on crater formation is relevant in the study of extraterrestrial craters. Gravitational effects can be tested experimentally using a centrifuge to increase gravitational accelerations.

A question of interest on explosion analysts is whether the material properties of the soil do become less important with increased charge size and increased gravity. Let’s consider the case in which the soil medium is characterized only by its densityρ. Thus, the crater volume V depends on the explosive, soil density ρ , gravity g, and the depth of burial d of the charge. In addition, the explicit role of material strength or cohesion has been tested and the strength—gravity transition is shown to be a function of charge size and soil strength.

We now describe our explosive in more detail than in previous models. To characterize an explosive, three independent variables are needed: size, energy field, and explosive densityδ. The size can be given as charge mass W, as charge energy E, or as the radius α of the spherical explosive. The energy yield can be given as a measure of the specific energy Q_e or the energy density per unit volume Q_v . The following equations relate the variables:

$$W = \frac{E}{Q_e}$$

$$Q_v = \delta Q_e$$

$$\alpha^3 = \left(\frac{3}{4\pi}\right)\left(\frac{W}{\delta}\right)$$

One choice of these variables leads to the model formulation

$$V = f(W, Q_e, \delta, \rho, g, d)$$

Because there are seven variables under consideration and the MLT system is being used, a dimensional analysis generally will result in four (7-3) dimensionless products. The dimensions of the variables are:

Variable	V	W	Q_e	δ	ρ	g	d
Dimension	L^3	M	L^2T^{-2}	ML^{-3}	ML^{-3}	LT^{-2}	L

Any product of the variables must be of the form

$$V^a W^b Q_e^c \delta^e \rho^f g^k d^m \qquad (2.22)$$

and hence have dimensions

$$(L^3)^a(M^b)(L^2T^{-2})^c(ML^{-3})^{e+f}(LT^{-2})^k(L)^m$$

Therefore, a product of the form (2.22) is dimensionless if and only if the exponents satisfy the following homogeneous system of equations:

$$\begin{array}{llll} M: & b & +e+f & =0 \\ L: & 3a+2c & -3e-3f+k+m & =0 \\ T: & -2c & -2k & =0 \end{array}$$

Solution to this system produces

$$b=\frac{k-m}{3}-a,\quad c=-k,\quad e=a-f+\frac{k-m}{3}$$

where a, f, k, and m are arbitrary. By setting one of these arbitrary exponents equal to 1 and the other three equal to 0, in succession, we obtain the following set of dimensionless products:

$$\frac{V\delta}{W},\ \left(\frac{g}{Q_s}\right)\left(\frac{W}{\delta}\right)^{\frac{1}{3}},\ d\left(\frac{\delta}{W}\right)^{1/3},\ \frac{\rho}{\delta}$$

(Convince yourself that these are dimensionless.) Because the dimensions of ρ and δ are equal, we can rewrite these dimensionless products as follows:

$$\pi_1=\frac{V\rho}{W}$$

$$\pi_2=\left(\frac{g}{Q_s}\right)\left(\frac{W}{\delta}\right)^{1/3}$$

$$\pi_3 = d(\frac{\rho}{W})^{1/3}$$

$$\pi_4 = \frac{\rho}{\delta}$$

So π_1 is consistent with the dimensionless product implied by equation (2.20). Then applying Buckingham’s theorem, we obtain the model

$$h(\pi_1, \pi_2, \pi_3, \pi_4) = 0 \qquad (2.23)$$

or

$$V = \frac{W}{\rho} H(\frac{gW^{1/3}}{Q_g \delta^{1/3}}, \frac{d\delta^{1/3}}{W^{1/3}}, \frac{\rho}{\delta})$$

Presenting the Results

For oil-base clay the value of ρ is approximately 1.53g/cm^2; for wet sand, 1.65; and for desert alluvium,1.60, For TNT, has the value 2.23g/cm^3. Thus, $0.69 < \pi_4 < 0.74$, so for simplicitywe can assume for these soils and TNT that π_4 is constant. Then, equation (8.23) becomes

$$h(\pi_1, \pi_2, \pi_3) = 0 \qquad (2.24)$$

R.M.Schmidt gathered experimental data to plot the surface described by equation (2.24). A plot of the surface is depicted in figure 8.12, showing the crater and volume parameterπ_1 as a function of the scaled energy charge π_2 and the depth of the burial parameter π_3. Cross-sectional data for the surface parallel to the π_1 π_3 plane when $\pi_2 = 1.15x10^{-6}$ are depicted in figure 2.13.

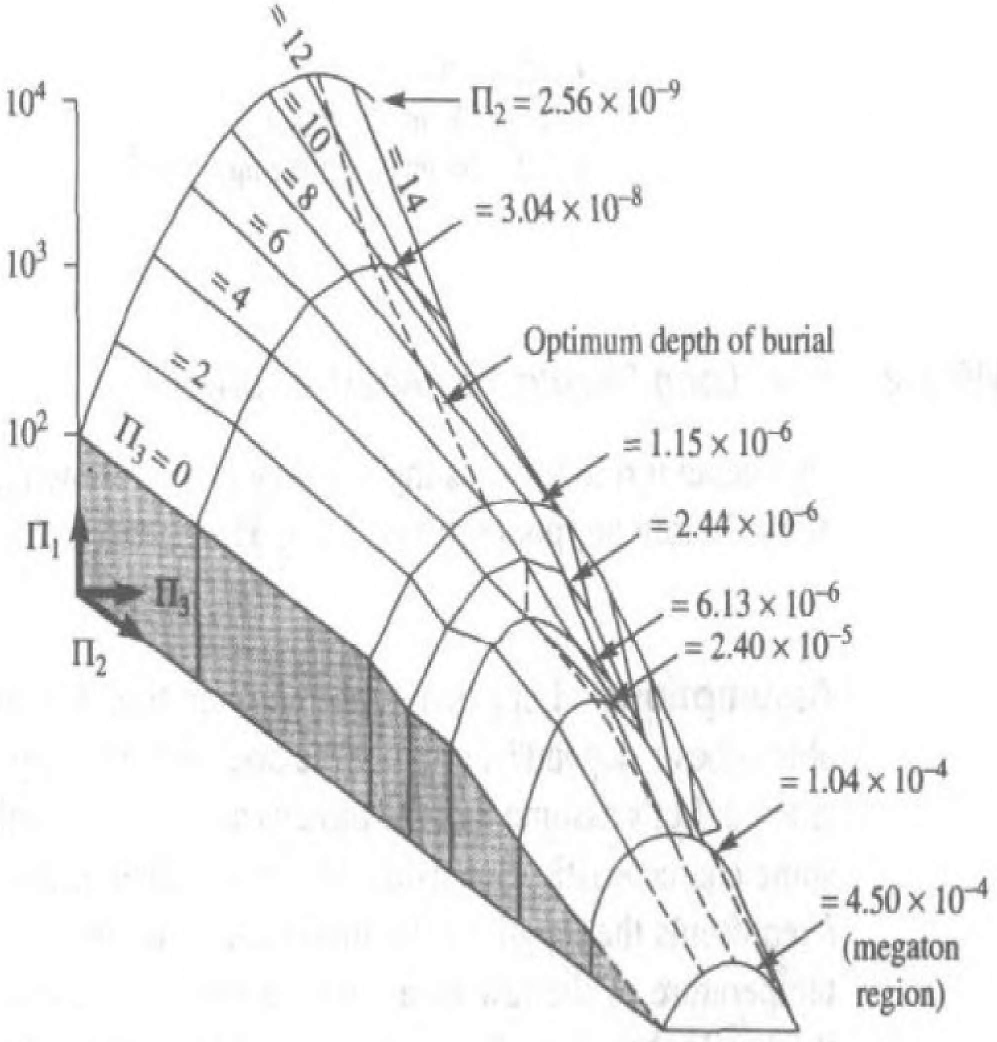


Figure 2.12 A plot of the surface $h(\pi_1,\pi_2,\pi_3) = 0$, showing the crater volume parameter π_1 as a function of gravity- scaled yieldπ_2 and depth of burial parameter π_3.

Experiments have shown that the physical effect of increasing gravity is to reduce crater volume for a given charge yield. This suggests that increased gravity can be compensated for by increasing the size of the charge to maintain the same cratering efficiency. Note also that Figures 2.12 and 2.13 can be used for prediction once an empirical interpolating model is constructed from the data. Holsapple and Schmidt (1982) extend these methods to impact cratering., and Housen, Schmidt, and Holsapple (1983) extend them to crater ejecta scaling.

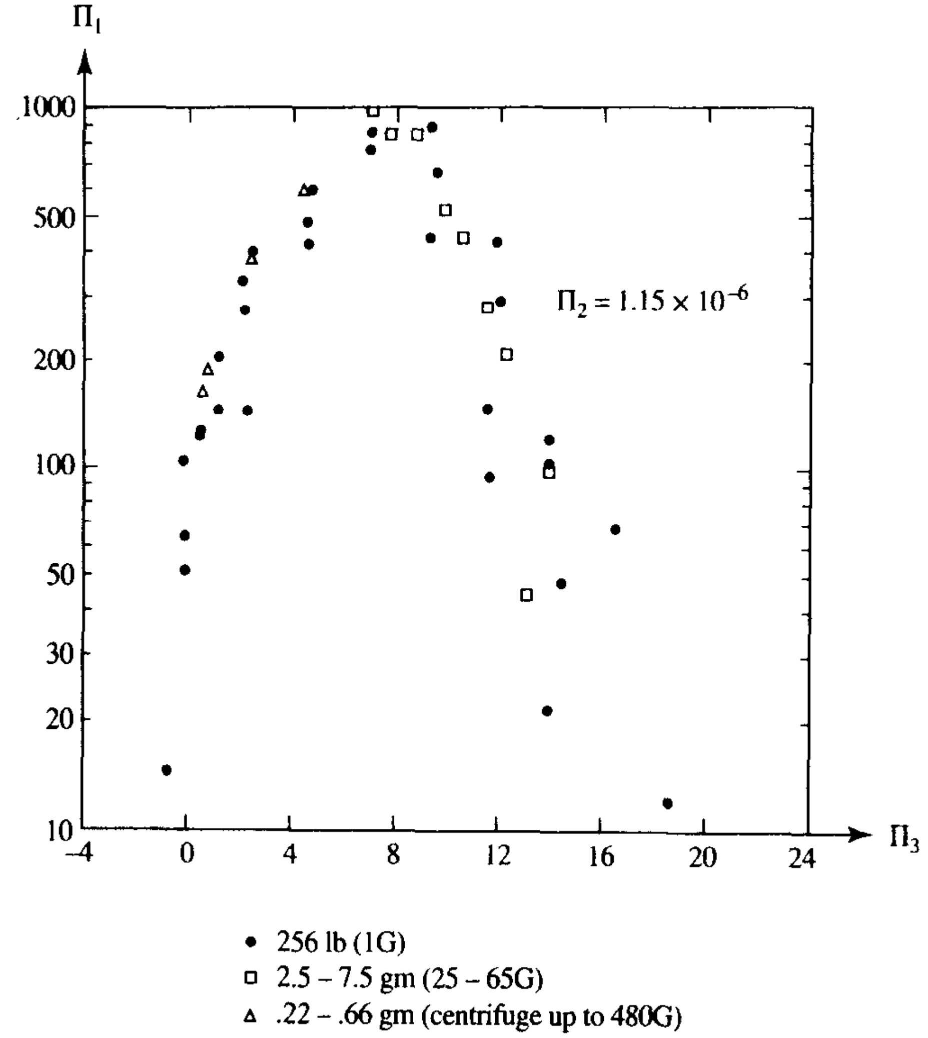


Figure 2.13

Data values for a cross section of the surface depicted in figure 2.12

Example 2 How Long Should You Roast a Turkey?

One general for roasting a turkey is the following: Set the oven to400^0F and allow 20min per pound for cooking. How good is this rule?

Assumptions Let t denote the cooking time for the turkey. Now, on what variables does t depend? Certainly the size of the turkey is a factor that must be considered. Let's assume that turkeys are geometrically similar and use l to denote some characteristic dimension of the uncooked meat; specifically, we assume that l represents the length of the turkey. Another factor is the difference between the temperature of the raw meat and the oven ΔT_m. (We know from experience that it takes longer to cook a bird that is nearly frozen than it does to cook one that is initially at room temperature.) Because the turkey will have to reach a certain interior temperature before it is considered fully cooked, the difference ΔT_c between the temperature of the cooked meat and the oven is a variable determining the cooking time. Finally, we know that different foods require different cooking times independent of size; it takes only 10min or so to bake a pan of cookies, whereas a roast beef or turkey requires several hours. A measure of the factor representing the differences between foods is the coefficient of heat conduction for a particular uncooked food. Let k denote the *coefficient of heat conduction* for a turkey. Thus, we have the following model formulation for the cooking time:

$$t = f(\Delta T_m, \Delta T_c, k, l)$$

Dimensional Analysis Consider the dimensions of the independent variables. The temperature variables ΔT_m and ΔT_c measure the energy per volume and therefore have the dimension $\frac{ML^2T^{-2}}{L^3}$, or simply$ML^{-1}T^{-2}$. Now, what about the heat conduction variable k? **Thermal conductivity** k is defined by the amount of energy crossing one unit cross-sectional area per second divided by the gradient perpendicular to the area. That is,

$$k = \frac{energy/(areaxtime)}{temperature/length}$$

Accordingly, the dimension of k is $\frac{(ML^2T^{-2})(L^{-2}T^{-1})}{(ML^{-1}T^{-2})(L^{-1})}$, or simply L^2T^{-1}. Our analysis gives the following table:

Variable	ΔT_m	ΔT_c	k	l	t
Dimension	$ML^{-1}T^{-2}$	$ML^{-1}T^{-2}$	L^2T^{-1}	L	T

Any product of the variables must be of the form

$$\Delta T_m{}^a \Delta T_c{}^b k^c l^d t^e \tag{2.25}$$

and hence have dimension

$$(ML^{-1}T^{-2})^a (ML^{-1}T^{-2})^b (L^2T^{-1})^c (L)^d (T)^e$$

Therefore, a product of the form (8.25) is dimensionless if and only the exponents satisfy

$$\begin{array}{llllll} M: & a+b & & & & =0 \\ L: & -a-b+2c & & +d & & =0 \\ T: & -2a-2b-c & & & +e & =0 \end{array}$$

Solution of this system of equations gives

$$a=-b, \quad c=e, \quad d=-2e$$

where b and e are arbitrary. If we set b=1, e=0, we obtain a=-1, c=0, and d=0; likewise, b=0, e=1 produces a=0, c=1, and d=-2. These independent solutions yield the complete set of dimensionless products:

$$\pi_1 = \Delta T_m^{-1} \Delta T_c \qquad and \qquad \pi_2 = k l^{-2} t$$

From Buckingham’s theorem, we obtain

$$h(\pi_1, \pi_2) = 0$$

or

$$t = \left(\frac{t^2}{k}\right) H\left(\frac{\Delta T_c}{\Delta T_m}\right) \tag{2.26}$$

The rule stated in our opening remarks gives the roasting time for the turkey in terms of its weight w. Let's assume the turkeys are geometrically similar, or $V \propto l^3$. If we assume the turkey is of constant density (which is not quite correct because the bones and flesh differ in density), then because weight is density times volume and volume is proportional to l^3, we get $w \propto l^3$. Moreover, if we set the oven to a constant baking temperature and specify that the turkey must initially be near room temperature (65^0F), then $\frac{\Delta T_c}{\Delta T_m}$ is a dimensionless constant. Combining these results with Equation (8.26), we get the proportionality

$$t \propto w^{2/3} \qquad (2.27)$$

because k is constant for turkeys. Thus, the required cooking time is proportional to weight raised to the two-thirds power. Therefore, if t_1 hours are required to cook a turkey weighing w_1 pounds and t_2 is the time for a weight of w_2 pounds,

$$\frac{t_1}{t_2} = (\frac{W_1}{w_2})^{2/3}$$

it follows that a doubling of the weight of a turkey increases the cooking time by the factor $2^{2/3} \approx 1.59$.

How does our result (8.27) compare to the rule stated previously? Assume that ΔT_m, ΔT_c, and k are independent of the length or weight of the turkey, and consider cooking a 23-lb turkey versus an 8-lb bird. According to our rule, the ratio of cooking times is given by

$$\frac{t_1}{t_2} = \left(\frac{20x23}{20x8}\right) = 2.875$$

On the other hand, from dimensionless analysis and equation (8.27),

$$\frac{t_1}{t_2} = (\frac{23}{8})^{2/3} \approx 2.02$$

Thus, the rule predicts it will take nearly three times as long to cook a 23-lb bird as it will to cook an 8-lb turkey. Dimensional analysis predicts it will take only twice as long. Which rule is correct? Why have so many cooks overcooked a turkey?

Testing the results Suppose that various sized turkeys are cooked in an oven preheated to 325^0F. The initial temperature of the turkeys is 65^0F. All the turkeys are removed from the oven when their internal temperature, measured by a meat thermometer, reaches 195^0F. The (hypothetical) cooking times for the various turkeys are recorded as follows:

W(lb)	5	10	15	20
t(hr)	2	3.4	4.5	5.4

A plot of t versus $w^{2/3}$ is shown in Figure 2.14. Because the graph approximates a straight line through the origin, we conclude that $t \propto w^{2/3}$, as predicted by our model.

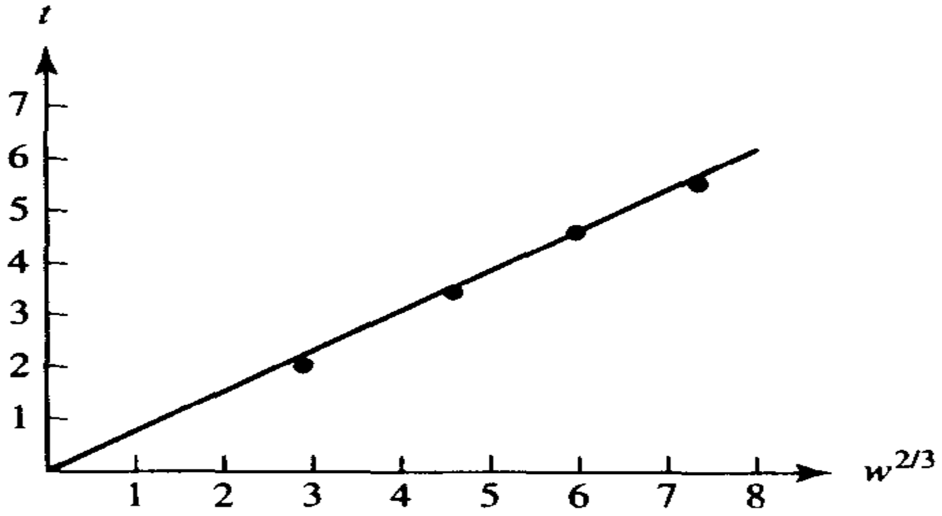


Figure 2.14 Plot of cooking times versus weight to the two-thirds power reveals the predicted proportionality

Problems2.4

1. (a) Use dimensional analysis to establish the cube-root law

$$r\left(\frac{\rho}{W}\right)^{1/3} = constant$$

for scaling of explosions, where r is the radius or depth of the crater, ρ is the density of the soil medium, and W the mass of the explosive.

(b) Use dimensional analysis to establish the one-fourth root law

$$r\left(\frac{\rho g}{E}\right)^{1/4} = constant$$

for scaling explosions, where r is the radius or depth of the crater, ρ is the density of the soil medium, g is gravity, and E is the charge energy of the explosive.

2. (a) Show that the products π_1, π_2, π_3, π_4 for the refined explosion model in the module are dimensionless products.

(b) Assume ρ is essentially constant for the soil being used and restrict the explosive to a specific type, say TNT. Under these conditions, $\frac{\rho}{\delta}$ is essentially constant, yielding$\pi_1 = f(\pi_2, \pi_3)$. You have collected the following data with $\pi_2 = 1.5x10^{-6}$:

π_3	0	2	4	6	8	10	12	14
π_1	15	150	425	750	825	425	250	90

i. Construct a scatterplot of π_1 versus π_3 . Does a trend exist?

ii. How accurate do you think the data are? Find an empirical model that captures the trend of the data with accuracy commensurate with your appraisal of the data.

iii. Use your empirical model to predict the volume of a crater using TNT in desert alluvium with (CGS system) $W = 1500g,\ \rho = \frac{1.53g}{cm^3},\qquad and\ \pi_3 = 12.5.$

1. Consider a zero-depth burst, spherical explosive in a soil medium. Assume the value of the crater volume V depends on the explosive, energy yield, and explosive energy, as well as on the strength Y of the soil (considered a resistance to pressure with dimensions $ML^{-1}T^{-2}$), soil density ρ, and gravity g. In this problem assume

$$V = f(W, Q_s, \delta, Y, \rho, g)$$

And use the following mass set of dimensionless products:

$$\pi_1 = \frac{V\rho}{W} \qquad \pi_2 = (\frac{g}{Q_s})(\frac{W}{\delta})^{1/3}$$

$$\pi_3 = \frac{Y}{\delta Q_s} \qquad \pi_4 = \frac{\rho}{\delta}$$

Self Test Exercises 2

I. Definitions and Terminologies

1. Define each of the following terms:
   (a) Dimensional analysis method
   (b) Dimensional compatibility
   (c) Dimensional constant
   (d) Complete set of solutions
   (e) Dimensionless products
   (f) Dimensionally homogeneous equation
2. If a function f under an investigation has n arguments, how many dimensionless products will be considered in MLT system of units, where
   (a) n=5?
   (b) n=7?
   (c) n=k? (k is a positive integer)
3. State Buckingham's theorem.
4. Describe the seven basic steps in the dimensional analysis methodology.

II. True/false items

1. A dimensional analysis provides qualitative information about the model.
2. Dimensional analysis is helpful in testing the validity of including or neglecting a particular factor.

3. Dimension representations in dimensional analysis are dependent on particular system of measurement.

III. Problems.

1. Find a dimensionless product relating the torque τ (ML^2T^{-2}) produced by an automobile engine, the engine's rotation rate ψ (T^{-1}), the volume V of air displaced by the engine, and the air density ρ.
2. Using dimensional analysis, find a proportinality relationship for the centrifugal force F of a particle in terms of its mass m, velocity v, and radius r of the curvature of its path.
3. The power P delivered to a pump depends on the specific weight w of the fluid pumped, the height h to which the fluid is pumped, and the fluid flow rate q in cubic feet per second. Use dimensional analysis to determine an equation for power.

