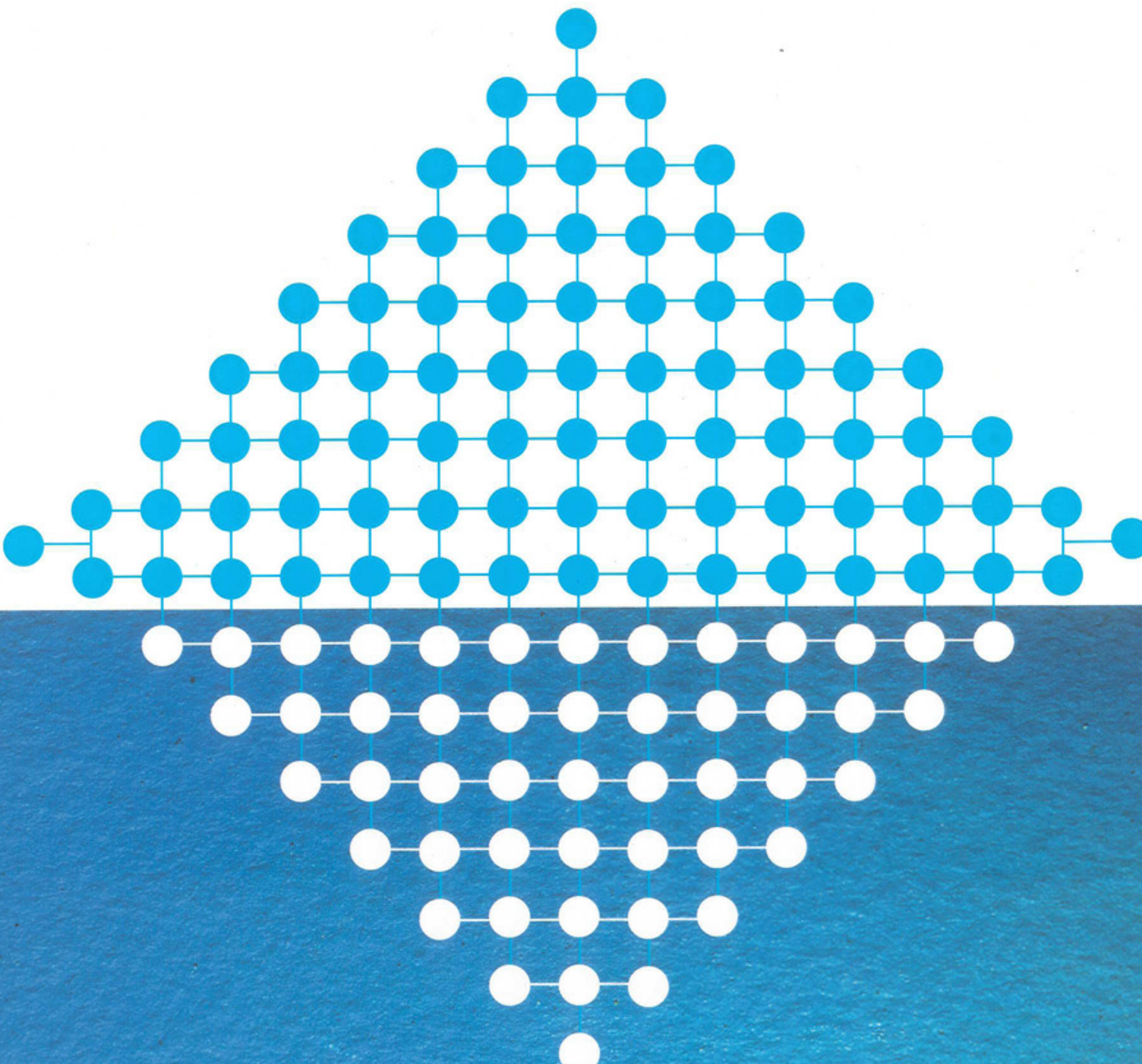
MATHEMATICAL LOGIC IN THE 20TH CENTURY



Gerald E. Sacks

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MATHEMATICAL LOGIC IN THE 20TH CENTURY

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For Jacob, Ella and Maggie

Introduction

Brevity is the soul.
H. M.

First come the disclaimers, then the rules for selecting the papers, the exceptions to the rules, the justifications of the exceptions, and finally some brief remarks on the papers. (But before all that, the point of it all. The original papers are studied in the hope of recovering early ideas lost in later expositions. Proofs are rare, but the ideas used in proofs are rarer still.) The title of this volume is too broad. Almost all of the papers belong to the second half of the twentieth century. The last decade of the twentieth century is lightly represented. Only so much can be forced into one volume. The first half of the last century is well represented elsewhere; now is too soon to reach conclusions about its final decade.

I have not read all the logic of the last century — far from it. And only a fraction of what was read was understood. The choices made were personal in nature. Who knows what "personal" means? The selection was certainly not based on an Olympian view of mathematical logic derived from a long and scholarly life of pondering the subject. Perhaps the choices cohere, if only because it is hard to see how it could be otherwise.

The selection rules were:

R1. No papers from before World War II.
R2. No long papers.
R3. At most one paper by any author.
R4. The paper was (and is) intellectually exciting then (and now).

The first three rules are justified above. They proved difficult to follow. Gödel's pre-war National Academy paper on the generalized continuum hypothesis is short, readable, more illuminating than his subsequent Princeton Mathematics orange book, and probably closer to his early thinking. His Academy paper, but not his orange book, mentions Russell's Axiom of Reducibility as a source of inspiration. The papers by Kleene and Tarski are much too long for this volumn, but Kleene is the father of recursion theory and Tarski of model theory. Why this particular paper by Kleene? Two reasons. He lifts the concepts of classical recursion theory to objects of finite type, and he shows that sets of non-negative integers are hyperarithmetic if and only if they are recursive in ${}^2E$, the type 2 object corresponding to the number quantifier. The title of Tarski's paper speaks for itself. Rule R4 was not violated, possibly a necessary fact.

Cohen has two papers intended for this volume, but they are in fact the two halves of one paper. For this work, he received the Fields Medal, the highest international award in mathematics.

Gödel's paper transformed set theory into a subject that welcomes a wide range of mathematical ideas. His use of the downward Skolem Löwenheim theorem inside

L is the beginning of fine structure theory. His paper combined with Cohen's puts Cantor's continuum problem outside the conventional realms (ZFC) of set theory.

Cohen's paper introduced the method of forcing, an essential technique with applications throughout logic. Forcing has unconscious precursors in recursion theory, for example the construction of a minimal Turing degree in Spector's paper.

Silver's paper proved (in ZFC) a new theorem of cardinal arithmetic at a time when such an outcome was thought unlikely because of the Gödel and Cohen results. He applied some ideas about ultrafilters to show: if the generalized continuum hypothesis holds below a singular cardinal κ of uncountable cofinality, then it holds at κ. This line of thought led to Shelah's pcf theory [I1], which yields estimates on the size of the power set of a singular cardinal of countable confinality. ([Im] is the mth item in the References at the end of this introduction. All other papers mentioned are from the Contents list for the volume.)

Choosing a paper by Shelah was a daunting task because of the large number of his contributions to model theory and set theory and the limit imposed by rule R2. His 1969 paper on stability can be seen as the beginning of his sweeping transformation of model theory.

The notion of ω-stability originated in Morley's proof that a countable theory categorical in some uncountable power is categorical in all uncountable powers. Morley's paper, building on Vaught's earlier paper, was the beginning of modern model theory. Vaught's paper went beyond immediate applications of compactness and stressed the notion of element type. For example he showed that the number of countable models up to isomorphism of a complete countable theory could not be two.

Jensen's covering theorem, in his paper with Devlin, makes a connection between sets of ordinals in V and L with the help of fine structure theory and the work of Silver [I2] and Solovay [I3] on $O^\sharp$.

The choice of Friedberg's paper on recursive enumeration came about as follows. He in [I4] and Mučnik independently solved Post's problem by introducing the priority method, a technique that dominates classical recursion theory to this day. By choosing Mučnik's version, I could satisfy rule R3 and still include Friedberg's construction of a maximal recursively enumerable set, a result that ignited interest in the lattice of recursively enumerable set under inclusion modulo finite sets. The lattice was initially studied in Post's paper on recursively enumerable sets and their decision problems. The lasting influence of Post's paper entitles him to be called the co-father, if there is such a thing, with Kleene of recursion theory. Soare's paper showed any two maximal sets are automorphic. His result is the reason that the lattice continues to be of interest.

Post's paper established the legitimacy of an intuitive approach to recursion theory: less equations and more words. Friedberg's paper is any early example of the intuitive style. Spector's paper adheres to Kleene's formal style, only because it was extracted from Spector's thesis supervised by Kleene.

Lachlan's paper introduced the so-called (but not by him) monstrous injury method, close to the final stage in the development of the Friedberg-Mučnik priority method.

Moschovakis's paper found a nearly paradoxical role for divergence in recursion theory and led to constructions of recursively enumerable sets in higher recursion theory in which divergence witnesses played as big a part as convergence witnesses.

Matijasevič's paper on the unsolvability of the Diophantine problem has historical antecedents as old as any in mathematics.

The underrepresentation of proof theory in this volume indicates nothing more than my own confusion over the subject. The most striking proof theorist I have met is Girard. His paper was chosen for its brevity and as an example of his unique mode of thought. In it he discusses his concept of dilator.

Kreisel, another proof theorist with his own mode of thought, is also included. His paper is a mixture of recursion theory, model theory, proof theory and other subjects hard to put a name to. He presents a compactness theorem for ω-logic based on his insight that generalizing the notion of finite is the key to extending various results in model theory and recursion theory. In this compactness theorem "hyperarithmetic" is the generalization of "finite".

Robinson's paper on non-standard analysis is a model theorist's way of making sense out of infinitesimals. Gödel thought it was of historic importance.

Wilkie's paper solves a long standing problem of Tarksi on the first order theory of the reals with the exponential function added.

The work of Zil'ber and Hruschovski bring ideas of geometry and stability to bear on model theory. Zil'ber's paper was chosen as a brief example of his approach, and Hruschovski's paper as a prime application of model theory to number theory.

H. Friedman's paper shows that Borel Determinateness (BD) cannot be proved without invoking objects of arbitrarily high countable rank despite the fact that BD is about Borel sets of reals, objects of rank 1. Later Martin [I5] proved BD by means of an induction that trades decreases in rank of Borel sets for increases in rank of objects.

Solovay's paper assumes the consistency of "there exists an inaccessible cardinal" and then demonstrates the consistency of "every set of reals is Lebesgue measurable and countable dependent choice". Later Shelah [I6] proved the converse.

Scott's paper showed the existence of a measurable cardinal implies the existence of a non-constructible set. This result, and its proof via ultrapowers, broke open the study of large cardinals.

Martin's paper used a measurable cardinal to establish the determinacy of analytic games. His argument needed only the sort of indiscernibles provided by $O^{\sharp}$. Later the converse was shown by Harrington [I7]. (Thus $O^{\natural}$ is equivalent to lightface Π^1_1 determinacy.) Martin's result eventually led to complex connections between determinateness and large cardinals obtained by Woodin, whose paper in this volume is a brief example of his unique insight.

Shoenfield's Σ^1_2 absoluteness result is a personal favorite. It has applications throughout logic. One example is the Slaman-Woodin proof [I8] of the definability of the double Turing jump.

My thanks to those who suggested papers for this volume. They insist on remaining anonymous.

Cambridge, Massachusetts
December 21, 2002

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Contents

THE INDEPENDENCE OF THE CONTINUUM HYPOTHESIS

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Communicated by Kurt Gödel, September 30, 1963

This is the first of two notes in which we outline a proof of the fact that the Continuum Hypothesis cannot be derived from the other axioms of set theory, including the Axiom of Choice. Since Gödel[3] has shown that the Continuum Hypothesis is consistent with these axioms, the independence of the hypothesis is thus established. We shall work with the usual axioms for Zermelo-Fraenkel set theory,[2] and by Z-F we shall denote these axioms without the Axiom of Choice, (but with the Axiom of Regularity). By a model for Z-F we shall always mean a collection of actual sets with the usual ϵ-relation satisfying Z-F. We use the standard definitions[3] for the set of integers ω, ordinal, and cardinal numbers.

THEOREM 1. *There are models for Z-F in which the following occur:*

(1) *There is a set a, $a \subseteq \omega$ such that a is not constructible in the sense of reference 3, yet the Axiom of Choice and the Generalized Continuum Hypothesis both hold.*

(2) *The continuum (i.e., $\mathcal{P}(\omega)$ where $\mathcal{P}$ means power set) has no well-ordering.*

(3) *The Axiom of Choice holds, but $\aleph_1 \neq 2^{\aleph_0}$.*

(4) *The Axiom of Choice for countable pairs of elements in $\mathcal{P}(\mathcal{P}(\omega))$ fails.*

Only part 3 will be discussed in this paper. In parts 1 and 3 the universe is well-ordered by a single definable relation. Note that 4 implies that there is no simple ordering of $\mathcal{P}(\mathcal{P}(\omega))$. Since the Axiom of Constructibility implies the Generalized Continuum Hypothesis,[3] and the latter implies the Axiom of Choice,[5] Theorem 1 completely settles the question of the relative strength of these axioms.

Before giving details, we sketch the intuitive ideas involved. The starting point is the realization[1, 4] that no formula $\alpha(x)$ can be shown from the axioms of Z-F to have the property that the collection of all x satisfying it form a model for Z-F in which the Axiom of Constructibility ($V = L$,[3]) fails. Thus, to find such models, it seems natural to strengthen Z-F by postulating the existence of a set which is a model for Z-F, thus giving us greater flexibility in constructing new models. (In the next paper we discuss how the question of independence, as distinct from that of models, can be handled entirely within Z-F.) The Löwenheim-Skolem theorem yields the existence of a countable model $\mathfrak{M}$. Let $\aleph_1$, $\aleph_2$, etc., denote the corresponding cardinals in $\mathfrak{M}$. Since $\mathfrak{M}$ is countable, there exist distinct sets $a_\delta \subseteq \omega$, $0 \leq \delta \leq \aleph_2$. Put $V = \{\langle a_\delta, a_{\delta'}\rangle | \delta < \delta'\}$. We form the model $\mathfrak{N}$ "generated" from $\mathfrak{M}$, a_δ, and V and hope to prove that in $\mathfrak{N}$ the continuum has cardinality at least $\aleph_2$. Of course, $\mathfrak{N}$ will contain many new sets and, if the a_δ are chosen indiscriminately, the set $\aleph_2$ (in $\mathfrak{M}$) may become countable in $\mathfrak{N}$. Rather than determine the a_δ directly, we first list all the countably many possible propositions concerning them and decide in advance which are to be true. Only those properties which are true in a "uniform" manner for "generic" subsets of ω in $\mathfrak{M}$ shall be true for the a_δ in $\mathfrak{N}$. For example, each a_δ contains infinitely many primes, has no asymptotic density, etc. If the a_δ are chosen in such a manner, no new information will be extracted from them in $\mathfrak{N}$ which was not already contained in $\mathfrak{M}$, so that, e.g., $\aleph_2$ will remain the second uncountable cardinal. The primitive conditions $n \in a_\delta$ are neither generically true nor false, and hence must be treated separately. Only when given a finite set of such conditions will we be able to speak of properties possibly being forced to hold for "generic" sets. The precise definition of forcing will be given in Definition 6.

From now on, let $\mathfrak{M}$ be a fixed countable model for Z-F, satisfying $V = L$, such that $x \in \mathfrak{M}$ implies $x \subset \mathfrak{M}$. If $\mathfrak{M}'$ is a countable model without this property, define Ψ by transfinite induction on the rank of x, so that $\Psi(x) = \{y | \exists z \in \mathfrak{M}', z \in x, \Psi(z) = y\}$; the image $\mathfrak{M}$ of $\mathfrak{M}'$ under Ψ is isomorphic to $\mathfrak{M}'$ with respect to $\in$ and satisfies our requirement. Thus, the ordinals in $\mathfrak{M}$ are truly ordinals. Let $\tau > 1$ be a fixed ordinal in $\mathfrak{M}$, $\aleph_\tau$ the corresponding cardinal in $\mathfrak{M}$, and let a_δ, $0 \leq \delta < \aleph_\tau$ be subsets of ω, not necessarily in $\mathfrak{M}$, $V = \{\langle a_\delta, a_{\delta'}\rangle | \delta < \delta'\}$.

LEMMA 1. *There exist unique functions j, K_1, K_2, N, from ordinals to ordinals definable in $\mathfrak{M}$ such that*

(1) *$j(\alpha + 1) > j(\alpha)$ and for all β such that $j(\alpha) + 1 < \beta < j(\alpha + 1)$ the map $\beta \rightarrow (N(\beta), K_1(\beta), K_2(\beta))$ is a 1-1 correspondence between all such β, and the set of all triples (i, γ, δ), $1 \leq i \leq 8$, $\gamma < j(\alpha)$, $\delta < j(\alpha)$. Furthermore, this map is order-preserving if the triples are given the natural ordering S (Ref. 3, p. 36).*

(2) *$j(0) = 3\aleph_\tau + 1, j(\alpha) = sup\{j(\beta) | \beta < \alpha\}$ if α is a limit ordinal.*

(3) *$N(j(\alpha)) = 0, N(j(\alpha) + 1) = 9, K_i = 0$ for these values.*

(4) *$N(\alpha)$, $K_i(\alpha)$ are zero if $\alpha \leq 3\aleph_\tau$.*

(5) *If β is as above, and $N(\beta) = i$, put $J(i, K_1(\beta), K_2(\beta), j(\alpha)) = \beta$. Also put $I(\beta) = j(\alpha)$.*

Definition 1: For α an ordinal in $\mathfrak{M}$, define F_α by means of induction as follows:

(1) $F_\alpha = \alpha$ if $\alpha \leq \omega$.

(2) For $\omega < \alpha < 3\aleph_\tau$, let F_α successively enumerate a_δ, the unordered pairs $(a_\delta, a_{\delta'})$ and the ordered pairs $\langle a_\delta, a_{\delta'}\rangle$ in any standard manner (e.g., the ordering R on pairs of ordinals def. 7.81[3]).

(3) For $\alpha = 3\aleph_\tau$, $F_\alpha = V$.

(4) For $\alpha > 3\aleph_\tau$, if $K_1(\alpha) = \beta$, $K_2(\alpha) = \gamma$

if $N(\alpha) = 0$, $F_\alpha = \{F_{\alpha'} | \alpha' < \alpha\}$

if $1 \leq i = N(\alpha) \leq 8$, $F_\alpha = \mathfrak{F}_i(F_\beta, F_\gamma)$

where $\mathfrak{F}_i$ are defined as follows (Def. 9.1[3]):

$$\begin{aligned}
\mathfrak{F}_1(x, y) &= \{x, y\}\\
\mathfrak{F}_2(x, y) &= \{\langle s, t\rangle | s \in t, \langle s, t\rangle \in x\}\\
\mathfrak{F}_3(x, y) &= x - y\\
\mathfrak{F}_4(x, y) &= \{\langle s, t\rangle | \langle s, t\rangle \in x, t \in y\}\\
\mathfrak{F}_5(x, y) &= \{s | s \in x, \exists t \langle t, s\rangle \in y\}\\
\mathfrak{F}_6(x, y) &= \{\langle s, t\rangle | \langle s, t\rangle \in x, \langle t, s\rangle \in y\}\\
\mathfrak{F}_7(x, y) &= \{\langle r, s, t\rangle | \langle r, s, t\rangle \in x, \langle t, r, s\rangle \in y\}\\
\mathfrak{F}_8(x, y) &= \{\langle r, s, t\rangle | \langle r, s, t\rangle \in x, \langle r, t, s\rangle \in y\}
\end{aligned}$$

if $N(\alpha) = 9$, $F_\alpha = \{F_{\alpha'} | \alpha' < \alpha, N(\alpha') = 9\}$. We also define $T_\alpha = \{F_\beta | \beta < \alpha\}$.

We have introduced the case $N = 9$ for the convenience of later arguments. It ensures that the ordinals in $\mathfrak{M}$ are listed among the F_α in a canonical manner. Observe that since $V = L$ holds in $\mathfrak{M}$, it is not difficult to see that this implies $\mathfrak{M} \subseteq \mathfrak{N}$.

Denote by $\mathfrak{N}$, the set of all F_α for $\alpha \in \mathfrak{M}$. Note that in $\mathfrak{N}$ each F_α is a collection of preceding F_β. We shall often write α in place of F_α if $\alpha \leq \omega$, and a_δ in place of $F_{\delta + \omega + 1}$, etc., if there is no danger of confusion. If $N(\alpha) = 9$, then the set F_α is defined independently of a_δ. We shall now examine statements concerning F_α before the a_δ are actually determined, and thus the F_α for a while shall be considered as merely formal symbols.

Definition 2: (1) $x \in y$, $x \in F_\alpha$, $F_\alpha \in x$, $F_\alpha \in F_\beta$ are formulas; (2) if φ and ψ are formulas, so are $\neg\varphi$ and $\varphi \ \&\ \psi$; and (3) only (1) and (2) define formulas.

Definition 3: A *Limited Statement* is a formula $\mathfrak{a}(x_1, \ldots, x_n)$ in which all variables are bound by a universal quantifier $(x_i)_\alpha$ or an existential quantifier $\exists_\alpha x_i$ placed in front of it, where α is an ordinal in $\mathfrak{M}$. An *Unlimited Statement* is the same except that no ordinals are attached to the quantifiers.

Our intention is that the variable x in $(x)_\alpha$ or $\exists_\alpha x$ is restricted to range over all F_β with $\beta < \alpha$. The symbol $=$ is not used since by means of the Axiom of Extensionality it can be avoided. We only consider statements in prenex form. Since it is clear how to reduce negations, conjunctions, etc., of such statements to prenex form, we shall not do so if there is no risk of confusion.

Definition 4: The rank of a limited statement $\mathfrak{a}$ is (α, r) if r is the number of quantifiers and α is the least ordinal such that for all β, $\beta < \alpha$ if F_β occurs in $\mathfrak{a}$, and $\beta \leq \alpha$ if $(x)_\beta$ or $\exists_\beta x$ occurs in $\mathfrak{a}$. We write $(\alpha, r) < (\beta, s)$ if $\alpha < \beta$ or $\alpha = \beta$ and $r < s$.

Thus, if rank $\mathfrak{a} = (\alpha, r)$, $\mathfrak{a}$ can be formulated in $\{F_\beta | \beta < \alpha\}$.

Definition 5: Let P denote a *finite* set of conditions of the form $n \in a_\delta$ or $\neg\, n \in a_\delta$ such that no condition and its negation are both included.

In the following definition, which is the key point of the paper, we shall define a certain concept for all limited statements by means of transfinite induction. The well-ordering we use is not, however, precisely the corresponding ordering of the ranks, but requires a slight modification. We say $\mathfrak{a}$ is of type $\mathfrak{R}$, if rank $\mathfrak{a} = (\alpha + 1, r)$, $(x)_{\alpha + 1}$ and $\exists_{\alpha + 1} x$ do not occur in $\mathfrak{a}$, and no expression of the form $F_\alpha \in (\cdot)$ occurs in $\mathfrak{a}$. We order the limited statements by saying, if rank $\mathfrak{a} = (\alpha, r)$ and rank $\mathfrak{b} = (\beta, s)$, $\mathfrak{a}$ precedes $\mathfrak{b}$ if and only if rank $\mathfrak{a} <$ rank $\mathfrak{b}$, unless $\alpha = \beta$ and one of the two statements $\mathfrak{a}$, $\mathfrak{b}$ is of type $\mathfrak{R}$ and the other is not of type $\mathfrak{R}$, in which case the former precedes the latter.

Definition 6: By induction, we define the concept of "P forces $\mathfrak{a}$" as follows:

I. If $r > 0$, P forces $\mathfrak{a} = (x)_\alpha \mathfrak{b}(x)$ if for all $P' \supset P$, P' does not force $\neg \mathfrak{b}(F_\beta)$ for $\beta < \alpha$. P forces $\exists_\alpha x \mathfrak{b}(x)$ if for some $\beta < \alpha$, P forces $\mathfrak{b}(F_\beta)$.

II. If $r = 0$, and $\mathfrak{a}$ has propositional connectives, P forces $\mathfrak{a}$ if for each component $F_\alpha \in F_\beta$ or $\neg F_\alpha \in F_\beta$ appearing in $\mathfrak{a}$, these, by case III of this definition, are forced to be true or their negations are forced to be true so that in the usual sense of the propositional calculus $\mathfrak{a}$ is true.

III. If $\mathfrak{a}$ is of the form $F_\alpha \in F_\beta$ or $\neg F_\alpha \in F_\beta$, we define P forces $\mathfrak{a}$ as follows:

(*i*) If $\alpha, \beta \leq 3\aleph_\tau$, then $\mathfrak{a}$ must hold as a formal consequence of P, i.e., P forces $\mathfrak{a}$, if $\mathfrak{a}$ is true whenever a_δ are distinct subsets of ω, satisfying P, different from any integer and ω.

(*ii*) $\neg F_\alpha \in F_\alpha$ is always forced.

(*iii*) If $\alpha < \beta$, $N(\beta) = i < 9$, $\beta > 3\aleph_\tau$, P forces $\mathfrak{a}$, where $\mathfrak{a} \equiv F_\alpha \in F_\beta$ or $\neg F_\alpha \in F_\beta$, if P forces ψ_i or $\neg \psi_i$, respectively, where ψ_i is the limited statement expressing the definition of F_β. That is, if $K_1(\beta) = \gamma$, $K_2(\beta) = \delta$:

(0) ψ_0 is vacuous and always forced.

(1) $\psi_1 \equiv F_\alpha = F_\gamma \vee F_\alpha = F_\delta$.

(2) $\psi_2 \equiv \exists_\beta x \, \exists_\beta y \, (F_\alpha = \langle x, y\rangle \,\&\, x \in y \,\&\, F_\alpha \in F_\gamma)$.

(3) $\psi_3 \equiv F_\alpha \in F_\gamma \,\&\, \neg F_\alpha \in F_\delta$.

(4) $\psi_4 \equiv \exists_\beta x \, \exists_\beta y \, (F_\alpha = \langle x, y\rangle \,\&\, F_\alpha \in F_\gamma \,\&\, y \in F_\delta)$.

(5) $\psi_5 \equiv \exists_\beta x \, (F_\alpha \in F_\gamma \,\&\, \langle x, F_\alpha\rangle \in F_\delta)$.

(6), (7), (8), similarly.

Here the use of ordered pairs must eventually be replaced by their definition, and the use of equality in $x = y$ is replaced by $(z)_\beta(z \in x \Longleftrightarrow z \in y)$.

(*iv*) If $\alpha < \beta$, $N(\beta) = 9$, $\beta > 3\aleph_\tau$, P forces $\mathfrak{a} \equiv F_\alpha \in F_\beta$ if for some $\beta' < \beta$, $N(\beta') = 9$, P forces $F_\alpha = F_{\beta'}$. P forces $\neg F_\alpha \in F_\beta$, if for all $\beta' < \beta$, $N(\beta') = 9$ and all $P' \supset P$, P' does not force $F_\alpha = F_{\beta'}$. Again the symbol $=$ is treated as before.

(*v*) If $\alpha > \beta$, we reduce the case $F_\alpha \in F_\beta$ to cases (*iii*) and (*iv*) treated above. We say P forces $F_\alpha \in F_\beta$ if for some $\beta' < \beta$, P forces $F_{\beta'} \in F_\beta$ and P forces $F_\alpha = F_{\beta'}$ (i.e., $(x)_\alpha(x \in F_\alpha \Longleftrightarrow x \in F_{\beta'})$ which is a statement of type $\mathfrak{R}$ and hence precedes $F_\alpha \in F_\beta$). We say P forces $\neg F_\alpha \in F_\beta$ if for all $\beta' < \beta$ and $P' \supset P$, P' does not force both $F_{\beta'} \in F_\beta$ and $F_{\beta'} = F_\alpha$.

The most important part of Definition 6 is I, the other parts are merely obvious derivatives of it.

Definition 7: If $\mathfrak{a}$ is an unlimited statement with r quantifiers, we define "P forces $\mathfrak{a}$" by induction on r. If $r = 0$, then $\mathfrak{a}$ is a limited statement. If $\mathfrak{a} \equiv (x)\,\mathfrak{b}(x)$, P forces $\mathfrak{a}$, if for all $P' \supset P$, and α, P' does not force $\neg \mathfrak{b}(F_\alpha)$. If $\mathfrak{a} \equiv \exists x\, \mathfrak{b}(x)$, P forces $\mathfrak{a}$ if for some α, P forces $\mathfrak{b}(F_\alpha)$.

In the proofs of Lemmas 2, 3, 4, and 5, we keep the same well-ordering on limited statements as in Definition 6, and proceed by induction.

LEMMA 2. *P does not force $\mathfrak{a}$ and $\neg \mathfrak{a}$, for any $\mathfrak{a}$ and P.*

Proof: Let $\mathfrak{a}$ be a limited statement with r quantifiers. If $r > 0$, and P forces both $\exists_\alpha x\, \mathfrak{b}(x)$ and $(x)_\alpha \neg \mathfrak{b}(x)$, then P must force $\mathfrak{b}(F_\beta)$ for $\beta < \alpha$ which means P cannot force $(x)_\alpha \neg \mathfrak{b}(x)$. Case II of Definition 6 will clearly follow from case III. Parts (*i*) and (*ii*) are trivial. If $\mathfrak{a}$ is in part (*iii*), then P forces $\mathfrak{a}$ if and only if P

forces a statement of lower rank and in this case the lemma follows by induction. In part (*iv*), if P forces $F_\alpha \epsilon F_\beta$, then for some $\beta' < \beta$, $N(\beta') = 9$, and P forces $F_\alpha = F_{\beta'}$ which means P can not force $\neg F_\alpha \epsilon F_\beta$. In part (*v*) if P forces $F_\alpha \epsilon F_\beta$, for some $\beta' < \beta$, P forces $F_{\beta'} \epsilon F_\beta$ and $F_\alpha = F_{\beta'}$ which again violates P forcing $\neg F_\alpha \epsilon F_\beta$. If $\mathfrak{a}$ is an unlimited statement, the lemma follows in the same manner by induction on the number of quantifiers.

LEMMA 3. *If P forces $\mathfrak{a}$ and $P' \supset P$, then P' forces $\mathfrak{a}$.*

Proof by induction as in Lemma 2.

LEMMA 4. *For any statement $\mathfrak{a}$ and condition P, there is $P' \supset P$ such that either P' forces $\mathfrak{a}$ or P' forces $\neg \mathfrak{a}$.*

Proof: Let $\mathfrak{a}$ be a limited statement with r quantifiers. If $r > 0$ and P does not force $\mathfrak{a} \equiv (x)_\alpha \mathfrak{b}(x)$, then for some $P' \supset P$, P' forces $\neg \mathfrak{b}(F_\beta)$, $\beta < \alpha$, which means P' forces $\neg \mathfrak{a}$. If $r = 0$, we may restrict ourselves to III, for if we enumerate the components of $\mathfrak{a}$, by defining a finite sequence P_n, $P_0 = P$ and $P_{n+1} \supset P_n$ we may successively force each component or its negation so that finally either $\mathfrak{a}$ or $\neg \mathfrak{a}$ is forced. Again, cases (*i*) and (*ii*) are trivially disposed of. Case (*iii*) is handled by induction as before. If $\mathfrak{a} \equiv F_\alpha \epsilon F_\beta$ is in case (*iv*) then if P does not force $\neg \mathfrak{a}$, for some $P' \supset P$ and $\beta' < \beta$, $N(\beta') = 9$, P' forces $F_\alpha = F_{\beta'}$ so P' forces $\mathfrak{a}$. If $\mathfrak{a} \equiv F_\alpha \epsilon F_\beta$ is in case (*v*) if P does not force $\neg \mathfrak{a}$, then for some $P' \supset P$, $\beta' < \beta$, P' forces $F_{\beta'} \epsilon F_\beta$ and $F_{\beta'} = F_{\alpha'}$ hence P' forces $\mathfrak{a}$. Unlimited statements are handled as before.

Definition 8: Enumerate all statements $\mathfrak{a}_n$, both limited and unlimited, and all ordinals α_n in $\mathfrak{M}$. Define P_{2n} as the first extension of P_{2n-1} which forces either $\mathfrak{a}_n$ or $\neg \mathfrak{a}_n$. Define P_{2n+1} as the first extension of P_{2n} which has the property that it forces $F_\beta \epsilon F_{\alpha_n}$ where β is the least possible ordinal for which there exists such an extension of P_{2n}, whereas if no such β exists, put $P_{2n+1} = P_{2n}$.

The sequence P_n is not definable in $\mathfrak{M}$. Since all statements of the form $n \epsilon a_\delta$ are enumerated, P_n clearly approach in an obvious sense, sets a_δ of integers. With this choice of a_δ, let $\mathfrak{N}$ be defined as the set of all F_α defined by Definition 1.

LEMMA 5. *All statements in $\mathfrak{N}$ which are forced by some P_n are true in $\mathfrak{N}$ and conversely.*

Proof: Let $\mathfrak{a}$ be a limited statement with r quantifiers. If $r > 0$, then if P_n forces $(x)_\alpha \mathfrak{b}(x)$, if $\beta < \alpha$, then some P_m must force $\mathfrak{b}(F_\beta)$ since no P_m can force $\neg \mathfrak{b}(F_\beta)$. By induction we have that $\mathfrak{b}(F_\beta)$ holds, so that $(x)_\alpha \mathfrak{b}(x)$ holds in $\mathfrak{N}$. If P_n forces $\exists_\alpha x\ \mathfrak{b}(x)$, for some $\beta < \alpha$, P_n forces $\mathfrak{b}(F_\beta)$ so by induction $\mathfrak{b}(F_\beta)$ holds and hence $\exists_\alpha x\ \mathfrak{b}(x)$ holds in $\mathfrak{N}$. Case II will clearly follow from case III and (*i*) and (*ii*) are trivial. If $\mathfrak{a}$ is $F_\alpha \epsilon F_\beta$ or $\neg F_\alpha \epsilon F_\beta$ in case (*iii*) then if P_n forces $\mathfrak{a}$, P_n forces precisely the statement which because of the definition of F_β is equivalent to $\mathfrak{a}$. In case (*iv*) if P_n forces $F_\alpha \epsilon F_\beta$, for some $\beta' < \beta$, $N(\beta') = 9$, P_n forces $F_{\beta'} = F_\alpha$, which therefore holds by induction in $\mathfrak{N}$. If P_n forces $\neg F_\alpha \epsilon F_\beta$, then for each $\beta' < \beta$, $N(\beta') = 9$, $F_\alpha = F_{\beta'}$ is not forced by any P_m so some P_m must force $F_\alpha \neq F_{\beta'}$ which proves $\neg F_\alpha \epsilon F_\beta$ holds in $\mathfrak{N}$. Similarly for case (*v*) and for unlimited statements. Since every statement or its negation is forced eventually, the converse is also true.

Lemma 5 is the justification of the definition of forcing since we can now throw back questions about $\mathfrak{N}$ to questions about forcing which can be formulated in $\mathfrak{M}$.

In the next paper, we shall prove that $\mathfrak{N}$ is a model for Z-F in which part 3 of Theorem 1 holds.

* The author is a fellow of the Alfred P. Sloan Foundation.

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*THE INDEPENDENCE OF THE CONTINUUM HYPOTHESIS, II**

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This paper is a continuation of reference 1, in which we began a proof of the fact that the Continuum Hypothesis cannot be derived from the other axioms of set theory, including the Axiom of Choice. We use the same notation as employed in reference 1.

THEOREM 2. *$\mathfrak{N}$ is a model for Z-F set theory.*

The proof will require several lemmas. The first two lemmas express the principle that forcing is a notion which is formalizable in the original model $\mathfrak{M}$.

LEMMA 6. *There is an enumeration* $\mathfrak{a}_\alpha$ *of all limited statements by means of the ordinal numbers of* $\mathfrak{M}$, *such that the usual formal operations performed on statements are expressible by means of definable functions in* $\mathfrak{M}$ *of the indices* α, *for example, forming negations, conjunctions, replacing variables by particular sets, etc. Furthermore, the ordering corresponds to the definition of forcing given by transfinite induction in Definition 6.*

LEMMA 7. *Let* $\mathfrak{a}(x,y)$ *be a fixed unlimited statement containing two unbound variables* x *and* y. *The relation* $\Phi_\mathfrak{a}(P,\alpha,\beta)$ *which says that* P *forces* $\mathfrak{a}(F_\alpha,F_\beta)$ *and* β *is the least such ordinal, is definable in* $\mathfrak{M}$.

This follows from the fact that using Lemma 6 the relation "P forces $\mathfrak{a}_\alpha$" can be formalized in Z-F as a statement about P and α. A given unlimited statement can also be handled since, after a finite number of replacements of variables, it is reduced to a limited statement.

Definition 9: For $\mathfrak{a}(x,y)$ as above, put $\Gamma_\mathfrak{a}(\alpha) = \sup\{\beta | \exists\ P, \alpha_1 < \alpha, \Phi_\mathfrak{a}(P,\alpha_1,\beta)\}$.

LEMMA 8. *Let* $\mathfrak{a}(x,y)$ *be a fixed unlimited statement,* α *an ordinal. For each* $\alpha' < \alpha$ *either there is no* F_β *such that* $\mathfrak{a}(F_{\alpha'},F_\beta)$ *or such an* F_β *exists with* $\beta \leq \Gamma_\mathfrak{a}(\alpha)$.

Proof: If β is the least ordinal such that $\mathfrak{a}(F_{\alpha'},F_\beta)$, then $\mathfrak{a}(F_{\alpha'},F_\beta)$ must be forced by some P_n which clearly implies $\beta < \Gamma_\mathfrak{a}(\alpha)$.

LEMMA 9. *Let* $\mathfrak{a}(x,y)$ *be an unlimited statement of the form*

$$Q_1x_1Q_2x_2, \ldots, Q_nx_n\mathfrak{b}(x,y,x_1, \ldots, x_n)$$

where $\mathfrak{b}$ *has no quantifiers and* Q_i *are either existential or universal quantifiers. In* $\mathfrak{N}$, *assume* $\mathfrak{a}$ *defines* y *as a single-valued function of* x. *Then for each* α *there exist ordinals* $\gamma_0, \ldots, \gamma_n$ *such that for* $x \in T_\alpha$, *there exist* $y \in F_{\gamma_0}$ *such that* $\mathfrak{a}(x,y)$ *and for* $\langle x,y\rangle$ *in* $T_\alpha \times T_{\gamma_0}$, *the statement* $\mathfrak{a}(x,y)$ *holds if and only if* $\tilde{\mathfrak{a}}(x,y)$ *holds where* $\tilde{\mathfrak{a}}$ *is the statement formed by restricting the quantifiers* Q_i *in* $\mathfrak{b}$ *to range over* F_{γ_i}.

Proof: Lemma 8 implies the existence of γ_0 such that for $x \in T_\alpha$, there is a $y \in F_{\gamma_0}$ such that $\mathfrak{a}(x,y)$. Define γ_k by induction as follows: let $g_k(x,y,x_1, \ldots, x_{k-1},z)$ be the condition

(*i*) if Q_k is universal,

$$\sim Q_{k+1}x_{k+1}, \ldots, Q_nx_n\mathfrak{b}(x,y,x_1, \ldots, x_{k-1},z,x_{k+1} \ldots x_n) \quad \text{or}$$

(*ii*) if Q_k is existential,

$$Q_{k+1}x_{k+1}, \ldots, Q_nx_n\mathfrak{b}(x,y,x_1, \ldots, x_{k-1},z,x_{k+1} \ldots x_n).$$

Lemma 8 implies that for some γ_k, for all $\langle x,y,x_1, \ldots, x_{k-1}\rangle \in T_\alpha \times F_{\gamma_0} \times \ldots \times F_{\gamma_{k-1}}$, either no z exists such that $g_k(x,y,x_1, \ldots, x_{k-1},z)$ or there is such a $z \in F_{\gamma_k}$. This clearly implies the lemma.

LEMMA 10. *The Axiom of Replacement holds in* $\mathfrak{N}$.

Proof: If $\mathfrak{a}(x,y)$ defines y as a single-valued function of x in $\mathfrak{N}$, then for any α if $D = \{x | \exists z, z \in F_\alpha \ \&\ \mathfrak{a}(z,x)\}$ then by Lemma 9, D is defined by a condition in which all variables are restricted to lie in fixed sets F_{γ_i}, which by the definition of the sets F_α implies that D is a set in $\mathfrak{N}$.

The only other axiom to verify which is nontrivial, is the Axiom of the Power Set. The proof we give follows closely the method in reference 2 used to prove that $V = L$ implies the Continuum Hypothesis.

LEMMA 11. *Let W be a set in* $\mathfrak{M}$, *consisting of conditions P, such that if* P_1 *and* P_2 *belong to W, then* $P_1 \cup P_2$ *is not an admissible condition (i.e., contains a contradiction). Then W is a countable set (in* $\mathfrak{M}$).

Proof: Define sequences n_k and P_j as follows. Put $n_1 = 1$ and P_1 the first P in W. (We assume the P are well-ordered.) If n_k and P_j for $j \leq n_k$ are defined, put R_k equal to the set of all conditions $n \in a_\delta$ or $\sim n \in a_\delta$ such that they or their negations are contained in some $P_j, j \leq n_k$. Let P_j, $n_k < j \leq n_{k+1}$, be finitely many P in W, such that for all P in W, $\exists j$, $n_k < j \leq n_{k+1}$ and P and P_j have precisely the same intersection with R_k. This is possible since R_k is a finite set. We claim that W consists only of the P_j. For if $P \in W$, then since P is a finite set of conditions, and $R_k \subseteq R_{k+1}$, there exists a k such that $P \cap R_k = P \cap R_{k+1}$. Let $n_k < j \leq n_{k+1}$, such that $P_j \cap R_k = P \cap R_k$. Then if P is not equal to P_j, since $P \cap R_{k+1} \subseteq P_j$ and $P_j \subseteq R_{k+1}$, $P \cup P_j$ is an admissible condition, which contradicts the hypothesis.

Definition 10: Put $C(P,\alpha) = \beta$ if P forces $F_\beta \in F_\alpha$ and for $P' \supset P$, $\gamma < \beta$, P' does not force $F_\gamma \in F_\alpha$. If no such β exists, put $C(P,\alpha) = 0$.

The function C is definable in $\mathfrak{M}$, by virtue of the general principle contained in Lemma 6.

LEMMA 12. *For any* α, *there are only countably many (in* $\mathfrak{M}$) β *such that for some* P, $C(P,\alpha) = \beta$.

Proof: For each such β, pick one P such that $C(P,\alpha) = \beta$. Then the set of all such P must be countable by Lemma 10.

LEMMA 13. *Let S be an infinite set of ordinals in* $\mathfrak{M}$. *There exists a set S′ of ordinals,* $S' \supset S$, $\bar{\bar{S'}} = \bar{\bar{S}}$ *such that S′ is closed under* $J(i,\alpha,\beta,\gamma)$, $K_i(\alpha)$, $C(P,\alpha)$, $I(\alpha)$, *for all P and* α, β, $\gamma \in S$. *Also* $\alpha \in S'$ *implies* $\alpha + 1 \in S'$.

The statement $\bar{\bar{S'}} = \bar{\bar{S}}$, above, means that with respect to $\mathfrak{M}$, the sets S and S' are of the same cardinality.

LEMMA 14. *Let S be a set of ordinals closed under the operations in Lemma 13, and such that if* $\alpha \leq 3\aleph_\tau$, $\alpha \in S$. *Then there is a map g mapping S 1-1 onto an initial segment of ordinals which preserves* J, K_i, I, N, *and such that if* $N(\alpha) = 0$ *(or 9),* $g(\alpha) = \beta$ *is the first ordinal such that* $N(\beta) = 0$ *(or 9) and* β *is greater than* $g(\alpha')$ *for* $\alpha' < \alpha$. *Also, g is the identity for* $\alpha \leq 3\aleph_\tau$.

Proof: S and g in the lemma refer to sets in the model $\mathfrak{M}$. We define g by transfinite induction. For $\alpha \leq 3\aleph_\tau$, let g be the identity. If g is defined for all β in S less than α, if $I(\alpha) = \alpha^3$ (i.e., $N(\alpha) = 0$), put $g(\alpha) = \sup\{g(\beta)|\beta < \alpha \text{ and } \beta \in S\}$. If $I(\alpha) = \beta < \alpha$, then if $N(\alpha) = 9$ (i.e., $\alpha = \beta + 1$), put $g(\alpha) = g(\beta) + 1$. If $i = N(\alpha)$, $1 \leq i \leq 8$, put $g(\alpha) = J(i,g(K_1(\alpha)), g(K_2(\alpha)), g(\beta))$. One can now show by induction that if $\alpha \in S$, $N(\alpha) = 0$, g maps the set of all $\beta < \alpha$ onto an initial segment. The lemma then easily follows.

LEMMA 15. *If we put* $G(F_\alpha) = F_{g(\alpha)}$ *for* α *in S, then G is an isomorphism with respect to* $\in$ *of* $A_1 = \{F_\alpha|\alpha \in S\}$ *onto* $A_2 = \{F_{g(\alpha)}|\alpha \in S\}$.

Proof: This follows by induction on α, in the same way as in 12.6 of reference 2. Observe that in examining the operations $\mathfrak{F}_4$ and $\mathfrak{F}_5$ we need the fact that if $F_\alpha \in A_1$ and is not empty, then it has a member in A_1 preceding it. This is true since S is closed under $C(P,\alpha)$, and $C(P,\alpha)$ for some P is the smallest β for which $F_\beta \in F_\alpha$, if $F_\alpha \neq \phi$.

LEMMA 16. *If* $F_\beta \subseteq F_\alpha$, *then for some* γ, $F_\beta = F_\gamma$, *where* $\bar{\bar{\gamma}} \leq \bar{\bar{\alpha}} + \aleph_\tau$ *in* $\mathfrak{M}$.

Proof: Let S contain all $\delta \leq \alpha$, all $\delta \leq 3\aleph_\tau$ and β, and be closed under the operations in Lemma 13. Let g be the corresponding isomorphism. Then clearly $g(\delta) = \delta$ if $\delta \leq \alpha$. Thus, by Lemma 15, if we put $\gamma = g(\beta)$, since $F_\beta \subseteq F_\alpha$, $F_\gamma = F_\beta$. Since g maps S onto an initial segment, $\bar{\bar{\gamma}} \leq \bar{\bar{S}}$ and so the lemma is proved.

LEMMA 17. *The Axiom of the Power Set holds in* $\mathfrak{N}$.

Proof: Since every subset of F_α is contained in F_β, where β is the first ordinal such that $N(\beta) = 0$ and $\bar{\bar{\beta}} > \bar{\bar{\alpha}} + \aleph_\tau$, it is clear that the power set of F_α occurs in $\mathfrak{N}$.

This completes the proof that $\mathfrak{N}$ is a model, the other axioms being trivially verified. Since rank $F_\alpha \leq \alpha$, $\mathfrak{N}$ contains no new ordinals.

LEMMA 18. *If* $N(\alpha) = N(\beta) = 9$, *and* $\bar{\bar{F}}_\alpha > \bar{\bar{F}}_\beta$ *in* $\mathfrak{M}$, *then* $\bar{\bar{F}}_\alpha > \bar{\bar{F}}_\beta$ *in* $\mathfrak{N}$.

Proof: The point of this lemma is that ordinals do not change their relative cardinality in the model $\mathfrak{N}$. The added complications in the definition of forcing due to $N(\alpha) = 9$ are compensated for in the proof of this lemma, in that as α runs through the ordinals with $N(\alpha) = 9$, F_α runs through the ordinals of $\mathfrak{M}$ in a manner independent of the sequence P_n. More exactly, the map $\alpha \rightarrow F_\alpha$ is an order-preserving map of the ordinals α, $N(\alpha) = 9$, onto all the ordinals of $\mathfrak{M}$.

Thus assume that some element in $\mathfrak{N}$ defines a relation $\varphi(x,y)$ on $F_\beta \times F_\alpha$ which is a single-valued function from F_β onto F_α. For each $\beta' < \beta$, $N(\beta') = 9$ consider the set $H_{\beta'}$ of all γ, $N(\gamma) = 9$, such that some P forces both $\varphi(F_{\beta'},F_\gamma)$ and $(x)[\varphi(F_{\beta'},x) \rightarrow x = F_\gamma]$. The set $H_{\beta'}$ exists in $\mathfrak{M}$ as does the map $\beta' \rightarrow H_{\beta'}$ since the notion of forcing is expressible in $\mathfrak{M}$. We shall now show that each $H_{\beta'}$ is countable in $\mathfrak{M}$. For each element in $H_{\beta'}$ choose a corresponding P which forces the above statements. By Lemma 11, it is sufficient to show that these P are mutually incompatible. If two such P corresponding to γ_1 and γ_2 were compatible, their union would force both $\varphi(F_{\beta'},F_{\gamma_1})$ and $(x)[\varphi(F_{\beta'},x) \rightarrow x = F_{\gamma_2}]$. Now since $\sim F_{\gamma_1} = F_{\gamma_2}$ is forced, taking into account that $\sim F_{\gamma_1} = F_{\gamma_2}$ involves only existential quantifiers, it follows that $\sim(\varphi(F_{\beta'},F_{\gamma_1}) \rightarrow F_{\gamma_1} = F_{\gamma_2})$ is forced, which is a contradiction. Thus the union of all the $H_{\beta'}$ is of cardinality $\bar{\bar{F}}_\beta$ in $\mathfrak{M}$, since we may clearly restrict ourselves to the case where F_β is infinite. If in $\mathfrak{N}$, $\varphi(F_{\beta'},F_\gamma)$ holds for some γ, $N(\gamma) = 9$ then since all true statements in $\mathfrak{N}$ are forced by some P, γ belongs to $H_{\beta'}$. Thus since φ is onto, the union of $H_{\beta'}$ must contain all $\gamma < \alpha$, $N(\gamma) = 9$ which is impossible since $\bar{\bar{F}}_\beta < \bar{\bar{F}}_\alpha$ in $\mathfrak{M}$.

LEMMA 19. *There is a statement* $\mathfrak{a}(x,y)$ *built up from the logical symbols and the set* V, *which expresses in* $\mathfrak{N}$ *the condition that* x *is an ordinal and* $F_x = y$. *Thus the Axiom of Choice holds in* $\mathfrak{N}$.

Proof: This is true because our construction differs from that of reference 2, merely in the introduction of the sets a_δ. If we use the set V, we can of course describe their ordering and so define the construction. We can thus well-order $\mathfrak{N}$ by saying F_α precedes F_β if $\alpha < \beta$ and $F_\beta \neq F_\gamma$ for $\gamma < \beta$.

LEMMA 20. *In* $\mathfrak{N}$, *we have* $\aleph_\tau \leq 2^{\aleph_0} \leq \aleph_{\tau+1}$.

