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Michel Brion
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Frobenius Splitting Methods in Geometry and Representation Theory

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Preface

In the 1980s, Mehta and Ramanathan made important breakthroughs in the study of Schubert varieties by introducing the notion of a Frobenius split variety and compatibly split subvarieties for algebraic varieties in positive characteristics. This was refined by Ramanan and Ramanathan via their notion of Frobenius splitting relative to an effective divisor.

Even though *most* of the projective varieties are *not* Frobenius split, those which are have remarkable geometric and cohomological properties, e.g., all the higher cohomology groups of ample line bundles are zero. Interestingly, many varieties where a linear algebraic group acts with a dense orbit turn out to be Frobenius split. This includes the flag varieties, which are split compatibly with their Schubert subvarieties, relative to a certain ample divisor; Bott–Samelson–Demazure–Hansen varieties; the product of two flag varieties for the same group G, which are split compatibly with their G-stable closed subvarieties; cotangent bundles of flag varieties; and equivariant embeddings of any connected reductive group, e.g., toric varieties.

The Frobenius splitting of the above mentioned varieties yields important geometric results: Schubert varieties have rational singularities, and they are projectively normal and projectively Cohen–Macaulay in the projective embedding given by any ample line bundle (in particular, they are normal and Cohen–Macaulay); the corresponding homogeneous coordinate rings are Koszul algebras; the intersection of any number of Schubert varieties is reduced; the full and subregular nilpotent cones have rational Gorenstein singularities; the equivariant embeddings of reductive groups have rational singularities. Moreover, their proofs are short and elegant.

Further remarkable applications of Frobenius splitting concern the representation theory of semisimple groups: the Demazure character formula; a proof of the Parthasarathy–Rango Rao–Varadarajan–Kostant conjecture on the existence of certain components in the tensor product of two dual Weyl modules; the existence of good filtrations for such tensor products and also for the coordinate rings of semisimple groups in positive characteristics, etc.

The technique of Frobenius splitting has proved to be so powerful in tackling numerous and varied problems in algebraic transformation groups that it has become an indispensable tool in the field. While much of the research has appeared in journals, nothing comprehensive exists in book form. This book systematically develops the theory from scratch. Its various consequences and applications to problems in algebraic group theory have been treated in full detail bringing the reader to the forefronts of the area. We have included a large number of exercises, many of them covering complementary material. Also included are some open problems.

This book is suitable for mathematicians and graduate students interested in geometric and representation-theoretic aspects of algebraic groups and their flag varieties. In addition, it is suitable for a slightly advanced graduate course on methods of positive characteristics in geometry and representation theory. Throughout the book, we assume some familiarity with algebraic geometry, specifically, with the contents of the first three chapters of Hartshorne's book [Har–77]. In addition, in Chapters 2 to 6 we assume familiarity with the structure of semisimple algebraic groups as exposed in the books of Borel [Bor–91] or Springer [Spr–98]. We also rely on some basic results of representation theory of algebraic groups, for which we refer to Jantzen's book [Jan–03]. We warn the reader that the text provides much more information than is needed for most applications. Thus, one should not hesitate to skip ahead at will, tracing back as needed.

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Michel Brion and Shrawan Kumar
September 2004

Notational Convention

Those exercises which are used in the proofs in the text appear with a star.

Frobenius Splitting Methods in Geometry and Representation Theory

Chapter 1

Frobenius Splitting: General Theory

Introduction

This chapter is devoted to the general study of Frobenius split schemes, a notion introduced by Mehta–Ramanathan and refined further by Ramanan–Ramanathan (see 1.C for more precise references).

Any scheme over a field of characteristic $p > 0$ possesses a remarkable endomorphism, the (absolute) Frobenius morphism F which fixes all the points and raises the functions to their p-th power. A scheme X is called Frobenius split (for short, split), if the p-th power map $\mathcal{O}_X \to F_*\mathcal{O}_X$ splits as a morphism of sheaves of $\mathcal{O}_X$-modules.

In Section 1.1, we give various examples of Frobenius split schemes. These include all nonsingular affine varieties (Proposition 1.1.6), their quotients by finite groups of order prime to p (Example 1.1.10.1), and also all projective spaces (Example 1.1.10.3). The existence of a splitting is preserved under taking images under certain morphisms (Lemmas 1.1.8 and 1.1.9). Thus, the total space of a line bundle over a split scheme is split (Lemma 1.1.11), and so is the affine cone over a split complete variety (Lemma 1.1.14). The compatibility of splittings with closed subschemes is also investigated.

Section 1.2 presents some of the fundamental properties of split schemes: they are reduced (Proposition 1.2.1) and not "too singular;" specifically, they are weakly normal (Proposition 1.2.5; which is a key step in the proof of normality of Schubert varieties presented in 3.2.2). Further, line bundles on projective split schemes satisfy remarkable homological properties: all the higher cohomology groups of ample line bundles vanish (Theorem 1.2.8), and the Kodaira vanishing theorem holds under the additional assumption of Cohen–Macaulayness (Theorem 1.2.9). We also present two relative vanishing results (Lemma 1.2.11 and Theorem 1.2.12), to be further developed in Section 1.3.

Section 1.3 is primarily devoted to establishing various geometric criteria for the existence of a splitting, including Proposition 1.3.11 and Theorem 1.3.8. The latter asserts that a complete nonsingular variety X is split if and only if there exists a global section φ of ω_X^{1-p}, and a closed point $x \in X$ with local coordinates $t_1, \ldots, t_n$, such that the monomial $(t_1 \cdots t_n)^{p-1}$ occurs in the local expansion of φ at x. Here ω_X denotes the (invertible) sheaf of differential forms of top degree, and ω_X^{1-p} is the $(1-p)$-th tensor power. This shows that complete split varieties are very special, e.g., among all the nonsingular projective curves, only the projective line and elliptic curves of Hasse invariant 1 are split.

These criteria rely on a closed formula in local coordinates for the trace map of the finite flat morphism F (Lemma 1.3.6). In turn, this formula is derived from results of Cartier on differential calculus in characteristic p, which are presented in detail in Section 1.3. This section ends with a version of the Grauert–Riemenschneider vanishing theorem for split varieties (Theorem 1.3.14) which will play an important role in proving that certain varieties admit "rational resolutions" (see Chapters 3 and 6). We also obtain a version of the Kawamata–Viehweg vanishing theorem in the presence of splitting (Theorem 1.3.16).

In Section 1.4, the notion of splitting relative to a divisor is developed. This yields versions of the vanishing theorems in Section 1.2, which apply to all semi-ample line bundles (Theorem 1.4.8; which is an essential ingredient in the proof of the Demazure character formula in Chapter 3).

Section 1.5 presents applications of splitting to syzygies. Specifically, if X is a normal projective variety, and $X \times X$ is split compatibly with the diagonal, then any ample line bundle on X is very ample, and X is projectively normal in the corresponding embedding θ into projective space (Corollaries 1.5.3 and 1.5.4). If, in addition, $X \times X \times X$ is split compatibly with the two partial diagonals Δ_{12} and Δ_{23}, then the image of X in the embedding θ is an intersection of quadrics (Proposition 1.5.8).

More generally, the existence of splittings of all products $X \times \cdots \times X$, compatible with all the partial diagonals $\Delta_{i,i+1}$, implies that the homogeneous coordinate ring of X is Koszul, i.e., the trivial module over this graded ring admits a linear resolution (Theorem 1.5.15). This result, together with its refinement to closed subschemes, is motivated by its applications to syzygies of flag varieties and their Schubert varieties, presented in Chapter 3.

The techniques of Frobenius splitting obviously involve positive characteristics in an essential way. But, some of their main applications concern certain varieties in characteristic 0, which are defined over the integers (possibly with finitely many primes inverted) and hence may be reduced modulo large primes. The relevant tools of semi-continuity are gathered in Section 1.6.

1.1 Basic definitions, properties, and examples

We begin by fixing notation and conventions on schemes; our basic reference is [Har–77]. Let k be an algebraically closed field of positive characteristic p. We will consider separated schemes of finite type over k; these will be called *schemes* for simplicity. A *variety* is an integral scheme (in particular, irreducible). The structure sheaf of a scheme X is denoted by $\mathcal{O}_X$, and the ideal sheaf of a closed subscheme Y is denoted by $\mathcal{I}_Y$.

Let X^{reg} be the regular locus of a scheme X. For a closed point $x \in X^{\mathrm{reg}}$, a *system of local coordinates* $(t_1, \dots, t_n)$ *of* X *at* x is a minimal system of generators of the maximal ideal of the local ring $\mathcal{O}_{X,x}$; then, n is the dimension of X at x. The choice of a system of local coordinates identifies the completed local ring $\hat{\mathcal{O}}_{X,x}$ with the ring of formal power series $k[[t_1, \dots, t_n]]$.

The monomial $t_1^{i_1} \dots t_n^{i_n}$ will be denoted by $t^{\mathbf{i}}$, where $\mathbf{i} = (i_1, \dots, i_n) \in \mathbb{N}^n$, $\mathbb{N}$ being the set of nonnegative integers. The monomial $(t_1 \dots t_n)^{p-1}$ will play a prominent role; we denote it by $t^{\mathbf{p-1}}$. We write $\mathbf{i} \leq \mathbf{p-1}$ if $i_j \leq p-1$ for $j = 1, \dots, n$.

Next, we introduce the Frobenius morphism. Consider first a commutative, associative k-algebra A. Then, the p-th power map

$$F : A \longrightarrow A, \quad a \mapsto a^p,$$

is a ring endomorphism, called the *Frobenius morphism*; its image is a subalgebra denoted by A^p. Notice that F is not a k-algebra homomorphism: it is semilinear with respect to the Frobenius endomorphism of k.

Here are some basic properties of the Frobenius morphism.

1.1.1 Lemma. *Let A be a localization of a finitely generated k-algebra. Then, the A^p-module A is finitely generated.*

If, in addition, A is regular, then the A^p-module A is locally free. Specifically, let $\mathcal{M}$ be a maximal ideal of A and let $t_1, \dots, t_n$ be a minimal set of generators of the maximal ideal $\mathcal{M}A_{\mathcal{M}}$ of the local ring $A_{\mathcal{M}}$; then, the monomials

$$t^{\mathbf{i}}, \ \mathbf{i} \leq \mathbf{p-1}$$

form a basis of the $(A_{\mathcal{M}})^p$-module $A_{\mathcal{M}}$.

Proof. In the case where A is the polynomial ring $k[t_1, \dots, t_n]$, we have $A^p = k[t_1^p, \dots, t_n^p]$, and the monomials $t^{\mathbf{i}}$, $\mathbf{i} \leq \mathbf{p-1}$, clearly form a basis of the A^p-module A.

Now, let A be a finitely generated algebra. Since A is a quotient of a polynomial ring, it follows that the A^p-module A is finitely generated. Further, let S be a multiplicative subset of A; then, $S^{-1}A = (S^p)^{-1}A$ and $(S^{-1}A)^p = (S^p)^{-1}A^p$. It follows that $(S^{-1}A)^p$-module $S^{-1}A$ is finitely generated.

For the second assertion, we may assume that A is local with maximal ideal $\mathcal{M}$. We claim that the monomials $t^{\mathbf{i}}$, $\mathbf{i} \leq \mathbf{p-1}$, generate the A^p-module A. By Nakayama's lemma, it suffices to show that their classes in the quotient $A/F(\mathcal{M})A$ are a basis of that space, where $F(\mathcal{M})$ consists of all p-th powers of elements of $\mathcal{M}$, and $F(\mathcal{M})A$ denotes the ideal generated by $F(\mathcal{M})$. Clearly,

$$F(\mathcal{M})A = (t_1^p, \dots, t_n^p),$$

where we denote by $(f_1, \dots, f_m)$ the ideal of A generated by $f_1, \dots, f_m \in A$. In particular, $F(\mathcal{M})A$ differs from the p-th power $\mathcal{M}^p$ of the ideal $\mathcal{M}$, consisting of all sums of products of p elements of $\mathcal{M}$. However, $F(\mathcal{M})A$ contains the pn-th power

$\mathcal{M}^{pn}$, as every monomial of total degree pn in $t_1, \ldots, t_n$ is divisible by t_i^p for some i. Since $A/\mathcal{M}^{pn} \simeq k[t_1, \ldots, t_n]/(t_1, \ldots, t_n)^{pn}$, it follows that

$$A/F(\mathcal{M})A = A/(F(\mathcal{M})A + \mathcal{M}^{pn}) \simeq k[t_1, \ldots, t_n]/(t_1^p, \ldots, t_n^p),$$

which implies our claim.

Next, we check that the monomials $t^{\mathbf{i}}$, $\mathbf{i} \leq \mathbf{p-1}$, are linearly independent over A^p. Consider a relation $\sum_{\mathbf{i}\leq\mathbf{p-1}} a_{\mathbf{i}}^p t^{\mathbf{i}} = 0$, where $a_{\mathbf{i}} \in A$ for all $\mathbf{i}$. Regard A as a subring of its completion $k[[t_1, \ldots, t_n]]$ and notice that $t^{p\mathbf{j}+\mathbf{i}} = t^{p\mathbf{j}'+\mathbf{i}'}$ implies that $\mathbf{i} = \mathbf{i}'$ and $\mathbf{j} = \mathbf{j}'$, whenever $\mathbf{i}, \mathbf{i}' \leq \mathbf{p-1}$. It follows that $a_{\mathbf{i}} = 0$. $\square$

We now extend the definition of Frobenius morphism to schemes.

1.1.2 Definition. Let X be a scheme; then, the *absolute Frobenius morphism*

$$F_X : X \longrightarrow X$$

is the identity on the underlying space of X, and the p-th power map on the structure sheaf $\mathcal{O}_X$.

We will abbreviate F_X by F if the reference to X is clear; likewise, we will denote the associated map

$$F_X^{\#} : \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$$

by $F^{\#}$. Then, $F^{\#}$ is just the p-th power map.

By Lemma 1.1.1, F is a finite morphism of schemes (but not of k-schemes), surjective on closed points. If, in addition, X is regular, then F is flat.

For any morphism $f : X \longrightarrow Y$ of schemes, the diagram

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "F_X"'] & Y \arrow[d, "F_Y"'] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

commutes.

Observe that, for a sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, $F_*\mathcal{F}$ equals $\mathcal{F}$ as sheaves of abelian groups, but the $\mathcal{O}_X$-module structure on $F_*\mathcal{F}$ is given by $f * s = f^p s$, for any local sections f of $\mathcal{O}_X$ and s of $\mathcal{F}$. In particular, $z * s = z^p s$, for any $z \in k$.

1.1.3 Definition. (i) A scheme X is *Frobenius split* (or simply *split*) if the $\mathcal{O}_X$-linear map $F^{\#} : \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$ splits, i.e., there exists an $\mathcal{O}_X$-linear map

$$\varphi : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$$

such that the composition $\varphi \circ F^{\#}$ is the identity map of $\mathcal{O}_X$. Any such φ is called a *splitting*.

(ii) A closed subscheme Y of X is *compatibly split* if there exists a splitting φ of X such that

$$\varphi(F_*\mathcal{I}_Y) \subset \mathcal{I}_Y.$$

(iii) More generally, closed subschemes $Y_1, \ldots, Y_m$ of X are *simultaneously compatibly split* (or simply *compatibly split*) if they are compatibly split by the same splitting of X.

1.1.4 Remarks. (i) A splitting of X is nothing but an endomorphism $\varphi : \mathcal{O}_X \longrightarrow \mathcal{O}_X$ of the sheaf $\mathcal{O}_X$, considered only as a sheaf of abelian groups on X, satisfying the following:

(a) $\varphi(f^p g) = f\varphi(g)$, for $f, g \in \mathcal{O}_X$, and
(b) $\varphi(1) = 1$.

Indeed, (a) is equivalent to the requirement that $\varphi \in \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$; and if (a) holds, then $\varphi(f^p) = f\varphi(1)$ for $f \in \mathcal{O}_X$. (Here and subsequently in the book we have abbreviated $\mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ as $\mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$.) In other words, for any $\varphi \in \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$, the composition $\varphi \circ F^\#$ is the multiplication by $\varphi(1)$, a regular function on X. Thus, φ is a splitting if and only if $\varphi(1) = 1$.

Assume now that X is a complete variety, so that every regular function on X is a constant. Thus, $\varphi \in \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$ is a nonzero scalar multiple of a splitting if and only if $\varphi(1)$ is not identically zero.

(ii) If φ compatibly splits a closed subscheme Y of X, then clearly φ induces a splitting φ_Y of Y, such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & F_*\mathcal{I}_Y \arrow[r] \arrow[d, "\varphi"'] & F_*\mathcal{O}_X \arrow[r] \arrow[d, "\varphi"'] & F_*\mathcal{O}_Y \arrow[r] \arrow[d, "\varphi_Y"'] & 0 \\
0 \arrow[r] & \mathcal{I}_Y \arrow[r] & \mathcal{O}_X \arrow[r] & \mathcal{O}_Y \arrow[r] & 0
\end{tikzcd}$$

commutes.

(iii) Let $\varphi \in \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$ be a splitting and let $\mathcal{I}$ be an ideal sheaf of $\mathcal{O}_X$. Then,

$$\mathcal{I} \subset \varphi(F_*\mathcal{I}),$$

since $f = \varphi(f^p)$ for any local section f of $\mathcal{O}_X$.

1.1.5 Example. The affine space $\mathbb{A}^n$ over k is split compatibly with all its coordinate subspaces. Indeed, consider an additive map

$$\varphi : k[t_1, \ldots, t_n] \longrightarrow k[t_1, \ldots, t_n]$$

such that $\varphi(f^p g) = f\varphi(g)$ for all f, g in $k[t_1, \ldots, t_n]$, and $\varphi(1) = 1$. By Remark 1.1.4 (i), any such φ uniquely extends to a splitting of $\mathbb{A}^n$ by setting $\varphi(\frac{f}{g}) = \frac{1}{g}\varphi(fg^{p-1})$, for a regular function $\frac{f}{g}$ on any open subset of $\mathbb{A}^n$, where $f, g \in k[t_1, \ldots, t_n]$.

Any such map φ is uniquely determined by its values at the monomials $t^{\mathbf{i}}$, $\mathbf{i} \leq \mathbf{p-1}$, and $\mathbf{i} \neq (0, \dots, 0)$. Conversely, assigning arbitrary polynomials to $t^{\mathbf{i}}$ for nonzero $\mathbf{i} \leq \mathbf{p-1}$ uniquely extends to a map φ satisfying the above conditions. Now, choose $\varphi(t^{\mathbf{i}}) = 0$ for all nonzero $\mathbf{i} \leq \mathbf{p-1}$. Then, we have for arbitrary $\mathbf{i} \in \mathbb{N}^n$:

$$\varphi(t^{\mathbf{i}}) = t^{\mathbf{i}/p} := \begin{cases} t^{\mathbf{j}} & \text{if } \mathbf{i} = p\mathbf{j},\ \mathbf{j} \in \mathbb{N}^n, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, φ maps every monomial to a monomial with the same support, or to 0. Thus, the ideal generated by any subset of the coordinate functions is invariant under φ.

This example may be generalized as follows.

1.1.6 Proposition. *Let X be a nonsingular affine variety and let Y be a closed nonsingular subvariety. Then, X is split compatibly with Y.*

Proof. We adapt the argument of Example 1.1.5. Let $X = \operatorname{Spec}(A)$, where A is a regular, finitely generated k-algebra, and let I be the ideal of Y in X. Let M be the set of all additive maps $\varphi : A \longrightarrow A$ such that $\varphi(f^p g) = f\varphi(g)$ for all $f, g \in A$, and let M_I be the subset of those φ such that $\varphi(I) \subset I$. Then, the splittings of X that are compatible with Y are those $\varphi \in M_I$ such that $\varphi(1) = 1$.

Note that M is an abelian group under pointwise addition; in fact, an A-module via $(f\varphi)(g) = f\varphi(g)$; further, M_I is an A-submodule. Any $\varphi \in M$ is uniquely determined by its values at a set of generators of the A^p-module A (and these values have to satisfy all the relations between these generators). By Lemma 1.1.1, it follows that the A-modules M and M_I are finitely generated.

The map

$$\epsilon : M_I \longrightarrow A, \quad \varphi \mapsto \varphi(1),$$

is A-linear, and X is split compatibly with Y if and only if ϵ is surjective. By the finiteness of the A-module M_I, this is equivalent to the surjectivity of ϵ after taking the completion of the localization at every maximal ideal of A. Thus, we are reduced to the case where $A = k[[t_1, \dots, t_n]]$ and the ideal I is generated by $t_1, \dots, t_m$ for some $m \leq n$. (Recall that the ideal of any nonsingular subvariety at any point of a nonsingular variety is generated by a subset of a suitably chosen system of local coordinates.) Now, the formula

$$\varphi\Big(\sum_{\mathbf{i}} a_{\mathbf{i}} t^{\mathbf{i}}\Big) = \sum_{\mathbf{i},\, p|\mathbf{i}} a_{\mathbf{i}}^{1/p}\, t^{\mathbf{i}/p}$$

defines a compatible splitting, as in Example 1.1.5. □

Another source of examples is the following.

1.1.7 Lemma. *(i) If a scheme X is split under $\varphi \in \operatorname{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$ compatibly with a closed subscheme Y, then for every open subscheme U of X, φ restricts to a splitting of U, compatible with $U \cap Y$.*

(ii) Conversely, if U is a dense open subscheme of a reduced scheme X, and if $\varphi \in \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$ restricts to a splitting of U, then φ is a splitting of X. If, in addition, Y is a reduced closed subscheme of X such that $U \cap Y$ is dense in Y and compatibly split by $\varphi|_U$, then Y is compatibly split by φ.

(iii) If X is a normal variety and U is an open subset with complement of codimension at least 2, then X is split if and only if U is. In fact, any splitting of U is the restriction of a unique splitting of X. In particular, X is split if and only if its regular locus is.

Proof. (i) Clearly, φ yields an $\mathcal{O}_U$-linear map $F_*\mathcal{O}_U \longrightarrow \mathcal{O}_U$. And since $\varphi(1) = 1$, this map is a splitting of U. It is compatible with $U \cap Y$, since $\varphi(F_*\mathcal{I}_Y) \subset \mathcal{I}_Y$.

(ii) The regular function $\varphi(1)$ equals 1 on U, and hence on the whole of X. (Indeed, we may regard $\varphi(1) - 1$ as a morphism from X to $\mathbb{A}^1$, and its fiber at 0 contains the dense open subset U). By Remark 1.1.4 (i), it follows that φ is a splitting of X. For the compatibility of this splitting with Y, we need to check that $\varphi(F_*\mathcal{I}_Y) \subset \mathcal{I}_Y$. Notice that $\varphi(F_*\mathcal{I}_Y)$ is a coherent ideal sheaf of $\mathcal{O}_X$, containing $\mathcal{I}_Y$ by Remark 1.1.4 (iii). Thus, we have $\varphi(F_*\mathcal{I}_Y) = \mathcal{I}_Z$ for some closed subscheme Z of X contained in Y. Further, $U \cap Z = U \cap Y$, since $\varphi|_U$ is compatible with $U \cap Y$. Thus, the ideal sheaf of Z (regarded as a closed subscheme of Y) vanishes on the dense subset $U \cap Y$. Since Y is reduced, this ideal sheaf vanishes on the whole of Y, so that $Z = Y$.

(iii) If X is split, then so is U by (i). Conversely, assume that U is split by ψ and denote the inclusion by $i : U \longrightarrow X$. Then, $i_*\mathcal{O}_U = \mathcal{O}_X$, since X is normal and $X \setminus U$ contains no divisors of X. Hence, the map

$$\varphi : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X, \quad f \mapsto i_*\psi i^\#(f),$$

is well defined, and extends $\psi : F_*\mathcal{O}_U \longrightarrow \mathcal{O}_U$. Further, φ is $\mathcal{O}_X$-linear (since ψ is $\mathcal{O}_U$-linear). Thus, φ is a splitting of X by (ii). The uniqueness of φ is easy to see from $i_*\mathcal{O}_U = \mathcal{O}_X$. ☐

Next, we show that the existence of a splitting is preserved by taking images under certain morphisms.

1.1.8 Lemma. *Let $f : X \longrightarrow Y$ be a morphism of schemes such that the map $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is an isomorphism. Let Z be a closed subscheme of X and let W be the scheme-theoretic image of Z in Y, cf. [Har–77, Chap. II, Exercise 3.11(d)]. Identifying $f_*\mathcal{O}_X$ with $\mathcal{O}_Y$ via $f^\#$, we have:*

(i) $\mathcal{I}_W = f_*\mathcal{I}_Z$.

(ii) If X is split, then so is Y. If, in addition, Z is compatibly split, then so is W.

Proof. (i) Since W is the scheme-theoretic image of Z, we have

$$\mathcal{I}_W = (f^\#)^{-1}(f_*\mathcal{I}_Z).$$

This proves (i), since $f^\#$ is an isomorphism.

(ii) Let $\varphi : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ be a splitting; consider the direct image

$$f_*\varphi : f_*F_*\mathcal{O}_X \longrightarrow f_*\mathcal{O}_X = \mathcal{O}_Y.$$

Since $f_*F_*\mathcal{O}_X = F_*(f_*\mathcal{O}_X) = F_*\mathcal{O}_Y$, we see that $f_*\varphi$ maps $F_*\mathcal{O}_Y$ to $\mathcal{O}_Y$, and 1 to 1. Thus, $f_*\varphi$ is a splitting of Y.

If, in addition, Z is compatibly split by φ, then $\varphi(F_*\mathcal{I}_Z) = \mathcal{I}_Z$, whence

$$(f_*\varphi)(F_*\mathcal{I}_W) = (f_*\varphi)(F_*(f_*\mathcal{I}_Z)) = f_*\mathcal{I}_Z = \mathcal{I}_W.$$

□

The following variant of Lemma 1.1.8 gives rise to further examples of split schemes.

1.1.9 Lemma. *Let $f : X \longrightarrow Y$ be a morphism of schemes such that the map $f^\# : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ splits as a morphism of $\mathcal{O}_Y$-modules. If X is split, then so is Y.*

Proof. Let $\varphi : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ be a splitting of X and let $\pi : f_*\mathcal{O}_X \longrightarrow \mathcal{O}_Y$ be a $\mathcal{O}_Y$-module splitting of $f^\#$. Then, the diagram

$$\begin{tikzcd} F_*\mathcal{O}_Y \arrow[r, "F_*f^\#"] & f_*F_*\mathcal{O}_X \arrow[d, "f_*\varphi"'] \\ \mathcal{O}_Y & f_*\mathcal{O}_X \arrow[l, "\pi"'] \end{tikzcd}$$

yields the desired candidate

$$\psi = \pi \circ (f_*\varphi) \circ F_*f^\#$$

for a splitting of Y. (One easily checks that $\psi : F_*\mathcal{O}_Y \longrightarrow \mathcal{O}_Y$ is a morphism of $\mathcal{O}_Y$-modules such that $\psi(1) = 1$.) □

1.1.10 Examples. (1) Let G be a finite group of automorphisms of a nonsingular affine variety $X = \mathrm{Spec}(A)$. Then, G acts on the algebra A, and the invariant subalgebra A^G is finitely generated [Eis–95, Theorem 13.17]. Let $Y = X/G := \mathrm{Spec}(A^G)$ be the corresponding affine variety. The inclusion of A^G into A yields the quotient morphism

$$f : X \longrightarrow Y.$$

Denote by d the order of G. If d is not divisible by p, then Y is split. Indeed, X is split by Proposition 1.1.6. Further, the inclusion

$$f^\# : A^G \longrightarrow A$$

is split by the projection

$$\pi : A \longrightarrow A^G, \;\; a \mapsto \frac{1}{d}\sum_{g\in G} g \cdot a,$$

so that Lemma 1.1.9 applies.

(2) More generally, consider a finite surjective morphism between varieties

$$f : X \longrightarrow Y,$$

where Y is assumed to be normal. Define the *degree* d of f as the degree of the field extension $k(X)/k(Y)$ and the *trace map* Tr as the trace of this field extension. Since $f_*\mathcal{O}_X$ is a sheaf of $\mathcal{O}_Y$-algebras finite over $\mathcal{O}_Y$, and contained in the constant sheaf $k(X)$,

$$\mathrm{Tr}(f_*\mathcal{O}_X) \subset \mathcal{O}_Y$$

(as the latter is integrally closed in $k(Y)$). Further, the restriction of Tr to $\mathcal{O}_Y$ is d times the identity map. If d is not divisible by p, then $\frac{1}{d}\,\mathrm{Tr}$ splits $f^\#$; thus, Y is split if X is.

(3) Let

$$f : E \longrightarrow X$$

be a vector bundle over a scheme X, with zero section E_0, and let

$$g : \mathbb{P}(E) \longrightarrow X$$

be the corresponding projective bundle (so that the fibers of g are the projective spaces consisting of the lines in the fibers of f). Consider the induced morphism

$$h : E \setminus E_0 \longrightarrow \mathbb{P}(E).$$

Then, the natural action of the multiplicative group $\mathbb{G}_m$ on $E \setminus E_0$ yields a grading of the sheaf $h_*\mathcal{O}_{E\setminus E_0}$, with degree zero component equal to $\mathcal{O}_{\mathbb{P}(E)}$. Thus, the map $h^\# : \mathcal{O}_{\mathbb{P}(E)} \longrightarrow h_*\mathcal{O}_{E\setminus E_0}$ splits as a morphism of $\mathcal{O}_{\mathbb{P}(E)}$-modules. Further, all the fibers of g being projective spaces, the map $g^\# : \mathcal{O}_X \longrightarrow g_*\mathcal{O}_{\mathbb{P}(E)}$ is an isomorphism.

By Lemmas 1.1.8 (ii) and 1.1.9, if E is split, then so are $\mathbb{P}(E)$ and X. As a consequence, the projective space $\mathbb{P}^n$ over k is split.

A converse is provided by the following.

1.1.11 Lemma. *Let X be a scheme, Y a closed subscheme, L a line bundle over X, and L_Y the preimage of Y in L. If X is split compatibly with Y, then L is split compatibly with its zero section and with L_Y.*

More generally, let E be a vector bundle over X, and let E_Y be the preimage of Y. If the associated projective bundle $\mathbb{P}(E)$ is split compatibly with $\mathbb{P}(E_Y)$, then E is split compatibly with its zero section and with E_Y. As a consequence, X is split compatibly with Y.

Proof. Denote by $\mathcal{L}$ the sheaf of local sections of the dual line bundle L^* and, for any integer ν, let $\mathcal{L}^\nu$ be the corresponding tensor power of the invertible sheaf $\mathcal{L}$. Then, the projection $f : L \longrightarrow X$ satisfies $f_*\mathcal{O}_L = \bigoplus_{\nu=0}^{\infty} \mathcal{L}^\nu$. Now, let $\varphi : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ be a splitting, compatible with Y. Define $\psi : F_*\mathcal{O}_L \longrightarrow \mathcal{O}_L$ by setting

$$\psi(g\sigma^\nu) = \begin{cases} \varphi(g)\sigma^{\nu/p} & \text{if } \nu \equiv 0 \pmod{p}, \\ 0 & \text{otherwise,} \end{cases}$$

for any local section g of $\mathcal{O}_X$ and local trivialization σ of $\mathcal{L}$.

We claim that ψ is well defined. Let σ_1 be another trivialization and g_1 another function, such that $g\sigma^\nu = g_1\sigma_1^\nu$. Then, $\sigma_1 = u\sigma$ for some local unit u in $\mathcal{O}_X$, and $g_1 = gu^{-\nu}$. Now,

$$\varphi(g_1)\sigma_1^{\nu/p} = \varphi(gu^{-\nu})u^{\nu/p}\sigma^{\nu/p} = \varphi(g)\sigma^{\nu/p},$$

which proves our claim.

One checks easily that ψ is a splitting of L, preserving the ideals of the zero section and of L_Y.

Now, let $f : E \longrightarrow X$ be a vector bundle, and let $O_{\mathbb{P}(E)}(-1)$ be the total space of the tautological line bundle over $\mathbb{P}(E)$. Then, we have a morphism $\pi : O_{\mathbb{P}(E)}(-1) \longrightarrow E$ (the blowing-up of the zero section E_0). This morphism is proper and satisfies $\pi_*\mathcal{O}_{O_{\mathbb{P}(E)}(-1)} = \mathcal{O}_E$. Further, E_0 is the scheme-theoretic image of the zero section of $O_{\mathbb{P}(E)}(-1)$ under π. By Lemma 1.1.8, it suffices to prove that $O_{\mathbb{P}(E)}(-1)$ is split, compatibly with its zero section and with $O_{\mathbb{P}(E_Y)}(-1)$. But, this follows from the first part of the lemma, since $\mathbb{P}(E)$ is split compatibly with $\mathbb{P}(E_Y)$. $\square$

To conclude this section, we show that the existence of a splitting is preserved under taking “affine cones.” Recall first the following.

1.1.12 Definition. An invertible sheaf $\mathcal{L}$ on a scheme X is called *semi-ample* if some positive power of $\mathcal{L}$ is generated by its global sections.

Consider a complete variety X equipped with a semi-ample invertible sheaf $\mathcal{L}$. Let L^{-1} be the total space of the dual line bundle, with projection $f : L^{-1} \longrightarrow X$ and zero section L_0^{-1}. Then, $f_*\mathcal{O}_{L^{-1}} = \bigoplus_{\nu=0}^{\infty} \mathcal{L}^\nu$, so that the graded algebra $\Gamma(L^{-1}, \mathcal{O}_{L^{-1}})$ equals

$$R(X, \mathcal{L}) := \bigoplus_{\nu=0}^{\infty} \Gamma(X, \mathcal{L}^\nu).$$

This is a graded algebra, with $R(X, \mathcal{L})_0 = \Gamma(X, \mathcal{O}_X) = k$.

1.1.13 Lemma. *(i) With the notation and assumptions as above, $R(X, \mathcal{L})$ is a finitely generated domain.*

Let $\hat{X}$ be the corresponding affine variety and let $0 \in \hat{X}$ be the closed point defined by the irrelevant ideal of $R(X, \mathcal{L})$. Let $\pi : L^{-1} \longrightarrow \hat{X}$ be the morphism corresponding to $\Gamma(L^{-1}, \mathcal{O}_{L^{-1}}) = \Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$. Then,

(ii) π is proper and satisfies $\pi_*\mathcal{O}_{L^{-1}} = \mathcal{O}_{\hat{X}}$. Further, $\pi^{-1}(0) = L_0^{-1}$ (as sets).

(iii) If, in addition, $\mathcal{L}$ is ample, then π restricts to an isomorphism $\pi^0 : L^{-1} \setminus L_0^{-1} \longrightarrow \hat{X} \setminus \{0\}$.

Proof. (i) Since L^{-1} is a variety, $R(X, \mathcal{L})$ is a domain. To show that it is finitely generated, choose a positive integer ν_o such that $\mathcal{L}^{\nu_o}$ is generated by its global sections.

Consider the inclusion

$$\bigoplus_{\nu=0}^{\infty} \mathcal{L}^{\nu\nu_o} \subset \bigoplus_{\nu=0}^{\infty} \mathcal{L}^{\nu},$$

and the corresponding morphism $\alpha : L^{-1} \longrightarrow L^{-\nu_o}$, a finite surjective map. Thus, we may regard $\mathcal{O}_{L^{-1}}$ as a coherent sheaf of algebras over $\mathcal{O}_{L^{-\nu_o}}$.

Let $V := \Gamma(X, \mathcal{L}^{\nu_o})$ and let $\mathbb{P}$ be the projective space of hyperplanes in V. Since $\mathcal{L}^{\nu_o}$ is generated by its global sections, we have a morphism $\varphi : X \longrightarrow \mathbb{P}$ such that $\mathcal{L}^{\nu_o} = \varphi^*\mathcal{O}_{\mathbb{P}}(1)$. Let $O_{\mathbb{P}}(-1)$ be the total space of the tautological line bundle over the projective space $\mathbb{P}$, with projection map $g : O_{\mathbb{P}}(-1) \longrightarrow \mathbb{P}$. Then, we have a Cartesian square

$$\begin{array}{ccc} L^{-\nu_o} & \xrightarrow{\psi} & O_{\mathbb{P}}(-1) \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle g}\big\downarrow \\ X & \xrightarrow{\varphi} & \mathbb{P}. \end{array}$$

Thus, ψ is proper, so that the composition $\gamma := \psi \circ \alpha$ is proper as well. As a consequence, $\gamma_*\mathcal{O}_{L^{-1}}$ is a coherent sheaf over $O_{\mathbb{P}}(-1)$. Further, we have a proper morphism $O_{\mathbb{P}}(-1) \to V^*$ (the blowing-up of the origin in V^*). As a consequence, $R(X, \mathcal{L}) = \Gamma(L^{-1}, \mathcal{O}_{L^{-1}}) = \Gamma(O_{\mathbb{P}}(-1), \gamma_*\mathcal{O}_{L^{-1}})$ is a finite module over the algebra of regular functions on V^*, that is, the symmetric algebra of V. In particular, the algebra $R(X, \mathcal{L})$ is finitely generated.

(ii) Let $\hat{X}^{\nu_o} :=$ Spec $R(X, \mathcal{L}^{\nu_o})$. Since the algebra $R(X, \mathcal{L})$ is integral over its subalgebra $R(X, \mathcal{L}^{\nu_o})$, we have a finite surjective morphism $\hat{X} \longrightarrow \hat{X}^{\nu_o}$. Its composition with π equals the composition of $\pi^{\nu_o} : L^{-\nu_o} \longrightarrow \hat{X}^{\nu_o}$ with the finite surjective morphism $\alpha : L^{-1} \longrightarrow L^{-\nu_o}$. Thus, to show that π is proper, we may replace $\mathcal{L}$ with $\mathcal{L}^{\nu_o}$, and hence assume that $\mathcal{L}$ is generated by its global sections. Then, with the preceding notation, the morphism $\hat{X} \longrightarrow V^*$ is finite, and its composition $L^{-1} \longrightarrow \hat{X} \longrightarrow V^*$ with π factors as $\psi : L^{-1} \longrightarrow O_{\mathbb{P}}(-1)$ (a proper morphism) followed by the blowing-up $O_{\mathbb{P}}(-1) \longrightarrow V^*$. Therefore, this composition is proper, and hence so is π.

To show the remaining assertions, note that, by definition, $\hat{X}$ is affine and $\Gamma(L^{-1}, \mathcal{O}_{L^{-1}})$ equals $\Gamma(\hat{X}, \mathcal{O}_{\hat{X}})$, so that the map $\mathcal{O}_{\hat{X}} \longrightarrow \pi_*\mathcal{O}_{L^{-1}}$ is an isomorphism. Moreover, since the elements of $\Gamma(X, \mathcal{L}^{\nu_o})$ have no common zeroes in X, the (set-theoretic) preimage of 0 under π is the zero section.

(iii) We may choose the positive integer ν_o such that $\mathcal{L}^{\nu_o}$ is very ample. Then, $\hat{X} \setminus \{0\}$ is covered by the affine open subsets $\hat{X}_\sigma := \{x \in \hat{X} \mid \sigma(x) \neq 0\}$, where $\sigma \in \Gamma(X, \mathcal{L}^{\nu_o})$ is nonzero (σ is to be thought of as a function on $\hat{X}$). Further, the preimage of $\hat{X}_\sigma$ under π is the pullback of $L^{-1} \setminus L_0^{-1}$ to the corresponding open subset $X_\sigma := \{x \in X \mid \sigma(x) \neq 0\}$ of X. Since $\mathcal{L}^{\nu_o}$ is ample, every subset X_σ is affine, and hence the pullback of $L^{-1} \setminus L_0^{-1}$ to X_σ is affine as well. Thus, π^0 is affine. But, since π is proper, it follows that π^0 must be finite. Finally, since $\pi_*\mathcal{O}_{L^{-1}} = \mathcal{O}_{\hat{X}}$, it follows that π^0 is an isomorphism. $\square$

1.1.14 Lemma. *Let $\mathcal{L}$ be a semi-ample invertible sheaf over a complete variety X, and let $\hat{X} :=$ Spec $R(X, \mathcal{L})$ be the corresponding "affine cone".*

If X is split, then $\hat{X}$ is split compatibly with the subvariety 0.

Conversely, if $\mathcal{L}$ is ample and $\hat{X}$ is split, then X is split as well.

Proof. If X is split, then by Lemma 1.1.11, the line bundle L^{-1} is split compatibly with its zero section. Further, the morphism $\pi : L^{-1} \longrightarrow \hat{X}$ is proper, maps the zero section to 0, and satisfies $\pi_*\mathcal{O}_{L^{-1}} = \mathcal{O}_{\hat{X}}$ by Lemma 1.1.13. Observe that the scheme-theoretic image of L_0^{-1} is the reduced subscheme 0 since L_0^{-1} is reduced, being isomorphic with X. By Lemma 1.1.8, it follows that $\hat{X}$ is split compatibly with 0.

For the converse, notice that $\hat{X} \setminus \{0\}$ is split as well. By Lemma 1.1.13 again, $\hat{X} \setminus \{0\} \simeq L^{-1} \setminus L_0^{-1}$ is equipped with a morphism $h : \hat{X} \setminus \{0\} \longrightarrow X$, such that $h^\#$ identifies $\mathcal{O}_X$ with the degree 0 component of the graded sheaf $h_*\mathcal{O}_{\hat{X}\setminus\{0\}}$. Thus, $h^\#$ splits, and our assertion follows from Lemma 1.1.9. □

1.1.E Exercises

In the following Exercises 1–4, X denotes a scheme endowed with invertible sheaves $\mathcal{L}_1, \ldots, \mathcal{L}_r$; the corresponding line bundles are denoted by $L_1, \ldots, L_r$ respectively.

(1) Assuming that X is split, show that the Whitney sum $L_1 \oplus \cdots \oplus L_r$ is split, compatibly with all partial sums.

(2*) Let $R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r) := \bigoplus \Gamma(X, \mathcal{L}_1^{\nu_1} \otimes \cdots \otimes \mathcal{L}_r^{\nu_r})$ (sum over all nonnegative integers $\nu_1, \ldots, \nu_r$). Assuming that X is a complete variety and that $\mathcal{L}_1, \ldots, \mathcal{L}_r$ are semi-ample, show that the algebra $R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r)$ is finitely generated.

The corresponding affine variety $\hat{X}$ is called the *multicone over X associated with* $\mathcal{L}_1, \ldots, \mathcal{L}_r$.

(3*) With the notation and assumptions of (2), generalize Lemma 1.1.13 to the morphism $\pi : L_1^{-1} \oplus \cdots \oplus L_r^{-1} \longrightarrow \hat{X}$.

(4) With the notation and combined assumptions of (1) and (2), show that $\hat{X}$ is split compatibly with all the multicones associated with subsets of $\{L_1, \ldots, L_r\}$.

(5) Let G be a linearly reductive group [MFK–94, Chapter 1] acting on a scheme X such that $f : X \longrightarrow Y$ is a good quotient, where *good quotient* means that f is an affine G-invariant morphism and the map $f^\# : \mathcal{O}_Y \to (f_*\mathcal{O}_X)^G$ is an isomorphism. Assume further that X is split. Then, show that Y is split.

Hint: Use the Reynolds operator of [loc cit.].

1.2 Consequences of Frobenius splitting

We begin with an easy but important observation.

1.2.1 Proposition. *Let X be a split scheme, then X is reduced.*

If the closed subschemes Y and Z are compatibly split, then so are their scheme-theoretic intersection $Y \cap Z$, their union $Y \cup Z$, and the irreducible components of all these schemes. In particular, the scheme-theoretic intersection $Y \cap Z$ is reduced.

Proof. Let φ be a splitting of X. Consider an affine open subscheme U of X, and a nilpotent element $f \in \Gamma(U, \mathcal{O}_U)$. Thus, there exists a positive integer ν such that $f^{p^\nu} = 0$. It follows that

$$f^{p^{\nu-1}} = (\varphi \circ F^\#)(f^{p^{\nu-1}}) = \varphi(f^{p^\nu}) = 0,$$

and hence, by induction, $f = 0$. Thus, X is reduced.

Recall that

$$\mathcal{I}_{Y\cap Z} := \mathcal{I}_Y + \mathcal{I}_Z \text{ and } \mathcal{I}_{Y\cup Z} := \mathcal{I}_Y \cap \mathcal{I}_Z.$$

If φ is compatible with Y and Z, then we have

$$\varphi(F_*\mathcal{I}_{Y\cup Z}) = \varphi(F_*(\mathcal{I}_Y \cap \mathcal{I}_Z)) \subset \varphi(F_*\mathcal{I}_Y) \cap \varphi(F_*\mathcal{I}_Z) = \mathcal{I}_Y \cap \mathcal{I}_Z = \mathcal{I}_{Y\cup Z};$$

and, similarly, $\varphi(F_*\mathcal{I}_{Y\cap Z}) \subset \mathcal{I}_{Y\cap Z}$. Thus, $Y \cup Z$ and $Y \cap Z$ are compatibly split by φ.

To complete the proof, it suffices to show that φ is compatible with every irreducible component of X. Let A be such a component and let B be the union of all the other components. Then, $\varphi(F_*\mathcal{I}_A)$ is a coherent sheaf of ideals of $\mathcal{O}_X$ containing $\mathcal{I}_A$. Further, $\varphi(F_*\mathcal{I}_A)$ and $\mathcal{I}_A$ coincide on $X \setminus B$, since $\mathcal{I}_A$ restricts to the zero sheaf on $X \setminus B = A \setminus B$. Since $X \setminus B$ is dense in A, it follows that $\varphi(F_*\mathcal{I}_A) = \mathcal{I}_A$ as in the proof of Lemma 1.1.7. □

1.2.2 Example. The affine plane $X = \mathbb{A}^2$ is split compatibly with the coordinate line $Y := (y = 0)$ (Example 1.1.5); it is also split compatibly with the nonsingular curve $Z := (y = x^2)$ by Proposition 1.1.6. But, Y and Z are not simultaneously compatibly split in X, since $Y \cap Z$ is not reduced.

Next, we obtain a restriction on the singularities of any split scheme. To formulate it, we need the following.

1.2.3 Definition. A morphism $f : Y \longrightarrow X$ between reduced schemes is *birational* if there exist dense open subsets $U \subset X$ and $V \subset Y$ such that f restricts to an isomorphism $V \longrightarrow U$.

A reduced scheme X is *weakly normal* if every finite birational bijective morphism $f : Y \longrightarrow X$ is an isomorphism. We refer to [AnBo–69], [Man–80] for more on this notion and on the (weaker) notion of semi-normality.

1.2.4 Examples. (1) A variety X is normal if and only if every finite birational morphism to X is an isomorphism [Har–77, Chap. II, Exercise 3.8]. Thus, normal varieties are weakly normal.

(2) The cuspidal cubic curve $X := (y^2 = x^3)$ in $\mathbb{A}^2$ is not weakly normal. (The morphism $\mathbb{A}^1 \longrightarrow X$, $t \longmapsto (t^2, t^3)$, is bijective, finite and birational, but is not an isomorphism.) But, the nodal cubic curve $X := (y^2 = x^2(x + 1))$ in $\mathbb{A}^2$ is weakly

normal if $p \neq 2$. (Let $f : Y \longrightarrow X$ be a finite birational bijective morphism. Then, the normalization $\eta : \mathbb{A}^1 \longrightarrow X,\ t \longmapsto (t^2-1, t(t^2-1))$, factors through f. Further, η is an isomorphism above $X - \{0, 0\}$, and the scheme-theoretic fiber $\eta^{-1}(0, 0)$ equals the reduced scheme $\{-1, 1\}$. It follows that the scheme-theoretic fibers of the factorization $\mathbb{A}^1 \longrightarrow Y$ are the same as those of η. Thus, f is an isomorphism.)

1.2.5 Proposition. *Every split scheme is weakly normal.*

Proof. Let X be a split scheme and let $f : Y \longrightarrow X$ be a finite birational bijective morphism. To check that f is an isomorphism, we may assume that X is affine. Then, Y is affine as well, since f is finite. Let $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$; then, A and B are finitely generated algebras with the same total quotient ring K (the localization at all nonzero divisors), and $A \subset B \subset K$, the A-module B being finitely generated. Let φ be a splitting of X. Then, φ extends uniquely to an additive map $\varphi : K \longrightarrow K$ satisfying $\varphi(x^p y) = x\varphi(y)$ and $\varphi(1) = 1$.

Consider the conductor

$$I := \{a \in A \mid aB \subset A\}.$$

This is a nonzero ideal of B contained in A. We claim that $\varphi(I) = I$. To check this, let $a \in I$ and $b \in B$, then $\varphi(a)b = \varphi(ab^p) \in \varphi(A) = A$, so that $\varphi(I) \subset I$. The opposite inclusion follows from Remark 1.1.4 (iii).

By the above claim and Proposition 1.2.1, the ring A/I is reduced. Likewise, if $b \in B$ and $b^p \in I$ then $b = \varphi(b^p) \in \varphi(I) = I$; it follows that B/I is reduced as well. The closed subset E of X corresponding to A/I consists of all those points where f is not birational, by definition of I. Thus, E contains no irreducible components of X. By the claim, X is split compatibly with E.

Assume that $I \neq A$ and let P be a minimal prime ideal of A over I. Then, P is the ideal of an irreducible component of E, so that $\varphi(P) = P$ by Proposition 1.2.1. Further, the localization $(A/I)_P$ is a field, and $(B/I)_P$ is a nontrivial, purely inseparable field extension (since B/I is reduced, and f restricts to a finite bijective, nowhere birational morphism $\mathrm{Spec}(B/I) \longrightarrow \mathrm{Spec}(A/I) = E$). Thus, there exists $b \in B_P$ such that its image $\bar{b} \in (B/I)_P$ satisfies $\bar{b}^p \in (A/I)_P$, but $\bar{b} \notin (A/I)_P$. Then, $b^p \in A_P$ and $b = \varphi(b^p) \in A_P$, a contradiction. Hence, $I = A$, and $B = A$. $\square$

Further consequences of the existence of a Frobenius splitting concern the vanishing of all higher cohomology groups of line bundles, or equivalently of invertible sheaves. To establish them, we need the following preliminary result.

1.2.6 Lemma. *Let $\mathcal{L}$ be an invertible sheaf on a scheme X. Then,*

$$F^*\mathcal{L} \simeq \mathcal{L}^p \text{ and } F_*(F^*\mathcal{L}) \simeq \mathcal{L} \otimes_{\mathcal{O}_X} F_*\mathcal{O}_X.$$

Proof. Recall that

$$F^*\mathcal{L} := F^{-1}\mathcal{L} \otimes_{F^{-1}\mathcal{O}_X} \mathcal{O}_X,$$

where the map $F^{-1}\mathcal{O}_X \longrightarrow \mathcal{O}_X$ arises from $F^\# : \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$. Now, F is the identity map on points, and $F^\#$ is the p-th power map. It follows that

$$F^*\mathcal{L} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X,$$

where $\mathcal{O}_X$ acts on itself by the p-th power map; in other words, $\sigma f \otimes g = \sigma \otimes f^p g$ for all local sections σ of $\mathcal{L}$, and f, g of $\mathcal{O}_X$. Thus, the map

$$F^*\mathcal{L} \longrightarrow \mathcal{L}^p, \ \sigma \otimes f \longmapsto \sigma^p f,$$

is well defined. Clearly, this map is $\mathcal{O}_X$-linear and surjective; this implies the first isomorphism. The second isomorphism follows from the projection formula. □

1.2.7 Lemma. *Let $\mathcal{L}$ be an invertible sheaf on a split scheme X.*
(i) If $H^i(X, \mathcal{L}^\nu) = 0$ for a fixed index i and all $\nu \gg 0$, then $H^i(X, \mathcal{L}) = 0$.
(ii) If a closed subscheme Y is compatibly split, and the restriction map $H^0(X, \mathcal{L}^\nu) \longrightarrow H^0(Y, \mathcal{L}^\nu)$ is surjective for all $\nu \gg 0$, then the restriction map $H^0(X, \mathcal{L}) \longrightarrow H^0(Y, \mathcal{L})$ is surjective. (Here and elsewhere, when no confusion is likely, we have abused the notation and denoted $\mathcal{L}_{|Y}$ by $\mathcal{L}$ itself.)

Proof. (i) Let $\varphi : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ be a splitting of $F^\# : \mathcal{O}_X \longrightarrow F_*\mathcal{O}_X$. Then, $\mathrm{id} \otimes \varphi$ splits the map

$$\mathrm{id} \otimes F^\# : \mathcal{L} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} F_*\mathcal{O}_X.$$

It follows that the induced map in cohomology

$$H^i(\mathrm{id} \otimes F^\#) : H^i(X, \mathcal{L}) \longrightarrow H^i(X, \mathcal{L} \otimes_{\mathcal{O}_X} F_*\mathcal{O}_X)$$

is split, and hence injective. But,

$$H^i(X, \mathcal{L} \otimes_{\mathcal{O}_X} F_*\mathcal{O}_X) \simeq H^i(X, F_*(F^*\mathcal{L})) \simeq H^i(X, F_*(\mathcal{L}^p)) \simeq H^i(X, \mathcal{L}^p)$$

by Lemma 1.2.6 and the fact that the morphism F is finite, hence affine. (Here the isomorphism on the extreme right is only semilinear.) This yields a split injection $H^i(X, \mathcal{L}) \longrightarrow H^i(X, \mathcal{L}^p)$ (as abelian groups). As a consequence, $H^i(X, \mathcal{L})$ is a direct factor of $H^i(X, \mathcal{L}^{p^\nu})$ for any positive integer ν.

(ii) We have a commutative diagram (of abelian groups)

$$\begin{array}{ccc} H^0(X, \mathcal{L}) & \longrightarrow & H^0(X, \mathcal{L}^p) \\ \downarrow & & \downarrow \\ H^0(Y, \mathcal{L}) & \longrightarrow & H^0(Y, \mathcal{L}^p), \end{array}$$

where the horizontal arrows are $H^0(\mathrm{id} \otimes F^\#)$, and the vertical arrows are the restriction maps. Since Y is compatibly split, the horizontal arrows are compatibly split as well. Thus, the surjectivity of $H^0(X, \mathcal{L}^p) \longrightarrow H^0(Y, \mathcal{L}^p)$ implies that of $H^0(X, \mathcal{L}) \longrightarrow H^0(Y, \mathcal{L})$. □

We now come to the main results of this section.

1.2.8 Theorem. *Let X be a proper scheme over an affine scheme, and let $\mathcal{L}$ be an ample invertible sheaf on X.*

(i) If X is split, then $H^i(X, \mathcal{L}) = 0$ for all $i \geq 1$.

(ii) If, in addition, a closed subscheme Y is compatibly split, then the restriction map $H^0(X, \mathcal{L}) \longrightarrow H^0(Y, \mathcal{L})$ is surjective, and $H^i(Y, \mathcal{L}) = 0$ for all $i \geq 1$. As a consequence, $H^i(X, \mathcal{I}_Y \otimes \mathcal{L}) = 0$ for all $i \geq 1$.

Proof. By the Serre vanishing theorem [Har–77, Chap. III, Proposition 5.3], we have for $\nu \gg 0$:

$$H^i(X, \mathcal{L}^\nu) = H^i(X, \mathcal{I}_Y \otimes \mathcal{L}^\nu) = H^i(Y, \mathcal{L}^\nu) = 0.$$

On the other hand, the short exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_Y \otimes \mathcal{L}^\nu \longrightarrow \mathcal{L}^\nu \longrightarrow \mathcal{L}^\nu|_Y \longrightarrow 0$$

yields the long exact sequence of cohomology groups

$$0 \to H^0(X, \mathcal{I}_Y \otimes \mathcal{L}^\nu) \to H^0(X, \mathcal{L}^\nu) \to H^0(Y, \mathcal{L}^\nu) \to H^1(X, \mathcal{I}_Y \otimes \mathcal{L}^\nu) \to \cdots.$$

It follows that the restriction map $H^0(X, \mathcal{L}^\nu) \longrightarrow H^0(Y, \mathcal{L}^\nu)$ is surjective for $\nu \gg 0$. Combined with Lemma 1.2.7, this implies the theorem. □

1.2.9 Theorem. *Let X be a split projective scheme equipped with an ample invertible sheaf $\mathcal{L}$.*

If X is Cohen–Macaulay with dualizing sheaf ω_X, then $H^i(X, \mathcal{L} \otimes \omega_X) = 0$ for all $i \geq 1$.

If, in addition, X is equidimensional, then $H^i(X, \mathcal{L}^{-1}) = 0$ for all $i \leq \dim(X) - 1$.

Proof. Assume that X is Cohen–Macaulay and equidimensional. Then, $H^i(X, \mathcal{L}^{-\nu}) = 0$ for all $\nu \gg 0$ and $i \leq \dim(X) - 1$, by [Har–77, Chap. III, Theorem 7.6]. Using Lemma 1.2.7, this implies the second assertion.

Now, the first assertion follows by applying Serre duality [Har–77, Chap. III, Corollary 7.7] to all connected components of X; these are equidimensional, since X is Cohen–Macaulay. □

1.2.10 Remarks. (i) In particular, the Kodaira vanishing theorem holds for every split projective nonsingular variety X, i.e., $H^i(X, \mathcal{L} \otimes \omega_X) = 0$ for any $i \geq 1$ and any ample invertible sheaf $\mathcal{L}$.

(ii) The vanishing of all the higher cohomology groups of all ample invertible sheaves on a given projective scheme X is a very strong condition that seldom holds. Consider, for example, the case where X is a projective nonsingular irreducible curve of genus g. If $g \geq 2$, then the invertible sheaf ω_X is ample [Har–77, Chap. IV, Corollary 3.3], and $H^1(X, \omega_X) \simeq H^0(X, \mathcal{O}_X)^* \simeq k$. Hence, X is not split.

As another class of examples, consider a hypersurface X of degree d in the projective space $\mathbb{P}^n$. Then, X is Cohen–Macaulay with dualizing sheaf $\mathcal{O}_X(d-n-1)$. Thus,

$$H^{n-1}(X, \mathcal{O}_X(1)) \simeq H^0(X, \mathcal{O}_X(-1) \otimes \omega_X)^* \simeq H^0(X, \mathcal{O}_X(d-n-2))^*$$

is nonzero whenever $d \geq n+2$; in this case, X is not split.

(iii) We saw in Proposition 1.1.6 that any nonsingular affine variety is split; but, this does not extend to nonsingular *quasi-affine* varieties. Consider, for example, a projective nonsingular irreducible curve X of genus $g \geq 2$, an ample invertible sheaf $\mathcal{L}$ on X, and the corresponding cone $\hat{X} := \operatorname{Spec} R(X, \mathcal{L})$ over X, with vertex 0. Then, $\hat{X} \setminus \{0\}$ is a nonsingular quasi-affine surface, which is not split by the proof of Lemma 1.1.14 and the above Remark (ii). In particular, notice that the normal affine surface $\hat{X}$ is not split as well.

Next, we obtain two "relative" vanishing results in the presence of Frobenius splitting.

1.2.11 Lemma. *Let $f : X \longrightarrow Y$ be a proper morphism of schemes, let D be a closed subscheme of X, and let $i \geq 1$. If X is split compatibly with D, and if $H^i(X_y, \mathcal{I}_D) = 0$ for all points $y \in Y$ (where y is not necessarily closed, and X_y denotes the scheme-theoretic fiber at y), then $R^i f_*(\mathcal{I}_D) = 0$.*

Proof. We may assume that Y is affine. Then, by [Har–77, Chap. III, Proposition 8.5], it suffices to prove the vanishing of $H^i(X, \mathcal{I}_D)$. By [loc cit., Chap. III, Theorem 8.8 and Remark 8.8.1], the $H^0(Y, \mathcal{O}_Y)$-module $H^i(X, \mathcal{I}_D)$ is finitely generated. Let $y \in Y$ be the generic point of an irreducible component of the support of this module. Then, the localization $H^i(X, \mathcal{I}_D)_y$ is a module of finite length over the local ring $\mathcal{O}_{Y,y}$. By the theorem on formal functions [loc cit., Chap. III, Theorem 11.1 and Remark 11.1.1], we have

$$(1) \qquad H^i(X, \mathcal{I}_D)_y = \lim_{\leftarrow s} H^i(X_s, \mathcal{I}_{D,s}),$$

where $\mathcal{I}_{D,s}$ is the pullback of $\mathcal{I}_D$ to $X_s := X \times_Y \operatorname{Spec}(\mathcal{O}_{Y,y}/\mathcal{M}_y^s)$, for $s \geq 1$. (Here $\mathcal{M}_y$ is the maximal ideal of $\mathcal{O}_{Y,y}$.)

We claim that the canonical map

$$H^i(X, \mathcal{I}_D)_y \longrightarrow H^i(X_s, \mathcal{I}_{D,s})$$

is injective for $s \gg 0$. To check this, denote by K_s the kernel of this map. Then, $(K_s)_{s \geq 1}$ is a decreasing sequence of $\mathcal{O}_{Y,y}$-submodules of $H^i(X, \mathcal{I}_D)_y$. Since the latter $\mathcal{O}_{Y,y}$-module has finite length, this sequence is constant for $s \gg 0$. But, $\bigcap_{s \geq 1} K_s = \{0\}$, whence the claim.

Thus, we may choose s such that $K_s = 0$. Consider the actions of the Frobenius morphism F on $H^i(X_s, \mathcal{I}_{D,s})$ and $H^i(X, \mathcal{I}_D)_y$; the latter action is injective, since the map $\mathcal{I}_D \longrightarrow F_* \mathcal{I}_D$ splits (as D is compatibly split in X). On the other hand, the action of F on $H^i(X_s, \mathcal{I}_{D,s})$ is nilpotent. Indeed, $F(\mathcal{M}_y^\nu \mathcal{I}_D) \subset \mathcal{M}_y^{p\nu}\mathcal{I}_D$ for any positive integer ν, so that F acts nilpotently on $H^i(X_s, \mathcal{M}_y \mathcal{I}_{D,s})$. Further,

$$H^i(X_s, \mathcal{I}_{D,s}/\mathcal{M}_y\mathcal{I}_{D,s}) \simeq H^i(X_y, \mathcal{I}_D)$$

vanishes by assumption; thus, the map

$$H^i(X_s, \mathcal{M}_y\mathcal{I}_{D,s}) \longrightarrow H^i(X_s, \mathcal{I}_{D,s})$$

is surjective.

Since the action of F on the subspace $H^i(X, \mathcal{I}_D)_y$ of $H^i(X_s, \mathcal{I}_{D,s})$ is injective and nilpotent, we conclude that $H^i(X, \mathcal{I}_D)_y = 0$. $\square$

The next relative vanishing theorem will be a key ingredient in obtaining versions of the Grauert–Riemenschneider and Kawamata–Viehweg theorems in the presence of a splitting (Theorems 1.3.14 and 1.3.16).

1.2.12 Theorem. *Let $f : X \longrightarrow Y$ be a proper morphism of schemes. Let D, resp. E, be a closed subscheme of X, resp. Y, and let $i \geq 1$ be such that:*

(i) D contains $f^{-1}(E)$ (set-theoretically),
(ii) $R^i f_*(\mathcal{I}_D)$ vanishes outside E,
(iii) X is split compatibly with D.

Then, $R^i f_*(\mathcal{I}_D) = 0$ everywhere.

Proof. We begin with reduction arguments similar to those of the proof of Lemma 1.2.11. We may assume that Y is affine; then, it suffices to prove the vanishing of $H^i(X, \mathcal{I}_D)$. We argue by contradiction, and assume that $H^i(X, \mathcal{I}_D)$ is nonzero. Then, this $H^0(Y, \mathcal{O}_Y)$-module is finitely generated, with support in E by assumption (ii). Choose an irreducible component of this support, with generic point y. Then, the localization $H^i(X, \mathcal{I}_D)_y$ is a nonzero module of finite length over $\mathcal{O}_{Y,y}$, and hence over its completion $\hat{\mathcal{O}}_{Y,y}$.

Choose a field of representatives K for the complete local ring $\hat{\mathcal{O}}_{Y,y}$; then, there exist $t_1, \ldots, t_n \in \mathcal{M}_y$ such that the natural map

$$K[[t_1, \ldots, t_n]] \longrightarrow \hat{\mathcal{O}}_{Y,y}$$

is surjective. Let $R := K[[t_1, \ldots, t_n]]$; this is a regular local ring, and $H^i(X, \mathcal{I}_D)_y$ becomes an R-module of finite length via the above homomorphism. The Frobenius morphism F and its iterates F^r act on R and on $H^i(X, \mathcal{I}_D)_y$; these actions are compatible, and the action of F on $H^i(X, \mathcal{I}_D)_y$ is split injective by the assumption (iii).

For any nonnegative integer r, let $R \otimes^{p^r} H^i(X, \mathcal{I}_D)_y$ denote the base change of $H^i(X, \mathcal{I}_D)_y$ under the endomorphism F^r of R; then, we have

$$a \otimes bm = ab^{p^r} \otimes m,$$

for all a, b in R and m in $H^i(X, \mathcal{I}_D)_y$. This yields maps (for any $r \geq 1$)

$$f_r : R \otimes^{p^r} H^i(X, \mathcal{I}_D)_y \longrightarrow R \otimes^{p^{r-1}} H^i(X, \mathcal{I}_D)_y, \;\; a \otimes m \mapsto a \otimes F(m),$$

that define a projective system

$$\cdots \longrightarrow R \otimes^{p^r} H^i(X, \mathcal{I}_D)_y \longrightarrow R \otimes^{p^{r-1}} H^i(X, \mathcal{I}_D)_y \longrightarrow \cdots \longrightarrow H^i(X, \mathcal{I}_D)_y,$$

and a projective limit

$$\lim_{\leftarrow r} R \otimes^{p^r} H^i(X, \mathcal{I}_D)_y.$$

On the other hand by (1.2.11.1),

$$H^i(X, \mathcal{I}_D)_y = \lim_{\leftarrow s} H^i(X_s, \mathcal{I}_s),$$

where $X_s := X \times_Y \mathrm{Spec}(\mathcal{O}_y/\mathcal{M}_y^s)$, and $\mathcal{I}_s = \mathcal{I}_{D,s}$ denotes the pullback of $\mathcal{I}_D$ to X_s. Since R is a finitely generated, free R-module under F^r by Lemma 1.1.1,

$$\begin{aligned}\lim_{\leftarrow r} R \otimes^{p^r} H^i(X, \mathcal{I}_D)_y &= \lim_{\leftarrow r} R \otimes^{p^r} \big(\lim_{\leftarrow s} H^i(X_s, \mathcal{I}_s)\big) \\ &= \lim_{\leftarrow r} \lim_{\leftarrow s} R \otimes^{p^r} H^i(X_s, \mathcal{I}_s) = \lim_{\leftarrow s} \lim_{\leftarrow r} R \otimes^{p^r} H^i(X_s, \mathcal{I}_s).\end{aligned}$$

Further, F acts nilpotently on $\mathcal{I}_s$ (indeed, some positive power of $\mathcal{I}_D$ is contained in $\mathcal{M}_y\mathcal{O}_X$, since $y \in E$ and $f^{-1}(E) \subset D$). It follows that

$$\lim_{\leftarrow r} R \otimes^{p^r} H^i(X_s, \mathcal{I}_s) = 0,$$

so that

$$\lim_{\leftarrow r} R \otimes^{p^r} H^i(X, \mathcal{I}_D)_y = 0.$$

On the other hand, all the R-modules in the above projective system have finite length, and every map

$$R \otimes^{p^r} H^i(X, \mathcal{I}_D)_y \longrightarrow H^i(X, \mathcal{I}_D)_y$$

is nonzero, since it sends every $1 \otimes m$ to $F^r(m)$, and since F^r is split injective on $H^i(X, \mathcal{I}_D)_y$. By Lemma 1.2.13, we obtain a contradiction to the assumption that $H^i(X, \mathcal{I}_D) \neq 0$. □

1.2.13 Lemma. *Let*

$$\cdots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow M_0$$

be a projective system of modules of finite length over a ring R, with transition maps

$$f_i^j : M_j \longrightarrow M_i, \;\; \textit{for } j \geq i \geq 0.$$

If $f_0^i \neq 0$ for all i, then $\lim_{\leftarrow r} M_r$ is nonzero.

Proof. Let

$$M_i^{\text{stab}} := \bigcap_{j \geq i} f_i^j(M_j).$$

Since M_i has finite length, we have $M_i^{\text{stab}} = f_i^j(M_j)$ for all $j \gg 0$. Thus,

$$f_i^{i+1}(M_{i+1}^{\text{stab}}) = f_i^{i+1} f_{i+1}^j(M_j) = f_i^j(M_j)$$

for all $j \gg 0$, so that $f_i^{i+1}(M_{i+1}^{\text{stab}}) = M_i^{\text{stab}}$. Therefore, $\{M_i^{\text{stab}}\}$ is a projective subsystem with nonzero surjective maps: its projective limit is a nonzero submodule of $\varprojlim_r M_r$. ☐

1.2.E Exercises

(1) Let X be a reduced scheme with normalization $f : Y \longrightarrow X$. Show that X is weakly normal if and only if $\mathcal{O}_X$, regarded as a subsheaf of $f_*\mathcal{O}_Y$, consists of those local sections that are constant on all set-theoretic fibers of f. In particular, weakly normal curves are those reduced curves having only ordinary multiple points as singularities.

(2) Show that every affine, weakly normal curve is split.

(3) Let $f : Y \longrightarrow X$ be a proper surjective morphism of varieties, such that (a) Y is normal, (b) the fibers of f are connected, and (c) X is split. Show that X is normal.

Hint: Factoring f through the normalization $\eta : \widetilde{X} \longrightarrow X$, show that η is bijective. Then, apply Proposition 1.2.5.

(4) Let X be a reduced scheme with normalization $f : Y \to X$. Let E be the closed subset of X where f is not an isomorphism, endowed with its reduced subscheme structure, and put $Z := f^{-1}(E)$. Show that any splitting φ of X is compatible with E and, moreover, it lifts to a splitting of Y, compatible with Z.

In particular, the normalization of a split scheme is split.

Hint: Reduce to the case where $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ are affine. Let I be the conductor as in the proof of Proposition 1.2.5. By that proof, $\varphi(I) = I$; further, I is the ideal of E in A, and of Z in B. Show that $I\varphi(b) \subset I$ for any $b \in B$. Deduce that $I\varphi(b)^N \subset I$ for any $N \geq 0$, and that $\varphi(b)$ is integral over A.

1.3 Criteria for splitting

In this section, we obtain several useful criteria for a given scheme X to be split.

Recall from Remark 1.1.4 (i) that a splitting of X is an $\mathcal{O}_X$-linear map $\varphi : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ such that $\varphi(1) = 1$. Therefore, to know if X is split and to determine all the splittings, we need to understand the *evaluation map*

$$\epsilon : \text{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \Gamma(X, \mathcal{O}_X), \quad \varphi \mapsto \varphi(1).$$

In fact this map is defined on sheaves:

$$\epsilon : \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{O}_X, \quad \varphi \mapsto \varphi(1).$$

Further, $\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ is a coherent sheaf of $F_*\mathcal{O}_X$-modules on X. Since F is a finite morphism by Lemma 1.1.1, there exists a unique coherent sheaf $F^!\mathcal{O}_X$ on X such that

$$\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) = F_*(F^!\mathcal{O}_X),$$

cf. [Har–77, Chap. II, Exercise 5.17 and Chap. III, Exercise 6.10].

If X is regular, then F is flat by Lemma 1.1.1. Then, the duality for the finite flat morphism F [Har–66] yields an isomorphism

$$F^!\mathcal{O}_X \simeq \mathcal{H}om_{\mathcal{O}_X}(F^*\omega_X, \omega_X)$$

(where ω_X denotes the dualizing sheaf of X), together with a trace map (defined in 1.3.5)

$$\tau : F_*\omega_X \longrightarrow \omega_X,$$

such that the evaluation map

$$\epsilon : F_*\mathcal{H}om_{\mathcal{O}_X}(F^*\omega_X, \omega_X) \simeq \mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \longrightarrow \mathcal{O}_X$$

may be identified with the map

$$\mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \longrightarrow \mathcal{E}nd_{\mathcal{O}_X}(\omega_X) \simeq \mathcal{O}_X, \; u \mapsto \tau \circ u.$$

Together with Lemma 1.2.6, from the duality as above, it follows that

$$F^!\mathcal{O}_X \simeq \omega_X^{1-p}.$$

In particular, this sheaf is invertible.

We will recover these results in a more direct and explicit way later in this section. We begin with the simplest

1.3.1 Example. Let $X = \mathbb{A}^n$ be the affine n-space over k, with coordinates $t_1, \ldots, t_n$. Recall that the $k[t_1, \ldots, t_n]$-module

$$\mathrm{Hom}(F_*\mathcal{O}_{\mathbb{A}^n}, \mathcal{O}_{\mathbb{A}^n}) = \Gamma(\mathbb{A}^n, F_*(F^!\mathcal{O}_{\mathbb{A}^n}))$$

is the space of all additive maps

$$\varphi : k[t_1, \ldots, t_n] \longrightarrow k[t_1, \ldots, t_n]$$

such that $\varphi(f^p g) = f\varphi(g)$ for all $f, g \in k[t_1, \ldots, t_n]$; where $k[t_1, \ldots, t_n]$ acts by

$$(f\varphi)(g) := f\varphi(g).$$

This space has another structure of $k[t_1, \ldots, t_n]$-module, via

$$(f * \varphi)(g) := \varphi(fg),$$

and the latter $k[t_1, \ldots, t_n]$-module is $\Gamma(\mathbb{A}^n, F^!\mathcal{O}_{\mathbb{A}^n})$ by [Har–77, Chap. II, Exercise 5.17]. Notice the relation $f^p * \varphi = f\varphi$ between the two $k[t_1, \ldots, t_n]$-module structures.

Let $\mathrm{Tr} : k[t_1, \ldots, t_n] \longrightarrow k[t_1, \ldots, t_n]$ be the unique additive map such that

$$\mathrm{Tr}(ct^{\mathbf{i}}) = \begin{cases} c^{1/p} t^{\mathbf{j}} & \text{if } \mathbf{i} = \mathbf{p-1} + p\mathbf{j} \text{ for some } \mathbf{j} \in \mathbb{N}^n, \\ 0 & \text{otherwise,} \end{cases}$$

for any $c \in k$ and $\mathbf{i} = (i_1, \ldots, i_n) \in \mathbb{N}^n$ (where we recall the notation $t^{\mathbf{i}} := t_1^{i_1} \cdots t_n^{i_n}$). Clearly, $\mathrm{Tr} \in \Gamma(\mathbb{A}^n, F^!\mathcal{O}_{\mathbb{A}^n})$. We claim that Tr is a free generator of this module (as a module over $k[t_1, \ldots, t_n]$ under $*$).

Let $\varphi \in \Gamma(\mathbb{A}^n, F^!\mathcal{O}_{\mathbb{A}^n})$. Set (identifying $\Gamma(\mathbb{A}^n, F^!\mathcal{O}_{\mathbb{A}^n})$ with $\mathrm{Hom}(F_*\mathcal{O}_{\mathbb{A}^n}, \mathcal{O}_{\mathbb{A}^n})$ as abelian groups as above)

$$f := \sum_{\mathbf{i} \le \mathbf{p-1}} \varphi(t^{\mathbf{i}})^p \, t^{\mathbf{p-1-i}}.$$

Then, one easily checks that $\varphi = f * \mathrm{Tr}$. Thus, Tr generates the $k[t_1, \ldots, t_n]$-module $\Gamma(\mathbb{A}^n, F^!\mathcal{O}_{\mathbb{A}^n})$. On the other hand, if $f * \mathrm{Tr} = 0$, then $0 = f^p * \mathrm{Tr} = f\,\mathrm{Tr}$, whence $f = 0$. This completes the proof of the claim.

Now, the evaluation map $\epsilon : \Gamma(\mathbb{A}^n, F^!\mathcal{O}_{\mathbb{A}^n}) \longrightarrow k[t_1, \ldots, t_n]$ is given by

$$\epsilon(f * \mathrm{Tr}) = (f * \mathrm{Tr})(1) = \mathrm{Tr}(f).$$

Thus, the map $f * \mathrm{Tr}$ splits $\mathbb{A}^n$ if and only if

(a) the monomial $t^{\mathbf{p-1}}$ occurs in f with coefficient 1, and
(b) f contains no monomial $t^{\mathbf{p-1}+p\mathbf{j}}$ where $\mathbf{j} \in \mathbb{N}^n$, $\mathbf{j} \neq 0$.

Observe that the splitting of Example 1.1.5 equals $t^{\mathbf{p-1}} * \mathrm{Tr}$.

We now aim at extending the results of Example 1.3.1 to all nonsingular varieties. To this end, we first develop some differential calculus for arbitrary schemes in characteristic p; we begin with the case of an affine scheme $X = \mathrm{Spec}(A)$. Let Ω^1_A be the A-module of *Kähler differentials* of A over k (cf. [Har–77, Chap. II, §8]) , equipped with the k-derivation

$$d : A \longrightarrow \Omega^1_A, \quad a \mapsto da.$$

Notice that $d(a^p) = pa^{p-1}da = 0$ for every $a \in A$, so that d is A^p-linear.

Next, let

$$\Omega^\bullet_A := \wedge^\bullet \Omega^1_A = \bigoplus_{i=0}^{\infty} \wedge^i \Omega^1_A$$

be the exterior algebra of Ω^1_A over A (where $\wedge^0\Omega^1_A := A$). Then, $\Omega^\bullet_A$ is the associative A-algebra generated by da ($a \in A$) with product $\wedge$ and relations: $da \wedge da = 0$, $d(ab) - a(db) - (da)b = 0$, for $a, b \in A$. For any $\alpha \in \Omega^i_A = \wedge^i\Omega^1_A$, and $\beta \in \Omega^j_A$, we have

$$\beta \wedge \alpha = (-1)^{ij}\alpha \wedge \beta,$$

that is, $\Omega^\bullet_A$ is a graded-commutative A-algebra.

The map $d : A \longrightarrow \Omega^1_A$ extends uniquely to a map, still denoted by $d : \Omega^\bullet_A \longrightarrow \Omega^\bullet_A$, such that

$$d(a_1 da_2 \wedge \cdots \wedge da_i) = da_1 \wedge da_2 \wedge \cdots \wedge da_i$$

for $a_1, \ldots, a_i \in A$. One easily checks that

$$d(\alpha \wedge \beta) = (d\alpha) \wedge \beta + (-1)^i \alpha \wedge d\beta,$$

for $\alpha \in \Omega^i_A$ and $\beta \in \Omega^\bullet_A$. So, d is a k-derivation of the graded-commutative algebra $\Omega^\bullet_A$. Further, d is A^p-linear and satisfies $d^2 = 0$. The complex $(\Omega^\bullet_A, d)$ is called the *De Rham complex* of A (over k). Here are some basic properties of its cohomology spaces.

1.3.2 Lemma. *The space*

$$Z^\bullet_A := \{\alpha \in \Omega^\bullet_A \mid d\alpha = 0\}$$

is a graded A^p-subalgebra of $\Omega^\bullet_A$, and the space

$$B^\bullet_A := d\Omega^{\bullet-1}_A = \{d\alpha \mid \alpha \in \Omega^{\bullet-1}_A\}$$

is a graded ideal of $Z^\bullet_A$.

Thus, the quotient

$$H^\bullet_A := Z^\bullet_A / B^\bullet_A$$

is a graded-commutative A^p-algebra.

If, in addition, A is a localization of a finitely generated algebra, then the A-module $\Omega^\bullet_A$ is finitely generated, and the A^p-modules $Z^\bullet_A$, $B^\bullet_A$ and $H^\bullet_A$ are finitely generated as well.

Proof. Clearly, $Z^\bullet_A$ and $B^\bullet_A$ are graded subspaces of $\Omega^\bullet_A$. Since d is an A^p-linear derivation, $Z^\bullet_A$ is an A^p-subalgebra, and contains $B^\bullet_A$ as an ideal.

For the second assertion, let A be a localization of an algebra generated by $t_1, \ldots, t_n$. Then, the A-module Ω^1_A is generated by $dt_1, \ldots, dt_n$. Thus, the A-module $\wedge^i\Omega^1_A$ is generated by $dt_{j_1} \wedge \cdots \wedge dt_{j_i}$, where $1 \leq j_1 < \cdots < j_i \leq n$. It follows that $\wedge^i\Omega^1_A = 0$ for all $i > n$, and that the A-module $\Omega^\bullet_A$ is finitely generated.

Since the A^p-module A also is finitely generated by Lemma 1.1.1, the same holds for the A^p-module $\Omega^\bullet_A$ and its submodules $Z^\bullet_A$ and $B^\bullet_A$. □

Next, we construct a homomorphism $\gamma : \Omega^\bullet_A \longrightarrow H^\bullet_A$. Consider the map

$$\gamma : A \longrightarrow \Omega^1_A, \quad a \mapsto a^{p-1}da.$$

1.3.3 Lemma. *With the notation as above, we have for all a, b in A:*
(i) $\gamma(ab) = a^p\gamma(b) + b^p\gamma(a)$.
(ii) $d\gamma(a) = 0$.
(iii) $\gamma(a+b) - \gamma(a) - \gamma(b) \in B^1_A$.

Proof. (i) and (ii) are straightforward. For (iii), notice that

$$d(a+b)^p = p(a+b)^{p-1}d(a+b) = p(a^{p-1}da + b^{p-1}db) + \sum_{i=1}^{p-1}\binom{p}{i}d(a^ib^{p-i})$$

in the space of Kähler differentials of the polynomial ring $\mathbb{Z}[a,b]$ over $\mathbb{Z}$, i.e., in $\mathbb{Z}[a,b]da \oplus \mathbb{Z}[a,b]db$. Since every binomial coefficient $\binom{p}{i}$, $1 \le i \le p-1$, is divisible by p, it follows that

$$(a+b)^{p-1}d(a+b) - a^{p-1}da - b^{p-1}db = dQ_p(a,b)$$

for some polynomial Q_p with integral coefficients. □

By Lemma 1.3.3, the composition

$$A \xrightarrow{\gamma} Z^1_A \longrightarrow Z^1_A/B^1_A = H^1_A$$

is a k-derivation, where A acts on itself by multiplication, and on H^1_A via $F : A \longrightarrow A^p$. We still denote this derivation by γ. Now, the universal property of Kähler differentials (cf. [Har–77, Chap. II, §8]) yields an A-linear map

$$\gamma : \Omega^1_A \longrightarrow H^1_A, \quad adb \longmapsto a^pb^{p-1}db \ (\mathrm{mod}\ dA).$$

Since every $\gamma(da)$ has square zero, we obtain an A-algebra homomorphism

$$\begin{array}{cccc} \gamma : & \Omega^\bullet_A & \longrightarrow & H^\bullet_A, \\ & a_1da_2 \wedge \cdots \wedge da_i & \longmapsto & a_1^pa_2^{p-1}da_2 \wedge \cdots \wedge a_i^{p-1}da_i \ (\mathrm{mod}\ B^\bullet_A), \end{array}$$

where A acts on $H^\bullet_A$ via $F : A \to A^p$.

1.3.4 Theorem. *If A is regular, then $\gamma : \Omega^\bullet_A \longrightarrow H^\bullet_A$ is an isomorphism.*

Proof. Using Lemma 1.3.2, it suffices to show that γ is an isomorphism after localization and completion at every maximal ideal. Hence, we may assume that $A = k[[t_1,\ldots,t_n]]$. We now argue by induction on n, the case where $n = 0$ being evident.

We check first that γ is surjective. Let $\alpha \in \Omega^i_A$ such that $d\alpha = 0$. Write

$$\alpha = \sum_{j=0}^{\infty} t_n^j(\alpha_j + \beta_j \wedge dt_n),$$

where $\alpha_j \in \Omega^i_{k[[t_1,\dots,t_{n-1}]]}$ and $\beta_j \in \Omega^{i-1}_{k[[t_1,\dots,t_{n-1}]]}$. Since $d\alpha = 0$, we obtain

$$d\alpha_j = 0 \text{ and } (-1)^i (j+1)\alpha_{j+1} + d\beta_j = 0 \tag{1}$$

for all $j \geq 0$. It follows that

$$t_n^{j+1}\alpha_{j+1} + t_n^j \beta_j \wedge dt_n = \frac{(-1)^{i-1}}{j+1} d(t_n^{j+1}\beta_j),$$

whenever $j+1$ is not divisible by p. Hence,

$$\alpha \equiv \sum_{j=0}^{\infty} t_n^{pj}\alpha_{pj} + t_n^{pj-1}\beta_{pj-1} \wedge dt_n \ (\text{mod } d\Omega_A^{i-1}),$$

where $\beta_{-1} := 0$. Since $d\alpha_{pj} = 0$ for all j, the image of $t_n^{pj}\alpha_{pj}$ in H^i_A lies in the image of γ by the induction hypothesis. The same holds for the image of $t_n^{pj-1}\beta_{pj-1} \wedge dt_n$ in H^i_A, since $\beta_{pj-1} \in \Omega^{i-1}_{k[[t_1,\dots,t_{n-1}]]}$ and $d\beta_{pj-1} = 0$ by (1). This proves the surjectivity of γ.

For the injectivity, let $\alpha \in \Omega^i_A$ such that $\gamma(\alpha) = 0$. Write as above $\alpha = \sum_{j=0}^{\infty} t_n^j(\alpha_j + \beta_j \wedge dt_n)$. Then,

$$\gamma(\alpha) = \sum_{j=0}^{\infty} t_n^{pj}(\gamma(\alpha_j) + t_n^{p-1}\gamma(\beta_j) \wedge dt_n) \ (\text{mod } B_A^\bullet)$$

is represented in Z^i_A by

$$\sum_{j=0}^{\infty} t_n^{pj}(\alpha'_j + t_n^{p-1}\beta'_j \wedge dt_n),$$

where $\alpha'_j := \gamma(\alpha_j) \in Z^i_{k[[t_1,\dots,t_{n-1}]]}$ and $\beta'_j := \gamma(\beta_j) \in Z^{i-1}_{k[[t_1,\dots,t_{n-1}]]}$. Since $\gamma(\alpha) = 0$, we have

$$\sum_{j=0}^{\infty} t_n^{pj}(\alpha'_j + t_n^{p-1}\beta'_j \wedge dt_n) \in B^i_A := d\Omega_A^{i-1}.$$

It follows as above that $\alpha'_j \in B^i_{k[[t_1,\dots,t_{n-1}]]}$ and $\beta'_j \in B^{i-1}_{k[[t_1,\dots,t_{n-1}]]}$ for all j. By the induction hypothesis, this implies $\alpha_j = \beta_j = 0$ for all j. □

The preceding constructions extend to any scheme X: they yield the sheaf of graded-commutative algebras $\Omega_X^\bullet$ of *Kähler differential forms* of X over k, endowed with a differential d of degree $+1$ such that the induced differential on $F_*\Omega_X^\bullet$ is $\mathcal{O}_X$-linear. By Lemma 1.3.2, the cohomology sheaves $\mathcal{H}^i F_*\Omega_X^\bullet$ are coherent sheaves of $\mathcal{O}_X$-modules; and Lemma 1.3.3 yields a unique homomorphism

$$\gamma : \Omega_X^\bullet \longrightarrow \bigoplus_{i=0}^{\infty} \mathcal{H}^i F_*\Omega_X^\bullet$$

of sheaves of graded-commutative $\mathcal{O}_X$-algebras, such that: $\gamma(f) = f^p$ and $\gamma(df) = f^{p-1}df \pmod{d\mathcal{O}_X}$, for any $f \in \mathcal{O}_X$.

If X is a nonsingular variety of dimension n, then the sheaf Ω^1_X is locally free of rank n, so that $\Omega^i_X = 0$ for all $i > n$; and $\Omega^n_X = \omega_X$, the dualizing sheaf of X. Further, γ is an isomorphism by Theorem 1.3.4.

1.3.5 Definition. Let X be a nonsingular variety of dimension n. Then, the inverse of the isomorphism γ is called the *Cartier operator*

$$C = \sum_{i=0}^{n} C_i : \bigoplus_{i=0}^{n} \mathcal{H}^i F_*\Omega^\bullet_X \longrightarrow \Omega^\bullet_X.$$

The composition of the quotient map $F_*\omega_X = F_*\Omega^n_X \longrightarrow F_*(\Omega^n_X/d\Omega^{n-1}_X)$ with $C_n : F_*(\Omega^n_X/d\Omega^{n-1}_X) \longrightarrow \Omega^n_X$ is, by definition, the *trace map*

$$\tau : F_*\omega_X \longrightarrow \omega_X.$$

Since γ is $\mathcal{O}_X$-linear, so is τ.

We now express the trace map in local coordinates.

1.3.6 Lemma. *Let X be a nonsingular variety of dimension n and let $t_1, \ldots, t_n$ be a system of local coordinates at $x \in X$. Then, the trace map at x is given by*

$$\tau(f dt_1 \wedge \cdots \wedge dt_n) = \mathrm{Tr}(f) dt_1 \wedge \cdots \wedge dt_n,$$

where $f \in \mathcal{O}_{X,x} \subset k[[t_1, \ldots, t_n]]$ and

$$\mathrm{Tr}\left(\sum_{\mathbf{i}} f_{\mathbf{i}} t^{\mathbf{i}}\right) := \sum f_{\mathbf{i}}^{1/p}\, t^{\mathbf{j}},$$

where the summation on the right side is taken over those $\mathbf{i}$ such that $\mathbf{i} = \mathbf{p} - \mathbf{1} + p\mathbf{j}$ for some $\mathbf{j} \in \mathbb{N}^n$. In particular, $\mathrm{Tr}(f) \in \mathcal{O}_{X,x}$.

Proof. Since $dt_1 \wedge \cdots \wedge dt_n$ is a generator of the stalk $\omega_{X,x}$; this identifies the completion of $\omega_{X,x}$ with $k[[t_1, \ldots, t_n]]$. Then, the completion of $\omega_{X,x}/d\Omega^{n-1}_{X,x}$ gets identified with $k[[t_1, \ldots, t_n]]/\mathcal{J}$, where $\mathcal{J}$ denotes the space spanned by all the partial derivatives of all formal power series, since

$$d(f dt_2 \wedge \cdots \wedge dt_n) = (\partial_{t_1} f) dt_1 \wedge \cdots \wedge dt_n, \text{ for } f \in k[[t_1, \ldots, t_n]].$$

Hence, $\mathcal{J}$ consists of all series $\sum_{\mathbf{i}} a_{\mathbf{i}} t^{\mathbf{i}}$, where $\mathbf{i} = (i_1, \ldots, i_n)$ is such that $i_j + 1$ is not divisible by p for some j; and the set of all series $\sum_{\mathbf{j}} a_{\mathbf{j}} t^{\mathbf{p}-\mathbf{1}+p\mathbf{j}}$ forms a system of representatives of $k[[t_1, \ldots, t_n]]/\mathcal{J}$. Now, the map

$$\gamma : k[[t_1, \ldots, t_n]] \longrightarrow k[[t_1, \ldots, t_n]]/\mathcal{J}$$

sends every f to the class of $t^{\mathbf{p}-\mathbf{1}} f^p \pmod{\mathcal{J}}$. This implies our formula. $\square$

Notice that the formula for Tr gives back the generator of $\Gamma(\mathbb{A}^n, F^!\mathcal{O}_{\mathbb{A}^n})$ found in Example 1.1.5. More generally, we have the following.

1.3.7 Proposition. *Let X be a nonsingular variety. Then, the map*

$$\iota : \mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$$

given by the equality in ω_X:

$$\iota(\psi)(f)\omega = \tau(f\psi(\omega)),$$

for all local sections $\psi \in \mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X)$, $f \in \mathcal{O}_X$, and local generator $\omega \in \omega_X$, is well defined and is an isomorphism of $F_*\mathcal{O}_X$-modules. Further, the diagram

$$\begin{array}{ccc} \mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) & \xrightarrow{\iota} & \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \\ {\scriptstyle\hat{\tau}}\big\downarrow & & {\scriptstyle\epsilon}\big\downarrow \\ \mathcal{E}nd_{\mathcal{O}_X}(\omega_X) & \xrightarrow{\simeq} & \mathcal{O}_X \end{array}$$

commutes, where ϵ is defined in the beginning of this section and $\hat{\tau}(\psi) := \tau \circ \psi$.

Proof. Let u be a local unit of $\mathcal{O}_X$ and let $\omega_1 = u\omega$. If $\iota(\psi)(f)\omega = \tau(f\psi(\omega))$, then

$$\iota(\psi)(f)\omega_1 = u\tau(f\psi(\omega)) = \tau(u^p f\psi(\omega)) = \tau(f\psi(u\omega)) = \tau(f\psi(\omega_1)).$$

Hence, ι is well defined. To check that it is $F_*\mathcal{O}_X$-linear, let $g \in \mathcal{O}_X$; then,

$$\iota(g\psi)(f)\omega = \tau(f(g\psi)(\omega)) = \tau(fg\psi(\omega)) = \iota(\psi)(fg)\omega.$$

Further, since $\epsilon(\iota(\psi)) = \iota(\psi)(1)$ and $\iota(\psi)(1)\omega = \tau(\psi(\omega))$, the diagram commutes.

We now check that ι is an isomorphism; for this, we argue in a system of local coordinates $t_1, \ldots, t_n$ at x. A local generator of the $F_*\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X)$ is the map ψ_0 given by

$$\psi_0(f dt_1 \wedge \cdots \wedge dt_n) = f^p dt_1 \wedge \cdots \wedge dt_n.$$

By the definition of ι, we have $\iota(\psi_0) = \mathrm{Tr}$; and the latter is a local generator of the $F_*\mathcal{O}_X$-module $\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X)$ by Example 1.3.1. □

Notice that

$$\mathcal{H}om_{\mathcal{O}_X}(\omega_X, F_*\omega_X) \simeq F_*\mathcal{H}om_{\mathcal{O}_X}(F^*\omega_X, \omega_X) \simeq F_*(\omega_X^{1-p}),$$

by the projection formula and Lemma 1.2.6. Together with the isomorphism ι of Proposition 1.3.7, this yields an isomorphism

$$\hat{\iota} : F_*(\omega_X^{1-p}) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X).$$

We say that an element $\varphi \in H^0(X, \omega_X^{1-p}) \simeq H^0(X, F_*(\omega_X^{1-p}))$ *splits* X, if $\hat{\iota}(\varphi)$ splits X.

We can now state an important characterization of split varieties, which follows immediately from Proposition 1.3.7 and Lemma 1.3.6.

1.3.8 Theorem. *Let X be a nonsingular variety. Then, via the above isomorphism $\hat{\iota}$, the evaluation map $\epsilon : \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{O}_X$ is identified with the map*

$$\hat{\tau} : F_*(\omega_X^{1-p}) \to \mathcal{O}_X$$

given at any closed point x by

$$\hat{\tau}(f\,(dt_1 \wedge \cdots \wedge dt_n)^{1-p}) = \mathrm{Tr}(f), \textit{ for all } f \in \mathcal{O}_{X,x}.$$

Here, $t_1, \ldots, t_n$ is a system of local coordinates at x, and Tr *is as defined in Lemma 1.3.6. Thus, an element $\varphi \in H^0(X, \omega_X^{1-p})$ splits X if and only if $\hat{\tau}(\varphi) = 1$.*

If X is complete (and nonsingular), then φ splits X if and only if the monomial $t^{\mathbf{p-1}}$ occurs with coefficient 1 *in the local expansion of φ at some (and hence every) closed point $x \in X$.*

1.3.9 Remarks. (i) In particular, a necessary condition for a nonsingular variety X to be split is the existence of nonzero sections of ω_X^{1-p}.

This yields another proof for the fact that nonsingular projective irreducible curves of genus $g \geq 2$, and nonsingular hypersurfaces of degree $d \geq n+2$ in $\mathbb{P}^n$ are not split (Remark 1.2.10).

(ii) Consider a complete, nonsingular variety X of dimension n. Then, the evaluation map $\epsilon : \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X)$ yields, by Serre duality, a map $H^n(X, \omega_X) \longrightarrow H^n(X, \omega_X \otimes F_*\mathcal{O}_X)$. Now, $\omega_X \otimes F_*\mathcal{O}_X \simeq F_*(F^*\omega_X) \simeq F_*(\omega_X^p)$, whence ϵ induces a map $H^n(X, \omega_X) \longrightarrow H^n(X, \omega_X^p)$, which turns out to be the pull-back F^*.

Thus, X is split if and only if the map

$$F^* : H^n(X, \omega_X) \longrightarrow H^n(X, \omega_X^p)$$

is nonzero.

For example, an elliptic curve X is split if and only if X is not supersingular (as defined in [Har–77, Chap. IV, §4]).

Next, we obtain a sufficient condition for the existence of a splitting, which is simpler than Theorem 1.3.8 and applies to many examples. To formulate this condition, we need the following.

1.3.10 Definition. Let X be a nonsingular variety of dimension n and let $Y_1, \ldots, Y_m$ be *prime divisors* in X, i.e., closed subvarieties of codimension 1.

We say that the scheme-theoretic intersection $Y_1 \cap \cdots \cap Y_m$ is *transversal* at a closed point $x \in X$ if (a) x is a nonsingular point of $Y_1, \ldots, Y_m$, and (b) the Zariski tangent space $T_x(Y_1 \cap \cdots \cap Y_m)$ equals $T_x(Y_1) \cap \cdots \cap T_x(Y_m)$, and has dimension $n - m$.

Equivalently, there exists a system of local coordinates $t_1, \ldots, t_n$ at x such that each Y_i has local equation $t_i = 0$.

1.3.11 Proposition. *Let X be a nonsingular variety of dimension n.*

If X is complete and if there exists $\sigma \in H^0(X, \omega_X^{-1})$ with divisor of zeros

$$(\sigma)_0 = Y_1 + \cdots + Y_n + Z,$$

where $Y_1, \ldots, Y_n$ are prime divisors intersecting transversally at a point x, and Z is an effective divisor (as defined in [Har–77, Chap. II, §6]) not containing x, then $\sigma^{p-1} \in H^0(X, \omega_X^{1-p})$ splits X compatibly with $Y_1, \ldots, Y_n$.

Conversely, if $\sigma \in H^0(X, \omega_X^{-1})$ is such that σ^{p-1} splits X, then the subscheme of zeros of σ is compatibly split. In particular, this subscheme is reduced.

Proof. Choose a system of local coordinates $t_1, \ldots, t_n$ at $x \in X$ such that each Y_i has local equation $t_i = 0$. Then, by our assumptions, the local expansion of σ at x is given by

$$t_1 \cdots t_n g(t_1, \ldots, t_n)(dt_1 \wedge \cdots \wedge dt_n)^{-1},$$

where $g(t_1, \ldots, t_n)$ is a formal power series with nonzero constant term. Thus, the coefficient of $t^{\mathbf{p-1}}$ in the series $t^{\mathbf{p-1}} g(t_1, \ldots, t_n)^{p-1}$ is nonzero as well. Hence, σ^{p-1} splits X by Theorem 1.3.8. Further, the corresponding splitting φ (via the isomorphism ι) satisfies

$$\varphi(f(t_1, \ldots, t_n)) = \operatorname{Tr}(t^{\mathbf{p-1}} g(t_1, \ldots, t_n)^{p-1} f(t_1, \ldots, t_n));$$

thus, $\varphi(t_i f(t_1, \ldots, t_n))$ is divisible by t_i for any i. In other words, φ is compatible with the zero loci of the coordinates at x, i.e., with $Y_1, \ldots, Y_n$ in a neighborhood of x. By 1.1.7, φ is compatible with $Y_1, \ldots, Y_n$ everywhere.

Conversely, let $\sigma \in H^0(X, \omega_X^{-1})$ be such that σ^{p-1} splits X, and choose a nonsingular closed point x of the zero scheme of σ. Let Y be the unique irreducible component of this zero scheme containing x, and let f be a local equation of Y at x. Since Y is nonsingular at x, we may choose a system of local coordinates $t_1, \ldots, t_n$ at x such that $f = t_1$. Let

$$t_1^m g(t_1, \ldots, t_n)(dt_1 \wedge \cdots \wedge dt_n)^{-1}$$

be the local expansion of σ at x, where $g(t_1, \ldots, t_n)$ is not divisible by t_1; then, $m \geq 1$ is the order of vanishing of σ along Y. Since σ^{p-1} splits X, the coefficient of $t^{\mathbf{p-1}}$ in $t_1^{m(p-1)} g(t_1, \ldots, t_n)^{p-1}$ is nonzero. Hence, $m = 1$, and the splitting of X by σ^{p-1} is compatible with Y at x. It follows that the zero scheme of σ is reduced (since it is a generically reduced hypersurface in a nonsingular variety). Applying Lemma 1.1.7 again, we conclude that our splitting is compatible with the subscheme of zeros of σ. □

1.3.12 Remark. Most of the results of this section adapt to any *normal* variety X, as follows. Let $i : X^{\mathrm{reg}} \longrightarrow X$ be the inclusion of the regular locus. Let

$$\omega_X := i_* \omega_{X^{\mathrm{reg}}}.$$

This is the *canonical sheaf* of X; we have

$$\omega_X = \mathcal{O}_X(K_X)$$

for some Weil divisor K_X on X, the *canonical divisor* (defined up to linear equivalence). If, in addition, X is Cohen–Macaulay, then ω_X is its dualizing sheaf.

For any integer ν, define the ν-th power of ω_X by

$$\omega_X^\nu = i_*(\omega_{X^{\mathrm{reg}}}^\nu);$$

then $\omega_X^\nu = \mathcal{O}_X(\nu K_X)$. Notice that $H^0(X, \omega_X^\nu) = H^0(X^{\mathrm{reg}}, \omega_{X^{\mathrm{reg}}}^\nu)$, and that the ν-th tensor power of ω_X has a natural map to ω_X^ν; but, this map need not be an isomorphism.