Chapter 3

GRAPHICAL METHODS

Objective:- at the end of this chapter student able to

- Know graphs to model real life activity
- Identify comparative statistics
- Answer stability question

3.1Using Graphs in Modeling

Graphs can be very useful in modeling if you are aware of their uses and limitations. Since many people expect either too much or too little from them, we discuss their uses and limitations before going into specific models.

People can take in an entire picture rather quickly and then deduce consequences by using their geometric intuition. It follows that graphs should be useful in conveying information. Those wonderful analog computers people carry in their skulls can rapidly locate certain patterns in visually presented data. One of the easiest to spot is a straight line. For this reason a variety of forms of graph paper (rectangular, polar, log-log, normal probability, etc.) are marketed so that plotted data will appear linear if the anticipated relationship exists.

Graphs are most useful in conveying qualitative relationships or approximate data which involve only a few variables. A graphical approach to a problem is most likely to be useful when not much information is available or when it is given in a rather imprecise form. Analytical methods are usually more appropriate when more precise information is available. In complex simulation models, graphs are frequently used to illustrate the qualitative behavior of several time varying endogenous variables simultaneously. This helps one obtain a qualitative feel for the behavior of a complicated simulation model.

So far we have talked about graphs primarily as a way of presenting data. Now let's consider some major roles graphs play in model formulation.

Since our imagination is limited to three dimensions, graphical representations of the interrelations of more than three variables are not directly useful. However, it is often possible to graph a function with most variables held fixed and then determine how the graph will change when one of the fixed variables is changed. This is the heart of the geometric approach to comparative statics which is discussed in Section 3.2. The differential calculus approach parallels the geometric arguments and provides a firm foundation for making statements when any number of variables is involved. The basic problem of comparative statics can be stated as follows : How does the equilibrium point of a system move when certain exogenous variables are changed ? For example, how will the output of a firm be affected by a higher tax rate?

Graphical methods are also useful in studying stability questions. The analytical treatment of local and global stability theory is not easy. Therefore it is desirable to use graphical methods whenever possible to suggest and perhaps prove results. Section 3.3 touches on this approach.

As a glance at the figures in this chapter shows, the intersections of curves are of major importance in comparative statics. This is because they determine the equilibrium points. A subtler observation is that slopes of curves play a central role in stability questions. The slope of a curve is a rate, and rates play a crucial role in stability theory.

Finally, graphical arguments are useful in optimization problems especially if the model is not quantitative.

3. 2. COMPARATIVE STATICS

The Nuclear Missile Arms Race

The United States and the U. S.S.R. both feel that they require a certain minimum number of intercontinental ballistic missiles (ICBMs) to avoid “nuclear blackmail." The idea is to ensure that enough missiles will survive a sneak attack so that “unacceptable damage " can be inflicted on the attacker. Given this philosophy, it is claimed by some and denied by others that the introduction of antiballistic missiles (ABMs) and/or multiple warheads on each missile (MIRVs) will cause both nations to increase their stock of missiles. Is this true? What about making missiles less vulnerable to attack by hardening silos or building missile firing submarines? The wrong answers to these questions could have drastic consequences. Who is right?

Obviously we cannot hope to settle the debate. However, a simple graphical model can shed some light on the problems involved and hopefully help lead to more intelligent debate. The following discussion is adapted from T. L. Saaty (1968, pp. 22-25).

We deal with two countries which we call country 1 and country 2.

Let x and y be the number of missiles possessed by countries 1 and 2, respectively. W e treat x and y as real numbers. O f course they are actually integers; but since they are large, the relative errors introduced by treating them as real numbers will be small; for example, the percentage difference between 500 and 500. 5 is quite small. For the time being we assume that all missiles are the same and are equally protected. From the above discussion it follows that there exist continuous, increasing functions f and g such that country 1 feels safe if and only if x>f(y) and country 2 feels safe if and only if y > g(x). These functions are plotted in Figure 1. The shaded region is the area in which armaments are stable, since both countries feel they have sufficient weapons to prevent a sneak attack. We consider questions such as: Does such a region actually exist? What effects do such things as ABMs, MIRVs, and so on, have on the point $A=(x_m, y_m)$?

First we show that the solid curves in Figure 1 are qualitatively correct. Let's look at things from the point of view of country 1. A certain number of missiles x_0 is needed to inflict what is considered unacceptable damage on country 2. When country 2 has no missiles, country 1 requires x_0.

We show that for any r > 0 the curve x = f(y) crosses the line y = rx. It suffices to show that there is a function x(r) such that, whenever $x \geq x(r)$ and y = rx, country 1 believes that it has enough missiles so that the number surviving a sneak attack by country 2 will be able to inflict unacceptable damage on country 2. In other words, country 1 wants to be practically certain of at least x_0 of its missiles surviving a sneak attack by country 2. Suppose that y = rx. To destroy the most missiles, country 2 should aim about r missiles at each of country 1 's missiles. Since a warhead may fail to reach and destroy its target, there is some probability, p(r) > 0, that a given missile belonging to country I will survive a sneak attack.

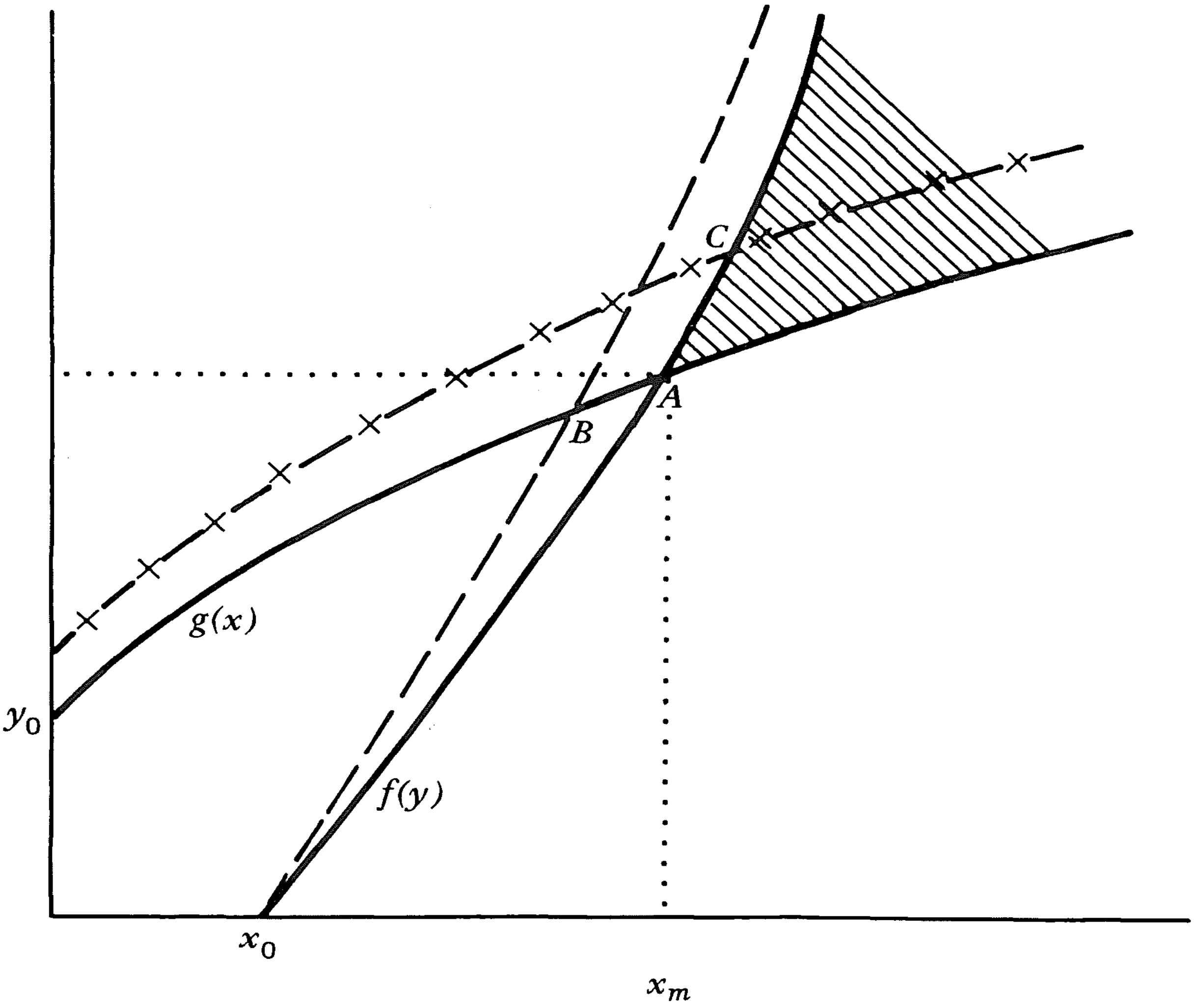


Figure1: Country 1 introduces ABMs. A = initial status (shaded area stable); B = Country 1 protects its missiles; C = country 1protects its cities. Axes show number of missiles.

Thus country 1 can expect $xp(r)$ missiles to survive. For large enough $x = x(r)$, this will exceed x_0 by an amount large enough to allow for uncertainties. This completes the proof that the curves intersect. Thus the curve x = f(y) starts at (x_0 , 0) and curves upward with a slope increasing to ∞ . By a symmetry argument, y = g(x) has the form shown, with a slope decreasing to 0.

Two such curves meet at exactly one point which we call (x_m, y_m), the minimum stable values for x and y.

This analysis applies to all the situations discussed below, so there is always a unique minimum stable point. We want to know how its position compares with (x_m, y_m).

Suppose the missiles of country 1 are made less vulnerable to sneak attack by the use of hardened silos, ABM protection, or some other means. This increases p(r), the probability that any given missile belonging to country 1 will survive a sneak attack. Hence the curve *f(y)* moves to the left with the point x_0 fixed. The shape of the curve is altered somewhat in the process. The new curve is shown dashed in Figure 1. We can see that both *countries require fewer missiles for stability.*

Suppose that country 1 protects its cities by some device such as ABMs. Country 2 now requires more than y_0 missiles to inflict unacceptable destruction on country 1 . Thus the curve g(x) moves upward as shown by the x - x - x curve in Figure 1. *Both countries require more missiles for stability.*

What happens if multiple warheads are installed? This situation is more complicated than the previous two. Suppose country 1 replaces the single warheads on each of its missiles with N warheads. It will then require that fewer of its missiles survive a sneak attack. (The number required is about x_0 / N .) Thus $x = f(y)$ moves to the left as in Figure 2. Country 2 will be faced with N times as many warheads in a sneak attack, so from its point of view the scale of the x axis has changed by about a factor of N, as shown in Figure 2. It appears that country 2 will require more missiles, and country 1 will require fewer; however, this depends on the detailed shape of the curves.

Therefore probabilistic models should be used instead of, or in conjunction with, graphical ones. This would require us to make more precise assumptions regarding the capabilities of the missiles, so we do not go into it here.

It seems unreasonable to assume that country 2 will not also develop and deploy multiple warheads if country 1 does. Therefore we should analyze the situation in which both countries deploy multiple warheads. There are two conflicting effects:

1. Since the axes measure missiles, the points [f(0), 0] and [0, g(O)] will move toward the origin, tending to decrease (x_m, y_m)
2. f(y) becomes more horizontal and g(x) becomes more vertical, tending to increase (x_m, y_m)

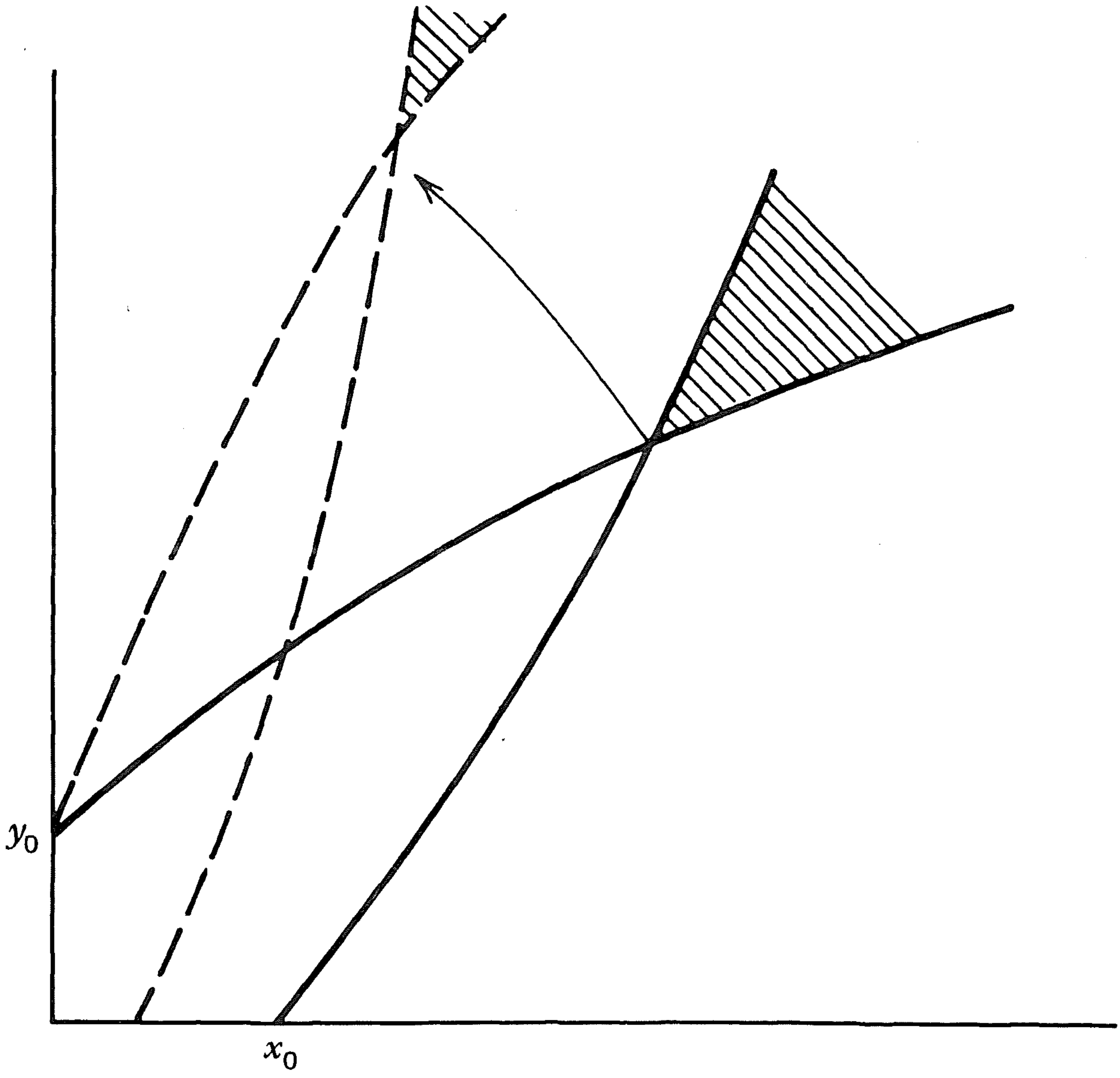


Figure 2 Country 1 introduces MIRVs. Axes show number of missiles.

We cannot decide without further information which effect will dominate.

In the above discussion, we assumed that all missiles were the same. This is unrealistic. If we drop this assumption, each country will change its strategy by aiming different numbers of missiles at the various enemy missiles. Of these, some targeting makes the expected surviving firepower a minimum. This targeting gives the curves for Figure 1, and the analysis proceeds as before.

Activity

1. Suppose that both countries install N warheads in each missile and that the new warheads are as effective as the old ones. Show that both countries will require more warheads.
2. Suppose a country is able to retarget missiles in flight so as to aim for missiles that previous warheads have failed to destroy. Discuss the effect.

3.3 STABILITY QUESTIONS

Cobweb Models in Economics

Definition:-The cobweb model is a graphical method for finding and testing fixed points for stability. Its graphs are called cobweb diagrams. This method is exceptional for the degree of visual insight that it gives, although to find a fixed point this way requires very precisely drawn graphs.

We consider the dynamics of supply and demand when there is a fairly constant time lag in production as, for example, in agriculture. It has been observed that there are fairly regular price fluctuations in such situations. This situation was studied by economists in the 1920s and 1930s.

When a commodity is marketed, the selling price is determined by the *demand curve.* This price is one of the factors producers use in determining how to alter production. In a " pure " situation, they produce the amount on the supply curve that corresponds to the present price. There we were interested in the intersection point of the curves. Thus (see
The following figure)

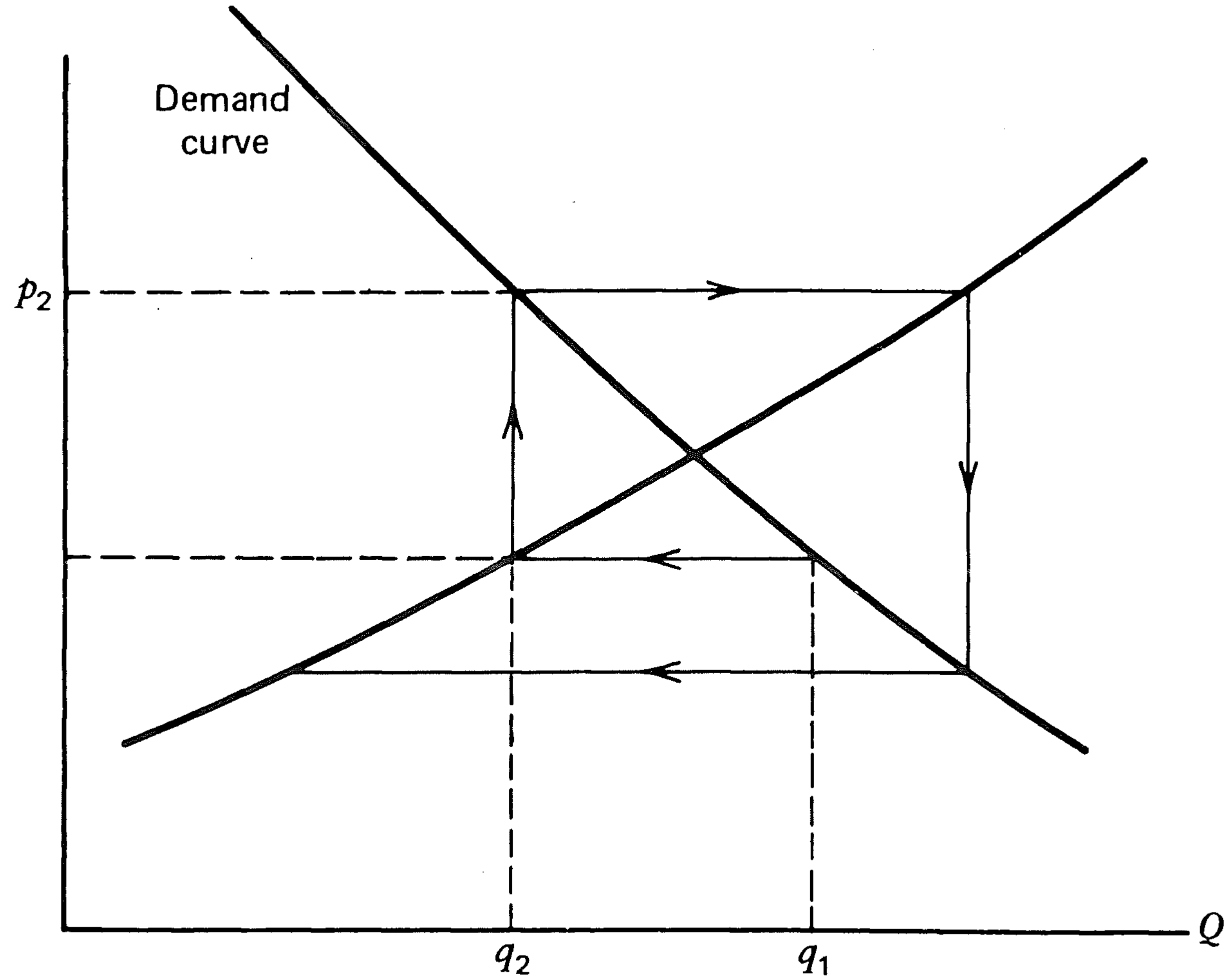


Figure3 The cobweb model

if the amount of potatoes produced in year 1 is q_1, the price per bushel will be p_1, As a result, farmers will decide to produce the amount q_2, in year 2, the market will set a price p_2 per bushel for this crop, and so on. Because of the picture, this idea is referred to as the cobweb theorem. In practice one does not know the supply and demand curves, but the above model predicts that the demand curve can be obtained by plotting (q_n, p_n) and the supply curve by plotting (q_n, p_{n-1}).

How realistic is this model? The existence of a supply curve assumes that producers can control output perfectly. This is not true in the agricultural sector where weather is very important, but it may be a reasonable approximation. If the supply and demand curves move erratically, the model will be upset. Changes in prices for other goods the supplier may produce, sudden changes in demand (e.g., the sale of wheat by the United States to the U. S.S.R. in 1972), and sudden changes in supply (e.g., crop blights) may cause this to happen. If the suppliers have some understanding of price fluctuations, they will not raise production levels much in spite of higher prices. However, this does not wreck the model. In this case the supply curve will be nearly independent of price near the equilibrium price, but the model will still apply. It predicts

small fluctuations in supply and a rapid approach to stability. Plot this. Ezekiel presented the material on U.S. potato production contained
in Table 1 . He obtained it from the Bureau of Agricultural Economics.

Discuss what should be used as "quantity" and what should be used as "price" in a cobweb plot and construct the plot. Should the model be modified because the yield per acre is not constant? What about the effect of population growth during the 15 year period? What about the effect of the Depression? Clearly there is a lot of noise (i.e., disturbances we can't hope to take into account in a simple model) in the data. Thus we should see if the data fit the model better than a random set of data would. Can you propose a method for doing this?

From the supply and demand curves near equilibrium it is easy to make a prediction concerning stability. If the negative of the demand curve's slope exceeds the slope of the supply curve, there will be instability; if it is less, stability. Convince yourself of this. Demand for some agricultural products is rather inflexible. When production is sensitive to price, the model predicts instability. The government can attempt to eliminate this by controlling production or prices. The former causes the supply curve to become vertical (or nearly so) above (and/or below) certain ranges of quantity. This keeps the instability from growing further.