Proof: By Lemma 18, the sets $\aleph_\lambda$ do not change in $\mathfrak{N}$. One can easily see that no P forces any two a_δ to be equal, hence they are distinct, which implies one half of the lemma. Our proof of the Power Set Axiom shows that every subset of ω is some F_α with $\bar{\bar{\alpha}} \leq \aleph_\tau$ or $\alpha < \aleph_{\tau+1}$. Thus Lemma 19 establishes a map of $\aleph_{\tau+1}$ onto $2^{\aleph_0}$.

We have now completed the proof of part 3 of Theorem 1. We now sketch the proof of one of the finer points involved.

LEMMA 21. *If in* $\mathfrak{M}$ $\aleph_\tau$ *is not the sum of countably many smaller cardinals, then* $2^{\aleph_0} = \aleph_\tau$ *in the model* $\mathfrak{N}$. *If it is, then* $2^{\aleph_0} = \aleph_{\tau+1}$.

Proof: The second part follows from Lemma 20, and the theorem of Koenig which says that the continuum is not a countable sum of smaller cardinals. Let $F_\alpha \subseteq \omega$. To prove the first part, let S be a set of indices containing all $\beta \leq \omega$, the ordinal α, and closed under J, K_i, I, $+ 1$, and C as before, and such that $\bar{\bar{S}} = \aleph_0$. The set S is definable in $\mathfrak{M}$ by virtue of the general principle of Lemma 6. For β in S define a new collection of sets G_β, defined by induction on β as follows. If $\beta \leq 3\aleph_\tau$, $G_\beta = F_\beta$. If $N(\beta) = 0$, put $G_\beta = \{G_{\beta'} | \beta' < \beta\}$, and if $N(\beta) = 9$, $G_\beta = \{G_{\beta'} | \beta' < \beta \;\&\; N(\beta') = 9\}$. If $1 \leq i = N(\beta) \leq 8$, $K_1(\beta) = \gamma_1$, $K_2(\beta) = \gamma_2$, put $G_\beta = \mathfrak{F}_i(G_{\gamma_1}, G_{\gamma_2})$. Then the correspondence $F_\beta \rightarrow G_\beta$ is an isomorphism with respect to ϵ. Clearly, $F_\alpha = G_\alpha$. Let ρ be an isomorphism with respect to ϵ of S onto a countable ordinal S'. Let $K_i' = \rho K_i \rho^{-1}$, and $N' = N\rho^{-1}$. Then our argument shows that F_α depends only upon S', K_i', N', $\rho(\alpha)$, and the set S $\cap$ $3\aleph_\tau$. The number of possible S' is $\aleph_1$. For each S', the number of possible K_i' and N' is $\aleph_1$, since $\aleph_0{}^{\aleph_0} = \aleph_1$ in $\mathfrak{M}$ and K_i', N' are definable in $\mathfrak{M}$. The number of countable subsets of $3\aleph_\tau$ is of cardinality $\aleph_\tau$, as follows easily from our hypothesis on $\aleph_\tau$ and the fact that the Generalized Continuum Hypothesis holds in $\mathfrak{M}$. Thus the number of possible F_α does not exceed $\aleph_\tau$ and the lemma is proved.

LEMMA 22. *If in* $\mathfrak{M}$ *the number of subsets of* $\aleph_\tau$ *of cardinality* $\aleph_1$ *is* $\aleph_\tau$, *then* $2^{\aleph_1} = \aleph_\tau$ *in* $\mathfrak{N}$.

Proof: This is very similar to Lemma 21. We merely demand that S contain all $\beta \leq \aleph_1$. The condition of the lemma may be rephrased by saying $\aleph_\tau$ is not cofinal with $\aleph_0$ or $\aleph_1$. In particular, τ may be 2.

This settles an old question of Lusin whether one can have $2^{\aleph_0} = 2^{\aleph_1}$ Other examples of this type presumably can be constructed with our method. In particular, one can construct models in which the set of constructible reals is countable, a countable union of countable sets is uncountable, etc.

We now give a short discussion of the question of how the above proof can be formalized. Let us denote by (Z-F)′ the axiom system obtained by adjoining to Z-F the axiom:

There exists a set $\mathfrak{M}$ *which is a model for Z-F.*

Observe that this axiom can be expressed as a single statement about $\mathfrak{M}$, because $\mathfrak{M}$ is a set. In the axiom system of Gödel-Bernays this would be still simpler, since only finitely many axioms are employed there. The classic argument of Gödel[2] shows that from (Z-F)′ one can deduce the existence of a set $\mathfrak{N}$ which is a model for Z-F and $V = L$. Similarly, the argument of this paper shows that (Z-F)′ implies the existence of a set $\mathfrak{N}$, which is a model for Z-F, the Axiom of Choice, and the negation of the Continuum Hypothesis. Since our additional axiom is quite readily acceptable to most mathematicians (being merely a formal expression of the Löwenheim-Skolem principle, and implied by well-known axioms such as the Axiom of an Inaccessible Cardinal), one can regard the unprovability of the Continuum Hypothesis as firmly established. However, the consistency of a formal system can also be regarded as a statement in elementary number theory, and one may ask for a proof within elementary number theory of various implications. If $(\text{Z-F})_1$ denotes Z-F with the Axiom of Choice and say $2^{\aleph_0} = \aleph_\tau$, the relevant question is, can we prove within number theory or, if need be, a system of higher type, the implication Con(Z-F) $\rightarrow$ Con(Z-F)$_1$. By using rather standard methods, we shall show how to prove the above implication purely within elementary number theory.

Let us enumerate the axioms of Z-F, A_n. For each n, there is in Z-F a proof of the existence of a countable set $\mathfrak{M}_n$ which satisfies the axioms A_j, $j \leq n$. Furthermore, the correspondence between n and the string of symbols corresponding to such a proof is expressible in number theory.

We may also assume by reference 2 that the axiom $V = L$ is valid in $\mathfrak{M}_n$. We now assert that the proof that $\mathfrak{N}$ is a model for A_j, $j \leq p$ as well as $2^{\aleph_0} = \aleph_\tau$ can be given under the assumption that $\mathfrak{M}$ is a set satisfying A_j, for $j \leq n$ where n is a suitable number greater than p, but still an arithmetical function of p. To see this, we observe that the notion of forcing for limited statements can in Z-F be formulated for unlimited statements as well and the basic lemmas may be proved, since no special properties of $\mathfrak{M}$ are used except the transitivity of $\mathfrak{M}$. To prove that the axioms of Z-F other than the Replacement Axiom holds in $\mathfrak{N}$, as well as $2^{\aleph_0} = \aleph_\tau$ requires only finitely many axioms to hold in $\mathfrak{M}$. Each instance of the Replacement Axiom to be proved in $\mathfrak{N}$ requires that a finite number of instances of replacement used in the proof of Lemma 8 hold in $\mathfrak{M}$. Which instances are sufficient is a simple function of the number of logical symbols used in the formula $\mathfrak{a}(x,y)$ discussed. Since any contradiction in (Z-F)$_1$ would involve only finitely many axioms and since we can prove the existence of a set $\mathfrak{N}$ satisfying these axioms, we would thus be led to a contradiction in Z-F itself. This mapping from contradictions in (Z-F)$_1$ to contradictions in (Z-F) is expressible in an elementary number-theoretic manner which is what was to be proved. In general the statement $2^{\aleph_0} = \aleph_\tau$, for τ in $\mathfrak{M}$, may not be capable of being expressed as a statement in Z-F or may have different interpretations in different countable models $\mathfrak{M}$ or $\mathfrak{N}$. If τ is a particular natural number or $\omega^2 + 1$, etc., then it can readily be expressed in Z-F and the proof sketched goes through.

The argument given in this paper to establish the independence of the Continuum Hypothesis will certainly carry over if one adjoins to Z-F the Axiom of an Inaccessible Cardinal. It seems probable to the author that the addition of any axiom of infinity, as the term is presently understood (i.e., of axioms such as those introduced by P. Mahlo and Azriel Levy), will not alter the situation.

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[2] Gödel, K., *The Consistency of the Continuum Hypothesis* (Princeton University Press, 1940).

[3] The definition of I in Lemma 1 is to be supplemented by the stipulation:

$$I(j(\alpha)) = I(j(\alpha) + 1) = j(\alpha).$$

MARGINALIA TO A THEOREM OF SILVER

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§ 0 Introduction

The singular cardinals problem, in its simplest form, asks whether the continuum hypothesis can hold below a singular cardinal β and fail at β. A variant of the question is whether we can have $2^{\underset{\smile}{\beta}} = \beta$ and $2^{\beta} > \beta^{+}$. Since forcing is the natural method for producing independence results, set theorists have concentrated on a more specific form of the problem: Given a transitive model M of ZF + GCH, can a positive solution be obtained by forcing over M with a set of conditions $\mathbb{P}$? This approach suggests a number of related problems: Is there a $\mathbb{P}$ which collapses β^{+} to β? Is there a $\mathbb{P}$ which makes an inaccessible cardinal singular?

Until very recently, there was a widespread assumption among set theorists that such sets of conditions do exist and merely awaited discovery. Then Silver challenged this assumption by proving it false. Specifically, Silver proved - in ZFC - that if the continuum hypothesis holds below a singular cardinal β of uncountable cofinality, then it holds at β. Thus, in many important cases, not only the narrower forcing problem but the general problem itself has a negative solution.

Much of the effort to produce a positive forcing solution centered on the attempt to exploit the properties of special ground models - either L or models containing large cardinals. The latter approach met with some success: Prikry, fr. ins., showed that a measurable cardinal can be turned into an ω cofinal cardinal. Magidor, starting with an elephantine cardinal, produced a model in which $2^{\omega_n} < \omega_\omega$ for $n < \omega$ and $2^{\omega_\omega} > \omega_{\omega+1}$. Jensen's efforts to produce a positive solution over L led to total failure. Silver's work then led him to consider the problem from a new perspective. He discovered that the statement "$0^{\#}$ does not exist" (henceforth abbreviated as $\neg 0^{\#}$) implies a negative solution to all cases of the singular cardinals problem. But then there cannot be a positive forcing solution over L, since every generic extension of L by a set of conditions satisfies $\neg 0^{\#}$.

Throughout this paper we assume ZFC. Our main theorem says, in effect, that if $\neg 0^{\#}$, then the "essential structure" of cardinalities and confinalities in L is retained in V.

Theorem 1. Assume $\neg 0^{\#}$. Let X be an uncountable set of ordinals. Then there is a constructible set Y s.t. $X \subset Y$ and $\overline{\overline{X}} = \overline{\overline{Y}}$.

Remark. By a theorem of Prikry, we cannot replace "uncountable" by "infinite" in Theorem 1.

Corollary 2. Assume $\neg 0^{\#}$. If $\tau \geq \omega_2$ is regular in L, then $cf(\tau) = \overline{\overline{\tau}}$.

Remark. By a theorem of Bukovski, we cannot replace ω_2 by ω_1 in Corollary 2.

The following corollary establishes a totally negative solution of the singular cardinals problem over L.

Corollary 3. Assume $\neg 0^{\#}$. Let β be a singular cardinal. Then

(a) β is singular in L

(b) $\beta^+ = \beta^{+L}$

(c) If $A \subset \beta$ s.t. $H_\beta = L_\beta[A]$, then $\mathcal{P}(\beta) \subset L[A]$.

(d) $cf(\beta) \leq \gamma < \beta \longrightarrow \beta^\gamma = 2^\gamma \cdot \beta^+$

(e) Let $\theta = 2^{\underset{\smile}{\beta}}$. Then

$$2^\beta = \begin{cases} \theta \text{ if } \forall \gamma < \beta \quad 2^\gamma = \theta \\ \theta^+ \text{ if not.} \end{cases}$$

The proofs of these corollaries are quite straightforward and will be left to the reader.

The above results are due to Jensen and were originally presented in three handwritten notes bearing the title of this paper. The proof given there was developed "piecewise" and contained many redundancies. The present streamlined proof is due chiefly to Devlin.

§ 1 The approach

From now on assume $\neg\, 0^{\#}$.

Def $\tau > \omega$ is suitable iff either $J_\tau \models$ (There is a largest cardinal) or else there are arbitrarily large $\gamma < \tau$ s.t. $cf(\gamma) > \omega$ and $J_\tau \models$ (γ is regular).

§ 0 Theorem 1 reduces to the statement:

Lemma 1. Let $\tau \geq \omega_2$ be a suitable cardinal in L. Let $X \subset \tau$ be cofinal in τ s.t. $\overline{\overline{X}} < \overline{\overline{\tau}}$. Then there is $Y \supset X$ s.t. $Y \in L$ and $\overline{\overline{Y}}^L < \tau$.

We first show that Lemma 1 implies § 0 Theorem 1. Suppose not. Let X be an uncountable set of ordinals for which the conclusion of §0 Theorem 1 fails. Choose $\tau = \text{lub}(X)$ minimal for such X. Then $\overline{\overline{X}} < \overline{\overline{\tau}}$, since otherwise the conclusion of § 0 Theorem 1 would hold with $Y = \tau$. Hence $\tau > \omega_2$. Now suppose that the conclusion of Lemma 1 held. There would then be $Z \in L$ s.t. $X \subset Z$ and $\overline{\overline{Z}}^L < \tau$. Let $\rho = \overline{\overline{Z}}^L$ and let $f : \rho \longleftrightarrow Z$ be constructible. Set $X' = f^{-1}{}''X$. Then $X' \subset \rho < \tau$. By the minimal choice of τ there is $Y' \in L$ s.t. $X' \subset Y'$ and $\overline{\overline{X'}} = \overline{\overline{Y'}}$. Hence $Y = f''Y'$ satisfies the conclusion of § 0 Theorem 1. Contradiction! Now suppose the conclusion of Lemma 1 to fail. Then τ is a cardinal in L, since otherwise the conclusion of Lemma 1 would hold with $Y = \tau$. But then τ is not suitable and, in particular, not a successor cardinal in L. Hence there are arbitrarily large $\gamma < \tau$ s.t. $\gamma > \omega_2$ and γ is a successor cardinal in L. But then γ

is suitable and hence $cf(\gamma) = \overline{\overline{\gamma}} > \omega$, since otherwise Lemma 1 would give $Y \subset \gamma$ s.t. $Y \in L$ and $\overline{\overline{Y}}^L < \gamma$, making γ singular in L. Hence τ is suitable. Contradiction! Q E D.

We now outline, very roughly, the method to be used in proving Lemma 1. Let $\tau \geq \omega_2$ be a suitable cardinal in L and let $X \subset \tau$ cofinally s.t. $\overline{\overline{X}} < \overline{\tau}$. We can easily construct a map $\pi : J_{\overline{\tau}} \longrightarrow_{\Sigma_1} J_\tau$ s.t.

(*) $\overline{\tau} < \tau$ is suitable

(**) $X \subset rng(\pi)$ (hence $rng(\pi) \cap \tau$ is cofinal in τ).

Suppose that $\overline{\tau}$ is not a cardinal in L. Then there is a least $\overline{\beta} \geq \overline{\tau}$ s.t. $\overline{\tau}$ is not a Σ_ω cardinal in $J_{\overline{\beta}}$ (i.e. there is a $J_{\overline{\beta}}$ definable map of some $\overline{\gamma} < \overline{\tau}$ onto $\overline{\tau}$ (allowing parameters)). But then there is a least $n \geq 1$ s.t. $\overline{\tau}$ is not a Σ_n cardinal in $J_{\overline{\beta}}$ (i.e. there is a $\Sigma_n(J_{\overline{\beta}})$ map from a subset of some $\overline{\gamma} < \overline{\tau}$ onto $\overline{\tau}$). We show that the map π "extends to $\overline{\beta}$" - i.e. there is $\tilde{\pi} \supset \pi$ s.t.

$\tilde{\pi} : J_{\overline{\beta}} \longrightarrow_{\Sigma_n} J_\beta$ for some $\beta \geq \tau$.

By the choice of $\overline{\beta}$, n, there exist $\overline{\gamma} < \overline{\tau}$, $\overline{p} \in J_{\overline{\beta}}$ s.t. each $x \in J_{\overline{\beta}}$ is $\Sigma_n(J_{\overline{\beta}})$ in parameters from $J_{\overline{\gamma}} \cup \{\overline{p}\}$. Let $\gamma = \pi(\overline{\gamma})$, $p = \tilde{\pi}(\overline{p})$. Then $\gamma < \tau$, $p \in J_\beta$. Since τ is a cardinal in L, there must be $\pi' : J_{\beta'} \longrightarrow_{\Sigma_n} J_\beta$ s.t. $\pi' \in L$, $\beta' < \beta$ and $J_\gamma \cup \{p\} \subset rng(\pi')$. But then $rng(\tilde{\pi}) \subset rng(\pi')$ since $\tilde{\pi}''(J_{\overline{\gamma}} \cup \{\overline{p}\}) \subset rng(\pi')$. Hence Lemma 1 holds with $Y = rng(\pi')$.

Now let $\overline{\tau}$ be a cardinal in L. The same proof which showed that π "extends to β" will, in this case, show that π "extends to ∞" - i.e. there is $\tilde{\pi} \supset \pi$ s.t. $\tilde{\pi} : L \longrightarrow_{\Sigma_1} L$. But that is a contradiction by the following well known lemma of Kunen:

<u>Lemma 2.</u> Let $\pi : L \longrightarrow_{\Sigma_1} L$ s.t. $\pi \neq id \restriction L$. Then $0^{\#}$ exists.

The cases: $cf(\tau) > \omega$, $cf(\tau) = \omega$ will be treated separately. The non ω cofinal case is the "natural" one, for we can then show that <u>every</u> $\pi : J_{\overline{\tau}} \longrightarrow_{\Sigma_1} J_\tau$ satisfying (*), (**) has the above extendability properties. In the ω cofinal case we shall have to resort to more or less unsavory legerdemain in order to show that π, $\tilde{\tau}$ with the extendability properties exist.

In proving the first extendability property, we shall not work directly with $J_{\bar\beta}$ but rather with $\langle J_{\bar\rho}, \bar{A}\rangle$, where $\bar\rho = \rho_{\bar\beta}^{n-1}$, $\bar{A} = A_{\bar\beta}^{n-1}$. We show that π extends to $\tilde\pi \supset \pi$ s.t. $\tilde\pi : \langle J_{\bar\rho}, \bar{A}\rangle \longrightarrow_{\Sigma_1} \langle J_\rho, A\rangle$ cofinally for some amenable $\langle J_\rho, A\rangle$. (Where "cofinally" means that $\tilde\pi$ " $\omega\bar\rho$ is cofinal in $\omega\rho$.) We then prove the existence of β s.t. $\rho = \rho_\beta^{n-1}$, $A = A_\beta^{n-1}$ (the same proof will show that $\tilde\pi$ extends to $\pi^* : J_{\bar\beta} \longrightarrow_{\Sigma_n} J_\beta$). This latter step is the main concern of § 2.

§ 2 Fine structure lemmas

For the basic theory of the fine structure, the reader is referred to [FS] trough § 4 or [Dev] Ch 7. ρ_α^n, A_α^n, p_α^n denote, as usual, the Σ_n projectum, the Σ_n standard code and the Σ_n standard parameter of α. We recall the following facts:

(1) ρ_α^n = the largest ρ s.t. $\langle J_\rho, A\rangle$ is amenable for all $A \in \Sigma_n(J_\alpha) \cap \mathcal{P}(J_\rho)$.

(2) $R \subset J_{\rho_\alpha^n}$ is $\Sigma_n(J_\alpha)$ iff R is $\Sigma_1(J_{\rho_\alpha^{n-1}}, A_\alpha^{n-1})$.

(3) $A_\alpha^0 = p_\alpha^0 = \varnothing$.

(4) Let $n \geq 1$ and let h be the canonical Σ_1 Skolem function for $\langle J_{\rho_\alpha^{n-1}}, A_\alpha^{n-1}\rangle$. Then ρ^n is the least ρ s.t. $J_{\rho_\alpha^{n-1}} = h''(\omega \times J_\rho \times \{p\})$ for some $p \in J_{\rho_\alpha^n}$ and p_α^n is the $<_J$ - least such p.

(5) $R \subset J_{\rho_\alpha^n}$ is $\Sigma_1(J_{\rho_\alpha^{n-1}}, A_\alpha^{n-1})$ in the parameter p_α^n iff R is rud in $\langle J_{\rho_\alpha^n}, A_\alpha^n\rangle$ (i.e. R is the intersection of $J_{\rho_\alpha^n}$ with a class rudimentary in A_α^n).

(6) Let $\pi : \langle J_{\bar\rho}, \bar A\rangle \longrightarrow_{\Sigma_i} \langle J_{\rho^n_\alpha}, A^n_\alpha\rangle$ $(i \geq 0)$. Then there is a unique $\bar\alpha \geq \bar\rho$ s.t. $\bar\rho = \rho^n_{\bar\alpha}$, $\bar A = A^n_{\bar\alpha}$. Moreover, there is a unique $\tilde\pi \supset \pi$ s.t. $\tilde\pi : J_{\bar\alpha} \longrightarrow_{\Sigma_{n+i}} J_\alpha$ and $\tilde\pi(p^j_{\bar\alpha}) = p^j_\alpha (j \leq n)$.

All of these facts are established in [Dev] and [FS]. The next result, though not explicit in our reference articles, does indead follow easily from the above facts.

<u>Def</u> Let $\alpha \leq \beta$, $0 \leq n \leq \omega$: $\omega\alpha$ is <u>a Σ_n cardinal</u> (<u>Σ_n regular</u>) <u>in</u> J_β iff there is no $\Sigma_n(J_\beta)$ function mapping a subset of some $\gamma < \omega\alpha$ onto (cofinally into) $\omega\alpha$.

α is <u>a cardinal (regular) in</u> J_β iff there is no $f \in J_\beta$ mapping a $\gamma < \omega\alpha$ onto (cofinally into) $\omega\alpha$. If α is a cardinal in J_β and $a \in J_\beta$ s.t. $a \subset J_\alpha$, then $\langle J_\alpha, a\rangle$ is amenable.

Clearly, being a cardinal (regular) in J_β is the same as being a Σ_0 cardinal (regular) in J_β.

<u>Lemma 1.</u> Let $n \geq 1$, $\alpha \leq \beta$.

(i) If $\omega\alpha$ is a Σ_{n-1} cardinal but not a Σ_n cardinal in J_β, then $\rho^n_\beta < \alpha \leq \rho^{n-1}_\beta$. Moreover $\omega\rho^n_\beta$ is the least $\gamma < \omega\alpha$ s.t. there is $\Sigma_n(J_\beta)$ map of a subset of γ onto $\omega\alpha$.

(ii) If $\rho^n_\beta < \alpha \leq \rho^{n-1}_\beta$ and α is regular in $J_{\rho^{n-1}_\beta}$, then $cf(\omega\alpha) = cf(\omega\rho^{n-1}_\beta)$.

Proof.

(i) $\rho^0_\beta = \beta \geq \alpha$. Using (4), (2) and the fact that for any ρ there is a $\Delta_1(J_\rho)$ map of $\omega\rho$ onto J_ρ, we get: $\rho^i_\beta \geq \alpha$ for $i < n$ (by induction on i). Hence $\rho^{n-1}_\beta \geq \alpha$. Now let ρ = the least ρ s.t. there is a $\Sigma_n(J_\beta)$ map of a subset of $\omega\rho$ onto $\omega\alpha$. Then $\rho < \alpha$. $\rho^n_\beta \geq \rho$ by (4), (2).

We claim: $\rho_\beta^n = \rho$. Let $f \in \Sigma_1(J_\beta)$ map a subset of $\omega\rho$ onto J_α. Then $f \notin J_\beta$ and hence $\langle J_\gamma, f\rangle$ is not amenable for $\alpha < \gamma \le \beta$. Hence $\rho_\beta^n \le \alpha$. Now set: $a = \{\nu \in \mathrm{dom}(f) \mid \nu \notin f(\nu)\}$. By a diagonal argument, $a \notin J_\alpha$. Hence $\langle J_\gamma, a\rangle$ is not amenable for $\rho < \gamma \le \alpha$. Hence $\rho_\beta^n \le \rho$. QED(i)

(ii) Set: $\rho = \rho_\beta^{n-1}$, $A = A_\beta^{n-1}$, $p = p_\beta^n$, $\gamma = \rho_\beta^n$.

Let h be the canonical Skolem function for $\langle J_\rho, A\rangle$. Define a map f from a subset of J_γ onto J_α by:

$f(\langle i,x\rangle) = h(i,x,p)$ if $x \in J_\gamma$ and $h(i,x,p) \in J_\alpha$

$f(u)$ undefined in all other cases.

Then f is $\Sigma_1(J_\rho, A)$ in a parameter q. Let

$$y = f(x) \longleftrightarrow \bigvee z\, F(z,y,x,q)$$

where F is Σ_0. Let $\lambda = cf(\omega\rho)$ and let $\langle \xi_\nu \mid \nu < \lambda\rangle$ be a monotone sequence converging to $\omega\rho$. Define $f_\nu (\nu < \lambda)$ by:

$$y = f_\nu(x) \longleftrightarrow y \in S_{\xi_\nu} \wedge \bigvee z \in S_{\xi_\nu}\, F(z,y,x,p).$$

Then $f_\nu \in J_\rho$ and f_ν maps a subset of J_γ into J_α.

Set: $\alpha_\nu = \sup(On \cap rng(f_\nu))$. Then $\alpha_\nu < \omega\alpha$ since $\omega\alpha$ is regular in J_ρ. But $\nu \le \eta \longrightarrow \alpha_\nu \le \alpha_\eta$, since $f_\nu \subset f_\eta$. Finally, $\sup_\nu \alpha_\nu = \omega\alpha$ since $\bigcup_\nu f_\nu = f$. QED

Carrying the proof of Lemma 1 (ii) a step further, we get the following rather technical lemma which will be of service to us in § 5.

Lemma 2. Let $\rho_\beta^{n-1} \ge \alpha > \rho_\beta^n$ where $\omega\alpha$ is regular in J_β. Let $\lambda = cf(\omega\alpha)$. Then there is a sequence $\langle f_\nu \mid \nu < \lambda\rangle$ s.t. $\{f_\nu \mid \nu < \lambda\} \subset J_\alpha$ and if $\pi : J_{\bar\alpha} \longrightarrow_{\Sigma_0} J_\alpha$ s.t. $\{f_\nu \mid \nu < \lambda\} \subset rng(\pi)$, then:

(a) There are unique $\tilde\pi \supset \pi$, $\bar\rho \ge \bar\alpha$, $\bar A \subset J_{\bar\rho}$ s.t. $p_\beta^n \in rng(\tilde\pi)$ and $\tilde\pi : \langle J_{\bar\rho}, \bar A\rangle \longrightarrow_{\Sigma_1} \langle J_{\rho_\beta^{n-1}}, A_\beta^{n-1}\rangle$.

(b) There is a unique $\bar\beta$ s.t. $\bar\rho = \rho^{n-1}_{\bar\beta}$, $\bar A = A^{n-1}_{\bar\beta}$.

(c) $\rho^n_{\bar\beta} < \bar\alpha$.

(d) If $\pi(\rho^n_{\bar\beta}) = \rho^n_\beta$, then $\tilde\pi(p^n_{\bar\beta}) = p^n_\beta$.

Proof.

We first prove the existence part of (a). Let ρ, A, p, γ, $\langle \xi_\nu | \ \nu < \lambda\rangle$ h, f, q, $\langle f_\nu | \ \nu < \lambda\rangle$ be as in the proof of Lemma 1 (ii). We note that $f_\nu \in J_\alpha$, since $f_\nu \in J_\rho$ is bounded in J_α and α is a cardinal in J_ρ. Set $Y = \mathrm{rng}(\pi) \cap J_\gamma$. $X = h''(\omega \times Y \times \{p\})$. Then $X \prec_{\Sigma_1} \langle J_\rho, A\rangle$. It is clear by the definition of f that $X \cap J_\alpha = f''Y$. Using this we get:

<u>Claim</u> $X \cap J_\alpha = \mathrm{rng}(\pi)$.

Proof.

($\subset$) Let $x \in X \cap J_\alpha$. Then $x = f(z)$ for a $z \in Y$. Hence $x = f_\nu(z)$ for some ν. Hence $x \in f''_\nu Y \subset \mathrm{rng}(\pi)$.

($\supset$) Let $x \in \mathrm{rng}(\pi)$. Let z = the $<_J$-least z s.t. $x = f(z)$. Then $x = f_\nu(z)$ for some ν. But then $z \in Y$ since z = the $<_J$-least z s.t. $x = f_\nu(z)$.

QED (Claim)

Now let $\tilde\pi : \langle J_{\bar\rho}, \bar A\rangle \overset{\sim}{\longleftrightarrow} \langle X, A \cap X\rangle$. Then $\bar\rho \geq \bar\alpha$ and $\tilde\pi \upharpoonright J_{\bar\alpha} = \pi$ by the claim. This proves the existence part of (a). The uniqueness part of (a) follows by the fact that if $\tilde\pi \supset \pi$ s.t. $\tilde\pi : \langle J_{\bar\rho}, \bar A\rangle \longrightarrow_{\Sigma_1} \langle J_\rho, A\rangle$ and $p \in \mathrm{rng}(\tilde\pi)$, then $\mathrm{rng}(\tilde\pi) = h''(\omega \times Y \times \{p\})$. (b) is immediate by fact (6) above. To prove (c), set $J_{\bar\gamma} = \pi^{-1}{}''J_\gamma$. Then $\bar\gamma < \alpha$. But $\rho^n_{\bar\beta} \leq \bar\gamma$, since, letting $\bar h = h_{\bar\rho, \bar A}$ be the canonical Σ_1 Skolem function for $\langle J_{\bar\rho}, \bar A\rangle$ and $\tilde\pi(\bar p) = p$, we have: $J_{\bar\rho} = \bar h''(\omega \times J_{\bar\gamma} \times \{\bar p\})$, since $\mathrm{rng}(\tilde\pi) = h''(\omega \times Y \times \{p\}) = \tilde\pi''\bar h''(\omega \times J_{\bar\gamma} \times \{\bar p\})$. We now prove (d). We have: $\bar\gamma = \rho^n_{\bar\beta}$ and $\pi(\bar\gamma) = \gamma$. Set $\bar p' = p^n_{\bar\beta}$, $p' = \tilde\pi(\bar p')$. Then $\bar p' \leq_J \bar p$, since $J_{\bar\rho} = \bar h''(\omega \times J_{\bar\gamma} \times \{\bar p\})$. But $p \leq_J p'$, since there is $\bar x \in J_{\bar\gamma}$ s.t. $\bar p = \bar h(i,\bar x,\bar p')$; hence $p = h(i,x,p')$, where $x = \pi(\bar x) \in J_\gamma$. Hence $h''(\omega \times J_\gamma \times \{p\}) = J_\rho$. QED

The main object of this section is to prove a sort of converse of fact (6) above. First a definition and a preliminary lemma.

<u>Def</u> An imbedding $\sigma : \langle J_\rho, A\rangle \longrightarrow_{\Sigma_1} \langle J_{\rho}', A'\rangle$ of one amenable structure into another is called <u>strong</u> iff whenever R is a well founded relation on J_ρ which is rud in $\langle J_\rho, A\rangle$ and R' is rud in $\langle J_\rho', A'\rangle$ by the same rud definition, then R' is well founded.

<u>Lemma 3.</u> Let i, n > 0 and suppose $\sigma : \langle J_{\rho^n_{\overline{\beta}}}, A^n_{\overline{\beta}}\rangle \longrightarrow_{\Sigma_i} \langle J_\rho, A\rangle$ is strong.

Then there are η, B, $\tilde{\sigma}$ s.t. $\tilde{\sigma} \supset \sigma$ and:

(i) $\rho = \rho^1_{\eta,B}$, $A = A^1_{\eta,B}$, $\tilde{\sigma}(p^{n-1}_{\overline{\beta}}) = p^1_{\eta\,B}$.

(ii) $\tilde{\sigma} : \langle J_{\rho^{n-1}_{\overline{\beta}}} , A^{n-1}_{\overline{\beta}} \rangle \longrightarrow_{\Sigma_{i+1}} \langle J_\eta,B\rangle$ is strong.

Proof of Lemma 3.

Set $\overline{\rho} = \rho^n_{\overline{\beta}}$, $\overline{A} = A^n_{\overline{\beta}}$, $\overline{\eta} = \rho^{n-1}_{\overline{\beta}}$, $\overline{B} = A^{n-1}_{\overline{\beta}}$, $\overline{p} = p^{n-1}_{\overline{\beta}}$.

Then $J_{\overline{\eta}} = h_{\overline{\eta},\overline{B}}$ "$(\omega \times J_{\overline{\rho}} \times \{\overline{p}\})$. (*We shall generally use* $h_{\eta B}$ *to denote* the canonical Σ_1 Skolem function of an amenable structure $\langle J_\eta,B\rangle$). Define $\overline{h}$ by

$$\overline{h}(\langle i,x\rangle) \simeq h_{\overline{\eta},\overline{B}}(i,x,p) \text{ if } x \in J_{\overline{\rho}}$$

$$\overline{h}(u) \text{ undefined otherwise.}$$

Define relations $\overline{D}$, $\overline{E}$, $\overline{I}$, $\overline{B}'$ on $J_{\overline{\rho}}$ by:

$$\overline{D} = \mathrm{dom}(\overline{h})$$

$$\overline{E} = \{\langle x,y\rangle \in \overline{D}^2 \mid \overline{h}(x) \in \overline{h}(y)\}$$

$$\overline{I} = \{\langle x,y\rangle \in \overline{D}^2 \mid \overline{h}(x) = \overline{h}(y)\}$$

$$\overline{B}' = \{x \in \overline{D} \mid \overline{h}(x) \in \overline{B}\}.$$

Since $\overline{D}$, $\overline{E}$, $\overline{I}$, $\overline{B}'$ are $\Sigma_1(J_{\overline{\eta}},\overline{B})$ in $\overline{p}$, they are rud in $\langle J_{\overline{\rho}}, \overline{A}\rangle$.
Let D, E, I, B' have the same rud definitions in $\langle J_\rho, A\rangle$. Then E is well founded, since $\overline{E}$ is well founded and σ is strong. Set:

$\bar{M} = \langle \bar{D}, \bar{I}, \bar{E}, \bar{B}' \rangle$, $M = \langle D, I, E, B' \rangle$. Let $\bar{T}$ be the Σ_1 satisfaction relation for the model $\bar{M}$. Then $\bar{T}(\phi, \langle \vec{x} \rangle) \longleftrightarrow \models^{\Sigma_1}_{\langle J_{\bar{\eta}}, \bar{B} \rangle} \phi[\bar{h}(\vec{x})]$, so $\bar{T}$ is $\Sigma_1(J_{\bar{\eta}}, \bar{B})$ in $\bar{p}$ and hence rud in $\langle J_{\bar{\rho}}, \bar{A} \rangle$. Let T have the same rud definition in $\langle J_\rho, A \rangle$.

<u>Fact 1.</u> T is the Σ_1 satisfaction relation for M.

Proof of Fact 1. We must show that:

$$T([v \in w], \langle x, y \rangle) \longleftrightarrow x \; E \; y$$

$$T([v = w], \langle x, y \rangle) \longleftrightarrow x \; I \; y$$

$$T(\overset{\circ}{A}(v), \langle x \rangle) \longleftrightarrow B' \; x$$

$$T(\phi \wedge \psi, \langle \vec{x} \rangle) \longleftrightarrow T(\phi, \langle \vec{x} \rangle) \wedge T(\psi, \langle \vec{x} \rangle)$$

$$T(\neg \phi, \langle \vec{x} \rangle) \longleftrightarrow \neg \, T(\phi, \langle \vec{x} \rangle)$$

$$T(\bigvee v \; \phi, \langle \vec{x} \rangle \longleftrightarrow \bigvee y \in D \; T(\phi, \langle y, \vec{x} \rangle).$$

All but the last equivalence are expressible as Π_1 statements in $\langle J_\rho, A \rangle$ and therefore hold since the corresponding Π_1 statements in $\langle J_{\bar{\rho}}, \bar{A} \rangle$ hold. To see the last equivalence, note that the relation $\models^{\Sigma_1}_{\langle J_{\bar{\eta}}, \bar{B} \rangle} \phi[y, \bar{h}(\vec{x})]$ is $\Sigma_1(J_{\bar{\eta}}, \bar{B})$ in $\bar{p}$; hence

$$\models_{\langle J_{\bar{\eta}}, \bar{B} \rangle} \bigvee v \; \phi[\bar{h}(\vec{x})] \longleftrightarrow \bigvee i < \omega \models_{\langle J_{\bar{\eta}}, \bar{B} \rangle} \phi[\bar{h}(\langle i, \vec{x} \rangle, \langle \vec{x} \rangle]$$

hence:

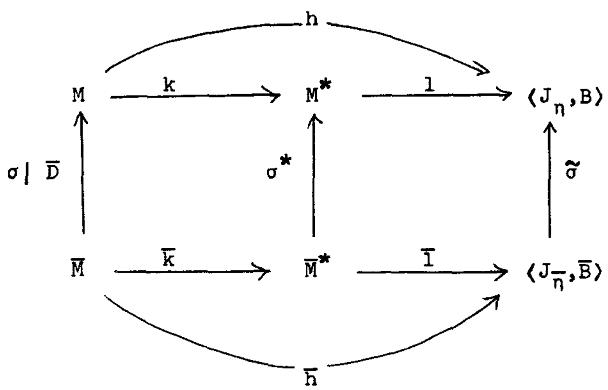
$$\bar{T}(\bigvee y \; \phi, \langle \vec{x} \rangle) \longleftrightarrow \bigvee i < \omega \; \bar{T}(\phi, \langle \langle i, \vec{x} \rangle, \langle \vec{x} \rangle \rangle).$$

But the last equivalence is expressible as a Π_1 statement in $\langle J_{\bar{\rho}}, \bar{A} \rangle$ (since $\bigvee i < \omega \; \bar{T}(\phi, \langle \langle i, \vec{x} \rangle, \langle \vec{x} \rangle \rangle)$ is rud in $\langle J_{\bar{\rho}}, \bar{A} \rangle$ in the parameter ω) and therefore carries up to $\langle J_\rho, A \rangle$. QED (Fact 1)

Since the satisfaction relations $\bar{T}$, T are rud in $\langle J_{\bar{\rho}}, \bar{A} \rangle$, $\langle J_\rho, A \rangle$ resp. by the same rud definitions and σ is Σ_i preserving, we have: $(\sigma \restriction \bar{D}) : \bar{M} \longrightarrow_{\Sigma_{i+1}} M$. So, in particular, M satisfies the identity axioms and the extensionality axiom, since $\bar{M}$ does. We may thus define the factor

model $\overline{M}^* = \overline{M}/\overline{I} = \langle \overline{D}^*, \overline{E}^*, \overline{B}^* \rangle$, $M^* = M/I = \langle D^*, E^*, B^* \rangle$.
Let $\overline{k} : \overline{M} \longrightarrow \overline{M}^*$, $k : M \longrightarrow M^*$ be the natural projections. $\overline{E}^*$, E^* are both well founded and extensional. Hence we may transitivise the models $\overline{M}^*$, M^* by Mostowski isomorphisms $\overline{1}$, 1. Clearly, $\overline{1} : \overline{M}^* \overset{\sim}{\longleftrightarrow} \langle J_{\overline{\eta}}, \overline{B} \rangle$ and $\overline{h} = \overline{1}\ \overline{k}$. Let $1 : M^* \overset{\sim}{\longleftrightarrow} \langle J_\eta, B \rangle$. Set $h = 1\ k$. Define $\sigma^* : \overline{M}^* \longrightarrow_{\Sigma_{i+1}} M^*$ by $\sigma^*\ \overline{k} = k\ \sigma$. Define $\tilde{\sigma} : \langle J_{\overline{\eta}}, \overline{B} \rangle \longrightarrow_{\Sigma_{i+1}} \langle J_\eta, B \rangle$ by $\tilde{\sigma}\ \overline{h} = h\ \sigma$.

Thus:



<u>Fact 2.</u> $\tilde{\sigma} \restriction J_{\overline{\rho}} = \sigma$

Proof of Fact 2.

By definition, $\overline{h}$ uniformises the relation

$\{\langle y, \langle i, x \rangle \rangle \mid x \in J_{\overline{\rho}} \wedge \models_{\langle J_{\overline{\eta}}, \overline{B} \rangle} \phi_i[y, \langle x, \overline{p} \rangle]\}$, where $\langle \phi_i \mid i < \omega \rangle$ is some fixed recursive enumeration of the Σ_1 formulae. Let $\phi_{j_0}(y,z)$ be the formula $\bigvee q(\langle y, q \rangle = x)$. Clearly, $\overline{h}(\langle j_0, x \rangle) = x$ for $x \in J_{\overline{\rho}}$. Set: $s(x) = \langle j_0, x \rangle$. Then s is rudimentary. $s \restriction J_\rho$ maps J_ρ into D, since $s \restriction J_{\overline{\rho}}$ maps $J_{\overline{\rho}}$ into $\overline{D}$. For x, y $\in J_\rho$ we have:

$$s(x)\ E\ s(y) \longleftrightarrow x \in y$$

$$s(x)\ I\ s(y) \longleftrightarrow x = y$$

$z\ E\ s(y) \longleftrightarrow \bigvee x \in y\quad z\ I\ s(x),$

since the corresponding formulae hold in $\langle J_{\bar\rho},\bar A\rangle$.

Thus $k\,s \restriction J_\rho$ maps $\in \restriction J_\rho$ isomorphically onto an initial segment of E^*. But then $h\,s \restriction J_\rho = 1k\,s \restriction J_\rho$ maps $\in \restriction J_\rho$ isomorphically onto an initial segment of $\in \restriction J_\eta$. Hence $h\,s \restriction J_\rho = \mathrm{id} \restriction J_\rho$. Clearly, $\bar h s \restriction J_{\bar\rho} = \mathrm{id} \restriction J_{\bar\rho}$. Hence, for $x \in J_{\bar\rho}$, we have $\tilde\sigma(x) = \tilde\sigma\,\bar h\,s(x) = h\,\sigma\,s(x) = h\,s\,\sigma(x) = \sigma(x)$.

QED (Fact 2)

Set $p = \tilde\sigma(\bar p)$.

Fact 3. $\langle i,x\rangle \in D \longrightarrow h(\langle i,x\rangle) = h_{\eta,B}(i,\langle x,p\rangle)$

Proof of Fact 3.

Let $\phi(y,w,u)$ be the canonical Σ_1 formula defining the relation $y = h_{\nu C}((w)_0,\langle (w)_1,u\rangle)$ for any $\langle J_\nu,C\rangle$. Let $\bar q \in \bar D$ be s.t. $\bar h(\bar q) = \bar p$ and set $q = \sigma(\bar q)$. Then $h(q) = p$. Then $\bigwedge y \in \bar D \models_{\bar M} \phi[y,s(y),\bar q]$, so

$\bigwedge y \in D \models_M \phi[y,s(y),q]$, since $\sigma \restriction \bar D : \bar M \longrightarrow_{\Sigma_2} M$. Hence

$\bigwedge y \in D \models_{\langle J_\eta,B\rangle} \phi[h(y), y, p]$. Hence $h(\langle i,x\rangle) = h_{\eta B}(i,\langle x,p\rangle)$ for $\langle i,x\rangle \in D$

QED (Fact 3)

We recall that by definition, if $\langle J_\nu,C\rangle$ is amenable and $\rho = \rho^1_{\nu,C}$, $p = p^1_{\nu,C}$, then $A^1_{\nu,C} = \{\langle i,x\rangle \mid x \in J_\rho \wedge \models_{\langle J_\nu,C\rangle} \phi_i[x,p]\}$, where $\langle \phi_i\rangle$ is a fixed recursive enumeration of the Σ_1 formulae.

Fact 4. $A = \{\langle i,x\rangle \mid x \in J_\rho \wedge \models_{\langle J_\eta,B\rangle} \phi_i[x,p]\}$.

Proof of Fact 4.

$\bar A = A^1_{\bar\eta,\bar B}$; hence: $\bigwedge i\ x(\bar A(i,x) \longleftrightarrow \models_{\bar M} \phi_i[s(x), \bar q])$.

Since $\sigma : \langle J_{\bar\rho}, \bar A\rangle \longrightarrow_{\Sigma_1} \langle J_\rho, A\rangle$, we conclude:

$\bigwedge i\ x(A(i,x) \longleftrightarrow \models_M \phi_i[s(x),q])$. Hence $A(i,x) \longleftrightarrow \models_{\langle J_\eta,B\rangle} \phi_i[x,p]$.

QED (Fact 4)

Fact 5. $\rho = \rho^1_{\eta,B}$.

Proof of Fact 5.

h is a $\Sigma_1(J_\eta,B)$ map of D onto J_η by Fact 3; but $D \subset J_\rho$, hence $\rho^1_{\eta,B} \le \rho$. On the other hand, $\langle J_\rho, A\rangle$ is amenable and every $\Sigma_1(J_\eta, B)$ subset P of J_ρ is Σ_1 in parameters from $J_\rho \cup \{p\}$ (by Fact 3), hence rud in $\langle J_\rho, A\rangle$ in parameters from J_ρ (by Fact 4). But then $\langle J_\rho, P\rangle$ is amenable for all such P. Hence $\rho \le \rho^1_{\eta,B}$. QED (Fact 5)

Fact 6. $p = p^1_{\eta,B}$

Proof.

$p^1_{\eta,B} \le_J p$ by Facts 3 and 5. Now let $p^1_{\eta,B} <_J p$. Then
$\bigvee i \bigvee q <_J p \bigvee x \in J_\rho \; h_{\eta,B}(i,\langle x,q\rangle) = p$; hence
$\bigvee i \bigvee \bar{q} <_J \bar{p} \bigvee x \in J_{\bar{\rho}} \; h_{\bar{\eta},\bar{B}}(i,\langle x,\bar{q}\rangle) = \bar{p}$ by the fact that $\tilde{\sigma}$ is Σ_1 preserving and $\text{rng}(\tilde{\sigma}) \cap J_\rho = \text{rng}(\sigma)$. Hence $p^1_{\bar{\eta},\bar{B}} < \bar{p}$. Contradiction!

QED (Fact 6)

Facts 4, 5, 6 immediately give:

Fact 7. $A = A^1_{\eta,B}$

All that remains to be proved is

Fact 8. $\tilde{\sigma}$ is strong.

Proof of Fact 8.

Let $\bar{R}$ be well founded and rud in $\langle J_{\bar{\eta}}, \bar{B}\rangle$. Let R have the same rud definition over $\langle J_\eta, B\rangle$. Set:

$\bar{R}' = \{\langle x,y\rangle \in \bar{D}^2 \mid \bar{h}(x) \; R \; \bar{h}(y)\}$

$R' = \{\langle x,y\rangle \in D^2 \mid h(x) \; R \; h(y)\}$.

Then $\bar{R}'$ is $\Sigma\,(J_{\bar{\eta}}, \bar{B})$ in $\bar{p}$ and R' is $\Sigma_1(J_\eta, B)$ in p by the same definition. Hence $\bar{R}'$ is rud in $\langle J_{\bar{\rho}}, \bar{A}\rangle$ and R' is rud in $\langle J_\rho, A\rangle$ by the same definition. But σ is strong; hence R' is well founded. But then R is well founded. QED

By finitely many iterations of Lemma 3 we get:

Lemma 4. Let $n > 0$ and suppose $\sigma : \langle J_{\rho^n_{\bar\beta}}, A^n_{\bar\beta}\rangle \longrightarrow_{\Sigma_1} \langle J_\rho, A\rangle$ is strong, where $\langle J_\rho, A\rangle$ is amenable. Then there is an ordinal β s.t. $\rho = \rho^n_\beta$, $A = A^n_\beta$.

Lemma 4 is the "converse" of Fact (6) announced earlier.

Remark. Though we shall not make use of the fact, notice that β above must be unique and that σ extends to a unique $\tilde\sigma : J_{\bar\beta} \longrightarrow_{\Sigma_{n+1}} J_\beta$ which preserves the first n standard parameters.

§ 3 The non ω cofinal case

Set $J_\infty = \bigcup_{\nu<\infty} J_\nu = L$.

Lemma 1. Let $\bar\tau$ be suitable s.t. $cf(\bar\tau) > \omega$. Let $\pi : J_{\bar\tau} \longrightarrow_{\Sigma_1} J_\tau$ cofinally (i.e. $\omega\tau = \sup (On \cap rng (\pi))$). Let $\bar\tau \le \bar\beta \le \infty$ where $\bar\beta$ is a limit ordinal and $\bar\tau$ is a cardinal in $J_{\bar\beta}$. Then there are $\beta \ge \tau$, $\tilde\pi \supset \pi$ s.t. $\tilde\pi : J_{\bar\beta} \longrightarrow_{\Sigma_1} J_\beta$ cofinally.

The proof stretches over several sublemmas. Assume for the moment that $1 < \tau \le \beta \le \infty$, where β is a limit ordinal.

Def $T = T^{\tau,\beta}$ = the collection of triples $t = \langle\delta_t, \mu_t, u_t\rangle$ s.t. $\delta_t < \tau$, $\mu_t < \beta$, $u_t \subset J_{\mu_t}$, $\bar{\bar{u}}_t < \omega$.

Define a partial ordering on T by $t \le t' \longleftrightarrow \delta_t \le \delta_{t'} \wedge \mu_t \le \mu_{t'} \wedge u_t \subset u_{t'}$. For $t \in T$ set:

X_t = the smallest $x \prec_\Sigma J_{\mu_t}$ s.t. $J_{\delta_t} \cup u_t \subset X$;

hence $X_t = h_{\mu_t}{}''(\omega \times J_{\delta_t} \times \{u_t\})$, where h_μ is the canonical Skolem function for J_μ. Clearly, $t \le t' \longrightarrow X_t \prec_{\Sigma_0} X_{t'}$. Set:

$\sigma_t : J_{\gamma_t} \overset{\sim}{\longleftrightarrow} X_t$; $\sigma_{tt'} = \sigma^{-1}_{t'}\,\sigma_t$ $(t \le t')$.

Then $J_{\gamma_t} \xrightarrow[\Sigma_0]{\sigma_{tt'}} J_{\gamma_{t'}} \xrightarrow[\Sigma_0]{\sigma_{t'}} J_\beta$ $(t \le t')$ and $\langle J_{\gamma_t}\rangle$, $\langle \sigma_{tt'}\rangle$ is a directed system whose limit is J_β, $\langle \sigma_t \rangle$. We note that $\sigma_t \in J_\beta$, since σ_t is the set of pairs $\langle h_{\mu_t}(i,z,u_t), h_{\gamma_t}(i,z, \sigma_t^{-1}(u_t))\rangle$ s.t. $z \in J_{\delta_t}$ and $\langle i,z,u_t\rangle \in \mathrm{dom}(h_{\mu_t})$. If $\mu_t < \tau$, the same argument shows: $\sigma_t \in J_\tau$. But then $\sigma_{tt'} \in J_\tau$ if $\gamma_{t'} < \tau$, since $\sigma_{tt'} = \sigma_s$, where $s = \langle \delta_t, \mu, \sigma_{t'}^{-1}(u_t)\rangle$ and $\mu = \sigma_{t'}^{-1}$ "$(X_{t'} \cap \mu_t)$.