By Lemma 1.1.7(iii) and Theorem 1.3.8, a normal variety X is split if and only if there exists $\varphi \in H^0(X, \omega_X^{1-p})$ such that $\hat{\tau}(\varphi) = 1$, where $\hat{\tau} : F_*(\omega_X^{1-p}) \longrightarrow \mathcal{O}_X$ is given by the formula of Theorem 1.3.8 at any point of X^{reg}.

If, in addition, X is complete, then every regular function on X^{reg} is constant. Hence, X is split if and only if there exists $\varphi \in H^0(X, \omega_X^{1-p})$ such that the monomial $t^{\mathbf{p-1}}$ occurs with coefficient 1 in the local expansion of φ at some nonsingular closed point.

We say that a normal variety Y is *Gorenstein* if its canonical sheaf ω_Y is invertible; equivalently, the canonical divisor is Cartier. (In the literature, sometimes Gorenstein varieties are assumed to be Cohen–Macaulay, but we do not require this assumption.)

Given a Gorenstein variety Y, a normal variety X, and a proper, birational morphism $f : X \longrightarrow Y$, the sheaves ω_X and $f^*\omega_Y$ coincide outside the exceptional locus of f. Recall that, by the *exceptional locus* of f, we mean the closed subset $X \setminus f^{-1}(U)$ of X, where U is the largest open subset of Y such that the restriction $f : f^{-1}(U) \to U$ is an isomorphism. Thus, we may write $\omega_X = (f^*\omega_Y)(D)$, where D is a Weil divisor on X supported in this exceptional locus. The divisor D is called the *discrepancy divisor* of f. Further, f is called *crepant* if D is trivial, i.e., $f^*\omega_Y = \omega_X$; then, X is Gorenstein as well.

With this terminology, we can state the following result that provides a partial converse to Lemma 1.1.8.

1.3.13 Lemma. *Let $f : X \longrightarrow Y$ be a crepant morphism, where X, Y and f are as above. If Y is split, then so is X.*

Proof. Note that $f^*(\omega_Y^\nu) = \omega_X^\nu$ for any integer ν. Further, as f is proper and birational, and Y is normal, we have $f_*\mathcal{O}_X = \mathcal{O}_Y$. Thus,

$$f_*(\omega_X^\nu) = f_*f^*(\omega_Y^\nu) = \omega_Y^\nu \otimes f_*\mathcal{O}_X = \omega_Y^\nu,$$

since ω_Y is invertible. This implies an isomorphism

$$H^0(X, \omega_X^{1-p}) \simeq H^0(Y, \omega_Y^{1-p}),$$

compatible with the maps $\hat{\tau}$. □

Next, we obtain versions of the Grauert–Riemenschneider and Kawamata–Viehweg vanishing theorems in the presence of Frobenius splitting.

1.3.14 Theorem. *Let X be a nonsingular variety and $f : X \longrightarrow Y$ a proper birational morphism. Assume that there exists $\sigma \in H^0(X, \omega_X^{-1})$ such that*

(i) σ^{p-1} splits X, and
(ii) σ vanishes identically on the exceptional locus of f.
(Then, the zero scheme of σ is reduced by Proposition 1.3.11). Let D be an effective subdivisor of $(\sigma)_0$, containing the exceptional locus. Then, $R^i f_*(\mathcal{O}_X(-D)) = 0$ for all $i \geq 1$. In particular, $R^i f_*(\omega_X) = 0$ for all $i \geq 1$.

Proof. We use the relative vanishing Theorem 1.2.12. Let E be the image in Y of the exceptional locus. Then, by assumption, D contains $f^{-1}(E)$ (set-theoretically). Further, $R^i f_*(\mathcal{O}_X(-D)) = 0$ outside E, since f is an isomorphism above $Y \setminus E$. Finally, X is split compatibly with D by assumption (i) and Proposition 1.3.11. Hence, the assertion follows from Theorem 1.2.12 and [Har–77, Chap. II, Proposition 6.18]. The "In particular" statement follows since $\mathcal{O}_X((\sigma)_0) \simeq \omega_X^{-1}$. □

1.3.15 Definition. Let $\mathcal{L}$ be a semi-ample invertible sheaf on a complete variety X of dimension n. Then, the ring $R(X, \mathcal{L}) = \bigoplus_{\nu=0}^{\infty} H^0(X, \mathcal{L}^\nu)$ is finitely generated (Lemma 1.1.13), and we have a morphism

$$\varphi : X \longrightarrow Y := \operatorname{Proj} R(X, \mathcal{L})$$

such that $\varphi_* \mathcal{O}_X = \mathcal{O}_Y$. In particular, φ is surjective, with connected fibers (which follows from [Har–77, Chap. III, Corollary 11.3] for projective X and from [Gro–61, Corollaire (4.3.2)] for X complete). The dimension of Y is called the *Kodaira dimension* of $\mathcal{L}$ and denoted by $\kappa(\mathcal{L})$; then, the general fibers of φ have dimension $n - \kappa(\mathcal{L})$. The *exceptional locus* of $\mathcal{L}$ is the set of those $x \in X$ such that $\dim \varphi^{-1}\varphi(x) > n - \kappa(\mathcal{L})$.

Note that $\kappa(\mathcal{L}) = n$ if $\mathcal{L}$ is ample; in this case, its exceptional locus is empty. On the other hand, since $\mathcal{L}$ is semi-ample, $\kappa(\mathcal{L}) = 0$ if and only if some positive power of $\mathcal{L}$ is trivial.

1.3.16 Theorem. *Let $\mathcal{L}$ be a semi-ample invertible sheaf on a complete nonsingular variety X of dimension n. Assume that there exists $\sigma \in H^0(X, \omega_X^{-1})$ such that*

(i) σ^{p-1} splits X, and
(ii) σ vanishes identically on the exceptional locus of $\mathcal{L}$.
Then, $H^i(X, \mathcal{L}^\nu \otimes \omega_X) = 0$ for all $i > n - \kappa(\mathcal{L})$ and $\nu \geq 1$. Equivalently, $H^i(X, \mathcal{L}^{-\nu}) = 0$ for all $i < \kappa(\mathcal{L})$ and $\nu \geq 1$.

Proof. We may assume that $\kappa(\mathcal{L}) > 0$, i.e., no positive power of $\mathcal{L}$ is trivial. With the notation of Lemma 1.1.13, consider the square

$$\begin{tikzcd} L^{-1} \setminus L_0^{-1} \arrow[r, "\pi^0"] \arrow[d, "f^0"'] & \hat{X} \setminus \{0\} \arrow[d, "g"] \\ X \arrow[r, "\varphi"] & Y, \end{tikzcd}$$

where f^0 is the restriction of $f : L^{-1} \to X$ to $L^{-1} \setminus L_0^{-1}$ and similarly π^0 is the restriction of $\pi : L^{-1} \to \hat{X}$ to $L^{-1} \setminus L_0^{-1}$. This square is commutative; the morphism π^0 is proper and surjective (by Lemma 1.1.13), and restricts to a finite map on any fiber of f^0 (since a positive power of $\mathcal{L}$ is generated by its global sections). Thus, the product map $\pi^0 \times f^0$ is quasi-finite. It follows that f^0 restricts to a quasi-finite map on every fiber of π^0, onto the corresponding fiber of φ. (Actually, this restriction is finite, since both fibers are complete.)

Let $Z \subset X$ be the zero scheme of σ, then $\mathcal{I}_Z \simeq \omega_X$. Consider the closed subscheme

$$D := L_0^{-1} \cup f^{-1}(Z)$$

of L^{-1}. Then, we have

$$\mathcal{I}_D = \mathcal{I}_{L_0^{-1}} \cap \mathcal{I}_{f^{-1}(Z)} \simeq \mathcal{I}_{L_0^{-1}} \otimes f^*\omega_X.$$

It follows that

$$f_*\mathcal{I}_D \simeq \bigoplus_{\nu=1}^{\infty} \mathcal{L}^\nu \otimes \omega_X.$$

Further, let E be the set of those $\xi \in \hat{X}$ such that $\dim \pi^{-1}(\xi) > n - \kappa(\mathcal{L})$. This is a closed subscheme of $\hat{X}$.

We check that the morphism $\pi : L^{-1} \longrightarrow \hat{X}$, the subschemes D and E, and the index $i > n - \kappa(\mathcal{L})$ satisfy the assumptions of Theorem 1.2.12.

Clearly, $\pi^{-1}(E)$ contains L_0^{-1}, and $\pi^{-1}(E) \setminus L_0^{-1}$ is the preimage under f^0 of the exceptional locus of $\mathcal{L}$ (as sets). Thus, $\pi^{-1}(E)$ is contained in D (as sets).

Since $i > n - \kappa(\mathcal{L})$ and all the fibers of π outside E have dimension $\leq n - \kappa(\mathcal{L})$, we obtain $R^i\pi_*(\mathcal{I}_D) = 0$ outside E.

Finally, since X is split compatibly with Z, Lemma 1.1.11 yields a splitting of L^{-1} compatible with L_0^{-1} and $f^{-1}(Z)$, hence with their union D.

Thus, Theorem 1.2.12 applies and yields $R^i\pi_*(\mathcal{I}_D) = 0$ everywhere. Since $\hat{X}$ is affine, it follows that $H^i(L^{-1}, \mathcal{I}_D) = 0$; thus, $H^i(X, f_*\mathcal{I}_D) = 0$ as f is an affine morphism. This yields the vanishing of $H^i(X, \mathcal{L}^\nu \otimes \omega_X)$ for all $\nu \geq 1$. By Serre duality, it follows that $H^j(X, \mathcal{L}^{-\nu}) = 0$ for all $j < \kappa(\mathcal{L})$. □

1.3.E Exercises

(1) Consider the projective space $\mathbb{P}^n$. Recall that $\omega_{\mathbb{P}^n} = \mathcal{O}(-n-1)$, and that $H^0(\mathbb{P}^n, \omega_{\mathbb{P}^n}^{1-p})$ is the space of all homogeneous polynomials of degree $(n+1)(p-1)$ in the variables $x_0, \ldots, x_n$. Show that $\varphi \in H^0(\mathbb{P}^n, \omega_{\mathbb{P}^n}^{1-p})$ splits $\mathbb{P}^n$ if and only if the monomial $(x_0 \cdots x_n)^{p-1}$ occurs with coefficient 1 in φ.

(2) We say that a nonsingular variety X is *split by a* $(p-1)$*-th power* if there exists $\sigma \in H^0(X, \omega_X^{-1})$ such that σ^{p-1} splits X. (For example, by the above exercise, $\mathbb{P}^n$ is split by the $p-1$-th power of $x_0 \cdots x_n$.)

Assuming that nonsingular X is complete and split, and that the multiplication map

$$H^0(X, \omega_X^{-1})^{\otimes p-1} \longrightarrow H^0(X, \omega_X^{1-p}), \quad \sigma_1 \otimes \cdots \otimes \sigma_{p-1} \mapsto \sigma_1 \cdots \sigma_{p-1},$$

is surjective; show that X is split by a $(p-1)$-th power.

Hint: Use the identity in $\mathbb{Z}[t_1, \ldots, t_n]$:

$$n!\, t_1 \cdots t_n = \sum_{1 \le i_1 < \cdots < i_j \le n} (-1)^{n-j} (t_{i_1} + \cdots + t_{i_j})^n .$$

(3*) Consider a nonsingular variety X, and a closed subscheme Y of pure codimension 1. Assume that X is split by $\varphi \in H^0(X, \omega_X^{1-p})$. Show that φ is compatible with Y if and only if $\varphi \in H^0(X, \omega_X^{1-p}((1-p)Y))$.

Next, take $X = \mathbb{P}^n$; then, Y is a hypersurface, with equation (say) f. Show that a splitting φ of $\mathbb{P}^n$ is compatible with Y, if and only if φ is divisible by f^{p-1}.

(4*) Let X be a nonsingular variety and Y a nonsingular prime divisor in X; denote by σ the canonical section of the invertible sheaf $\mathcal{O}_X(Y)$ (cf. [Ful–98, §B.4.5]). Let $\psi \in H^0(Y, \omega_Y^{1-p})$; assume that ψ admits a lift $\widetilde{\psi}$ under the natural map $H^0(X, \omega_X^{1-p}((1-p)Y)) \longrightarrow H^0(Y, \omega_Y^{1-p})$ (induced by the isomorphism $\omega_X(Y) \otimes_{\mathcal{O}_X} \mathcal{O}_Y \simeq \omega_Y$).

If ψ splits Y, show that the product $\sigma^{p-1}\widetilde{\psi} \in H^0(X, \omega_X^{1-p})$ splits X compatibly with Y and the induced splitting of Y coincides with that by ψ.

In particular, if Y is split by a $(p-1)$-th power and the restriction map $H^0(X, \omega_X^{-1}(-Y)) \longrightarrow H^0(Y, \omega_Y^{-1})$ is surjective, then X is split by a $(p-1)$-th power, compatibly with Y.

(5) Let Y be a nonsingular irreducible hypersurface of degree d in $X = \mathbb{P}^n$, where $n \ge 2$. Show that the map $H^0(X, \omega_X^{1-p}((1-p)Y)) \longrightarrow H^0(Y, \omega_Y^{1-p})$ is surjective. So, by Exercise 4, if Y is split then it is compatibly split in $\mathbb{P}^n$; thus, $d \le n+1$ in this case.

In the case where $d = n+1$, show that Y given by an equation f is split if and only if the monomial $(x_0 \cdots x_n)^{p-1}$ occurs with nonzero coefficient in f^{p-1}.

(6*) Let X be a *toric variety*, i.e., a normal variety containing a torus $T \simeq (\mathbb{G}_m)^n$ as an open subset, such that the action of T on itself by multiplication extends to an action on X. Let $\partial X = X \setminus T$ be the *boundary* of X. Show that X is split compatibly with ∂X.

Hint: Let $t_1, \ldots, t_n$ be the coordinates on T coming from $\mathbb{G}_m$; then,

$$\theta = \frac{dt_1 \wedge \cdots \wedge dt_n}{t_1 \cdots t_n}$$

is a rational section of ω_X, having a pole of order 1 along each irreducible component of ∂X [Ful–93, §4.3]. Thus, $\theta^{-1} \in H^0(X, \omega_X^{-1}(-\partial X))$; and $\theta_{|T}^{1-p}$ splits T by Example 1.3.1.

Using the classification of toric varieties in terms of fans [Ful–93, §§1.4, 5.1], show that any closed T-stable subvariety of X is the (set-theoretic) intersection of a family of irreducible components of ∂X. Deduce thus that θ^{1-p} compatibly splits all the closed T-stable subvarieties.

Also, show that the above splitting is the unique T-invariant splitting of X.

(7) Let $f : X \longrightarrow Y$ be a morphism of nonsingular varieties, such that $f^{\#} : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is an isomorphism. Then, we have a canonical map of $\mathcal{O}_Y$-modules

$$f_*\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_Y}(F_*\mathcal{O}_Y, \mathcal{O}_Y), \quad \varphi \longmapsto f_*\varphi,$$

(see the proof of Lemma 1.1.8). In view of the isomorphism of $\mathcal{O}_X$-modules

$$\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \simeq F_*(\omega_X^{1-p})$$

(Theorem 1.3.8) and the analogous isomorphism for Y, this yields a map of $\mathcal{O}_Y$-modules

$$f_*(F_*(\omega_X^{1-p})) \longrightarrow F_*(\omega_Y^{1-p}),$$

compatible with the evaluation maps.

Assume now that Y is split, and the induced map

$$H^0(X, \omega_X^{1-p}) \longrightarrow H^0(Y, \omega_Y^{1-p})$$

is surjective. Then, show that X is split.

(8) Let X_1, X_2 be split schemes, with splittings φ_1, φ_2. Show that the tensor product

$$\varphi : F_*\mathcal{O}_{X_1\times X_2} \longrightarrow \mathcal{O}_{X_1\times X_2}, \ f_1 \otimes f_2 \longmapsto \varphi_1(f_1) \otimes \varphi_2(f_2)$$

is a splitting of $X_1 \times X_2$. If, in addition, φ_1, resp. φ_2, is compatible with a closed subscheme $Y_1 \subset X_1$, resp. $Y_2 \subset X_2$, then show that φ is compatible with both $X_1 \times Y_2$ and $Y_1 \times X_2$.

If both X_1 and X_2 are nonsingular, then each $\varphi_i \in \operatorname{Hom}(F_*\mathcal{O}_{X_i}, \mathcal{O}_{X_i})$ corresponds to $\sigma_i \in H^0(X_i, \omega_{X_i}^{1-p})$. Show that φ corresponds to $\sigma_1 \otimes \sigma_2 \in H^0(X_1 \times X_2, \omega_{X_1\times X_2}^{1-p})$.

(9) Show that, in general, an affine Gorenstein variety is *not* split.

Hint: Take a homogeneous polynomial of degree d in n variables such that the corresponding affine hypersurface $X \subset \mathbb{A}^n$ is normal. Then, X is Gorenstein but not split if $d > n$.

(10) Let X be a nonsingular variety of dimension n and $D \subset X$ a reduced effective divisor. We say that D has *residually normal crossing* at $x \in X$ if there exists a system of local coordinates $t_1, \dots, t_n$ at x and functions $f_0, \dots, f_{n-1}$ in $\widehat{\mathcal{O}}_{X,x}$ such that:

(i) f_0 is a local equation of D at x, and

(ii) $f_i \equiv t_{i+1} f_{i+1} \mod (t_1, \dots, t_i)$, for $i = 0, 1, \dots, n-1$, where we set $f_n = 1$.

Show that any reduced divisor with n irreducible components through x, such that they intersect transversally at x, has residually normal crossing. Show that the converse holds for $n = 2$, but not for $n \geq 3$.

(11) Let X be a complete nonsingular variety. Assume that there exist $\sigma \in H^0(X, \omega_X^{-1})$ and $x \in X$ such that the divisor $(\sigma)_0$ is reduced and has residually normal crossing at x (in the sense of the preceding exercise). Then, show that σ^{p-1} splits X.

(12) Let X be a nonsingular variety, Y a nonsingular closed subvariety of codimension $d \geq 2$, $\widetilde{X}$ the blowing-up of X with center Y, and E the exceptional divisor. Assume that X is split by $\sigma \in H^0(X, \omega_X^{1-p})$ and denote by $\mathrm{ord}_Y(\sigma)$ the order of vanishing of σ along Y. Then, show that:

(i) $\mathrm{ord}_Y(\sigma) \leq d(p-1)$.

(ii) $\mathrm{ord}_Y(\sigma) \geq (d-1)(p-1)$ iff σ lifts to a (unique) splitting $\widetilde{\sigma}$ of $\widetilde{X}$, where "σ lifts to $\widetilde{\sigma}$" means that the splitting of X induced from $\widetilde{\sigma}$ via Lemma 1.1.8(ii) is σ.

(iii) The splitting $\widetilde{\sigma}$ of $\widetilde{X}$ is compatible with E iff $\mathrm{ord}_Y(\sigma) \geq d(p-1)$.

We say that Y is *compatibly split by σ with maximal multiplicity* if $\mathrm{ord}_Y(\sigma) = d(p-1)$. (Observe that, in this case, Y is automatically compatibly split by σ by using (ii) and (iii) and Lemma 1.1.8(ii).)

(13) Let $f : X \to Y$ be a proper morphism between nonsingular varieties such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Let Z be a nonsingular closed subvariety of X such that f is smooth at some point of Z. If Z is compatibly split in X with maximal multiplicity (in the sense of the preceding exercise), then show that the induced splitting of Y has maximal multiplicity along the nonsingular locus of $f(Z)$.

1.4 Splitting relative to a divisor

In this section, we present a useful refinement of the notion of Frobenius splitting, which yields stronger versions of the vanishing Theorem 1.2.8.

1.4.1 Definition. (i) Let X be a scheme and D an effective Cartier divisor on X, with support $\mathrm{Supp}(D)$ and canonical section σ. Then, X is *Frobenius split relative to D* (or simply *D-split*) if there exists a $\mathcal{O}_X$-linear map

$$\psi : F_*(\mathcal{O}_X(D)) \longrightarrow \mathcal{O}_X$$

such that the composition

$$\varphi := \psi \circ F_*(\sigma) \in \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$$

is a splitting of X; in this case, ψ is called a *D-splitting*. In particular, it is split if it is D-split. From now on, we will abbreviate $F_*(\mathcal{O}_X(D))$ by $F_*\mathcal{O}_X(D)$. Thus, we have a commutative diagram

$$\begin{tikzcd} F_*\mathcal{O}_X \arrow[rr, "F_*(\sigma)"] \arrow[dr, "\varphi"'] & & F_*\mathcal{O}_X(D) \arrow[dl, "\psi"] \\ & \mathcal{O}_X. & \end{tikzcd}$$

(ii) Further, let Y be a closed subscheme of X; then, Y is *compatibly D-split* (or X is *D-split compatibly with Y*) if: (a) Supp(D) contains no irreducible component of Y, and (b) Y is compatibly split by φ.

(iii) More generally, closed subschemes $Y_1, \ldots, Y_m$ of X are *simultaneously compatibly D-split* (or simply *compatibly D-split*) if they are compatibly D-split by the same D-splitting.

1.4.2 Remarks. (i) Any $\psi \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X(D), \mathcal{O}_X)$ is a D-splitting if and only if $\psi(F_*(\sigma)) = 1$.

(ii) If E is another effective Cartier divisor on X such that $D - E$ is effective, then every D-splitting yields an E-splitting, as follows. Let τ, resp. η, be the canonical section of E, resp. $D - E$. We regard τ as an $\mathcal{O}_X$-linear map $\mathcal{O}_X \longrightarrow \mathcal{O}_X(E)$. Similarly, since $\mathcal{O}_X(D) \simeq \mathcal{O}_X(E) \otimes_{\mathcal{O}_X} \mathcal{O}_X(D - E)$, we may regard η as an $\mathcal{O}_X$-linear map $\mathcal{O}_X(E) \longrightarrow \mathcal{O}_X(D)$. Then, $\sigma : \mathcal{O}_X \longrightarrow \mathcal{O}_X(D)$ is the composition

$$\mathcal{O}_X \xrightarrow{\ \tau\ } \mathcal{O}_X(E) \xrightarrow{\ \eta\ } \mathcal{O}_X(D).$$

It follows that $\psi \circ F_*(\eta) : F_*\mathcal{O}_X(E) \longrightarrow \mathcal{O}_X$ is an E-splitting.

1.4.3 Proposition. *Let D be an effective Cartier divisor on a scheme X, let Y be a closed subscheme of X and let $\psi \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X(D), \mathcal{O}_X)$. If ψ is a D-splitting of X compatible with Y, then $\psi(F_*\mathcal{I}_Y(D)) = \mathcal{I}_Y$, and ψ induces a $D \cap Y$-splitting of Y, where $D \cap Y$ denotes the pullback of D to Y (which is also the scheme theoretic intersection of Y with D regarded as a closed subscheme of X).*

Proof. Consider the sheaf $\psi(F_*\mathcal{I}_Y(D))$. This is a coherent subsheaf of $\psi(F_*\mathcal{O}_X(D)) = \mathcal{O}_X$, i.e., a coherent sheaf of ideals of $\mathcal{O}_X$. Let Z be the corresponding closed subscheme of X. Since

$$\psi(F_*\mathcal{I}_Y(D)) \supset \psi(F_*(\sigma)F_*\mathcal{I}_Y) = \varphi(F_*\mathcal{I}_Y) = \mathcal{I}_Y,$$

with equality outside Supp(D), we have $Z \subset Y$ and $Z \setminus \mathrm{Supp}(D) = Y \setminus \mathrm{Supp}(D)$. But, Supp$(D)$ contains no irreducible component of Y; hence, $Z = Y$, and $\psi(F_*\mathcal{I}_Y(D)) = \mathcal{I}_Y$. It follows that ψ induces a $D \cap Y$-splitting of Y. □

Likewise, we obtain the following generalizations of Lemmas 7.3.5 and 1.1.8.

1.4.4 Lemma. *(i) If a scheme X is D-split under $\psi \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X(D), \mathcal{O}_X)$ compatibly with a closed subscheme Y, then for every open subscheme U of X, ψ restricts to a $D \cap U$-splitting of U, compatible with $Y \cap U$.*

(ii) Conversely, if U is a dense open subscheme of a reduced scheme X, and if $\psi \in \mathrm{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X(D), \mathcal{O}_X)$ restricts to a $D \cap U$-splitting of U, then ψ is a D-splitting of X. If, in addition, Y is a reduced closed subscheme of X such that $U \cap Y$ is dense in Y and compatibly $D \cap U$-split by $\psi_{|U}$, then Y is compatibly D-split by ψ.

(iii) Let X be a normal variety and U an open subset of X, with complement of codimension at least 2. Then, X is D-split if and only if U is $D \cap U$-split. In fact, any $D \cap U$-splitting of U is the restriction of a unique D-splitting of X.

1.4.5 Lemma. *Let $f : X \longrightarrow Y$ be a proper morphism of schemes such that the map $f^{\#} : \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ is an isomorphism and let E be an effective Cartier divisor on Y, with pullback D in X.*

(i) If ψ is a D-splitting of X, then $f_*\psi$ is an E-splitting of Y.

(ii) If, in addition, a closed subscheme Z of X is compatibly D-split by ψ, then $f(Z)$ is compatibly E-split by $f_*\psi$.

(Note that under assumption (ii), no irreducible component of Z is contained in $\mathrm{Supp}(D)$, so that no irreducible component of $f(Z)$ is contained in $\mathrm{Supp}(E)$.)

1.4.6 Lemma. *Let X be D-split by ψ, compatibly with closed subschemes Y and Z. Then, $Y \cup Z$ and its irreducible components are D-split by ψ. If, in addition, no irreducible component of $Y \cap Z$ is contained in* $\mathrm{Supp}(D)$*, then $Y \cap Z$ is also D-split by ψ.*

Next, we obtain a generalization of Lemma 1.2.7.

1.4.7 Lemma. *Let X be a scheme equipped with an effective Cartier divisor D and with an invertible sheaf $\mathcal{L}$.*

(i) If X is D-split and if

$$H^i(X, \mathcal{L}^{p^\nu}((p^{\nu-1} + p^{\nu-2} + \cdots + 1)D)) = 0$$

for a fixed index i and some $\nu \geq 1$, then $H^i(X, \mathcal{L}) = 0$.

(ii) If a closed subscheme Y is compatibly D-split, and if the restriction map

$$H^0(X, \mathcal{L}^{p^\nu}((p^{\nu-1}+p^{\nu-2}+\cdots+1)D)) \longrightarrow H^0(Y, \mathcal{L}^{p^\nu}((p^{\nu-1}+p^{\nu-2}+\cdots+1)D))$$

is surjective for some $\nu \geq 1$, then the restriction map $H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L})$ is surjective.

Proof. We adapt the proof of Lemma 1.2.7.

(i) Let $\psi : F_*\mathcal{O}_X(D) \longrightarrow \mathcal{O}_X$ be a D-splitting. Then, $\mathrm{id} \otimes \psi$ splits the map

$$\mathrm{id} \otimes (F_*(\sigma) \circ F^{\#}) : \mathcal{L} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} F_*\mathcal{O}_X(D).$$

It follows that the induced map in cohomology

$$H^i(\mathrm{id} \otimes (F_*(\sigma) \circ F^{\#})) : H^i(X, \mathcal{L}) \longrightarrow H^i(X, \mathcal{L} \otimes_{\mathcal{O}_X} F_*\mathcal{O}_X(D))$$

is split, and hence injective. But,

$$\begin{aligned} H^i(X, \mathcal{L} \otimes_{\mathcal{O}_X} F_*\mathcal{O}_X(D)) &\simeq H^i(X, F_*((F^*\mathcal{L})(D))) \\ &\simeq H^i(X, F_*(\mathcal{L}^p(D))) \simeq H^i(X, \mathcal{L}^p(D)). \end{aligned}$$

This yields a split injection of abelian groups

$$H^i(X, \mathcal{L}) \longrightarrow H^i(X, \mathcal{L}^p(D)).$$

Iterating this process and composing all the resulting maps, we obtain an injective additive map

$$H^i(X, \mathcal{L}) \longrightarrow H^i(X, \mathcal{L}^{p^\nu}((p^{\nu-1} + p^{\nu-2} + \cdots + 1)D))$$

for every $\nu \geq 1$. This proves (i).

(ii) We have a commutative diagram (of abelian groups)

$$\begin{array}{ccc} H^0(X, \mathcal{L}) & \longrightarrow & H^0(X, \mathcal{L}^p(D)) \\ \downarrow & & \downarrow \\ H^0(Y, \mathcal{L}) & \longrightarrow & H^0(Y, \mathcal{L}^p(D)), \end{array}$$

where the horizontal arrows are $H^0(\mathrm{id} \otimes (F_*(\sigma) \circ F^\#))$, and the vertical arrows are the restriction maps. Since Y is compatibly D-split, the horizontal arrows are compatibly split as well. Thus, the surjectivity of $H^0(X, \mathcal{L}^p(D)) \longrightarrow H^0(Y, \mathcal{L}^p(D))$ implies that of $H^0(X, \mathcal{L}) \longrightarrow H^0(Y, \mathcal{L})$. Iterating this argument as in (i) completes the proof of (ii). □

1.4.8 Theorem. *Let X be a proper scheme over an affine scheme; let $\mathcal{L}$ be a semi-ample invertible sheaf on X, and let D be an ample effective Cartier divisor on X.*

(i) If X is D-split, then $H^i(X, \mathcal{L}) = 0$ for all $i \geq 1$.

(ii) If a closed subscheme Y is compatibly D-split, then the restriction map $H^0(X, \mathcal{L}) \longrightarrow H^0(Y, \mathcal{L})$ is surjective, and $H^i(Y, \mathcal{L}) = 0$ for all $i \geq 1$. As a consequence, $H^i(X, \mathcal{I}_Y \otimes \mathcal{L}) = 0$ for all $i \geq 1$.

Proof. Since $\mathcal{L}$ is semi-ample and D is ample, $\mathcal{L}^{p^\nu}((p^{\nu-1} + p^{\nu-2} + \cdots + 1)D)$ is ample for every positive integer ν (this follows from the definition of ampleness in [Har–77, Chap. II, §7]). The assertions now follow from Lemma 1.4.7 and Theorem 1.2.8. □

1.4.9 Remark. In fact, Theorem 1.4.8 extends to any invertible sheaf $\mathcal{L}$ such that $\mathcal{L} \otimes \mathcal{M}$ is ample for any ample invertible sheaf $\mathcal{M}$ on X. Such a sheaf is called *numerically effective*, or *nef* for brevity; cf. [Har–70, Chapter 1] for other characterizations and examples of nef invertible sheaves.

Next, we obtain a criterion for D-splitting of a scheme X, generalizing Theorem 1.3.8. For this, we consider the evaluation map

$$\epsilon_D : \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X(D), \mathcal{O}_X) \longrightarrow \mathcal{O}_X, \quad \psi \mapsto \psi(F_*(\sigma)).$$

Notice that X is D-split under ψ if and only if $\epsilon_D(\psi) = 1$.

By [Har–77, Chap. III, Exercise 6.10], applied to the finite morphism F, we have an isomorphism

(1) $$\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X(D), \mathcal{O}_X) \simeq F_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), F^!\mathcal{O}_X)$$

sending a local section ψ to the local homomorphism

$$F_*\mathcal{O}_X(D) \longrightarrow F_*\big(F^!\mathcal{O}_X\big) = \mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X), \quad s \mapsto (f \mapsto \psi(fs)).$$

If X is a nonsingular variety, then $F^!\mathcal{O}_X \simeq \omega_X^{1-p}$ (see Section 1.3), so that

$$F_*\mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X(D), F^!\mathcal{O}_X) \simeq F_*(\omega_X^{1-p}(-D)). \tag{2}$$

This isomorphism (under the identification (1)) fits into the commutative diagram

$$\begin{tikzcd}
\mathcal{H}om_{\mathcal{O}_X}(F_*\mathcal{O}_X(D), \mathcal{O}_X) \arrow[rr] \arrow[dr, "\epsilon_D"'] & & F_*(\omega_X^{1-p}(-D)) \arrow[dl, "\tau \circ F_*(\sigma)"] \\
& \mathcal{O}_X, &
\end{tikzcd}$$

where $\hat{\tau} : F_*(\omega_X^{1-p}) \to \mathcal{O}_X$ is as in Theorem 1.3.8. Together with Exercise 1.3.E.3, this implies the following refinement of Theorem 1.3.8.

1.4.10 Theorem. *Let X be a nonsingular variety and D an effective divisor on X. Then, $\varphi \in H^0(X, \omega_X^{1-p}(-D))$ provides a D-splitting of X under the above identifications (1) and (2) if and only if $\hat{\tau}(\varphi\sigma) = 1$.*

In particular, if X is split by $\varphi \in H^0(X, \omega_X^{1-p})$, then X is split relative to the divisor of zeros of φ.

As another consequence, $\varphi \in H^0(X, \omega_X^{1-p})$ splits X compatibly with D if and only if φ provides a $(p-1)D$-splitting of X.

The last assertion of the above theorem and the proof of Lemma 1.4.7 imply the following result, which will be used in Chapter 3.

1.4.11 Lemma. *Let X be a nonsingular variety, split compatibly with an effective divisor D. By Proposition 1.2.1, D is reduced; let $D = \sum_{j=1}^r D_j$ be its decomposition into prime divisors. Then, for any integer $\nu \geq 1$, any integers $0 \leq a_1, \ldots, a_r < p^\nu$ and any invertible sheaf $\mathcal{L}$ on X, there is a split injection of abelian groups:*

$$H^i(X, \mathcal{L}) \longrightarrow H^i(X, \mathcal{L}^{p^\nu}(\sum_{j=1}^r a_j D_j)),$$

for all $i \geq 0$.

Proof. By Theorem 1.4.10, X is $(p-1)(\sum_{j=1}^r D_j)$-split. Thus, it is also $\sum_{j=1}^r a_{j,1} D_j$-split for any $0 \leq a_{1,1}, \ldots, a_{r,1} < p$ by Remark 1.4.2 (ii). Arguing as in the proof of Lemma 1.4.7, this yields a split injection

$$H^i(X, \mathcal{L}) \longrightarrow H^i(X, \mathcal{L}^p(\sum_{j=1}^r a_{j,1} D_j)).$$

Iterating ν times, we get a split injection

$$H^i(X, \mathcal{L}) \longrightarrow H^i(X, \mathcal{L}^{p^\nu}(\sum_{j=1}^{r}(p^{\nu-1}a_{j,1} + p^{\nu-2}a_{j,2} + \cdots + a_{j,\nu})D_j)),$$

for any $0 \le a_{j,\ell} < p$. To complete the proof, note that any number $0 \le a_j < p^\nu$ can be written as $p^{\nu-1}a_{j,1} + p^{\nu-2}a_{j,2} + \cdots + a_{j,\nu}$ for some (unique) $0 \le a_{j,\ell} < p$. $\square$

We also record the following stronger version of the first part of Proposition 1.3.11, which follows from the first part of Proposition 1.3.11, Theorem 1.4.10 and Remark 1.4.2(ii).

1.4.12 Proposition. *Let X be a complete nonsingular variety of dimension n. If there exists $\sigma \in H^0(X, \omega_X^{-1})$ such that its divisor of zeroes*

$$(\sigma)_0 = Y_1 + \cdots + Y_n + Z,$$

where $Y_1, \ldots, Y_n$ are prime divisors intersecting transversally at a closed point x, and Z is an effective divisor not containing x, then σ^{p-1} provides $(p-1)Z$-splitting of X, compatible with $Y_1, \ldots, Y_n$.

1.4.13 Remark. As in Remark 1.3.12, the results of this section extend to the setting of normal varieties. Given such a variety X and an effective Weil divisor D on X, we say that X is D-split, if U is $D \cap U$-split for some nonsingular open subset U of X, with complement of codimension at least 2. (This definition makes sense in view of Lemma 1.4.4.)

1.4.E Exercises

(1) Consider a hyperplane H and a hypersurface Y of degree d in $\mathbb{P}^n$.

If $d \le n$ and Y is general (that is, Y belongs to a certain nonempty open subset of the projective space of hypersurfaces of degree d), show that $\mathbb{P}^n$ is H-split compatibly with Y.

If $d \ge n + 1$, show that $\mathbb{P}^n$ is not H-split compatibly with Y.

(2) Let X be a normal variety which is split compatibly with an effective Weil divisor D. Let E be an effective Weil divisor on X such that Supp(E) is contained in Supp(D), and let $\mathcal{L}$ be an invertible sheaf on X. Then, show the existence of a split injection $H^i(X, \mathcal{L}) \to H^i(X, \mathcal{L}^{p^\nu}(E))$ for all $i \ge 0$ and $\nu \gg 0$.

(3) With the notation and assumptions of the preceding exercise, assume further that X is proper over an affine variety, E is ample, and $\mathcal{L}$ is semi-ample. Then, show that the conclusion of Theorem 1.4.8(i) holds.

(4) Let $\mathcal{L}$ be a semi-ample invertible sheaf on a projective toric variety X. Deduce from Exercises 1.3.E.6 and 1.4.E.3 that $H^i(X, \mathcal{L}) = 0$ for all $i \ge 1$.

(5) Show that any split Gorenstein variety X is split relative to $-K_X$ for some canonical divisor K_X.

1.5 Consequences of diagonal splitting

Consider an invertible sheaf $\mathcal{L}$ on a complete variety X. If $\mathcal{L}$ is semi-ample (in the sense of Definition 1.1.12), then the graded algebra $R(X, \mathcal{L}) := \bigoplus_{\nu=0}^{\infty} \Gamma(X, \mathcal{L}^{\nu})$ is finitely generated by Lemma 1.1.13. We will derive some nice presentations of this algebra from the splitting properties of products $X \times \cdots \times X$, compatibly with certain partial diagonals.

Indeed, the multiplication of sections may be interpreted geometrically as the restriction to the diagonal, as follows. Let $\mathcal{M}$ be another invertible sheaf on X. Consider the multiplication map

$$\begin{array}{rccc} m = m(\mathcal{L}, \mathcal{M}) : & \Gamma(X, \mathcal{L}) \otimes \Gamma(X, \mathcal{M}) & \longrightarrow & \Gamma(X, \mathcal{L} \otimes \mathcal{M}) \\ & s \otimes t & \longmapsto & (x \longmapsto s(x) \otimes t(x)). \end{array}$$

Consider also the product $X \times X$ with projections $p_1, p_2 : X \times X \longrightarrow X$, and the diagonal embedding $i : X \longrightarrow X \times X$ with image Δ. Then, $\mathcal{L} \boxtimes \mathcal{M} := p_1^* \mathcal{L} \otimes p_2^* \mathcal{M}$ is an invertible sheaf on $X \times X$, such that $\Gamma(X \times X, \mathcal{L} \boxtimes \mathcal{M}) = \Gamma(X, \mathcal{L}) \otimes \Gamma(X, \mathcal{M})$ and $i^*(\mathcal{L} \boxtimes \mathcal{M}) = \mathcal{L} \otimes \mathcal{M}$. Further, the multiplication $m(\mathcal{L}, \mathcal{M})$ may be identified with the restriction map

$$\Gamma(X \times X, \mathcal{L} \boxtimes \mathcal{M}) \longrightarrow \Gamma(\Delta, \mathcal{L} \boxtimes \mathcal{M}).$$

This easily implies the following.

1.5.1 Proposition. *Let X be a complete variety. With the notation as above, $m(\mathcal{L}, \mathcal{M})$ is surjective, if either*

(a) $\mathcal{L}$, $\mathcal{M}$ are ample and Δ is compatibly split in $X \times X$, or
(b) $\mathcal{L}$, $\mathcal{M}$ are semi-ample and Δ is compatibly $X \times D$-split in $X \times X$ for some ample effective Cartier divisor D on X.

Proof. If (a) holds, then the proposition follows from Theorem 1.2.8.

If (b) holds, then by Lemma 1.4.7, it suffices to show that

$$m(\mathcal{L}^r, \mathcal{M}^r(sD)) : \Gamma(X, \mathcal{L}^r) \otimes \Gamma(X, \mathcal{M}^r(sD)) \longrightarrow \Gamma(X, \mathcal{L}^r \otimes \mathcal{M}^r(sD))$$

is surjective for all $r, s \geq 1$. Since $\mathcal{M}$ is semi-ample and D is ample, $\mathcal{M}^r(sD)$ is ample as well. By the (a) part, it follows that $m(\mathcal{L}^r, \mathcal{M}^r(sD))$ is surjective for ample $\mathcal{L}$. Therefore, $m(\mathcal{L}, \mathcal{M})$ is surjective if $\mathcal{L}$ is ample. Exchanging both factors of $X \times X$, we see that it is $D \times X$-split compatibly with Δ, so that $m(\mathcal{L}, \mathcal{M})$ is surjective if $\mathcal{M}$ is ample. In particular, $m(\mathcal{L}^r, \mathcal{M}^r(sD))$ is surjective under assumption (b); this completes the proof. $\square$

1.5.2 Corollary. *Let $\mathcal{L}$ be an invertible sheaf on a complete variety X. Then, the graded algebra $R(X, \mathcal{L})$ is generated by its subspace $\Gamma(X, \mathcal{L})$ of elements of degree 1, if either*

(a) $\mathcal{L}$ is ample and Δ is compatibly split in $X \times X$, or

(b) $\mathcal{L}$ is semi-ample and Δ is compatibly $X \times D$-split in $X \times X$ for some ample effective Cartier divisor D on X.

Moreover, we have the following.

(a′) Assume (a) and assume further that the splitting of $X \times X$ is compatible also with $Y \times X$ for a closed subvariety Y of X. Then, the restriction $R(X, \mathcal{L}) \to R(Y, \mathcal{L})$ is surjective.
(b′) Assume (b) and assume further that the $X \times D$-splitting of $X \times X$ is compatible also with $Y \times X$ for a closed subvariety Y of X such that Y is not contained in $\mathrm{Supp}(D)$. *Then, again, the restriction $R(X, \mathcal{L}) \to R(Y, \mathcal{L})$ is surjective.*

Thus, in either of cases (a′) or (b′), the algebra $R(Y, \mathcal{L})$ is generated by degree 1 elements.

Proof. By Proposition 1.5.1, the multiplication map

$$m(\mathcal{L}, \mathcal{L}^\nu) : \Gamma(X, \mathcal{L}) \otimes \Gamma(X, \mathcal{L}^\nu) \longrightarrow \Gamma(X, \mathcal{L}^{\nu+1})$$

is surjective for all $\nu \geq 1$. By induction on ν, it follows that the multiplication map $\Gamma(X, \mathcal{L})^{\otimes \nu} \longrightarrow \Gamma(X, \mathcal{L}^\nu)$ is surjective as well.

Assume now that the splitting of $X \times X$ is compatible with $Y \times X$. Since $\Delta \simeq X$ and $\Delta \cap (Y \times X) \simeq Y$ via the first projection, it follows that X is split compatibly with Y. Hence, in case (a), the restriction maps $\Gamma(X, \mathcal{L}^\nu) \longrightarrow \Gamma(Y, \mathcal{L}^\nu)$ are surjective for all $\nu \geq 1$ by Theorem 1.2.8. In case (b), X is D-split compatibly with Y, so that the surjectivity follows from Theorem 1.4.8. □

1.5.3 Corollary. *Let X be a complete variety.*

(a) If $X \times X$ is split compatibly with Δ, then any ample invertible sheaf on X is very ample.
(b) If $X \times X$ is $X \times D$-split compatibly with Δ for some ample effective Cartier divisor D, then any semi-ample invertible sheaf on X is generated by its global sections.

Proof. (a) Let $\mathcal{L}$ be an ample invertible sheaf on X; then, $\mathcal{L}$ is globally generated by Corollary 1.5.2. Consider the standard morphism

$$\varphi : X \longrightarrow \mathbb{P}(\Gamma(X, \mathcal{L})^*).$$

Choose $\nu \geq 1$ such that $\mathcal{L}^\nu$ is very ample and let

$$\varphi_\nu : X \longrightarrow \mathbb{P}(\Gamma(X, \mathcal{L}^\nu)^*)$$

be the corresponding closed immersion. Since the multiplication map

$$\Gamma(X, \mathcal{L})^{\otimes \nu} \longrightarrow \Gamma(X, \mathcal{L}^\nu)$$

is surjective, the corresponding map $X \longrightarrow \mathbb{P}((\Gamma(X, \mathcal{L})^{\otimes \nu})^*)$ is a closed immersion. But, this map factors through φ followed by the Segre embedding $\mathbb{P}(\Gamma(X, \mathcal{L})^*) \longrightarrow \mathbb{P}((\Gamma(X, \mathcal{L})^{\otimes \nu})^*)$, so that φ is a closed immersion as well.

(b) Let $\mathcal{L}$ be a semi-ample invertible sheaf on X; then, Corollary 1.5.2 implies that the global sections of $\mathcal{L}$ have no common zeroes in X. □

Recall that a closed subvariety $X \subset \mathbb{P}^n$ is said to be *projectively normal*, also called *arithmetically normal*, if the affine cone over X in $\mathbb{A}^{n+1}$ is normal. Then, X itself is normal, but there are closed normal varieties in projective space which are not projectively normal [Har–77, Chap. I, Exercise 3.18]. In fact, a closed normal variety $X \subset \mathbb{P}^n$ is projectively normal if and only if the restriction map

$$\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\nu)) \longrightarrow \Gamma(X, \mathcal{O}_X(\nu))$$

is surjective for any $\nu \geq 0$ [loc cit., Chap. II, Exercise 5.14]. Since the multiplication map $S^\nu(\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))) \longrightarrow \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\nu))$ is an isomorphism [loc cit., Chap. III, Theorem 5.1], this amounts in turn to the surjectivity of $\Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \longrightarrow \Gamma(X, \mathcal{O}_X(1))$, together with the generation of the algebra $R(X, \mathcal{O}_X(1))$ by its elements of degree 1.

Likewise, a closed subscheme $X \subset \mathbb{P}^n$ is said to be *arithmetically Cohen–Macaulay* (also called *projectively Cohen–Macaulay*) if its affine cone is Cohen–Macaulay. Then, again, X is Cohen–Macaulay, but the converse does not hold in general. By [Eis–95, Exercise 18.16], an equidimensional closed subscheme $X \subset \mathbb{P}^n$ is arithmetically Cohen–Macaulay if and only if it satisfies the following conditions:

(a) The natural map

$$k[t_0, \ldots, t_n]/I_X \longrightarrow \bigoplus_{\nu \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(\nu))$$

is surjective, where I_X denotes the homogeneous ideal of X. (This map is always injective, cf. [Har–77, Chap. II, Exercise 5.10].)

(b) $H^i(X, \mathcal{O}_X(\nu)) = 0$ for $1 \leq i < \dim(X)$ and all $\nu \in \mathbb{Z}$.

(On the other hand, X is Cohen–Macaulay if and only if $H^i(X, \mathcal{O}_X(\nu)) = 0$ for all $i < \dim(X)$ and $\nu \ll 0$, by [Har–77, Chap. III, Proof of Theorem 7.6].)

Any very ample invertible sheaf $\mathcal{L}$ on a projective variety X yields a projective embedding $X \subset \mathbb{P}(\Gamma(X, \mathcal{L})^*)$. By the preceding discussion, X is projectively normal in this embedding if and only if it is normal, and the algebra $R(X, \mathcal{L})$ is generated by its elements of degree 1. Moreover, by the above discussion, a projectively normal, Cohen–Macaulay closed subvariety $X \subset \mathbb{P}^n$ is arithmetically Cohen–Macaulay if and only it satisfies (b) as above. Together with Lemma 1.2.7, Theorem 1.4.8 and Corollary 1.5.2, this implies the following.

1.5.4 Corollary. *If X is a normal projective variety, and $X \times X$ is split compatibly with Δ, then X is projectively normal in $\mathbb{P}(\Gamma(X, \mathcal{L})^*)$ for any (very) ample invertible sheaf $\mathcal{L}$.*

If, in addition, X is Cohen–Macaulay and $X \times X$ is $X \times D$-split compatibly with Δ for some ample effective Cartier divisor D, then X is arithmetically Cohen–Macaulay in $\mathbb{P}(\Gamma(X, \mathcal{L})^*)$ for any ample invertible sheaf $\mathcal{L}$.

We have seen that the existence of a splitting of X^2 compatible with the diagonal implies that $R(X, \mathcal{L})$ is generated in degree 1 for ample $\mathcal{L}$ (Corollary 1.5.2). Likewise, we will show that the existence of a splitting of X^3 compatible with the partial diagonals implies that the relations of $R(X, \mathcal{L})$ are generated in degree 2. For this, we need the following.

1.5.5 Definition. Let $R = \bigoplus_{\nu=0}^{\infty} R_\nu$ be a commutative, graded k-algebra with $R_0 = k$, and let $M = \bigoplus_{\nu=-\infty}^{\infty} M_\nu$ be a graded R-module.

(i) R is *quadratic* if it is generated as a k-algebra by its subspace R_1 of degree 1 elements, and the kernel of the multiplication map

$$T(R_1) := \bigoplus_{\nu=0}^{\infty} R_1^{\otimes \nu} \longrightarrow R$$

is generated by its subspace of degree 2 elements, as an ideal of the tensor algebra $T(R_1)$ (called the *ideal of relations*).

(ii) M is *quadratic* if it is generated by M_0 as an R-module, and the kernel of the multiplication map $R \otimes M_0 \longrightarrow M$ is generated by its subspace of degree 1 elements, as an R-module (called the *module of relations*), where $R \otimes M_0$ is an R-module under the multiplication on the first factor. In particular, for a quadratic M, $M_\nu = 0$ for $\nu < 0$.

1.5.6 Remarks. (i) For any k-vector space V, the symmetric algebra $S(V)$ is quadratic, since it is the quotient of $T(V)$ by the ideal generated by $x \otimes y - y \otimes x$, $x, y \in V$. Further, the $S(V)$-module $S(W)$ is quadratic, for any quotient space W of V.

The definitions of quadratic algebras and modules make sense, more generally, for noncommutative graded rings. In our commutative setting, we may replace the tensor algebra with the symmetric algebra in (i). But, as we will see, it is easier to handle relations in the tensor algebra.

(ii) Let R be a quadratic algebra. Let $V = R_1$ (the space of generators) and let W be the kernel of the multiplication map $V \otimes V \longrightarrow R_2$ (the space of quadratic relations). Then, R is the quotient of the tensor algebra $T(V)$ by its graded two-sided ideal $T(V)WT(V)$. Thus, the multiplication map

$$\pi_\nu : V^{\otimes \nu} \longrightarrow R_\nu$$

is surjective, with kernel

$$\ker(\pi_\nu) = \sum_{i=1}^{\nu-1} V^{\otimes i-1} \otimes W \otimes V^{\otimes \nu-i-1}.$$

For any quadratic R-module M, similarly let $W(M)$ be the kernel of the multiplication map $V \otimes M_0 \longrightarrow M_1$, and let $\pi_\nu(M) : V^{\otimes \nu} \otimes M_0 \longrightarrow M_\nu$ be the multiplication. Then,

$$\ker(\pi_\nu(M)) = \sum_{i=1}^{\nu} V^{\otimes i-1} \otimes W(M) \otimes V^{\otimes \nu-i}.$$

(iii) Let R be a quadratic algebra and $I \subset J \subset R$ two homogeneous ideals, giving rise to surjective maps

$$R \longrightarrow R/I =: S \longrightarrow R/J =: T.$$

Then, S and T are graded algebras, generated by their elements of degree 1. If the R-modules S and T are quadratic, then the ideals I and J are also generated by their elements of degree 1. In this case, the algebras S and T are quadratic, and T is a quadratic S-module.

We now obtain a criterion for algebras or modules to be quadratic.

1.5.7 Lemma. *Let R be a graded k-algebra with $R_0 = k$. For any triple (ν_1, ν_2, ν_3) of nonnegative integers, let K_{ν_1,ν_2,ν_3} be the kernel of the multiplication map*

$$m_{\nu_1,\nu_2,\nu_3} : R_{\nu_1} \otimes R_{\nu_2} \otimes R_{\nu_3} \longrightarrow R_{\nu_1+\nu_2+\nu_3}$$

and let $K_{\nu_1,\nu_2} := K_{\nu_1,\nu_2,0}$. Then, R is quadratic if and only if m_{ν_1,ν_2,ν_3} is surjective and

$$K_{\nu_1,\nu_2,\nu_3} = K_{\nu_1,\nu_2} \otimes R_{\nu_3} + R_{\nu_1} \otimes K_{\nu_2,\nu_3},$$

for all triples (ν_1, ν_2, ν_3) of nonnegative integers.

Further, let M be a graded R-module and similarly let $K_{\nu_1,\nu_2,\nu_3}(M)$ be the kernel of the multiplication map

$$m_{\nu_1,\nu_2,\nu_3}(M) : R_{\nu_1} \otimes R_{\nu_2} \otimes M_{\nu_3} \longrightarrow M_{\nu_1+\nu_2+\nu_3}.$$

Then, M is quadratic if and only if $m_{\nu_1,\nu_2,\nu_3}(M)$ is surjective and

$$K_{\nu_1,\nu_2,\nu_3}(M) = K_{\nu_1,\nu_2} \otimes M_{\nu_3} + R_{\nu_1} \otimes K_{\nu_2,\nu_3}(M),$$

for all triples of nonnegative integers, where $K_{\nu_2,\nu_3}(M) := K_{\nu_2,\nu_3,0}(M)$.

Proof. We give the argument for algebras, the case of modules being similar. Let R be a graded algebra generated by $R_1 =: V$, and let $W \subset V^{\otimes 2}$ be the subspace of quadratic relations. Let ν_1, ν_2, ν_3 be positive integers with sum ν. Then, K_{ν_1,ν_2,ν_3} is the image of $\ker(\pi_\nu)$ under the map

$$\pi_{\nu_1} \otimes \pi_{\nu_2} \otimes \pi_{\nu_3} : V^{\otimes \nu} \longrightarrow R_{\nu_1} \otimes R_{\nu_2} \otimes R_{\nu_3}.$$

If R is quadratic, then we have

$$\begin{aligned}\ker(\pi_\nu) &= \sum_{i=1}^{\nu-1} V^{\otimes i-1} \otimes W \otimes V^{\otimes \nu-i-1} \\ &= V^{\otimes \nu_1} \otimes \ker(\pi_{\nu_2+\nu_3}) + \ker(\pi_{\nu_1+\nu_2}) \otimes V^{\otimes \nu_3}.\end{aligned}$$

It follows that $K_{\nu_1,\nu_2,\nu_3} = K_{\nu_1,\nu_2} \otimes R_{\nu_3} + R_{\nu_1} \otimes K_{\nu_2,\nu_3}$.

For the converse, consider the multiplication $m : R \otimes V \longrightarrow R$ and its degree ν component $m_\nu : R_\nu \otimes V \longrightarrow R_{\nu+1}$, where $\nu \geq 2$. Then, $\ker(m_\nu)$ is the image of $K_{1,\nu-1,1}$ under the map

$$m_{\nu-1} \otimes \mathrm{id} : R_1 \otimes R_{\nu-1} \otimes R_1 \longrightarrow R_\nu \otimes R_1.$$

Since $K_{1,\nu-1,1} = K_{1,\nu-1} \otimes R_1 + R_1 \otimes K_{\nu-1,1}$, it follows that $\ker(m_\nu) = (m_{\nu-1} \otimes \mathrm{id})(R_1 \otimes \ker(m_{\nu-1}))$. By induction, we conclude that the graded R-module $\ker(m)$ is generated by W. □

1.5.8 Proposition. *Let $\mathcal{L}$ be an invertible sheaf on a complete variety X, and let $Z \subset Y$ be closed subvarieties of X. Consider the triple product $X^3 = X \times X \times X$, with partial diagonals $\Delta_{12} := (x_1 = x_2)$ and $\Delta_{23} := (x_2 = x_3)$. Assume that either*

(a) $\mathcal{L}$ is ample and X^3 is split compatibly with $Y \times X^2$, $Z \times X^2$, $\Delta_{1,2}$, $\Delta_{2,3}$, or
(b) $\mathcal{L}$ is semi-ample and X^3 is $X^2 \times D$-split compatibly with $Y \times X^2$, $Z \times X^2$, $\Delta_{1,2}$, $\Delta_{2,3}$, for some ample effective Cartier divisor D on X. Moreover, Z is not contained in $\mathrm{Supp}(D)$.

Then, the algebra $R(X, \mathcal{L})$ is quadratic, and $R(Y, \mathcal{L})$, $R(Z, \mathcal{L})$ are quadratic modules over $R(X, \mathcal{L})$ under the restriction, so that they are quadratic algebras as well.

Thus, in case (a), Y embeds into the projective space $\mathbb{P}(\Gamma(Y, \mathcal{L})^*)$, its homogeneous ideal is generated by quadratic forms, and the homogeneous ideal of Z in Y is generated by linear forms.

Proof. By intersecting with $\Delta_{23} \simeq X^2$, we see that $X, Y, \mathcal{L}$ and $X, Z, \mathcal{L}$ satisfy the assumptions of Corollary 1.5.2. Thus, the algebras $R = R(X, \mathcal{L})$, $S = R(Y, \mathcal{L})$ and $T = R(Z, \mathcal{L})$ are generated by their elements of degree 1, and the restrictions $R \longrightarrow S$, $R \longrightarrow T$ are surjective, so that $S \longrightarrow T$ is surjective as well. Further, $\mathcal{L}$ is very ample in case (a), resp. generated by its global sections in case (b), by Corollary 1.5.3.

We now reduce case (b) to case (a), as follows. Let $X' := \mathrm{Proj}(R)$ with its ample invertible sheaf $\mathcal{L}'$, and let $f : X \longrightarrow X'$ be the natural map (see 1.3.15). Then, $f_*\mathcal{O}_X = \mathcal{O}_{X'}$ and $\mathcal{L} = f^*\mathcal{L}'$, so that $R(X', \mathcal{L}') = R$. Similarly, since $R \longrightarrow S$ is surjective, we also have $f_*\mathcal{O}_Y = \mathcal{O}_{Y'}$, where $Y' := \mathrm{Proj}(S)$; thus, $f(Y) = Y'$, and $R(Y', \mathcal{L}') = S$. Likewise, we obtain for $Z' := \mathrm{Proj}(T)$ that $f(Z) = Z'$ and $R(Z', \mathcal{L}') = T$. Further, by Lemma 1.1.8, X'^3 is split compatibly with $Y' \times X'^2$, $Z' \times X'^2$, Δ'_{12}, Δ'_{23}.

Next, we show that R is quadratic by checking that the criterion of Lemma 1.5.7 applies. With the notation of that lemma, we have

$$R_{\nu_1} \otimes R_{\nu_2} \otimes R_{\nu_3} = \Gamma(X^3, \mathcal{L}^{\nu_1} \boxtimes \mathcal{L}^{\nu_2} \boxtimes \mathcal{L}^{\nu_3}),$$

$$R_{\nu_1+\nu_2+\nu_3} = \Gamma(\Delta_{123}, \mathcal{L}^{\nu_1} \boxtimes \mathcal{L}^{\nu_2} \boxtimes \mathcal{L}^{\nu_3}),$$

where $\Delta_{123} := (x_1 = x_2 = x_3)$ is the small diagonal in X^3. Thus, we obtain

$$K_{\nu_1,\nu_2,\nu_3} = \Gamma(X^3, \mathcal{I}_{\Delta_{123}} \otimes (\mathcal{L}^{\nu_1} \boxtimes \mathcal{L}^{\nu_2} \boxtimes \mathcal{L}^{\nu_3})),$$

$$K_{\nu_1,\nu_2} \otimes R_{\nu_3} = \Gamma(X^3, \mathcal{I}_{\Delta_{12}} \otimes (\mathcal{L}^{\nu_1} \boxtimes \mathcal{L}^{\nu_2} \boxtimes \mathcal{L}^{\nu_3})),$$
$$R_{\nu_1} \otimes K_{\nu_2,\nu_3} = \Gamma(X^3, \mathcal{I}_{\Delta_{23}} \otimes (\mathcal{L}^{\nu_1} \boxtimes \mathcal{L}^{\nu_2} \boxtimes \mathcal{L}^{\nu_3})).$$

Further, $\Delta_{123} = \Delta_{12} \cap \Delta_{23}$ as schemes, by the assumption of compatible splitting and Proposition 1.2.1. So, $\mathcal{I}_{\Delta_{123}} = \mathcal{I}_{\Delta_{12}} + \mathcal{I}_{\Delta_{23}}$ fits into the Mayer–Vietoris exact sequence

$$0 \longrightarrow \mathcal{I}_{\Delta_{12} \cup \Delta_{23}} \longrightarrow \mathcal{I}_{\Delta_{12}} \oplus \mathcal{I}_{\Delta_{23}} \longrightarrow \mathcal{I}_{\Delta_{123}} \longrightarrow 0.$$

Tensoring this sequence with $\mathcal{L}^{\nu_1} \boxtimes \mathcal{L}^{\nu_2} \boxtimes \mathcal{L}^{\nu_3}$ and taking cohomology yields the exact sequence:

$$\begin{aligned}(K_{\nu_1,\nu_2} \otimes R_{\nu_3}) \oplus (R_{\nu_1} \otimes K_{\nu_2,\nu_3}) &\longrightarrow K_{\nu_1,\nu_2,\nu_3} \\ &\longrightarrow H^1(X^3, \mathcal{I}_{\Delta_{12} \cup \Delta_{23}} \otimes (\mathcal{L}^{\nu_1} \boxtimes \mathcal{L}^{\nu_2} \boxtimes \mathcal{L}^{\nu_3})).\end{aligned}$$

Further, $H^1(X^3, \mathcal{I}_{\Delta_{12} \cup \Delta_{23}} \otimes (\mathcal{L}^{\nu_1} \boxtimes \mathcal{L}^{\nu_2} \boxtimes \mathcal{L}^{\nu_3})) = 0$ for any positive integers ν_1, ν_2, ν_3, by Theorem 1.2.8. Therefore, R is quadratic by Lemma 1.5.7.

One checks similarly that the R-module S is quadratic by using the fact that the small diagonal of Y equals $(Y \times X^2) \cap \Delta_{12} \cap \Delta_{23}$. Likewise, T is a quadratic S-module. The final assertion of the Proposition follows from Remark 1.5.6 (iii). ☐

We now investigate higher syzygies, in relation to the splitting properties of multiple products. For this, we present the following algebraic notion.

1.5.9 Definition. Let $R = \bigoplus_{\nu=0}^{\infty} R_\nu$ be a graded k-algebra with $R_0 = k$, and let $M = \bigoplus_{\nu=-\infty}^{\infty} M_\nu$ be a graded R-module.

(i) R is *Koszul* if the trivial R-module k admits a graded R-module resolution

$$\cdots \longrightarrow L^2 \longrightarrow L^1 \longrightarrow L^0 \longrightarrow k \longrightarrow 0,$$

where each L^ν is a graded free R-module, generated by its subspace of degree ν.
(ii) M is *Koszul* if it admits a graded R-module resolution

$$\cdots \longrightarrow L^2 \longrightarrow L^1 \longrightarrow L^0 \longrightarrow M \longrightarrow 0,$$

where each L^ν is a graded free R-module, generated by its subspace of degree ν.

1.5.10 Remarks. (i) Let V be a k-vector space, with symmetric algebra $S(V)$. Then, the Koszul complex

$$\cdots \longrightarrow S(V) \otimes \wedge^\nu(V) \longrightarrow \cdots \longrightarrow S(V) \otimes V \longrightarrow S(V) \longrightarrow k \longrightarrow 0$$

is a graded free resolution of the $S(V)$-module k. Thus, the algebra $S(V)$ is Koszul. Likewise, the $S(V)$-module $S(W)$ is Koszul, for any quotient space W of V.

(ii) Consider a graded algebra R and the minimal graded free resolution

$$\cdots \longrightarrow L^2 \longrightarrow L^1 \longrightarrow L^0 = R \longrightarrow k \longrightarrow 0$$

of the trivial R-module k. Then, one easily checks that each L^ν (the ν-th syzygy module) is generated by its subspace of degree $\geq \nu$. As a consequence, R is Koszul if and only if each L^ν is generated by its degree ν component. Since

$$\mathrm{Tor}_\nu^R(k, k) = L^\nu \otimes_R k,$$

this amounts to each graded space $\mathrm{Tor}_\nu^R(k, k)$ being concentrated in degree ν.

Likewise, a graded R-module M is Koszul if and only if each graded space $\mathrm{Tor}_\nu^R(M, k)$ is concentrated in degree ν.

(iii) Clearly, any Koszul module is quadratic. We show that this also holds for algebras.

1.5.11 Lemma. *Any Koszul algebra is quadratic.*

Proof. Let R be a Koszul algebra and put $R_+ := \bigoplus_{\nu=1}^\infty R_\nu$; this is the irrelevant ideal of R. Choose a graded subspace $V \subset R_+$ which generates the ideal R_+ and which is minimal for this property, i.e., the induced map $V \to R_+ \otimes_R k = R_+/(R_+)^2$ is an isomorphism. Consider the multiplication map

$$m : R \otimes V \longrightarrow R.$$

Let $W \subset R \otimes V$ be a graded subspace which generates the R-module $\ker(m)$ and which is minimal in the above sense. Then, we have an exact sequence of graded R-modules

$$R \otimes W \longrightarrow R \otimes V \longrightarrow R \longrightarrow k \longrightarrow 0,$$

which can be completed to a minimal graded free resolution of k. Therefore, V, resp. W, is concentrated in degree 1, resp. 2. Since V generates the irrelevant ideal R_+, it also generates the algebra R. Thus, $V = R_1$. Now, the exact sequence

$$W \longrightarrow V \otimes V = R_1^{\otimes 2} \longrightarrow R_2 \longrightarrow 0$$

identifies W with the space of quadratic relations.

Assume that W does not generate the ideal of relations. Then, there exists a homogeneous relation $x \in V^{\otimes \nu}$ of degree $\nu > 2$, not belonging to the ideal generated by relations of smaller degree. The multiplication map $\pi_\nu : V^{\otimes \nu} \longrightarrow R_\nu$ factors as

$$V^{\otimes \nu} \xrightarrow{\pi_{\nu-1} \otimes \mathrm{id}} R_{\nu-1} \otimes V \xrightarrow{m_{\nu-1}} R_\nu,$$

so that $(\pi_{\nu-1} \otimes \mathrm{id})(x) \in \ker(m_{\nu-1})$, where $m_{\nu-1}$ is the multiplication map. Hence, $(\pi_{\nu-1} \otimes \mathrm{id})(x) \in (m_{\nu-2} \otimes \mathrm{id})(R_{\nu-2} \otimes W)$. As a consequence, there exists $y \in V \otimes \ker(\pi_{\nu-1})$ such that $(\pi_{\nu-1} \otimes \mathrm{id})(x) = (\pi_{\nu-1} \otimes \mathrm{id})(y)$. Thus, $x - y \in \ker(\pi_{\nu-1}) \otimes V$, and $x \in \ker(\pi_{\nu-1}) \otimes V + V \otimes \ker(\pi_{\nu-1})$, a contradiction. $\square$

Next, we will obtain a criterion for quadratic algebras or modules to be Koszul. To formulate this criterion, we introduce additional notation.

Let R be a quadratic algebra with spaces V of generators of degree 1, and W of quadratic relations. For any positive integer ν, let

$$U^{(\nu)} := V^{\otimes \nu} \text{ and } U_i^{(\nu)} := V^{\otimes i-1} \otimes W \otimes V^{\otimes \nu-i-1} \quad (1 \leq i \leq \nu - 1).$$

Also, let $U^{(0)} := k$. Then, the multiplication map $U^{(\nu)} \longrightarrow R_\nu$ is surjective and its kernel is $U_1^{(\nu)} + \cdots + U_{\nu-1}^{(\nu)}$ (Remark 1.5.6(ii)).

For any quadratic R-module M, denote by $W(M)$ the kernel of the multiplication map $V \otimes M_0 \longrightarrow M_1$, and similarly let

$$U^{(\nu)}(M) := V^{\otimes \nu} \otimes M_0, \quad U_i^{(\nu)}(M) := V^{\otimes i-1} \otimes W(M) \otimes V^{\otimes \nu-i}.$$

Then, each $U_i^{(\nu)}(M)$ can be identified with a subspace of $U^{(\nu)}(M)$; further, the multiplication map $U^{(\nu)}(M) \longrightarrow M_\nu$ is surjective, with kernel $U_1^{(\nu)}(M) + \cdots + U_\nu^{(\nu)}(M)$.

As a final preparation, recall that a set $\mathcal{U}$ of subspaces of a k-vector space U is a *lattice* if $\mathcal{U}$ is stable under finite intersections and sums. The lattice $\mathcal{U}$ is *distributive* if

$$U_1 \cap (U_2 + U_3) = (U_1 \cap U_2) + (U_1 \cap U_3)$$

for all U_1, U_2, U_3 in $\mathcal{U}$.

We now formulate our criterion for a quadratic algebra to be Koszul.

1.5.12 Lemma. *With the notation as above, a quadratic algebra R is Koszul if for any $\nu \geq 1$, the lattice of subspaces of $U^{(\nu)}$ generated by $U_1^{(\nu)}, \ldots, U_{\nu-1}^{(\nu)}$ is distributive.*

Likewise, a quadratic R-module M is Koszul if for any $\nu \geq 1$, the lattice of subspaces of $U^{(\nu)}(M)$ generated by $U_1^{(\nu)}(M), \ldots, U_\nu^{(\nu)}(M)$ is distributive.

Proof. We give the argument in the case of algebras; the case of modules is similar and left to the reader. For any $\nu \geq 2$, let

$$K_\nu^\nu := U_1^{(\nu)} \cap \cdots \cap U_{\nu-1}^{(\nu)},$$

and put $K_0^0 := k$, $K_1^1 := V$. Then, each K_ν^ν is a subspace of $U^{(\nu)}$. Let $K^\nu := R \otimes K_\nu^\nu$; this is a graded free R-module, where K_ν^ν is assigned degree ν. The map

$$d : R \otimes U^{(\nu)} \longrightarrow R \otimes U^{(\nu-1)}, \;\; x \otimes v_1 \otimes \cdots \otimes v_\nu \mapsto x v_1 \otimes v_2 \otimes \cdots \otimes v_\nu,$$

is R-linear and preserves degrees; one easily checks that $d(K^\nu) \subset K^{\nu-1}$ and $d^2(K^\nu) = 0$ for all ν. Thus, $(K^\bullet, d)$ is a complex of graded free R-modules. It suffices to show that this complex is a resolution of k.

For this, we decompose $K^\bullet$ into its homogeneous components $K_\nu^\bullet$. This splits up $(K^\bullet, d)$ into subcomplexes (for any $\nu \geq 1$)

$$0 \longrightarrow K_\nu^\nu \longrightarrow R_1 \otimes K_{\nu-1}^{\nu-1} \longrightarrow \cdots \longrightarrow R_{\nu-1} \otimes K_1^1 \longrightarrow R_\nu \longrightarrow 0$$

with entries $K_\nu^i := R_{\nu-i} \otimes K_i^i$. Writing

$$R_i = V^{\otimes i} / \sum_{j=1}^{i-1} V^{\otimes j-1} \otimes W \otimes V^{\otimes i-j-1}$$

and

$$K_{\nu-i}^{\nu-i} = \bigcap_{j=1}^{\nu-i-1} V^{\otimes j-1} \otimes W \otimes V^{\otimes \nu-i-j-1},$$

we obtain

$$K_\nu^{\nu-i} = U_{i+1} \cap \cdots \cap U_{\nu-1} / (U_1 + \cdots + U_{i-1}) \cap U_{i+1} \cap \cdots \cap U_{\nu-1},$$

where we set for simplicity $U_j^{(\nu)} =: U_j$. Further, the differential $d_\nu^i : K_\nu^{\nu-i} \to K_\nu^{\nu-i-1}$ is induced by the inclusion $U_{i+1} \cap \cdots \cap U_{\nu-1} \subset U_{i+2} \cap \cdots \cap U_{\nu-1}$. Thus, its kernel is

$$(U_1 + \cdots + U_i) \cap U_{i+1} \cap \cdots \cap U_{\nu-1} / (U_1 + \cdots + U_{i-1}) \cap U_{i+1} \cap \cdots \cap U_{\nu-1},$$

while the image of d_ν^{i-1} is

$$U_i \cap \cdots \cap U_{\nu-1} / (U_1 + \cdots + U_{i-1}) \cap U_{i+1} \cap \cdots \cap U_{\nu-1}.$$

It follows that $(K_\nu^\bullet, d_\nu)$ is exact if and only if

$$\begin{aligned}(U_1 + \cdots + U_i) \cap U_{i+1} \cap \cdots \cap U_{\nu-1} = \\ (U_i \cap \cdots \cap U_{\nu-1}) + \big((U_1 + \cdots + U_{i-1}) \cap U_{i+1} \cap \cdots \cap U_{\nu-1}\big),\end{aligned}$$

for $0 \le i \le \nu$. This is the distributivity condition for $U_{i+1} \cap \cdots \cap U_{\nu-1}$, U_i and $U_1 + \cdots + U_{i-1}$. □

1.5.13 Lemma. *Let R be a Koszul algebra, $I \subset J \subset R$ two homogeneous ideals, and $S := R/I$, $T := R/J$. If the graded algebras S and T are Koszul as R-modules, then they are Koszul algebras as well, and T is a Koszul S-module.*

Proof. We first show that the algebra S is Koszul. For this, we use the homology spectral sequence

$$E_{i,j}^2 = \mathrm{Tor}_i^S(\mathrm{Tor}_j^R(S,k),k) \Rightarrow \mathrm{Tor}_{i+j}^R(k,k).$$

Since S acts on each $\mathrm{Tor}_j^R(S,k)$ via its quotient $S/S_+ = k$, we have

$$E_{i,j}^2 = \mathrm{Tor}_i^S(k,k) \otimes \mathrm{Tor}_j^R(S,k).$$

If S is not Koszul, then there exist an index i and a degree $\nu \neq i$ such that the subspace $\mathrm{Tor}_i^S(k,k)_\nu$ is nonzero. Let i_o be the minimal such index; then, $(E_{i_o,0}^2)_\nu \neq 0$. But, since i_o is minimal, $\mathrm{Tor}_i^S(k,k)$ is concentrated in degree i for $i < i_o$. Further, $\mathrm{Tor}_j^R(S,k)$

is concentrated in degree j for all j, since the R-module S is Koszul. Hence, $E^2_{i,j}$ is concentrated in degree $i + j$ whenever $i + j < i_o$. The differential

$$d_r : E^r_{i,j} \longrightarrow E^r_{i-r,j+r-1}$$

preserves degrees, so that $d_r((E^r_{i_o,0})_\nu) = 0$ for all $r \geq 2$. Therefore, $(E^\infty_{i_o,0})_\nu = (E^2_{i_o,0})_\nu$ is nonzero, i.e., $\mathrm{Tor}^R_{i_o}(k,k)_\nu \neq 0$. But, this contradicts the assumption that the algebra R is Koszul.

Thus, both S and T are Koszul algebras. To show that the S-module T is Koszul as well, we use the homology spectral sequence

$$E^2_{i,j} = \mathrm{Tor}^S_i(\mathrm{Tor}^R_j(S,k), T) \Rightarrow \mathrm{Tor}^R_{i+j}(k,T).$$

Likewise, we have

$$E^2_{i,j} = \mathrm{Tor}^S_i(k,T) \otimes \mathrm{Tor}^R_j(S,k),$$

and the same arguments complete the proof. □

Before applying these algebraic results to diagonal splitting, we need a geometric lemma. Consider a scheme X, an invertible sheaf $\mathcal{L}$ on X, and a family $\mathcal{S}$ of closed subschemes of X such that $X \in \mathcal{S}$, and $\mathcal{S}$ is stable under finite intersections and finite unions. Let $U := \Gamma(X,\mathcal{L})$ and, for any $Y \in \mathcal{S}$, let U_Y be the kernel of the restriction map $\Gamma(X,\mathcal{L}) \longrightarrow \Gamma(Y,\mathcal{L})$. In other words, $U_Y = \Gamma(X, \mathcal{I}_Y \otimes \mathcal{L})$.

1.5.14 Lemma. *With the notation as above, assume that any subscheme $Y \in \mathcal{S}$ is reduced, and that $H^1(Y_2, \mathcal{I}_{Y_1} \otimes \mathcal{L}) = 0$ whenever $Y_1 \subset Y_2$ and $Y_1, Y_2 \in \mathcal{S}$. Then, the U_Y, $Y \in \mathcal{S}$, form a distributive lattice of subspaces of U.*

Proof. We claim that $U_{Y_1} + U_{Y_2} = U_{Y_1 \cap Y_2}$ for all Y_1, Y_2 in $\mathcal{S}$. For this, as in the proof of Proposition 1.5.8, we consider the Mayer–Vietoris exact sequence

$$0 \longrightarrow \mathcal{I}_{Y_1 \cup Y_2} \longrightarrow \mathcal{I}_{Y_1} \oplus \mathcal{I}_{Y_2} \longrightarrow \mathcal{I}_{Y_1 \cap Y_2} \longrightarrow 0.$$

Tensoring this sequence with $\mathcal{L}$ and taking cohomology, we obtain

$$0 \longrightarrow U_{Y_1 \cup Y_2} \longrightarrow U_{Y_1} \oplus U_{Y_2} \longrightarrow U_{Y_1 \cap Y_2} \longrightarrow H^1(X, \mathcal{I}_{Y_1 \cup Y_2} \otimes \mathcal{L}).$$

Further, this H^1 vanishes, since $Y_1 \cup Y_2 \in \mathcal{S}$. This proves our claim.

Now, $U_{Y_1 \cup Y_2} = U_{Y_1} \cap U_{Y_2}$ for all $Y_1, Y_2 \in \mathcal{S}$. Moreover, for $Y \in \mathcal{S}$, we have $Y \cup (Y_1 \cap Y_2) = (Y \cup Y_1) \cap (Y \cup Y_2)$ as subsets of X, and hence as reduced subschemes. Together with the claim, it follows that

$$U_Y \cap (U_{Y_1} + U_{Y_2}) = (U_Y \cap U_{Y_1}) + (U_Y \cap U_{Y_2}).$$

□

For any positive integer ν, let X^ν be the ν-fold product $X \times \cdots \times X$ (ν times); for $1 \le i \le \nu - 1$, let

$$\Delta_{i,i+1} = \{(x_1, \ldots, x_\nu) \in X^\nu \mid x_i = x_{i+1}\}$$

be the corresponding partial diagonal in X^ν.

1.5.15 Theorem. *Let X be a complete variety, $\mathcal{L}$ an invertible sheaf on X, and $Z \subset Y$ closed subvarieties of X. Assume that either*

(a) $\mathcal{L}$ is ample and X^ν is split compatibly with $Y \times X^{\nu-1}$, $Z \times X^{\nu-1}$, $\Delta_{1,2}, \ldots, \Delta_{\nu-1,\nu}$ for any $\nu \ge 1$, or
(b) $\mathcal{L}$ is semi-ample and X^ν is $X^{\nu-1} \times D$-split compatibly with $Y \times X^{\nu-1}$, $Z \times X^{\nu-1}$, $\Delta_{1,2}, \ldots, \Delta_{\nu-1,\nu}$ for any $\nu \ge 1$, where D is an ample effective Cartier divisor on X. Moreover, Z is not contained in $\mathrm{Supp}(D)$.

Then, $R(X, \mathcal{L})$ is a Koszul algebra, and $R(Y, \mathcal{L})$, $R(Z, \mathcal{L})$ are Koszul modules over $R(X, \mathcal{L})$. Thus, the algebras $R(Y, \mathcal{L})$, $R(Z, \mathcal{L})$ and the $R(Y, \mathcal{L})$-module $R(Z, \mathcal{L})$ are Koszul as well.

Proof. By Proposition 1.5.8, the algebra $R := R(X, \mathcal{L})$ is quadratic, and the R-modules $S := R(Y, \mathcal{L})$, $T := R(Z, \mathcal{L})$ are quadratic as well. Further, case (b) reduces to the case (a), as in the proof of that Proposition.

We now show that the algebra R is Koszul. For this, we will apply Lemma 1.5.14 to the scheme X^ν, the invertible sheaf $\mathcal{L}^{\boxtimes \nu}$, and the smallest subset $\mathcal{S}$ of closed subschemes of X^ν which contains X^ν, $Y \times X^{\nu-1}$, $\Delta_{1,2}, \ldots, \Delta_{\nu-1,\nu}$, and which is stable under finite unions and finite intersections. By our assumptions and Proposition 1.2.1, all subschemes in $\mathcal{S}$ are compatibly split, and hence reduced. Further, $H^1(Y_2, \mathcal{I}_{Y_1} \otimes \mathcal{L}^{\boxtimes \nu}) = 0$ for any $Y_1 \subset Y_2$ in $\mathcal{S}$, by Theorem 1.2.8. Thus, the assumptions of Lemma 1.5.14 are satisfied.

By this lemma, the subspaces $V^{\otimes i-1} \otimes W \otimes V^{\nu-i-1}$ of $V^{\otimes \nu}$ generate a distributive lattice, where $V := H^0(X, \mathcal{L})$ and $W := \ker\big(H^0(X^2, \mathcal{L}^{\boxtimes 2}) \to H^0(\Delta, \mathcal{L}^2)\big)$. Together with Lemma 1.5.12, this implies that the algebra R is Koszul.

Since the small diagonal in Y^ν equals $(Y \times X^{\nu-1}) \cap \Delta_{1,2} \cap \cdots \cap \Delta_{\nu-1,\nu}$, the same arguments show that the R-module $S = R(Y, \mathcal{L})$ is Koszul as well. Likewise, the R-module T is Koszul. By Lemma 1.5.13, it follows that the algebras S, T and the S-module T are Koszul. $\square$

1.5.E Exercises

(1) Let X be a complete variety and $\mathcal{L}$ a globally generated invertible sheaf on X. Then, if $X \times X$ is split compatibly with its diagonal, show that the ring $R(X, \mathcal{L})$ is generated in degree 1. In particular, if X is normal, then its image in $\mathbb{P}(\Gamma(X, \mathcal{L})^*)$ is projectively normal.

Hint: Reduce to the case where $\mathcal{L}$ is ample by using the construction of Lemma 1.1.13.

In the following exercises, let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be ample invertible sheaves on a complete variety X and let $Z \subset Y \subset X$ be closed subvarieties. Let $R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r)$ be the algebra introduced in Exercise 1.1.E.2. This algebra is multigraded by r-tuples of nonnegative integers $(\nu_1, \ldots, \nu_r)$; we will consider its grading by the total degree $\nu_1 + \cdots + \nu_r$.

(2) If $X \times X$ is split compatibly with Δ and $Y \times X$, then show that the algebra $R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r)$ is generated in degree 1 and that the restriction map

$$R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r) \longrightarrow R(Y; \mathcal{L}_1, \ldots, \mathcal{L}_r)$$

is surjective.

(3) If X^3 is split compatibly with $Y \times X^2$, Δ_{12} and Δ_{23}, then show that the algebra $R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r)$ is quadratic, and $R(Y; \mathcal{L}_1, \ldots, \mathcal{L}_r)$ is a quadratic module over $R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r)$.

(4) If X^ν is split compatibly with $Y \times X^{\nu-1}$, $\Delta_{12}, \ldots,$ and $\Delta_{\nu-1,\nu}$ for all $\nu \geq 1$, show that the algebra $R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r)$ is Koszul, and $R(Y; \mathcal{L}_1, \ldots, \mathcal{L}_r)$ is a Koszul module over $R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r)$.

(5) Adapt the preceding exercises to semi-ample invertible sheaves and splitting relative to an ample effective Cartier divisor.

1.6 From characteristic p to characteristic 0

In this section, we collect some results which will allow one to apply the positive characteristic techniques of Frobenius splitting to certain schemes in characteristic zero. *Contrary to our assumption in earlier sections, by schemes in this section we will mean separated schemes of finite type over* $\mathrm{Spec}(\mathbb{Z})$*, or, more generally, over* $\mathrm{Spec}(\mathbb{Z}[S^{-1}])$*, where S denotes a set of prime numbers.*

Given a scheme $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{Z}[S^{-1}])$, we will often assume that quasi-coherent sheaves over $\mathcal{X}$ are flat over $\mathrm{Spec}(\mathbb{Z}[S^{-1}])$. This assumption is not very restrictive, as shown by the following.

1.6.1 Remarks. (i) Since $\mathbb{Z}[S^{-1}]$ is a principal ideal domain, flatness is equivalent to being torsion-free.
(ii) If $\mathcal{F}$ is coherent, then there exists a finite set S' of primes such that the sheaf $\mathcal{F}_{\mathbb{Z}[(S \cup S')^{-1}]}$ (obtained by base change) is flat, cf. [Eis–95, Theorem 14.4].

If S consists of all the primes except a unique one p, then we denote $\mathbb{Z}[S^{-1}] = \mathbb{Z}_{(p)}$, the ring of rational numbers with denominators prime to p. The spectrum of this discrete valuation ring consists of two points: the closed point $p\mathbb{Z}_{(p)}$ with residue field $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$, and the generic point 0 with residue field $\mathbb{Q}$.

Thus, given a scheme $\mathcal{X}$ over $\mathbb{Z}_{(p)}$ and a quasi-coherent sheaf $\mathcal{F}$ on $\mathcal{X}$, we have the *special fiber* $\mathcal{F}_p$ (also called the *reduction mod* p), resp. the *generic fiber* $\mathcal{F}_{\mathbb{Q}}$; these are quasi-coherent sheaves over $\mathcal{X}_p$, resp. $\mathcal{X}_{\mathbb{Q}}$. By the base change with the algebraic closure $\bar{\mathbb{F}}_p$, resp. $\bar{\mathbb{Q}}$, we obtain the geometric fiber $\mathcal{F}_{\bar{p}}$, resp. $\mathcal{F}_{\bar{\mathbb{Q}}}$.

By [Har–77, Chap. III, Proposition 9.3], we have for any $i \geq 0$:

$$H^i(\mathcal{X}_p, \mathcal{F}_p) \otimes_{\mathbb{F}_p} \bar{\mathbb{F}}_p = H^i(\mathcal{X}_{\bar{p}}, \mathcal{F}_{\bar{p}}),$$

and a similar statement holds over $\bar{\mathbb{Q}}$. Together with the semicontinuity theorem [Har–77, Chap. III, Theorem 12.8], this implies the following.

1.6.2 Proposition. *Let $\mathcal{X}$ be a projective scheme over $\mathbb{Z}_{(p)}$, let $\mathcal{F}$ be a coherent sheaf on $\mathcal{X}$, flat over $\mathbb{Z}_{(p)}$, and let $i \geq 0$. If $H^i(\mathcal{X}_{\bar{p}}, \mathcal{F}_{\bar{p}}) = 0$, then $H^i(\mathcal{X}_{\bar{\mathbb{Q}}}, \mathcal{F}_{\bar{\mathbb{Q}}}) = 0$.*

In particular, for a scheme $\mathcal{X}$ which is projective and flat over $\mathbb{Z}_{(p)}$, together with a coherent, $\mathcal{O}_{\mathcal{X}}$-torsion-free sheaf $\mathcal{F}$, if $H^i(\mathcal{X}_{\bar{p}}, \mathcal{F}_{\bar{p}}) = 0$, then $H^i(\mathcal{X}_{\bar{\mathbb{Q}}}, \mathcal{F}_{\bar{\mathbb{Q}}}) = 0$.

Applying the above proposition to the sheaf $\mathcal{F} = \mathcal{I}_{\mathcal{Y}} \otimes \mathcal{L}$, we obtain the following.

1.6.3 Corollary. *Let $\mathcal{X}$ be a projective and flat scheme over $\mathbb{Z}_{(p)}$, let $\mathcal{Y} \subset \mathcal{X}$ be a closed subscheme and let $\mathcal{L}$ be an invertible sheaf over $\mathcal{X}$ such that the following conditions are satisfied:*

(a) $H^1(\mathcal{X}_{\bar{p}}, \mathcal{L}_{\bar{p}}) = 0$.
(b) The restriction map $H^0(\mathcal{X}_{\bar{p}}, \mathcal{L}_{\bar{p}}) \longrightarrow H^0(\mathcal{Y}_{\bar{p}}, \mathcal{L}_{\bar{p}})$ is surjective.

Then, the restriction map $H^0(\mathcal{X}_{\bar{\mathbb{Q}}}, \mathcal{L}_{\bar{\mathbb{Q}}}) \longrightarrow H^0(\mathcal{Y}_{\bar{\mathbb{Q}}}, \mathcal{L}_{\bar{\mathbb{Q}}})$ is surjective.

1.6.4 Proposition. *Let $\mathcal{X}$ be a closed subscheme of $\mathbb{P}^n_{\mathbb{Z}_{(p)}}$, flat over $\mathbb{Z}_{(p)}$. If $\mathcal{X}_{\bar{p}}$ is equidimensional, then $\mathcal{X}_{\bar{\mathbb{Q}}}$ is equidimensional of the same dimension.*

If, in addition, $\mathcal{X}_{\bar{p}}$ is Cohen–Macaulay, resp. arithmetically Cohen–Macaulay, then $\mathcal{X}_{\bar{\mathbb{Q}}}$ is Cohen–Macaulay, resp. arithmetically Cohen–Macaulay.

Proof. The assertion on equidimensionality follows from the theorem on fiber dimensions of flat morphisms [Eis–95, Theorem 10.10].

If $\mathcal{X}_{\bar{p}}$ is equidimensional and Cohen–Macaulay, then $H^i(\mathcal{X}_{\bar{p}}, \mathcal{O}(\nu)) = 0$ for any $i < \dim(\mathcal{X}_{\bar{p}})$ and $\nu \ll 0$, cf. [Har–77, Chap. III, Theorem 7.6]. Thus, the same vanishing holds over $\bar{\mathbb{Q}}$ by Proposition 1.6.2. Now, the proof of [Har–77, Chap. III, Theorem 7.6] implies that $\mathcal{X}_{\bar{\mathbb{Q}}}$ is Cohen–Macaulay.

Next, by the criterion after Corollary 1.5.3, $\mathcal{X}_{\bar{p}}$ is arithmetically Cohen–Macaulay if and only if the natural map

$$\bar{\mathbb{F}}_p[t_0, \ldots, t_n]/I_{\mathcal{X}_{\bar{p}}} \longrightarrow \bigoplus_{\nu \in \mathbb{Z}} H^0(\mathcal{X}_{\bar{p}}, \mathcal{O}(\nu))$$

is surjective, and $H^i(\mathcal{X}_{\bar{p}}, \mathcal{O}(\nu)) = 0$ for all $1 \leq i < \dim(\mathcal{X}_{\bar{p}})$ and all ν. The first condition is equivalent to the vanishings: $H^0(\mathcal{X}_{\bar{p}}, \mathcal{O}(\nu)) = 0$ for all $\nu < 0$, and $H^1(\mathbb{P}^n_{\bar{\mathbb{F}}_p}, \mathcal{I}_{\mathcal{X}_{\bar{p}}}(\nu)) = 0$ for all $\nu \geq 0$. By Proposition 1.6.2 again, it follows that $\mathcal{X}_{\bar{\mathbb{Q}}}$ is arithmetically Cohen–Macaulay. $\square$

1.6.5 Proposition. *Let $\mathcal{X}$ be a scheme over $\mathbb{Z}[S^{-1}]$, where S is a finite set of primes. If $\mathcal{X}_{\bar{p}}$ is reduced for all $p \gg 0$, then $\mathcal{X}_{\bar{\mathbb{Q}}}$ is reduced.*

Proof. We may assume that $\mathcal{X}$ is affine, and (enlarging S) that it is flat over $\mathbb{Z}[S^{-1}]$ and $\mathcal{X}_{\bar{p}}$ is reduced for all $p \notin S$; then, $\mathcal{X}_p$ is reduced as well. Let $\mathcal{X} = \mathrm{Spec}(A)$; then, A is finitely generated as a $\mathbb{Z}[S^{-1}]$-algebra, and hence as a ring. Moreover, A/pA is reduced for all $p \notin S$. Let N be the ideal of A consisting of all the nilpotent elements. Then, $N \subset pA$ for any $p \notin S$; thus, $N = pN$ since p is a nonzero divisor in A.

If $N \neq 0$, then N contains a proper subideal N' such that the A-module N/N' is isomorphic to A/I, for some proper ideal I. Then, A/I is a $\mathbb{Q}$-algebra, since any $p \in S$ is invertible in A, and any $p \notin S$ satisfies $N = pN$. On the other hand, since I is a proper ideal of A, it is contained in some maximal ideal $\mathcal{M}$. Now, $A/\mathcal{M}$ is a finitely generated ring and also a $\mathbb{Q}$-algebra. But, this is impossible in view of the general form of the Nullstellensatz [Eis–95, Theorem 4.19]. Thus, $N = 0$, that is, $\mathcal{X}$ is reduced. Hence, $\mathcal{X}_{\mathbb{Q}}$ is reduced, and since $\bar{\mathbb{Q}}$ is separable over $\mathbb{Q}$, $\mathcal{X}_{\bar{\mathbb{Q}}}$ is reduced as well. $\square$

1.6.6 Corollary. *Let $\mathcal{X}$ be a scheme over $\mathbb{Z}[S^{-1}]$, where S is a finite set of primes; let $\mathcal{Y}$, $\mathcal{Z}$ be closed subschemes. If the scheme-theoretic intersection $\mathcal{Y}_{\bar{p}} \cap \mathcal{Z}_{\bar{p}}$ is reduced for all $p \gg 0$, then $\mathcal{Y}_{\bar{\mathbb{Q}}} \cap \mathcal{Z}_{\bar{\mathbb{Q}}}$ is reduced.*

Proof. Again, we may assume that $\mathcal{X}$ is affine. Put $\mathcal{X} = \mathrm{Spec}(A)$ and let I, J be the ideals of A corresponding to $\mathcal{Y}$, $\mathcal{Z}$. Then, $\mathcal{X}_p := \mathrm{Spec}(A/pA)$, $\mathcal{Y}_p = \mathrm{Spec}(A/I + pA)$ and $\mathcal{Z}_p = \mathrm{Spec}(A/J + pA)$, so that $(\mathcal{Y} \cap \mathcal{Z})_p = \mathcal{Y}_p \cap \mathcal{Z}_p$. It follows that $(\mathcal{Y} \cap \mathcal{Z})_{\bar{p}} = \mathcal{Y}_{\bar{p}} \cap \mathcal{Z}_{\bar{p}}$, and likewise for $\bar{\mathbb{Q}}$. Now, applying Proposition 1.6.5 to $\mathcal{Y} \cap \mathcal{Z}$ completes the proof. $\square$

1.6.E Exercises

In the following exercises, X denotes a scheme of finite type over a field K of characteristic 0.

(1) Show that there exists a subring R of K, finitely generated as a $\mathbb{Z}$-algebra, and a scheme X_R of finite type over $\mathrm{Spec}(R)$, such that

$$X = X_R \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(K).$$

(2) Let X, R be as above and let $\mathfrak{m}$ be a maximal ideal of R. Show that the field $R/\mathfrak{m}$ is finite. Show also that any sufficiently large prime number p is the characteristic of $R/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$.

Then, $X_R \underset{\mathrm{Spec}(R)}{\times} \mathrm{Spec}(R/\mathfrak{m})$ is called a *reduction mod p* of X, and denoted by X_p.

(3) Let X be a nonsingular projective variety of dimension n over K and let $\mathcal{L}$ be an ample invertible sheaf on X. Show that X_p is a nonsingular projective variety of dimension n for all $p \gg 0$. Show also that (for suitable R) $\mathcal{L}_p$ exists and is an ample invertible sheaf on X_p for $p \gg 0$.

(4) Let $X, n, \mathcal{L}$ be as in the above exercise (3). Show that the map

$$F^* : H^n(X_p, \mathcal{L}_p^{-1}) \to H^n(X_p, \mathcal{L}_p^{-p})$$

is injective for $p \gg 0$.

Hint: Using the notation of Section 1.3, show that there are exact sequences of coherent sheaves on X_p for $i = 0, 1, \dots, n$:

(a) $0 \to F_*Z^i \to F_*\Omega^i \xrightarrow{d} F_*B^{i+1} \to 0$,

(b) $0 \to F_*B^i \to F_*Z^i \to \Omega^i \to 0$, where $\Omega^i := \Omega^i_{X_p}$.

Further, show that (a) for $i = 0$ may be identified with

(c) $0 \to \mathcal{O}_{X_p} \xrightarrow{F^\#} F_*\mathcal{O}_{X_p} \xrightarrow{d} F_*B^1 \to 0$.

Using (c), show that the desired result follows from the vanishing of $H^{n-1}(X_p, \mathcal{L}_p^{-1} \otimes F_*B^1)$. Using (b), the latter follows in turn from the vanishings of $H^{n-2}(X_p, \mathcal{L}_p^{-1} \otimes \Omega^1)$ and $H^{n-1}(X_p, \mathcal{L}_p^{-1} \otimes F_*Z^1)$. Deduce from the Kodaira–Akizuki–Nakano theorem [EsVi–92, Corollary 6.4] that

$$H^{n-2}(X_p, \mathcal{L}_p^{-1} \otimes \Omega^1) = 0 \qquad \text{for } p \gg 0.$$

Also, show that the vanishing of $H^{n-1}(X_p, \mathcal{L}_p^{-1} \otimes F_*Z^1)$ follows from those of $H^{n-1}(X_p, \mathcal{L}_p^{-1} \otimes F_*\Omega^1) \simeq H^{n-1}(X_p, \mathcal{L}_p^{-p} \otimes \Omega^1)$ and $H^{n-2}(X_p, \mathcal{L}_p^{-1} \otimes F_*B^2)$. Complete the argument by induction.

(5) Recall that a nonsingular projective variety X over K is called *Fano* if ω_X^{-1} is ample. Under this assumption, deduce from the above exercise that X_p is split for all $p \gg 0$.

Hint: Use Remark 1.3.9 (ii).

1.C. Comments

The notion of Frobenius splitting was introduced by Mehta–Ramanathan in their seminal article [MeRa–85]. Ramanan–Ramanathan [RaRa–85] refined it further by introducing the notion of splitting relative to a divisor.

By [Kun–69], a Noetherian local ring A of characteristic p is regular if and only if it is reduced and flat over A^p. This yields a stronger version of Lemma 1.1.1; the present version is sufficient for our purposes. Most of the subsequent results in Section 1.1 are due to Mehta–Ramanathan [MeRa–85], although we have not seen Proposition 1.1.6 and Lemma 1.1.14 explicitly stated in the literature.

Proposition 1.2.1 is due to Ramanathan [Ram–85]; Proposition 1.2.5 was first formulated in [Mat–89a]. It is an immediate consequence of the following result of Itoh [Ito–83] and Yanagihara [Yan–83]. A reduced affine scheme $X = \mathrm{Spec}(A)$ with normalization $Y = \mathrm{Spec}(B)$ is weakly normal if and only if: $b \in B$ and $b^p \in A$ imply that

$b \in A$. But, the proof presented here yields more information on the normalization of split schemes, see the result in Exercise 1.2.E.4 (due to J.F. Thomsen).

The cohomology vanishing results 1.2.7, 1.2.8 and 1.2.9 are due to Mehta–Ramanathan [MeRa–85]. Kodaira obtained his vanishing theorem for ample invertible sheaves on complex projective manifolds; this theorem fails for certain (nonsplit) nonsingular projective varieties in positive characteristic, cf. [Ray–78], [Lau–92] and [LaRa–97] for specific examples.

A slightly weaker form of Lemma 1.2.11 is due to Mehta–Srinivas [MeSr–89]; and Theorem 1.2.12 is due to Mehta–van der Kallen [MeVa–92b]. The result of Exercise 1.2.E.3 was obtained by Mehta–Srinivas [MeSr–87] in their proof of the normality of Schubert varieties.

The criterion for Frobenius splitting in terms of the sections of ω_X^{1-p} (as in Theorem 1.3.8) was first obtained in [MeRa–85], via Serre duality for projective nonsingular varieties. Our approach via duality for the Frobenius morphism follows [Van–93]; it is valid for any nonsingular (not necessarily projective) variety. The results 1.3.2–1.3.5 on differential calculus in positive characteristic are due to Cartier [Car–57]. They play a fundamental role in several other applications of positive characteristic methods in algebraic geometry. See, e.g., [DeIl–87].

The role of the Cartier operator may be replaced with a version of the change of variables formula in positive characteristic obtained by Mathieu in [Mat–87]; this is developed in [Mat–00]. Remark 1.3.9 and Proposition 1.3.11 are due to Mehta–Ramanathan [MeRa–85]. Lemma 1.3.13 was first obtained in [KuTh–01] as a crucial step towards the splitting of certain Hilbert schemes; see Chapter 7.

Theorem 1.3.14, due to Mehta–van der Kallen [MeVa–92b], is a version for split varieties of the Grauert–Riemenschneider vanishing theorem: $R^i f_*(\omega_X) = 0$ for all $i \geq 1$ whenever $f : X \to Y$ is a proper birational morphism between complex algebraic varieties and X is nonsingular. See, e.g., [EsVi–92, p. 59]. This result does not extend to positive characteristic in general, as follows from the failure of the Kodaira vanishing theorem. Likewise, Theorem 1.3.16 is a version of the Kawamata–Viehweg vanishing theorem, see [loc cit., 5.12].

The study of splittings of hypersurfaces in projective spaces, sketched in Exercises 1.3.E.3, 1.3.E.5 and 1.4.E.1, is due to Kock [Koc–97]. He showed that any split complete intersection in $\mathbb{P}^n$ is compatibly split, and arises as an intersection of irreducible components of a hypersurface $(f = 0)$ of degree $n + 1$ such that f^{p+1} splits $\mathbb{P}^n$.