Activity

Discus on the effect of price control on cobweb models

Phase planes

The previous model dealt with the stability of a difference equation. A similar procedure is used for differential equations. This requires the notion of a phase plane, suppose we are dealing with the two equations

$$x' = f(x, y), \quad y' = g(x, y) \qquad (1)$$

At each point (x, y) in the x - y plane we can plot a vector proportional to (x', y'). This is called the direction field of (1). To graph a solution of (1) we then start at an initial point and follow a path parallel to the direction field. (Since the direction field varies from point to point, the path is usually curved.) The speed is determined by the magnitude of the vector tangent to the path at that point. If we start at a point with If=g=0, we will not move from it. Such points are called equilibrium points.

Since we have only crude information about f and g, our phase plane diagrams cannot be this detailed. To answer stability questions it is often sufficient to plot the two curves f = 0 and g = 0 and indicate roughly the vectors (x', y') in the neighborhood of these curves. The intersections of the curves are the equilibrium points of (1). The curve f = 0 divides space into two regions such that x' > 0 in one and x' < 0 in the other. If you determine which region is which for f = 0, and likewise for g = 0, the rest will be easy.

The vectors cross f = 0 vertically, and the direction will be upward if and only if g > O. Similarly, they cross g = 0 horizontally, and the direction will be rightward if and only if f> o. See Figure for an example. In plotting f = 0 and g = 0, it is helpful to determine the slopes of the curves.

This can be done by implicit differentiation: For f = 0,

$\frac{dy}{dx} = -\frac{\partial f / \partial x}{\partial f / \partial y}$, and similarly for g = O. It is important to remember that the partial derivatives for the slope of f= 0 are evaluated at values of x and y at which x is at equilibrium; that is, x’=0O. (This is important in determining the sign of $\partial f / \partial x$).The partial derivatives also help decide which region corresponds to f > 0 and which to f < 0 : f > 0 to the right of (or above) f = 0 if and only if $\partial f / \partial x > 0$ (or $\partial f / \partial y > 0$).).

Small - Group Dynamics

You wish to set up a local committee to help elect a candidate to office. What keeps a group together and working? Does more work improve a task oriented group or harm it? Very little mathematical modeling has been done in this area and, unfortunately, the following is rather crude and lacking in practical advice.

We want to study the stability and comparative statics of a group which has a required activity imposed from the outside (a task). The model is taken from H. Simon (1952), who based it on a nonmathematical model proposed by G. C. Homans (1950).

There are four basic functions of time:

I(t), the intensity of *interaction* among the group members.

F(t), the level of *friendliness* among the group members .

A(t), the amount of *activity* within the group.

E(t), the amount of activity imposed on the group by the external environment. The variables can be treated as averages over all group members or as some overall measure for the entire group. We regard I, F, and A as endogenous variables and E as an exogenous variable which we generally treat as being constant.

To make the concepts more concrete, let's consider an example. The imposed activity E is the laying in of firewood. The group may be engaged in this for wages, or they may be friends preparing for winter. The various activities A include locating wood sources, sawing logs, stacking logs, and setting up a football pool. Note that some activities may not be directed toward the externally imposed task. G. C. Homans says, “By our definition interaction takes place when the action of one man sets off the action of another.” "Action" here refers to activity, so that activity is required for interaction, but not conversely-a person can work alone. The many situations in our example that involve interaction include discussing where to obtain wood, working opposite ends of a saw while cutting logs, passing wood from one person to another in stacking, and conversing idly. Some of the interaction is necessary, but a lot of it can be reduced considerably. The same is true of activity, as any efficiency expert knows ; however, this may involve changes in habit patterns and so require more time.

There are three relations on which the model is based:

1. I(t) depends on A(t) and F(t) in such a way that it increases if either A or F does. The adjustment is practically instantaneous.
2. F(t) depends on l(t). It tends to increase when it is too low for the present level of interaction and to decrease if there is not enough interaction to sustain its present level. This adjustment requires time, and the rate of adjustment is greater when the discrepancy between present and equilibrium levels is greater.

Chapter 4

Application of Mathematical Modeling

Introduction

How can we construct and use models in the mathematical world to help us better understand real-world system? Before discussing how we link the two worlds together, let's consider what we mean by a real-world system and why we would be interested in constructing a mathematical model for a system in the first place.

A **system** is an assemblage of objects joined in some regular interaction or independence. The modeler is interested in understanding how a particular system works, what causes changes in the system, and how sensitive the system is to certain changes.

In this section we will discuss different applications of modeling process to express real world phenomenon.

Objectives

At the end of this chapter you will be able to:

- apply proportionality concept in mathematical model forming
- apply similarity concept in mathematical model forming
- form mathematical model by using differential equation
- form mathematical model by using system of differential equations

4.1 Modeling using Proportionality

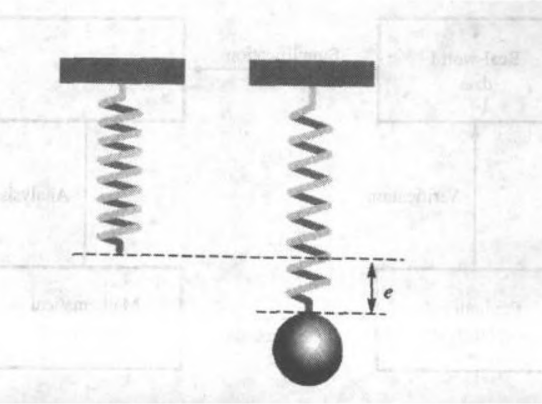
We introduced the concept of proportionality in chapter one in this section we use the concept of proportionality in model formation.

Example1: Testing for Proportionality

Consider a spring-mass system, such as the one showing in figure 1.2. We conduct an experiment to measure the stretch of the spring as a function of the mass (measured as weight) placed on the spring. Consider the data collected for this experiment, dis playing in the table below a scatter plot graph of the stretch or elongation of the spring versus the mass or weight placed on it reveals an approximate straight line passing through the origin.

The data appear to follow the proportionality rule that elongation e is proportional to the mass m, or symbolically, $e \alpha m.$ the straight line appears to pass through the origin.

Figure 4.1 spring-mass system



This geometric understanding allows us to look at the data to determine if proportionality is a reasonable simplifying assumption and, if so, to estimate the slope k. In this case, the assumption is valid, so we estimate the constant of proportionality by picking the two points (200, 3.25) and (300, 4.875) as lying along the straight line. We calculate the slope of the line joining this points as

$$slope = \frac{4.875 - 3.25}{300 - 200} = 0.01625$$

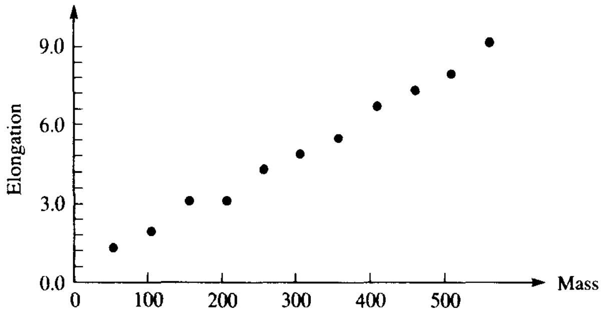


Figure 4.2Data from spring-mass system

Thus the constant of proportionality is approximately 0.0163 and we estimate our model as

$$e = 0.0163m$$

Now $y \alpha x$ if and only if y=kx for some constant k>0 (1)

Of course, if $y \alpha x$, then $x \alpha y$ because the constant k in equation (1) is greater than zero and then $x = \frac{1}{k} y$. The following are other examples of proportionality relationships:

$y \alpha x^2$ if and only if $y = k_1 x^2$ for k_1 a constant (2)

$y \alpha \ln x$ if and only if $y = k_2 \ln x$ for k_2 a constant (3)

$y \alpha e^x$ if and only if $y = k_3 e^x$ for k_3 a constant (4)

In equation (2), $y = kx^2$, k>0, so we also have $x \alpha y^{1/2}$ because $x = (\frac{1}{\sqrt{k}})y^{1/2}$. This leads us to consider how to link proportionalities together, a transitive rule for proportionality:

$$y \alpha x \text{ and } x \alpha z, \text{ then } y \alpha z$$

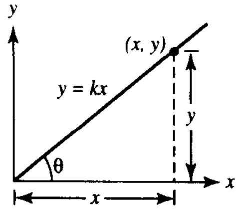


Figure 4.3 Geometrical interpretation of $y \alpha x$

Thus, any variables proportional to the same variables are proportional to one another.

Now let's explore a geometric interpretation of proportionality. In equation (1), $y = kx$ yields $k = y/x$. Thus, k may be interpreted as the tangent of the angle θ depicted in Figure 4.3, and the relation $y \alpha x$ defines a set of points along a line in the plane with angle of inclination θ.

Comparing the general form of a proportionality relationship $y = kx$ with the equation for a straight line $y = mx + b$, we can see that the graph of a proportionality relationship is a line (possibly extended) passing through the origin. If we plot the proportionality variables for Models (2)-(4), we obtain the straight line graphs presented in Figure 4.4.

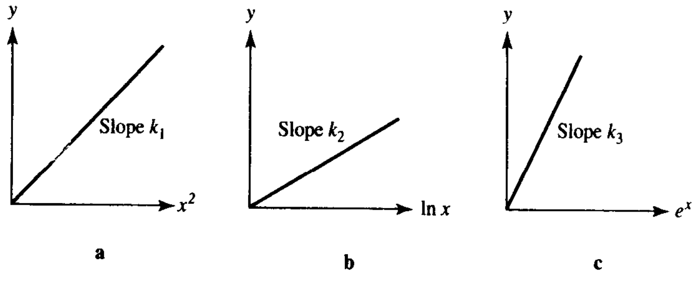


Figure 4.4 Geometrical interpretations of models (a) (2), (b) (3), (c) (4)

Remark:-it is important to note that not just any straight line represents a proportionality relationship: the y- intercept must be zero so that the line passes through the origin.

Example: suppose we are interested in predicting the volume of water displaced by a boat as it is loaded with cargo. Because a floating object displaces a volume of water equal to its weight, we might be tempted to assume that the total volume y of displaced water is proportional to the weight x of the added cargo. However, there is a flaw with that assumption because the unloaded boat already displaces a volume of water equal to its weight. Although the graph of total volume of displaced water versus weight of added cargo is given by a straight line, it is not given by a line passing through the origin (Figure 4.5), so the proportionality assumption is incorrect.

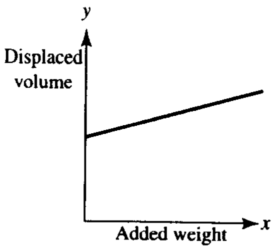


Figure 4.5 A straight –line relationship between displaced volume and total weight, but it is not a proportionality because the line fails to pass through the origin.

Example: Kepler's Third Law

To assist in further understanding the idea of proportionality, let's examine one of the famous proportionalities from Table 1, Kepler's third law. In 1601, the German astronomer Johannes Kepler became director of the Prague Observatory.

Kepler's had formulated his first two laws on the relative motion of the planet:

i) Each planet moves along an ellipse with the sun at one focus.

ii) For each planet, the line from the sun to the planet sweeps areas in equal times.

Activity

1. Show graphically the meaning of the proportionality $y \propto u/v$

2. If an architectural drawing is scaled so that 0.75cm represents 4m, what length represents 27m?

3. Determine whether the following data support a proportionality argument for $y \propto z^{1/2}$. If so, estimate the slope.

y	3.5	5	6	7	8
z	3	6	9	12	15

4. A new planet is discovered beyond Pluto at a mean distance to sun of 4004 millions miles. Using Kepler's third law, determine an estimate for the time T to travel a round the sun in an orbit.

4.2 Modeling Using Geometric Similarity

Geometric similarity is a concept related to proportionality and can be useful to simplify the mathematical modeling process.

Definition: - Two objects are said to be geometrically similar if there is a one to one correspondence between points of the objects such that the ratio of distance between corresponding points is constant for all possible pair's points.

For example, consider the two boxes depicted in Figure 2.2.1. Let l denote the distance between the points A and B in (a), and Let l' be the distance between the corresponding points A' and B' in (b).Other corresponding points in the two figure, and associated distance between the points, are marked the same way. For the boxes to be geometrically similar, it must be true that

$$\frac{l}{l'} = \frac{w}{w'} = \frac{h}{h'} = k. \text{for some constant k>0}$$

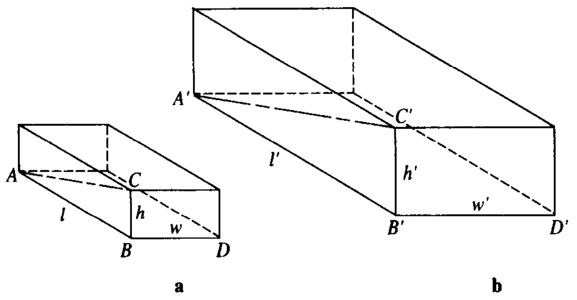


Figure4.6 Two geometrically similar objects

Let's interpret the last result geometrically. In Figure 4.6, consider the triangles ABC and A'B'C'. If the two boxes are geometrically similar, this triangle must be similar. The same argument can be applied to any corresponding triangles, such as CBD and C'B'D'. Thus, corresponding angles are equal for objects that are geometrically similar. In other words the shape is the same for two geometrically similar objects, and one object is simply an enlarged copy of the other.

One advantage that results when two objects are similar is a simplification in certain computations, such as volume and surface area.

For example for the boxes in Figure 2.2.1, consider the following argument for the ratio of the volumes V and V':

$$\frac{V}{V'} = \frac{lwh}{l'w'h'} = k^3 \tag{5}$$

Similarly, the ratio of their total surface areas S and S' is given by

$$\frac{S}{S'} = \frac{2lh + 2wh + 2wl}{2l'h' + 2w'h' + 2w'l'} = k^2 \tag{6}$$

Not only are these ratios immediately known once the scaling factor k has been specified but also the surface area and volume may be expressed as proportionality in terms of some selected characteristic dimensions. Let's select the length l as the characteristic dimension. Then with $l/l' = k$, we have

$$\frac{S}{S'} = k^2 = \frac{l^2}{l'^2}$$

Therefore, $\frac{S}{l^2} = \frac{S'^2}{l'^2} = cons\tan t$ holds for any two geometrically similar objects. This is, surface area is always proportional to the square of the characteristic dimension length:

$$S \propto l^2$$

Likewise, volume is proportional to the length cubed:

$$V \propto l^3$$

Thus, if we are interested in some function depending on an object's length, surface area, and volume, for example:

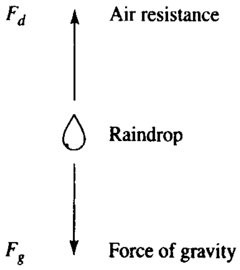
$$y = f(l,S,V)$$

We could express all the function arguments in terms of some selected characteristic dimension, such as length, giving

$$y = (l,l^2,l^3)$$

Geometric similarity is a powerful simplifying assumption.

Example: Raindrops from a Motionless Cloud



Suppose we are interested in the terminal velocity of a raindrop from a motionless cloud. Examining the free-body diagram, the only force acting on the raindrop are gravity and drag.

Assume that the atmospheric drag on the raindrop is proportional to its surface area S times the square of its speed v. The mass m of the raindrop is proportional to the weight of the raindrop (assuming constant gravity in Newton's second law)

$$F = F_g - F_d = ma \tag{7}$$

Under terminal velocity ($v = v_t$), we have a=0 so equation (7) reduced to

$$F_g - F_d = 0$$

Or

$$F_g = F_d$$

We are assuming that $F_d \propto Sv^2$ and that F_g is proportional to weight w . Since $m \propto w$, we have

$$F_g \propto m.$$

Next we assume all the raindrops are geometrically similar. This assumption allows us to relate area and volume so that

$$S \propto l^2 \text{ and } V \propto l^3$$

For any characteristic dimension *l*. Thus, $l \propto S^{1/2} \propto V^{1/3},$ which implies

$$S \propto V^{2/3}$$

Because weight and mass are proportional to volume, the transitive rule for proportionality gives

$$S \propto m^{2/3}$$

From the equation $F_g = F_d$, we now have $m \propto m^{2/3} v_t^{\ 2}$. Solving for the terminal velocity, we have

$$m^{\frac{1}{3}} \propto v_t^2 \text{ or } m^{\frac{1}{6}} \propto v_t$$

Therefore, the terminal velocity of the raindrop is proportional to its mass raised to the one-sixth power.

Activity

1. Consider a 20-kg pink flamingo that stands 3 m in height and has legs that are 2m in length. Model the height and leg length of a 100kg flamingo. What assumption is necessary? Are they reasonable assumptions?
2. A circle of radius r increases by 5% by what percentage the area of the circle increases?
3. How fast is the volume of a rectangle box changing when the length is 6cm, the width is 5cm, and the depth is 4cm, if the length and depth are both increasing at a rate of 1cm/s and the width is decreasing at a rate of 2cm/s?
4. How fast is the surface area of a cube changing when the volume of the cube is 64m^{2}and is increasing at 2m^3/s?

4.3 Modeling Using Differential Equations

Introduction

Many phenomena can be described in a general way by saying that rates of change of the endogenous variables depend on past and present values of the variables. These situations lead to models involving differential and difference equations.

Section objectives

At the end of this section you will able to:

- Apply derivatives as a rate change means
- Apply derivatives as a slope of the tangent line
- Apply First order differential equation in modeling real world phenomenon
- Use higher order differential equations in mathematical modeling

On this section we have information relating a rate of change of a dependent variable with respect to one or more independent variables and are interested in discovering the function relating the variables. For example, if P *represents the number of people in a large population with respect to time* t then it is reasonable to assume that the rate of change of the population with respect to time depends on the current size of P as well as other factors like immigration, emigration, age, gender and so on. For ecological, economical, and other importance reasons, it is desirable to determine a relation between P and t to make prediction about P. If the present population size is denoted by $P(t)$ and the population size at time $t+\Delta t$ is $P(t+\Delta t)$, then the change in population ΔP during that time period Δt is given by

$$\Delta P = P(t+\Delta t) - P(t) \qquad (1)$$

By assuming all other factors listed above neglected: $\Delta P \propto P$. and we can assume that during a unit time period a certain percentage of the population reproduces while a certain percent age dies. Suppose the constant of proportionality k is expressed as a percentage per unit time. Then our proportionality assumption gives

$$\Delta P = P(t+\Delta t) - P(t) = kP\Delta t \qquad (2)$$

Equation (2) is a difference equation in which we are treating a discrete set of time period rather than allowing t to vary continuously over some interval.

Assume that t does vary continuously so that we can take advantage of the calculus. Division of equation (2) by Δt gives

$$\frac{\Delta P}{\Delta t} = \frac{P(t+\Delta t) = P(t)}{\Delta t} = kP \tag{3}$$

Next, allow Δt to approach zero. The definition of the derivatives gives the differential equation

$$\lim_{\Delta t \to 0} \frac{\Delta P}{\Delta t} = \frac{dP}{dt} = kP$$

Where dP/dt represents the instantaneous rate of change.

In modeling the derivatives is used in two distinct roles:

1. To represent the instantaneous rate of change in continuous problems.
2. To approximate an average rate of change in discrete problems.

The advantage of approximating an average rate of change by a derivative is that the calculus often helps in uncovering a functional relationship between the variables under investigation. The interpretation of the derivatives as an instantaneous rate of change is useful in many modeling applications. The geometrical interpretation of the derivative as the slope of the line tangent to the curve is useful for constructing numerical solutions.

Activity:

1. Let's briefly review the derivatives as the slope of the tangent line to the curve from the calculus.

4.3.1 The Derivative as a Rate of Change

The origin of the derivative lies in human kind's curiosity about motion and our need to develop a deeper understanding of motion. The search for the laws governing planetary motion, the study of the pendulum and its application to clock building, and the law governing the flight of a cannonball and so on.

Consider a particle whose distance s from a fixed position depends on time t. Let the graph in Figure 4.7 represent the distance s as a function of time t, and let *(t_1, s_1)* and *(t_2, s_2)* denotes two points on the graph

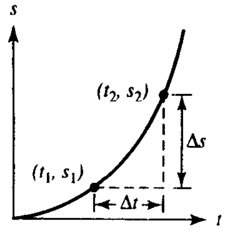


Figure 4.7 Graph of distance s as a function of time t

Define $\Delta t = t_2 - t_1$, $\Delta s = s_2 - s_1$, and from the ratio $\Delta s / \Delta t$ Note that this ratio represents a rate: an increment of distance traveled Δs over some increment of time Δt. That is, the ratio $\Delta s / \Delta t$ represents the average velocity during the time period in question. Now the derivatives ds/dt evaluated at t=t_1 is defined as

$$\left.\frac{ds}{dt}\right|_{t=t_1} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} \tag{4}$$

Discuss on what occurs as $\Delta t \to 0$?

Using the interpretation of average velocity, we can see that at each state of using a smaller Δt we are computing the average velocity over smaller and smaller intervals with left endpoints at t_1 until, in the limit, we have the instantaneous velocity at $t=t_1$. If we think of the motion of a moving vehicle, this instantaneous velocity would correspond to the exact reading of its speedometer at the instant t_1.

4.3.2The Derivatives as the Slope of the Tangent Line

Let's consider another interpretation of the derivative. We consider *s(t)* simply as a curve. Let examine a set of secant lines each emanating from the point *A*= *(t_1, s(t_1))* on the curve. To each secant there corresponds a pair of increments $(\Delta t_i, \Delta s_i)$ as shown in Figure 4.8

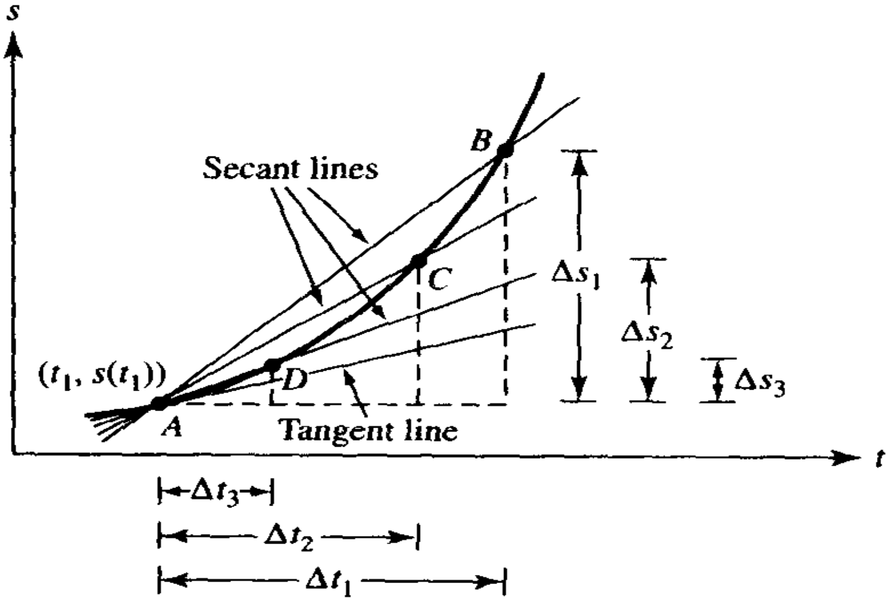


Figure 4.8 The slope of each secant line approximates the slope of the tangent line to the curve at the point A.