We also note that σ_t is describable as the unique $\sigma : J_{\gamma_t} \longrightarrow_{\Sigma_0} L$ s.t. $\sigma \upharpoonright J_{\delta_t} = \mathrm{id} \upharpoonright J_{\delta_t}$ and $\sigma(\sigma_t^{-1}(u_t)) = u_t$. To see this, note that

(*) $J_{\gamma_t} \models \phi(x) \longrightarrow L \models \phi(\sigma(\vec{x}))$ for all $\vec{x} \in J_{\gamma_t}$ and Σ_1 formulae ϕ.

Now let $h = h_\infty$ be the canonical Σ_1 Skolem function for L. Let $x \in J_{\gamma_t}$. Then $x = h_{\gamma_t}(i,z, \sigma_t^{-1}(u_t))$ for some $i < \omega$, $z \in J_{\delta_t}$. By (*) we have: $\sigma(h_{\gamma_t}(i,z, \sigma_t^{-1}(u_t))) = h(i, z, u_t)$. Hence σ is unique.

<u>Lemma 1.1.</u> $\omega\tau$ is a cardinal in J_β iff $\bigwedge t \in T$ $\gamma_t < \tau$.

Proof.

$(\longleftarrow)$ Suppose $\omega\tau$ is not a cardinal in J_β. Then there are $\mu < \beta$, $f \in J_\mu$ s.t. f maps a $\delta < \tau$ onto J_τ. Hence $J_\tau \subset X_{\langle \delta,\mu,\{f\}\rangle}$ and $\gamma_{\langle \delta,\mu,\{f\}\rangle} \ge \tau$.

QED $(\longleftarrow)$

$(\longrightarrow)$ We may assume $X_t = J_{\gamma_t}$, since otherwise this holds with t replaced by $t' = \langle \delta_t, \gamma_t, \sigma_t^{-1}(u_t)\rangle$. But then $h_{\mu_t} \in J_\beta$ and $J_{\gamma_t} = h_{\mu_t}$"$(\omega \times J_{\delta_t} \times \{u_t\})$. It follows that an $f \in J_\beta$ maps δ_t onto $\omega\gamma_t$. Hence $\gamma_t < \tau$, since $\omega\tau$ is a cardinal in J_β. QED

Now let $\omega\bar{\tau}$ be a cardinal in $J_{\bar{\beta}}$, $\pi : J_{\bar{\tau}} \longrightarrow_{\Sigma_1} J_\tau$ cofinally, $T = T^{\bar{\tau},\bar{\beta}}$.

For $t, t' \in T$, $t \le t'$ set:

$$\delta_t^* = \delta_t^{*(\pi)} = \pi(\delta_t)$$
$$\gamma_t^* = \gamma_t^{*(\pi)} = \pi(\gamma_t)$$
$$\sigma_{tt'}^* = \sigma_{tt'}^{*(\pi)} = \pi(\sigma_{tt'}).$$

Then $\sigma^*_{tt'} : J_{\gamma^*_t} \longrightarrow_{\Sigma_0} J_{\gamma^*_{t'}}$, $\sigma^*_{tt'} \upharpoonright J_{\delta^*_t} = \mathrm{id} \upharpoonright J_{\delta^*_t}$ and $\langle J_{\gamma^*_t} \rangle$, $\langle \sigma^*_{tt'} \rangle$ is a directed system. Define $M = M^{\bar\beta,\pi}$, $\sigma^*_t = \sigma^{*(\bar\beta,\pi)}_t$ $(t \in T)$ by: $M, \langle \sigma^*_t \rangle$ = the direct limit of $\langle J_{\gamma^*_t} \rangle$, $\langle \sigma^*_{tt'} \rangle$. We assume w. l. o. g. that $\sigma^*_t \upharpoonright J_{\delta^*_t} = \mathrm{id} \upharpoonright J_{\delta^*_t}$ (hence $J_\tau \subset M$ since $\tau = \sup_t \delta^*_t$).

Define $\tilde\pi = \tilde\pi^{(\bar\beta)} : J_{\bar\beta} \longrightarrow_{\Sigma_1} M$ by:

$$\begin{array}{ccc} J_{\gamma^*_t} & \xrightarrow{\ \sigma^*_t\ } & M \\ {\scriptstyle \pi \upharpoonright J_\gamma} \uparrow & & \uparrow {\scriptstyle \tilde\pi} \\ J_{\gamma_t} & \xrightarrow[\ \sigma_t\ ]{} & J_{\bar\beta} . \end{array}$$

Then $\tilde\pi \supset \pi$, since for $x \in J_{\bar\tau}$, there is $t \in T$ s.t. $x \in J_{\delta_t}$ and hence $\tilde\pi(x) = \sigma^*_t \, \pi \, \sigma^{-1}_t(x) = \pi(x)$. We note that $\tilde\pi''\omega\bar\beta$ lies cofinally in $\{x \mid M \models x \in \mathrm{On}\}$, since if $x \in \mathrm{On}$ in M, $x = \sigma^*_t(n)$, then $\sigma^*_s(\pi\, \sigma^{-1}_s(\mu_t)) > x$ in M, where $s = \langle \delta_t, \mu_t + 1, u_t \cup \{\mu_t\} \rangle$. We also note that M satisfies the Π_2 statement "I am a J_α", since $M, \langle \sigma^*_t \rangle$ is the limit of $\langle J_{\gamma^*_t} \rangle$, $\langle \sigma^*_{tt'} \rangle$ and each $J_{\gamma^*_t}$ satisfies it. Hence if M were transitive we could conclude: $\bigvee \beta \le \infty \; M = J_\beta$.

<u>Lemma 1.2.</u> $\{y \mid M \models y \in x\}$ is a set for $x \in M$.

Proof.

We assume $\bar\beta = \infty$, since otherwise M is a set and there is nothing to prove. We first note:

(1) If $t \in T$, then $\sigma^*_t = \tilde\sigma_t$, where $\tilde\sigma_t = \{\langle y,x \rangle \mid M \models y = \tilde\pi(\sigma_t)(x)\}$.

Proof of (1).

Since M satisfies "I am a J_α", we can define its canonical Σ_1 Skolem function h. Then h, $h_{\gamma^*_t}$ have the same Σ_1 definition. But $J_{\gamma^*_t} =$

$h_{\gamma_t^*}$"$(\omega \times J_{\delta_t^*} \times \{\sigma_t^{*-1}(u_t^*)\})$, where $u_t^* = \tilde{\pi}(u_t)$, since $\pi(h_{\gamma_t}) = h_{\gamma_t^*}$ and $\pi(\sigma_t^{-1}(u_t)) = \sigma_t^{*-1}(u_t^*)$.

By our previous argument, we conclude that $\sigma_t^* = \tilde{\sigma}_t$ = the unique $\sigma : J_{\gamma_t^*} \longrightarrow_{\Sigma_0} M$ s.t. $\sigma(\sigma_t^{*-1}(u_t^*)) = u_t^*$. QED (1)

Now set: $\tilde{J}_\kappa = \{y \mid M \models y \in J_{\tilde{\pi}(\kappa)}\}$. It suffices to show that $\tilde{J}_\kappa$ is a set for arbitrarily large κ. We show:

(2) If $\kappa > \tau$ is regular, then $\tilde{J}_\kappa \subset \bigcup_{t \in T \cap J_\kappa} \mathrm{rng}(\sigma_t^*)$

Proof of (2).

Let $t \in T$. We shall construct $t' \in T \cap J_\kappa$ s.t. $\mathrm{rng}(\sigma_t^*) \cap \tilde{J}_\kappa \subset \mathrm{rng}(\sigma_{t'}^*)$. Since κ is regular, there is $\eta < \kappa$ s.t. $\mathrm{rng}(\sigma_t) \cap J_\kappa \subset J_\eta$. Set: $Y = h_{\mu_t}$"$(\omega \times J_\eta \times \{u_t\})$; $\sigma : J_{\mu'} \overset{\sim}{\longleftrightarrow} Y$; $\sigma(u') = u_t$; $t' = \langle \delta_t, \mu', u\rangle$. Then $t' \in T \cap J_\kappa$ and $\mathrm{rng}(\sigma_t) \cap J_\kappa \subset \mathrm{rng}(\sigma_{t'})$. Hence $\mathrm{rng}(\tilde{\sigma}_t) \cap \tilde{J}_\kappa \subset \mathrm{rng}(\tilde{\sigma}_{t'})$ and the conclusion follows by (1). QED

By Lemma 1.2. we may assume w. l. o. g. that the well founded core of M is transitive. Thus M is a transitive class if M is well founded and, in fact, $\forall\beta \leq \infty\ M = J_\beta$, since M satisfies "I am a J_α". We complete the proof of Lemma 1 by showing:

<u>Lemma 1.3.</u> If $\bar{\tau}$ is suitable and $\mathrm{cf}(\bar{\tau}) > \omega$, then $\forall\beta \leq \infty\ M = J_\beta$.

Proof of Lemma 1.3.

As remarked, we need only show that M is well founded. Suppose not. Then there are $x_i \in M$ s.t. $x_0 \ni x_1 \ni \dots$. We may suppose that $x_i \in \mathrm{rng}(\sigma_{t_i}^*)$, where $t_i \leq t_{i+1}$, $\gamma_{t_i} < \delta_{t_{i+1}}$ and $t_i \in u_{t_{i+1}}$. Then the system $\langle J_{\gamma_{t_i}^*}\rangle$, $\langle \sigma_{t_i t_j}^*\rangle$ has a limit which is not well founded. On the other hand, $\langle J_{\gamma_t}\rangle$, $\langle \sigma_{t_i t_j}\rangle$ has a well founded limit, since $\sigma_{t_j}\sigma_{t_i t_j} = \sigma_{t_i}$ and $\sigma_{t_i} : J_{\gamma_{t_i}} \longrightarrow_{\Sigma_0} J_{\bar{\beta}}$, where J_β is well founded. Let N, $\langle \sigma_i \rangle$ = the limit of $\langle J_{\gamma_{t_i}}\rangle$, $\langle \sigma_{t_i t_j}\rangle$. Since N is well founded, we may assume it to be transitive. Hence $N = J_\gamma$ for some γ.

Claim $J_\gamma, \sigma_i \in J_{\bar\tau}$.

Proof.

We first note:

(1) $\sigma_i \in J_\gamma$, since $\sigma_i = \sigma_{t_i'}$ where $t_i' = \sigma_{i+1}\sigma^{-1}_{t_{i+1}}(t_i)$.

Since $cf(\tau) > \omega$, we have:

(2) $\sup_i \delta_{t_i} < \bar\tau$.

Let $\delta = \sup_i \delta_{t_i}$. Pick $\rho > \delta$ s.t. $\rho \le \bar\tau$, ρ is regular in $J_{\bar\tau}$ (hence in $J_{\bar\beta}$) and $cf(\rho) > \omega$ (such ρ exists by our assumptions on $\bar\tau$). It is clear that $\gamma \le \sup_i \mu_{t_i} \le \beta$; hence $\sigma_i \in J_{\bar\beta}$ by (1). But $dom(\sigma_i) = J_{\gamma_{t_i}}$ and $\bar J_{\gamma_{t_i}} = \bar\delta_{t_i}$ in $J_{\bar\tau}$; hence

(3) $rng(\sigma_i) \cap \rho$ is bounded in ρ, since $\delta_i < \rho$ and ρ is regular in $J_{\bar\tau}$.

Set: $\eta = \bigcup_i rng(\sigma_i) \cap \rho = J_\gamma \cap \rho$. Then $\eta < \rho$ since $cf(\rho) > \omega$.

Hence $\gamma = \eta < \rho$ and $J_\gamma \in J_{\bar\tau}$. Hence $\sigma_i \in J_{\bar\tau}$ by (1).

QED (Claim)

Now set: $\sigma_i^* = \pi(\sigma_i)$, $\gamma^* = \pi(\gamma)$. Then $\sigma_i^* : J_{\gamma^*_{t_i}} \longrightarrow_{\Sigma_0} J_{\gamma^*}$ and $\sigma_j^* \sigma^*_{t_i t_j} = \sigma_i^*$.

Hence $\langle J_{\gamma^*_t}\rangle, \langle\sigma^*_{t_i t_j}\rangle$ has a well founded limit. Contradiction! QED

This proves Lemma 1. As an immediate corollary we have:

<u>Corollary 2.</u> Let $\bar\tau$ be suitable s.t. $cf(\bar\tau) > \omega$. Let $\pi : J_{\bar\tau} \longrightarrow_{\Sigma_1} J_\tau$ s.t. $\pi \ne id \restriction J_{\bar\tau}$. Then $\bar\tau$ is not a cardinal in L.

Proof.

Suppose not. Then π extends to $\tilde\pi : L \longrightarrow_{\Sigma_1} L$. Hence $\tilde\pi \ne id \restriction L$ and $0^{\#}$ exists by Kunen's lemma. Contradiction! QED

<u>Note.</u> Corollary 2 could also have been proven by an ultrapower construction. (In Ch. 17 of [Dev] the existence of $0^{\#}$ is derived from a slightly stronger assumption. That proof can be adapted virtually without change; only the proof that the ultrapower is well founded (p.200) needs

amendment.)

Lemma 3. Let $\tau \geq \omega_2$ be a suitable cardinal in L s.t. $cf(\tau) > \omega$. Then the conclusion of § 1 Lemma 1 holds.

Proof.

Let $X \subset \tau$ cofinally s.t. $\bar{\bar{X}} < \bar{\bar{\tau}}$. We wish to construct $Y \in L$ s.t. $X \subset Y$ and $\bar{\bar{Y}}^L < \tau$. Since τ is suitable, we may assume w. l. o. g. That either τ is a successor cardinal in L or there are arbitrarily large $\gamma \in X$ s.t. γ is regular in L and $cf(\gamma) > \omega$. Define sets $Z_i \prec J_\tau (i \leq \omega_1)$ by:

Z_0 = the smallest $Z \prec J_\tau$ s.t. $X \subset Z$

Z_{i+1} = the smallest $Z \prec J_\tau$ s.t. $Z_i \cup Z_i^* \subset Z$

where Z_i^* = the set of limit points $< \tau$ of $\tau \cap Z_i$.

$Z_\lambda = \bigcup_{i<\lambda} Z_i$ for limit λ.

Set $Z = Z_{\omega_1}$. Then

(a) $Z \prec J_\tau$

(b) $\bar{\bar{Z}} = \omega_1 \cdot \bar{\bar{X}} < \bar{\bar{\tau}}$

(c) If $\gamma \in Z$ is a regular cardinal in L, then either $Z \cap \gamma$ is cofinal in γ or else $cf(| Z \cap \gamma |) = \omega_1$.

Let $\pi : J_{\bar{\tau}} \overset{\sim}{\longleftrightarrow} Z$. By (b) we have $\bar{\tau} < \bar{\bar{\tau}}$. By (c) and the above assumptions on X we have: $\bar{\tau}$ is suitable. Since π is cofinal in τ and $cf(\tau) > \omega$, we have: $cf(\bar{\tau}) > \omega$. By Lemma 2 it follows that $\bar{\tau}$ is not a cardinal in L. Let $\bar{\beta}$ be the least $\bar{\beta} \geq \bar{\tau}$ s.t. $\bar{\tau}$ is not a Σ_ω cardinal in $J_{\bar{\beta}}$. Let n be the least $n \geq 1$ s.t. $\bar{\tau}$ is not a Σ_n cardinal in $J_{\bar{\beta}}$. Then $\rho_{\bar{\beta}}^n < \bar{\tau} \leq \rho_{\bar{\beta}}^{n-1}$.

Set: $\bar{\rho} = \rho_{\bar{\beta}}^{n-1}$, $\bar{A} = A_{\bar{\beta}}^{n-1}$ $\bar{\gamma} = \rho_{\bar{\beta}}^n$, $\bar{p} = p_{\bar{\beta}}^n$. By § 2 Lemma 1 , we have $cf(\bar{\rho}) > \omega$, since $\bar{\gamma} < \eta \leq \bar{\rho}$ for some $\eta \leq \bar{\tau}$ s.t. η is regular in J_β and $cf(\eta) > \omega$. Hence $\bar{\rho}$ is a limit ordinal and Lemma 1 gives us $\rho \geq \tau$, $\tilde{\pi} \supset \pi$ s.t. $\tilde{\pi} : J_{\bar{\rho}} \longrightarrow_{\Sigma_1} J_\rho$ cofinally. Set: $A = \bigcup_{\nu<\bar{\rho}} \tilde{\pi}(\bar{A} \cap \nu)$. Then $\langle J_\rho, A\rangle$ is amenable and $\tilde{\pi} : \langle J_{\bar{\rho}}, \bar{A}\rangle \longrightarrow_{\Sigma_1} \langle J_\rho, A\rangle$ cofinally.

Claim $\bigvee \beta\ (\rho = \rho_\beta^{n-1} \wedge A = A_\beta^{n-1})$.

If $n = 1$, then $A = \emptyset = A^0_\rho$ and $\rho = \rho^0_\rho$. Now let $n > 1$. By § 2 Lemma 4 it suffices to show that $\tilde{\pi}$ is strong. Suppose not. Then there are $\overline{R}$, R s.t.

(a) $\overline{R} \subset J^2_{\overline{\rho}}$ is rud in $\langle J_{\overline{\rho}}, \overline{A}\rangle$ and $R \subset J^2_\rho$ is rud in $\langle J_\rho, A\rangle$ by the same rud definition.

(b) $\overline{R}$ is well founded but R is not.

Then there are $x_i \in J_\rho$ s.t. $x_{i+1}\ R\ x_i (i<\omega)$. Since $cf(\overline{\rho}) > \omega$ and $\tilde{\pi}$ is cofinal in ρ, there is $\eta = \tilde{\pi}(\overline{\eta})$ s.t. $\{x_i \mid i < \omega\} \subset J_\eta$. Then $R \cap J_\eta$ is not well founded. But $\overline{R} \cap J_{\overline{\eta}}$ is well founded and $\overline{\rho}$ is admissible. Hence there is $\overline{f} \in J_{\overline{\rho}}$ s.t. $\overline{f} : J_{\overline{\eta}} \longrightarrow \overline{\rho}$ and $x\overline{R}y \longrightarrow \overline{f}(x) < \overline{f}(y)$. Let $f = \tilde{\pi}(\overline{f})$. Then $f : J_\eta \longrightarrow \rho$ and $xRy \longrightarrow f(x) < f(y)$. Hence $R \cap J_\eta$ is well founded. Contradiction ! QED (Claim)

Set: $\gamma = \pi(\overline{\gamma})$, $p = \tilde{\pi}(\overline{p})$. Let h, $\overline{h}$ be the canonical Σ_1 Skolem functions for $\langle J_\rho, A\rangle$, $\langle J_{\overline{\rho}}, \overline{A}\rangle$ resp. Set $Y = h''(\omega \times J_\gamma \times \{p\})$. Then $Y \in L$. Since $\gamma < \tau$ and τ is a cardinal in L, we have: $\overline{\overline{Y}}^L < \tau$. By the definition of $\overline{\gamma}$, $\overline{p}$ we have: $J_{\overline{\rho}} = \overline{h}''(\omega \times J_{\overline{\gamma}} \times \{\overline{p}\})$. But $\pi''J_{\overline{\gamma}} \subset J_\gamma$; hence $X \subset \tilde{\pi}''J_\rho = h''(\omega \times (\pi''J_{\overline{\gamma}}) \times \{p\}) \subset Y$. QED

§ 4 Vicious sequences

Troughout this section we assume that $\nu < \tau$ and $\pi : J_\nu \longrightarrow_{\Sigma_1} J_\tau$ cofinally. We wish to examine more closely the circumstances under which $M^{\beta\pi}$ can fail to be well founded. To this end we define:

Def $\Theta = \Theta(\pi) \simeq$ the least limit ordinal $\Theta \geq \nu$ s.t. ν is a cardinal in J_Θ and $M^{\Theta\pi}$ is not well founded.

§ 3 Lemma 1 says that Θ does not exist if ν is suitable and $cf(\nu) > \omega$. It is clear that, if Θ does exist, then $\Theta > \nu$. Moreover $\Theta < \infty$, since otherwise $M^{\kappa\pi}$ would be well founded, where κ is the first regular cardinal $> \nu$. But then $M^{\kappa\pi} = J_{\kappa^*}$ for some κ^* and there is $\tilde{\pi} \supset \tilde{\pi}^{(\kappa)} \supset \pi$ s.t. $\tilde{\pi} : J_\kappa \longrightarrow J_{\kappa^*}$ cofinally, contradicting § 3 Corollary 2.

Now suppose that Θ exists. Let $T = T^{\nu,\Theta}$, $M = M^{\Theta,\pi}$, $\sigma^*_t = \sigma^{*(\Theta,\pi)}$, $\sigma^*_{tt'} = \sigma^{*}_{tt'}{}^{(\pi)}$, $\tilde{\pi} = \tilde{\pi}^{(\Theta)}$. There must be a sequence $\langle t_i, x_i\rangle (i < \omega)$ s.t.

(a) $t_i \in T$, $t_i \leq t_{i+1}$, $t_i \in u_{t_{i+1}}$, $\gamma_{t_i} < \delta_{t_{i+1}}$

(b) If $J_\nu \models$ (α is the largest cardinal then $\delta_{t_0} > \alpha$

(c) $x_i \in J_{\gamma^*_{t_i}}$ s.t. $x_{i+1} \in \sigma^*_{t_i t_{i+1}}(x_i)$, (hence $\sigma^*_{t_{i+1}}(x_{i+1}) \in \sigma^*_{t_i}(x_i)$ $(i < \omega)$).

We refer to any sequence satisfying (a) - (c) as a <u>vicious sequence</u> for π. Note that $\sup_i \mu_{t_i}$ is a limit ordinal if $\langle t_i, x_i\rangle$ is vicious, since $\mu_{t_{i+1}} > \mu_{t_i}$. If $\beta \geq \nu$ is any limit ordinal s.t. $\beta \geq \sup_i \mu_{t_i}$ and ν is a cardinal in β, then $M^{\beta\pi}$ is not well founded. But then $\sup_i \mu_{t_i} > \nu$, since otherwise $J_\tau = M^{\nu\pi}$ would not be well founded. Hence $\Theta = \sup_i \mu_{t_i}$ for any vicious $\langle t_i, x_i\rangle$.

We define a <u>canonical vicious sequence</u> $\langle t_i, x_i\rangle = \langle t_i^\pi, x_i^\pi\rangle (i < \omega)$ as follows:

t_i = the $<_J$-least $t \in T$ s.t.
there is a vicious sequence $\langle t'_k, x'_k\rangle$ $(k < \omega)$ with
$\langle t'_j, x'_j\rangle = \langle t_j, x_j\rangle$ for $j < i$ and $t'_i = t$.

x_i = the $<_J$-least $x_i \in J_{\gamma^*_{t_i}}$ s.t.
there is a vicious sequence $\langle t'_k, x'_k\rangle$ $(k < \omega)$ with
$\langle t'_j, x'_j\rangle = \langle t_j, x_j\rangle$ for $j \leq i$.

<u>Lemma 1.</u> $\bigcup_{i<\omega} \mathrm{rng}(\sigma_{t_i}) = J_\Theta$

Proof.

Set $X = \bigcup_{i<\omega} X_{t_i}$. Since $X_{t_i} \prec_{\Sigma_1} J_{\mu_{t_i}}$ and $\sup_i \mu_{t_i} = \Theta$, we have $X \prec_{\Sigma_1} J_\Theta$.

Let $\sigma : J_\lambda \xleftrightarrow{\sim} X$. Then $\lambda \leq \Theta$. We know that $t_i \in X$, since $t_i \in u_{t_{i+1}} \subset X_{t_{i+1}}$.

Claim $\sigma^{-1}(t_i) = t_i$.

Proof.

Let $t_i' = \sigma^{-1}(t_i)$. Then $X_{t_i'} = h_{\mu_{t_i'}}"(\omega \times J_{\delta_{t_i}} \times \{\sigma^{-1}(u_{t_i})\}) = \sigma^{-1}"X_t$.

Hence $\gamma_{t_i'} = \gamma_{t_i}$ and $\sigma_{t_i't_j'} = \sigma_{t_it_j}$ $(i \le j)$. But then $\gamma^*_{t_i'} = \gamma^*_{t_i}$, $\sigma^*_{t_i't_j'} = \sigma^*_{t_it_j}$ and it follows easily that $\langle t_i', x_i\rangle (i < \omega)$ is vicious.

But $t_i' \le_J t_i$. Hence $t_i' = t_i$ by the minimal choice of t_i.

QED (Claim)

But then $\lambda = \sup_i \mu_{t_i} = \theta$ and $X_{t_i} = h_{t_i}"(\omega \times J_{\delta_{t_i}} \times \{u_{t_i}\}) = \sigma^{-1}"X_{t_i}$.

Hence $J_\theta = J_\lambda = \sigma^{-1}"X = \bigcup_i \sigma^{-1}"X_{t_i} = \bigcup_i X_{t_i} = X$. QED

Corollary 2. If ν is suitable, then $\sup_i \delta_{t_i} = \nu$.

Proof.

ν is either a successor cardinal or a limit cardinal in J_θ. In the first case, $\delta_{t_i} > \alpha$ by definition, where ν succeeds α in J_θ. But then $\eta_i = X_{t_i} \cap \nu$ is transitive. $\eta_i < \nu$, since ν is regular in J_θ. Clearly $\sup_i \eta_i = \nu$, since $\bigcup_i X_{t_i} = J_\theta$. But $\delta_i \le \eta_i \le \gamma_i < \delta_{i+1}$; hence $\sup_i \delta_i = \nu$.

Now let ν be a limit cardinal in J_θ. Let $\delta = \sup_i \delta_{t_i} < \nu$. By suitability, there is $\gamma > \delta$ s.t. γ is regular in J_θ, $\gamma < \nu$ and $cf(\gamma) > \omega$. Set $\eta_i = \sup(\gamma \cap X_{t_i})$. Then $\eta_i < \gamma$ by the regularity of γ. But then $\eta < \gamma$, where $\eta = \sup_i \eta_i$, since $cf(\gamma) > \omega$. Hence $\eta \notin \bigcup_i X_{t_i} = J_\theta$.
Contradiction! QED

Remark Using Corollary 2, it would be easy to show that $M^{\theta,\pi} = \bigcup_i$ rng $(\sigma^*_{t_i})$ if ν is suitable, but we shall not need this.

Def $v_i = v_i^\pi = \langle \delta_{t_i}, \gamma_{t_i}, \sigma^{-1}_{t_i}(u_{t_i})\rangle$ $(i < \omega)$.

Then $v_i \in J_\nu$ $(i < \omega)$. The sequence $\langle v_i \rangle$ gives "complete information" about J_θ, since J_θ, $\langle \sigma_{t_i}\rangle$ = the limit of $\langle J_{\gamma_{t_i}}\rangle$, $\langle \sigma_{t_it_j}\rangle$ and the maps

$\sigma_{t_i t_j}$ are recoverable from the v_i by: $\sigma_{t_i t_j} = \sigma_s$ where $s = \sigma_{t_j}^{-1}(t_i) \in \sigma_{t_j}^{-1}(u_j)$ $(i < j)$.

We use this to prove:

Lemma 3. Let ν be suitable. Let $\bar{\pi} : J_{\bar{\nu}} \longrightarrow_{\Sigma_1} J_\nu$ s.t. $\{v_i \mid i < \omega\} \subset \mathrm{rng}(\bar{\pi})$. Then $\Theta(\pi\bar{\pi})$ exists (hence $\langle t_i^{\pi\bar{\pi}}, x_i^{\pi\bar{\pi}} \rangle$ exists) and $x_i^{\pi\bar{\pi}} = x_i$ $(i < \omega)$.

Proof.

$\sigma_{t_i t_j} \in \mathrm{rng}(\bar{\pi})$ since $\sigma_{t_i t_j}$ is canonically recoverable from the v_i. Set:

$\bar{\sigma}_{ij} = \bar{\pi}^{-1}(\sigma_{t_i t_j})$, $\bar{\gamma}_i = \bar{\pi}^{-1}(\gamma_{t_i})$, $\bar{\delta}_i = \bar{\pi}^{-1}(\delta_{t_i})$.

Then $\bar{\sigma}_{ij} : J_{\bar{\gamma}_i} \longrightarrow_{\Sigma_0} J_{\bar{\gamma}_j}$ $(i \le j < \omega)$ is a directed system s.t. $\bar{\sigma}_{ij} \restriction J_{\bar{\delta}_i} = \mathrm{id} \restriction J_{\bar{\delta}_i}$. Let U, $\langle \bar{\sigma}_i \rangle$ be the limit of $\langle J_{\bar{\gamma}_i} \rangle$, $\langle \bar{\sigma}_{ij} \rangle$. We may assume w. l. o. g. that $\sigma_i \restriction J_{\bar{\delta}_i} = \mathrm{id} \restriction J_{\bar{\delta}_i}$. But $\sup_i \bar{\delta}_i = \bar{\nu}$, since $\sup_i \delta_{t_i} = \nu$ and hence $J_{\bar{\nu}} \subset U$. Define $\hat{\pi} : U \longrightarrow_{\Sigma_0} J_\Theta$ by:

$$\begin{array}{ccc} U & \xrightarrow{\hat{\pi}} & J_\Theta \\ {\scriptstyle \bar{\sigma}_i} \uparrow & & \uparrow {\scriptstyle \sigma_{t_i}} \\ J_{\bar{\gamma}_i} & \xrightarrow[\bar{\pi} \restriction J_{\bar{\gamma}_i}]{} & J_{\gamma_{t_i}} \end{array}$$

It is easily seen that $\hat{\pi} \supset \bar{\pi}$. U is well founded, since $\hat{\pi}$ imbeds it into J_Θ, and satisfies "I am a J_α". Hence we may assume $U = J_{\bar{\Theta}}$ for some $\bar{\Theta}$. Set $\bar{t}_i = \hat{\pi}^{-1}(t_i) = \bar{\sigma}_{i+1}\, \bar{\pi}^{-1}\, \sigma_{t_{i+1}}^{-1}(t_i)$. Since $\sup_i \mu_{t_i} = \Theta$ and $\mu_{t_i} = \hat{\pi}(\mu_{\bar{t}_i})$, it follows that $\hat{\pi}''\bar{\Theta}$ is cofinal in Θ. Hence $\hat{\pi} : J_{\bar{\Theta}} \longrightarrow_{\Sigma_1} J_\Theta$ cofinally. Clearly, $\bar{\Theta} = \sup_i \mu_{\bar{t}_i}$.

(1) If $t \in \bar{T} = T^{\bar{\Theta}\bar{\nu}}$, then $\hat{\pi}(\sigma_t) = \sigma_{\hat{\pi}(t)}$ (hence $\pi(\gamma_t) = \gamma_{\hat{\pi}(t)}$ and $\pi(\sigma_{tt'}) = \sigma_{\hat{\pi}(t)\hat{\pi}(t')}$ for $t, t' \in \bar{T}$, $t \le t'$).

Proof of (1).

$\sigma = \sigma_t$, $u = \sigma^{-1}(u_t)$, $\gamma = \gamma_t$ are uniquely characterized by:

(a) $J_\gamma = h_\gamma{}''(\omega \times J_{\delta_t} \times \{u\})$

(b) $\sigma : J_\gamma \longrightarrow_{\Sigma_1} J_{\mu_t}$

(c) $\sigma \upharpoonright J_{\delta_t} = \mathrm{id} \upharpoonright J_{\delta_t}$, $\sigma(u) = u_t$. QED (1)

(2) $\bar{\gamma}_i = \gamma_{\bar{t}_i}$, $\bar{\sigma}_i = \sigma_{\bar{t}_i}$ (hence $\bar{\sigma}_{ij} = \sigma_{t_i t_j}$).

Proof of (2)

$\pi(\gamma_{\bar{t}_i}) = \gamma_{t_i} = \pi(\bar{\gamma}_i)$ by (1); hence $\gamma_{\bar{t}_i} = \bar{\gamma}_i$. Set: $\bar{u} = \pi^{-1}\, \sigma_{t_i}^{-1}(u_{t_i})$.

Then $\hat{\pi}\, \bar{\sigma}_i(\bar{u}) = \sigma_{t_i}\pi(\bar{u}) = u_{t_i}$ and $\hat{\pi}\, \sigma_{\bar{t}_i}(\bar{u}) = \hat{\pi}(\sigma_{\bar{t}_i})\pi(\bar{u}) = \sigma_{t_i}\pi(\bar{u}) = u_{t_i}$ by (1). Hence $\bar{\sigma}_i(\bar{u}) = \sigma_{\bar{t}_i}(\bar{u}) = u_{\bar{t}_i}$. But then $\bar{\sigma}_i = \sigma_{\bar{t}_i}$ = the unique $\sigma : J_{\bar{\gamma}_i} \longrightarrow_{\Sigma_0} L$ s.t. $\sigma \upharpoonright J_{\bar{\delta}_i} = \mathrm{id} \upharpoonright J_{\bar{\delta}_i}$ and $\sigma(\bar{u}) = u_{\bar{t}_i}$. QED (2)

(3) $\bar{\Theta} = \Theta(\pi\bar{\pi})$.

Proof.

Let $\bar{\rho} = \Theta(\pi\bar{\pi})$.

($\bar{\Theta} \geq \bar{\rho}$) $\sigma^{*(\pi\bar{\pi})}_{\bar{t}_i\bar{t}_j} = \pi\bar{\pi}(\sigma_{\bar{t}_i\bar{t}_j}) = \pi(\sigma_{t_i t_j}) = \sigma^*_{t_i t_j}$ by (2). But then $\sigma^*_{\bar{t}_{i+1}}(x_{i+1}) \in \sigma^*_{\bar{t}_i}(x_i)$ in $M^{\bar{\Theta},\pi\bar{\pi}}$, where $\sigma^*_{\bar{t}_i} = \sigma^{*(\bar{\Theta},\pi\bar{\pi})}_{\bar{t}_i}$ and $M^{\bar{\Theta},\pi\bar{\pi}}$ is not well founded.

($\bar{\rho} \geq \bar{\Theta}$) suppose not. Let $\langle \bar{s}_i, y_i \rangle$ be vicious for $\pi\bar{\pi}$ and set $\rho = \hat{\pi}(\bar{\rho})$, $s_i = \hat{\pi}(\bar{s}_i)$. By the above argument, $\langle s_i, y_i \rangle$ is vicious for π. Hence $\Theta(\pi) = \sup_i \mu_{s_i} \leq \rho < \Theta$. Contradiction ! QED (3)

(4) $\langle \bar{t}_i, x_i \rangle = \langle t_i^{\pi\bar{\pi}}, x_i^{\pi\bar{\pi}} \rangle$.

Proof.

$\langle \bar{t}_i, x_i \rangle$ is vicious for $\pi\bar{\pi}$ by the above argument. But (1), (2) and the minimal choice of $\langle t_i, x_i\rangle$, $\langle \bar{t}_i, x_i\rangle$ must be chosen minimally.

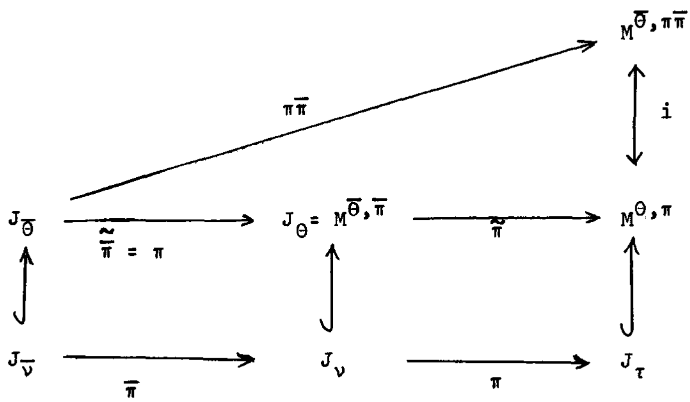
QED

<u>Note</u> We could have carried the proof of Lemma 2 a bit further to show:

(a) $M^{\Theta,\bar{\pi}} = J_\Theta$ and $\tilde{\bar{\pi}}(\Theta) = \hat{\pi}$

(b) There is $i : M^{\bar{\Theta},\pi\bar{\pi}} \stackrel{\sim}{\longleftrightarrow} M^{\Theta,\pi}$ s.t. $\widetilde{\pi\bar{\pi}}(\bar{\Theta}) = i\,\tilde{\pi}\hat{\pi}$.

Thus:



§ 5 <u>The ω cofinal case</u>

Let $\tau \geq \omega_2$ be an ω cofinal suitable cardinal in L. Let $X \subset \tau$ be cofinal in τ s.t. $\bar{\bar{X}} < \bar{\bar{\tau}}$. As before, we suppose w. l. o. g. that, if τ is not a successor cardinal in L, there are arbitrarily large $\gamma \in X$ s.t. γ is regular in L and $\mathrm{cf}(\gamma) > \omega$. We wish to show that there is $Y \in L$ s.t. $X \subset Y$ and $\bar{\bar{Y}}^L < \tau$. Obviously, it suffices to prove this in a generic extension of the universe. Since $\bar{\bar{X}} < \bar{\bar{\tau}}$, $\tau \geq \omega_2$ and τ is singular (in V), there is a regular $\kappa \geq \omega_2$ s.t. $\bar{\bar{X}} < \kappa < \tau$. But we may then assume that $\bar{\bar{\tau}} = \kappa$, since if this is not true already, we can make it true by generically collapsing τ to κ. Now let k map κ onto J_τ. For $\gamma < \kappa$ set: Y_γ = the smallest $Y \prec_{\Sigma_\omega} J_\tau$ s.t. $X \cup \gamma \cup k''\gamma \subset Y$.

Set: $\Gamma = \{\alpha < \kappa \mid \alpha = Y_\alpha \cap \kappa\}$. Then Γ is cub in κ. For $\alpha, \beta \in \Gamma$, $\alpha \leq \beta$ set: $\pi_\alpha : J_{\tau_\alpha} \xleftrightarrow{\sim} Y_\alpha$; $\pi_{\alpha\beta} = \pi_\beta^{-1} \pi_\alpha$. Then $J_{\tau_\alpha} \xrightarrow[\Sigma_\omega]{\pi_{\alpha\beta}} J_{\tau_\beta} \xrightarrow[\Sigma_\omega]{\pi_\beta} J_\tau$, and J_τ, $\langle\pi_\alpha\rangle$ is the limit of the directed system $\langle J_{\tau_\alpha}\rangle$, $\langle\pi_{\alpha\beta}\rangle$. Clearly, $\pi_\alpha \restriction J_\alpha = \mathrm{id} \restriction J_\alpha$ and $\pi_\alpha(\alpha) = \kappa$. Also, $\bar{\bar{Y}}_\alpha < \kappa$ for $\alpha \in \Gamma$ and κ is regular. Hence for each $\alpha \in \Gamma$ there is $\beta \in \Gamma$ s.t. $\tau_\alpha < \beta$ and $Y_\alpha^* \subset Y_\beta$, where Y_α^* is the set of limit points $< \tau$ of $\tau \cap Y_\alpha$. Let Γ_0 be the set of $\alpha \in \Gamma$ s.t. $cf(\alpha) > \omega$, α is a limit point of Γ, and $\tau_\beta < \alpha$, $Y_\beta^* \subset Y_\alpha$ for all $\beta \in \Gamma \cap \alpha$. Then Γ_0 is stationary in κ. It follows by the argument of § 3 Lemma 3 that τ_α is suitable for $\alpha \in \Gamma_0$. Clearly $cf(\tau_\alpha) = \omega$, since $\tau \cap Y_\alpha$ is cofinal in τ.

<u>Lemma 1.</u> $\{\alpha \in \Gamma_0 \mid \Theta(\pi_\alpha) \text{ exists}\}$ is not stationary in κ.

Proof.

Suppose not. For $\alpha \in \Gamma_0$ s.t. $\Theta(\pi_\alpha)$ exists set: $\Theta_\alpha = \Theta(\pi_\alpha)$, $\langle t_i^\alpha, x_i^\alpha\rangle = \langle t_i^{\pi_\alpha}, x_i^{\pi_\alpha}\rangle$, $v_i^\alpha = v_i^{\pi_\alpha}$. Since $cf(\alpha) > \omega$, there is $\beta \in \Gamma \cap \alpha$ s.t. $\{v_i^\alpha \mid i < \omega\} \subset rng(\pi_{\beta\alpha})$. Let $f(\alpha)$ be the least such β. Then f is regressive and hence there is β_0 s.t. $\Delta = \{\alpha \mid f(\alpha) = \beta_0\}$ is stationary. But then $\{u_i^\beta \mid i < \omega\} \subset rng(\pi_\alpha)$ for $\alpha, \beta \in \Delta$, $\alpha \leq \beta$ and hence $x_i^\alpha = x_i^\beta$ by § 4 Lemma 3. Set $x_i = x_i^\alpha$ $(\alpha \in \Delta)$. Since $cf(\kappa) > \omega$ and $J_\tau = \bigcup_{\alpha\in\Delta} rng(\pi_\alpha)$, there is $\alpha \in \Delta$ s.t. $\{x_i \mid i < \omega\} \subset rng(\pi_\alpha)$. Then $\sigma_{t_{i+1}^\alpha} \pi_\alpha^{-1}(x_{i+1}) \in \sigma_{t_i^\alpha} \pi_\alpha^{-1}(x_i)$ $(i < \omega)$ and J_{Θ_α} is not well founded. Contradiction ! QED

<u>Lemma 2.</u> If τ is a limit cardinal in L, then the conclusion of § 1 Lemma 1 holds.

Proof.

Pick $\alpha \in \Gamma_0$ s.t. $\Theta(\pi_\alpha)$ does not exist. Set $\bar{\tau} = \tau_\alpha$, $\pi = \pi_\alpha$. Then $\bar{\tau}$ is not a cardinal in L, since otherwise $M^{\kappa,\pi}$ would be well founded. But then $M^{\kappa,\bar{\tau}} = J_{\kappa^*}$ for some κ^* and $\tilde{\pi} : J_\kappa \longrightarrow_{\Sigma_1} J_{\kappa^*}$ cofinally, where $\tilde{\pi} = \tilde{\pi}^{(\kappa)} \supset \pi$, violating § 3 Lemma 2. As in § 3 Lemma 3 set: $\bar{\beta}$ = the least $\bar{\beta} \geq \bar{\tau}$ s.t. $\bar{\tau}$ is not a Σ_ω cardinal in $J_{\bar{\beta}}$; n = the least $n \geq 1$ s.t. $\bar{\tau}$ is not a Σ_n cardinal in $J_{\bar{\beta}}$, $\bar{\rho} = \rho_{\bar{\beta}}^{n-1}$, $\bar{A} = A_{\bar{\beta}}^{n-1}$, $\bar{\gamma} = \rho_{\bar{\beta}}^{n}$. Then $\bar{\rho} \geq \bar{\tau} > \bar{\gamma}$. By suitability, there must then be η s.t. $\bar{\rho} > \eta > \bar{\gamma}$, η is regular in $J_{\bar{\rho}}$ and $cf(\eta) > \omega$. Hence $cf(\bar{\rho}) > \omega$. We can then finish the proof exactly as in § 3 Lemma 3. QED

Lemma 3. If τ is a successor cardinal in L, then the conclusion of § 1 Lemma 1 holds.

Proof.

Set: $\Gamma_1 = \{\alpha \in \Gamma_0 \mid \Theta(\pi_\alpha)$ does not exist$\}$. Then Γ_1 is stationary. As above, τ_α is not a cardinal in L for $\alpha \in \Gamma_1$. Set: β_α = the least $\beta \geq \tau_\alpha$ s.t. τ_α is not a Σ_ω cardinal in J_β; $n = n_\alpha$ = the least $n \geq 1$ s.t. τ_α is not a Σ_n cardinal in J_β; $\rho_\alpha = \rho_{\beta_\alpha}^{n-1}$, $A_\alpha = A_{\beta_\alpha}^{n-1}$, $p_\alpha = p_{\beta_\alpha}^{n}$ $\gamma_\alpha = \rho_{\beta_\alpha}^{n}$. Then $\rho_\alpha \geq \tau_\alpha > \gamma_\alpha$ and τ_α is a successor cardinal, hence regular, in J_{ρ_α}. Hence $cf(\omega\rho_\alpha) = cf(\tau_\alpha) = \omega$.

Let f_i^α $(i < \omega)$ be as in § 2 Lemma 2. Since $cf(\alpha) > \omega$ for $\alpha \in \Gamma_1$, there is $\beta \in \Gamma \cap \alpha$ s.t. $\{\gamma_\alpha\} \cup \{f_i^\alpha \mid i < \omega\} \subset rng(\pi_{\beta\alpha})$. Set $g = \langle \beta, \pi_{\beta\alpha}^{-1}(\gamma_\alpha)\rangle$ where β is the least such ordinal. Then g is regressive. Hence there are β', γ' s.t. $\Delta = \{\alpha \in \Gamma_1 \mid g(\alpha) = \langle \beta', \gamma'\rangle\}$ is stationary. But for $\alpha, \beta \in \Delta$, $\alpha \leq \beta$ we then have $n_\alpha = n_\beta$, $\pi(\gamma_\alpha) = \gamma_\beta$ and there is a unique $\tilde{\pi}_{\alpha\beta} \supset \pi_{\alpha\beta}$ s.t. $\tilde{\pi}_{\alpha\beta} : \langle J_{\rho_\alpha}, A_\alpha\rangle \longrightarrow_{\Sigma_1} \langle J_{\rho_\beta}, A_\beta\rangle$ and $\tilde{\pi}_{\alpha\beta}(p_\alpha) = p_\beta$.

By the uniqueness of the $\tilde{\pi}_{\alpha\beta}$ it follows that if $\alpha, \beta, \gamma \in \Delta$, $\alpha \leq \beta \leq \gamma$, then $\tilde{\pi}_{\beta\gamma}\tilde{\pi}_{\alpha\beta} = \tilde{\pi}_{\alpha\gamma}$. Let M, $\langle \tilde{\pi}_\alpha \mid \alpha \in \Delta\rangle$ be the direct limit of $\langle\langle J_{\rho_\alpha}, A_\alpha\rangle \mid \alpha \in \Delta\rangle$, $\langle \tilde{\pi}_{\alpha\beta} \mid \alpha, \beta \in \Delta$ and $\alpha \leq \beta\rangle$.

M is well founded, since if $x_{i+1} \in x_i$ in M $(i < \omega)$, there must be $\alpha \in \Delta$ s.t. $\{x_i \mid i < \omega\} \subset rng(\tilde{\pi}_\alpha)$. But then $\tilde{\pi}_\alpha(x_{i+1}) \in \tilde{\pi}_\alpha(x_i)(i < \omega)$. Contradiction ! M satisfies "I am a J_α" and hence we may assume: $M = \langle J_\rho, A\rangle$ for some ρ. Then $\rho \geq \tau$ and $\langle J_\rho, A\rangle$ is amenable. Fix $\alpha \in \Delta$

and set: $\overline{\tau} = \tau_\alpha$, $\overline{\beta} = \beta_\alpha$, $\overline{\rho} = \rho_\alpha$, $\overline{A} = A_\alpha$, $\overline{\gamma} = \gamma_\alpha$, $\pi = \pi_\alpha$, $\tilde{\pi} = \tilde{\pi}_\alpha$. It is enough to show that $\rho = \rho_\beta^{n-1}$, $A = A_\beta^{n-1}$ for some β, for we can then finish the proof exactly as in § 3 Lemma 3. But for this, it suffices to show that the map $\tilde{\pi}$ is strong. $\tilde{\pi}$ will be strong, however, if α is a chosen sufficiently large. To see this, let $R_n (n < \omega)$ enumerate the relations rud in $\langle J_\rho, A\rangle$ which are not well founded. Let $\overline{R}_n$ have the same rud definition in $\langle J_{\overline{\rho}}, \overline{A}\rangle$. For $n < \omega$ choose $\langle x_n^i \mid i < \omega\rangle$ s.t. $x_n^{i+1}\ R_n\ x_n^i\ (i < \omega)$. Set $Y = \{x_n^i \mid i, n < \omega\}$. Then $Y \subset \mathrm{rng}(\tilde{\pi})$ for sufficiently large α. But then $\tilde{\pi}^{-1}(x_n^{i+1})\ \overline{R}_n\ \tilde{\pi}^{-1}(x_n^i)$ and $\overline{R}_n$ is not well founded. Now let $\overline{R}$ be well founded and rud in $\langle J_{\overline{\rho}}, \overline{A}\rangle$. Let R be rud in $\langle J_\rho, A\rangle$ by the same rud definition. Then $\overline{R} \neq \overline{R}_n$ and hence $R \neq R_n$ $(n < \omega)$. Hence R is well founded. QED

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THREE THEOREMS ON RECURSIVE ENUMERATION. I. DECOMPOSITION. II. MAXIMAL SET. III. ENUMERATION WITHOUT DUPLICATION

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In this paper we shall prove three theorems about recursively enumerable sets. The first two answer questions posed by Myhill [1].

The three proofs are independent and can be presented in any order. Certain notations will be common to all three. We shall denote by "R_e" the set enumerated by the procedure of which e is the Gödel number. In describing the construction for each proof, we shall suppose that a clerk is carrying out the simultaneous enumeration of $R_0, R_1, R_2, \ldots,$ in such a way that at each step only a finite number of sets have begun to be enumerated and only a finite number of the members of any set have been listed. (One plan the clerk can follow is to turn his attention at Step a to the enumeration of R_e where $e+1$ is the number of prime factors of a. Then each R_e receives his attention infinitely often.) We shall denote by "R_e^a" the set of numbers which, at or before Step a, the clerk has listed as members of R_e. Obviously, all the R_e^a are finite sets, recursive uniformly in e and a. For any a we can determine effectively the highest e for which R_e^a is not empty, and for any a, e we can effectively find the highest member of R_e^a, just by scanning what the clerk has done by Step a. Additional notations will be introduced in the proofs to which they pertain.

Theorem 1. *Every nonrecursive recursively enumerable set is the union of two disjoint nonrecursive recursively enumerable sets.*

Proof. Let us be given the Gödel number u of a recursively enumerable set R_u. We shall show how to enumerate two disjoint sets P and Q whose union is R_u. Then we shall prove that neither P nor Q is recursive if R_u is not recursive.

The enumeration of P and Q will be carried out *pari passu* with the clerk's enumeration of the R_e's. "P^a" or "Q^a" will denote the set of numbers which, at or before Step a, have been made members of P or Q, respectively. A number e will be called *satisfied* at Step a if R_e^a intersects both P^a and Q^a. (If e ever is satisfied, R_e cannot be the complement of either P or Q.)