The result of Exercise 1.3.E.4 yields an inductive construction of a splitting of a variety X, starting with a splitting of a complete intersection in X. This construction has been used in several contexts, including the splittings of the moduli space of rank-2 vector bundles on a generic curve [MeRam–96] and of the wonderful compactifications of certain homogeneous spaces [Str–87], [BrIn–94], [DeSp–99]. See also Chapter 6.

The notions and results of Exercises 1.3.E.10–13 are due to Lakshmibai–Mehta–Parameswaran [LMP–98]. For further results in this direction for the flag varieties, see the comments to Chapter 2.

The notion of splitting relative to a divisor was introduced in [RaRa–85] (also see

[Ram–87]). Most of the exposition in 1.4 is taken from [Ram–87].

The first applications of diagonal splitting to syzygies (1.5.1)–(1.5.8) are due to Ramanathan [Ram–87]. They were developed further in [KeRa–87] and [InMe–94a,b] to yield versions of Theorem 1.5.15 and Exercises 1.5.E. The present version was obtained later in the unpublished e-print [Bez–95] from which our exposition is taken. Koszul graded algebras appear in algebraic geometry (sometimes under the name "wonderful"), cf. [Kem–90], [Kem–92] and in representation theory [BGSo–96].

The results of Section 1.6 are quite standard though we could not locate them in the literature explicitly stated.

There are examples of smooth projective Fano varieties X (i.e., ω_X^{-1} is ample) such that X is *not* split. Such examples are provided by G/P, for some *nonreduced* parabolic subgroup schemes P in a semisimple algebraic group G (cf. [Lau–93]). However, the reduction mod p of any Fano variety is split for $p \gg 0$, as shown in Exercises 1.6.E.

Finally, let us mention some recent developments based on the notions of F-rationality, F-regularity and tight closure, which are closely related to the Frobenius splitting. We refer to the exposition [Smi–01] for an excellent survey of these topics, and for further references.

Chapter 2

Frobenius Splitting of Schubert Varieties

Introduction

The main aim of this chapter is to prove that the flag varieties G/P are split compatibly splitting all the Schubert subvarieties. Similarly, it is proved that the product variety $G/P \times G/Q$ is split compatibly splitting all the G-Schubert subvarieties. In fact, these varieties are shown to have D-splittings for certain ample divisors D. More specifically, the content of this chapter is as follows.

Section 2.1 is devoted to establishing the basic notation associated to semisimple groups. By G we mean a connected, simply-connected, semisimple algebraic group over an algebraically closed field k of characteristic $p > 0$. We fix a Borel subgroup B, and a maximal torus $T \subset B$ with the associated Weyl group W. Let $B \subset P$ be a parabolic subgroup. For any $w \in W$, we have the Schubert variety X_w^P and also the opposite Schubert variety $\widetilde{X}_w^P$ in $X^P := G/P$. This notation will be followed throughout the book.

Section 2.2 starts off with the definition and well known elementary properties of the Bott–Samelson–Demazure–Hansen (for short, BSDH) varieties, including the determination of their canonical bundles (Proposition 2.2.2). By using a general criterion of splitting proved in Chapter 1 (Proposition 1.3.11), these BSDH varieties are proved to be split compatibly splitting all the BSDH subvarieties (Theorem 2.2.3). As an immediate consequence, one obtains the important result due to Mehta–Ramanathan that the flag varieties X^P are split (for any parabolic subgroup P) compatibly splitting all the Schubert subvarieties X_w^P. In fact, as proved by Ramanan–Ramanathan, these varieties are shown to be simultaneously $(p-1)\partial^- X^P$-split (Theorem 2.2.5), where $\partial^- X^P$ is the reduced divisor of X^P defined as the complement of the big open cell $U^- P/P$. As another consequence of the splitting of the BSDH varieties, it is shown that the product variety $\mathcal{X}^{P,Q} := X^P \times X^Q$ is split compatibly splitting all the G-Schubert subvarieties

$\mathcal{X}_w^{P,Q}$ under the diagonal action of G, where P, Q are any parabolic subgroups of G (Corollary 2.2.7).

Section 2.3 is devoted to determining all possible splittings of $X = G/B$ is terms of the Steinberg module St. There is the canonical multiplication map $m : \mathrm{St} \otimes \mathrm{St} \to H^0(X, \omega_X^{1-p})$. Since St is self-dual, there exists a G-invariant nondegenerate bilinear form $\chi : \mathrm{St} \otimes \mathrm{St} \to k$. Then, as proved by Lauritzen–Thomsen, it is shown that $m(f)$ splits X for $f \in \mathrm{St} \otimes \mathrm{St}$ iff $\chi(f) \neq 0$ (Corollary 2.3.5). To prove this, we first show explicitly that $m(f_+ \otimes f_-)$ provides a $\big((p-1)(\partial^- X + \partial X)\big)$-splitting of X, $(p-1)\partial^- X$-splitting all the Schubert subvarieties X_w and $(p-1)\partial X$-splitting all the opposite Schubert subvarieties $\widetilde{X}_w$ simultaneously (Theorem 2.3.1), where f_+ (resp. f_-) is the highest (resp. lowest) weight vector of St and ∂X is the complement of Bw_oB/B (w_o being the longest element of W).

If we take $f \in \mathrm{St} \otimes \mathrm{St}$ such that $\chi(f) \neq 0$ and $m(f) = \sigma^{p-1}$ for a section σ of an appropriate homogeneous line bundle on X, then the splitting of X provided by $m(f)$ is a $(p-1)Z(\sigma)$-splitting, compatibly splitting $Z(\sigma)$, where $Z(\sigma)$ is the zero scheme of σ (Proposition 2.3.7). This simple result immediately gives that $\mathcal{X} = X \times X$ is $(p-1)D'$-split such that all the G-Schubert subvarieties $\mathcal{X}_w$ are compatibly $(p-1)D'$-split, D' being the reduced divisor $\partial X \times X \cup X \times \partial^- X$ (Theorem 2.3.8). In fact, a slightly sharper result is proved in Theorem 2.3.8.

Similar results are obtained for X^P and $\mathcal{X}^{P,Q}$ by considering the canonical morphisms $X \to X^P$ and $\mathcal{X} \to \mathcal{X}^{P,Q}$ (Theorem 2.3.2 and Corollary 2.3.9). As an immediate consequence of the above, one obtains that any subscheme of X^P, obtained by taking unions and intersections of the Schubert subvarieties and opposite Schubert subvarieties of X^P, is reduced (Corollary 2.3.3). Also, one obtains a certain analogue of these results for the n-fold products $(G/P)^n$ (Theorem 2.3.10).

In Exercise 2.3.E.3, a proof is outlined to show that St is an irreducible self-dual G-module.

2.1 Notation

We begin by fixing notation and reviewing some known facts on algebraic groups; we refer the reader to [Bor–91], [Spr–98] for details.

Let H be an affine algebraic group over k. A scheme X together with an action of H is called an *H-scheme* if the action map $\theta : H \times X \to X$ is algebraic. By an *H-linearized sheaf* (also called *H-equivariant sheaf*) on X we mean a quasi-coherent sheaf $\mathcal{S}$ of $\mathcal{O}_X$-modules on X together with an isomorphism $\phi : \theta^*(\mathcal{S}) \simeq \pi_2^*(\mathcal{S})$ of $\mathcal{O}_{H\times X}$-modules, where $\pi_2 : H \times X \to X$ is the projection onto the second factor. The isomorphism ϕ must be "associative" in the sense that it satisfies the usual cocycle condition

$$(\pi_{23}^*\phi) \circ ((I \times \theta)^*\phi) = (m \times I)^*\phi$$

on $H \times H \times X$, where I is the identity map, $m : H \times H \to H$ is the multiplication map, and π_{23} is the projection onto the second and third factor.

For a finite-dimensional vector space V over k, a morphism of algebraic groups $\rho : H \to \operatorname{Aut} V$ is called a *rational* (or *algebraic*) *representation* of H in V. More generally, an abstract representation of H in a (not necessarily finite-dimensional) k-vector space V is called *rational* if, for any $v \in V$, there exists a finite-dimensional H-stable subspace $M_v \subset V$ containing v such that the representation $\rho_{|M_v}$ is rational.

Also, recall the definition of the *hyperalgebra* of H from [Jan–03, Part I, §7.7], which we will denote by $\mathfrak{U}_H$. (In [loc cit.] it is called the algebra of distributions of H and denoted by Dist (H).)

We now consider a connected, simply-connected, semisimple algebraic group G over an algebraically closed field k and denote $B \subset G$, resp. $T \subset B$, a fixed Borel subgroup, resp. a maximal torus. Let U be the unipotent radical of B, so that $B = TU$. Let $W := N(T)/T$ be the Weyl group of G, where $N(T)$ is the normalizer of T in G. We denote the Lie algebras of G, B, T, U respectively by $\mathfrak{g}$, $\mathfrak{b}$, $\mathfrak{t}$, $\mathfrak{u}$. In fact, we will denote the Lie algebra of any closed subgroup of G by the corresponding lower case Gothic character.

For the hyperalgebra $\mathfrak{U}_G$ of G, there is a canonical isomorphism

$$\mathfrak{U}_G \simeq k \otimes_{\mathbb{Z}} U_{\mathbb{Z}}(\mathfrak{g}^{\mathbb{C}}),$$

where $U_{\mathbb{Z}}(\mathfrak{g}^{\mathbb{C}})$ is the Kostant $\mathbb{Z}$-form of the enveloping algebra $U(\mathfrak{g}^{\mathbb{C}})$ over $\mathbb{C}$, and $\mathfrak{g}^{\mathbb{C}}$ is the Lie algebra over $\mathbb{C}$ of the corresponding complex algebraic group $G(\mathbb{C})$.

Let $X^*(T)$ denote the *group of characters* of T, i.e., the group of algebraic group morphisms $T \to \mathbb{G}_m$. Similarly, let $X_*(T)$ denote the *group of cocharacters* of T, i.e., the group of algebraic group morphisms $\mathbb{G}_m \to T$. Then, $X^*(T)$ and $X_*(T)$ are both free abelian groups of rank ℓ, where ℓ is the dimension of T (which is called the *rank* of G). Moreover, the standard pairing $\langle \cdot, \cdot \rangle : X^*(T) \times X_*(T) \to \mathbb{Z}$, $(\lambda \circ \mu^\vee)(z) = z^{\langle \lambda, \mu^\vee \rangle}$, for $z \in \mathbb{G}_m$, $\lambda \in X^*(T)$, $\mu^\vee \in X_*(T)$, is perfect, i.e., identifies $X^*(T)$ with $\operatorname{Hom}_{\mathbb{Z}}(X_*(T), \mathbb{Z})$.

Let $\Delta \subset X^*(T)$ be the set of *roots* (i.e., the set of nonzero weights for the adjoint action of T on $\mathfrak{g}$) and $\Delta^+ \subset \Delta$ the set of *positive roots* with respect to the choice of B, i.e., Δ^+ is the set of weights for the action of T on $\mathfrak{u}$. The set of *negative roots* is $\Delta^- := -\Delta^+$, associated with the opposite Borel subgroup $B^- = TU^-$. For any $\alpha \in \Delta^\pm$, we have the *root subgroup* $U_\alpha \subset U^\pm$ normalized by T, together with an algebraic group isomorphism $\varepsilon_\alpha : \mathbb{G}_a \to U_\alpha$ such that $t\varepsilon_\alpha(z)t^{-1} = \varepsilon_\alpha(\alpha(t)z)$, for $t \in T$ and $z \in \mathbb{G}_a$. Moreover, the multiplication map $\prod_{\alpha \in \Delta^\pm} U_\alpha \to U^\pm$ is a T-equivariant isomorphism of varieties for any prescribed ordering of $\Delta^\pm$.

Let $\{\alpha_1, \ldots, \alpha_\ell\} \subset \Delta^+$ be the set of *simple roots*, and $\{s_1, \ldots, s_\ell\} \subset W$ the corresponding set of *simple reflections*. These generate the group W; given $w \in W$, a decomposition $w = s_{i_1} \cdots s_{i_n}$ into a product of simple reflections is called a *reduced expression* also called a *reduced decomposition* if n is minimal among all such decompositions. Then, n is called the *length* of w and denoted $\ell(w)$. Moreover,

$$\Delta^+ \cap w^{-1}(\Delta^-) = \{\alpha_{i_n}, s_{i_n}(\alpha_{i_{n-1}}), \ldots, s_{i_n} \cdots s_{i_2}(\alpha_{i_1})\}, \tag{1}$$

and these roots are distinct. There exists a unique element w_o of maximal length, characterized by $w_o(\Delta^+) = \Delta^-$.

For any $w \in W$ with a representative $\dot{w}$ in $N(T)$, the double coset $B\dot{w}B$ depends only on w; we denote it by BwB. By the Bruhat decomposition, G is the disjoint union of the locally closed subsets BwB, where $w \in W$; moreover, the map

$$(U \cap wU^-w^{-1}) \times B \longrightarrow BwB, \quad (u, b) \mapsto u\dot{w}b \tag{2}$$

is an isomorphism, and the multiplication map

$$\prod_{\alpha \in \Delta^+ \cap w(\Delta^-)} U_\alpha \longrightarrow U \cap wU^-w^{-1}, \tag{3}$$

under any ordering of $\Delta^+ \cap w(\Delta^-)$, is an isomorphism as well. In particular, Bw_oB is an open subset of G, isomorphic to $U \times B$. Recall that for any simple reflection s_i and $w \in W$, we have

$$(Bs_iB) \cdot (BwB) = \begin{cases} Bs_iwB, & \text{if } \ell(s_iw) > \ell(w) \\ (Bs_iwB) \cup (BwB), & \text{otherwise.} \end{cases} \tag{4}$$

Any subgroup P of G which contains B is called a *standard parabolic subgroup*, and any subgroup of G which is conjugate to a standard parabolic subgroup is called a *parabolic subgroup*. It is well known that any parabolic subgroup is closed. Moreover, the standard parabolic subgroups are in bijective correspondence with the subsets I of $\{1, \dots, \ell\}$, under

$$I \rightsquigarrow P_I := \bigsqcup_{w \in W_I} BwB, \tag{5}$$

where W_I denotes the subgroup of W generated by the s_i, $i \in I$. In particular, $P_\emptyset = B$, $P_{\{1,\dots,\ell\}} = G$ and there are exactly 2^ℓ standard parabolic subgroups of G. For any $1 \le i \le \ell$, the subgroup $P_i = P_{\{i\}} := B \cup Bs_iB$ is called a *standard minimal parabolic subgroup* of G.

For any closed subgroup H of G, the coset space G/H acquires a natural structure of a quasi-projective variety such that the action of G on G/H via the left multiplication is algebraic and the projection map $\pi_H : G \to G/H$ is a smooth morphism. Under this structure, the parabolic subgroups are precisely those closed (reduced) subgroups such that G/P is a projective variety.

In particular, G/B is a projective variety, the (full) *flag variety* of G, parametrizing all the Borel subgroups. By the Bruhat decomposition, G/B is the disjoint union of the *Bruhat cells* $C_w := BwB/B$; by (2)-(3), C_w is a locally closed subset isomorphic to $U \cap wU^-w^{-1}$, an affine space of dimension $\ell(w)$. The closure of C_w in G/B equipped with the reduced subscheme structure is called the *Schubert variety* X_w. This closed B-stable subvariety of G/B is the disjoint union of the Bruhat cells C_x with $x \le w$, where $\le$ denotes the *Bruhat–Chevalley order* on W. Let $w = s_{i_1} \dots s_{i_n}$ be a reduced expression. Then, $x \le w$ iff x is obtained from w by deleting some s_{i_j}'s, i.e., there exists $1 \leq j_1 < \cdots < j_p \leq n$ such that $x = s_{i_{j_1}} \dots s_{i_{j_p}}$ (cf. [Spr–98, Proposition 8.5.5]). The "*boundary*"

(6) $$\partial X_w := X_w \setminus C_w = \bigsqcup_{x<w} C_x$$

is the union of all the Schubert subvarieties of codimension one in X_w. In particular, $G/B = X_{w_o}$, and its boundary $\partial G/B$ is the union of all the *Schubert divisors* $X_{w_o s_1}, \dots, X_{w_o s_\ell}$. On the other hand, the *Schubert curves* (i.e., the one-dimensional Schubert varieties) are the $X_{s_i} = P_i/B$, isomorphic to the projective line.

Likewise, we have the *opposite Bruhat cell* $\widetilde{C}_w := B^- wB/B$, with closure the *opposite Schubert variety* $\widetilde{X}_w$, and boundary $\partial^- \widetilde{X}_w := \widetilde{X}_w \setminus \widetilde{C}_w$. Since $B^- = w_o B w_o$, each $\widetilde{C}_w$ is an affine space of codimension $\ell(w)$ in G/B, and $\widetilde{X}_w = \bigsqcup_{x \geq w} \widetilde{C}_x$, $\partial^- \widetilde{X}_w = \bigsqcup_{x>w} \widetilde{C}_x$. In particular, $\widetilde{C}_1 = U^- B/B \simeq U^-$ is open in G/B, so that $\pi_B : G \to G/B$ is a locally trivial principal B-bundle. Moreover, $\partial^- G/B = \partial^- \widetilde{X}_1$ is the union of the *opposite Schubert divisors* $\widetilde{X}_{s_1}, \dots, \widetilde{X}_{s_\ell}$.

More generally, for any standard parabolic subgroup P and any $w \in W$, we define the Bruhat cell $C_w^P := BwP/P \subset G/P$, the Schubert variety $X_w^P := \overline{C_w^P}$, its boundary $\partial X_w^P := X_w^P \setminus C_w^P$, and similarly the opposite Bruhat cell $\widetilde{C}_w^P := B^- wP/P$, opposite Schubert variety $\widetilde{X}_w^P := \overline{\widetilde{C}_w^P}$ and boundary $\partial^- \widetilde{X}_w^P$. Then, $\widetilde{C}_1^P = B^- P/P$ is isomorphic to the unipotent radical of the opposite parabolic subgroup P^-. As above, it follows that $\pi_P : G \to G/P$ is a locally trivial principal P-bundle. Moreover, the morphism

$$f_P : G/B \to G/P$$

is a locally trivial fibration with fiber P/B, and each X_w^P (resp. $\widetilde{X}_w^P$) is the scheme-theoretic image of X_w (resp. $\widetilde{X}_w$) under f_P.

The set of *simple coroots* $\{\alpha_1^\vee, \dots, \alpha_\ell^\vee\} \subset X_*(T)$ forms a basis of $X_*(T)$ (cf. [Spr–98, 7.4.3] for the definition of coroots). The dual basis of $X^*(T)$ is, by definition, the *fundamental weights* $\{\chi_i\}_{i \leq i \leq \ell}$. A weight $\lambda \in X^*(T)$ is called *dominant* if $\langle \lambda, \alpha_i^\vee \rangle \geq 0$ for all the simple coroots $\alpha_i^\vee$; equivalently, if all the coefficients of λ in the basis of fundamental weights are nonnegative. The set of dominant weights will be denoted by $X^*(T)^+$. We put $\rho := \chi_1 + \cdots + \chi_\ell$; then ρ equals the half sum of all the positive roots.

The group of characters $X^*(B)$ can be identified with $X^*(T)$ under the restriction map. Thus, any character λ of T can uniquely be extended to a character (still denoted by) λ of B. Let k_λ be the associated one-dimensional representation of B. Then, we have the G-equivariant line bundle

(7) $$\mathcal{L}(\lambda) := G \times_B k_{-\lambda} \to G/B$$

associated to the locally trivial principal B-bundle $\pi_B : G \to G/B$ via the representation $k_{-\lambda}$ of B. (We use the additive notation for $X^*(T)$, thus $-\lambda$ denotes the character λ^{-1}.) In fact, any (not necessarily G-equivariant) line bundle on G/B is isomorphic to $\mathcal{L}(\lambda)$, for some $\lambda \in X^*(T)$. More generally, for any rational B-module V, by $\mathcal{L}(V)$

we mean the G-equivariant vector bundle $G \times_B V \to G/B$ associated to the principal B-bundle $\pi_B : G \to G/B$. Thus, in this notation, $\mathcal{L}(\lambda) = \mathcal{L}(k_{-\lambda})$. It is easy to see that, as a G-equivariant line bundle,

(8) $$\omega_{G/B} \simeq \mathcal{L}(-2\rho).$$

For an arbitrary λ, the space of global sections $H^0(G/B, \mathcal{L}(\lambda))$ is a finite-dimensional rational G-module, which is nonzero if and only if λ is dominant; then this space contains a unique B-stable line, and the corresponding weight is $-w_o(\lambda)$ (since any U-invariant section is determined by its value at the point w_oB). Moreover, all the weights of $H^0(G/B, \mathcal{L}(\lambda))$ are $\leq -w_o(\lambda)$. Recall that, for $\lambda, \mu \in X^*(T)$, by definition, $\lambda \leq \mu$ if and only if $\mu - \lambda \in \sum_{i=1}^{\ell} \mathbb{N}\alpha_i$. The dual module $V(\lambda) := H^0(G/B, \mathcal{L}(\lambda))^*$ contains a B-stable line with weight λ and all the weights of $V(\lambda)$ are $\leq \lambda$. This module is called the *Weyl module* with highest weight λ.

2.2 Frobenius splitting of the BSDH varieties $Z_{\mathfrak{w}}$

We follow the notation as above. In particular, G is a connected, simply-connected, semisimple algebraic group over an algebraically closed field k of any characteristic $p > 0$. For subsections 2.2.1–2.2.2, we could take $p = 0$ as well. Varieties are reduced and irreducible schemes as earlier.

Let $w \in W$ and choose a reduced expression $w = s_{i_1} s_{i_2} \cdots s_{i_n}$. Then, we have by 2.1.4, $BwB = (Bs_{i_1}B) \cdot (Bs_{i_2}B) \cdot \cdots \cdot (Bs_{i_n}B) = Bs_{i_1}Bs_{i_2}B \cdots Bs_{i_n}B$, and it follows (cf. [Spr–98, p. 150]) that

$$X_w = P_{i_1} P_{i_2} \cdots P_{i_n}/B.$$

Further, the product map $P_{i_1} \times P_{i_2} \times \cdots \times P_{i_n} \to X_w$ is invariant under the action of B^n from the right via

(∗) $$(p_1, \ldots, p_n) \odot (b_1, \ldots, b_n) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{n-1}^{-1} p_n b_n).$$

This motivates the following.

2.2.1 Definition. (Bott–Samelson–Demazure–Hansen variety) Let $\mathfrak{w} = \big(s_{i_1}, \ldots, s_{i_n}\big)$ be any ordered sequence of simple reflections in W, called a *word* in W.

Define the *Bott–Samelson–Demazure–Hansen* (for short *BSDH*) variety $Z_{\mathfrak{w}}$ as the orbit space

(1) $$Z_{\mathfrak{w}} := P_{\mathfrak{w}}/B^n,$$

where B^n acts on the product variety $P_{\mathfrak{w}} := P_{i_1} \times \cdots \times P_{i_n}$ via (∗) as above.

Now, we put a smooth projective variety structure on $Z_{\mathfrak{w}}$ such that the orbit map $\pi_{\mathfrak{w}} : P_{\mathfrak{w}} \to Z_{\mathfrak{w}}$ is a locally trivial principal B^n-bundle. Define the B^n-equivariant morphism of varieties $\phi_n : G^n \to G^n$ by

(2) $$(g_1, \ldots, g_n) \mapsto (g_1, g_1 g_2, \ldots, g_1 \cdots g_{n-1} g_n),$$

where B^n acts on the domain via

$$(g_1, \dots, g_n) \odot (b_1, \dots, b_n) = (g_1 b_1, b_1^{-1} g_2 b_2, \dots, b_{n-1}^{-1} g_n b_n) \tag{3}$$

and on the range via the right multiplication componentwise:

$$(g_1, \dots, g_n) \cdot (b_1, \dots, b_n) = (g_1 b_1, g_2 b_2, \dots, g_n b_n), \tag{4}$$

for $g_j \in G$ and $b_j \in B$.

It is easy to see that ϕ_n is an isomorphism with the inverse given by $(h_1, \dots, h_n) \mapsto (h_1, h_1^{-1} h_2, h_2^{-1} h_3, \dots, h_{n-1}^{-1} h_n)$. We put the unique reduced scheme structure on $Z_{\mathfrak{w}}$ so that the horizontal maps in the following diagram are closed embeddings:

$$\begin{array}{ccc} P_{\mathfrak{w}} & \xrightarrow{\phi_{n|P_{\mathfrak{w}}}} & G^n \\ \downarrow{\scriptstyle \pi_{\mathfrak{w}}} & & \downarrow \\ Z_{\mathfrak{w}} & \xrightarrow[\phi_{\mathfrak{w}}]{} & (G/B)^n, \end{array}$$

where

$$\phi_{\mathfrak{w}}[p_1, \dots, p_n] = (p_1 B, p_1 p_2 B, \dots, p_1 \cdots p_n B) \tag{5}$$

and $[p_1, \dots, p_n]$ denotes the B^n-orbit through $(p_1, \dots, p_n)$. Thus, $Z_{\mathfrak{w}}$ is a projective variety and $\pi_{\mathfrak{w}}$ is the pullback of the locally trivial principal B^n-bundle $G^n \to (G/B)^n$. In particular, $\pi_{\mathfrak{w}}$ is a locally trivial principal B^n-bundle. Thus, $Z_{\mathfrak{w}}$ is smooth, since $P_{\mathfrak{w}}$ is smooth. The left multiplication of P_{i_1} on the first factor makes $Z_{\mathfrak{w}}$ into a P_{i_1}-variety. The projection of $\phi_{\mathfrak{w}}$ on the last factor gives rise to the P_{i_1}-equivariant morphism $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to G/B$, i.e.,

$$\theta_{\mathfrak{w}}[p_1, \dots, p_n] = p_1 \cdots p_n B. \tag{6}$$

Since $P_i = B \cup B s_i B$ and the map $U_{\alpha_i} \times B \to B s_i B$, $(u, b) \mapsto u \dot{s}_i b$ is an isomorphism by (2.1.2), we see that

$$Z_{\mathfrak{w}}^o := \big((B s_{i_1} B) \times \cdots \times (B s_{i_n} B)\big)/B^n$$

is an open subset of $Z_{\mathfrak{w}}$ isomorphic to $\prod_{j=1}^n U_{\alpha_{i_j}}$, an affine n-space.

For any subsequence $J : 1 \le j_1 < j_2 < \cdots < j_m \le n$, there is a closed embedding

$$\begin{gathered} i = i_{\mathfrak{w}_J, \mathfrak{w}} : Z_{\mathfrak{w}_J} \to Z_{\mathfrak{w}}, \\ \big[p_{j_1}, \dots, p_{j_m}\big] \mapsto \big[1, \dots, 1, p_{j_1}, \dots, p_{j_m}, 1, \dots, 1\big], \end{gathered} \tag{7}$$

where $\mathfrak{w}_J$ is the subsequence $\big(s_{i_{j_1}}, \dots, s_{i_{j_m}}\big)$ of $\mathfrak{w}$, p_{j_q} is put in the j_q-th slot and the remaining slots are filled by 1. Clearly, i is a morphism and it is easy to see that it is injective. Moreover, since $Z_{\mathfrak{w}_J}$ is a projective variety, Im i is closed [Har–77, Chap. II, Theorem 4.9]. Finally, to prove that $i : Z_{\mathfrak{w}_J} \to \operatorname{Im} i$ is an isomorphism with the closed subvariety structure on Im i, it suffices to consider the morphism

$$B^{j_1-1} \times P_{i_{j_1}} \times B^{j_2-j_1-1} \times P_{i_{j_2}} \times \cdots \times B^{j_m-j_{m-1}-1} \times P_{i_{j_m}} \times B^{n-j_m}$$
$$\to P_{i_{j_1}} \times \cdots \times P_{i_{j_m}},$$
$$(p_1, \ldots, p_n) \mapsto \big(p_1 \ldots p_{j_1}, p_{j_1+1} \cdots p_{j_1+j_2}, \ldots, p_{j_{m-1}+1} \cdots p_{j_m}\big),$$

which descends to give the inverse of i on Im i.

For any $1 \le m \le n$, consider the subsequence $\mathfrak{w}[m] := \big(s_{i_1}, \ldots, s_{i_m}\big)$ of $\mathfrak{w}$. Then, there is a morphism $\psi_{\mathfrak{w},m} : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}[m]}$, given by $[p_1, \ldots, p_n] \mapsto [p_1, \ldots, p_m]$. For $m = n-1$, we abbreviate $\psi_{\mathfrak{w},m}$ by $\psi_{\mathfrak{w}}$. Thus,

$$\psi_{\mathfrak{w},m} = \psi_{\mathfrak{w}[m+1]} \circ \cdots \circ \psi_{\mathfrak{w}[n-2]} \circ \psi_{\mathfrak{w}[n-1]} \circ \psi_{\mathfrak{w}}. \tag{8}$$

We next show that $\psi_{\mathfrak{w}}$ is a locally trivial $\mathbb{P}^1$-fibration: Let $\mathfrak{v} := \mathfrak{w}[n-1]$. Consider the locally trivial principal B-bundle

$$P_{\mathfrak{v}}/(B^{n-2} \times 1) \xrightarrow{\pi'_{\mathfrak{v}}} Z_{\mathfrak{v}} = P_{\mathfrak{v}}/B^{n-1}.$$

Then, we have an isomorphism γ from the associated fiber bundle (with fiber P_{i_n}/B) to $Z_{\mathfrak{w}}$ making the following diagram commutative:

$$\begin{tikzcd} \Big(P_{\mathfrak{v}}/(B^{n-2} \times 1)\Big) \times_B P_{i_n}/B \arrow[rr, "\gamma", "\sim"'] \arrow[dr, "\bar{\pi}_{\mathfrak{v}}"'] & & Z_{\mathfrak{w}} \arrow[dl, "\psi_{\mathfrak{w}}"] \\ & Z_{\mathfrak{v}}, & \end{tikzcd}$$

where $\bar{\pi}_{\mathfrak{v}}$ is induced from $\pi'_{\mathfrak{v}}$ and γ is induced from the map

$$\big((p_1, \ldots, p_{n-1}) \mod B^{n-2}, \bar{p}_n\big) \mapsto [p_1, \ldots, p_n],$$

for $(p_1, \ldots, p_{n-1}) \in P_{\mathfrak{v}}$ and $\bar{p}_n \in P_{i_n}/B$. By constructing the inverse of γ explicitly, it is easy to see that γ is an isomorphism. Thus, $\psi_{\mathfrak{w}}$ is a locally trivial $\mathbb{P}^1$-fibration since so is $\bar{\pi}_{\mathfrak{v}}$. Hence, by (8), $\psi_{\mathfrak{w},m}$ is a smooth morphism. Furthermore, we have the following commutative diagram:

$$\begin{tikzcd} Z_{\mathfrak{w}} \arrow[r, "\theta_{\mathfrak{w}}"] \arrow[d, "\psi_{\mathfrak{w}}"'] & G/B \arrow[d, "f_{i_n}"] \\ Z_{\mathfrak{v}} \arrow[r, "f_{i_n} \circ \theta_{\mathfrak{v}}"] & G/P_{i_n}, \end{tikzcd} \tag{$\mathcal{D}$}$$

where $\mathfrak{v} := \mathfrak{w}[n-1]$ and $f_i = f_{P_i} : G/B \to G/P_i$ is the canonical morphism. In fact, by Exercise 2.2.E.1, $Z_{\mathfrak{w}}$ is the fiber product $Z_{\mathfrak{v}} \times_{G/P_{i_n}} G/B$ via the above diagram $(\mathcal{D})$. The fibration $\psi_{\mathfrak{w}}$ admits a section $\sigma_{\mathfrak{w}} : Z_{\mathfrak{v}} \to Z_{\mathfrak{w}}$, $[p_1, \dots, p_{n-1}] \mapsto [p_1, \dots, p_{n-1}, 1]$.

For any $\lambda \in X^*(T)$, define the line bundle $\mathcal{L}_{\mathfrak{w}}(\lambda)$ on $Z_{\mathfrak{w}}$ as the pullback of the line bundle $\mathcal{L}(\lambda)$ on G/B via the morphism $\theta_{\mathfrak{w}}$. More generally, for any algebraic B-module V, let $\mathcal{L}_{\mathfrak{w}}(V)$ be the vector bundle on $Z_{\mathfrak{w}}$ obtained as the pullback of the homogeneous vector bundle $\mathcal{L}(V)$ on G/B via $\theta_{\mathfrak{w}}$.

We define the "boundary" $\partial Z_{\mathfrak{w}}$ of $Z_{\mathfrak{w}}$ by

$$\partial Z_{\mathfrak{w}} := \bigcup_{j=1}^{n} Z_{\mathfrak{w}(j)} \tag{9}$$

equipped with the closed reduced subscheme structure, where $\mathfrak{w}(j)$ is the subsequence $(s_{i_1}, \dots, \hat{s}_{i_j}, \dots, s_{i_n})$ and $Z_{\mathfrak{w}(j)}$ is identified with a divisor of $Z_{\mathfrak{w}}$ via the embedding $i_{\mathfrak{w}(j),\mathfrak{w}}$ of $Z_{\mathfrak{w}(j)}$. Then, $Z_{\mathfrak{w}(1)}, \dots, Z_{\mathfrak{w}(n)}$ are the irreducible components of $\partial Z_{\mathfrak{w}}$; they are nonsingular prime divisors with normal crossings in $Z_{\mathfrak{w}}$; and, as schemes,

$$Z_{\mathfrak{w}_J} \simeq \bigcap_{j \notin J} Z_{\mathfrak{w}(j)}, \text{ for any } J \subset \{1, \dots, n\}. \tag{10}$$

This can be proved by considering the pullback via the smooth morphism $\pi_{\mathfrak{w}} : P_{\mathfrak{w}} \to Z_{\mathfrak{w}}$. In particular,

$$\bigcap_{i=1}^{n} Z_{\mathfrak{w}(i)} = \{[1, \dots, 1]\}. \tag{11}$$

Moreover, $Z_{\mathfrak{w}}^o = Z_{\mathfrak{w}} \setminus \partial Z_{\mathfrak{w}}$.

If $\mathfrak{w}$ is a reduced sequence, i.e., if $s_{i_1} \cdots s_{i_n}$ is a reduced expression in W, then $\theta_{\mathfrak{w}}(Z_{\mathfrak{w}}) = X_w$, where $w := s_{i_1} \cdots s_{i_n}$. Moreover, $\theta_{\mathfrak{w}}(Z_{\mathfrak{w}}^o) = Bs_{i_1}B \cdots Bs_{i_n}B/B$ equals the Bruhat cell C_w, and $\theta_{\mathfrak{w}}$ restricts to an isomorphism $Z_{\mathfrak{w}}^o \to C_w$. Thus, $\theta_{\mathfrak{w}}$ is a desingularization of the Schubert variety X_w.

The following proposition is crucially used to show that $Z_{\mathfrak{w}}$ is split.

2.2.2 Proposition. *Let $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ be any sequence of simple reflections of W. Then, the canonical bundle $\omega_{Z_{\mathfrak{w}}}$ of $Z_{\mathfrak{w}}$ is given by*

$$\omega_{Z_{\mathfrak{w}}} \simeq \mathcal{O}_{Z_{\mathfrak{w}}}(-\partial Z_{\mathfrak{w}}) \otimes \mathcal{L}_{\mathfrak{w}}(-\rho). \tag{1}$$

In fact, as B-equivariant line bundles,

$$\omega_{Z_{\mathfrak{w}}} \simeq \mathcal{O}_{Z_{\mathfrak{w}}}(-\partial Z_{\mathfrak{w}}) \otimes \mathcal{L}_{\mathfrak{w}}(-\rho) \otimes \mathbf{k}_{-\rho}, \tag{2}$$

where $\mathbf{k}_{-\rho}$ is the trivial line bundle on $Z_{\mathfrak{w}}$ equipped with the B-equivariant line bundle structure coming from the representation $k_{-\rho}$ of B.

Proof. We prove (1) by induction on $\ell(\mathfrak{w}) := n$. For $n = 1$, $Z_{\mathfrak{w}} = P_{i_1}/B \simeq \mathbb{P}^1$ and hence $\omega_{Z_{\mathfrak{w}}} \simeq \mathcal{O}_{\mathbb{P}^1}(-2x_o)$, for any point $x_o \in \mathbb{P}^1$ (cf. [Har–77, Chap. II, Example 8.20.1]). It can easily be seen that $\mathcal{L}_{\mathfrak{w}}(-\rho) \simeq \mathcal{O}_{\mathbb{P}^1}(-x_o)$. So, (1) follows in this case.

Recall the $\mathbb{P}^1$-fibration $\psi_{\mathfrak{w}} : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}(n)}$ and the section $\sigma_{\mathfrak{w}}$ from 2.2.1. By induction, we assume the validity of (1) for $Z_{\mathfrak{w}(n)}$. Further, it is easy to see that the line bundle $\mathcal{L}_{\mathfrak{w}}(\rho)$ is of degree 1 along the fibers of $\psi_{\mathfrak{w}}$. Now, (1) follows from [Kum–02, Lemmas A.18 and A.16] by observing that

$$\sigma_{\mathfrak{w}}^* \mathcal{L}_{\mathfrak{w}}(\rho) = \mathcal{L}_{\mathfrak{w}(n)}(\rho). \tag{3}$$

Since $Z_{\mathfrak{w}}$ is projective, it is easy to see that for any two B-equivariant line bundles $\mathcal{L}_1$ and $\mathcal{L}_2$ such that $\mathcal{L}_1$ and $\mathcal{L}_2$ are isomorphic as nonequivariant line bundles, there exists a character $\lambda \in X^*(T)$ such that $\mathcal{L}_1 \simeq \mathcal{L}_2 \otimes \mathbf{k}_\lambda$ as B-equivariant line bundles. So, to prove (2), it suffices to show that the actions of T on the fibers of the two sides of (2) over $[1, \dots, 1]$ are given by the same character. The latter is easy to verify from the definition of the T-actions on the two sides of (2). □

As a corollary of the above proposition, we obtain the following.

2.2.3 Theorem. *With the notation as in Proposition 2.2.2, $Z_{\mathfrak{w}}$ is split by σ^{p-1}, where $\sigma \in H^0(Z_{\mathfrak{w}}, \omega_{Z_{\mathfrak{w}}}^{-1})$ vanishes on all the divisors $Z_{\mathfrak{w}(j)}$, $1 \le j \le n$.*

Thus, for any subsequence $\mathfrak{w}_J$ of $\mathfrak{w}$, $Z_{\mathfrak{w}_J}$ is compatibly split by σ^{p-1}, where $Z_{\mathfrak{w}_J}$ is identified with its image in $Z_{\mathfrak{w}}$ via the closed embedding $i_{\mathfrak{w}_J,\mathfrak{w}}$.

Proof. Let $\sigma' \in H^0\Big(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}(\partial Z_{\mathfrak{w}})\Big)$ be the canonical section [Har–77, Chap. II, §7], with divisor of zeros

$$(\sigma')_0 = \partial Z_{\mathfrak{w}}.$$

Since $H^0(G/B, \mathcal{L}(\rho)) \neq \{0\}$, by choosing a G-translate if needed, we get a section $\sigma'' \in H^0(G/B, \mathcal{L}(\rho))$ such that $\sigma''(1 \cdot B) \neq 0$. Now, by Proposition 2.2.2, the section $\sigma' \otimes \theta_{\mathfrak{w}}^* \sigma''$ provides a section of $\omega_{Z_{\mathfrak{w}}}^{-1}$. Thus, the first part of the theorem follows from Proposition 1.3.11 together with (2.2.1.11).

Let $\mathfrak{w}_J$ be the subsequence $\left(s_{i_{j_1}}, \dots, s_{i_{j_m}}\right)$. Then, by (2.2.1.10) and Proposition 1.2.1 we conclude that $Z_{\mathfrak{w}_J}$ is compatibly split. (We can also use Proposition 1.2.1 to conclude that the scheme-theoretic intersection of the right side of (2.2.1.10) is reduced.) □

2.2.4 Remark. Let $\sigma'' \in H^0(G/B, \mathcal{L}(\rho))$ be a section such that $\sigma''(1 \cdot B) \neq 0$ and let $(\sigma'')_0$ be the associated divisor of zeros. Let $D_{\sigma''}$ be the pullback divisor $\theta_{\mathfrak{w}}^*((\sigma'')_0)$. Then, by Proposition 1.4.12, Lemma 1.4.6, and the above proof, we see that $Z_{\mathfrak{w}}$ is $(p-1)D_{\sigma''}$-split compatibly $(p-1)D_{\sigma''}$-splitting all the $Z_{\mathfrak{w}_J}$.

As a consequence of Theorem 2.2.3, we get the following important result. We will study all possible splittings of G/P in the next section.

2.2.5 Theorem. *Let $P \subset G$ be any standard parabolic subgroup. Then, the flag variety G/P is $(p-1)\partial^- G/P$-split compatibly $(p-1)\partial^- G/P$-splitting all the Schubert subvarieties X_w^P (for any $w \in W$).*

Proof. We first consider the case $P = B$. Take a reduced expression $w_o = s_{i_1} \cdots s_{i_N}$. Let $\mathfrak{w}_o$ be the sequence $(s_{i_1}, \ldots, s_{i_N})$. Then, by 2.2.1, $\theta_{\mathfrak{w}_o} : Z_{\mathfrak{w}_o} \to G/B$ is a (surjective) birational morphism. Thus, by Zariski's main theorem [Har–77, Chap. III, Proof of Corollary 11.4],

$$\theta_{\mathfrak{w}_o *} \mathcal{O}_{Z_{\mathfrak{w}_o}} = \mathcal{O}_{G/B}. \tag{1}$$

Moreover, for any $w \in W$, there exists a subsequence $\mathfrak{w} = (s_{i_{j_1}}, \ldots, s_{i_{j_m}})$ of $\mathfrak{w}_o$ such that $w = s_{i_{j_1}} \cdots s_{i_{j_m}}$ is a reduced expression (as $w \leq w_o$). Hence, by 2.2.1,

$$\theta_{\mathfrak{w}_o}(Z_{\mathfrak{w}}) = X_w, \tag{2}$$

where $Z_{\mathfrak{w}}$ is identified with a subvariety of $Z_{\mathfrak{w}_o}$ via the embedding $i_{\mathfrak{w},\mathfrak{w}_o}$.

Thus, the theorem for $P = B$ follows from (1)–(2) together with Exercise 2.2.E.4, Theorem 2.2.3, Remark 2.2.4 and Lemma 1.4.5. Now, the theorem for an arbitrary P follows from that for B by using Lemma 1.4.5 again and observing the following: For the projection $f = f_P : G/B \to G/P$,

$$f_* \mathcal{O}_{G/B} = \mathcal{O}_{G/P}. \tag{3}$$

This follows easily since f is a locally trivial fibration with fiber the projective variety P/B. Moreover, f being a smooth morphism, $f^*(\partial^- G/P)$ is the reduced divisor $f^{-1}(\partial^- G/P) \subset \partial^- G/B$. In particular, by Remark 1.4.2(ii), G/B is $(p-1)\, f^*(\partial^- G/P)$-split. $\square$

2.2.6 Definition. Consider the flag variety $G/P \times G/Q$ for any standard parabolic subgroups P, Q. Then, it is a G-variety under the diagonal action of G. Moreover, there is a G-equivariant isomorphism

$$\xi : G \times_P (G/Q) \to G/P \times G/Q, \quad (g, g'Q) \mapsto (gP, gg'Q). \tag{1}$$

From this it is easy to see that any G-stable closed irreducible subset of $G/P \times G/Q$ is given by $\xi(G \times_P Y)$, for a closed irreducible P-stable subset Y of G/Q. In particular, taking $P = Q = B$, the closed irreducible G-stable subsets of $G/B \times G/B$ are precisely of the form $\{\xi(G \times_B X_w)\}_{w \in W}$. Denote, for any $w \in W$,

$$\mathcal{X}_w := \xi(G \times_B X_w)$$

equipped with the structure of a closed subvariety of $G/B \times G/B$. Then, any closed irreducible G-stable subset of $G/P \times G/Q$ is of the form $f_{P,Q}(\mathcal{X}_w)$, for some (but not necessarily unique) $w \in W$, where $f_{P,Q} : G/B \times G/B \to G/P \times G/Q$ is the projection. Observe that $\mathcal{X}_e$ is the diagonal in $G/B \times G/B$. We denote $\mathcal{X}_w^{P,Q} := f_{P,Q}(\mathcal{X}_w)$ again equipped with the structure of a closed subvariety of $G/P \times G/Q$. These are called *G-Schubert varieties* in $G/P \times G/Q$.

2.2.7 Corollary. *For any standard parabolic subgroups $P, Q \subset G$, the variety $G/P \times G/Q$ is split compatibly splitting all the G-Schubert varieties $\mathcal{X}_w^{P,Q}$, $w \in W$.*

Proof. We first consider the case $P = Q = B$. Fix a reduced decomposition $w_o = s_{i_1} \cdots s_{i_N}$ and let $\mathfrak{w}_o$ be the word $\left(s_{i_1}, \ldots, s_{i_N}\right)$. Let $Z'_{\mathfrak{w}_o}$ be the pullback principal B-bundle

$$\begin{tikzcd} Z'_{\mathfrak{w}_o} \arrow[r, "\theta'_{\mathfrak{w}_o}"] \arrow[d, "\pi'"'] & G \arrow[d, "\pi"] \\ Z_{\mathfrak{w}_o} \arrow[r, "\theta_{\mathfrak{w}_o}"'] & G/B. \end{tikzcd}$$

Since $\theta_{\mathfrak{w}_o}$ is birational (2.2.1), so is $\theta'_{\mathfrak{w}_o}$. Consider the morphism

$$\theta : Z \to G \times_B G/B, \ \theta(z', z) = (\theta'_{\mathfrak{w}_o}(z'), \theta_{\mathfrak{w}_o}(z)),$$

where $Z := Z'_{\mathfrak{w}_o} \times_B Z_{\mathfrak{w}_o}$. Then, θ is a birational (surjective) morphism and hence

(1) $$\theta_*(\mathcal{O}_Z) = \mathcal{O}_{G \times_B G/B}.$$

Further, it is easy to see that there is a canonical isomorphism

(2) $$Z \simeq Z_{(\mathfrak{w}_o, \mathfrak{w}_o)},$$

where $(\mathfrak{w}_o, \mathfrak{w}_o)$ is the word $\left(s_{i_1}, \ldots, s_{i_N}, s_{i_1}, \ldots, s_{i_N}\right)$.

By Theorem 2.2.3, Z is split compatibly splitting all the subvarieties $Z_{\mathfrak{w}_J}$, for any subsequence $\mathfrak{w}_J$ of $(\mathfrak{w}_o, \mathfrak{w}_o)$. Since any Schubert variety X_w is the image of a subvariety $Z_{\mathfrak{w}_J} \subset Z_{\mathfrak{w}_o}$ under $\theta_{\mathfrak{w}_o}$, for some subword $\mathfrak{w}_J$ of $\mathfrak{w}_o$ (2.2.5.2), we obtain from Lemma 1.1.8 that $G \times_B G/B$ is split compatibly splitting each $G \times_B X_w$. Now, using the isomorphism

$$\xi : G \times_B G/B \simeq G/B \times G/B \qquad (2.2.6.1),$$

we get the corollary for the case $P = Q = B$. The general case follows from this case by using Lemma 1.1.8 again. $\square$

2.2.E Exercises

For the following exercises, the characteristic of k is arbitrary (including 0).

(1*) With the notation as in 2.2.1, show that for any sequence $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$ of simple reflections, $Z_{\mathfrak{w}}$ is the fiber product $Z_{\mathfrak{w}[n-1]} \times_{G/P_{i_n}} G/B$ via the diagram $(\mathcal{D})$ of 2.2.1.

(2) Show that any opposite Schubert divisor $\widetilde{X}_{s_i}$ intersects the Schubert curve X_{s_i} transversally at the unique point $s_i B$, and intersects no other Schubert curve. Show

also that the Picard group $\mathrm{Pic}(G/B)$ is freely generated by the classes $[\widetilde{X}_{s_1}], \ldots, [\widetilde{X}_{s_\ell}]$ of the opposite Schubert divisors. Deduce the equality in $\mathrm{Pic}(G/B)$:

$$[\mathcal{L}] = (\mathcal{L} \cdot X_{s_1})[\widetilde{X}_{s_1}] + \cdots + (\mathcal{L} \cdot X_{s_\ell})[\widetilde{X}_{s_\ell}]$$

for the class of any line bundle $\mathcal{L}$ on G/B, where $(\mathcal{L} \cdot X_{s_i})$ denotes the degree of the restriction of $\mathcal{L}$ to $X_{s_i} \simeq \mathbb{P}^1$.

(3) Deduce from Exercise (2) the equality

$$[\mathcal{L}(\lambda)] = \langle \lambda, \check{\alpha}_1 \rangle [\widetilde{X}_{s_1}] + \cdots + \langle \lambda, \check{\alpha}_\ell \rangle [\widetilde{X}_{s_\ell}]$$

in $\mathrm{Pic}(G/B)$. Show that this group is isomorphic to $X^*(T)$ via $\lambda \mapsto [\mathcal{L}(\lambda)]$. If λ is dominant, show that any nonzero $\sigma \in H^0(G/B, \mathcal{L}(\lambda))$ satisfies

$$[(\sigma)_0] = \langle \lambda, \check{\alpha}_1 \rangle [\widetilde{X}_{s_1}] + \cdots + \langle \lambda, \check{\alpha}_\ell \rangle [\widetilde{X}_{s_\ell}],$$

where $(\sigma)_0$ is the divisor of zeroes of σ.

In fact, if σ is an eigenvector with respect to the action of B^-, then show that the above equality holds as divisors (not merely as divisor classes).

(4*) Let $P = P_I$ be a standard parabolic subgroup of G. Generalize the results of Exercises (2) and (3) to G/P.

Define $\rho_P \in X^*(T)$ by $\rho_P(\alpha_i^\vee) = 0$ if $\alpha_i \in I$, and $\rho_P(\alpha_i^\vee) = 1$ for all the other simple coroots. Then, show that ρ_P extends to a character of P. Observe that $\rho_B = \rho$.

Show further that the divisor of any B^--eigenvector in $H^0(G/P, \mathcal{L}^P(\rho_P))$ is precisely equal to $\partial^- G/P$, where $\mathcal{L}^P(\rho_P)$ is the line bundle on G/P associated to the character $-\rho_P$.

(5*) Let P be any standard parabolic subgroup of G. Show that, for the divisor $\partial^- G/P \subset G/P$,

$$\mathcal{O}_{G/P}(\partial^- G/P) \simeq \mathcal{L}^P(\rho_P),$$

where $\rho_P \in X^*(P)$ is defined above. Similarly,

$$\mathcal{O}_{G/P}(\partial G/P) \simeq \mathcal{L}^P(\rho_P).$$

(6*) Let $\lambda \in X^*(T)$ be a dominant weight. Show that the tensor product $H^0(G/B, \mathcal{L}(\lambda)) \otimes H^0(G/B, \mathcal{L}(-w_o\lambda))$ has a unique nonzero G-invariant vector (up to nonzero scalar multiples).

Hint: Use the Frobenius reciprocity [Jan–03, Part I, Proposition 3.4] and [Jan–03, Part I, Proposition 5.12(a)] to conclude that

$$\begin{aligned}\mathrm{Hom}_G\Big(&H^0(G/B, \mathcal{L}(\lambda))^*,\ H^0(G/B, \mathcal{L}(-w_o\lambda))\Big)\\ &\simeq \mathrm{Hom}_B\big(H^0(G/B, \mathcal{L}(\lambda))^*,\ k_{w_o\lambda}\big).\end{aligned}$$

(7*) For any G-module M and B-module V, show that there is a G-module isomorphism

$$\xi : M \otimes H^0(G/B, \mathcal{L}(V)) \to H^0(G/B, \mathcal{L}(M \otimes V)),$$

$$\xi(m \otimes \sigma)(gB) = [g, g^{-1}m \otimes \bar{\sigma}(g)],$$

for $m \in M$ and $\sigma \in H^0(G/B, \mathcal{L}(V))$, where $\sigma(gB) = [g, \bar{\sigma}(g)]$.

2.3 Some more splittings of G/B and $G/B \times G/B$

We give a self-contained and an entirely different proof of Theorem 2.2.5 in this section. In fact, we determine all possible splittings of G/B.

We continue to follow the same notation as in the beginning of this chapter. In particular, G is a connected, simply-connected, semisimple algebraic group over an algebraically closed field k of characteristic $p > 0$. But we abbreviate G/B, resp. G/P, by X, resp. X^P, and $H^0(G/B, \mathcal{L}(\lambda))$ by $H^0(\lambda)$ in the rest of this chapter.

The G-module $H^0((p-1)\rho)$ is called the *Steinberg module*, denoted by St. Let f_+, resp. f_-, be a nonzero highest, resp. lowest, weight vector of St; then the weight of $f_\pm$ is $\pm(p-1)\rho$.

Observe that the multiplication of sections gives rise to the morphism of G-modules

$$m : H^0(\lambda) \otimes H^0(\mu) \to H^0(\lambda + \mu),$$

for $\lambda, \mu \in X^*(T)^+$. In particular, we have a map

$$m : \mathrm{St} \otimes \mathrm{St} \to H^0(2(p-1)\rho).$$

Furthermore,

$$H^0(2(p-1)\rho) = H^0(X, \omega_X^{1-p})$$

by (2.1.8). Thus, the splittings of X are elements of this space, by Theorem 1.3.8.

2.3.1 Theorem. *The section $m(f_+ \otimes f_-)$ provides, up to a nonzero scalar multiple, a $\big((p-1)(\partial^- X + \partial X)\big)$-splitting of X.*

Further, this splitting compatibly $(p-1)\partial^- X$-splits all the Schubert subvarieties X_w and $(p-1)\partial X$-splits all the opposite Schubert varieties $\widetilde{X}_w$ $(w \in W)$.

Proof. Let $f := f_+ \otimes f_-$. Then, the section $m(f)$ is given by

(1) $$m(f)(gB) = \big(g, f_+(gv_+) \cdot f_-(gv_+)\big) \mod B,$$

for $gB \in X$, where v_+ is a highest weight vector of the Weyl module $V((p-1)\rho) := \mathrm{St}^*$.

For $g \in U^-$, (1) simplifies to

(2) $$m(f)(gB) = \big(g, f_+(gv_+)\big) \mod B.$$

Order the positive roots $\Delta^+ = \{\beta_1, \dots, \beta_N\}$ and choose isomorphisms $\varepsilon_i = \varepsilon_{-\beta_i} : \mathbb{G}_a \to U_{-\beta_i}$ as in Section 2.1. Then, there exists a root vector $f_{\beta_i} \in \mathfrak{g}_{-\beta_i}$ (called a *Chevalley generator*) such that for any algebraic representation V of G, $v \in V$ and $z \in \mathbb{G}_a$,

$$\varepsilon_i(z)\, v = \sum_{m \geq 0} z^m \big(f_{\beta_i}^{(m)} \cdot v\big), \tag{3}$$

where $f_{\beta_i}^{(m)}$ denotes the m-th divided power of f_{β_i} (cf. [Jan–03, Part I, §§7.8–7.12]).

Recall the variety isomorphism (2.1.3):

$$\varepsilon : k^N \xrightarrow[\sim]{} U^-, \ \ \varepsilon(z_1, \dots, z_N) = \varepsilon_1(z_1) \cdots \varepsilon_N(z_N).$$

Thus, under the identification ε, $m(f)_{|(U^- B/B)}$ can be written as

$$\begin{aligned} &m(f)(z_1, \dots, z_N) \\ &= \Bigg(\varepsilon(z_1, \dots, z_N),\, f_+\Bigg(\sum_{(k_1,\dots,k_N) \in \mathbb{Z}_+^N} z_1^{k_1} \cdots z_N^{k_N}\, f_{\beta_1}^{(k_1)} \cdots f_{\beta_N}^{(k_N)} v_+\Bigg)\Bigg) \quad \text{mod } B. \end{aligned}$$

Thus, trivializing the line bundle $\mathcal{L}(2(p-1)\rho)$ over $U^- B/B$, the section $m(f)$ corresponds to the function

$$(z_1, \dots, z_N) \mapsto \sum_{(k_1,\dots,k_N) \in \mathbb{Z}_+^N} z_1^{k_1} \cdots z_N^{k_N}\, f_+\Big(f_{\beta_1}^{(k_1)} \cdots f_{\beta_N}^{(k_N)} v_+\Big).$$

By Exercise 2.3.E.2,

$$f_{\beta_1}^{(p-1)} \cdots f_{\beta_N}^{(p-1)} v_+ = v_-,$$

up to a nonzero scalar multiple, where v_- is a lowest weight vector of $V((p-1)\rho)$. Thus, the coefficient of $z_1^{p-1} \cdots z_N^{p-1}$ in the above function is nonzero and hence, by Theorem 1.3.8, $m(f)$ provides a splitting of X (up to a scalar multiple).

We next calculate the divisor Z of the zeroes of the section $m(f)$.

Under the multiplication map

$$\begin{aligned} &\varphi : H^0(\rho)^{\otimes p-1} \to H^0((p-1)\rho) = \mathrm{St}, \\ &\varphi\big(\bar{v}_+^{\otimes p-1}\big) = f_+ \quad \text{and} \quad \varphi\big(\bar{v}_-^{\otimes p-1}\big) = f_-, \end{aligned} \tag{4}$$

for a highest, resp. lowest, weight vector $\bar{v}_+$, resp. $\bar{v}_-$, of $H^0(\rho)$. From (1),

$$m(f)_{|\partial^- X \cup \partial X} \equiv 0.$$

Moreover, from (4),

$$m(f) = \sigma^{p-1}, \tag{5}$$

for a section $\sigma \in H^0(2\rho)$.

Hence, the divisor $Z - (p-1)(\partial^- X + \partial X)$ is effective. Thus, by Theorem 1.4.10, $m(f)$ provides a $(p-1)(\partial^- X + \partial X)$-splitting of X.

We now prove the compatible splitting of X_w. By Proposition 1.3.11, the zero scheme of σ is compatibly split and thus reduced. In particular, by Proposition 1.2.1, each Schubert divisor $X_{w_o s_i}$ (for any simple reflection s_i) is compatibly split. Now, let X_w be a Schubert variety of codimension ≥ 2. By [BGG–75, Lemma 10.3] there exist distinct v_1, v_2 in W such that $w < v_1$, $w < v_2$ and $\ell(v_1) = \ell(v_2) = \ell(w) + 1$. It follows that X_w set-theoretically is an irreducible component X'_w of the intersection $X_{v_1} \cap X_{v_2}$. Thus, by decreasing induction on $\ell(w)$ using Proposition 1.2.1 again, X'_w is compatibly split and hence reduced. Since X_w is reduced by definition, the schemes X_w and X'_w coincide. This proves the compatible splitting of X_w. Since no X_w is contained in Supp $\partial^- X$, each X_w is compatibly $(p-1)\partial^- X$-split. The proof for $\widetilde{X}_w$ is the same. □

We now get the following parabolic analogue of Theorem 2.3.1, which is a slight strengthening of Theorem 2.2.5.

2.3.2 Theorem. *For any standard parabolic subgroup P of G, the flag variety X^P is $\big((p-1)(\partial^- X^P + \partial X^P)\big)$-split, compatibly $(p-1)\partial^- X^P$-splitting all the Schubert varieties X^P_w and $(p-1)\partial X^P$-splitting all the opposite Schubert varieties $\widetilde{X}^P_w$ (for any $w \in W$).*

Proof. The theorem follows by applying Lemma 1.4.5 and Theorem 2.3.1 to the morphism $X \to X^P$ (see the last part of the proof of Theorem 2.2.5). □

As an immediate consequence of Theorem 2.3.2 and Proposition 1.2.1, we get the following.

2.3.3 Corollary. *Let $P \subset G$ be a standard parabolic subgroup. Let $\{Y_i\}_{1 \leq i \leq m}$ and $\{Z_j\}_{1 \leq j \leq n}$ be any collections of Schubert varieties and opposite Schubert varieties in X^P, i.e., $\{Y_i, Z_j; 1 \leq i \leq m, 1 \leq j \leq n\} \subset \big\{X^P_v, \widetilde{X}^P_w; v, w \in W\big\}$, and let $Y = Y_1 \cup \cdots \cup Y_m$, $Z = Z_1 \cup \cdots \cup Z_n$ be their unions equipped with the reduced scheme structures. Then, the scheme-theoretic intersection $Y \cap Z$ is reduced.*

This corollary remains true in characteristic 0 *by Corollary 1.6.6.*

2.3.4 Definition. Recall from [Jan–03, Part II, §§3.18 and 10.1] that the Steinberg module St is irreducible and self-dual (see also Exercise 2.3.E.3). Fixing a G-module isomorphism $\bar{\chi}\colon \mathrm{St} \to \mathrm{St}^*$, which is unique up to a scalar multiple, we get a G-invariant nondegenerate bilinear form

$$\chi : \mathrm{St} \otimes \mathrm{St} \to k, \quad \chi(v \otimes w) = \bar{\chi}(v)w. \tag{1}$$

As another corollary of Theorem 2.3.2, we determine all possible splittings of X. Analogous splittings of X^P will be given in Chapter 5.

2.3.5 Corollary. *For any $f \in \mathrm{St} \otimes \mathrm{St}$, $m(f)$ splits X up to a nonzero scalar multiple iff $\chi(f) \neq 0$.*

Proof. Recall from Section 1.3 the natural isomorphism

$$\mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X) \simeq F_*H^0(X, \omega_X^{1-p}). \tag{1}$$

This yields a natural k-linear isomorphism

$$\mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)^{[-1]} \simeq H^0(2(p-1)\rho), \tag{2}$$

where, for a k-vector space V, $V^{[-1]}$ denotes the k-vector space with the same underlying abelian group as V, whereas the k-linear structure is twisted as

$$z \odot v = z^{1/p} v, \quad \text{for } z \in k \text{ and } v \in V. \tag{3}$$

Since the isomorphism (1) is natural, it is easy to see that under the canonical G-structures on $F_*\mathcal{O}_X$ and $\mathcal{O}_X$, the isomorphism $R^{[-1]} \simeq H^0(2(p-1)\rho)$ (as in (2)) is G-equivariant, where $R := \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$. Define the k-linear G-module map

$$e : R^{[-1]} \to k, \; e(\sigma) = \sigma(1)^p, \quad \text{for } \sigma \in R,$$

where $\sigma(1) \in k$ is the constant function $\sigma_{|\mathcal{O}_X} : \mathcal{O}_X \to \mathcal{O}_X$. (This is the p-th power of the evaluation map ϵ considered in Section 1.3.) By the definition of splitting, any $\sigma \in R$ splits X up to a nonzero scalar multiple iff $e(\sigma) \neq 0$.

Under the above identification (2), composing m with e, we get a k-linear G-module map

$$\chi' : \mathrm{St} \otimes \mathrm{St} \to k.$$

Thus, $\chi' = z_o\chi$, for some $z_o \in k$. Moreover, by Theorem 2.3.1, $z_o \neq 0$. (An alternative proof for the surjectivity of χ' is outlined in Exercise 2.3.E.1.) This proves the corollary. □

2.3.6 Remark. As we will see in the next chapter, the map m is surjective (Theorem 3.1.2(c)). Thus, all possible splittings of X are provided by Corollary 2.3.5.

2.3.7 Proposition. *Let $f \in \mathrm{St} \otimes \mathrm{St}$ be such that $\chi(f) \neq 0$ and $m(f) = \sigma^{p-1}$ for a section $\sigma \in H^0(2\rho)$ (e.g., $f = \bar{v}_+^{p-1} \otimes \bar{v}_-^{p-1}$, where $\bar{v}_\pm$ is as in (2.3.1.4)). Then, the splitting of X provided by $m(f)$ up to a nonzero scalar multiple (Corollary 2.3.5) is, in fact, a $(p-1)Z(\sigma)$-splitting, where $Z(\sigma)$ is the divisor of zeroes of σ.*

Moreover, $Z(\sigma)$ is compatibly split. In particular, it is a reduced scheme.

Proof. By Corollary 2.3.5, $m(f)$ splits X. By Theorem 1.4.10, $m(f)$ provides a $(p-1)Z(\sigma)$-splitting and, by Proposition 1.3.11, $Z(\sigma)$ is compatibly split. □

Recall the definition of the map ξ from (2.2.6.1). As a corollary of Proposition 2.3.7, we get the following.

2.3.8 Theorem. *The product variety $\mathcal{X} := X \times X$ is $(p-1)D$-split, where D is the reduced divisor $\xi(G \times_B \partial X) \cup (\partial X \times X) \cup (X \times \partial^- X)$. Further, this splitting compatibly splits the reduced subscheme D.*

Moreover, all the G-Schubert varieties $\mathcal{X}_w$, $w \in W$, are compatibly $(p-1)D'$-split, where D' is the reduced divisor $(\partial X \times X) \cup (X \times \partial^- X)$.

In particular, $\mathcal{X}$ is $(X \times \partial^- X)$-split compatibly with the diagonal Δ.

Proof. Let G^2 be the product group $G \times G$ with the Borel subgroup $B^2 := B \times B$. We apply Proposition 2.3.7 with G replaced by G^2. The Steinberg module St^2 for G^2 is given by

$$\mathrm{St}^2 = H^0\big(\mathcal{X}, \mathcal{L}((p-1)\,\rho \boxtimes (p-1)\rho)\big) \simeq \mathrm{St} \otimes \mathrm{St},$$

where $\mathcal{L}(\lambda \boxtimes \mu)$ denotes the line bundle $\mathcal{L}(\lambda) \boxtimes \mathcal{L}(\mu)$ on $\mathcal{X}$.

By Exercise 2.2.E.6, St^2 has a unique G-invariant nonzero vector f_o up to scalar multiples. In fact, f_o is the $(p-1)$-th power of a G-invariant $\bar{f}_o \in H^0(\mathcal{X}, \mathcal{L}(\rho \boxtimes \rho))$. Also, consider $f_+ \otimes f_- \in \mathrm{St}^2$, where $f_\pm$ is defined just before Theorem 2.3.1. Then, it is easy to see that

$$\chi^2(f_o \otimes (f_+ \otimes f_-)) \neq 0, \tag{1}$$

where $\chi^2 : \mathrm{St}^2 \otimes \mathrm{St}^2 \to k$ is the G^2-invariant pairing. In particular, by Corollary 2.3.5, σ splits $\mathcal{X}$, where $\sigma := m^2(f_o \otimes (f_+ \otimes f_-))$ and

$$m^2 : H^0\big(\mathcal{X}, \mathcal{L}((p-1)\rho \boxtimes (p-1)\rho)\big)^{\otimes 2} \to H^0\big(\mathcal{X}, \mathcal{L}(2(p-1)\rho \boxtimes 2(p-1)\rho)\big)$$

is the standard multiplication map. Also, by (2.3.1.4), f_+, resp. f_-, is a $(p-1)$-th power of a section $\bar{v}_+$, resp. $\bar{v}_-$, $\in H^0(\rho)$.

We next calculate the zero set of σ: By an analogue of (2.3.1.1), the zero set

$$Z(f_+ \otimes f_-) = (\partial X \times X) \cup (X \times \partial^- X) \text{ set-theoretically.} \tag{2}$$

Also, clearly $Z(f_o)$ is a G-stable subset of $\mathcal{X}$. We next show that $(1, \dot{w}) \in Z(f_o)$ for any $w \neq w_o$, $\dot{w}$ being a representative of w in $N(T)$:

For any $t \in T$,

$$\begin{aligned}(t \cdot f_o)(1, \dot{w}) &= t \cdot (f_o(1, \dot{w})) \\ &= e^{-(p-1)(\rho + w\rho)}(t)\, f_o(1, \dot{w}).\end{aligned} \tag{3}$$

But since f_o is G-invariant, we get

$$e^{-(p-1)(\rho+w\rho)}(t)\, f_o(1, \dot{w}) = f_o(1, \dot{w}), \quad \text{for all } t \in T. \tag{4}$$

Since the W-isotropy of ρ is trivial and $w_o\rho = -\rho$, (4) forces

$$f_o(1, \dot{w}) = 0, \text{ for all } w \neq w_o.$$

The G-orbit closure of $(1, \dot{w})$ being $\mathcal{X}_w$, we get that

$$(5) \qquad Z(f_o) \supseteq \bigcup_{w \neq w_o} \mathcal{X}_w.$$

Moreover, $f_o(1, \dot{w}_o) \neq 0$. Otherwise, $Z(f_o) \supseteq \mathcal{X}_{w_o} = \mathcal{X}$, forcing $f_o \equiv 0$, a contradiction. Thus, we have equality:

$$(6) \qquad Z(f_o) = \bigcup_{w \neq w_o} \mathcal{X}_w = \xi\big(G \times_B \partial X\big) \quad \text{(set-theoretically).}$$

Combining (2) and (6), we get

$$(7) \qquad Z(\sigma) = D \quad \text{(set-theoretically).}$$

Since $\sigma = \sigma_o^{p-1}$ for a section $\sigma_o \in H^0\big(\mathcal{X}, \omega_{\mathcal{X}}^{-1}\big)$, and σ splits $\mathcal{X}$, by Proposition 1.3.11, the zero scheme $Z(\sigma_o)$ of σ_o is reduced. Thus, as divisors,

$$(\sigma)_0 = (p-1)D.$$

Now, by Theorem 1.4.10, σ provides a $(p-1)D$-splitting of $\mathcal{X}$ and, by Proposition 1.3.11, σ compatibly splits D. Thus, by Proposition 1.2.1, each $\mathcal{X}_w$ is compatibly split (see the proof of Theorem 2.3.1). Finally, since no $\mathcal{X}_w$ is contained in D', $\mathcal{X}_w$'s are compatibly $(p-1)D'$-split. □

As an immediate corollary of Theorem 2.3.8 and Lemma 1.4.5, we get the following.

2.3.9 Corollary. *For any standard parabolic subgroups $P, Q \subset G$, the variety $X^P \times X^Q$ is $(p-1)((\partial X^P \times X^Q) \cup (X^P \times \partial^- X^Q))$-split, compatibly $(p-1)((\partial X^P \times X^Q) \cup (X^P \times \partial^- X^Q))$-splitting all the G-Schubert varieties $\mathcal{X}_w^{P,Q}$, $w \in W$.*

To study the defining ideal of the Schubert varieties in Section 3.5, we will need the following.

2.3.10 Theorem. *Let $P \subset G$ be any standard parabolic subgroup. Then, for any $n \geq 1$, $(X^P)^n$ is $(p-1)((X^P)^{n-1} \times \partial^- X^P)$-split, compatibly $(p-1)((X^P)^{n-1} \times \partial^- X^P)$-splitting all the following subvarieties:*

$$\big\{X_w^P \times (X^P)^{n-1}, (X^P)^q \times \mathcal{X}_w^{P,P} \times (X^P)^{n-2-q};\ w \in W, 0 \leq q \leq n-2\big\}.$$

Proof. We first consider the case $P = B$ and use Proposition 2.3.7 for G replaced by G^n. The idea of the proof is similar to that of the proof of Theorem 2.3.8, so we will be brief. Similar to the pairing χ (see (2.3.4.1)), consider the G^n-invariant pairing

$$\chi_n : \mathrm{St}^{\otimes n} \otimes \mathrm{St}^{\otimes n} \to k, \quad \text{defined by}$$
$$\chi_n(f_1 \otimes \cdots \otimes f_n, g_1 \otimes \cdots \otimes g_n) = \chi(f_1 \otimes g_1) \cdots \chi(f_n \otimes g_n).$$

Similarly, consider the multiplication map

$$m_n : \mathrm{St}^{\otimes n} \otimes \mathrm{St}^{\otimes n} := H^0\big(X^n, \mathcal{L}((p-1)\rho \boxtimes \cdots \boxtimes (p-1)\rho)\big)^{\otimes 2} \longrightarrow H^0\big(X^n, \mathcal{L}(2(p-1)\rho \boxtimes \cdots \boxtimes 2(p-1)\rho)\big).$$

Now, we define an element $\theta_n \in \mathrm{St}^{\otimes n} \otimes \mathrm{St}^{\otimes n}$. Its definition depends upon whether n is even or odd. If $n = 2m$, define the element

$$\theta_n = f_o^{\otimes m} \otimes \big(f_+ \otimes f_o^{\otimes m-1} \otimes f_-\big) \in \mathrm{St}^{\otimes n} \otimes \mathrm{St}^{\otimes n},$$

where f_o, $f_\pm$ are as in the proof of Theorem 2.3.8. If $n = 2m+1$, define

$$\theta_n = (f_o^{\otimes m} \otimes f_-) \otimes (f_+ \otimes f_o^{\otimes m}) \in \mathrm{St}^{\otimes n} \otimes \mathrm{St}^{\otimes n}.$$

In either case, the zero set of $m_n(\theta_n)$ is equal to the following closed subset of X^n:

$$Z_n := \Bigg(\bigcup_{\substack{1\le i\le \ell,\\ 0\le q\le n-2}} X^q \times \mathcal{X}_{w_o s_i} \times X^{n-2-q}\Bigg) \cup (X^{n-1} \times \partial^- X) \cup (\partial X \times X^{n-1}).$$

It is easy to see that $\chi_n(\theta_n) \neq 0$ and, moreover, $m_n(\theta_n) = \bar\theta_n^{p-1}$ for some section $\bar\theta_n \in H^0\big(X^n, \mathcal{L}(2\rho \boxtimes \cdots \boxtimes 2\rho)\big)$. Thus, by Proposition 2.3.7, $m_n(\theta_n)$ provides a $(p-1)\,Z(\bar\theta_n)$-splitting of X^n, where $Z(\bar\theta_n)$ is the zero scheme of the section $\bar\theta_n$. Moreover, $Z(\bar\theta_n)$ is a reduced subscheme of X^n. Thus, $m_n(\theta_n)$ provides a $(p-1)\big(X^{n-1} \times \partial^- X\big)$-splitting of X^n compatibly $(p-1)\big(X^{n-1} \times \partial^- X\big)$-splitting all the (reduced) subvarieties $X^q \times \mathcal{X}_w \times X^{n-2-q}$, and $X_w \times X^{n-1}$, $w \in W$.

Now, the case of an arbitrary P follows from that of B, by considering the morphism $X^n \to (X^P)^n$ and applying Lemma 1.4.5. □

2.3.E Exercises

(1*) Use the following result (∗) of [And–80a], [Hab–80] to give an alternative proof for the surjectivity of the map $\chi' : \mathrm{St} \otimes \mathrm{St} \to k$ defined in the proof of Corollary 2.3.5.

As G-equivariant $\mathcal{O}_X$-modules, for $X = G/B$,

$$F_*\big(\mathcal{L}((p-1)\rho)\big) \simeq \mathrm{St} \otimes_k \mathcal{O}_X. \tag{$*$}$$

Hint: Consider the maps

$$\mathrm{St} = \mathrm{Hom}_{\mathcal{O}_X}\big(\mathcal{O}_X, \mathcal{L}((p-1)\rho)\big) \xrightarrow{\beta_1} \mathrm{Hom}_{\mathcal{O}_X}\big(F_*\mathcal{O}_X, F_*(\mathcal{L}((p-1)\rho))\big),\ \ \sigma \mapsto F_*\sigma,$$

and

$$\mathrm{St}^* \underset{\sim}{\xrightarrow{\beta_2}} \mathrm{Hom}_{\mathcal{O}_X}\big(F_*(\mathcal{L}((p-1)\rho)), \mathcal{O}_X\big) \simeq \mathrm{Hom}_{\mathcal{O}_X}(\mathrm{St} \otimes_k \mathcal{O}_X, \mathcal{O}_X).$$

Now, define the k-linear G-module map

$$\hat{\chi} : \mathrm{St} \otimes \mathrm{St}^* \to R^{[-1]}, \; f \otimes v \mapsto \beta_2(v) \circ \beta_1(f),$$

where $R := \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$ is as in the proof of Corollary 2.3.5. Show that for any nonzero $f \in \mathrm{St}$, there exists $v \in \mathrm{St}^*$ such that $\hat{\chi}(f \otimes v)_{|\mathcal{O}_X}$ is nonzero. Prove the uniqueness of $\hat{\chi}$ by using (2.3.5.2) and the Frobenius reciprocity. Finally, use the uniqueness of $\hat{\chi}$ to show that, under the identification (2.3.5.2), $\hat{\chi} = m$ up to a nonzero scalar multiple, where m is the multiplication map.

(2*) Let $w_o = s_{i_N} \cdots s_{i_1}$ be a reduced decomposition of the longest element of the Weyl group W. This gives an enumeration of the positive roots $\{\beta_1, \ldots, \beta_N\}$, where $\beta_j := s_{i_N} \cdots s_{i_{j+1}} \alpha_{i_j}$. Let f_β be a Chevalley generator of the negative root space $\mathfrak{g}_{-\beta}$ (see the proof of Theorem 2.3.1). Then, for any $m \geq 1$, show that in the Weyl module $V(m\rho) := H^0(m\rho)^*$,

$$v_- = f_{\beta_1}^{(m)} \cdots f_{\beta_N}^{(m)} \cdot v_+$$

up to a nonzero scalar multiple, where v_+, resp. v_-, is a highest, resp. lowest, weight vector of $V(m\rho)$.

Prove further that $f_{\beta_1}^{(m)} \cdots f_{\beta_N}^{(m)} \cdot v_+$ is independent of the ordering of positive roots for $m = p-1$.

Hint: Show by induction on j that (up to a nonzero scalar multiple)

$$(s_{i_j} \cdots s_{i_1}) v_+ = f^{(m)}_{s_{i_j} \cdots s_{i_2} \alpha_{i_1}} \cdot f^{(m)}_{s_{i_j} \cdots s_{i_3} \alpha_{i_2}} \cdots f^{(m)}_{\alpha_{i_j}} v_+.$$

(3*) Using Frobenius splitting of G/B, show that the Steinberg module $\mathrm{St} := H^0((p-1)\rho)$ is an irreducible, self-dual G-module.

More generally, show that $H^0((p^r-1)\rho)$ is an irreducible, self-dual G-module, for any $r \geq 1$.

Hint: Use the identification for any smooth variety X (as in Section 1.3): $\mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X) \simeq F_* H^0(X, \omega_X^{1-p})$, and the multiplication map $m : \mathrm{St} \otimes \mathrm{St} \to H^0\big(G/B, \omega_{G/B}^{1-p}\big)$ to get a G-invariant pairing $\chi : \mathrm{St} \otimes \mathrm{St} \to k$. Show that this pairing is nondegenerate by showing that, for a nonzero highest, resp. lowest, weight vector f_+, resp. f_-, $m(f_+ \otimes f_-)$ splits G/B. Finally, to prove the irreducibility of St, use the isomorphism (induced by χ) $\mathrm{St} \underset{\sim}{\to} \mathrm{St}^*$ and the fact that St^* is generated, as a G-module, by its highest weight vector (cf. [Jan–03, Part II, Lemma 2.13(b)]).

2.C. Comments

The most important contributions to the results in this chapter are due to Mehta, Ramanan and Ramanathan.

The Bott–Samelson–Demazure–Hansen varieties were first introduced by Bott–Samelson [BoSa–58] in a differential geometric and topological context; Demazure [Dem–74] and Hansen [Han–73] adapted the construction in algebro-geometric situation and used it to desingularize the Schubert varieties and to determine the Chow group of G/B. Proposition 2.2.2 for the case when $\mathfrak{w}$ comes from a reduced decomposition of the longest element of W is due to Mehta–Ramanathan [MeRa–85]. This proposition, in general, is due to Ramanathan [Ram–85] (though he only determined it nonequivariantly). Theorem 2.2.3 is due to Mehta–Ramanathan [MeRa–85]. The splitting of G/P compatibly splitting the Schubert subvarieties was proved by Mehta–Ramanathan [MeRa–85]. The $(p-1)\partial^- G/P$ splitting as in Theorem 2.2.5 is due to Ramanan–Ramanathan [RaRa–85]. Corollary 2.2.7 is due to Mehta–Ramanathan [MeRa–88], though the proof given here is slightly different and is due to Mathieu [Mat–89b]. A special case of Corollary 2.2.7 was earlier obtained by Ramanathan [Ram–87]. For Exercise 2.2.E.7, see, e.g., [Jan–03, Part I, Proposition 3.6].

Theorems 2.3.1 and 2.3.2, apart from providing an explicit splitting, are slightly stronger than Theorem 2.2.5. In their present form they are proved in [Ram–87]. However, the very simple and direct proof we give here is new (as far as we know). Corollary 2.3.3 is due to Ramanathan [Ram–85]. This was conjectured earlier by Lakshmibai–Seshadri [LaSe–86] and proved by them for classical groups. For arbitrary groups but for "special" Schubert varieties it was proved by Kempf [Kem–76a]. Corollary 2.3.5 and Proposition 2.3.7 are due to Lauritzen–Thomsen [LaTh–97] (see also the works [Kan–94b, 95]). Both of these results play a crucial role in the proofs of Theorems 2.3.8 and 2.3.10. Theorem 2.3.8 and its Corollary 2.3.9 in their full strength do not seem to be available in the literature. However, as mentioned above, slightly weaker results are available in [Ram–87] and [MeRa–88]. Theorem 2.3.10 for an arbitrary n is essentially contained in [LaTh–97]; the case $n = 3$ is due to Ramanathan [Ram–87] (see also [InMe–94a] for a weaker result). A version of Theorem 2.3.10 can also be found in [Bez–95].

The alternative proof of the surjectivity of χ' outlined in Exercise 2.3.E.1 is due to Lauritzen–Thomsen [LaTh–97]. Of course, the fact that St is a self-dual and irreducible G-module is well known, and has played a fundamental role in several important problems (cf. [Jan–03, Part II, Chap. 10]). However, the proof outlined in Exercise 2.3.E.3 using Frobenius splitting is due to Mehta–Venkataramana [MeVe–96].

Recently most of the results of this chapter have been obtained purely algebraically using quantized enveloping algebras at a p-th root of unity by Kumar–Littelmann [KuLi–02].

It may be mentioned that Mehta–Ramadas [MeRam–96] proved that for a generic irreducible projective curve X of genus g over an algebraically closed field of characteristic $p \geq 5$, the moduli space of rank-2 parabolic bundles on X is split.

For any line bundle $\mathcal{L}$ on a smooth toric variety, Thomsen [Tho-00a] has proved that the direct image $F_*(\mathcal{L})$ is a direct sum of some explicitly determined line bundles (see also [Bog-98] for some generalizations of this result).

It is conjectured in [LMP–98] that for any parabolic subgroup P of G, $G/P \times G/P$

admits a splitting which has the "maximum" multiplicity, $(p-1)\dim G/P$, along the diagonal Δ. By Exercise 1.3.E.12, this conjecture is equivalent to the conjecture that the blowing-up of $G/P \times G/P$ along its diagonal is split compatibly with the exceptional divisor. This conjecture implies Wahl's conjecture in characteristic $p > 0$, which asserts that the restriction map

$$H^0\big(G/P \times G/P, \mathcal{I}_\Delta \otimes (\mathcal{L}^P(\lambda) \boxtimes \mathcal{L}^P(\mu))\big) \to H^0\big(G/P, \Omega^1_{G/P} \otimes \mathcal{L}^P(\lambda + \mu)\big)$$

is surjective for any ample line bundles $\mathcal{L}^P(\lambda)$ and $\mathcal{L}^P(\mu)$ on G/P, where $\mathcal{I}_\Delta$ is the ideal sheaf of Δ in $G/P \times G/P$ and $\Omega^1_{G/P}$ is the sheaf of differential 1-forms on G/P. This conjecture in characteristic 0 for an arbitrary G/P was proved by Kumar [Kum–92]; and in an arbitrary characteristic by Mehta–Parameswaran [MePa–97] for G/P the Grassmannians.

Also, it is an open question whether the Schubert varieties X_w^P are diagonally split.

Further, it is not known if all the homogeneous spaces G/H are split, where G is any connected linear algebraic group and H is a closed connected subgroup which is reduced as a subscheme.

Chapter 3

Cohomology and Geometry of Schubert Varieties

Introduction

The main aim of this chapter is to derive various algebro-geometric and representation-theoretic consequences of the Frobenius splitting results proved in the last chapter.

By the general cohomological properties of a D-split projective variety Y, where D is an ample divisor on Y, and compatibly D-split subvariety $Z \subset Y$ proved in Chapter 1, together with the Frobenius splitting properties of the flag varieties $X^P := G/P$ and their double analogues $\mathcal{X}^{P,Q} := G/P \times G/Q$ obtained in Chapter 2, we immediately obtain the following fundamental result. For dominant characters λ, μ of P and $w \in W$, one has the cohomology vanishing: $H^i\big(X_w^P, \mathcal{L}_w^P(\lambda)\big) = H^i\big(\mathcal{X}_w^{P,Q}, \mathcal{L}_w^{P,Q}(\lambda \boxtimes \mu)\big) = 0$ for all $i > 0$. Moreover, the restriction maps

$$H^0\big(G/P, \mathcal{L}^P(\lambda)\big) \to H^0\big(X_w^P, \mathcal{L}_w^P(\lambda)\big) \text{ and}$$
$$H^0\big(G/P \times G/Q, \mathcal{L}^{P,Q}(\lambda \boxtimes \mu)\big) \to H^0\big(\mathcal{X}_w^{P,Q}, \mathcal{L}_w^{P,Q}(\lambda \boxtimes \mu)\big)$$

are both surjective (Theorems 3.1.1 and 3.1.2). We also prove that for any sequence of simple reflections $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$ and any $1 \le q \le r \le n$ such that the subsequence $(s_{i_q}, \ldots, s_{i_r})$ is reduced, the cohomology $H^i\big(Z_{\mathfrak{w}}, \mathcal{L}\big(\sum_{j=q}^{r} -Z_{\mathfrak{w}(j)}\big)\big) = 0$ for all $i > 0$ and any globally generated line bundle $\mathcal{L}$ on the BSDH variety $Z_{\mathfrak{w}}$, where $Z_{\mathfrak{w}(j)}$ is the j-th divisor defined in 2.2.1 (Theorem 3.1.4). A systematic study of line bundles on $Z_{\mathfrak{w}}$ is made in Exercise 3.1.E.3.

In Section 3.2 we prove that any Schubert variety X_w^P is normal by making use of the splitting of X_w^P (Theorem 3.2.2). We give two other proofs of normality of X_w^P, one using the H^0-surjectivity result mentioned above (Remark 3.2.3) and the other outlined in Exercise 3.2.E.1 which does *not* use Frobenius splitting. We also prove that the linear system on X_w^P, given by any ample line bundle, embeds X_w^P as a projectively normal variety (Theorem 3.2.2), the proof of which uses the H^0-surjectivity mentioned above.

Section 3.3 is devoted to the proof of the Demazure character formula (Theorem 3.3.8). Recall that the Demazure character formula gives the T-character of $H^0(X_w, \mathcal{L}_w(\lambda))$ for any dominant character λ in terms of the Demazure operators (defined in 3.3.6). This is achieved by showing that, for a reduced decomposition $w = s_{i_1} \cdots s_{i_n}$ with the associated word $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$, the canonical morphism $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ is rational (Theorem 3.3.4(b)). In addition, we need to use the cohomology vanishing result for X_w proved in Section 3.1 (mentioned above). We also show that the canonical morphism $\pi : X_w \to X_w^P$ is rational; in particular, for a locally free sheaf $\mathcal{S}$ on X_w^P, $H^i(X_w^P, \mathcal{S}) \simeq H^i(X_w, \pi^*\mathcal{S})$ (Theorem 3.3.4(a)). The same result is true for the canonical morphism between G-Schubert varieties (Exercise 3.3.E.3). Moreover, it is proved that the B-module $H^0(X_w, \mathcal{L}_w(\lambda))$ is isomorphic to the dual of the Demazure submodule $V_w(\lambda)$ of the Weyl module $V(\lambda)$, where $V_w(\lambda)$ is generated as a B-module by a weight vector of weight $w\lambda$.

In Section 3.4 we show, by using the Frobenius splitting of $Z_{\mathfrak{w}}$ proved in Section 2.2, that the morphism $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ (as in the above paragraph) is a rational resolution (Theorem 3.4.3). Moreover, if $X_w \to X_w^P$ is birational, then the composite map $Z_{\mathfrak{w}} \to X_w^P$ is again a rational resolution. In particular, any X_w^P is Cohen–Macaulay. In addition, X_w^P is projectively Cohen–Macaulay in the projective embedding given by any ample line bundle (Corollary 3.4.4). As another consequence of the result that $\theta_{\mathfrak{w}}$ is a rational resolution, we prove that for any dominant characters $\lambda_1, \ldots, \lambda_r$ of P, the multicone $C\big(X_w^P; \mathcal{L}_w^P(\lambda_1), \ldots, \mathcal{L}_w^P(\lambda_r)\big)$, defined in Exercise 1.1.E.2, admits a rational resolution (Theorem 3.4.7). An expression for the canonical sheaf of X_w is given in Exercise 3.4.E.1.

Finally in Section 3.5, we study the defining ideals of Schubert varieties X_w^P with respect to any line bundle $\mathcal{L}_w^P(\lambda)$. It is shown that for any dominant character λ of P and any $v \leq w \in W$, the line bundle $\mathcal{L}_w^P(\lambda)$ on X_w^P is normally presented and X_v^P is linearly defined in X_w^P with respect to $\mathcal{L}_w^P(\lambda)$ (Theorem 3.5.2). Similar results are available for G-Schubert varieties $\mathcal{X}_w^{P,Q}$ (Exercise 3.5.E.2). Further, the homogeneous coordinate ring $R(X_w^P, \mathcal{L}_w^P(\lambda)) := \bigoplus_{m \geq 0} H^0(X_w^P, \mathcal{L}_w^P(m\lambda))$ is shown to be Koszul and, for any $v \leq w$, $R(X_v^P, \mathcal{L}_v^P(\lambda))$ is a Koszul module over $R(X_w^P, \mathcal{L}_w^P(\lambda))$ under the standard restriction (Theorem 3.5.3). The main ingredient in the proof of both of these results is the Frobenius splitting property of the n-fold product $(G/P)^n$ obtained in Chapter 2, specifically Theorem 2.3.10. Analogues of both of these results are true for multihomogeneous coordinate rings of X_w^P with respect to line bundles $\mathcal{L}_w^P(\lambda_1), \ldots, \mathcal{L}_w^P(\lambda_r)$ for dominant characters $\lambda_1, \ldots, \lambda_r$ of P (Exercise 3.5.E.1).

Notation. We continue to follow the notation from Section 2.1. In particular, G is a semisimple, connected, simply-connected algebraic group over an algebraically closed field k of characteristic $p > 0$.

Let $X^*(P)$ be the character group of $P = P_I$ which can canonically be identified, under the restriction, as the subgroup of $X^*(T)$ consisting of those λ such that $\langle \lambda, \alpha_i^\vee \rangle = 0$, for all $i \in I$. For any rational P-module V, by $\mathcal{L}^P(V)$ we mean the G-equivariant vector bundle $G \times_P V \to G/P$ associated to the locally trivial principal P-bundle

$\pi_P : G \to G/P$. For $\lambda \in X^*(P)$, we denote $\mathcal{L}^P(k_{-\lambda})$ by $\mathcal{L}^P(\lambda)$. Then, any (not necessarily G-equivariant) line bundle on G/P is isomorphic to $\mathcal{L}^P(\lambda)$, for some $\lambda \in X^*(P)$. As in Section 2.1, we shall abbreviate $\mathcal{L}^B(V)$ by $\mathcal{L}(V)$.

The restriction of $\mathcal{L}(\lambda)$, resp. $\mathcal{L}^P(\lambda)$, to X_w, resp. X_w^P, is denoted by $\mathcal{L}_w(\lambda)$, resp. $\mathcal{L}_w^P(\lambda)$. Then, any line bundle on X_w^P is isomorphic to $\mathcal{L}_w^P(\lambda)$, for some $\lambda \in X^*(P)$.

For $\lambda \in X^*(P)$, $\mu \in X^*(Q)$, and $w \in W$, let $\mathcal{L}_w^{P,Q}(\lambda \boxtimes \mu)$ be the line bundle $\mathcal{L}^P\big(k_{-\lambda}\big) \boxtimes \mathcal{L}^Q\big(k_{-\mu}\big)$ restricted to $\mathcal{X}_w^{P,Q}$. If $P = Q = B$, we abbreviate $\mathcal{L}_w^{P,Q}(\lambda \boxtimes \mu)$ by $\mathcal{L}_w(\lambda \boxtimes \mu)$. If $w = w_o$, we abbreviate $\mathcal{L}_w^{P,Q}(\lambda \boxtimes \mu)$ by $\mathcal{L}^{P,Q}(\lambda \boxtimes \mu)$.

3.1 Cohomology of Schubert varieties

For any standard parabolic subgroup $P = P_I$ of G, define

$$\delta_P = \rho + w_o^P \rho \in X^*(P) \subset X^*(T),$$

where w_o^P is the longest element of the Weyl group $W_P := W_I$ of P. Then,

$$\omega_{G/P} \simeq \mathcal{L}^P(-\delta_P).$$

Clearly, $\delta_P - 2\rho_P$ is a dominant weight, where $\rho_P \in X^*(P)$ is defined in Exercise 2.2.E.4.

We have the following important result on the cohomology of Schubert varieties.

3.1.1 Theorem. *Let P be any standard parabolic subgroup of G. Then, for any dominant $\lambda \in X^*(P)$ and $w \in W$,*

(a) $H^i\big(X_w^P, \mathcal{L}_w^P(\lambda)\big) = 0$, for all $i > 0$.

(b) The restriction map $H^0\big(G/P, \mathcal{L}^P(\lambda)\big) \to H^0\big(X_w^P, \mathcal{L}_w^P(\lambda)\big)$ is surjective.

For $X_w^P = G/P$, we have the following stronger vanishing:

(c) For any $\lambda \in X^*(P)$ such that $\lambda + \rho_P$ is dominant, $H^i(G/P, \mathcal{L}^P(\lambda)) = 0$, for all $i > 0$.

Proof. By Theorem 2.2.5, G/P is $(p-1)\partial^- G/P$-split, compatibly $(p-1)\partial^- G/P$-splitting all the Schubert subvarieties X_w^P. Moreover, by Exercises 2.2.E.5 and 3.1.E.1, $\partial^- G/P$ is an ample divisor on G/P. By Exercise 3.1.E.1, for any dominant $\lambda \in X^*(P)$, $\mathcal{L}^P(\lambda)$ is semi-ample. So, the statements (a) and (b) of the theorem follow from Theorem 1.4.8.

We now prove the (c) part. By Serre duality,

$$H^i\Big(G/P, \mathcal{L}^P(\lambda)\Big) \simeq H^{n-i}\Big(G/P, \mathcal{L}^P(-\lambda - \delta_P)\Big)^*,$$

where $n := \dim G/P$. Since $\lambda + \rho_P$ is dominant by assumption, $\mathcal{L}^P(\lambda + 2\rho_P)$ is ample and hence so is $\mathcal{L}^P(\lambda + \delta_P)$. Thus, the (c) part follows from Theorem 1.2.9. $\square$

3.1.2 Theorem. *Let P, Q be two standard parabolic subgroups and let $\lambda \in X^*(P)$, $\mu \in X^*(Q)$ be dominant weights. Then, for any $w \in W$,*

(a) $H^i\big(\mathcal{X}_w^{P,Q}, \mathcal{L}_w^{P,Q}(\lambda \boxtimes \mu)\big) = 0$, *for all* $i > 0$.

(b) The restriction map

$$H^0\big(G/P \times G/Q, \mathcal{L}^{P,Q}(\lambda \boxtimes \mu)\big) \to H^0\big(\mathcal{X}_w^{P,Q}, \mathcal{L}_w^{P,Q}(\lambda \boxtimes \mu)\big)$$

is surjective.

(c) In particular, for dominant $\lambda, \mu \in X^*(P)$, *the product map*

$$H^0\big(G/P, \mathcal{L}^P(\lambda)\big) \otimes H^0\big(G/P, \mathcal{L}^P(\mu)\big) \to H^0\big(G/P, \mathcal{L}^P(\lambda + \mu)\big)$$

is surjective.

Proof. By Corollary 2.3.9, $G/P \times G/Q$ is $(p-1)\big(\partial G/P \times G/Q \cup G/P \times \partial^- G/Q\big)$-split, compatibly $(p-1)(\partial G/P \times G/Q \cup G/P \times \partial^- G/Q))$-splitting any G-Schubert variety $\mathcal{X}_w^{P,Q}$. Thus, the (a) and (b) parts of the theorem follow from Theorem 1.4.8 coupled with Exercises 2.2.E.5 and 3.1.E.1.

The (c) part is a special case of the (b) part by taking $Q = P$ and $w = e$. □

3.1.3 Remark. (a) Let P be any standard parabolic subgroup of G and let $X_{w_1}^P, \ldots, X_{w_q}^P$ be a collection of Schubert subvarieties of G/P. Let $Y := \bigcup_{i=1}^q X_{w_i}^P$ be the union taken with the reduced scheme structure. Then, the same proof as that of Theorem 3.1.1 gives that for any dominant $\lambda \in X^*(P)$,

(1) $$H^i\big(Y, \mathcal{L}^P(\lambda)_{|Y}\big) = 0, \quad \text{for all } i > 0, \text{ and}$$

(2) $$H^0(G/P, \mathcal{L}^P(\lambda)) \to H^0\big(Y, \mathcal{L}^P(\lambda)_{|Y}\big) \quad \text{is surjective.}$$

(b) Similarly, for any standard parabolic subgroups $P, Q \subset G$ and any dominant weights $\lambda \in X^*(P)$, $\mu \in X^*(Q)$, we have

(3) $$H^i\big(\mathcal{Y}, \mathcal{L}^{P,Q}(\lambda \boxtimes \mu)_{|\mathcal{Y}}\big) = 0 \quad \text{for all } i > 0 \text{ and}$$

(4) $$H^0\big(G/P \times G/Q, \mathcal{L}^{P,Q}(\lambda \boxtimes \mu)\big) \to H^0\big(\mathcal{Y}, \mathcal{L}^{P,Q}(\lambda \boxtimes \mu)_{|\mathcal{Y}}\big) \text{ is surjective,}$$

where $\mathcal{Y}$ is any union (with the reduced scheme structure) of G-Schubert subvarieties of $G/P \times G/Q$.

3.1.4 Theorem. *Let* $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$ *be any word and let* $1 \le q \le r \le n$ *be integers such that the subword* $(s_{i_q}, \ldots, s_{i_r})$ *is reduced. Then, for any globally generated line bundle* $\mathcal{L}$ *on* $Z_{\mathfrak{w}}$,

(1) $$H^i\Big(Z_{\mathfrak{w}}, \mathcal{L}\Big(\sum_{j=q}^{r} -Z_{\mathfrak{w}(j)}\Big)\Big) = 0, \quad \textit{for all } i > 0,$$

where $Z_{\mathfrak{w}}$ is the BSDH variety and $Z_{\mathfrak{w}(j)}$ are its divisors (2.2.1).
Also,

$$H^i(Z_{\mathfrak{w}}, \mathcal{L}) = 0, \quad \textit{for all } i > 0. \tag{2}$$

Before we prove the theorem, we need the following lemma.

3.1.5 Lemma. *With the notation and assumptions as in the above theorem, there exist integers $m_1, \ldots, m_n$ and $m \geq 0$ such that*

(a) $m_j \geq 0$ for $j \notin \{q, q+1, \ldots, r\}$,

(b) $m_j \leq 0$ for $j \in \{q, q+1, \ldots, r\}$ and $m_r = -1$, and

(c) the line bundle $\mathcal{O}\Big(\sum_{j=1}^n m_j Z_{w(j)}\Big) \otimes \mathcal{L}_{\mathfrak{w}}(m\rho)$ is globally generated, where we have abbreviated $\mathcal{O}_{Z_{\mathfrak{w}}}$ by $\mathcal{O}$.

Proof. Let $\mathfrak{v} := \mathfrak{w}[n-1] = (s_{i_1}, \ldots, s_{i_{n-1}})$ and let $\psi = \psi_{\mathfrak{w}} : Z_{\mathfrak{w}} \to Z_{\mathfrak{v}}$ be the $\mathbb{P}^1$-fibration as in 2.2.1.

We first show that for any $s \geq 4$, the line bundle $\mathcal{F}_s := \mathcal{L}_{\mathfrak{w}}(s\rho) \otimes \mathcal{O}(Z_{\mathfrak{w}(n)}) \otimes \psi^*\mathcal{L}_{\mathfrak{v}}(-\rho)$ is globally generated. Since ψ is the pullback (Exercise 2.2.E.1)

$$\begin{array}{ccc} Z_{\mathfrak{w}} & \longrightarrow & G/B \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle f_{i_n}} \\ Z_{\mathfrak{v}} & \longrightarrow & G/P_{i_n}, \end{array}$$

the relative tangent bundle T_ψ of ψ is given by $\mathcal{L}_{\mathfrak{w}}(\alpha_{i_n})$. On the other hand,

$$\begin{aligned} \text{(1)} \quad \mathcal{L}_{\mathfrak{w}}(\alpha_{i_n}) \simeq T_\psi &= \omega_{Z_{\mathfrak{w}}}^{-1} \otimes \psi^*\omega_{Z_{\mathfrak{v}}} \\ &\simeq \mathcal{L}_{\mathfrak{w}}(\rho) \otimes \mathcal{O}(Z_{\mathfrak{w}(n)}) \otimes \psi^*(\mathcal{L}_{\mathfrak{v}}(-\rho)), \text{ by Proposition 2.2.2.} \end{aligned}$$

Now, for $s \geq 4$, $(s-1)\rho + \alpha_{i_n}$ is dominant, thus $\mathcal{F}_s$ $(s \geq 4)$ is globally generated by (1).