The lines AB, AC, and AD are secant lines. As $\Delta t \to 0,$ these secant lines approach the line tangent to the curve at the point A. Because the slope of each secant is $\Delta s/\Delta t$, we may interpret the derivative as the slope of the line tangent to the curve *s(t)* at the point A.

4.3.3 Some Mathematical Models Related to first order differential equations

I. Newton's Law of Cooling

According to Newton's empirical law of cooling, "the rate at which a body cools is proportional to the difference between the temperature of the body and the temperature of the surrounding medium."

To model this physical law if we let T(t) represent the temperature of the body at any time t, T_m represent the constant temperature of the surrounding medium, and dT/dt represent the rate at which a body cools, then Newton's law of cooling translates into the mathematical statement

$$\frac{dT}{dt} \propto T - T_m \text{ or } \frac{dT}{dt} = k(T - T_m) \qquad (5)$$

where k is a constant of proportionality. Since we have assumed the body is cooling, we must have $T > T_m$, and so it stands to reason that k<0.

Example :- Cooling of a cake

When a cake is removed from an oven, its temperature is measured at $300^0 F$. Three minute later its temperature is $200^0 F$. How long will it take for the cake to cool off to a room temperature of $70^0 F$?

Solution:- In (5) we make the identification $T_m = 70.$ We must then solve the initial-value problem

$$\frac{dT}{dt} = k(T - 70), \; T(0) = 300 \qquad (6)$$

And determine the value of k so that $T(3) = 200.$

Equation (6) is both linear and separable. Separating variables,

$$\frac{dT}{T - 70} = kdt$$

yields $\ln|T - 70| = kt + c_1,$ and so $T = 70 + c_2 e^{kt}.$ When t=0,T=300. So that $300 = 70 + c_2$ gives

$c_2 = 230$ and, therefore, $T = 70 + 230e^{kt}.$

Finally, the measurement $T(3) = 200.$ leads to $e^{3k} = \frac{13}{23}$ or $k = \frac{1}{3}\ln\frac{13}{23} = -0.19018.$

Thus

$$T(t) = 70 + 230e^{-0.1918t}. \qquad (7)$$

We note that (7) furnishes no finite solutions to $T(t) = 70.$ since $\lim_{t\to\infty} T(t) = 70.$

Activity

1. A thermometer is taken from an inside room to the outside, where the air temperature is $5^0F.$ After 1 minute the thermometer reads $55^0F.$ and after 5 minute the reading is $30^0F.$ What is the initial temperature of the room?

2. A thermometer is removed from a room where the air temperature is $70^0F.$ to the outside, where the temperature is $10^0F.$ After ½ minute the thermometer reads $50^0F.$ What is the reading at t=1min? how long will it take for the thermometer to reach 15^0F ?

II. Population Growth

How many people will there be in the population of a certain country in n years? How many births? In this section we build the simplest possible model for answering these questions.

Problem Identification: suppose we know the population at some given time, for example,P_0 at time $t=t_0$, and we are interested in predicting the population P at some future time $t=t_1$. In other words, we want to find a population function P(t) for $t_0 \leq t \leq t_1$ satisfying $P(t_0) = P_0$.

Assumptions: Consider some factors that pertain to population growth. Two obvious ones are the birthrate and the death rate. The birth rate and death rate are determined by different factors. The birth rate is influenced by infant mortality rate, attitude toward and availability of contraceptives, attitudes toward abortion, health care during pregnancy, and so forth. The death rate is affected by sanitations and public health, wars, pollutions, medicines, diet, psychological stress and anxiety, and so forth. Other factors that influence population growth in a given region are immigration and emigration, living space restrictions, availability of food and water, and epidemics.

For our model, let's neglect all these latter factors. Now we will consider only the birthrate and death rate. Because knowledge and technology have helped humankinds diminish the death rate below the birthrate, human populations have tended to grow.

Let's begin by assuming that a small unit time period a percentage b of the population is newly born. And a percentage c of the population dies.

So the new population $P(t+\Delta t)$ is the old population *P(t)* plus the number of birth minus the number of death during the time period Δt.

Symbolically $P(t+\Delta t) = P(t) + bP(t)\Delta t - cP(t)\Delta t$ or

$$\frac{\Delta P}{\Delta t} = bP - cP = (b-c)P = kP$$

From our assumptions the average rate of change of the population over an interval is proportional to the size of the population.

Using the instantaneous rate of change to approximate the average rate of change, we have the following differential equation model:

$$\frac{dP}{dt} = kP, \qquad P(t_0) = P_0, \quad t_0 \le t \le t_1 \tag{8}$$

where (for growth) k is a positive constant.

Solving the Model:- We can separate the variables and rewrite equation (8) by moving all terms involving *P* and *dP* to one side of the equation and all terms in *t* and *dt* to the other. This gives

$$\frac{dP}{P} = kdt,$$

Integration of both sides of this last equation yields

$$\ln P = kt + C \tag{9}$$

For some constant C. applying the condition $P(t_0) = P_0$ to equation (9) to find C results in

$$C = \ln P_0 - kt_0$$

Then, substitution for C in to equation (9) gives

$$\ln P = kt + \ln P_0 - kt$$

Or, simplifying algebraically,

$$\ln\frac{P}{P_0} = k(t - t_0)$$

Finally, we obtain the solution

$$P(t) = P_0 e^{k(t-t_0)} \qquad (10)$$

Equation (10), known as the Malthusian model of population growth, predicts that the population grows exponentially with time.

Verifying the Model:- Because $\ln\frac{P}{P_0} = k(t - t_0)$, our model predicts that if we plot $\ln P / P_0 \; versus \; t - t_0$, a straight line passing through the origin with slope k should result. However, if we plot the population data for the United States for several years, the model does not fit very well, especially in the later years. In fact, the 1990 census for the population of the US was 248,710,000, and in 1970 it was 203,211,926. Substituting this values in equation (10) we can get the value of k as

$$\frac{248{,}710{,}000}{203{,}211{,}926} = e^{k(1990-1970)}$$

Thus,

$$k = (\frac{1}{20})\ln\frac{248{,}710{,}000}{203{,}211{,}926} \approx 0.01$$

That is during the 20-year period from 1970-1990, population in the US was increasing at average rate of 1%per year. We can use this information together with equation (10) to predict the population for 2000, in this case t_0=1990, P_0=248,710,000 and k=0.01 yields

$$P(2000) = 248{,}710{,}000e^{0.01(2000-1990)} = 303{,}775{,}080$$

The 2000 census for the population of the US was 281,400,000. Thus our prediction is off the mark by approximately 8%. We can probably live with that magnitude of error, but let's look into the disaster future. Our model predicts that the population of the US will be 55,209 billion in the year 2300, a population that far exceed current estimates of the maximum sustainable population of the entire planet.

We are forced to conclude that our model is unreasonable over the long term.

Activity

Based on Malthusian model of population growth do the following activities

1. The population of a certain community is known to increase at a rate proportional to the number of people present at any time, if the population is doubled in 5 years, how long will it take to triple? To quadruple?
2. Suppose it is known that the population of the community in problem 1 is 10,000 after 3 years. What was the initial population? What will be the population in 10 years?
3. The population of Ethiopia increase at a rate proportional to the number of its inhabitant present at any time t. If the population of Ethiopia was 40 million in 1980 and 52 million in 1990, what will be the population of Eth. In 2010?

Refining the Model to Reflect Limited Growth

Let’s consider that the proportionality factor k, measuring the rate of population growth in equation (8)is now no longer constant but a function of the population. As the population increases and gets closer to the maximum population M, the rate k decreases. One simple sub-model for k is the linear one

$$k=r(M-P), \qquad r>0 \text{ where r is a constant.}$$

Substitution in to equation (8) leads to

$$\frac{dP}{dt} = r(M-P)P, \tag*{(11)*}$$

$$\text{Or} \quad \frac{dP}{P(M-P)} = rdt \tag{12}$$

Again we assume the initial condition $P(t_0)=P_0$. And Equation (11) is referred to ***logistic growth***.

By using partial fraction

$$\frac{1}{P(M-P)}=\frac{1}{M}\left(\frac{1}{P}+\frac{1}{M-P}\right)$$

Thus, Equation (12) can be rewritten as

$$\frac{dP}{P}+\frac{dP}{M-P}=rMdt$$

which integrates to

$$\ln P-\ln\left|M-P\right|=rMt+C \tag{13}$$

For some arbitrary constant C. Using the initial condition, we evaluate C in the case P<M:

$$C=\ln\frac{P_0}{M-P}-rMt_0$$

Substituting in to Equation (13) and simplifying gives

$$\ln\frac{P}{M-P}-\ln\frac{P_0}{M-P}=rM(t-t_0)$$

Or $$\ln\frac{P(M-P_0)}{P_0(M-P)}=rM(t-t_0)$$

Exponentiation both sides of this equation gives

$$\frac{P(M-P_0)}{P_0(M-P)}=e^{rM(t-t_0)}$$

Then,

$$P_0Me^{rM(t=t_0)}=P(M-P_0)+P_0Pe^{rM(t-t_0)}$$

So that solving for the population P gives

$$P(t) = \frac{P_0 M e^{rM(t-t_0)}}{M - P_0 + P_0 e^{rM(t-t_0)}}$$

To estimate P as $t \to \infty$, we rewrite this last equation as

$$P(t) = \frac{P_0 M}{[P_0 + (M - P_0)e^{-rM(t-t_0)}]} \tag{14}$$

The graph of the limited growth Equation (14) is shown in Figure 4.9 for the case P<M. Such a curve is called a **logistic curve.**

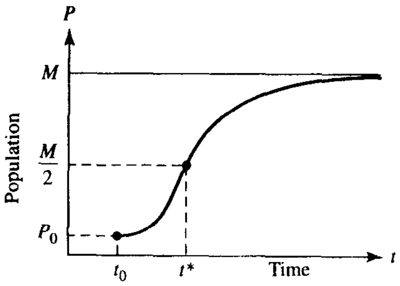


Figure 4.9 Graph of the limited growth model.

Activity

Based on logistic growth Equation (11) and its limiting solution Equation (14) answer the following question.

1. For what value of P the maximum rate of growth occur?
2. Show that the population P in the logistic equation reaches half the maximum population M at time t* given by

$$t^* = t_0 - (1/r\,M)\ln[P_0/(M - P_0)]$$

3. Consider the solution of Equation (11) evaluate the constant C in Equation (13) in the case that P>M for all t.

III. Prescribing Drug Dosage

The problem of how much a drug dosage to prescribe and how often the dosage should be administered is an important one in pharmacology. For most drugs there is a concentration below which the drug is ineffective and a concentration above which the drug is dangerous.

Problem Identification:- How can the doses and the time between doses be adjusted to maintain a safe but effective concentration of the drug in the blood.

The concentration in the blood resulting from a single dose of a drug normally decreases with time as the drug is eliminated from the body (Figure 4.10).

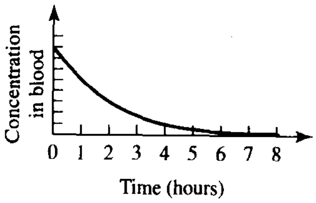


Figure 4.10The concentration of a drug in the bloodstream decreases with time

We are interested in what happens to the concentration of the drug in the blood as doses are given at regular intervals.

Now our aim is to Model this pharmacological idea in to Mathematical concept as follow

Let *H* denotes the highest save level of the drug in the bloodstream and

L denotes the lowest effective level of the drug in the bloodstream; it would be desirable to prescribe a dosage C_0 with time *T* between doses so that the concentration of the drug in the blood stream remains between *L* and *H* over each dose period.

Let's consider several ways in which the drugs might be administered. In Figure 4.11a the time between doses is such that effectively there is no buildup of the drug in the system. On the other hand in Figure 4.11b the interval between doses relative to the amount administered and the decay rate of the concentration is such that a residual concentration exists at each time the drug is taken.

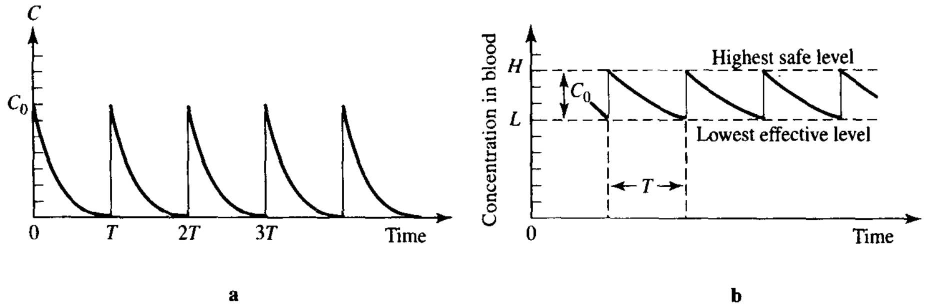


Figure 4.11 Residual build up depends on the time interval between administrations of drug doses

Our ultimate goal in prescribing drugs is to determine dose *amounts* and *intervals* between doses and thereafter the concentration is maintained between *L* and *H,* as shown in Figure 4.12

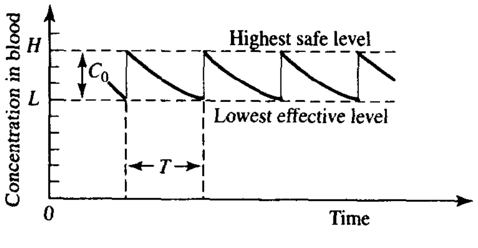


Figure 4.12 Safe but effective levels of drug in the blood: C_0 is the change in concentration produced by one dose and T is the time interval between doses

Assumption:-To solve the problem we have identified, let's consider the factors that determine the concentration *C(t)* of the drug in the blood stream at any time t.

We begin with

$$C(t) = f(decayrate, assimilati\ on\ rate, dosage\ amount, dosage\ \mathrm{int}\, erval, ...)$$

And various other factors, including body weight and blood volume. To simplify our assumptions, let's assume body weight and blood volume are constants.

Next we determine sub models for decay rate and assimilation rate.

Sub-model for Decay Rate:-Consider the elimination of the drug from the bloodstream. This is probably a discrete phenomenon, but let's approximates it by a continuous function**.** Clinical experiments have revealed that the decrease in the concentration of a drug in the blood stream will be proportional to the concentration.

Mathematically this assumption means: $C'(t) = -kC(t)$ (15)

In this formula k is a positive constant called the **elimination constant** of the drug. Notice $C'(t)$ is negative; it is to describe the decreasing concentration. In equation (15) the quantities measured as follows: time t in hours, C(t) is (mg/ml), $C'(t)$ is mg/ml.hr, and k is hr^{-1}.

Assume that the concentration H and L can be determined experimentally for a given population, such as an age group. Then set the drug concentration for a single dose at the level

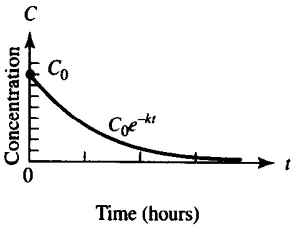
$$C_0 = H - L \tag{16}$$

If we assume that C_0 is the concentration at t=0, then we have the model

$$\frac{dC}{dt} = -kC, \qquad C(0) = C_0 \tag{17}$$

The variables can be separable in Equation (17). The solution of the model gives

$$C(t) = C_0 e^{-kt} \tag{18}$$



The graph of C(t) looks like the one in Figure 4.13

Figure 4.13 Exponential model for decay of drug concentration with time

Sub-model for Assimilation Rate:-Having made an assumption about how drug concentration decreases with time, let's consider how they increase again when drugs are administered. Our initial assumption is that when a drug is taken, it is diffused rapidly throughout the blood that the graph looks vertical. That is, we assume an instantaneous rise in concentration whenever a drug is administered.

Now let's see how the drug accumulates in the bloodstream with repeated doses.

Drug Accumulation with Repeated Doses:- Consider what happens to the concentration *C(t)* when a dose that is capable of raising the concentration by C_0 *mg/ml each* time it is given is administered regularly at fixed time intervals of length *T*.

Suppose at time t=0 the first dose is administered. According to model (18),

After T hours have elapsed, the residual $R_1 = C_0 e^{-kT}$ remains in the blood, and then the second dose is administered. Because of our assumption concerning the increase in drug concentration, the level jumps to $C_1 = C_0 + C_0 e^{-kT}$ then after T hours elapse again, the residual $R_2 = C_1 e^{-kT} = C_0 e^{-kT} + C_0 e^{-2kT}$ remains in the blood. In similar fashion this continues for n successive time we determine a formula for the nth residual R_n.

$$\begin{aligned} R_n &= C_0 e^{-kT} + C_0 e^{-2kT} + \ldots + C_0 e^{-nkT} \\ &= C_0 e^{-kT}(1 + r + r^2 + \ldots + r^{n-1}) \end{aligned} \tag{19}$$

Where $r = e^{-kT}$.

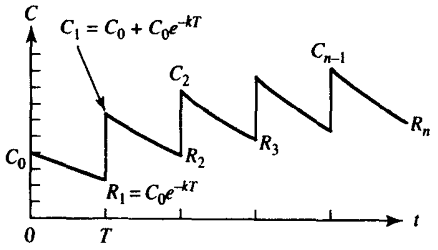


Figure 4.14 One possible effect of repeating equal doses.

Algebraically,

$$1 + r + r^2 + ... + r^{n-1} = \frac{1 - r^n}{1 - r}$$

So substitution for r in Equation (19)gives the result

$$R_n = \frac{C_0 e^{-kT}(1 - e^{-nkT})}{1 - e^{-kT}} \tag{20}$$

When $n \to \infty$, the number e^{-nkT} is close to 0. As a result, the sequence of R_n's has a limiting value, which we call R:

$$R = \lim_{n \to \infty} R_n = \frac{C_0 e^{-kT}}{1 - e^{-kT}}$$

or

$$R = \frac{C_0}{e^{kT} - 1} \tag{21}$$

In summary, if a dose is capable of raising the concentration by C_0 mg/ml is repeated at intervals of T hours, then the limiting value R of the residual concentration is given by Equation (21)

Determining the Dose Schedule:- The concentration C_{n-1} at the beginning of the nth interval is given by

$$C_{n-1} = C_0 + R_{n-1} \tag{22}$$

C_{n-1} to approach H as n becomes large. That is,

$$H = \lim_{n\to\infty} C_{n-1} = \lim_{n\to\infty}(C_0 + R_{n-1}) = C_0 + R$$

Combining this last result with $C_0 = H - L$ yields

$$R = L \tag{23}$$

A meaningful way to examine what happens to the residual concentration R for different intervals T between doses is to examine R in comparison with C_0, the change in concentration to each dose. To make this comparison, we form the dimensionless ratio

$$\frac{R}{C_0} = \frac{1}{e^{kT} - 1} \tag{24}$$

Then substitution of $R = L$ and $C_0 = H - L$ in Equation (21) yields

$$L = \frac{H - L}{e^{kT} - 1}$$

We then solve the preceding equation for e^{kT} to obtain

$$e^{kT} = H / L$$

Taking the logarithm of both sides of this last equation and dividing the result by k gives the desired dose schedule:

$$T = \frac{1}{k}\ln\frac{H}{L} \tag{25}$$

To reach an effective level rapidly, administer a dose, often called a *loading dose*, which will immediately produce a blood concentration of H mg/ml. This medication can be followed every $T = \frac{1}{k}\ln\frac{H}{L}$ hour by a dose that raises the concentration by $C_0 = H - L$ mg/ml.

Activity

Based on Prescribing Drug Dosage model try to answer the following

1. Discuss how the elimination constant k in Equation (15) could be obtained experimentally for a given drug.
2. (a) If $k=0.05hr^{-1}$and the highest safe concentration is e times the lowest effective concentration, find the length of time between repeated doses that will ensure safe but effective concentrations.

   (b) Does part a give enough information to determine the size of each dose?
3. Suppose $k=0.01hr^{-1}$ and T=10hr.Find the smallest n such that $R_n>0.5R$.
4. Given H=2mg/ml, L=05mg/ml, and $k=0.02hr^{-1}$, suppose concentration below L are not only ineffective but also harmful. Determine a scheme for administering this drug (in terms of concentration and times of dosages).

4.4 Modeling with Higher-order differential Equations

In this section we are going to consider several linear dynamical systems in which each mathematical model is a second order differential equation with constant coefficients

$$a_2 \frac{d^2 y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = g(t).$$

Recall that the function g is the input or forcing function of the system. A solution of the differential equation on an interval containing t_0 and satisfying prescribed initial conditions $y(t_0)=y_0$, $y'(t_0)=y_1$ is the output or response of the system.

4.4.1 Spring/Mass Systems: Free Un-damped Motion

Hooke's Law: Suppose, as in Figure 4.15, that a mass m_1 is attached to a flexible spring suspended from a rigid support. When m_1 is replaced with a different mass m_2, the amount of stretch, or elongation, of the spring will of course be different.

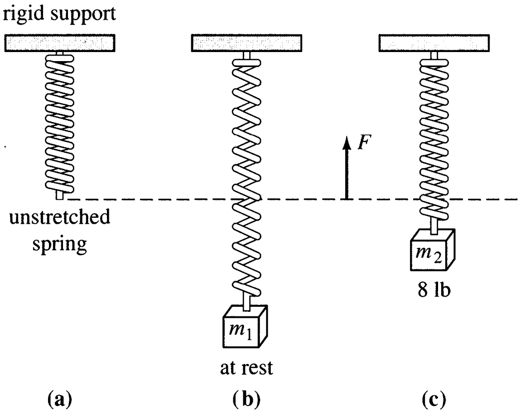


Figure 4.15

By Hooke's law the spring itself exerts a restoring force F opposite to the direction of elongation and proportional to the amount s of elongation.

Simply stated, $F = ks,$ where k is a constant of proportionality called the **spring constant**. Although masses with different weights stretch a spring by different amounts, the spring is essentially characterized by the number k. For example, if a mass weighing 10kg stretches a spring $1/2$ cm, then 10 = k(1/2) implies k = 20 kg/cm. Necessarily then, a mass weighing, say, 8 kg stretches the same spring 2/5cm.