Every time the clerk lists a new member of R_u, we shall put that number either into P or into Q. Suppose the number is n, and it is listed at Step a; that is, $n \in R_u^a - R_u^{a-1}$. If every e such that $n \in R_e^a$ is already satisfied, then we put n into P. Otherwise, we *attack* the lowest unsatisfied e such that $n \in R_e^a$. If R_e^a, for the e under attack, intersects neither P^{a-1} nor Q^{a-1},

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then we put n into P, so that $R_e{}^a$ intersects P^a. Otherwise we put n into P or Q according as $R_e{}^a \cap P^{a-1}$ or $R_e{}^a \cap Q^{a-1}$ is empty, so that e now becomes satisfied.

Now, neither P nor Q is recursive if R_u is not recursive. For if either one of P or Q is recursive, its complement is some recursively enumerable set R_c. This implies that

(1) R_c includes the complement of R_u, and

(2) c never becomes satisfied.

Note that no number is attacked more than twice, for after two attacks it is satisfied, and only unsatisfied numbers are attacked. Therefore there is a step a_0 in our construction after which no number $\leq c$ is ever attacked. Then after Step a_0, the clerk never lists a member of R_u that has previously been listed as a member of R_c. For if he did, c being unsatisfied by (2), we should attack either c or some $e < c$.

This argument gives us a way of enumerating the complement of R_u. Simply write down every number that the clerk lists as a member of R_c, *provided* that it was not previously listed as a member of R_u *and* that it still has not been listed as a member of R_u by Step a_0. The resulting set contains all the non-members of R_u, by (1); and it contains no members of R_u, by the argument of the preceding paragraph. Since its complement is recursively enumerable, R_u is recursive. This proves the theorem.

THEOREM 2. *There exists a recursively enumerable set M (a "maximal" set) such that*

(3) *$\bar{M}$ is infinite, and*

(4) *there is no recursively enumerable set R such that $R \cap \bar{M}$ and $\bar{R} \cap \bar{M}$ are both infinite.*

PROOF. As we construct M, we shall use the notation "M^a", to denote the set of numbers which we have put into M at or before Step a. At Step a, for any particular e, n, we say that n inhabits the i-th e-state if $n \notin M^a$ and $i = \Sigma_{n \in R_{e'}{}^a, e' \leqq e} 2^{e-e'}$ where the summation is made over all $e' \leqq e$ such that $n \in R_{e'}{}^a$. The i-th e-state is *lower* than the j-th if $i < j$. It is evident that at the beginning of the construction every number inhabits the 0-th e-state. As the construction proceeds, a number n may sometimes move from a lower to a higher, but never from a higher to a lower e-state. If n is eventually put into M, n ceases then to inhabit any e-state. If $n \in \bar{M}$, n will always inhabit some e-state, and since there are only 2^{e+1} e-states, n must eventually inhabit some e-state which it continues to inhabit forever. n is then called a *resident* of that e-state. An e-state is *well-resided* if it has an infinite number of residents. An e-state is *well-habited* if each of an infinite set of numbers inhabits it at one step or another.

M is constructed as follows. When, at Step a, the clerk lists any number n, not in M^{a-1}, as a new member of any set R_{e_0}, we examine every $e \geqq e_0$ such that $n \in R_e{}^a$. There are only finitely many such e. For each e examined, if more than e numbers $< n$ inhabit the e-state which is one lower than that of n, we put all but the e lowest of those numbers into M.

An example may prevent misunderstanding. Suppose the clerk lists 21 as a new member of R_2. Suppose that 21 has previously been listed as a member of R_0, of R_3, and of R_5, but not of M, nor of any other R_e. Then 21 is now in the 1-st 0-state, in the 2-nd 1-state, in the 5-th 2-state, in the 11-th 3-state, in the 22-nd 4-state, and in the 45-th 5-state. The values of e to be examined are 2, 3, and 5. Suppose that, of the numbers less than 21, only 6 is in the 4-th 2-state; 1, 9, 11, and 13 are in the 10-th 3-state; and 4, 8, 10, 16, 18, 19, and 20 are in the 44-th 5-state. Then 13, 19, and 20 are the numbers we must put into M.

M shall contain only those numbers which are put into it according to the foregoing prescription. To show that M is maximal, we prove two lemmas.

LEMMA 1. *For each c, the highest well-habited c-state has at least c residents.*

PROOF. The 0-th c-state is well-habited. Hence there exists a highest well-habited c-state, say the i_m-th. Then only finitely many numbers are fated ever to inhabit a c-state higher than the i_m-th. Let a_0 be a step in the construction at which each of these numbers has either entered the c-state in which it is to reside or been put into M, so that after Step a_0 no number enters a c-state higher than the i_m-th. Since the i_m-th c-state is well-habited and only a finite number of its inhabitants can have been put into M by Step a_0, there are infinitely many numbers which, at some step subsequent to a_0, will be found inhabiting the i_m-th c-state. Let n_0 be one of the c lowest of these numbers. By hypothesis, n_0 will never leave the i_m-th c-state for a higher one. Therefore n_0 is a resident of the i_m-th c-state, unless $n_0 \in M$. But the act of putting n_0 into M requires that, at some step $a > a_0$, a number $n > n_0$ is listed as a member of some R_{e_0}, and for some $e \geqq e_0$ there are at least e numbers $< n_0$ in the same e-state as n_0, and n is then in the next higher e-state. This is impossible. For if $e \geqq c$, all numbers in the same e-state as n_0 are also in the same c-state, and not more than $c-1$ of these are less than n_0. And if $e < c$, then $e_0 < c$, so that n, by being newly listed in R_{e_0}, has entered a new c-state, and this must be higher than the i_m-th c-state if n is in a higher e-state than n_0. This, by hypothesis, cannot occur after Step a_0. Therefore n_0 cannot be put into M and must be a resident of the i_m-th c-state. Since n_0 could have been any of c numbers, the i_m-th c-state has at least c residents.

LEMMA 2. *For any c, not more than one c-state is well-resided.*

PROOF. Let the highest well-habited c-state be the i_m-th. Then higher c-states, not being well-habited, cannot be well-resided. On the other hand, if $i < i_m$, the i-th c-state has no more than c residents. For supposing it has more, let n_0 be the $(c+1)$-th lowest resident of the i-th c-state. Let Step a_0 be a step at which this c-state is already inhabited by its $c+1$ lowest residents. Only finitely many numbers, at Step a_0, are in c-states higher than the i-th. But infinitely many numbers are fated eventually to inhabit the i_m-th c-state. Therefore infinitely many numbers must, after Step a_0, pass from a c-state $\leqq$ the i-th to one $>$ the i-th. Let the first number $> n_0$ that does so be called n. n can make this transition only by being listed, at some step a, as a new member of some R_{e_0} with $e_0 \leqq c$. n is not in M^{a-1}, or it could not be in any c-state. Consider the lowest e such that $R_e{}^a$ contains n but not n_0. This e must exist and be $\leq c$, since n is now in a higher c-state than n_0. For the same reason, no $R_{e'}{}^a$, where $e' < e$, contains n_0 but not n. Therefore the e-state of n_0 is one lower than that of n. Moreover, since by hypothesis n was not in a c-state higher than that of n_0 *before* n was listed as a member of $R_{e_0}{}^a$, e_0 must be $\leqq e$. Finally, the c lowest residents of the i-th c-state are all lower than n_0 and are all in the same e-state as n_0, since $e \leqq c$. Therefore all the conditions are satisfied for putting n_0 into M. This contradicts the hypothesis that n_0 is a resident of the i-th c-state. Therefore the i-th c-state does not, in fact, have more than c residents and is consequently not well-resided. This proves the lemma.

Since every resident is a member of $\bar{M}$, Lemma 1 implies that $\bar{M}$ has at least c members, for arbitrary c. Hence $\bar{M}$ is infinite and (3) is satisfied. For any R_e, all the members of $R_e \cap \bar{M}$ are residents of odd-numbered e-states, and all the members of $\bar{R}_e \cap \bar{M}$ are residents of even-numbered e-states. Therefore Lemma 2 implies (4). The theorem is proved.[1]

THEOREM 3. *There exists a sequence of $S_0, S_1, S_2, \ldots$ of uniformly recursively enumerable sets in which every recursively enumerable set occurs once and only once.*

PROOF. We shall give a procedure for listing simultaneously the members of $S_0, S_1, S_2, \ldots$ just as the clerk is listing the members of $R_0, R_1, R_2, \ldots$, $S_x{}^a$ shall be the set of numbers listed in S_x at or before Step a.

During the construction we shall establish certain conceptual relationships between values of e and values of x. Under certain circumstances we shall

[1] It can easily be proved that any maximal set is hyper-hyper-simple. Hence our construction of M answers Post's question ([2], p. 313) as to the existence of such sets. Namely, assume that M is *not* hyper-hyper-simple. Then, by definition, there exists an infinite recursively enumerable set $\{a_1, a_2, \ldots\}$ such that the sets R_{a_1}, R_{a_2}, ... are all finite and mutually exclusive, and such that each contains some member of $\bar{M}$. Let $S = R_{a_2} \cup R_{a_4} \cup R_{a_6} \cup \ldots$. Then $S \cup \bar{M}$ and $\bar{S} \cup \bar{M}$ are both infinite, which is impossible. Q.E.D.

call a certain x the *follower* of a certain e. This means that we intend, if convenient, to make S_x identical to R_e. At times we shall *release* an x; that is, we shall make it cease to be the follower of some e. This x will thereafter be *free*, and it will never again be the follower of any e. However, e, having lost its follower, may acquire a new one later. If some x, having been made a follower of some e, remains its follower forever, never being freed, then x will be called a *loyal* follower of e. If x is eventually freed, it will be called a *disloyal* follower. At any step the values of x which have not yet been followers of any e will be called *unused*. 0 shall always be unused, and S_0 shall be the empty set.

Each step a shall be dedicated to the *pursuit* of a certain e, according to some plan which causes every e to be pursued an infinite number of times during the construction. (For example, we may use the plan suggested above for the clerk, in which e is chosen so that $e+1$ is the number of prime factors of a.)

At any step a, let e_a be the value of e which is being pursued. One of three cases arises.

CASE 1: e_a has *a* follower, x, and there is some $e < e_a$ such that $R_e{}^a \cap L(x)$ is identical to $R_{e_a}{}^a \cap L(x)$, where $L(x)$ is the set of all integers less than x.

Then we release x.

CASE 2: Case 1 does not hold, and there exists an x such that $R_{e_a}{}^a$ is identical to $S_x{}^{a-1}$. Furthermore, either this x is the follower of some $e \leqq e_a$, or $x = 0$, or else x is free; if x is free, either $x \leqq e_a$ or x has previously been displaced by e_a. (See Case 3, Act C.)

Then we do nothing.

CASE 3: neither Case 1 nor Case 2 holds.

Then we perform four acts, some of which may be vacuous.

A. If e_a has no follower, we make the lowest unused x, other than 0, the follower of e_a.

B. We put into S_x, where x is the follower of e_a, all the members of $R_{e_a}{}^a$. (This makes $S_x{}^a$ identical to $R_{e_a}{}^a$, for there is no way in which S_x could previously have acquired any members that were not yet in R_{e_a}, either while x was unused or while x was the follower of e_a.)

C. If, for some $x' \neq x$, $S_{x'}{}^{a-1}$ is identical to $R_{e_a}{}^a$, then we put into $S_{x'}$ the lowest number not yet listed as a member of any R or S. This makes $S_{x'}{}^a$ different from $S_x{}^a$ and from all the other S's. When we perform this act, we say that x' is being *displaced* by e_a.

D. If the x' of Act C is the follower of some e', we release it.

LEMMA 3. *Let $\bar{e}$ be the smallest Gödel number of $R_{\bar{e}}$; i.e., let $R_{\bar{e}}$ differ from R_e for every $e < \bar{e}$. Then there exists an x such that S_x is identical to $R_{\bar{e}}$.*

PROOF. $\bar{e}$ cannot have an infinite number of disloyal followers. For eventually, for each $e < \bar{e}$, one of the two sets R_e and $R_{\bar{e}}$ must acquire a member n never to be acquired by the other. Thereafter, $R_e{}^a$ and $R_{\bar{e}}{}^a$ can never be identical, and $R_e{}^a \cap L(x)$ cannot be identical to $R_{\bar{e}}{}^a \cap L(x)$ for any $x > n$. When such an n has appeared for every $e < \bar{e}$, $\bar{e}$ can no longer lose any follower through Act D with $e_a < e$, and if Case 1 continues to occur with $e_a = \bar{e}$, $\bar{e}$ must eventually be found with a follower x greater than all the n's, so that Case 1 is impossible. Subsequently, any follower that $\bar{e}$ has will be loyal.

Let a_0 be a step after which $\bar{e}$ is destined never to lose a follower. Suppose Case 3 occurs infinitely often with $e_a = \bar{e}$. Then the first time it occurs after Step a_0, $\bar{e}$ acquires a follower through Act A, or else it already has one. In either case, this follower will be loyal. Call it x. S_x will be identical to $R_{\bar{e}}$, for no number will be made a member of S_x unless the clerk has already listed it in $R_{\bar{e}}$, and every number that the clerk lists in $R_{\bar{e}}$ will be put into S_x at the next occurrence of Case 3 with $e_a = \bar{e}$.

Suppose, on the other hand, that Case 3 occurs only finitely often with $e_a = \bar{e}$. Since Case 1 does not arise after Step a_0, Case 2 must occur infinitely often with $e_a = \bar{e}$. Each time Case 2 occurs, there must be an x such that $S_x{}^{a-1}$ is identical to $R_{\bar{e}}{}^a$. This x must either be a follower of some $e \leqq \bar{e}$, or be itself $\leq \bar{e}$, or have been displaced by $\bar{e}$ at a previous step. We shall show that x can take only a finite number of different values in occurrences of Case 2 with $e_a = \bar{e}$.

If $e < \bar{e}$, eventually either $R_{\bar{e}}$ has acquired a member that will never be acquired by R_e, or R_e has acquired a member that will never be acquired by $R_{\bar{e}}$. In the first case, $S_x{}^{a-1}$ can never subsequently be identical to $R_{\bar{e}}{}^a$ when x is a follower of e, for $S_x{}^{a-1}$ will have no members not in $R_e{}^{a-1}$. In the second case, if e subsequently acquires a new follower x, S_x will immediately acquire all the members listed already in R_e, and $S_x{}^{a-1}$ cannot thereafter be identical to $R_{\bar{e}}{}^a$. In either case, only a finite number of different followers of e can serve as the x in Case 2 with $e_a = \bar{e}$.

If $e = \bar{e}$, e has only a finite number of different followers during the construction, as we have already shown.

Only a finite number of values of x are $\leqq \bar{e}$.

Since, by hypothesis, Case 3 arises only finitely often with $e_a = \bar{e}$, only a finite number of x's are ever displaced by $\bar{e}$.

Therefore only a finite number of different numbers can serve as the x in Case 2 with $e_a = \bar{e}$. But Case 2 arises infinitely often. Therefore there is a single x such that $S_x{}^{a-1}$ is identical to $R_e{}^a$ on an infinite number of occasions. Then S_x must be identical to $R_{\bar{e}}$, for otherwise one of the two sets would eventually acquire a member never to be acquired by the other, and then $S_x{}^{a-1}$ could never again be identical to $R_{\bar{e}}{}^a$. This proves Lemma 3.

COROLLARY TO LEMMA 3. *0 is the only value of x that remains forever unused.*

For whenever a new x becomes used, the lowest unused $x > 0$ is selected. Therefore, if any $x_0 > 0$ were forever unused, then all $x \geqq x_0$ would remain forever unused. But if x is forever unused, then S_x is empty. Thus any R_e that differed from S_x for all $x < x_0$ would differ from all S_x, contrary to Lemma 3.

LEMMA 4. *If $x \neq x'$, then S_x and $S_{x'}$ are not the same finite set.*

PROOF. If S_x is finite, then $S_x{}^a$ is identical to S_x for sufficiently high a. Similarly for $S_{x'}$, $S_{x'}{}^a$. But $S_x{}^a$ is never identical to $S_{x'}{}^a$ if $x \neq x'$, except while both x and x' are unused. The reader may verify this by examining Case 3; it is contrived so that all the $S_x{}^a$ for different used x's are different, provided that this was true of $a-1$. By the corollary to Lemma 3, x and x' cannot both remain unused indefinitely. Therefore, for all sufficiently high a, $S_x{}^a$ differs from $S_{x'}{}^a$, and hence S_x differs from $S_{x'}$.

LEMMA 5. *If $x \neq x'$, then S_x and $S_{x'}$ are not the same infinite set.*

PROOF. If $x = 0$, S_x is empty, not infinite. If $x > 0$, then x is eventually a follower of some e, by the corollary to Lemma 3. If x is disloyal, then after x is released S_x can acquire a new member only when x is displaced. x can be displaced only once by each $e < x$ and never by any $e \geqq x$. Therefore S_x is finite if x is disloyal. Hence, if S_x is infinite, x must be a loyal follower of some e. Similarly, if $S_{x'}$ is infinite, x' is a loyal follower of some e'. A single e cannot have more than one loyal follower, since it cannot have more than one follower at a time. Hence, if $x \neq x'$, then $e \neq e'$. We may suppose arbitrarily that $e < e'$. Case 3 must arise infinitely often with $e_a = e$; otherwise S_x could not be infinite. Therefore every number listed in R_e is subsequently listed in S_x, and so S_x is identical to R_e. Similarly, $S_{x'}$ is identical to $R_{e'}$. Therefore, if S_x and $S_{x'}$ are identical, R_e and $R_{e'}$ must be identical. But then $R_e{}^a \cap L(x')$ must be identical to $R_{e'}{}^a \cap L(x')$ for all sufficiently high a. Once this is true, Case 1 will arise the next time that $e_a = e'$, and x' will be released. This contradicts the assumption that x' is a loyal follower of e'. Therefore S_x and $S_{x'}$ cannot be the same infinite set.

Every recursively enumerable set R occurs in the sequence $S_0, S_1, S_2, \ldots$. Simply let $\bar{e}$ be the lowest e for which R_e is identical to R, and apply Lemma 3. But no set occurs more than once in the sequence $S_0, S_1, \ldots$, by Lemmas 4 and 5. The theorem is proved.

COROLLARY TO THEOREM 3. *There exists a sequence of uniformly partial recursive functions which contains every partial recursive function once and only once.*

PROOF. Instead of letting the R's be all recursively enumerable sets, let them be all partial recursive functions, expressed as sets of ordered pairs. Construct the S's as in Theorem 3, except that in Act C of Case 3 the new member of $S_{x'}$ must be an ordered pair which differs *in its argument member* from any ordered pair previously listed in any R or S. The S's will be the desired sequence of partial recursive functions expressed as sets of ordered pairs.

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HIGHER SET THEORY AND MATHEMATICAL PRACTICE *

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Introduction

When we examine the classical set-theoretic foundations of mathematics, we see that the only sets that play a role are sets of restricted type; at the risk of understatement, only sets of rank $< \omega + \omega$. Further examination reveals four fundamental principles about sets used: the existence of an infinite set; the existence of the power set of any set; every property determines a subset of any set; and the axiom of choice. The theory based on these four principles is known as Zermelo set theory together with the axiom of choice, and is written **Z** in this paper. Then **Z** adequately formalizes mathematical practice (excluding modern set theory) in an elegant and straightforward way.

In modern set theory, however, the object of study is the notion (or notions) of set of transfinite rank. Whether or not there is a single meaningful notion of set of transfinite type, rather than, instead only a multitute of notions of set obtained by prescribing a definite "number" of iterations of the power set operation, remains a controversial issue. In any case, what is completely clear is that no notion of: set of arbitrary transfinite type, or even notions of set obtained by some definite iteration (beyond $\omega + \omega$) of the power set operation, is relevant, as of now, to mathematical practice, or even understood by mathematicians. We refer to this characteristic aspect of modern set theory, the consideration of sets of transfinite rank, or of sets obtained by more than finite-

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ly many iterations of the power set operation applied to the hereditarily finite sets, as higher set theory.

What is the significance of this sociology for us? It suggests to us consideration of the following conjecture:

*) every sentence of mathematical discourse (excluding, of course, higher set theory) which can be decided using fundamental principles about sets of *transfinite* rank (like: **Z** consists of fundamental principles about sets of rank $< \omega + \omega$), can already be decided in mathematical practice.

It is beyond the scope of this paper to thoroughly discuss whether certain formal systems do or do not codify fundamental principles about sets of transfinite rank, but certain cases are clear cut. (It is, of course, the case that no one today knows how to provide a theoretical description of what is a fundamental principle and what is not; a general theory of notions and principles is nowhere in sight). That **Z** codifies fundamental principles about sets of transfinite rank is clear, even though it was intended to codify only fundamental principles about sets of rank $< \omega + \omega$. That the theory $\mathbf{Z}(\Omega) = \mathbf{Z}$ together with "there is a rank function defined on every countable well-ordering" does, is fairly clear cut. That, say, Zermelo-Fraenkel set theory together with the existence of a measurable cardinal, or, say, Zermelo-Fraenkel together with the existence of nonconstructible sets of natural numbers *does not* is also fairly clear cut. There is nothing in the phrase "set of transfinite rank" which even remotely suggests that all sets are constructible or that all cardinals are nonmeasurable.

With these rough guidelines in mind, the reader can appreciate the following important open question, which has turned out to be connected with attempts at settling *):

**) are there fundamental principles about sets of transfinite rank which refute or prove the axiom of constructibility?

No answer to **) is in sight.

Perhaps some more rough guidelines may be useful in helping the reader appreciate *). Clearly Con(**Z**) can be proved in $\mathbf{Z}(\Omega)$ but not in

Z itself. Does this constitute a refutation of *)? No, because Con(Z) is really about (formal systems of) set theory of rank $< \omega + \omega$, and to *understand what* a set of rank $< \omega + \omega$ is, one has to go beyond use of sets of rank $< \omega + \omega$, and so, go beyond (our model of) mathematical practice. Thus Con(Z) is considered outisde of mathematical discourse.

The main obstacle in obtaining a genuine negative solution to *) is that the only sentences of mathematical discouse which are known to be independent of Z at the same time which have proofs in higher set theory (even using, say, the existence of a measurable cardinal) are also known to imply, within Z, the existence of nonconstructible sets; so, if one wishes to solve *) using such sentences, then one will also have to solve **).

Our approach avoids this nonconstructible trouble by producing a sentence of mathematical discouse about Borel sets which is Π^1_3 (hence provably relativizes to constructible sets) and giving a proof of independence of this Π^1_3 sentence from Z and conjecturing that this Π^1_3 sentence is provable within Z(Ω). That the Π^1_3 sentence is provable within Z(Ω) seems like a reasonable conjecture because of

1) examination of the proofs of independence given here;
2) the Π^1_3 sentence is known to be provable using the existence of Ramsey cardinals (D.Martin [4]);
3) this proof of Martin uses partition properties of cardinals directly, and the cardinal of $V(\Omega)$ is the first cardinal satisfying certain important weaker partition properties.

The Π^1_3 sentence under investigation here is Borel determinateness, written here as $(\forall\alpha)(D(\alpha))$, (see Definitions 1.4 and 1.5). Our independence result from Z is given in the Corollary to Theorem 1.6. Actually, the independence proofs work equally well for the following consequence of Borel determinateness, which *reads* like (but by our independence proof *is* not) a standard Theorem in the classical theory of the Borel hierarchy: to every Borel set $Y \subset 2^\omega \times 2^\omega$ there is a continuous function F which either uniformizes Y or uniformizes $[(f, g): (g, f) \notin Y]$; see Section 4 for elaboration.

The paper is organized as follows. In Section 1 we proceed directly to the many independence result which is Theorem 1.6 (and Corollary), making use of detailed information about the model, $L^{\omega+\omega}$, (see Definition 1.16) of **Z** used in the independence proof. Section 2 is entirely devoted to an outline of a proof of this detailed information. Thus Section 1 comprises the body of the independence proof, and Section 2 comprises the routine detailed machinery needed. Section 3 considers various refinements, including the independence from 2nd-order arithmetic of determinateness for $G_{\delta\sigma\ \delta\sigma}$ sets; this is to be compared with M.Davis [2], which gives a mathematical practice type proof of determinateness for $G_{\delta\sigma}$ sets (easily formalizable in 2nd-order arithmetic). Neither our independence methods nor the methods of [2] (or any other mathematical practice methods) seem to apply to $G_{\delta\sigma\delta}$.

Apparently, determinateness was first introduced by Gale and Stewart in [3]. Determinateness in various forms (for analytic sets, projective sets, ordinal definable sets, all sets, to mention some divisions) have been under intensive investigation in recent years. For a recent survey, see A.Mathias [5].

Section 1

The purpose of this Section is to prove Theorem 1.6 and its Corollary.

We let ω be [0, 1, 2, ...], 2^ω be the set of all functions from ω into [0, 1], and Ω be the first uncountable ordinal.

The Borel subsets of 2^ω are the least σ-algebra containing all open and closed subsets of 2^ω. It is well known that the Borel subsets of 2^ω are just those subsets which lie in some B_α, $\alpha < \Omega$, as defined below. But first we define the open subsets of 2^ω.

Definition 1.1. We say $Y \subset 2^\omega$ is *open* if and only if $(\forall x)(x \in Y \rightarrow \rightarrow (\exists n \in \omega)(\forall y \in 2^\omega)((\forall m \leq n)(y(m) = x(m)) \rightarrow y \in Y))$. We say $Y \subset 2^\omega$ is *closed* if and only if $2^\omega - Y$ is open.

Definition 1.2. Define $B_1 = [Y \subset 2^\omega : Y$ is open or Y is closed$]$, $B_{\alpha+1} = [Y \subset 2^\omega : Y$ is the intersection of some countable (or finite) subset of B_α or Y is the union of some countable subset of $B_\alpha]$, $B_\lambda = \bigcup_{\alpha<\lambda} B_\alpha$, where $\alpha, \lambda < \Omega$, λ is a limit ordinal.

We can associate in informal terms, to each $Y \subset 2^\omega$, a discrete two-person game of infinite duration. The players are designated I, II. The players alternately produce (or play) either 0 or 1, starting with I. If the resulting element of 2^ω is in Y then I is considered the winner; if not, then II is. The question arises as to whether there is a perfect strategy for winning available to one of the two players.

We now wish to give the well known formal analysis of the above.

Definition 1.3. A 0, 1*-sequence* is a function s whose domain is an initial segment (possibly empty) of ω and whose range is a subset of [0, 1]. We write $\ln(s)$ to be such that $\mathrm{Dom}(s) = [i : i < \ln(s)]$. If s, t are 0, 1-sequences then we say t extends s if and only if $\ln(s) \leq \ln(t)$ and $(\forall i < \ln(s))(s(i) = t(i))$. If s is a 0, 1-sequence and $f \in 2^\omega$ then f extends s means that $(\forall i)(i < \ln(s) \rightarrow s(i) = f(i))$.

Definition 1.4. Let $Y \subset 2^\omega$. We write $S(Y, \mathrm{I}, f)$ if and only if

1) f is a function from the 0, 1-sequences into $[0, 1]$,
2) $(\forall g \in 2^\omega)(\lambda n(g((n-1)/2)$ if n is odd; $f(g \restriction [i: i \leqslant n/2])$ if n is even$) \in Y)$.

We write $S(Y, \text{II}, g)$ if and only if
1) g is a function from the 0, 1-sequences into $[0, 1]$
2) $(\forall f \in 2^\omega)(\lambda n(f(n/2)$ if n is even; $g(f \restriction [i: i < (n+1)/2])$ if n is odd$) \in 2^\omega - Y)$.

We write $D(Y)$ if and only if $(\exists f)(S(Y, \text{I}, f) \vee S(Y, \text{II}, f))$.

Thus $S(Y, \text{I}, f)$ expresses that f is a winning strategy for I in the game associated with Y; $S(Y, \text{II}, f)$ for II. And $D(Y)$ expresses that either I or II has a winning strategy.

In this paper we are only concerned with $D(Y)$ for Borel Y.

Definition 1.5. Let $1 < \alpha < \Omega$. Then $D(\alpha)$ means $(\forall Y \in B_\alpha)(D(Y))$.

We use some notions from ordinary recursion theory.

Definition 1.6. For $f \in 2^\omega$ we write φ_e^f for the eth partial function of one argument on ω that is partial recursive in f, according to some customary enumeration. We write $g \leq_T f$ for $(\exists e)(g = \varphi_e^f)$. We write $g =_T f$ for $g \leq_T f \;\&\; f \leq_T g$, and we write $f <_T g$ for $f \neq_T g \;\&\; f \leq_T g$.

Thus $g \leq_T f$ is read "g is partial recursive in f". The T stands for Turing.

Definition 1.7. We write $J(f)$ for the Turing jump of $f \in 2^\omega$. Define $J^{n+1}(f) = J(J^n(f))$, $0 < n$. Define $J^\omega(f) = \lambda m((J^a(f))(b)$ if $0 < a$, $0 \leq b$ and $m = 2^a 3^b$; 0 otherwise).

Definition 1.8. A *Turing set* is a $Y \subset 2^\omega$ such that $(\forall f)(\forall g)((f \in Y \;\&\; f =_T g) \rightarrow g \in Y)$. A Turing cone is a $Y \subset 2^\omega$ such that $(\exists f \in 2^\omega)(\forall g)(g \in Y \equiv f \leq_T g)$.

Unless we specify otherwise, whenever we quantify over functions we are quantifying only over 2^ω.

We now present a theorem of D.Martin modified and specialized to suit our purposes.

Theorem 1.1. *Suppose* $(\forall\alpha)(D(\alpha))$. *Then for all* α, *every Turing set* $Y \in B_\alpha$ *either contains or is disjoint from a Turing cone.*

Proof. Take X as $[f\colon \lambda n(f(2n)) \in Y \;\&\; \lambda n(f(2n+1)) \leq_T \lambda n(f(2n))]$. If $S(X, \mathrm{I}, g)$, then $[\alpha \in 2^\omega : h \leq_T \alpha] \subset Y$. If $S(X, \mathrm{II}, g)$, then $[\alpha \in 2^\omega : h \leq_T \alpha] \cap Y = \phi$.

Definition 1.9. LST is the *language of set theory*; i.e. the predicate calculus with equality (=) and membership (∈).

Definition 1.10. **Z** is *Zermelo set theory*, a theory in LST, whose non-logical axioms are

1) $(\exists y)(\forall z)(z \in y \equiv z \subset x)$

2) $(\exists z)(\forall w)(w \in z \equiv (w = x \vee w = y))$

3) $x = y \equiv (\forall z)(z \in x \equiv z \in y)$

4) $(\exists x)(\forall y)(y \in x \equiv (y \in a \;\&\; F))$, where F is a formula in LST which does not mention x free

5) $(\exists y)(\forall z)(z \in y \equiv (\exists w)(z \in w \;\&\; w \in x))$

6) $(\exists x)(\phi \in x \;\&\; (\forall y)(y \in x \rightarrow (\exists z)(z \in x \;\&\; (\forall w)(w \in z \equiv (w \in y \vee w = y)))))$. Here $z \subset y$ is an abbreviation for $(\forall x)(x \in z \rightarrow x \in y)$, and $\phi \in x$ is an abbreviation for $(\exists y)(\forall z)(z \notin y \;\&\; y \in x)$

7) $x \neq \phi \rightarrow (\exists y)(y \in x \;\&\; (\forall y)(z \in x \rightarrow z \notin y))$

8) the Axiom of Choice.

We now describe the model of **Z** we will use in this Section, and which we analyze in Section 2.

Definition 1.11. If x is a set then ϵ_x is the *binary relation* on x given by $\epsilon_x(a, b) \equiv (a \in x \;\&\; b \in x \;\&\; a \in b)$.

Definition 1.12. Define $V(0) = \phi$, $V(\alpha + 1) = \mathrm{P}(V(\alpha))$, $V(\lambda) = \bigcup_{\alpha<\lambda} V(\alpha)$, where $\mathrm{P}(x)$ is $[\, y : y \subset x\,]$ and λ is a limit ordinal.

Definition 1.13. A structure is a system (A, R), where A is a nonempty set, R is a binary relation on A. An assignment in (A, R) is a function $f: \omega \to A$ with finite range. We write $\mathrm{Sat}((A, R), F, f)$ to express that the formula F of LST holds in the structure (A, R) when ϵ is interpreted as R, = as equality, and each free variable v_i in F is interpreted as $f(i)$. If F has no free variables then we may write $\mathrm{Sat}((A, R), F)$.

Definition 1.14. For structures (A, R), (B, S) we write $\mathrm{Inj}(f, (A, R), (B, S))$ to express that $f: A \to B$, f 1–1, and $(\forall x, y \in A)(R(x, y) \equiv S(f(x), f(y)))$. We write $\mathrm{Iso}(f, (A, R), (B, S))$ if the above holds and f is onto. We write $(A, R) \approx (B, S)$ for $(\exists f)(\mathrm{Iso}(f, (A, R), (B, S)))$.

Definition 1.15. For structures (A, R) we take $\mathrm{FODO}((A, R)) = [\, x \subset A$: for some formula F and assignment f we have $x = [\, y\colon \mathrm{Sat}((A, R), F, f_y^0)\,]\,]$, where $f_y^0(i) = f(i)$ if $i \neq 0$; y if $i = 0$.

FODO stands for "first order definable over".
Often we abbreviate (x, ϵ_x) by (x, ϵ).

Definition 1.16. Define $L(0) = V(\omega)$, $L(\alpha + 1) = \mathrm{FODO}((L(\alpha), \epsilon_{L(\alpha)}))$, $L(\lambda) = \bigcup_{\alpha<\lambda} L(\alpha)$, where λ is a limit ordinal. Define $L^{\omega+\omega}(0) = \phi$, $L^{\omega+\omega}(\alpha + 1) = \mathrm{FODO}((L^{\omega+\omega}(\alpha), \epsilon)) \cap V(\omega + \omega)$, $L^{\omega+\omega}(\lambda) = \bigcup_{\alpha<\lambda} L^{\omega+\omega}(\alpha)$, where λ is a limit ordinal. Define $L^{\omega+\omega} = [\, x\colon (\exists\alpha)(x \in L^{\omega+\omega}(\alpha))\,]$.

Thus our L is the usual constructible hierarchy.

Lemma 1.2.1. *Each $L^{\omega+\omega}(\alpha)$ is transitive. In addition, $L^{\omega+\omega}$ is transitive.*

Lemma 1.2.2. *For all transitive sets x and all $f\colon \omega \to x$ with finite range we have* $\mathrm{Sat}((x, \epsilon), v_0 \subset v_1, f) \equiv f(0) \subset f(1)$.

Lemma 1.2.3. *$V(\omega + \omega)$ is closed under subset and power set and union.*

Theorem 1.2. *$L^{\omega+\omega}$ satisfies Z.*

Proof. There is an α such that $L^{\omega+\omega}(\alpha) = L^{\omega+\omega}(\alpha + 1)$. Choose α least with this property. 3), 5), and 6) follow from the lemmas; to check 1), 2), and 4), note first that $L^{\omega+\omega}(\alpha) = L^{\omega+\omega}$. For 1), note that $[x \in L^{\omega+\omega}(\alpha): x \subset y] \in L^{\omega+\omega}(\alpha + 1)$ for any $y \in L^{\omega+\omega}(\alpha)$. For 2), note that $[x \in L^{\omega+\omega}(\alpha): x = y \vee x = y] \in L^{\omega+\omega}(\alpha + 1)$ for any y, $z \in L^{\omega+\omega}(\alpha)$. For 4), note that $[y \in a: \text{Sat}((L^{\omega+\omega}(\alpha), \epsilon), F, f_y^0)] \in L^{\omega+\omega}(\alpha + 1)$ for all $a \in L^{\omega+\omega}(\alpha)$, all assignments f, all formulae F in LST.

Definition 1.17. We assume a fixed primitive recursive total one-one onto Gödel numbering of the formulae in LST. We let 'φ' be the Gödel number of φ. Let (A, R) be a structure. We write $\text{Def}((A, R), n, x)$ if and only if n is the Godel number of the formula $F(v_0)$ with only the free variables shown and x is the unique element of A with $\text{Sat}((A, R), F(v_0), \lambda n(x))$, and furthermore n is the least integer with this property that x is the unique element of A with $\text{Sat}((A, R), F(v_0), \lambda n(x))$.

Definition 1.18. Let (A, R) be a structure. Then we let $\text{Th}((A, R))$ be $[n: n$ is the Gödel number of the sentence F and $\text{Sat}((A, R), F)]$.

Definition 1.19. If $x \subset \omega$ then we write $\text{Ch}(x)$ for $\lambda n(1$ if $n \in x$; 0 if $n \notin x)$.

We need to draw on one fact about the construction of $L^{\omega+\omega}$; Section 2 is devoted to a detailed outline of a proof of the following.

Theorem 2. *There are formulae $\varphi_1(v_0, v_1)$, $\varphi_2(v_0, v_1)$, and $\varphi_3(v_0, v_1)$ in* LST *with only the free variables shown such that for each $x \subset \omega$, $x \in L^{\omega+\omega}$, there is a limit ordinal λ such that*

1) $x \in L^{\omega+\omega}(\lambda)$

2) $(\forall y \in L^{\omega+\omega}(\lambda))(\exists n)(\text{Def}((L^{\omega+\omega}(\alpha), \epsilon)n, y))$

3) $\mathrm{Th}((L^{\omega+\omega}(\lambda), \epsilon)) \in L^{\omega+\omega}(\lambda + 2)$

4) $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), \varphi_1(v_0, v_1), f)$ *if and only if* $(\mu\beta)(f(0) \in L^{\omega+\omega}(\beta)) < (\mu\beta)(f(1) \in L^{\omega+\omega}(\beta))$

5) $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), \varphi_2(v_0, v_1), f)$ *if and only if* $(\mu\beta)(f(0) \in L^{\omega+\omega}(\beta)) = (\mu\beta)(f(1) \in L^{\omega+\omega}(\beta))$

6) $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), \varphi_3(v_0, v_1), f)$ *if and only if* $f(1) = (\mu n \in \omega)(f(0) \in V(\omega + n))$.

We make the following Definition 1.21 modelled after Theorem 2, using the φ_1, φ_2, and φ_3 of the statement of that Theorem.

Definition 1.20. We fix a structure (A^0, R^0) such that $A^0 = [i : i \text{ is odd}]$, R^0 is a recursive relation, and (A^0, R^0) is isomorphic to $(V(\omega), \epsilon)$. By $\bar{n}$ we mean that element of A^0 which is satisfied, in (A^0, R^0), to be n.

Definition 1.21. A *towered structure* is a structure (A, R) such that

1) $A \subset \omega$ and the relation $x \sim y \equiv \mathrm{Sat}((A, R), \varphi_2(v_0, v_1), \lambda n(x \text{ if } n = 0; y \text{ if } n \neq 0))$ is an equivalence relation on A

2) the relation $x < y \equiv \mathrm{Sat}((A, R), \varphi_1(v_0, v_1), \lambda n(x \text{ if } n = 0; y \text{ if } n \neq 0))$ has that $(\forall x, y \in A)((x < y \;\&\; \sim y < x) \vee (y < x \;\&\; \sim x < y) \vee (x \sim y \;\&\; \sim x < y \;\&\; \sim y < x))$ and $(\forall x, y, z \in A)(((x \sim z \;\&\; x < y) \to z < y) \;\&\; ((x \sim z \;\&\; y < x) \to y < z))$, and $<$ has no maximal element

3) $A^0 = [i : i \in A \;\&\; (\forall j)(\sim j < i)]$, $R^0 = R \restriction A^0$

4) we have $(\forall x \in A)(\exists !\, y)(\mathrm{Sat}((A, R), \varphi_3(v_0, v_1), \lambda n(x \text{ if } n = 0; y \text{ if } n \neq 0))$, and so we let F be given by $(\forall x \in A)(\mathrm{Sat}((A, R), \varphi_3(v_0, v_1), \lambda n(x \text{ if } n = 0; F(x) \text{ if } n \neq 0))$. Then we want $(\forall x \in A)(\exists n)(F(x) = \bar{n})$, and $(\forall x \in A^0)(F(x) = \bar{0})$

5) $(\forall x \in A - A^0)(F(x) = \bar{n}$ where n is the least integer greater than every i such that $(\exists y)(R(y, x) \;\&\; F(y) = \bar{i}))$

6) suppose $x \in A$. Then $\mathrm{FODO}(([i: i < x], R \upharpoonright [i: i < x])) =$
$[z \subset [i: i < x] : (\exists j)((j < x \vee j \sim x) \,\&\, z = [k: R(k, j)]))]$

7) $(\forall x, y \in A)(R(x, y) \to x < y)$

8) (A, R) satisfies the axiom of extensionality

9) $(\forall i \in A - A^0)(\exists j)(\mathrm{Def}((A, R), j, 2j) \,\&\, i = 2j)$

10) $[i: i \in \mathrm{Th}((A, R))] \in \mathrm{FODO}(\mathrm{FODO}((A, R)), \epsilon)$

11) for all nonempty $x \subset A$ with $\mathrm{Ch}(x) \leq_T J(J^\omega(\mathrm{Ch}(\mathrm{Th}((A, R)))))$ there exists a $y \in x$ such that for all $z \in x$ we have $\sim z < y$.

We presume knowledge of the effective Borel hierarchy. In particular, we will make use of the notion of: being in $B_{\omega+\omega}$ with recursive code.

Lemma 1.3.1. $[f \in 2^\omega : f$ codes $\mathrm{Th}((A, R))$ for some towered structure $(A, R)]$ *is in* $B_{\omega+\omega}$ *with recursive code. In other words,* $\mathcal{S} = [f \in 2^\omega :$ $f = \mathrm{Ch}(\mathrm{Th}((A, R)))$ for some towered structure $(A, R)]$ *is in* $B_{\omega+\omega}$ *with recursive code.*

Proof. A more detailed proof of a more delicate version of this is given as Lemma 3.2.2; we will only mention some basic points for this present version. To "test" whether $f \in \mathcal{S}$ first construct the relational structure (A, R) given by $A^0 \subset A$, $R^0 = R \upharpoonright A^0$, $A - A^0 = [2i: i$ is the Gödel number of some formula $F(v_0)$ such that '$(\exists!\, v_0)(F(v_0))$' $\in [k: f(k) = 1]$ and $(\forall j < i)$ (if j is the Gödel number of some formula $G(v_0)$ then '$(\exists!\, v_0)(G(v_0)) \,\&\, (\exists v_0)(G(v_0) \,\&\, F(v_0))$' $\in [k: f(k) = 0]\,]$, $R(2i, 2j)$, for $2i, 2j \in A$, holds if and only if for the corresponding F, G we have '$(\exists v_0)(F(v_0) \,\&\, (\exists v_1)(G(v_1) \,\&\, v_0 \in v_1))$' $\in [k: f(k) = 1]$, $R(2i, 2j + 1)$ is always false, $R(2i + 1, 2j)$ holds if and only if '$(\exists v_0)(P(v_0) \,\&$ $(\exists v_1)(G(v_1) \,\&\, v_0 \in v_1))$' $\in [k: f(k) = 1]$, where P is the *canonical* definition of $2i + 1$ in (A^0, R^0). Then check whether clauses 1) – 11) hold for this (A, R). It is clear that if there is any (A, R) with $\mathrm{Th}((A, R)) = [k: f(k) = 1]$ it must be this (A, R) above.

Lemma 1.3.2. *If* $Y \subset 2^\omega$ *is in* $B_{\omega+\omega}$ *with recursive code then* $Y \cap L^{\omega+\omega}$ *must be in* $L^{\omega+\omega}$ *and* $L^{\omega+\omega}$ *must satisfy that* $Y \cap L^{\omega+\omega}$ *is in* $B_{\omega+\omega}$ *with recursive code.*

Proof. This is a well known absoluteness property of the effective Borel hierarchy.

Theorem 1.3. *$\mathcal{S} \cap L^{\omega+\omega} \in L^{\omega+\omega}$ and is satisfied in $L^{\omega+\omega}$ to be an element of $B_{\omega+\omega}$ with recursive code.*

Theorem 1.4. *For all $f \in 2^\omega \cap L^{\omega+\omega}$ there is a $g \in \mathcal{S} \cap L^{\omega+\omega}$ such that $f \leq_T g$.*

Proof. Take this f. Let $x = [k : f(k) = 1]$. Choose λ according to Theorem 2. We must choose the appropriate towered structure $(A, R) \approx (L^{\omega+\omega}(\alpha), \epsilon)$. We will define a g such that $\text{Iso}(g, (L^{\omega+\omega}(\lambda), \epsilon), (A, R))$. Take $g \upharpoonright V(\omega)$ to be the isomorphism from $(V(\omega), \epsilon)$ onto (A^0, R^0). For $y \in L^{\omega+\omega}(\lambda) - V(\omega)$ take $g(y)$ to be $2n$ where $\text{Def}((L^{\omega+\omega}(\lambda), \epsilon), n, y)$. Take R to be the relation on $\text{Rng}(g)$ induced by g. Conditions 1) – 10) in the definition of towered structure are easily verified. Condition 11) also is satisfied since $<$ will be a well-founded relation.

Definition 1.22. Let $f, g \in 2^\omega$. The join of f, g, written (f, g), is $\lambda n(f(n/2)$ if n is even; $g((n-1)/2)$ if n is odd).

Lemma 1.5.1. *Suppose (A, R), (B, S) are towered structures such that* $\text{Ch}(\text{Th}((A, R))) \leq_T J(\text{Ch}(\text{Th}((B, S))))$ *and* $\text{Ch}(\text{Th}((B, S))) \leq_T J(\text{Ch}(\text{Th}((A, R))))$. *Then either $(\exists f)(\text{Iso}(f, (A, R), (B, S)))$ or $(\exists f)(\text{Inj}(f, (A, R), (B, S))$ and $(\exists x \in B)(\text{Rng}(f) = [y \in B : y < x]$, where $<$ is as in (B, S) as in* Definition 1.21)), *or $(\exists f)(\text{Inj}(f, (B, S), (A, R))$ and $(\exists x \in A)(\text{Rng}(f) = [y \in A : y < x]$, where $<$ is as in (A, R) as in* Definition 1.21)).

Proof. Let $T_1 = \text{Th}((A, R))$, $T_2 = \text{Th}((B, S))$. Let $\sim_1, <_1, F_1$ be as in Definition 1.21 for (A, R); $\sim_2, <_2, F_2$ be as in Definition 1.21 for (B, S).

Define the predicate $P(n, i, j)$ by recursion on n. $P(0, i, j) \equiv i \in A^0$ & $i = j$. $P(n+1, i, j) \equiv F_1(i) = F_2(j) = \overline{n+1}$ & $(\forall a)(R(a, i) \rightarrow (\exists b)(\exists k)(S(b, j)$ & $P(k, a, b)$ & $F_1(a) = F_2(b) = \bar{k}))$ & $(\forall a)(S(a, j) \rightarrow (\exists b)(\exists k)(R(b, i)$ & $P(k, b, a)$ & $F_2(a) = F_1(b) = \bar{k}))$. It is easily seen that, uniformly, for each k, the relation $P(k, a, b)$ is recursive in

$J^k((\mathrm{Ch}(T_1), \mathrm{Ch}(T_2)))$. Hence, uniformly, for each k, the relation $P(k, a, b)$ is recursive in both $J^{k+1}(\mathrm{Ch}(T_1))$ and $J^{k+1}(\mathrm{Ch}(T_2))$.

We now wish to prove by induction on n that for each i there is at most one j such that $P(n, i, j)$. The case $n = 0$ is trivial. Suppose true for all $k \leq n$ and let $P(n+1, i, j), P(n+1, i, a)$. Let $S(x, j)$. Then $F_2(x) = \bar{k}$ for some $k \leq n$. Then for some $x_0 \in A$ we have $P(k, x_0, x)$ and $R(x_0, i)$. Hence by $P(n+1, i, a)$ we must have for some $y \in B, P(k, x_0, y)$ and $S(y, a)$. But since $k \leq n$ we must have $x = y$. So $S(x, a)$. Hence $(\forall x)(S(x, j) \rightarrow S(x, a))$. Symmetrically, $(\forall x)(S(x, a) \rightarrow S(x, j))$. So $a = j$, and we are done.

Symmetrically, for each j there is at most one i such that $P(n, i, j)$.

Clearly $(P(n, i, j)\ \&\ R(a, i)) \rightarrow (\exists b)(\exists k)(P(k, a, b)\ \&\ S(b, j))$; the only nontrivial case is when $j \in A^0$, in which case $a \in A^0$ by clause 7) of Definition 1.21. Also $(P(n, i, j)\ \&\ S(a, j)) \rightarrow (\exists b)(\exists k)(P(k, b, a)\ \&\ R(b, i))$.

Thus roughly speaking, P defines a partial isomorphism between (A, R) and (B, S).

Consider $K = [\, i \in A : (\forall j)(j \sim_1 i \rightarrow (\exists n)(\exists m)(\exists a)(\exists b)(P(n, i, a)\ \&\ P(m, j, b)\ \&\ a \sim_2 b\ \&\ (\forall c)(c \sim_2 b \rightarrow (\exists d)(\exists r)(d \sim_1 i\ \&\ P(r, d, c)))\ \&\ (\forall c)(c <_2 b \rightarrow (\exists d)(\exists r)(d <_1 i\ \&\ P(r, d, c)))))\,]$. Then clearly $\mathrm{Ch}(A - K) \leq_T J(J^\omega(\mathrm{Ch}(T_1)))$. We now break into cases.

Case 1. $A - K = \phi, (\forall j \in B)(\exists n)(\exists i)(P(n, i, j))$. Then obviously $(A, R) \approx (B, S)$, given by P.

Case 2. $A - K = \phi, (\exists j \in B)(\forall n)(\forall i)(\sim P(n, i, j))$. Note that then $\mathrm{Ch}([\, j \in B : (\forall n)(\forall i)(\sim P(n, i, j))\,]) \leq_T J(J^\omega(\mathrm{Ch}(T_2)))$ and is nonempty. Choose $x \in B$ with $(\forall n)(\forall i)(\sim P(n, i, j))\ \&\ (\forall y < x)(\exists n)(\exists i)(P(n, i, j))$. Then since $K = A$ we must have that $(\forall j)[(\exists n)(\exists i)(P(n, i, j)) \rightarrow j <_2 x]$. Hence set $f(i)$ to be the unique j such that $(\exists n)(P(n, i, j))$. Then $\mathrm{Inj}(f, (A, R), (B, S))\ \&\ \mathrm{Rng}(f) = [\, j : j < x\,]$.