Now, we come to the proof of the lemma. Assume first that $r < n$. By induction on n, we can choose integers $m_1, \ldots, m_{n-1}, m$ satisfying all three properties (a)–(c) of this lemma for $\mathfrak{w}$ replaced by $\mathfrak{v}$. In particular, the pullback line bundle $\mathcal{O}\Big(\sum_{j=1}^{n-1} m_j Z_{\mathfrak{w}(j)}\Big) \otimes \psi^*\mathcal{L}_{\mathfrak{v}}(m\rho)$ is globally generated. Now, choose $s \geq 0$ such that $\mathcal{L}_{\mathfrak{w}}(s\rho) \otimes \mathcal{O}(Z_{\mathfrak{w}(n)}) \otimes \psi^*\mathcal{L}_{\mathfrak{v}}(-\rho)$ is globally generated. Since $\mathcal{L}(\rho)$ is ample on G/B (Exercise 3.1.E.1), this is possible. Thus, $\mathcal{O}\Big(\sum_{j=1}^{n} m_j Z_{\mathfrak{w}(j)}\Big) \otimes \mathcal{L}_{\mathfrak{w}}(sm\rho)$ is globally generated, where $m_n := m$. So, this case is taken care of.

Now, consider the case $r = n$. We freely follow the notation and results from Exercise 3.1.E.3. Since the line bundles $\mathcal{O}_{\mathfrak{w}}(\delta_n)$ and $\mathcal{O}(Z_{\mathfrak{w}(n)})$ both are of degree 1 along the fibers $\mathbb{P}^1$ of ψ, we can write

$$\mathcal{L}_{\mathfrak{w}}(\chi_{i_n}) = \mathcal{O}_{\mathfrak{w}}(\delta_n) = \mathcal{O}\Big(Z_{\mathfrak{w}(n)} + \sum_{j=1}^{n-1} b_j Z_{\mathfrak{w}(j)}\Big). \tag{2}$$

Considering the restriction of the equation (2) to the fibers of $\psi_{\mathfrak{w},q-1} : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}[q-1]}$, we get

$$\mathcal{L}_{\mathfrak{u}}(\chi_{i_n}) = \mathcal{O}_{Z_{\mathfrak{u}}}\left(Z_{\mathfrak{u}(n-q+1)} + \sum_{j=1}^{n-q} b_{j+q-1} Z_{\mathfrak{u}(j)} \right),$$

where $\mathfrak{u}$ is the subword $(s_{i_q}, \ldots, s_{i_n})$. Thus, from Exercise 3.1.E.3(e), we get $b_j \geq 0$ for $q \leq j \leq n-1$. This is the place where we have used the assumption that $\mathfrak{u}$ is a reduced word.

Now, the line bundle

$$\mathcal{L}_{\mathfrak{w}}(\rho - \chi_{i_n}) \simeq \mathcal{L}_{\mathfrak{w}}(\rho) \otimes \mathcal{O}\left(-Z_{\mathfrak{w}(n)} - \sum_{j=1}^{n-1} b_j \, Z_{\mathfrak{w}(j)} \right)$$

is globally generated since it is the pullback of globally generated line bundle $\mathcal{L}(\rho - \chi_{i_n})$. Finally, by Exercise 3.1.E.3(f), there exist integers $a_1, \ldots, a_{q-1} > 0$ such that $\mathcal{O}_{Z_{\mathfrak{x}}}\left(\sum_{j=1}^{q-1} a_j \, Z_{\mathfrak{x}(j)}\right)$ is ample on $Z_{\mathfrak{x}}$; in particular, it is globally generated, where $\mathfrak{x} := (s_{i_1}, \ldots, s_{i_{q-1}})$. Now, taking a large enough $b > 0$ such that $ba_j \geq b_j$ for all $1 \leq j \leq q-1$, we get the lemma in this case as well. □

Now, we are ready to prove Theorem 3.1.4.

3.1.6 *Proof of Theorem 3.1.4.* We first prove (3.1.4.2), i.e., $H^i(Z_{\mathfrak{w}}, \mathcal{L}) = 0$, for $i > 0$. By the proof of Theorem 2.2.3, there exists a section $\sigma \in H^0\big(Z_{\mathfrak{w}}, \omega^{-1}_{Z_{\mathfrak{w}}}\big)$ such that σ^{p-1} splits $Z_{\mathfrak{w}}$ and the zero scheme $Z(\sigma)$ of σ is given by

$$Z(\sigma) = \sum_{j=1}^{n} Z_{\mathfrak{w}(j)} + D, \tag{1}$$

where D is the divisor of a section of $\mathcal{L}_{\mathfrak{w}}(\rho)$. By Exercise 3.1.E.3(f), there exist integers $a_1, \ldots, a_n > 0$ such that $\mathcal{S} := \mathcal{O}\left(\sum_{j=1}^{n} a_j \, Z_{\mathfrak{w}(j)}\right)$ is ample on $Z_{\mathfrak{w}}$. Take $m > 0$ such that each $a_j < p^m$. Then, by (1) and Lemma 1.4.11, there exists an injection of abelian groups

$$H^i(Z_{\mathfrak{w}}, \mathcal{L}) \hookrightarrow H^i\left(Z_{\mathfrak{w}}, \mathcal{L}^{p^m} \otimes \mathcal{S}\right). \tag{2}$$

Since $\mathcal{L}$ is globally generated and $\mathcal{S}$ is ample, $\mathcal{L}^{p^m} \otimes \mathcal{S}$ is ample as well. Thus, by Theorem 1.2.8, $H^i\big(Z_{\mathfrak{w}}, \mathcal{L}^{p^m} \otimes \mathcal{S}\big) = 0$ for all $i > 0$, and hence $H^i(Z_{\mathfrak{w}}, \mathcal{L}) = 0$ for $i > 0$, by (2).

Now, we come to the proof of (3.1.4.1). Let m_j and m be the integers as in Lemma 3.1.5. Choose $t > 0$ such that $|m_1|, \ldots, |m_n|, m < p^t$. Then, by (1) and Lemma

1.4.11, we have an injection of abelian groups:

$$H^i\Big(Z_{\mathfrak{w}}, \mathcal{L}\Big(-\sum_{j=q}^{r} Z_{\mathfrak{w}(j)}\Big)\Big) \hookrightarrow H^i\Big(Z_{\mathfrak{w}}, \mathcal{M}\Big(-\sum_{j=q}^{r-1} Z_{\mathfrak{w}(j)}\Big)\Big), \tag{3}$$

where $\mathcal{M} := \mathcal{L}^{p^t} \otimes \mathcal{L}_{\mathfrak{w}}\big((p^t-1-m)\rho\big) \otimes \Big[\mathcal{O}\Big(\sum_{j=1}^{n} m_j Z_{\mathfrak{w}(j)}\Big) \otimes \mathcal{L}_{\mathfrak{w}}(m\rho)\Big]$.

By Lemma 3.1.5, $\mathcal{M}$ is globally generated (since $\mathcal{L}_{\mathfrak{w}}\big((p^t-1-m)\rho\big)$ is globally generated, being the pullback of a globally generated line bundle). Thus, by induction on $r-q$, the right side of (3) is 0 for any $i > 0$. This proves that the left side of (3) is 0 as well, thus completing the proof of the theorem. □

3.1.7 Remark. Taking $\mathfrak{w} = (s_i, s_i)$ and $q = 1$, $r = 2$, it is easy to see that the assumption in Theorem 3.1.4 that $(s_{i_q}, \dots, s_{i_r})$ be reduced is essential in general.

3.1.E Exercises

For the following exercises (1), (3) and (4), the characteristic of k is arbitrary.

(1*) Show that the homogeneous line bundle $\mathcal{L}^P(\lambda)$ on G/P, for $\lambda \in X^*(P)$, is ample if and only if $\lambda - \rho_P$ is a dominant weight. Moreover, in this case, it is very ample.

Further, for $\lambda \in X^*(P)$, $\mathcal{L}^P(\lambda)$ is globally generated if and only if λ is dominant.

(2) For a vector space V over a field k of characteristic $p > 0$, recall the definition of $V^{[-1]}$ from the proof of Corollary 2.3.5. For any dominant $\lambda \in X^*(T)$, show that the k-linear map

$$\varphi_\lambda : H^0(G/B, \mathcal{L}(\lambda))^{[-1]} \otimes H^0(G/B, \mathcal{L}((p-1)\rho)) \\ \to H^0(G/B, \mathcal{L}(p\lambda + (p-1)\rho)), \ \sigma' \otimes \sigma'' \mapsto \sigma'^p \sigma'',$$

is an isomorphism of G-modules.

Hint: Show first, by the Weyl dimension formula, that the domain and the range of φ_λ have the same dimensions. Next, if λ is of the form $(p^r-1)\rho$ for some $r \geq 1$, then show that φ_λ is an isomorphism by using the irreducibility of $H^0\big(G/B, \mathcal{L}((p^{r+1}-1)\rho)\big)$ (Exercise 2.3.E.3). Now, for any dominant λ, choose r large enough such that $(p^r-1)\rho - \lambda$ is dominant. Take a nonzero section $s \in H^0\big(G/B, \mathcal{L}((p^r-1)\rho - \lambda)\big)$. Tensoring with s^p gives an injection $H^0\big(G/B, \mathcal{L}(\lambda)\big)^{[-1]} \to H^0\big(G/B, \mathcal{L}((p^r-1)\rho)\big)^{[-1]}$. Use this to conclude that φ_λ is injective and hence an isomorphism.

(3*) *Line bundles on* $Z_{\mathfrak{w}}$: Let $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ be any word. Recall the definition of the BSDH variety $Z_{\mathfrak{w}}$ and the morphism $\psi_{\mathfrak{w},m} : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}[m]}$, for any $1 \leq m \leq n$ from 2.2.1, where $\mathfrak{w}[m]$ is the subword $(s_{i_1}, \dots, s_{i_m})$. Define the line bundle

$$\mathcal{O}_{\mathfrak{w}}(\delta_m) := \psi^*_{\mathfrak{w},m}\big(\mathcal{L}_{\mathfrak{w}[m]}(\chi_{i_m})\big), \quad \text{for any } 1 \leq m \leq n, \tag{1}$$

where χ_{i_m} is the i_m-th fundamental weight (as in Section 2.1). Thus, $\mathcal{O}_{\mathfrak{w}}(\delta_n) = \mathcal{L}_{\mathfrak{w}}(\chi_{i_n})$.

Now, for any $(j_1, \dots, j_n) \in \mathbb{Z}^n$, define the line bundle

$$\mathcal{O}_{\mathfrak{w}}(j_1, \dots, j_n) := \bigotimes_{m=1}^{n} \big(\mathcal{O}_{\mathfrak{w}}(\delta_m)^{\otimes j_m}\big). \tag{2}$$

Also, recall the definition of the divisors $Z_{\mathfrak{w}(m)}$ of $Z_{\mathfrak{w}}$, $1 \le m \le n$, from 2.2.1. Prove the following:

(a) The line bundles $\mathcal{O}_{\mathfrak{w}}(\delta_m)$, $1 \le m \le n$, are globally generated. Moreover, $\mathcal{O}_{\mathfrak{w}}(\delta_n)$ is of degree 1 along the fibers $\mathbb{P}^1$ of $\psi_{\mathfrak{w},n-1}$.

Also, the divisor $Z_{\mathfrak{w}(n)}$ has degree 1 along the fibers of $\psi_{\mathfrak{w},n-1}$.

(b) By induction on n, show that

$$\operatorname{Pic} Z_{\mathfrak{w}} = \bigoplus_{m=1}^{n} \mathbb{Z}\,\mathcal{O}_{\mathfrak{w}}(\delta_m), \quad \text{and also}$$
$$\operatorname{Pic} Z_{\mathfrak{w}} = \bigoplus_{m=1}^{n} \mathbb{Z}\,\mathcal{O}_{Z_{\mathfrak{w}}}(Z_{\mathfrak{w}(m)}).$$

(c) Show that $\mathcal{O}_{\mathfrak{w}}(j_1, \dots, j_n)$ is very ample on $Z_{\mathfrak{w}}$ iff each $j_m > 0$.

Thus, ample line bundles on $Z_{\mathfrak{w}}$ are very ample.

Hint: Consider the morphism $f_{\mathfrak{w}} : Z_{\mathfrak{w}} \to G/Q_{i_1} \times \dots \times G/Q_{i_n}$, $[p_1, \dots, p_n] \mapsto \big(p_1 Q_{i_1}, p_1 p_2 Q_{i_2}, \dots, p_1 \cdots p_n Q_{i_n}\big)$, where Q_{i_j} is the maximal parabolic subgroup of G such that $s_{i_j} \notin W_{Q_{i_j}}$. Then, $f_{\mathfrak{w}}$ is a closed embedding.

Show now that for any $(j_1, \dots, j_n) \in \mathbb{Z}^n$,

$$f_{\mathfrak{w}}^*\Big(\mathcal{L}^{Q_{i_1}}(j_1\chi_{i_1}) \boxtimes \dots \boxtimes \mathcal{L}^{Q_{i_n}}(j_n\chi_{i_n})\Big) = \mathcal{O}_{\mathfrak{w}}(j_1, \dots, j_n).$$

Thus, conclude that for $j_1, \dots, j_n > 0$, $\mathcal{O}_{\mathfrak{w}}(j_1, \dots, j_n)$ is very ample on $Z_{\mathfrak{w}}$.

For the converse part, prove that the restriction $\mathcal{O}_{\mathfrak{w}}(j_1, \dots, j_n)_{|Z_{\mathfrak{w}(1)}} = \mathcal{O}_{\mathfrak{w}(1)}(j_2, \dots, j_n)$. Thus, by induction on n, $j_2, \dots, j_n > 0$. Also, if $s_{i_1} \neq s_{i_2}$, prove that $\mathcal{O}_{\mathfrak{w}}(j_1, \dots, j_n)_{|Z_{\mathfrak{w}(2)}} = \mathcal{O}_{\mathfrak{w}(2)}(j_1, j_3, \dots, j_n)$. Thus, $j_1 > 0$ as well. Finally, if $s_{i_1} = s_{i_2}$, use the decomposition $Z_{\mathfrak{w}} \simeq (P_{i_1}/B) \times Z_{\mathfrak{w}(1)}$ to conclude that $j_1 > 0$.

(d) Use (c) to show that the line bundle $\mathcal{O}_{\mathfrak{w}}(j_1, \dots, j_n)$ on $Z_{\mathfrak{w}}$ is globally generated iff each $j_m \ge 0$.

Hint: Use the fact that the tensor product of an ample line bundle with a globally generated line bundle is ample.

(e) Let $\mathfrak{w}$ be a reduced word. Then, the line bundle $\mathcal{L} := \mathcal{O}_{Z_{\mathfrak{w}}}\left(\sum_{m=1}^{n} j_m\, Z_{\mathfrak{w}(m)}\right)$ is effective iff each $j_m \geq 0$.

Hint: If each $j_m \geq 0$, $\mathcal{L}$ is clearly effective. Conversely, if $\mathcal{L}$ is effective, take a B-invariant line M in $H^0(Z_{\mathfrak{w}}, \mathcal{L})$. Conclude that the zero scheme $Z(\sigma)$ of any nonzero $\sigma \in M$ satisfies $Z(\sigma) = \sum_{m=1}^{n} j_m\, Z_{\mathfrak{w}(m)}$, for some $j_m \geq 0$.

(f) Show that there exist integers $j_1, \ldots, j_n > 0$ such that $\mathcal{O}_{Z_{\mathfrak{w}}}\left(\sum_{m=1}^{n} j_m\, Z_{\mathfrak{w}(m)}\right)$ is ample on $Z_{\mathfrak{w}}$.

Hint: Let $\mathfrak{v} = \mathfrak{w}[n-1]$. By induction, $\mathcal{O}_{Z_{\mathfrak{v}}}\left(\sum_{m=1}^{n-1} j_m\, Z_{\mathfrak{v}(m)}\right)$ is ample for some $j_1, \ldots, j_{n-1} > 0$. Now, $Z_{\mathfrak{w}(n)}$ has degree 1 along the fibers of $\psi_{\mathfrak{w},n-1}$. Thus, for $q > 0$ sufficiently large, $\mathcal{O}_{Z_{\mathfrak{w}}}\left(\sum_{m=1}^{n-1} j_m q Z_{\mathfrak{w}(m)} + Z_{\mathfrak{w}(n)}\right)$ is ample on $Z_{\mathfrak{w}}$.

(4) Prove the analogue of Theorem 3.1.4 for $\mathcal{Z}_{\mathfrak{w}} := G \times_B Z_{\mathfrak{w}}$. More specifically, for any word $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$ and $1 \leq q \leq r \leq n$ such that the subword $(s_{i_q}, \ldots, s_{i_r})$ is reduced, prove that

$$H^i\left(\mathcal{Z}_{\mathfrak{w}}, \mathcal{L} \otimes \mathcal{O}\Big(\sum_{j=q}^{r} -\mathcal{Z}_{\mathfrak{w}(j)}\Big)\right) = 0, \quad \text{for all } i > 0,$$

and any globally generated line bundle $\mathcal{L}$ on $\mathcal{Z}_{\mathfrak{w}}$, where $\mathcal{Z}_{\mathfrak{w}(j)} := G \times_B Z_{\mathfrak{w}(j)}$.

3.2 Normality of Schubert varieties

We continue to follow the notation as in 2.1. As preparation for the proof of Theorem 3.2.2, we begin with the following.

3.2.1 Proposition. *Let $v \in W$, s a simple reflection, and $P = B \cup BsB$ the corresponding standard minimal parabolic subgroup. Then, the product morphism $f : P \times_B X_v \to PX_v$ has connected fibers, and satisfies $R^i f_* \mathcal{O}_{P \times_B X_v} = 0$ for all $i \geq 1$.*

Proof. If $sv < v$, then $PX_v = X_v$, $P \times_B X_v \simeq P/B \times X_v$, and f identifies with the projection $P/B \times X_v \to X_v$ with fiber $P/B \simeq \mathbb{P}^1$. Both the assertions are clear in that case; thus, we may assume that $sv > v$. Then, by equation (2.1.4), $PX_v = X_w$, where $w := sv$.

For the first assertion, let $x \in X_w$. Then,

$$f^{-1}(x) \simeq \{p \in P \mid p^{-1}x \in X_v\}/B,$$

where the right action of B is given by $p \cdot b = pb$. In particular, $f^{-1}(x)$ is a closed subscheme of $P/B \simeq \mathbb{P}^1$. By the P-equivariance of f, to prove the connectedness of $f^{-1}(x)$, we may assume that $x \in W$ and, moreover, $x < sx \leq w$. Then, $f^{-1}(x)$ is stable under the standard action of T on P/B. So, if this fiber is not connected, then it consists of the T-fixed points B and sB in P/B. Thus, $x < sx \le v$, so that $x^{-1}(\alpha) > 0$ (where α is the simple root corresponding to s), and hence $sU_\alpha x$ is contained in X_v. Hence, $f^{-1}(x)$ also contains $U_\alpha sB$, a contradiction.

For the second assertion, since all the fibers of f have dimension ≤ 1, we may assume that $i = 1$. Now, factor f as

$$\iota : P \times_B X_v \to P/B \times X_w,\ (p, x)B \mapsto (pB, px)$$

(a closed immersion) followed by the projection $\pi : P/B \times X_w \to X_w$. Then, $R^1 f_* \mathcal{O}_{P\times_B X_v} = R^1\pi_*(\iota_*\mathcal{O}_{P\times_B X_v})$, by [Har–77, Chap. III, Exercise 4.1]. Further, the surjection $\mathcal{O}_{P/B\times X_w} \to \iota_*\mathcal{O}_{P\times_B X_v}$ and the vanishing of $R^2\pi_*\mathcal{F}$ for any coherent sheaf $\mathcal{F}$ on $P/B \times X_w$ yield a surjection

$$R^1\pi_*(\mathcal{O}_{P/B\times X_w}) \to R^1\pi_*(\iota_*\mathcal{O}_{P\times_B X_v}).$$

Now, $R^1\pi_*(\mathcal{O}_{P/B\times X_w}) = 0$, since $H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$. □

Recall the notion of projectively normal subvarieties $X \subset \mathbb{P}^N$ from Section 1.5. Observe that, by Exercise 3.1.E.1, $\mathcal{L}^P(\lambda + \rho_P)$ is very ample on G/P for any dominant $\lambda \in X^*(P)$. In particular, $\mathcal{L}^P_w(\lambda + \rho_P)$ is very ample on X^P_w.

3.2.2 Theorem. *For any standard parabolic subgroup P of G, and any $w \in W$, the Schubert variety X^P_w is normal.*

Moreover, for any dominant $\lambda \in X^*(P)$, the linear system on X^P_w given by $\mathcal{L}^P_w(\lambda + \rho_P)$ embeds X^P_w as a projectively normal variety. In particular,

$$X^P_w \simeq \mathrm{Proj}\big(R(X^P_w, \mathcal{L}^P_w(\lambda + \rho_P))\big).$$

Proof. We first prove the normality of X^P_w. From the fibration $\pi : G/B \to G/P$ with smooth fiber P/B, the normality of X^P_w is equivalent to the normality of $\pi^{-1}(X^P_w)$. Since $\pi^{-1}(X^P_w)$ is a B-stable closed subvariety of G/B, it is of the form X_v, for some $v \in W$. Thus, it suffices to prove the normality of X_w. We prove this by induction on $\ell(w)$. Of course, for $\ell(w) = 0$, since X_w is a point, it is normal. So, take a $w \in W$ with $\ell(w) > 0$ and let s_i be a simple reflection such that $s_i w < w$. Let $P_i = B \cup Bs_iB$ be the corresponding standard minimal parabolic subgroup. Then, the product morphism

$$f : P_i \times_B X_{s_i w} \to X_w$$

is birational; and by Proposition 3.2.1, it has connected fibers. Moreover, $X_{s_i w}$ is normal by the induction assumption. Thus, the normalization $\theta : \widetilde{X}_w \to X_w$ is bijective (use [Har–77, Chap. II, Exercise 3.8]). But, X_w is split, and hence weakly normal by Proposition 1.2.5. So, θ is an isomorphism, and X_w is normal.

The second part of the theorem means that the embedding $X^P_w \to \mathbb{P}(E^*)$, $x \mapsto H^0(x, \mathcal{L}^P_w(\lambda + \rho_P)_{|x})^*$, is projectively normal, where $E := H^0(X^P_w, \mathcal{L}^P_w(\lambda + \rho_P))$.

By the normality of X_w^P and the characterization of projectively normal varieties given before Corollary 1.5.4, it suffices to show that the product map

$$H^0\big(X_w^P, \mathcal{L}_w^P(n\mu)\big) \otimes H^0\big(X_w^P, \mathcal{L}_w^P(m\mu)\big) \to H^0\big(X_w^P, \mathcal{L}_w^P((n+m)\mu)\big)$$

is surjective for all $n, m \geq 0$, where $\mu := \lambda + \rho_P$. But this follows from Theorem 3.1.2(c) coupled with Theorem 3.1.1(b). $\square$

3.2.3 Remark. An alternative proof of the normality of the Schubert varieties X_w, $w \in W$, can be given as follows.

Let s_i be a simple reflection such that $ws_i < w$. Consider the fibration $\pi_i : G/B \to G/P_i$, where P_i is the minimal parabolic subgroup corresponding to the singleton $I = \{i\}$. Then, $\pi_{i|X_w} \to X_w^{P_i}$ is a $\mathbb{P}^1$-fibration and $\pi_{i|X_{ws_i}} : X_{ws_i} \to X_w^{P_i}$ is a birational (surjective) morphism (cf. [Kem–76a, §2, Lemma 1]). Thus, X_w is normal iff $X_w^{P_i}$ is normal. By induction, X_{ws_i} is normal, so the normality of $X_w^{P_i}$ is equivalent to

$$(1) \qquad (\bar{\pi}_i)_* \mathcal{O}_{X_{ws_i}} = \mathcal{O}_{X_w^{P_i}},$$

where $\bar{\pi}_i := \pi_{i|X_{ws_i}}$.

For any dominant $\lambda \in X^*(P_i)$, the map

$$(\bar{\pi}_i)^* : H^0\Big(X_w^{P_i}, \mathcal{L}_w^{P_i}(\lambda)\Big) \to H^0\big(X_{ws_i}, \mathcal{L}_{ws_i}(\lambda)\big)$$

is surjective since we have the commutative diagram:

$$\begin{array}{ccc} H^0\big(G/P_i, \mathcal{L}^{P_i}(\lambda)\big) & \xrightarrow{\pi_i^*} & H^0\big(G/B, \mathcal{L}(\lambda)\big) \\ \big\downarrow & & \big\downarrow \\ H^0\big(X_w^{P_i}, \mathcal{L}_w^{P_i}(\lambda)\big) & \xrightarrow{\bar{\pi}_i^*} & H^0\big(X_{ws_i}, \mathcal{L}_{ws_i}(\lambda)\big), \end{array}$$

where the vertical maps are the canonical restriction maps, which are surjective by Theorem 3.1.1(b). Moreover, π_i^* is an isomorphism, since the fibration π_i has connected projective fibers. Thus, $\bar{\pi}_i^*$ is surjective, and hence an isomorphism (for any dominant $\lambda \in X^*(P_i)$). Hence, (1) follows from Lemma 3.3.3(b), proving the normality of X_w.

3.2.E Exercises

(1) Let k be an algebraically closed field of arbitrary characteristic. We give here the outline for an alternative proof of the normality of the Schubert varieties X_w without using the Frobenius splitting methods.

Prove the normality of X_w, $w \in W$, by downward induction on $\ell(w)$. Let $\theta : \widetilde{X}_w \to X_w$ be the normalization of X_w. For any simple reflection s_i such that

$v := s_i w > w$, consider the product map $f : Z = P_i \times_B X_w \to X_v$, which is a birational morphism. Also, let $\Theta : \widetilde{Z} = P_i \times_B \widetilde{X}_w \to Z$ be the morphism induced from θ. Set $\widetilde{f} := f \circ \Theta$. For any B-equivariant sheaf $\mathcal{S}$ on X_w, let $\operatorname{Ind}_B^{P_i} \mathcal{S}$ be the P_i-equivariant sheaf on Z induced from $\mathcal{S}$. The exact sequence of B-equivariant sheaves:

$$0 \to \mathcal{O}_{X_w} \to \theta_* \mathcal{O}_{\widetilde{X}_w} \to \mathcal{Q} := (\theta_* \mathcal{O}_{\widetilde{X}_w})/\mathcal{O}_{X_w} \to 0$$

gives rise to an exact sequence of P_i-equivariant sheaves on Z:

$$0 \to \mathcal{O}_Z \to \Theta_* \mathcal{O}_{\widetilde{Z}} \to \operatorname{Ind}_B^{P_i}(\mathcal{Q}) \to 0.$$

Then, we have the following property (a) by Proposition 3.2.1.

(a) $R^i f_* \mathcal{O}_Z = 0$, for all $i \geq 1$.

Thus, we get an exact sequence of P_i-equivariant sheaves on X_v:

$$0 \to f_* \mathcal{O}_Z \to f_*(\Theta_* \mathcal{O}_{\widetilde{Z}}) \to f_*\big(\operatorname{Ind}_B^{P_i}(\mathcal{Q})\big) \to 0.$$

(b) Show that $f_*\big(\operatorname{Ind}_B^{P_i}(\mathcal{Q})\big) = 0$, by showing that $f_* \mathcal{O}_Z = f_*(\Theta_* \mathcal{O}_{\widetilde{Z}}) = \mathcal{O}_{X_v}$. (Use the normality of X_v by induction.)

Let $\Sigma \subset X_w$ be the locus of nonnormal points of X_w. Then, Σ is the support of the sheaf $\mathcal{Q}$; it is B-stable. Let X_u be an irreducible component of Σ. Define a subsheaf $\mathcal{Q}_u \subset \mathcal{Q}$ on X_w by

$$\Gamma(V, \mathcal{Q}_u) := \big\{\sigma \in \Gamma(V, \mathcal{Q}) : \Gamma(V, \mathcal{I}_{X_u}) \cdot \sigma = 0\big\},$$

for any open subset $V \subset X_w$, where $\mathcal{I}_{X_u}$ is the ideal sheaf of X_u in X_w. By definition, the sheaf $\mathcal{Q}_u$ is killed by $\mathcal{I}_{X_u}$, so that it may be regarded as a sheaf on X_u.

Choose a standard minimal parabolic subgroup P_j such that $P_j X_u \neq X_u$, i.e., $s_j u > u$. Then, show that

(c) $s_j w > w$, and thus we could take $s_i = s_j$.

By the (b) part and the exactness of $\operatorname{Ind}_B^{P_j}$, show that

(d) for the morphism $f' : P_j \times_B X_u \to X_{s_j u}$,

$$f'_*(\operatorname{Ind}_B^{P_j}(\mathcal{Q}_u)) = 0.$$

Finally, show that

(e) the support of $\operatorname{Ind}_B^{P_j}(\mathcal{Q}_u)$ is the whole of $P_j \times_B X_u$.

This contradicts (d) and thus Σ is empty, proving that X_w is normal.

3.3 Demazure character formula

3.3.1 Definition. Let $f : X \to Y$ be a morphism of schemes. Following [Kem–76a, page 567], f is called a *rational morphism* if the induced map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ is an isomorphism and the direct images $R^i f_*\mathcal{O}_X$ vanish for $i > 0$. (Kempf calls it a trivial morphism, but we prefer to call it a rational morphism.)

The following lemma follows immediately from the Leray spectral sequence [God–58, Chap. II, Theorem 4.17.1] and the projection formula [Har–77, Chap. III, Exercise 8.3].

3.3.2 Lemma. *Let $f : X \to Y$ be a rational morphism between schemes. Then, for any locally free sheaf $\mathcal{S}$ on Y,*

$$H^i(Y, \mathcal{S}) \xrightarrow{\sim} H^i(X, f^*\mathcal{S}), \quad \textit{for all } i \geq 0.$$

If, in addition, $g : Y \to Z$ is another rational morphism, then $g \circ f : X \to Z$ is rational as well.

The next lemma is very useful in proving that certain morphisms are rational.

3.3.3 Lemma. *Let $f : X \to Y$ be a morphism between projective schemes and let $\mathcal{L}$ be an ample invertible sheaf on Y.*

(a) *Assume that $H^q(X, f^*\mathcal{L}^n) = 0$, for all $q > 0$ and all sufficiently large n. Then,*

$$R^q f_*\mathcal{O}_X = 0, \quad \textit{for all } q > 0.$$

(b) *Assume that f is surjective and*

$$H^0(Y, \mathcal{L}^n) \to H^0(X, f^*\mathcal{L}^n)$$

is an isomorphism for all sufficiently large n. (We do not impose the assumption as in (a).) Then,

$$f_*\mathcal{O}_X = \mathcal{O}_Y.$$

Proof. (a) The $E_2^{p,q}$ term of the Leray spectral sequence for the morphism f and the sheaf $f^*\mathcal{L}^n$ on X is given by

$$\begin{aligned} E_2^{p,q} &= H^p\big(Y, R^q f_*(f^*\mathcal{L}^n)\big) \\ &\simeq H^p\big(Y, (R^q f_*\mathcal{O}_X) \otimes \mathcal{L}^n\big), \end{aligned}$$

by the projection formula. Thus, by [Har–77, Chap III, Theorem 8.8 and Proposition 5.3], $E_2^{p,q} = 0$ for all $p > 0$ provided $n \gg 0$. In particular, for $n \gg 0$,

$$H^q(X, f^*\mathcal{L}^n) \simeq H^0\big(Y, (R^q f_*\mathcal{O}_X) \otimes \mathcal{L}^n\big),$$

which is 0 for $q > 0$ (by assumption). By the definition of an ample invertible sheaf (cf. [Har–77, Definition on page 153]), for any coherent sheaf $\mathcal{S}$ on Y, $\mathcal{S} \otimes \mathcal{L}^n$ is globally generated for large enough n. Thus,

$$R^q f_* \mathcal{O}_X = 0 \quad \text{for } q > 0.$$

(b) Consider the sheaf exact sequence on Y:

$$0 \to \mathcal{O}_Y \to f_* \mathcal{O}_X \to \mathcal{Q} \to 0,$$

where $\mathcal{Q}$, by definition, is the quotient sheaf $f_* \mathcal{O}_X / \mathcal{O}_Y$. Tensoring this sequence over $\mathcal{O}_Y$ with the locally free sheaf $\mathcal{L}^n$ and taking cohomology (and using the projection formula), we get

$$0 \to H^0(Y, \mathcal{L}^n) \to H^0(X, f^* \mathcal{L}^n) \to H^0(Y, \mathcal{Q} \otimes \mathcal{L}^n) \to H^1(Y, \mathcal{L}^n) \to \dots .$$

But, since $\mathcal{L}$ is ample, $H^1(Y, \mathcal{L}^n) = 0$, for all $n \gg 0$. In particular, by the assumption, $H^0(Y, \mathcal{Q} \otimes \mathcal{L}^n) = 0$ for all $n \gg 0$. Now, by [Har–77, Chap. III, Theorem 8.8], $f_* \mathcal{O}_X$, and hence $\mathcal{Q}$, is a coherent sheaf on Y. But then, since $\mathcal{L}$ is ample, we conclude that $\mathcal{Q}$ itself is 0, i.e., $\mathcal{O}_Y = f_* \mathcal{O}_X$, proving the lemma. □

3.3.4 Theorem. *(a) For any standard parabolic subgroup P of G and any $w \in W$, the canonical morphism $\pi : X_w \to X_w^P$ is rational.*

In particular, for any locally free sheaf $\mathcal{S}$ on X_w^P, and any $i \geq 0$,

$$\pi^* : H^i(X_w^P, \mathcal{S}) \to H^i(X_w, \pi^* \mathcal{S})$$

is an isomorphism.

(See Exercise 3.3.E.3 for the corresponding result for $\mathcal{X}_w^{P,Q}$.)

(b) Let $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ be a reduced word and let $a(\mathfrak{w}) := s_{i_1} \cdots s_{i_n} \in W$. Then, the standard morphism $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ ((2.2.1.6)) is rational, where $w := a(\mathfrak{w})$. Thus, for any locally free sheaf $\mathcal{S}$ on X_w and any $i \geq 0$,

$$\theta_{\mathfrak{w}}^* : H^i(X_w, \mathcal{S}) \to H^i(Z_{\mathfrak{w}}, \theta_{\mathfrak{w}}^* \mathcal{S})$$

is an isomorphism.

Proof. (a) By the argument used in Remark 3.2.3, for any dominant $\lambda \in X^*(P)$,

$$\pi^* : H^0(X_w^P, \mathcal{L}_w^P(\lambda)) \to H^0(X_w, \mathcal{L}_w(\lambda))$$

is surjective and, π being surjective, of course it is injective. Moreover, by Theorem 3.1.1(a), $H^i(X_w, \mathcal{L}_w(\lambda)) = 0$ for all $i > 0$. Thus, the (a) part follows from Lemmas 3.3.2 and 3.3.3.

(b) We argue by induction on $n = \ell(w)$; if $n = 1$, then $\theta_{\mathfrak{w}}$ is an isomorphism. For an arbitrary reduced $\mathfrak{w}$, let $\mathfrak{v} = (s_{i_2}, \dots, s_{i_n})$ and $v = a(\mathfrak{v})$. Then, $\mathfrak{v}$ is a reduced word, and $\theta_{\mathfrak{w}}$ factors as

$$P_{i_1} \times_B \theta_{\mathfrak{v}} : P_{i_1} \times_B Z_{\mathfrak{v}} \to P_{i_1} \times_B X_v$$

followed by the product morphism $f : P_{i_1} \times_B X_v \to X_w$. By induction, the morphism $\theta_{\mathfrak{v}}$ is rational; thus it follows easily that $P_{i_1} \times_B \theta_{\mathfrak{v}}$ is rational as well. On the other hand, f is rational by normality of X_w and Proposition 3.2.1. Thus, $\theta_{\mathfrak{w}}$ is rational by Lemma 3.3.2. $\square$

3.3.5 Remark. Theorem 3.3.4(b) can also be obtained immediately by using (3.1.4.2) and Lemma 3.3.3(a). But we still need to use the normality of X_w.

Moreover, Theorem 3.3.4(b) is true for an arbitrary $\mathfrak{w}$ by Exercise 3.3.E.2.

3.3.6 Definition. (Demazure operators) For any simple reflection s_i, $1 \le i \le \ell$, following Demazure, define the $\mathbb{Z}$-linear operator $D_{s_i} : A(T) \to A(T)$ by

$$D_{s_i}(e^\lambda) = \frac{e^\lambda - e^{s_i\lambda - \alpha_i}}{1 - e^{-\alpha_i}}, \quad \text{for } e^\lambda \in X^*(T),$$

where $A(T) := \mathbb{Z}[X^*(T)]$ is the group algebra of the character group $X^*(T)$ and α_i is the i-th simple (positive) root. It is easy to see that $D_{s_i}(e^\lambda) \in A(T)$. In fact, one has the following simple lemma.

3.3.7 Lemma.

$$D_{s_i}(e^\lambda) = \begin{cases} e^\lambda + e^{\lambda-\alpha_i} + \cdots + e^{s_i\lambda}, & \textit{if } \langle\lambda, \alpha_i^\vee\rangle \ge 0 \\ 0, & \textit{if } \langle\lambda, \alpha_i^\vee\rangle = -1 \\ -(e^{\lambda+\alpha_i} + \cdots + e^{s_i\lambda-\alpha_i}), & \textit{if } \langle\lambda, \alpha_i^\vee\rangle < -1. \end{cases}$$

Now, for any word $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$, define $D_{\mathfrak{w}} = D_{s_{i_1}} \circ \cdots \circ D_{s_{i_n}} : A(T) \to A(T)$.

The ring $A(T)$ admits an involution defined by $\overline{e^\lambda} = e^{-\lambda}$. We denote $\overline{D_{\mathfrak{w}}(e^\lambda)}$ by $\bar{D}_{\mathfrak{w}}(e^\lambda)$.

Now, we are ready to prove the following Demazure character formula. For any finite-dimensional representation M of B, by $\mathcal{L}_{\mathfrak{w}}(M)$ we mean the pullback vector bundle $\theta_{\mathfrak{w}}^*(\mathcal{L}(M))$.

3.3.8 Theorem. *For any (not necessarily reduced) word $\mathfrak{w}$ and any finite-dimensional representation M of B, we have*

$$\text{(1)} \qquad \chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(M)) = \bar{D}_{\mathfrak{w}}(\overline{\operatorname{ch} M}), \qquad \textit{as elements of } A(T),$$

where $\chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(M)) := \sum_p (-1)^p \operatorname{ch} H^p(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(M)) \in A(T)$ and, for any finite-dimensional T-module N, $\operatorname{ch} N$ denotes its formal T-character.

In particular, for $\lambda \in X^*(T)$ and any reduced word $\mathfrak{w}$ with $w = a(\mathfrak{w})$,

$$\text{(2)} \qquad \chi(X_w, \mathcal{L}_w(\lambda)) = \bar{D}_{\mathfrak{w}}(e^\lambda).$$

Hence, if $\lambda \in X^*(T)^+$,

$$\text{(3)} \qquad \operatorname{ch} H^0(X_w, \mathcal{L}_w(\lambda)) = \bar{D}_{\mathfrak{w}}(e^\lambda).$$

Proof. For any exact sequence

$$0 \to M_1 \to M \to M_2 \to 0 \tag{4}$$

of finite-dimensional representations of B we have, from the corresponding long exact cohomology sequence ($\mathcal{L}_{\mathfrak{w}}$ being an exact functor),

$$\chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(M)) = \chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(M_1)) + \chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(M_2)). \tag{5}$$

We prove (1) by induction on the length n of $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$. If $n = 1$, (1) follows for any one-dimensional representation M of B from Exercise 3.3.E.1. Now, by (5), we get the validity of (1) for general M (in the case $n = 1$), since any nonzero B-module M has a B-fixed line by using the Borel fixed point theorem (cf. [Spr–98, Theorem 6.2.6]). Assume the validity of (1) for $\mathfrak{w}[n-1]$ by induction (and any M). The Leray spectral sequence for the fibration $\psi = \psi_{\mathfrak{w},n-1} : Z_{\mathfrak{w}} \to Z_{\mathfrak{w}[n-1]}$ takes the form

$$E_2^{p,q} = H^p\left(Z_{\mathfrak{w}[n-1]}, \mathcal{L}_{\mathfrak{w}[n-1]}(H^q(P_{i_n}/B, \mathcal{L}_{s_{i_n}}(M)))\right),$$

and it converges to $H^{p+q}(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(M))$.

From this it is easy to see that

$$\begin{aligned}\sum_{p,q}(-1)^{p+q} \operatorname{ch} H^p&\left(Z_{\mathfrak{w}[n-1]}, \mathcal{L}_{\mathfrak{w}[n-1]}(H^q(P_{i_n}/B, \mathcal{L}_{s_{i_n}}(M)))\right)\\ &= \chi(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(M)).\end{aligned} \tag{6}$$

But, by the induction hypothesis and the case $n = 1$, the left side of (6) is given by

$$\begin{aligned}\sum_{q}(-1)^{q} \chi&\left(Z_{\mathfrak{w}[n-1]}, \mathcal{L}_{\mathfrak{w}[n-1]}(H^q(P_{i_n}/B, \mathcal{L}_{s_{i_n}}(M)))\right)\\ &= \bar{D}_{\mathfrak{w}[n-1]}(\bar{D}_{s_{i_n}}(\overline{\operatorname{ch} M}))\\ &= \bar{D}_{\mathfrak{w}}(\overline{\operatorname{ch} M}).\end{aligned}$$

This, together with (6), proves (1) for $\mathfrak{w}$ and thereby completes the induction.

(2) follows from (1) by Theorem 3.3.4(b) and (3) follows from (2) by Theorem 3.1.1(a). □

The following corollary follows immediately from (3.3.8.2).

3.3.9 Corollary. *For any reduced word* $\mathfrak{w}$, *the operator* $D_{\mathfrak{w}} : A(T) \to A(T)$ *depends only on the Weyl group element* $a(\mathfrak{w})$.

For $w \in W$, *we set* $D_w = D_{\mathfrak{w}}$ *for any reduced word* $\mathfrak{w}$ *with* $a(\mathfrak{w}) = w$.

(This corollary also admits a purely algebraic proof.)

3.3.10 Definition. For $\lambda \in X^*(T)^+$ recall some elementary properties of the Weyl module $V(\lambda) := H^0(G/B, \mathcal{L}(\lambda))^*$ from Section 2.1. For $w \in W$, define the *Demazure module* $V_w(\lambda)$ as the B-submodule of $V(\lambda)$ generated by a nonzero vector of weight $w\lambda$ in $V(\lambda)$. Observe that, since the λ-weight space $V(\lambda)_\lambda$ of $V(\lambda)$ is one-dimensional, so is

$$\dim V(\lambda)_{w\lambda} = 1. \tag{1}$$

Thinking of the line bundle $\mathcal{L}(\lambda)$ as $G \times_B \big(\mathrm{Hom}(k_\lambda, k)\big)$, the identification $\theta : V(\lambda)^* \to H^0(G/B, \mathcal{L}(\lambda))$ is explicitly given by

$$\theta(f)(gB) = (g, \bar{f}_g) \bmod B, \quad \text{for } g \in G \text{ and } f \in V(\lambda)^*, \tag{2}$$

where $\bar{f}_g : k_\lambda = V(\lambda)_\lambda \to k$ is defined by $\bar{f}_g(v_\lambda) = f(gv_\lambda)$.

The following corollary follows from Theorems 3.3.8 and 3.1.1(b) for $P = B$.

3.3.11 Corollary. *For $\lambda \in X^*(T)^+$ and $w \in W$, as B-modules,*

$$H^0(X_w, \mathcal{L}_w(\lambda))^* \simeq V_w(\lambda). \tag{1}$$

In particular, for any reduced word $\mathfrak{w}$ with $w = a(\mathfrak{w})$,

$$\mathrm{ch}\, V_w(\lambda) = D_{\mathfrak{w}}(e^\lambda). \tag{2}$$

Proof. By Theorem 3.1.1(b), the restriction map (under the identification θ)

$$\gamma : V(\lambda)^* \to H^0\big(X_w, \mathcal{L}_w(\lambda)\big)$$

is surjective. Since $B \cdot \bar{w} \subset X_w$ is a dense (open) subset, where $\bar{w}$ is the coset wB,

$$\begin{aligned} \operatorname{Ker} \gamma &= \{f \in V(\lambda)^* : \gamma(f)_{|B\cdot\bar{w}} \equiv 0\} \\ &= \{f \in V(\lambda)^* : f_{|V_w(\lambda)} \equiv 0\}. \end{aligned}$$

From this (1) follows and (2) follows from (3.3.8.3). □

3.3.E Exercises

For the following exercises, the characteristic of k is arbitrary.

(1*) Let P_i be any minimal parabolic subgroup of G and let $\lambda \in X^*(T) = X^*(B)$. Then, show that with the notation as in Theorem 3.3.8,

$$\chi(P_i/B, \mathcal{L}(k_\lambda)) = \bar{D}_{s_i}(e^{-\lambda}).$$

(2*) Let $\mathfrak{w}$ be any, not necessarily reduced, word. Then, the image of the morphism $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to G/B$ is an irreducible, closed, B-stable subset of G/B. Thus, $\operatorname{Im} \theta_{\mathfrak{w}} = X_w$, for some $w \in W$. Show that

$$\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w \quad \text{is a rational morphism.}$$

Hint: Consider the fibration $Z_{\mathfrak{w}} \to P_{i_1}/B$ with fiber $Z_{\mathfrak{w}'}$, where $\mathfrak{w} = (s_{i_1}, \dots, s_{i_n})$ and $\mathfrak{w}' := (s_{i_2}, \dots, s_{i_n})$. Assume the validity of the exercise for $\mathfrak{w}'$ by induction on the length n of $\mathfrak{w}$.

(3*) Prove the analogue of Theorem 3.3.4(a) for X_w replaced by $\mathcal{X}_w$. More specifically, show that for any standard parabolic subgroups P and Q of G and any $w \in W$, the canonical morphism $\pi : \mathcal{X}_w \to \mathcal{X}_w^{P,Q}$ is rational. In particular, for any locally free sheaf $\mathcal{S}$ on $\mathcal{X}_w^{P,Q}$ and any $i \geq 0$,

$$\pi^* : H^i(\mathcal{X}_w^{P,Q}, \mathcal{S}) \to H^i(\mathcal{X}_w, \pi^*\mathcal{S})$$

is an isomorphism.

3.4 Schubert varieties have rational resolutions

3.4.1 Definition. A proper birational morphism $f : X \to Y$ of varieties is called a *rational resolution* if X is nonsingular, $f_*\mathcal{O}_X = \mathcal{O}_Y$ and

$$R^i f_*(\mathcal{O}_X) = R^i f_*(\omega_X) = 0, \text{ for all } i > 0.$$

If such a resolution exists, then Y is said to *admit a rational resolution*. Recall that in characteristic 0, the requirement $R^i f_*(\omega_X) = 0$ is automatically satisfied by the Grauert–Riemenschneider vanishing theorem, cf. [GrRi–70] or [EsVi–92, p. 59].

A fundamental property of such resolutions is the following well known result (cf. [KKMS–73, p. 50–51]).

3.4.2 Lemma. *Let $f : X \to Y$ be a rational resolution. Then, Y is Cohen–Macaulay with dualizing sheaf $f_*\omega_X$.*

Proof. The assertion being local in Y, we may assume that Y is a closed subvariety of a nonsingular affine variety Z. Let $\iota : Y \to Z$ be the inclusion, and put $g = \iota \circ f$. Then, $g_*\mathcal{O}_X = \iota_*\mathcal{O}_Y$, and $R^i g_*\mathcal{O}_X = 0$ for all $i \geq 1$. Applying the local duality theorem [Har–66] to the proper morphism g and the sheaves $\mathcal{O}_X$ and ω_Z, we obtain

$$\begin{aligned} RHom(\iota_*\mathcal{O}_Y, \omega_Z) &= RHom(Rg_*\mathcal{O}_X, \omega_Z) = Rg_*RHom(\mathcal{O}_X, g^!\omega_Z) \\ &= Rg_*g^!\omega_Z = Rg_*\omega_X[-d] = g_*\omega_X[-d], \end{aligned}$$

that is, $Ext^i_Z(\iota_*\mathcal{O}_Y, \omega_Z) = 0$ for all $i \neq d$, and $Ext^d_Z(\iota_*\mathcal{O}_Y, \omega_Z) = g_*\omega_X$, where $d := \operatorname{codim} Y = \dim X - \dim Z$. This means that Y is Cohen–Macaulay with dualizing sheaf $f_*\omega_X$. □

3.4.3 Theorem. *For any reduced $\mathfrak{w}$, the resolution $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to X_w$ is rational, where $w := a(\mathfrak{w})$. If, in addition, the natural map $X_w \to X_w^P$ is birational for a standard parabolic subgroup P of G, then the composition $\theta_{\mathfrak{w}}^P : Z_{\mathfrak{w}} \to X_w^P$ is a rational resolution as well.*

Proof. By Theorem 3.3.4, it suffices to show the vanishing of $R^i\theta_{\mathfrak{w}*}\omega_{Z_{\mathfrak{w}}}$, and likewise for $\theta_{\mathfrak{w}}^P$, for $i > 0$. But by Theorem 2.2.3, $Z_{\mathfrak{w}}$ is split by σ^{p-1}, where $\sigma \in H^0(Z_{\mathfrak{w}}, \omega_{Z_{\mathfrak{w}}}^{-1})$ vanishes identically on the exceptional loci of $\theta_{\mathfrak{w}}$ and of $\theta_{\mathfrak{w}}^P$. So, the desired vanishing follows from Theorem 1.3.14. □

3.4.4 Corollary. *Any Schubert variety $X_w^P \subset G/P$ is Cohen–Macaulay.*

Moreover, for any ample line bundle $\mathcal{L} = \mathcal{L}^P(\lambda)$ on G/P, X_w^P is projectively Cohen–Macaulay in the projective embedding given by $\mathcal{L}_w := \mathcal{L}_{|X_w^P}$. In particular, for any such $\mathcal{L}$, $H^i\big(X_w^P, \mathcal{L}_w^{-1}\big) = 0$, for all $i < \dim X_w^P$.

Proof. The first assertion follows from Lemma 3.4.2 and Theorem 3.4.3.

To prove the second assertion, since X_w^P is projectively normal by Theorem 3.2.2, it suffices to show (by the discussion before Corollary 1.5.4) that

$$H^i(X_w^P, \mathcal{L}_w^n) = 0, \quad \text{for all } 0 < i < \dim X_w^P \text{ and all } n \in \mathbb{Z}.$$

Since X_w^P is Cohen–Macaulay, this holds for all $0 \le i < \dim X_w^P$ and all $n \ll 0$ (by [Har–77, Chap. III, Theorem 7.6(b)]) and hence for all $n < 0$, by the splitting of X_w^P. For $n \ge 0$, this follows from Theorem 3.1.1(a).

The "In particular" statement follows from Theorem 1.2.9. This proves the corollary. □

3.4.5 Remark. The assertion that for any $w \in W$ and dominant regular $\lambda \in X^*(T)$,

$$H^i(X_w, \mathcal{L}_w(-\lambda)) = 0, \text{ for all } i < \ell(w),$$

can also be obtained immediately from Theorems 3.1.4 and 3.3.4(b) as follows. By Theorem 3.3.4(b),

$$H^i(X_w, \mathcal{L}_w(-\lambda)) \simeq H^i(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(-\lambda)),$$

where $\mathfrak{w}$ is a reduced word with $a(\mathfrak{w}) = w$. By Serre duality and Proposition 2.2.2,

$$\begin{aligned} H^i\big(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(-\lambda)\big) &\simeq H^{\ell(w)-i}\big(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda) \otimes \omega_{Z_{\mathfrak{w}}}\big)^* \\ &= H^{\ell(w)-i}\Big(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}(\lambda - \rho) \otimes \mathcal{O}\big(-\sum_{j=1}^{n} Z_{\mathfrak{w}(j)}\big)\Big)^* \\ &= 0\,, \text{ by (3.1.4.1).} \end{aligned}$$

Let X be a complete variety and let $\mathcal{L}_1, \dots, \mathcal{L}_r$ be semi-ample line bundles on X. Then, recall the definition of the multicone $C(X; \mathcal{L}_1, \dots, \mathcal{L}_r) := \operatorname{Spec} R(X; \mathcal{L}_1, \dots, \mathcal{L}_r)$ from Exercise 1.1.E.2.

3.4.6 Proposition. *Let X be a projective variety admitting a rational resolution and let $\mathcal{L}_1, \dots, \mathcal{L}_r$ be semi-ample line bundles on X such that $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_r$ is ample. Then, the multicone $C(X; \mathcal{L}_1, \dots, \mathcal{L}_r)$ admits a rational resolution if the following conditions (a)-(b) are satisfied for all $m_1, \dots, m_r \geq 0$.*

(a) $H^i\big(X, \mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_r^{m_r}\big) = 0$, *for all* $i \geq 1$.

(b) $H^i\big(X, \mathcal{L}_1^{-1-m_1} \otimes \cdots \otimes \mathcal{L}_r^{-1-m_r}\big) = 0$, *for all* $i < \dim X$.

Proof. Let $\mathbb{V}$ be the total space of the vector bundle $\mathcal{L}_1^{-1} \oplus \cdots \oplus \mathcal{L}_r^{-1}$ on X, with the projection map

$$f : \mathbb{V} \to X.$$

Then,

$$f_* \mathcal{O}_{\mathbb{V}} = \bigoplus_{m_1, \dots, m_r \geq 0} \mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_r^{m_r},$$

so that $\Gamma(\mathbb{V}, \mathcal{O}_{\mathbb{V}}) = R(X; \mathcal{L}_1, \dots, \mathcal{L}_r)$. This yields a dominant morphism

$$\pi : \mathbb{V} \to C(X; \mathcal{L}_1, \dots, \mathcal{L}_r).$$

Observe that π is projective (hence surjective) and satisfies $\pi_* \mathcal{O}_{\mathbb{V}} = \mathcal{O}_{C(X; \mathcal{L}_1, \dots, \mathcal{L}_r)}$, by Exercise 1.1.E.3. Thus, the fibers of π are connected by [Har–77, Chap. III, Corollary 11.3]. On the other hand, the assumption that $\mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_r$ is ample easily implies that $\dim \mathbb{V} = \dim C(X; \mathcal{L}_1, \dots, \mathcal{L}_r)$. It follows that π is birational.

The assumption (a) amounts to $H^i(\mathbb{V}, \mathcal{O}_{\mathbb{V}}) = 0$ for all $i \geq 1$. Since $C(X; \mathcal{L}_1, \dots, \mathcal{L}_r)$ is affine, it follows that $R^i \pi_* \mathcal{O}_{\mathbb{V}} = 0$ for all $i \geq 1$. Moreover, since X is Cohen–Macaulay (Lemma 3.4.2), $\mathbb{V}$ is Cohen–Macaulay as well, with dualizing sheaf

$$\omega_{\mathbb{V}} = f^*(\omega_X \otimes \mathcal{L}_1 \otimes \cdots \otimes \mathcal{L}_r).$$

Now, (b), together with Serre duality, implies that $R^i \pi_* \omega_{\mathbb{V}} = 0$ for all $i \geq 1$.

On the other hand, any rational resolution $\varphi : \widetilde{X} \to X$ yields a rational resolution $\psi : \varphi^* \mathbb{V} \to \mathbb{V}$. By Lemma 3.4.2 and the Grothendieck spectral sequence [Gro–57], $\pi \circ \psi : \varphi^* \mathbb{V} \to C(X; \mathcal{L}_1, \dots, \mathcal{L}_r)$ is a rational resolution. □

As a consequence of the above proposition and Theorem 3.4.3, we obtain the following.

3.4.7 Theorem. *Let P be any standard parabolic subgroup of G and let $X_w^P \subset G/P$ be a Schubert variety. Then, for any dominant $\lambda_1, \dots, \lambda_r \in X^*(P)$, the multicone $C\big(X_w^P; \mathcal{L}_w^P(\lambda_1), \dots, \mathcal{L}_w^P(\lambda_r)\big)$ admits a rational resolution.*

Proof. By taking a larger parabolic $Q \supset P$ (if needed) and using Theorem 3.3.4(a), we can assume that $\lambda_1 + \cdots + \lambda_r - \rho_P$ is dominant, where ρ_P is as in Exercise 2.2.E.4. In view of Proposition 3.4.6 and Theorem 3.4.3, it suffices to show that for any dominant $\lambda \in X^*(P)$, the following are satisfied.

(1) $$H^i\big(X_w^P, \mathcal{L}_w^P(\lambda)\big) = 0, \quad \text{for } i > 0, \text{ and}$$

(2) $$H^i\big(X_w^P, \mathcal{L}_w^P(-\lambda - \rho_P)\big) = 0, \quad \text{for all } i < \dim X_w^P.$$

Now, (1) is nothing but Theorem 3.1.1(a) and (2) follows from Corollary 3.4.4. □

3.4.E Exercises

For the following exercises, the characteristic of k is arbitrary.

(1) Let $X_w \subset G/B$ be any Schubert variety. Then, show that the canonical sheaf of X_w is given by $\mathcal{O}_{X_w}(-\partial X_w) \otimes \mathcal{L}_w(-\rho)$, where ∂X_w is the union of codimension one Schubert subvarieties of X_w.

(2) Show that for any $w \in W$, standard parabolic subgroups P, Q of G and any dominant $\lambda \in X^*(P)$, $\mu \in X^*(Q)$, the linear system on $\mathcal{X}_w^{P,Q}$ given by $\mathcal{L}_w^{P,Q}\big((\lambda + \rho_P) \boxtimes (\mu + \rho_Q)\big)$ embeds $\mathcal{X}_w^{P,Q}$ as a projectively normal and projectively Cohen–Macaulay variety.

3.5 Homogeneous coordinate rings of Schubert varieties are Koszul algebras

3.5.1 Definition. Let $\mathcal{L}$ be a line bundle on a scheme X. Then, consider the $\mathbb{Z}_+$-graded algebra

$$R(X, \mathcal{L}) := \bigoplus_{m \geq 0} H^0(X, \mathcal{L}^m)$$

with respect to the line bundle $\mathcal{L}$. The line bundle $\mathcal{L}$ on X is said to be *normally presented* if the canonical $\mathbb{Z}_+$-graded algebra homomorphism

$$\xi : \bigoplus_{m \geq 0} S^m(H^0(X, \mathcal{L})) \to R(X, \mathcal{L})$$

is surjective and the kernel $\operatorname{Ker}(\xi)$ is generated as an ideal in the symmetric algebra $S(H^0(X, \mathcal{L}))$ by its elements of degree 2; that is, $R(X, \mathcal{L})$ is a quadratic algebra in the sense of Definition 1.5.5.

If Y is a closed subscheme of X, then Y is said to be *linearly defined in X with respect to $\mathcal{L}$* if the restriction map

$$R(X, \mathcal{L}) \to R(Y, \mathcal{L}_{|Y})$$

is surjective and its kernel is generated by its degree 1 elements. If $H^0(Y, \mathcal{O}_Y)$ is one-dimensional, Y is linearly defined in $X$with respect to $\mathcal{L}$ if and only if $R(Y, \mathcal{L}_{|Y})$ is a quadratic $R(X, \mathcal{L})$-module in the sense of Definition 1.5.5.

3.5.2 Theorem. *Let $P \subset G$ be any standard parabolic subgroup and let $v \le w \in W$. Then, for any dominant $\lambda \in X^*(P)$, the line bundle $\mathcal{L}_w^P(\lambda)$ on X_w^P is normally presented. Moreover, X_v^P is linearly defined in X_w^P with respect to $\mathcal{L}_w^P(\lambda)$.*

Proof. By Proposition 1.5.8, it suffices to show that $(G/P)^3$ is $\big((G/P)^2 \times \partial^- G/P\big)$-split compatibly splitting $X_w^P \times (G/P)^2$, $X_v^P \times (G/P)^2$, $\mathcal{X}_e^{P,P} \times G/P$, $G/P \times \mathcal{X}_e^{P,P}$. Now, applying Theorem 2.3.10, we get such a splitting of $(G/P)^3$. Since X_v^P and X_w^P are both linearly defined in G/P, X_v^P is linearly defined in X_w^P (Remark 1.5.6(iii)). □

As another consequence of Theorem 2.3.10, we get the following strengthening of Theorem 3.5.2. Recall that the Koszul algebras and Koszul modules are defined in 1.5.9.

3.5.3 Theorem. *Let $P \subset G$ be any standard parabolic subgroup. Then, for any $v \le w \in W$ and dominant $\lambda \in X^*(P)$, the algebra $R(X_w^P, \mathcal{L}_w^P(\lambda))$ is Koszul. In particular, $R(G/P, \mathcal{L}^P(\lambda))$ is a Koszul algebra. Moreover, $R(X_v^P, \mathcal{L}_v^P(\lambda))$ is a Koszul module over $R(X_w^P, \mathcal{L}_w^P(\lambda))$.*

Proof. Apply Theorem 1.5.15 together with Theorem 2.3.10. □

Even though the results in this chapter were obtained under the assumption of positive characteristic, most of them remain true in characteristic 0 and follow from the corresponding results in positive characteristic by applying the semicontinuity theorems from Section 1.6 (specifically Proposition 1.6.2 and Corollary 1.6.3). More precisely, we state the following.

3.5.4 Theorem. *Theorems 3.1.1, 3.1.2, 3.1.4, 3.2.2, 3.3.4, 3.3.8, 3.4.3 and 3.4.7; Remark 3.1.3; and Corollaries 3.3.11 and 3.4.4 remain true over an algebraically closed field of an arbitrary characteristic.*

Proof. For Theorems 3.1.1, 3.1.2 and Remark 3.1.3, use these results in characteristic $p > 0$ and Proposition 1.6.2 and Corollary 1.6.3. Similarly, for Theorem 3.1.4 use this result in characteristic $p > 0$ and Proposition 1.6.2 together with Exercise 3.1.E.3(d). For the normality of X_w^P (Theorem 3.2.2), follow the same argument as in Remark 3.2.3; alternatively the proof outlined in Exercise 3.2.E.1 works over any k. The projective normality of X_w^P (Theorem 3.2.2) follows by the same argument. Theorem 3.3.4 follows by the same argument since Proposition 3.2.1, Lemmas 3.3.2 and 3.3.3 are characteristic free. Theorems 3.3.8, 3.4.3 and Corollary 3.3.11 follow by the same proof; observe that $R^i\theta_{\mathfrak{w}*}\omega_{Z_{\mathfrak{w}}}$ is automatically zero in characteristic 0 (Definition 3.4.1). Corollary 3.4.4, first part, follows by the same argument since Lemma 3.4.2 is characteristic free; and the second part follows by the same argument once we use Proposition 1.6.2. Theorem 3.4.7 follows by the same argument since Proposition 3.4.6 is characteristic free. □

3.5.5 Remark. All the above results (with possibly an exception of Remark 3.1.3) in characteristic 0 can also be proved directly by characteristic 0 methods (cf. [Kum–87, 88, 02]).

3.5.E Exercises

(1) Let $\mathcal{L}_1, \ldots, \mathcal{L}_r$ be line bundles on a scheme X. Then, the $\mathbb{Z}_+^r$-graded algebra

$$R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r) = \bigoplus_{(m_1,\ldots,m_r)\in\mathbb{Z}_+^r} H^0(X, \mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_r^{m_r})$$

is called the *multihomogeneous coordinate ring* of X with respect to the line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r$. Any nonzero element of $H^0(X, \mathcal{L}_1^{m_1} \otimes \cdots \otimes \mathcal{L}_r^{m_r})$ is said to be of *total degree* $\sum m_i$. Consider the canonical $\mathbb{Z}_+^r$-graded algebra homomorphism

$$\psi : \bigoplus_{(m_1,\ldots,m_r)\in\mathbb{Z}_+^r} S^{m_1}\big(H^0(X, \mathcal{L}_1)\big) \otimes \cdots \otimes S^{m_r}\big(H^0(X, \mathcal{L}_r)\big) \to R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r).$$

Then, the line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r$ on X are said to be *normally presented*, if ψ is surjective and its kernel is generated by its elements of total degree 2.

Let Y be a closed subscheme of X. If the restriction map $R(X; \mathcal{L}_1, \ldots, \mathcal{L}_r) \to R(Y; \mathcal{L}_{1|Y}, \ldots, \mathcal{L}_{r|Y})$ is surjective and its kernel is generated as an ideal by its elements of total degree 1, then Y is said to be *linearly defined* in X with respect to the line bundles $\mathcal{L}_1, \ldots, \mathcal{L}_r$.

Let $P \subset G$ be a standard parabolic subgroup and let $\lambda_1, \ldots, \lambda_r$ be dominant weights in $X^*(P)$. Then, for any $v \leq w \in W$, prove the following:

(a) The line bundles $\mathcal{L}_w^P(\lambda_1), \ldots, \mathcal{L}_w^P(\lambda_r)$ on X_w^P are normally presented.

(b) X_v^P is linearly defined in X_w^P with respect to the line bundles $\mathcal{L}_w^P(\lambda_1), \ldots, \mathcal{L}_w^P(\lambda_r)$.

(c) The ring $R\big(X_v^P; \mathcal{L}_v^P(\lambda_1), \ldots, \mathcal{L}_v^P(\lambda_r)\big)$ is Koszul, and it is a Koszul module over $R\big(X_w^P; \mathcal{L}_w^P(\lambda_1), \ldots, \mathcal{L}_w^P(\lambda_r)\big)$.

Hint: Use the Exercises 1.5.E.

(2) Show that for any standard parabolic subgroups P, Q of G and any dominant $\lambda \in X^*(P)$, $\mu \in X^*(Q)$, the line bundle $\mathcal{L}_w^{P,Q}(\lambda \boxtimes \mu)$ on $\mathcal{X}_w^{P,Q}$ is normally presented for any $w \in W$. Moreover, for $v \leq w$, $\mathcal{X}_v^{P,Q}$ is linearly defined in $\mathcal{X}_w^{P,Q}$ with respect to $\mathcal{L}_w^{P,Q}(\lambda \boxtimes \mu)$.

Hint: Take an appropriate splitting of X^3 and use Proposition 1.5.8, where $X = G/P \times G/Q$. This splitting is slightly different from that of Theorem 2.3.10, but obtained by a similar method.

(3) Show that for $X = G/B$, X^3 does *not* admit a splitting such that all the partial diagonals are compatibly split.

Hint: Take $X = \mathbb{P}^1$ and consider the ample line bundle $\mathcal{L}$ on X^3 with multidegree $(1, 1, 1)$. Then, show that the restriction map $H^0(X^3, \mathcal{L}) \to H^0(Y, \mathcal{L})$ is not surjective, where Y is the union of the three partial diagonals.

Alternatively, show that the intersection of any partial diagonal with the union of the remaining two partial diagonals is not reduced.

3.C. Comments

In their full generality Theorems 3.1.1(a)–(b) and 3.1.2(c) are due to Andersen [And–85] and Ramanan–Ramanathan [RaRa–85]; and Remark 3.1.3(a) is due to Ramanathan [Ram–85]. For an alternative proof of Theorem 3.1.2(c), see [Ram–87]. Theorem 3.1.1(a)–(b) in the case when $\mathcal{L}^P(\lambda)$ is ample was earlier obtained by Mehta–Ramanathan [MeRa–85] and Theorem 3.1.1(c) follows readily from [loc cit.]. Also recall that Theorem 3.1.1(a)–(b) for "special" Schubert varieties was obtained by Kempf [Kem–76a] and so was their normality and Cohen–Macaulayness; and different (from that of Kempf) proofs of Theorem 3.1.1(a) for $X_w^P = G/B$ were given by Andersen [And–80a] and Haboush [Hab–80]. Theorem 3.1.2 in characteristic 0 was obtained first by Kumar [Kum-88] and over an arbitrary characteristic it can be deduced from [MeRa–88]. Theorem 3.1.4 in characteristic 0 (and for line bundles $\mathcal{L}$ on $Z_{\mathfrak{w}}$ which are pullback from the globally generated line bundles on G/B) is due to Kumar [Kum-87]. In its present form (in an arbitrary characteristic) it is due to Lauritzen–Thomsen [LaTh–04]. Exercise 3.1.E.3 is taken from [loc cit.] and Exercise 3.1.E.1 is taken from [Jan–03, Part II, §§ 4.4 and 8.5].

Theorem 3.2.2 in its full generality is due to Ramanan–Ramanathan [RaRa–85]. The normality of X_w^P was also proved by Andersen [And–85]. The proof of the normality of X_w^P given in 3.2.2 is influenced by the proof due to Mehta–Srinivas [MeSr–87]; the proof given in [RaRa–85] is outlined in 3.2.3; and yet another proof of the normality of X_w^P in an arbitrary characteristic due to Seshadri [Ses–87] (though with some simplifications) is outlined in Exercise 3.2.E.1. This proof of Seshadri predates all the other proofs of normality in an arbitrary characteristic. It can be modified to prove the normality and Cohen–Macaulayness of a certain class of subvarieties of G/P which includes all the Schubert subvarieties (cf. [Bri–03b]).

As is well known (and pointed out by V. Kac), the original proof of the Demazure character formula (3.3.8.3) as in [Dem–74] has a serious gap. Subsequently, Joseph [Jos–85] proved the Demazure character formula in characteristic 0 for "large" dominant weights. Since the validity of the Demazure character formula over k (for any algebraically closed field k) for large powers of an ample line bundle on G/B is equivalent to the normality of the Schubert varieties over k, Joseph's above cited work provided the first proof of the normality of Schubert varieties over k of characteristic 0. Now, Theorem 3.1.1(a)–(b) over any k for $P = B$ implies the Demazure character formula over k and thus the results of Ramanan–Ramanathan [RaRa–85] provide a proof of the Demazure character formula in an arbitrary characteristic. Similarly, the normality of the Schubert varieties over k together with the validity of Theorem 3.1.1(a)–(b) over k for $P = B$ and only for ample line bundles $\mathcal{L}(\lambda)$ again imply the Demazure character formula over k. Thus, the works [Ses–87] and [MeRa–85] together provide another proof of the Demazure character formula. Yet another proof of the Demazure character formula is due to Andersen [And–85]. Still another proof of the Demazure character formula in characteristic 0 was given by Kumar [Kum–87] crucially using Theorem

3.1.4. Lemma 3.3.3(a) is due to Kempf (cf. [Dem–74]) and 3.3.3(b) is taken from [Kum–87].

Theorem 3.4.3 and Corollary 3.4.4 in their full generality are due to Ramanathan [Ram–85]; Theorem 3.4.3 in characteristic 0 was also obtained by Andersen [And–85]. ([Mehta–Srinivas [MeSr–89] gave a different proof of the Cohen–Macaulayness of X_w^P.) Such results were proved earlier by Hochster [Hoch–73], Kempf [Kem–76a,b], Laksov [Lak–72], Musili [Mus–72] for Schubert varieties in Grassmannians; by de Concini–Lakshmibai [DeLa–81] for the Schubert varieties X_w^P for P of "classical" type; by Musili–Seshadri [MuSe–83] for $X_w \subset SL_n/B$. Proposition 3.4.6 and Theorem 3.4.7 are due to Kempf–Ramanathan [KeRa–87] and so is Exercise 3.5.E.1(a)–(b). Theorem 3.5.2 and Exercise 3.4.E.1 are due to Ramanathan [Ram–87], however his proof of Theorem 3.5.2 is different. Recall that by a well known result of Kostant, the full ideal of G/P inside $\mathbb{P}(H^0(G/P, \mathcal{L}^P(\lambda))^*)$, for an ample line bundle $\mathcal{L}^P(\lambda)$, is generated by an "explicit" set of quadratic equations generalizing the Plücker equations (cf. [Gar–82], [Kum–02, §10.1]). Theorem 3.5.3 and Exercise 3.5.E.1(c) are due to Inamdar–Mehta [InMe–94a,94b]. For Theorems 3.5.2 and 3.5.3, also see [Bez–95]. Earlier, Kempf [Kem–90] had proved that the homogeneous coordinate ring of a Grassmannian in its Plücker embedding is Koszul.

Several of the results of this chapter for the case of classical groups were obtained earlier by the Standard Monomial Theory developed by Seshadri–Lakshmibai–Musili (cf. the survey article [LaSe–91]).

It may be mentioned that Kashiwara [Kas–93] has given a proof of the Demazure character formula using his crystal base and Littelmann [Lit–98] has given another proof using his "LS path model." Many of the results of this chapter have been obtained algebraically via the quantum groups at roots of unity by Kumar–Littelmann [KuLi–02].

It will be very interesting to see if the results of the Standard Monomial Theory (as completed by Littelmann [Lit–94, 95, 98]) can be recovered by Frobenius splitting methods. Some results in this direction have been obtained by Brion–Lakshmibai [BrLa–03], where the classical groups are handled.

Chapter 4

Canonical Splitting and Good Filtration

Introduction

This chapter is devoted to the study of B-canonical splittings of a B-scheme and its various consequences, including the existence of good filtrations for the space of global sections of G-linearized line bundles on G-schemes admitting B-canonical splittings. In addition, we prove the Parthasarathy–Ranga Rao–Varadarajan–Kostant (for short PRVK) conjecture and its refinement (proved by Kumar in characteristic 0 and Mathieu in characteristic p).

Section 4.1 is devoted to the study of B-canonical splittings. We begin by defining the notion of a B-canonical Frobenius-linear endomorphism of a commutative B-algebra R over k. Let $\mathrm{End}_F(R)$ be the additive group of all the Frobenius-linear endomorphisms of R. Then, it is canonically an R-module with an action of B. It is shown that the B-canonical Frobenius-linear endomorphisms ϕ of R arise from B-module maps $\mathrm{St} \otimes k_{(p-1)\rho} \to \mathrm{End}_F(R)$ (Lemma 4.1.2). Further, as shown in Proposition 4.1.8, any B-canonical $\phi \in \mathrm{End}_F(R)$ takes B-submodules of R to B-submodules. In fact, if R is a G-algebra, it is shown that any B-canonical ϕ takes G-submodules of R to G-submodules (Proposition 4.1.10). The notion of B-canonical Frobenius-linear endomorphism of a B-algebra can easily be "sheafified" to allow one to define the notion of B-canonical splittings of a B-scheme X. It is shown that the flag varieties G/P admit a unique B-canonical splitting. Moreover, this compatibly splits all the Schubert subvarieties X_w^P.

Further, it is shown that for a B-scheme X which admits a B-canonical splitting σ, the G-scheme $\widetilde{X} := G \times_B X$ admits a B-canonical splitting $\widetilde{\sigma}$ extending the original B-canonical splitting on $X = e \times X \subset \widetilde{X}$ (Proposition 4.2.17). Moreover, if σ compatibly splits a closed B-subscheme Y of X, then $\widetilde{\sigma}$ compatibly splits all the closed subschemes $\{\overline{BwB} \times_B X, G \times_B Y; w \in W\}$ (Exercise 4.1.E.4). In Exercise 4.1.E.2, the BSDH varieties are asserted to admit B-canonical splittings compatibly splitting the BSDH subvarieties. Also, for any B-equivariant morphism $f : X \to Y$ between B-schemes such that X admits a B-canonical splitting and $f_*\mathcal{O}_X = \mathcal{O}_Y$, then the induced splitting of Y is B-canonical (Exercise 4.1.E.3). Let X, Y be two G-schemes which admit B-canonical splittings. Then, so is their product $X \times Y$ (Exercise 4.1.E.5). In particular, for any parabolic subgroups P_i, $1 \le i \le n$, $G/P_1 \times \cdots \times G/P_n$ admits a B-canonical splitting under the diagonal action of B.

Recall that a filtration $F^0 = (0) \subset F^1 \subset \cdots$ of a G-module M by G-submodules F^i is called a good filtration if the successive quotients are isomorphic to direct sums of dual Weyl modules $H^0(G/B, \mathcal{L}(\lambda))$. The relevance of this property is, of course, in characteristic $p > 0$. The modules which admit a good filtration have many features akin to characteristic 0 theory. This makes such modules "easier" to handle. It is shown that the $G \times G$-module $k[G]$ admits a good filtration for the $G \times G$-action (Theorem 4.2.5). This is used to give the most useful cohomological criterion to decide when a G-module M admits a good filtration. It is shown that M admits a good filtration iff $\mathrm{Ext}^1_G(V(\lambda), M) = 0$ for all the Weyl modules $V(\lambda)$ (Theorem 4.2.7 and Remark 4.2.8).

We now come to the most important representation-theoretic result for G-schemes X which admit B-canonical splittings, proved by Mathieu. This result asserts that for any such X (i.e., X is a G-scheme admitting a B-canonical splitting), and any G-linearized line bundle $\mathcal{L}$ on X, the G-module $H^0(X, \mathcal{L})$ admits a good filtration (Theorem 4.2.13). In fact, it is this result which makes the notion of B-canonical splittings a very useful tool in characteristic $p > 0$ representation theory.

For the reader's convenience, we break the proof of this theorem into several steps. The basic idea of the proof is as follows. We first take the product G-scheme $Y := G/B \times X$ under the diagonal action of G and consider its open subset $\overset{\circ}{Y} := Bw_oB/B \times X$. Let $\widetilde{\mathcal{L}}$ be the line bundle $\varepsilon \boxtimes \mathcal{L}$ on Y, ε being the trivial line bundle on G/B. Let $\overset{\circ}{\mathcal{C}}$ be the graded B-algebra $\bigoplus_{n \ge 0} \overset{\circ}{\mathcal{C}}_n$, where $\overset{\circ}{\mathcal{C}}_n := H^0(\overset{\circ}{Y}, \widetilde{\mathcal{L}}^n)$ and $\mathcal{C}$ its graded subalgebra $\bigoplus_{n \ge 0} \mathcal{C}_n$, where $\mathcal{C}_n := H^0(Y, \widetilde{\mathcal{L}}^n)$, which actually is a graded G-algebra. Fix a height function $h : X^*(T) \to \mathbb{R}$ (as in 4.2.1) and $\lambda \in X^*(T)^+$ and let $\mathcal{F}_{n\lambda}(\mathcal{C}_n)$, resp. $\mathcal{F}^-_{n\lambda}(\mathcal{C}_n)$, be the largest B-submodule of $\mathcal{C}_n$ such that each weight μ of $\mathcal{F}_{n\lambda}(\mathcal{C}_n)$, resp. $\mathcal{F}^-_{n\lambda}(\mathcal{C}_n)$, satisfies $h(\mu) \le h(n\lambda)$, resp. $h(\mu) < h(n\lambda)$; $\mathcal{F}_{n\lambda}(\overset{\circ}{\mathcal{C}}_n)$ and $\mathcal{F}^-_{n\lambda}(\overset{\circ}{\mathcal{C}}_n)$ are defined similarly. Now, consider the graded G-subalgebra $\mathcal{C}(\lambda) := \bigoplus_n \mathcal{F}_{n\lambda}(\mathcal{C}_n)$ of $\mathcal{C}$ and G-stable graded ideal $\mathcal{C}(\lambda)^- = \bigoplus_n \mathcal{F}^-_{n\lambda}(\mathcal{C}_n)$ of $\mathcal{C}(\lambda)$; $\overset{\circ}{\mathcal{C}}(\lambda)$ and $\overset{\circ}{\mathcal{C}}(\lambda)^-$ are defined similarly. Finally, consider the quotient algebras $C(\lambda) := \mathcal{C}(\lambda)/\mathcal{C}(\lambda)^-$ and $\overset{\circ}{C}(\lambda) := \overset{\circ}{\mathcal{C}}(\lambda)/\overset{\circ}{\mathcal{C}}(\lambda)^-$. Then, $C(\lambda)$ is a graded G-algebra such that the component $C(\lambda)_n$ is $n\lambda$-isotypical as a B-module. Similarly, $\overset{\circ}{C}(\lambda)$ is a graded $(\mathfrak{U}_G, B)$-algebra such that $\overset{\circ}{C}(\lambda)_n$ is $n\lambda$-isotypical, where $\mathfrak{U}_G$ is the hyperalgebra of G. We next show, using the B-canonical splitting of X, that the algebras $C(\lambda)$ and $\overset{\circ}{C}(\lambda)$ are reduced (i.e., they do not contain nonzero nilpotent elements). Further, we show that $\overset{\circ}{C}(\lambda)$ is an injective B-module. Finally, we show that $C(\lambda)_1 = \mathcal{D}_\lambda(C(\lambda)_1)$

$:= H^0(G/B, \mathcal{L}((C(\lambda)_1)_{w_o\lambda}))$. This shows that $\mathcal{C}_1$ has a filtration such that the successive quotients are isomorphic to $\{\mathcal{D}_\lambda(C(\lambda)_1)\}_{\lambda\in X^*(T)^+}$, proving that $\mathcal{C}_1$ admits a good filtration, thereby finishing the proof of the theorem.

As an immediate consequence of the above result, one obtains that the tensor product G-module $H^0(G/B, \mathcal{L}(\lambda)) \otimes H^0(G/B, \mathcal{L}(\mu))$ admits a good filtration (Corollary 4.2.14), originally proved (in almost all the cases) by Donkin by long case-by-case analysis. Similarly, for a parabolic subgroup P of G with the Levi subgroup L_P, the G-module $H^0(G/B, \mathcal{L}(\lambda))$ admits a good filtration as a L_P-module (Corollary 4.2.15).

Section 4.3 is devoted to the proof of the PRVK conjecture (proved by Kumar in characteristic 0 and Mathieu in characteristic p). It asserts that for $\lambda, \mu \in X^*(T)^+$ and $w \in W$, there exists a unique nonzero G-module homomorphism (unique up to scalar multiples) $V(\overline{-\lambda - w\mu}) \to H^0(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))$, where $\overline{-\lambda - w\mu}$ is the unique dominant weight in the W-orbit of $-\lambda - w\mu$ and $\mathcal{X}_w \subset G/B \times G/B$ is the G-Schubert variety as in Section 2.2 (Theorem 4.3.2). Moreover, the dual $H^0(\mathcal{X}_w, \mathcal{L}(\lambda \boxtimes \mu))^*$ is canonically isomorphic to the G-submodule of $V(\lambda) \otimes V(\mu)$ generated by $v_\lambda \otimes v_{w\mu}$. Apart from the above identification, which relies on the H^0-surjectivity result of Section 3.1, the main ingredients of the proof are: (1) The identification of $H^0(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))$ with $H^0(G/B, \mathcal{L}(k_{-\lambda} \otimes V_w(\mu)^*))$, $V_w(\mu) \subset V(\mu)$ being the Demazure submodule, and (2) a result of Joseph and Polo on the annihilator of the B-module $V_w(\mu)$ (Proposition 4.3.1).

In fact, we prove a refinement of the above result due to Kumar asserting that for λ, μ and w as above, let $\{W_\lambda w_1 W_\mu, W_\lambda w_2 W_\mu, \ldots, W_\lambda w_n W_\mu\}$ be the distinct double cosets in W such that $\overline{\lambda + w_i\mu} = \overline{\lambda + w\mu}$ for all i. Then,

$$\dim \operatorname{Hom}_G\Big(V\big(\overline{-\lambda - w\mu}\big), H^0\Big(\bigcup_{i=1}^m \mathcal{X}_{w_i}^{P_\lambda, P_\mu}, \mathcal{L}^{P_\lambda, P_\mu}(\lambda \boxtimes \mu)\Big)\Big) = m,$$

for all $1 \le m \le n$, where W_λ is the isotropy of λ and P_λ is the parabolic subgroup $BW_\lambda B$ (Theorem 4.3.5). Thus, the dual Weyl module $H^0(G/B, \mathcal{L}(\overline{-\lambda - w\mu}))$ appears in $V(\lambda)^* \otimes V(\mu)^*$ with multiplicity at least n. These results immediately imply the corresponding results in characteristic 0 by semicontinuity (Theorem 4.3.8). In Exercise 4.3.E.1, a formula for the Euler–Poincaré characteristic of $\mathcal{X}_w$ with coefficients in any line bundle $\mathcal{L}_w(\lambda \boxtimes \mu)$ is given in terms of the Demazure operators generalizing a well-known result of Brauer.

4.1 Canonical splitting

We follow the notation as in Section 2.1. In particular, G is a connected, simply-connected, semisimple algebraic group over an algebraically closed field k of characteristic $p > 0$, B is a Borel subgroup of G and $T \subset B$ a maximal torus. For any root β, let U_β be the corresponding root subgroup. Then, as in 2.1, there exists an algebraic group isomorphism $\varepsilon_\beta : \mathbb{G}_a \to U_\beta$ satisfying

$$t\varepsilon_\beta(z)t^{-1} = \varepsilon_\beta(\beta(t)z),$$

for $z \in \mathbb{G}_a$ and $t \in T$. For any $\beta \in \Delta^+$, similar to the root vector $f_\beta \in \mathfrak{g}_{-\beta}$ (as in (2.3.1.3)), there exists a root vector $e_\beta \in \mathfrak{g}_\beta$ such that for any algebraic representation V of G, $v \in V$ and $z \in \mathbb{G}_a$,

$$\varepsilon_\beta(z)\, v = \sum_{m \geq 0} z^m \left(e_\beta^{(m)} \cdot v\right),$$

where $e_\beta^{(m)}$ denotes the m-th divided power of e_β. If β is a simple root α_i, we abbreviate e_β by e_i. Similarly, we abbreviate f_β by f_i.

4.1.1 Definition. Let R be a (not necessarily finitely generated) commutative associative k-algebra with multiplicative identity 1. Then, an additive map $\phi : R \to R$ is called a *Frobenius-linear endomorphism* if it satisfies the following:

$$\text{(1)} \qquad \phi(a^p b) = a\phi(b), \quad \text{for } a, b \in R.$$

Let $\mathrm{End}_F(R)$ be the additive group of all the Frobenius-linear endomorphisms of R. Then, $\mathrm{End}_F(R)$ is an R-module under

$$\text{(2)} \qquad (a * \phi)(b) = \phi(ab), \quad \text{for } a, b \in R \text{ and } \phi \in \mathrm{End}_F(R).$$

In particular, k acts on $\mathrm{End}_F(R)$ via

$$\text{(3)} \qquad (z * \phi)(b) = \phi(zb) = z^{1/p}\phi(b).$$

Assume now that R is a B-algebra, i.e., B acts algebraically on R via k-algebra automorphisms (in particular, B acts locally finitely on R). Then, B acts k-linearly on $\mathrm{End}_F(R)$ via

$$\text{(4)} \qquad (x * \phi)(a) = x(\phi(x^{-1}a)), \quad \text{for } x \in B, a \in R \text{ and } \phi \in \mathrm{End}_F(R).$$

A Frobenius-linear endomorphism $\phi \in \mathrm{End}_F(R)$ is called *B-canonical* if the following two conditions are satisfied:

(c_1) $t * \phi = \phi$, for all $t \in T$, and

(c_2) For any simple root α_i, $1 \leq i \leq \ell$, there exist $\phi_{i,j} \in \mathrm{End}_F(R)$, $0 \leq j \leq p-1$, such that

$$\text{(5)} \qquad \varepsilon_{\alpha_i}(z) * \phi = \sum_{j=0}^{p-1} z^j * \phi_{i,j}, \text{ for all } z \in k.$$

In fact, if we only require that $\phi_{i,j} : R \to R$ are additive maps satisfying (5), then automatically $\phi_{i,j} \in \mathrm{End}_F(R)$. It is easy to see from (c_1) that $\phi(R_\lambda) \subset R_{\lambda/p}$, for any $\lambda \in X^*(T)$, where R_λ is the weight space of R corresponding to the weight λ. In particular,

$$\text{(6)} \qquad \phi(R_\lambda) = 0 \quad \text{unless } \frac{\lambda}{p} \in X^*(T).$$

4.1.2 Lemma. *Let R be a B-algebra and let $\phi \in \operatorname{End}_F(R)$. Assume further that the B-submodule of $\operatorname{End}_F(R)$ generated by ϕ is a finite-dimensional algebraic B-module.*

(1) *Then, ϕ is B-canonical iff $e_i^{(n)} * \phi = 0$ for all $1 \leq i \leq \ell$ and $n \geq p$ and, moreover, ϕ is T-invariant,*

where $e_i^{(n)} * \phi$ denotes the action of $e_i^{(n)}$ on ϕ obtained by differentiating the action of $\varepsilon_{\alpha_i}(\mathbb{G}_a)$ on ϕ.

Thus, ϕ is B-canonical iff there exists a B-module (k-linear) map

$$\theta_\phi : \mathrm{St} \otimes k_{(p-1)\rho} \to \operatorname{End}_F(R) \ \text{ such that } \ \theta_\phi(f_- \otimes f_+) = \phi,$$

where k_λ (for any $\lambda \in X^*(T)$) is the one-dimensional B-module as in Section 2.1 associated to the character λ, f_- is a nonzero lowest weight vector of St *and $0 \neq f_+ \in k_{(p-1)\rho}$.*

Proof. For any $1 \leq i \leq \ell$, $\phi \in \operatorname{End}_F(R)$ and $z \in k$,

$$\varepsilon_{\alpha_i}(z) * \phi = \sum_{n \geq 0} z^n * \left(e_i^{(n)} * \phi\right). \tag{2}$$

From this (1) follows.

Let $\mathfrak{U}_U$ be the hyperalgebra of the unipotent radical U of B, i.e., it is the subalgebra of $\mathfrak{U}_G$ generated by $\{e_\beta^{(n)}; \beta \in \Delta^+, n \in \mathbb{Z}_+\}$. Then, the map $\gamma : \mathfrak{U}_U \to \mathrm{St}$, $a \mapsto a \cdot f_-$, is a surjective $\mathfrak{U}_U$-module map with kernel precisely equal to the left ideal $\sum_{\substack{1 \leq i \leq \ell \\ n \geq p}} \mathfrak{U}_U \cdot e_i^{(n)}$.

From the above description of $\operatorname{Ker}\gamma$, the second part of the lemma follows.

(Observe that $f_- \otimes f_+$ is T-invariant.) $\square$

4.1.3 Remarks. (a) In the above lemma, assume that R is a finitely generated k-algebra. Then, for any $\phi \in \operatorname{End}_F(R)$, the B-submodule of $\operatorname{End}_F(R)$ generated by ϕ is automatically a finite-dimensional algebraic B-module. To see this take a finite-dimensional B-submodule V of R such that the multiplication map $R^p \otimes_k V \to R$ is surjective for the subalgebra $R^p := \{a^p : a \in R\}$. This is possible since R is a finitely generated k-algebra and hence a finitely generated R^p-module (Lemma 1.1.1). Thus, under the restriction map, $\operatorname{End}_F(R)$ is a B-submodule of $\operatorname{Hom}_k(V^{[1]}, R)^{[-1]}$. From this the remark follows.

(b) By the identities (4.1.2.1)–(4.1.2.2), for any B-canonical $\phi \in \operatorname{End}_F(R)$ such that the B-submodule of $\operatorname{End}_F(R)$ generated by ϕ is a finite-dimensional algebraic B-module (for a B-algebra R), and any $1 \leq i \leq \ell$, $0 \leq j \leq p-1$,

$$\phi_{i,j} = e_i^{(j)} * \phi, \tag{1}$$

where $\phi_{i,j}$ is as in Definition 4.1.1.

Moreover,

$$e_i^{(j)} * \phi = 0, \quad \text{for } j \geq p. \tag{2}$$

We now "sheafify" the Definition 4.1.1 for any B-scheme X as follows.

4.1.4 Definition. Recall from 1.1.2 that for any scheme X, the absolute Frobenius morphism $F : X \to X$ gives rise to an $\mathcal{O}_X$-module structure $F^\# : \mathcal{O}_X \to F_*\mathcal{O}_X$. Let $\mathrm{End}_F(X) := \mathrm{Hom}(F_*\mathcal{O}_X, \mathcal{O}_X)$ be the additive group of all the $\mathcal{O}_X$-module maps $F_*\mathcal{O}_X \to \mathcal{O}_X$. As in 1.1.2, $F_*\mathcal{O}_X$ can canonically be identified with $\mathcal{O}_X$ as a sheaf of abelian groups on X. Under this identification, however, the $\mathcal{O}_X$-module structure is given by $f \odot g := f^p g$, for $f, g \in \mathcal{O}_X$. We define the $\mathcal{O}_X$-module structure on $\mathrm{End}_F(X)$ by $(f * \psi)s = \psi(fs)$, for $f, s \in \mathcal{O}_X$ and $\psi \in \mathrm{End}_F(X)$. In particular, the k-linear structure on $\mathrm{End}_F(X)$ is given by

$$(z * \psi)s = \psi(zs) = z^{1/p}\psi(s),$$

for $z \in k$, $\psi \in \mathrm{End}_F(X)$ and $s \in \mathcal{O}_X$.

If X is an H-scheme for an algebraic group H, then H acts k-linearly on $\mathrm{End}_F(X)$ by

$$(h * \psi)s = h(\psi(h^{-1}s)), \quad \text{for } h \in H, \psi \in \mathrm{End}_F(X) \text{ and } s \in F_*\mathcal{O}_X,$$

where the action of H on $F_*\mathcal{O}_X$ is defined to be the standard action of H on $\mathcal{O}_X$ under the identification $F_*\mathcal{O}_X = \mathcal{O}_X$ (as sheaves of abelian groups). Moreover, for $h \in H$, $f \in \mathcal{O}_X$ and $\psi \in \mathrm{End}_F(X)$,

$$h * (f * \psi) = (hf) * (h * \psi).$$

Let X be a B-scheme and let $\phi \in \mathrm{End}_F(X)$. Then, ϕ is called *B-canonical* if it satisfies the following:

(a) ϕ is T-invariant, i.e.,

$$t * \phi = \phi, \qquad \text{for all } t \in T.$$

(b) For any simple root α_i, $1 \le i \le \ell$, there exist $\phi_{i,j} \in \mathrm{End}_F(X)$, $0 \le j \le p-1$, such that

$$\text{(1)} \qquad \varepsilon_{\alpha_i}(z) * \phi = \sum_{j=0}^{p-1} z^j * \phi_{i,j}, \qquad \text{for all } z \in \mathbb{G}_a.$$

In fact, as in 4.1.1, if we only require that $\phi_{i,j} : \mathcal{O}_X \to \mathcal{O}_X$ are additive maps satisfying (1), then automatically $\phi_{i,j} \in \mathrm{End}_F(X)$.

A splitting $\phi \in \mathrm{End}_F(X)$ (see Remark 1.1.4(i)) is called a *B-canonical splitting* if ϕ is B-canonical.

As in 4.1.1, it is easy to see from (a) that, under the identification $F_*\mathcal{O}_X = \mathcal{O}_X$, $\phi(\mathcal{O}_X(\lambda)) \subset \mathcal{O}_X\left(\frac{\lambda}{p}\right)$, for any $\lambda \in X^*(T)$, where $\mathcal{O}_X(\lambda) \subset \mathcal{O}_X$ denotes the subsheaf of T-eigenfunctions corresponding to the weight λ, i.e., on any T-stable open subset V of X, $\mathcal{O}_X(\lambda)(V) = \{f \in \mathcal{O}_X(V) : t \cdot f = \lambda(t)f, \text{ for all } t \in T\}$. In particular,

$$\text{(2)} \qquad \phi(\mathcal{O}_X(\lambda)) = 0, \quad \text{unless } \frac{\lambda}{p} \in X^*(T).$$

4.1.5 Remark. Instead of the terminology "B-canonical," the term "semi-B-invariant" would have been more appropriate.

Recall [Kem–78, §11] that for a B-scheme X and a B-linearized quasi-coherent sheaf $\mathcal{S}$ on X, there is a natural action of the divided power $e_\alpha^{(n)}$ on $\mathcal{S}$ for any $n \geq 0$ and any root vector $e_\alpha \in \mathfrak{g}_\alpha$ (for $\alpha \in \Delta^+$).

Since $\mathrm{End}_F(X)$ is an algebraic B-module by [Har–77, Chap. III, Exercise 6.10(a)] and [Kem–78, Theorem 11.6(a)], following the same proof as that of Lemma 4.1.2, we get the following.

4.1.6 Lemma. *Let X be a B-scheme and let $\phi \in \mathrm{End}_F(X)$.*

(1) *Then, ϕ is B-canonical iff $e_i^{(n)} * \phi = 0$ for all $1 \leq i \leq \ell$ and $n \geq p$ and, moreover, ϕ is T-invariant.*

Thus, ϕ is B-canonical iff there exists a B-module (k-linear) map

$$\theta_\phi : \mathrm{St} \otimes k_{(p-1)\rho} \to \mathrm{End}_F(X)$$

such that $\theta_\phi(f_- \otimes f_+) = \phi$.

4.1.7 Remark. Similar to Remark 4.1.3(b), we see that for a B-canonical $\phi \in \mathrm{End}_F(X)$,

$$\phi_{i,j} = e_i^{(j)} * \phi, \quad \text{for any } 1 \leq i \leq \ell, 0 \leq j \leq p-1 \text{ and} \tag{1}$$

$$e_i^{(j)} * \phi = 0, \quad \text{for all } j \geq p, \tag{2}$$

where $\phi_{i,j}$ is as in Definition 4.1.4(b).

4.1.8 Proposition. *Let R be a B-algebra and let $\phi \in \mathrm{End}_F(R)$ be B-canonical. Then, for any $n \geq 0$ and $1 \leq i \leq \ell$,*

$$\phi\big(e_i^{(pn)} s\big) = e_i^{(n)} \phi(s), \ \textit{for all } s \in R. \tag{1}$$

In particular, ϕ takes B-submodules to B-submodules.

Proof. Since ϕ is B-canonical, for any $z \in k$ and $s \in R$,

$$\big(\varepsilon_{\alpha_i}(-z^p) * \phi\big)s = \sum_{j=0}^{p-1} (-z)^j \phi_{i,j}(s), \tag{2}$$

for some $\phi_{i,j} \in \mathrm{End}_F(R)$. On the other hand

$$\begin{aligned}
\big(\varepsilon_{\alpha_i}(-z^p) * \phi\big)s &= \varepsilon_{\alpha_i}(-z^p)\big(\phi(\varepsilon_{\alpha_i}(z^p)s)\big) \\
&= \sum_{m,n \geq 0} (-1)^m z^{pm} e_i^{(m)} \big(\phi(z^{pn} e_i^{(n)} s)\big) \\
&= \sum_{m,n \geq 0} (-1)^m z^{pm+n} e_i^{(m)} \big(\phi(e_i^{(n)} s)\big).
\end{aligned} \tag{3}$$

Since the highest power of z in the right side of (2) is $p-1$, we see from (3) (by collecting terms involving z^d for $d \in p\mathbb{Z}_+$) that

$$\phi(s) = \sum_{m,n\geq 0} (-1)^m z^{pm+pn} e_i^{(m)}\big(\phi(e_i^{(pn)}s)\big). \tag{4}$$

Moreover, by (2)–(4), for any $0 \leq j \leq p-1$,

$$(-1)^j \phi_{i,j}(s) = \phi(e_i^{(j)}s). \tag{5}$$

Now,

$$\varepsilon_{\alpha_i}(-z^p)\Big(\sum_{n\geq 0}\phi\Big((z^p)^{pn}\, e_i^{(pn)}s\Big)\Big) = \sum_{m,n\geq 0}(-1)^m z^{pm+pn} e_i^{(m)}\big(\phi(e_i^{(pn)}s)\big). \tag{6}$$

Combining (6) with (4) we get

$$\begin{aligned}\sum_{n\geq 0} z^{pn}\,\phi(e_i^{(pn)}s) &= \varepsilon_{\alpha_i}(z^p)\,\phi(s)\\ &= \sum_{n\geq 0} z^{pn} e_i^{(n)}(\phi(s)).\end{aligned}$$

Equating the coefficients of the above equality, we get

$$\phi(e_i^{(pn)}s) = e_i^{(n)}(\phi(s)), \text{ for all } n \geq 0.$$

This proves (1).

By (1), we see that the image of any B-submodule N of R under ϕ is closed under each $e_i^{(n)}$ and thus under $\varepsilon_{\alpha_i}(z)$, for any $z \in k$. Moreover, ϕ being T-equivariant, $\phi(N)$ is stable under T. Thus, $\phi(N)$ is stable under B; B being generated by T and $\{\varepsilon_{\alpha_i}(z); z \in k, 1 \leq i \leq \ell\}$. This proves the proposition. □

4.1.9 Remarks. (a) For any B-algebra R and B-canonical $\phi \in \mathrm{End}_F(R)$, we have, by the identity (4.1.8.5),

$$\phi_{i,j}(s) = (-1)^j\,\phi(e_i^{(j)}s), \quad \text{for any } 1 \leq i \leq \ell,\ 1 \leq j \leq p-1 \text{ and } s \in R, \tag{1}$$

where $\phi_{i,j}$ is as in (4.1.1.5).

(b) We have the following converse of Proposition 4.1.8. Let R be a B-algebra and let $\phi \in \mathrm{End}_F(R)$ be a T-invariant element such that ϕ satisfies (4.1.8.1) for all

$1 \le i \le \ell$ and all $n \ge 0$. Then, ϕ is B-canonical. To show this, by (4.1.8.3),

$$\begin{aligned}\big(\varepsilon_{\alpha_i}(-z^p) * \phi\big)s &= \sum_{\substack{m,n\ge 0\\ 0\le j\le p-1}} (-1)^m\, z^{pm+pn+j}\, e_i^{(m)}\big(\phi(e_i^{(pn)} e_i^{(j)} s)\big)\\ &= \sum (-1)^m\, z^{p(m+n)}\, e_i^{(m)} e_i^{(n)}\big(z^j \phi(e_i^{(j)} s)\big)\\ &= \Bigg(\sum_{d\ge 0}\Bigg(\sum_{m=0}^{d}(-1)^m \binom{d}{m}\Bigg) e_i^{(d)} z^{pd}\Bigg)\Bigg(\sum_{j=0}^{p-1} z^j \phi(e_i^{(j)} s)\Bigg)\\ &= \sum_{j=0}^{p-1} z^j\, \phi(e_i^{(j)} s).\end{aligned}$$

This proves that ϕ is B-canonical.