Newton's Second Law After a mass m is attached to a spring; it stretches the spring by an amount *s* and attains a position of equilibrium at which its weight W is balanced by the restoring force $ks.$ Recall that weight is defined by *W = mg*, where mass is measured in slugs, kilograms,or grams and g = 32 ft/s ^{2} , 9.8 m/s ^{2} , or 980 cm/s ^{2} , respectively. As indicated in Figure 2.4.2(b), the condition of equilibrium is $mg = ks.$ or $mg - ks = 0.$ If the mass is displaced by an amount x from its equilibrium position, the restoring force of the spring is then *k(x + s).* Assuming that there are no retarding forces acting on the system and assuming that the mass vibrates free of other external forces-**free motion**-we can equate Newton's second law with the net, or resultant, force of the restoring force and the weight:

$$m\frac{d^2x}{dt^2} = -k(s+x) + mg = -kx + \underbrace{mg - ks}_{zero} = -kx \qquad (1)$$

The negative sign in (1) indicates that the restoring force of the spring acts opposite to the direction of motion. Furthermore, we can adopt the convention that displacements measured below the equilibrium position are positive. See Figure 4.16

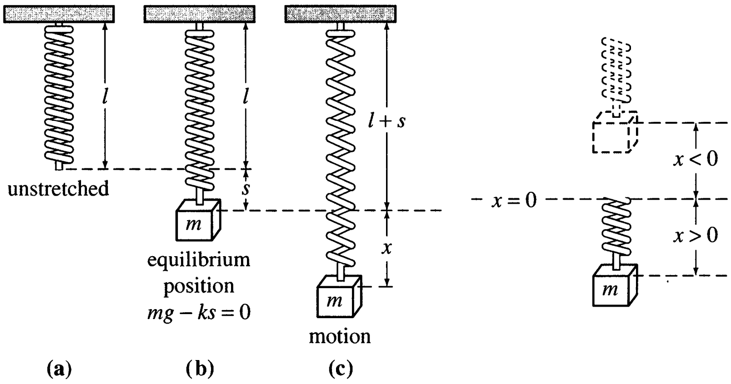


Figure 4.16 Mass spring system

Differential Equation of Free Undamped Motion: By dividing (1) by the mass m we obtain the second-order differential equation $d^2x/dt^2 + (k/m)x = 0$ or

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \tag{2}$$

Where $\omega^2 = k/m.$. Equation (2) is said to describe **simple harmonic motion** or **free undamped motion**. Two obvious initial conditions associated with (2) are $x(0) = \alpha$, the amount of initial displacement, and $x'(0) = \beta$, the initial velocity of the mass. For example, if $\alpha > 0, \beta > 0,$ the mass starts from a point below the equilibrium position with an imparted upward velocity. If $\alpha < 0, \beta = 0,$ the mass is released from *rest* from a point $|\alpha|$ units above the equilibrium position, and so on.

Solution and Equation of Motion: To solve equation (2) we note that the solutions of the auxiliary equation $m^2 + \omega^2 = 0$ are the complex numbers $m_1 = \omega i, m_2 == \omega i$. We find the general solution of (2) to be

$$x(t) = c_1 \cos \omega t + c_2 \sin \omega t. \tag{3}$$

The period of free vibrations described by (3) is $T = 2\pi/\omega$, and the frequency is $f = 1/T = \omega/2\pi$. For example, for $x(t) = 2\cos 3t - 4\sin 3t.$the period is $2\pi/3$ and the frequency is $3/2\pi..$ The former number means that the graph of x(t) repeats every $2\pi/3$ units; the latter number means that there are 3 cycles of the graph every 2π units or, equivalently, that

the mass undergoes $3/2\pi.$complete vibrations per unit time. In addition, it can be shown that the period $2\pi/3$ is the time interval between two successive maxima of x(t). Keep in mind that a maximum of x(t) is a positive displacement corresponding to the mass's attaining a maximum distance below the equilibrium position, whereas a minimum of x(t) is a negative displacement corresponding to the mass's attaining a maximum height above the equilibrium position. We refer to either case as an extreme displacement of the mass. Finally, when the initial conditions are used to determine the constants c_1 and c_2 in (3), we say that the resulting particular solution or response is the equation of motion.

Example 1 Interpretation of an IVP

Solve and interpret the initial-value problem

$$\frac{d^2x}{dt^2} + 16x = 0, \qquad x(0) = 10, \;\; x'(0) = 0.$$

SOLUTION The problem is equivalent to pulling a mass on a spring down 10 units below the equilibrium position, holding it until t=0, and then releasing it from rest. Applying the initial conditions to the solution

$$x(t) = c_1 \cos 4t + c_2 \sin 4t.$$

gives $x(0) = c_1.1 + c_2.0$ so that c₁=10. Hence

$$x(t) = 10\cos 4t + c_2 \sin 4t$$

From $x'(t) = -40\sin 4t + 4c_2 \cos 4t$ we see that $x'(0) = 0 = 4c_2.1$ and so c₂=0. Therefore the equation of motion is $x(t) = 10\cos 4t.$

The solution clearly shows that once the system is set in motion, it stays in motion, with the mass bouncing back and forth 10 units on either side of the equilibrium position x=0. As shown in Figure 4.17 the period of oscillation is $2\pi/4 = \pi/2$ s.

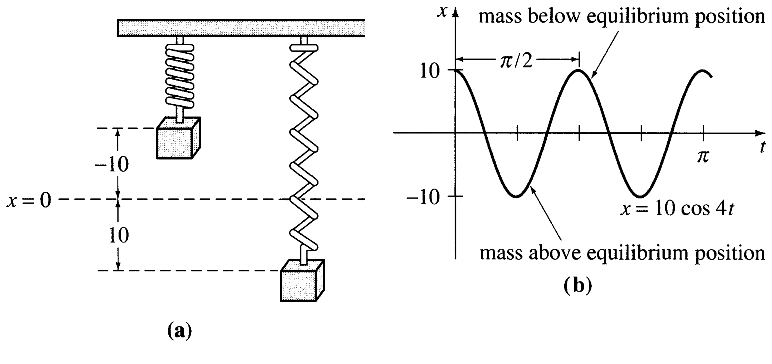


Figure 4.17

EXAMPLE2 Free Un-damped Motion

A mass weighing 2 pounds stretches a spring 6 inches. At t= 0 the mass is released from a point 8 inches below the equilibrium position with an upward velocity of 4/3ft/s. Determine the equation of free motion.

SOLUTION Because we are using the engineering system of units, the measurements given in terms of inches must be converted into feet: 6 in. = 1/2ft; 8 in. = 2/3ft. In addition, we must convert the units of weight given in pounds into units of mass. From m = W/g we have $m = \frac{2}{32} = \frac{1}{16}$ slug. Also, from Hooke's law, 2 = k(1/2) implies that the spring constant is k = 4 lb/ft. Hence (1) gives

$$\frac{1}{16}\frac{d^2x}{dt^2} = -4x \quad or \quad \frac{d^2x}{dt^2} + 64x = 0$$

The initial displacement and initial velocity are $x(0) = \frac{2}{3}, x'(0) = -\frac{4}{3}$, where the negative sign in the last condition is a consequence of the act that the mass is given an initial velocity in the negative, or upward, direction.

Now $\omega^2 = 64 \; or \; \omega = 16$, so that the general solution of the differential equation is

$$x(t) = c_1 \cos 8t + c_2 \sin 8t. \tag{4}$$

Applying the initial conditions to x(t) and x' (t) gives $c_1 = \frac{2}{3} \; and \; c_2 = -\frac{1}{6}$. Thus the equation of motion is

$$x(t) = \frac{2}{3}\cos 8t - \frac{1}{6}\sin 8t. \tag{5}$$

Alternative Form of x(t) When, $c_1 \neq 0 \; and \; c_2 \neq 0,$ the actual **amplitude A** of free vibrations is not obvious from inspection of Equation(3). For example, although the mass in Example 2 is initially displaced 2/3 foot beyond the equilibrium position, the amplitude of vibrations is a number larger than 2/3. Hence it is often convenient to convert a solution of form (3) to the simpler form

$$x(t) = A\sin(\omega t + \phi), \tag{6}$$

where $A=\sqrt{c_1{}^2+c_2{}^2}$ and ϕ is a phase angle defined by

$$\left.\begin{aligned}\sin\phi &= \frac{c_1}{A}\\ \cos\phi &= \frac{c_2}{A}\end{aligned}\right\}\tan\phi=\frac{c_1}{c_2}, \tag{7}$$

To verify this we expand (6) by the addition formula for the sine function:

$$A\sin\omega t\cos\phi + A\cos\omega t\sin\phi = (A\sin\phi)\cos\omega t + (A\cos\phi)\sin\omega t. \tag{8}$$

It follows from Figure 2.4.5 that if ϕ is defined by

$$\sin\phi=\frac{c_1}{\sqrt{c_1{}^2+c_2{}^2}}=\frac{c_1}{A},\quad \cos\phi=\frac{c_2}{\sqrt{c_1{}^2+c_2{}^2}}=\frac{c_2}{A},$$ then (8) becomes

$$A\frac{c_1}{A}\cos\omega t + A\frac{c_2}{A}\sin\omega t = c_1\cos\omega t + c_2\sin\omega t = x(t).$$

4.4.2 Spring/Mass Systems: Free Damped Motion

The concept of free harmonic motion is somewhat unrealistic since the motion described by equation (1) assumes that there are no retarding forces acting on the moving mass. Unless the mass is suspended in a perfect vacuum, there will be at least a resisting force due to the surrounding medium. As Figure 4.18 shows, the mass could be suspended in a viscous medium or connected to a dashpot damping device.

Differential Equation of Free Damped Motion In the study of mechanics, damping forces acting on a body are considered to be proportional to a power of the instantaneous velocity. In particular, we shall assume throughout the subsequent discussion that this force is given by a constant multiple of dx/dt. When no other external forces are impressed on the system, it follows from Newton's second law that

$$m\frac{d^2x}{dt^2}=-kx-\beta\frac{dx}{dt}, \tag{9}$$

where β is a positive damping constant and the negative sign is a consequence of the fact that the damping force acts in a direction opposite to the motion.

Dividing (10) by the mass m, we find the differential equation of **free damped motion** is $d^2x/dt^2 + (\beta/m)dx/dt + (k/m)x = 0.$ or

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0, \tag{10}$$

Where $$2\lambda = \frac{\beta}{m}, \quad \omega^2 = \frac{k}{m} \tag{11}$$

The symbol 2λ is used only for algebraic convenience since the auxiliary equation is $m^2 + 2\lambda m + \omega^2 = 0$ and the corresponding roots are then $m_1 = -\lambda + \sqrt{\lambda^2 - \omega^2}$, $m_2 = -\lambda - \sqrt{\lambda^2 - \omega^2}$, .

We can now distinguish three possible cases depending on the algebraic sign of $\lambda^2 - \omega^2$. Since each solution contains the damping factor $e^{-\lambda t}, \lambda > 0.$ the displacements of the mass become negligible for large time.

CASE I: $\lambda^2 - \omega^2 > 0$ In this situation the system is said to be over-damped since the damping coefficient β is large when compared to the spring constant k. The corresponding solution of (10) is $x(t) = c_1 e^{m_1 t} + c_2 e^{m_2 t}$ or

$$x(t) = e^{-\lambda t}(c_1 e^{\sqrt{\lambda^2 - \omega^2 t}} + c_2 e^{-\sqrt{\lambda^2 - \omega^2 t}}). \tag{12}$$

This equation represents a smooth and non oscillatory motion. Figure 4.19 shows two possible graphs of x(t).

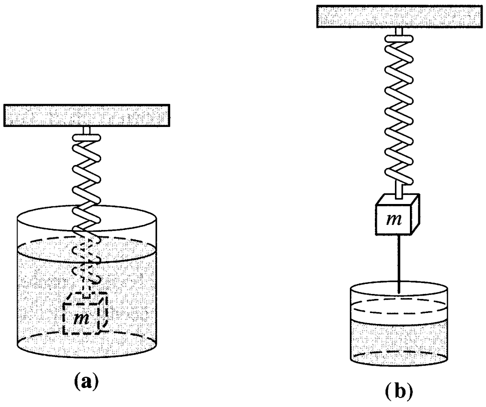


Figure 4.18 mass suspended in a viscous medium

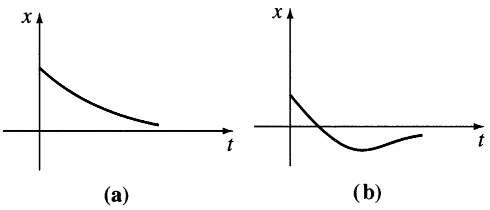


Figure 4.19 two possible graphs of x(t) in Equation (12)

Case II: $\lambda^2 - \omega^2 = 0$ The system is said to be **critically damped** since any slight decrease in the damping force would result in oscillatory motion. The general solution of (10) is $x(t) = c_1 e^{m_1 t} + c_2 t e^{m_1 t}$ or

$$x(t) = e^{-\lambda t}(c_1 + c_2 t). \tag{13}$$

Notice that the motion is quite similar to that of an over damped system. It is also apparent from (13) that the mass can pass through the equilibrium position at most one time.

CASE III: $\lambda^2 - \omega^2 < 0$ In this case the system is said to be **under- damped** since the damping coefficient is small compared to the spring constant. The roots m_l and m_2 are now complex:

$$m_1 = -\lambda + \sqrt{\omega^2 - \lambda^2}\, i, \qquad m_1 = -\lambda - \sqrt{\omega^2 - \lambda^2}\, i.$$

Thus the general solution of Equation (10) is

$$x(t) = e^{-\lambda t}(c_1 \cos\sqrt{\omega^2 - \lambda^2}t + c_2 \sin\sqrt{\omega^2 - \lambda^2}t). \qquad (14)$$

As indicated in Figure 2.4.8 the motion described by (15) is oscillatory; but because of the coefficient $e^{-\lambda t}$, the amplitudes of vibration $\rightarrow 0\, as\;\; t \rightarrow \infty$.

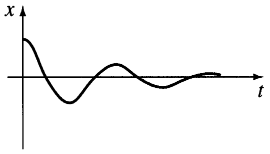


Figure 4.20

EXAMPLE 4 Over-damped Motions

It is readily verified that the solution of the initial-value problem

$$\frac{d^2x}{dt^2} + 5\frac{dx}{dt} + 4x = 0, \qquad x(0) = 1,\ x'(0) = 1$$

$$x(t) = \frac{5}{3}e^{-t} - \frac{2}{3}e^{-4t} \qquad (15)$$

The problem can be interpreted as representing the over-damped motion of a mass on a spring. The mass starts from a position 1 unit below the equilibrium position with a downward velocity of 1 ft/s.

To graph x(t) we find the value of t for which the function has an extremum-that is, the value of time for which the first derivative (velocity) is zero. Differentiating (15) gives $x'(t) = -\frac{5}{3}e^{-t} + \frac{8}{3}e^{-4t}$ so that $x'(t) = 0$ implies $e^{3t} = \frac{8}{5} or\ t = \frac{1}{3}\ln\frac{8}{5} = 0.157$. It follows from the first derivative test, as well as our physical intuition, that x(0.157)=1.069 ft is actually a maximum. In other words, the mass attains an extreme displacement of 1.069 feet below the equilibrium position.

EXAMPLE 5 Critically Damped Motions

An 8-pound weight stretches a spring 2 feet. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, model the equation of motion if the weight is released from the equilibrium position with an upward velocity of 3 ft/s.

SOLUTION From Hooke's law we see that 8=k(2)gives k=4 lb/ft and that *W=mg* gives $m = \frac{8}{32} = \frac{1}{4} slugs$. The differential equation of motion is then

$$\frac{1}{4}\frac{d^2x}{dt^2} = -4x - 2\frac{dx}{dt} \ or \ \frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 16x = 0 \tag{16}$$

The auxiliary equation for (16) is $m^2 + 8m + 16 = (m+4)^2 = 0$ so that $m_1 = m_2 = -4$. Hence the system is critically damped and

$$x(t) = c_1e^{-4t} + c_2te^{-4t}. \tag{17}$$

Applying the initial conditions x(0)=0 and x'(0)=-3, we find, in turn, that C_l =0 and C_2=- 3. Thus the equation of motion is modeled as

$$x(t) = -3te^{-4t}. \tag{18}$$

Example 6 Under-damped Motion

A 16-pound weight is attached to a 5-foot-Iong spring. At equilibrium the spring measures 8.2 feet. If the weight is pushed up and released from rest at a point 2 feet above the equilibrium position, find the displacements x(t) if it is further known that the surrounding medium offers a resistance numerically equal to the instantaneous velocity.

SOLUTION The elongation of the spring after the weight is attached is 8.2-5=3.2ft, so it follows from Hooke's law that 16=k(3.2) or k =5 lb/ft. In addition, m=16/32=1/2 slug so that the differential equation is given by

$$\frac{1}{2}\frac{d^2x}{dt^2} = -5x - \frac{dx}{dt} \; or \; \frac{d^2x}{dt^2} + 2\frac{dx}{dt} + 10x = 0 \tag{19}$$

Proceeding, we find that the roots of $m^2 + 2m + 16 = 0 \; are \; m_1 = -1 + 3i \; and \; m_2 = -1 - 3i$ which then implies the system is under damped and

$$x(t) = e^{-t}(c_1 \cos 3t + c_2 \sin 3t). \tag{20}$$

Finally, the initial conditions x(0)=-2 and x’(0)=0 yields c_1=-2 and c_2=-2/3, so the equation of motion is

$$x(t) = e^{-t}(-2\cos 3t - \frac{2}{3}\sin 3t) \tag{21}$$

4.4.3 Spring/Mass Systems: Driven Motion

Differential Equation of Driven Motion with Damping Suppose we now take into consideration an external force *f(t)* acting on a vibrating mass on a spring. For example, f(t) could represent a driving force causing an oscillatory vertical motion of the support of the spring. See Figure 4.18. The inclusion of f(t) in the formulation of Newton's second law gives the differential equation of **driven** or **forced motion**:

$$m\frac{d^2x}{dt^2} = -kx - \beta\frac{dx}{dt} + f(t) \tag{22}$$

Dividing (24) by m gives

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t) \tag{23}$$

where F(t) == f(t)/m and, as in the preceding section, $2\lambda = \beta / m, \; \omega^2 = k / m$.

To solve the latter non homogeneous equation we can use either the method of undetermined coefficients or variation of parameters.

EXAMPLE 7 Interpretation of an Initial-Value Problem

Interpret and solve the initial-value problem I

$$\frac{1}{5}\frac{d^2x}{dt^2}+1.2\frac{dx}{dt}+2x=5\cos 4t, \qquad x(0)=\frac{1}{2},\ x'(0)=1 \tag{24}$$

SOLUTION We can interpret the problem to represent a vibrational system consisting of a mass (m=1/5slug or kilogram) attached to a spring (k = 2 lb/ft or N/m). The mass is released from rest1/2 unit (foot or meter) below the equilibrium position. The motion is damped $(\beta = 1.2)$ and is being driven by. An external periodic $(T = \pi / 2s)$ force beginning at t = 0. Intuitively we would expect that even with damping the system would remain in motion until such time as the forcing function was "turned off," in which case the amplitudes would diminish.

However, as the problem is given, $f(t) = 5\cos 4t$ will remain "on" forever.

We first multiply the differential equation in (24) by 5 and solve

$$\frac{d^2x}{dt^2}+6\frac{dx}{dt}+10x=0,$$

by the usual methods. Since $m_1 = -3+i \ \ and \ \ m_2 = -3-i$, it follows that

$$x_c(t) = e^{-3t}(c_1 \cos t - c_2 \sin t).$$

Using the method of undetermined coefficients, we assume a particular solution of the form $x_p(t) = A\cos 4t + B\sin 4t).$ Now

$$x'_p(t) = -4A\sin 4t + 4B\cos 4t,\ x''_p(t) = -16A\cos 4t - 16B\sin 4t.$$

so that $x_p''+6x_p'+10x_p = (-6A+24B)\cos 4t + (-24A-6B)\sin 4t = 25\cos 4t$

The resulting system of equations -6A + 24B = 25, - 24A - 6B = 0

yields A = -25/102 and B = 50/51 . It follows that

$$x\ \ (t) = e^{-3t}(c_1 \cos t - c_2 \sin t) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$$

When we set t = 0 in the above equation, we obtain $c_1 = \frac{38}{51}$. By differentiating the expression and then setting t = 0, we also find that $c_2 = -\frac{86}{51}$

Therefore the equation of motion is $x\ (t) = e^{-3t}(\frac{38}{51}\cos t - \frac{86}{51}\sin t) - \frac{25}{102}\cos 4t + \frac{50}{51}\sin 4t$

Activity

1. A 1-kilogram mass is attached to a spring whose constant is 16 N/m,and the entire system is then submerged in a liquid that imparts a damping force numerically equal to 10 times the instantaneous velocity. Model equations of motion if

   (a) the weight is released from rest 1 meter below the equilibrium position and

   (b) the weight is released 1 meter below the equilibrium position with an upward velocity of 12 m/s.

2. A 4-foot spring measures 8 feet long after an 8-pound weight is attached to it. The medium through which the weight moves offers a resistance numerically equal to $\sqrt{2}$ times the instantaneous velocity. Model equation of motion if the weight is released from the equilibrium position with a downward velocity of 5 ft/s. Find the time at which the weight attains its extreme displacement from the equilibrium position. What is the position of the weight at this instant?

CHAPTER 5

Basic optimization

Introduction

In the previous Chapters we used calculus to solve the optimization problem related to modeling. Although we formulated several optimization problems resulting from the first criterion to minimize the sum of the absolute deviations, we were unable to solve the resulting mathematical problem. In this chapter we study several search techniques that allow us to find good solutions, and we examine many other optimization problems as well.

For example, given a collection of *m* data points (x_i, y_i) , $i = 1,2 ,\ldots,m$, fit the collection to that line $y = ax + b$ (determined by the parameters a and *b)* that minimizes the greatest distance r_{max} between any data point (x_i, y_i) and its corresponding point $(x_i, ax_i + b)$ on the line. That is, the largest absolute deviation,

$$r = \text{Maximum}\ \{|y_i - y(x_i)|\}$$

is minimized over the entire collection of data points. This criterion defines the optimization problem to ,

$$Minimize\ r$$

$$Subjectto: \quad \left.\begin{aligned} r - r_i \geq 0 \\ r + r_i \geq 0 \end{aligned}\right\} \ for\, i = 1, 2, \ldots, m$$

which is a *linear program* for many applications. You will learn how to solve linear programs geometrically and algebraically in this chapter. We begin by providing a general classification of discrete optimization problems. Our emphasis is on *model formulation, which* will allow you additional practice on the first several steps of the modeling process while simultaneously providing a preview of the kinds of problems you will learn to solve in advanced mathematics courses.