Case 3. $A - K \neq \phi$, and $(\exists x)(x \in A - K\ \&\ (\forall y)(y <_1 x \rightarrow y \in K)\ \&\ x \notin A^0)$. Fix this x. Note $\mathrm{Ch}([\, j \in B : (\forall n)(\forall i)(i <_1 x \rightarrow \sim P(n, i, j))\,]) \leq_T J^\omega(\mathrm{Ch}(T_2))$. If $(\forall j \in B)(\exists n)(\exists i)(i <_1 x\ \&\ P(n, i, j))$ then take $f(j)$ to be the unique i such that $(\exists n)(P(n, i, j))$. Then $\mathrm{Inj}(f, (B, S), (A, R))\ \&\ \mathrm{Rng}(f) = [\, y : y <_1 x\,]$. If

$(\exists j \in B)(\forall n)(\forall i)(i <_1 x \to \sim P(n, i\ j))$, then choose $y \in B$ such that $(\forall n)(\forall i)(i <_1 x \to \sim P(n, i, y))$ and $(\forall j <_2 y)(\exists n)(\exists i)(i <_1 x$ & $P(n, i, j))$. Now note that $([i : i <_1 x], R \restriction [i : i <_1 x]) \approx ([j : j <_2 y], S \restriction [j : j <_2 y])$ and let f be the isomorphism given by $f(i) =$ the unique j such that $(\exists n)(P(n, i, j))$. We obtain a contradiction by showing that $x \in K$. It suffices to show that $(\forall a)(a \sim_1 x \to (\exists n)(\exists b)(P(n, a, b)$ & $b \sim_2 y))$ & $(\forall a)(a \sim_2 y \to (\exists n)(\exists b)(P(n, b, a)$ & $b \sim_1 x))$. By symmetry it suffices to obtain the first conjunct. Let $a \sim_1 x$. Then $[i : R(i, a)] \in \mathrm{FODO}([i : i <_1 x], R \restriction [i : i <_1 x])$. In particular let G be a formula and g an assignment such that $[i : R(i, a)] = [i : \mathrm{Sat}(([i : i <_1 x], R \restriction [i : i <_1 x]), G, g_i^0]$. Now there must be a k such that $F_1(a) = \overline{k+1}$. Choose the unique $a^* \in B$ such that $[j : S(j, x^*)] = [j : \mathrm{Sat}(([j : j <_2 y], S \restriction [j : j <_2 y]), G, (f \circ g)_j^0)]$. Then since f is an isomorphism we must have $a^* \notin [j : j <_2 y]$ since $a \notin [i : i <_1 x]$. But $a^* \in \mathrm{FODO}([j : j <_2 y], S \restriction [j : j <_2 y])$, and so we have $a^* \sim_2 y$. Also since f is an isomorphism, we have that $\mathrm{Rng}(f \restriction [i : R(i, a)]) = [j : S(j, a^*)]$, and hence by the way f is defined, we have $P(k+1, a, a^*)$.

Case 4. $A - K \neq \phi$, and $(A - K) \cap A^0 \neq \phi$. But this is obviously impossible since $A^0 \subset K$.

Lemma 1.5.2. *Let* (A, R), (B, S) *be towered structures,* $\mathrm{Inj}(f, (A, R), (B, S))$, $x \in B$, $\mathrm{Rng}(f) = [i : i <_2 x]$, *where* $<_2$ *refers to* (B, S). *Then* $J(\mathrm{Ch}(\mathrm{Th}((A, R)))) <_T \mathrm{Ch}(\mathrm{Th}((B, S)))$.

Proof. We use the notation of the proof of Lemma 1.5.1. Fix f, x. Note that $<_2$ has no maximum element. Let $x_1 =$ any $<_2$-least element of $[i : x <_2 i]$. Let $x_2 =$ any $<_2$-least element of $[i : x <_2 i]$. Then $[\bar{i} : i \in \mathrm{Th}((A, R))] \in \mathrm{FODO}(\mathrm{FODO}((A, R)), \epsilon)$ as in 10) of Definition 1.21. Hence there is a $y \sim_2 x_2$ with $S(z, y) \equiv z$ is some $\bar{i}$ with $i \in \mathrm{Th}((A, R))$. Next it is easy to find a formula $P(v_0, v_1)$ such that $\mathrm{Sat}((B, S), P(v_0, v_1), f_y^1) \equiv f(0)$ is some $\bar{j}$ with $J^2(\mathrm{Ch}(\mathrm{Th}(A, R)))(j) = 1$. Hence clearly $J^2(\mathrm{Ch}(\mathrm{Th}((A, R)))) \leq_T \mathrm{Ch}(\mathrm{Th}(B, S))$, since $(\exists n)\,\mathrm{Def}((B, S), n, y)$. Since $J(\mathrm{Ch}(\mathrm{Th}((A, R)))) <_T J^2(\mathrm{Ch}(\mathrm{Th}(A, R))))$, we must have $J(\mathrm{Ch}(\mathrm{Th}((A, R)))) <_T \mathrm{Ch}(\mathrm{Th}((B, S)))$.

Lemma 1.5.3. *Suppose* (A,R), (B,S) *are towered structures such that* $\mathrm{Ch}(\mathrm{Th}((A,R))) \leq_T J(\mathrm{Ch}(\mathrm{Th}((B,S))))$ *and* $\mathrm{Ch}(\mathrm{Th}((B,S))) \leq_T J(\mathrm{Ch}(\mathrm{Th}((A,R))))$. *Then* $(A\ R) = (B,S)$.

Proof. Assume hypotheses. Then either $(\exists f)(\mathrm{Iso}(f, (A,R), (B,S)))$ or $(\exists f)(\mathrm{Inj}(f, (A,R), (B,S))$ and $(\exists x \in B)(\mathrm{Rng}(f) = [\, y \in B : y <_2 x \,]))$, or vice versa. The latter two cases contradict our hypothesis by Lemma 1.5.2. Hence $\mathrm{Iso}(f, (A,R), (B,S))$ for some f. Hence $\mathrm{Th}((A,R) = \mathrm{Th}((B,S))$, and so obviously for all i, $\mathrm{Def}((A,R)\ i, x) \equiv \mathrm{Def}((B,S), i, f(x))$. Hence by clause 9) of Definition 1.21, f must be the identity. Hence $(A,R) = (B,S)$, and we are done.

Theorem 1.5. *For all* $f \in 2^\omega \cap L^{\omega+\omega}$ *there is a* g *such that* $f \leq_T g$ *and* $(\forall \alpha \in 2^\omega)(g =_T \alpha \rightarrow \alpha \in (2^\omega - \mathcal{S}) \cap L^{\omega+\omega})$.

Proof. Fix $f \in 2^\omega \cap L^{\omega+\omega}$. By Theorem 1.4, choose $h \in \mathcal{S} \cap L^{\omega+\omega}$ with $f \leq_T h$, and let $[\, i : h(i) = 1 \,] = \mathrm{Th}((A,R))$, where (A,R) is a towered structure. Then $J(h) \in L^{\omega+\omega}$ and so $(\forall \alpha)(\alpha =_T J(h) \rightarrow \alpha \in L^{\omega+\omega})$. Clearly $f \leq_T J(h)$. Now $J(h) \leq_T J(h)$ and $h \leq_T J(J(h))$, and so by Lemma 1.5.3 there must not be a towered (B,S) with $J(h) =_T \mathrm{Th}((B,S))$. In other words, $(\forall \alpha)(g =_T \alpha \rightarrow \alpha \in 2^\omega - \mathcal{S})$.

Theorem 1.6. $L^{\omega+\omega}$ *satisfies that there exists an element of* $B_{\omega+\omega}$ *with recursive code which is a Turing set but does not contain nor is disjoint from a Turing cone. In particular,* $L^{\omega+\omega}$ *satisfies* $\sim D(\omega+\omega)$ *by* Theorem 1.1.

Proof. Take the Turing set X to be $[\, f \in 2^\omega : (\exists g \in \mathcal{S})(f =_T g) \,]$. Then using Theorem 1.3 it is easily seen that $X \cap L^{\omega+\omega} \in L^{\omega+\omega}$ and is satisfied to be an element of $B_{\omega+\omega}$ with recursive code and to be a Turing set. From Theorem 1.4 one has that X is satisfied to intersect every Turing cone, because of the absoluteness of Turing reducibility. By Theorem 1.5, X is satisfied to not contain any Turing cone.

Corollary. *By* Theorem 1.2, $D(\omega+\omega)$ *is not provable in* **Z**.

Section 2

We have defined **Z** in Definition 1.10, and $L^{\omega+\omega}(\alpha)$, $L^{\omega+\omega}$ in Definition 1.6, and have remarked that each $L^{\omega+\omega}(\alpha)$ is transitive and that $\mathrm{Sat}((L^{\omega+\omega}, \epsilon), F, f)$ for all $F \in \mathbf{Z}$ and assignments f (see Definition 1.13). Furthermore, we have the special structure (A^0, R^0) of Definition 1.20.

The purpose of this Section is to give a detailed outline of a proof of the fact about the $L^{\omega+\omega}(\alpha)$ needed in Section 1; namely, Theorem 2.

Definition 2.1. We let $\langle x, y\rangle = [\,x, [\,x, y\,]\,]$. We write $\mathrm{Fcn}(x)$ for $(\forall y \in x)(\exists a)(\exists b)(y = \langle a, b\rangle)$ & $(\forall a)(\forall b)(\forall c)((\langle a, b\rangle \in x$ & $\langle a, c\rangle \in x) \to b = c)$. We write $\mathrm{Dom}(x)$ for $[\,a\colon (\exists b)(\langle a, b\rangle \in x)]$, $\mathrm{Rng}(x)$ for $[\,a\colon (\mathrm{E}b)(\langle b, a\rangle \in x)]$. We let $(\;) = \phi$, $(x) = [\langle 0, x\rangle]$, $(x_0, ..., x_k) = [\langle i, x_i\rangle\colon 0 \le i \le k\,]$. We write $\ln((x_0, ..., x_{k-1})) = k$, $(x_0, ..., x_{k-1})(i) = x_i$, $i < k$. We take $\mathrm{Seq}(x) = [\,y\colon \mathrm{Fcn}(y)$ & $(\exists k \in \omega)(k \neq \phi$ & $\mathrm{Dom}(y) = k)$ & $\mathrm{Rng}(y) \subset x\,]$. We take $a_0 * a_1 * ... * a_k$, for $a_i \in \mathrm{Seq}(x)$, to be the result of concatenation.

Definition 2.2. We assume a one-one Gödel numbering from formulae onto ω. A formula is a formula using $\forall$, $\exists$, &, $\vee$, $\sim$, $\in$, $=$, $v_0, v_1, ...$. For formulae F we let 'F' be the Gödel number of F. For $n \in \omega$ we let $|n|$ be that formula with Gödel number n.

Definition 2.3. We write $\mathrm{LO}(x)$, (x is a linear ordering) for $x = \langle A, R\rangle$ and $A \neq \phi$ and $R \subset [\langle a, b\rangle\colon a \in A$ & $b \in A\,]$ and $A \cap V(\omega) = \phi$ and (A, R) constitutes a linear ordering on all of A. We write $A = \mathrm{Field}(x)$, $R = \mathrm{Rel}_1(x)$.

Definition 2.4. If $\mathrm{LO}(x)$ we take $O(x, y) \equiv y \in A$ & $(\forall z)(\langle z, y\rangle \notin \mathrm{Rel}_1(x))$, $\mathrm{Suc}(x, y, z) \equiv \langle z, y\rangle \in R_1$ & $\sim (\exists a)(\langle z, a\rangle \in R_1$ & $\langle a, y\rangle \in R_1)$, $\mathrm{Lim}(x, y) \equiv y \in A$ & $(\forall z)(\langle z, y\rangle \in R_1 \to (\exists a)(\langle z, a\rangle \in R_1$ & $\langle a, y\rangle \in R_1))$.

Definition 2.5. We write $\mathrm{CS}(x)$, (x is a coded structure), for $x = \langle A, R\rangle$ and $A \neq \phi$ and $R \subset [\langle a, b\rangle : a \in A$ & $b \in A\,]$. We write $A = \mathrm{Field}(x)$, and whenever we write $\mathrm{CS}(x)$, we write Rel_2 for R.

Definition 2.6. We write SLO(x), (x is a structured linear ordering), for $x = (F, \langle A, R_1\rangle, \langle A, R_2\rangle)$ and LO$(\langle A, R_1\rangle)$ and CS$(\langle A, R_2\rangle)$, and $F\colon A \to \omega$. We write Field$(x) = A$, Rel$_1(x) = R_1$, Rel$_2(x) = R_2$, Fn$(x) = F$.

Definition 2.7. We write Sati(x, n, y) for CS(x) & $n \in \omega$ & $y \in$ Seq(x) & $y = (a_0, ..., a_k)$, $0 \leq k$, & $x = \langle A, R\rangle$ & Sat$((A, R), |n|, f)$, where $f(i) = y(i)$ for $i < \ln(y)$; $y(\ln(y) - 1)$ for $i \geq \ln(y)$.

Definition 2.8. Let K be the least class satisfying

1) $A^0 \subset K$. See Definition 1.20

2) whenever $a_0, ..., a_k \in K$, $0 \leq k$, $n \in \omega$, $x \notin V(\omega)$ we have $(x, n, a_0, ..., a_k) \in K$. Let F_0 be the function on K given by $F_0(n) = n$ if $n \in A^0$; $F_0((x, n, a_0, ..., a_k)) = ([1], x, n) * F_0(a_0) * ... * F_0(a_k) * * ([2])$.

Lemma 2.1. *F_0 is a one-one function on K.*

Proof. We prove by induction on $\ln(s)$ that if $F_0(y_1) = s$, $F_0(y_2) = s$, then $y_1 = y_2$. Assume $F_0(y_1) = s$, $F_0(y_2) = s$. Then if s is not a sequence then $s = n$ for some $n \in A^0$, in which case $y_1 = y_2 = n$. So s is a sequence. Clearly s must be of the form $([1], x, n) * F_0(a_0) * ... * F_0(a_k) * ([2])$. Now we must show that the $a_0, ..., a_k, x, n$ above are unique. Let $s = ([1], y, m) * F_0(b_0) * ... * F_0(b_r) * ([2])$. Obviously $x = y$, $n = m$. If $F_0(a_0) \in A^0$ then obviously $F_0(b_0) \in A^0$ and $F_0(a_0) = F_0(b_0)$. If $F_0(a_0) \notin A^0$ then $F_0(a_0)$ starts with $[1]$ and ends with $[2]$, and no $[1]$ or $[2]$ occurs in between. Therefore $F_0(a_0) = F_0(b_0)$, and so on. So we obtain that $k = r$ and each $F_0(a_i) = F_0(b_i)$. Since each $F(a_i)$ has shorter length than s, we are done by induction hypothesis.

Definition 2.9. We write $<(x, a, b)$ for LO(x) & $a, b \in$ Seq(Field(x)) & a comes before b in the lexicographic ordering on Seq(Field(x)) induced by x.

Definition 2.10. We write Defn(x, a, k) for SLO(x) and $a = (n, b_0, ..., b_m)$, $0 \leq m$, and each $b_i \in$ Field(x) and $Y = [b\colon$ Sati$(\langle$Field(x),

$\mathrm{Rel}_2(x)\rangle, n, (b, b_0, ..., b_m))]$ satisfies the following conditions:

a) the range of $\mathrm{Fn}(x) \upharpoonright Y$ contains $k-1$ as an element and is a subset of k and $k \in \omega - [0]$,

b) $Y \neq [b: \langle b, c\rangle \in \mathrm{Rel}_2(x)]$ for all $c \in \mathrm{Field}(x)$,

c) $Y \neq [b: \mathrm{Sati}(\langle \mathrm{Field}(x), \mathrm{Rel}_2(x)\rangle, r, (b, b_0, ..., b_m))]$ for all $r < n$,

d) $Y \neq [b: \mathrm{Sati}(\langle \mathrm{Field}(x), \mathrm{Rel}_2(x)\rangle, n, (b, c_0, ..., c_r))]$ whenever $< (x, (c_0, ..., c_r), (b_0, ..., b_m))$.

Definition 2.11. We write $\mathrm{CHY}(x,f)$ for

1) $\mathrm{LO}(x)$

2) $\mathrm{Fcn}(f)$ & $\mathrm{Dom}(f) = \mathrm{Field}(x)$ & $(\forall y)(y \in \mathrm{Field}(x) \to (\mathrm{SLO}(f(y))$ & $\mathrm{Field}(f(y)) \subset A^0 \cup \mathrm{Seq}(V(\omega) \cup x)))$

3) $0(x,y) \to f(y) = (F, \langle A, R_1\rangle, \langle A, R_2\rangle)$, where $A = A^0$, $R_2 = R^0$, $R_1 = \epsilon \upharpoonright A^0$, $F(a) = 0$ for all $a \in A$

4) $\mathrm{Suc}(x, a, b) \to f(a) = (F, \langle A, R_1\rangle, \langle A, R_2\rangle)$, where $A = \mathrm{Field}(f(b)) \cup [([1], b, n) * b_0 * ... * b_m * ([2]): \mathrm{Defn}(f(b), (n, b_0, ..., b_m), k)$ for some $k]$, $R_1 = \mathrm{Rel}_1(f(b)) \cup [\langle a, s\rangle: a \in \mathrm{Field}(f(b))$ & $s \in A\text{-}\mathrm{Field}(f(b))] \cup [\langle a, s\rangle: a, s \in A\text{-}\mathrm{Field}(f(b))$ & $a = ([1], b, n) * b_0 * ... * b_m * ([2])$ & $s = ([1], b, m) * c_0 * ... * c_r([2])$ & $(n < m \vee < ((\mathrm{Field}(f(b)), \mathrm{Rel}_1(f(b))), (b_0, ..., b_m), (c_0, ..., c_r)))]$, $R_2 = \mathrm{Rel}_2(f(b)) \cup [\langle a, s\rangle: a \in \mathrm{Field}(f(b))$ & $s \in A\text{-}\mathrm{Field}(f(b))$ & $s = ([1], b, n) * b_0 * ... * b_m * ([2])$ & $\mathrm{Sati}(\langle \mathrm{Field}(f(b)), \mathrm{Rel}_2(f(b))\rangle, n, (a, b_0, ..., b_m))]$, $F(a) = \mathrm{Fn}(f(b))(a)$ if $a \in \mathrm{Field}(f(b))$; if $a \in A\text{-}\mathrm{Field}(f(b))$, $a = ([1], b, n) * b_0 * ... * b_m * ([2])$, then $F(a) = k$ where $\mathrm{Defn}(f(b), (n, b_0, ..., b_m), k)$

5) $\mathrm{Lim}(x, a) \to f(a) = (F, \langle A, R_1\rangle, \langle A, R_2\rangle)$, where F, A, R_1, R_2 are the unions, over those b with $\langle b, a\rangle \in \mathrm{Rel}_1(x)$, of $\mathrm{Fn}(f(b))$, $\mathrm{Field}(f(b))$, $\mathrm{Rel}_1(f(b))$, $\mathrm{Rel}_2(f(b))$, respectively. $\mathrm{CHY}(x,f)$ reads "f is a coded hierarchy on x".

Definition 2.11. A limit ordinal λ is an ordinal > 0 with no immediate predecessor. Whenever we write λ we mean a limit ordinal.

Lemma 2.2. *There is a formula* $P_1(v_0, v_1, v_2)$ *and a sentence* Q_1 *such that for all* λ *we have* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), Q_1)$, *and for all transitive sets* A *such that* $\mathrm{Sat}((A, \epsilon), Q_1)$ *we have* : $\mathrm{Sat}((A, \epsilon), P_1, f) \equiv \mathrm{Sati}(f(0), f(1), f(2))$, *for all assignments* f *in* A, *and* $\mathrm{Sat}((A, \epsilon), (\forall v_0)(\exists x)(\forall y)(y \in x \equiv (y = \langle v_1, v_2\rangle \;\&\; P_1(v_0, v_1, v_2))))$.

Lemma 2.3. *There is a formula* $P_2(v_0, v_1, v_2)$ *and a sentence* Q_2 *such that for all* λ *we have* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), Q_2)$, *and for all transitive sets* A *such that* $\mathrm{Sat}((A, \epsilon), Q_2)$ *we have* $\mathrm{Sat}((A, \epsilon), P_2, f) \equiv\, <(f(0), f(1), f(2))$, *for all assignments* f *in* A, *and have also* $\mathrm{Sat}((A, \epsilon), (\forall v_0)(\exists x)(\forall y)(y \in x \equiv (y = \langle v_1, v_2\rangle \;\&\; P_2(v_0, v_1, v_2))))$.

Lemma 2.4. *There is a formula* $P_3(v_0, v_1, v_2)$ *and a sentence* Q_3 *such that for all* λ *we have* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), Q_3)$, *and for all transitive sets* A *with* $\mathrm{Sat}((A, \epsilon), Q_3)$ *we have*: $\mathrm{Sat}((A, \epsilon), P_3, f) \equiv \mathrm{Defn}(f(0), f(1), f(2))$, *for all assignments* f *in* A, *and* $\mathrm{Sat}((A, \epsilon), (\forall v_0)(\exists x)(\forall y)(y \in x \equiv (y = \langle v_1, v_2\rangle \;\&\; P_2(v_0, v_1, v_2))))$.

Lemma 2.5. *There is a formula* $P_4(v_0, v_1)$ *and a sentence* Q_4 *such that for all* λ *we have* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), Q_4)$ *and for all transitive sets* A *with* $\mathrm{Sat}((A, \epsilon), Q_4)$ *we have* $\mathrm{Sat}((A, \epsilon), P_4, f) \equiv \mathrm{CHY}(f(0), f(1))$, *for all assignments* f *in* A.

Definition 2.12. We write $\mathrm{WO}(x)$ for $\mathrm{LO}(x) \;\&\; (\forall y \subset \mathrm{Field}(x))(y \neq \phi \rightarrow (\exists a \in y)(\forall b \in y)(\langle b, a\rangle \notin \mathrm{Rel}_1(x)))$. We write $(A, R) \approx (B, S)$ for $(\exists f)(\mathrm{Iso}(f, (A, R), (B, S))$. If $\mathrm{LO}(x)$ and $a \in \mathrm{Field}(x)$, then we write x_a for $[b : \langle b, a\rangle \in \mathrm{Rel}_1(x)]$.

Lemma 2.6. *For all* $x \in V(\omega + \omega)$ *with* $\mathrm{WO}(x)$ *there is a unique* f *such that* $\mathrm{CHY}(x, f) \;\&\; f \in V(\omega + \omega)$. *Furthermore,*

1) *for all* $a \in \mathrm{Field}(x)$ *we have that* $(\exists\, ! g_a)(\mathrm{Iso}(g_a, (\mathrm{Field}(f(a)), \mathrm{Rel}_2(f(a))), (L^{\omega+\omega}(\beta), \epsilon)))$, *where* $(x_a, \mathrm{Rel}_1 \restriction x_a) \approx (\beta, \epsilon)$
2) *for all* $a \in \mathrm{Field}(x)$ *and for all* $b \in \mathrm{Field}(f(a))$ *we have that* $\mathrm{Fn}(f(a))(b) = \mu n(g_a(b) \in V(\omega + n))$
3) *for all* $a \in \mathrm{Field}(x)$ *we have* $\mathrm{WO}(\langle \mathrm{Field}(f(a)), \mathrm{Rel}_1(f(a))\rangle)$.

Lemma 2.7. *Let* LO(x), (Field(x), Rel$_1(x)) \approx (\alpha, \epsilon), x \in L^{\omega+\omega}(\beta)$. *Then* $(\exists f)$(CHY(x,f) & $f \in L^{\omega+\omega}(\beta + \alpha + \omega)$). *Furthermore for each* $a \in$ Field(x) *and* k *there is a* $g_a^k \in L^{\omega+\omega}(\beta + \alpha + \omega)$ *such that* Iso$(g_a^k$, (Field$(f(a)) \cap [b\colon \mathrm{Fn}(f(a))(b) \leq k]$, Rel$_2(f(a)) \upharpoonright$ Field$(f(a)) \cap [b\colon \mathrm{Fn}(f(a))(b) \leq k]$), $(L^{\omega+\omega}(\gamma) \cap V(\omega + k), \epsilon)$), *and* $L^{\omega+\omega}(\gamma) \cap V(\omega + k) \in L^{\omega+\omega}(\beta + \alpha + \omega)$, where $(\gamma, \epsilon) \approx (x_a, \mathrm{Rel}_1(x) \upharpoonright x_a)$.

Proof. Fix β. Then argue by induction on α. The basis case is trivial. Argue the limit case through use of Lemma 2.6, which gives unicity below the limit and which assures that the types needed are bounded below by $V(\omega + n_0)$, and by Lemma 2.5, which gives a first-order description below the limit. Argue the successor case by Lemma 2.4.

The g_a^k are developed by induction on k.

Definition 2.13. We say $L^{\omega+\omega}(\alpha)$ is pure just in case $\omega < \alpha$ and for all $\beta < \alpha$ there is an $x \in L^{\omega+\omega}(\alpha)$ with LO(x) and $(\beta, \epsilon) \approx$ (Field(x), Rel$_1(x)$), and for all $\beta < \alpha$ we have $L^{\omega+\omega}(\beta) \neq L^{\omega+\omega}(\beta + 1)$.

Lemma 2.8. *Let* $L^{\omega+\omega}(\alpha)$ *be pure,* $(\forall \beta < \alpha)(\beta + \beta < \alpha)$, Sat$((L^{\omega+\omega}(\alpha), \epsilon)$, WO$(v_0)$, $\lambda k(x))$. *Then either* WO(x) *or for all* $\beta < \alpha$ *there is an* $a \in$ Field(x) *with* $(\beta, \epsilon) \approx ([b\colon \langle b, a\rangle \in \mathrm{Rel}_1(x)], \mathrm{Rel}_1(x) \upharpoonright [b\colon \langle b, a\rangle \in \mathrm{Rel}_1(x)])$.

Proof. Let $x \in L^{\omega+\omega}(\alpha)$, Sat$((L^{\omega+\omega}(\alpha), \epsilon)$, WO$(v_0)$, $\lambda k(x))$, and assume $\beta < \alpha$, $\sim$ WO(x), and β is the order type of the maximal well-ordered initial segment of (Field(x), Rel$_1(x)$). We wish to obtain a contradiction. By purity, let $y \in L(\alpha)$ have LO(y) & $(\beta, \epsilon) \approx$ (Field(y), Rel$_1(y)$), and choose $\gamma < \alpha$ with $x, y \in L^{\omega+\omega}(\gamma)$. Then a straightforward inductive argument will reveal the existence of an isomorphism from the ordering defined by y onto the maximal well-ordered initial segment of the ordering defined by x, which lies in $L^{\omega+\omega}(\gamma + \beta + \omega)$. But then Sat$((L^{\omega+\omega}(\gamma + \beta + \omega), \epsilon)$, $\sim$ WO(v_0), $\lambda k(x))$, and hence Sat$((L^{\omega+\omega}(\alpha), \epsilon)$, $\sim$ WO(v_0), $\lambda k(x))$, which is a contradiction.

Lemma 2.9. *Let* $L^{\omega+\omega}(\alpha)$ *be pure,* $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha + 1)$, $(\forall \beta < \alpha)(\beta + \beta < \alpha)$, *and* Sat$((L^{\omega+\omega}(\alpha), \epsilon)$, WO$(v_0)$, $\lambda n(x))$. *Then*

$(\exists f \in L^{\omega+\omega}(\alpha))(\mathrm{CHY}(x,f))$ *if and only if* $(\exists \beta < \alpha)((\mathrm{Field}(x), \mathrm{Rel}_1(x)) \approx (\beta, \epsilon))$.

Proof. Suppose $\sim \mathrm{WO}(x)$. Then by Lemma 2.8 the maximal well-ordered initial segment of x must be at least α. Note that we can define $g_a^k \in L^{\omega+\omega}(\alpha)$ as in Lemma 2.7, for each $a \in \mathrm{Field}(x)$, even though $\sim \mathrm{WO}(x)$. In fact, let $x \in L^{\omega+\omega}(\beta)$. Then the g_a^k are in $L^{\omega+\omega}(\beta+\omega)$. Consider $S = [a \in \mathrm{Field}(x)\colon (\exists k)(\exists b \in \mathrm{Rng}(g_a^k))(\forall c)(\langle c, a\rangle \in \mathrm{Rel}_1(x) \rightarrow (\forall p)(b \notin \mathrm{Rng}(g_c^p)))]$. Then clearly S contains the initial segment of x of type α. Now, S is in $L^{\omega+\omega}(\beta+\omega+\omega)$. If α is the type of the maximal well-ordered initial segment of x then, since $\mathrm{WO}(x)$ holds in $L^{\omega+\omega}(\alpha)$, we must have $(\exists a \in S)$ (a is beyond the maximal well-ordered initial segment of x). If there is a well-ordered initial segment of x of type $\alpha+1$ then since $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha+1)$, we must again have $(\exists a \in S)$ (a is beyond the maximal well-ordered initial segment of x). Fixing this a, form $g_a^k \in L^{\omega+\omega}(\beta+\omega)$. Then by definition of S, we will have a $y \in L^{\omega+\omega}(\beta+\omega)$ which does not lie in $L^{\omega+\omega}(\alpha)$, which is a contradiction. The converse is by Lemma 2.7.

Lemma 2.10. *There is a sentence* Q_5 *such that*

1) *for all pure* $L^{\omega+\omega}(\alpha)$ *with* $(\forall \beta < \alpha)(\beta+\beta < \alpha)$ *and* $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha+1)$ *we have* $\mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), Q_5)$

2) *if* A *is transitive and* $\mathrm{Sat}((A, \epsilon), Q_5)$ *and for all assignments* f *in* A, $\mathrm{Sat}((A, \epsilon), (\exists v_1)(P_4(v_0, v_1)), f) \rightarrow \mathrm{WO}(f(0))$, *then* $(\exists \beta)(A = L^{\omega+\omega}(\beta) \;\&\; (\forall \gamma)(\gamma < \beta \rightarrow \gamma+\gamma < \beta))$.

Lemma 2.11. *There is a formula* $P_5(v_0, v_1)$ *such that for all pure* $L^{\omega+\omega}(\alpha)$ *with* $(\forall \beta < \alpha)(\beta+\beta < \alpha)$ *and* $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha+1)$ *we have* $\mathrm{WO}(\langle A, R\rangle)$, *where* $A = L^{\omega+\omega}(\alpha)$ *and* $R = [\langle a, b\rangle\colon \mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), P_5, \lambda n(a \text{ if } n = 0;\ b \text{ if } n \neq 0))]$.

Proof. We will just define the R. Take $R = [\langle g_y^k(a), g_y^p(b)\rangle\colon (\exists x)(\exists y)(\exists f)(\mathrm{WO}(x) \;\&\; f \in L^{\omega+\omega}(\alpha) \;\&\; \mathrm{CHY}(x,f) \;\&\; y \in \mathrm{Field}(x) \;\&\; a, b \in \mathrm{Field}(f(y)) \;\&\; \langle a, b\rangle \in \mathrm{Rel}_1(f(y)) \;\&\; \mathrm{Fn}(f(y)(a) = k \;\&\; \mathrm{Fn}(f(y))(b) = p)]$. Of course, g_y^k, g_y^p depend on x, f as in Lemma 2.7.

Lemma 2.12. *Let $L^{\omega+\omega}(\alpha)$ be pure,* $(\forall \beta < \alpha)(\beta + \beta < \alpha)$, $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha + 1)$, $x \in L^{\omega+\omega}(\alpha + 1)$, *where* $x = [a\colon \mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), F, \lambda n(a))]$. *Then there is a transitive set* $A \subset L^{\omega+\omega}(\alpha)$ *such that*

1) $\mathrm{Sat}((A, \epsilon), Q_4 \ \& \ Q_5)$

2) $\mathrm{TC}(x) \subset A \ \& \ x \in A$ *and* $(\forall a \in x)(\mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), F, \lambda n(a)) \equiv \mathrm{Sat}((A, \epsilon), F, \lambda n(a)))$

3) $\mathrm{Sat}((A, \epsilon), (\forall v_0)(\exists v_1)(P_4(v_0, v_1)) \to \mathrm{WO}(v_0)))$

4) *for all* $y \in A$ *we have* $[\mathrm{Sat}((A, \epsilon), \mathrm{WO}(v_0), \lambda n(y)) \equiv \mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), \mathrm{WO}(v_0), \lambda n(y))] \ \& \ [\mathrm{Sat}((A, \epsilon), (\exists f)(P_4(y, f)), \lambda n(y)) \equiv \mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), (\exists f)(P_4(y, f)), \lambda n(y))]$

5) *there is a partial function G which is from the cartesian product of* ω *with* $\mathrm{TC}(x)$ *onto A and a formula* $P_6(v_0, v_1, v_2, v_3)$ *such that* $G(a, b) = c$ *if and only if* $\mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), P_6(v_0, v_1, v_2, v_3), \lambda n(a \text{ if } n = 0; b \text{ if } n = 1; c \text{ if } n = 2; x \text{ if } n > 2))$.

Proof. Using Lemma 2.11, employ a standard closure of $\mathrm{TC}(x) \cup [x]$ under the Skolem functions for the finite number of formulae needed. This can be described in $L^{\omega+\omega}(\alpha)$ because of the bound in complexity of the formulae. Then perform the isomorphy onto the transitive set A. This isomorphism can also be described in $L^{\omega+\omega}(\alpha)$, and will result in a subset of $L^{\omega+\omega}(\alpha)$. This isomorphism will carry well-orderings into well-orderings.

Lemma 2.13. *Let $L^{\omega+\omega}(\alpha)$ be pure,* $(\forall \beta < \alpha)(\beta + \beta < \alpha)$. *Furthermore, suppose* $L^{\omega+\omega}(\alpha + 1) - L^{\omega+\omega}(\alpha) \neq \phi$. *Then there is a partial function G, and P_6 such that* 5) *in* Lemma 2.12 *holds and* $A = L^{\omega+\omega}(\alpha)$.

Proof. Choose A as in Lemma 2.12, using any $x \in L^{\omega+\omega}(\alpha + 1) - L^{\omega+\omega}(\alpha)$ of the form $[a\colon \mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), F, \lambda n(a))]$. Such an x can be found by Lemma 2.11. It suffices to prove that $A = L^{\omega+\omega}(\alpha)$. Note that by Lemma 2.10 we have $A = L^{\omega+\omega}(\beta)$ for some β. Note by 2) of Lemma 2.12 that $x \in L^{\omega+\omega}(\beta + 1)$. Hence $\alpha = \beta$.

Lemma 2.14. *Let $L^{\omega+\omega}(\alpha)$ be pure,* $(\forall \beta < \alpha)(\beta + \beta < \alpha)$, $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha + 1)$. *Then* $L^{\omega+\omega}(\alpha + 1)$ *is pure.*

Proof. We use the G, P_6 of Lemma 2.13, for some $x \in L^{\omega+\omega}(\alpha + 1) - L^{\omega+\omega}(\alpha)$, and P_5 of Lemma 2.11. It suffices to produce a linear ordering $y \in L^{\omega+\omega}(\alpha + 1)$ with $(\alpha, \epsilon) \approx (\mathrm{Field}(y), \mathrm{Rel}_1(y))$. Take $y = \langle A, R\rangle$, where $A = \mathrm{Dom}(G)$, $R = [\langle(x_1, y_1), (x_2, y_2)\rangle : (x_1, y_1), (x_2, y_2) \in A$ & $\mathrm{Sat}((L^{\omega+\omega}(\alpha), \epsilon), P_5(v_0, v_1), \lambda n(G(x_1, y_1)$ if $n = 0$; $G(x_2, y_2)$ if $n > 0))]$. If this $\langle A, R\rangle$ is longer than (α, ϵ) then take the appropriate initial segment; this $\langle A, R\rangle$ must be a well-ordering.

Lemma 2.15. *If $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha + 1)$ and $\omega < \alpha$ then $L^{\omega+\omega}(\alpha + 1)$ and $L^{\omega+\omega}(\alpha)$ are pure.*

Proof. Straightforward from Lemma 2.14 by transfinite induction.

Lemma 2.16. *Suppose $L^{\omega+\omega}(\alpha) \neq L^{\omega+\omega}(\alpha + 1)$. Then $L^{\omega+\omega}(\alpha \times \omega) \neq L^{\omega+\omega}((\alpha \times \omega) + 1)$.*

Proof. Suppose $L^{\omega+\omega}(\alpha \times \omega) = L^{\omega+\omega}((\alpha \times \omega) + 1)$. By Lemma 2.15, there is a well-ordering in $L^{\omega+\omega}(\alpha + 1)$ of type α. Hence there is a well-ordering $y \in L^{\omega+\omega}(\alpha \times \omega)$ of type $(\alpha \times \omega) + 1$. Since $(L^{\omega+\omega}(\alpha + \omega), \epsilon)$ satisfies Z, there must be an $f \in L^{\omega+\omega}(\alpha \times \omega)$ with $\mathrm{CHY}(y, f)$. Hence $\mathrm{TC}(f) \in L^{\omega+\omega}(\alpha \times \omega)$ since $(L^{\omega+\omega}(\alpha \times \omega), \epsilon)$ satisfies Z. In addition $(L^{\omega+\omega}(\alpha \times \omega), \epsilon)$ must satisfy that every set has smaller cardinality than $\mathrm{TC}(f)$. But $(L^{\omega+\omega}(\alpha \times \omega), \epsilon)$ satisfies the power set axiom and Cantor's Theorem, and so we have a contradiction.

Lemma 2.17. *Let $y \subset \omega$, $y \in L^{\omega+\omega}$. Then there is a λ such that $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda + 1)$ and $y \in L^{\omega+\omega}(\lambda)$ and a formula $P_7(v_0, v_1, v_2)$ such that* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), (\forall v_1)(\exists ! v_0)(v_0 \in \omega \;\&\; P_7(v_0, v_1, v_2)), \lambda n(z))$, *for some $z \in L^{\omega+\omega}(\lambda)$.*

Proof. Choose α least such that $y \in L^{\omega+\omega}(\alpha)$, $\omega < \alpha$. Then $\alpha = \beta + 1$. Set $\lambda = \beta \times \omega$. Note that by Lemma 2.16, $L^{\omega+\omega}(\lambda)$ satisfies the hypotheses of Lemma 2.12, using y for x. Using Lemma 2.10, the resulting A must be $L^{\omega+\omega}(\lambda)$. Using the P_6 of Lemma 2.12 one easily constructs the desired P_7 since $\mathrm{TC}(y) = \omega$, or y is finite.

Lemma 2.18. *Let $y \subseteq \omega, y \in L^{\omega+\omega}$. Then there is a λ such that $L^{\omega+\omega}(\lambda) \neq L^{\omega+\omega}(\lambda + 1)$ and $y \in L^{\omega+\omega}(\lambda)$ and a formula $P_8(v_0, v_1)$ such that* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), (\forall v_1)(\exists ! v_0)(v_0 \in \omega \;\&\; P_8(v_0, v_1)))$.

Proof. Take λ, P, as in Lemma 2.17. Note that $L^{\omega+\omega}(\lambda)$ satisfies the hypotheses of Lemma 2.11. Using the P_5 of Lemma 2.11, take $P_8(v_0, v_1)$ to be $(\exists v_2)((\forall v_1)(\exists ! v_0)(v_0 \in \omega \;\&\; P_7(v_0, v_1, v_2))\;\&$ $(\forall v_4)(P_5(v_4, v_1) \rightarrow \sim (\forall v_1)(\exists ! v_0)(v_0 \in \omega \;\&\; P_7(v_0, v_1, v_4)))\;\&$ $P_7(v_0, v_1, v_2))$.

Lemma 2.19. *Suppose* $P_9(v_0, v_1)$ *is a formula such that* $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), (\forall v_1)(\exists ! v_0)(v_0 \in \omega \;\&\; P_9(v_0, v_1)))$. *Then* $\mathrm{Th}((L^{\omega+\omega}(\lambda), \epsilon)) \in L^{\omega+\omega}(\lambda + 2)$.

Proof. Note that there must be an $(\omega, R) \approx (L^{\omega+\omega}(\lambda), \epsilon)$ such that $R \in L^{\omega+\omega}(\lambda + 1)$. In addition, every set of natural numbers arithmetical in R will be in $L^{\omega+\omega}(\lambda + 1)$. Hence straightforwardly, $\mathrm{Th}((L^{\omega+\omega}(\lambda), \epsilon)) \in L^{\omega+\omega}(\lambda + 2)$.

Combining Lemmas 2.17 and 2.18, we immediately have:

Theorem 2. *There are formulae* $\varphi_1(v_0, v_1)$, $\varphi_2(v_0, v_1)$, *and* $\varphi_3(v_0, v_1)$ *in* LST *with only the free variables shown such that for each* $x \subset \omega$, $x \in L^{\omega+\omega}$ *there is a limit ordinal* λ *such that*

1) $x \in L^{\omega+\omega}(\lambda)$

2) $(\forall y \in L^{\omega+\omega}(\lambda))(\exists n)(\mathrm{Def}((L^{\omega+\omega}(\alpha), \epsilon), n, y))$

3) $\mathrm{Th}((L^{\omega+\omega}(\lambda), \epsilon)) \in L^{\omega+\omega}(\lambda + 2)$

4) $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), \varphi_1(v_0, v_1), f)$ *if and only if* $(\mu\beta)(f(0) \in L^{\omega+\omega}(\beta)) < (\mu\beta)(f(1) \in L^{\omega+\omega}(\beta))$

5) $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), \varphi_2(v_0, v_1), f)$ *if and only if* $(\mu\beta)(f(0) \in L^{\omega+\omega}(\beta)) = (\mu\beta)(f(1) \in L^{\omega+\omega}(\beta))$

6) $\mathrm{Sat}((L^{\omega+\omega}(\lambda), \epsilon), \varphi_3(v_0, v_1), f)$ *if and only if* $f(1) = (\mu n \in \omega)(f(0) \in V(\omega + n))$.

Section 3

In this Section we discuss various refinements of Theorem 1.6 and its Corollary.

We assume familiarity with the hierarchy of numerical formulae with one function parameter ranging over ω^ω.

Definition 3.1. A *towered * structure* is a structure (A, R) such that clauses 1) – 10) of Definition 1.21 hold and in addition, for each Π^0_1 predicate $Q(n, f)$ we have $(\exists n)(n \in A \ \& \sim Q(n, J^\omega(\mathrm{Ch}(\mathrm{Th}(A, R)))))) \rightarrow$ $(\exists n)(n \in A \ \& \sim Q(n, J^\omega(\mathrm{Ch}(\mathrm{Th}(A, R)))) \ \& \ (\forall m)(m < n \rightarrow$ $Q(m, J^\omega(\mathrm{Ch}(\mathrm{Th}((A, R))))))$. Define $\mathcal{S}^* = [\mathrm{Ch}(\mathrm{Th}((A, R))) : (A, R)$ is a towered* structure$]$.

Lemma 3.1.1. *$L^{\omega+\omega}$ satisfies that $\mathcal{S}^* \cap L^{\omega+\omega}$ is an element of $B_{\omega+3}$ with recursive code.*

Proof. Routine counting of quantifiers and comparison with the Borel hierarchy.

Lemma 3.1.2. *Suppose (A, R), (B, S) are towered* structures such that* $\mathrm{Ch}(\mathrm{Th}((A, R))) \leq_T J(\mathrm{Ch}(\mathrm{Th}((B, S))))$ *and* $\mathrm{Ch}(\mathrm{Th}((B, S))) \leq_T$ $J(\mathrm{Ch}(\mathrm{Th}((A, R))))$. *Then either* $(\exists f)(\mathrm{Iso}(f, (A, R), (B, S)))$ *or* $(\exists f)(\mathrm{Inj}(f, (A, R)(B, S))$ *and* $(\exists x \in B)(\mathrm{Rng}(f) = [y \in B : y < x]$, *where $<$ is as in (B, S) as in* Definition 3.1 (*which refers back to* Definition 1.21))), *or* $(\exists f)(\mathrm{Inj}(f, (B, S), (A, R))$ *and* $(\exists x \in A)(\mathrm{Rng}(f) =$ $[y \in A : y < x]$, *where $<$ is as in (A, R) in* Definition 1.21)).

Proof. This is the analogue to Lemma 1.5.1, and is proved exactly the same way, moticing that, for instance, the K of that proof is defined by a Π^0_1 predicate $Q(n, J^\omega(\mathrm{Ch}((A, R))))$.

Arguing as in Section 1, we have

Theorem 3.1. *$L^{\omega+\omega}$ satisfies "there exists an element $Y \in B_{\omega+3}$, with recursive code, such that $\sim D(Y)$". Hence the assertion in quotes is consistent with* **Z**.

Proof. Consider the game given by $Y \subset 2^\omega$, where $Y = [f \in 2^\omega : \lambda n(f(2n)) \in \mathcal{S}$ & $\lambda n(f(2n+1)) \leq_T \lambda n(f(2n))]$.

Definition 3.2. Define $L^\alpha(0) = V(\omega)$, $L^\alpha(\beta+1) = \mathrm{FODO}((L^\alpha(\beta), \epsilon)) \cap V(\alpha)$, $L^\alpha(\lambda) = \bigcup_{\beta<\lambda} L^\alpha(\beta)$, where λ is a limit ordinal. Define $L^\alpha = [x : (\exists \beta)(x \in L^\alpha(\beta))]$.

For the moment, let us concentrate on the case $\alpha = \omega + 1$.

Now we cannot directly speak of Borel subsets of 2^ω and determinateness within $L^{\omega+1}$. What we do is to consider formulae $P(v_0)$ and associate the sentence P^* which naturally formalizes the assertion that $D([f : f \in 2^\omega$ & $P(f)])$. In particular we shall construct a numerical formula $P(f)$ which is in prenex form and has 5 quantifiers (numerical, of course) such that the corresponding sentence P^* fails in $(L^{\omega+1}, \epsilon)$. Thus we can say that, in the appropriate sense, $L^{\omega+1}$ satisfies that "there is a $Y \in B_5$ with recursive code such that $\sim D(Y)$". However, with L^α, where $\omega + 1 < \alpha$, no such devices of expression are needed.

Lemma 3.2.1. *There are formulae $\psi_1(v_0, v_1)$, and $\psi_2(v_0, v_1)$ in* LST *with only the free variables shown such that for each $x \subset \omega$, $x \in L^{\omega+1}$, there is a limit ordinal λ such that*

1) $x \in L^{\omega+1}(\lambda)$

2) $(\forall y \in L^{\omega+1}(\lambda))(\exists n)(\mathrm{Def}((L^{\omega+1}(\alpha), \epsilon), n, y))$

3) $\mathrm{Th}((L^{\omega+1}(\lambda), \epsilon)) \in L^{\omega+1}(\lambda + 2)$

4) $\mathrm{Sat}((L^{\omega+1}(\lambda), \epsilon), \varphi_1(v_0, v_1), f)$ *if and only if* $(\mu\beta)(f(0) \in L^{\omega+1}(\beta)) < (\mu\beta)(f(1) \in L^{\omega+1}(\beta))$

5) $\mathrm{Sat}((L^{\omega+1}(\lambda), \epsilon), \varphi_2((v_0, v_1), f)$ *if and only if* $(\mu\beta)(f(0) \in L^{\omega+1}(\beta)) = (\mu\beta)(f(1) \in L^{\omega+1}(\beta))$

6) $(\forall x \in L^{\omega+1}(\lambda))(x \subset V(\omega))$.

Proof. The proof is like the proof of Theorem 2. One uses standard pairing and inverse pairing functions on $V(\omega)$ to code everything as a subset of $V(\omega)$.

In the following, we use φ_1, and φ_2 as in the statement of Theorem 3.2.1.

Definition 3.3. A *towered⁻ structure* is a structure (A, R) such that

1) $A \subset \omega$ and the relation $x \sim y \equiv \mathrm{Sat}((A,R), \varphi_2(v_0, v_1), \lambda n(x \text{ if } n = 0; y \text{ if } n \neq 0))$ is an equivalence relation on A

2) the relation $x < y \equiv \mathrm{Sat}((A,R), \varphi_1(v_0, v_1), \lambda n(x \text{ if } n = 0; y \text{ if } n \neq 0))$ has that $(\forall x, y \in A)((x < y \mathbin{\&} \sim y < x) \vee (y < x \mathbin{\&} \sim x < y) \vee (x \sim y \mathbin{\&} \sim x < y \mathbin{\&} \sim y < x))$ and $(\forall x, y, z \in A)(((x \sim z \mathbin{\&} x < y) \to z < y) \mathbin{\&} ((x \sim z \mathbin{\&} y < x) \to y < z))$, and $<$ has no maximal element

3) $A^0 = [\, i : i \in A \mathbin{\&} (\forall j)(\sim j < i)\,]$, $R^0 = R \restriction A^0$

4) we have $(\forall x \in A)(\forall y)(R(y, x) \to y \in A^0)$

5) suppose $x \in A$. Then $\mathrm{FODO}(([\, i : i < x\,], R \restriction [\, i : i < x\,])) = [\, z \subset [\, i : i < x\,] : (\exists j)(j < x \vee j \sim x) \mathbin{\&} z = [\, k : R(k, j)\,]\,)]$

6) (A, R) satisfies the axiom of extensionality

7) $(\forall i \in A - A^0)(\mathrm{Def}((A,R), i, 2i))$

8) for some k we have that for all $x \in A$ there exists a prenex formula φ with only free variable v_0 and with only k alterations of quantifiers such that $\mathrm{Sat}((A,R), (\exists\, ! v_0)(\varphi) \mathbin{\&} \varphi, \lambda n(x))$

9) for each Π^0_3 predicate $Q(n, f)$ we have $(\exists n)(n \in A \mathbin{\&} \sim Q(n, \mathrm{Ch}(\mathrm{Th}((A,R))))) \to (\exists n)(n \in A \mathbin{\&} \sim Q(n, \mathrm{Ch}(\mathrm{Th}((A,R)))) \mathbin{\&} (\forall m)(m < n \to Q(m, \mathrm{Ch}(\mathrm{Th}((A,R))))))$. Define $\mathcal{S}^- = [\, \mathrm{Ch}(\mathrm{Th}((A,R)) \text{ is a towered}^- \text{ structure}\,]$.