As in Section 2.1, let B^- be the Borel subgroup of G opposite to B containing the maximal torus T. Then, similar to the notion of a Frobenius-linear endomorphism $\phi \in \mathrm{End}_F(R)$ of a B-algebra R to be B-canonical, we have the notion of B^--canonical $\phi \in \mathrm{End}_F(R)$ of a B^--algebra R.

4.1.10 Proposition. *Let R be a G-algebra and let $\phi \in \mathrm{End}_F(R)$ be B-canonical. Assume further that the G-submodule of* $\mathrm{End}_F(R)$ *generated by ϕ is a finite-dimensional algebraic G-module. (This condition is redundant if R is a finitely generated k-algebra, (see Remark 4.1.3(a).)) Then, ϕ is automatically B^--canonical.*

In particular, ϕ takes G-submodules to G-submodules.

Proof. By (4.1.3.2), for $1 \le i \le \ell$,

(1) $$e_i^{(j)} * \phi = 0, \quad \text{for any } j \ge p.$$

Moreover, since ϕ is T-invariant, by Exercise 4.1.E.1,

$$f_i^{(j)} * \phi = 0, \quad \text{for any } j \ge p.$$

Thus, by (4.1.2.1), ϕ is B^--canonical.

Since the group G is generated by its subgroups B and B^-, by Proposition 4.1.8, ϕ takes G-submodules to G-submodules. □

The following example provides an important class of B-algebras.

4.1.11 Example. Let X be a scheme and let $\mathcal{L}$ be a line bundle on X. Consider the $\mathbb{Z}_+$-graded algebra (as considered in 1.1.12):

(1) $$R_{\mathcal{L}} = R(X, \mathcal{L}) := \bigoplus_{n\ge 0} H^0(X, \mathcal{L}^n)$$

under the standard product obtained by the multiplication of sections.

If X is a H-scheme and $\mathcal{L}$ is a H-linearized line bundle on X, then each $H^0(X, \mathcal{L}^n)$ is an algebraic H-module (cf. [Kem–78, Theorem 11.6]) and thus $R_{\mathcal{L}}$ is a H-algebra, where H is any affine algebraic group.

4.1.12 Definition. Let X be a scheme and let $\phi \in \mathrm{End}_F(X)$. Then, for any line bundle $\mathcal{L}$ on X, ϕ gives rise to an additive map $\phi_{\mathcal{L}} : R_{\mathcal{L}} \to R_{\mathcal{L}}$ as follows. (In fact, $\phi_{\mathcal{L}}$ is obtained from a sheaf morphisms $\widetilde{\phi}_{\mathcal{L}^m} : \mathcal{L}^{pm} \to \mathcal{L}^m$ of sheaves of abelian groups by taking global sections.)

If n is not divisible by p, we set

$$(1) \qquad \phi_{\mathcal{L}_{|H^0(X,\mathcal{L}^n)}} \equiv 0.$$

On the other hand if $n = pm$, as in Section 1.2, there is an isomorphism of sheaves of abelian groups on X

$$\xi_{\mathcal{L}^m} : F_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L}^m \xrightarrow{\sim} \mathcal{L}^{pm}, \ f \otimes s \mapsto fs^p,$$

for $f \in F_*\mathcal{O}_X \simeq \mathcal{O}_X$ and $s \in \mathcal{L}^m$. Thus, the Frobenius-linear endomorphism $\phi : F_*\mathcal{O}_X \to \mathcal{O}_X$ gives rise to the $\mathcal{O}_X$-module map

$$\phi \otimes I_{\mathcal{L}^m} : F_*\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{L}^m \to \mathcal{L}^m$$

and thus using the isomorphism $\xi_{\mathcal{L}^m}$ of sheaves of abelian groups, we get a sheaf morphism (of sheaves of abelian groups)

$$\widetilde{\phi}_{\mathcal{L}^m} : \mathcal{L}^{pm} \to \mathcal{L}^m.$$

Taking the global sections, we get a homomorphism of abelian groups $H^0(X, \mathcal{L}^{pm}) \to H^0(X, \mathcal{L}^m)$. This is, by definition, the map $\phi_{\mathcal{L}_{|H^0(X,\mathcal{L}^{pm})}}$.

As we will see in the proof of Lemma 4.1.13, $\phi_{\mathcal{L}} \in \mathrm{End}_F(R_{\mathcal{L}})$.

4.1.13 Lemma. *Let X be a B-scheme and let $\mathcal{L}$ be a B-linearized line bundle on X. Then, for any B-canonical $\phi \in \mathrm{End}_F(X)$, the induced map $\phi_{\mathcal{L}} : R_{\mathcal{L}} \to R_{\mathcal{L}}$ is B-canonical.*

Moreover, if ϕ is a splitting, then so is $\phi_{\mathcal{L}}$ (i.e., $\phi_{\mathcal{L}}(1) = 1$).

Proof. We first prove that $\phi_{\mathcal{L}}$ is Frobenius-linear, i.e., for $a \in H^0(X, \mathcal{L}^n)$, $b \in H^0(X, \mathcal{L}^m)$,

$$(1) \qquad \phi_{\mathcal{L}}(a^p b) = a\, \phi_{\mathcal{L}}(b).$$

If m is not divisible by p, then $\phi_{\mathcal{L}}(a^p b) = 0 = \phi_{\mathcal{L}}(b)$ and thus (1) is satisfied. So, assume that m is divisible by p. Take a nowhere vanishing section s of $\mathcal{L}$ on a small enough open set $V \subset X$. Then, any section a of $H^0(V, \mathcal{L}^n)$ can be written as $a = fs^n$, for some $f \in H^0(V, \mathcal{O}_X)$, and also $b \in H^0(V, \mathcal{L}^m)$ can be written as $b = gs^m$, for some $g \in H^0(V, \mathcal{O}_X)$. Thus, denoting $\mathcal{L}_{|V}$ by $\mathcal{L}_V$,

$$\begin{aligned}\phi_{\mathcal{L}_V}(a^p b) &= \phi_{\mathcal{L}_V}(f^p g\, s^{pn+m}) \\ &= \phi(f^p g)\, s^{n+\frac{m}{p}} \\ &= f\phi(g)\, s^{n+\frac{m}{p}} \\ &= a\phi_{\mathcal{L}_V}(b).\end{aligned}$$

This proves (1).

Since ϕ is B-canonical (in particular, T-invariant), it is easy to see that $\phi_{\mathcal{L}}$ is T-invariant. So, we just need to check the property (c_2) as in 4.1.1. As above, take a nowhere vanishing section s of $\mathcal{L}$ on a small enough open set $V \subset X$. Then, any section $a \in H^0(V, \mathcal{L}^{mp})$ can be written as $a = fs^{mp}$, for some $f \in H^0(V, \mathcal{O}_X)$. Thus, for any $1 \le i \le \ell$ and $z \in k$,

$$\begin{aligned}\big(\varepsilon_{\alpha_i}(z) * \phi_{\mathcal{L}}\big)a &= \varepsilon_{\alpha_i}(z)\big(\phi_{\mathcal{L}}(\varepsilon_{\alpha_i}(-z)a)\big) \\ &= \varepsilon_{\alpha_i}(z)\Big(\phi_{\mathcal{L}}\big((\varepsilon_{\alpha_i}(-z)\, f)(\varepsilon_{\alpha_i}(-z)s)^{mp}\big)\Big) \\ &= \varepsilon_{\alpha_i}(z)\Big(\big(\phi(\varepsilon_{\alpha_i}(-z)\, f)\big)(\varepsilon_{\alpha_i}(-z)s)^{m}\Big) \\ &= \big((\varepsilon_{\alpha_i}(z) * \phi)\, f\big)\, s^m \\ &= \sum_{j=0}^{p-1}(\phi_{i,j}(z^j f))\, s^m, \quad \text{since } \phi \text{ is } B\text{-canonical} \\ &= \sum_{j=0}^{p-1}\phi^{\mathcal{L}}_{i,j}(z^j a),\end{aligned}$$

where the Frobenius-linear endomorphism $\phi^{\mathcal{L}}_{i,j} : R_{\mathcal{L}} \to R_{\mathcal{L}}$ is, by definition, $(\phi_{i,j})_{\mathcal{L}}$. This proves the lemma. $\square$

4.1.14 Lemma. *Let X be a smooth H-scheme for an algebraic group H. Then, the isomorphism (defined in 1.3.7)*

$$\bar{\iota} : H^0(X, \omega_X^{1-p}) \to \mathrm{End}_F(X)$$

is a k-linear H-module isomorphism, where the notation $\mathrm{End}_F(X)$ is as defined in 4.1.4.

Proof. Let $\{t_1, \ldots, t_n\}$ be a system of local parameters on an open set $V \subset X$ and let dT be the volume form $dt_1 \wedge \cdots \wedge dt_n$ on V. Then, for any $s \in H^0(X, \omega_X^{1-p})$, by definition

$$\bar{\iota}(s)\, f = \frac{\tau(f\theta_s dT)}{dT}, \quad \text{for } f \in \Gamma(V, \mathcal{O}_X),$$

where $s_{|V} = \theta_s(dT)^{1-p}$ and τ is the trace map defined in 1.3.5 (see also Lemma 1.3.6). By the definition of τ, it is easy to see that $\bar{\iota}(s) \in \mathrm{End}_F(X)$. Moreover, from the definition of $\bar{\iota}$, it follows easily that it is a k-linear H-module map. □

4.1.15 Theorem. *For any standard parabolic subgroup $P \subset G$, there exists a unique (up to nonzero scalar multiples) nonzero B-canonical $\phi \in \mathrm{End}_F(G/P)$.*

Moreover, this ϕ is a splitting of G/P (up to scalar multiples) compatibly splitting all the Schubert subvarieties and the opposite Schubert subvarieties $\{X_w^P, \widetilde{X}_w^P\}_{w \in W}$.

Proof. As in the beginning of Section 3.1,

$$(1) \qquad \omega_{G/P} \simeq \mathcal{L}^P\left(-\delta_P\right), \text{ as } G\text{-equivariant bundles,}$$

where $\delta_P := \rho + w_o^P \rho$, w_o^P being the longest element of the Weyl group W_P of P.

Thus, by Lemmas 4.1.6 and 4.1.14, it suffices to prove (for the first part of the theorem) that the dimension of the B-module maps

$$(2) \qquad \dim \mathrm{Hom}_B\big(\mathrm{St} \otimes k_{(p-1)\rho}, H^0(G/P, \mathcal{L}^P((p-1)\delta_P))\big) = 1.$$

Now,

$$\begin{aligned} &\mathrm{Hom}_B\big(\mathrm{St} \otimes k_{(p-1)\rho}, H^0(G/P, \mathcal{L}^P((p-1)\delta_P))\big) \\ &\quad \simeq \mathrm{Hom}_B\big(\mathrm{St}, H^0(G/P, \mathcal{L}^P((p-1)\delta_P)) \otimes k_{-(p-1)\rho}\big) \\ &\quad \simeq \mathrm{Hom}_G\big(\mathrm{St}, H^0(G/P, \mathcal{L}^P((p-1)\delta_P)) \otimes \mathrm{St}\big), \\ &\qquad \text{by [Jan–03, Part I, Propositions 3.4 and 3.6]} \\ (3) &\quad \simeq \mathrm{Hom}_P(\mathrm{St}, k_{-(p-1)\delta_P} \otimes \mathrm{St}), \quad \text{again by [loc cit.].} \end{aligned}$$

But St is generated as a B-module; in particular, as a P-module, by its lowest weight vector f_- of weight $-(p-1)\rho$. Moreover, the weight space of St corresponding to the weight $(p-1)\delta_P - (p-1)\rho$ is one-dimensional since

$$(p-1)\,\delta_P - (p-1)\,\rho = (p-1)\,w_o^P\rho.$$

Thus, by (3), we get

$$(4) \qquad \dim \mathrm{Hom}_B\big(\mathrm{St} \otimes k_{(p-1)\rho}, H^0(G/P, \mathcal{L}^P((p-1)\delta_P))\big) \le 1,$$

proving the uniqueness of ϕ.

To prove the existence of ϕ, we first take $P = B$. By Theorem 2.3.1 and Lemmas 4.1.6 and 4.1.14, there exists a B-canonical splitting of G/B compatibly splitting all the $X_w, \widetilde{X}_w$. But then, by Exercise 4.1.E.3, G/P admits a B-canonical splitting compatibly splitting all the $X_w^P, \widetilde{X}_w^P$ by Lemma 1.1.8. This proves the reverse inequality in (4) and so the theorem is fully established. □

4.1.16 Definition. Let X be a scheme and let $Y \subset X$ be a closed subscheme. Then, $\phi \in \mathrm{End}_F(X)$ is said to be *compatible with* Y if

$$\phi(F_*\mathcal{I}_Y) \subset \mathcal{I}_Y, \tag{1}$$

where $\mathcal{I}_Y \subset \mathcal{O}_X$ is the ideal sheaf of Y. Clearly such a ϕ induces $\phi_Y \in \mathrm{End}_F(Y)$.

Assume further that X is a B-scheme and $Y \subset X$ a (closed) B-stable subscheme. Then, for a B-canonical $\phi \in \mathrm{End}_F(X)$ which is compatible with Y, the induced endomorphism $\phi_Y \in \mathrm{End}_F(Y)$ is again B-canonical. To see this, let $\mathrm{End}_F(X, Y)$ be the B-submodule of $\mathrm{End}_F(X)$ consisting of those $\phi \in \mathrm{End}_F(X)$ such that ϕ is compatible with Y. Then, the induced map

$$\mathrm{End}_F(X, Y) \to \mathrm{End}_F(Y), \ \phi \mapsto \phi_Y,$$

is a B-module map. Thus, if $\phi \in \mathrm{End}_F(X, Y)$ is B-canonical, so is ϕ_Y by Lemma 4.1.6.

For a B-scheme X, let $\mathrm{End}_F^{\mathrm{can}}(X)$ be the linear subspace of $\mathrm{End}_F(X)$ consisting of B-canonical Frobenius-linear endomorphisms of X.

4.1.17 Proposition. *Let X be a B-scheme. Then, there exists a "natural" injective map (described in the proof below)*

$$\Theta : \mathrm{End}_F^{\mathrm{can}}(X) \to \mathrm{End}_F^{\mathrm{can}}(\widetilde{X}),$$

where $\widetilde{X} := G \times_B X$.

Moreover, Θ takes a B-canonical splitting σ of X to a B-canonical splitting $\widetilde{\sigma}$ of $\widetilde{X}$ which compatibly splits $X = e \times X \subset \widetilde{X}$ and such that $\widetilde{\sigma}_{|X} = \sigma$.

In fact, for any $\sigma \in \mathrm{End}_F^{\mathrm{can}}(X)$, $\Theta(\sigma)$ is compatible with X and $\Theta(\sigma)_{|X} = \sigma$.

Proof. By Lemma 4.1.6, we have the isomorphism

$$\mathrm{End}_F^{\mathrm{can}}(X) \simeq \mathrm{Hom}_B(\mathrm{St}, \mathrm{End}_F(X) \otimes k_{-(p-1)\rho}),$$

taking $\phi \mapsto \bar{\theta}_\phi$ such that $\bar{\theta}_\phi(f_-) = \phi \otimes \mathbf{1}$, where $f_- \in \mathrm{St}$ is a nonzero lowest weight vector, $\mathbf{1}$ is a nonzero element of $k_{-(p-1)\rho}$ and $\mathrm{Hom}_B(M, N)$ denotes the space of all the k-linear B-module homomorphisms from M to N. Together with the Frobenius reciprocity, this yields the isomorphism

$$\mathrm{End}_F^{\mathrm{can}}(X) \simeq \mathrm{Hom}_G(\mathrm{St}, \mathrm{Ind}_B^G(\mathrm{End}_F(X) \otimes k_{-(p-1)\rho})),$$

where, for a B-module M, $\mathrm{Ind}_B^G(M)$ denotes the space of global sections $H^0(G/B, \mathcal{L}(M))$. Composing the above isomorphism with the evaluation at f_-, we obtain the injective map

$$\Phi : \mathrm{End}_F^{\mathrm{can}}(X) \to \mathrm{Ind}_B^G(\mathrm{End}_F(X) \otimes k_{-(p-1)\rho})$$

of weight $-(p-1)\rho$ with respect to T.

We provide a geometric interpretation of Φ as follows. Let X' be the scheme associated with the ringed space $(X, \mathcal{O}_X^p)$. Then, X' is a B-scheme; we put $\widetilde{X}' := G \times_B X'$. This is a G-scheme, equipped with the projection $\pi : \widetilde{X}' \to G/B$ and the inclusion $i : X' \to \widetilde{X}'$. We may regard $\widetilde{X}'$ as the scheme associated with the ringed space $(\widetilde{X}, \mathcal{O}_{G/B} \otimes_{\mathcal{O}_{G/B}^p} \mathcal{O}_{\widetilde{X}}^p)$. (Indeed, both are G-schemes mapping to G/B, with fiber at the base point the B-scheme X'.) So, the structure sheaf $\mathcal{O}_{\widetilde{X}'}$ strictly contains $\mathcal{O}_{\widetilde{X}}^p$.

Note that i^* yields an equivalence from the category of G-linearized coherent sheaves on $\widetilde{X}'$, to the category of B-linearized coherent sheaves on X'. This is proved in [Bri–03b, Lemma 2]. Here is the construction of its inverse. Consider the pullback $\mathcal{G}$ of a B-linearized coherent sheaf $\mathcal{F}$ on X' to $G \times X'$ under the second projection. Then, $\mathcal{G}$ is $G \times B$-linearized, for the action of $G \times B$ on $G \times X'$ by $(g, b)(h, x) = (ghb^{-1}, bx)$. Since the quotient $G \times X' \to G \times_B X' = \widetilde{X}'$ is a locally trivial B-bundle, $\mathcal{G}$ descends to a G-linearized sheaf on $\widetilde{X}'$. One checks that the pullback of this sheaf to X' (identified with $B \times_B X'$) is the original sheaf $\mathcal{F}$. The inverse of i^* will be denoted Ind_B^G; then

$$\Gamma(\widetilde{X}', Ind_B^G \mathcal{F}) = \mathrm{Ind}_B^G \, \Gamma(X', \mathcal{F}),$$

for any B-linearized coherent sheaf $\mathcal{F}$ on X'.

Put $\mathcal{L} := \pi^* \mathcal{L}((p-1)\rho)$; this is a G-linearized line bundle on $\widetilde{X}'$. Further, $i^* \mathcal{L} = \mathcal{O}_{X'} \otimes k_{-(p-1)\rho} = \mathcal{O}_X^p \otimes k_{-(p-1)\rho}$. Thus,

$$i^* Hom_{\mathcal{O}_{\widetilde{X}'}}(\mathcal{O}_{\widetilde{X}}, \mathcal{L}) = Hom_{\mathcal{O}_{X'}}(\mathcal{O}_X, i^* \mathcal{L}) = End_F(X) \otimes k_{-(p-1)\rho},$$

and hence

$$Hom_{\mathcal{O}_{\widetilde{X}'}}(\mathcal{O}_{\widetilde{X}}, \mathcal{L}) = Ind_B^G (End_F(X) \otimes k_{-(p-1)\rho}).$$

Taking global sections, we obtain an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\widetilde{X}'}}(\mathcal{O}_{\widetilde{X}}, \mathcal{L}) = \mathrm{Ind}_B^G (\mathrm{End}_F(X) \otimes k_{-(p-1)\rho}).$$

Thus, we may regard Φ as an injective map

$$\mathrm{End}_F^{\mathrm{can}}(X) \to \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}'}}(\mathcal{O}_{\widetilde{X}}, \mathcal{L}),$$

still denoted by Φ. For any $\sigma \in \mathrm{End}_F^{\mathrm{can}}(X)$, its image $\Phi(\sigma)$ is a T-eigenvector of weight $-(p-1)\rho$, killed by all $e_i^{(n)}$, $n \geq p$. This follows from the definition of Φ, since $e_i^{(n)} f_- = 0$ in St for all $n \geq p$.

Next, recall that the canonical map $\mathcal{O}_{G/B} \otimes_k \mathrm{St} \to \mathcal{L}((p-1)\rho)$ (obtained from the definition $\mathrm{St} := \Gamma(G/B, \mathcal{L}((p-1)\rho))$ restricts to an isomorphism of sheaves of $\mathcal{O}_{G/B}^p$-modules $\mathcal{O}_{G/B}^p \otimes \mathrm{St} \simeq \mathcal{L}((p-1)\rho)$ (this is a reformulation of Exercise 2.3.E.1). This yields an isomorphism of G-linearized sheaves of $\mathcal{O}_{\widetilde{X}}^p$-modules:

$$\mathcal{O}_{\widetilde{X}}^p \otimes \mathrm{St} \simeq \mathcal{L}.$$

Let u be a T-eigenvector of weight $2(p-1)\rho$ in the restricted enveloping algebra of U. Then, u is unique up to scalars; it is U-invariant, and maps f_- to f_+ (see, e.g., Exercise

2.3.E.2 for the latter assertion). Thus, u acting on $\mathcal{L}$ maps $\mathcal{O}_{\widetilde{X}}^p \otimes \mathrm{St}$ to $\mathcal{O}_{\widetilde{X}}^p \otimes kf_+$ yielding the map

$$\mathrm{Hom}_{\mathcal{O}_{\widetilde{X}'}}(\mathcal{O}_{\widetilde{X}}, \mathcal{L}) \to \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}}^p}(\mathcal{O}_{\widetilde{X}}, \mathcal{O}_{\widetilde{X}}^p \otimes kf_+) = \mathrm{End}_F(\widetilde{X}) \otimes k_{(p-1)\rho}.$$

(Here we have used the fact that the action of u on $\mathcal{L}$ is $\mathcal{O}_{\widetilde{X}}^p$-linear since any root vector kills $\mathcal{O}_{\widetilde{X}}^p$.) We thus obtain a map

$$\Psi : \mathrm{Hom}_{\mathcal{O}_{\widetilde{X}'}}(\mathcal{O}_{\widetilde{X}}, \mathcal{L}) \to \mathrm{End}_F(\widetilde{X})$$

which is U-invariant, of weight $(p-1)\rho$. So, $\Theta := \Psi \circ \Phi$ maps $\mathrm{End}_F^{\mathrm{can}}(X)$ to $\mathrm{End}_F^{\mathrm{can}}(\widetilde{X})$.

Now, let $\sigma \in \mathrm{End}_F^{\mathrm{can}}(X)$ be a splitting of X, that is, $\sigma(1) = 1$. Then, one checks that $\Phi(\sigma)(1) = f_-$, where $f_- \in \mathrm{St} = \Gamma(G/B, \mathcal{L}((p-1)\rho))$ is regarded as an element of $\Gamma(\widetilde{X}', \mathcal{L})$. It follows that $\Theta(\sigma)(1) = 1$. Thus, $\Theta(\sigma)$ is a splitting of $\widetilde{X}$.

Further, by construction, $\Phi(\sigma) : \mathcal{O}_{\widetilde{X}} \to \mathcal{L}$ maps the ideal sheaf $\mathcal{I}_X$ to $\mathcal{I}_{X'}\mathcal{L}$, and the map induced from $\Phi(\sigma)$ on $\mathcal{O}_X$ identifies with $\sigma \otimes f_- : \mathcal{O}_X \to i^*\mathcal{L} = \mathcal{O}_X^p \otimes k_{-(p-1)\rho}$. Finally, as $\mathcal{I}_{X'}\mathcal{L}$ is stable under U, and u is in the restricted enveloping algebra of U, we have that u maps $\mathcal{I}_{X'}\mathcal{L}$ to the kernel of the induced map $u(\mathcal{L}) \to u(i^*\mathcal{L}) = i^*u(\mathcal{L})$, that is, to the kernel of the map $\mathcal{O}_{\widetilde{X}}^p \otimes kf_+ \to \mathcal{O}_X^p \otimes kf_+$. It follows that $\Theta(\sigma)$ maps $\mathcal{I}_X$ to $\mathcal{I}_X^p$, and restricts to σ on X. In particular, Θ is injective, proving the proposition. $\square$

4.1.18 Remarks. (i) Even though we do not prove it, the map Θ induces a bijection between the B-canonical splittings of X and those B-canonical splittings of $\widetilde{X}$ which compatibly split $e \times X \subset \widetilde{X}$ (cf. [Mat–00, §5]).

(ii) There is a different proof of Proposition 4.1.17 given in [Van–01, §4], which relies on representation-theoretic methods.

4.1.E Exercises

(1*) Let V be an algebraic representation of $\mathrm{SL}_2(k)$, where k is any algebraically closed field. Let $v_o \in V$ be fixed under the maximal torus $T := \left\{\begin{pmatrix} z & 0 \\ 0 & z^{-1}\end{pmatrix} : z \in k^*\right\}$. Assume that $e^{(n)}v_o = 0$ for all $n \geq m$ (for a fixed positive integer m). Then, prove that $f^{(n)}v_o = 0$, for all $n \geq m$, where $e := \begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}$, $f := \begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix} \in sl_2(k)$.

(2) For any sequence $\mathfrak{w} = (s_{i_1}, \ldots, s_{i_n})$ of simple reflections, show that the BSDH variety $Z_{\mathfrak{w}}$ admits a unique B-canonical splitting, compatibly splitting all the subvarieties $Z_{\mathfrak{w}_J}$ for any subsequence $\mathfrak{w}_J$ of $\mathfrak{w}$.

Hint: This exercise can be obtained easily by using Proposition 4.1.17 and Remark 4.1.18(i). However, we outline a direct proof. For the existence, by Lemmas 4.1.6 and 4.1.14, it suffices to construct a B-module map

$$\phi : \mathrm{St} \otimes k_{(p-1)\rho} \to H^0\big(Z_{\mathfrak{w}}, \omega_{Z_{\mathfrak{w}}}^{1-p}\big),$$

such that $\phi(f_-\otimes f_+)$ is a splitting of $Z_{\mathfrak{w}}$, where $f_\pm$ is as in Lemma 4.1.2. By Proposition 2.2.2, as B-linearized line bundles,

$$\omega_{Z_{\mathfrak{w}}}^{1-p} \simeq \mathcal{O}_{Z_{\mathfrak{w}}}\big[(p-1)\partial Z_{\mathfrak{w}}\big] \otimes \mathcal{L}_{\mathfrak{w}}((p-1)\rho) \otimes \mathbf{k}_{(p-1)\rho}.$$

The map $\theta_{\mathfrak{w}} : Z_{\mathfrak{w}} \to G/B$, defined by (2.2.1.6), induces the B-module map

$$\theta_{\mathfrak{w}}^* : \mathrm{St} \to H^0(Z_{\mathfrak{w}}, \mathcal{L}_{\mathfrak{w}}((p-1)\rho)).$$

Also, consider the canonical section $\sigma \in H^0(Z_{\mathfrak{w}}, \mathcal{O}_{Z_{\mathfrak{w}}}[(p-1)\partial Z_{\mathfrak{w}}])$ (with the associated divisor of zeroes $(\sigma)_0 = (p-1)\partial Z_{\mathfrak{w}}$). It is easy to see that σ is B-invariant. Combine $\theta_{\mathfrak{w}}^*$ and the section σ to construct ϕ. Now, show that $\phi(f_-\otimes f_+)$ is a splitting of $Z_{\mathfrak{w}}$ compatibly splitting all the subvarieties $Z_{\mathfrak{w}_J}$ by using Proposition 1.3.11. To prove the uniqueness assertion, use Exercises 1.3.E.3, 3.3.E.2 and Proposition 2.2.2.

(3*) Let $f : X \to Y$ be a B-equivariant morphism of B-schemes such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume that X admits a B-canonical splitting. Then, show that the induced splitting of Y given by Lemma 1.1.8 is again B-canonical.

(4) With the notation and assumptions as in Proposition 4.1.17, show that for any B-canonical splitting σ of X, which compatibly splits a closed B-stable subscheme $Y \subset X$, the induced B-canonical splitting $\widetilde{\sigma} := \Theta(\sigma)$ of $\widetilde{X} := G \times_B X$ compatibly splits the closed subschemes $\{\overline{BwB} \times_B X, G \times_B Y\}_{w\in W}$.

(5) Let X, Y be two G-schemes which admit B-canonical splittings. Then, show that the product G-scheme $X \times Y$ also admits a B-canonical splitting. (This, in general, is false for B-schemes.)

Hint: Construct a G-module map $\mathrm{St}\otimes\mathrm{St} \to \mathrm{St}^{\otimes 2}\otimes\mathrm{St}^{\otimes 2}$ under the diagonal action of G.

(6*) Analogous to Theorem 4.1.15, show that, for any parabolic subgroups P, Q of G, $G/P \times G/Q$ admits a B-canonical splitting, compatibly splitting all the G-Schubert subvarieties $\{\mathcal{X}_w^{P,Q}\}_{w\in W}$.

(7) Let $n \geq 1$. Show that the splitting of $(G/B)^n$ given by $m_n(\theta_n)$ in the proof of Theorem 2.3.10 is a B-canonical splitting. Recall from the proof of Theorem 2.3.10 that this splitting compatibly splits all the subvarieties $\big\{X_w \times X^{n-1}, X^q \times \mathcal{X}_w \times X^{n-2-q};\ w \in W, 0 \leq q \leq n-2\big\}$.

Hint: Construct a G-module map $\mathrm{St}\otimes\mathrm{St} \to \mathrm{St}^{\otimes n}\otimes\mathrm{St}^{\otimes n}$ under the diagonal action of G.

4.2 Good filtrations

We continue to follow the notation from Section 2.1. In particular, G is a connected, simply-connected, semisimple algebraic group over an algebraically closed field k of characteristic $p > 0$, $B \subset G$ a Borel subgroup with the unipotent radical U and let $T \subset B$ be a maximal torus.

All the modules in this section will be assumed to be rational, so we will abbreviate rational H-modules simply by H-modules, for any affine algebraic group H.

4.2.1 Definition. For any dominant $\lambda \in X^*(T)$, define the (rational) G-module

$$\nabla(\lambda) := H^0(G/B, \mathcal{L}(-w_o\lambda)), \tag{1}$$

and recall the definition and elementary properties of the Weyl module

$$V(\lambda) := \nabla(-w_o\lambda)^*, \tag{2}$$

from Section 2.1, where w_o is the longest element of W. We have by Theorem 3.3.8,

$$\operatorname{ch} \nabla(\lambda) = \operatorname{ch} V(\lambda) = D_{\mathfrak{w}_o}(e^\lambda), \tag{3}$$

where $\mathfrak{w}_o$ is a reduced word with $a(\mathfrak{w}_o) = w_o$.

A filtration of a G-module V by G-submodules:

$$F^0 = (0) \subset F^1 \subset F^2 \subset \cdots$$

is called a *good filtration* of V if

(c$_1$) $\bigcup_j F^j = V$, and

(c$_2$) For any $j \geq 1$, as G-modules,

$$F^j/F^{j-1} \simeq \bigoplus_\lambda \nabla(\lambda) \otimes_k A(\lambda, j), \tag{4}$$

for some trivial G-modules $A(\lambda, j)$, where the summation runs over the set of dominant integral weights $X^*(T)^+$.

Choose an injective additive map, called a *height function*, $h : X^*(T) \to \mathbb{R}$ such that

$$h(\alpha_i) > 0 \quad \text{for all the simple roots } \alpha_i. \tag{5}$$

Since the root lattice $\bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i$ is of finite index (say d) in $X^*(T)$, the injectivity of h is equivalent to the condition that $\{h(\alpha_1), \ldots, h(\alpha_\ell)\}$ are linearly independent over $\mathbb{Q}$. Since $h(\alpha_i) > 0$,

$$h(\lambda) < h(\mu) \quad \text{for } \lambda < \mu. \tag{6}$$

Moreover, since for any $\lambda \in X^*(T)^+$, $d\lambda = \sum n_i\alpha_i$ with $n_i \in \mathbb{Z}_+$ (Exercise 4.2.E.4), we get that $h(X^*(T)^+)$ is a discrete subset of $\mathbb{R}_+$. This allows us to totally order the set $X^*(T)^+$ as

$$\{\lambda_1 = 0, \lambda_2, \lambda_3, \ldots\} \text{ such that } h(\lambda_i) < h(\lambda_{i+1}), \text{ for } i \geq 1. \tag{7}$$

For any B-module M, define a filtration of M (depending upon the choice of the height function h) as follows:

$$M^0 = (0) \subset M^1 \subset M^2 \subset \cdots, \tag{8}$$

where $M^j = M^j(h)$ is the largest B-submodule of M such that any weight λ of M^j satisfies:

$$h(\lambda) \leq h(\lambda_j). \tag{9}$$

Clearly, $\bigcup M^j = M$. The filtration $(M^j)_{j\geq 0}$ is called the *h-canonical filtration* of M. When the reference to the height function h is clear, we simply write the *canonical filtration* and drop the adjective "h."

For any B-module M and $\lambda \in X^*(T)$, let $M_\lambda^{(B)}$ be the B-eigenspace of M with weight λ, i.e.,

$$M_\lambda^{(B)} := \{m \in M : bm = \lambda(b)m, \text{ for all } b \in B\}. \tag{10}$$

Then, for a G-module M and $\lambda \in X^*(T)^+$, by the Frobenius reciprocity, there exists a k-linear isomorphism

$$i_\lambda : \operatorname{Hom}_G(V(\lambda), M) \xrightarrow{\sim} M_\lambda^{(B)}, \; \phi \mapsto \phi(v_\lambda), \tag{11}$$

where v_λ is a nonzero B-eigenvector of $V(\lambda)$ with weight λ.

For any $\lambda \in X^*(T)^+$ and G-module M, set

$$h^0(M, \lambda) := \dim M_\lambda^{(B)} = \dim \operatorname{Hom}_G(V(\lambda), M), \text{ and} \tag{12}$$

$$h^1(M, \lambda) := \dim \operatorname{Ext}^1_G(V(\lambda), M), \tag{13}$$

where the k-space $\operatorname{Ext}^1_G(N, M)$, for G-modules N, M, has for its underlying set the set of isomorphism classes of extensions of rational G-modules:

$$0 \to M \to \widetilde{M} \to N \to 0.$$

Equivalently, it is the "Ext^1" functor in the category of rational G-modules. We allow $h^0(M, \lambda)$ and $h^1(M, \lambda)$ to be ∞.

Define a partial order $\leq$ in the group algebra $A(T) := \mathbb{Z}[X^*(T)]$ by declaring

$$\sum_{\lambda \in X^*(T)} a_\lambda e^\lambda \leq \sum_{\lambda \in X^*(T)} b_\lambda e^\lambda \Leftrightarrow a_\lambda \leq b_\lambda \text{ for all } \lambda. \tag{14}$$

A (rational) T-module M is called an *admissible T-module* if all the weight spaces of M are finite-dimensional. For an admissible T-module M, we define its *formal character* ch by

$$\operatorname{ch} M = \sum_{\lambda \in X^*(T)} (\dim M_\lambda) e^\lambda \in \hat{A}(T), \tag{15}$$

where M_λ is the weight space of M corresponding to the weight λ and $\hat{A}(T)$ is the set of all the formal linear combinations $\sum_{\lambda \in X^*(T)} n_\lambda e^\lambda$ with $n_\lambda \in \mathbb{Z}$ (where we allow infinitely many of n_λ's to be nonzero).

4.2.2 Lemma. *Let M be a G-module and let $\lambda \in X^*(T)^+$.*

(a) If $\mathrm{Ext}^1_G(V(\lambda), M) \neq 0$, *then there is a weight μ of M such that $\mu > \lambda$.*

(b) If $\mathrm{Ext}^1_G(M, \nabla(\lambda)) \neq 0$, *then again there is a weight μ of M such that $\mu > \lambda$. In particular, for any $\lambda, \mu \in X^*(T)^+$, we have*

(c) $\mathrm{Ext}^1_G(V(\lambda), \nabla(\mu)) = 0$.

Proof. (a) Consider a nontrivial extension in the category of rational G-modules:

$$(1) \qquad 0 \to M \xrightarrow{i} \widetilde{V(\lambda)} \xrightarrow{\pi} V(\lambda) \to 0\,.$$

Take a preimage $\widetilde{v}_\lambda \in \widetilde{V(\lambda)}$ of the highest weight vector $v_\lambda \in V(\lambda)$ such that $\widetilde{v}_\lambda$ is a weight vector (of weight λ). Then, $\widetilde{v}_\lambda$ can not be a B-eigenvector; for otherwise, the sequence (1) would split. Thus, the B-submodule of $\widetilde{V(\lambda)}$ generated by $\widetilde{v}_\lambda$ must contain nonzero weight vectors v of weight $\mu > \lambda$. Of course, $v \in i(M)$, proving (a).

To prove (b), take a nontrivial extension

$$(2) \qquad 0 \to \nabla(\lambda) \xrightarrow{i} \widetilde{M} \to M \to 0.$$

Choose a T-module projection $\beta : \widetilde{M} \to k_{w_o\lambda}$ such that $\beta \circ i \neq 0$. Assume, if possible, that there does not exist any weight μ of M such that $\mu > \lambda$. Then, it is easy to see that β is a B-module map. On inducing β, it gives rise to a G-module map $\hat{\beta} : \widetilde{M} \to \nabla(\lambda)$. By the Frobenius reciprocity, $\mathrm{Hom}_G(\nabla(\lambda), \nabla(\lambda)) \simeq \mathrm{Hom}_B(\nabla(\lambda), k_{w_o\lambda})$ is one-dimensional and hence $\hat{\beta} \circ i = I_{\nabla(\lambda)}$ (up to a nonzero scalar multiple), splitting (2). This is a contradiction, proving (b). (Observe that, if M is finite-dimensional, (a) and (b) are equivalent by the duality of the Ext functor.)

(c) Assume, if possible, that $\mathrm{Ext}^1_G(V(\lambda), \nabla(\mu)) \neq 0$. Then, by the (a) part, $\mu > \lambda$ and, by the (b) part, $\lambda > \mu$. This is a contradiction, proving (c). □

4.2.3 Proposition. *Let M be a G-module. Then, for the canonical filtration $(M^j)_{j \geq 0}$ of M, we have the following.*

(a) Each M^j is a G-submodule of M.

(b) There is a G-module embedding, for any $j \geq 1$,

$$\pi_j : \frac{M^j}{M^{j-1}} \hookrightarrow \nabla(\lambda_j) \otimes_k M^{(B)}_{\lambda_j},$$

where $M^{(B)}_{\lambda_j}$ is equipped with the trivial G-module structure.

In particular, if M is finite-dimensional,

$$(1) \qquad \mathrm{ch}\, M \leq \sum_{\lambda \in X^*(T)^+} h^0(M, \lambda)\, \mathrm{ch}\, \nabla(\lambda).$$

(c) The filtration $(M^j)_j$ is a good filtration of M iff each π_j is an isomorphism.

In particular, if M is finite-dimensional, $(M^j)_j$ is a good filtration of M iff we have equality in (1).

Proof. (a) Let $\mathfrak{U}_G, \mathfrak{U}_B, \mathfrak{U}_{U^-}$ be the hyperalgebras over k of G, B and U^- respectively (Section 2.1). By an analogue of the Poincaré–Birkhoff–Witt theorem (cf. [Jan–03, Part II, §1.12]),

$$\mathfrak{U}_G = \mathfrak{U}_{U^-} \cdot \mathfrak{U}_B,$$

and thus, M^j being a B-submodule,

$$\mathfrak{U}_G \cdot M^j = \mathfrak{U}_{U^-} \cdot M^j.$$

Since $h(\alpha_i) > 0$, any weight λ of $\mathfrak{U}_{U^-} \cdot M^j$ satisfies

$$h(\lambda) \leq h(\lambda') \leq h(\lambda_j), \text{ for some weight } \lambda' \text{ of } M^j.$$

Moreover, since $\mathfrak{U}_G \cdot M^j$ is a G-submodule of M; in particular, a B-submodule, by the maximality of M^j we get $M^j = \mathfrak{U}_G \cdot M^j$ and thus M^j is a G-submodule of M.

(b) Consider the T-module projection onto the $w_o\lambda_j$-weight space $\hat{\pi}_j : M^j \to M^j_{w_o\lambda_j}$. Since M^j is a G-module and no weight of M^j is $> \lambda_j$ by (4.2.1.9), we get that $\hat{\pi}_j$ is a B-module map, where the range $M^j_{w_o\lambda_j}$ is equipped with the B-module structure via the character $w_o\lambda_j$ of B. Moreover, as vector spaces,

$$M^j_{w_o\lambda_j} \simeq M^j_{\lambda_j} = (M^j)^{(B)}_{\lambda_j} = M^{(B)}_{\lambda_j}. \tag{2}$$

Thus, on inducing $\hat{\pi}_j$, we get a G-module map $\bar{\pi}_j : M^j \to \nabla(\lambda_j) \otimes_k M^{(B)}_{\lambda_j}$, where $M^{(B)}_{\lambda_j}$ is equipped with the trivial G-module structure. It is easy to see that $\bar{\pi}_{j|_{M^j_{w_o\lambda_j}}} : M^j_{w_o\lambda_j} \to \nabla(\lambda_j)_{w_o\lambda_j} \otimes_k M^{(B)}_{\lambda_j}$ is bijective and hence so is $\bar{\pi}_{j|_{M^j_{\lambda_j}}} : M^j_{\lambda_j} \to \nabla(\lambda_j)_{\lambda_j} \otimes_k M^{(B)}_{\lambda_j}$. Let $K := \operatorname{Ker} \bar{\pi}_j$. Let μ be a weight of K and let μ^+ be the dominant weight in the W-orbit of μ. Then, K being a G-module, μ^+ also is a weight of K. Since $\bar{\pi}_{j|_{M^j_{\lambda_j}}}$ is injective, $\mu^+ \neq \lambda_j$ and thus $h(\mu^+) < h(\lambda_j)$. From this we get that $h(\mu) \leq h(\lambda_{j-1})$, proving that $K \subset M^{j-1}$.

Conversely, if possible, assume that $M^{j-1} \not\subset K$. Then, $\bar{\pi}_j(M^{j-1}) \neq 0$. Choose a B-eigenvector of weight λ in $\bar{\pi}_j(M^{j-1})$. Then, by the definition of M^{j-1},

$$h(\lambda) \leq h(\lambda_{j-1}) < h(\lambda_j). \tag{3}$$

But, as observed in Section 2.1, $\nabla(\lambda_j)$ has a unique B-eigenvector (up to scalar multiples) and it is of weight λ_j. This contradicts (3), showing that $M^{j-1} \subset K$. Thus,

$$\operatorname{Ker} \bar{\pi}_j = M^{j-1},$$

proving (b).

(c) If each π_j is an isomorphism, (M^j) is a good filtration of M by definition.

Conversely, assume that (M^j) is a good filtration of M. Since $\nabla(\lambda_j)$ has a unique B-stable line and it has weight λ_j, we get from the injectivity of π_j that, as G-modules,

$$\frac{M^j}{M^{j-1}} \simeq \nabla(\lambda_j) \otimes A_j, \quad \text{for some trivial } G\text{-modules } A_j. \tag{4}$$

Moreover, from the isomorphism $\bar{\pi}_{j_{|M^j_{\lambda_j}}} : M^j_{\lambda_j} \xrightarrow{\sim} \nabla(\lambda_j)_{\lambda_j} \otimes M^{(B)}_{\lambda_j}$ proved in the (b) part and (2), it is easy to see that π_j induces an isomorphism between B-eigen spaces:

$$\left(\frac{M^j}{M^{j-1}}\right)^{(B)}_{\lambda_j} \simeq \left(\nabla(\lambda_j) \otimes M^{(B)}_{\lambda_j}\right)^{(B)}_{\lambda_j}. \tag{5}$$

By Lemma 4.2.2(b) together with (4) and [Jan–03, Part I, Lemma 4.17], the image of π_j is a G-module direct summand in $\nabla(\lambda_j) \otimes M^{(B)}_{\lambda_j}$. But, since π_j is injective, by using (5) we get that π_j is an isomorphism. This proves the proposition. □

4.2.4 Definition. (a) For any $\lambda \in X^*(T)^+$ and G-module M admitting a good filtration, the dimension of $M^{(B)}_\lambda$ is called the *multiplicity* of $\nabla(\lambda)$ in M. The multiplicity of $\nabla(\lambda)$ in M is the number of times it occurs in any good filtration of M (Exercise 4.2.E.5); in particular, this number is independent of the choice of the good filtration of M.

(b) For any algebraic group H, define the $H \times H$-module structure on the affine coordinate ring $k[H]$ as follows:

$$\big((h_1, h_2)\cdot f\big)(h) = f(h_1^{-1} h h_2), \text{ for } h, h_1, h_2 \in H \text{ and } f \in k[H]. \tag{1}$$

The restriction of this action to $H \times 1$, resp. $1 \times H$, is called the *left*, resp. *right*, *regular representation* of H, and denoted respectively by $k[H]_\ell$ and $k[H]_r$. Thus, $k[H]_\ell$ denotes the representation of H in $k[H]$ defined by

$$(h_1 \cdot f)(h) = f(h_1^{-1} h). \tag{2}$$

Let M be a (rational) H-module and let M^{triv} be the trivial H-module with the same underlying vector space as M. Then, there exists an H-module embedding $\varepsilon : M \to k[H]_\ell \otimes M^{\text{triv}}$ defined as follows. View $k[H]_\ell \otimes M^{\text{triv}}$ as the space of morphisms $\phi : H \to M$ under the H-module structure given by

$$(h_1 \cdot \phi)h = \phi(h_1^{-1} h), \text{ for } h, h_1 \in H.$$

Now, define ε by

$$\varepsilon(m)h = h^{-1} m, \text{ for } h \in H, m \in M.$$

Clearly, ε is an H-module embedding.

In the following, G is as in the beginning of this section.

4.2.5 Theorem. *The $G \times G$-module $k[G]$ admits a filtration by $G \times G$-submodules:*

$$\mathcal{F}^0 = 0 \subset \mathcal{F}^1 \subset \mathcal{F}^2 \subset \cdots,$$

such that $\bigcup_j \mathcal{F}^j = k[G]$ and, for all $j \geq 1$,

$$\frac{\mathcal{F}^j}{\mathcal{F}^{j-1}} \simeq \nabla(\mu_j) \otimes \nabla(-w_o\mu_j), \quad \text{as } G \times G\text{-modules},$$

where the first copy of G acts only on the first factor $\nabla(\mu_j)$ and the second copy of G acts only on the second factor and $\{\mu_j\}$ is some bijective enumeration of $X^*(T)^+$.

In particular, $\{\mathcal{F}^j\}_j$ is a good filtration of the $G \times G$-module $k[G]$.

Proof. We apply Proposition 4.2.3 for the group G replaced by $G \times G$. We take $B \times B$, resp. $T \times T$, for the Borel subgroup, resp. maximal torus, of $G \times G$. Choose a height function $\widetilde{h} : X^*(T) \times X^*(T) \to \mathbb{R}$ as in 4.2.1. This gives rise to the canonical filtration of the $G \times G$-module $M = k[G]$ by $G \times G$-submodules

$$M^0 = (0) \subset M^1 \subset M^2 \subset \cdots.$$

Then, by Proposition 4.2.3(b), we have a $G \times G$-module embedding

$$\frac{M^j}{M^{j-1}} \hookrightarrow [\nabla(\lambda_j) \otimes \nabla(\mu_j)] \otimes M^{(B\times B)}_{(\lambda_j,\mu_j)}, \tag{1}$$

where $\{(\lambda_j, \mu_j)\}_{j\geq 1}$ is the enumeration of $X^*(T)^+ \times X^*(T)^+$ given by (4.2.1.7).

We next prove that for any $(\lambda, \mu) \in X^*(T)^+ \times X^*(T)^+$,

$$h^0(M, (\lambda, \mu)) = 0 \ \text{ if } \mu \neq -w_o\lambda, \text{ and} \tag{2}$$

$$h^0(M, (\lambda, -w_o\lambda)) = 1. \tag{3}$$

Since $UTw_oU \subset G$ is an open (dense) subset, $M \hookrightarrow k[UTw_oU]$, where Tw_o denotes $T\dot{w}_o$ for any coset representative $\dot{w}_o$ of w_o in $N(T)$. Thus, the $U \times U$-invariants

$$M^{U\times U} \hookrightarrow k[UTw_oU]^{U\times U} \simeq k[Tw_o] \overset{\beta}{\underset{\sim}{\to}} k[T] = \bigoplus_{\lambda\in X^*(T)} k_\lambda,$$

where the isomorphism β is induced from the variety isomorphism $T \to Tw_o$, $t \mapsto t\dot{w}_o$. Observe next that, for the character $\lambda : T \to k^*$, $\beta^{-1}(\lambda)$ is an eigenvector for the action of $T \times T$ with weight $(-\lambda, w_o\lambda)$ since

$$\begin{aligned}\big((t_1, t_2) \cdot (\beta^{-1}\lambda)\big)(s\dot{w}_o) &= (\beta^{-1}\lambda)(t_1^{-1}s\dot{w}_o t_2)\\ &= (\beta^{-1}\lambda)(t_1^{-1}s\dot{w}_o t_2\dot{w}_o^{-1}\dot{w}_o)\\ &= \lambda(t_1^{-1})(\beta^{-1}\lambda)(s\dot{w}_o)(w_o\lambda)(t_2).\end{aligned}$$

This shows that, for $(\lambda, \mu) \in X^*(T)^+ \times X^*(T)^+$,

(4) $$\dim M^{(B\times B)}_{(\lambda,\mu)} = 0 \ \text{ unless } \mu = -w_o\lambda, \text{ and}$$

(5) $$\dim M^{(B\times B)}_{(\lambda,-w_o\lambda)} \leq 1.$$

For $\lambda \in X^*(T)^+$, let v_λ be a nonzero highest weight vector of the Weyl module $V(\lambda)$ and let $f_{-w_o\lambda} \in V(\lambda)^*$ be a nonzero B-eigenvector (of weight $-w_o\lambda$). Consider the function $\theta_\lambda : G \to k$ defined by $g \mapsto f_{-w_o\lambda}(g^{-1}v_\lambda)$. Then, θ_λ is a $B \times B$-eigenvector of weight $(\lambda, -w_o\lambda)$ and thus we have equality in (5), i.e.,

(6) $$\dim M^{(B\times B)}_{(\lambda,-w_o\lambda)} = 1 \ \text{ for any } \lambda \in X^*(T)^+.$$

Taking the associated Chevalley group scheme $G_{\mathbb{Z}}$ over $\mathbb{Z}$ (i.e., the split form of G over $\mathbb{Z}$), we have, for any ring R,

$$R[G_R] = \mathbb{Z}[G_{\mathbb{Z}}] \otimes_{\mathbb{Z}} R\,,$$

where G_R is the group of R-rational points of $G_{\mathbb{Z}}$ and $R[G_R]$ is its affine coordinate ring over R. Thus, in our notation, $G_k = G$.

Analogous to the filtration $\{M^j\}_{j\geq 0}$ of $k[G]$, we have the filtration $\{M^j(\mathbb{Q})\}_j$ of $\mathbb{Q}[G_{\mathbb{Q}}]$. Now, for any $j \geq 0$, set

(7) $$M^j(\mathbb{Z}) := M^j(\mathbb{Q}) \cap \mathbb{Z}[G_{\mathbb{Z}}].$$

Then, $M^j(\mathbb{Z})$ is a finitely generated (and hence free) $\mathbb{Z}$-module with

(8) $$\operatorname{rank} M^j(\mathbb{Z}) = \dim_{\mathbb{Q}} M^j(\mathbb{Q}).$$

Clearly, $\mathbb{Z}[G_{\mathbb{Z}}]/M^j(\mathbb{Z})$ is torsion free and thus we have a canonical injection

$$M^j(\mathbb{Z}) \otimes_{\mathbb{Z}} R \hookrightarrow R[G_R].$$

In particular,

(9) $$M^j(\mathbb{Z}) \otimes_{\mathbb{Z}} k \hookrightarrow M^j.$$

By (1), (4) and (6), for any $j \geq 1$,

(10) $$M^j \neq M^{j-1} \iff \mu_j = -w_o\lambda_j,$$

and, similarly,

(11) $$M^j(\mathbb{Q}) \neq M^{j-1}(\mathbb{Q}) \iff \mu_j = -w_o\lambda_j.$$

Let $r(1)$ be the smallest integer such that $M^{r(1)} \neq 0$, $r(2)$ be the smallest integer such that $M^{r(2)} \supsetneq M^{r(1)}$, and so on. Set $\mathcal{F}^j = M^{r(j)}$ for any $j \geq 1$. Then, by (1), (4), (6),

(12) $$\pi_j : \frac{\mathcal{F}^j}{\mathcal{F}^{j-1}} \hookrightarrow \nabla(\mu_j) \otimes \nabla(-w_o\mu_j),$$

where $\{\mu_1, \mu_2, \ldots\}$ is a bijective enumeration of $X^*(T)^+$.

Similarly, letting

$$\mathcal{F}^j(\mathbb{Q}) = M^{r(j)}(\mathbb{Q}),$$

by (10), (11),

$$\mathcal{F}^{j-1}(\mathbb{Q}) \subsetneq \mathcal{F}^j(\mathbb{Q}), \tag{13}$$

and, moreover, we have a $G_{\mathbb{Q}} \times G_{\mathbb{Q}}$-module injection

$$\pi_j^{\mathbb{Q}} : \frac{\mathcal{F}^j(\mathbb{Q})}{\mathcal{F}^{j-1}(\mathbb{Q})} \hookrightarrow \nabla^{\mathbb{Q}}(\mu_j) \otimes \nabla^{\mathbb{Q}}(-w_o\mu_j), \tag{14}$$

where $\nabla^{\mathbb{Q}}(\mu_j)$ is the same as $\nabla(\mu_j)$ over the base field $\mathbb{Q}$. Since $\nabla^{\mathbb{Q}}(\mu_j)\otimes\nabla^{\mathbb{Q}}(-w_o\mu_j)$ is an irreducible $G_{\mathbb{Q}} \times G_{\mathbb{Q}}$-module, by (13), $\pi_j^{\mathbb{Q}}$ is an isomorphism for all $j \geq 1$.

Assume now by induction that $\mathcal{F}^i = M^{r(i)}(\mathbb{Z}) \otimes_{\mathbb{Z}} k$, and π_i is an isomorphism for all $i \leq j-1$. Thus, $\pi_j^{\mathbb{Q}}$ being an isomorphism, by (8), (9) and (12), we get that $\mathcal{F}^j = M^{r(j)}(\mathbb{Z}) \otimes_{\mathbb{Z}} k$, and π_j is an isomorphism. This completes the induction and proves the theorem. $\square$

4.2.6 Corollary. *For any finite-dimensional G-module M,*

$$\operatorname{ch} M \geq \sum_{\lambda \in X^*(T)^+} \big(h^0(M,\lambda) - h^1(M,\lambda)\big) \operatorname{ch} \nabla(\lambda). \tag{1}$$

Observe that, since M is finite-dimensional, $h^0(M,\lambda)$ is nonzero only for finitely many $\lambda \in X^*(T)^+$ and also, by Lemma 4.2.2(a), $h^1(M,\lambda)$ is nonzero only for finitely many λ. Moreover, for any λ, $h^0(M,\lambda)$ is clearly finite and $h^1(M,\lambda)$ is finite by (2)–(3) below.

Proof. By 4.2.4, there is a G-module embedding $\varepsilon : M \hookrightarrow k[G]_\ell \otimes M^{\mathrm{triv}}$. Since M is finite-dimensional, there exists a large enough $\mathcal{F}^{j_o}$ such that $\varepsilon(M) \subset \mathcal{F}_\ell^{j_o} \otimes M^{\mathrm{triv}}$, where $\{\mathcal{F}^j\}$ is the filtration as in Theorem 4.2.5 and $\mathcal{F}_\ell^j$ denotes $\mathcal{F}^j$ considered only as a $G = G \times e$-module.

The exact sequence

$$0 \to M \to \mathcal{F}_\ell^{j_o} \otimes M^{\mathrm{triv}} \to Q \to 0,$$

where $Q := (\mathcal{F}_\ell^{j_o} \otimes M^{\mathrm{triv}})/\varepsilon(M)$, gives rise to the exact sequence (for any $\lambda \in X^*(T)^+$):

(2)

$$0 \to \operatorname{Hom}_G(V(\lambda), M) \to \operatorname{Hom}_G(V(\lambda), \mathcal{F}_\ell^{j_o} \otimes M^{\mathrm{triv}}) \to$$
$$\operatorname{Hom}_G(V(\lambda), Q) \to \operatorname{Ext}^1_G(V(\lambda), M) \to \operatorname{Ext}^1_G(V(\lambda), \mathcal{F}_\ell^{j_o} \otimes M^{\mathrm{triv}}) \to \cdots.$$

By Theorem 4.2.5 and Lemma 4.2.2(c),

(3) $$\operatorname{Ext}^1_G(V(\lambda), \mathcal{F}^{j_o}_\ell \otimes M^{\text{triv}}) = 0.$$

Thus,

(4) $$h^0(M, \lambda) - h^0(\mathcal{F}^{j_o}_\ell \otimes M^{\text{triv}}, \lambda) + h^0(Q, \lambda) - h^1(M, \lambda) = 0.$$

Since $\{\mathcal{F}^j\}$ is a good filtration of the $G \times G$-module $k[G]$ and also it is part of the canonical filtration of $k[G]$ (see Theorem 4.2.5 and its proof), we get by Proposition 4.2.3(c),

(5) $$\operatorname{ch} \mathcal{F}^{j_o}_\ell \otimes M^{\text{triv}} = \sum_{\lambda \in X^*(T)^+} h^0(\mathcal{F}^{j_o}_\ell \otimes M^{\text{triv}}, \lambda) \operatorname{ch} \nabla(\lambda),$$

and, by Proposition 4.2.3(b),

(6) $$\operatorname{ch} Q \leq \sum_{\lambda \in X^*(T)^+} h^0(Q, \lambda) \operatorname{ch} \nabla(\lambda).$$

Combining (4)–(6), (1) follows. □

The following cohomological criterion for the existence of good filtrations is very useful.

4.2.7 Theorem. *For any finite-dimensional G-module M, the following are equivalent:*

(a) The canonical filtration of M is a good filtration.
(b) There exists a good filtration of M.
(c) $\operatorname{Ext}^1_G(V(\lambda), M) = 0$, *for all* $\lambda \in X^*(T)^+$.

Thus, for two finite-dimensional G-modules M_1, M_2, $M_1 \oplus M_2$ admits a good filtration iff both of M_1 and M_2 admit good filtrations.

Proof. Of course (b) is a particular case of (a).

(b)⇒(c): Let $F^0 = (0) \subset F^1 \subset F^2 \subset \cdots$ be a good filtration of M. Then, considering the long exact Ext sequence associated to the short exact sequence:

$$0 \to F^{j-1} \to F^j \to F^j/F^{j-1} \to 0,$$

and using Lemma 4.2.2(c), we get that

$$\operatorname{Ext}^1_G(V(\lambda), F^j) = 0 \text{ for all } j \text{ and any } \lambda \in X^*(T)^+.$$

Thus, (c) follows by taking large enough j such that $F^j = M$.

(c)⇒(a): By Corollary 4.2.6, since $h^1(M, \lambda) = 0$ by assumption,

$$\operatorname{ch} M \geq \sum_{\lambda \in X^*(T)^+} h^0(M, \lambda) \operatorname{ch} \nabla(\lambda).$$

Thus, (a) follows by applying Proposition 4.2.3 (b)–(c). □

4.2.8 Remark. The above theorem continues to hold for an arbitrary (rational) G-module M. The implication (b) $\Rightarrow$ (c) follows by the same proof given above together with [Jan–03, Part I, Lemma 4.17]. For the proof of the implication (c)$\Rightarrow$(a), consider the canonical filtration $(M^j)_{j\geq 0}$ of M. It is easy to see that for any $j \geq 0$ and $\lambda \in X^*(T)^+$,

$$\operatorname{Hom}_G\big(V(\lambda), M\big) \to \operatorname{Hom}_G\big(V(\lambda), M/M^j\big)$$

is surjective. Thus, we get $\operatorname{Ext}^1_G\big(V(\lambda), M^j\big) = 0$.

Consider the embedding, as in Proposition 4.2.3(b),

$$\pi_j : \frac{M^j}{M^{j-1}} \hookrightarrow \nabla(\lambda_j) \otimes M^{(B)}_{\lambda_j}$$

and let Q_j be the cokernel. By (4.2.3.5) and (4.2.1.11), the induced map

$$\operatorname{Hom}_G\Big(V(\lambda), \frac{M^j}{M^{j-1}}\Big) \to \operatorname{Hom}_G\big(V(\lambda), \nabla(\lambda_j) \otimes M^{(B)}_{\lambda_j}\big)$$

is surjective and thus

(1) $$\operatorname{Hom}_G\big(V(\lambda), Q_j\big) \hookrightarrow \operatorname{Ext}^1_G\Big(V(\lambda), \frac{M^j}{M^{j-1}}\Big).$$

Assume by induction that π_m is an isomorphism for all $1 \leq m \leq j-1$ and use the following vanishing for any $\lambda, \mu \in X^*(T)^+$:

(2) $$\operatorname{Ext}^q_G\big(V(\lambda), \nabla(\mu)\big) = 0 \text{ for all } q > 0$$

(cf. [Jan–03, Part II, Proposition 4.13]) to show that

(3) $$\operatorname{Ext}^1_G\Big(V(\lambda), \frac{M^j}{M^{j-1}}\Big) = 0.$$

To prove (3), consider the long exact Ext sequence corresponding to the short exact sequence:

$$0 \to M^{j-1} \to M^j \to M^j/M^{j-1} \to 0.$$

Thus, from (1), we get that $\operatorname{Hom}_G(V(\lambda), Q_j) = 0$ and thus π_j is an isomorphism. This proves that (M^j) is a good filtration of M.

4.2.9 Definition. Recall the definition of hyperalgebra $\mathfrak{U}_H$ from Section 2.1. A $\mathfrak{U}_G$-module M is called a $(\mathfrak{U}_G, B)$-*module* if the $\mathfrak{U}_B$-action on M "integrates" to a rational B-module structure on M, i.e., M is a B-module such that the associated action of $\mathfrak{U}_B$ coincides with the restriction of the $\mathfrak{U}_G$-action. For a $(\mathfrak{U}_G, B)$-module M, let $M_{\text{int}} \subset M$ be the subspace of $\mathfrak{U}_G$-finite vectors of M, i.e.,

(1) $$M_{\text{int}} := \{v \in M : \dim(\mathfrak{U}_G \cdot v) < \infty\}.$$

Then, by Verma's conjecture proved in [CPS–80] and [Sul–78], M_{int} is the biggest $\mathfrak{U}_G$-submodule such that the $\mathfrak{U}_G$-action integrates to a rational G-action.

For a B-module M and $\lambda \in X^*(T)$, M is called λ-*isotypical* if all the B-eigenvectors of M have λ as their weight.

For $\lambda \in X^*(T)$, define the tensor product B-module

$$I(\lambda) = k[B/T] \otimes k_\lambda, \tag{2}$$

where $k[B/T]$ is a B-module under the left multiplication of B on B/T. Then, $I(\lambda)$ has a unique line k_λ of B-eigenvectors. In particular, it is an indecomposable B-module. Moreover, from the B-equivariant fibration $B \to B/T$, we get

$$k[B] = \bigoplus_{\mu \in X^*(T)} I(\mu). \tag{3}$$

Further, for any algebraic group H, $k[H]$ is an injective H-module in the category of rational H-modules (cf. [Jan–03, Part I, Proposition 3.10]). This follows since, by the Frobenius reciprocity [Jan–03, Part I, Proposition 3.4(b)], for any H-module M,

$$\operatorname{Hom}_H(M, k[H]) \simeq \operatorname{Hom}_k(M, k)$$

under $\theta \mapsto e \circ \theta$, where $e : k[H] \to k$ is the evaluation at 1. Thus, $I(\lambda)$ is an injective B-module, and hence it is the injective hull (cf. [Jan–03, Part I, §§3.16–3.17]) in the category of rational B-modules of the one-dimensional B-module k_λ .

In fact, $I(\lambda)$ acquires the structure of a $(\mathfrak{U}_G, B)$-module by identifying

$$I(\lambda) \simeq H^0\big(Bw_oB/B, \mathcal{L}(-w_o\lambda)\big)$$

under $B/T \to Bw_oB/B$, $b \mod T \mapsto bw_o \mod B$ and applying [Kem–78, §11].

4.2.10 Lemma. *Let $\lambda \in X^*(T)$ and let M, N be two λ-isotypical $(\mathfrak{U}_G, B)$-modules (i.e., M, N are λ-isotypical as B-modules). Then, any B-module map $f : M \to N$ is automatically a $\mathfrak{U}_G$-module map.*

In particular, a λ-isotypical B-module M admits at most one $(\mathfrak{U}_G, B)$-module structure extending the original B-module structure.

Proof. Since M, N are $\mathfrak{U}_G$-modules and $\mathfrak{U}_G$ is a Hopf algebra, the space of all the k-linear maps $\operatorname{Hom}_k(M, N)$ is canonically a $\mathfrak{U}_G$-module. Let f be a B-module map and assume that it is not a $\mathfrak{U}_G$-module map. Then, f being a B-module map, there exists a negative simple root vector $f_i = f_{\alpha_i}$ such that $f_i^{(m)} \cdot f \neq 0$ for some $m \geq 1$. Take the smallest $m_o \geq 1$ such that $f_o := f_i^{(m_o)} \cdot f \neq 0$. Then, since $f_i^{(m)}$ commutes with $e_{\alpha_j}^{(n)}$ for any $i \neq j$,

$$e_{\alpha_j}^{(n)} \cdot f_o = 0 \ \text{ for all } n \geq 1 \text{ and } j \neq i. \tag{1}$$

Moreover, by the $s\ell_2$ commutation relation [Hum–72, Lemma 26.2],

$$e_{\alpha_i}^{(n)} \cdot f_o = 0 \ \text{ for all } n \geq 1. \tag{2}$$

Thus, f_o is U-invariant and has weight $-m_o\alpha_i$, where U is the unipotent radical of B. Take a nonzero B-eigenvector $v_o \in f_o(M)$ (of weight λ). Then, $f_o^{-1}(kv_o)$ contains a nonzero B-eigenvector of weight $\lambda + m_o\alpha_i$. This is a contradiction since each B-eigenvector of M has weight λ. Thus, f is a $\mathfrak{U}_G$-module map. $\square$

4.2.11 Definition. Let $\lambda \in X^*(T)^+$. For a G-module M which is λ-isotypical as a B-module, define

$$\mathcal{D}_\lambda(M) := H^0\big(G/B, \mathcal{L}(M_{w_o\lambda})\big),$$

where the $w_o\lambda$-weight space $M_{w_o\lambda}$ of M is regarded as a B-module under the trivial action of U. Then,

$$\mathcal{D}_\lambda(M) \simeq \nabla(\lambda) \otimes_k M^{\mathrm{triv}}_{w_o\lambda}, \quad \text{as } G\text{-modules},$$

where $M^{\mathrm{triv}}_{w_o\lambda}$ is the space $M_{w_o\lambda}$ with the trivial G-module structure. Thus, $\mathcal{D}_\lambda(M)$ is λ-isotypical. Moreover, the projection onto the $w_o\lambda$-weight space $\pi : M \to M_{w_o\lambda}$ induces a G-module map $i_M : M \to \mathcal{D}_\lambda(M)$ (since M is λ-isotypical, π is a B-module map). Further, i_M is injective. Otherwise, $\operatorname{Ker} i_M$ would contain a nonzero vector of weight $w_o\lambda$ (since $\operatorname{Ker} i_M$ would contain a B-eigenvector of weight λ), a contradiction. Since the G-module map i_M restricted to the weight spaces corresponding to the weight $w_o\lambda$ is an isomorphism, we get that i_M restricts to an isomorphism

$$M_\lambda \simeq \big(\mathcal{D}_\lambda(M)\big)_\lambda. \tag{1}$$

Thus, i_M induces an isomorphism between U-invariants:

$$M^U \simeq \mathcal{D}_\lambda(M)^U. \tag{2}$$

4.2.12 Proposition. *Let $\lambda \in X^*(T)^+$ and let $R = \bigoplus_{n\geq 0} R_n$ be a commutative associative reduced graded G-algebra. Assume further that each R_n is $n\lambda$-isotypical as a B-module and that*

$$R_1 \neq \mathcal{D}_\lambda(R_1). \tag{1}$$

Then, there exists $y \in \mathcal{D}_{n\lambda}(R_n)\backslash R_n$ for some $n \geq 0$ such that $y^p \in R_{np}$.

(Observe that $\mathcal{D}R := \bigoplus_{n\geq 0} \mathcal{D}_{n\lambda}(R_n)$ is canonically a graded G-algebra canonically containing R as a graded G-subalgebra. Moreover, $\mathcal{D}R$ is a reduced algebra.)

Proof. Writing $(R_1)_{w_o\lambda}$ as a direct sum of one-dimensional spaces, decompose $\mathcal{D}_\lambda(R_1)$ as a direct sum of G-submodules each isomorphic with $\nabla(\lambda)$. By the assumption (1), there exists a G-submodule $N \subset \mathcal{D}_\lambda(R_1)$ such that $N \simeq \nabla(\lambda)$ and N is not contained in R_1. Take a nonzero $x \in N_{w_o\lambda} \subset R_1$. Since R is reduced, the subalgebra R^x of R generated by x is the polynomial algebra $k[x]$ on one generator x. Let $\mathcal{D}R^x$ be the graded subalgebra $\bigoplus_{n\geq 0} H^0(G/B, \mathcal{L}(R^x_n))$ of $\mathcal{D}R$, where R^x_n is regarded as a B-module under the trivial action of U. Then, $(\mathcal{D}R^x)_1 = N$ and

$$\mathcal{D}R^x \simeq \bigoplus_{n\geq 0} \nabla(n\lambda), \quad \text{as graded algebras}. \tag{2}$$

Consider the subalgebra $S^x := \mathcal{D}R^x \cap R$ of R. Then, we claim that the algebras $\mathcal{D}R^x$ and S^x are finitely generated and the morphism $\theta : \mathrm{Spec}(\mathcal{D}R^x) \to \mathrm{Spec}\, S^x$, induced from the inclusion $S^x \hookrightarrow \mathcal{D}R^x$, is finite and bijective. Let $\mathcal{A} \subset \mathcal{D}R^x$ be the subalgebra generated by $\bigoplus_{w\in W} N_{w\lambda}$ and let $\mathcal{A}_+$ be the irrelevant ideal of $\mathcal{A}$. Since $\mathcal{L}(\lambda)$ is semi-ample (Exercise 3.1.E.1), by Lemma 1.1.13(i), the algebra $\mathcal{D}R^x$ is a finitely generated domain.

Moreover, the spectrum Spec $(\mathcal{D}R^x)$ can be identified with the subset $G\cdot v_+ \cup \{0\} \subset \nabla(\lambda)^*$, the affine cone over the image of $G/B \hookrightarrow \mathbb{P}(\nabla(\lambda)^*)$, $gB \mapsto g[v_+]$, where v_+ is a highest weight vector of $\nabla(\lambda)^*$. Using the following variant of the Bruhat decomposition:

$$G = \bigcup_{w\in W} wU^-B,$$

for any $g \in G$ we can find $w = w_g \in W$ such that

$$gv_+(wx) \neq 0.$$

Since $wx \in \mathcal{A}_+$, the radical $\sqrt{\mathcal{A}_+ \cdot \mathcal{D}R^x} = \mathcal{D}R^x_+$, where $\mathcal{D}R^x_+$ is the irrelevant ideal of $\mathcal{D}R^x$. Since $\mathcal{D}R^x$ is a finitely generated algebra; in particular, noetherian, we get that $\left(\sqrt{\mathcal{A}_+ \cdot \mathcal{D}R^x}\right)^n \subset \mathcal{A}_+ \cdot \mathcal{D}R^x$ for some $n \geq 1$. Thus, $\mathcal{D}R^x/\left(\mathcal{A}_+ \cdot \mathcal{D}R^x\right)$ is a finite-dimensional vector space over k. This proves that $\mathcal{D}R^x$ is a finite $\mathcal{A}$-module. Since $N_{w\lambda} \subset R_1$, we have $\mathcal{A} \subset S^x$. In particular, $\mathcal{D}R^x$ is a finite S^x-module and thus S^x is a finitely generated algebra. This shows that the canonical map $\theta : \mathrm{Spec}\,(\mathcal{D}R^x) \to \mathrm{Spec}\, S^x$ is finite and surjective. To show that θ is injective, it suffices to observe that the stabilizers of the lines kv_+ and kv'_+ in G are the same parabolic subgroups, where v'_+ is a highest weight vector of $(N \cap R_1)^*$.

We first prove the proposition assuming that the algebra S^x is not normal. Take its normalization $\bar{S}^x$. Since $\mathcal{D}R^x$ is normal (Theorem 3.2.2), we get

$$S^x \subset \bar{S}^x \subset \mathcal{D}R^x.$$

Moreover, from the bijectivity of θ and since $\mathcal{D}R^x$ is a finite S^x-module, we get that the induced map $i : \mathrm{Spec}\,(\bar{S}^x) \to \mathrm{Spec}\,(S^x)$ is bijective and, of course, i (being the normalization) is an isomorphism on a nonempty open subset. In fact, by the G-equivariance, i is an isomorphism outside $\{0\}$. Thus, we get $S^x_n = \bar{S}^x_n$ for all large enough n. Take a homogeneous $y \in \bar{S}^x \backslash S^x$ of maximal degree. Then, $y^p \in R$ and $y \in \mathcal{D}R\backslash R$, proving the proposition in the case $S^x \neq \bar{S}^x$.

So, assume now that S^x is normal, i.e., $S^x = \bar{S}^x$ and consider the field K_{S^x} and $K_{\mathcal{D}R^x}$ of fractions of the domains S^x and $\mathcal{D}R^x$ respectively. Since $(\mathcal{D}R^x)_1 = N$ and N is not contained in R_1, $S^x \neq \mathcal{D}R^x$, so is $K_{S^x} \neq K_{\mathcal{D}R^x}$. But the extension $K_{S^x} \subset K_{\mathcal{D}R^x}$ is purely inseparable, since θ is a bijection. In particular, there exists a power q of p with $K^q_{\mathcal{D}R^x} \subset K_{S^x}$. Thus, we can find $y \in \mathcal{D}R^x\backslash S^x$ such that $y^p \in K_{S^x}$. But S^x being normal, we have $y^p \in S^x$, proving the proposition in this case as well. □

4.2.13 Theorem. *Let X be a G-scheme which admits a B-canonical splitting. Then, for any G-linearized line bundle $\mathcal{L}$ on X, the G-module $H^0(X, \mathcal{L})$ admits a good filtration.*

Proof. Let Y be the product space $G/B \times X$ under the diagonal action of G. Then, Y has an open dense B-stable subset $Y^o := Bw_oB/B \times X$. Moreover, there is a B-equivariant biregular isomorphism

$$B/T \to Bw_oB/B, \; bT \mapsto bw_oB. \tag{1}$$

Consider the G-linearized line bundle $\widetilde{\mathcal{L}} := \varepsilon \boxtimes \mathcal{L}$ on Y, where ε is the trivial line bundle on G/B. Then, of course,

$$H^0(X, \mathcal{L}^n) \simeq H^0(Y, \widetilde{\mathcal{L}}^n), \quad \text{as } G\text{-modules.} \tag{2}$$

Define the graded B-algebra

$$\overset{\circ}{\mathcal{C}} := \bigoplus_{n\geq 0} \overset{\circ}{\mathcal{C}}_n, \quad \text{where } \overset{\circ}{\mathcal{C}}_n := H^0(Y^o, \widetilde{\mathcal{L}}^n),$$

and the graded G-algebra

$$\mathcal{C} := \bigoplus_{n\geq 0} \mathcal{C}_n, \quad \text{where } \mathcal{C}_n := H^0(Y, \widetilde{\mathcal{L}}^n).$$

Then, of course, $\mathcal{C}$ is a graded subalgebra of $\overset{\circ}{\mathcal{C}}$. By [Kem–78, §11], $\overset{\circ}{\mathcal{C}}$ is a $(\mathfrak{U}_G, B)$-algebra.

We now break the proof of the theorem in several steps.

Step 1: *Construction of the algebras $C(\lambda)$ and $\overset{\circ}{C}(\lambda)$.*

Fix a height function $h : X^*(T) \to \mathbb{R}$ as in 4.2.1. For any $\lambda \in X^*(T)^+$ define the following graded subalgebras of $\mathcal{C}$ and $\overset{\circ}{\mathcal{C}}$ respectively.

$$\mathcal{C}(\lambda) := \bigoplus_n \mathcal{F}_{n\lambda}(\mathcal{C}_n), \quad \text{and}$$

$$\overset{\circ}{\mathcal{C}}(\lambda) := \bigoplus_n \mathcal{F}_{n\lambda}(\overset{\circ}{\mathcal{C}}_n),$$

where $\mathcal{F}_{n\lambda}(\mathcal{C}_n)$ is the largest B-submodule of $\mathcal{C}_n$ such that each weight μ of $\mathcal{F}_{n\lambda}(\mathcal{C}_n)$ satisfies

$$h(\mu) \leq h(n\lambda),$$

and $\mathcal{F}_{n\lambda}(\overset{\circ}{\mathcal{C}}_n)$ is defined similarly. We define the B-stable ideals $\mathcal{C}(\lambda)^-$ of $\mathcal{C}(\lambda)$ and $\overset{\circ}{\mathcal{C}}(\lambda)^-$ of $\overset{\circ}{\mathcal{C}}(\lambda)$ by

$$\mathcal{C}(\lambda)^- := \bigoplus_n \mathcal{F}^-_{n\lambda}(\mathcal{C}_n), \quad \text{and}$$

$$\overset{\circ}{\mathcal{C}}(\lambda)^- := \bigoplus_n \mathcal{F}^-_{n\lambda}(\overset{\circ}{\mathcal{C}}_n),$$

where $\mathcal{F}^-_{n\lambda}(\mathcal{C}_n)$ is the largest submodule of $\mathcal{C}_n$ such that each weight μ of $\mathcal{F}^-_{n\lambda}(\mathcal{C}_n)$ satisfies

$$h(\mu) < h(n\lambda)$$

and similarly for $\mathcal{F}^-_{n\lambda}(\overset{\circ}{\mathcal{C}}_n)$.

By Proposition 4.2.3(a), $\mathcal{C}(\lambda)$ is a graded G-algebra and $\mathcal{C}(\lambda)^-$ is a G-stable graded ideal. Moreover, by Exercise 4.2.E.1, $\overset{\circ}{\mathcal{C}}(\lambda)$ is a $(\mathfrak{U}_G, B)$-algebra and $\overset{\circ}{\mathcal{C}}(\lambda)^-$ is a $(\mathfrak{U}_G, B)$-stable ideal of $\overset{\circ}{\mathcal{C}}(\lambda)$. Consider the quotient algebras

$$C(\lambda) := \mathcal{C}(\lambda)/\mathcal{C}(\lambda)^-, \quad \text{and}$$
$$\overset{\circ}{C}(\lambda) := \overset{\circ}{\mathcal{C}}(\lambda)/\overset{\circ}{\mathcal{C}}(\lambda)^-.$$

Then, $C(\lambda)$ is a graded G-algebra such that the homogeneous component $C(\lambda)_n$ of $C(\lambda)$ of degree n is $n\lambda$-isotypical as a B-module. Similarly, $\overset{\circ}{C}(\lambda)$ is a graded $(\mathfrak{U}_G, B)$-algebra such that $\overset{\circ}{C}(\lambda)_n$ is $n\lambda$-isotypical as a B-module. Clearly, $C(\lambda)$ is a graded subalgebra of $\overset{\circ}{C}(\lambda)$.

Step 2: *The algebras $C(\lambda)$ and $\overset{\circ}{C}(\lambda)$ are reduced.*

Let ϕ be a B-canonical splitting of X. Then, ϕ induces a B-canonical Frobenius-linear endomorphism $\phi_{\overset{\circ}{C}(\lambda)}$ of the B-algebra $\overset{\circ}{C}(\lambda)$ keeping the subalgebra $C(\lambda)$ stable. To see this, observe first that, by Proposition 4.1.17, Y admits a B-canonical splitting. Thus, by Lemma 4.1.13, ϕ induces a B-canonical $\phi_{\overset{\circ}{\mathcal{C}}} \in \mathrm{End}_F(\overset{\circ}{\mathcal{C}})$. Since $\phi_{\overset{\circ}{\mathcal{C}}}$ is T-invariant and takes B-submodules to B-submodules (Proposition 4.1.8), it induces a B-canonical $\phi_{\overset{\circ}{C}(\lambda)} \in \mathrm{End}_F(\overset{\circ}{C}(\lambda))$. Moreover, $\phi_{\overset{\circ}{C}(\lambda)}(1) = 1$. It is easy to see that $\phi_{\overset{\circ}{C}(\lambda)}(C(\lambda)) \subset C(\lambda)$. Let $\phi_{C(\lambda)} := (\phi_{\overset{\circ}{C}(\lambda)})_{|C(\lambda)}$. Using the Frobenius-linear endomorphism $\phi_{\overset{\circ}{C}(\lambda)}$, we immediately obtain that $\overset{\circ}{C}(\lambda)$ is reduced and hence so is $C(\lambda)$.

Step 3: *$\overset{\circ}{C}(\lambda)$ is an injective B-module.*

We prove that, for any $n \geq 0$, the n-th graded component $\overset{\circ}{C}(\lambda)_n$ is an injective B-module (which is $n\lambda$-isotypical).

For any B-module M, there is a B-module isomorphism

$$\xi : k[B/T] \otimes M^{\mathrm{triv}} \simeq k[B/T] \otimes M$$

defined as follows, where M^{triv} is the same as M as a T-module, but U acts trivially on M^{triv}. For $m \in M$, let M_m be the (finite-dimensional) B-submodule of M generated by m. Take a basis $\{m_i\}$ of M_m and the dual basis $\{m_i^*\}$ of M_m^*. Now, define

$$\xi(f \otimes m) = \sum f \cdot \theta_i^m \otimes m_i, \tag{3}$$

where $\theta_i^m(u) = m_i^*(um)$, for $u \in U \simeq B/T$. Since $k[Bw_oB/B] \simeq k[B/T]$ ((4.2.13.1)) has a unique B-eigenvector, from the isomorphism ξ we conclude that

$$\mathcal{F}_{n\lambda}(\overset{\circ}{\mathcal{C}}_n) \simeq \xi\left(k[Bw_oB/B] \otimes \left(\bigoplus_{\substack{\mu\in X^*(T)\\ h(\mu)\le h(n\lambda)}} \left(\mathcal{C}_n^{\mathrm{triv}}\right)_\mu\right)\right). \tag{4}$$

Thus,

$$\overset{\circ}{C}(\lambda)_n \simeq \xi\left(k[Bw_oB/B] \otimes \left(\mathcal{C}_n^{\mathrm{triv}}\right)_{n\lambda}\right). \tag{5}$$

In particular, by 4.2.9 and the isomorphism ξ, $\overset{\circ}{C}(\lambda)_n$ is an injective B-module.

Thus, the B-module inclusion $C(\lambda)_n \subset \overset{\circ}{C}(\lambda)_n$ extends to a B-module map $\theta : \mathcal{D}_{n\lambda}(C(\lambda)_n) \to \overset{\circ}{C}(\lambda)_n$, where we identify $C(\lambda)_n \hookrightarrow \mathcal{D}_{n\lambda}(C(\lambda)_n)$ via $i_{C(\lambda)_n}$ (as in 4.2.11). In fact, θ is unique. For let θ', θ'' be two such extensions. Then, $\theta' - \theta''$ is a B-module map which is identically zero on $C(\lambda)_n$. By (4.2.11.1), the $n\lambda$-weight space $(C(\lambda)_n)_{n\lambda} = (\mathcal{D}_{n\lambda}(C(\lambda)_n))_{n\lambda}$. Thus, $\mathrm{Im}(\theta' - \theta'') \cap \left((\overset{\circ}{C}(\lambda)_n)_{n\lambda}\right) = (0)$. Since $\overset{\circ}{C}(\lambda)_n$ is $n\lambda$-isotypical, this forces $\theta' = \theta''$. Further, θ is injective by using (4.2.11.2).

By Lemma 4.2.10, θ is a $\mathfrak{U}_G$-module map. So, we can canonically identify

$$C(\lambda)_n \subset \mathcal{D}_{n\lambda}(C(\lambda)_n) \subset \overset{\circ}{C}(\lambda)_n.$$

Step 4: *We have the following:*

$$C(\lambda)_1 = \mathcal{D}_\lambda(C(\lambda)_1). \tag{6}$$

If (6) were false, there exists an element $x \in \mathcal{D}_{n\lambda}(C(\lambda)_n)\backslash C(\lambda)_n$ by Proposition 4.2.12 for some $n \geq 0$ such that $x^p \in C(\lambda)_{np}$. Applying $\phi_{\overset{\circ}{C}(\lambda)}$ we get $\phi_{\overset{\circ}{C}(\lambda)}(x^p) = x \in C(\lambda)_n$, which is a contradiction to the choice of x. Thus, (6) is proved.

This shows that $\mathcal{C}_1$ has a filtration such that the successive quotients are isomorphic with $\{\mathcal{D}_\lambda(C(\lambda)_1)\}_{\lambda\in X^*(T)^+}$. Thus, $\mathcal{C}_1$ admits a good filtration, proving the theorem by using the identification (2). ☐

4.2.14 Corollary. *For $\lambda, \mu \in X^*(T)^+$, the tensor product $\nabla(\lambda)\otimes\nabla(\mu)$ admits a good filtration, where $\nabla(\lambda)$ is defined by (4.2.1.1). More generally, for any $w \in W$, the G-module $H^0\left(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\right)$ admits a good filtration.*

Proof. By Theorem 4.1.15, there exists a B-canonical splitting of G/B and thus, by Proposition 4.1.17, there exists a B-canonical splitting of $G \times_B (G/B) \simeq G/B \times G/B$ (alternatively use Exercise 4.1.E.6). Now, apply Theorem 4.2.13 to the G-variety $G/B \times G/B$ under the diagonal action of G together with the G-linearized line bundle $\mathcal{L}\left((-w_o\lambda) \boxtimes (-w_o\mu)\right)$. This proves the first part of the corollary.

The more general statement follows by the same argument. Use Exercise 4.1.E.6 and Theorem 4.2.13. ☐

Even though we defined earlier the notion of good filtration of a G-module for a semisimple, simply-connected G, the same definition works for any connected reductive G. Similarly, the definition of B-canonical splittings extends without change to a Borel subgroup of any reductive G. Then, Theorem 4.2.13 remains true (with the same proof) for any reductive G.

As another consequence of Theorem 4.2.13, we obtain the following.

4.2.15 Corollary. *Let P be a standard parabolic subgroup of G with the Levi component $L_P \supset T$. Then, for any $\lambda \in X^*(T)^+$, the G-module $\nabla(\lambda)$ admits a good filtration considered as a module for the reductive group L_P.*

More generally, for any P-stable closed reduced subscheme $Y \subset G/B$, $H^0(Y, \mathcal{L}(\lambda)_{|Y})$ admits a good filtration considered as a module for the group L_P.

Proof. By Theorem 4.1.15, there exists a B-canonical splitting of G/B compatibly splitting Y. Of course, a B-canonical splitting is in particular a B_{L_P}-canonical splitting, where $B_{L_P} \subset B$ is a Borel subgroup of L_P. Thus, the corollary follows from Theorem 4.2.13. $\square$

4.2.E Exercises

(1*) Let $h : X^*(T) \to \mathbb{R}$ be a height function as in 4.2.1 and let M be a $(\mathfrak{U}_G, B)$-module. For any $\lambda \in X^*(T)^+$, consider the largest B-submodule $\mathcal{F}_\lambda(M)$ of M such that each weight μ of $\mathcal{F}_\lambda(M)$ satisfies

$$h(\mu) \leq h(\lambda).$$

Then, show that $\mathcal{F}_\lambda(M)$ is a $(\mathfrak{U}_G, B)$-submodule of M.

(2*) Let X be a G-scheme and $Y \subset X$ a closed G-stable subscheme. Assume that X admits a B-canonical splitting compatibly splitting Y. Then, show that for any G-linearized line bundle $\mathcal{L}$ on X, the kernel of the restriction map $H^0(X, \mathcal{L}) \to H^0(Y, \mathcal{L}_{|Y})$ admits a good filtration.

Hint: Follow the proof of Theorem 4.2.13.

(3) Show that if M admits a good filtration then, for any $\lambda \in X^*(T)^+$,

$$\mathrm{Ext}^q_G(V(\lambda), M) = 0, \text{ for all } q > 0.$$

Use this to prove that for an exact sequence of G-modules:

$$0 \to V' \to V \to V'' \to 0,$$

such that V' and V admit good filtrations, then so does V''. Similarly, if V' and V'' admit good filtrations, then so does V.

(4*) Show that for any $\lambda \in X^*(T)^+$, $d\lambda = \sum_i n_i\alpha_i$ with $n_i \in \mathbb{Z}_+$, where d is the index of the root lattice $\bigoplus_i \mathbb{Z}\alpha_i$ in $X^*(T)$.

(5*) Let M be a G-module admitting a good filtration. Show that, for any $\lambda \in X^*(T)^+$, $\dim M_\lambda^{(B)}$ $(= \dim \operatorname{Hom}_G(V(\lambda), M)$ by (4.2.1.11)) is the number of times $\nabla(\lambda)$ appears in any good filtration of M.

4.3 Proof of the PRVK conjecture and its refinement

In this section k is taken to be an algebraically closed field of any characteristic (including 0). We follow the notation as in Section 2.1. In particular, G is a connected, simply-connected, semisimple algebraic group over k, $B \subset G$ a Borel subgroup with the unipotent radical U, $T \subset B$ a maximal torus and W the associated Weyl group. Also recall that $\mathfrak{U}_G$ denotes the hyperalgebra of G. For any $\beta \in \Delta^+$, $e_\beta \in \mathfrak{g}_\beta$ is a root vector as in the beginning of Section 4.1. For any $\lambda \in X^*(T)^+$, $V(\lambda) := H^0(G/B, \mathcal{L}(\lambda))^*$ is the Weyl module. For $\theta \in X^*(T)$, there exists a unique $\bar{\theta} \in X^*(T)^+$ in the W-orbit of θ, i.e.,

$$W \cdot \theta \cap X^*(T)^+ = \{\bar{\theta}\}.$$

In this section, when we refer to a specific result from Chapter 3, we also mean to use the corresponding result in characteristic 0 as given in Theorem 3.5.4. Also, in characteristic 0, any filtration of a G-module by G-submodules is, of course, automatically a good filtration.

We begin by recalling the following result on the structure of the Demazure module $V_w(\mu)$ (Definition 3.3.10) as a $\mathfrak{U}_U$-module, where $\mathfrak{U}_U$ is the hyperalgebra of U.

4.3.1 Proposition. *For $\mu \in X^*(T)^+$ and $w \in W$, consider the $\mathfrak{U}_U$-module map*

$$\xi : \mathfrak{U}_U \to V_w(\mu), \quad a \to a \cdot v_{w\mu},$$

where $v_{w\mu}$ is a nonzero vector of $V_w(\mu)$ of weight $w\mu$. (Recall from (3.3.10.1) that $v_{w\mu}$ is unique up to scalar multiples.) Then, ξ is surjective and Ker ξ *is generated as a left $\mathfrak{U}_U$-ideal by the elements $\{e_\beta^{(m)}; \beta \in \Delta^+, m \geq \delta_\beta(w\mu)\}$, where $\delta_\beta(\nu)$ for any $\nu \in X^*(T)$ is defined by*

$$\delta_\beta(\nu) = 1 + \max\{-\langle \nu, \beta^\vee \rangle, 0\}. \tag{1}$$

In particular, for any B-module N, we have a k-linear isomorphism

(2) $\operatorname{Hom}_B(V_w(\mu), N) \xrightarrow{\sim}$

$$\{x \in N_{w\mu} : e_\beta^{(m)} x = 0 \textit{ for all } \beta \in \Delta^+ \textit{ and } m \geq \delta_\beta(w\mu)\},\ f \mapsto f(v_{w\mu}).$$

Proof. (*an indication*) Since, by definition, $V_w(\mu)$ is generated by the vector $v_{w\mu}$ as a U-module, by [Jan–03, Part I, Lemma 7.15], the map ξ is surjective. Moreover, for any $\beta \in \Delta^+$ and any $m \geq \delta_\beta(w\mu)$, it is easy to see that

$$\langle w\mu + m\beta, w\mu + m\beta \rangle > \langle \mu, \mu \rangle.$$

But, any weight λ of $V(\mu)$ satisfies $\langle\lambda,\lambda\rangle \le \langle\mu,\mu\rangle$. Thus, the elements $e_\beta^{(m)}$ do belong to Ker ξ.

Let $\mathcal{I}$ be the left ideal of $\mathfrak{U}_U$ generated by the elements $e_\beta^{(m)}$ as β runs over Δ^+ and m runs over $m \ge \delta_\beta(w\mu)$, and let Q be the quotient $\mathfrak{U}_U$-module $\mathfrak{U}_U/\mathcal{I}$. Then, ξ induces the (surjective) $\mathfrak{U}_U$-module map $\bar{\xi} : Q \to V_w(\mu)$. In particular,

(3) $$\dim Q \ge \dim V_w(\mu).$$

Moreover, it is easy to see that Q is a finite-dimensional vector space over k and it is a $(\mathfrak{U}_U, T)$-module under the adjoint action of T. Thus, the $\mathfrak{U}_U$-module structure on Q "integrates" to give a U-module structure (cf. [CPS–80, Theorem 9.4]). Let $\mathbf{1}$ denote the coset $1+\mathcal{I}$ of Q. Then, $\mathbf{1}$ generates Q as a $\mathfrak{U}_U$-module, and thus as a U-module. Further, since $e_\beta^{(m)} \in \mathcal{I}$ for all $\beta \in \Delta^+ \cap w(\Delta^+)$ and $m \ge 1$, it follows that $\mathbf{1}$ is fixed by the subgroup $U \cap wUw^{-1}$ of U.

We next construct an injective $\mathfrak{U}_U$-module map

$$\theta : Q^* \to H^0(C_w, \mathcal{L}(\mu)_{|C_w})$$

as follows. Recall from Section 2.1 that C_w denotes the Schubert cell BwB/B, isomorphic to $U/U \cap wUw^{-1}$ under $u \mapsto uwB$. Fix a representative $\dot{w}$ of w in $N(T)$. For any $f \in Q^*$ and $x = u\dot{w}b \in BwB$ for $u \in U$ and $b \in B$, define $\theta(f)(x) = \mu(b) f(u \cdot \mathbf{1})$. Then, $\theta(f)$ is well defined (i.e., it does not depend on the choices of u and b) and, moreover, from its equivariance properties it gives a section of $\mathcal{L}(\mu)$ over C_w. Since $\mathbf{1}$ generates Q as a U-module, the map θ is injective.

Finally, one proves that for any $f \in Q^*$, the section $\theta(f)$ extends to a section of $\mathcal{L}_w(\mu)$ on the closure X_w of C_w. One checks that $\theta(f)$ extends to any Schubert cell C_v of codimension 1 in X_w, and hence $\theta(f)$ extends to X_w by the normality of X_w (cf. [Pol–89, Proof of Proposition 2.1] for the details). In particular,

(4) $$\dim Q = \dim Q^* \le \dim H^0(X_w, \mathcal{L}_w(\mu)) = \dim V_w(\mu),$$

where the last equality follows from Corollary 3.3.11.

Combining (3) and (4), we get that $\dim Q = \dim V_w(\mu)$. Thus, $\bar{\xi}$ is forced to be an isomorphism, proving the proposition. ☐

We come to the proof of the Parthasarathy–Ranga Rao–Varadarajan–Kostant (for short PRVK) conjecture.

4.3.2 Theorem. *Let $\lambda, \mu \in X^*(T)^+$ and $w \in W$ and let $M := \mathfrak{U}_G \cdot (v_\lambda \otimes v_{w\mu})$ be the G-submodule of the tensor product $V(\lambda) \otimes V(\mu)$ generated by $v_\lambda \otimes v_{w\mu}$. Then, setting $\theta = \lambda + w\mu$, and $\theta' = \overline{(-\theta)}$,*

(1) $$\dim \operatorname{Hom}_G\big(V(\theta'), M^*\big) = 1.$$

In fact, this G-module map is induced from a surjective G-module map $\psi : M \twoheadrightarrow V(\bar{\theta})$, composed with the G-module map $q : V(\bar{\theta}) \to V(\theta')^*$ (constructed in the proof below) and then taking duals.

Proof. Assertion I. Analogous to Corollary 3.3.11, we first make the identification

(2) $$M^* \simeq H^0\big(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big), \quad \text{as } G\text{-modules.}$$

By Theorem 3.1.2(b), the restriction map

$$\gamma : V(\lambda)^* \otimes V(\mu)^* = H^0\big(G/B \times G/B, \mathcal{L}(\lambda \boxtimes \mu)\big) \to H^0\big(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big)$$

is surjective. Since $G \cdot (1, \bar{w}) \subset \mathcal{X}_w$ is a dense (open) subset, where G acts diagonally on $G/B \times G/B$ and $\bar{w}$ is the coset wB,

$$\begin{aligned}\operatorname{Ker}\gamma &= \{f \in V(\lambda)^* \otimes V(\mu)^* : \gamma(f)_{|G\cdot(1,\bar{w})} \equiv 0\}\\ &= \{f \in (V(\lambda) \otimes V(\mu))^* : f_{|M} \equiv 0\}, \text{ by (3.3.10.2).}\end{aligned}$$

Thus, we get an exact sequence

$$0 \to \Big(\frac{V(\lambda) \otimes V(\mu)}{M}\Big)^* \to (V(\lambda) \otimes V(\mu))^* \to H^0\big(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big) \to 0.$$

From this we get (2).

Assertion II. There is a canonical G-module isomorphism

(3) $$H^0\big(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big) \simeq H^0\big(G/B, \mathcal{L}(k_{-\lambda} \otimes V_w(\mu)^*)\big).$$

From the fibration $\pi_1 : \mathcal{X}_w \to G/B$ obtained from the projection onto the first factor: $\mathcal{X}_w \subset G/B \times G/B \to G/B$, we get that

$$H^0\big(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big) \simeq H^0\big(G/B, \pi_{1*}(\mathcal{L}_w(\lambda \boxtimes \mu))\big).$$

But since π_1 is a G-equivariant morphism under the diagonal action of G on $\mathcal{X}_w$, $\pi_{1*}(\mathcal{L}_w(\lambda \boxtimes \mu))$ is a G-equivariant vector bundle on G/B associated to the tensor product B-module $k_{-\lambda} \otimes H^0(X_w, \mathcal{L}_w(\mu))$. So, the assertion (3) follows from Corollary 3.3.11.

Assertion III. For any $\delta \in X^*(T)^+$,

(4) $$\operatorname{Hom}_G(V(\delta), M^*) \simeq \operatorname{Hom}_B\big(k_\lambda \otimes V_w(\mu), V(\delta)^*\big).$$

Combining (2)–(3), we get

$$\begin{aligned}\operatorname{Hom}_G(V(\delta), M^*) &\simeq \operatorname{Hom}_G\big(V(\delta), H^0(G/B, \mathcal{L}(k_{-\lambda} \otimes V_w(\mu)^*))\big)\\ &\simeq \operatorname{Hom}_B(V(\delta), k_{-\lambda} \otimes V_w(\mu)^*),\\ &\qquad\qquad \text{by Frobenius reciprocity}\\ &\simeq \operatorname{Hom}_B(k_\lambda \otimes V_w(\mu), V(\delta)^*).\end{aligned}$$

This proves (4).

Assertion IV.

$$\dim \operatorname{Hom}_B(k_\lambda \otimes V_w(\mu), V(\theta')^*) = 1. \tag{5}$$

By (4.3.1.2), we get

$$\begin{aligned} \operatorname{Hom}_B\big(k_\lambda \otimes V_w(\mu), V(\theta')^*\big) &= \operatorname{Hom}_B\big(V_w(\mu), k_{-\lambda} \otimes V(\theta')^*\big) \\ &\simeq \big\{v \in (V(\theta')^*)_\theta : e_\beta^{(m)} v = 0 \ \text{ for all } \beta \in \Delta^+ \text{ and } m \geq \delta_\beta(w\mu)\big\}. \end{aligned} \tag{6}$$

Since the $\bar{\theta}$-weight space of $V(\theta')^*$ is one-dimensional, so is

$$\dim(V(\theta')^*)_\theta = 1. \tag{7}$$

Consider the quotient map $q_{\theta'} : V(\theta') \twoheadrightarrow L(\theta')$, where $L(\theta')$ is the irreducible G-module with highest weight θ'. Now, $L(\theta')^* \simeq L(\bar{\theta})$. Thus, we get the G-module maps

$$V(\bar{\theta}) \xrightarrow{q_{\bar{\theta}}} L(\bar{\theta}) \simeq L(\theta')^* \hookrightarrow V(\theta')^*,$$

where the last map is the dual of $q_{\theta'}$. Let $q : V(\bar{\theta}) \to V(\theta')^*$ be the composite map. Then, being extremal weight, q induces an isomorphism of one-dimensional T-modules

$$V(\bar{\theta})_\theta \xrightarrow{\sim} (V(\theta')^*)_\theta.$$

Now, applying Proposition 4.3.1 to the G-module $V(\bar{\theta})$ and extremal weight vector $v_\theta \in V(\bar{\theta})_\theta$, we get $e_\beta^{(m)} v_\theta = 0$ for all $\beta \in \Delta^+$ and $m \geq \delta_\beta(\theta)$. In particular, by virtue of the G-module map q and the above isomorphism, for $v \in (V(\theta')^*)_\theta$,

$$e_\beta^{(m)} v = 0 \ \text{ for all } \beta \in \Delta^+ \text{ and } m \geq \delta_\beta(\theta). \tag{8}$$

Next, for any $\beta \in \Delta^+$ such that $\langle \theta, \beta^\vee \rangle \leq 0$,

$$\begin{aligned} \delta_\beta(\theta) &:= 1 - \langle \theta, \beta^\vee \rangle \\ &= 1 - \langle \lambda + w\mu, \beta^\vee \rangle \\ &= 1 - \langle w\mu, \beta^\vee \rangle - \langle \lambda, \beta^\vee \rangle \\ &= \delta_\beta(w\mu) - \langle \lambda, \beta^\vee \rangle. \end{aligned}$$

Thus, for any $\beta \in \Delta^+$,

$$\delta_\beta(\theta) \leq \delta_\beta(w\mu). \tag{9}$$

Combining (6)–(9), we get

$$\dim \operatorname{Hom}_B\big(k_\lambda \otimes V_w(\mu), V(\theta')^*\big) = \dim(V(\theta')^*)_\theta = 1,$$

proving (5).