5.1 An Overview of Discrete Optimization Modeling

To provide a framework for discussing a class of discrete optimization problems, we offer a basic model for such problems. The problems are classified according to the various characteristics of the basic model that are possessed by the particular problem. We also discuss variations from the basic model. The basic model is

$$\text{Optimize } f_j(\mathbf{X}) \text{ for } j \text{ in } J$$

Subject to

$$g_i(\mathbf{X}) \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} b_i \text{ for all } i \text{ in } I \tag{5.1}$$

Now let's explain the notation. To optimize means to maximize *or* minimize. The subscript j indicates that there may be one or more functions to optimize. The functions are distinguished by the integer subscripts that belong to the finite set *J*. We seek the vector X_0 giving the optimal value for the set of functions $f_j(X)$. The various components of the vector *X* are called the decision variables of the model. Whereas the functions f_j *(X)* are called the objective functions. By *subject to,* we connote that there may be certain side conditions that must be met. For example, if the objective is to minimize costs of producing a particular product, it might be specified that all contractual obligations for the product be met as side conditions. Side conditions are typically called constraints. The integer subscript i indicates that there may be one or more constraint relationships that must be satisfied. A constraint may be an equality (such as precisely meeting the demand for a product) or inequality (such as not exceeding budgetary limitations or providing the minimal nutritional requirements in a diet problem). Finally, each constant $\boldsymbol{b_i}$ represents the level that the associated constraint function $g_j(X)$ must achieve and, because of the way optimization problems are typically written, is often called the right-hand side in the model. Thus, the solution vector X_0 must optimize each of the objective functions f_j(X) and simultaneously satisfy each constraint relationship. We now consider one simplistic problem illustrating *the* basic ideas.

EXAMPLE 1 Determining a Production Schedule

A carpenter makes tables and bookcases. He is trying to determine how many of each type of furniture he should make each week. The carpenter wishes to determine a weekly production schedule for tables and bookcases that maximizes his profits. It costs $5 and $7 to produce tables and bookcases, respectively. The revenues are estimated by the expressions

$50x_1 - 0.2x_1^2$,where x_1 is the number of tables produced per week and

$65x_2 - 0.3x_2^2$, where x_2 is the number of bookcases produced per week

In this example, the problem is to decide how many tables and bookcases to make every week. Consequently, the decision variables are the quantities of tables and bookcases to be made per week. We assume this is a schedule so non integer values of tables and bookcases make sense. The objective function is a nonlinear express ion representing the net weekly profit to be realized from selling the tables and bookcases. Profit is revenue minus costs. The profit function is, $f\left(x_1, x_2\right) = 50x_1 - 0.2x_1^2 + 65x_2 - 0.3x_2^2 - 5x_1 - 7x_2$

There are no constraints in this problem.

Let's consider a variation w the previous scenario. The carpenter realizes a net unit profit of $25 per table and $30 per bookcase. He is trying to determine how many of each piece of furniture he should make each week He has up to 600 board feet of lumber to devote weekly to the project and up to 40 hr of labor. He can use lumber and labor productively elsewhere if they are not used in the production of tables and bookcases. He estimates that it requires 20 board feet of lumber and 5 hr of labor to complete a table and 30 board feet of lumber and 4 hr of labor for a bookcase. Moreover, he has signed contracts to deliver four tables and two bookcases every week. The carpenter wishes to determine a weekly production schedule for tables and bookcases that maximizes his profits. The formulation yields

$$\text{Maximize } 25x_1 + 30x_2$$

subject to

$$\begin{aligned} 20x_1 + 30x_2 &\le 600 && \text{(lumber)} \\ 5x_1 + 4x_2 &\le 4 && \text{(labor)} \\ x_1 &\ge 4 && \text{(contract)} \\ x_2 &\ge 2 && \text{(contract)} \end{aligned}$$

5.1.1 Classifying Some Optimization Problems

There are various ways of classifying optimization problems. These classifications are not meant to be mutually exclusive but to describe certain mathematical characteristics possessed by the problem under investigation. We now describe several of these classifications.

An optimization problem is said to be unconstrained if there are no constraints and constrained if one or more side conditions are present. The first production schedule problem described in Example 1 illustrates an unconstrained problem.

An optimization problem is said to be a linear program if it satisfies the following properties:

1. There is a unique objective function.
2. Whenever a decision variable appears in either the objective function or one of the constraint functions, it must appear only as a power term with an exponent of 1, possibly multiplied by a constant.
3. No term in the objective function or in any of the constraints can contain products of the decision variables.
4. The coefficients of the decision variables in *the* objective *function* and each constraint are constant.
5. The decision variables are permitted to assume fractional as well as integer values.

These properties ensure, among other things, that the effect of any decision variable is *proportional* to its value. Let's examine each property more closely.

Property 1 limits the problem to a single objective function. Problems with more than one objective function are called multi objective or goal programs. **Properties 2 and 3** are self-explanatory, and any optimization problem that fails to satisfy either one of them is said to be nonlinear. The first production schedule objective function had both decision variables as squared terms and thus violated **Property 2**. **Property 4** is quite restrictive for many scenarios you might wish to model. Consider examining the amount of board feet and labor required to make tables and bookcases. It might be possible to know exactly the number of board feet and labor required to produce each item and incorporate these into constraints. Often, however, ills impossible to predict precisely the required values in advance (consider trying to predict the market price of corn), or the coefficients represent average values with rather large deviations from the actual values occurring in practice. The coefficients may be time dependent as well; lime-dependent problems in a certain class are called dynamic programs. If the coefficients are not constant but instead are probabilistic in nature, the problem is classified as a stochastic prog ram. Finally, if one or more of the decision variables are restricted to integer values (hence violating **Property 5**). The resulting problem is called an integer program (or a mixed integer program if the integer restriction applies to only a subset of the decision variables). In the variation of the production scheduling problem. It makes sense *to* allow fractional numbers of tables and bookcases in determining a weekly schedule because they can be completed during the following week.

5.1.2 Unconstrained Discrete Optimization Problem

A criterion considered for fitting a model to data points is minimizing the sum of absolute deviations. For the model y = *f(x),* if $y(x_i)$ represents the function evaluated at $x = x_i$, and (x_i, y_i) denotes the corresponding data point for i = 1,2.... ,*m* points, then this criterion can be formulated as follows: Find the parameters of the model y= f(x) to

$$\text{Optimize } f_j(\mathbf{X}) \text{ for } j \text{ in } J$$

Subject to

$$g_i(\mathbf{X}) \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} b_i \text{ for all } i \text{ in } I$$

This last condition illustrates an unconstrained Optimization problem. Because the derivative of the function being minimized fails to be continuous (because of the presence of the absolute value), it is impossible to solve this problem with a straightforward application of the elementary calculus.

In the next several sections we focus our attention on solving linear programming problems, first geometrically and then by the Simplex Method.

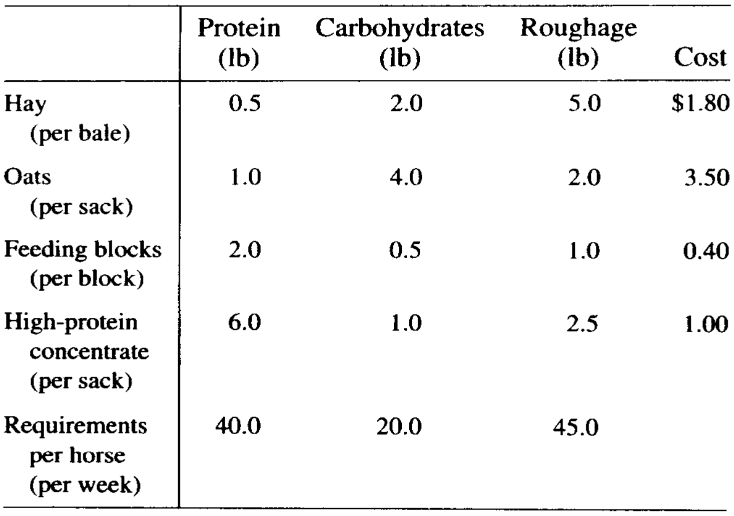
Use the model-building process described in the previous chapters to analyze the following scenarios. You may find it helpful to answer the following questions in words before formulating the optimization model:

(a) Identify the decision variables: What decision is to be made?
(b) Formulate the objective function: How do these decisions affect the objective?

c) Formulate the constraint set: What constraints must be satisfied? Be sure to consider whether negative values of the decision variables are allowed by the problem, and ensure they are so constrained if required.

After constructing the model, check the assumptions for a linear program and compare the form of the model to the examples in this section. Try to determine which method of optimization may be applied to obtain a solution.

1. ***Nutritional Requirements***—*A* rancher has determined that the minimum weekly nutritional requirements for an average-sized horse include 40 lb of protein, 20 lb of carbohydrates, and 45 lb of roughage. These are obtained from the following sources in varying amounts at the prices indicated:



	Protein (lb)	Carbohydrates (lb)	Roughage (lb)	Cost
Hay (per bale)	0.5	2.0	5.0	$1.80
Oats (per sack)	1.0	4.0	2.0	3.50
Feeding blocks (per block)	2.0	0.5	1.0	0.40
High-protein concentrate (per sack)	6.0	1.0	2.5	1.00
Requirements per horse (per week)	40.0	20.0	45.0	

Formulate a mathematical model to determine how to meet the minimum nutritional requirements at minimum cost.

5.2 Linear Programming I: Geometric Solutions

Consider using the Chebyshev criterion to fit the model y = *cx* to the following data set:

x	1	2	3
y	2	5	8

The optimization problem that determines the parameter c to minimize the largest absolute deviation $r_i = |y_i - y(x_i)|$ (residual or error) is the linear program

$$\text{Minimize } r$$

subject to

$$\left.\begin{array}{lr} r - (2 - c) \geq 0 & \text{(constraint 1)} \\ r + (2 - c) \geq 0 & \text{(constraint 2)} \\ r - (5 - 2c) \geq 0 & \text{(constraint 3)} \\ r + (5 - 2c) \geq 0 & \text{(constraint 4)} \\ r - (8 - 3c) \geq 0 & \text{(constraint 5)} \\ r + (8 - 3c) \geq 0 & \text{(constraint 6)} \end{array}\right\} \tag{5.2}$$

In this section we solve this problem geometrically.

5.2.1 Interpreting a Linear Program Geometrically

Linear programs can include a set of constraints that are linear equations *or linear* inequalities. Of course, in the case of two decision variables, equality requires that solutions to the linear program lie precisely on the line representing the equality. What about inequalities? To gain some insight, consider the constraints

$$\begin{aligned} x_1 + 2x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned} \tag{5.3}$$

The non negativity constraints $x_1, x_2 \geq 0$ mean that possible solutions lie in the first quadrant. The inequality $x_1 + 2x_2 \leq 4$ divides the first quadrant into two regions. The *feasible* region is the half-space in which the constraint is satisfied. The feasible region can be found by graphing the equation $x_1 + x_2 = 4$ and determining which half-plane is feasible, as shown in Figure 5.1.

If the feasible half-plane fails to be obvious, choose a convenient point (such as the origin) and substitute it into the constraint to determine if it is satisfied. If it is, then all points on the same side of the line as this point will also satisfy the constraint.

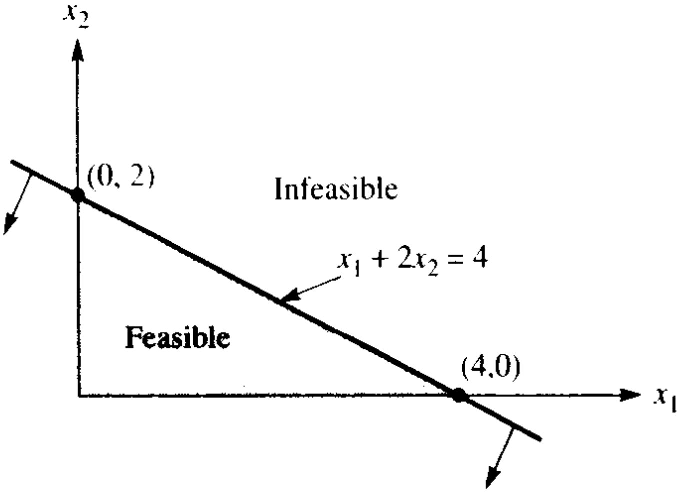


Figure 5.1
The feasible region for the constraints $x_i + 2x_2 \leq 4$, $x_1, x_2 \geq 0$

A linear program has the important property that the points satisfying the constraints form a *convex set,* which is a set in which any two of its points are joined by a straight line segment, all of whose points lie within the set. The set depicted in Figure 5.2a fails to be convex, whereas the set in Figure 5.2b is convex.

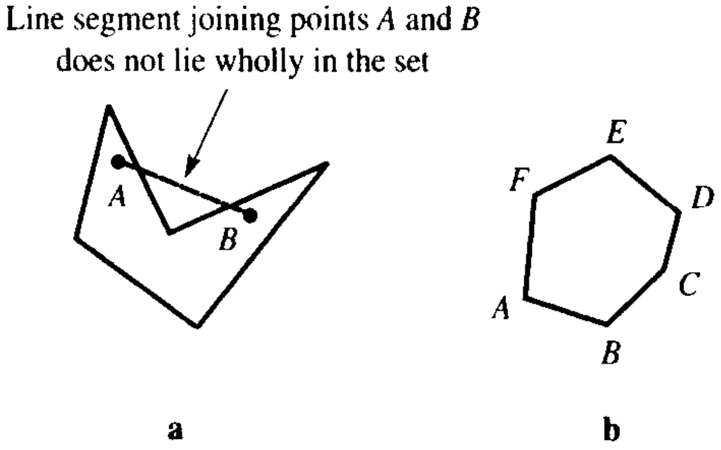


Figure 5.2
The set shown in **a** is not convex, whereas the set shown in **b** is convex

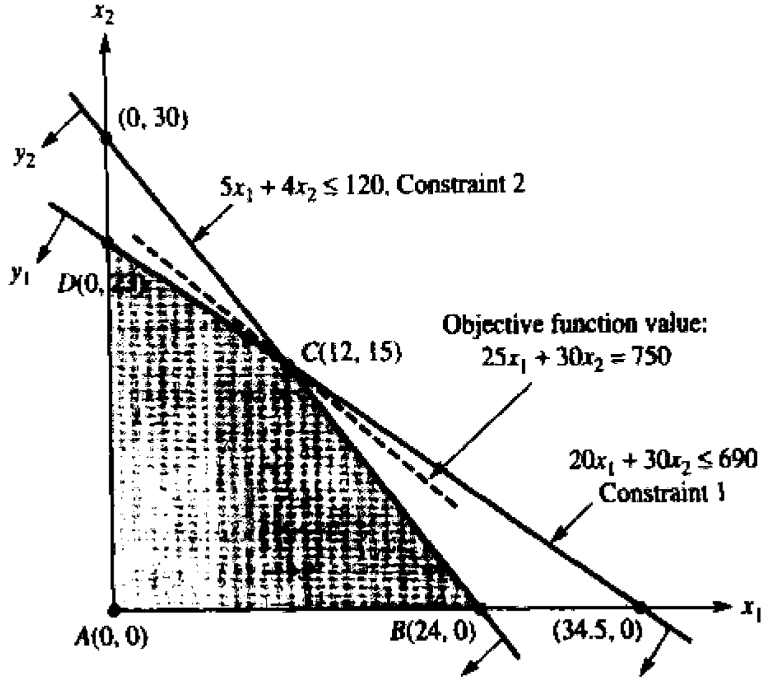
An extreme point (corner point) of a convex set is any boundary point in the convex set that is the unique intersection point of two of the (straight line) boundary segments. In Figure 5.2b, points A—F are extreme points. Let’s now find the feasible region and the optimal solution for the carpenter's problem formulated in Example 1 in Section 5.1.

EXAMPLE 1 ***The Carpenter's Problem***

The convex set for the constraints in the carpenter's problem is graphed and given by the polygon region ABCD in Figure 5.3. Note that there are six intersection points of the constraints, but only four of these points (namely, A-D) satisfy all of the constraints and hence belong to the convex set. The points A - D are the extreme points of the polygon. The variables y_1 and y_2 will be explained later.

If an optimal solution to a linear program exists, it must occur among the extreme points of the convex set formed by the set of constraints. The values of the objective function (profit for the Carpenter's problem) at the extreme points *are*

Figure 5.3
The set of points satisfying the constraints of the carpenter's problem form a convex set



Extreme point	Objective function value
A (0, 0)	\$0
B (24, 0)	600
C (12, 15)	750
D (0, 23)	690

Thus, the carpenter should make 12 tables and is bookcases each week to earn a maximum weekly profit of $750. We provide further geometrical evidence later in this section that extreme point C is optimal.

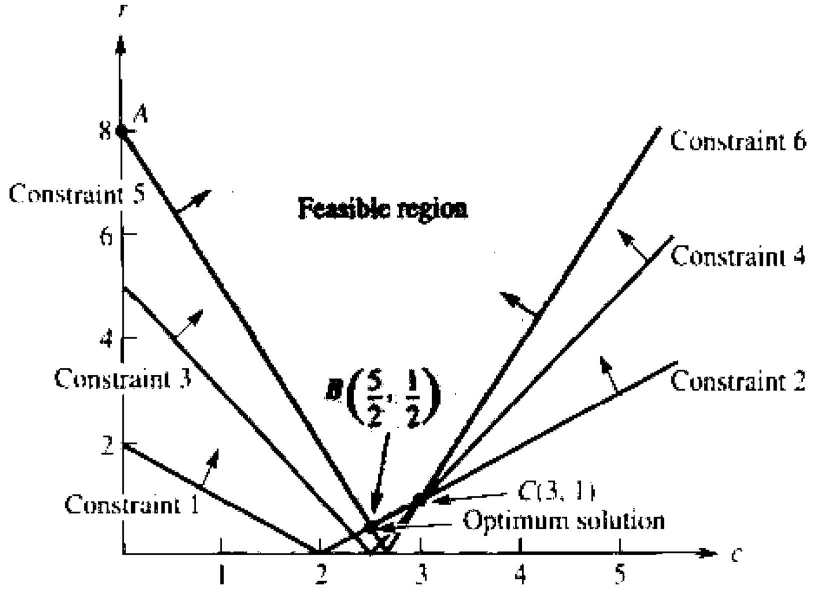
Before considering a second example, let's summarize the ideas presented thus far. The constraint set to a linear program is a convex set, which generally contains an infinite number of feasible points to the linear program. If an optimal solution to the linear program exists, it must be taken on at one or more of the extreme points. Thus, to find an optimal solution, we choose from among all the extreme points the one with the best value for the objective function.

EXAMPLE 2 A Data-Fitting Problem

Let's now solve the linear program represented by Equation (5.2). Given the model $y = cx$ and the data set

x	1	2	3
y	2	5	8

Figure 5.4
The feasible region for fitting $y = cx$ to a collection of data



we wish to find a value for c such that the resulting largest absolute deviation r is as small as possible. In Figure 5.4 we graph the set of six constraints

$$r - (2 - c) \geq 0 \qquad \text{(constraint 1)}$$

$$r + (2 - c) \geq 0 \qquad \text{(constraint 2)}$$

$$r - (5 - 2c) \geq 0 \qquad \text{(constraint 3)}$$

$$r + (5 - 2c) \geq 0 \qquad \text{(constraint 4)}$$

$$r - (8 - 3c) \geq 0 \qquad \text{(constraint 5)}$$

$$r + (8 - 3c) \geq 0 \qquad \text{(constraint 6)}$$

by first graphing the equations

$$r - (2 - c) = 0 \qquad \text{(constraint 1 boundary)}$$

$$r + (2 - c) = 0 \qquad \text{(constraint 2 boundary)}$$

$$r - (5 - 2c) = 0 \qquad \text{(constraint 3 boundary)}$$

$$r + (5 - 2c) = 0 \qquad \text{(constraint 4 boundary)}$$

$$r - (8 - 3c) = 0 \qquad \text{(constraint 5 boundary)}$$

$$r + (8 - 3c) = 0 \qquad \text{(constraint 6 boundary)}$$

We note that constraints 1,3, and 5 are satisfied above and to the right of the graph of their boundary equations. Similarly, constraints 2, 4, and 6 are satisfied above and to the left of their boundary equations. To convince your self, pick a point (such as the origin) and determine if the point satisfies the constraint. If it does, it must be in the feasible region determined by the constraint.

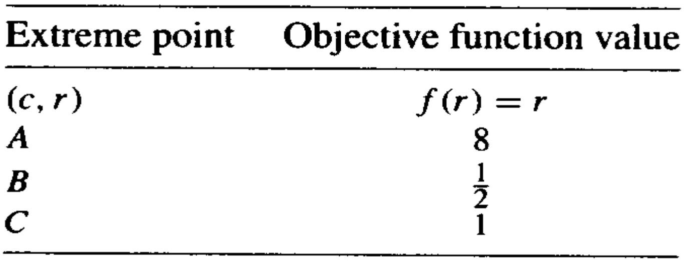
The intersection of all the feasible regions for constraints 1—6 form a convex set in the *c, r* plane, with extreme points labeled *A*—*C* in Figure 5.4. The point *A* is the intersection of constraint 5 and the r axis: r - (8 - 3c) = 0 and $c = 0$, or $A = (0, 8)$. Similarly, *B* is the intersection of constraints 5 and 2:

$$r - (8 - 3c) = 0 \quad \text{or} \quad r + 3c = 8$$

$$r + (2 - c) = 0 \quad \text{or} \quad r - c = -2$$

yielding $c = \frac{5}{2}$ and $r = \frac{1}{2}$, or $B = (\frac{5}{2}, \frac{1}{2})$. Finally, *C* is the intersection of constraints 2 and 4 yielding $C = (3, 1)$. Note that the set is *unbounded.* If an optimal solution to the problem exists, at least one extreme point must take on the optimal solution. We now evaluate the objective function *f(r)* = *r* at each of the three extreme points.