Lemma 3.2.2. *$[\, f \in 2^\omega : f$ codes $\mathrm{Th}((A,R))$ for some towered⁻ structure $(A,R)\,]$ is in B_5 with recursive code. In other words $\mathcal{S}^- = [\, f \in 2^\omega : f = \mathrm{Ch}(\mathrm{Th}((A,R)))$ for some towered⁻ structure $(A,R)\,]$ is in B_5 with recursive code.*

Proof. We define $f \in \mathcal{S} \equiv P_1(f)$ & $P_2(f)$ & $P_3(f)$ & $P_4(f)$ & $P_5(f)$ & $P_6(f)$ & $P_7(f)$ & $P_8(f)$ & $P_9(f)$, where $P_1(f)$ is '$(\forall x)(\varphi_2(x,x))$ & $(\forall x)(\forall y)(\varphi_2(x,y) \equiv \varphi_2(y,x))$ & $(\forall x)(\forall z)((\varphi_2(x,y)$ & $\varphi_2(y,z)) \to \varphi_2(x,z))$' $\in [i : f(i) = 1]$; $P_2(f)$ is '$(\forall x)(\forall y)((\varphi_1(x,y)$ & $\sim \varphi_1(y,x)) \vee (\varphi_1(y,x)$ & $\sim \varphi_1(x,y)) \vee (\varphi_2(x,y)$ & $\sim \varphi_1(x,y)$ & $\sim \varphi_1(y,x)$ & $(\forall x)(\forall y)(\forall z)(((\varphi_2(x,z)$ & $\varphi_1(x,y)) \to \varphi_1(z,y))$ & $((\varphi_2(x,z)$ & (y,z) & $(y,x) \to \varphi_1(y,z)))$ & $\sim (\exists x)(\forall y)(\varphi_1(y,x) \vee \varphi_2(x,y))$' $\in [i : f(i) = 1]$; $P_3(f)$ is '$(\forall x)(x \in V(\omega) \equiv (\forall y)(\varphi_1(x,y) \vee \varphi_2(x,y)))$ & $(\exists x)(x = V(\omega))$' $\in [i : f(i) = 1]$; $P_4(f)$ is '$(\forall x)(\forall y)(y \in x \to y \in V(\omega))$' $\in [i : f(i) = 1]$; $P_6(f)$ is '$(\forall x)(\forall y)(\forall z)(z \in x \equiv z \in y) \to x = y$)' $\in [i : f(i) = 1]$; $P_7(f)$ is "for each sentence $\exists v_0(\varphi)$ such that f('$(\exists ! v_0)(\varphi)$') = 1 we have that for some formula ψ with only the free variable v_0, '$\exists v_0(\varphi$ & $\psi)$ & $(\exists ! v_0)(\psi)$' $\in [i : f(i) = 1]$" & $[F$: 'F' $\in [i : f(i) = 1]$ is a consistent set of sentences in LST"; $P_5(f)$ is "for each formula φ with only the free variable v_1 such that f('$(\exists ! v_1)(\psi)$') = 1 we have that '$(\exists v_0)(\exists v_1)(\varphi(v_0)$ & $\psi(v_1)$ & $(\varphi_1(v_1, v_0) \vee \varphi_2(v_1, v_0)))$' $\in [i : f(i) = 1]$ if and only if there exists a formula ψ_1 with free variables $v_2, \ldots, v_k, v_{k+1}$ such that '$(\exists v_0)(\exists v_1)(\exists v_2) \ldots (\exists v_k)(\forall v_{k+1})(\varphi(v_0)$ & $\psi(v_1)$ & $\varphi_1(v_2, v_0)$ & ... & $\varphi_1(v_k, v_0)$ & $(v_{k+1} \in v_1 \equiv (\varphi_1(v_{k+1}, v_0)$ & $\psi^*)))$' $\in [i : f(i) = 1]$, where ψ^* is the result of relativizing the quantifiers in ψ to those y with $\varphi_1(y, v_0)$"; $P_8(f)$ is "for some k we have that for all formulae P with only the free variable v_0 such that f('$(\exists ! v_0)(P)$') = 1 there is a formula ψ with free variable only v_0 and which is prenex and only has k alterations of quantifiers such that f('$(\exists v_0)(P$ & $\psi)$') = 1; $P_9(f)$ is $(\forall k)[(\exists n)(A(n)$ & $\sim Q(k,n,f)) \to (\exists n)(A(n)$ & $\sim Q(k,n,f)$ & $(\forall m)(B(m,n) \to Q(k,m,f)))]$, where Q is a complete Π^0_3 predicate, $A(n)$ is "n is odd or (n is even & $|n/2|$ is P with only free variable v_0 and f('$(\exists ! v_0)(P)$') = 1 & $(\forall m < n/2)$ $(\sim(|m|$ has only free variable v_0 and is, say, $Q(v_0)$, and f('$(\forall v_0)(Q(v_0) \equiv P(v_0))$ & $(\exists ! v_0)(Q)$') = 1)))", $B(m,n)$ is "$A(m)$ & $A(n)$ & $|m/2|$ is P & $|n/2|$ is Q & '$(\exists v_0)(\exists v_1)(P(v_0)$ & $Q(v_1)$ & $\varphi_1(v_0, v_1))$' $\in [i : f(i) = 1]$".

To show that this is the desired conjunction, we must show that, for the corresponding (A, R) to f, as in the proof of Lemma 1.3.1, that (A, R) is a towered$^-$ structure. To do this, one proves by induction on the complexity of a formula F that for all assignments g in (A, R), we have $\mathrm{Sat}((A,R), F, g) \equiv$ '$(\exists v_{i_1})(\exists v_{i_2}) \ldots (\exists v_{i_j})(G_{i_1}(v_{i_1})$ & ... &

$G_{i_j}(v_{i_j})$ & F)' $\in [i: f(i) = 1]$, where $G_{i_k}(v_0)$ is $|g(i_k)$ if v_0 is even; $G_{i_k}(v_0)$ is the *canonical* definition of $g(i_k)$ in (A^0, R^0) if $g(i_k)$ is odd; and $v_{i_1}, \ldots, v_{i_j}$ is a complete list of the free variables in F.

Theorem 3.2. $L^{\omega+1}$ *satisfies* "there exists an element $Y \in B_5$, with recursive code, such that $\sim D(Y)$".

Proof. Proceed as in Section 1. The predicate defining the set K of the proof of Lemma 1.5.1 is replaced by a Π^0_3 predicate since one needs to consider $P(n, i, j)$ only for $n = 0, 1$.

We can state an independence result corresponding to Theorem 3.2.

Definition 3.3. We let Z(2) be

1) $(\exists x)(x = V(\omega))$

2) $(\forall y)(y \subset V(\omega))$

3) $(\forall z)(z \in x \equiv z \in y) \rightarrow x = y$

4) $x \neq \phi \rightarrow (\exists y)(y \in x \;\&\; (\forall z)(z \in x \rightarrow z \notin y))$

5) $(\exists y)(\forall z)(z \in y \equiv (\exists w)(z \in w \;\&\; w \in x))$

6) $(\forall x)(\exists y)(\forall z)(z \in y \equiv (F \;\&\; z \in x))$, where F is a formula not containing y free

7) $(\forall x)(\exists y)(P(x, y)) \rightarrow (\forall x)(\exists f)([n: (\exists k)(f(0, k) = n)] = x \;\&$ $(\forall m)(P([n: (\exists k)(f(m, k) = n)], [n: (\exists k)(f(m+1, k) = n)])))$, where P is a formula which does not mention f free.

It is well known that $L^{\omega+1}$ satisfies Z(2). The dependent choices principle 7) can be seen to hold using the definable well-ordering of $L^{\omega+1}$. For a discussion of the ramified analytical hierarchy, $L^{\omega+1}$, see Boyd, Hensel, and Putnam [1].

Theorem 3.3. Z(2) *is consistent with* "there exists an element $Y \in B_5$, with recursive code, such that $\sim D(Y)$".

Extensions of these independence results can be obtained for certain stronger theories than **Z**. Rather than give a systematic formulation, we given an example of what can be done.

Definition 3.4. We let $\mathbf{Z}(L)$ be **Z** together with $(\exists x)(\exists \alpha)(\alpha = \Omega^L$ & $x = V(\alpha)$, where Ω^L is the first constructible uncountable ordinal). Naturally, we assume some standard formulation of the constructible hierarchy appropriate to **Z**.

Theorem 3.4. $\mathbf{Z}(L)$ *is consistent with* "$(\exists \alpha)(\sim D(\alpha))$".

Proof. Using the Skolem-Lowenheim theorem, choose β countable such that L^β possesses a well-ordering of type β and no well-ordering of ω of type β and a well-ordering on ω of type any $\alpha < \beta$. That is, β is countable and is Ω in L^β. It is not known whether $(\exists \alpha)(\sim D(\alpha))$ holds in L^β. But instead pass to the generic extension of L^β obtained by adjoining a generic well-ordering y of ω of type β. In this extension we have $\mathbf{Z}(L)$. In addition, we can carry out the independence techniques of this paper using $L^\beta(y)$ instead of L^β, where $L^\beta(y)$ is the same as L^β except that $L^\beta(0) = V(\omega) \cup [y]$. The resulting Borel set will have code recursive in y.

We can turn Theorems 3.1 – 3.4 into proofs of consistency from deteminateness. We make use of the usual way of formalizing the constructible hierarchy within set theories, such as the ones being considered, based on sets of restricted type. This formalization is done by means of the predicate $\mathrm{CHY}^+(x,f)$, which is the same as the $\mathrm{CHY}(x,f)$ of Section 2 except that no type restrictions are placed in the successor case. In addition we shall use $\mathrm{CODE}(f,y)$, $\mathrm{CODE}^+(f,y)$ to mean, respectively, that $(\exists x)(\mathrm{CHY}(x,f)$ & y is coded by $f)$, $(\exists x)(\mathrm{CHY}^+(x,f)$ & y is coded by $f)$. Thus, $L^{\omega+\omega}$ was $[y : (\exists f)(\mathrm{CODE}(f,y))]$, and $L = [y : (\exists f)(\mathrm{CODE}^+(f,y))]$.

Lemma 3.5.1. *The following can be proved respectively, in* $\mathbf{Z}(2)$ *and in* **Z** *without the power set axiom:* $(\mathrm{CHY}(x,f)$ & $\mathrm{CODE}(f,y)) \to (\exists g)(\mathrm{CHY}^+(x,g)$ & $\mathrm{CODE}^+(g,y))$, $(\mathrm{CHY}(x,f)$ & $\mathrm{CODE}(f,y)$ & $f \in V(\omega+\omega)) \to (\exists g)(\mathrm{CHY}^+(x,g)$ & $\mathrm{CODE}^+(g,y)))$.

Lemma 3.5.2. *Shoenfield's absoluteness theorem*, (see Shoenfield [7]) *is provable in* **Z** *without the power set axiom.*

Theorem 3.5. **Z** *without the power set axiom* + $D(\omega + 3)$ *proves the consistency of* **Z**.

Proof. The assertion that $D(Y)$ holds for all $Y \in B_{\omega+3}$ with recursive code is Σ^1_2 in the analytical hierarchy, and is therefore subject to Shoenfield's theorem. Hence in **Z** without power set + $D(\omega + 3)$ we can prove that every $Y \in B_{\omega+3}$ with recursive code has a constructible winning strategy. Now we can formalize the proof of Theorem 3.1, so that we obtain within **Z** without power set, that $(\exists x)(\exists f)(\exists y)(\mathrm{CHY}^+(x,f)$ & $\mathrm{CODE}^+(f,y)$ & $(\forall g)(\sim \mathrm{CODE}(g,y)))$. Fix such a well-ordering x. Then, arguing in **Z** without power set, we have that all of $L^{\omega+\omega}$ is coded in the f with $\mathrm{CHY}^+(x,f)$. Using this f, we can straightforwardly give a model of **Z** and hence derive the consistency of **Z**.

We may similarly obtain

Theorem 3.6. **Z**(2) + $D(5)$ *proves the consistency of* **Z**(2).

The level of the Borel hierarchy jumps up by one if we want to consider sets of Turing degree.

Theorem 3.7. **Z** *without the power set axiom* + "every Turing set $Y \in B_{\omega+4}$ either contains or is disjoint from a Turing cone" *proves the consistency of* **Z**. **Z**(2) + "every Turing set $Y \in B_6$ contains or is disjoint from a Turing cone" *proves the consistency of* **Z**(2).

In fact Theorems 3.5, 3.6, and 3.7 can be sharpened in the following way: our proofs actually produce specific subsets Y of 2^ω, and so the respective hypotheses may be weakened in the respective theorems by using the respective Y instead of using all Y at the respective level of the Borel hierarchy.

Section 4

Here we wish to mention some possibilities for future research.

What is the formal relation between the questions about the Borel hierarchy studied here and the commonly considered axioms and hypotheses in set theory? At one extreme, as far as we know, even D(5) may not be derivable from Morse-Kelley set theory together with the 2nd-order reflection principle *. At another extreme, it may be that Z together with $(\forall x)$ (if x is a well-ordering on ω then the cumulative hierarchy exists up through x) is sufficient to derive $(\forall \alpha)(D(\alpha))$.

What is the relation between Borel determinateness, (written $(\forall \alpha)(D(\alpha))$), and "every Borel set of Turing degrees contains or is disjoint from a Turing cone?"

It is easily seen that the following can be derived from Borel determinateness: for every Borel $Y \subset 2^\omega \times 2^\omega$ either Y can be uniformized by a Borel function or $[(f,g): (g,f) \notin Y]$ can be uniformized by a Borel function. A Borel function is just a subset, X, of $2^\omega \times 2^\omega$ such that $(\forall f \in 2^\omega)(\exists !g \in 2^\omega)((f,g) \in X)$. A Borel function X uniformizes Y just in case $(\forall f \in 2^\omega)(\exists !g)((f,g) \in X \; \& \; (f,g) \in Y)$. In fact, a Y can be found which is continuous. So we have

I. to every Borel set $Y \subset 2^\omega \times 2^\omega$ there is a Borel function F which either uniformizes Y or uniformizes $[(f,g): (g,f) \notin Y]$

II. there is an ordinal $\alpha < \Omega$ such that to every Borel set $Y \subset 2^\omega \times 2^\omega$ there is a Borel function $F \in B_\alpha$ which either uniformizes Y or uniformizes $[(f,g): (g,f) \notin Y]$

III. to every Borel set $Y \subset 2^\omega \times 2^\omega$ there is a continuous function F which either uniformizes Y or uniformizes $[(f,g): (g,f) \notin Y]$

IV. Borel determinateness.

What is the relation between I–IV? Of course we have IV $\rightarrow$ III $\rightarrow$

* D.A.Martin has recently derived D(4) from MK + 2nd-order reflector principle (unpublished).

II → I. It seems reasonable to hope for a mathematician's proof of I, but beware of II! Our results can be seen to carry over to obtain the independence of II from $Z(L)$ using α-degrees, $\alpha < \Omega$.

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INTRODUCTION TO $\prod_2^1$-LOGIC

The increasing success of set-theory as a framework for mathematics has been responsible for the fact that a certain number of simple finitistic facts about the mathematical universe have been neglected. Worse, the set-theoretic conception of mathematics has pervaded proof-theory: but if set-theory is a very successful technical tool, it is simply of no interest from the viewpoint of foundations, since it takes as primitive what one wants to analyze: the infinite, and especially the most mysterious of all infinite sets, namely the set $\mathbb{N}$ of integers. Hence, it is more reasonable to take our intuition from traditional proof-theory, and of course to use set-theoretic techniques, when necessary.

$\prod_2^1$-logic starts with a conception of mathematics which belongs to the family styled as "potential infinity standpoint". When doing $\prod_2^1$-logic, we always deal with ideal limits of some work in progress: this is more precisely expressed by preservation of direct limits. In other terms, $\prod_2^1$-logic is mostly algebraic. Here lies the *subtlety* of the theory.

The viewpoint of potential infinity has not been taken very seriously outside some philosophical circles; this is because the principles considered are always so terribly weak that one cannot use the corresponding systems. The other feature of $\prod_2^1$-logic is *strength*, which comes from its logical complexity, which is greater than the complexities of the existing logics, namely $\sum_1^0$ and $\prod_1^1$. For instance, the principle of induction on dilators states something stronger than usual transfinite induction, but the "finitary control" is not lost....

The general aim of $\prod_2^1$-logic is to rebuild some parts of mathematics (at least in mathematical logic) by making more explicit the finitary contents of such and such construction of the actual infinite kind. The general possibility of such a study is due to the strength of the concepts. The philosophical interest is not dubious. The mathematical interest could simply lie in the conceptual simplification, giving thus a new approach to old things....

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1. SYSTEMS OF DENOTATIONS

Systems of *ordinal notations* are familiar from proof-theoretical practice; the idea is to represent a given ordinal by means of well-formed expressions of a recursive language. The study of general systems of notations (i.e., not the construction of specific ones, connected with particular applications) is not very interesting; in fact a system of ordinal notations is nothing but a recursive well-order, and the theory of these objects is well-known. This shows that the idea of ordinal notation does not lead to any original concept.

But let's turn our attention towards practice: when we construct systems of ordinal notations, we usually use *normal form theorems*. This enables us to represent ordinals by means of smaller ones: in order to find a notation for x, we use the normal form theorem, and in case the parameters of this normal form are all $<x$, we can replace them by their already constructed notations, and so we get a notation for x....

Many normal form theorems are known; let us mention

(i) if $z < x^2$, then z can be uniquely written as $x.x_0 + x_1$, with $x_0, x_1 < x$

(ii) any ordinal $z < 10^x$ can be uniquely written as

$$10^{x_0}.a_0 + \cdots + 10^{x_{n-1}}.a_{n-1}$$

with $x_{n-1} < \cdots < x_0 < x$ and $1 \leq a_0, \ldots, a_{n-1} < 10$. This is the famous Cantor Normal Form Theorem, in base 10.

We select as object of study the normal form theorems. We say that the expressions occurring as normal forms are *denotations*, and normal form theorems will therefore be identified with *systems of denotations*; we only have to find out which specific properties of such systems are of interest for a general theory.

A basic distinction is the distinction *skeleton/parameters*, i.e., between the static and dynamic parts of a denotation. The skeleton is a fixed configuration, in which we have places for ordinal parameters, for instance $10^*.7 + 10^*.4 + 10^*.9$; the ordinal parameters are supposed to take rather "arbitrary" values. What is the exact amount of freedom the parameters have? Let's go back to our examples:

(i) Here we must distinguish between two kinds of parameters, on one hand x and on the other hand x_0, x_1; x is needed to determine with respect to which x^2, z is analyzed, and x_0, x_1 are uniquely determined once z and x are fixed. x_0, x_1, x can take arbitrary values provided $x_0, x_1 < x$.

(ii) Here the denotation does not depend on the choice of an x such that $z < 10^x$; the parameters $x_0, \ldots, x_{n-1}$ are arbitrary, but must form a strictly decreasing sequence.

We can without loss of generality, assume that our skeleton is of the form $(C; *, \ldots, *; *)$: C followed by $n+1$ places; the ultimate place (after ";") is to be filled with an x corresponding to the bound x^2 in example (i) and 10^x in example (ii): in this last example, we are therefore adding this x which was not actually needed. The other places must be filled by ordinals $< x$, and by looking at example (ii) we see that the only reasonable requirement is the following:

If $(C; x_0, \ldots, x_{n-1}; x)$ is a denotation, and the sequence $y_0, \ldots, y_{n-1}, y$ is isomorphic to $x_0, \ldots, x_{n-1}, x$ then $(C; y_0, \ldots, y_{n-1}; y)$ is a denotation. From this we see that we can, without loss of generality, assume that the order type of the sequence of parameters is always the same (otherwise, divide your given denotation into as many "sub-denotations" as we have possible order types for the parameters). Finally, once the order type is fixed, there is no need for having the same parameters occurring twice, and so we can assume that the parameters are all distinct, and arranged in increasing order, for instance:

(i) The denotation $z = x.x_0 + x_1$ corresponds in fact to three sub-denotations:

(1) when $x_0 < x_1$ $\quad z = (C_1; x_0, x_1; x)$
(2) when $x_0 = x_1$ $\quad z = (C_2; x_0; x)$
(3) when $x_0 > x_1$ $\quad z = (C_3; x_1, x_0; x)$.

In the three cases, the parameters must form a strictly increasing sequence.

(ii) The denotation $10^{x_0}.a_0 + \cdots + 10^{x_{n-1}}.a_{n-1}$ can be rewritten as

$$(C_{a_0, \ldots, a_{n-1}}; x_{n-1}, \ldots, x_0; x).$$

In the two cases we use symbols $C_1, C_2, C_3, C_{a_0, \ldots, a_{n-1}}$ for skeletons; it is immediate that the choice of these symbols is irrelevant, the only principle being that we distinguish different kinds of denotations by different pairs $(C; n+1)$, $n+1$ being the number of parameters; in particular, if we want to render unique the part C of a denotation, we can decide to systematically use, instead of the rather arbitrary symbol C in $(C; x_0, \dots, x_{n-1}; x)$, the ordinal z_0 defined by $z_0 = (C; 0, \dots, n-1; n) : (z_0; x_0, \dots, x_{n-1}; x)$. For instance, this would lead to the following choices:

(i) $C_1 = (C_1; 0, 1; 2) = 2.0 + 1 = 1$
$C_2 = (C_2; 0; 1) = 1.0 + 0 = 0$
$C_3 = (C_3; 0, 1; 2) = 2.1 + 0 = 2$

(ii) $C_{a_0, \dots, a_{n-1}} = (C_{a_0, \dots, a_{n-1}}; 0, \dots, n-1; n)$
$= 10^{n-1}.a_{n-1} + \cdots + 10^0.a_0,$

i.e., the integer written $a_{n-1} \cdots a_0$ in number base 10.

The system of denotations will be well-determined when we know the set of all possible denotations, and when we know how to compare any two such denotations. Let's go back once more to our examples: when we want to compare $x.x_0 + x_1$ and $x.x'_0 + x'_1$ (the same x!), then it is sufficient to compare x_0 and x'_0, then x_1 and x'_1. The same type of process is used to compare Cantor Normal Forms: they are compared by means of the comparison of the coefficients. Hence we see that the relative order of two denotations (with the same x) only depends on the relative order of their coefficients (and not on the actual values of these coefficients).

Our discussion has been sufficiently complete so that we can now give a general definition of a denotation system:

1.1. DEFINITION. A *denotation system D* consists in:
(i) For all x an ordinal $D(x)$
(ii) For all x and $z < D(x)$ an expression of z of the form

$$z = (C; x_0, \dots, x_{n-1}; x)$$

with $x_0 < \cdots < x_{n-1} < x$.
Such an expression is called a *denotation*; if $z = (C; x_0, \dots, x_{n-1}; x)$, we say that $(C; x_0, \dots, x_{n-1}; x)$ *denotes* z.

The properties of denotation systems are the following:

(DS1) Distinct ordinals are denoted by distinct denotations.

(DS2) If $(C; x_0, \dots, x_{n-1}; x)$ is a denotation and $y_0 < \cdots y_{n-1} < y$, then $(C; y_0, \dots, y_{n-1}; y)$ is a denotation as well.

(DS3) Assume that $z = (C; x_0, \dots, x_{n-1}; x)$, $w = (C'; y_0, \dots, y_{m-1}; x)$, $z' = (C; x'_0, \dots, x'_{n-1}; x')$ and $w' = (C'; y'_0, \dots, y'_{m-1}; x')$; assume that the following hold:

(i) For all $i < n$, $j < m$, $x_i < y_j \rightarrow x'_i < y'_j$.
(ii) For all $i < n$, $j < m$, $x_i = y_j \rightarrow x'_i = y'_j$.
(iii) For all $i < n$, $j < m$, $x_i > y_j \rightarrow x'_i > y'_j$.

Then $z < w \rightarrow z' < w'$.

We usually identify two denotation systems when they only differ by the choice of the parts C of the denotations; anyway, recall that one can always assume that $C = (C; 0, \ldots, n-1; n)$.

2. DILATORS

In the sequel we shall use the category *ON* of ordinals, where the morphisms are given by the sets $I(x, y)$ of strictly increasing functions from x to y.

2.1. DEFINITION. A *dilator* is a functor F from *ON* to *ON* preserving direct limits and pull-backs.

In order to understand this definition, a certain familiarity with category theory is perhaps necessary; let us explain what the definition concretely means:

(i) To say that F is a functor from *ON* to *ON* means that we are given

- for any ordinal x, an ordinal $F(x)$
- for all ordinals x, y and any $f \in I(x, y)$, a function $F(f) \in I(F(x), F(y))$

and that: $F(fg) = F(f)F(g)$, $F(E_x) = E_{F(x)}$. (fg denotes the composition of f with g, E_x is the identity of x, i.e., the identity map of x.)

(ii) To say that F preserves direct limits means that, given $z < F(x)$, it is possible to find an integer n, together with $f \in I(n, x)$, such that $z \in rg(F(f))$.

(iii) To say that F preserves pull-backs means that, for all f_1, f_2, f_3, morphisms in *ON* with the same target y, we have

$$rg(f_3) = rg(f_1) \cap rg(f_2) \rightarrow rg(F(f_3)) = rg(F(f_1)) \cap rg(F(f_2)).$$

Examples of dilators are manifold; let us give two examples, which are closely related to the denotation systems considered in section 1:

(i) The dilator Id^2 is defined by:

$$Id^2(x) = x^2$$
$$Id^2(f)(x.x_0 + x_1) = y.f(x_0) + f(x_1) \text{ when } f \in I(x, y).$$

(ii) The dilator 10^{Id} is defined by:

$$10^{Id}(x) = 10^x$$
$$10^{Id}(f)(10^{x_0}.a_0 + \cdots + 10^{x_{n-1}}.a_{n-1})$$
$$= 10^{f(x_0)}.a_0 + \cdots + 10^{f(x_{n-1})}.a_{n-1}.$$

It is easily checked that these two definitions are definitions of dilators; moreover, we see that the definition of $F(f)$ essentially depends on the denotations of ordinals $< F(x)$! This indeed is a general situation:

2.2. THEOREM. Assume that D is a denotation system, and define F as follows:

- $F(x) = D(x)$
- $F(f)((C; x_0, \ldots, x_{n-1}; x)) = (C; f(x_0), \ldots, f(x_{n-1}); y)$ when $f \in I(x, y)$.

Then F is a dilator.

Proof. It is easy to see that F is a functor from ON to ON; if $z < F(x) = D(x)$, write $z = (C; x_0, \ldots, x_{n-1}; x)$, and define $f \in I(n, x)$ by $rg(f) = \{x_0, \ldots, x_{n-1}\}$; then $z = F(f)((C; 0, \ldots, n-1; n))$: this establishes preservation of direct limits; finally observe that, when $f \in I(x, y)$, a point $z < F(x) = D(x)$ is in the range of $F(f)$ iff all the coefficients $x_0, \ldots, x_{n-1}$ of its denotation $z = (C; x_0, \ldots, x_{n-1}; x)$ belong to $rg(f)$: from this we easily obtain preservation of pull-backs.

Hence, to each denotation system, we can attach a dilator; but the converse is also true: every dilator induces a denotation system. This rests upon the following result:

2.3. THEOREM. Assume that F is a dilator, and that x is an ordinal and $z < F(x)$; then z can be written as:

(1) $\quad z = (z_0; x_0, \ldots, x_{n-1}; x).$

This means that, if we define $f \in I(n, x)$ by $rg(f) = \{x_0, \ldots, x_{n-1}\}$, then $z = F(f)(z_0)$ (hence $z_0 < F(n)$).

Furthermore, if among all solutions of (i) n is chosen minimum, then this representation is unique.

Proof. The existence of solutions to (1) is due to preservation of direct limits; the unicity comes from the fact that if f_1, f_2 are two functions such that $z \in rg(F(f_i))$ $(i = 1, 2)$, then we have $z \in rg(F(f_3))$, where f_3 is defined by $rg(f_3) = rg(f_1) \cap rg(f_2)$, by preservation of pull-backs: hence if the cardinal of $rg(f)$ is minimum, f is uniquely determined, and since $F(f)$ is injective, z_0 is unique as well.

This normal form theorem obviously enables us to associate to F a denotation system D (easy verification); furthermore

2.4. THEOREM. The processes of sections 2.2 and 2.3 are reciprocal. In other terms, dilators and denotation systems can be identified.

3. THE ALGEBRAIC THEORY OF DILATORS

Dilators (equivalently denotation systems) are very interesting because of their important *algebraic* features. By algebraic, I essentially mean the aspects of the theory which are not essentially connected to well-foundedness.

A typical remark is that, once we know the values of a dilator $F(n)$, $F(f)$ $(f \in I(n, m))$ on the integers and morphisms of integers, then we know it everywhere. This is clear from the viewpoint of denotations: $F(x)$ is the set of all formal expressions $(z_0; x_0, \ldots, x_{n-1}; x)$, where $n \in \mathbb{N}$, $x_0 < \cdots < x_{n-1} < x$, and $z_0 < F(n)$ is such that z_0 cannot be written as $F(f)(z_1)$, for some $f \neq E_n$. In order to compare $(z_0; x_0, \ldots, x_{n-1}; x)$ with $(z_1; y_0, \ldots, y_{m-1}; x)$, all we have to do is to find integers $p_0, \ldots, p_{n-1}$, and $q_0, \ldots, q_{m-1}$ such that the sequences $x_0, \ldots, x_{n-1}, y_0, \ldots, y_{m-1}, x$ and $p_0, \ldots, p_{n-1}, q_0, \ldots, q_{m-1}, n+m$ have the same order type, and to compare $(z_0; p_0, \ldots, p_{n-1}; n+m)$ with $(z_1; q_0, \ldots, q_{m-1}; n+m) \ldots$.

A particularly important case is when the restriction of our dilator to integers and morphisms of integers take values which are integers and morphisms of integers (*weakly finite* dilators); such dilators can be encoded by functions from $\mathbb{N}$ to $\mathbb{N}$, and when the encoding function is recursive, so is by definition the dilator. We therefore see that a recursive dilator enables us to compute $F(x)$ "recursively", effectively in the datum x. In particular dilators, as functions from *On* to *On*, are a typical example of the viewpoint of *potential infinity*, applied to ordinals: if we consider our datum (the ordinal x) as the (ideal) direct limit of integers, then the value $F(x)$ is the (ideal) direct limit of the values of F on these integers.

The algebraic viewpoint is the part of the theory which does not bother too much about well-foundedness, or if one prefers the theory of *predilators*. Predilators are functors from the category OL of linear orders (morphisms: $I(x, y)$ as in ON) to itself, preserving direct limits and pull-backs, and the order relation between morphisms: if $f, g \in I(x, y)$ are such that $f \leq g$ (i.e., $f(z) \leq g(z)$ for all $z < x$), then $F(f) \leq F(g)$. It is clear that the same process used above to reconstruct a dilator from its restriction to integers could be used to define it on linear orders! The property $f \leq g \rightarrow F(f) \leq F(g)$ is just the fact that denotations are increasing in the coefficients, i.e., $x_0 \leq y_0, \ldots, x_{n-1} \leq y_{n-1}$ implies $(z_0; x_0, \ldots, x_{n-1}; x) \leq (z_0; y_0, \ldots, y_{n-1}; x)$. Hence dilators can be considered as predilators. We now consider the question of determining all predilators.

3.1. DEFINITION. Let F be a dilator (or a predilator); then the *trace* of F is the set $\mathrm{Tr}(F)$ of all pairs $(z_0; n)$ such that $z = (z_0; 0, \ldots, n-1; n)$. (If we define, in the obvious way, the notion of subdilator, then the subdilators of F are in 1–1 correspondence with the subsets of $\mathrm{Tr}(F)$.) The *dimension* of F is by definition the cardinal of $\mathrm{Tr}(F)$.

One can easily show, that, up to isomorphism, the notions of dilator and predilator coincide in finite dimension.

The first obvious question is to determine all dilators of dimension 1; an equivalent formulation is the following:

How do we compare two denotations

$$(z_0; x_0, \ldots, x_{n-1}) \text{ and } (z_0; x'_0, \ldots, x'_{n-1}; x)?$$

The answer is simple.

3.2. THEOREM. There is a permutation σ of the integers $\{0, \ldots n-1\}$ such that, if $x_{\sigma(0)} = x'_{\sigma(0)}, \ldots, x_{\sigma(p-1)} = x'_{\sigma(p-1)}$ and $x_{\sigma(p)} < x'_{\sigma(p)}$, then

$$(z_0; x_0, \ldots, x_{n-1}; x) < (z_0; x'_0, \ldots, x'_{n-1}; x).$$

Conversely, if σ is a permutation of $\{0, \ldots n-1\}$ one can find a dilator of dimension 1 such that the comparison of denotation w.r.t. this dilator is done as just explained.

The next step is the characterization of all dilators of dimension 2; since a dilator of dimension 2, say F, contains exactly two (maybe isomorphic) subdilators of dimension 1, the main problem is to determine the way these two subdilators must be "bridged" together to form a dilator of dimension 2. An equivalent problem is the following: How do we compare denotations

$(z_0; x_0, \ldots, x_{n-1}; x)$ and $(z'_0; x'_0, \ldots, x'_{m-1}; x)$, when $(z_0; n) \neq (z_0; m)$?

3.3. THEOREM. Let σ and τ be the permutations governing the $(z_0; n)$ and the $(z'_0; m)$-denotations by means of section 3.2, then it is possible to find an integer p, and a number ϵ taking one of the values $+1, -1$, such that:

(i) For all $i, j < p$, $\sigma(i) < \sigma(j) \leftrightarrow \tau(i) < \tau(j)$.

(ii) In order to compare $t = (z_0; x_0, \ldots, x_{n-1}; x)$ with $t' = (z'_0, \ldots, z'_{m-1}; x)$ two possibilities may occur:

- if for some $i < p$, $x_{\sigma(i)} \neq x'_{\tau(i)}$, choose i_0 minimum with this property;

then if $x_{\sigma(i_0)} < x'_{\tau(i_0)}$, $t < t'$, and if $x'_{\tau(i_0)} < x_{\sigma(i_0)}$, $t' < t$.

- otherwise, $t < t'$ if $\epsilon = +1$, $t' < t$ if $\epsilon = -1$.

Conversely, given σ, τ, ϵ and p such that (i) holds, then one can find a two-dimensional dilator F such that the comparison of denotations w.r.t. F is governed by the principle just given.

The permutation governing the $(z_0; n)$-denotations is denoted $\sigma_{z_0,n}$ and the pair $(p; \epsilon)$ governing the comparison between $(z_0; n)$-denotations and $(z'_0; m)$-denotations is denoted $\S(z_0; n, z'_0; m)$.

The last step is the determination of all three-dimensional dilators, if $\mathrm{Tr}(F) = \{a, b, c\}$ has three elements, then we can consider the three subsets of cardinal 2 of $\mathrm{Tr}(F)$, and each of these subsets determines a two-dimensional subdilator of F; the problem is to determine the way three subdilators are bridged together. The answer is simple:

3.4. THEOREM. If $\S(a, b) = (p; +1)$, $\S(b, c) = (q, +1)$, then $\S(a, c) = (\inf(p, q), +1)$. Conversely, given permutations $\sigma_a, \sigma_b, \sigma_c$, and values for $\S(a, b)$, $\S(b, c)$, $\S(a, c)$ just as above, then, if these data are enough to define the two-dimensional dilators corresponding to $\{a, b\}$ and $\{b, c\}$, then there is a (unique) dilator of dimension 3 corresponding to these data.

We claim that we have completely determined all predilators; more precisely

(i) a predilator is completely determined when we know its trace the permutations $\sigma_{z_0,n}$ and the data $\S(z_0; n, z'_0; m)$. Here remark the rather surprising result that in order to compare denotations the parameters must be compared according to a certain order of importance (given by the permutations). Of course for Cantor Normal Forms, this is known to be like that, but nothing in the general definition of denotation systems could let us imagine that there was such a simple universal solution for the comparison of denotations!

(ii) conversely, given any set (supposed to be isomorphic with the trace of F) X, together with permutations σ_x for $x \in X$ and data $\S(x, x')$ for $x, x' \in X$, $x \neq x'$, then, provided these data verify the general compatibility conditions obtained in sections 3.3 and 3.4, then there exists a (unique up to isomorphism) predilator F which has precisely these data. The reason is quite simple: from that, we can define a set of denotations $F(x)$ for any linear order x, together with functions $F(f) \in I(F(x), F(y))$ when $f \in I(x, y)$; we also define relations R_x on $F(x)$, and the nontrivial part of the proof essentially amounts to prove that R_x is a linear order; but the axioms for linear orders involve at most three different elements at a time (in the case of transitivity), and the property follows from the similar property of a ≤ 3-dimensional subdilator corresponding to the points of the trace involved in these at most three points!

The algebraic aspect of the theory enables us to represent dilators (or denotation systems) by means of certain trees with ordinal branchings; such tress are called *dendroids*, and may be very useful in giving a more explicit description of the data implicit in the dilators.

4. DILATORS AS WELL-ORDERED CLASSES

It is clear from the algebraic analysis of predilators that this concept has no logical complexity (for instance, to be a recursive predilator is a Π^0_1 formula); but it is easily checked that the concept of dilator has the logical complexity Π^1_2. This complexity cannot be significantly lowered, because of the

4.1. THEOREM. The set of (indices of) recursive dilators is Π^1_2-complete; more precisely, if A is a Π^1_2 formula, one can build, primitive recursively in A, a predilator F_A such that

$$A \leftrightarrow \text{“}F_A \text{ is a dilator”}.$$

In particular, the problem for a given predilator, to be (isomorphic to) a dilator, cannot by any means be solved by "algebraic" methods, since the logical complexity of this question is terrible!

Important properties of the dilator F are connected to its value $F(On)$ on the ordinal-class On: when F is a dilator, $F(On)$ can be defined as a linear order on a proper class (take for instance the class of all formal denotations $(z_0; x_0, \ldots, x_{n-1}; On)$, where $x_0, \ldots, x_{n-1}$ is a strictly increasing sequence of ordinals, and $(z_0; n) \in \mathrm{Tr}(F)$), and this linear order is easily shown to be well-founded, so we speak of an *ordinal class*. In fact $F(On)$ will be a proper class exactly when F is nonconstant.

The main idea is to use the algebraic features developed in section 3 in order to introduce a strong principle, the principle of induction up to $F(On)$, which will be called *induction on dilators*. Of course, one could write a principle of induction on the class of all formal denotations $(z_0; x_0, \ldots, x_{n-1}; On)$, but this is not so interesting: we would like to associate an unique dilator (called a predecessor of F) at each point of $F(On)$. The situation is similar to what is currently done with well-founded trees, where we associate a subtree to each node....

Our first tool will be the *classification of dilators*; for this we define the concept of a sum $\sum_{i<x} F_i$ of dilators, and we prove:

4.1. THEOREM. Any dilator can uniquely be written as a sum $F = \sum_{i<x} F_i$ of *connected* dilators (i.e., dilators which are $\neq 0$, and not themselves sums).

4.2. DEFINITION. Let F be a dilator, and let $F = \sum_{i<x} F_i$ its decomposition in section 4.1:

- (i) if $x = 0$ (hence F is the null dilator 0), F is said to be *of kind 0*
- (ii) if x is limit (not necessarily denumerable), F is *of kind ω*
- (iii) if $x = x' + 1$, and $F_{x'}$ is the constant 1, F is *of kind 1*
- (iv) if $x = x' + 1$ and $F_{x'}$ is not 1, F is *of kind Ω*.

The first three kinds correspond to the familiar decomposition of ordinals into 0, limits and successors; kind Ω is a new thing, with no analogue in the case of ordinals. The classification is simply related to the values $F(On)$ by means of the

4.3. PROPOSITION. If F is a dilator, then

(i)	F is of kind 0	iff	$F(On) = 0$
(ii)	F is of kind ω	iff	$F(On)$ is limit of cofinality $< On$
(iii)	F is of kind 1	iff	$F(On)$ is successor
(iv)	F is of kind Ω	iff	$F(On)$ is limit of cofinality On.

But of course our definition of kind does not refer at all to On! Let's go back to our original motivation, and recall that we want to define, for all $z \in F(On)$, a unique dilator F_z, which will be called a predecessor of F. In order to do this, it will be enough to define the predecessors F_z when z varies through a cofinal subset of $F(On)$; if the definition is such that $z < z' \to F_z$ predecessor of $F_{z'}$, then it will be possible, to define the predecessors of F by a transitivity argument.

The kinds 0, ω and 1 are not problematic: in all three cases, the problem is answered by the requirement

(1) F is a predecessor of $F + F'$ when $F' \neq 0$.

But in the case F is of kind Ω, (1) does not give us enough predecessors; here we must use a subtler form of decomposition, based on the idea of separation of variables: let us say that F is a bilator when F is a functor from ON^2 to ON such that:

(i) F is nonconstant in the second argument
(ii) F preserves direct limits and pull-backs
(iii) if we denote, when $y \leq y'$; by $E_{yy'}$ the canonical embedding from y to y'; then $F(E_x, E_{yy}) = E_{F(x,y)F(x,y')}$.

4.4. THEOREM. There exist isomorphisms SEP (separation) and UN (unification) which are reciprocal, and which identify the category ΩDIL of dilators of kind Ω and the category BIL of dilators.

Proof. The construction of SEP and UN is not so simple; it essentially makes use of the algebraic analysis of section 3, in particular the properties of the coefficient $x_{\sigma(0)}$ is a denotation.

We can now complete our solution: for this observe that, when F is of kind Ω, then we have $F(On) = \text{SEP}(F)(On, On)$, so the values

SEP(F)(On, x) ($x \in On$) form a closed confinal subset of $F(On)$, and we thus say that:

(2) SEP(F)(., x) is a predecessor of F, for all $x \in On$.

All predecessors of F are obtained by means of transitivity:

(3) If H is a predecessor of G, if G is a predecessor of F, then H is a predecessor of F.

The relation "is a predecessor of" is not a linear order; but for a given F, the class of predecessors of F is a well-ordered (so is linearly ordered) class of order type $F(On)$.

The principle of induction on dilators states that the predecessor relation is well-founded:

4.5. THEOREM. Assume that χ is a property of dilators, and that:

(i) $\chi[0]$
(ii) $\chi[F] \rightarrow \chi[F+1]$
(iii) for x limit, if $\chi[\sum_{i<x'} F_i]$ for all $x' < x$, then $\chi[\sum_{i<x} F_i]$
(iv) for F of kind Ω, if for all $x \in On$, χ[SEP(., x)] then $\chi[F]$ then $\chi[F]$ holds for all F.

A typical application is the construction of the functor Λ, which maps the category DIL of dilators into itself. In fact Λ is a main theme, on which many variations can be done, the essential idea being the use of *primitive recursion on dilators*, which has the same relation to induction on dilators as primitive recursion has to usual induction.

Among all possible variants, let us select one: fix a *flower* F_0 (i.e., a dilator enjoying $F_0(E_{xx'}) = E_{F_0(x)F_0(x')}$), and define, when F is a dilator, a flower ΛF, by induction on F:

$\Lambda 0 = Id$ (the identity functor)
$\Lambda F = \Lambda F' \circ F_0$ when $F = F' + 1$ is of kind 1
$\Lambda F = \prod_{i<x} \Lambda F_i$ when $F = \sum_{i<x} F_i$ is of kind ω. ($\prod$ is an infinite composition.)
$\Lambda F(x) = (\Lambda \mathrm{SEP}(F)(., x))(0)$ when F is of kind Ω.

(In fact, our definition has been slightly cheated, and we have omitted some parts of it, for instance the value of $\Lambda F(f)$ when F is of kind Ω; but this rough description is essentially correct!)

In fact, it seems that Λ is a very important object in proof-theory; in particular, many existing works in proof-theory using traditional methods can be restated in a more satisfactory way, by means of the functor Λ.

5. Λ AND TRADITIONAL PROOF-THEORY

A very important, although cryptic, work in proof-theory is the well-known interpretation of analysis, due to Spector. The main idea of this work is the interpretation of comprehension axioms by means of principles of *Bar-recursion*: these principles enable us to construct functionals by recursion over well-founded trees, but the structure of the branchings may be of a rather uneven kind. Let us concentrate upon the most interesting case: so called Bar-recursion of type 2 (Bar-recursion of type n would be analyzed by means of $\prod^1_n$-logic!). Here the well-founded trees are continuity trees associated with functionals of type 3, and this means that the trees are made of finite sequences of functionals of type 2. Spector's result enables us to analyze $\prod^1_1$-comprehension by means of Bar-recursion of type 2. But, if Spector's interpretation is a very often quoted result, it is never used: this is due to

(i) the use of functional interpretation, which is a very boring technique

(ii) the use of intuitionistic framework, which is typical of a certain ideological approach to the subject, but has nothing to do with the heart of the matter

(iii) the fact that Bar-recursion in itself is very hard to use: in spite of its theoretic strength, the only existing works on Bar-reeursion are works giving *models* for it, and by no means applications of what could have been one of the main tools in logic!

Now observe that Λ is indeed a particular case of Bar-recursion of type 2; the reason is quite simple:

– Λ is defined by induction on dilators, i.e., by an induction up to the "ordinal" $F(On)$

– the ordinal class $F(On)$ can be represented (using the technique of dendroids) as a tree with ordinal branchings: the branchings may be full branchings!

– now, traditional methods of logic enable us to replace ordinals by type 2 functionals, and so we can replace our tree by a well-founded tree with type 2 branchings, . . .

But recursion (more generally induction) on dilators is significantly simpler than the corresponding principles on trees with type 2 branchings: Bar-recursion of type 2 (more generally so called *Bar-induction* of type 2). This is due to the rather simple algebraic features of dilators: for instance the ordinal branchings in the tree associated with $F(On)$ are of a very simple nature, whereas we don't know enough from the rough continuity properties in the general case of Bar-recursion of type 2. Now it seems reasonable that the principle of induction on dilators is of the same "strength" as the principle of Bar-induction, and that Λ can replace Bar-recursion (of type 2). There are many evidences for that, but a precise proof has never been given. If this is true (and this can hardly be false), we can claim that

"Λ is a civilized version of Bar-recursion of type 2".

Another traditional technique of proof-theory is the use of Bachmann collections, which are ordinals (the *height* of the collection) equipped with *fundamental sequences* whose length is less than the *type* of the collection. When x is a cardinal (or better: the recursive analogue of a cardinal), and B is a Bachmann collection of type x, then one defines a normal function ϕ_B, from x to x, as follows:

$\phi_B(z) = z$ when $B = 0$
$\phi_{B+1}(z) = \phi_B(f_0(z))$ (f_0 is a fixed normal function from x to x)
$\phi_B(z)$ enumerates the points which are in all sets $rg(\phi_{B_i})$, when B is a supremum of a family $(B_i)_{i<x'<x}$
$\phi_B(z) = \phi_{B_z}(0)$ when the height of B is of cofinality x, and B_z is obtained by restricting B to $[|B|]z$. ($|B|$ denotes as usual the height of B.)

The technical development of Bachmann collections is limited by the fact that the construction of such collections is very painful. For instance the ordinal $\phi_B(z)$ can in turn be equipped (when z is a cardinal) with a structure of Bachmann collection of type z, but this is more simple to state than to do! The literature on the subject has suffered from the fact that too many verifications of boring properties were needed.

Dilators give us a new approach to these topics, using the slogan:

a dilator = a Bachmann collection in each type.

Concretely this means that if D is a dilator and x is a cardinal, then the techniques of section 4 enable us to equip $D(x)$ with a structure of Bachmann collection of type x. The most remarkable achievement here is that of course everything is already encoded by the restriction of D to integers! For instance, if D is of kind Ω, then $D(x)$ is of cofinality x, and we can define the fundamental sequence $[D(x)]z = \mathrm{SEP}(D)(x, z)\ldots$.

Now, if we compute the values $\Lambda D(z)$ and $\Phi_{D(x)}(z)$, for some "cardinal" $x > z$, then we get (provided $F_0(t) = f_0(t)$ for all $t < x$)

(1) $\Lambda D(z) = \Phi_{D(x)}(z)$.

In particular, one can say that Λ gives a more interesting framework for Bachmann collections and hierarchies.

A typical application of the equation (1) is the comparison of hierarchies: one can consider the following two hierarchies of recursive functions indexed by elements of Kleene's O, i.e., by Bachmann collections of type ω:

$$\begin{array}{rll} & \lambda_0(n) = n & \gamma_0(n) = 0 \\ & \lambda_{x+1}(n) = \lambda_x(n+1) & \gamma_{x+1}(n) = \gamma_x(n) + 1 \\ x \text{ limit} \rightarrow & \lambda_x(n) = \lambda_{[x]n}(n) & \gamma_x(n) = \gamma_{[x]n}(n). \end{array}$$

Traditional proof-theory has much to do with λ: for instance every provably total recursive function of PA (Peano arithmetic) is bounded by a function λ_x, for some $x < \epsilon_0$, equipped with the familiar fundamental sequences.

This fact (together with related facts coming essentially from the work of Gentzen) was responsible for the common belief that

ϵ_0 is the ordinal of arithmetic.

But how many steps are needed in order to exhaust all provably total functions of AP by means of γ? Surely more than ϵ_0! In fact by using the equation (1) and the obvious relations between λ and (a variant of) ϕ, then it is possible to give a precise relation between λ and γ:

(2) $\lambda_{D(\omega)} = \gamma_{\Lambda D(\omega)}$.

In particular, applying this to dilators D such that $D(\omega) < \epsilon_0$, it follows that the ordinal of AP, computed in terms of γ, is the Howard ordinal η_0, which is traditionally associated with ID_1; similar results can be obtained for current theories, by means of the equation (2). (In fact the situation is a bit more complicated, because we must use here variants of the concept of dilators, e.g., the notion of ladder, and also that we are using a variant of Λ adapted to ladders; but this is only a technical remark of no deep meaning.)

This estimation for the "γ-ordinal" of arithmetic renders suspect the claim that ϵ_0 is "the" ordinal of arithmetic; ϵ_0 is surely one of the main ordinals connected with arithmetic, but η_0 is another one! In fact, for *practical* purposes, ϵ_0 is the best ordinal, since one can very easily handle the hierarchy λ (using the functional equation $\lambda_x \circ \lambda_y = \lambda_{x+y}$) and if for instance we want to compute explicit bounds, etc. . . λ will be the best choice; but for *theoretical* purposes, γ is a nicer hierarchy, which takes into account every single step in the computation, and with this hierarchy, we certainly reach the ultimate step-by-step construction of recursive functions. The distinction is akin to the distinction between usual proofs (nicer in practice) and cut-free proofs (the best in theory), and in fact, γ is connected to a cut-elimination theorem for arithmetic assigning η_0 as a bound for the cut-free proofs obtained!

6. β-PROOFS

If we keep in mind the obvious analogy with ω-logic, we have so far developed the concepts which correspond to the notions of well-founded tree (and its variants: recursive well-orders, Kleene's O . . . .); but ω-logic also contains an original notion of proof, known as the ω-rule. It turns out that $\prod^1_2$-logic also has its own notion of proof, the *β-rule*.

6.1. DEFINITION. A *β-model* (of a theory T in a language $\mathbb{L}$ containing a type o for the ordinals, together with a predicate $<$ taking two type o arguments, and with no function letters: we speak of β-language, β-theory) of a β-theory T is a model $\mathbb{M}$ of T such that:

(i) the interpretation $\mathbb{M}(o)$ of the type o is an ordinal $x = |\mathbb{M}|$; contrarily to the tradition, $|\mathbb{M}| = 0$ is allowed.

(ii) the interpretation $\mathbb{M}(<)$ of the symbol $<$ is just the usual order of $|\mathbb{M}|$.

It is easily checked that the set of all (indices of) formulas which are valid in all β-models of T is $\prod^1_2$ in a code of T; moreover it is possible to choose T finite such that this set is $\prod^1_2$-complete. Hence the question of validity in all β-models of a given β-theory is therefore a very natural one. But this question, raised by Mostowski, was unanswered for a long time: how to characterize syntactically the set of formulas valid in all β-models of T? There was even a "proof" that no such characterization is possible!