Finally, the assertions III–IV put together prove (1). The "In fact" part of the theorem follows from the above proof of Assertion IV. $\square$

4.3.3 Remark. By (4.3.2.2),

$$\big(\mathfrak{U}_G \cdot (v_\lambda \otimes v_{w\mu})\big)^* \simeq H^0(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)).$$

Moreover, by Corollary 4.2.14, the G-module $H^0(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))$ admits a good filtration. From the above theorem we see that the multiplicity of $\nabla(\theta')$ in $H^0(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))$ is exactly one (Exercise 4.2.E.5). In particular, by Exercise 4.2.E.2, the multiplicity of $\nabla(\overline{\lambda + w\mu})$ in $\nabla(\lambda) \otimes \nabla(\mu)$ is at least one. This multiplicity is often more than 1, as we will see in the refinement of Theorem 4.3.2 proved below.

4.3.4 Definition. Fix $\lambda, \mu \in X^*(T)^+$ and let W_λ, resp. W_μ, be the stabilizer of λ, resp. μ, in W. Then, the map $W \to X^*(T)^+$, $w \mapsto \overline{\lambda + w\mu}$, factors through the double coset set to give the map

$$\eta : W_\lambda \backslash W / W_\mu \to X^*(T)^+.$$

As is well known (cf. [Bou–81, Chap. V, Proposition 3.3.1]), W_λ is generated by the simple reflections it contains. Let P_λ be the parabolic subgroup $BW_\lambda B$. Then, the double coset set $W_\lambda \backslash W / W_\mu$ bijectively parametrizes the G-orbits in $G/P_\lambda \times G/P_\mu$ under the diagonal actional of G. The correspondence is given by

$$W_\lambda w W_\mu \mapsto G \cdot (1 \bmod P_\lambda, w \bmod P_\mu).$$

From the Bruhat decomposition and the product formula (2.1.4) together with the isomorphism (2.2.6.1) we indeed see that the above correspondence is bijective.

The following theorem provides a refinement of Theorem 4.3.2.

4.3.5 Theorem. *For $\lambda, \mu \in X^*(T)^+$, $w \in W$ and $1 \le m \le n$,*

$$\dim \operatorname{Hom}_G\Big(V(\theta'), H^0\big(\mathcal{Y}_m, \mathcal{L}^{P_\lambda, P_\mu}(\lambda \boxtimes \mu)_{|\mathcal{Y}_m}\big)\Big) = m, \tag{1}$$

where $\eta^{-1}\big(\eta(W_\lambda w W_\mu)\big) = \{W_\lambda w_1 W_\mu, \ldots, W_\lambda w_n W_\mu\}$, $\mathcal{Y}_m := \bigcup_{i=1}^m \mathcal{X}_{w_i}^{P_\lambda, P_\mu}$, and $\theta' = \overline{-(\lambda + w\mu)}$ (in particular, $n := \#\eta^{-1}\big(\eta(W_\lambda w W_\mu)\big)$).

Moreover, the G-module $L_m := H^0\big(\mathcal{Y}_m, \mathcal{L}^{P_\lambda, P_\mu}(\lambda \boxtimes \mu)_{|\mathcal{Y}_m}\big)$ admits a good filtration. Thus, $\nabla(\theta')$ appears in L_m with multiplicity exactly equal to m.

Observe that L_m is a G-module quotient of the tensor product G-module $\nabla(-w_o\lambda) \otimes \nabla(-w_o\mu)$ by (3.1.3.4) and Theorem 3.3.4(a).

As a preparation for the proof of the above theorem, we first prove the following.

4.3.6 Lemma. *Take any $\lambda, \mu \in X^*(T)^+$, $w \in W$ and assume that w is of minimal length in its double coset $W_\lambda w W_\mu$ (even though we do not need it, such a w is unique). Then, for any $u < w$,*

$$\operatorname{Hom}_G\big(V(\theta'), H^0(\mathcal{X}_u, \mathcal{L}_u(\lambda \boxtimes \mu)\big) = 0,$$

where (as in the above theorem) $\theta' := \overline{(-\theta)}$ and $\theta := \lambda + w\mu$.

Proof. Consider the exact sequence of G-modules

$$0 \to K \to H^0(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)) \xrightarrow{\gamma} H^0(\mathcal{X}_u, \mathcal{L}_u(\lambda \boxtimes \mu)) \to 0,$$

where K is, by definition, the kernel of the restriction map γ. For any $\delta \in X^*(T)^+$, considering the corresponding long exact $\mathrm{Ext}^*_G(V(\delta), -)$ sequence and the Ext^1_G vanishing result as in Theorem 4.2.7, together with Exercise 4.2.E.2, we get that the induced map

$$(1) \quad f : \mathrm{Hom}_G\big(V(\delta), H^0(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))\big) \to \mathrm{Hom}_G\big(V(\delta), H^0(\mathcal{X}_u, \mathcal{L}_u(\lambda \boxtimes \mu))\big)$$

is surjective.

By (4.3.2.2) and (4.3.2.4),

$$(2) \qquad \mathrm{Hom}_G\big(V(\delta), H^0(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))\big) \simeq \mathrm{Hom}_B(k_\lambda \otimes V_w(\mu), V(\delta)^*),$$

and a similar statement with w replaced by u. Moreover, by Assertion IV of the proof of Theorem 4.3.2, there exists a unique B-module map (up to scalar multiples)

$$\xi : k_\lambda \otimes V_w(\mu) \to V(\theta')^*.$$

Of course, ξ takes the B-module generator $\mathbf{1}_\lambda \otimes v_{w\mu}$ of $k_\lambda \otimes V_w(\mu)$ to the unique vector (up to scalar multiples) in $(V(\theta')^*)_\theta$.

Assume now that $u \to w$ (i.e., $\ell(u) = \ell(w) - 1$ and $u \leq w$). Let β be the positive root such that $s_\beta u = w$. In this case,

$$(3) \qquad e_\beta^{(a)} v_{w\mu} = v_{u\mu}, \quad \text{where } a := \langle u\mu, \beta^\vee \rangle.$$

(Observe that $a \geq 0$ by [Bou–81, Chap. VI, Proposition 1.6.17], since $u^{-1}\beta \in \Delta^+$.) To prove (3), we first observe that for any $i > 0$,

$$f_\beta^{(i)} v_{w\mu} = w\big(e^{(i)}_{-w^{-1}\beta} v_\mu\big) = 0, \quad \text{since } w^{-1}\beta = -u^{-1}\beta \in \Delta^-,$$

where $f_\beta \in \mathfrak{g}_{-\beta}$ is a negative root vector as in the proof of Theorem 2.3.1.

Now, using the hyperalgebra $\mathfrak{U}_{\mathrm{SL}_2}$ corresponding to the root β, (3) follows since $w\mu + a\beta = u\mu$.

In view of (1)–(3), the lemma is equivalent to the following assertion:

$$(4) \qquad e_\beta^{(a)} \cdot \big((V(\theta')^*)_\theta\big) = 0.$$

From the G-module map $q : V(\bar{\theta}) \to V(\theta')^*$ defined in the Assertion IV of the proof of Theorem 4.3.2, to prove (4), it suffices to show that

$$(5) \qquad e_\beta^{(a)} \cdot \big(V(\bar{\theta})_\theta\big) = 0.$$

Since w is of smallest length in its double coset $W_\lambda w W_\mu$, $s_\beta \notin W_\lambda$ and thus $\langle \lambda, \beta^\vee \rangle \geq 1$. Also, since $u^{-1}w \notin W_\mu$, we get $a > 0$. Thus, (5) follows from Proposition 4.3.1. Now, for any $u < w$, there exists $u \leq u' \to w$. Thus, the lemma follows for u from u' and (1). (Use (1) for w replaced by u'.) □

4.3.7 Corollary. *(of the above proof) For any $\lambda, \mu, \theta \in X^*(T)^+$ and $u \leq w \in W$, the restriction map*

$$\operatorname{Hom}_B(k_\lambda \otimes V_w(\mu), \nabla(\theta)) \to \operatorname{Hom}_B(k_\lambda \otimes V_u(\mu), \nabla(\theta))$$

is surjective.

Proof. We get the corollary from (4.3.6.1) and the identification (4.3.6.2). □

We now come to the proof of Theorem 4.3.5.

Proof. By Exercise 4.1.E.6, $\mathcal{Y}_m$ admits a B-canonical splitting. Thus, by Theorem 4.2.13, the G-module L_m admits a good filtration.

We next prove (1) of Theorem 4.3.5 by induction on m. The case $m = 1$ follows from Theorem 4.3.2 together with (4.3.2.2) and Exercise 3.3.E.3. We assume now the validity of (1) for $\mathcal{Y}_m$ by induction on m and prove the same for $\mathcal{Y}_{m+1}$.

We have the sheaf exact sequence on $\mathcal{Y}_{m+1}$:

(1)
$$0 \to \mathcal{I}_{\mathcal{Y}_m}(\mathcal{Y}_{m+1}) \to \mathcal{O}_{\mathcal{Y}_{m+1}} \to \mathcal{O}_{\mathcal{Y}_m} \to 0,$$

where $\mathcal{I}_{\mathcal{Y}_m}(\mathcal{Y}_{m+1})$ is the ideal sheaf of the closed subscheme $\mathcal{Y}_m$ in $\mathcal{Y}_{m+1}$. Abbreviate the line bundle $\mathcal{L}^{P_\lambda, P_\mu}(\lambda \boxtimes \mu)$ on $G/P_\lambda \times G/P_\mu$ by $\mathcal{L}$. Then, tensoring the sequence (1) with the line bundle $\mathcal{L}_{|\mathcal{Y}_{m+1}}$ and taking cohomology, we obtain the following long exact sequence:

(2)
$$\begin{aligned} 0 \to H^0\Big(\mathcal{Y}_{m+1}, \mathcal{I}_{\mathcal{Y}_m}(\mathcal{Y}_{m+1}) \otimes \mathcal{L}_{|\mathcal{Y}_{m+1}}\Big) &\to H^0\Big(\mathcal{Y}_{m+1}, \mathcal{L}_{|\mathcal{Y}_{m+1}}\Big) \\ \xrightarrow{\gamma} H^0\Big(\mathcal{Y}_m, \mathcal{L}_{|\mathcal{Y}_m}\Big) \to 0, \end{aligned}$$

where the surjectivity of the restriction map γ follows from (3.1.3.4).

But, as is fairly easy to see,

$$\mathcal{I}_{\mathcal{Y}_m}(\mathcal{Y}_{m+1}) \simeq \mathcal{I}_{\mathcal{X}^P_{w_{m+1}} \cap \mathcal{Y}_m}(\mathcal{X}^P_{w_{m+1}}),$$

where we abbreviate $\mathcal{X}^{P_\lambda, P_\mu}_{w_i}$ by $\mathcal{X}^P_{w_i}$, the intersection $\mathcal{Y}_m \cap \mathcal{X}^P_{w_{m+1}}$ is the scheme theoretic intersection and the sheaf $\mathcal{I}_{\mathcal{X}^P_{w_{m+1}} \cap \mathcal{Y}_m}(\mathcal{X}^P_{w_{m+1}})$, which is defined on $\mathcal{X}^P_{w_{m+1}}$, is extended to the whole of $\mathcal{Y}_{m+1}$ by defining it to be zero on the open set $\mathcal{Y}_{m+1} \backslash \mathcal{X}^P_{w_{m+1}}$. In particular,

(3)
$$\begin{aligned} H^0\Big(\mathcal{Y}_{m+1}, \mathcal{I}_{\mathcal{Y}_m}(\mathcal{Y}_{m+1}) \otimes \mathcal{L}_{|\mathcal{Y}_{m+1}}\Big) \\ \simeq H^0\Big(\mathcal{X}^P_{w_{m+1}}, I_{\mathcal{X}^P_{w_{m+1}} \cap \mathcal{Y}_m}(\mathcal{X}^P_{w_{m+1}}) \otimes \mathcal{L}_{|\mathcal{X}^P_{w_{m+1}}}\Big). \end{aligned}$$

Similarly, the sheaf exact sequence

$$0 \to \mathcal{I}_{\mathcal{X}^P_{w_{m+1}} \cap \mathcal{Y}_m}(\mathcal{X}^P_{w_{m+1}}) \to \mathcal{O}_{\mathcal{X}^P_{w_{m+1}}} \to \mathcal{O}_{\mathcal{X}^P_{w_{m+1}} \cap \mathcal{Y}_m} \to 0$$

gives rise to the long exact sequence

$$\text{(4)}\quad 0 \to H^0\Big(\mathcal{X}^P_{w_{m+1}}, \mathcal{I}_{\mathcal{X}^P_{w_{m+1}}\cap \mathcal{Y}_m}(\mathcal{X}^P_{w_{m+1}}) \otimes \mathcal{L}_{|_{\mathcal{X}^P_{w_{m+1}}}}\Big) \to H^0\Big(\mathcal{X}^P_{w_{m+1}}, \mathcal{L}_{|_{\mathcal{X}^P_{w_{m+1}}}}\Big)$$
$$\xrightarrow{\gamma'} H^0\Big(\mathcal{X}^P_{w_{m+1}} \cap \mathcal{Y}_m, \mathcal{L}_{|_{\mathcal{X}^P_{w_{m+1}}\cap\mathcal{Y}_m}}\Big) \to 0.$$

Again the surjectivity of the restriction map γ' follows from (3.1.3.4).

We further claim that

$$(*)\qquad \mathcal{X}^P_{w_i} \text{ is not contained in } \mathcal{X}^P_{w_j} \text{ for } i \neq j.$$

If possible, assume that

$$\mathcal{X}^P_{w_i} \subset \mathcal{X}^P_{w_j} \text{ for some } i \neq j.$$

We can of course take w_j to be of minimal length in its double coset. Then, under the canonical map $\pi : G/B \times G/B \to G/P_\lambda \times G/P_\mu$,

$$\pi(\mathcal{X}_{w_j}) = \mathcal{X}^P_{w_j} \text{ and } \pi(\mathcal{X}_u) = \mathcal{X}^P_{w_i}$$

for some $u \leq w_j$. Moreover, $u \neq w_j$ since $\mathcal{X}^P_{w_i} \neq \mathcal{X}^P_{w_j}$ as $W_\lambda\backslash W/W_\mu$ bijectively parametrizes the G-orbits in $G/P_\lambda \times G/P_\mu$ (4.3.4).

By Exercise 3.3.E.3,

$$\text{(5)}\qquad H^0(\mathcal{X}_u, \mathcal{L}_u(\lambda \boxtimes \mu)) \simeq H^0\Big(\mathcal{X}^P_{w_i}, \mathcal{L}^P_{w_i}(\lambda \boxtimes \mu)\Big).$$

Thus, by the (already established) case $m = 1$ of this theorem, $\mathrm{Hom}_G\big(V(\theta'), H^0(\mathcal{X}_u, \mathcal{L}_u(\lambda\boxtimes\mu))\big)$ is nonzero, contradicting Lemma 4.3.6 for $u < w_j$. Thus, the claim $(*)$ is established.

Thus, $\mathcal{X}^P_{w_{m+1}} \cap \mathcal{Y}_m$ being reduced and G-stable,

$$\mathcal{X}^P_{w_{m+1}} \cap \mathcal{Y}_m = \bigcup \mathcal{X}^P_u,$$

where the above union is taken over some $u \in W$ with $u < w_{m+1}$ and w_{m+1} is chosen to be of smallest length in its double coset $W_\lambda w_{m+1} W_\mu$. In particular,

$$H^0\Big(\mathcal{X}^P_{w_{m+1}} \cap \mathcal{Y}_m, \mathcal{L}_{|_{\mathcal{X}^P_{w_{m+1}}\cap\mathcal{Y}_m}}\Big) \hookrightarrow \bigoplus_u H^0(\mathcal{X}^P_u, \mathcal{L}_{|_{\mathcal{X}^P_u}}).$$

Thus, by Lemma 4.3.6 and Exercise 3.3.E.3,

$$\text{(6)}\qquad \mathrm{Hom}_G\Big(V(\theta'), H^0\Big(\mathcal{X}^P_{w_{m+1}} \cap \mathcal{Y}_m, \mathcal{L}_{|_{\mathcal{X}^P_{w_{m+1}}\cap\mathcal{Y}_m}}\Big)\Big) = 0.$$

From the exact sequence (4), using (6) and Theorem 4.2.7, we get

$$\text{(7)}\quad \mathrm{Hom}_G\Big(V(\theta'), H^0\Big(\mathcal{X}^P_{w_{m+1}}, \mathcal{I}_{\mathcal{X}^P_{w_{m+1}}\cap\mathcal{Y}_m}(\mathcal{X}^P_{w_{m+1}}) \otimes \mathcal{L}_{|_{\mathcal{X}^P_{w_{m+1}}}}\Big)\Big)$$
$$\simeq \mathrm{Hom}_G\Big(V(\theta'), H^0\Big(\mathcal{X}^P_{w_{m+1}}, \mathcal{L}_{|_{\mathcal{X}^P_{w_{m+1}}}}\Big)\Big),$$

and

$$\text{(8)} \qquad \operatorname{Ext}^1_G\Big(V(\theta'),\, H^0\Big(\mathcal{X}^P_{w_{m+1}}, \mathcal{I}_{\mathcal{X}^P_{w_{m+1}}\cap \mathcal{Y}_m}(\mathcal{X}^P_{w_{m+1}}) \otimes \mathcal{L}_{|_{\mathcal{X}^P_{w_{m+1}}}}\Big)\Big) = 0.$$

Hence, from the exact sequence (2), using (3) and (7)–(8), we get the exact sequence:

$$0 \to \operatorname{Hom}_G\Big(V(\theta'),\, H^0\Big(\mathcal{X}^P_{w_{m+1}}, \mathcal{L}_{|_{\mathcal{X}^P_{w_{m+1}}}}\Big)\Big) \to$$
$$\operatorname{Hom}_G\big(V(\theta'),\, H^0(\mathcal{Y}_{m+1}, \mathcal{L}_{|_{\mathcal{Y}_{m+1}}})\big) \to \operatorname{Hom}_G\big(V(\theta'),\, H^0(\mathcal{Y}_m, \mathcal{L}_{|_{\mathcal{Y}_m}})\big) \to 0.$$

By induction on m, the last space is m-dimensional and the first space is one-dimensional (for the case $m = 1$ of the theorem). Thus,

$$\dim \operatorname{Hom}_G\Big(V(\theta'),\, H^0\big(\mathcal{Y}_{m+1}, \mathcal{L}_{|_{\mathcal{Y}_{m+1}}}\big)\Big) = m + 1,$$

proving (1) of Theorem 4.3.5.

The assertion that $\nabla(\theta')$ appears in $H^0\big(\mathcal{Y}_m, \mathcal{L}_{|_{\mathcal{Y}_m}}\big)$ with multiplicity m follows immediately from (1) and Exercise (4.2.E.5). This completes the proof of the theorem. □

Specializing Theorems 4.3.2 and 4.3.5 in characteristic 0, we obtain the following result proving the original PRVK conjecture and its refinement. We follow the same notation as in Theorems 4.3.2 and 4.3.5; $U(\mathfrak{g})$ denotes the enveloping algebra of the Lie algebra $\mathfrak{g}$ of G, which is the same as $\mathfrak{U}_G$ in characteristic 0.

4.3.8 Theorem. *Let the base field be any algebraically closed field of characteristic 0. For any $\lambda, \mu \in X^*(T)^+$, and $w \in W$, the G-submodule $U(\mathfrak{g})\,(v_\lambda \otimes v_{w\mu})$ of the tensor product G-module $V(\lambda) \otimes V(\mu)$ contains a unique copy of $V(\overline{\lambda + w\mu})$.*

In fact, for any $1 \le m \le n$, $H^0\big(\mathcal{Y}_m, \mathcal{L}^{P_\lambda, P_\mu}(\lambda \boxtimes \mu)_{|_{\mathcal{Y}_m}}\big)^*$, which is a G-submodule of $V(\lambda) \otimes V(\mu)$, contains exactly m copies of $V(\overline{\lambda + w\mu})$.

In particular, the multiplicity $m_w(\lambda, \mu)$ of $V(\overline{\lambda + w\mu})$ in $V(\lambda) \otimes V(\mu)$ satisfies

$$\text{(1)} \qquad m_w(\lambda, \mu) \ge \#\eta^{-1}\big(\eta(W_\lambda w W_\mu)\big).$$

Thus, the total number $m(\lambda, \mu)$ of irreducible components in $V(\lambda) \otimes V(\mu)$ (counted with multiplicities) satisfies

$$\text{(2)} \qquad m(\lambda, \mu) \ge \# W_\lambda \backslash W / W_\mu.$$

4.3.9 Remark. The inequality (4.3.8.1) is often strict and hence so is (4.3.8.2). This is illustrated by the following example of $G = G_2$ and $\lambda = \mu = \rho$. In this case, the full decomposition of the tensor product (over $\mathbb{C}$):

$$V(\rho) \otimes V(\rho) = \bigoplus_{\nu \in X^*(T)^+} m_\nu V(\nu)$$

is given by the following table. We follow the convention of indexing the simple roots as in [Bou–81, Planche IX].

ν	2ρ	$3\chi_2$	$5\chi_1$	$3\chi_1+\chi_2$	$\chi_1+2\chi_2$	$4\chi_1$	$2\chi_1+\chi_2$
m_ν	1	1	1	2	1	2	3
$\#\eta^{-1}(\nu)$	1	1	1	2	0	1	0

$2\chi_2$	$3\chi_1$	ρ	$2\chi_1$	χ_2	χ_1	0
2	3	2	2	2	1	1
1	0	2	0	1	1	1

The above chart also shows that there are some components in $V(\rho) \otimes V(\rho)$ which are *not* of the form $\overline{\rho + w\rho}$.

To generate many more examples where the inequality (4.3.8.2) is strict (and for an arbitrary $\mathfrak{g}$), take any $\mu \in X^*(T)^+$ and $\lambda = n\lambda'$, with λ' regular and n a large enough integer (depending upon μ).

4.3.E Exercises

(1) For any $\lambda, \mu \in X^*(T)$ (not necessarily dominant), and any $w \in W$, prove that

$$\chi\big(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu)\big) = \bar{D}_{w_o}(e^\lambda \cdot D_w(e^\mu)),$$

where w_o is the longest element of W, D_w is as in Corollary 3.3.9 and $\chi(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))$ denotes $\sum_i (-1)^i \operatorname{ch} H^i(\mathcal{X}_w, \mathcal{L}_w(\lambda \boxtimes \mu))$. In particular, for $\lambda, \mu \in X^*(T)^+$, this specializes to the following result due to Brauer:

$$\operatorname{ch}(V(\lambda) \otimes V(\mu)) = D_{w_o}(e^\lambda \cdot D_{w_o} e^\mu).$$

4.C. Comments

The notion of canonical splitting was introduced by Mathieu and most of the results in Section 4.1, including Exercises 4.1.E.2–4, are due to him [Mat–90a] (see also [Mat-00]). However, Mathieu takes the characterization (4.1.6.1) as the definition of a B-canonical splitting. The definition of B-canonical splitting we give in 4.1.1 is taken from [Van–93, Definition 4.3.5]. In [loc cit.] van der Kallen has studied filtrations of B-modules (including the notion of excellent filtrations) in relation to the F-splitting. However, in our book we focus on filtrations of G-modules. For a slightly different exposition of some of the results in Sections 4.1 and 4.2, we refer to [Jan–03, Chap. G].

Lemma 4.2.2 and Exercise 4.2.E.3 are due to Cline–Parshall–Scott–van der Kallen [CPSV–77]. Theorem 4.2.5 is due independently to Donkin [Don–88] and Koppinen [Kop-84]. The equivalence of (b) and (c) in Theorem 4.2.7 is due to Donkin [Don–81] and its extension to an arbitrary G-module as in Remark 4.2.8 is due to Friedlander [Fri–85] who also introduced the canonical filtration. Proposition 4.2.3, Corollary 4.2.6, the equivalence of (a) and (b) in Theorem 4.2.7, Lemma 4.2.10, Proposition

4.2.12 and Theorem 4.2.13 are all due to Mathieu [Mat–90a, 00]. The first part of Corollary 4.2.14 is due to Wang [Wan–82] for $p = \text{char.}\,k$ large, and, in general, due to Donkin [Don–85] except for $p = 2$ if G has a component of type E_7 or E_8. Also, the first part of Corollary 4.2.15 is due to him [loc cit.] under the same restriction on p. His proof involves an elaborate case-by-case analysis. In contrast, both the parts of Corollaries 4.2.14–15 follow immediately from Mathieu's uniform result Theorem 4.2.13. For other accounts of Mathieu's proof of Corollaries 4.2.14–15 we refer to [Van–93] and [Kan–94a]. Exercise 4.2.E.5 is taken from [Don–85, 12.1.1]. There are other subsequent proofs via quantum groups: by Paradowski [Par–94] using Lusztig's canonical basis; and Kaneda [Kan–98] using Lusztig's results on based modules.

In the nineteen sixties, Parthasarathy–Ranga Rao–Varadarajan (for short PRV) conjectured (unpublished) that for any $\lambda, \mu \in X^*(T)^+$ and $w \in W$, the irreducible G-module over complex numbers $V(\overline{\lambda + w\mu})$ occurs in the tensor product $V(\lambda) \otimes V(\mu)$. They proved this in the case where w is the longest element of the Weyl group [PRV–67]. Then, Kostant (in the mid eighties) came up with a more precise form of their conjecture suggesting that $V(\overline{\lambda + w\mu})$ should occur with multiplicity exactly one in the G-submodule of $V(\lambda) \otimes V(\mu)$ generated by the vector $v_\lambda \otimes v_{w\mu}$ (again over $\mathbb{C}$), to which we refer as the Parthasarathy–Ranga Rao–Varadarajan–Kostant (for short PRVK) conjecture. Theorems 4.3.2 and 4.3.8 prove this conjecture and its analogue in an arbitrary characteristic. It was proved by Kumar [Kum-88] in characteristic 0 using characteristic 0 methods and was extended by Mathieu [Mat–89b] to an arbitrary characteristic. Proposition 4.3.1 is due to Joseph [Jos–85] in characteristic 0 and was extended to an arbitrary characteristic by Polo [Pol–89].

The refinement of Theorem 4.3.2 as in Theorem 4.3.5 was proved by Kumar [Kum–89] in characteristic 0. This refinement in characteristic 0 was conjectured by D.N. Verma (unpublished) after the work [Kum-88] appeared. Its extension to characteristic p (as in Theorem 4.3.5) appears here for the first time (to our knowledge). The proof given here is a slight modification of the original proof given in [Kum–89]. The table as in Remark 4.3.9 is taken from [Kum–88]. Exercise 4.3.E.1 is taken from [loc cit.].

Subsequently, other proofs of the original PRV conjecture appeared. Lusztig mentioned in 1989 that his results on the intersection homology of generalized Schubert varieties associated to affine Kac–Moody groups give a proof of the PRV conjecture. Rajeswari [Raj–91] gave a proof for classical G using Standard Monomial Theory; Littelmann [Lit–94] gave a proof using his LS path models.

Let G be a connected semisimple group and H a closed connected reductive subgroup. Then, (G, H) is called a *Donkin pair* if any G-module with a good filtration admits a good filtration as an H-module. Brundan [Bru–98] conjectured that if G is simply-connected and either H is the centralizer of a graph automorphism of G; or H is the centralizer of an involution of G and characteristic is at least three, then (G, H) is a Donkin pair. Combining Corollaries 4.2.14, 4.2.15 and [Bru–98], the conjecture was proved by van der Kallen [Van–01] by some case-by-case analysis. It is desirable to give a case-free proof of Brundan's conjecture.

Chapter 5

Cotangent Bundles of Flag Varieties

Introduction

The main aim of this chapter is to give a family of Frobenius splittings of the cotangent bundle $T^*(G/P)$ of any flag variety G/P due to Kumar–Lauritzen–Thomsen and thus obtain a cohomology vanishing result for $T^*(G/P)$ with coefficients in the line bundles obtained by pullback from G/P. This cohomology vanishing result is applied to study the geometry of nilpotent and subregular cones.

Let $P \subset G$ be any parabolic subgroup with unipotent radical U_P. We begin by showing that the canonical bundle of the G-variety $\mathfrak{X}_P := G \times_P U_P$ is G-equivariantly trivial, where P acts on U_P via conjugation. Thus, a splitting of $\mathfrak{X}_P$ can be thought of as a regular function on $\mathfrak{X}_P$. If the characteristic p of k is a good prime for G (for the classical groups all the odd primes are good; see 5.1.8 for a complete list of good primes), then by Proposition 5.1.9 and Corollary 5.1.11, the cotangent bundle $T^*(G/P)$ is G-equivariantly isomorphic to $\mathfrak{X}_P$. Now, consider the map $\psi_P : \mathrm{St} \otimes \mathrm{St} \to k[\mathfrak{X}_P]$, defined by $\psi_P(v_1 \otimes v_2)(g, u) = \chi(v_1 \otimes gug^{-1}v_2)$ for $v_1, v_2 \in \mathrm{St}$, $g \in G$ and $u \in U_P$, where $\chi : \mathrm{St} \otimes \mathrm{St} \to k$ is a G-invariant nondegenerate bilinear form. Then, the main result of this section, Theorem 5.1.3, asserts that for any $f \in \mathrm{St} \otimes \mathrm{St}$, $\psi_P(f)$ splits $\mathfrak{X}_P$ iff $\chi(f) \neq 0$. This result is obtained by comparing the splittings of G/P with those of $\mathfrak{X}_P$ (Lemma 5.1.5) and then studying the splittings of G/P. As an immediate corollary of this result (Theorem 5.1.3), one obtains that for any p which is a good prime for G, the cotangent bundle $T^*(G/P)$ is split. In Example 5.1.15, we explicitly work out the example of $G = \mathrm{SL}_n(k)$ recovering in this case the splitting of $T^*(G/B)$ given by Mehta–van der Kallen. This splitting compatibly splits the subvarieties $G \times_B \mathfrak{u}_P$ for any parabolic subgroup P, where $\mathfrak{u}_P = \mathrm{Lie}\, U_P$ (Exercise 5.1.E.6). In Exercises 5.1.E, we assert that the varieties G, $G \times_B B$ and $G \times_B \mathfrak{b}$ are B-canonically split; in fact, G is $B \times B$-canonically split. (The canonical splitting of the first two can also be obtained from Theorem 6.1.12 in the next chapter.)

For any parabolic subgroup $P \subset G$, let $\pi_P : T^*(G/P) \to G/P$ be the standard projection. For the Borel subgroup B, we abbreviate π_B by π. In Section 3.2 we prove that if the characteristic p of k is a good prime for G, then $H^i(T^*(G/B), \pi^*\mathcal{L}(\lambda)) = 0$ for all $i > 0$ and $\lambda \in \mathcal{C} := \{\mu \in X^*(T) : \langle \mu, \beta^\vee \rangle \geq -1$ for all the positive roots $\beta\}$ (Theorem 5.2.1). This is the main result of this section. Here is an outline of its proof. Since the morphism π is affine, we can make the identification

$$H^i\big(T^*(G/B), \pi^*\mathcal{L}(\lambda)\big) \simeq H^i\big(G/B, \mathcal{L}(S(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda)\big),$$

where $\mathfrak{u} = \operatorname{Lie} U_B$. Assume first that λ is dominant. By using the Koszul resolution corresponding to the exact sequence of B-modules:

$$0 \to (\mathfrak{b}/\mathfrak{u})^* \to \mathfrak{b}^* \to \mathfrak{u}^* \to 0,$$

we show that the vanishing of $H^i\big(G/B, \mathcal{L}(S(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda)\big)$ follows provided we show the vanishing of $H^i\big(G/B, \mathcal{L}(S(\mathfrak{b}^*)) \otimes \mathcal{L}(\lambda)\big)$. Now, the vanishing of the latter (for dominant λ) follows from the diagonality of the Hodge cohomology: $H^i\big(G/B, \Omega^j_{G/B}\big) = 0$ for $i \neq j$; splitting of $T^*(G/B)$; and the Koszul resolution corresponding to the exact sequence:

$$0 \to (\mathfrak{g}/\mathfrak{b})^* \to \mathfrak{g}^* \to \mathfrak{b}^* \to 0,$$

where $\Omega^j_{G/B}$ is the sheaf of j-forms in G/B. Now, the result for general $\lambda \in \mathcal{C}$ follows from the dominant case by using the following simple result (Lemma 5.2.4). For a simple root α and any $\lambda \in X^*(T)$ such that $\langle \lambda, \alpha^\vee \rangle = -1$,

$$H^i\big(G/B, \mathcal{L}(V) \otimes \mathcal{L}(\lambda)\big) = 0, \text{ for any } i \geq 0 \text{ and any } P_\alpha\text{-module } V.$$

We prove a slightly weaker P-analogue of the above main theorem. For any parabolic subgroup $P \subset G$ and any ample line bundle $\mathcal{L}^P(\lambda)$ on G/P,

$$H^i\big(T^*(G/P), \pi_P^*\mathcal{L}^P(\lambda)\big) = 0, \text{ for any } i > 0 \text{ (Theorem 5.2.11).}$$

It is natural to conjecture that this vanishing remains true for any dominant $\lambda \in X^*(P)$.

As a consequence of the above vanishing theorem for B and making use of the above two Koszul resolutions, we obtain the Dolbeault vanishing (Theorem 5.2.9). For any $\lambda \in \mathcal{C}$ and p a good prime for G,

$$H^i\big(G/B, \Omega^j_{G/B} \otimes \mathcal{L}(\lambda)\big) = 0 \quad \text{for any } i > j.$$

Finally, in Section 5.3, we use the main cohomology vanishing result of Section 5.2 to show that the nilpotent cone and the subregular cone of $\mathfrak{g}$ are normal Gorenstein varieties with rational singularities (again under the assumption that p is a good prime for G). Also, by Exercises 5.3.E, the closure of $\mathrm{SL}_n(k)$-conjugacy class of any nilpotent matrix $N \in s\ell_n(k)$ is a normal Gorenstein variety with rational singularities.

Notation. We follow the notation from Section 2.1. In particular, G is a connected, simply-connected, semisimple algebraic group over an algebraically closed field k of characteristic $p > 0$. We fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$

and let $U \subset B$ be the unipotent radical of B. Let $B \subset P$ be a (standard) parabolic subgroup with the unipotent radical U_P and the Levi component L_P containing T; so that P is the semidirect product $P = L_P \ltimes U_P$. Let U_P^- be the unipotent radical of the opposite parabolic P^-. We denote the Lie algebras of G, P, B, T, U, U_P, L_P by the corresponding Gothic characters: $\mathfrak{g}, \mathfrak{p}, \mathfrak{b}, \mathfrak{t}, \mathfrak{u}, \mathfrak{u}_P, \mathfrak{l}_P$, respectively.

By a *volume form* on a smooth variety X, we mean a nowhere vanishing differential form of top degree on X.

5.1 Splitting of cotangent bundles of flag varieties

We begin with the following lemma valid in any characteristic (including 0).

5.1.1 Lemma. *Let P be any parabolic subgroup of G and let P act on U_P by the conjugation action and on $\mathfrak{u}_P$ by the adjoint action. Let G act on $G \times_P U_P$ and $G \times_P \mathfrak{u}_P$ via the left multiplication on the first factor. Then, the canonical line bundles of the G-varieties $G \times_P U_P$ and $G \times_P \mathfrak{u}_P$ are G-equivariantly trivial.*

Proof. Abbreviate $G \times_P U_P$ by $\mathfrak{X}_P$ and let $\bar{\pi}_P : \mathfrak{X}_P \to G/P$ be the projection. Then, the canonical bundle $\omega_{\mathfrak{X}_P}$ of $\mathfrak{X}_P$ is G-equivariantly isomorphic to the tensor product of $(\bar{\pi}_P)^*\omega_{G/P}$ with the relative canonical bundle $\omega_{\bar{\pi}_P}$ of the fibration $\bar{\pi}_P$. Now, as in the beginning of Section 3.1, as G-bundles,

(1) $$\omega_{G/P} \simeq \mathcal{L}^P(-\delta_P).$$

Also, as G-bundles,

(2) $$\omega_{\bar{\pi}_P} \simeq G \times_P \omega_{U_P}.$$

But, as P-bundles,

(3) $$\omega_{U_P} \simeq \mathbf{k}_{-\delta_P},$$

where $\mathbf{k}_{-\delta_P}$ denotes the trivial line bundle on U_P together with the action of P via its character $-\delta_P$. Thus, as G-bundles,

(4) $$G \times_P \omega_{U_P} \simeq (\bar{\pi}_P)^*(\mathcal{L}^P(\delta_P)).$$

Combining (1), (2) and (4), we get the lemma for $G \times_P U_P$. The proof for $G \times_P \mathfrak{u}_P$ is similar. $\square$

5.1.2 The map ψ_P. As earlier in 2.3.4, the Steinberg module

$$\mathrm{St} := H^0(G/B, \mathcal{L}((p-1)\rho))$$

has a G-invariant nondegenerate bilinear form

(1) $$\chi : \mathrm{St} \otimes \mathrm{St} \to k,$$

which is unique up to a scalar multiple.

Define a k-linear G-module map

$$\psi_P : \mathrm{St} \otimes \mathrm{St} \to k[\mathfrak{X}_P] \tag{2}$$

by $\psi_P(v_1 \otimes v_2)(g, u) = \chi(v_1 \otimes gug^{-1}v_2)$, for $g \in G$, $u \in U_P$ and $v_1, v_2 \in \mathrm{St}$, where $\mathfrak{X}_P := G \times_P U_P$. Clearly, $\psi_P(v_1 \otimes v_2)$ is a well defined regular function on the quotient variety $\mathfrak{X}_P$.

By Lemma 5.1.1, there exists a G-invariant (nowhere vanishing) volume form $\theta_{\mathfrak{X}_P}$ on the (smooth) variety $\mathfrak{X}_P$. In fact, any two volume forms on $\mathfrak{X}_P$ are nonzero scalar multiples of each other, since there are no nonconstant regular functions $\mathfrak{X}_P \to k^*$. (Use, e.g., the fact that $U_P^- \times U_P$ is dense open in $\mathfrak{X}_P$.) Thus, the map $\mathrm{St} \otimes \mathrm{St} \to H^0\left(\mathfrak{X}_P, \omega_{\mathfrak{X}_P}^{1-p}\right)$, $f \mapsto \psi_P(f)\,\theta_{\mathfrak{X}_P}^{1-p}$, is a k-linear G-module map. Recall from 1.3.7 that there is a canonical identification $H^0(X, \omega_X^{1-p}) \simeq \mathrm{End}_F(X)$, for any smooth scheme X.

5.1.3 Theorem. *For any $f \in \mathrm{St} \otimes \mathrm{St}$, $\psi_P(f)\,\theta_{\mathfrak{X}_P}^{1-p} \in H^0\left(\mathfrak{X}_P, \omega_{\mathfrak{X}_P}^{1-p}\right)$ splits $\mathfrak{X}_P$ up to a nonzero scalar multiple iff $\chi(f) \neq 0$.*

In particular, $\mathfrak{X}_P$ admits a B-canonical splitting.

Before we come to the proof of the theorem, we need the following preparatory work.

5.1.4 The map ϕ_P. Let the standard parabolic subgroup P be given by $P = P_I$ for a subset $I \subset \{1, \ldots, \ell\}$, and set $\Delta_I^{\pm} := \Delta^{\pm} \cap \left(\oplus_{i \in I} \mathbb{Z}\alpha_i\right)$. Recall the definition of $\delta_P \in X^*(P)$ from the beginning of Section 3.1. Then,

$$\delta_P = \sum_{\alpha \in \Delta^+ \setminus \Delta_I^+} \alpha\,. \tag{1}$$

Choose T-eigenfunctions $\{x_\alpha\}_{\alpha \in \Delta^+ \setminus \Delta_I^+} \subset k[U_P]$ with $x_\alpha(1) = 0$ and x_α of weight $-\alpha$ such that, as T-algebras over k,

$$k[U_P] \simeq k[x_\alpha]_{\alpha \in \Delta^+ \setminus \Delta_I^+}. \tag{2}$$

Similarly, choose T-eigenfunctions $\{y_\alpha\}_{\alpha \in \Delta^+ \setminus \Delta_I^+} \subset k[U_P^-]$ with $y_\alpha(1) = 0$ and y_α of weight α such that, as T-algebras over k,

$$k[U_P^-] \simeq k[y_\alpha]_{\alpha \in \Delta^+ \setminus \Delta_I^+}. \tag{3}$$

These are guaranteed by [Spr–98, Lemma 8.2.2], also see Section 2.1. It is easy to see that the ideal $\langle x_\alpha^p \rangle_{\alpha \in \Delta^+ \setminus \Delta_I^+} \subset k[U_P]$ is P-stable under the conjugation action of P on U_P. Moreover, by (1)–(2), the P-module $k[U_P]/\langle x_\alpha^p \rangle$ has all its weights $\geq$

$-(p-1)\delta_P$ and the weight space of $k[U_P]/\langle x_\alpha^p\rangle$ corresponding to the weight $-(p-1)\delta_P$ is one-dimensional spanned by $\prod_{\alpha\in\Delta^+\setminus\Delta_I^+} x_\alpha^{p-1}$. (Observe that $k[U_P]/\langle x_\alpha^p\rangle$ is the coordinate ring of the first Frobenius kernel of U_P, cf. [Jan–03, Part I, Chap. 9].) Let

$$\phi_P : k[U_P] \to k_{-(p-1)\delta_P} \tag{4}$$

be the composition of the quotient map $k[U_P] \to k[U_P]/\langle x_\alpha^p\rangle$ with the T-equivariant projection onto the lowest weight space spanned by the vector $\prod_{\alpha\in\Delta^+\setminus\Delta_I^+} x_\alpha^{p-1}$. Further, the above map ϕ_P is a P-module map since the sum Σ of the weight spaces in $k[U_P]/\langle x_\alpha^p\rangle$ of weight $> -(p-1)\delta_P$ is stable under the action of P. (Clearly, Σ is stable under the action of B. Further, any simple reflection s_i, for $i \in I$, fixes δ_P, thus keeps Σ stable. By (2.1.5) this shows that Σ is P-stable.)

Inducing the map ϕ_P, we get the G-module map

$$H^0(\phi_P) : k[\mathfrak{X}_P] = H^0\big(G/P, \mathcal{L}(k[U_P])\big) \to H^0\big(G/P, \mathcal{L}^P((p-1)\delta_P)\big).$$

Also, define the G-module map

$$\bar{\psi} : \mathrm{St}\otimes\mathrm{St} \to k[G],\ \bar{\psi}(v_1\otimes v_2)g = \chi(v_1\otimes gv_2), \tag{5}$$

where G acts on $k[G]$ via the conjugation action. Restricting $\bar{\psi}$ to U_P, we get the P-module map

$$\bar{\psi}_P : \mathrm{St}\otimes\mathrm{St} \to k[U_P]. \tag{6}$$

Inducing $\bar{\psi}_P$ we get the G-module map (see Exercise 2.2.E.7)

$$H^0(\bar{\psi}_P) : \mathrm{St}\otimes\mathrm{St} \to H^0\big(G/P, \mathcal{L}(k[U_P])\big) = k[\mathfrak{X}_P]. \tag{7}$$

By Exercise 5.1.E.3, the above map

$$H^0(\bar{\psi}_P) = \psi_P, \tag{8}$$

where ψ_P is the map as in (5.1.2.2).

The composite P-module map $\phi_P \circ \bar{\psi}_P : \mathrm{St}\otimes\mathrm{St} \to k_{-(p-1)\delta_P}$ induces the G-module map

$$\eta_P := H^0(\phi_P \circ \bar{\psi}_P) : \mathrm{St}\otimes\mathrm{St} \to H^0\big(G/P, \mathcal{L}^P((p-1)\delta_P)\big). \tag{9}$$

Recall from the beginning of Section 3.1 that $\mathcal{L}^P(-\delta_P) \simeq \omega_{G/P}$.

5.1.5 Lemma. *For any $f \in \mathrm{St}\otimes\mathrm{St}$, $\eta_P(f)$ splits G/P up to a nonzero scalar multiple iff $\psi_P(f)\,\theta_{\mathfrak{X}_P}^{1-p}$ splits $\mathfrak{X}_P$ up to a nonzero scalar multiple, where ψ_P is the map defined in (5.1.2.2).*

Proof. For any $f \in \mathrm{St} \otimes \mathrm{St}$, write

$$\psi_P(f)(y,x) = \sum_{\mathbf{n},\mathbf{m} \in \mathbb{Z}_+^{N_I}} c_{\mathbf{n},\mathbf{m}} y^{\mathbf{n}} x^{\mathbf{m}}, \tag{1}$$

for $(y,x) \in U_P^- \times U_P$, where $c_{\mathbf{n},\mathbf{m}} = c_{\mathbf{n},\mathbf{m}}(f) \in k$, $N_I := |\Delta^+ \backslash \Delta_I^+|$, $\mathbf{n} = (n_\alpha)_{\alpha \in \Delta^+ \backslash \Delta_I^+}$, $\mathbf{m} = (m_\alpha)$ and $y^{\mathbf{n}} := \prod_\alpha y_\alpha^{n_\alpha}$, $x^{\mathbf{m}} := \prod_\alpha x_\alpha^{m_\alpha}$. Then, $\psi_P(f)\,\theta_{\mathfrak{X}_P}^{1-p}$ splits $\mathfrak{X}_P$ up to a nonzero scalar multiple iff

$$c_{\mathbf{p-1}+p\mathbf{n},\mathbf{p-1}+p\mathbf{m}} = 0 \tag{2}$$

if at least one of $\mathbf{n}$ or $\mathbf{m}$ is nonzero and

$$c_{\mathbf{p-1},\mathbf{p-1}} \neq 0, \tag{3}$$

where, as in Chapter 1, $\mathbf{p-1}$ is the constant sequence $(p-1, p-1, \ldots, p-1) \in \mathbb{Z}_+^{N_I}$ (use Lemma 1.1.7(ii) and Example 1.3.1). Since $\psi_P(f)$ lies in the image of $k[G] \otimes (\mathrm{St} \otimes \mathrm{St}) \xrightarrow{I \otimes \bar{\psi}_P} k[G] \otimes k[U_P]$ (by (5.1.4.8)) and any weight of $\mathrm{St} \otimes \mathrm{St}$ is $\geq -2(p-1)\rho$, $c_{\mathbf{n},\mathbf{m}} = 0$ unless

$$-\mathrm{wt}(\mathbf{m}) \geq -2(p-1)\rho, \text{ i.e., } \mathrm{wt}(\mathbf{m}) \leq 2(p-1)\rho,$$

where

$$\mathrm{wt}(\mathbf{m}) := \sum_{\alpha \in \Delta^+ \backslash \Delta_I^+} m_\alpha \alpha.$$

Thus,

$$c_{\mathbf{n},\mathbf{p-1}+p\mathbf{m}} = 0, \quad \text{if } \mathbf{m} \text{ is nonzero} \tag{4}$$

(since $\mathrm{wt}(\mathbf{p-1} + p\mathbf{m}) \not\leq 2(p-1)\rho$ if $\mathbf{m}$ is nonzero).

Assume first that $\eta_P(f)$ splits G/P and consider the coefficients $c_{\mathbf{n},\mathbf{p-1}}$. Since $\eta_P(f)$ splits G/P; in particular, $\eta_P(f)_{|U_P^-}$ splits U_P^-. By the definition of the map η_P as in (5.1.4.9) and Example 1.3.1, for nonzero $\mathbf{n}$,

$$c_{\mathbf{p-1}+p\mathbf{n},\mathbf{p-1}} = 0, \qquad \text{and } \; c_{\mathbf{p-1},\mathbf{p-1}} \neq 0. \tag{5}$$

(Use the composite map $k[G] \otimes (\mathrm{St} \otimes \mathrm{St}) \xrightarrow{I \otimes \bar{\psi}_P} k[G] \otimes k[U_P] \xrightarrow{I \otimes \phi_P} k[G] \otimes k_{-(p-1)\delta_P}$.) Thus, (2) and (3) are established, proving that $\psi_P(f)\,\theta_{\mathfrak{X}_P}^{1-p}$ splits $\mathfrak{X}_P$ up to a nonzero scalar multiple.

Conversely, assume that $\psi_P(f)\,\theta_{\mathfrak{X}_P}^{1-p}$ splits $\mathfrak{X}_P$. Then, (2) and (3) are satisfied; in particular, (5) is satisfied. Thus, $\eta_P(f)_{|U_P^-}$ splits U_P^- up to a nonzero scalar multiple and hence $\eta_P(f)$ splits G/P up to a nonzero scalar multiple by Lemma 1.1.7(ii). $\square$

5.1.6 Theorem. *For any $f \in \mathrm{St} \otimes \mathrm{St}$, $\eta_P(f)$ splits G/P up to a nonzero scalar multiple iff $\chi(f) \neq 0$.*

Proof. We first consider the case $P = B$ and show that $\eta = \eta_B$ is nonzero. To show this, it suffices to show that the composite map $\phi_B \circ \bar{\psi}_B : \mathrm{St} \otimes \mathrm{St} \to k_{-2(p-1)\rho}$ (see 5.1.4) is nonzero.

By Theorem 2.3.1, $m(f_+ \otimes f_-)$ splits $X = G/B$ up to a nonzero scalar multiple, where f_-, resp. f_+, is a lowest, resp. highest, weight vector of St and m is the multiplication map $\mathrm{St} \otimes \mathrm{St} \to H^0\big(G/B, \mathcal{L}(2(p-1)\rho)\big)$ as in the beginning of Section 2.3. Now, by (2.3.1.2), under the standard trivialization of $\mathcal{L}(2(p-1)\rho)$ over U^-, $m(f_+ \otimes f_-)_{|U^-}$ is given by

(1) $$g \mapsto \chi(gf_+ \otimes f_+), \quad \text{for } g \in U^- .$$

(Observe that, by the definition of χ as in 2.3.4, $\chi(gf_+ \otimes f_+) = f_+(gv_+)$, where f_+ corresponds to v_+ under the identification $\bar{\chi} : \mathrm{St} \to \mathrm{St}^*$.) Since $m(f_+ \otimes f_-)$ splits X; in particular, $m(f_+ \otimes f_-)_{|U^-}$ splits the open subset $U^- \subset X$. Thus, the monomial $y^{\mathbf{p-1}}$ occurs with nonzero coefficient in $m(f_+ \otimes f_-)_{|U^-}$ (where the notation $y^{\mathbf{p-1}}$ is as in the proof of Lemma 5.1.5). Conjugating this by the longest element w_o of the Weyl group W, we get that $x^{\mathbf{p-1}}$ occurs with nonzero coefficient in the function

$$g \mapsto \chi(gf_- \otimes f_-), \ g \in U.$$

This proves that

(2) $$\phi_B \circ \bar{\psi}_B(f_- \otimes f_-) \neq 0.$$

Thus, the G-module map $\eta : \mathrm{St} \otimes \mathrm{St} \to H^0(G/B, \mathcal{L}(2(p-1)\rho))$ is nonzero.

By the Frobenius reciprocity,

(3) $$\begin{aligned}\dim_k \mathrm{Hom}_G\Big(\mathrm{St} \otimes \mathrm{St}, H^0(G/B, \mathcal{L}(2(p-1)\rho))\Big) \\ = \dim_k \mathrm{Hom}_B\Big(\mathrm{St} \otimes \mathrm{St}, k_{-2(p-1)\rho}\Big) = 1,\end{aligned}$$

since the weight space of $\mathrm{St} \otimes \mathrm{St}$ corresponding to the weight $-2(p-1)\rho$ is one-dimensional and it is of smallest weight.

Thus,

(4) $$m = \eta \qquad \text{up to a nonzero scalar multiple.}$$

Thus, the proposition for $P = B$ follows from Corollary 2.3.5.

We now come to the general P. By the above case (i.e., $P = B$) and Lemma 5.1.5, $\psi_B(f_+ \otimes f_-)\,\theta_{\mathfrak{X}_B}^{1-p}$ splits $\mathfrak{X}_B := G \times_B U$ up to a nonzero scalar multiple. We claim, in fact, that $\psi_P(f_+ \otimes f_-)\,\theta_{\mathfrak{X}_P}^{1-p}$ splits $\mathfrak{X}_P$ for any P up to a nonzero scalar multiple.

For any root β, as in Section 2.1, there exists an algebraic group isomorphism $\varepsilon_\beta : \mathbb{G}_a \to U_\beta$ onto the root subgroup U_β satisfying

(5) $$t\varepsilon_\beta(z)t^{-1} = \varepsilon_\beta(\beta(t)z), \quad \text{for all } t \in T, z \in \mathbb{G}_a.$$

Order the roots $\Delta^+ \backslash \Delta_I^+ = \{\beta_1, \dots, \beta_{N_I}\}$.

Now, define the variety isomorphisms (Section 2.1)

(6) $$\varepsilon : k^{N_I} \to U_P, \quad \varepsilon(t_1, \dots, t_{N_I}) = \varepsilon_{\beta_1}(t_1) \cdots \varepsilon_{\beta_{N_I}}(t_{N_I}),$$

and

(7) $$\bar{\varepsilon} : k^{N_I} \to U_P^-, \quad \bar{\varepsilon}(s_1, \dots, s_{N_I}) = \varepsilon_{-\beta_1}(s_1) \cdots \varepsilon_{-\beta_{N_I}}(s_{N_I}).$$

For $g = \bar{\varepsilon}(\mathfrak{s}) \in U_P^-$, write

(8) $$g^{-1} f_+ = \sum_{\mathbf{n} \in \mathbb{Z}_+^{N_I}} \mathfrak{s}^{\mathbf{n}} v_{\mathbf{n}}, \quad \text{for some } v_{\mathbf{n}} \in \mathrm{St},$$

where $\mathfrak{s} := (s_1, \dots, s_{N_I})$ and $\mathfrak{s}^{\mathbf{n}} := s_1^{n_1} \cdots s_{N_I}^{n_{N_I}}$. For a T-eigenvector v, let weight(v) denote its weight. Then, it is easy to see that $v_{\mathbf{n}}$ is a weight vector with

(9) $$\operatorname{weight}(v_{\mathbf{n}}) = (p-1)\rho - \sum_{i=1}^{N_I} n_i \beta_i.$$

Similarly, write for $u = \varepsilon(\mathrm{t}) \in U_P$,

(10) $$u\, f_- = \sum_{\mathbf{m} \in \mathbb{Z}_+^{N_I}} \mathrm{t}^{\mathbf{m}} w_{\mathbf{m}}, \quad \text{for some } w_{\mathbf{m}} \in \mathrm{St}\,.$$

Then,

(11) $$\operatorname{weight}(w_{\mathbf{m}}) = -(p-1)\rho + \sum_{i=1}^{N_I} m_i \beta_i.$$

Thus, for $g \in U_P^-$ and $u \in U_P$,

(12) $$\chi(f_+ \otimes gug^{-1} f_-) = \chi(g^{-1} f_+ \otimes uf_-) = \sum_{\mathbf{n},\mathbf{m}} \mathfrak{s}^{\mathbf{n}}\, \mathrm{t}^{\mathbf{m}}\, \chi(v_{\mathbf{n}} \otimes w_{\mathbf{m}}).$$

We claim that, for the constant sequence $\mathbf{p} - \mathbf{1} = (p-1, \dots, p-1) \in \mathbb{Z}_+^{N_I}$,

(13) $$v_{\mathbf{p}-\mathbf{1}+p\mathbf{n}} = 0 = w_{\mathbf{p}-\mathbf{1}+p\mathbf{n}} \qquad \text{if } \mathbf{n} \neq 0.$$

By (9) and (5.1.4.1),

$$\begin{aligned}\text{weight}\,(v_{\mathbf{p-1}+p\mathbf{n}}) &= (p-1)\rho - (p-1)\delta_P - \sum_{i=1}^{N_I} pn_i\beta_i\\ &= -(p-1)\,w_o^P\rho - \sum_{i=1}^{N_I} pn_i\beta_i,\end{aligned}$$

where w_o^P is the longest element of the Weyl group W_P of P. Since w_o^P permutes the roots in $\Delta^+\backslash\Delta_I^+$, the weight space of St corresponding to the weight $-(p-1)\,w_o^P\rho - \sum_{i=1}^{N_I} pn_i\beta_i$ is zero-dimensional. This proves the first equality of (13). The second equality follows similarly. We next show that $v_{\mathbf{p-1}}$ and $w_{\mathbf{p-1}}$ are both nonzero; thus they are the unique (up to nonzero multiples) extremal weight vectors of St corresponding to the weights $-(p-1)\,w_o^P\rho$ and $(p-1)\,w_o^P\rho$ respectively.

Analogous to ε, define a variety isomorphism $\varepsilon' : k^{N-N_I} \to U(L_P)$, where $N := |\Delta^+|$ and $U(L_P) := U\cap L_P$ is the unipotent radical of the Borel subgroup $B\cap L_P$ of L_P. Similarly, define

$$\bar\varepsilon' : k^{N-N_I} \to U(L_P)^-.$$

For $g = \bar\varepsilon(\mathfrak{s})\,\bar\varepsilon'(\mathfrak{s}') \in U^-$, where $\mathfrak{s}' = \big(s_{N_I+1},\dots,s_N\big)$, and $\mathfrak{s}\in k^{N_I}$, write

(14) $$g^{-1}f_+ = \sum_{\mathbf{n}\in\mathbb{Z}_+^{N_I}} \mathfrak{s}^{\mathbf{n}}\big(\bar\varepsilon'(\mathfrak{s}')^{-1}\,v_{\mathbf{n}}\big),$$

for some $v_{\mathbf{n}}\in$ St. Similarly, for $u = \varepsilon'(\mathfrak{t}')\varepsilon(\mathfrak{t})\in U$, write

(15) $$uf_- = \sum_{\mathbf{m}\in\mathbb{Z}_+^{N_I}} \mathfrak{t}^{\mathbf{m}}\big(\varepsilon'(\mathfrak{t}')\,w_{\mathbf{m}}\big),$$

for some $w_{\mathbf{m}}\in$ St. Since $\psi_B(f_+\otimes f_-)\,\theta_{\mathfrak{X}_B}^{1-p}$ splits $\mathfrak{X}_B$ up to a nonzero scalar multiple, the coefficient of $\mathfrak{s}^{\mathbf{p-1}}\,\mathfrak{s}'^{\mathbf{p-1}}\,\mathfrak{t}^{\mathbf{p-1}}\,\mathfrak{t}'^{\mathbf{p-1}}$ in $\chi(g^{-1}f_+\otimes uf_-)$ is nonzero (see the proof of Lemma 5.1.5). This is possible only if $v_{\mathbf{p-1}}$ and $w_{\mathbf{p-1}}$ are both nonzero (use (14) and (15)). Thus, $\chi\big(v_{\mathbf{p-1}}\otimes w_{\mathbf{p-1}}\big)\neq 0$ (since $v_{\mathbf{p-1}}$ and $w_{\mathbf{p-1}}$ are extremal weight vectors of opposite weights). This proves (by virtue of (12)) that the monomial $\mathfrak{s}^{\mathbf{p-1}}\,\mathfrak{t}^{\mathbf{p-1}}$ occurs with nonzero coefficient in the function $(g,u)\mapsto \chi(f_+\otimes gug^{-1}f_-)$, for $(g,u)\in U_P^-\times U_P$. Combining this with (13), we get that $\psi_P(f_+\otimes f_-)\,\theta_{\mathfrak{X}_P}^{1-p}$ splits $\mathfrak{X}_P$ up to a nonzero scalar multiple (see the proof of Lemma 5.1.5). Thus, by Lemma 5.1.5, $\eta_P(f_+\otimes f_-)$ splits G/P up to a nonzero scalar multiple.

To complete the proof of the theorem, we follow the argument as in the proof of Corollary 2.3.5. As in (2.3.5.2), since $\omega_Y\simeq\mathcal{L}^P(-\delta_P)$ (see the beginning of Section 3.1), identify (via a k-linear G-module isomorphism)

$$H^0\Big(Y,\mathcal{L}^P((p-1)\,\delta_P)\Big)\simeq \mathrm{Hom}\Big(F_*\mathcal{O}_Y,\mathcal{O}_Y\Big)^{[-1]},$$

where $Y = G/P$. Further, consider the k-linear G-module map

$$e : \mathrm{Hom}(F_*\mathcal{O}_Y, \mathcal{O}_Y)^{[-1]} \to k$$

defined by $e(\sigma) = \sigma(1)^p$. Then, since Y is irreducible and projective, we have that $\sigma \in \mathrm{Hom}(F_*\mathcal{O}_Y, \mathcal{O}_Y)$ splits Y (up to a nonzero scalar multiple) iff $e(\sigma) \neq 0$. Combining the above two maps, we get a k-linear G-module map

$$\theta : H^0\big(Y, \mathcal{L}^P((p-1)\delta_P)\big) \to k$$

and thus a k-linear G-module map

$$\theta \circ \eta_P : \mathrm{St} \otimes \mathrm{St} \to k.$$

Moreover, since $\eta_P(f_+ \otimes f_-)$ splits Y up to a nonzero scalar multiple, $\theta \circ \eta_P(f_+ \otimes f_-) \neq 0$. Thus, $\theta \circ \eta_P = \chi$ (up to a nonzero scalar multiple) and hence $\eta_P(f)$ splits G/P up to a nonzero scalar multiple iff $\chi(f) \neq 0$. This completes the proof of the theorem. $\square$

5.1.7 *Proof of Theorem 5.1.3.* Combining Lemma 5.1.5 and Theorem 5.1.6, the first part of Theorem 5.1.3 follows immediately. For the "In particular" statement, the splitting of $\mathfrak{X}_P$ given by $\psi_P(f_- \otimes f_+)\theta_{\mathfrak{X}_P}^{1-p}$ provides a B-canonical splitting of $\mathfrak{X}_P$ (up to a nonzero scalar multiple) by Lemmas 4.1.14 and 4.1.6. $\square$

5.1.8 Definition. Let G be a connected, simply-connected, *simple* algebraic group. Then, a prime p is said to be a *good prime for G* if p is coprime to all the coefficients of the highest root of G written in terms of the simple roots. A prime which is not a good prime for G is called a *bad prime for G*. A prime p is called good for a connected, simply-connected, semisimple algebraic group if p is good for all its simple components.

For simple G of type A_ℓ no prime is bad; for G of type B_ℓ, C_ℓ, D_ℓ only $p = 2$ is a bad prime; for G of type E_6, E_7, F_4, G_2 only $p = 2, 3$ are bad primes; and for G of type E_8 only $p = 2, 3, 5$ are bad primes.

5.1.9 Proposition. *Let G be a connected, simply-connected, semisimple algebraic group over any algebraically closed field k. Assume that the characteristic of k is* 0 *or a good prime for G. Then, there exists a B-equivariant isomorphism of varieties $\xi : U \xrightarrow{\sim} \mathfrak{u}$ which restricts to a P-equivariant isomorphism $\xi_P : U_P \xrightarrow{\sim} \mathfrak{u}_P$, for any standard parabolic subgroup P of G.*

Proof. The existence of a B-equivariant isomorphism ξ taking 1 to 0 is proved in [Spr–69, Proposition 3.5]. We now prove that ξ restricts to an isomorphism ξ_P. By [Spr–98, Exercise 8.4.6(5)], there exists a one-parameter subgroup $\gamma : \mathbb{G}_m \to B$ such that:

(1) U_P is the set of those $g \in G$ such that $\gamma(z)g\gamma(z)^{-1}$ has limit 1 when z tends to zero, and

(2) $\mathfrak{u}_P$ is the set of those $x \in \mathfrak{g}$ such that $\mathrm{Ad}(\gamma(z))x$ has limit 0 when z tends to 0.

Using (1)–(2) and the B-equivariance of ξ, it follows easily that ξ takes U_P surjectively onto $\mathfrak{u}_P$. From this, of course, we get that ξ_P is an isomorphism. Since ξ_P is a B-equivariant morphism between affine P-varieties, it is automatically P-equivariant (because P/B is projective). $\square$

A proof for the following can be found in [Spr–69, Lemma 4.4].

5.1.10 Proposition. *Let G and k be as in the above proposition. Assume, in addition, that G does not have simple components of type A_ℓ. Then, there exists a nondegenerate symmetric G-invariant bilinear form on $\mathfrak{g}$.*

Also, there exists a nondegenerate symmetric $GL_n(k)$-invariant bilinear form on $M_n(k) = \mathrm{Lie}\, GL_n(k)$, for any k.

As an immediate consequence of the above proposition, we get the following.

5.1.11 Corollary. *Let G and k be as in Proposition 5.1.9. Then, for any parabolic subgroup $P \subset G$, the cotangent bundle $T^*(G/P)$ is G-equivariantly isomorphic to the homogeneous vector bundle $G \times_P \mathfrak{u}_P$.*

As a corollary of Theorem 5.1.3, we get the following.

5.1.12 Corollary. *Let G be as in the beginning of this chapter and assume that the characteristic of k is a good prime for G. Then, for any standard parabolic subgroup $P \subset G$, the cotangent bundle $T^*(G/P)$ is B-canonically split.*

Proof. By Corollary 5.1.11, $T^*(G/P) \simeq G \times_P \mathfrak{u}_P$. Further, by Proposition 5.1.9,

(1) $$G \times_P \mathfrak{u}_P \simeq G \times_P U_P.$$

Thus, the corollary follows from Theorem 5.1.3. $\square$

Let $\bar{\pi}_P : \mathfrak{X}_P \to G/P$ and $\pi_P : T^*(G/P) \to G/P$ be the canonical projections. Then, both of these morphisms are G-equivariant. The following corollary follows immediately by combining Theorems 5.1.3 and 4.2.13.

5.1.13 Corollary. *For any (not necessarily dominant) $\lambda \in X^*(P)$, the G-module $H^0\big(\mathfrak{X}_P, \bar{\pi}_P^*\mathcal{L}^P(\lambda)\big)$ admits a good filtration.*

In particular, if the characteristic of k is a good prime for G, then $H^0\big(T^*(G/P), \pi_P^*\mathcal{L}^P(\lambda)\big)$ admits a good filtration.

5.1.14 Homogeneous splittings of $T^*(G/P)$. With the notation and assumptions of Corollary 5.1.11, the ring of functions

(1) $$k[T^*(G/P)] \simeq k[G \times_P \mathfrak{u}_P] \simeq [k[G] \otimes S(\mathfrak{u}_P^*)]^P,$$

where P acts on $k[G]$ via the right regular representation.

(2) $$(p \cdot f)g = f(gp), \text{ for } f \in k[G], p \in P \text{ and } g \in G,$$

and the action of P on $S(\mathfrak{u}_P^*)$ is induced from the adjoint action.

Taking the standard grading $S(\mathfrak{u}_P^*) = \bigoplus_{d\geq 0} S^d(\mathfrak{u}_P^*)$, we get a grading on $k[T^*(G/P)]$ via the isomorphism (1) declaring $k[G]$ to have degree 0. For any $d \geq 0$, let

$$\kappa_d : k[T^*(G/P)] \to [k[G] \otimes S^d(\mathfrak{u}_P^*)]^P$$

be the projection onto the d-th homogeneous component.

Let Y be a smooth variety together with a (nowhere vanishing) volume form θ_Y. Of course, in general, θ_Y does not exist. Then, a function $f \in k[Y]$ is called a *splitting function* (with respect to θ_Y) if $f\theta_Y^{1-p} \in H^0\big(Y, \omega_Y^{1-p}\big)$ splits Y up to a nonzero scalar multiple. In the case θ_Y is unique up to a scalar multiple (e.g., $T^*(G/P)$ by 5.1.2), we can talk of a splitting function (without any reference to θ_Y).

If $f \in k[T^*(G/P)]$ is a splitting function, then so is $\kappa_{(p-1)N_I} f$, where $N_I :=$ dim G/P (as earlier), since $(\kappa_{(p-1)N_I} f)_{|U_P^- \times \mathfrak{u}_P}$ is a splitting function by Example 1.3.1. In particular, by Theorem 5.1.3, for any $f \in \mathrm{St} \otimes \mathrm{St}$ such that $\chi(f) \neq 0$, the function $\kappa_{(p-1)N_I}(\psi_P(f))$ is a splitting function of $T^*(G/P)$, where we consider the function $\psi_P(f) \in k[G \times_P U_P]$ as a function on $T^*(G/P)$ via the identification $G \times_P U_P \simeq T^*(G/P)$ given by (5.1.12.1) and Corollary 5.1.11. These splitting functions $\kappa_{(p-1)N_I}(\psi_P(f))$ have an advantage of being homogeneous in the fiber direction; thus they give rise to a splitting of the projectivized cotangent bundle $\mathbb{P}(T^*(G/P))$ consisting of lines in $T^*(G/P)$ (Example 1.1.10(3)).

5.1.15 Example. We consider the example of $G = \mathrm{SL}_{n+1}(k)$, where k is an algebraically closed field of an arbitrary characteristic $p > 0$. In this case, the Springer isomorphism (Proposition 5.1.9)

$$\xi : U \to \mathfrak{u} \qquad \text{is explicitly given by} \qquad A \mapsto A - I,$$

where U, resp. $\mathfrak{u}$, is the set of upper triangular unipotent matrices, resp. upper triangular nilpotent matrices, and I is the identity $(n+1) \times (n+1)$ matrix. We consider the element $f_- \otimes f_+ \in \mathrm{St} \otimes \mathrm{St}$. Then, the function $\psi_B(f_- \otimes f_+)$, under the identification $G \times_B U \simeq G \times_B \mathfrak{u}$, is given by

(1) $$\psi_B(f_- \otimes f_+)(g, A) = \chi(f_- \otimes g(I + A)g^{-1} f_+), \text{ for } g \in G, A \in \mathfrak{u}.$$

We identify the above function more explicitly as follows. Let $V = k^{n+1}$ be the standard representation of $\mathrm{SL}_{n+1}(k)$ with standard basis $\{e_1, \dots, e_{n+1}\}$. Then, for any $1 \leq i \leq n$, $\wedge^i V$ is a Weyl module with the i-th fundamental weight χ_i as its highest weight and highest weight vector $e_1 \wedge \cdots \wedge e_i$. Moreover, we have a G-module embedding (St sitting as the "Cartan" piece)

(2) $$i : \mathrm{St} \hookrightarrow \big(V \otimes \wedge^2 V \otimes \cdots \otimes \wedge^n V\big)^{\otimes p-1},$$

taking $f_+ \mapsto \big(e_1 \otimes (e_1 \wedge e_2) \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_n)\big)^{\otimes p-1}$. The existence of i follows from the self-duality of St and [Jan–03, Part II, Lemma 2.13(a)]. Moreover, since St is irreducible (Exercise 2.3.E.3), i is an embedding.

Similarly, replacing B by the opposite Borel B^-, there is a G-module embedding

$$i' : \mathrm{St} \hookrightarrow \big(V^* \otimes \wedge^2(V^*) \otimes \cdots \otimes \wedge^n(V^*)\big)^{\otimes p-1}$$

taking $f_- \mapsto \big(e_1^* \otimes (e_1^* \wedge e_2^*) \otimes \cdots \otimes (e_1^* \wedge \cdots \wedge e_n^*)\big)^{\otimes p-1}$, where $\{e_1^*, \ldots, e_{n+1}^*\}$ is the dual basis of $V^* : e_i^*(e_j) = \delta_{i,j}$. Further, the G-invariant pairing $\chi : \mathrm{St} \otimes \mathrm{St} \to k$ is the restriction of the pairing

$$\big(V^* \otimes \wedge^2(V^*) \otimes \cdots \otimes \wedge^n(V^*)\big)^{\otimes p-1} \otimes \big(V \otimes \wedge^2 V \otimes \cdots \otimes \wedge^n V\big)^{\otimes p-1} \to k$$

induced by the standard pairing $V^* \otimes V \to k$. So,

$$\begin{aligned} \psi_B(f_- \otimes f_+)(g, A) &= \chi(f_- \otimes g(I+A)g^{-1} f_+) \\ &= (\det M_1 \det M_2 \cdots \det M_n)^{p-1}, \end{aligned} \tag{3}$$

where $M = M(g, A)$ is the matrix $g(I + A)g^{-1}$ and, for any $1 \le i \le n$, M_i is the matrix obtained from M by taking the first i rows and the first i columns.

For any fixed g, by (3), the map $A \mapsto \psi_B(f_- \otimes f_+)(g, A)$ is a (not necessarily homogeneous) polynomial on $\mathfrak{u}$ of degree $(p-1)(1+2+\cdots+n) = (p-1)\frac{n(n+1)}{2} = (p-1)\dim G/B$. The splitting function $\kappa_{(p-1)N}(\psi_B(f_- \otimes f_+))$ (where $N := \dim G/B$) on $T^*(G/B)$ was originally used by [MeVa-92a] to split $T^*(\mathrm{SL}_{n+1}/B)$.

5.1.E Exercises

In the following exercises (1), (2) and (5), k is of arbitrary characteristic $p \ge 0$. Moreover, unless otherwise stated, G denotes a connected, simply-connected, semisimple algebraic group.

(1) Show that there are no nonconstant regular maps $G \to k^*$ for any connected, semisimple algebraic group over k. In particular, up to a nonzero scalar multiple, there exists a unique volume form on G.

Hint: Any (connected) unipotent subgroup H being isomorphic, as a variety, to an affine space, there are no nonconstant regular maps $H \to k^*$.

(2) More generally, for any connected algebraic group G, any regular map $f : G \to k^*$ with $f(1) = 1$ is a character.

This is a result due to Rosenlicht.

(3*) Show that the map $H^0(\bar{\psi}_P)$ as in (5.1.4.7) is the same as the map ψ_P as in (5.1.2.2).

(4) Show that the function $\phi : G \to k$ defined by $g \mapsto \chi(f_+ \otimes g f_-)\,\chi(f_+ \otimes g^{-1} f_-)$ is a splitting function, where f_+, f_- are highest and lowest weight vectors respectively of St and χ is the pairing as in (5.1.2.1). (By Exercise (1), there exists a unique volume from θ_G on G up to a nonzero scalar multiple.) Moreover, the associated splitting of G is $B \times B$-canonical under the action $(b_1, b_2) \cdot g = b_1 g b_2^{-1}$.

Hint: First show that $\theta_{G|_{U^-UT}} = t^{-\rho}\theta_{U^-} \wedge \theta_U \wedge \theta_T$, where θ_T is the volume form $dt_1 \wedge \cdots \wedge dt_\ell$; t_i being the function t^{χ_i} for the i-th fundamental weight χ_i. In particular, for $u_- \in U^-$, $u_+ \in U$ and $t \in T$,

$$(\phi\theta_G^{1-p})(u_-u_+t) = \chi(u_-^{-1}f_+ \otimes u_+f_-)t^{(p-1)\rho}\left(\theta_{U^-}(u_-) \wedge \theta_U(u_+) \wedge \theta_T(t)\right)^{\otimes 1-p}.$$

Now, use Theorem 5.1.3.

In fact, consider the linear function $\Phi : (\mathrm{St} \boxtimes \mathrm{St}) \otimes (\mathrm{St} \boxtimes \mathrm{St}) \to k[G]$ defined by

$$\Phi(v \boxtimes v' \otimes w \boxtimes w') = \chi(v \otimes gv')\chi(w \otimes gw').$$

Then, show that $\Phi(v \boxtimes v' \otimes w \boxtimes w')$ is a splitting function iff $\chi(v \otimes w)\chi(v' \otimes w') \neq 0$.

(5) Let P be any parabolic subgroup of G. Show that the canonical bundle of $G \times_P P$ is G-equivariantly trivial, where P acts on P via the conjugation. Prove the same result for $G \times_P \mathfrak{p}$.

Show further that $G \times_P \mathfrak{p}$ has a unique volume form (up to a nonzero scalar multiple), but it is false for $G \times_P P$ for any parabolic subgroup $P \subsetneq G$.

Hint: For the first part use ideas similar to the proof of Lemma 5.1.1. For the nonuniqueness part, observe that any nontrivial character χ of P induces a nonconstant regular function on $G \times_P P$ (as χ is invariant under conjugation).

(6) Let $G = \mathrm{SL}_{n+1}(k)$, where k is an algebraically closed field of any characteristic $p > 0$. Show that the splitting of $T^*(G/B)$ given by the function $h = \kappa_{(p-1)N}(\psi_B(f_- \otimes f_+))$ (as in 5.1.15), i.e., the splitting $h\theta_{T^*(G/B)}^{1-p}$ of $T^*(G/B)$ compatibly splits all the subvarieties $G \times_B \mathfrak{u}_P \subset G \times_B \mathfrak{u} \simeq T^*(G/B)$, for any standard parabolic subgroup $P \subset G$.

(7) By Exercise (5), there exists a volume form θ on $G \times_B B$ such that $\theta_{|U^- \times TU} = \theta_{U^-} \wedge \theta_T \wedge \theta_U$. Show that $\alpha\theta^{1-p}$ provides a B-canonical splitting of $G \times_B B$ up to a nonzero scalar multiple, where α is the function given by $\alpha(g, b) = \chi(f_- \otimes gbg^{-1}f_+)$, for $g \in G, b \in B$.

In fact, for any $f = \sum v_i \otimes w_i \in \mathrm{St} \otimes \mathrm{St}$ such that $\chi(f) \neq 0, \alpha_f(g, b) := \sum_i \chi(v_i \otimes gbg^{-1}w_i)$ is a splitting function on $G \times_B B$ with respect to the volume form θ.

Hint: Use Theorem 5.1.3.

(8) Assume that p is a good prime for G and, moreover, G does not have any components of the form $\mathrm{SL}_n(k)$. Recall that there exists a G-equivariant morphism (cf. [BaRi–85, Proposition 9.3.3]) $\xi_G : G \to \mathfrak{g}$, taking $1 \mapsto 0$, such that the differential

$$(*) \qquad (d\xi_G)_1 : \mathfrak{g} \to \mathfrak{g}$$

is the identity map. By [Spr–98, Proposition 8.4.5], there exists a one-parameter subgroup $\gamma : \mathbb{G}_m \to B$ such that B, resp. $\mathfrak{b}$, is the set of those $g \in G$, resp. $x \in \mathfrak{g}$, such that $\gamma(z)g\gamma(z)^{-1}$, resp. $\mathrm{Ad}(\gamma(z))x$, has a limit when z tends to zero. Using such a γ, show that ξ_G restricts to a morphism $\xi = \xi_B : B \to \mathfrak{b}$.

Then, ξ induces a map at the level of the top degree forms

$$\xi^* : H^0(\mathfrak{b}, \omega_{\mathfrak{b}}) \to H^0(B, \omega_B),$$

where ω denotes the canonical bundle. Fix a volume form $\theta_{\mathfrak{b}}$ on $\mathfrak{b}$ and a volume form $\theta_B = \theta_T \theta_U$ on $B = TU$. Then, $\xi^*(\theta_{\mathfrak{b}}) = \Phi\, \theta_B$, for some $\Phi \in k[B]$. Moreover, $\Phi(1) \neq 0$, by $(*)$.

The map ξ induces an isomorphism of algebras

$$\hat{\xi}^* : \widehat{k[\mathfrak{b}]} \xrightarrow{\sim} \widehat{k[B]},$$

where $\widehat{k[\mathfrak{b}]}$, resp. $\widehat{k[B]}$, is the completion of $k[\mathfrak{b}]$ at the maximal ideal corresponding to 0, resp. of $k[B]$ at 1.

Write

$$\alpha(g, b) = \sum_j \alpha'_j(g)\, \alpha''_j(b),$$

for some $\alpha'_j \in k[G]$ and $\alpha''_j \in k[B]$, where $\alpha : G \times_B B \to k$ is as in Exercise (7). Define the regular function $\hat{\alpha} : G \times \mathfrak{b} \to k$ by

$$\hat{\alpha} = \sum_j \alpha'_j \otimes \Big(\big(\hat{\xi}^*\big)^{-1} \big(\alpha''_j\, \Phi^{p-1}\big) \Big)_o,$$

where $(\cdot)_o$ denotes the homogeneous component of degree $(p-1)\dim B$. Now, prove the following.

(a) $\hat{\alpha}$ descends to a function on $G \times_B \mathfrak{b}$.
Hint: Show that Φ is B-invariant under the adjoint action.

(b) $\hat{\alpha}$ provides a B-canonical splitting of $G \times_B \mathfrak{b}$ with respect to the unique (up to a nonzero scalar multiple) volume form on $G \times_B \mathfrak{b}$.

(c) This splitting of $G \times_B \mathfrak{b}$ descends to a splitting of $\mathfrak{g}$ under the map $G \times_B \mathfrak{b} \to \mathfrak{g}$, $(g, x) \mapsto (\operatorname{Ad} g)x$.

Hint: Use the arguments in [MeVa-92a, Proof of Theorem 4.3] to show that it suffices to prove that $\hat{\alpha}$ descends to a function on $\mathfrak{g}$. To prove the latter assertion, show that the composite map $G \times B \xrightarrow{\pi_B} B \xrightarrow{\Phi} k$ descends to a map on G via the map $G \times B \to G$, $(g, b) \mapsto gbg^{-1}$, where π_B is the projection onto the B-factor.

(d) Let $P \subset G$ be any standard parabolic subgroup. Give a construction similar to that given above to split $G \times_P \mathfrak{u}_P$ using the splittings of $G \times_P U_P$ provided by Theorem 5.1.3.

(e) The splitting of $G \times_B B$ given by $\alpha\theta^{1-p}$ (up to a nonzero scalar multiple) as in Exercise (7) descends to a splitting of G via the map $G \times_B B \to G$, $(g, b) \mapsto gbg^{-1}$.

(9) Show the following negative results.

(a) Let $U_i \subset U$ be the unipotent radical of a standard minimal parabolic subgroup P_i of G. Then, none of the splittings of $G \times_B U$ provided by Theorem 5.1.3 compatibly split $G \times_B U_i$.

Hint: Show that, for any $f \in \mathrm{St} \otimes \mathrm{St}$ such that $\chi(f) \neq 0$, $\psi_B(f)$ does not identically vanish on $G \times x$, for any $x \in U$. Now, use Exercise 1.3.E.3.

(b) For any simple reflection s_i, $1 \leq i \leq \ell$, let $D_i \subset G/B$ be the corresponding Schubert divisor $\overline{Bw_o s_i B/B}$. Then, none of the splittings of $G \times_B U$ provided by Theorem 5.1.3 give a $\hat{D}_i$-splitting of $G \times_B U$, where $\hat{D}_i := \pi^{-1}(D_i)$; π being the quotient map $G \times_B U \to G/B$.

Hint: Show that, for any $f \in \mathrm{St} \otimes \mathrm{St}$, if $\psi_B(f)$ vanishes identically on $\hat{D}_i$ then it vanishes identically on $G \times_B U_i$ as well. Now, use the (a) part.

(10) Let $p \neq 2$. Then, show that for $G = \mathrm{SO}_n(k)$ and $\mathrm{Sp}_{2n}(k)$, the Cayley transform $C : U \to \mathfrak{u}$, $g \mapsto (1-g)(1+g)^{-1}$, is a B-equivariant isomorphism. So, C provides an explicit Springer isomorphism in these cases.

Hint: In fact, C is defined on square matrices x such that $1 + x$ is invertible. Moreover, for such an x, $1 + C(x)$ is again invertible and $C^2 x = x$.

(11) Let $B \subsetneqq P \subset G$ be a parabolic subgroup. Fix a volume form θ on $G \times_P P$ (which exists by Exercise (5)). Then, show that (up to a nonzero scalar multiple)

$$\theta_{|_{U_P^- \times (L_P^- \times U \times T)}} = t^{\beta} t^{-\rho} \theta_{U_P^-} \wedge \theta_{L_P^-} \wedge \theta_U \wedge \theta_T,$$

for some character t^{β} of T which extends to a character of L_P, where $L_P^- := L_P \cap U^-$.

Show further that, for any $f \in \mathrm{St} \otimes \mathrm{St}$, $\psi(f)\theta^{1-p}$ is *not* a splitting of $G \times_P P$, where $\psi(v \otimes w)(g, p) = \chi(v \otimes gpg^{-1}w)$.

Hint: For the first part, take the right invariant volume form θ_P on P. Then, $\theta_{|_{U_P^- \times P}} = \gamma\, \theta_{U_P^-} \wedge \theta_P$, where γ is a nowherevanishing function on P, thus descends to a function on P/U_P. Now, use Exercise (1).

For the second part, take weight vectors $v, w \in \mathrm{St}$. Then, for $g \in U_P^-$, $h \in L_P^-$, $u \in U$, $t \in T$,

$$\psi(v \otimes w)\,(g, hut) = \chi(g^{-1}v \otimes hut\, g^{-1}w).$$

The coefficient of the monomial $t^{(p-1)\rho}$ in $\big(\psi(v \otimes w)(g, hut)\big) \cdot t^{(p-1)(\rho-\beta)}$ is given by $\chi\Big(g^{-1}v \otimes hu(g^{-1}w)_{(p-1)\beta}\Big)$, where $(g^{-1}w)_{(p-1)\beta}$ denotes the component of $g^{-1}w$ in the weight space corresponding to the weight $(p-1)\beta$. Now, the monomial $\prod_{\alpha \in \Delta^+} x_\alpha^{p-1}$ (5.1.4) would occur in $\chi\big(g^{-1}v \otimes hu(g^{-1}w)_{(p-1)\beta}\big)$ with nonzero coefficient only if the weight spaces in St corresponding to the weights $(p-1)\beta$ and $(p-1)\beta + 2(p-1)\rho$ are both nonzero. This is possible only if $\beta = -\rho$. But $t^{-\rho}$ is not a character of L_P unless $P = B$.

5.2 Cohomology vanishing of cotangent bundles of flag varieties

We continue to follow the same notation as in the beginning of this chapter.

Let $\pi : T^*(G/B) \to G/B$ denote the projection and let

$$\mathcal{C} := \{\mu \in X^*(T) : \langle \mu, \beta^\vee \rangle \geq -1, \quad \text{for all } \beta \in \Delta^+\}.$$

We begin with the statement of one of the main results of this section.

5.2.1 Theorem. *Let the characteristic of k be a good prime for G. Then, for any* $\lambda \in \mathcal{C}$,

$$H^i\big(T^*(G/B), \pi^*\mathcal{L}(\lambda)\big) = 0, \quad \textit{for all } i > 0.$$

Before we come to the proof of the theorem, we need several preparatory results.

5.2.2 Lemma. *Let the characteristic of k be a good prime for G, or* 0*. Then, for any parabolic subgroup* $P \subset G$ *and any vector bundle* $\mathcal{V}$ *on* G/P*, there is a canonical isomorphism*

$$H^i\big(T^*(G/P), \pi_P^*\mathcal{V}\big) \simeq H^i\big(G/P, \mathcal{L}^P(S(\mathfrak{u}_P^*)) \otimes \mathcal{V}\big), \textit{ for all } i \geq 0,$$

where $S(\cdot)$ *denotes the symmetric algebra and* $\pi_P : T^*(G/P) \to G/P$ *is the projection.*

Proof. By Corollary 5.1.11, the cotangent bundle

(1) $$T^*(G/P) \simeq G \times_P \mathfrak{u}_P, \qquad \text{as } G\text{-equivariant vector bundles.}$$

(Here we have used the assumption that the characteristic of k is a good prime for G, or 0.)

The projection $\pi_P : T^*(G/P) \to G/P$ is clearly an affine morphism. Now, we use the projection formula [Har–77, Chap. II, Exercise 5.1(d)] and a degenerate case of the Leray spectral sequence [Har–77, Chap. III, Exercise 8.2] to get

$$H^i\big(T^*(G/P), \pi_P^*\mathcal{V}\big) \simeq H^i\big(G/P, \pi_{P*}(\mathcal{O}_{T^*(G/P)}) \otimes \mathcal{V}\big).$$

The fiber of π_P at the base point eP can be identified with $\mathfrak{u}_P$ (under the identification (1)). Thus, from the G-equivariance, the sheaf $\pi_{P*}\big(\mathcal{O}_{T^*(G/P)}\big)$ can be identified with the homogeneous vector bundle $\mathcal{L}^P(S(\mathfrak{u}_P^*))$. □

5.2.3 Definition. (Koszul resolution) Let

$$0 \to V' \overset{p_1}{\to} V \overset{p_2}{\to} V'' \to 0$$

be a short exact sequence of vector spaces (over any field k). For any $n > 0$, consider the sequence:

$$
\begin{aligned}
&0 \to \wedge^n(V') \to \cdots \to S^{n-i}(V) \otimes \wedge^i(V') \xrightarrow{\delta_{i-1}} S^{n-i+1}(V) \otimes \wedge^{i-1}(V') \\
(1) \qquad &\to \cdots \to S^{n-1}(V) \otimes V' \xrightarrow{\delta_0} S^n(V) \xrightarrow{\hat{p}_2} S^n(V'') \to 0,
\end{aligned}
$$

where $\hat{p}_2$ is induced by the map p_2 and $\delta_{i-1} : S^{n-i}(V) \otimes \wedge^i(V') \to S^{n-i+1}(V) \otimes \wedge^{i-1}(V')$ is defined by

$$
(2) \qquad \delta_{i-1}(f \otimes v_1 \wedge \cdots \wedge v_i) = \sum_{j=1}^{i} (-1)^{j-1}(p_1 v_j) f \otimes v_1 \wedge \cdots \wedge \hat{v}_j \wedge \cdots \wedge v_i.
$$

Then, as is well known, the above sequence (1) is an exact complex called the *Koszul complex* (cf. [Ser–89, Chap. IV.A]). The direct sum of the resolutions (1) over all $n \geq 0$ provides a resolution of the $S(V)$-module $S(V'') \simeq S(V)/V' \cdot S(V)$, called the *Koszul resolution* of the $S(V)$-module $S(V'')$.

5.2.4 Lemma. *Let k be a field of an arbitrary characteristic (including 0). Let P_α be the standard minimal parabolic subgroup corresponding to a simple root α and let $\lambda \in X^*(T)$ satisfy $\langle \lambda, \alpha^\vee \rangle = -1$. Then, for any P_α-module V,*

$$
H^j\big(G/B, \mathcal{L}(V) \otimes \mathcal{L}(\lambda)\big) = 0, \quad \textit{for all } j \geq 0.
$$

Proof. For the projection $f : G/B \to G/P_\alpha$, since $f^*\mathcal{L}^{P_\alpha}(V) \simeq \mathcal{L}(V)$, by the projection formula [Har–77, Chap. III, Exercise 8.3],

$$
(1) \qquad R^j f_*(\mathcal{L}(V) \otimes \mathcal{L}(\lambda)) \simeq \mathcal{L}^{P_\alpha}(V) \otimes R^j f_*(\mathcal{L}(\lambda)).
$$

But, from the G-equivariance, $R^j f_*(\mathcal{L}(\lambda))$ is the homogeneous vector bundle on G/P_α associated to the P_α-module $H^j(P_\alpha/B, \mathcal{L}(\lambda)_{|P_\alpha/B})$. Since $\mathcal{L}(\lambda)_{|P_\alpha/B}$, by assumption, is a line bundle of degree -1 on the projective line $\mathbb{P}^1 \simeq P_\alpha/B$,

$$
H^j\big(P_\alpha/B, \mathcal{L}(\lambda)_{|P_\alpha/B}\big) = 0, \quad \text{for all } j \geq 0.
$$

Thus, by (1),

$$
R^j f_*(\mathcal{L}(V) \otimes \mathcal{L}(\lambda)) = 0, \quad \text{for all } j \geq 0.
$$

Hence, the lemma follows from the Leray spectral sequence associated to the morphism f. $\square$

5.2.5 Corollary. *Let k be a field of an arbitrary characteristic. Suppose that $\lambda \in \mathcal{C}$ and $\langle \lambda, \alpha^\vee \rangle = -1$ for a simple root α. Then, $s_\alpha \lambda \in \mathcal{C}$ and*

$$
(1) \qquad H^i\big(G/B, \mathcal{L}(S^n(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda)\big) \simeq H^i\big(G/B, \mathcal{L}(S^{n-1}(\mathfrak{u}^*)) \otimes \mathcal{L}(s_\alpha\lambda)\big)
$$

for all $i \geq 0$ and $n > 0$.

Proof. As s_α permutes $\Delta^+\backslash\{\alpha\}$ and $s_\alpha\alpha = -\alpha$, we get that $s_\alpha\lambda \in \mathcal{C}$.

For the isomorphism (1), apply the Koszul resolution (5.2.3.1) to the short exact sequence of B-modules:

$$0 \to k_{-\alpha} \to \mathfrak{u}^* \to \mathfrak{u}^*_{P_\alpha} \to 0,$$

to obtain the exact sequence of B-modules:

$$0 \to S^{n-1}(\mathfrak{u}^*) \otimes k_{-\alpha} \to S^n(\mathfrak{u}^*) \to S^n(\mathfrak{u}^*_{P_\alpha}) \to 0.$$

Thus, we get an exact sequence of homogeneous vector bundles on G/B:

$$0 \to \mathcal{L}(S^{n-1}(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda+\alpha) \to \mathcal{L}(S^n(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda) \to \mathcal{L}(S^n(\mathfrak{u}^*_{P_\alpha})) \otimes \mathcal{L}(\lambda) \to 0.$$

Now, taking the associated long exact cohomology sequence and applying Lemma 5.2.4 (to get the vanishing $H^j\big(G/B, \mathcal{L}(S^n(\mathfrak{u}^*_{P_\alpha})) \otimes \mathcal{L}(\lambda)\big) = 0$, for all $j \geq 0$), we obtain (1). (Observe that, by assumption since $\langle\lambda, \alpha^\vee\rangle = -1$, $s_\alpha\lambda = \lambda + \alpha$.) $\square$

5.2.6 Relative Kähler Differentials. For a scheme over any field k, recall the definition of the sheaf of Kähler differentials $\Omega^1_X = \Omega_{X/k}$ from [Har–77, Chap. II, §8]. For any k-morphism $f : X \to Y$ between schemes, the *sheaf of relative Kähler differentials* $\Omega^1_{X/Y}$ (which we will abbreviate as Ω^1_f) is defined as the quotient sheaf $\Omega^1_X / i(f^*\Omega^1_Y)$, where $i : f^*\Omega^1_Y \to \Omega^1_X$ is the canonical map. Thus, these fit into an exact sequence of sheaves on X:

(1) $$f^*\Omega^1_Y \xrightarrow{i} \Omega^1_X \to \Omega^1_f \to 0.$$

Assume further that X and Y are smooth varieties and f is a smooth morphism. In this case, i is injective and Ω^1_X and Ω^1_f are locally free $\mathcal{O}_X$-modules (cf. [Har–77, Chap. III, Proposition 10.4]). Fix $j \geq 0$ and let $\Omega^j_X := \wedge^j_{\mathcal{O}_X}(\Omega^1_X)$ be the sheaf of differential j-forms on X. Then, by (1) and [Har–77, Chap. II, Exercise 5.16], there exists a decreasing filtration $\big\{\mathcal{F}^f_i(\Omega^j_X)\big\}_{i\geq 0}$ of Ω^j_X by locally free $\mathcal{O}_X$-submodules such that the associated graded sheaf

(2) $$\mathrm{Gr}(\mathcal{F}^f(\Omega^j_X)) \simeq \bigoplus_{i\geq 0} f^*(\Omega^i_Y) \otimes \Omega^{j-i}_f,$$

where $\Omega^s_f := \wedge^s_{\mathcal{O}_X}(\Omega^1_f)$.

For dominant regular λ, the following lemma is a special case of the Serre vanishing theorem [Har–77, Chap. III, Proposition 5.3].

5.2.7 Lemma. *Let λ be a dominant weight. Then, there exists $m_o = m_o(\lambda)$ such that for any $i > j$,*

(1) $$H^i\big(G/B, \Omega^j_{G/B} \otimes \mathcal{L}(m\lambda)\big) = 0, \quad \textit{for all } m \geq m_o.$$

Proof. For $\lambda = 0$, from the diagonality of the Hodge cohomology [Jan–03, Part II, Proposition 6.18],

$$H^i\big(G/B, \Omega^j_{G/B}\big) = 0 \quad \text{for all } i \neq j. \tag{2}$$

Thus, (1) is true in this case.

For $\lambda \neq 0$, let $B \subset P \subset G$ be the (unique) parabolic subgroup such that $\mathcal{L}(\lambda)$ is the pullback of an ample line bundle $\mathcal{L}^P(\lambda)$ on G/P via the projection $f : G/B \to G/P$. (Of course, $P = P_I$, where $I := \{1 \leq q \leq \ell : \lambda(\alpha_q^\vee) = 0\}$.) Applying (5.2.6.2) to the (smooth) morphism f, to prove (1), it suffices to show that for $i > j$,

$$H^i\big(G/B, f^*\Omega^r_{G/P} \otimes \Omega_f^{j-r} \otimes \mathcal{L}(m\lambda)\big) = 0, \quad \text{for all } 0 \leq r \leq j \text{ and all } m \geq m_o. \tag{3}$$

By the projection formula, the $E_2^{s,t}$ term of the Leray spectral sequence for the fibration f may be written as

$$E_2^{s,t} = H^s\big(G/P, \Omega^r_{G/P} \otimes \mathcal{L}^P(m\lambda) \otimes R^t f_* \Omega_f^{j-r}\big),$$

which converges to $H^*\big(G/B, f^*\Omega^r_{G/P} \otimes \Omega_f^{j-r} \otimes \mathcal{L}(m\lambda)\big)$.

Since $\mathcal{L}^P(\lambda)$ is an ample line bundle on G/P, by the Serre vanishing theorem [Har–77, Chap. III, Proposition 5.3], there exists m_o such that for all $m \geq m_o$,

$$H^s\big(G/P, \Omega^r_{G/P} \otimes \mathcal{L}^P(m\lambda) \otimes R^t f_* \Omega_f^{j-r}\big) = 0, \tag{4}$$

for all $s > 0$, and all $j, r, t \geq 0$.

Further, $R^t f_* \Omega_f^{j-r}$ is the homogeneous vector bundle on G/P associated to the P-module $H^t\big(P/B, \Omega^{j-r}_{P/B}\big)$. Again, the diagonality of the Hodge cohomology (this time for P/B) gives

$$H^t\big(P/B, \Omega^{j-r}_{P/B}\big) = 0, \quad \text{unless } t = j - r. \tag{5}$$

Combining (4) and (5), we obtain

$$H^s\big(G/P, \Omega^r_{G/P} \otimes \mathcal{L}^P(m\lambda) \otimes R^t f_* \Omega_f^{j-r}\big) = 0,$$

unless $s = 0$ and $t = j - r$. Thus,

$$E_2^{s,t} = 0, \quad \text{unless } s = 0 \text{ and } t = j - r.$$

Hence,

$$H^i\big(G/B, f^*\Omega^r_{G/P} \otimes \Omega_f^{j-r} \otimes \mathcal{L}(m\lambda)\big) = 0,$$

for all $m \geq m_o$, unless $i = j - r$. This proves (3) and thus (1) for any $i > j$. $\square$

With this preparation, we are ready to prove Theorem 5.2.1.