Figure 5.5



Extreme point	Objective function value
(c, r)	$f(r) = r$
A	8
B	$\frac{1}{2}$
C	1

The extreme point with the smallest value of *r* is the extreme point *B* with coordinates $\left(\frac{5}{2}, \frac{1}{2}\right)$. Thus, c = $25x_1 + 30x_2$ is the optimal value of *c*. No other value of *c* will result in a largest absolute deviation as small as $|r_{\max}| = \frac{1}{2}$

.

5.2.2 Empty and Unbounded Feasible Regions

We have been careful to say that if an optimal solution to the linear program exists, at least one of the extreme points must *lake* on the optimal value for the objective function. When does an optimal solution fail to exist? Moreover, when does more than *one* optimal solution exist?

If the feasible region is empty, no feasible solution can exist. For example, given the constraints

$$x_1 \leq 3 \qquad \& \quad x_1 \geq 5$$

there is no value of x_1 that satisfies both of them. We say that such constraint sets are *inconsistent.*

There is another reason an optimal solution may fail to exist, that is when the feasible region is *unbounded* (in the sense that either x_1 or x_2 can become arbitrarily large). Then it would be impossible to

$$\text{Maximize } x_1 + x_2$$

over the feasible region because x_1 and x_2 can take on arbitrarily large values. Note, however, that even though the feasible region is unbounded, an optimal solution *does* exist for the objective function we considered in Example 2. So it is not *necessary* for the feasible region to be bounded for an optimal solution to exist.

Level Curves of the Objective Function

Consider again the carpenter's problem. The objective function is $25x_1 + 30x_2$ and in Figure 5.6 we plot the lines

$$25x_1 + 30x_2 = 650$$

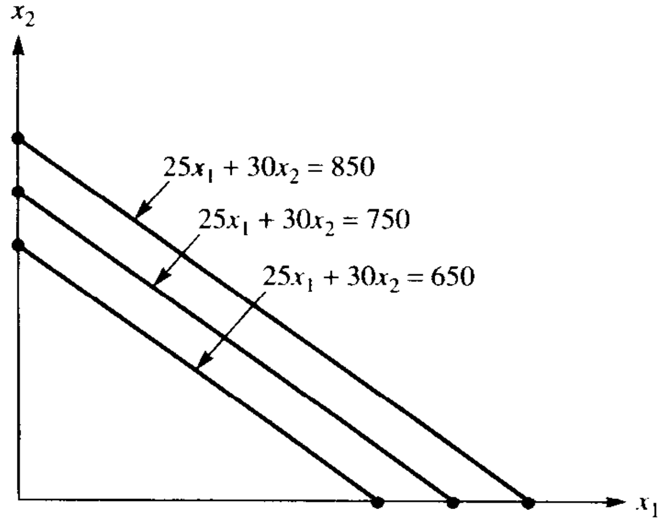
$$25x_1 + 30x_2 = 750$$

$$25x_1 + 30x_2 = 850$$

in the first quadrant

Note that the objective function has constant values along these line segments. The line segments are called level curves of the objective function. As we move in a direction perpendicular to these line segments, the objective function either increases or decreases.

Figure 5.6
The level curves of the objective function f are parallel line segments in the first quadrant; the objective function either increases or decreases as we move in a direction perpendicular to the level curves



THEOREM 1

Suppose the feasible region of a linear program is a nonempty and bounded convex set. Then the objective function must attain both a maximum and mini mum value occurring at extreme points of the region. If the feasible region is unbounded, the objective function need not assume its optimal values. If either a maximum or minimum does exist, ii must occur at one of the extreme points.

The power of this theorem is that it guarantees an optimal solution to a linear program from among the extreme points of a bounded nonempty convex set.

Problems

1. Consider a company that carves wooden soldiers. The company specializes in two main *types:* Confederate and Union soldiers. The profit for each is \$28 and \$30, respectively. It requires 2 units of lumber, 4 hr of carpentry, and 2 hr of finishing completing a Confederate soldier. It requires 3 units of lumber, *3.5* hr of carpentry, and 3 hr of finishing completing a Union soldier. Each week the company has 100 Units of lumber delivered. There are 120 hr of carpenter machine time available and 90 hr of finishing time available. Determine the number of each wooden soldier to produce to maximize weekly profits.

2. Solve the following problems using graphical analysis:

$$a)\ Maximize\ x+y$$
$$Suject\ to:\ x+y \le 6$$
$$3x-y \le 9$$
$$x, y \ge 0$$

$$b)\ Minimize\ x+y$$
$$Suject\ to:\ x+y \ge 6$$
$$3x-y \ge 9$$
$$x, y \ge 0$$

$$c)\ Maximize\ 25x_1+30x_2$$
$$Suject\ to:\ 20x_1+30x_2 \le 690$$
$$5x_1+4x_2 \le 120 \quad \& \ x_1, x_2 \ge 0$$

5.3 Linear Programming II: Algebraic Solutions

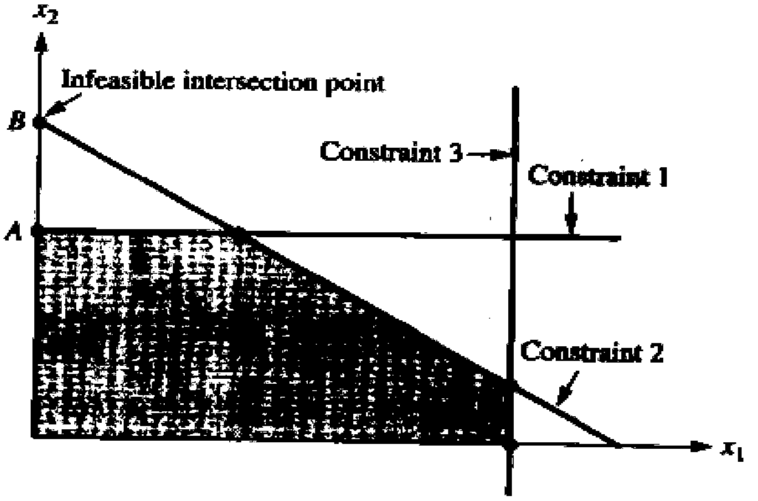
The graphical solution to the carpenter's problem suggests a rudimentary procedure for finding an optimal solution to a linear program with a nonempty and bounded feasible region:

1. Find all intersection points of the constraints.
2. Determine which intersection points, if any, are feasible to obtain the extreme points.
2. Evaluate the objective function at each extreme point.
3. Choose the extreme point(s) with the largest (or smallest) value for the objective function.

To implement this procedure algebraically, we must characterize the intersect ion points and the extreme points.

The convex set depicted in Figure 5.7 consists of three linear constraints (plus the two non negativity constraints).The nonnegative variables y_1, y_2 & y_3 indicated in the figure measure the degree by which a point satisfies each of the constraints I, 2, and 3, respectively. The variable y_i is added to the left side of inequality constraint *i* to convert it to an equality. Thus, $y_2 = 0$ characterizes those points that lie precisely on constraint 2, and a negative value for y_2 indicates the violation of constraint 2. Likewise, the decision variables x_1 and x_2 are constrained to nonnegative values. Thus, the values of the decision variables x_1 & x_2 measure the degree of satisfaction of the non negativity constraints, $x_1 \ge, 0\ x_2 \ge 0$. Note that along the x_1 axis, the decision variable x_2 is 0. Now consider the values for the entire set of variables { x_1,x_2,y_1,y_2,y_3}. If two of the variables simultaneously have the value 0, then we have characterized an intersection point in the x_Ix_2 plane. All (possible) intersection points can be determined systematically by setting all possible distinguishable pairs of the five variables to zero and solving for the remaining three dependent variables. If a solution to the resulting system of equations exists, then it must be an intersection point, which may or may not be a feasible solution. A negative value for any of the five variables indicates that a constraint is not satisfied. Such an intersection point would be infeasible. For example, the intersection point B, where $y_2=0$ and $x_1=0$, gives a negative value for y_1 and hence is not feasible. Let's illustrate the procedure by solving the carpenter's problem algebraically.

Figure 5.7
The variables x_1, x_2, y_1, y_2, and y_3 measure the satisfaction of each of the constraints; intersection point A is characterized by $y_1 = x_1 = 0$; intersection point B is not feasible because y_1 is negative; the intersection points surrounding the shaded region are all feasible because none of the five variables is negative there



EXAMPLE 1 Solving the Carpenters Problem Algebraically

The carpenter's model is :

$$Maximize\, 25x_1 + 30x_2$$

$$Suject\,to:\; 20x_1 + 30x_2 \le 690\left(lumber\right)$$

$$5x_1 + 4x_2 \le 120\left(labor\right)\; \&\, x_1, x_2 \ge 0\left(nonnegativity\right)$$

We convert each of the first two inequalities to equations by adding new nonnegative "slack" variables y_1 & y_2. If either y_1 or y_2 is negative, the constraint is not satisfied. Thus, the problem becomes

$$Maximize\, 25x_1 + 30x_2$$

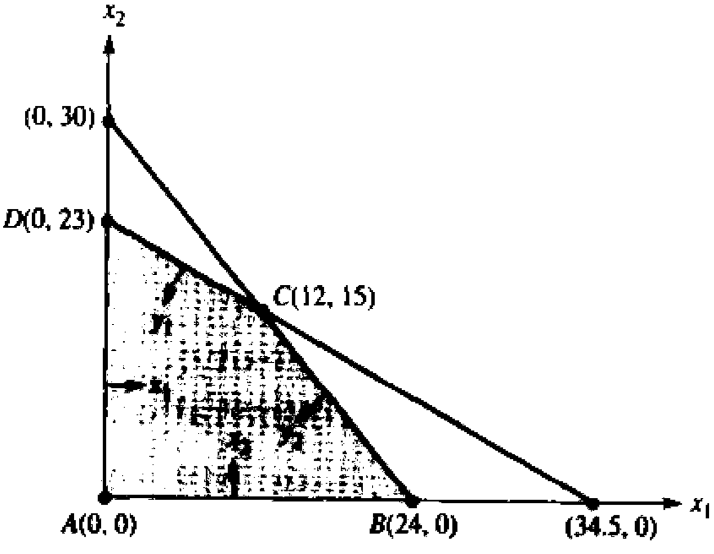
$$Suject\,to: \; 20x_1 + 30x_2 + y_1 \leq 690$$

$$5x_1 + 4x_2 + y_2 \leq 120 \quad \& \quad x_1, x_2, y_1, y_2 \geq 0$$

We now consider the entire set of four variables $\{x_1,x_2,x_3,x_4\}$, which are interpreted geometrically in Figure 5.8. To determine a possible intersection point in the x_1x_2 plane, assign two of the four variables the value zero. There are $\frac{4!}{2!2!} = 6$ possible intersection points to consider in this way (four variables taken two at a time). Let's begin by assigning the variables x_1 and x_2 the value zero, resulting in the following set of equations:

$$y_1 = 690, \qquad y_2 = 120$$

Figure 5.8
The variables $\{x_1, x_2, y_1, y_2\}$ measure the satisfaction of each constraint; an intersection point is characterized by setting two of the variables to zero



which is a feasible intersection point *A(0,*0) because all four variables are nonnegative.

For the second intersection point we choose the variables x and y and set them to zero, resulting in the system

$$30x_2 = 690$$
$$4x_2 + y_2 = 120$$

that has solution x_2 = 23 and y_2 = 28, which is also a feasible intersection point *D(0,* 23).

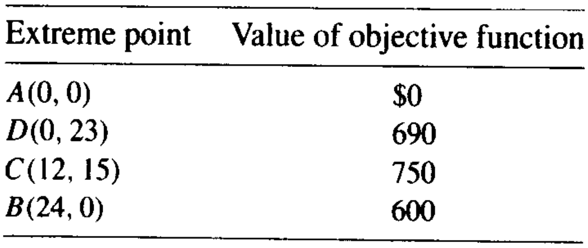
For the third intersection point we choose x_1 and y_2 and set them to zero, yielding the system

$$30x_2 + y_1 = 690$$
$$4x_2 = 120$$

with solution x_2 = 30 and y_1 = -210. Thus, the first Constraint is violated by 210 units, indicating that the intersection point (0, 30) is infeasible. In a similar manner, choosing y_1 and y_2 and setting them to zero gives x_1=12 and x_2 = 15, corresponding to the intersection point *C (12,* 15), which is feasible. Our fifth choice is to choose the variables x_2 and y_1 and set them to zero, giving values of x_1, = 34.5 and y_2 = -52.5, so the second constraint is not satisfied. Thus, the intersection point (34.5, 0) is infeasible.

Finally we determine the sixth intersection point by setting the variables x_2 and y_2 zero to determine x_1 = 24 and y_1 = 210: therefore, the intersection point *B(24,* 0) is feasible.

In summary, of the six possible intersection points in the x_Ix_2 plane, four were found to be feasible. For the four we find the value of the objective function by substitution:



Extreme point	Value of objective function
$A(0, 0)$	\$0
$D(0, 23)$	690
$C(12, 15)$	750
$B(24, 0)$	600

Our procedure determines that the optimum solution w maximize the profit is x_1 = 12 and x_2 = 15. That is, the carpenter should make 12 tables and is bookcases for a maximum profit of $750.

Problems

1. How many possible intersection points are there in the following cases?

   (a) 2 decision variables and 5 $\leq$ inequalities

   (b) 2 decision variables and 10 $\leq$ inequalities

   (c) 25 decision variables and 50 $\leq$ inequalities

5.4 Linear Programming Ill: The Simplex Method

So far we have learned to find an optimal extreme point by searching among all possible intersection points associated with the decision and slack variables. Can we reduce the number of intersection points we actually consider in our search? Certainly, once finding an initial feasible intersection point, we need not consider a potential intersection point that fails to improve the value of the objective function. Can we test the optimality of our current solution against other possible if intersection points? Even if an intersection point promises to be more optimal than the current extreme point, it is of no interest if it violates one or more of the constraints. Is there a test to determine if a proposed intersection point is feasible? The Simplex Method, developed by George Dantzig, incorporates both *optimality* and *feasibility* tests to find the optimal solution(s) to a linear program (if one exists).

An optimality test shows whether or not an intersection point corresponds to a value of the objective function better than the best value found so far. A feasibility test determines whether the proposed intersection point is feasible. To implement the Simplex Method we first separate the decision and slack variables into two non overlapping sets that we call the independent and dependent sets. For the particular linear programs we consider, the original independent set will consist of the decision variables, and the slack variables will belong to the dependent set.

5.4.1 Steps of the Simplex Method

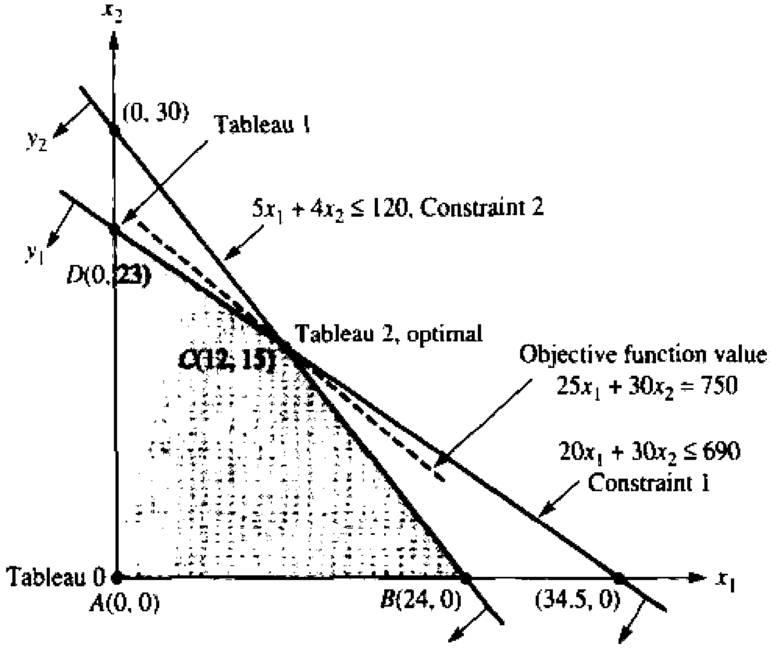
1. **Tableau Format**: Place the linear program in Tableau Format, as explained later.
2. **Initial Extreme Point**: The Simplex Method begins with a known extreme point, usually the origin (0, 0).
3. **Optimality Test**: Determine if an adjacent intersection point improves the value of the objective function. If not, the current extreme point is optimal. If an improvement is possible, the optimality test determines which variable currently in the independent set (having value zero) should *enter* the dependent set and become nonzero.
4. **Feasibility Test**: To find a new intersection point, one of the variables in the dependent se must *exit* to allow the entering variable from Step 3 to become dependent. The feasibility test determines which current dependent variable to choose for exiting, ensuring feasibility.

5. **Pivot**: Form a new equivalent system of equations by eliminating the new dependent variable from the equations that do not contain the variable that exited in Step 4. Then set the new independent variables to zero in the new system to find the values of the new dependent variables, thereby determining an intersection point.

6. **Repeat Steps 3 - 5** until an optimal extreme point is found.

Before detailing each of the preceding steps, let's examine the carpenter's problem (Figure 5.9). The origin is an extreme point, so we choose it as our starting point. Thus, x_1 and x_2 are the current arbitrary independent variables and assigned the value zero, whereas y_1 and Y_2 are the current dependent variables with values of 690 and 120, respectively. The optimality test determines if a current independent variable assigned the value zero could improve the value of the objective function if it is made dependent and positive. For example, either x_1 or x_2, if made positive, would improve the objective function value. (They have positive coefficients in the objective function we are trying to maximize.) Thus, the optimality test determines a promising variable to enter the dependent set. Later, we give a rule of thumb for choosing which independent variable to enter when more than one candidate exists. In the carpenter's problem at hand, we select x_2 as the new dependent variable.

Figure 5.9
The set of points satisfying the constraints of a linear program (the shaded region) form a convex set



The variable chosen for entry into the dependent set by the optimality condition replaces one of the current dependent variables. The feasibility condition determines which exiting variable this entering variable replaces. Basically, the entering variable replaces whichever current dependent variable can assume a zero value while maintaining nonnegative values for all the remaining dependent variables. That is, the feasibility condition ensures that the new intersection point will be feasible and hence an extreme point. In Figure 5.9, the feasibility test would lead us to the intersection point (0,23), which is feasible, and not to (0, 30), which is infeasible. Thus, 2 replace y_1 as a dependent or nonzero variable. Therefore, x_2 enters and y_1 exits the set of dependent variables.

5.4.2 Computational Efficiency

The feasibility test does not require actual computation of the values of the dependent variables when selecting an exiting variable for replacement. Instead, we will see that an appropriate exiting variable is selected by quickly determining whether any variable becomes negative if the dependent variable being considered for replacement is assigned the value zero (a ratio test that will be explained later). If any variable would become negative, then the dependent variable under consideration cannot be replaced by the entering variable if feasibility is to be maintained. Once a set of dependent variables corresponding to a more optimal extreme point is found from the optimality and feasibility tests, the values of the new dependent variables are determined by pivoting. The pivoting process essentially solves an equivalent system of equations for the new dependent variables after the exchange of the entering and exiting variables in the dependent set. The values of the new dependent variables are obtained by assigning the independent variables the value zero. Note that only one dependent variable is replaced at each stage. *Geometrically, the Simplex Method proceeds from an initial extreme point to an adjacent extreme point until no adjacent extreme point is more optimal.* At that time, the current extreme point is an optimal solution. We now detail the steps of the Simplex Method.

STEP 1 Tableau Format Many formats exist for implementing the Simplex Method. The format we use assumes the objective function is to be maximized and that the constraints are less than or equal to inequalities. (If the problem is no expressed initially in this format it can easily be changed to this format.) For the carpenter's example, the problem is to

$$\text{Maximize } 25x_1 + 30x_2$$

subject to

$$20x_1 + 30x_2 \leq 690$$

$$5x_1 + 4x_2 \leq 120$$

$$x_1, x_2 \geq 0$$

Next we adjoin a new constraint to ensure that any solution improves the best value of the objective function found so far. Take the initial extreme point as the origin, where the value of the objective function is zero. We want to constrain the objective function to be better than its current value, so we require $25x_1 + 30x_2 \geq 0$

Because all the constraints must be ≤ inequalities, multiply the new constraint by -1 and adjoin it to the original constraint set:

$$\begin{aligned} 20x_1 + 30x_2 &\leq 690 \quad &&\textbf{(constraint 1, lumber)} \\ 5x_1 + 4x_2 &\leq 120 \quad &&\textbf{(constraint 2, labor)} \\ -25x_1 - 30x_2 &\leq 0 \quad &&\textbf{(objective function constraint)} \end{aligned}$$

The Simplex Method implicitly assumes that all variables are nonnegative, so we do not repeat the non negativity constraints in the remainder of the presentation.

Next, we convert each inequality to equality by adding a *nonnegative* new variable y_i (or z) called a *slack variable* because it measures the slack or degree of satisfaction of the constraint. A negative value for y_i indicates the constraint is not satisfied. (We use the variable z for the objective function constraint to avoid confusion with the other constraints.) This process gives the *augmented constraint*

$$\begin{aligned} 20x_1 + 30x_2 + y_1 &= 690 \\ 5x_1 + 4x_2 + y_2 &= 120 \\ -25x_1 - 30x_2 + z &= 0 \end{aligned}$$

where the variables x_1,x_2,y_1,y_2 are nonnegative. The value of the variable z represents the value of the objective function, as we shall see later. (Note from the last equation, $z = 25x_1 + 30x_2$ is the value of the objective function.)

STEP 2 Initial extreme point Because there are two decision variables, all possible intersection points lie in the x_1x_2 plane and can he determined by setting two of the variables $\{x_1,x_2,y_1,y_2\}$ to zero. (The variable *z* is *always* a dependent variable and represents the value of the objective function at the extreme point in question.) The origin is feasible and corresponds to the extreme point characterized by ***$x_1 = x_2 = 0$, $y_1 = 690$, and $y_2 = 120$.*** Thus, x_1 and x_2 are independent variables assigned the value 0; y_1, y_2, and z are dependent variables whose values are then determined. As we will see, conveniently records the current value of the objective function at the extreme points of the convex set in the x_1x_2 plane as we compute them by elimination.