In order to find the solution, the best thing is first to use a very trivial answer to the question: to say that A is valid in all β-models of T means that for all $x \in On$, A is valid in all β-models $\mathbb{M}$ of T, with $|\mathbb{M}| = x$. Now we can (at least when x is denumerable, in fact for all x) apply an analogue of the ω-completeness theorem for such an x: this shows that A is provable by means of the x-rule:

$$\frac{\cdots A[\bar{y}] \cdots \quad \text{all } y < x}{\forall a^o A[a]}$$

and the axioms $\bar{y} < \bar{y}'$ and $\sim(\bar{y}' < \bar{y})$ of the atomic diagram of x.

If we call such a proof a x-proof, then A is valid in all β-models of the theory T iff for all x, A has an x-proof. Hence our first and trivial answer to the question of Mostowski is:

A is valid in all β-models of T iff there is a family $(P_x)_{x \in On}$ such that for all x, P_x is a x-proof of A.

But such a family (P_x) can by no means be considered as an acceptable syntactic object! Unless we find a way to make effective such families. Now, observe that, if $f \in I(x, y)$ and P is a y-proof, we can (in certain cases) define a x-proof $f^{-1}(P)$. $f^{-1}(P)$ is obtained by means of the *mutilation process*:

(i) In P remove all premises of y-rules whose index is not in $rg(f)$ and the subproofs above such premises.

(ii) If then all ordinal parameters which remain are in $rg(f)$, then replace systematically $\overline{f(t)}$ by $\bar{t}$: the result is the x-proof $f^{-1}(P)$.

This definition can be used to define a category PF_T, where the objects are pairs (x, P), with P a x-proof in T, and the morphisms from (x, P) to (y, Q) are all $f \in I(x, y)$ such that $f^{-1}(Q) = P$.

Going back to our problem, we require that the family $(P_x)_{x \in On}$ is functorial in the following sense: $P(x) = P_x$, $P(f) = f$ defines a functor from ON to PF_T. Such a functor is called a *β-proof of A*. Equivalently when $f \in I(x, y)$, $f^{-1}(P_y) = P_x$. It is easily checked that, as a functor, β-proofs preserve direct limits and pull-backs, and in particular, a β-proof is uniquely determined by its restriction $(P_n)_{n<\omega}$ or by P_ω alone! So it will make sense to speak of a recursive, prim. rec. β-proof... and this means that the concept of β-proof is a perfectly acceptable notion of proof as a syntactic object! In fact a β-proof is as "syntactic" as an ω-proof, and it is not exaggerated to say that the

β-proofs are more "finitary" than ω-proofs.... Of course, one can show that:

6.2. THEOREM. A closed formula A is valid in all β-models of a recursive theory T iff there is a recursive β-proof of A in T.

Proof. We introduce PF'_T as PF_T, but here the objects are preproofs, i.e., the proofs are not necessarily well-founded; we produce a preβ-proof of A in T, (P_x), with the property that P_x is well-founded iff A is valid in all β-models $\mathbb{M}$ of T, with $|\mathbb{M}| \leq x$....

The mathematical structure of β-proofs is very close to the structure of dilators (more precisely: dendroids).

A priori the existence of the β-rule is promising w.r.t. the question of cut-elimination, with its corollary, the subformula property: take for instance Peano arithmetic; by ω-completeness, every true formula of PA has a recursive ω-proof: simply because such formulas are $\prod_1^1$. In particular, this fact will still hold in any extension of PA, and this is the essential reason why we obtain a principle of purity of methods for ω-proofs of arithmetic formulas in any ω-consistent extension of PA. But if we turn our attention towards formulas which are not $\prod_1^1$, typically formulas involving a negative occurrence of some inductive definition, then ω-logic cannot be used to obtain a proof of such formulas, when true: this means that we will obtain more formulas of that kind by adding new axioms, and so purity of methods cannot be expected in that case! But if we change the logical framework and replace the ω-rule by the β-rule, then since such formulas are $\prod_2^1$, then by an argument analogue to what we said for PA, purity of methods can be expected! Now, for the proof-theorist, purity of methods is called the subformula property, and usually follows from a cut-elimination theorem. In particular, it will be possible to prove a cut-elimination result for theories of inductive definitions.

7. INDUCTIVE LOGIC

As just explained, the methods of usual logic as well as the methods of ω-logic, are not enough to obtain cut-elimination for inductive definitions; in fact Martin-Löf, who used usual logic was only able to eliminate cuts from proofs of $\sum_1^0$ sequents, whereas the German school (Pohlers, Buchholz, Sieg...) working within the framework of ω-logic, was only able to eliminate cuts from proofs of $\prod_1^1$ sequents, i.e., when the inductively defined predicate only occurs positively in the conclusion.

Our notations will be as follows: Φ is a positive operator $\Phi[X, x]$ in the language $\mathbb{L}$; $I\Phi^z$ stands for the z^{th} iterate of Φ, whereas $D\Phi$ stands for the iteration of Φ up to the closure ordinal of the inductive definition. (In other terms, $I\Phi^z$ stands for $\mathbb{W}_{z'<z}\Phi[I\Phi^{z'}, .]$, and $D\Phi$ stands for $I\Phi^{On}$.)

In any language where ordinals $\leq z$ are available, then the symmetric rules:

$$(1)\qquad \frac{\Gamma \mid\!\!\rightarrow \Phi[I\Phi^{z'}, t], \Delta \quad \text{some } z' < z}{\Gamma \mid\!\!\rightarrow t \in I\Phi^z, \Delta}$$

$$(2)\qquad \frac{\Gamma, \Phi[\mathrm{I}\Phi^z, \mathrm{t}] \mid\!\!\rightarrow \Delta \cdots\cdots \quad \text{all } z' < z}{\Gamma, \mathrm{t} \in \mathrm{I}\Phi^z \mid\!\!\rightarrow \Delta}$$

exactly express the construction of the $I\Phi^z$'s; if x and y are ordinals and $x < y$, then the asymmetric rules:

$$(3)\qquad \frac{\Gamma \mid\!\!\rightarrow \Phi[I\Phi^{z'}, t], \Delta \quad \text{some } z' < y}{\Gamma \mid\!\!\rightarrow t \in D\Phi, \Delta}$$

$$(4)\qquad \frac{\Gamma, \Phi[I\Phi^{z'}, t] \mid\!\!\rightarrow \Delta \cdot \cdots\cdot \quad \text{all } z' < x}{\Gamma, t \in D\Phi \mid\!\!\rightarrow \Delta}$$

express the construction of $D\Phi$; the rule (3) says that $I\Phi^y \subset D\Phi$, whereas the rule (4) expresses that $D\Phi \subset I\Phi^x$. By transitivity of inclusion (i.e., by the cut-rule!) we obtain $I\Phi^y \subset I\Phi^x$, and since $x < y$, the ordinal x must be $\geq$ the closure ordinal of Φ. Of course, for most of the choices of ordinals x and y, the system just written, say $T[x, y]$, is simply inconsistent!

The systems $T[x, y]$ are bridged together by means of the β-rule; if D is a dilator of the form $Id + 1 + D'$, we can restrict our attention to so called D-proofs, which are families (P_x) such that for all x, P_x is a proof in the system $T[x, D(x)]$, with obvious mutilation conditions....

7.1. THEOREM. Assume that $\Gamma \mid\!\!\rightarrow \Delta$ is a sequent which is true in all models where $D\Phi$ is interpreted in the standard way: then there exists a recursive $Id+1$-proof of $\Gamma \vert\!\!\rightarrow \Delta$.

Proof. The result combines the interpretation of the rules (1)–(4) with the β-completeness theorem....

Now the calculus just sketched enjoys cut-elimination; but observe that such a result cannot be proved by too simple ways. For instance, one cannot prove cut-elimination in the theories $T[x, y]$, simply because for many values of x and y, these theories are inconsistent, and inconsistency is incompatible with cut-elimination. The only possibility is to use the given D-proof *as a whole*. The crucial case in the cut-elimination theorem is the reduction of a cut whose main formula is of the form $t \in D\Phi$; the question is nontrivial, since the rules (3) and (4) are not symmetric; in particular, (4) has not enough premises compared to (3). In order to have enough premises in (4), one uses the fact that the data are in fact functorially dependant on x, and that x can therefore be replaced by any expression $D(x)$, by means of a composition with D. This is a way to restore the symmetry between the rule, and the iteration of this process finally yields full cut-elimination. But the way the result has been proved has made D change: we start with a D-proof and finally get a D_1-proof.

7.2. THEOREM. Assume that P is a D-proof of a sequent which does not contain $I\Phi^*$; then one can construct effectively in the data a dilator D_1 and a D_1-proof of the same sequent, which is cut-free.

The cut-elimination theorem for inductive definitions is deeply related to Λ; in fact one would show that bounds on the cut-free proof of section 7.2 can be obtained by means of Λ. Following the same line of thought, it would be possible to show that $\prod^1_1$-CA is formally equivalent (over a system of analysis only containing $\prod^0_1$-CA) to the fact that the functor Λ (viewed as a recursive functor from predilators to predilators) sends dilators on dilators, and also to the cut-elimination Theorem 7.2.

8. APPLICATIONS TO GENERALIZED RECURSION

The results just obtained can be applied to generalized recursion: for instance if f is a function from ω_1^{CK} to itself which is $\sum^s$ over $L_{\omega_1^{CK}}$, then f can be expressed as the Skolem function of a true formula B:

$$\forall x \in D\Phi \quad \exists y \in D\Phi \quad A[x, y]$$

where the formula A is arithmetical, and for a well-chosen Φ.

Now, if we apply sections 7.1 and 7.2, we find a recursive dilator D and a recursive D-proof of B. Now it is easy to see, by replacing in this cut-free proof all positive occurrences of $D\Phi$ by $I\Phi^{D(z)}$ and all negative occurrences of $D\Phi$ by $I\Phi^z$, that, for all $z \in On$, the formula

$$\forall x \in I\Phi^z \quad \exists y \in I\Phi^{D(z)} \quad A[x, y]$$

is valid. This forces $f(z) < D(z+1)$ for all $z < \omega_1^{CK}$ (for technical reasons the majoration is only effective for z infinite). Hence, any ω_1^{CK}-recursive function is bounded, for all values $\geq \omega$, by a recursive dilator.

The importance of this result is that generalized recursion makes use of many noneffective schemes, for instance infinite μ-operators . . . but the computation of $D(z)$ is perfectly effective in the data D and z: the result is therefore a reduction of the class of algorithms that may be used to define ω_1^{CK}-recursion in the total case. In fact, if we are only concerned with the rate of growth of the generalized functions (and in generalized recursion, this is the crucial thing), then we can simply take recursive dilators, which are truly recursive. We therefore succeed in eliminating the actual infinite from ω_1^{CK}-recursion!

The result just given generalizes: there is an ordinal s_0 such that, for all $x < s_0$, the following holds:

8.1. THEOREM. All total $\sum^1$ functions from x^+ to x^+ (the next admissible) are bounded, for all arguments $\geq x$, by a recursive dilator

$$\forall z(x \leq z < x^+ \rightarrow f(z) \leq F(z)).$$

In fact, the result of section 8.1 is true for a cofinal subset of the first stable σ_0. Observe that the result implies the equality

$$x^+ = \Xi_1(x)$$

where Ξ_1 is the sum of all recursive dilators.

The question of the generalization of such results to other kinds of admissibles (not only successors) has led to a certain number of developments. For instance, let G be the dilator (flower) obtained by iterating Ξ_1, i.e., $G(x+1) = G(x) + \Xi_1(G(x))$, etc. . .; for all $x < s_0$, we have $G(x) = \omega_x^{CK}$; consider the functor Λ, with $\Lambda 1 = G$; then

8.2. THEOREM. Assume that f is a total $\sum^1$ function from i_0^{CK} to itself (the first recursively inaccessible); then there exists a recursive dilator D such that $f(z) < D(z)$ for all $z < i_0^{CK}$.

Many other results following these lines have been found: this is one of the most active directions of $\prod_2^1$ logic!

9. DESCRIPTIVE SET-THEORY

Here not so many things have been accumulated, but the results already obtained are rather encouraging.

Basis and uniformization theorems play an important role in descriptive set-theory; but, as soon as the logical complexity of the sets involved becomes not too small, special axioms are needed (e.g., determinacy assumptions). There is a reasonable ground to think that $\prod_2^1$-logic and its generalizations (dilators of finite type, or *ptykes*) can be of some use to clarify some of these questions.

The Novikoff–Kondo–Addison theorem enables one to uniformize a $\sum_2^1$ graph by a $\sum_2^1$ function. We can translate this question in terms of dilators: if we want to select a point in a nonvoid $\sum_2^1$ set of reals, this essentially amounts, given a predilator P, which is not a dilator, to select an ordinal x, together with a s.d.s. in $P(x)$, in such a way that the selected data are (encoded by) a $\prod_1^1$ singleton. In fact, such a selection is very easy and natural, and the crucial point is that ordinals are linearly ordered (in the selection of x).

Now, if we turn our attention towards uniformization of a $\sum_3^1$ graph, then, in analogy to the $\sum_2^1$ case, it is natural to consider the question of selecting, given a preptyx P of type $(o \to o) \to o$ (i.e., a functor from DIL to OL preserving direct limits and pull-backs) which is not a ptyx, a dilator D together with a s.d.s. in $P(D)$. In fact the main part of the argument used in the $\sum_2^1$ case is still valid, the only problem being that dilators are not linearily ordered (problem of the selection of D).

Here come the sharps: the real $o^{\#}$ can be used to define a dilator, that we shall note by o^*; now one easily proves the following result:

9.1. THEOREM. Assume that D and D' are recursive dilators; then

(i) $D(o^*(0)) = D'(o^*(0)) \to D \circ o^* = D' \circ o^*$

(ii) $D(o^*(0)) < D'(o^*(0)) \to \exists E(D \circ o^* + E = D' \circ o^*)$.

In other terms, recursive dilators, when composed with o^*, form a linearity ordered set, with respect to the relation of subsum.

Of course, the argument relativizes to an arbitrary real a: in Theorem 9.1, D and D' are now recursive in a, and o^* is replaced by a^*.

Now, all the a^*, when $a \in \mathbb{R}$, can be arranged into a direct system, and one easily shows that the direct limit $\overset{+}{\Omega}$ of this system is a dilator. Now, one proves, using the relativization of Theorem 9.1:

9.2. THEOREM. Dilators with a denumerable trace, when composed with $\overset{+}{\Omega}$, are linearly ordered by the subsum relation.

This result is enough to obtain uniformization: we can select a unique dilator of the form $D \circ \overset{+}{\Omega}$, and when we encode the solution, we finally uniformize by means of a $\sum_4^1$ function. The result is of course not new, but the method of proof makes use of $\prod_2^1$ logic.

Now the question is to find similar linearization principles for higher type *p*tykes; such results could lead to objects which would be analogues of sharps "of higher types", objects that set-theorists are seeking, but have not yet been able to produce. We hope that the conceptual clarification that seems to arise from the use of $\prod_2^1$ logic in this matter could be of essential interest here.

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CONSISTENCY-PROOF FOR THE GENERALIZED CONTINUUM-HYPOTHESIS[1]

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If M is an arbitrary domain of things in which a binary relation ϵ is defined, call "*propositional function over M*" any expression φ containing (besides brackets) only the following symbols: 1. Variables x, y .... whose range is M. 2. Symbols $a_1 \ldots a_n$ denoting[2] individual elements of M (referred to in the sequel as "*the constants of* φ"). 3. ϵ. 4. $\sim$ (not), $\vee$ (or). 5. Quantifiers for the above variables x, y ...* Denote by M' the set of all subsets of M defined by prop. funct. $\varphi(x)$ over M. Call a function f with s variables a "*function in M*" if for any elements $x_1 \ldots x_s$ of M

$f(x_1 \ldots x_s)$ is defined and is an element of M. If $\varphi(x)$ is a prop. funct. over M with the following normal form:

$$(x_1 \ldots x_n)\,(\exists y_1 \ldots y_m)\,(z_1 \ldots z_k)\,(\exists u_1 \ldots u_e) \ldots . \quad L(xx_1 \ldots x_n y_1 \ldots y_m z_1 \ldots z_k u_1 \ldots u_e \ldots .)$$

(L containing no more quantifiers) and if $a \in M$, then call "*Skolem-functions for φ and a*" any functions $f_1 \ldots f_m\ g_1 \ldots g_e \ldots$ in M with resp. $n \ldots n$, $n + k \ldots n + k \ldots$ variables such that for any elements $x_1 \ldots x_n z_1 \ldots z_k \ldots$ of M the following is true:

$$L(ax_1 \ldots x_n\, f_1(x_1 \ldots x_n) \ldots f_m(x_1 \ldots x_n)\, z_1 \ldots z_k\; g_1(x_1 \ldots x_n z_1 \ldots z_k) \ldots g_e(x_1 \ldots x_n z_1 \ldots z_k) \ldots .)$$

The proposition $\varphi(a)$ is then equivalent with the existence of Skolem-fnct. for φ and a.

Now define: $M_0 = \{\Lambda\}$, $M_{\alpha+1} = M_\alpha'$, $M_\beta = \sum_{\alpha<\beta} M_\alpha$ for limit numbers β. Call a set x "*constructible*," if there exists an ordinal α such that $x \in M_\alpha$ and "*constructible of order α*" if $x \in M_{\alpha+1} - M_\alpha$. It follows immediately that: $M_\alpha \subset M_\beta$ *and* $M_\alpha \in M_\beta$ *for* $\alpha < \beta$ and that:

TH. 1. *$x \in y$ implies that the order of x is smaller than the order of y for any constr. sets x, y.*

It is easy to define a well-ordering of all constr. sets and to associate with each constr. set (of an arbitrary order α) a uniquely determined prop. fnct. $\varphi_\alpha(x)$ over M_α as its "*definition*" and furthermore to associate with each pair φ_α, a (consisting of a prop. fnct. φ_α over M_α and an element a of M_α for which $\varphi_\alpha(a)$ is true) uniquely determined "*designated Skolem-fnct. for φ_α, a.*"[3]

TH. 2. *Any constr. subset m of M_{ω_α} has an order $< \omega_{\alpha+1}$ (i.e., a constr. set, all of whose elements have orders $< \omega_\alpha$ has an order $< \omega_{\alpha+1}$).*

PROOF: Define a set K of constr. sets, a set O of ordinals and a set F of Skolem-fnct. by the following postulates I–VII:

I. $M_{\omega_\alpha} \subseteq K$ and $m \in K$.

II. If $x \in K$, the order of x belongs to O.

III. If $x \in K$, all constants occurring in the definition of x belong to K.

IV. If $\alpha \in O$ and $\varphi_\alpha(x)$ is a prop. fnct. over M_α all of whose constants belong to K then:

1. The subset of M_α defined by φ_α belongs to K.

2. For any $y \in K \cdot M_\alpha$ the design . Skolem-fnct. for φ_α and y or $\sim \varphi_\alpha$ and y (according as $\varphi_\alpha(y)$ or $\sim \varphi_\alpha(y)$) belong to F.

V. If $f \in F$, $x_1 \ldots x_n \in K$ and $(x_1 \ldots x_n)$ belongs to the domain of definition of f, then $f(x_1 \ldots x_n) \in K$.

VI. If $x, y \in K$ and $x - y \neq \Lambda$ the first[4] element of $x - y$ belongs to K.

VII. No proper subsets of K, O, F satisfy I–VI.

TH. 3. *If* $x \neq y$ *and* $x, y \in K \cdot M_{\alpha+1}$, *then there exists a* $z \in K \cdot M_\alpha$ *such that* $z \in x - y$ or $z \in y - x$.[13]
(follows from VI and Th. 1.)

TH. 4.[5] $\overline{\overline{K + O + F}} = \aleph_\alpha$
since $\overline{\overline{M}}_{\omega_\alpha} = \aleph_\alpha$ *and* $K + O + F$ *is obtained from* $M_{\omega_\alpha} + \{m\}$ *by forming the closure with respect to the operations expressed by* II–VI.

Now denote by η the order type of O and by $\bar{\alpha}$ the ordinal corresponding to α in the similar mapping of O on the set of ordinals $< \eta$. Then we have:

TH. 5. *There exists a one to one mapping* x' *of* K *on* M_η *such that* $x \in y \equiv x' \in y'$ *for* $x, y \in K$ *and* $x' = x$ *for* $x \in M_{\omega_\alpha}$.

PROOF: The mapping x' (which will carry over the elements of order α of K exactly into all constr. sets of order $\bar{\alpha}$ for any $\alpha \in O$) is defined by transfinite induction on the order, i.e., we assume that for some $\alpha \in O$ an isomorphic[6] mapping f of $K \cdot M_\alpha$ on $M_{\bar{\alpha}}$[7] has been defined and prove that it can be extended to an isomorphic mapping g of $K \cdot M_{\alpha+1}$ on $M_{\bar{\alpha}+1}$[8] in the following way: At first those prop. fnct. over M_α whose constants belong to K (hence to $K \cdot M_\alpha$) can be mapped in a one to one manner on all prop. fnct. over $M_{\bar{\alpha}}$ by associating with a prop. fnct. φ_α over M_α having the constants $a_1 \ldots a_n$ the prop. fnct. $\varphi_{\bar{\alpha}}$ over $M_{\bar{\alpha}}$ obtained from φ_α by replacing a_i by a_i^1 and the quantifiers with the range M_α by quantifiers with the range $M_{\bar{\alpha}}$. Then we have:

TH. 6. $\varphi_\alpha(x)$ $\varphi_{\bar{\alpha}}(x^1)$ *for any* $x \in K \cdot M_\alpha$.

PROOF: If $\varphi_\alpha(x)$ is true, the design . Skolem-fnct. for φ_α and x exist, belong to F (by IV, 2) and are functions in $K \cdot M_\alpha$ (by $\overline{\text{V}}$). Hence they are carried over by the mapping f into functions in $M_{\bar{\alpha}}$ which are Skolem-functions for $\varphi_{\bar{\alpha}}$, x^1, because the mapping f is isomorphic with respect to ϵ. Hence $\varphi_\alpha(x) \supset \varphi_{\bar{\alpha}}(x^1)$.

$\sim\varphi_\alpha(x) \supset \sim\varphi_{\bar{\alpha}}(x^1)$ is proved in the same way.

Now any φ_α over M_α whose constants belong to K, defines an element of $K \cdot M_{\alpha+1}$ by IV, 1 and any element b of $K \cdot M_{\alpha+1}$ can be defined by such a φ_α (if $b \in M_{\alpha+1} - M_\alpha$ this follows by III, if $b \in M_\alpha$ then "$x \in b$" is such a φ_α). Hence the above mapping of the φ_α on the $\varphi_{\bar{\alpha}}$ gives a mapping g of all elements of $K \cdot M_{\alpha+1}$ on all elements of $M_{\bar{\alpha}+1}$ with the following properties:

A. *g is singlevalued*, because if φ_α, ψ_α define the same set, we have $\varphi_\alpha(x) \equiv \psi_\alpha(x)$ for $x \in M_\alpha \cdot K$, hence $\varphi_{\bar{\alpha}}(x^1) \equiv \psi_{\bar{\alpha}}(x^1)$ by Th. 6, i.e., $\varphi_{\bar{\alpha}}$ and $\psi_{\bar{\alpha}}$ also define the same set.

B. $x \in y \equiv x^1 \in g(y)$ *for* $x \in K \cdot M_\alpha$, $y \in K \cdot M_{\alpha+1}$.
(by Th. 6)

C. *g is one to one*, because if $x, y \in K \cdot M_{\alpha+1}$, $x \neq y$ then by Th. 3 there is a $z \in (x - y) + (y - x)$, $z \in K \cdot M_\alpha$, hence $z^1 \in [g(x) - g(y)] + [g(y) - g(x)]$ by B. Hence $g(x) \neq g(y)$.

D. *g is an extension of the mapping f, i.e.,* $g(x) = x^1$ *for* $x \in K \cdot M_\alpha$.

PROOF: For any $b \in K \cdot M_\alpha$ a corresponding φ_α which defines it is $x \in b$, hence $\varphi_{\bar{\alpha}}$ is $x \in b^1$ hence $g(b) = b^1$.

E. *g maps* $K \cdot M_\alpha$ *exactly on* $M_{\bar{\alpha}}$ *(by D)*[9] *and therefore,* $K(M_{\alpha+1} - M_\alpha)$ *on* $M_{\bar{\alpha}+1} - M_{\bar{\alpha}}$ *by C.*

F. *g is isomorphic for* ϵ, *i.e.,* $g(x) \in g(y) \equiv x \in y$ *for any* $x, y \in K \cdot M_{\alpha+1}$.

PROOF: If $x \in K \cdot M_\alpha$, this follows from B and D, if $x \in K \cdot (M_{\alpha+1} - M_\alpha)$ then $g(x) \in M_{\bar{\alpha}+1} - M_{\bar{\alpha}}$ by E, hence both sides of the equivalence are false by Th. 1.

By D and F, g is the desired extension of f and hence the existence of an isomorphic mapping x' of K on M_η follows by complete induction. Furthermore since all ordinals $< \omega_\alpha$ belong to O (by I, II) we have $\bar{\beta} = \beta$ for $\beta < \omega_\alpha$ from which it follows easily that $x = x'$ for $x \in M_{\omega_\alpha}$. This finishes the proof of Th. 5.

Now in order to prove Th. 2 consider the set m' corresponding to m in the isomorphic mapping of K on M_η. Its order is $< \eta < \omega_{\alpha+1}$, because $m' \in M_\eta$ and $\bar{\bar{\eta}} = \bar{\bar{O}} \leq \aleph_\alpha$ by Th. 4. Since $x \in m \equiv x' \in m'$ for $x \in K$, we have $x \in m \equiv x \in m'$ for $x \in M_{\omega_\alpha}$ by Th. 5. Since furthermore $m \subseteq M_{\omega_\alpha}$ it follows that $m = m' \cdot M_{\omega_\alpha}$, i.e., m is an intersection of two sets of order $< \omega_{\alpha+1}$, which implies trivially that it has an order $< \omega_{\alpha+1}$.

TH. 7. M_{ω_ω} *considered as a model for set-theory satisfies all axioms of Zermelo*[10] *except perhaps the axiom of choice and* M_Ω *(*Ω *being the first inaccessible number) satisfies in addition the axiom of substitution, if in both cases "definite Eigenschaft" resp. "definite Relation" is identified with "prop. fnct. over the class of all sets" (with one resp. two free variables).*

Sketch of proof for M_{ω_ω}: ax. I, II are trivial, ax. VII is satisfied by $Z = M_\omega$, ax. III–V have the form $(\exists x)(u)[u \in x \equiv \varphi(u)]$, where the φ are certain prop. fnct. over M_{ω_ω}. Hence, by def. of $M_{\alpha+1}$ there exist sets x in $M_{\omega_\omega+1}$ satisfying the axioms. But from Th. 1 and Th. 2 it follows easily, that the order of x is smaller than ω_ω for the particular φ under consideration, so that there exist sets x in the model satisfying the axioms.

For M_Ω ax. I–V and VII are proved in exactly the same way and the axiom of subst. is proved by the same method as ax. III–V. Now denote by "A" the proposition "There exist no non-constructible sets"[11] by "R" the axiom of choice and by "C" the proposition "$2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for any ordinal α." Then we have:

TH. 8. A $\supset$ R *and* A $\supset$ C.

A $\supset$ R follows because for the constr. sets a well-ordering can be defined and A $\supset$ C holds by Th.2, because $\bar{\bar{M}}_{\omega_\alpha} = \aleph_\alpha$.

Now the notion of "constr. set" can be defined and its theory developed in the formal systems of set theory themselves. In particular Th. 2 and, therefore, Th. 8 can be proved from the axioms of set theory. Denote the notion of "constr. set" relativized for a model M of set theory (i.e., defined in terms of the ϵ-relation of the model) by *constr.*$_M$, then we have:

TH. 9. *Any element of M_{ω_ω} (resp. M_Ω) is constr.$_{M_{\omega_\omega}}$ (resp. constr.$_{M_\Omega}$); in other words: A is true in the models M_{ω_ω} and M_Ω.*

The proof is based on the following two facts: 1. The operation M' (defined on p. 220) is absolute in the sense that the operation relativized for the Model M_{ω_ω}, applied to an $x \epsilon M_{\omega_\omega}$ gives the same result as the original operation (similarly for M_Ω). 2. The set N_α which has as elements all the M_β (for $\beta < \alpha$) is constr.$_{M_{\omega_\omega}}$ for $\alpha < \omega_\omega$ and constr.$_{M_\Omega}$ for $\alpha < \Omega$, as is easily seen by an induction on α. From Th. 9 and the provability (from the axioms of set theory) of Th. 8 it follows:

TH. 10. *R and C are true for the models M_{ω_ω} and M_Ω.*

The construction of M_{ω_ω} and M_Ω and the proof for Th. 7 and Th. 9 (therefore also for Th. 10) can (after certain slight modifications)[12] be accomplished in the resp. formal systems of set theory (without the axiom of choice), so that a contradiction derived from C, R, A and the other axioms would lead to a contradiction in set theory without C, R, A.

[1] This paper gives a sketch of the consistency proof for propositions 1, 2 of *Proc. Nat. Acad. Sci.*, **24**, 556 (1938), if T is Zermelo's system of axioms for set theory (*Math. Ann.*, **65**, 261) with or without axiom of substitution and if Zermelo's notion of "Definite Eigenschaft" is identified with "propositional function over the system of all sets." Cf. the first definition of this paper.

[2] It is assumed that for any element of M a symbol denoting it can be introduced.

[3] At first with each φ_α an equivalent normal form of the above type has to be associated, which can easily be done.

[4] In the well-ordering of the constr. sets.

[5] $\overline{\overline{m}}$ means "power of m."

[6] I.e., $x \epsilon y \equiv f(x) \epsilon f(y)$. In the following proof $f(x)$ is abbreviated by x^1.

[7] I.e., of the elements of order $<\alpha$ of K on the elements of order $<\overline{\alpha}$ of M_η.

[8] I.e., of the elements of order $\leq\alpha$ of K on the elements of order $\leq\overline{\alpha}$ of M_η.

[9] Because f maps $K \cdot M_\alpha$ on $M_{\overline{\alpha}}$ by induct. assumpt.

[10] Cf. *Math. Ann.*, **65**, 261 (1908).

[11] In order to give A an intuitive meaning, one has to understand by "sets" all objects obtained by building up the simplified hierarchy of types on an empty set of individuals (including types of arbitrary transfinite orders).

[12] In particular for the system without the axiom of substitution we have to consider instead of M_{ω_ω} an isomorphic image of it (with some other relation R instead of the ϵ-relation), because M_{ω_ω} contains sets of infinite type, whose existence cannot be proved without the axiom of subst. The same device is needed for proving the consistency of prop. 3, 4 of the paper quoted in footnote 1.

[13] Th. 3, 4, 5, are lemmas for the proof of Th. 2.

* Unless explicitly stated otherwise "prop. fnct." always means "propositional function with one free variable."

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THE MORDELL-LANG CONJECTURE FOR FUNCTION FIELDS

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1. Introduction

In [La65], Lang formulated a hypothesis including as special cases the Mordell conjecture concerning rational points on curves, and the Manin-Mumford conjecture on torsion points of Abelian varieties. Sometimes generalized to semi-Abelian varieties, and to positive characteristic, this has been called the Mordell-Lang conjecture; see [AV91] and [La91]. It is essentially a finiteness statement on the intersection of a subvariety of a semi-Abelian variety with a subgroup of finite rank. We prove here the function-field version of the conjecture, in any characteristic.

In characteristic 0, the Mordell-Lang conjecture was proved in a sequence of papers by Raynaud, Faltings and Vojta, at least in the case of Abelian varieties or finitely generated groups. The full result was proved over number fields, and these cases were inferred for function fields using a specialization argument; see [La91] for a description. For Abelian varieties in characteristic 0, a quite different argument was found by Buium; this inspired our approach. In positive characteristic, many cases were proved in [AV91]. This paper presents a uniform proof incorporating all cases, using model theoretic ideals. We describe the strategy following the statement of the result (equivalent to the statement in [AV91]).

In this statement, and in the entire paper, we use the language of varieties rather than schemes. We refer to the absolute Zariski topology unless otherwise stated, and generally use terms in their geometric sense (over an algebraically closed field).

Let K/k be a field extension, k algebraically closed. Call a group Γ *p'-finitely generated* if $Q_p \otimes \Gamma$ is finitely generated as a Q_p-module, where $Q_p = \mathbf{Q}$ if $p = 0$, and $Q_p = \{m/n \in \mathbf{Q} : n \text{ prime to } p\}$ if $p > 0$. This condition is of course valid for finitely generated Abelian groups, and for prime-to-p-torsion groups. Recall that a semi-Abelian variety is an extension of an Abelian variety by an algebraic torus.

Theorem 1.1. *Let S be a semi-Abelian variety defined over K, and X a subvariety of S. Let Γ be a p'-finitely generated subgroup of S. Suppose $X \cap \Gamma$ is Zariski dense in X. Then there exists a semi-Abelian variety S_0 defined over k, a subvariety X_0 of S_0 defined over k, and a rational homomorphism h from a group subvariety of S into S_0, such that X is a translate of $h^{-1}(X_0)$.*

Call a subvariety of S satisfying the conclusion *special*. The theorem then states that for any subvariety Z of S, $Z \cap \Gamma$ is contained in a finite union of special

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subvarieties of Z. (To see this, apply the theorem to the irreducible components X of the Zariski closure of $Z \cap \Gamma$.) If S is an Abelian variety with K/k trace 0, the special subvarieties are just the cosets of Abelian subvarieties of S. In this case the theorem reads as follows.

Corollary 1.2. *Let k be an algebraically closed field, $\underline{A}$ an Abelian variety with no nonzero homomorphic images defined over k, $\underline{X}$ a subvariety of $\underline{A}$, and Γ a p'-finitely generated subgroup of A. Then $\underline{X}(K) \cap \Gamma = \underline{Y}(K) \cap \Gamma$, where $\underline{Y}$ is a (reducible) subvariety of $\underline{X}$, equal to a finite union of translates of Abelian subvarieties of $\underline{A}$.*

See also §6 for some quantitative statements, one in the presence of a discrete valuation of K/k and hence a distance function, another giving an exponential bound on the number of translates in 1.2 in terms of the rank of Γ (the other data being fixed), in fixed positive characteristic.

We remark here that the characteristic zero geometric Mordell-Lang conjecture can be deduced by specialization arguments from the positive characteristic case. This was suggested in [La91], following the description of Voloch's theorem; Lang says one would need to know something on existence of ordinary specializations, but that's because of the ordinariness hypothesis in Voloch's result, no longer present. (Some other issues arising from the higher dimension and non-finite-generation need to be addressed, but this can be done using methods of Neron and Raynaud.) However we will not take this route, and will give a direct proof in characteristic zero. The exponential growth with rk(Γ) alluded to above does not appear to ascend to characteristic zero by this method.

Our opening move is taken from Buium ([Bu92]), who proved certain cases of the theorem, including Corollary 1.2 in characteristic 0, using differential algebra. A differential field should be regarded as an infinite-dimensional object, and in particular an Abelian variety can no longer be viewed as finite-dimensional in an absolute sense. However, using a homomorphism introduced by Manin ([Ma58], [Ma63]), Buium points out that Γ is contained in a certain finite-dimensional group. At this point he proceeds to use some of his theory of finite-dimensional Kolchin closed groups (but also some analysis). Our proof uses instead the model theory of abstract finite-dimensional groups ("groups of finite Morley rank").

An algebraic structure (e.g. a group with a distinguished subset) is said to have finite Morley dimension if one can assign an integer dimension to the "definable subsets" so that certain natural conditions are satisfied. Such structures have been analyzed, in the abstract, by model theorists, motivated initially by categoricity questions. Examples are provided by algebraic varieties over an algebraically closed field, with the usual dimension theory. However we apply the theory to the kernel of the Manin homomorphism and certain related groups, that do not a priori carry the structure of an algebraic variety. Following some reductions, we apply a general dichotomy theorem of B. Zilber and the author. This theorem implies that an enriched group satisfying the appropriate dimension-theoretic axioms is either a module over a certain local ring, with no additional structure, or it carries the structure of an algebraic group over an algebraically closed field. This dichotomy leads to the two kinds of subvarieties mentioned implicitly in the theorem: group subvarieties, and ones arising from varieties defined over the (algebraically closed) constant field.

In positive characteristic, we do not explicitly use differential algebra. We work instead with fields K of some fixed finite dimension over K^p, endowed with a distinguished basis for K/K^p. The role of the kernel of the Manin homomorphism is played by the group $p^\infty A(F)$ of infinitely p-divisible points; F is a large field that will be described below. It can be shown that for each n, there is a map from $A(F)$ into a vector group over the truncated Witt vector ring $W_n(F)$ whose kernel is $p^n A(F)$ (up to finite index). We note that Manin asks in [Ma58] for a positive characteristic analog of the Manin homomorphism. We are not certain if we have found the "correct" analog. For us only the kernel $p^\infty A(F)$ plays a role; we show in §2 that it has finite Morley dimension, and enjoys a dimension theory as in the characteristic 0 case. Beyond this point, the proof is uniform in the different characteristics. (Indeed it appears that the divergence in the proofs in the different characteristics is due merely to an accident of the historical development of model theory. Distinguishing a basis for K/K^p has the effect of fixing also a stack of Hasse derivations. One expects that quantifier elimination and elimination of imaginaries hold already in the differential language, without the distinguished basis, and in this language the proof should become entirely uniform with respect to the characteristic.)

We refer the reader to the short expository paper [NeP89] for basic model-theoretic notions, and to [Sa72] for proofs. The paper [HZ] also contains a short section summarizing some of the relevant definitions.

We proceed to describe sequentially the organization of the paper. We work throughout in a universal domain F. If the characteristic is 0, F is a field with a distinguished derivation. If it is p, F is a field with a distinguished p-basis (i.e. a distinguished basis of F over F^p); the basis is assumed finite, with p^ν elements. In either case we make the following assumption:

(*) Let F_0 be a countable differential field (respectively, a countable field of characteristic p with distinguished p-basis of size p^ν). Then F_0 embeds into F (preserving the derivation, or the p-basis). Moreover, any such embedding is unique, up to an automorphism of F.

Such structures are called λ-saturated ($\lambda > \omega$). They can be shown to exist by standard model theoretic methods; see [Sa72] (§16, §40) and [Del88], or [RR75]. (They also enjoy strong uniqueness properties, whose discussion however would be irrelevant here.) One immediate example of their usefulness for us is furnished by the following observation. Whereas in general the group of infinitely-p-divisible points of an Abelian variety over a separably closed field may be trivial, in the saturated model the group is large, and reflects to a certain extent the properties of the ambient Abelian variety.

In §2 we discuss the appropriate dimension theory, and describe a context in which it applies, over separably closed fields. In particular we deduce that the group of infinitely-p-divisible points mentioned above falls into this framework (Lemma 2.15).

In §3 we state the main theorem in the language of differentially closed fields (following Buium's lead), or in the language of separably closed fields (in characteristic $p > 0$). We show that it implies Theorem 1.1 as stated.

In §4 we develop the required theory of Abelian groups of finite Morley dimension. The results of this section apply of course to commutative algebraic groups, but the main issues dealt with here do not exist in that context. Better examples may be found in the domain of complex tori; these can have some subquotients that are Abelian varieties and others that are not; the interaction between them, in the general context, forms the subject of study of the section.

In §5 we prove the main theorem. It is here that we use the powerful results from [HZ]. These results apply at or near dimension one, and it is the theory of §4 that permits the reduction to that level.

In §6 we give a quantitative variant of the result, conjectured by Voloch, in the presence of a valuation and a corresponding local proximity function. In the situation of Corollary 1.2, we show that the distance from a point of Γ to X can be bounded in terms of its distance to Y.

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2. Thin types and Zariski geometries

The goal of this section is to introduce the various notions of finite dimension, and to show that the types inside the kernel group admit a dimension theory of the required type. The facts in characteristic 0 are largely classical model theory, see e.g. [Sa72]. The fact that minimal types are Zariski in the sense below (after removing a finite number of singularities) is shown in [HS]. Following the initial definitions, we will therefore concentrate on characteristic $p > 0$.

We work in a universal domain U. acl denotes algebraic closure in the model theoretic sense; so for countable X, $\mathrm{acl}(X)$ is the set of elements of U whose orbit under $\mathrm{Aut}(U/X)$ is finite. $\mathrm{dcl}(X)$ is the set of elements definable over X, or the set of elements of U fixed by $\mathrm{Aut}(U/X)$. If U is an algebraically closed field and $\langle X\rangle$ is the subfield generated by a subset X of U, then $\mathrm{acl}(X)$ is the algebraic closure of $\langle X\rangle$ in U, and $\mathrm{dcl}(X)$ is the perfect closure of $\langle X\rangle$.

We need to lightly modify standard model theoretic usage in order to make the language of Morley dimension apply in our context.

Definition 2.1. Let P be the solution set to a set of formulas of size $\leq \lambda$, in a λ^+-saturated model. A *definable subset of* P is a set of the form $D\cap P$, where D is a definable set (perhaps with parameters outside P). P is *minimal* if every definable subset of P is finite or cofinite. Let B be any set over which P is defined; the following notions are relative to B. If A is a small set of elements of P, let the rank $\mathrm{rk}_P(A)$ be the size of any maximal algebraically independent subset of A. In general, let $\mathrm{rk}_P(A) = \mathrm{rk}_P(\mathrm{acl}(A)\cap P)$. If $q = \mathrm{tp}(a_1, \ldots, a_m)$, let $\mathrm{rk}^P(q) = \mathrm{rk}(\{a_1, \ldots, a_m\})$. If D is a definable or ∞-definable subset of P^n, define $\mathrm{rk}^P(D) = \max\{\mathrm{rk}(q) : q$ a type in $D\}$; an element of maximal rank of D is called *generic*. We omit P if its identity is clear.

P is called *pluriminimal* if it is a finite union of minimal types. It is called *semi-minimal* if there exists a minimal type Q and a finite set F such that $P \subseteq \mathrm{acl}(F, Q)$. Similarly define semi-pluriminimal.

Morley dimension is defined recursively as follows. P is said to have dimension -1 iff it is empty. P has dimension k if for some integer m, P cannot be split into $m+1$ definable subsets none of which have dimension $\leq (k-1)$. The smallest such m is called the Morley degree, or multiplicity, of P. (If U is an algebraically closed field, this agrees with Zariski dimension.)

Lemma 2.2 ("internal Morley dimension"). *Let P be minimal, and let D be a definable subset of P^n of rank k. There exists an integer $m =$ $\mathrm{Mult}(D)$ such that D cannot be split into $m+1$ pairwise disjoint definable subsets of rank k.*

Proof. It suffices to show that there are only finitely many types q in D with $\mathrm{rk}(q) = k$. Let $p^{[k]}$ be the type of a generic k-tuple from P. For each subset s of $\{1, \ldots, n\}$ of size k, consider the partial type $q_s(x_1, \ldots, x_n)$ asserting that $p^{[k]}$ holds of $(x_i : i \in s)$, $(x_1, \ldots, x_n) \in D$, and each $x_i \in P$. Let a_i $(i \in s)$ realize $p^{[k]}$. Then there are only finitely many choices of a_i $(i \notin s)$ such that $(a_i : i = 1, \ldots, n)$ realizes q_s. For otherwise there would be such a choice $(a_1, \ldots, a_n)$ with some a_j $(j \notin s)$ nonalgebraic over $(a_i : i \in s)$, so $\mathrm{rk}(a_1, \ldots, a_n) \geq k + 1$, contradicting $\mathrm{rk}(D) = k$. Thus each q_s has only finitely many complete extensions. Since every rank-k extension of D extends some q_s, there are only finitely many of these.

The lemma shows that for definable subsets of P^n, dimension equals rank. (If P is minimal.) □

Definition 2.3. Let T be a theory with quantifier elimination and P a minimal type. P is called *Zariski* if (i)–(iii) hold. (iii) is referred to as the "dimension theorem".

Call a subset of P^n *closed* if it is defined by a positive, quantifier-free formula. Call it *irreducible* if it is not the union of two proper closed subsets.

(i) Every closed set in P^n is the union of finitely many closed, irreducible sets.

(ii) If X is a proper subset of Y, both closed subsets of P^n, and Y is irreducible, then $\mathrm{rk}(X) < \mathrm{rk}(Y)$.

(iii) If X is a closed, irreducible subset of P^n, $\mathrm{rk}(X) = m$, and Y is a diagonal $x_i = x_j$, then $X \cap Y$ is the union of closed irreducible sets of dimension at least $m - 1$.

Remark 2.4. Under these conditions, the collection of closed sets of P^n defines a Noetherian topology, and the dimension of a closed set with respect to this topology equals its rank. For every definable set Y there exists a proper closed subset F of $\mathrm{cl}(Y)$ such that $Y - F = \mathrm{cl}(Y) - F$.

Proof. The first statement is clear: given an infinite descending chain of closed sets, the first can be assumed irreducible, hence the second has smaller rank, beginning an infinite descent of ranks. If X is irreducible, of rank k, one shows $\dim(X) \geq k$ by induction on k: X has a subset of rank $k - 1$; this subset can be chosen closed, irreducible, hence by induction has dimension $k - 1$, so X has dimension $\geq k$. The other inequality is immediate from (ii). For the final statement, note that Y is a finite union of sets $Y_i = H_i - F_i$, with H_i closed irreducible and F_i a proper closed subset. We may assume Y is not the union of fewer of these. Let $H = \bigcup_i H_i$, $F = \bigcup_i F_i$. Then $\mathrm{cl}(Y) = H$; and some H_i is not contained in any other H_j, hence not in F_i, and being irreducible, not in F. Thus F is a proper subset of H, and $H - F \subseteq Y$. □

Lemma 2.5. *Let T be a stable theory with a minimal type P. Assume P is Zariski and not locally modular. Then T interprets a field F, with definable subfields F_α, such that $\bigcap_\alpha F_\alpha$ is minimal, and nonorthogonal to P.*

Proof. This is proved in [HZ] when P is *strongly* minimal (i.e. it is the solution set of a single formula), but the proof goes through, and gives a type-definable field F^*, minimal and nonorthogonal to P. By [Hr90], there exists a definable field F and definable subfields F_α, such that $\bigcap_\alpha F_\alpha = F^*$.

We require here only the results of §6 of [HZ]. This section is written largely axiomatically, the axioms having been proved in §4 and §5. At the end of this section (2.20), we will indicate how one can shortcut this and directly prove the axioms in the present contex. □

2.6 **Conventions on separably closed fields.** Let $T(p,\nu)$ be the theory of separably closed fields F of char $p > 0$, of finite dimension p^ν over F^p. (ν is the Ersov invariant.) We endow F with a basis $e_1, \ldots, e_{p^\nu}$ of F over F^p. The language is the language of rings with p^ν distinguished constants, for the e_i, as well as function symbols for certain definable functions λ_n, described in 2.7(a) below. We work in a universal domain $U = F$ for this theory (a countably saturated model). The notions of substructure and algebraic closure will be relative to F, in this language.

Usually we will denote algebraic varieties defined over F by an underlined capital letter, such as $\underline{V}$, and the group of F-points of $\underline{V}$ by V. We often implicitly assume that $\underline{V}$ is given with an affine chart, and so we discuss coordinates of elements of $\underline{V}$; in particular the set of F-points makes sense. We can also apply the Frobenius to $\underline{V}$. The variety obtained in this way will be denoted $\mathrm{Fr}\,\overline{V}$. If V is a definable group, the set of p-th powers of elements of V will be denoted V^p. $[V]^p$ denotes the set of p-tuples from V. When there is no danger of confusion with the previous two meanings, we revert to the notation V^p. $T(p,\nu)$ admits quantifier elimination and elimination of imaginaries (see [Del88]).

Fact 2.7 ([Del88]). In $T(p,\nu)$ there exist (basic) definable functions $\lambda_n : F \to F^{p^{\nu n}}$ with the following properties:

(a) λ_n is the inverse of the bijective morphism $r(x_1, \ldots, x_{p^{\nu n}}) = \sum x_i^{p^n} e_{n,i}$ for some designated basis $\{e_{n,i}\}$ of F over F^{p^n}. In particular, $\lambda_n = r_n \circ \lambda_{n+1}$ for a certain polynomial function $r_n : [F]^{p^{\nu(n+1)}} \to [F]^{p^{\nu n}}$.

(b) Any atomically definable subset of $[F]^k$ is, for some n, the pullback by λ_n^{-1} of some (possibly reducible) subvariety U of $[F^{p^{\nu n}}]^k$. Let $\mathrm{dcl}_f(X)$ denote the field generated by $X \cup \{e_1, \ldots, e_{p^\nu}\}$, and $\mathrm{acl}_f(X)$ the relative field-theoretic algebraic closure of $\mathrm{dcl}_f(X)$ in the field F.

(c) If $a \in \mathrm{acl}(b_1, \ldots, b_d)$, then $a \in \mathrm{acl}_f(\{\lambda_n b_i : i \le d, n < \infty\})$.

(d) If $a \in \mathrm{dcl}(b_1, \ldots, b_d)$, then $a \in \mathrm{dcl}_f(\{\lambda_n b_i : i \le d, n < \infty\})$.

Proof. (c) $\mathrm{acl}_f(\{\lambda_n b_i : i \le d, n < \infty\})$ is an elementary substructure of F, so it is (model-theoretically) algebraically closed.

(d) Suppose $a \in \mathrm{dcl}(b_1, \ldots, b_d)$. Let $k = \mathrm{dcl}_f(\{\lambda_n b_i : i \le d, n < \infty\})$. Note that k is a perfect field. The separable closure k_s of k is an elementary submodel of F, so $a \in k_s$, and every automorphism of k_s fixing k also fixes a. Hence $a \in k$. □

Definition 2.8. $\mathrm{tp}(c/B)$ is (k-)*thin* if $\mathrm{dcl}(c, B)$ is a field extension of $\mathrm{dcl}(B)$ of finite transcendence degree (at most k).

Lemma 2.9. *Let K be a countable subfield of F, $K = \mathrm{dcl}(K)$. $\mathrm{tp}(a/K)$ is k-thin iff the following condition holds:*

($*$) for all powers q of $p = \mathrm{char}(F)$ there exists a subfield L of F containing K and of transcendence degree $\le k$ over K, such that $a \in KL^q$.

Proof. If $\mathrm{tp}(a/K)$ is k-thin, given $q = p^n$, let $L = K(\lambda_n(a))$. Then since each basis element of F/F^q lies in K, $a \in KL^q$. For the converse, we require a claim.

Claim. If $a \in KL^q$ and $q = p^{n+m}$, then each coordinate of $\lambda_n(a)$ is in KL^{p^m}.