5.2.8 *Proof of Theorem 5.2.1.* We begin by proving the following for any dominant λ.

$$H^i(G/B, \mathcal{L}(S(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda)) = 0, \quad \text{for all } i > 0. \tag{1}$$

Consider the Koszul resolution (5.2.3.1) for any $n > 0$:

$$\begin{aligned} 0 \to \wedge^n((\mathfrak{b}/\mathfrak{u})^*) \to \cdots \to S^{n-s}(\mathfrak{b}^*) \otimes \wedge^s((\mathfrak{b}/\mathfrak{u})^*) \xrightarrow{\delta_{s-1}} \cdots \\ \to S^{n-1}(\mathfrak{b}^*) \otimes (\mathfrak{b}/\mathfrak{u})^* \xrightarrow{\delta_0} S^n(\mathfrak{b}^*) \to S^n(\mathfrak{u}^*) \to 0, \end{aligned} \tag{2}$$

corresponding to the short exact sequence of B-modules:

$$0 \to (\mathfrak{b}/\mathfrak{u})^* \to \mathfrak{b}^* \to \mathfrak{u}^* \to 0.$$

Let $K_s := \text{Image } \delta_s$. Then, the exact sequence (2) breaks up into several short exact sequences of B-modules (for any $s \geq 1$):

$$0 \to K_s \to S^{n-s}(\mathfrak{b}^*) \otimes \wedge^s((\mathfrak{b}/\mathfrak{u})^*) \xrightarrow{\delta_{s-1}} K_{s-1} \to 0, \text{ and} \tag*{$(3)_s$}$$

$$0 \to K_0 \to S^n(\mathfrak{b}^*) \to S^n(\mathfrak{u}^*) \to 0. \tag{4}$$

Fix a dominant λ and assume that

$$H^i(G/B, \mathcal{L}(S(\mathfrak{b}^*)) \otimes \mathcal{L}(\lambda)) = 0 \quad \text{for all } i > 0. \tag{5}$$

Considering the long exact cohomology sequences corresponding to the short exact sequences $(3)_s$ and (4) of B-modules tensored with the B-module $k_{-\lambda}$ and using assumption (5) and the fact that $\mathfrak{b}/\mathfrak{u}$ is a trivial B-module, we see that for any $i > 0$ and any $s \geq 1$,

$$\begin{aligned} H^{i+1}(G/B, \mathcal{L}(K_s) \otimes \mathcal{L}(\lambda)) &\simeq H^i(G/B, \mathcal{L}(K_{s-1}) \otimes \mathcal{L}(\lambda)), \text{ and} \\ H^{i+1}(G/B, \mathcal{L}(K_0) \otimes \mathcal{L}(\lambda)) &\simeq H^i(G/B, \mathcal{L}(S^n(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda)). \end{aligned}$$

Thus, by iteration, for any $s \geq 0$,

$$H^i(G/B, \mathcal{L}(S^n(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda)) \simeq H^{i+s+1}(G/B, \mathcal{L}(K_s) \otimes \mathcal{L}(\lambda)). \tag{6}$$

Since $K_s = 0$ for large enough s, from (6), we deduce that

$$H^i(G/B, \mathcal{L}(S^n(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda)) = 0, \quad \text{for any } i > 0. \tag{7}$$

Thus, (1) follows for (dominant) λ if we prove (5), which we prove now.

Again, fix any $n > 0$ and consider the Koszul resolution:

$$\begin{aligned} 0 \to \wedge^n((\mathfrak{g}/\mathfrak{b})^*) \to \cdots \to S^{n-s}(\mathfrak{g}^*) \otimes \wedge^s((\mathfrak{g}/\mathfrak{b})^*) \xrightarrow{\hat{\delta}_{s-1}} \cdots \\ \to S^{n-1}(\mathfrak{g}^*) \otimes (\mathfrak{g}/\mathfrak{b})^* \xrightarrow{\hat{\delta}_0} S^n(\mathfrak{g}^*) \to S^n(\mathfrak{b}^*) \to 0, \end{aligned} \tag{8}$$

corresponding to the short exact sequence of B-modules:

$$0 \to (\mathfrak{g}/\mathfrak{b})^* \to \mathfrak{g}^* \to \mathfrak{b}^* \to 0.$$

By Lemma 5.2.7, there exists m_o such that

$$(9) \qquad H^t(G/B, \Omega^j_{G/B} \otimes \mathcal{L}(m\lambda)) = 0, \text{ for } t > j \text{ and all } m \geq m_o.$$

Thus, by [Jan–03, Part I, Proposition 4.8],

$$(10) \qquad H^t\Big(G/B, \mathcal{L}\big(S(\mathfrak{g}^*) \otimes \wedge^j((\mathfrak{g}/\mathfrak{b})^*)\big) \otimes \mathcal{L}(m\lambda)\Big) = 0,$$

for all $t > j$ and $m \geq m_0$, since

$$(11) \qquad \Omega^j_{G/B} \simeq \mathcal{L}\big(\wedge^j((\mathfrak{g}/\mathfrak{b})^*)\big).$$

Considering the short exact sequences similar to $(3)_s$ and (4) (obtained from the Koszul resolution (8)) and using (10) we obtain (similarly to (6)), for any $s \geq 0$ and $i > 0$,

$$(12) \qquad H^i\big(G/B, \mathcal{L}(S^n(\mathfrak{b}^*)) \otimes \mathcal{L}(m\lambda)\big) \simeq H^{i+s+1}\big(G/B, \mathcal{L}(\hat{K}_s) \otimes \mathcal{L}(m\lambda)\big),$$

where $\hat{K}_s :=$ Image $\hat{\delta}_s$. Taking s large enough, we obtain (for any $n > 0$)

$$(13) \qquad H^i\big(G/B, \mathcal{L}(S^n(\mathfrak{b}^*)) \otimes \mathcal{L}(m\lambda)\big) = 0, \text{ for all } i > 0 \text{ and } m \geq m_o.$$

The above vanishing for $n = 0$ is a particular case of (9).

This proves (5) for λ replaced by $m\lambda$ (for any $m \geq m_o$) and thus we obtain

$$(14) \qquad H^i\big(G/B, \mathcal{L}(S(\mathfrak{u}^*)) \otimes \mathcal{L}(m\lambda)\big) = 0, \text{ for all } i > 0 \text{ and } m \geq m_o.$$

(Observe that, so far, we have not used the assumption that the characteristic of k is a good prime for G, which we use now.) By Lemma 5.2.2,

$$(15) \qquad H^i\big(T^*(G/B), \pi^*\mathcal{L}(m\lambda)\big) \simeq H^i\big(G/B, \mathcal{L}(S(\mathfrak{u}^*)) \otimes \mathcal{L}(m\lambda)\big).$$

Thus, by (14), $H^i(T^*(G/B), \pi^*\mathcal{L}(m\lambda)) = 0$, for all $i > 0$ and $m \geq m_o$. Since $T^*(G/B)$ is split (Corollary 5.1.12), by Lemma 1.2.7(i),

$$H^i\big(T^*(G/B), \pi^*\mathcal{L}(\lambda)\big) = 0.$$

This proves the theorem for dominant λ.

Finally, we prove the theorem for an arbitrary $\lambda \in \mathcal{C}$. Take $n \geq 0$ and assume by induction that $H^i\big(G/B, \mathcal{L}(S^j(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda)\big) = 0$, for all $i > 0$ and $j < n$. If λ is dominant, then (as proved above) $H^i\big(G/B, \mathcal{L}(S^n(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda)\big) = 0$. So, assume that $\lambda \in \mathcal{C}$ is not dominant. Then, there exists a simple root α such that $\langle \lambda, \alpha^\vee \rangle = -1$. For $n = 0$, $H^i(G/B, \mathcal{L}(\lambda)) = 0$ by Lemma 5.2.4. For $n > 0$, by Corollary 5.2.5,

$$H^i\big(G/B, \mathcal{L}(S^n(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda)\big) \simeq H^i\big(G/B, \mathcal{L}(S^{n-1}(\mathfrak{u}^*)) \otimes \mathcal{L}(s_\alpha\lambda)\big),$$

and $s_\alpha\lambda \in \mathcal{C}$. Thus, by induction, $H^i\big(G/B, \mathcal{L}(S^n(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda)\big) = 0$.

This finishes the proof of the theorem by Lemma 5.2.2. □

As a consequence of Theorem 5.2.1 and its proof, we obtain the following Dolbeault vanishing.

5.2.9 Theorem. *Let the characteristic of k be a good prime for G. Then, for any $\lambda \in \mathcal{C}$,*

$$H^i\big(G/B, \Omega^j_{G/B} \otimes \mathcal{L}(\lambda)\big) = 0, \textit{ for all } i > j. \tag{1}$$

Proof. We first show that

$$H^i\big(G/B, \mathcal{L}(S(\mathfrak{b}^*)) \otimes \mathcal{L}(\lambda)\big) = 0, \text{ for all } i > 0. \tag{2}$$

As in the proof of Theorem 5.2.1, consider the exact sequence of B-modules:

$$0 \to (\mathfrak{b}/\mathfrak{u})^* \to \mathfrak{b}^* \to \mathfrak{u}^* \to 0.$$

Associated to the above sequence, for any $r \geq 0$, there is a decreasing filtration (by B-submodules):

$$S^r(\mathfrak{b}^*) = \mathcal{F}_0 \supset \mathcal{F}_1 \supset \cdots \supset \mathcal{F}_r \supset \mathcal{F}_{r+1} = (0)$$

such that

$$\mathrm{Gr}\,\mathcal{F} = \bigoplus_{j\geq 0} S^j((\mathfrak{b}/\mathfrak{u})^*) \otimes S^{r-j}(\mathfrak{u}^*)$$

(cf. [Har–77, Chap. II, Exercise 5.16]).

Thus, to prove (2), it suffices to show that

$$H^i\big(G/B, \mathcal{L}(S((\mathfrak{b}/\mathfrak{u})^*)) \otimes \mathcal{L}(S(\mathfrak{u}^*)) \otimes \mathcal{L}(\lambda)\big) = 0, \text{ for all } i > 0. \tag{3}$$

But since $\mathfrak{b}/\mathfrak{u}$ is a trivial B-module, (3) follows from Theorem 5.2.1 and Lemma 5.2.2.

Fix $n \geq 1$ and assume by induction that

$$H^i\big(G/B, \mathcal{L}(\wedge^j((\mathfrak{g}/\mathfrak{b})^*)) \otimes \mathcal{L}(\lambda)\big) = 0 \text{ for all } i > j \text{ and } j < n. \tag{4}$$

(Observe that $H^i(G/B, \mathcal{L}(\lambda)) = 0$, for $i > 0$, as this is a particular case of (3).)

Recall the Koszul resolution from (5.2.8.8):

$$0 \to \wedge^n((\mathfrak{g}/\mathfrak{b})^*) \to \cdots \to S^{n-s}(\mathfrak{g}^*) \otimes \wedge^s((\mathfrak{g}/\mathfrak{b})^*) \xrightarrow{\hat{\delta}_{s-1}} \cdots \to$$
$$S^{n-1}(\mathfrak{g}^*) \otimes (\mathfrak{g}/\mathfrak{b})^* \xrightarrow{\hat{\delta}_0} S^n(\mathfrak{g}^*) \to S^n(\mathfrak{b}^*) \to 0.$$

As in 5.2.8, break this into short exact sequences (for any $s \geq 1$):

$$0 \to \hat{K}_s \to S^{n-s}(\mathfrak{g}^*) \otimes \wedge^s((\mathfrak{g}/\mathfrak{b})^*) \xrightarrow{\hat{\delta}_{s-1}} \hat{K}_{s-1} \to 0, \text{ and} \tag{5$_s$}$$

$$0 \to \hat{K}_0 \to S^n(\mathfrak{g}^*) \to S^n(\mathfrak{b}^*) \to 0, \tag{6}$$

where $\hat{K}_s :=$ Image $\hat{\delta}_s$. Then,

$$\hat{K}_n = 0 \quad \text{and} \quad \hat{K}_{n-1} \simeq \wedge^n((\mathfrak{g}/\mathfrak{b})^*).$$

From the long exact cohomology sequences associated to $(5)_s$ tensored with $k_{-\lambda}$ (for $1 \le s \le n-1$), and using the induction hypothesis (4), we get (for any $i > n$):

$$H^i\big(G/B, \mathcal{L}(\wedge^n((\mathfrak{g}/\mathfrak{b})^*)) \otimes \mathcal{L}(\lambda)\big) \simeq H^{i-1}\big(G/B, \mathcal{L}(\hat{K}_{n-2}) \otimes \mathcal{L}(\lambda)\big) \simeq$$
$$\cdots \simeq H^{i-n+1}\big(G/B, \mathcal{L}(\hat{K}_0) \otimes \mathcal{L}(\lambda)\big) \simeq H^{i-n}\big(G/B, \mathcal{L}(S^n(\mathfrak{b}^*)) \otimes \mathcal{L}(\lambda)\big),$$

where the last isomorphism is obtained from (6). The last term is 0 by (2) and hence so is the first term. This completes the induction and we get

$$H^i\big(G/B, \mathcal{L}(\wedge^j((\mathfrak{g}/\mathfrak{b})^*)) \otimes \mathcal{L}(\lambda)\big) = 0, \text{ for all } i > j.$$

Now, the theorem follows from (5.2.8.11). □

5.2.10 Remark. We obtained the Dolbeault vanishing Theorem 5.2.9 from Theorem 5.2.1 and its proof. We can reverse the steps in the proof of Theorem 5.2.9 and obtain Theorem 5.2.1 as a consequence.

We extend a slightly weaker version of Theorem 5.2.1 to an arbitrary parabolic subgroup.

5.2.11 Theorem. *Assume that the characteristic of k is a good prime for G. Then, for any standard parabolic subgroup $P \subset G$ and a dominant weight $\lambda \in X^*(P)$ such that $\mathcal{L}^P(\lambda)$ is ample on G/P (equivalently, if $\lambda - \rho_P$ remains dominant; see Exercise 3.1.E.1),*

(1) $$H^i\big(T^*(G/P), \pi_P^* \mathcal{L}^P(\lambda)\big) \simeq H^i\big(G/P, \mathcal{L}^P(S(\mathfrak{u}_P^*)) \otimes \mathcal{L}^P(\lambda)\big) = 0,$$

for all $i > 0$.

Proof. The first isomorphism in (1) of course is a particular case of Lemma 5.2.2.

Regard $T^*(G/P) \simeq G \times_P \mathfrak{u}_P$ (Corollary 5.1.11) as a closed subvariety of

(2) $$G \times_P \mathfrak{g} \simeq G/P \times \mathfrak{g},$$

where the last isomorphism is given by $(g, x) \mapsto (gP, \operatorname{Ad} g(x))$. Then, the projection onto the second factor gives rise to a proper morphism $\alpha : T^*(G/P) \to \mathfrak{g}$. Let $\epsilon_{\mathfrak{g}}$ be the trivial line bundle on $\mathfrak{g}$. Then, $\pi_P^* \mathcal{L}^P(\lambda)$ can be identified with the restriction of the product line bundle $\mathcal{L}^P(\lambda) \boxtimes \epsilon_{\mathfrak{g}}$ to $T^*(G/P)$ under the identification (2). In particular, $\pi_P^* \mathcal{L}^P(\lambda)$ is ample on $T^*(G/P)$. Thus, the vanishing of $H^i\big(T^*(G/P), \pi_P^* \mathcal{L}^P(\lambda)\big)$, for $i > 0$, follows from Theorem 1.2.8(i) and Corollary 5.1.12. This proves the theorem. □

The following result follows from the corresponding result in characteristic $p > 0$ and the semicontinuity theorem.

5.2.12 Theorem. *Theorems 5.2.1, 5.2.9 and 5.2.11 remain true over an algebraically closed field of characteristic* 0.

Proof. For Theorem 5.2.1, resp. 5.2.11, in characteristic 0, use Lemma 5.2.2, Theorem 5.2.1, resp. 5.2.11, and Proposition 1.6.2. For Theorem 5.2.9 in characteristic 0, use the same proof as that of Theorem 5.2.9 (and use Theorem 5.2.1 in characteristic 0). □

5.2.E Exercises

(1) Let X be a smooth projective variety over any field k and let T_X be its tangent bundle. Assume that there is an exact sequence of vector bundles:

$$0 \to K \to \varepsilon_X \to T_X \to 0,$$

where ε_X is a trivial vector bundle on X (for some vector bundle K on X). Then, for any vector bundle $\mathcal{S}$ on X, prove that the following two assertions are equivalent for any fixed $t \geq 0$.

(a) $H^q(X, \Omega_X^p \otimes \mathcal{S}) = 0$, for all $q - p > t$.

(b) $H^q(X, S^j(K^*) \otimes \mathcal{S}) = 0$, for all $j \geq 0$ and $q > t$.

Hint: Use an appropriate Koszul resolution and ideas similar to those used in the proofs of Theorems 5.2.1 and 5.2.9.

(2) Give an alternative proof of Theorem 5.2.11 along the following lines.

Considering the Koszul resolution corresponding to the short exact sequence of P-modules:

$$0 \to (\mathfrak{g}/\mathfrak{u}_P)^* \to \mathfrak{g}^* \to \mathfrak{u}_P^* \to 0,$$

first show that, for any fixed $i \geq 0$, if

$$H^{i+j}\big(G/P, \mathcal{L}(\wedge^j((\mathfrak{g}/\mathfrak{u}_P)^*)) \otimes \mathcal{L}^P(\lambda)\big) = 0, \text{ for all } j \geq 0,$$

then so is

$$H^i(T^*(G/P), \pi_P^* \mathcal{L}^P(\lambda)) = 0.$$

Now, since $\mathcal{L}^P(\lambda)$ is an ample line bundle on G/P, there exists a large enough m_o such that $H^i\big(G/P, \mathcal{L}(\wedge^j((\mathfrak{g}/\mathfrak{u}_P)^*)) \otimes \mathcal{L}^P(m\lambda)\big) = 0$, for all $i > 0$, $j \geq 0$ and $m \geq m_o$. Thus, we get the vanishing

$$H^i(T^*(G/P), \pi_P^*\mathcal{L}^P(m\lambda)) = 0, \text{ for all } i > 0.$$

Finally, use the splitting of $T^*(G/P)$ to complete the proof.

5.3 Geometry of the nilpotent and subregular cones

In this section we assume that G is a simple (connected, simply-connected) group and the characteristic of k is a good prime for G or 0. Recall that by a desingularization, or resolution, of a variety Y, we mean a nonsingular variety $\widetilde{Y}$ together with a proper birational morphism $\widetilde{Y} \to Y$. We will also use the notion of a rational resolution, defined in 3.4.1.

We now review basic definitions and properties of the unipotent, resp. nilpotent varieties and their subregular varieties as given in, e.g., [Hum–95b, Chapter 6]. Let $\mathcal{U}$ be the *unipotent variety* of G, i.e., the closed subset of G (with the reduced scheme structure) consisting of all the unipotent elements of G. Then, $\mathcal{U}$ is irreducible, and normal by [Spr–69, §1]. Further, the map

$$\phi : G \times_B U \to \mathcal{U}, \quad (g, u) \mapsto gug^{-1},$$

is a G-equivariant desingularization, called the *Springer resolution*, where G acts on $\mathcal{U}$ via conjugation. The unipotent variety $\mathcal{U}$ contains a dense open G-orbit $\mathcal{U}^{\text{reg}}$ consisting of the regular unipotent elements. Moreover, the complement $\mathfrak{S} := \mathcal{U} \setminus \mathcal{U}^{\text{reg}}$ is an irreducible closed subset of $\mathcal{U}$, which we endow with the (reduced) subvariety structure. The subvariety $\mathfrak{S}$ is called the *subregular variety*. It contains a dense open G-orbit $\mathfrak{S}^{\text{reg}}$.

Similarly, let $\mathcal{N} \subset \mathfrak{g}$ be the *nilpotent cone* consisting of all the nilpotent elements. Then, $\mathcal{N}$ is closed, irreducible, and invariant under scalar multiplication. We endow $\mathcal{N}$ with the closed subvariety structure; then, $\mathcal{N}$ is normal as well. In fact, $\mathcal{U}$ and $\mathcal{N}$ are isomorphic as G-varieties under the adjoint actions (cf. [BaRi–85, Corollary 9.3.4]). As a consequence, the map

$$\widetilde{\phi} : G \times_B \mathfrak{u} \to \mathcal{N}, \quad (g, X) \mapsto \operatorname{Ad} g \cdot X,$$

is a G-equivariant resolution, called the *Springer resolution*. Further, $\mathcal{N}$ contains a dense open G-orbit $\mathcal{N}^{\text{reg}}$ consisting of the regular nilpotent elements. Moreover, the complement $\mathcal{S} := \mathcal{N} \setminus \mathcal{N}^{\text{reg}}$ is an irreducible closed subset of $\mathcal{N}$, invariant under scalar multplication, and containing a dense open G-orbit $\mathcal{S}^{\text{reg}}$. Endowed with the (reduced) subvariety structure, $\mathcal{S}$ is called the *subregular cone*. By [BaRi–85, Corollary 9.3.4] again, $\mathfrak{S}$ and $\mathcal{S}$ are isomorphic as G-varieties.

Let $P = P_\alpha$ be the minimal parabolic subgroup associated to a short simple root α. Then, $\phi_{|G \times_B U_P}$ factors through

$$\phi_\alpha : G \times_P U_P \to \mathcal{U},$$

with image exactly equal to $\mathfrak{S}$.

Similarly, the restriction of $\widetilde{\phi}$ to $G \times_B \mathfrak{u}_P$ factors through $G \times_P \mathfrak{u}_P$ to give the map

$$\widetilde{\phi}_\alpha : G \times_P \mathfrak{u}_P \to \mathcal{N},$$

with image exactly equal to $\mathcal{S}$.

5.3.1 Lemma. *Let α be a short simple root. Then, all the maps ϕ, ϕ_α, $\widetilde{\phi}$ and $\widetilde{\phi}_\alpha$ are projective morphisms.*

Further, ϕ_α and $\widetilde{\phi}_\alpha$ are both birational onto their images $\mathfrak{S}$ and $\mathcal{S}$ respectively.

Proof. The map ϕ is the composition

$$G \times_B U \hookrightarrow G \times_B G \simeq G/B \times G \xrightarrow{\pi_2} G\,,$$

where the first map is the standard closed embedding, the second isomorphism is defined by $(g_1, g_2) \mapsto (g_1 B, g_1 g_2 g_1^{-1})$ and the last map π_2 is the projection onto the second factor. Since π_2 is a projective morphism, so is ϕ. The assertion that the maps ϕ_α, $\widetilde{\phi}$ and $\widetilde{\phi}_\alpha$ are projective morphisms can be proved similarly.

It is a consequence of [Ste–74, Theorem 2 on page 153] that ϕ_α is bijective over $\mathfrak{S}^{\text{reg}}$. Indeed, by [loc cit.], any element $g \in \mathfrak{S}^{\text{reg}}$ lies in the unipotent radical of a unique conjugate of P_α. Now, to show that ϕ_α is birational, it suffices to recall from [SpSt–70] that the orbit maps are separable for unipotent conjugacy classes. The proof for $\widetilde{\phi}_\alpha$ is similar. □

5.3.2 Theorem. *The nilpotent cone $\mathcal{N}$ and the subregular cone $\mathcal{S}$ are normal Gorenstein varieties. Further, $\widetilde{\phi}$ and $\widetilde{\phi}_\alpha$ are rational resolutions, where α is any short simple root.*

Proof. As mentioned above, the unipotent variety $\mathcal{U}$, and hence the nilpotent cone $\mathcal{N}$, is normal. We begin by proving that $\widetilde{\phi}$ is a rational resolution. In characteristic 0, this follows at once from the triviality of the canonical bundle of $G \times_B \mathfrak{u}$ (Lemma 5.1.1) together with the Grauert–Riemenschneider vanishing theorem (cf. [GrRi–70]). In positive characteristics, by Lemma 5.1.1 again, it suffices to show that $R^i \widetilde{\phi}_*(\mathcal{O}_{G \times_B \mathfrak{u}}) = 0$ for all $i > 0$. Since $\mathcal{N}$ is affine, this is equivalent to

$$H^i(G \times_B \mathfrak{u}, \mathcal{O}_{G \times_B \mathfrak{u}}) = 0, \text{ for all } i > 0,$$

cf. [Har–77, Chap. III, Proposition 8.5]. But, $G \times_B \mathfrak{u} = T^*(G/B)$ by Corollary 5.1.11; thus, Theorem 5.2.1 yields the desired vanishing. This completes the proof of the rationality of the resolution $\widetilde{\phi}$. Together with Lemma 3.4.2 and the triviality of the canonical bundle of $G \times_B \mathfrak{u}$ again, it follows that the nilpotent cone $\mathcal{N}$ is Gorenstein.

Next, we turn to the subregular cone. We first show that the natural restriction map

(1) $$k[G \times_B \mathfrak{u}] \to k[G \times_B \mathfrak{u}_P] \text{ is surjective,}$$

where $P = P_\alpha$ is the minimal parabolic subgroup corresponding to any short simple root α.

Fix $n > 0$ and consider the exact sequence of B-modules (proof of Corollary 5.2.5):

$$0 \to S^{n-1}(\mathfrak{u}^*) \otimes k_{-\alpha} \to S^n(\mathfrak{u}^*) \to S^n(\mathfrak{u}_P^*) \to 0.$$

This gives rise to the long exact cohomology sequence:

(2) $$\begin{aligned} 0 &\to H^0\big(G/B, \mathcal{L}(S^{n-1}(\mathfrak{u}^*)) \otimes \mathcal{L}(\alpha)\big) \to H^0\big(G/B, \mathcal{L}(S^n(\mathfrak{u}^*))\big) \to \\ &H^0\big(G/B, \mathcal{L}(S^n(\mathfrak{u}_P^*))\big) \to H^1\big(G/B, \mathcal{L}(S^{n-1}(\mathfrak{u}^*)) \otimes \mathcal{L}(\alpha)\big) \to \cdots . \end{aligned}$$

By [Bou–81, Page 278], any positive short root belongs to $\mathcal{C}$, where $\mathcal{C}$ is defined in the beginning of Section 5.2. By Theorem 5.2.1 in characteristic $p > 0$ and Theorem 5.2.12 in characteristic 0, and Lemma 5.2.2, for any $i > 0$,

$$H^i\big(G/B, \mathcal{L}(S(\mathfrak{u}^*)) \otimes \mathcal{L}(\alpha)\big) = 0, \text{ since } \alpha \in \mathcal{C}. \tag{3}$$

In particular, we get the surjection

$$H^0(G/B, \mathcal{L}(S(\mathfrak{u}^*))) \twoheadrightarrow H^0(G/B, \mathcal{L}(S(\mathfrak{u}_P^*))).$$

Using a degenerate case of the Leray spectral sequence for the affine morphisms $G \times_B \mathfrak{u} \to G/B$ and $G \times_B \mathfrak{u}_P \to G/B$ (see the proof of Lemma 5.2.2), we get (1).

We now show that $\mathcal{S}$ is normal. Considering the $\mathbb{P}^1$-fibration $G \times_B \mathfrak{u}_P \to G \times_P \mathfrak{u}_P$, we get

$$k[G \times_B \mathfrak{u}_P] \simeq k[G \times_P \mathfrak{u}_P]. \tag{4}$$

Consider the commutative diagram:

$$\begin{tikzcd}
k[G \times_B \mathfrak{u}] \arrow[d] & k[\mathcal{N}] \arrow[l, "\widetilde{\phi}^*"', "\sim"] \arrow[d] \\
k[G \times_P \mathfrak{u}_P] & k[\mathcal{S}], \arrow[l, "\widetilde{\phi}_\alpha^*"']
\end{tikzcd}$$

where the left vertical map is induced from the natural restriction followed by the isomorphism (4) and thus is surjective (by (1)), and the right vertical map is induced from the inclusion. The map $\widetilde{\phi}^*$ is an isomorphism since $\mathcal{N}$ is normal and $\widetilde{\phi}$ is birational. Thus, $\widetilde{\phi}_\alpha^*$ is surjective, and, of course, it is injective since $\widetilde{\phi}_\alpha$ is surjective. Now, the affine variety $G \times \mathfrak{u}_P$ being normal, $k[G \times \mathfrak{u}_P]$ is integrally closed in its quotient field. In particular, $k[G \times_P \mathfrak{u}_P] \simeq k[G \times \mathfrak{u}_P]^P$ is integrally closed in its quotient field, proving that the affine variety $\mathcal{S}$ is normal.

Finally, we prove that $\widetilde{\phi}_\alpha$ is a rational resolution. As in the case of $\widetilde{\phi}$, in characteristic 0, this follows from the Grauert–Riemenschneider vanishing theorem. Thus, we may assume that the characteristic is positive. As for $\widetilde{\phi}$ again, it suffices to prove that

$$H^i\big(G \times_P \mathfrak{u}_P, \mathcal{O}_{G \times_P \mathfrak{u}_P}\big) = 0, \text{ for all } i > 0. \tag{5}$$

Further, by Corollary 5.1.11 and Lemma 5.2.2,

$$\begin{aligned} H^i\big(G \times_P \mathfrak{u}_P, \mathcal{O}_{G \times_P \mathfrak{u}_P}\big) &\simeq H^i\big(G/P, \mathcal{L}^P(S(\mathfrak{u}_P^*))\big) \\ &\simeq H^i\big(G/B, \mathcal{L}(S(\mathfrak{u}_P^*))\big), \end{aligned} \tag{6}$$

where the last isomorphism follows from Theorem 3.3.4(a). Now, the vanishing

$$H^i\big(G/B, \mathcal{L}(S(\mathfrak{u}_P^*))\big) = 0, \text{ for all } i > 0, \tag{7}$$

follows from the long exact sequence (2) and the vanishing (3) and using the following consequence of Theorem 5.2.1 and Lemma 5.2.2:

$$H^i\big(G/B, \mathcal{L}(S(\mathfrak{u}^*))\big) = 0, \text{ for all } i > 0. \tag{8}$$

Thus, we obtain (5) (from (6) and (7)), thereby showing that $\widetilde{\phi}_\alpha$ is a rational resolution. Finally, the assertion that $\mathcal{S}$ is Gorenstein follows again from Lemma 3.4.2, since the canonical bundle of $G \times_P \mathfrak{u}_P$ is trivial by Lemma 5.1.1. □

5.3.3 Remark. Let $\{\alpha_{i_1}, \ldots, \alpha_{i_m}\}$ be a set of mutually orthogonal short simple roots and let $P = P_I$ be the standard parabolic subgroup corresponding to the subset $I := \{i_1, \ldots, i_m\}$. Then, Thomsen [Tho–00b] has proved that, if the characteristic of k is a good prime for G, the closure $\overline{G \cdot \mathfrak{u}_P}$ is a normal, Gorenstein variety which admits a rational resolution. This generalizes the corresponding result in characteristic 0 by Broer [Bro–94].

5.3.E Exercises

(1) Let $G = \mathrm{SL}_n(k)$. Take any nilpotent matrix $N \in s\ell_n(k)$ and let $C(N)$ be its G-conjugacy class under the adjoint action.

(a) Using the normality of the full nilpotent cone $\mathcal{N} \subset s\ell_n(k)$, show that the closure $\overline{C(N)} \subset s\ell_n(k)$ is a normal variety.

Also, show that the ring of regular functions $k[\overline{C(N)}]$ admits a good filtration as a G-module.

Hint: Use the fact that there exists a parabolic subgroup $P = P_N$ such that $\phi_P : G \times_P \mathfrak{u}_P \to \overline{C(N)}$, $(g, x) \mapsto \operatorname{Ad} g(x)$, is a proper birational morphism (cf. [Hum–95b, Proposition 5.5]). Moreover, by a result of Spaltenstein (cf. [MeVa-92a, Theorem 4.8]), all the fibers of ϕ_P are connected. Now, use Exercises 5.1.E.6 and 1.2.E.3.

(b) Show that the above map $\phi_P : G \times_P \mathfrak{u}_P \to \overline{C(N)}$ is a rational resolution. Hence, prove that $\overline{C(N)}$ is Gorenstein.

Hint: Consider the splitting of $G \times_B \mathfrak{u}_P$ given by Exercise 5.1.E.6, which of course descends to give a splitting of $G \times_P \mathfrak{u}_P$. Now, apply Theorem 1.3.14 to the morphism ϕ_P by showing that the above splitting satisfies the hypotheses of the theorem. Finally, use Lemma 5.1.1 and the normality of $\overline{C(N)}$ to conclude that ϕ_P is a rational resolution. To prove that $\overline{C(N)}$ is Gorenstein, use Lemmas 3.4.2 and 5.1.1.

5.C. Comments

All of the results of Section 5.1 are due to Kumar–Lauritzen–Thomsen [KLT–99], except for Propositions 5.1.9 and 5.1.10 which are due to Springer [Spr–69] and the explicit splitting of the cotangent bundle $T^*(G/B)$ for $G = \mathrm{SL}_n(k)$ given in Example 5.1.15 which is due to Mehta–van der Kallen [MeVa–92a]. Lemma 5.1.1 in the special case

$G = \mathrm{SL}_n(k)$ and $P = B$, and Exercise 5.1.E.6 are also due to Mehta–van der Kallen [MeVa–92a]. Exercises 5.1.E.4 (the first part), 7, 8 (except for (d), (e)) are taken from [KLT–99] and we have learnt of Exercises 5.1.E.4 (the "in fact" part) and 5.1.E.8(e) from Thomsen.

Theorem 5.2.1 was obtained by Andersen–Jantzen [AnJa–84] by some case-by-case analysis for the case where $p > h$ (h being the Coxeter number of G) and either $\lambda = 0$ or λ is strongly dominant (i.e., $\langle \lambda, \alpha_i^\vee \rangle \geq h - 1$ for all the simple coroots $\alpha_i^\vee$). For $p \geq h-1$ and all the components of G classical or G_2, they proved this theorem for any dominant λ. Theorem 5.2.1 in full generality in characteristic 0 was proved by Broer [Bro–93]. In fact, Broer showed that $\mathcal{C}$ is precisely the set of weights for which the vanishing as in Theorem 5.2.1 holds (in characteristic 0). Lemma 5.2.4 is the simple key lemma in Demazure's very simple proof of the Borel–Weil–Bott theorem [Dem–76]. Corollary 5.2.5 is essentially due to Broer [Bro–94]. Theorem 5.2.9, which is equivalent to Theorem 5.2.1 by using an appropriate Koszul resolution (see Remark 5.2.10), is due to Broer [Bro–97] in characteristic 0 and Kumar–Lauritzen–Thomsen [KLT–99] in characteristic p. In fact, Broer proved the corresponding result in characteristic 0 for an arbitrary G/P and for an arbitrary $\lambda \in X^*(P) \cap \mathcal{C}$. Theorem 5.2.11 is again due to Broer [Bro–94] in characteristic 0 (in fact he proves it for an arbitrary $\lambda \in X^*(P)^+$) and Kumar–Lauritzen–Thomsen [KLT–99] in characteristic p, although the proof given here is new.

Theorem 5.3.2 for the subregular cone in characteristic 0 is due to Broer [Bro–93]. As observed in [KLT–99], his proof carries over to give the same result in characteristic p once one uses the cohomology vanishing Theorem 5.2.1. The normality of the nilpotent cone $\mathcal{N}$ in characteristic 0 is a classical result due to Kostant [Kos–63]; and Hesselink [Hes–76] proved that $\widetilde{\phi}$ is a rational resolution of $\mathcal{N}$ in characteristic 0. Exercise 5.3.E.1 is taken from [Don–90], [MeVa–92a]. In characteristic 0 this was proved by Kraft–Procesi [KrPr–79].

We believe that Theorem 5.2.11 should remain true for any dominant $\lambda \in X^*(P)$. This will follow if, e.g., we can prove that the cotangent bundle $T^*(G/P)$ of the flag variety G/P is split relative to an ample divisor.

Chapter 6

Equivariant Embeddings of Reductive Groups

The main result of this chapter asserts that any equivariant embedding of a connected reductive group G admits a canonical splitting which compatibly splits all the $G \times G$-orbit closures. Here, by an equivariant embedding of G we mean a normal $G \times G$-variety containing an open orbit isomorphic to G itself, where $G \times G$ acts on G by left and right multiplications.

This result is first established for a special class of embeddings: the wonderful compactifications of adjoint semisimple groups. Any such group G_{ad} admits a projective nonsingular equivariant embedding $\mathbf{X}$ such that the complement of the open orbit is a union of nonsingular prime divisors, the boundary divisors, intersecting transversally. Further, the $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$-orbit closures in $\mathbf{X}$ are the partial intersections of boundary divisors, and the intersection of all of these divisors is the unique closed orbit. The construction and main properties of $\mathbf{X}$, due to de Concini–Procesi in characteristic 0, were extended to positive characteristic by Strickland who also obtained its splitting.

Another special class of embeddings consists of toric varieties, which are precisely the equivariant embeddings of tori. For these, the existence of an invariant splitting compatible with all the orbit closures is easily established (Exercise 1.3.E.6). The case of arbitrary equivariant embeddings of reductive groups combines the features of wonderful embeddings and those of toric varieties, as will be made more precise in this chapter. We assume some familiarity with the theory of toric varieties, for which we use [Ful–93] as a general reference.

Section 6.1 begins by constructing the wonderful compactification $\mathbf{X}$ and studying its line bundles. In particular, the Picard group of $\mathbf{X}$ is shown to be isomorphic to the weight lattice of G_{ad} such that the globally generated, resp. ample, line bundles correspond to the dominant, resp. regular dominant, weights. Then, it is shown that $\mathbf{X}$ admits a canonical splitting, compatible with all the boundary divisors and also with the Schubert divisors and opposite Schubert divisors (Theorem 6.1.12). As consequences, the vanishing of the higher cohomology groups of globally generated line bundles is obtained (Corollary 6.1.13), and good filtrations for the spaces of global sections of line bundles over **X** are constructed (Corollary 6.1.14).

Section 6.2 is devoted to some results of Rittatore on equivariant embeddings of a connected reductive group G. A special role is played by those embeddings that admit an equivariant morphism to the wonderful compactification of the associated adjoint group. Such embeddings are called toroidal, as their $G \times G$-orbit structure turns out to be that of a toric variety (Proposition 6.2.3). A combinatorial classification of toroidal embeddings by fans with support in the negative Weyl chamber is obtained (Proposition 6.2.4). Also, it is shown that any equivariant embedding admits an equivariant resolution of singularities by a toroidal embedding (Proposition 6.2.5). Then, it is shown that any equivariant embedding X of G admits a canonical splitting (Theorem 6.2.7), compatible with the boundary divisors, the Schubert divisors and the opposite Schubert divisors. An important consequence is the rationality (in the sense of Definition 3.4.1) of any toroidal resolution $\pi : \widetilde{X} \to X$ (Corollary 6.2.8). In particular, X is Cohen–Macaulay.

In Subsection 6.2.C, these results are applied to the normal reductive monoids, i.e., to the normal varieties endowed with an associative multiplication and with a unit element such that the group of invertible elements is reductive. Indeed, the normal (reductive) monoids M with unit group G turn out to be exactly the affine equivariant embeddings of G (Proposition 6.2.12). Since these admit a canonical splitting, Theorem 4.2.13 implies that the coordinate ring $k[M]$ has a good filtration as a $G \times G$-module. The associated graded module is shown to be the direct sum of $\nabla(\lambda) \boxtimes \nabla(-w_o\lambda)$, where λ runs over the weights of T in the coordinate ring $k[\overline{T}]$, $\overline{T}$ being the closure in M of the maximal torus T (Theorem 6.2.13). Again, these results are due to Rittatore, generalizing earlier works of Doty, Renner, and Vinberg.

When applied to $M = G$, this yields an alternative proof of the existence of a good filtration of the $G \times G$-module $k[G]$ with the associated graded module being the direct sum of all the $\nabla(\lambda) \boxtimes \nabla(-w_o\lambda)$ (Theorem 4.2.5). As another application, it is shown that the closure of a maximal torus in any equivariant embedding of G is normal (Corollary 6.2.14).

6.1 The wonderful compactification

6.1.A Construction

We follow the notation as in Section 2.1. In particular, G denotes a connected, simply-connected, semisimple algebraic group over an algebraically closed field k of characteristic $p \geq 0$. Let G_{ad} be the corresponding adjoint group. We have a homomorphism of algebraic groups $\pi : G \to G_{\mathrm{ad}}$ which is the quotient by the scheme-theoretic center Z of G, a finite (possibly non-reduced) subgroup-scheme of G. In fact, Z is the subgroup-scheme of T obtained as the intersection of the kernels of all the simple roots; the Lie algebra $\mathfrak{z}$ of Z is the center of $\mathfrak{g}$. Further, Z is reduced if and only if p does not divide the index of the root lattice in the weight lattice of G.

We shall construct a "nice" compactification **X** of G_{ad} equipped with an action of the group $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$ extending its action on G_{ad} by left and right multiplication. In other words, **X** is an equivariant compactification of the homogeneous space

$$G_{\mathrm{ad}} = (G_{\mathrm{ad}} \times G_{\mathrm{ad}})/\operatorname{diag}(G_{\mathrm{ad}}) = (G \times G)/((Z \times Z)\operatorname{diag}(G)),$$

where $\operatorname{diag}(G_{\mathrm{ad}})$ denotes the diagonal in $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$, and similarly for $\operatorname{diag}(G)$.

The idea is to embed G_{ad} into the projective linear group $\mathrm{PGL}(M)$, where M is a suitable G-module, and to regard $\mathrm{PGL}(M)$ as an open stable subset of the projective space $\mathbb{P}\operatorname{End}(M)$ of the space of endomorphisms of M, equipped with the action of $\mathrm{PGL}(M) \times \mathrm{PGL}(M)$ via left and right composition. Thus, the closure of G_{ad} in $\mathbb{P}\operatorname{End}(M)$ is an equivariant projective compactification of G_{ad}.

Our first result specifies the properties of M that we will use.

6.1.1 Lemma. *Given a regular dominant weight λ, there exists a rational, finite-dimensional G-module $M = M(\lambda)$ satisfying the following properties:*

(i) The T-eigenspace M_λ of weight λ is a line of B-eigenvectors. All the other weights of M are $< \lambda$.
(ii) $\mathfrak{g}_{-\alpha} M_\lambda \neq 0$ for all the positive roots α. The morphism $G/B \to \mathbb{P}(M)$, $gB \mapsto g \cdot M_\lambda$ is a closed immersion.
(iii) Let $M^*_{-\lambda}$ be the T-eigenspace of weight $-\lambda$ in the dual module M^*. Then, $M^*_{-\lambda}$ is a line of B^--eigenvectors. The morphism $G/B^- \to \mathbb{P}(M^*)$, $gB^- \mapsto g \cdot M^*_{-\lambda}$ is a closed immersion.
(iv) The action of G, resp. $\mathfrak{g}$, on $\mathbb{P}(M)$ factors through a faithful action of G_{ad}, resp. $\mathfrak{g}_{\mathrm{ad}}$.

Proof. Recall that a rational, finite-dimensional G-module M is called *tilting* if both M and M^* admit good filtrations. By [Don–93] (cf. also [Mat–00]), there exists a unique indecomposable tilting module $M(\lambda)$ with highest weight λ. Further, $M(\lambda)$ satisfies (i) and (ii), and $M(\lambda)^* \simeq M(-w_o\lambda)$. In particular, the highest weight line in $M(\lambda)^*$ satisfies (ii). Applying w_o to this line yields (iii).

Since all the weights μ of $M(\lambda)$ are such that $\mu - \lambda$ is in the root lattice, the group scheme Z acts trivially on $\mathbb{P}(M)$. Thus, the action of G factors through an action of G_{ad}. This action is faithful, since $\mathbb{P}(M)$ contains a G-orbit isomorphic to G/B. The Lie algebra assertion is proved similarly. This completes the proof of (iv). $\square$

6.1.2 Remark. In characteristic 0, we may take for $M(\lambda)$ the simple G-module with highest weight λ. In characteristic $p > 0$, we may take for $M((p-1)\rho)$ the Steinberg module St, which is a tilting module by 2.3.4.

Fix λ, M as in Lemma 6.1.1, and consider the $G \times G$-module $\operatorname{End}(M) \simeq M^* \otimes M$. Let $h \in \operatorname{End}(M)$ be the identity, with image $[h]$ in the projectivization $\mathbb{P}\operatorname{End}(M)$.

6.1.3 Lemma. *The orbit $(G \times G) \cdot [h]$ (with its structure of locally closed reduced subscheme of $\mathbb{P}\operatorname{End}(M)$) is isomorphic to the homogeneous space $G \times G/((Z \times Z)\operatorname{diag}(G)) \simeq G_{\mathrm{ad}}$.*

Proof. The isotropy group $(G \times G)_{[h]}$ consists of those pairs (g_1, g_2) such that $g_1 g_2^{-1}$ acts on M by a scalar. By Lemma 6.1.1 (iv), this is equivalent to $g_1 g_2^{-1} \in Z$, i.e., $(g_1, g_2) \in (Z \times Z)\,\mathrm{diag}(G)$. Likewise, the isotropy Lie algebra $(\mathfrak{g}_{\mathrm{ad}} \times \mathfrak{g}_{\mathrm{ad}})_{[h]}$ (for the induced action of $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$) consists of those pairs (x_1, x_2) such that $x_1 - x_2$ acts on M by a scalar. By Lemma 6.1.1 (iv) again, this is equivalent to $x_1 = x_2$. Thus, the orbit map $G \times G \to (G \times G) \cdot [h]$ factors through an isomorphism $G \times G/((Z \times Z)\,\mathrm{diag}(G)) \simeq (G \times G) \cdot [h]$. $\square$

Next, we put $\mathbb{P} := \mathbb{P}\,\mathrm{End}(M)$ and we denote by $\mathbf{X}$ the closure in $\mathbb{P}$ of $(G \times G) \cdot [h]$. By Lemma 6.1.3, $\mathbf{X}$ is a projective compactification of G_{ad}, equivariant with respect to $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$. To study the structure of $\mathbf{X}$, we begin by analyzing the closure of the orbit $(T \times T) \cdot [h]$. This is motivated by the following result, where G is any connected reductive group.

6.1.4 Lemma. *Let X be a $G \times G$-variety and let $x \in X$. Assume that the orbit $(G \times G) \cdot x$ is open in X and that the isotropy group $(G \times G)_x$ contains* $\mathrm{diag}(G)$. *Put $X' := \overline{(T \times T) \cdot x}$. Then, the following hold:*

(i) Any $G \times G$-orbit in X meets X'.
(ii) X contains only finitely many $G \times G$-orbits.
(iii) X is complete if and only if X' is complete.

Proof. (i) We follow the argument as in the proof of the Hilbert–Mumford criterion in [MFK–94]. The map $\varphi : G \to X$, $g \mapsto (g, 1) \cdot x$, is a $G \times G$-equivariant dominant morphism since $\mathrm{diag}(G) \subset (G \times G)_x$. Further, φ restricts to a dominant $T \times T$-equivariant morphism $T \to X'$.

Given any point $y \in X$, we can find a nonsingular irreducible curve C, a point $z \in C$, and a morphism $\psi : C \setminus \{z\} \to G$ such that the composition $\varphi \circ \psi : C \setminus \{z\} \to X$ extends to a morphism $C \to X$ sending z to y. This yields a commutative diagram

$$\begin{array}{ccc} C \setminus \{z\} & \xrightarrow{\psi} & G \\ \downarrow & & \downarrow{\scriptstyle\varphi} \\ C & \longrightarrow & X. \end{array}$$

Choose a local coordinate t of C at z. This defines an isomorphism of the completion of the local ring $\mathcal{O}_{C,z}$ with the power series ring $k[[t]]$ and, in turn, a commutative diagram

$$\begin{array}{ccccc} \mathrm{Spec}\,k((t)) & \longrightarrow & C \setminus \{z\} & \xrightarrow{\psi} & G \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle\varphi} \\ \mathrm{Spec}\,k[[t]] & \longrightarrow & C & \longrightarrow & X. \end{array}$$

Thus, we obtain a point $g(t) \in G(k((t)))$ such that $\varphi(g(t))$ is defined at $t = 0$ and $\varphi(g(0)) = y$. Now, recall the decomposition

$$G(k((t))) = G(k[[t]])\,T(k((t)))\,G(k[[t]]), \tag{1}$$

cf. [IwMa–65]. Write accordingly $g(t) = g_1(t)h(t)g_2(t)^{-1}$. Then, $(g_1(0), g_2(0))$ are in $G \times G$, and $y = (g_1(0), g_2(0)) \cdot \varphi(h(0))$. Thus, the $G \times G$-orbit of y meets $\overline{\varphi(T)} = X'$.

(ii) By (i), it suffices to show that X' contains only finitely many T-orbits (for the left action of T). But, this follows from the fact that the normalization of X' is a toric variety for a quotient of T.

(iii) Since X' is closed in X, the completeness of X implies that of X'. For the converse, by the valuative criterion of properness (cf. [Har–77, Chap. II, Theorem 4.7 and Exercise 4.11]), it suffices to show that every morphism $\operatorname{Spec} k((t)) \to X$ extends to a morphism $\operatorname{Spec} k[[t]] \to X$. But, this follows from (1) as in the proof of (i). $\square$

Returning to our simply-connected, semisimple group G, we put $T_{\text{ad}} := \pi(T)$, this is the quotient of T by its subgroup-scheme Z. Then, the simple roots $\alpha_1, \ldots, \alpha_\ell$ form a basis of the character group $X^*(T_{\text{ad}})$. We may identify T_{ad} with the orbit $(T \times T) \cdot [h]$ by Lemma 6.1.3; let $\overline{T_{\text{ad}}}$ be the closure in $\mathbf{X}$ of this orbit. Write $h = \sum_\mu h_\mu$, where each h_μ is a T-eigenvector of weight μ for the action of $1 \times T$ on $\operatorname{End}(M)$. If (m_i) is a basis of T-eigenvectors of M, and (m_i^*) is the dual basis of M^*, then $h_\mu = \sum m_i^* \otimes m_i$ (the sum over those i such that m_i has weight μ). Together with Lemma 6.1.1, this immediately implies the following.

6.1.5 Lemma. *(i) $h_\lambda = m_\lambda^* \otimes m_\lambda$, where $m_\lambda \in M_\lambda$ and $m_\lambda^* \in M_{-\lambda}^*$ satisfy $\langle m_\lambda^*, m_\lambda \rangle = 1$. In particular, $h_\lambda \neq 0$.*
(ii) $h_{\lambda-\alpha} \neq 0$ for all the positive roots α.
(iii) If $h_\mu \neq 0$ then $\mu \leq \lambda$.

Note that $m_\lambda \otimes m_\lambda^*$, regarded as an element of $M \otimes M^* = \operatorname{End}(M)^*$, is an eigenvector of $B \times B^-$. As earlier, put $\mathbb{P}\operatorname{End}(M) =: \mathbb{P}$ and denote by $\mathbb{P}_0$ the complement in $\mathbb{P}$ of the hyperplane $(m_\lambda \otimes m_\lambda^* = 0)$. Then, $\mathbb{P}_0$ is isomorphic to the affine hyperplane $(m_\lambda \otimes m_\lambda^* = 1)$ of $\operatorname{End}(M)$. Also, put

$$\mathbf{X}_0 := \mathbf{X} \cap \mathbb{P}_0 \text{ and } \overline{T_{\text{ad},0}} := \overline{T_{\text{ad}}} \cap \mathbb{P}_0. \tag{1}$$

These are affine open subsets of $\mathbf{X}$ and $\overline{T_{\text{ad}}}$ respectively; stable under $B \times B^-$ and $T \times T$ respectively.

6.1.6 Lemma. *(i) Embed T_{ad} into affine space $\mathbb{A}^\ell$ by*

$$t \mapsto (\alpha_1(t^{-1}), \ldots, \alpha_\ell(t^{-1})).$$

Then, the inclusion of T_{ad} into $\overline{T_{\text{ad},0}}$ extends to an isomorphism

$$\gamma : \mathbb{A}^\ell \to \overline{T_{\text{ad},0}}.$$

(ii) The diagonal subgroup $\operatorname{diag}(W)$ *of $W \times W$ acts on $\overline{T_{\text{ad}}}$, and $\overline{T_{\text{ad}}} = \operatorname{diag}(W) \cdot \overline{T_{\text{ad},0}}$. In particular, $\overline{T_{\text{ad}}}$ is a nonsingular toric variety (for T_{ad}) with fan consisting of the Weyl chambers and their faces.*

Proof. (i) For any $t \in T$,

$$(1 \times t)h = \sum_{\mu} \mu(t)h_{\mu} = \lambda(t) \sum_{(n_1,\dots,n_\ell)\in\mathbb{N}^\ell} \alpha_1(t^{-n_1}) \cdots \alpha_\ell(t^{-n_\ell}) h_{\lambda - n_1\alpha_1 - \cdots - n_\ell\alpha_\ell}$$

by Lemma 6.1.5. Thus, the surjective map $T \to (T \times T)[h]$, $t \mapsto (1, t)[h]$ induces a morphism $\gamma : \mathbb{A}^\ell \to \overline{T_{\mathrm{ad},0}}$. Since each $h_{\lambda-\alpha_i}$ is nonzero, γ is an isomorphism.

(ii) By (i), $\overline{T_{\mathrm{ad},0}}$ is the toric variety associated with the negative Weyl chamber. On the other hand, the Weyl group W acts diagonally on T_{ad} and hence on $\overline{T_{\mathrm{ad}}}$. It follows that $\overline{T_{\mathrm{ad}}}$ contains the toric variety associated with the fan of Weyl chambers. The latter toric variety being complete, it equals $\overline{T_{\mathrm{ad}}}$. □

The main ingredient in the analysis of $\mathbf{X}$ is the following description of the $B \times B^-$-action on $\mathbf{X}_0$.

6.1.7 Proposition. *The map*

$$\Gamma : U \times U^- \times \mathbb{A}^\ell \to \mathbf{X}_0, \quad (u, v, z) \mapsto (u, v) \cdot \gamma(z),$$

is a $U \times U^-$-equivariant isomorphism, where γ is the isomorphism of Lemma 6.1.6, and $U \times U^-$ acts on $U \times U^- \times \mathbb{A}^\ell$ via multiplication componentwise on the first two factors. As a consequence, $\mathbf{X}_0$ is isomorphic to an affine space.

Proof. Note that the restriction $U \times U^- \times T_{\mathrm{ad}} \to \mathbf{X}_0$ of Γ is just the product map $(u, v, t) \mapsto utv^{-1}$ in G_{ad}. By the Bruhat decomposition, this restriction is an isomorphism onto the open cell in G_{ad}. In particular, Γ is birational.

To show that Γ is an isomorphism, it suffices to construct a $U \times U^-$-equivariant morphism $\beta : \mathbf{X}_0 \to U \times U^-$ such that

$$(\beta \circ \Gamma)(u, v, t) = (u, v) \text{ on } U \times U^- \times T_{\mathrm{ad}}. \tag{1}$$

Indeed, this equality, together with the $U \times U^-$-equivariance of β, implies that Γ induces an isomorphism $U \times U^- \times \beta^{-1}(1, 1) \to \mathbf{X}_0$. In particular, the scheme-theoretic fiber $\beta^{-1}(1, 1)$ is a variety (since $\mathbf{X}_0$ is), of dimension $\dim \mathbf{X}_0 - \dim U - \dim U^- = \ell$. By (1), this fiber contains $\Gamma(1 \times 1 \times \mathbb{A}^\ell) = \gamma(\mathbb{A}^\ell) = \overline{T_{\mathrm{ad},0}}$ as a closed subset. But, both have the same dimension ℓ, so that $\beta^{-1}(1, 1) = \overline{T_{\mathrm{ad},0}}$.

To construct β, we regard $\mathbb{P}$ as a space of rational maps

$$\varphi : \mathbb{P}(M) - - \to \mathbb{P}(M).$$

Then, $\mathbb{P}_0$ consists of those rational maps in $\mathbb{P}$ that are defined at $[m_\lambda]$ and, moreover, send this point to a point of $\mathbb{P}(M)_0 := (m_\lambda^* \neq 0)$. Now, any $\varphi \in G_{\mathrm{ad}}$ is a regular self-map of $\mathbb{P}(M)$ preserving the subvariety $G \cdot [m_\lambda] \simeq G/B$. Thus, each element of $\mathbf{X}_0$ defines a rational self-map of G/B. Note that

$$G \cdot [m_\lambda] \cap \mathbb{P}(M)_0 = U^- \cdot [m_\lambda] \simeq U^-,$$

by the Bruhat decomposition again. Hence, any $\varphi \in \mathbf{X}_0$ maps $[m_\lambda]$ to $u^- \cdot [m_\lambda]$ for a unique $u^- = u^-(\varphi) \in U^-$, and the map $\varphi \mapsto u^-(\varphi)$ is a morphism. Likewise, regarding $\mathbb{P} = \mathbb{P}\,\mathrm{End}(M) = \mathbb{P}\,\mathrm{End}(M^*)$ as a space of rational self-maps of $\mathbb{P}(M^*)$ yields a morphism $\mathbf{X}_0 \to U$. The product morphism $\beta : \mathbf{X}_0 \to U \times U^-$ is clearly $U \times U^-$-equivariant and sends any $\varphi \in T_{\mathrm{ad}}$ to $(1, 1)$. Thus, β satisfies (1). □

We now come to the main result of this section.

6.1.8 Theorem. *(i)* $\mathbf{X}$ *is covered by the* $G \times G$*-translates of* $\mathbf{X}_0$*. In particular,* $\mathbf{X}$ *is nonsingular.*
(ii) The boundary $\partial\mathbf{X} := \mathbf{X} \setminus G_{\mathrm{ad}}$ *is the union of* ℓ *nonsingular prime divisors* $\mathbf{X}_1, \ldots, \mathbf{X}_\ell$ *with normal crossings.*
(iii) For each subset $I \subset \{1, \ldots, \ell\}$*, the intersection* $\mathbf{X}_I := \bigcap_{i \in I} \mathbf{X}_i$ *is the closure of a unique* $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$*-orbit* $\mathcal{O}_I$*. Conversely, any* $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$*-orbit in* $\mathbf{X}$ *equals* $\mathcal{O}_I$ *for a unique* I*. Further,* $\overline{\mathcal{O}_I} \supseteq \mathcal{O}_J$ *if and only if* $I \subset J$*.*

In particular, $\mathbf{X}$ *contains a unique closed orbit*

$$\mathbf{Y} := \mathcal{O}_{\{1,\ldots,\ell\}} = \mathbf{X}_1 \cap \cdots \cap \mathbf{X}_\ell,$$

which is isomorphic to $G/B \times G/B$*.*
(iv) $\mathbf{X}$ *is independent of the choices of* λ *and* M*.*

The variety $\mathbf{X}$ *is called the* wonderful compactification *of the semisimple adjoint group* G_{ad}.

Proof. (i) By Lemma 6.1.6 (ii), $\overline{T_{\mathrm{ad}}} = \bigcup_{w \in W} (w, w) \cdot \overline{T_{\mathrm{ad},0}}$. Further, $\mathbf{X} = (G \times G) \cdot \overline{T_{\mathrm{ad}}}$ by Lemma 6.1.4 (i). Thus, $\mathbf{X} = (G \times G) \cdot \mathbf{X}_0$.

For (ii) and (iii), notice that the complement of T_{ad} in $\mathbb{A}^\ell$, under the embedding of Lemma 6.1.6 (i), is the union of the coordinate hyperplanes. Thus, it is the disjoint union of the orbits $T_{\mathrm{ad}} \cdot [h_I]$, where $I \subset \{1, \ldots, \ell\}$ is not empty, and $[h_I] \in \mathbb{A}^\ell$ has i-th coordinate 0 if $i \in I$, and 1 otherwise. Further, $h_I \in \mathrm{End}(M)$ is the projection to the sum of those weight subspaces M_μ, where $\lambda - \mu \in \sum_{i \notin I} \mathbb{N}\alpha_i$. In particular, $[h_I] \notin G_{\mathrm{ad}}$. Together with Proposition 6.1.7, it follows that Γ restricts to an isomorphism $U \times U^- \times (\mathbb{A}^\ell \setminus T_{\mathrm{ad}}) \to \partial\mathbf{X} \cap \mathbf{X}_0$. This readily implies our assertions.

(iv) Let $\mathbf{X}', \mathbf{X}''$ be two compactifications of G_{ad} associated with different choices of λ and M. Let $\mathbf{X}$ be the closure of G_{ad} embedded diagonally in $\mathbf{X}' \times \mathbf{X}''$. Then, $\mathbf{X}$ is a projective $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$-equivariant compactification of G_{ad} equipped with equivariant projections $\pi' : \mathbf{X} \to \mathbf{X}'$, $\pi'' : \mathbf{X} \to \mathbf{X}''$. By Lemma 6.1.6 (ii), the closures of T_{ad} in $\mathbf{X}', \mathbf{X}''$ are isomorphic, so that the closure of T_{ad} in $\mathbf{X}$ is mapped isomorphically to both. Let $\mathbf{X}_0$ be the preimage of $\mathbf{X}'_0$ under π'. Then, using Proposition 6.1.7, we obtain an isomorphism $\mathbf{X}_0 \simeq U \times U^- \times \overline{T_{\mathrm{ad},0}}$. Thus, π' is an isomorphism over $\mathbf{X}'_0$. Since this open subset meets all the $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$-orbits in $\mathbf{X}'$, π' is an isomorphism everywhere. □

6.1.B Line bundles

We begin with a very simple description of the Picard group $\mathrm{Pic}(\mathbf{X})$, regarded as the group of linear equivalence classes of divisors on $\mathbf{X}$.

6.1.9 Lemma. *The irreducible components of* $\mathbf{X} \setminus \mathbf{X}_0$ *are the prime divisors* $\overline{Bs_iB^-}$, *where, as in 2.1,* $s_i \in W$ *denotes the simple reflection associated with* α_i. *Further, the abelian group* $\mathrm{Pic}(\mathbf{X})$ *is freely generated by the classes of these divisors.*

Proof. Since $\mathbf{X}_0$ is affine, $\mathbf{X}\setminus\mathbf{X}_0$ has pure codimension 1 by [Har–70, Chap. II]. Further, $\mathbf{X} \setminus \mathbf{X}_0$ contains no $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$-orbit by Theorem 6.1.8(i). Thus, $G_{\mathrm{ad}} \setminus \mathbf{X}_0$ is dense in $\mathbf{X}\setminus\mathbf{X}_0$. But, $G_{\mathrm{ad}} \cap \mathbf{X}_0 = UT_{\mathrm{ad}}U^-$ by Proposition 6.1.7. Now, the Bruhat decomposition of G_{ad} yields the first assertion.

Let D be a divisor in $\mathbf{X}$, then $D \cap \mathbf{X}_0$ is principal as $\mathbf{X}_0$ is an affine space. Thus, D is linearly equivalent to a combination of the $\overline{Bs_iB^-}$. The corresponding coefficients are unique, since any regular invertible function on $\mathbf{X}_0$ is constant. □

6.1.10 Definition. We put

$$\mathbf{D}_i := \overline{Bs_iw_oB} = (1, w_o) \cdot \overline{Bs_iB^-} \tag{1}$$

and

$$\widetilde{\mathbf{D}}_i := (w_o, w_o) \cdot \mathbf{D}_i = (w_o, 1) \cdot \overline{Bs_iB^-} = \overline{B^-w_os_iB^-}. \tag{2}$$

We call $\mathbf{D}_1, \ldots, \mathbf{D}_\ell$ the *Schubert divisors* and $\widetilde{\mathbf{D}}_1, \ldots, \widetilde{\mathbf{D}}_\ell$ the *opposite Schubert divisors* in $\mathbf{X}$.

Note that $\mathbf{D}_i$ and $\widetilde{\mathbf{D}}_i$ are both linearly equivalent to $\overline{Bs_iB^-}$, since $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$ acts trivially on $\mathrm{Pic}(\mathbf{X})$. Thus, the classes of the (opposite) Schubert divisors form a basis of $\mathrm{Pic}(\mathbf{X})$.

We now obtain another description of $\mathrm{Pic}(\mathbf{X})$, regarded as the group of isomorphism classes of invertible sheaves on $\mathbf{X}$. Since G is semisimple and simply-connected, and $\mathbf{X}$ is nonsingular (and hence normal), any invertible sheaf $\mathcal{L}$ over $\mathbf{X}$ admits a unique $G\times G$-linearization (cf. [Dol–03, Theorem 7.2]). Thus, the restriction of $\mathcal{L}$ to the (unique) closed $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$-orbit $\mathbf{Y} \simeq G/B \times G/B$ is isomorphic as a $G \times G$-linearized invertible sheaf to $\mathcal{L}(\lambda) \boxtimes \mathcal{L}(\mu)$, where $\lambda, \mu \in X^*(T)$ are uniquely determined. We put

$$\mathcal{L}_Y(\lambda) := \mathcal{L}(-w_o\lambda) \boxtimes \mathcal{L}(\lambda). \tag{3}$$

Also, we denote by τ_i the canonical section of the invertible sheaf $\mathcal{O}_{\mathbf{X}}(\mathbf{D}_i)$ and by σ_i the canonical section of $\mathcal{O}_{\mathbf{X}}(\mathbf{X}_i)$, where $\mathbf{X}_i$ is the boundary divisor defined in Theorem 6.1.8 (ii). Then, τ_i is a $B \times B$-eigenvector (for the action of $G \times G$ on $\Gamma(\mathbf{X}, \mathcal{O}_{\mathbf{X}}(\mathbf{D}_i))$; whereas σ_i is $G \times G$-invariant. We may now state the following.

6.1.11 Proposition. *(i) The restriction map* $\mathrm{res}_{\mathbf{Y}} : \mathrm{Pic}(\mathbf{X}) \to \mathrm{Pic}(\mathbf{Y})$ *is injective, and its image consists precisely of the classes* $\mathcal{L}_{\mathbf{Y}}(\lambda)$, $\lambda \in X^*(T)$.

For $\lambda \in X^*(T)$, *let* $\mathcal{L}_{\mathbf{X}}(\lambda)$ *be the unique element of* $\mathrm{Pic}(\mathbf{X})$ *such that* $\mathrm{res}_{\mathbf{Y}}\, \mathcal{L}_{\mathbf{X}}(\lambda) = \mathcal{L}_{\mathbf{Y}}(\lambda)$. *Thus, the map* $X^*(T) \to \mathrm{Pic}(\mathbf{X})$, $\lambda \mapsto \mathcal{L}_{\mathbf{X}}(\lambda)$ *is an isomorphism of groups.*

(ii) $\mathcal{O}_{\mathbf{X}}(\mathbf{D}_i) = \mathcal{L}_{\mathbf{X}}(\chi_i)$ *and* $\mathcal{O}_{\mathbf{X}}(\mathbf{X}_j) = \mathcal{L}_{\mathbf{X}}(\alpha_j)$, *for* $i, j = 1, \ldots, \ell$, *where* $\{\chi_i\}_{1\le i\le\ell}$ *are the fundamental weights (Section 2.1).*

(iii) For $\lambda \in X^*(T)$, $\mathcal{L}_{\mathbf{X}}(\lambda)$ is globally generated, resp. ample, if and only if λ is dominant, resp. regular dominant.

(iv) For any dominant $\lambda \in X^*(T)$, $\mathcal{L}_{\mathbf{X}}(\lambda)$ has a nonzero global section τ_λ such that τ_λ is a $B \times B$-eigenvector of weight $(\lambda, -w_o\lambda)$. Further, τ_λ is unique up to nonzero scalar multiples; and its divisor is given by

(1) $$(\tau_\lambda)_0 = \sum_{i=1}^{\ell} \langle \lambda, \check{\alpha}_i \rangle \mathbf{D}_i.$$

In particular, $\tau_{\chi_i} = \tau_i$ (up to a nonzero scalar multiple).

(v) A canonical divisor $K_{\mathbf{X}}$ for $\mathbf{X}$ is given by

(2) $$K_{\mathbf{X}} = -2\sum_{i=1}^{\ell} \mathbf{D}_i - \sum_{j=1}^{\ell} \mathbf{X}_j.$$

Equivalently,

(3) $$\omega_{\mathbf{X}} = \mathcal{L}_{\mathbf{X}}(-2\rho - \alpha_1 - \cdots - \alpha_\ell).$$

Proof. Note that the closure in G of Bs_iw_oB is the divisor of a nonzero regular function on G, which is unique (up to a nonzero scalar multiple) and which is a $B \times B$-eigenvector of weight $(\chi_i, -w_o\chi_i)$. Thus, the canonical section τ_i of $\mathcal{O}_{\mathbf{X}}(\mathbf{D}_i)$ is a $B \times B$-eigenvector of the same weight. It follows that (see Section 2.1)

$$\operatorname{res}_{\mathbf{Y}} \mathcal{O}_{\mathbf{X}}(\mathbf{D}_i) = \mathcal{L}_{\mathbf{Y}}(\chi_i).$$

Together with Lemma 6.1.9, this implies (i) and the first assertion of (ii). The second assertion of (ii) follows from the structure of $\mathbf{X}_0$ (Proposition 6.1.7).

We now prove (iii) and (iv) simultaneously. Let $\lambda \in X^*(T)$. If $\mathcal{L}_{\mathbf{X}}(\lambda)$ is globally generated, resp. ample, then so is its restriction to $\mathbf{Y}$. Thus, λ is dominant, resp. regular dominant (Exercise 3.1.E.1). For the converse, note first that since $\mathbf{D}_i$ contains no $G \times G$-orbit, the $G \times G$-translates of the canonical section τ_i have no common zeroes. Thus, $\mathcal{O}_{\mathbf{X}}(\mathbf{D}_i)$ is globally generated. Now, write λ as

$$\lambda = \sum_{i=1}^{\ell} \langle \lambda, \check{\alpha}_i \rangle \chi_i.$$

In particular, for dominant λ, all the coefficients are nonnegative. It follows that $\mathcal{L}_{\mathbf{X}}(\lambda)$ is globally generated and admits a global section

$$\tau_\lambda = \prod_{i=1}^{\ell} \tau_i^{\langle \lambda, \check{\alpha}_i \rangle},$$

which is a $B \times B$-eigenvector of weight $(\lambda, -w_o\lambda)$, satisfying (1). If $\tau \in \Gamma(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda))$ is another $B \times B$-eigenvector of the same weight, then the quotient $\tau\tau_\lambda^{-1}$ is a rational function on $\mathbf{X}$ which is $B \times B$-invariant. Since $\mathbf{X}$ contains an open $B \times B$-orbit, this rational function must be constant, i.e., τ is a scalar multiple of τ_λ. This proves (iv).

To complete the proof of (iii), it remains to show that $\mathcal{L}_{\mathbf{X}}(\lambda)$ is ample for regular dominant λ. For this, choose a very ample invertible sheaf $\mathcal{L} = \mathcal{L}_{\mathbf{X}}(\mu)$. Then, $N\lambda - \mu$ is dominant for $N \gg 0$. Thus, $\mathcal{L}_{\mathbf{X}}(\lambda)^{\otimes N} \otimes \mathcal{L}^{-1}$ is globally generated, so that $\mathcal{L}_{\mathbf{X}}(\lambda)^{\otimes N}$ is very ample.

For (v), note that $\mathbf{Y}$ is the transversal intersection of $\mathbf{X}_1, \dots, \mathbf{X}_\ell$ (Theorem 6.1.8). Thus, the normal bundle of $\mathbf{Y}$ in $\mathbf{X}$ equals $\mathcal{O}_{\mathbf{Y}}(\mathbf{X}_1 + \cdots + \mathbf{X}_\ell) = \mathcal{L}_{\mathbf{Y}}(\alpha_1 + \cdots + \alpha_\ell)$. So, the adjunction formula [Har–77, Chap. II, Proposition 8.20] yields

$$\mathrm{res}_{\mathbf{Y}}\, \omega_{\mathbf{X}} = \omega_{\mathbf{Y}} \otimes \mathcal{L}_{\mathbf{Y}}(-\alpha_1 - \cdots - \alpha_\ell).$$

Since $\omega_{\mathbf{Y}} = \mathcal{L}_{\mathbf{Y}}(-2\rho)$ (by 2.1.8), this proves (3) and hence (2). □

6.1.C Canonical splitting

By Theorem 4.1.15, the flag variety G/B admits a B-canonical splitting which is compatible with all the Schubert subvarieties and all the opposite Schubert subvarieties. We generalize this to the wonderful compactification $\mathbf{X}$ of G_{ad}.

6.1.12 Theorem. *The wonderful compactification* $\mathbf{X}$ *admits a* $B \times B$*-canonical splitting given by the* $(p-1)$*-th power of a global section of* $\omega_{\mathbf{X}}^{-1}$*. This splitting can be chosen to be compatible with all the boundary divisors* $\mathbf{X}_i$ *and with all the* $\mathbf{D}_i$ *and* $\widetilde{\mathbf{D}}_i$ *simultaneously.*

Proof. By Proposition 6.1.11 (iv), the $G \times G$-module $H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}((p-1)\rho))$ contains a unique line of $B \times B$-eigenvectors of weight $((p-1)\rho, (p-1)\rho)$. By the Frobenius reciprocity and self-duality of St (Exercise 2.3.E.3), this yields a $G \times G$-module map

$$\gamma : \mathrm{St} \boxtimes \mathrm{St} \to H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}((p-1)\rho))$$

such that $\gamma(f^+ \boxtimes f^+) = \tau_{(p-1)\rho}$, where $f^+ \in \mathrm{St}$ is a highest weight vector, and $\tau_{(p-1)\rho}$ is a section as in Proposition 6.1.11 (iv). We thus obtain a homomorphism of $G \times G$-modules

$$\begin{array}{rccc} \gamma^2 : & (\mathrm{St} \boxtimes \mathrm{St})^{\otimes 2} & \to & H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(2(p-1)\rho)) \\ & (x_1 \boxtimes y_1) \otimes (x_2 \boxtimes y_2) & \mapsto & \gamma(x_1 \boxtimes y_1) \cdot \gamma(x_2 \boxtimes y_2), \end{array}$$

where the dot denotes the multiplication of sections. Let $f_- := w_o f^+$ be a lowest weight vector in St and put $\tau := \gamma^2((f^- \boxtimes f^-) \otimes (f^+ \boxtimes f^+))$. Then,

$$\tau = \gamma(f^- \boxtimes f^-) \cdot \gamma(f^+ \boxtimes f^+) = \tau_{-(p-1)\rho} \cdot \tau_{(p-1)\rho},$$

where $\tau_{-(p-1)\rho} := (w_o, w_o) \cdot \tau_{(p-1)\rho}$. In particular, τ is nonzero; by Proposition 6.1.11 (iv), its divisor is $(p-1)\sum_{i=1}^{\ell}(\mathbf{D}_i + \widetilde{\mathbf{D}}_i)$.

Now, consider $\prod_{i=1}^{\ell} \sigma_i^{p-1}$, a $G \times G$-invariant global section of the invertible sheaf $\mathcal{O}_{\mathbf{X}}((p-1)(\mathbf{X}_1 + \cdots + \mathbf{X}_\ell)) = \mathcal{L}_{\mathbf{X}}((p-1)(\alpha_1 + \cdots + \alpha_\ell))$. The composition of the multiplication by this section with γ^2 is a homomorphism of $G \times G$-modules

$$\begin{aligned}\theta : (\mathrm{St} \boxtimes \mathrm{St})^{\otimes 2} \longrightarrow & H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}((p-1)(2\rho + \alpha_1 + \cdots + \alpha_\ell))) \\ &= H^0(\mathbf{X}, \omega_{\mathbf{X}}^{1-p}) = \mathrm{End}_F(\mathbf{X}),\end{aligned}$$

where the last equality follows from Lemma 4.1.14. Further,

$$\sigma := \theta((f^- \boxtimes f^-) \otimes (f^+ \boxtimes f^+)) = \tau \prod_{i=1}^{\ell} \sigma_i^{p-1}$$

is nonzero; thus, $\sigma \in \mathrm{End}_F(X)$ is $B \times B$-canonical (use Lemma 4.1.6) and it is a splitting (up to a nonzero scalar multiple) by successively using Exercise 1.3.E.4. Regarding σ as a global section of $\omega_{\mathbf{X}}^{1-p}$, its divisor is

$$(\sigma)_0 = (p-1) \sum_{i=1}^{\ell} (\mathbf{X}_i + \mathbf{D}_i + \widetilde{\mathbf{D}}_i).$$

Thus, σ is the $(p-1)$-th power of a section of $\omega_{\mathbf{X}}^{-1}$. Hence, the splitting given by σ is compatible with $\mathbf{X}_i$, $\mathbf{D}_i$ and $\widetilde{\mathbf{D}}_i$ by Proposition 1.3.11. □

6.1.13 Corollary. *Let Y be a closed $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$-stable reduced subscheme of $\mathbf{X}$ and let λ be a dominant weight. Then, the restriction map*

$$\mathrm{res}_Y : H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) \to H^0(Y, \mathcal{L}_{\mathbf{X}}(\lambda)_{|Y})$$

is surjective. Further, $H^i(Y, \mathcal{L}_{\mathbf{X}}(\lambda)_{|Y}) = 0$ for all $i \geq 1$.

Proof. By Theorems 1.4.10 and 6.1.12, $\mathbf{X}$ is split relative to the divisor $\sum_{i=1}^{\ell} \mathbf{D}_i$. This divisor is ample by Proposition 6.1.11 (ii) and (iii), and contains no irreducible component of Y since it contains no $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$-orbit. Further, since each $\mathbf{X}_i$ is compatibly split, then so is Y by Theorem 6.1.8 (iii). Now, the assertions follow from Theorem 1.4.8. □

Another consequence of Theorem 6.1.12 is the existence of a good filtration for the $G \times G$-module $H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda))$, where λ is an arbitrary weight (Theorem 4.2.13). In fact, such a filtration may be constructed from the geometry of $\mathbf{X}$ as follows.

Let $\mathcal{I}_{\mathbf{Y}}$ be the ideal sheaf of the closed orbit $\mathbf{Y}$, with positive powers $\mathcal{I}_{\mathbf{Y}}^n$. Put

$$F_n H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) := H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda) \otimes \mathcal{I}_{\mathbf{Y}}^n),$$

the subspace of sections that vanish with order $\geq n$ along $\mathbf{Y}$. This yields a decreasing filtration of $H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda))$ by $G \times G$-submodules; furthermore, $F_n H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) = 0$ for $n \gg 0$ (depending on λ).

We now construct a refinement of this filtration. Since $\mathbf{Y}$ is the transversal intersection of the boundary divisors $\mathbf{X}_1, \ldots, \mathbf{X}_\ell$ (Theorem 6.1.8), the $\mathcal{O}_{\mathbf{X}}$-module $\mathcal{I}_{\mathbf{Y}}$ is generated by $\sigma_1, \ldots, \sigma_\ell$. Further, these are $G \times G$-invariant and form a regular sequence. Thus, $\mathcal{I}_{\mathbf{Y}}^n$ is generated by the monomials

$$\sigma^{\mathbf{n}} := \prod_{i=1}^{\ell} \sigma_i^{n_i},$$

where $\mathbf{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ and $n = |\mathbf{n}| := \sum_{i=1}^{\ell} n_i$. The multiplication by $\sigma^{\mathbf{n}}$ defines an injective map of $G \times G$-modules

$$\sigma^{\mathbf{n}} : H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda - \sum n_i \alpha_i)) \to H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)).$$

Let $F_{\mathbf{n}} H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda))$ be the image of $\sigma^{\mathbf{n}}$; this is a $G \times G$-submodule of $F_n H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda))$, where $n = |\mathbf{n}|$.

6.1.14 Corollary. *With the notation as above,*

$$(1) \qquad F_n H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) = \sum_{|\mathbf{n}|=n} F_{\mathbf{n}} H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)).$$

Furthermore, the associated graded of the filtration $\{F_n H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda))\}_{n \geq 0}$ *satisfies*

$$(2) \qquad \mathrm{gr}_n\, H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) = \bigoplus_{\mu} H^0(G/B, \mathcal{L}(-w_o \mu)) \boxtimes H^0(G/B, \mathcal{L}(\mu))$$

as $G \times G$*-modules, where the sum is taken over those dominant weights* $\mu = \lambda - \sum n_i \alpha_i$ *such that* $(n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$ *and* $\sum n_i = n$. *In particular, as* $G \times G$*-modules,*

$$\mathrm{gr}\, H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) = \bigoplus_{\mu \leq \lambda} H^0(G/B, \mathcal{L}(-w_o \mu)) \boxtimes H^0(G/B, \mathcal{L}(\mu)).$$

Proof. From the exact sequence of sheaves on $\mathbf{X}$:

$$0 \to \mathcal{L}_{\mathbf{X}}(\lambda) \otimes \mathcal{I}_{\mathbf{Y}}^{n+1} \to \mathcal{L}_{\mathbf{X}}(\lambda) \otimes \mathcal{I}_{\mathbf{Y}}^{n} \to \mathcal{L}_{\mathbf{X}}(\lambda) \otimes \mathcal{I}_{\mathbf{Y}}^{n}/\mathcal{I}_{\mathbf{Y}}^{n+1} \to 0,$$

we see that $\mathrm{gr}_n\, H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda))$ injects into $H^0(\mathbf{Y}, \mathcal{L}_{\mathbf{Y}}(\lambda) \otimes \mathcal{I}_{\mathbf{Y}}^{n}/\mathcal{I}_{\mathbf{Y}}^{n+1})$. Since $\mathcal{I}_{\mathbf{Y}}$ is generated by the regular sequence $\{\sigma_1, \ldots, \sigma_\ell\}$, we obtain

$$\mathcal{I}_{\mathbf{Y}}^{n}/\mathcal{I}_{\mathbf{Y}}^{n+1} = \bigoplus_{\sum n_i = n} (\prod \sigma_i^{n_i}) \mathcal{L}_{\mathbf{Y}}(-\sum n_i \alpha_i),$$

so that

$$(3) \qquad H^0(\mathbf{Y}, \mathcal{L}_{\mathbf{Y}}(\lambda) \otimes \mathcal{I}_{\mathbf{Y}}^{n}/\mathcal{I}_{\mathbf{Y}}^{n+1}) = \bigoplus_{\sum n_i = n} (\prod \sigma_i^{n_i}) H^0(\mathbf{Y}, \mathcal{L}_{\mathbf{Y}}(\lambda - \sum n_i \alpha_i)).$$

Furthermore, since $\mathbf{Y} \simeq G/B \times G/B$ and $\mathcal{L}_{\mathbf{Y}}(\mu) \simeq \mathcal{L}(-w_o\mu) \boxtimes \mathcal{L}(\mu)$ for any weight μ (6.1.10), the space $H^0(\mathbf{Y}, \mathcal{L}_{\mathbf{Y}}(\mu))$ is nonzero if and only if μ is dominant; then,

$$\mathrm{res}_{\mathbf{Y}} : H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\mu)) \to H^0(\mathbf{Y}, \mathcal{L}_{\mathbf{Y}}(\mu))$$

is surjective by Corollary 6.1.13. Therefore, for any $\mathbf{n} = (n_1, \ldots, n_\ell) \in \mathbb{N}^\ell$,

$$\mathrm{res}_{\mathbf{Y}} : F_{\mathbf{n}} H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) \to H^0(\mathbf{Y}, \mathcal{L}_{\mathbf{Y}}(\lambda - \sum n_i\alpha_i))$$

is surjective, where $F_{\mathbf{n}} H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda))$ is identified with $H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda - \sum n_i\alpha_i))$ via $\sigma^{\mathbf{n}}$. This implies

$$F_n H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) = F_{n+1} H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) + \sum_{|\mathbf{n}|=n} F_{\mathbf{n}} H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)).$$

Replacing n by $n+1, n+2, \ldots$ and using the inclusion $F_{\mathbf{n}'} H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) \subset F_{\mathbf{n}} H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda))$ if $\mathbf{n}' \geq \mathbf{n}$ for the product ordering of $\mathbb{N}^n$, we obtain

$$F_n H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) = F_{n+t} H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) + \sum_{|\mathbf{n}|=n} F_{\mathbf{n}} H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda))$$

for any $t \geq 1$. Since $F_{n+t} H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) = 0$ for $t \gg 0$, this implies (1).

To prove (2), using (1), we obtain a surjective map

$$\bigoplus_{\sum n_i = n} H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda - \sum n_i\alpha_i)) \longrightarrow \mathrm{gr}_n\, H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)),$$

and, by (3), an injective map

$$\iota_n : \mathrm{gr}_n\, H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda)) \longrightarrow \bigoplus_{\sum n_i = n} H^0(\mathbf{Y}, \mathcal{L}_{\mathbf{Y}}(\lambda - \sum n_i\alpha_i)).$$

The composition of these two maps is the direct sum of the restriction maps

$$\mathrm{res}_{\mathbf{Y}} : H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(\lambda - \sum n_i\alpha_i)) \longrightarrow H^0(\mathbf{Y}, \mathcal{L}_{\mathbf{Y}}(\lambda - \sum n_i\alpha_i)),$$

which are all surjective. Thus, ι_n is an isomorphism. ☐

6.1.15 Remark. The wonderful compactification $\mathbf{X}$ is not always diagonally split. For example, consider the group $G = \mathrm{SL}_n(k)$ over k of characteristic $p = 2$ and any $n \geq 4$. Then, for the line bundle $\mathcal{L} = \mathcal{L}_{\mathbf{X}}(\chi_2)$, $H^0(\mathbf{X}, \mathcal{L})$ can be identified with the subspace V of the polynomial ring $k[x_{i,j}]_{1 \leq i,j \leq n}$ spanned by all the 2×2 minors of the matrix $(x_{i,j})_{1 \leq i,j \leq n}$. Moreover, the algebra $R(\mathbf{X}, \mathcal{L})$ can be identified with the integral closure of the subalgebra of $k[x_{i,j}]_{1 \leq i,j \leq n}$ generated by V. Let S be the subalgebra generated by V. Then, by [Brun–91, Remarks 5.2], the ring S is not normal; in particular, S is a proper subring of $R(\mathbf{X}, \mathcal{L})$. Thus, by Exercise 1.5.E.1, $\mathbf{X}$ is not diagonally split.

We learnt from de Concini that, for k of arbitrary characteristic p and for the group $G = \mathrm{SL}_{2p}(k)$, the product map

$$H^0(\mathbf{X}, \mathcal{L}_\mathbf{X}(\chi_p))^{\otimes 2} \longrightarrow H^0(\mathbf{X}, \mathcal{L}_\mathbf{X}(2\chi_p))$$

is not surjective. In particular, in this case, $\mathbf{X}$ is not diagonally split (use Exercise 1.5.E.1).

It may be mentioned that Kannan [Kann–02] shows that, for any dominant weights λ and μ, the product map

$$H^0(\mathbf{X}, \mathcal{L}_\mathbf{X}(\lambda)) \otimes H^0(\mathbf{X}, \mathcal{L}_\mathbf{X}(\mu)) \longrightarrow H^0(\mathbf{X}, \mathcal{L}_\mathbf{X}(\lambda + \mu))$$

is surjective in characteristic 0. However, his claim of the same result in characteristic p is incorrect, as the above examples show.

6.1.E Exercises

(1) Let $\mathcal{M}$ be the submonoid of $X^*(T)$ generated by the simple roots $\alpha_1, \ldots, \alpha_\ell$ and the fundamental weights $\chi_1, \ldots, \chi_\ell$. Show that the invertible sheaves on $\mathbf{X}$ admitting nonzero global sections are precisely those $\mathcal{L}_\mathbf{X}(\lambda)$, where $\lambda \in \mathcal{M}$.

(2) Define

$$R(\mathbf{X}) := \bigoplus_{\lambda \in X^*(T)} H^0(\mathbf{X}, \mathcal{L}_\mathbf{X}(\lambda)).$$

Then, by the above exercise, $R(\mathbf{X})$ is a k-algebra graded by the monoid $\mathcal{M}$. We may regard $R(\mathbf{X})$ as the multihomogeneous coordinate ring of $\mathbf{X}$ (see Exercise 3.5.E.1).

Now, show that the sections $\sigma_1, \ldots, \sigma_\ell$, regarded as homogeneous elements of $R(\mathbf{X})$ of degrees $\alpha_1, \ldots, \alpha_\ell$, form a regular sequence in $R(\mathbf{X})$, and that the quotient $R(\mathbf{Y}) := R(\mathbf{X})/(\sigma_1, \ldots, \sigma_\ell)$ satisfies

$$R(\mathbf{Y}) \simeq \bigoplus_{\mu} H^0(\mathbf{Y}, \mathcal{L}_\mathbf{Y}(\mu)),$$

where the sum runs over all the dominant weights μ.

(3) Put $\mathcal{L}_i := \mathcal{L}_\mathbf{X}(\chi_i)$ for $1 \le i \le \ell$. Show that the algebra $R(\mathbf{Y})$ is generated by its subspaces $H^0(\mathbf{Y}, \mathcal{L}_{1|\mathbf{Y}}), \ldots, H^0(\mathbf{Y}, \mathcal{L}_{\ell|\mathbf{Y}})$. Deduce then that the algebra $R(\mathbf{X})$ is generated by its subspaces $H^0(\mathbf{X}, \mathcal{L}_1), \ldots, H^0(\mathbf{X}, \mathcal{L}_\ell)$, together with $\sigma_1, \ldots, \sigma_\ell$. In particular, the algebras $R(\mathbf{X})$, $R(\mathbf{Y})$ are finitely generated.

(4) Let $\hat{\mathbf{Y}}$ be the affine scheme corresponding to $R(\mathbf{Y})$. Show that $\hat{\mathbf{Y}}$ is the multicone over $\mathbf{Y}$ associated with $\mathcal{L}_{1|\mathbf{Y}}, \ldots, \mathcal{L}_{\ell|\mathbf{Y}}$ (as defined in Exercise 1.1.E.2). This yields a proper birational morphism $\pi : L_1^{-1} \oplus \cdots \oplus L_r^{-1} \to \hat{\mathbf{Y}}$, where L_i is the line bundle corresponding to the invertible sheaf $\mathcal{L}_{i|\mathbf{Y}}$. Show that π is a rational resolution.

(5) Deduce from (4) that the rings $R(\mathbf{Y})$ and $R(\mathbf{X})$ are Cohen–Macaulay.

(6) For any subset I of $\{1, \ldots, \ell\}$, show that the ideal of $R(\mathbf{X})$ generated by σ_i, $i \in I$, is prime. In particular, $R(\mathbf{X})$ is a domain; show that it is normal.

(7) Show that the localization $R(\mathbf{X})[\sigma_1^{-1}, \dots, \sigma_\ell^{-1}]$ is isomorphic to the ring of regular functions on $G \times_Z T$, the quotient of $G \times T$ by the diagonal action of the center Z of G.

6.2 Reductive embeddings

In this section, G denotes a connected reductive algebraic group, Z its scheme-theoretic center, and $G_{\mathrm{ad}} := G/Z$ the corresponding adjoint semisimple group. Let $\pi : G \to G_{\mathrm{ad}}$ be the quotient map and put $T := \pi^{-1}(T_{\mathrm{ad}})$, $B := \pi^{-1}(B_{\mathrm{ad}})$, etc. Then, $T_{\mathrm{ad}} = T/Z$, $B_{\mathrm{ad}} = B/Z$, etc.

6.2.A Toroidal embeddings

6.2.1 Definition. An *equivariant embedding* of G is a normal variety X equipped with an action of $G \times G$ and containing the homogeneous space $G = (G \times G)/\operatorname{diag}(G)$ as an open orbit.

In other words, X is a normal $G \times G$-variety containing a point x such that the orbit $(G \times G)x$ is open and isomorphic to G via the orbit map. For brevity, we say that X is a *reductive embedding*, or a *G-embedding* if we wish to specify the group G.

Since G is an affine open subset of X, the complement $\partial X := X \setminus G$ has pure codimension 1, cf. [Har–70, Chap. II]. Thus, all the irreducible components of $\partial X := X \setminus G$ are prime divisors in X, called the *boundary divisors*.

If $G = T$ is a torus, then the left multiplication by any $t \in T$ equals the right multiplication by t^{-1}. Thus, the T-embeddings are just the toric varieties for T. Returning to an arbitrary G, we now introduce a class of G-embeddings which turn out to be closely related to toric varieties.

6.2.2 Definition. A G-embedding X is *toroidal* if the quotient map $\pi : G \to G_{\mathrm{ad}}$ extends to a morphism from X to the wonderful compactification $\mathbf{X}$ of G_{ad}.

Then, this extension $\pi : X \to \mathbf{X}$ is unique and $G \times G$-equivariant, where $G \times G$ acts on $\mathbf{X}$ through its quotient $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$. We put $X_0 := \pi^{-1}(\mathbf{X}_0)$; this is a $B \times B^-$-stable open subset of X. Also, put $X' := \overline{(T \times T) \cdot x}$ (i.e., X' is the closure of T in X), and $X'_0 := \pi^{-1}(\overline{T_{\mathrm{ad},0}})$. We may now formulate the following generalization of Proposition 6.1.7 and Theorem 6.1.8.

6.2.3 Proposition. *Let X be a toroidal G-embedding. Then, the following properties hold.*

(i) The map $U \times U^- \times X'_0 \to X_0$, $(u, v, z) \mapsto (u, v) \cdot z$ is an isomorphism. Furthermore, the irreducible components of $X \setminus X_0$ are precisely $\overline{B s_i B^-}$, where $i = 1, \dots, \ell$; they contain no $G \times G$-orbit.
(ii) X'_0 meets any $G \times G$-orbit in X along a unique $T \times T$-orbit. Furthermore, $X' = \operatorname{diag}(W) \cdot X'_0$, so that X' is a toric variety for T, with a compatible action of W.

(iii) Any $G \times G$-orbit closure in X is the intersection of the boundary divisors which contain it.
(iv) X is nonsingular (resp. complete, quasi-projective) if and only if X' is.
(v) If X is complete, then any closed $G \times G$-orbit is isomorphic to $G/B \times G/B$.

Proof. (i) The first assertion follows from the corresponding statement for $\mathbf{X}$ (Proposition 6.1.7). It implies that $X \setminus X_0$ contains no $G \times G$-orbit. As $X \setminus X_0$ is of pure codimension 1, it follows that $X \setminus X_0$ is the closure of $G \setminus X_0$. On the other hand, $G \cap X_0 = UTU^-$, since $G_{\mathrm{ad}} \cap \mathbf{X}_0 = UT_{\mathrm{ad}}U^-$ and π extends the quotient map $G \to G_{\mathrm{ad}}$. By the Bruhat decomposition of G, the irreducible components of $G \setminus X_0$ are the closures of Bs_iB^- in G. This completes the proof of (i).

(ii) Since X is normal, then so is X'_0 by (i). Furthermore, $\mathrm{diag}(W)$ acts on X', and by Lemma 6.1.6(ii),

$$X' \subset \pi^{-1}(\overline{T_{\mathrm{ad}}}) = \mathrm{diag}(W)X'_0.$$

Thus, $X' = \mathrm{diag}(W)X'_0$. It follows that X' is normal, and hence it is a toric variety for T.

By (i), X_0 meets all the $G \times G$-orbits in X. Furthermore,

$$\partial X \cap X_0 = (U \times U^-)\partial X'_0,$$

where $\partial X'_0 := \partial X \cap X'_0$. Hence, each boundary divisor of X meets $\partial X'_0$ along a unique boundary divisor of X'_0. Since every orbit closure in a toric variety is the intersection of the boundary divisors which contain it, it follows readily that X'_0 meets any $G \times G$-orbit along a unique $T \times T$-orbit.

(iii) follows from (i) and (ii), together with the corresponding result for toric varieties.

(iv) By (i) and (ii), X is nonsingular if and only if X' is. On the other hand, X is complete if and only if X' is, by Lemma 6.1.4 (iii).

If X is quasi-projective, then so is its closed subset X'. Conversely, if X' is quasi-projective, then so is X'_0. Thus, a positive linear combination of the boundary divisors of X'_0 is ample. Now, the same linear combination of the corresponding boundary divisors of X is a $G \times G$-invariant divisor, and, moreover, by (i) and (ii), it is ample relative to π. Since $\mathbf{X}$ is projective, it follows that X is quasi-projective.

(v) follows easily from the corresponding assertion for $\mathbf{X}$ (Theorem 6.1.8). □

Next, we obtain a classification of toroidal G-embeddings in terms of toric varieties equipped with a compatible action of the Weyl group.

6.2.4 Proposition. *(i) Any toroidal G-embedding X is uniquely determined by its associated toric variety X'. The latter admits a morphism to $\overline{T_{\mathrm{ad}}}$, which is equivariant with respect to the actions of T and W.*
(ii) Any toric variety for T, equipped with a compatible action of W and with an equivariant morphism to $\overline{T_{\mathrm{ad}}}$, arises from a toroidal G-embedding.
(iii) The toroidal G-embeddings are classified by the fans in $X_*(T)_{\mathbb{R}} := X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$ with support in the negative Weyl chamber. The nonsingular embeddings correspond to those fans whose all the cones are generated by subsets of bases of $X_*(T)$.

Proof. (i) follows from Proposition 6.2.3 (i) by arguing as in the proof of the uniqueness of wonderful compactification (Theorem 6.1.8 (iv)).

(ii) may be deduced from the embedding theory of spherical homogeneous spaces, cf. [Kno–91]. However, we sketch a direct proof as follows.

Let X' be a toric variety for T equipped with a compatible action of W and with an equivariant morphism to $\overline{T_{\mathrm{ad}}}$. Let X'_0 be the preimage of $\overline{T_{\mathrm{ad},0}}$ under this morphism. Then, $X' = W \cdot X'_0$, and X'_0 is a toric variety whose fan Σ is contained in the negative Weyl chamber; the fan of X' consists of all the cones $w\sigma$, where $w \in W$ and $\sigma \in \Sigma$. Fix such a σ and denote by $\overline{T}_\sigma$ the corresponding affine toric variety. Then, the ring $k[\overline{T}_\sigma]$, regarded as a subring of $k[T]$, is generated by a finite set F of characters; these also generate the dual cone C of σ. Since σ is contained in the negative Weyl chamber, C contains all the negative roots.

Choose a regular dominant weight λ such that $\lambda+\varphi$ is regular dominant for all $\varphi \in F$. Choose G-modules $M(\lambda)$, $M(\lambda+\varphi)$ $(\varphi \in F)$ satisfying the properties of Lemma 6.1.1. (This lemma, though stated for semisimple groups, holds for all connected reductive groups, cf. [Don–93].) Let $h(\lambda) \in \mathrm{End}(M(\lambda))$ be the identity and define likewise $h(\lambda + \varphi) \in \mathrm{End}(M(\lambda + \varphi))$. Consider the point

$$h := h(\lambda) + \sum_{\varphi\in F} h(\lambda + \varphi)$$

of the $G \times G$-module

$$\mathrm{End}\big(M(\lambda) \oplus \bigoplus_{\varphi\in F} M(\lambda + \varphi)\big),$$

and the orbit closure $\overline{(G \times G)[h]}$ in the projectivization of this module. Let X_σ be the open subset of $\overline{(G \times G)[h]}$, where the projection to $\mathbb{P}\,\mathrm{End}(M(\lambda))$ is defined. Clearly, X_σ is a quasi-projective $G \times G$-variety equipped with an equivariant morphism $\pi : X_\sigma \to \mathbf{X}$ via the projection to $\mathbb{P}\,\mathrm{End}(M(\lambda))$. Using the fact that C contains all the negative roots, one checks that $\pi^{-1}(\mathbf{X}_0) \simeq U \times U^- \times \overline{T}_\sigma$. As a consequence, $\pi^{-1}(\mathbf{X}_0)$ is normal; hence, so is X_σ, since $\pi^{-1}(\mathbf{X}_0)$ contains no $G \times G$-orbit. Thus, X_σ is a toroidal G-embedding. The fan of its associated toric variety consists of the cones $w\sigma$ $(w \in W)$ and their faces.

Finally, one checks that X_σ, $\sigma \in \Sigma$, can be glued into the desired toroidal G-embedding.

(iii) follows from (i) and (ii), together with the classification of toric varieties. □

Next, we show that any G-embedding admits a toroidal resolution.

6.2.5 Proposition. *For any G-embedding X, there exists a nonsingular quasi-projective toroidal G-embedding $\widetilde{X}$ and a projective morphism $f : \widetilde{X} \to X$ extending the identity map of G.*

Proof. The quotient $\pi : G \to G_{\mathrm{ad}}$ may be regarded as a rational map $X - - \to \mathbf{X}$. The normalization of the graph of this rational map yields a toroidal G-embedding K equipped with an equivariant projective morphism to X. Now, the toric variety

K_0' admits an equivariant resolution by a (nonsingular) quasi-projective toric variety. Together with Proposition 6.2.4, this yields the desired resolution $\widetilde{X}$ of X. $\square$

We now extend the description of the canonical divisor of $\mathbf{X}$ (Proposition 6.1.11 (v)) to any G-embedding.

6.2.6 Proposition. *Let X be a G-embedding, $X_1, \dots, X_n$ its boundary prime divisors, and put $D_i = \overline{Bs_i w_o B}$ for $i = 1, \dots, \ell$. Then, a canonical divisor for X is given by*

(1)
$$K_X = -2\sum_{i=1}^{\ell} D_i - \sum_{j=1}^{n} X_j.$$

Proof. We may replace X with any open $G \times G$-stable subset with complement of codimension ≥ 2; hence, we may assume that X is nonsingular. Next, observe that the indeterminacy locus of the rational map $X - - \to \mathbf{X}$ is $G \times G$-stable and has codimension ≥ 2 in X. (Indeed, consider the graph Z of this rational map, with projection $\pi : Z \to X$. Then, π is a projective birational morphism ; since X is normal, the exceptional locus of π has codimension ≥ 2 by Zariski's main theorem.) Hence, we may also assume that X is toroidal.

Now, consider the open subset

$$UTU^- \simeq U \times T \times U^-$$

of $G \subset X$. Its dualizing sheaf is freely generated by $\sigma := \theta_U \wedge \theta_T' \wedge \theta_{U^-}$, where θ_U, θ_{U^-} are the unique (up to nonzero scalar multiples) volume forms on U, U^- respectively, and $\theta_T' := \frac{dt_1 \wedge \cdots \wedge dt_r}{t_1 \cdots t_r}$. Here $t_1, \dots, t_r$ are the coordinate functions on $T \simeq (\mathbb{G}_m)^r$. Note that σ is a volume form on UTU^-, invariant under $(U \times U^-)\,\mathrm{diag}(T)$, where $\mathrm{diag}(T)$ denotes the diagonal subgroup of $T \times T$. We may regard σ as a rational section of ω_X with zeros and poles along the irreducible components of $X \setminus UTU^-$, i.e., along X_i and $\overline{Bs_jB^-} = (1, w_o) \cdot D_j$.