STEP 3 The optimality test for choosing an entering variable In the preceding form at, a negative coefficient in the last (or objective function) equation indicates that the corresponding variable could improve the current objective function value. Thus, the coefficients -*25* and -30 indicate that either x_1 or x_2 could enter and improve the current objective function value of $z = 0$. (The current constraint corresponds to $z = 25x_1 + 30x_2 \geq 0$, with x_1 and x_2 currently independent and 0.) When more than one candidate exists for the entering variable, a rule of thumb for selecting the variable to enter the dependent set is to select that variable with the largest (in absolute value) negative coefficient in the objective function row. If no negative coefficients exist, the current solution is optimal. In the case at hand, *we* choose x_2 as the new entering variable. (The procedure is inexact because at this stage we do not know what values the entering variable can assume.)

STEP 4 The feasibility condition for choosing an exiting variable The entering variable x_2 (in our example) must replace either y_1, or y_2 as a dependent variable (because z *always* remains the third dependent variable). To determine which of these variables is to exit the dependent set, first divide the right-hand side values 690 and 120 (associated with the original constraint inequalities) by the components for the entering variable in each inequality (30 and *4*, respectively, in our example) to obtain the ratios $\frac{690}{30} = 23$ & $\frac{120}{4} = 30$. From the subset of ratios that arc positive (both in this case), the variable corresponding to the minimum ratio is chosen for replacement (y_1, which corresponds to 23 in this case). *The ratios represent the value the entering variable would obtain if the corresponding exiling variable were assigned the value* 0. Thus, only positive values are considered and the smallest positive value is chosen so as no to drive any variable negative. For instance, if y_2 were chosen as the exiting variable and assigned the value 0, then x_2 would assume a value 30 as the new dependent variable. However, then y_1 would be negative, indicating that the intersection point (0,30) does not satisfy the first constraint. Note that the intersection point (0,30) is not feasible in Figure 5.9. The minimum positive ratio rule illustrated previously obviates enumeration of any infeasible intersection points. In the case at hand, the dependent variable corresponding to the smallest ratio 23 is y_1, so it becomes the exiting variable.

Thus, x_2, y_2 and z form the new set of dependent variables and x_1 and $y_1$form the new set of independent variables.

STEP 5 Pivoting to solve for the new dependent variable values Next we derive a new (equivalent) system of equations by eliminating the entering variable x_2 in all the equations of the previous system that do not contain the exiting variable y_1. There are numerous gays to execute this step, such as the method of elimination used in Section 5.3. Then we find the values of the dependent variables, x_2, y_2 , and *z* when the independent variables x_1 and y_1 are assigned the value 0 in the new system of equations. This is called the pivoting procedure. The values of x_1 and x_2 give the new extreme point (x_1,x_2), and *z* is the (improved) value of the objective function at that point.

After performing the pivot, the optimality test is applied again to determine if another candidate entering variable exists. If so, choose an appropriate one and apply the feasibility lest to choose an exiting variable. Then the pivoting procedure is performed again. *The* process is repeated until no variable has a negative coefficient in the objective function row. We now summarize the procedure and use it to solve the carpenter’s problem.

Summary of the Simplex Method

STEP 1 Place the problem in Tableau Format. Adjoin slack variables as needed to convert inequality constraints to equalities. Remember that all variables are nonnegative. Include the objective function constraint as the last constraint, including its slack variable z.

STEP 2 Find one initial extreme point. (For the problems we consider, the origin will be an extreme point.)

STEP 3 Apply the optimality test. Examine the last equation (which corresponds to the objective function). If all its coefficients are nonnegative, then stop; The current extreme point is optimal. Otherwise, *some* variables have negative coefficients, so choose the variable with the largest (in absolute value) negative coefficient as the new entering variable.

STEP 4 Apply the feasibility test. Divide the current right-hand-side values by the corresponding coefficient values of the entering variable in each equation.

Choose the exiting variable to be the one corresponding to the smallest positive ratio after this division.

STEP 5 Pivot. Eliminate the entering variable from all the equations that do not contain the exiting variable. Then assign the value 0 to the variables in the new independent set (consisting of the exited variable and the variables remaining after the entering variable has left to become dependent). The resulting values give the new extreme point $(x_1. x_2)$ and objective function value z for that point.

STEP 6 Repeat Steps 3 - 5 until an optimal extreme point is found.

EXAMPLE 1 The Carpenter's Problem Revisited

STEP 1 The Tableau Format gives

$$20x_1 + 30x_2 + y_1 = 690$$

$$5x_1 + 4x_2 + y_2 = 120$$

$$-25x_1 - 30x_2 + z = 0$$

STEP 2 The origin (0,0) is an initial extreme point for which the independent variables are $x_1 = x_2 = 0$, and the dependent variables *are* $y_1 = 690$, $y_2 = 120$, and $z = 0$.

STEP 3 We apply the optimality test to choose 2 as the variable entering the dependent set because it corresponds to the negative coefficient with the largest absolute value.

STEP 4 Applying the feasibility test, *we* divide the right-hand-side values 690 and 120 by the components for the entering variable x_2 in each equation (30 and 4,respectively), yielding the ratios $\frac{690}{30} = 23$ & $\frac{120}{4} = 30.$ & z= 0. The smallest positive ratio is 23, corresponding to the first equation that has the slack variable y_1. Thus, we choose y_1 as the exiting dependent variable.

STEP 5 We pivot to find the values of the new dependent variables x_2, y_2 and z when the independent variables x_1 and y_1 are set to the value 0. After eliminating the new dependent variable x_2 from each previous equation that does not contain the exiting variable y_1 we obtain the equivalent system

$$\begin{array}{lllll} \frac{2}{3}x_1 & +x_2 & +\frac{1}{30}y_1 & & =23 \\ \frac{7}{3}x_1 & & -\frac{2}{15}y_1 & +y_2 & =28 \\ -5x_1 & & +y_1 & \quad +z & =690 \end{array}$$

Setting $x_1=y_1=0$, we determine $x_2 = 23$, $y_2 = 28$, and $z = 690$. These results give the extreme point (0, 23) where the value of the objective function is $z = 690$.

Applying the optimality test again, we set that the current extreme point (0,23) is not optimal (because there is a negative coefficient -5 in the last equation corresponding to the variable x_1). Before continuing, observe that we really do not need to write out the entire symbolism of the equations in each step. We merely need to know the coefficient values associated with the variables in each of the equations together with the right-hand side. A table format, or *tableau,* is commonly used to record these numbers. We illustrate the completion of the carpenter's problem using this format, where the headers of each column designate the variables; the abbreviation RHS is the value of the right-hand side. We begin with Tableau 0, corresponding to the initial extreme point at the origin.

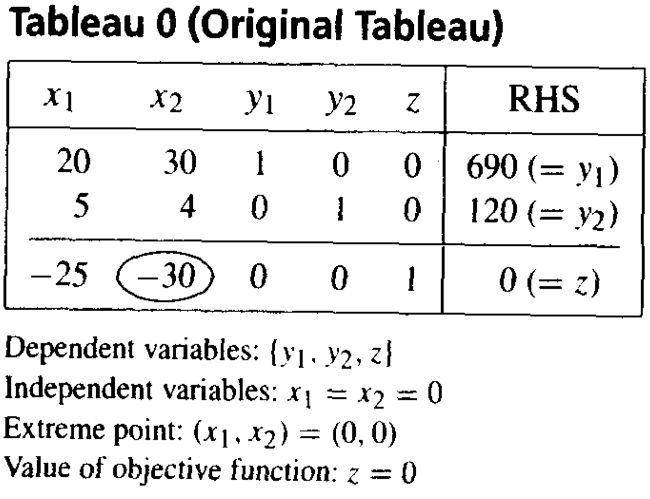


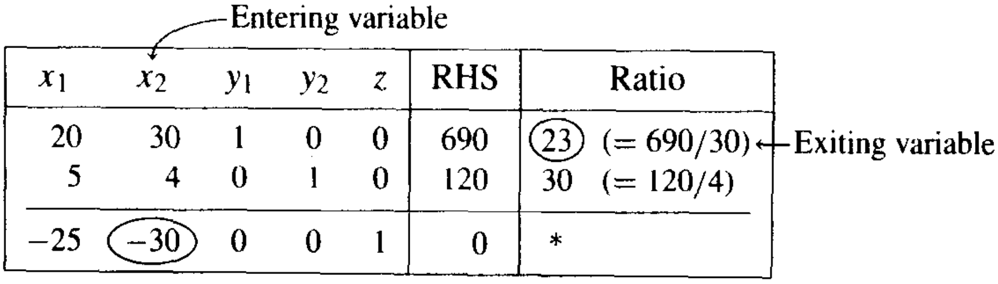
Tableau 0 (Original Tableau)

x_1	x_2	y_1	y_2	z	RHS
20	30	1	0	0	690 (= y_1)
5	4	0	1	0	120 (= y_2)
−25	$\enclose{circle}{-30}$	0	0	1	0 (= z)

Dependent variables: $\{y_1, y_2, z\}$
Independent variables: $x_1 = x_2 = 0$
Extreme point: $(x_1, x_2) = (0, 0)$
Value of objective function: $z = 0$

Optimality Test The entering variable is x_2 (corresponding to -30 in the last row).

Feasibility Test Compute the ratios for the RHS divided by the coefficients in the column labeled x_2 to determine the minimum positive ratio.



Entering variable

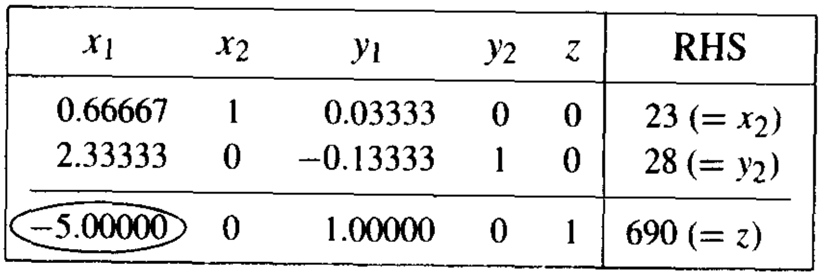
x_1	x_2	y_1	y_2	z	RHS	Ratio
20	30	1	0	0	690	23 (= 690/30)
5	4	0	1	0	120	30 (= 120/4)
−25	−30	0	0	1	0	*

Exiting variable

Choose y_1, corresponding to the minimum positive ratio 23 as the exiting variable.

Pivot Divide the row containing the exiting variable (the first row in this case) by the coefficient of the entering variable in that row (the coefficient of x_2 in this case), giving a coefficient of 1 for the entering variable in this row. Then eliminate the entering variable x_2 from the remaining rows (which do not contain the exiting variable y_1 and have 0 coefficient for it). The results are summarized in the next tableau, where we use five-place decimal approximations for the numerical values.

Tableau 1



x_1	x_2	y_1	y_2	z	RHS
0.66667	1	0.03333	0	0	23 (= x_2)
2.33333	0	−0.13333	1	0	28 (= y_2)
−5.00000	0	1.00000	0	1	690 (= z)

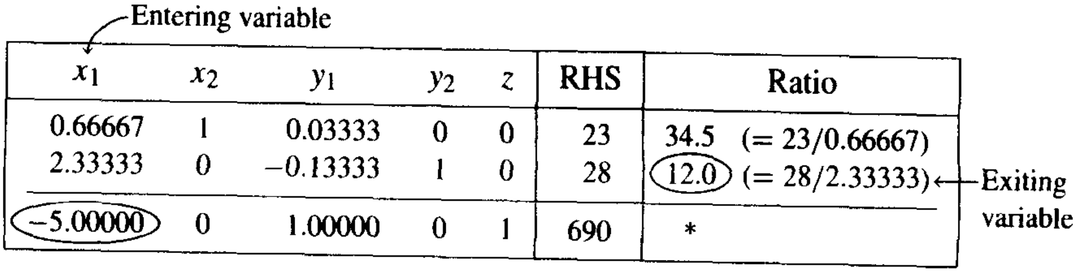
Dependent variables: $\{x_2, y_2, z\}$
Independent variables: $x_1 = y_1 = 0$
Extreme point: $(x_1, x_2) = (0, 23)$
Value of objective function: $z = 690$

The pivot determines that the new dependent variables have the values , $x_2 = 23$, $y_2 = 28$, and $z = 690$.

Optimality Test The entering variable is x_1(corresponding to the coefficient -5 in the last row).

Feasibility Test Compute the ratios for the RHS.

Entering variable



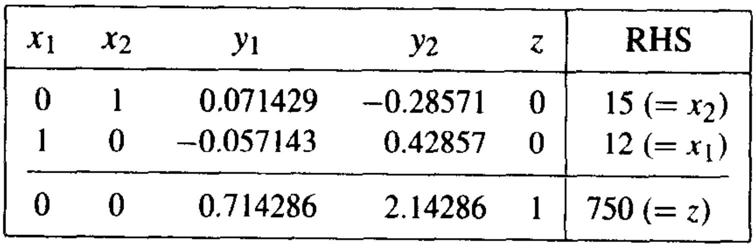
x_1	x_2	y_1	y_2	z	RHS	Ratio
0.66667	1	0.03333	0	0	23	34.5 (= 23/0.66667)
2.33333	0	−0.13333	1	0	28	12.0 (= 28/2.33333)
−5.00000	0	1.00000	0	1	690	*

Exiting variable

Choose y_2 as the exiting variable because it corresponds to the minimum positive ratio 12.

Pivot Divide the row containing the exiting variable (the second row in this case) by the coefficient of the entering variable in that row (the coefficient of x_1 in this *case),* giving a coefficient of 1 for the entering variable in this row. Then eliminate the entering variable x_1 from the remaining rows (which do not contain the exiting variable y_2 and have a zero coefficient for it). The results are summarized in the next tableau.

Tableau 2



x_1	x_2	y_1	y_2	z	RHS
0	1	0.071429	−0.28571	0	15 (= x_2)
1	0	−0.057143	0.42857	0	12 (= x_1)
0	0	0.714286	2.14286	1	750 (= z)

Dependent variables: $\{x_2, x_1, z\}$
Independent variables: $y_1 = y_2 = 0$
Extreme point: $(x_1, x_2) = (12, 15)$
Value of objective function: $z = 750$

Optimality Test Because there are no negative coefficients in the bottom row, x_1= 12 and x_2 = 15 gives the optimal solution z = \$750 for the objective function. Note that starting with an initial extreme point; we had to enumerate only two of the possible six intersection points. The power of the Simplex Method is its reduction of the computations required to find an optimal extreme point.

***EXAMPLE 2** Using the Tableau Format*

Solve the problem

$$\begin{aligned} &Maximize\, 3x_1 + x_2 \\ &\quad Suject\, to:\ 2x_1 + x_2 \le 6 \\ &\qquad\qquad x_1 + 3x_2 \le 9 \\ &\qquad\qquad x_1, x_2 \ge 0 \end{aligned}$$

The problem in Tableau Format is

$$\begin{aligned} 2x_1 + x_2 + y_1 &= 6 \\ x_1 + 3x_2 + y_2 &= 9 \\ -3x_1 - x_2 + z &= 0 \quad , where\, x_1, x_2, y_1, y_2\, \&\, z \ge 0. \end{aligned}$$

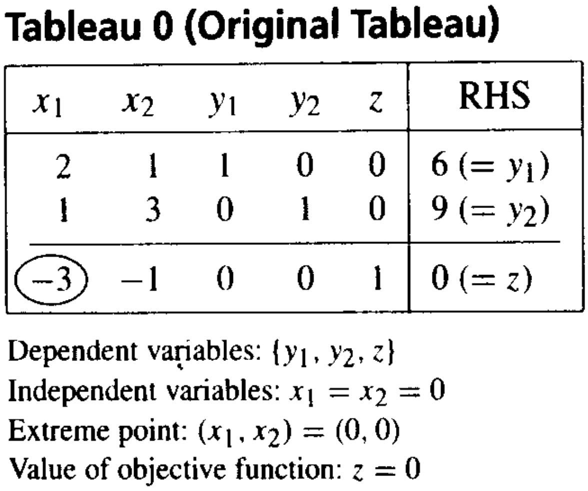


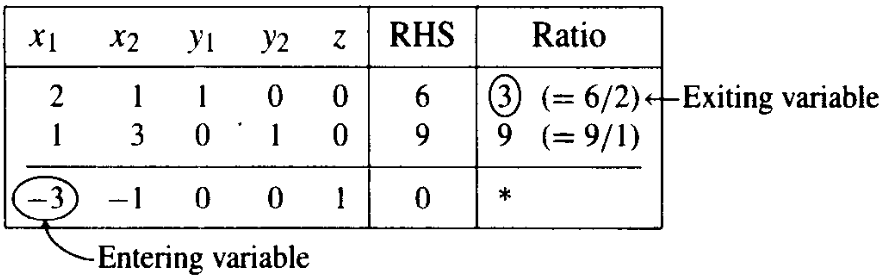
Tableau 0 (Original Tableau)

x_1	x_2	y_1	y_2	z	RHS
2	1	1	0	0	$6\ (= y_1)$
1	3	0	1	0	$9\ (= y_2)$
−3	−1	0	0	1	$0\ (= z)$

Dependent variables: $\{y_1, y_2, z\}$
Independent variables: $x_1 = x_2 = 0$
Extreme point: $(x_1, x_2) = (0, 0)$
Value of objective function: $z = 0$

Optimality Test The entering variable is x_1 (corresponding to -3 in the bottom row).

Feasibility Test Compute the ratios of the RHS divided by the column labeled x_1 to determine the minimum positive ratio.



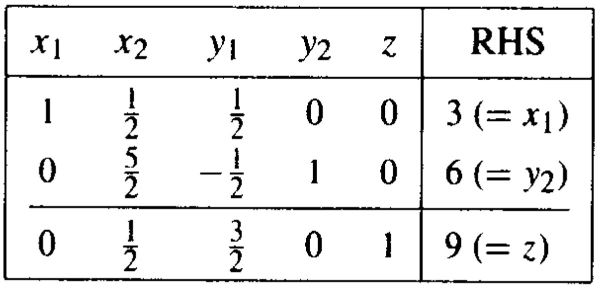
x_1	x_2	y_1	y_2	z	RHS	Ratio
2	1	1	0	0	6	③ (= 6/2) ← Exiting variable
1	3	0	1	0	9	9 (= 9/1)
−3	−1	0	0	1	0	*

Entering variable

Choose y_1 corresponding to the minimum positive ratio 3 as the exiting variable.

Pivot Divide the row containing the exiting variable (the first row in this case) by the coefficient of the entering variable in that row (the coefficient of x_1 in this case), giving a coefficient of 1 for the entering variable in this row. Then eliminate the entering variable x_1 from the remaining rows (which do not contain the exiting variable y_1 and have a zero coefficient for it).The results are summarized in the next tableau.

Tableau 1



x_1	x_2	y_1	y_2	z	RHS
1	$\frac{1}{2}$	$\frac{1}{2}$	0	0	3 (= x_1)
0	$\frac{5}{2}$	$-\frac{1}{2}$	1	0	6 (= y_2)
0	$\frac{1}{2}$	$\frac{3}{2}$	0	1	9 (= z)

Dependent variables: $\{x_1, y_2, z\}$
Independent variables: $x_2 = y_1 = 0$
Extreme point: $(x_1, x_2) = (3, 0)$
Value of objective function: $z = 9$

The pivot determine that the dependent variables have the values x_1= 3 , y_2=6 , & z=9.

Optimality Test There are no negative coefficients in the bottom row. Thus, x_1=3 , x_2=0 is an extreme point giving the optimal objective function value z=9.

Remarks We have assumed that the origin is a feasible extreme point. If it is not, then some extreme point must be found before the Simplex Method can be used as presented. We have also assumed that the linear program is not degenerate in the sense that no more than two constraints intersect at the same point. These restrictions and other topics are studied in more advanced discrete optimization courses.

Self Test Exercises.

1. Use the Simplex Method to resolve Problems 1 &2 in Section 5.2.
2. Solve the following. (Use simplex method)

a. $Min. z = 2x_1 + 2x_2$

$subject\,to: \quad 2x_1 + 4x_2 \geq 1,\ x_1 + 2x_2 \geq 1, 2x_1 + x_2 \geq 1\ \ and\ \ x_1, x_2 \geq 0$

b. $Max. Z = 3x_1 + 4x_2 + x_3 + 7x_4$

subjects to:- $8x_1 + 3x_2 + 4x_3 + x_4 \leq 7,\ 2x_1 + 6x_2 + x_3 + 5x_4 \leq 3$

$x_1 + 4x_2 + 5x_3 + 2x_4 \leq 8,\ and\ \ x_1, x_2,, x_3, x_4 \geq 0$

$c.\quad Max.z = 8x_1 + 19x_2 + 7x_3$

$subject \quad to: 3x_1 + 4x_2 + x_3 \leq 25, x_1 + 3x_2 + 3x_3 \leq 50 \; and \quad x_1, x_2, x_3 \geq 0$

$d.\quad Max.z = x_1 + x_2 + 3x_3$

$subject \quad to: 3x_1 + 2x_2 + x_3 \leq 3, 2x_1 + x_2 + 2x_3 \leq 2 \; and \quad x_1, x_2, x_3 \geq 0$

$e.\quad Max.z = 4x_1 + 3x_2 + 4x_3 + 6x_4$

$subject \quad to: x_1 + 2x_2 + 2x_3 + 4x_4 \leq 80, 2x_1 + 2x_3 + x_4 \leq 60, \; 3x_1 + 3x_2 + x_3 + x_4 \leq 80 \, and \quad x_1, x_2,$

$f.\quad Max.z = 2x_1 + 4x_2 + x_3 + 4x_4$

$subject \quad to: x_1 + 3x_2 + x_4 \leq 4, 2x_1 + x_2 \leq 3, \; x_2 + 4x_3 + x_4 \leq 3 \, and \quad x_1, x_2, x_3, x_4 \geq 0$

$g.\quad Max.z = 8x_1 + 11x_2$

$subject \quad to: 3x_1 + x_2 \; \leq 7, x_1 + 3x_2 \leq 8, \; and \quad x_1, x_2 \geq 0$

Reference

1. J. Berry & K. Houston, *Mathematical Modeling*, Edward Arnold, London, 1995
2. J. N. Kapoor, Mathematical Modeling, Wiley, 2000
3. Charles R. Mac Cluer, *Industrial mathematics, modeling in industry, science and government,* Prentice Hall, 2000
4. Glyn James, *Advanced modern engineering mathematics,* 2nd ed, prentice Hall, 1999.
5. F. R. Giordano, M. D. Weir and W. P. Fox, *A first course in mathematical modeling*, 2dn ed, 1997.
6. Friedman and W. Littman, *Industrial mathematics*, SIAM Pub., 1994
7. Dennis G.Zill and R.Cullen, *Differential Equations with Boundary value Problems,*Brooks/Cole Pub.2000.

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