By Proposition 6.2.3 and the structure of the dualizing sheaf of a toric variety, σ has poles of order 1 along each of $X_1, \dots, X_n$ (see Exercise 1.3.E.6). To complete the proof, it suffices to check that σ has a pole of order 2 along any divisor $(1, w_o) \cdot D_j$. To prove this, denote by P_j the minimal parabolic subgroup $B \cup Bs_jB$, by Q_j the opposite parabolic subgroup containing T, and by L_j their common Levi subgroup. Then, the multiplication map $R_u(P_j) \times L_j \times R_u(Q_j) \to G$ is an open immersion, and its image meets D_j. Using the $U \times U^-$-invariance of σ, we may replace G with L_j, and hence assume that G has semisimple rank 1. Then, $G/B^- = \mathbb{P}^1$ and $\sigma = \tau \otimes \pi^*\eta$ under the decomposition $\omega_G = \omega_\pi \otimes \pi^*\omega_{G/B^-}$, where $\pi : G \to G/B^-$ is the projection, τ is a nowhere vanishing section of ω_π, and η is a rational differential form on $\mathbb{P}^1$ having a pole of order 2 at the U-fixed point. $\square$

6.2.B Canonical splitting

We begin by generalizing Theorem 6.1.12 to all the (normal) G-embeddings. Since these are possibly singular, we use the notation and results of Remark 1.3.12 concerning splittings of normal varieties.

6.2.7 Theorem. *Any G-embedding X admits a $B \times B$-canonical splitting given by the $(p-1)$-th power of a global section of ω_X^{-1}. This splitting is compatible with all the $G \times G$-orbit closures and with the Schubert divisors $D_i := \overline{Bs_i w_o B}$ and the opposite Schubert divisors $\widetilde{D}_i := (w_o, w_o) \cdot D_i$.*

Proof. By Lemma 1.1.8 and Proposition 6.2.5, we may assume that X is nonsingular, toroidal, and quasi-projective. Then, Proposition 6.2.4 implies that X admits an equivariant completion by a nonsingular toroidal projective G-embedding; thus, we may also assume that X is projective.

Let $X_1, \ldots, X_n$ be the boundary divisors of X and let $\sigma_1, \ldots, \sigma_n$ be the canonical sections of the associated invertible sheaves $\mathcal{O}_X(X_1), \ldots, \mathcal{O}_X(X_n)$, respectively. Let $\tau \in H^0(\mathbf{X}, \mathcal{L}_{\mathbf{X}}(2(p-1)\rho))$ be as in the proof of Theorem 6.1.12. Since $(\tau)_0 = \sum_{i=1}^{\ell}(p-1)(\mathbf{D}_i + \widetilde{\mathbf{D}}_i)$, τ lifts to a global section $\widetilde{\tau}$ of the invertible sheaf $\mathcal{O}_X(\sum_{i=1}^{\ell}(p-1)(D_i + \widetilde{D}_i))$. Put $\widetilde{\sigma} := \widetilde{\tau}\prod_{i=1}^{n}\sigma_i^{p-1}$. Then, by Proposition 6.2.6, $\widetilde{\sigma}$ is a nonzero section of ω_X^{1-p}. Arguing as in the proof of Theorem 6.1.12, one checks that $\widetilde{\sigma}$ yields the desired splitting of X. $\square$

6.2.8 Corollary. *Let X be a G-embedding and let $f : \widetilde{X} \to X$ be a toroidal resolution as in Proposition 6.2.5. Then, $f_*(\mathcal{O}_{\widetilde{X}}) = \mathcal{O}_X$ and $R^i f_*(\mathcal{O}_{\widetilde{X}}) = R^i f_*(\omega_{\widetilde{X}}) = 0$ for all $i \geq 1$, i.e., f is a rational resolution.*

Proof. Since X is normal, $f_*(\mathcal{O}_{\widetilde{X}}) = \mathcal{O}_X$. To show the remaining assertions, we begin by reducing to the case where X is projective. For this, note that the normal $G \times G$-variety X is covered by $G \times G$-stable open subsets admitting locally closed embeddings into projectivizations of $G \times G$-modules. Further, since the assertions are local on X, we may replace X by any of these open subsets, and then by the normalization of its closure. Thus, we may assume that X is projective.

We now show that $H^i(\widetilde{X}, \widetilde{\mathcal{L}}) = 0$ for any $i \geq 1$ and any semi-ample invertible sheaf $\widetilde{\mathcal{L}}$. For this, note that $\widetilde{X}$ is projective (since X and f are projective), so that it admits an ample effective $B \times B^-$-invariant divisor A. Then, A must have support in $\widetilde{X} \setminus UTU^-$, the complement of the open $B \times B^-$-orbit. Now, $\widetilde{X}$ is split compatibly with its reduced divisor $\widetilde{X} \setminus UTU^-$, as a consequence of Theorem 6.2.7. Write $A = \sum_{j=1}^{n} a_j A_j$, where the a_j are nonnegative integers and the A_j are certain irreducible components of $\widetilde{X} \setminus UTU^-$. Then, we have a split injection $H^i(\widetilde{X}, \widetilde{\mathcal{L}}) \longrightarrow H^i(\widetilde{X}, \widetilde{\mathcal{L}}^{p^\nu}(A))$, whenever $a_1, \ldots, a_n < p^\nu$ (Lemma 1.4.11). Further, $\widetilde{\mathcal{L}}^{p^\nu}(A)$ is ample, since $\widetilde{\mathcal{L}}$ is semi-ample and A is ample. So, Theorem 1.2.8 yields the desired vanishing.

Taking $\widetilde{\mathcal{L}} = f^*\mathcal{L}$, where $\mathcal{L}$ is an ample invertible sheaf on X, and using Lemma 3.3.3(a), we obtain that $R^i f_*(\mathcal{O}_{\widetilde{X}}) = 0$ for all $i \geq 1$.

Further, $\widetilde{X}$ is split by σ^{p-1}, where σ is a global section of $\omega_{\widetilde{X}}^{-1}$ vanishing identically on all the boundary divisors; and the exceptional locus of f is contained in the union of these divisors. Thus, we may apply Theorem 1.3.14 which yields $R^i f_*(\omega_{\widetilde{X}}) = 0$ for $i \geq 1$. □

The preceding corollary, combined with Lemma 3.4.2, implies the following.

6.2.9 Corollary. *Any G-embedding is Cohen–Macaulay.*

6.2.C Reductive monoids

We begin with a brief discussion of linear algebraic monoids, referring to [Put–88] for details.

6.2.10 Definition. A *linear algebraic monoid* is an affine variety M endowed with a morphism $m : M \times M \to M$, $(x, y) \mapsto x \cdot y$, such that the multiplication m is associative and admits a unit element.

The *unit group* of M is the group $G(M)$ consisting of all the invertible elements; this is a linear algebraic group, open in M. In particular, $G(M)$ is irreducible and hence connected.

For example, the space $\mathrm{End}(V)$ of all linear endomorphisms of a finite-dimensional vector space V is a linear algebraic monoid with unit group $\mathrm{GL}(V)$. In fact, any linear algebraic monoid M admits a closed embedding into some $\mathrm{End}(V)$ which is compatible with the multiplication and satisfies $G(M) = M \cap \mathrm{GL}(V)$.

6.2.11 Definition. A linear algebraic monoid M is called *reductive* if its unit group $G(M)$ is reductive; M is called *normal* if its underlying variety is normal.

6.2.12 Proposition. *The normal (reductive) monoids with unit group G are precisely the affine G-embeddings.*

Proof. Let X be an affine G-embedding; we regard its coordinate ring $k[X]$ as a subring of $k[G]$, stable under the $G \times G$-action. The multiplication $m : G \times G \to G$ yields an algebra homomorphism

$$m^{\#} : k[G] \to k[G \times G] \simeq k[G] \otimes k[G],$$

the comultiplication. Since m extends to morphisms $G \times X \to X$ and to $X \times G \to X$, the map $m^{\#}$ restricts to an algebra homomorphism

$$k[X] \to (k[G] \otimes k[X]) \cap (k[X] \otimes k[G]) \subset k[G] \otimes k[G].$$

But, $(k[G] \otimes k[X]) \cap (k[X] \otimes k[G]) = k[X] \otimes k[X]$ by Exercise 6.2.E.1. Thus, $m^{\#}$ restricts to a morphism $k[X] \to k[X \times X]$, i.e., m extends to a morphism $X \times X \to X$. It follows that X is a linear algebraic monoid and that G is an open subgroup of its unit group $G(X)$. Thus, G is also closed in $G(X)$. Since $G(X)$ is irreducible, we conclude that $G = G(X)$.

Conversely, any normal (reductive) monoid M with unit group G is clearly an affine G-embedding for the action of $G \times G$ by left and right multiplication. □

Next, we obtain a structure theorem for normal monoids with unit group G and a description of their coordinate rings as $G \times G$-modules.

6.2.13 Theorem. *Let M be a normal (reductive) monoid with unit group G; let $\overline{T}$ be the closure of T in M and let $C = C_M$ be the convex cone in $X^*(T)_{\mathbb{R}}$ generated by all the weights of T in the coordinate ring $k[\overline{T}]$, where T acts on $\overline{T}$ via the left multiplication. Then, the following hold.*

(i) $\overline{T}$ is an affine toric variety for T, and C is a W-stable rational polyhedral cone with nonempty interior.

(ii) The $G \times G$-module $k[M]$ admits a filtration with associated graded

$$\bigoplus_{\lambda \in C \cap X^*(T)^+} H^0(G/B, \mathcal{L}(-w_o\lambda)) \boxtimes H^0(G/B, \mathcal{L}(\lambda)). \tag{1}$$

(iii) The assignment $M \mapsto C_M$ yields a bijection from the isomorphism classes of normal (reductive) monoids with unit group G, to the W-stable rational polyhedral cones in $X^*(T)_{\mathbb{R}}$ with nonempty interior.

Proof. We prove (i) and (ii) simultaneously. By Theorems 4.2.13 and 6.2.7 and Proposition 6.2.12, the $G \times G$-module $k[M]$ admits a good filtration. Write the associated graded as a direct sum of tensor products

$$H^0(G/B, \mathcal{L}(-w_o\lambda)) \boxtimes H^0(G/B, \mathcal{L}(\mu))$$

with corresponding multiplicities $m_{\lambda,\mu}$. Then, by Exercise 4.2.E.5, $m_{\lambda,\mu}$ is the dimension of the space of $B \times B^-$-eigenvectors in $k[M]$ with weight $(\lambda, -\mu)$; i.e., $m_{\lambda,\mu}$ is the multiplicity of the weight $(\lambda, -\mu)$ in the invariant subalgebra $k[M]^{U \times U^-}$ of $k[M]$.

We determine these multiplicities in terms of the geometry of M as follows. Let $\mathcal{O}_1, \dots, \mathcal{O}_n$ be the $G \times G$-orbits of codimension 1 in M. Together with G, they form an open $G \times G$-stable subset $X \subset M$ whose complement has codimension ≥ 2. Further, X is a toroidal G-embedding, since the rational map $M - - \to \mathbf{X}$ is clearly $G \times G$-equivariant and is defined in codimension 1 (see the proof of Proposition 6.2.6). Let X_0 be the open subset of the toroidal G-embedding X defined in 6.2.2. Then, $G \cup X_0$ is a $B \times B^-$-stable open subset of X whose complement has codimension ≥ 2. Thus, $k[M] \simeq k[G \cup X_0]$ via restriction. It follows that $k[M]^{U \times U^-}$ consists of those elements of $k[G]^{U \times U^-}$ that extend to X_0 or, equivalently, to X_0', since $X_0 \simeq U \times U^- \times X_0'$ by Proposition 6.2.3 (i).

The fan of the toric variety X_0' consists of n rays (1-dimensional cones), generated by indivisible one-parameter subgroups $\theta_1, \dots, \theta_n$ lying in the negative Weyl chamber. On the other hand, any $T \times T$-eigenvector $f \in k[G]^{U \times U^-}$ has weight $(\lambda, -\lambda)$ for some dominant weight λ, and f is determined by λ up to a scalar multiple, by Theorem 4.2.5. Then, f extends to X'_0 if and only if

$$\langle \lambda, \theta_i \rangle \geq 0 \text{ for } i = 1, \ldots, n. \tag{2}$$

Thus, we have proved that the weights of $k[M]^{U \times U^-}$ are the pairs $(\lambda, -\lambda)$, where λ is a dominant weight satisfying (2); all such weights have multiplicity 1. Let σ be the cone of $X^*(T)_{\mathbb{R}}$ generated by these weights λ. Then, σ is a finite intersection of closed rational half-spaces, and hence a rational polyhedral convex cone. Further, the interior of σ is nonempty. Indeed, the algebras $k[M]$ and $k[G]$ have the same quotient field, so that the subalgebras $k[M]^{U \times U^-}$ and $k[G]^{U \times U^-}$ have the same quotient field as well. Thus, any dominant weight is a difference of two weights in σ. Next, let $C = W\sigma$; this is also a rational polyhedral convex cone (Exercise 6.2.E.2) with nonempty interior.

We now show that $C \cap X^*(T)$ is the set of weights of $T \times 1$ in $k[\overline{T}]$. Indeed, any such weight lies in C, as the $T \times T$-module $k[\overline{T}]$ is a quotient of $k[M]$, and all the T-weights of $H^0(G/B, \mathcal{L}(-w_o\lambda))$ lie in the convex hull of the orbit $W\lambda$. On the other hand, any weight $\lambda \in \sigma$ is also a weight of $k[\overline{T}]$, since the restriction map $k[M]^{U \times U^-} \to k[\overline{T}]$ is injective (Proposition 6.2.3(i)). Since $\overline{T}$ is diag(W)-stable, it follows that any weight $\lambda \in C$ occurs in $k[\overline{T}]$. In particular, $\overline{T}$ is normal; it is the affine toric variety with cone the dual of the cone C. This completes the proof of (i) and (ii).

For (iii), note that M is uniquely determined by C as $k[M] = k[X]$, where X is the toroidal G-embedding associated with the extremal rays of the intersection σ of C with the positive Weyl chamber. Further, given a W-stable rational polyhedral cone $C \subset X^*(T)_{\mathbb{R}}$ with nonempty interior, let σ be its intersection with the positive Weyl chamber and let X be the toroidal G-embedding defined as above. Then, $k[X]$ is an integrally closed $G \times G$-stable subalgebra of $k[G]$; by the preceding arguments, this subalgebra admits a good filtration whose associated graded is given by (1). It follows that the algebra $k[X]$ is finitely generated with quotient field $k(G)$; thus, the corresponding affine variety is a normal reductive monoid with unit group G and cone C. □

6.2.14 Corollary. *For any G-embedding X, the closure of T in X is normal.*

Proof. In the case where X is affine, the assertion follows from Proposition 6.2.12 and Theorem 6.2.13. We now show how the general case reduces to this one.

First, arguing as in the proof of Corollary 6.2.8, we may reduce to the case where X is projective. Then, we can find a very ample, G-linearized invertible sheaf $\mathcal{L}$ on X such that X is projectively normal in the corresponding projective embedding. The connected reductive group $\hat{G} := G \times \mathbb{G}_m$ acts on the affine cone $\hat{X}$, which, in fact, is an affine $\hat{G}$-embedding. Therefore, the closure in $\hat{X}$ of $\hat{T} := T \times \mathbb{G}_m$ (a maximal torus of $\hat{G}$) is normal. In other words, the closure of T in X is projectively normal in the embedding associated with $\mathcal{L}$. □

6.2.E Exercises

(1*) Let E be a vector space over a field K and let $F \subset E$ be a subspace. Show that $(E \otimes_K F) \cap (F \otimes_K E) = F \otimes_K F$, where the intersection is taken in $E \otimes_K E$.

(2*) Let $\sigma \subset X^*(T)_{\mathbb{R}}$ be the intersection of the positive Weyl chamber with finitely many closed half-spaces $\theta_i \geq 0$, where $\theta_i \in X_*(T)_{\mathbb{R}}$ lies in the negative Weyl chamber. Show that the subset $W\sigma \subset X^*(T)_{\mathbb{R}}$ is a rational polyhedral convex cone.

(3) Let $\hat{\mathbf{X}}$ be the affine variety associated with the algebra $R(\mathbf{X})$ considered in Exercises 6.1.E; we may regard $\hat{\mathbf{X}}$ as the "universal" multicone over $\mathbf{X}$.

Show that $\hat{\mathbf{X}}$ is a normal reductive monoid with unit group $G \times_Z T$ (the quotient of $G \times T$ by the center Z of G embedded diagonally).

Hence, Corollary 6.2.9 yields another proof for the Cohen–Macaulayness of $\hat{\mathbf{X}}$ (Exercise 6.1.E.5).

(4) Let $\pi : \hat{\mathbf{X}} \to \mathbb{A}^\ell$ be the morphism associated with the regular functions $\sigma_1, \ldots, \sigma_\ell$ on $\hat{\mathbf{X}}$ (defined in 6.1.10). Show that π is flat with reduced and irreducible fibers; the fibers over $T_{\mathrm{ad}} \subset \mathbb{A}^\ell$ (embedded via the characters $\alpha_1, \ldots, \alpha_\ell$) are all isomorphic to G.

(5) Regarding $\mathbb{A}^\ell$ as a monoid under the pointwise multiplication, show that π is a morphism of linear algebraic monoids. Also, show that π is universal for morphisms from $\mathbb{A}^\ell$ to commutative algebraic monoids, i.e., for any morphism of linear algebraic monoids $\varphi : \hat{\mathbf{X}} \to A$, where A is commutative, there exists a unique morphism of linear algebraic monoids $\bar{\varphi} : \mathbb{A}^\ell \to A$ such that $\varphi = \bar{\varphi} \circ \pi$.

6.C. Comments

In characteristic 0, the results of 6.1 (except for Theorem 6.1.12) were obtained by de Concini–Procesi [DePr–83] for the wonderful compactification of any adjoint symmetric space, i.e., of any homogeneous space $G_{\mathrm{ad}}/G_{\mathrm{ad}}^\theta$, where θ denotes an involutive automorphism of G_{ad} and G_{ad}^θ denotes its fixed point subgroup. This includes the space $G_{\mathrm{ad}} = (G_{\mathrm{ad}} \times G_{\mathrm{ad}})/\operatorname{diag}(G_{\mathrm{ad}})$, since $\operatorname{diag}(G_{\mathrm{ad}})$ is the fixed point subgroup of the involution of $G_{\mathrm{ad}} \times G_{\mathrm{ad}}$ exchanging the two factors.

Then, Strickland [Str–87] extended these results to positive characteristics and obtained the existence of a splitting which is compatible with all the boundary divisors for the wonderful compactification of G_{ad}. This was generalized by de Concini-Springer [DeSp–99] to all the adjoint symmetric spaces in characteristic $p \neq 2$ (cf. also [Fal–97]).

The exposition in 6.1.A and 6.1.B follows rather closely [DeSp–99]. The arguments of 6.1.C are adapted from [BrPo–00]. In [loc cit.], it is observed that the splitting of Theorem 6.1.12 is also compatible with the closures $\overline{BwB}$, $w \in W$, called the large Schubert varieties. Further, the description of line bundles over $\mathbf{X}$ and the structure of their spaces of global sections are generalized to all the large Schubert varieties, which are shown to be normal and Cohen–Macaulay.

It is tempting to extend these results to the wonderful compactification $\mathbf{X}$ of an arbitrary adjoint symmetric space, by replacing the large Schubert varieties with the orbit closures of a Borel subgroup B. However, examples show that $\mathbf{X}$ may not be split compatibly with B-orbit closures, and that some of these are neither normal nor Cohen–Macaulay. We refer to [Bri–01] and [Pin–01] for such examples, and to [Bri–03b] for some positive results concerning a class of varieties that includes Schubert varieties and large Schubert varieties.

All the results of Section 6.2 are taken from [Rit–98] and [Rit–03], where these results are deduced from the Luna-Vust theory (generalized by Knop [Kno–91] in an arbitrary characteristic) of equivariant embeddings of spherical homogeneous spaces (i.e., those homogeneous spaces under a connected reductive group G that contain an open orbit of a Borel subgroup). Actually, in characteristic 0, the results of Section 6.2 (except for Theorem 6.2.7) extend to equivariant embeddings of spherical homogeneous spaces, cf. [BrIn–94], where it is also shown that the reduction mod. p of any such embedding is split for $p \gg 0$, compatibly with all the G-orbit closures. Here, we have attempted to give self-contained proofs in the setting of reductive group embeddings.

The classification of reductive monoids M (Theorem 6.2.13 (iii)) was obtained by Renner [Ren–85] under the additional assumption that M has a 1-dimensional center; whereas the structure of their coordinate ring as a representation (Theorem 6.2.13 (ii)) is due to Doty [Dot–99] under the same assumption. In characteristic 0, Proposition 6.2.12 and a different version of Theorem 6.2.13 (iii) are in [Vin–95]. The "universal" multicone $\hat{\mathbf{X}}$ studied in Exercises 6.1.E and 6.2.E is nothing but the "enveloping semigroup" of [loc cit.], as proved in [Rit–01].

Given a flag variety $X = G/B$ and an orbit closure $Y \subset X$ of a spherical subgroup of G, it is not known under what conditions is X split compatibly with Y. Similarly, it is not known when Y is normal or Cohen–Macaulay.

Chapter 7

Hilbert Schemes of Points on Surfaces

Introduction

The main aim of this chapter is to prove the following result of Kumar–Thomsen. For any nonsingular split surface X, the Hilbert scheme $X^{[n]}$ (parametrizing length-n subschemes of X) is split as well. Here, as earlier in this book, by split we mean Frobenius split. The proof relies on some results of Fogarty on the geometry of $X^{[n]}$ and a study of the Hilbert–Chow morphism $\gamma : X^{[n]} \longrightarrow X^{(n)}$, where $X^{(n)}$ denotes the n-fold symmetric product of X (parametrizing effective 0-cycles of degree n), and γ maps any length-n subscheme to its underlying cycle.

In Section 7.1, some fundamental properties of symmetric products $X^{(n)}$ are established, where X is an arbitrary quasi-projective scheme. In particular, it is shown that $X^{(n)}$ is a Gorenstein, $\mathbb{Q}$-factorial variety of dimension nd, if X is a nonsingular variety of dimension d (Lemmas 7.1.7 and 7.1.9). Further, the singular locus of $X^{(n)}$ is the complement of the locus $X^{(n)}_{**}$ of sum of n distinct points of X, if $d \geq 2$ (Lemma 7.1.6). On the other hand, every $X^{(n)}$ is nonsingular if $d = 1$ (Exercise 7.1.E.5).

Section 7.2 presents some general results on Hilbert schemes of points on quasi-projective schemes: their existence (Theorem 7.2.3) and the description of their Zariski tangent spaces (Lemma 7.2.5). Also, the punctual Hilbert scheme $X^{[n]}_x$ (parametrizing length-n subschemes supported at a given point x) is introduced, and it is shown that $X^{[n]}_x$ is projective and connected (Proposition 7.2.9).

The Hilbert–Chow morphism is introduced and studied in Section 7.3, again in the setting of quasi-projective schemes. The existence of a projective morphism of schemes $\gamma : X^{[n]} \longrightarrow X^{(n)}$ which yields the cycle map on closed points is deduced from work of Iversen (Theorem 7.3.1). The fibers of γ are products of punctual Hilbert schemes; their connectedness implies that $X^{[n]}$ is connected if X is (Corollary 7.3.4). Then, the loci $X^{[n]}_{**}$, resp. $X^{[n]}_{*}$, consisting of subschemes supported at n distinct points, resp. at least $n-1$ distinct points, are considered. In particular, it is shown that $X^{[n]}_{*}$ is a nonsingular variety of dimension nd, and the complement $X^{[n]}_{*} \setminus X^{[n]}_{**}$ is a nonsingular prime divisor, if X is a nonsingular variety of dimension d (Lemma 7.3.5).

Section 7.4 is devoted to Hilbert schemes of points on a nonsingular surface X. Each $X^{[n]}$ is shown to be a nonsingular variety of dimension $2n$ (Theorem 7.4.1) and the Hilbert–Chow morphism is shown to be birational, with exceptional set being a prime divisor (Proposition 7.4.5). Finally, the Hilbert–Chow morphism is shown to be crepant (Theorem 7.4.6), a result due to Beauville in characteristic 0 and to Kumar–Thomsen in characteristic $p \geq 3$.

Section 7.5 begins with the observation that any symmetric product of a split quasi-projective scheme is split as well (Lemma 7.5.1). Together with the crepantness of the Hilbert–Chow morphism, this implies the splitting of $X^{[n]}$, where X is a nonsingular split surface (Theorem 7.5.2). In turn, this yields the vanishing of higher cohomology groups of any ample invertible sheaf on $X^{[n]}$, if X is split and proper over an affine variety (Corollary 7.5.4). This applies, in particular, to the nonsingular projective split surfaces and also to the nonsingular affine surfaces (since these are split by Proposition 1.1.6). Further, we obtain a relative vanishing result for the Hilbert–Chow morphism (Corollary 7.5.5).

Notation. Throughout this chapter, X denotes a quasi-projective scheme over an algebraically closed field k of characteristic $p \geq 0$, and n denotes a positive integer. By schemes, as earlier in the book, we mean Noetherian separated schemes over k; their closed points will just be called points.

7.1 Symmetric products

The symmetric group S_n acts on the n-fold product $X^n = X \times \cdots \times X$ by permuting the factors. Let

$$X^{(n)} := X^n / S_n$$

be the set of orbits, with quotient map

$$\pi : X^n \longrightarrow X^{(n)}.$$

We endow $X^{(n)}$ with the quotient topology, i.e., a subset is open if and only if its preimage under π is. In particular, π is continuous. On the topological space $X^{(n)}$, we have the sheaf of rings $\pi_* \mathcal{O}_{X^n} = \pi_*(\mathcal{O}_X^{\boxtimes n})$. Since π is invariant, S_n acts on this sheaf. Let

$$\mathcal{O}_{X^{(n)}} := (\pi_* \mathcal{O}_{X^n})^{S_n}$$

be the subsheaf of S_n-invariants, then $(X^{(n)}, \mathcal{O}_{X^{(n)}})$ is a ringed space.

More generally, any sheaf $\mathcal{F}$ on X yields a sheaf $\mathcal{F}^{\boxtimes n}$ on X^n endowed with an action of S_n, and hence a sheaf

$$\mathcal{F}^{(n)} := (\pi_*(\mathcal{F}^{\boxtimes n}))^{S_n}$$

of $\mathcal{O}_{X^{(n)}}$-modules.

7.1.1 Lemma. *(i) With the preceding notation,* $(X^{(n)}, \mathcal{O}_{X^{(n)}})$ *is a quasi-projective scheme, and* π *is a finite surjective morphism. Moreover,* π *satisfies the following universal property:*

For any S_n*-invariant morphism* $\varphi : X^n \longrightarrow Z$*, where* Z *is a scheme (with trivial action of* S_n*), there exists a unique morphism* $\psi : X^{(n)} \longrightarrow Z$ *such that* $\varphi = \psi \circ \pi$.
(ii) If Y *is an open, resp. closed, subscheme of* X*, then* $Y^{(n)}$ *is an open, resp. closed, subscheme of* $X^{(n)}$.
(iii) If $\mathcal{F}$ *is a coherent sheaf on* X*, then the sheaf* $\mathcal{F}^{(n)}$ *on* $X^{(n)}$ *is coherent.*
(iv) If, in addition, $\mathcal{F}$ *is invertible, then so is* $\mathcal{F}^{(n)}$*, and*

$$\mathcal{F}^{\boxtimes n} \simeq \pi^* \mathcal{F}^{(n)}.$$

Further, if $\mathcal{F}$ *is ample, resp. very ample, then so is* $\mathcal{F}^{(n)}$.

Proof. (i) First we consider the case where X is projective. Then, let $\mathcal{L}$ be a very ample invertible sheaf on X. Then, $\mathcal{L}^{\boxtimes n}$ is a very ample invertible sheaf on X^n, and S_n acts on the graded algebra

$$R := R(X^n, \mathcal{L}^{\boxtimes n}) = \bigoplus_{\nu=0}^{\infty} \Gamma(X^n, (\mathcal{L}^{\boxtimes n})^{\nu})$$

by automorphisms. The invariant subalgebra

$$S := R^{S_n}$$

is also graded. Since the algebra R is finitely generated (Lemma 1.1.13(i)), then so is S, and the S-module R is finite, cf. [Eis–95, Exercise 13.2 and Theorem 13.17]. This yields a finite surjective morphism

$$\bar{\pi} : X^n = \mathrm{Proj}(R) \longrightarrow \mathrm{Proj}(S).$$

We claim that π may be identified with this morphism.

To see this, let σ be a nonzero global section of $\mathcal{L}$. Then, $\tau := \sigma^{\boxtimes n}$ is a nonzero S_n-invariant global section of $\mathcal{L}^{\boxtimes n}$, thus, an element of degree 1 in S. The open affine subset $\mathrm{Proj}(S)_\tau$ is the spectrum of the subring $S[\tau^{-1}]_0$ of homogeneous elements of degree 0 in the localization $S[\tau^{-1}]$. Moreover,

$$\bar{\pi}^{-1}(\mathrm{Proj}(S)_\tau) = \mathrm{Proj}(R)_\tau = (X^n)_\tau = (X_\sigma)^n,$$

where X_σ denotes the complement of the zero subscheme of σ. Therefore, the natural map

$$\mathcal{O}_{\mathrm{Proj}(S)_\tau} \longrightarrow (\bar{\pi}_* \mathcal{O}_{(X_\sigma)^n})^{S_n}$$

is an isomorphism. Thus, by [Eis–95, Proposition 13.10], it follows that the topological space $\mathrm{Proj}(S)_\tau$ is the orbit space $(X^n)_\tau / S_n$. To complete the proof (in the case where X is projective), observe that X^n is covered by its open affine S_n-stable subschemes

$(X^n)_\tau = (X_\sigma)^n$. Indeed, any finite subset of a projective space is contained in the complement of some hyperplane.

By the above proof, $X^{(n)}$ is a projective scheme, and π is a finite surjective morphism; its universal property is evident. Note that the twisting sheaf $\mathcal{O}_{\mathrm{Proj}(S)}(1)$ is invertible, and that

$$\mathcal{L}^{\boxtimes n} \simeq \pi^* \mathcal{O}_{\mathrm{Proj}(S)}(1).$$

Next, let $Y \subset X$ be a closed subscheme, where X is still assumed to be projective; then, we obtain an injective morphism $Y^{(n)} \longrightarrow X^{(n)}$. We check that this morphism is a closed immersion. For this, by taking an affine open cover, we may reduce as above to the case where $X = \mathrm{Spec}(A)$ is affine. Then, $Y = \mathrm{Spec}(B)$, where $B = A/I$ for some ideal I of A. Therefore, $X^{(n)} = \mathrm{Spec}((A^{\otimes n})^{S_n})$ and $Y^{(n)} = \mathrm{Spec}((B^{\otimes n})^{S_n})$. We may write $A = C \oplus I$, where $C \simeq B$ is a subspace of A. Then,

$$A^{\otimes n} = C^{\otimes n} \oplus I_n,$$

where $I_n := \bigoplus_{i=0}^{n-1} A^{\otimes i} \otimes I \otimes A^{\otimes n-i-1}$ is an S_n-stable subspace of $A^{\otimes n}$. Thus, the map

$$(A^{\otimes n})^{S_n} \longrightarrow (B^{\otimes n})^{S_n}$$

is surjective, as desired.

In the case where X is quasi-projective, we may write $X = \bar{X} \setminus Y$, where $\bar{X}$ is projective and $Y \subset \bar{X}$ is a closed subscheme. Then, one checks that $X^{(n)} = \bar{X}^{(n)} \setminus Y^{(n)}$ (as ringed spaces).

(ii) The assertion for open subschemes follows from the definition and universal property of π; the assertion for closed subschemes has been established in the proof of (i).

(iii) Since $\mathcal{F}^{\boxtimes n}$ is coherent on X^n and π is finite, $\pi_*(\mathcal{F}^{\boxtimes n})$ is a coherent sheaf on $X^{(n)}$. Thus, its subsheaf of S_n-invariants is coherent as well.

(iv) Let again $\mathcal{L}$ be a very ample invertible sheaf on X. Then, by [Har–77, Chap. II, Theorem 7.6], there exists a positive integer m such that the invertible sheaf $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^m$ is very ample. As observed in the proof of (i), both $\mathcal{L}^{\boxtimes n}$ and $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^m)^{\boxtimes n}$ are pullbacks of invertible sheaves on $X^{(n)}$. Thus, there exists an invertible sheaf $\mathcal{G}$ on $X^{(n)}$ such that

$$\mathcal{F}^{\boxtimes n} \simeq \pi^* \mathcal{G}.$$

By the projection formula, it follows that $\pi_*(\mathcal{F}^{\boxtimes n}) \simeq \mathcal{G} \otimes_{\mathcal{O}_{X^{(n)}}} \pi_* \mathcal{O}_{X^n}$. Taking S_n-invariants, we obtain $\mathcal{F}^{(n)} \simeq \mathcal{G}$, which completes the proof of the first part of (iv).

If $\mathcal{F}$ is very ample, then so is $\mathcal{F}^{(n)}$ by the proof of (i). Now, if $\mathcal{F}$ is ample, then $\mathcal{F}^\nu$ is very ample for some $\nu \geq 1$. But, $(\mathcal{F}^\nu)^{(n)} \simeq (\mathcal{F}^{(n)})^\nu$ by the preceding argument, so that $\mathcal{F}^{(n)}$ is ample as well. $\square$

7.1.2 Definition. The scheme $X^{(n)}$ is called the n-fold *symmetric product* of X.

The points of $X^{(n)}$ may be regarded as 0-cycles on X as follows. Recall that the *group of* 0*-cycles* on X, denoted $Z_0(X)$, is the free abelian group on all points. In other words, a 0-cycle on X is a finite formal sum $z = \sum n_i x_i$, where the x_i are points of X, and the n_i are integers. The *degree* of z is, by definition, $\sum n_i$, and z is *effective* if all n_i are nonnegative. Now, for any point $(x_1, \dots, x_n)$ of X^n, its S_n-orbit $\pi(x_1, \dots, x_n)$ is uniquely determined by $x_1 + \cdots + x_n \in Z_0(X)$. This allows us to identify the points of $X^{(n)}$ with the effective 0-cycles of degree n on X.

7.1.3 Lemma. *Let $n_1, \dots, n_r$ be positive integers with sum n and let $x_1, \dots, x_r$ be distinct points of X. Then, the map $\pi : X^n \longrightarrow X^{(n)}$ factors through a morphism $X^{(n_1)} \times \cdots \times X^{(n_r)} \longrightarrow X^{(n)}$ which is étale at the point $n_1 x_1 + \cdots + n_r x_r$.*

Proof. This follows from the argument in [Mum–70, pp. 68–69]. Specifically, as in the proof of Lemma 7.1.1, we may reduce to the case where X is affine. Then, $X^n = \operatorname{Spec}(R)$ for some k-algebra R and $X^{(n)} = \operatorname{Spec}(S)$, where $S := R^{S_n}$. Let $z := n_1 x_1 + \cdots + n_r x_r$ and $\mathcal{M}$ the corresponding maximal ideal of S; let $\hat{R}$, $\hat{S}$ be the completions of R, S respectively, for the $\mathcal{M}$-adic topology. Then, $\hat{S} = \hat{\mathcal{O}}_{X^{(n)},z}$ by definition, and the map $\hat{S} \otimes_S R \longrightarrow \hat{R}$ is an isomorphism, since R is finite over S.

The assumption that S is the ring of S_n-invariants in R is equivalent to S being the kernel of the (S-module) map

$$R \longrightarrow \prod_{\sigma \in S_n} R, \quad f \longmapsto (\sigma(f) - f)_{\sigma \in S_n}.$$

Since $\hat{S}$ is flat over S, it follows that $\hat{S}$ is the kernel of the corresponding map $\hat{S} \otimes_S R \longrightarrow \prod_{\sigma \in S_n} \hat{S} \otimes_S R$. As a consequence, the isomorphism $\hat{S} \otimes_S R \simeq \hat{R}$ restricts to an isomorphism

$$\hat{S} \simeq \hat{R}^{S_n}.$$

The prime ideals of R containing $\mathcal{M}$ are exactly the maximal ideals of the points of the set-theoretic fiber $\pi^{-1}(z)$; this yields an isomorphism

$$\hat{R} \simeq \prod_{y \in \pi^{-1}(z)} \hat{\mathcal{O}}_{X^n,y},$$

which is equivariant for the action of S_n by permuting the points of $\pi^{-1}(z)$. Moreover, the isotropy group of

$$y := (x_1 \ (n_1 \text{ times}), \dots, x_r \ (n_r \text{ times}))$$

is the product $S_{n_1} \times \cdots \times S_{n_r}$. Thus, taking S_n-invariants in $\hat{R}$ yields an isomorphism

$$\hat{S} \simeq \hat{\mathcal{O}}_{X^n,y}^{S_{n_1} \times \cdots \times S_{n_r}}.$$

Since

$$\mathcal{O}_{X^n,y}^{S_{n_1} \times \cdots \times S_{n_r}} = \mathcal{O}_{X^{(n_1)} \times \cdots \times X^{(n_r)},(n_1 x_1, \dots, n_r x_r)},$$

the proof is completed. □

We shall apply this lemma to describe the loci of $X^{(n)}$ where all the points are distinct, or at most two may coincide. Specifically, for $1 \le i < j \le n$, let $\Delta_{i,j}$ be the partial diagonal $(x_i = x_j)$ in X^n, and put

$$X^n_s := \bigcup_{i<j} \Delta_{i,j}.$$

This is a closed S_n-stable subscheme of X^n; we put

$$X^n_{**} := X^n \setminus X^n_s,$$

an open S_n-stable subscheme of X^n. The points of X^n_{**} are the n-tuples of distinct points of X. The image of X^n_{**} in $X^{(n)}$ will be denoted $X^{(n)}_{**}$; this is an open subscheme of $X^{(n)}$. Likewise, we denote $X^{(n)}_s$ the image of X^n_s in $X^{(n)}$, a closed subscheme of $X^{(n)}$. Now, Lemma 7.1.3 implies the following.

7.1.4 Lemma. *With the preceding notation, π restricts to an étale surjective morphism $X^n_{**} \longrightarrow X^{(n)}_{**}$. Thus, if X is nonsingular, then so is $X^{(n)}_{**}$.*

Likewise, consider the partial diagonals $\Delta_{i,j,k} := (x_i = x_j = x_k)$, where $1 \le i < j < k \le n$. Let X^n_* be the complement of their union, which is a S_n-stable open subscheme of X^n, containing X^n_{**}. Put $X^{(n)}_* := \pi(X^n_*)$; this is an open subscheme of $X^{(n)}$ with points being the 0-cycles $x_1 + x_2 + \cdots + x_n$, where only x_1 and x_2 may coincide. Also, put $X^{(n)}_{s*} := X^{(n)}_* \cap X^{(n)}_s$, a closed subscheme of $X^{(n)}_*$, with points being the 0-cycles $2x_1 + x_3 + \cdots + x_n$, where $x_1, x_3, \ldots, x_n$ are pairwise distinct.

For example, if $n = 2$ then $X^2_* = X^2$, and $X^2_{s*} = X^2_s$ is just the diagonal.

Finally, let $(X^{(2)} \times X^{(n-2)})_*$ be the image in $X^{(2)} \times X^{(n-2)}$ of the open subscheme $X^n \setminus \bigcup_{(i,j)\neq(1,2)} \Delta_{i,j}$. Since this subscheme is invariant under $S_2 \times S_{n-2}$, $(X^{(2)} \times X^{(n-2)})_*$ is open in $X^{(2)} \times X^{(n-2)}$. Its points are the pairs $(x_1 + x_2, x_3 + \cdots + x_n)$, where only x_1 and x_2 may coincide.

7.1.5 Lemma. *The quotient morphism $X^n_* \longrightarrow X^{(n)}_*$ factors through an étale surjective morphism*

$$(X^{(2)} \times X^{(n-2)})_* \longrightarrow X^{(n)}_*, \ (x_1 + x_2, x_3 + \cdots + x_n) \longmapsto x_1 + \cdots + x_n,$$

which restricts to an isomorphism

$$(X^{(2)}_s \times X^{(n-2)})_* \simeq X^{(n)}_{s*},$$

where $(X^{(2)}_s \times X^{(n-2)})_* := (X^{(2)}_s \times X^{(n-2)}) \cap (X^{(2)} \times X^{(n-2)})_*$. Moreover, the restriction of the quotient $X^2 \longrightarrow X^{(2)}$ to the diagonal induces a bijective morphism $X \longrightarrow X^{(2)}_s$, which is an isomorphism if $p \neq 2$, and which may be identified with the Frobenius morphism $F : X \longrightarrow X$ if $p = 2$.

Proof. By Lemma 7.1.3, the quotient map $\pi : X^n \longrightarrow X^{(n)}$ factors through a surjective morphism $X^{(2)} \times X^{(n-2)} \longrightarrow X^{(n)}$. Let $(X^{(2)} \times X^{n-2})_*$ be the preimage of $(X^{(2)} \times X^{(n-2)})_*$ in $X^{(2)} \times X^{n-2}$. Then, by Lemmas 7.1.4 and 7.1.3, the morphisms $(X^{(2)} \times X^{n-2})_* \longrightarrow (X^{(2)} \times X^{(n-2)})_*$ and $(X^{(2)} \times X^{n-2})_* \longrightarrow X^{(n)}_*$ are étale and surjective. Thus, the morphism $(X^{(2)} \times X^{(n-2)})_* \longrightarrow X^{(n)}_*$ is étale and surjective as well. Its pullback to $X^{(n)}_{s*}$ yields an étale morphism

$$(X^{(2)}_s \times X^{(n-2)})_* \longrightarrow X^{(n)}_{s*}, \ (2x_1, x_3 + \cdots + x_n) \longmapsto 2x_1 + x_3 + \cdots + x_n,$$

which is bijective on points, and hence an isomorphism.

Clearly, the quotient map $\pi : X^2 \longrightarrow X^{(2)}$ maps the diagonal Δ bijectively to $X^{(2)}_s$. If $p \neq 2$, then the restriction $\Delta \longrightarrow X^{(2)}_s$ is an isomorphism by Exercise 7.1.E.1. To check the assertion for $p = 2$, we may reduce to the case where $X = \mathbb{A}^d$ by using Lemma 7.1.1 (ii). Then, $X^2 = \mathbb{A}^{2d}$ with coordinates $t_1, \dots, t_d, u_1, \dots, u_d$ such that the nontrivial element $\sigma \in S_2$ exchanges each t_i with u_i. Now, the S_2-invariants in the polynomial ring $k[t_1, \dots, t_d, u_1, \dots, u_d]$ are spanned by the invariant monomials (that is, by the monomials in the products $t_i u_i$), together with the sums $M + \sigma(M)$ where M is a non-invariant monomial. So, the restrictions of these S_2-invariants to the diagonal $(t_1 = u_1, \dots, t_d = u_d)$ are just the polynomials in $t_1^2, \dots, t_d^2$. □

Note also that π restricts to a finite surjective morphism $\Delta_{1,2} \longrightarrow X^{(n)}_s$. Together with Lemma 7.1.5 and purity of the branch locus, this yields the following.

7.1.6 Lemma. *If X is a variety of dimension d, then $X^{(n)}$ is a variety of dimension nd, and $X^{(n)}_s$ is a subvariety of codimension d, containing $X^{(n)}_{s*}$ as a dense open subset.*

If, in addition, X is nonsingular, then so is $X^{(n)}_{s*}$; but $X^{(n)}_*$ is singular along $X^{(n)}_{s*}$ when $d \geq 2$. In fact, the singular locus of $X^{(n)}$ is then $X^{(n)}_s$.

(If X is nonsingular of dimension 1, then $X^{(n)}$ is nonsingular as well (Exercise 7.1.E.5).)

Next, we assume that X is a normal variety. Recall from 1.3.12 that the canonical sheaf ω_X is defined as the sheaf $i_* \omega_{X^{\mathrm{reg}}}$, where $i : X^{\mathrm{reg}} \longrightarrow X$ denotes the inclusion of the nonsingular locus, and $\omega_{X^{\mathrm{reg}}}$ denotes the sheaf of differential forms of top degree. Then, ω_X is *divisorial*, i.e., it is the sheaf of local sections of a Weil divisor (the canonical divisor, uniquely defined up to linear equivalence). Recall from 1.3.12 that X is Gorenstein if ω_X is invertible, that is, if the canonical divisor is Cartier.

Note that each X^n is normal with canonical sheaf $\omega_X^{\boxtimes n}$; as a consequence, X^n is Gorenstein if and only if X is.

7.1.7 Lemma. *(i) If X is a normal variety, then $X^{(n)}$ is a normal variety as well. Moreover, for any divisorial sheaf $\mathcal{F}$ on X, the sheaf $\mathcal{F}^{(n)}$ is divisorial.*
(ii) If, in addition, $\dim(X) \geq 2$, *then $\omega_{X^{(n)}} \simeq (\omega_X)^{(n)}$.*
(iii) If X is Gorenstein, then so is $X^{(n)}$.

Proof. (i) For the normality of $X^{(n)}$, we may assume that it is affine; then X is affine, since π is finite. Now, the assertion follows from Exercise 7.1.E.2.

Let $\mathcal{F}$ be a divisorial sheaf on X. Then, there exist an open subscheme U of X, and an invertible sheaf $\mathcal{L}$ on U such that $X \setminus U$ has codimension at least 2 and $\mathcal{F} = i_*\mathcal{L}$, where $i : U \longrightarrow X$ denotes the inclusion. This yields open immersions $i^n : U^n \longrightarrow X^n$ and $i^{(n)} : U^{(n)} \longrightarrow X^{(n)}$, and (by Lemma 7.1.1) an invertible sheaf $\mathcal{M}$ on $U^{(n)}$ such that $\mathcal{L}^{\boxtimes n} \simeq \pi^*\mathcal{M}$. Moreover, $\mathcal{F}^{\boxtimes n} = (i^n)_*\mathcal{L}^{\boxtimes n}$, so that

$$\pi_*(\mathcal{F}^{\boxtimes n}) = (i^{(n)})_*\pi_*(\mathcal{L}^{\boxtimes n}) = (i^{(n)})_*(\mathcal{M} \otimes_{\mathcal{O}_{U^{(n)}}} \mathcal{O}_{U^n}).$$

Therefore, $\mathcal{F}^{(n)} = (i^{(n)})_*\mathcal{M}$. Since $X^{(n)} \setminus U^{(n)}$ has codimension at least 2 in $X^{(n)}$, it follows that $\mathcal{F}^{(n)}$ is divisorial.

(ii) To check this isomorphism between divisorial sheaves, we may replace $X^{(n)}$ with any open subset U such that the codimension of $X^{(n)} \setminus U$ is at least 2. By Lemma 7.1.6 and the assumption on $\dim(X)$, we may take $U = (X^{\mathrm{reg}})^{(n)}_{**}$; in particular, we may assume that X is nonsingular. Then, the restriction $X^n_{**} = \pi^{-1}(U) \longrightarrow U$ is étale by Lemma 7.1.4. Thus, we have an isomorphism $\pi^*\omega_U \simeq \omega_{X^n_{**}}$. Applying π_* and taking S_n-invariants yields the desired isomorphism.

(iii) follows from Lemma 7.1.1 (iv) and the second part of this lemma. □

7.1.8 Definition. A normal variety X is called *$\mathbb{Q}$-factorial at a point x*, if the divisor class group of the local ring $\mathcal{O}_{X,x}$ is torsion. If this holds at all points, then X is called *$\mathbb{Q}$-factorial*. This is equivalent to the following condition:

For any divisor D of X, there exists a positive integer N such that the divisor ND is Cartier.

7.1.9 Lemma. *If X is nonsingular, then $X^{(n)}$ is $\mathbb{Q}$-factorial.*

Proof. We may assume that $\dim(X) \geq 2$. As in the proof of Lemma 7.1.3, we consider a point y of X^n, and its image $z = n_1x_1 + \cdots + n_rx_r$ in $X^{(n)}$, where the points $x_1, \ldots, x_r$ are distinct. We put $\Gamma := S_{n_1} \times \cdots \times S_{n_r}$ and $N := |\Gamma| = n_1! \cdots n_r!$. Then, the ring $\mathcal{O}^{\Gamma}_{X^n,y}$ is étale over its subring $\mathcal{O}_{X^{(n)},z}$, and both are normal. Thus, the divisor class group of $\mathcal{O}_{X^{(n)},z}$ injects into that of $\mathcal{O}^{\Gamma}_{X^n,y}$ by [Bou–98, Chap. VII, §1.10]. Further, $\mathcal{O}^{\Gamma}_{X^n,y} = \mathcal{O}_{X^{(n_1)}\times\cdots\times X^{(n_r)},(n_1x_1,\ldots,n_rx_r)}$. Thus, it suffices to show that $X^{(n_1)} \times \cdots \times X^{(n_r)}$ is $\mathbb{Q}$-factorial at $(n_1x_1, \ldots, n_rx_r)$.

Let

$$\varphi : X^n \longrightarrow X^{(n_1)} \times \cdots \times X^{(n_r)}$$

be the quotient map. Let D be a prime divisor in $X^{(n_1)} \times \cdots \times X^{(n_r)}$ containing the point $(n_1x_1, \ldots, n_rx_r)$. Since φ is étale in codimension 1, we may define the pullback $\varphi^*(D)$, a multiplicity-free sum of prime divisors in X^n. Clearly, $\varphi^*(D)$ is invariant under Γ. Let $f \in \mathcal{O}_{X^n,y}$ be a local equation of $\varphi^*(D)$. Then, $\prod_{\gamma\in\Gamma}(\gamma \cdot f)$ is a local equation of $N\varphi^*(D)$, and belongs to $\mathcal{O}^{\Gamma}_{X^n,y}$. Thus, ND admits a local equation at $(n_1x_1, \ldots, n_rx_r)$. □

7.1.10 Remarks. (i) More generally, for any quasi-projective scheme Y endowed with an action of a finite group G, we may consider the quotient map $\pi : Y \longrightarrow Y/G$ and the quotient topology on Y/G. Then, the ringed space $(Y/G, (\pi_* \mathcal{O}_Y)^G)$ is again a quasi-projective scheme. Further, π is a finite, surjective morphism, universal for G-invariant morphisms with source Y.

If, in addition, Z is an open G-stable subscheme of Y, then Z/G is an open subscheme of Y/G (for these results, see, e.g., [Mum–70, Chap. II, §7]).

On the other hand, if Z is a closed G-invariant subscheme of Y, then the map $Z/G \longrightarrow Y/G$ is a closed immersion when the order of G is prime to p, but not in general (Exercise 7.1.E.1).

(ii) For G, Y, π as above, and a point y of Y with isotropy group G_y, the map π factors through a morphism $Y/G_y \longrightarrow Y/G$ which is étale at the image of y (as may be shown by the argument of Lemma 7.1.3). In particular, π is étale at the points with trivial isotropy group.

(iii) For G, Y, π as above, if Y is a normal variety, then so is Y/G (Exercise 7.1.E.2). If, in addition, Y is $\mathbb{Q}$-factorial and π is étale in codimension 1, then Y/G is $\mathbb{Q}$-factorial by the proof of Lemma 7.1.9.

7.1.E Exercises

(1*) Let R be a ring, I an ideal, and G a finite group of automorphisms of R that leaves I stable. If the order of G is prime to the characteristic of R, show that the map $R^G \longrightarrow (R/I)^G$ is surjective.

Deduce that the map $Z/G \longrightarrow Y/G$ is a closed immersion, for a quasi-projective scheme Y endowed with an action of a finite group G and a closed G-stable subscheme $Z \subset Y$, if the order of G is prime to p. Show by examples that the latter assumption cannot be omitted.

(2*) Let R be a normal domain and let G be a finite group of automorphisms of R. Show that the invariant subring R^G is a normal domain.

(3) Regarding the affine n-space as the space of polynomials in one variable t of degree at most n and constant term 1, show that the map

$$\pi : (\mathbb{A}^1)^n \longrightarrow \mathbb{A}^n, \ (x_1, \ldots, x_n) \longmapsto (1 + tx_1) \cdots (1 + tx_n)$$

factors through an isomorphism $(\mathbb{A}^1)^{(n)} \simeq \mathbb{A}^n$.

(4) Likewise, show that the map

$$\pi : ([x_1, y_1], \ldots, [x_n, y_n]) \longmapsto [(y_1 + tx_1) \cdots (y_n + tx_n)]$$

factors through an isomorphism $(\mathbb{P}^1)^{(n)} \simeq \mathbb{P}^n$.

(5*) Let X be a nonsingular curve (in particular, X is quasi-projective). Show that $X^{(n)}$ is nonsingular; deduce that $\pi : X^n \longrightarrow X^{(n)}$ is flat. Also, show that

$$\omega_{X^{(n)}} \simeq (\pi_*(\omega_{X^n}(-X^n_s)))^{S_n}.$$

7.2 Hilbert schemes of points

7.2.1 Definition. A *length-n subscheme of* X is a finite subscheme Y such that the k-vector space $\Gamma(Y, \mathcal{O}_Y)$ has dimension n.

Examples are the reduced unions of n distinct points $x_1, \ldots, x_n$ of X. More generally, we associate to any length-n subscheme Y its connected components $Y_1, \ldots, Y_r$; then each Y_i is a subscheme of finite length, supported at a unique point x_i. We say that the length n_i of Y_i is the *multiplicity of* Y *at* x_i. Then, $\Gamma(Y, \mathcal{O}_Y) = \mathcal{O}_{Y,x_1} \times \cdots \times \mathcal{O}_{Y,x_r}$ and $n_i = \dim_k(\mathcal{O}_{Y,x_i})$ for $1 \le i \le r$; thus, $n = n_1 + \cdots + n_r$.

Next, we define a relative version of length-n subschemes.

7.2.2 Definition. A *flat family of length-n subschemes* of X over a scheme S is a closed subscheme $\mathcal{Y} \subset X \times S$, finite over S via the restriction $\pi : \mathcal{Y} \longrightarrow S$ of the projection $X \times S \longrightarrow S$, and such that $\pi_* \mathcal{O}_{\mathcal{Y}}$ is a locally free $\mathcal{O}_S$-module of rank n.

Equivalently, π is finite and flat, with fibers being length-n subschemes of X.

Given such a family $\mathcal{Y} \subset X \times S$ and a morphism of schemes $f : S' \longrightarrow S$, we can form the cartesian square

$$\begin{array}{ccc} \mathcal{Y} \times_S S' & \xrightarrow{\pi'} & S' \\ {\scriptstyle f'}\big\downarrow & & {\scriptstyle f}\big\downarrow \\ \mathcal{Y} & \xrightarrow{\pi} & S. \end{array}$$

Then, $\pi' : \mathcal{Y} \times_S S' \longrightarrow S'$ is a flat family of length-n subschemes of X over S', called the *pullback* of π under f.

We may now formulate the following fundamental result, which is a special case of the existence of the Hilbert scheme of a quasi-projective scheme [Gro–62]. For the proof, we refer to loc cit.

7.2.3 Theorem. *Fix $n \ge 1$ and let X be a quasi-projective scheme. Then, there exists a unique scheme $X^{[n]}$, together with a flat family $\pi_n : \widetilde{X}^{[n]} \longrightarrow X^{[n]}$ of length-n subschemes of X, satisfying the following universal property:*

For any flat family $\pi : \mathcal{Y} \longrightarrow S$ of length-n subschemes of X, there exists a unique morphism $f : S \to X^{[n]}$ such that π is the pullback of π_n under f.

Further, $X^{[n]}$ and $\widetilde{X}^{[n]}$ are quasi-projective. In fact, if $X \longrightarrow \bar{X}$ is an open immersion into a projective scheme, then $\bar{X}^{[n]}$ is projective, and $X^{[n]}$ can be identified with the open subscheme of $\bar{X}^{[n]}$ parametrizing length-n subschemes with support in X.

7.2.4 Definition. The scheme $X^{[n]}$ is called the *Hilbert scheme of n points* on X. The morphism $\pi_n : \widetilde{X}^{[n]} \longrightarrow X^{[n]}$ is called the *universal family*.

The universal property of $X^{[n]}$ implies that its points are precisely the length-n subschemes of X. This property also allows to determine its Zariski tangent spaces as follows.

7.2.5 Lemma. *For any point Y of $X^{[n]}$, the Zariski tangent space $T_Y X^{[n]}$ equals*

$$\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{I}_Y, \mathcal{O}_Y) = \operatorname{Hom}_{\mathcal{O}_Y}(\mathcal{I}_Y/\mathcal{I}_Y^2, \mathcal{O}_Y).$$

As a consequence,

$$T_Y X^{[n]} = T_{Y_1} X^{[n_1]} \oplus \cdots \oplus T_{Y_r} X^{[n_r]},$$

where $Y_1, \ldots, Y_r$ are the connected components of Y, and $n_1, \ldots, n_r$ the corresponding multiplicities.

Proof. Let $S := \operatorname{Spec}(k[\varepsilon]/(\varepsilon^2))$ with (unique) point s. Then, the Zariski tangent space $T_Y X^{[n]}$ consists of those morphisms $S \to X^{[n]}$ that map s to Y. By Theorem 7.2.3, these morphisms may be identified with the closed subschemes $\mathcal{Y} \subset X \times S$, finite and flat over S, with fiber Y at s. These are the infinitesimal deformations of Y in X, classified by $\operatorname{Hom}_X(\mathcal{I}_Y, \mathcal{O}_Y)$, cf. [Har–77, Chap. III, Exercise 9.7]. □

Next, we study the connected length-n subschemes. We begin with the following easy result.

7.2.6 Lemma. *Let Y be a length-n subscheme of X, supported at a unique point x. Then,*

$$\mathcal{M}_x^n \subset \mathcal{I}_Y \subset \mathcal{M}_x,$$

where $\mathcal{M}_x$ denotes the maximal ideal of the local ring $\mathcal{O}_{X,x}$ of X at x.

Proof. Clearly, $\mathcal{I}_Y \subset \mathcal{M}_x$. To show the other inclusion, consider the local algebra $R := \Gamma(Y, \mathcal{O}_Y) \simeq \mathcal{O}_{X,x}/\mathcal{I}_Y$, its maximal ideal $\mathcal{M} := \mathcal{M}_x/\mathcal{I}_Y$, and its positive powers $\mathcal{M}^\nu$; these form a decreasing sequence of subspaces of R. By Nakayama's lemma, $\mathcal{M}^{\nu+1} \neq \mathcal{M}^\nu$ unless $\mathcal{M}^{\nu+1} = 0$. Since the dimension of R as a k-vector space is n, it follows that $\mathcal{M}^n = 0$. □

7.2.7 Definition. Let x be a point of X and let $X_x := \operatorname{Spec}(\mathcal{O}_{X,x}/\mathcal{M}_x^n)$ (a finite-length subscheme of X, supported at x). The n-th *punctual Hilbert scheme* of X at x is defined to be the Hilbert scheme $X_x^{[n]}$.

In the case where $n = 2$, Lemma 7.2.6 readily implies a complete description of $X_x^{[n]}$.

7.2.8 Lemma. *The ideal sheaves of length-2 subschemes of X supported at x are exactly the preimages in $\mathcal{M}_x$ of hyperplanes in $\mathcal{M}_x/\mathcal{M}_x^2$. Thus, $X_x^{[2]}$ is isomorphic to $\mathbb{P}(T_x X)$, the projective space of lines in $T_x X$.*

Returning to arbitrary lengths, we have the following.

7.2.9 Proposition. *$X_x^{[n]}$ is projective and connected.*

Proof. Let $R := \mathcal{O}_{X,x}/\mathcal{M}_x^n$, a local k-algebra with maximal ideal $\mathcal{M} := \mathcal{M}_x/\mathcal{M}_x^n$. By Lemma 7.2.6, the ideals of length-n subschemes of X_x are exactly those linear subspaces $I \subset \mathcal{M}$ of codimension $n-1$ such that $\mathcal{M}I \subset I$. Since $1+x$ is invertible for any $x \in \mathcal{M}$, the latter condition amounts to $(1+\mathcal{M})I = I$. This realizes the underlying set of $X_x^{[n]}$ as the subset of the Grassmannian variety, $\mathrm{Grass}^{n-1}(\mathcal{M})$, consisting of the subspaces of codimension $n-1$, fixed by the action of the group $1+\mathcal{M}$ via multiplication in R. Since $\mathrm{Grass}^{n-1}(\mathcal{M})$ is projective, $X_x^{[n]}$ is projective as well.

To show that $X_x^{[n]}$ is connected, note that the group $1+\mathcal{M}$ is abelian and has a decreasing filtration by the subgroups $1+\mathcal{M}^\nu$. Since $\mathcal{M}^n = 0$, this filtration is finite; its successive quotients

$$(1+\mathcal{M}^\nu)/(1+\mathcal{M}^{\nu+1}) \simeq \mathcal{M}^\nu/\mathcal{M}^{\nu+1}$$

are finite-dimensional k-vector spaces, thus, products of copies of the additive group $\mathbb{G}_a$. So, the proof will be completed by the following result.

7.2.10 Lemma. *Let X be a complete connected scheme equipped with an action of the additive group $\mathbb{G}_a$. Then, the fixed point subscheme $X^{\mathbb{G}_a}$ is connected.*

Proof. By the Borel fixed point theorem (cf. [Bor–91, Theorem 10.4]), any non-empty complete scheme with an action of $\mathbb{G}_a$ contains fixed points. Thus, we may reduce to the case where X is irreducible. We argue by induction on the dimension d of X.

If $d = 1$, consider the normalization

$$f : \widetilde{X} \to X.$$

Then, $\widetilde{X}$ is a complete nonsingular irreducible curve, and the action of $\mathbb{G}_a$ lifts to $\widetilde{X}$ so that f is equivariant. Hence, either $\mathbb{G}_a$ fixes $\widetilde{X}$ pointwise, or $\widetilde{X}$ is a projective line and $\mathbb{G}_a$ acts by translations. In both cases, $\widetilde{X}^{\mathbb{G}_a}$ is connected. Since $X^{\mathbb{G}_a}$ equals $f(\widetilde{X}^{\mathbb{G}_a})$, it is connected as well.

If $d > 1$, then there exists a nonconstant $\mathbb{G}_a$-invariant rational function f on X. Let $\mathbb{G}_a$ act on $X \times \mathbb{P}^1$ via its action on X and the trivial action on $\mathbb{P}^1$, and let $\widetilde{X}$ be the closure in $X \times \mathbb{P}^1$ of the subset of pairs (x, t) such that: f is defined at x, and $t = f(x)$. Then, $\widetilde{X}$ is a complete variety, stable under $\mathbb{G}_a$; the first projection $\pi : \widetilde{X} \longrightarrow X$ is $\mathbb{G}_a$-equivariant, surjective and birational, and the second projection

$$\widetilde{f} : \widetilde{X} \longrightarrow \mathbb{P}^1$$

is a surjective $\mathbb{G}_a$-invariant morphism. Consider the Stein factorization

$$\widetilde{X} \xrightarrow{\ \varphi\ } C \xrightarrow{\ \psi\ } \mathbb{P}^1$$

of $\widetilde{f}$, where $C := \mathrm{Spec}_{\mathcal{O}_{\mathbb{P}^1}}(\widetilde{f}_*\mathcal{O}_{\widetilde{X}})$. Then, ψ is finite and surjective. Further, all the fibers $\varphi^{-1}(c)$ are non-empty and connected of dimension $d-1$. Note that $\mathbb{G}_a$ acts on C and that both φ and ψ are equivariant (with the trivial action of $\mathbb{G}_a$ on $\mathbb{P}^1$). But, since ψ

is finite and $\widetilde{f} = \psi \circ \varphi$ is invariant, it follows that $\mathbb{G}_a$ fixed C pointwise. Thus, φ maps $\widetilde{X}^{\mathbb{G}_a}$ onto C with fibers $\varphi^{-1}(c)^{\mathbb{G}_a}$; these are connected by the induction assumption. By Exercise 7.2.E.1, it follows that $\widetilde{X}^{\mathbb{G}_a}$ is connected. Hence, $X^{\mathbb{G}_a} = \pi(\widetilde{X}^{\mathbb{G}_a})$ is connected as well. $\square$

7.2.E Exercises

(1*) Let $f : X \to Y$ be a proper surjective morphism of schemes. Assume that Y and all the fibers of f are connected. Then, show that X is connected.

Hint: Use the Stein factorization.

In the exercises below, we shall sketch a description of the punctual Hilbert schemes in terms of linear algebra; the case where X is a surface will be developed further in Exercises 7.4.E.

(2*) Let x be a nonsingular point of X. Show that $X_x^{[n]}$ is isomorphic to $(\mathbb{A}^d)_0^{[n]}$, where d denotes the dimension of X at x, and 0 denotes the origin of the affine space $\mathbb{A}^d$.

(3*) Let Y be a point of $(\mathbb{A}^d)_0^{[n]}$. Then, $\Gamma(Y, \mathcal{O}_Y)$ is a module of length n over the local ring

$$R := k[x_1, \ldots, x_d]/(x_1, \ldots, x_d)^n,$$

generated by one element (for example, the identity); we say that this module admits a *cyclic vector*. Moreover, the annihilator of this vector is the ideal of Y.

Conversely, let V be an R-module of length n admitting a cyclic vector v. Show that the annihilator of v is the ideal of a point of $(\mathbb{A}^d)_0^{[n]}$.

(4*) Let $\mathcal{N}_{d,n}$ be the subset of $M_n(k)^d$ consisting of d-tuples of nilpotent, pairwise commuting $n \times n$ matrices, where $M_n(k)$ is the space of all the $n \times n$ matrices over k. For any $(A_1, \ldots, A_d) \in \mathcal{N}_{d,n}$, note that the subring $k[A_1, \ldots, A_d] \subset M_n(k)$ is a quotient of R, where R is as in the above exercise; this yields an R-module structure on k^n. Let $Z_{d,n} \subset \mathcal{N}_{d,n} \times k^n$ be the subset of those $(A_1, \ldots, A_d; v)$ such that v is a cyclic vector for the R-module k^n.

Show that $\mathcal{N}_{d,n}$ is closed in $M_n(k)^d$, and that $Z_{d,n}$ is open in $\mathcal{N}_{d,n} \times k^n$. Also, show that the map $f : Z_{d,n} \longrightarrow (\mathbb{A}^d)_0^{[n]}$, which takes any $(A_1, \ldots, A_d; v)$ to the zero subscheme of the annihilator of v in R, is a morphism.

(5*) The group $\mathrm{GL}_n(k)$ acts in $M_n(k)^d \times k^n$ by

$$g \cdot (A_1, \ldots, A_d; v) = (gA_1g^{-1}, \ldots, gA_dg^{-1}; gv),$$

and this action leaves $Z_{d,n}$ stable. Show that f is invariant under $\mathrm{GL}_n(k)$, with its (set-theoretic) fibers being exactly the orbits.

(6*) Show that the isotropy group of any point of $Z_{d,n}$ is trivial; prove the same for the isotropy Lie algebra. Deduce that the quotient $Z_{d,n} \longrightarrow Z_{d,n}/\,\mathrm{GL}_n(k)$ exists and is a principal $\mathrm{GL}_n(k)$-bundle, and that f factors through a bijective morphism $Z_{d,n}/\,\mathrm{GL}_n(k) \longrightarrow (\mathbb{A}^d)_0^{[n]}$. Conclude that

$$\dim(X_x^{[n]}) = \dim(Z_{d,n}) - n^2,$$

for any nonsingular point $x \in X$.

7.3 The Hilbert–Chow morphism

Any length-n subscheme Y of X defines an effective 0-cycle of degree n,

$$[Y] = n_1 x_1 + \cdots + n_r x_r,$$

where $x_1, \ldots, x_r$ are the (distinct) points of Y, with respective multiplicities $n_1, \ldots, n_r$. This yields a cycle map

$$X^{[n]} \longrightarrow X^{(n)}, \quad Y \mapsto [Y],$$

a surjective map of sets.

7.3.1 Theorem. *There exists a canonical morphism of schemes*

$$\gamma = \gamma_n : X^{[n]} \longrightarrow X^{(n)}$$

having the cycle map as underlying map of sets. Further, γ is projective.

Proof. The first assertion follows from [Ive–70, II.2, II.3]. Specifically, any flat family $\mathcal{Y} \subset X \times S$ of length-n subschemes is a n-fold section of the projection $X \times S \longrightarrow S$, in the sense of [loc cit., Definition II.3.1]. So, by [loc cit., II.2], this family yields a canonical morphism from S to the n-fold symmetric product $\mathcal{Y}_S^{(n)}$, the image of the n-fold product

$$\mathcal{Y} \times_S \mathcal{Y} \times_S \cdots \times_S \mathcal{Y} \subset X^n \times S$$

in $X^{(n)} \times S$. Since $\mathcal{Y}_S^{(n)}$ is a closed subscheme of $X^{(n)} \times S$, we obtain a canonical morphism $S \longrightarrow X^{(n)}$. Taking $S = X^{[n]}$ yields $\gamma : X^{[n]} \longrightarrow X^{(n)}$. On the other hand, taking S to be a point yields a length-n subscheme Y of X, and a point of $X^{(n)}$, which is nothing but $[Y]$ by [loc cit., II.4. Appendix]. It follows that $\gamma(Y) = [Y]$, by the compatibility property of [loc cit., II.2.3].

If X is projective, then so is γ, since $X^{[n]}$ is projective in this case. For an arbitrary X, let $X \longrightarrow \bar{X}$ be an open immersion into a projective scheme. By [loc cit., II.2], this yields a Cartesian diagram

$$\begin{tikzcd} X^{[n]} \arrow[r] \arrow[d, "\gamma"'] & \bar{X}^{[n]} \arrow[d, "\bar{\gamma}"] \\ X^{(n)} \arrow[r] & \bar{X}^{(n)}, \end{tikzcd}$$

where $\bar{\gamma}$ is projective; thus, so is γ. $\square$

7.3.2 Definition. The morphism γ of Theorem 7.3.1 is called the *Hilbert–Chow morphism.*

Now, Proposition 7.2.9 and Theorem 7.3.1 imply the following.

7.3.3 Lemma. *Let $n_1, \dots, n_r$ be positive integers with sum n, and let $x_1, \dots, x_r$ be distinct points of X. Then, the fiber of γ at the point $n_1x_1 + \cdots + n_rx_r$ equals $X_{x_1}^{[n_1]} \times \cdots \times X_{x_r}^{[n_r]}$. As a consequence, all the fibers of γ are connected.*

If X is connected, then $X^{(n)}$ is connected as well. Combined with Lemma 7.3.3 and Exercise 7.2.E.1, this yields

7.3.4 Corollary. *If X is connected, then so is $X^{[n]}$.*

Taking the preimages under γ of $X_{**}^{(n)}, X_*^{(n)}, X_s^{(n)}, X_{s*}^{(n)}$, we obtain open subschemes $X_{**}^{[n]} \subset X_*^{[n]}$ of $X^{[n]}$, a closed subscheme $X_s^{[n]}$, and a locally closed subscheme $X_{s*}^{[n]} := X_*^{[n]} \cap X_s^{[n]}$ respectively. Likewise,

$$(X^{[2]} \times X^{[n-2]})_* := (\gamma_2 \times \gamma_{n-2})^{-1}((X^{(2)} \times X^{(n-2)})_*)$$

is an open subscheme of $X^{[2]} \times X^{[n-2]}$; its points consist of the pairs (Y, Z) where $Y \in X^{[2]}$ and $Z \in X_{**}^{[n-2]}$ have disjoint supports.

We now obtain an analogue of Lemmas 7.1.4, 7.1.5 and 7.1.6 for these subschemes.

7.3.5 Lemma. *Let X be a nonsingular variety of dimension d. Then, we have the following.*

(i) $\gamma' : X_{**}^{[n]} \longrightarrow X_{**}^{(n)}$ is an isomorphism, where γ' is the restriction of γ. As a consequence, $X_{**}^{[n]}$ is a nonsingular variety of dimension nd.

(ii) $X_*^{[n]}$ is also a nonsingular variety of dimension nd. In particular, $X^{[2]}$ is a nonsingular variety of dimension $2d$.

(iii) The map

$$u : (X^{[2]} \times X^{[n-2]})_* \longrightarrow X_*^{[n]}, \quad (Y, Z) \longmapsto Y \cup Z,$$

is an étale surjective morphism, which restricts to an isomorphism

$$(X_s^{[2]} \times X^{[n-2]})_* \simeq X_{s*}^{[n]}$$

compatible with the isomorphism $(X_s^{(2)} \times X^{(n-2)})_* \simeq X_{s*}^{(n)}$ (Lemma 7.1.5), where $(X_s^{[2]} \times X^{[n-2]})_* := (X_s^{[2]} \times X^{[n-2]}) \cap (X^{[2]} \times X^{[n-2]})_*$.

(iv) $X_{s*}^{[n]}$ is a nonsingular prime divisor of $X_*^{[n]}$, and $-X_{s*}^{[n]}$ is ample relative to the restriction $\gamma|_{X_*^{[n]}} : X_*^{[n]} \longrightarrow X_*^{(n)}$. Moreover, any (set-theoretic) fiber F of $\gamma|_{X_{s*}^{[n]}}$ is isomorphic to $\mathbb{P}^{d-1}$. Under this isomorphism, the determinant of the normal sheaf $\mathcal{N}_{F,X_*^{[n]}}$ becomes $\mathcal{O}_{\mathbb{P}^{d-1}}(-2)$.

Proof. (i) Clearly, γ' is bijective. Moreover, $X^{(n)}_{**}$ is a nonsingular variety of dimension nd by Lemma 7.1.4; and the Zariski tangent space of $X^{[n]}_{**}$ at any point has dimension nd as well by Lemma 7.2.5. Thus, $X^{[n]}_{**}$ is also a nonsingular variety of dimension nd. To complete the proof, we construct the inverse of γ' as follows.

Let $\mathcal{Y} \subset X \times X^n$ be the union of the partial diagonals $(x = x_1), \ldots, (x = x_n)$. Clearly, $\mathcal{Y}$ is finite over X^n, and the intersection $\mathcal{Y}_{**} := \mathcal{Y} \cap (X \times X^n_{**})$ is a flat family of length-n subschemes of X. This yields a morphism $\alpha : X^n_{**} \longrightarrow X^{[n]}$ which is S_n-invariant, with image $X^{[n]}_{**}$, and hence a morphism $\beta : X^{(n)}_{**} \longrightarrow X^{[n]}_{**}$ which is the desired inverse.

(ii) Since $X^{(n)}_*$ is connected and γ is proper with connected fibers, $X^{[n]}_*$ is connected by Exercise 7.2.E.1. Thus, it suffices to show that $T_Y X^{[n]}$ has dimension nd, for any point Y of $X^{[n]}_*$. By (i), we may assume that $Y \in X^{[n]}_{s*}$. Write $[Y] = 2x_1 + x_3 + \cdots + x_n$, where $x_1, x_3, \ldots, x_n$ are distinct. Then, by Lemma 7.2.5,

$$T_Y X^{[n]} = T_Z X^{[2]} \oplus T_{x_3} X \oplus \cdots \oplus T_{x_n} X,$$

where Z is a point of $X^{[2]}_{x_1}$. By Lemma 7.2.8, there exist local coordinates $t_1, \ldots, t_d$ at x_1 such that $\mathcal{I}_Z$ is generated by $t_1, \ldots, t_{d-1}, t_d^2$. Since the latter form a regular sequence, their classes modulo $\mathcal{I}_Z^2$ form a basis of the $\Gamma(Z, \mathcal{O}_Z)$-module $\mathcal{I}_Z/\mathcal{I}_Z^2$. Further, since the vector space $\Gamma(Z, \mathcal{O}_Z)$ has dimension 2, it follows that the k-vector space $\mathrm{Hom}_{\mathcal{O}_Z}(\mathcal{I}_Z/\mathcal{I}_Z^2, \mathcal{O}_Z) = T_Z X^{[2]}$ has dimension $2d$. This completes the proof.

(iii) Let $\mathcal{Y} \subset X \times S$, resp. $\mathcal{Z} \subset X \times T$, be a flat family of length-2, resp. length-$(n-2)$, subschemes. Let $\mathcal{Y}_T$, $\mathcal{Z}_S$ be their pullbacks to $S \times T$. If $\mathcal{Y}_T \cap \mathcal{Z}_S$ is empty, then $\mathcal{Y}_T \cup \mathcal{Z}_S$ is a flat family of length-n subschemes. It follows that the map u is a morphism. By construction and Lemma 7.2.5, the differential of u at any point is an isomorphism. Therefore, u is étale, since its source and target are nonsingular by (ii). Over $X^{[n]}_{s*}$, u induces a bijective map, and hence an isomorphism. The compatibility follows from the construction.

(iv) By using (iii), we may reduce to the case where $n = 2$. Then, $X^{[2]}_{s*} = X^{[2]}_s = \gamma^{-1}(X^{(2)}_s)$; its pullback along the morphism $X \longrightarrow X^{(2)}_s$ parametrizes the pairs (Y, x) where Y is a length-2 subscheme of X supported at x. By Lemma 7.2.8, this pullback is isomorphic to $\mathbb{P}(T_X)$, the projective bundle of lines in the Zariski tangent space to X. Together with Lemma 7.1.5, it follows that $X^{[2]}_s$ is a nonsingular prime divisor in $X^{[2]}$, and that the (set-theoretic) fibers of γ over $X^{(2)}_s$ are isomorphic to $\mathbb{P}^{d-1}$.

By (i), (ii) and Theorem 7.3.1, $\gamma : X^{[2]} \longrightarrow X^{(2)}$ is a projective, birational morphism. Thus, it is the blowing-up of $X^{(2)}$ along a closed subscheme Z, and the ideal sheaf of $\gamma^{-1}(Z)$ is a relatively ample invertible sheaf, by (i), (ii) and [Har-77, Chap. II, §7]. But, the exceptional divisor $\gamma^{-1}(Z)$ must be a positive multiple of $X^{[2]}_s$, so that $-X^{[2]}_s$ is relatively ample.

Let $x \in X$ with local coordinates $t_1, \ldots, t_d$ and consider the point

$$Y := \mathrm{Spec}(\mathcal{O}_{X,x}/(t_1, \ldots, t_{d-1}, t_d^2))$$

of $X_s^{[2]}$. Let $F = \mathbb{P}(T_x X) \simeq \mathbb{P}^{d-1}$ be the corresponding fiber of γ; we may regard $t_1, \ldots, t_d$ as homogeneous coordinates on F. Consider the family

$$\mathcal{Y} := \operatorname{Spec}(\mathcal{O}_{X,x}[a_1, \ldots, a_d, b_1, \ldots, b_d]/\mathcal{I}),$$

where $\mathcal{I}$ is the ideal generated by

$$t_1 + a_1 t_d + b_1, \ldots, t_{d-1} + a_{d-1} t_d + b_{d-1}, t_d^2 + a_d t_d + b_d$$

together with all the monomials of degree 2 in a_i and b_j. This is a flat family of length-2 subschemes of X over $\operatorname{Spec}(k[a_i, b_j]/(a_i a_j, a_i b_j, b_i b_j))$, with fiber Y at the (unique) point. Further, the differential of the induced morphism

$$\operatorname{Spec}(k[a_i, b_j]/(a_i a_j, a_i b_j, b_i b_j)) \to X^{[2]}$$

at the unique point yields an isomorphism $k^{2d} \simeq T_Y X^{[2]}$, where a_i, b_j are the coordinate functions on k^{2d}. Under this identification, the tangent subspace to the fiber F is given by $(b_1 = \cdots = b_{d-1} = b_d = a_d = 0)$. This shows that the normal bundle to $F \simeq \mathbb{P}^{d-1}$ is homogeneous, and that its determinant is isomorphic to $\mathcal{O}_{\mathbb{P}^{d-1}}(-2)$. $\square$

7.3.E Exercises

(1) Let X be a nonsingular curve. Show that the Hilbert–Chow morphism $\gamma : X^{[n]} \longrightarrow X^{(n)}$ is an isomorphism, and that the quotient morphism $\pi : X^n \longrightarrow X^{(n)}$ can be identified with the universal family.

In the following sequence of exercises, we assume that X is a nonsingular variety, and $p \neq 2$. The aim behind the following exercises is to study the geometry of $X_*^{[n]}$ in more detail, beginning with $X^{[2]}$.

(2) Let $\pi_2 : \widetilde{X}^{[2]} \longrightarrow X^{[2]}$ be the universal family. Show that $\widetilde{X}^{[2]}$ is a variety, and that π_2 is separable. Thus, we may write

$$\pi_{2*}\mathcal{O}_{\widetilde{X}^{[2]}} = \mathcal{O}_{X^{[2]}} \oplus \mathcal{L},$$

where $\mathcal{L}$ is the kernel of the trace map ; this is an invertible sheaf over $X^{[2]}$. We define an action of S_2 on the sheaf $\pi_{2*}\mathcal{O}_{\widetilde{X}^{[2]}}$ by letting the nontrivial element σ act by 1 on $\mathcal{O}_{X^{[2]}}$, and -1 on $\mathcal{L}$. Show that S_2 acts by automorphisms of the sheaf of algebras $\pi_{2*}\mathcal{O}_{\widetilde{X}^{[2]}}$. In other words, S_2 acts on $\widetilde{X}^{[2]}$, and π_2 is the quotient morphism.

(3) Let $\pi_1 : \widetilde{X}^{[2]} \longrightarrow X$ be the standard projection and let

$$\gamma_2 := (\pi_1, \pi_1 \circ \sigma) : \widetilde{X}^{[2]} \longrightarrow X^2.$$

Show that the diagram

$$\begin{array}{ccc} \widetilde{X}^{[2]} & \xrightarrow{\gamma_2} & X^2 \\ {\scriptstyle \pi_2}\big\downarrow & & {\scriptstyle \pi}\big\downarrow \\ X^{[2]} & \xrightarrow{\gamma} & X^{(2)} \end{array}$$

commutes, and that γ_2 is an isomorphism above the complement of the diagonal $\Delta \subset X^2$.

(4) Show that the square of the ideal sheaf $\mathcal{I}_\Delta^2 \subset \mathcal{O}_{X^2}$ is generated by $\mathcal{I}_{X_s^{(2)}} \subset \mathcal{O}_{X^{(2)}}$. As the pullback of $\mathcal{I}_{X_s^{(2)}}$ in $X^{[2]}$ via γ is invertible (Lemma 7.3.5), it follows that the pullback of $\mathcal{I}_\Delta$ in $\widetilde{X}^{[2]}$ is invertible as well, so that γ_2 factors through a morphism $\beta : \widetilde{X}^{[2]} \longrightarrow \mathrm{Bl}_\Delta(X^2)$ (the blowing-up of X^2 along the diagonal). Using Zariski's main theorem, show that β is an isomorphism.

(5) Show that $\gamma : X^{[2]} \longrightarrow X^{(2)}$ is the blowing-up of $X^{(2)}$ along $X_s^{(2)}$. Deduce that the restriction of the Hilbert–Chow morphism $X_*^{[n]} \longrightarrow X_*^{(n)}$ is the blowing-up of $X_*^{(n)}$ along $X_{s*}^{(n)}$.

7.4 Hilbert schemes of points on surfaces

In this section, X denotes a nonsingular surface, i.e., a nonsingular variety of dimension 2; then, X is quasi-projective (cf. [Har–77, Chap. II, Remark 4.10.2] and the references therein).

7.4.1 Theorem. *$X^{[n]}$ is a nonsingular variety of dimension $2n$.*

Proof. By Corollary 7.3.4, $X^{[n]}$ is connected; and by Lemma 7.3.5, it contains an open subvariety $X_*^{[n]}$ of dimension $2n$. Thus, it suffices to show that the dimension of the Zariski tangent space $T_Y X^{[n]}$ at any point Y of $X^{[n]}$ is at most $2n$. By decomposing Y into connected components and using Lemma 7.2.5, we reduce further to proving the following.

7.4.2 Lemma. *Let $(R, \mathcal{M})$ be a regular local ring of dimension 2 and let I be an ideal of R such that the R-module R/I has finite length. Then,*

$$\ell(\mathrm{Hom}_R(I, R/I)) \leq 2\ell(R/I),$$

where $\ell(M)$ denotes the length of an R-module M.

Proof. Since the R-module R/I is Artinian, its depth at $\mathcal{M}$ is zero. By the Auslander-Buchsbaum formula [Eis–95, Theorem 19.9], it follows that R/I has projective dimension 2. So, the surjection $R \longrightarrow R/I$ fits into an exact sequence of R-modules

$$0 \longrightarrow R^r \longrightarrow R^s \longrightarrow R \longrightarrow R/I \longrightarrow 0,$$

where r and s are positive integers. Considering ranks, we obtain $s = r + 1$, whence an exact sequence

$$0 \longrightarrow R^r \longrightarrow R^{r+1} \longrightarrow I \longrightarrow 0.$$

It yields an exact sequence

$$\begin{aligned}0 \longrightarrow \mathrm{Hom}_R(I, R/I) \longrightarrow \mathrm{Hom}_R(R^{r+1}, R/I) \longrightarrow \mathrm{Hom}_R(R^r, R/I)\\ \longrightarrow \mathrm{Ext}^1_R(I, R/I) \longrightarrow \mathrm{Ext}^1_R(R^{r+1}, R/I),\end{aligned}$$

that is,

$$0 \longrightarrow \mathrm{Hom}_R(I, R/I) \longrightarrow (R/I)^{r+1} \longrightarrow (R/I)^r \longrightarrow \mathrm{Ext}^1_R(I, R/I) \longrightarrow 0.$$

Therefore, the R-module $\mathrm{Ext}^1_R(I, R/I)$ has finite length, and we obtain by the additivity of the length function:

$$\ell(\mathrm{Hom}_R(I, R/I)) = \ell(R/I) + \ell(\mathrm{Ext}^1_R(I, R/I)).$$

Likewise, the exact sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0$$

and the vanishing of $\mathrm{Ext}^i_R(R, -)$ for $i \geq 1$ yield an isomorphism

$$\mathrm{Ext}^1_R(I, R/I) \simeq \mathrm{Ext}^2_R(R/I, R/I).$$

Thus, it suffices to show the inequality

$$\ell(\mathrm{Ext}^2_R(R/I, R/I)) \leq \ell(R/I).$$

For this, we use the exact sequence

$$\mathrm{Ext}^2_R(R/I, R) \longrightarrow \mathrm{Ext}^2_R(R/I, R/I) \longrightarrow \mathrm{Ext}^3_R(R/I, I)$$

together with the vanishing of $\mathrm{Ext}^3_R(R/I, -)$ (since R/I has projective dimension 2) to obtain

$$\ell(\mathrm{Ext}^2_R(R/I, R/I)) \leq \ell(\mathrm{Ext}^2_R(R/I, R)).$$

Further, $\mathrm{Ext}^2_R(R/\mathcal{M}, R) \simeq R/\mathcal{M}$ and $\mathrm{Ext}^i_R(R/\mathcal{M}, R) = 0$ for $i \neq 2$ (as seen from the Koszul resolution of the R-module $R/\mathcal{M}$). Therefore, by induction on $\ell(M)$, any R-module M of finite length satisfies

$$\ell(\mathrm{Ext}^2_R(M, R)) \leq \ell(M).$$

Taking $M = R/I$ yields the desired inequality. □

Next, we determine the dimensions of the fibers of the Hilbert–Chow morphism; for this, we first record the following result proved in [Iar–77] (an alternative proof is sketched in Exercises 7.4.E.1–3).

7.4.3 Lemma. *For any point x of X, the n-th punctual Hilbert scheme $X^{[n]}_x$ has dimension $n-1$.*

Together with Lemma 7.3.3, this readily implies the following.

7.4.4 Lemma. *(i) The fiber $\gamma^{-1}(n_1x_1 + \cdots + n_rx_r)$, where $x_1, \ldots, x_r$ are distinct points of X, has dimension $n - r$.*
(ii) $X^{[n]} \setminus X_*^{[n]}$ has codimension 2 in $X^{[n]}$.

In turn, this lemma has important geometric consequences:

7.4.5 Proposition. *$\gamma : X^{[n]} \longrightarrow X^{(n)}$ is birational, with exceptional set $X_s^{[n]}$. The latter is a prime divisor of $X^{[n]}$, and its negative is γ-ample.*

Proof. The first assertion follows from Lemmas 7.3.5(i), 7.4.4, and the irreducibility of $X^{[n]}$ (which is a consequence of Theorem 7.4.1).

Since $X^{[n]}$ is nonsingular and $X^{(n)}$ is $\mathbb{Q}$-factorial (Lemma 7.1.9), the exceptional set of γ has pure codimension 1, cf. [Deb–01, 1.40]. Together with Lemma 7.4.4 (ii), it follows that $X_{s*}^{[n]}$ is dense in $X_s^{[n]}$. But, $X_{s*}^{[n]}$ is irreducible by Lemma 7.3.5, so that $X_s^{[n]}$ is irreducible as well. Finally, the γ-ampleness of $-X_s^{[n]}$ is obtained by the proof of Lemma 7.3.5 (iv). □

Recall from 1.3.12 that a proper, birational morphism $f : Y \longrightarrow Z$ between Gorenstein varieties is called crepant if $f^*\omega_Z \simeq \omega_Y$. Now, $X^{(n)}$ is Gorenstein by Lemma 7.1.7, and $X^{[n]}$ is nonsingular by Theorem 7.4.1, so that the following statement makes sense.

7.4.6 Theorem. *The morphism $\gamma : X^{[n]} \longrightarrow X^{(n)}$ is crepant.*

Proof. By Lemma 7.4.4 (ii), it suffices to check that the restriction $\gamma|_{X_*^{[n]}}$ is crepant. Now, by Lemma 7.3.5, this restriction is an isomorphism outside $X_{s*}^{[n]}$, which is a nonsingular prime divisor of $X_*^{[n]}$. So, there exists an integer t such that

$$\omega_{X_*^{[n]}} \simeq (\gamma^*\omega_{X_*^{(n)}})(tX_{s*}^{[n]}).$$

To determine t, we restrict both sides to an exceptional fiber F, i.e., to a (set-theoretic) fiber at a point of $X_{s*}^{(n)}$; recall from Lemma 7.3.5 (iv) that $F \simeq \mathbb{P}^1$. We have the equality of degrees:

$$\deg_F(\omega_{X_*^{[n]}}) = \deg_F\big((\gamma^*\omega_{X_*^{(n)}})(tX_{s*}^{[n]})\big) = \deg_F(\gamma^*\omega_{X_*^{(n)}}) + t\,(X_{s*}^{[n]}, F).$$

Further, $\deg_F(\gamma^*\omega_{X_*^{(n)}}) = 0$ since γ maps F to a point; and the intersection number $(X_{s*}^{[n]}, F)$ is negative since $-X_{s*}^{[n]}$ is relatively ample (by Lemma 7.4.5). On the other hand, by [Har–77, Chap. II, Proposition 8.20],

$$\deg_F(\omega_{X_*^{[n]}}) = \deg_F(\omega_F) - \deg_F\Big(\bigwedge^{2n-1} \mathcal{N}_{F,X_*^{[n]}}\Big),$$

where $\mathcal{N}_{F,X_*^{[n]}}$ denotes the normal sheaf. Moreover, $\deg_F(\omega_F) = -2$ as $F \simeq \mathbb{P}^1$, and $\deg_F(\bigwedge^{2n-1}\mathcal{N}_{F,X_*^{[n]}}) = -2$ by Lemma 7.3.5 (iv). It follows that $\deg_F(\omega_{X_*^{[n]}}) = 0$, so that $t = 0$. □

7.4.E Exercises

(1*) Let A be a nilpotent $n \times n$ matrix with coefficients in k. Let $t^{d_1}, \ldots, t^{d_s}$ be the elementary divisors of k^n as a $k[t]$-module, where t acts on k^n as A. For any exponent i, let r_i be the number of d_j's equal to i.

Show that the centralizer of A in $M_n(k)$ is isomorphic to the direct product of a nilpotent ideal of dimension $\sum_i (\sum_{j \geq i} r_j)^2 - r_i^2$, with the product of the matrix algebras $M_{r_i}(k)$.

(2*) Let $\mathcal{N}_{2,n}(A)$ be the set of those pairs (A_1, A_2) of commuting nilpotent $n \times n$ matrices, such that A_1 is conjugate to A. Show that $\mathcal{N}_{2,n}(A)$ is a locally closed subvariety of $M_n(k) \times M_n(k)$, of dimension $n^2 - \sum_i r_i$.

(3*) With the notation of Exercise 7.2.E.4, show that $\dim(Z_{2,n}) = n^2 + n - 1$. Using Exercise 7.2.E.6, this immediately gives $\dim(X_x^{[n]}) = n - 1$, for any nonsingular point x in a surface X.

7.5 Splitting of Hilbert schemes of points on surfaces

In this section, k denotes an algebraically closed field of characteristic $p > 0$.

7.5.1 Lemma. *Let X be a quasi-projective scheme. If X is split, then any symmetric product $X^{(n)}$ is split.*

Proof. Let $\varphi : F_*\mathcal{O}_X \longrightarrow \mathcal{O}_X$ be a splitting. Then, $\varphi^{\boxtimes n} : F_*\mathcal{O}_{X^n} \longrightarrow \mathcal{O}_{X^n}$ is a splitting of X^n, equivariant for the action of S_n. Thus, $\varphi^{\boxtimes n}$ restricts to a map $(F_*\mathcal{O}_{X^n})^{S_n} \longrightarrow \mathcal{O}_{X^n}^{S_n}$ that yields a splitting of $X^{(n)}$. □

Next, recall from Lemma 1.3.13 that for any crepant morphism $f : Y \longrightarrow Z$, Y is split if Z is. Together with Theorem 7.4.6, this yields the following main result of this section.

7.5.2 Theorem. *Let X be a nonsingular surface. If X is split, then $X^{[n]}$ is split as well.*

7.5.3 Remarks. (i) The list of split surfaces includes:
• the nonsingular affine surfaces (Proposition 1.1.6),
• the toric surfaces (Exercise 1.3.E.6); in particular, all rational ruled surfaces,
• those projective nonsingular surfaces with trivial canonical class that are *ordinary*, i.e., the map $F^* : H^2(X, \mathcal{O}_X) \longrightarrow H^2(X, \mathcal{O}_X)$ is nonzero (Remark 1.3.9 (ii)). In particular, all ordinary K3 or abelian surfaces are split.

On the other hand, non-ordinary projective surfaces with trivial canonical class are not split, as well as the projective surfaces with Kodaira dimension at least 1 (Remark 1.3.9 (i)).

(ii) For split X, one would like $X^{[n]}$ to be split compatibly with its exceptional divisor $X_s^{[n]}$. But, this does not always hold. Consider, for example, $X = C \times C$ where C is an ordinary elliptic curve. Then, X is split, but $X^{[2]}$ is not split compatibly with

$X_s^{[2]}$. Otherwise, $X^{(2)}$ would be split compatibly with $X_s^{(2)}$ (Lemma 1.1.8). Thus, the restriction map $H^0(X^{(2)}, \mathcal{L}^{(2)}) \longrightarrow H^0(X_s^{(2)}, \mathcal{L}^{(2)})$ would be surjective for any ample invertible sheaf $\mathcal{L}$ on X (Theorem 1.2.8). As a consequence, the multiplication map

$$H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{L}) \longrightarrow H^0(X, \mathcal{L}^2)$$

would be surjective as well. But, this fails for $\mathcal{L} = \mathcal{O}_C(D)^{\boxtimes 2}$, where D is a divisor of degree 2 on C.

Theorem 7.5.2 implies the following global vanishing result.

7.5.4 Corollary. *Let X be a nonsingular surface which is split and which is proper over an affine variety. Then, $H^i(X^{[n]}, \mathcal{L}) = 0$ for any $i \geq 1$ and for any ample invertible sheaf $\mathcal{L}$ on $X^{[n]}$.*

Proof. By assumption, there exists a proper morphism $f : X \longrightarrow Y$, where Y is an affine variety. Then, $f^n : X^n \longrightarrow Y^n$ is also proper, so that the induced morphism $f^{(n)} : X^{(n)} \longrightarrow Y^{(n)}$ is proper as well (since the quotient maps $X^n \longrightarrow X^{(n)}$, $Y^n \longrightarrow Y^{(n)}$ are finite and surjective). Since the Hilbert–Chow morphism $\gamma : X^{[n]} \longrightarrow X^{(n)}$ is proper, it follows that $X^{[n]}$ is proper over the affine scheme $Y^{(n)}$. Now, the corollary is a consequence of Theorems 1.2.8 and 7.5.2. □

Another consequence of Theorem 7.5.2 is the following relative vanishing result for the Hilbert–Chow morphism.

7.5.5 Corollary. $R^i\gamma_*\mathcal{O}_{X^{[n]}}(-\nu X_s^{[n]}) = 0$ *for any nonsingular surface X and all $i \geq 1$, $\nu \geq 1$.*

Proof. Since X is quasi-projective, any finite subset is contained in some open affine subset. Therefore, $X^{(n)}$ is covered by its open affine subsets $U^{(n)}$, where U runs over the open affine subsets of X. Thus, we may assume that X is affine, and hence split. Further, $X^{[n]}$ is proper over the affine variety $X^{(n)}$ by Theorem 7.3.1, and the divisor $-X_s^{[n]}$ is ample by Proposition 7.4.5. Hence, Theorem 1.2.8 (i) yields: $H^i(X^{[n]}, \mathcal{O}_{X^{[n]}}(-\nu X_s^{[n]})) = 0$ for any $i \geq 1$ and $\nu \geq 1$, which is equivalent to the desired vanishing, since $X^{(n)}$ is affine. □

7.5.6 Remark. In characteristic 0, Corollary 7.5.5 is a direct consequence of Theorem 7.4.6, Lemma 7.1.7(iii) and the Grauert–Riemenschneider vanishing theorem (since $-X_s^{[n]}$ is a γ-ample divisor by Proposition 7.4.5), which imply also that $R^i\gamma_*\mathcal{O}_{X^{[n]}} = 0$ for any $i \geq 1$.

7.C. Comments.

The results of Section 7.1 are classical, and many of them extend to quotients by arbitrary finite groups, at least in characteristic 0. But, we could not locate an appropriate reference for the symmetric products in positive characteristics, thus we have endeavoured to give a detailed exposition.

The construction of Hilbert schemes is due to Grothendieck [Gro–62]. He obtained the existence of a universal flat family of subschemes of a fixed projective space $\mathbb{P}^r$, having a fixed Hilbert polynomial P; the base of this family is a projective scheme, the Hilbert scheme $\mathrm{Hilb}^P(\mathbb{P}^r)$. Taking P to be the constant polynomial equal to n yields the existence of the Hilbert scheme $\mathrm{Hilb}^n(\mathbb{P}^r) = (\mathbb{P}^r)^{[n]}$ parametrizing length-n subschemes of $\mathbb{P}^r$ and, in turn, the existence of $X^{[n]}$ for an arbitrary quasi-projective X (Theorem 7.2.3).

The description of the Zariski tangent spaces to $X^{[n]}$ (Lemma 7.2.5) is also a special case of a result of Grothendieck [Gro–62]. The subsequent results of Section 7.2 are due to Fogarty [Fog–68] and [Fog–73]; a stronger version of Lemma 7.2.10 is due to Horrocks [Hor–69].

The approach to punctual Hilbert schemes of affine spaces in terms of linear algebra, sketched in Exercises 7.2.E, is developed by Nakajima in [Nak–99].

Our definition of the Hilbert–Chow morphism is based on results of Iversen [Ive-70]; it yields a refinement of the morphism constructed in [Fog–68], which had for its source the reduced subscheme of the Hilbert scheme, and for its target the Chow variety of effective 0-cycles. See [Nee–91] for another construction of the Hilbert–Chow morphism, in the case of a projective space.

Lemma 7.3.3 and Corollary 7.3.4 are originally in [Fog–68]; Lemma 7.3.5 is a version of Lemma 4.4 in [Fog–73], but some details in [loc cit.] are not clear to us.

The fundamental Theorem 7.4.1 is again due to Fogarty [Fog–68]; our presentation follows the original proof closely. In characteristic 0, Lemma 7.4.3 was obtained by Briançon [Bria–77] in a stronger form: the n-th punctual Hilbert scheme $X_x^{[n]}$ is irreducible, of dimension $n-1$ (cf. also [Gran–83]). This stronger result continues to hold in characteristic $p > n$, cf. [Iar–77]; where it is also shown that Lemma 7.4.3 holds in an arbitrary characteristic (without the irreducibility of $X_x^{[n]}$).

As shown by Baranovsky [Bar–01], the irreducibility of $X_x^{[n]}$ (in an arbitrary characteristic) is equivalent to the irreducibility of the space $\mathcal{N}_{2,n}$ of pairs of commuting nilpotent $n \times n$ matrices. Basili [Bas–03] established directly the irreducibility of $\mathcal{N}_{2,n}$ in characteristic 0 and in characteristic $p \geq n/2$. The case of an arbitrary characteristic is due to Premet [Pre–03]; in fact, his main result describes all the irreducible components of the space of pairs of commuting nilpotent elements of the Lie algebra of any semisimple group G in good characteristic.

Proposition 7.4.5 is stated in [Fog–73]. Theorem 7.4.6 is classical in characteristic 0; in fact, symmetric products of nonsingular surfaces admit a unique crepant resolution given by the Hilbert–Chow morphism, cf. [FuNa–04]. Theorem 7.4.6 in positive characteristic is due to Kumar–Thomsen [KuTh–01]; the proof presented here is somewhat different from theirs.

The results of Section 7.5 are all taken from [KuTh–01], except for Corollary 7.5.5. Haiman [Hai–98] conjectures the vanishing of higher cohomology groups of all the tensor powers of the tautological vector bundle (i.e., the image of the structure sheaf of the universal family) on $(\mathbb{A}^2)^{[n]}$, and he presents remarkable combinatorial consequences of this conjecture. He proved his conjecture in characteristic 0 [Hai–02]. Further cohomology vanishing results for Hilbert schemes of points on nonsingular projective surfaces are due to Danila [Dan–01, 04], again in characteristic 0.

It is not known if the Hilbert–Chow morphism for a nonsingular surface is a rational morphism. (This would, in particular, imply that symmetric products of nonsingular surfaces are Cohen–Macaulay.) Further, it is not known if the total space of the universal family over the Hilbert scheme of a smooth split surface is again split. This question is motivated by the vanishing theorems of Danila and Haiman mentioned above.

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