Proof. We have $a = f(b)/g(b)$, where $f, g \in K[X]$, a polynomial ring, and b is from L^q. Let $c \in L^{p^m}$ be such that $c^{p^n} = b$. We can write $a = f(b)g(b)^{p^n-1}/g(b)^{p^n}$, so we may assume $g \in K[X]^{p^n}$. Let $\{e_i\}$ be the chosen basis for F/F^{p^n}, and write $f = \sum e_i f_i^{[p^n]}$, where $f_i \in F[X]$ and $f_i^{[p^n]}$ is the result of taking the p^n-th power of each coefficient. Since $K = \mathrm{dcl}(K)$, $f_i \in K[X]$ for each i. Clearly

$$\begin{aligned}\lambda_n(a) = \lambda_n(f(b))/g(b) &= \lambda_n\left(\sum e_i f_i^{[p^n]}(c^{p^n})\right)/g(b)\\ &= \lambda_n\left(\sum e_i f_i(c)^{p^n}\right)/g(b) = (f_1(c)/g(b), \dots, f_{p^{\nu n}}(c)/g(b)).\end{aligned}$$

Evidently all the coordinates are in KL^{p^m}.

Now if $(*)$ holds, let L_n witness $(*)$ for $q = p^n$. Then $\operatorname{tr\,deg}(\mathrm{dcl}(a)/K) = \sup \operatorname{tr\,deg} K(\lambda_n(a))/K \le \operatorname{tr\,deg} L_n/K \le k$. □

Lemma 2.10. (i) *If* $\mathrm{tp}(a/K)$ *is* k*-thin,* $K \subseteq K'$*, then* $\mathrm{tp}(a/K')$ *is* k*-thin.*

(ii) *If* $\mathrm{tp}(a_i/K)$ *is* k_i*-thin,* $a \in \mathrm{acl}_f(a_1, \dots, a_m)$*,* $k = \sum k_i$*, then* $\mathrm{tp}(a/K)$ *is* k*-thin.*

(iii) *If* $\mathrm{tp}(a/K)$ *is* k*-thin, then so is* $\mathrm{tp}(\lambda_n a/K)$*.*

(iv) *If* $\mathrm{tp}(a_i/K)$ *is* k_i*-thin,* $a \in \mathrm{acl}(a_1, \dots, a_m)$*,* $k = \sum k_i$*, then* $\mathrm{tp}(a/K)$ *is* k*-thin.*

Proof. (i), (ii), (iii) are clear. (iv) follows from (ii), (iii), and (c) of Fact 2.7. □

Recall the definition of $U(a/B)$ ([Las]). We say that $U(a/B) = 0$ if $a \in \mathrm{acl}(B)$; $U(a/B) \le n+1$ if for all B' containing B such that a forks with B' over B, $U(a/B') \le n$; $U(a/B) = n$ if it is $\le n$ but not $\le (n-1)$. In the present context, "a forks with B' over B" means that $\mathrm{dcl}(a, B)$ is *not* free from $\mathrm{dcl}(B')$ over $\mathrm{dcl}(B)$; see [Del88]. Note that $U(a/B) = 1$ iff $\mathrm{tp}(a/\mathrm{acl}(B))$ is minimal.

Lemma 2.11. *If* $\mathrm{tp}(a/B)$ *is* k*-thin, then* $U(a/B) \le k$*.*

Proof. By induction on k, this is immediate from the above definitions. □

Proposition 2.12. *A thin minimal type in* $T(p, \nu)$ *is Zariski.*

Proof. The assumption here is that the type is complete over some base substructure K.

The first claim takes place over an algebraically closed field.

Claim 2.12.1. Let U, V be irreducible smooth varieties of the same dimension, and $f : U \to V$ a finite rational map (defined everywhere on U). Let C be a closed irreducible subset of V^n. Then all components of $f^{-1}C$ have the same dimension (equal to $\dim(C)$).

Proof. The induced map $f : U^n \to V^n$ satisfies the same assumptions as f, so we may assume $n = 1$. Note that the graph F of f is a closed irreducible subset of $U \times V$ (because we have a surjective rational map from U to F). $f^{-1}C$ is isomorphic to $F \cap (U \times C)$. Let X be a component of $F \cap (U \times C)$. By the dimension theorem on $U \times V$,

$$\begin{aligned}\dim X &\ge \dim F + \dim(U \times C) - \dim(U \times V)\\ &= \dim(U) + \dim(U) + \dim(C) - 2\dim(U) = \dim(C).\end{aligned}$$

On the other hand the projection is a finite map from X to C, so $\dim X \le \dim C$. □

We now work in a universal domain for $T(p,\nu)$. Using elimination of imaginaries and the existence of a definable pairing function, it suffices to consider 1-types. Let P be a 1-type over K, K an algebraically closed, countable substructure of the universal domain.

Let P_n be the Zariski closure of $\lambda_n P$, a subvariety of $[F]^{p^{\nu n}}$. r_n maps P_{n+1} to P_n. Since P is thin, for large enough n, say $n \geq N\#$, r_n is finite-to-one above some Zariski open subset P_n^* of P_n. We may choose P_n^* smooth; and by defining P_n^* inductively, we can arrange that r_n carries P_{n+1}^* into P_n^*. Let $e = \dim P_n$ for large n. If $U \subseteq P_n^k$, write $\lambda_n^* U$ for $\{x \in P^k : \lambda_n x \in U\}$, where $\lambda_n(x_1,\ldots,x_k) = (\lambda_n x_1,\ldots,\lambda_n x_k)$.

Let X be a closed subset of P^k. We will show that (i) of the definition of Zariski geometry holds by induction on $\mathrm{rk}(X)$. Let X_n be the Zariski closure in $[P_n]^k$ of $\lambda_n X$. Then for all large enough n, $X = \lambda_n^* X_n$.

Claim 2.12.2. $\dim(X_n) = e\cdot \mathrm{rk}(X)$ for large n. (More specifically, for any $n \geq N\#$ such that $X = \lambda_n^* X_n$.)

Proof. Let $(a_1,\ldots,a_k)$ be a point of X, of rank $d = \mathrm{rk}(X)$. Say $a_1,\ldots,a_d$ are algebraically independent, so $a \in \mathrm{acl}(a_1,\ldots,a_d)$. By (c), for all k and j, $\lambda_k a_j \in \mathrm{acl}_f(\{\lambda_n a_i : i \leq d, n < \infty\})$. But since P is thin, for some $N\#$, $\mathrm{acl}_f(\{\lambda_n a_i : n < \infty\}) = \mathrm{acl}_f(\lambda_{N\#} a_i)$. Thus for $n \geq N\#$, $\lambda_n a_i \in \mathrm{acl}_f(\lambda_n a_1,\ldots,\lambda_n a_d)$. Since this holds for any $(a_1,\ldots,a_k) \in X$, by compactness it holds in one of a finite number of ways, and so persists to the Zariski closure: $y_i \in \mathrm{acl}_f(y_1,\ldots,y_d)$ for any $(y_1,\ldots,y_k)$ in the Zariski closure X_n of $\lambda_n X$, after some permutation of indices. Thus $\dim X_n \leq d\cdot\dim(P_n)$. On the other hand for $i \leq d$, the algebraic locus of $\lambda_n a_i$ over $\{\lambda_n a_1,\ldots,\lambda_n a_{i-1}\}$ contains $\lambda_n b$ for all but finitely many $b \in P$; hence it equals P_n. Thus

$$\dim X_n = d\cdot\dim(P_n) = de = e\cdot\mathrm{rk}(X). \quad \square$$

We consider only $n \geq N\#$. Write $X_n = X_n(1)\cup\cdots\cup X_n(m_n)\cup Y_n$, where $X_n(i)$ are the distinct components of X_n of dimension de. Let $X(n;i) = \lambda_n^* X_n(i)$. Then $X_n(i)$ is the Zariski closure of $\lambda_n X(n;i)$. By 2.12.2 applied to $X(n;i)$, we have $\mathrm{rk}\,X(n;i) = \dim X_n(i)/e = \mathrm{rk}(X)$. Similarly the intersection of $X(n;i)$ with $X(n;i')$ $(i \neq i')$ has smaller rank. By Lemma 2.2, the number m_n is bounded independently of n. By Claim 1, $r_n^{-1}X_n(i)$ is a union of some of the components $X_{n+1}(j)$. So m_n is nondecreasing with n; so for n above some N^* $(\geq N\#)$ it reaches a constant maximum m^*. It follows that for $n \geq N^*$, $r_n^{-1}X_n(i)$ equals some $X_{n+1}(j)$; we recursively renumber so that $r_n^{-1}X_n(i) = X_{n+1}(i)$. So $X(N^*;i) = X(N^*+1;i) = \cdots \stackrel{\mathrm{def}}{=} X(i)$. Let $Y = \lambda_{N^*}^* Y_{N^*}$. Then $X = X(1)\cup\cdots\cup X(m^*)\cup Y$, Y is closed of rank smaller than $\mathrm{rk}(X)$, and $X(i)$ is closed of rank equal to $\mathrm{rk}(X)$. It remains only to show that $X(i)$ is irreducible. Suppose $X(i) = U\cup V$, with U, V closed. Pick $n \geq N^*$ such that $U = \lambda_n^* U_n$, $V = \lambda_n^* V_n$, with U_n, V_n the Zariski closures in P_n^{*k} of $\lambda_n U$, $\lambda_n V$ respectively. Then $X_n(i) = U_n \cup V_n$. Since $X_n(i)$ is irreducible, it equals one of them; say $X_n(i) = U_n$. Then in $[P]^k$,

$$X_i = \lambda_n^* X_n(i) = \lambda_n^* U_n = U.$$

We have shown that any closed set X is a finite union of irreducible components. Further the proof showed that if X is irreducible, then $X = \lambda_n^* X_n$ for large n, where X_n is Zariski closed and irreducible of dimension $e\cdot\mathrm{rk}(X)$. So if Y is irreducible, and X is a proper subset of Y, then (with the parallel notation) X_n must be a proper subset of Y_n, so $e \cdot \mathrm{rk}(X) < e \cdot \mathrm{rk}(Y)$, and $\mathrm{rk}(X) < \mathrm{rk}(Y)$.

It remains to prove the "dimension theorem" 2.3(iii). Let X, X' be closed irreducible subsets of P^k, $\mathrm{rk}\, X = d$, $\mathrm{rk}\, X' = d'$, and let Y be a component of $X \cap X'$. We must show: $\mathrm{rk}\, Y \geq d + d' - k$. Let the notation X_n, X'_n, etc. be as above, and let n be large. Let $Y_n\#$ be the component of $X_n \cap X'_n$ containing $\lambda_n Y$. We have $\dim(X_n) = de$, $\dim(X'_n) = d'e$. By the dimension theorem for the ke-dimensional smooth algebraic variety P_n^{*k},

$$\dim Y_n\# \geq de + d'e - ke = e(d + d' - k).$$

Evidently $r_n^{-1} Y_n\# = Y_{n+1}\#$. Hence $\dim Y_n\#$ is nondecreasing with n, so we choose n large enough that $\dim Y_n\# = \dim Y_N\#$ for $N \geq n$. However, we do not yet know that $\lambda^n Y$ is Zariski dense in $Y_n\#$. (And this would not be true without thinness.) The problem may arise that for generic c in $Y_n\#$, for some $q = p^l$ and some polynomials f_i over K, $\sum f_i(c)^q e_{l,i} = 0$; in this case, every F-point c' of $Y_n\#$ must also satisfy $f_i(c') = 0$ for each i, and this may force c' to lie on a proper subvariety of C_n. To rule out this scenario we apply Claim 2 again.

Let X be a set of variables appropriate for describing elements of P_n^k; $X = (X_1, \ldots, X_k)$ where each $X_i = X_{i,1}, \ldots, X_{i,p^n}$. Let $I\#, I, I'$ denote the ideals of $K[X]$ vanishing on $Y_n\#, X_n, X'_n$ respectively.

Claim 2.12.3. If $\sum f_i^q e_{l,i} \in I(Y_n\#)$, then $f_i \in I(Y_n\#)$ for each i.

Proof. $I\#$ is one of the prime components of $\sqrt{(I + I')}$, so there exists $h \notin I\#$ and an integer s such that $(h(\sum f_i^q e_{l,i}))^s \in I + I'$. By enlarging s, we may assume it is a power of p. Replacing q by qs, h by h^s, and l by l' (where $p^{l'} = qs$), we may assume $s = 1$ (note that $e_{l,i}^s$ are some of the $e_{l',j}$). Multiplying by h^{q-1} we have $h^q(\sum f_i^q e_{l,i}) \in I + I'$, so $\sum (f_i h)^q e_{l,i} \in I + I'$. Now if we show that $f_i h \in I\#$ for each i, then also $f_i \in I\#$, as $I\#$ is a prime ideal. Thus we may take $h = 1$. So $\sum f_i^q e_{l,i} \in I + I'$. Say

(i) $\sum f_i^q e_{l,i} = g + g'$, $g \in I$, $g' \in I'$.

At this point we make a change of variables. Let $N = n + l$, and let $r = r_n r_{n+1} \cdots r_{N-1}$, so that $\lambda_n = r \lambda_N$. Let $Y = (Y_1, \ldots, Y_{p^N})$ be a set of variables appropriate for elements of P_N. Let $r^* : K[X] \to K[Y]$ be dual to r; it carries $K[X]$ into $K[Y^q]$. Since r takes X_N into X_n, r^* takes I into $I(X_N)$, and similarly I' into $I(X'_N)$. Thus $r^*(g) \in I(X_N)$, $r^*(g') \in I(X'_N)$. We have:

(ii) $\sum r^*(f_i)^q e_{l,i} = r^*(g) + r^*(g')$.

Now we can decompose $r^*(g) = \sum e_{l,i} H_i$, with $H_i \in K^q[Y]$. Since $r^*(g)$ is in $K[Y^q]$, so is each H_i; so we may write $r^*(g) = \sum e_{l,i} h_i^q$, with $h_i \in K[Y]$. Similarly $r^*(g') = \sum e_{l,i} h_i'^q$. Since $r^*(g) \in I(X_N)$, and $\lambda_N(X)$ is Zariski dense in X_n, we have $h_i \in I(X_N)$ for each i (as was argued above). Similarly $r^*(g') = \sum e_{l,i} h_i'^q$, with $h'_i \in I(X'_N)$. Now

(iii) $\sum r^*(f_i)^q e_{l,i} = \sum (h_i + h'_i)^q e_{l,i}$.

Comparing coefficients of each monomial and using the fact that the $e_{l,i}$ are linearly independent over K^q, this equality of polynomials implies that $r^*(f_i) = h_i + h'_i$ for each i. Thus $r^*(f_i) \in I(X_N) + I(X'_N) \subseteq I(Y_N\#)$. But $\dim(Y_N\#) = \dim(Y_n\#)$, and r is generically finite-to-one, so $rY_N\#$ is Zariski dense in $Y_n\#$. Thus r^* induces a 1-1 map from $K[X]/I(Y_n\#)$ to $K[Y]/I(Y_N\#)$, so $r^*(f_i) \in I(Y_N\#)$ implies $f_i \in I(Y_n)\# = I\#$. This finishes the proof of the claim. □

It follows from the claim that $\lambda_n(Y)$ is Zariski dense in $Y_n\#$, so $Y_n\# = Y_n$, and $\mathrm{rk}(Y)\cdot e = \dim(Y_n) = \dim(Y_n\#) \geq e(d+d'-k)$, so $\mathrm{rk}(Y) \geq d+d'-k$, as required by the dimension theorem. □

Corollary 2.13. *Let $T(p,\nu)$ be the theory of separably closed fields F of* char $p > 0$, *Ersov invariant $\nu < \infty$. Let P be a minimal thin type in $T(p,\nu)$. If P is not locally modular, then P is nonorthogonal to $F^{p^\infty} = \bigcap\{F^q\colon q \text{ a power of } p\}$.*

Proof. By [Wood79] $T(p,\nu)$ is stable. By Lemma 2.5 we may assume $P = \bigcap_n P_n$, where P_n is a subfield of some definable field K. By [Mes], 3.6, K is definably isomorphic to a finite extension of F, hence is a subfield of $F^{1/q}$ for some q. So we may assume K is a subfield of F. By [Mes], 3.1, each P_n contains some F^q. Hence P contains F^{p^∞}. Since P is minimal, $P = F^{p^\infty}$. □

By an ∞-definable subgroup of a group G we will mean the intersection of countably many definable subgroups. A result from [Hr90] (mentioned above for fields) states that an ∞-definable subset, which is also a subgroup, is an ∞-definable subgroup in this sense.

Remark 2.14. The following will emerge in §5: Let F be a saturated model of $T(p,\nu)$. Let $\underline{G}$ be a connected algebraic group defined over F, $G = \underline{G}(F)$, and A an ∞-definable subgroup of G. Assume A has no proper nontrivial definable subgroups, and has thin generic type. Then either A is minimal and locally modular, or $\underline{G}$ is isogenous (as an algebraic group) to a group $\underline{H}$ defined over F^{p^∞}, by an isomorphism carrying A to $H(F^{p^\infty})$.

Let F be a saturated separably closed field. The following lemma shows (using 2.12) that the group of infinitely-divisible points of a commutative algebraic group over F has finite Morley dimension.

Lemma 2.15. *Let $\underline{G}$ be a k-dimensional commutative algebraic group defined over F, $G = \underline{G}(F)$. Let $A = p^\infty G = \bigcap_n p^n G$. Then any generic type of A is k-thin.*

Proof. Say A is defined over $K = \mathrm{dcl}(K)$. Let $a \in A$. Given $q = p^n$, let $a = qb$ with $b \in G$. By Weil's theorem on symmetric functions, $a \in K(b^q)$ (this is in fact just the content of Lemma 4 in Chapter 1 of [Weil48]). This proves the criterion of Lemma 2.9. □

The following lemmas will inform the ensuing discussion but will not be explicitly used; they are included here in order to clarify the picture.

Lemma 2.16. *Let $\underline{G}$ be a simple Abelian variety defined over F, but not isomorphic to one defined over F^{p^∞}. Let $G = \underline{G}(F)$ and $A = p^\infty G$. Then A has no proper infinite definable subgroups.*

Proof. We work over a relatively algebraically closed substructure, over which $\underline{G}$ is defined. Let $e = \dim(\underline{G})$. It suffices to show that A is contained in every infinite definable subgroup of G. Let H be such a definable subgroup. Let $G_n = \lambda_n G$, and let $\underline{H}_n$ be the Zariski closure of $\lambda_n H$. Then for large enough n, $H = G \cap \lambda_n^{-1}\underline{H}_n(F)$. We will say something about F, G, A, and H in turn.

Let $K = F^{p^{-n}}$. Then $K = F[e_1^{p^{-n}},\dots,e_\nu^{p^{-n}}]$. Let $\underline{F}$ be the algebraic closure of F, and let $\underline{R}$ be the ring $\underline{F}[e_1^{p^{-n}},\dots,e_\nu^{p^{-n}}]$. Then $\underline{R}$ can be viewed as an affine algebraic ring, and K as the set of F-rational points of $\underline{R}$. Note that $\underline{R}$ is isomorphic, over $\underline{F}$, to $\underline{F}[u_1,\dots,u_\nu]$, where the u_i are commuting infinitesimals: $u_i^{p^n} = 0$.

By taking p^n-th roots of points of $\lambda_n G$, we identify $\lambda_n G$ with the set of K-points of $\underline{G}_1 = \mathrm{Fr}^{-n}(\underline{G})$. Viewing K as $\underline{R}(F)$ as above, we see that $\lambda_n G$ can be identified with $\underline{G}_1(\underline{R}(F))$. Now the "composition" $\underline{G}_2 = \underline{G}_1(\underline{R})$ is another group scheme, and since $\underline{R}(\underline{F})$ is isomorphic to $\underline{F}[u_1, \ldots, u_\nu]$, $\underline{G}_2$ is an extension of $\underline{G}_1$ by a commutative unipotent group $\underline{G}_u$. Let $\underline{G}_3$ be the closed subgroup of $p^\infty \underline{G}_2$. Then $\underline{G}_3$ meets $\underline{G}_u$ in a finite group. Hence $\underline{G}_3$ is isogenous to $\underline{G}_1$ (over $\underline{F}$). So $\underline{G}_3$ is a simple Abelian variety of dimension e.

Let $\underline{A}_n$ be the Zariski closure of $\lambda_n(A)$. Then $\lambda_n(A) \subseteq p\underline{A}_n$, so $p\underline{A}_n = \underline{A}_n$, and hence $\underline{A}_n \subseteq p^\infty \underline{G}_2(\underline{F}) = \underline{G}_3(\underline{F})$. Since $\underline{G}_3$ is simple, $\underline{A}_n = \underline{G}_3$.

It follows that $\underline{H}_n \cap \underline{A}_n$ is finite, or else $\underline{H}_n$ contains $\underline{A}_n$. In the first case, H contains only finitely many p^n-th powers. In particular H^{p^n} is finite. Since the p^n-torsion points of $\underline{G}$ are finite in number, H is finite; a contradiction. So $\underline{H}_n$ contains $\underline{A}_n$, hence $H = G \cap \lambda_n^{-1} \underline{H}_n(F)$ contains A. □

Notation. $G_{[q]} = \{x \in G : qx = 0\}$.

Lemma 2.17. *Let $\underline{G}$ be a commutative algebraic group, and A an ∞-definable subgroup of $G = \underline{G}(F)$ of finite Morley dimension, or just: with no properly descending sequence of definable subgroups. Then $A \subseteq p^\infty G + G_{[p^n]}$ for some n.*

Proof. The chain of subgroups $A \cap p^n G$ must stabilize. So $A \cap p^n G = A \cap p^\infty G$ for some n. Hence if $a \in A$, then $p^n a \in p^\infty G$. Using saturation of F, $p^\infty G$ is p-divisible; so $p^n a = p^n b$ for some $b \in p^\infty G$. Thus $p^n(a - b) = 0$, so $a - b \in G_{[p^n]}$, and $a = b + (a - b)$. □

Lemma 2.18. *Let $\underline{G}$ be a connected algebraic group defined over F, $G = \underline{G}(F)$. Then G is connected.*

Proof. Let H be a definable subgroup of G of finite index. Let $G_n = \lambda_n G$; then G_n can be endowed with a group structure in such a way that λ_n is an isomorphism; and $G_n = \underline{G}_n(F)$, where $\underline{G}_n$ is an algebraic group, isomorphic over F^a to a power of $\underline{G}$. Let $\underline{H}_n$ be the Zariski closure of $\lambda_n H$. Then for large enough n, $H = G \cap \lambda_n^{-1} \underline{H}_n(F)$. $\lambda_n H$ has finite index in $\lambda_n G$, hence $\underline{H}_n$ has finite index in $\underline{G}_n$. But $\underline{G}_n$ is connected; so $\underline{H}_n = \underline{G}_n$, and $H = G$. □

Remark 2.19. Let $\underline{G}$ be a simple Abelian variety, defined over F, not isomorphic to one defined over F^{p^∞}, $G = G(F)$. Let $A = p^\infty G$. In 2.15 we showed that A has finite Morley dimension, and in 2.16 that A is minimal as a group. This suggests that $\dim(A) = 1$. This will indeed be shown in §5 as a consequence of modularity; we do not know a direct proof.

2.20 **Guide to §6 of** [HZ]. This is intended for the reader who wishes to obtain 2.5 as efficiently as possible, reading only §6 of [HZ] and one preceding page. §6.1 is motivation and includes no results. §6.2 is written for minimal types of stable theories, and can be read directly. In §6.3 one assumes in addition a notion of specialization and of a regular specialization between tuples of elements of the minimal type, satisfying certain axioms. We will immediately give a definition of these that may be used when D is a thin type in a separably closed field. In Lemmas 6.8 and 6.10, using §6.2, one obtains an Abelian group of dimension one. Then 6.9 and 6.11 work with the group elements and provide the required field.

The group obtained is again a thin type of U-rank one in a separably closed field, and so the same definitions and axioms of specialization may be used. The dimension theorem 2.3(iii) has been proved only for complete types P. However, if D is an ∞-definable group of dimension one, one may choose a complete type P contained in D, and a generic element a of P; then D is covered by P and $(P+a)$, translation by a gives a homeomorphism in every dimension, and the dimension theorem thus holds in P and $P+a$ and hence in D.

It remains to define the notion of specialization and regular specialization, and prove the axioms used in §6.3 of [HZ]. Let D be a thin type in a separably closed field; we suppose D is either a complete type over some base set, or else an ∞-definable group of dimension one. The notion of closed set used in this section gives rise to one specialization: $a \to b$ if b lies in every 0-definable closed set in which a lies. We can also describe this directly. Given an element a of D^n, write λa for the sequence $(a, \lambda_1 a, \lambda_2 a, \ldots)$. We write $a \to b$ if λa specializes to λb in the ordinary field-theoretic sense. In other words, any polynomial with coefficients in the base field vanishing on any $\lambda_n a$, also vanishes on $\lambda_n b$; or again, $\lambda_n b$ is an element of the locus of $\lambda_n a$. We say that the specialization is *regular* if for each n, $\lambda_n b$ is a nonsingular point on the locus of $\lambda_n a$.

We now indicate the proof of the axioms 6.6 of [HZ].

(1) is trivial.

(2) follows from the same fact in algebraically closed fields, applied to $\lambda a, \lambda b$.

(3) This follows from 2.3(iii): a' is any generic point of the component of the intersection of locus(a) with the diagonal $x(1) = x(2)$ containing the locus of a''.

(4) and (5) require reading the page preceding §6, starting with the definition of "good" specialization. The amalgamation Lemma 5.14 in [HZ] for regular specializations follows from the same lemma in the case of algebraically closed fields, applied to $\lambda a, \lambda b$, etc. Observe that amalgamating two fields over a (λ-closed) substructure of a separably closed field can never create inseparability. Now follow the proof of 5.14 for good specializations, and of 5.15 in [HZ]

(6) This states that the graph of addition is closed; indeed by results from [Hr90] or [Mes], addition can be taken to be given locally by rational functions.

(7) The first statement follows from the fact that a product of smooth varieties is smooth. For the second let a be a generic point of D and a' any point of D; we need to know that $\lambda_n a'$ is a nonsingular point of the locus of a. In case D is a complete type, a' must also be generic, so this is trivial. In case D carries a group structure, all points on the locus are smooth.

3. Manin's homomorphism and Buium's reduction

In this section we show that the main Theorem follows from a slightly different version, in which Γ is replaced by a certain definable subgroup. For Abelian varieties in characteristic 0, this was observed by Buium, and forms the basis of his approach in [Bu92], [Bu93].

We work in a universal domain F for differential fields of characteristic 0, or for fields with a distinguished p-basis $\{e_i\}_{i=1,\ldots,p^\nu}$. Definable or ∞-definable sets are understood in the sense of F, as is Morley dimension, etc.

We will eventually prove:

Theorem 5.9. *Let K be either a separably closed field of characteristic $p > 0$, with a finite p-basis fixed, or a differentially closed field of* char 0. *Let $k = \bigcap_n K^{p^n}$ if $p > 0$, $k = \{x : Dx = 0\}$ if the characteristic is* 0. *Let $\underline{S}$ be a semi-Abelian variety defined over K, $\underline{X}$ a subvariety, $S = \underline{S}(K)$, $X = \underline{X}(K)$, and let Γ_n $(n = 1, 2, \ldots)$*

be a descending sequence of definable subgroups of S, such that $\bigcap_n \Gamma_n(F)$ has finite dimension. Assume that for each n, for some coset C_n of Γ_n, $X \cap C_n$ is Zariski dense in $\underline{X}$. Then there exists a semi-Abelian variety $\underline{S}_0$ defined over k, a subvariety $\underline{X}_0$ of $\underline{S}_0$ defined over k, and a rational homomorphism h from a group subvariety of $\underline{S}$ into $\underline{S}_0$, such that $\underline{X} = h^{-1}(\underline{X}_0) + c$ for some c.

We now argue that this is sufficient. Let S, X, k, Γ be as in 1.1. We choose a finitely generated field extension L of k, such that S, X and some p'-generators of Γ are defined over L. In characteristic p, $[L : L^p]$ is finite, and $\bigcap_n L^{p^n} = k$. The same remains true for the separable closure K of L. In characteristic 0, we endow L with a differential structure over k such that k is the field of constants, and we let K be a differential closure. It remains only to replace Γ by an appropriate definable subgroup.

In characteristic 0 we use:

Lemma 3.1. *Let S be a semi-Abelian variety defined over F. Let Γ be a p'-finitely-generated subgroup of S. Then there exists a subgroup of S of finite dimension containing Γ.*

Proof. This is proved in [Bu93] for Abelian varieties; the proof goes through in the semi-Abelian case. □

In characteristic p, we let $\Gamma_n = p^n S(K)$. The fact that the intersection of the groups Γ_n has finite dimension was shown in the previous section. If Γ is a p'-finitely generated subgroup of $S(K)$, then $(\Gamma/p^n\Gamma)$ is finite, so Γ meets only finitely many cosets of Γ_n. Hence some coset of Γ_n meets X in a Zariski dense set.

In either case, 1.1 follows from 5.9.

4. Abelian groups of finite Morley dimension

We work in this section with the category of ∞-definable groups and morphisms within some saturated stable structure $\mathbb{C}$. All groups are assumed to have finite Morley dimension.

A group G of finite Morley dimension always has a maximal connected semi-pluriminimal subgroup $S_1(G)$. For algebraic groups, we always have $S_1(G) = G^0$ (the connected component). In general this fails however. In Proposition 4.3 we show how to reduce certain questions about definable subsets of G to similar questions in $S_1(G)$ and in proper quotients of G; for example, a definable subset of G containing no cosets of infinite subgroups of G is contained in finitely many cosets of $S_1(G)$. We then proceed to analyze semi-pluriminimal groups; they are an "almost" direct sum of pairwise orthogonal definable subgroups, each of which is semi-minimal.

Certain basic notions of algebraic groups generalize to the present context; one must give definitions that do not rely on the Zariski topology, but only on dimension theory for constructible sets. The definitions and facts in 4.0 are due to Zilber and Poizat; see [NeP89].

Definition 4.0. (a) An ∞-definable group G is *connected* if it has no definable subgroups of finite index.

(b) Let G be an ∞-definable group, and X a definable subset of G. The stabilizer $\mathrm{Stab}(X)$ is

$$\{g \in G : \dim((X - gX) \cup (gX - X)) < \dim(X)\}.$$

It is a definable subset of G.

(c) An ∞-definable subset X of G is *indecomposable* if whenever X is contained in a finite union of cosets of a definable subgroup H of G, it is contained in a single coset. If X is a complete type of degree 1, then it is indecomposable. If X is indecomposable, then the coset generated by it is generated in finitely many steps.

Definition 4.1. Let G be an Abelian group, ∞-definable over a base set B, and X a definable set over B. G is X-rigid if every connected definable subgroup of G can be defined over a set C independent from $X(\mathbb{C})$ over B. G is *rigid* if every connected definable subgroup is defined over $\operatorname{acl}(B)$. Equivalently, every connected definable subgroup is defined over some fixed countable set. G is *strongly rigid* if the same holds for arbitrary (not necessarily connected) subgroups.

Definition 4.2 (of "full" orthogonality of two ∞-definable sets X, Y). X, Y are orthogonal if for any algebraically closed $B \subseteq \mathbf{C}^{\mathrm{eq}}$ with X, Y defined over B, any $b \in X$ and any $c \in Y$, $\operatorname{tp}(b/B) \cup \operatorname{tp}(c/B)$ implies $\operatorname{tp}(bc/B)$.

In our applications of 4.3 below, A will be the maximal connected semi-pluri-strongly minimal subgroup of G, and will be strongly rigid.

Proposition 4.3. *Let G be an Abelian group of finite Morley dimension, A a non-trivial connected definable subgroup, and X an (∞-) definable subset of G of multiplicity* 1. *Assume:*

(i) *A is G/A-rigid.*

(ii) *There is no definable group $A' \supset A$ with A'/A infinite and $A' \subseteq \operatorname{acl}(Y, A, C)$ for some minimal Y and finite C.*

(iii) $\operatorname{Stab}(X) \cap A$ *is finite.*

Then X is contained in a single coset of A, up to a set of smaller dimension.

Proof. We may assume G, A, X are 0-definable. It is convenient to replace X by the corresponding complete type, i.e. to remove from X all 0-definable subsets of X of smaller rank. So we must now show that X is contained in a single coset of A. Let $\theta : G \to G/A$ be the canonical homomorphism, and let $X/A = \theta(X)$. For $b \in (X/A)$, let $A(b) = \theta^{-1}(b)$, a coset of A.

Let b be an element of X/A. Then $X \cap A(b) \neq \emptyset$. Let U be a nonempty subset of $A(b)$, defined over $\{b\} \cup A$, of least possible dimension and multiplicity. For $c \in A$, we can consider the translate $U + c$ of U within $A(b)$. Then $(U + c) \cap U$ is defined over $\{b\} \cup A$. Necessarily either $U \cap (U + c)$ or $U - (U + c)$ has smaller dimension or multiplicity than U; so one of them is empty. It follows that U is a coset of some subgroup K of A (namely $K = \{c \in A : U = U + c\}$). By considering intersections of U with A-translates of X, we see that every A-translate of U meets X trivially or is contained in X; so $X \cap A(b)$ is K-invariant.

By the rigidity assumption, K^o is defined over a set F_0 orthogonal to G/A; so b remains a generic element of X/A over F_0. Thus $X \cap A(b')$ is invariant under translation by K^o, for generic $b' \in (X/A)$. So $K^o \subseteq \operatorname{Stab}(X)$. By (iii), K is finite.

Since U is a coset of K, it is also finite. Recalling that U is defined over $\{b\} \cup A$, we have $U \subseteq \operatorname{acl}(b, A)$. Every element of $A(b)$ has the form $a + x$ for some $a \in A$, $x \in U$, so $A(b) \subseteq \operatorname{acl}(b, A)$.

If $\operatorname{rk}(b/F_0) = 0$, then X/A is finite, and having multiplicity 1, it consists of a single element; in other words X is contained in a single coset. Otherwise, we will get a contradiction. Increase F_0 to F_1 so that $\operatorname{rk}(b/F_1) = 1$, and let Y be the locus of b over F_1, and $X' = \{x \in X : x + A \in Y\}$. Then $A(b) \subseteq \operatorname{acl}(b, A)$ for $b \in Y$, so $X' \subseteq \operatorname{acl}(Y \cup A)$. By the indecomposability theorem, for some finite

m, $\{\sum n_i y_i : (y_1, \ldots, y_m) \in Y^m, (n_1, \ldots, n_m) \in \mathbf{Z}^m, \sum_i n_i = 0\}$ is a subgroup of G/A. So $\{a + \sum n_i b_i : a \in A, (b_1, \ldots, b_m) \in X'^m, (n_1, \ldots, n_m) \in \mathbf{Z}^m, \sum_i n_i = 0\}$ is a subgroup of G, and evidently it contains A and is contained in $\mathrm{dcl}(A \cup X') \subseteq \mathrm{acl}(Y \cup A)$. This contradicts assumption (ii). □

Various formulations of this proposition lift easily to the superstable context. We include a variation using full orthogonality, though it will not be used for the proof of the main theorem.

Proposition 4.4. *Let G be an Abelian group of finite Morley dimension, X a definable subset of G, and A a connected definable subgroup. Assume A is orthogonal to G/A. Then X is a finite union of definable subsets X_i with the following property: For some definable subgroup H_i of G, X_i is a union of cosets of $(H_i \cap A)^o$, and is contained in a single coset of $H_i + A$.*

Proof. Let $G^* = G/A$. Each element $b \in G^*$ can be thought of as a coset of A in G, which we denote as $A(b)$. There is no loss of generality in assuming $\mathrm{acl}^{\mathrm{eq}}(\emptyset) = \mathrm{dcl}^{\mathrm{eq}}(\emptyset)$, i.e. all types over $\emptyset$ are stationary. Let $X\# \subseteq G^*$ be the solution set of a complete type over $\mathrm{acl}^{\mathrm{eq}}(\emptyset)$. We will find a 0-definable subgroup H of G such that $\{x \in X : x + A \in X\#\}$ is contained in a coset of $H + A$, and is a union of cosets of $(H \cap A)^o$. By compactness, there exists a 0-definable set X^* containing $X\#$ such that $\{x \in X : x + A \in X^*\}$ is contained in a coset of $H + A$ (and of course this set is still closed under translation by elements of $(H \cap A)^o$). Thus G^* can be covered with 0-definable sets with this property. The statement of the proposition follows by another application of compactness.

Pick $b \in X\#$, and also $g \in A(b)$. Let $S = \{c \in A : g + c \in X\}$. By stable definability, S is definable with parameters C from A. We have

(*) For some $g \in A(b)$, $A(b) \cap X = S + g$.

By the orthogonality assumption, $\mathrm{tp}(b'/C)$ does not depend on $b' \in X\#$. Thus (*) holds for any $b \in X^{\#}$. Let $K = \{c \in A : c + S = S\}$. K is a definable subgroup of A. Now if $S + g = S + g'$, then $g - g' \in K$. Thus we have a definable map $f : X\# \to G/K$ (given by: $f(b) = g + K$ iff $A(b) \cap X = S + g$). f is a section of the natural projection $\pi : (G/K) \to (G/A)$.

Claim. Let C^* be the coset generated by $X\#$ in G^*. Then f extends to an affine homomorphism from C^* to G.

Proof. For large enough odd n, and $(a_1, \ldots, a_n) \in X\#^n$ generic, $\sum(-1)^i a_i$ is a generic element of the connected, definable coset C^* ("indecomposability theorem"). It suffices to show that for all n there exists a constant χ_n such that for all generic $(a_1, \ldots, a_n) \in X\#^n$, $f(\sum' a_i) - \sum' f(a_i) = \chi_n$, where $\sum'$ denotes the alternating sum. Let $h(a_1, \ldots, a_n) = f(\sum' a_i) - \sum' f(a_i)$. Then since f is a section of π, $h(a_1, \ldots, a_n) = 0 \pmod{A/K}$, i.e. $h(a_1, \ldots, a_n) \in A/K$. Thus h is a map from $X\#^n$ into A/K; since they are orthogonal, h is generically constant.

Thus there exists a connected definable subgroup H^* of G^* and a nontrivial definable group homomorphism $h : H^* \to G/K$. Let H be a subgroup of G containing K, such that $H/K = hH^*$. Note that $(H + A)/A = H^*$.

Claim. $(H \cap A)^o \subseteq K$.

Proof. Clearly $H \cap A \supseteq K$. We must show that $(H \cap A)/K$ is finite. The homomorphism h induces an isomorphism between $(H \cap A)/K$ and $h^{-1}((H \cap A)/K)/\mathrm{Ker}(h)$, a quotient of two definable subgroups of G^*. By the orthogonality assumption, both sides must be finite.

This finishes the proof. □

Definition 4.5. If A is a definable group and X a minimal set, then there exists a unique maximal connected subgroup C of A such that $C \subseteq \text{acl}(X, F)$ for some finite F. We denote this group by $A\langle X\rangle$.

Lemma 4.6. *Suppose A is connected, and $A \subseteq \text{acl}(Y)$ for some pluriminimal Y. Then A is isogenous to a direct sum of pairwise orthogonal semi-minimal groups (namely to the direct sum of the various nontrivial subgroups $A\langle X\rangle$, X minimal).*

Proof. Clearly if X is orthogonal to $X_1 \cup \cdots \cup X_k$, then

$$A\langle X\rangle \cap (A\langle X_1\rangle + \cdots + A\langle X_k\rangle)$$

is finite. If X_i, X_j are nonorthogonal, then $A\langle X_i\rangle = A\langle X_j\rangle$. Thus the group B generated by all $A\langle X\rangle$ (X minimal) is a finite orthogonal sum. We must show $B = A$. Let c be a generic element of $A, b = c + B \in A/B$. We have $c \in \text{acl}(y_1, \ldots, y_k)$ for some y_j from Y, hence $c \in \text{acl}(b, y_1, \ldots, y_k)$. Otherwise, minimizing k, we may assume $y_j \notin \text{acl}(b, y_1, \ldots, y_{j-1})$ for each j. So $\{b, y_1, \ldots, y_k\}$ is an independent set. In particular, $c \notin \text{acl}(y_1, \ldots, y_k)$.

Let Z be the locus of c over $\{y_1, \ldots, y_k\}$. Then Z is infinite, but any element c of Z is algebraic over $c + B$. Replace Z by a minimal ∞-definable subset Z'; it has the same property. $A\langle Z'\rangle$ is nontrivial, so it is contained in B. But $Z'/A\langle Z'\rangle$ must be finite, while Z'/B is infinite, a contradiction. □

Lemma 4.7. *Suppose A is connected, semi-minimal, and locally modular. Then A is isogenous to a direct sum of minimal subgroups.*

Proof. Let $\{A_i\}$ be a maximal set of minimal subgroups of A, such that the map from $\bigoplus_i A_i$ to A has finite kernel. Let B be the image of this map, and suppose for contradiction that $B \neq A$. As in 4.6 one finds a minimal $Z \subseteq A$ such that Z/B is infinite. But by [HP86] Z is a coset of a definable subgroup C of A (up to a finite number of points). C must be minimal, and $(C + B)/B$ is infinite, a contradiction. □

The following lemma is a special case of results from [Hr90]; we include a proof for the reader's convenience.

Lemma 4.8. *Let A be a semi-minimal group. Suppose A is nonorthogonal to an ∞-definable set D. Then there exists a group B with $B \subseteq \text{dcl}(D)$ and a definable surjective homomorphism $h : A \to B$, with finite kernel.*

Proof. We have $A \subseteq \text{acl}(Y)$ for some minimal Y; Y is necessarily nonorthogonal to D, so $Y \subseteq \text{acl}(F, D)$ for some finite F, and hence $A \subseteq \text{acl}(F, D)$. Let a be a generic point of A over F, and let $\phi(x, y)$ be a formula over F such that $\phi(a, d)$ holds for some $d \in D^n$, and for any d', $\phi(x, d')$ has only finitely many solutions. Let $C(a) = \{d \in D^n : \phi(a, d)\}$. By stability, $C(a)$ can be defined with parameters from D; let b be a canonical parameter for $C(a)$. Then $b = g(a)$ for some F-definable function g. We may assume $g(x) \in \text{dcl}(D)$ for all x.

Let $K = \{x \in A$: for generic $a \in A$, $g(a + x) = g(a)\}$. Let $a_0, \ldots, a_{2r}$ be mutually independent generic elements of A, $r = \text{rk}(A)$. If $x - y \notin K$, then since some a_i is generic over $x, y, g(x + a_i) \neq g(y + a_i)$. Thus the function $f(x) = (g(x + a_0), \ldots, g(x + a_{2r}))$ is 1-1 modulo K. The image B of A by f is contained in $\text{dcl}(A)$, and may be endowed with a group structure so that f is an isomorphism. By stability (cf. [NeP89]), B and its group structure are definable with parameters from D. □

Lemma 4.9. *Assume: Whenever $H_1 \subset H_2$ are definable subgroups of G, there exists a nonzero element of H_2/H_1 algebraic over any base of definition for H_1. Then G is strongly rigid.*

Proof. We assume G is defined over $\operatorname{acl}(\emptyset)$, and show every definable subgroup of G is defined over $\operatorname{acl}(\emptyset)$. Let $H(a)$ be an a-definable subgroup. Let H_1 be the intersection of $H(a')$ over all a' realizing $\operatorname{tp}(a/\operatorname{acl}(\emptyset))$. If $H_1 = H(a)$ we are done. Otherwise there exists a nonzero $b \in H(a)/H_1$ with b algebraic over $\emptyset$. Necessarily $b \in H(a')$ for all a', a contradiction. □

The following lemma is well known.

Lemma 4.10. *Let k be an algebraically closed field and G a semi-Abelian variety defined over k. Then $G(k)$ is strongly rigid.*

Proof. All torsion points are algebraic, and 4.9 applies. □

Lemma 4.11. *Let A be a locally modular group. Then A is rigid.*

Proof. [HP86]. □

Lemma 4.12. *Let A, B be orthogonal subgroups of a group G, and let $X \subseteq (A+B)$ be the solution set of a complete type over* $\operatorname{acl}(\emptyset)$. *Then X has the form $U + V$, $U \subseteq A$, $V \subseteq B$.*

Proof. Let (a, b) be a point of X. Let U be the locus of a and V the locus of b. Then by definition of orthogonality, for any $a' \in U$ and $b' \in V$, (a', b') has the same type as (a, b); so $U + V \subseteq X$. □

Lemma 4.13. *Suppose A, B are orthogonal (strongly) rigid groups. Then their product is (strongly) rigid.*

Proof. Applying 4.12 to the generic type, a connected definable subgroup of $A \times B$ is a product of definable subgroups of A and of B. Thus an arbitrary definable subgroup of $A \times B$ lies between two definable subgroups $C \subseteq D$ with $[D : C]$ finite, and C, D each a product of subgroups of A and of B. □

Lemma 4.14. *Suppose F is a finite subgroup of A, and A/F is (strongly) rigid. Then A is (strongly) rigid.*

Proof. Let $H(a)$ be a definable subgroup of A. Then $J = H(a) + F$ does not depend on a (if $\operatorname{tp}(a/\operatorname{acl}(\emptyset))$ is given). Hence $H(a)$ is a subgroup of J containing the connected component H of J. But H has finite index in J, so there are only finitely many intermediate subgroups. □

Lemma 4.15. *Let $A = A_1 + A_2$, where A_1, A_2 are orthogonal semi-minimal subgroups, and A_1 is of linear type. Let $X \subseteq A$ be the solution set of a complete type over* $\operatorname{acl}(\emptyset)$. *Assume* $\operatorname{Stab}(X)$ *is finite. Then X is contained in a coset of A_2.*

Proof. By 4.12 we have $X = U_1 + U_2$, with $U_i \subseteq A_i$. So $\operatorname{Stab}_{A_1}(U_1) \subseteq \operatorname{Stab}(X)$. Since A_1 is locally modular, U_1 is contained in a coset of $\operatorname{Stab}(U_1)$; so U_1 is finite, and being the solution set of a complete type, it has one point. Thus X is a translate of U_2. □

5. The main theorem

K is either a separably closed field of characteristic $p > 0$, with a finite p-basis fixed, or a differentially closed field of char 0. In either case K is assumed saturated (i.e. we work in the universal domain). Let $k = \bigcap_n K^{p^n}$ if $p > 0$, and $k = \{x : Dx = 0\}$ if the characteristic is 0. In either case k is an algebraically closed field. We will say that a type has *finite dimension* if it has finite Morley dimension and, when $p > 0$, is thin.

Our goal is the following version of the main theorem.

Theorem 5.1. *Assume K is saturated. Let S be a semi-Abelian variety defined over K, and X a subvariety of S. Let Γ be an ∞-definable subgroup of S, of finite dimension. Assume $X \cap \Gamma$ is Zariski dense in X. Then there exists a semi-Abelian variety S_0 defined over k, a subvariety X_0 of S_0 defined over k, and a rational homomorphism h from a group subvariety of S into S_0, such that $X = h^{-1}(X_0) + c$ for some c.*

Definition. We call a function h p-rational if $p = 0$ and it is rational, or when $p > 0$ if it is the composition of a rational function with some negative power of Frobenius.

Fact 5.2. If $S \subseteq k^n$ is a (relatively) definable subset, then S is constructible (i.e. definable in the field structure $(k, +, \cdot)$ with parameters). If h is a definable map on S, then S may be split into finitely many constructible sets, on each of which h is a p-rational function.

Proof. This follows from quantifier elimination, and the fact that every automorphism of a (differential) field extends to its separable (resp. differential) closure. □

Fact 5.3. Let L be an ∞-definable field with minimal generic type. Then L is definably isomorphic to k.

Proof. This is part of the theses of Sokolovic [So92] when $p = 0$, and of Messmer [Mes] when $p > 0$. □

Lemma 5.4. *Let X be a Zariski minimal type. Then either X is locally modular, or X is nonorthogonal to k.*

Proof. When X is strongly minimal, i.e. it is the solution set of a single formula, it is proved in [HZ] that there exists a field L satisfying the hypotheses of 5.3, and nonorthogonal to X. The proof goes through in the general case. By 5.3, we may take $L = k$. □

Lemma 5.5. *Let A be a semi-minimal group. Then either A is locally modular, or there exists an algebraic group $\underline{H}$ defined over k and a definable surjective group homomorphism $h : A \to \underline{H}(k)$, with finite kernel.*

Proof. By 5.4, if A is not locally modular, it is nonorthogonal to k. By 4.8, there exists a definable surjective group homomorphism h with finite kernel, such that $B = hA \subseteq \mathrm{dcl}(k)$. By Fact 5.2, B is definable in the structure $(k, +, \cdot)$. By [NeP89] (see chapter on Weil's theorem), there exists an algebraic group $\underline{H}$ defined over k such that B is definably isomorphic to $\underline{H}(k)$. □

Proposition 5.6. *Let $\underline{G}$ be a semi-Abelian variety defined over K. Let $G = \underline{G}(K)$. Let A be a semi-minimal definable subgroup of G, Zariski dense in $\underline{G}$. Then either*

A is locally modular, or there exists an algebraic group $\underline{H}$ defined over k and a bijective rational homomorphism $\underline{h} : \underline{G} \to \underline{H}$, with $\underline{h}(A) = \underline{H}(k)$.

Proof. Suppose A is not locally modular. Then by 5.5 there exists an algebraic group $\underline{H}$ defined over k and a definable surjective group homomorphism $h : A \to \underline{H}(k)$, with finite kernel of size n, say. Consider $R = \{(nx, y) : h(x) = y\}$. If $(u, 0) \in R$, then $u = nx$, $h(x) = 0$, so x is in the kernel and $u = 0$. Thus R defines a homomorphism g from $h(A)$ to G. If $(0, y) \in \underline{R}$, then y is the image of an n-torsion point of A; so g has finite kernel, and is surjective.

By 5.2, g is given by a p-rational map $\underline{g}$. Since $h(A)$ is Zariski dense in $\underline{H}$, g defines a homomorphism $\underline{g}$ on $\underline{H}$ into $\underline{G}$. Let R be the smallest closed subgroup of H with H/R semi-Abelian; then R is defined over k; H/R is strongly rigid by 4.10, so $\mathrm{Ker}(\underline{g})/R$ is defined over k; hence $\mathrm{Ker}(\underline{g})$ is defined over k. Since k is algebraically closed, there exists an algebraic group $\underline{H}^*$ defined over k and a surjective p-rational map $f : \underline{H} \to \underline{H}^*$ whose kernel is $\mathrm{Ker}(\underline{g})$. We get an induced p-rational map $\underline{g}^* : \underline{H}^* \to \underline{G}$ with $\underline{g} = \underline{g}^* f$. Further f carries $\underline{H}(k)$ to $\underline{H}^*(k)$. Thus we may assume $\mathrm{Ker}(\underline{g})$ is trivial.

The image $\underline{g}(\underline{H})$ contains A, which is Zariski dense in $\underline{G}$, so it equals $\underline{G}$. Let $\underline{h}$ be the inverse map to $\underline{g}$. Then $\underline{h}$ is p-rational. Composing with a power of Frobenius (and changing $\underline{H}$ appropriately), we may assume $\underline{h}$ is rational. □

Lemma 5.7. *Let $\underline{G}$ be a semi-Abelian variety defined over $K, G = \underline{G}(K)$. Let A be a semi-pluriminimal definable subgroup of G. Then A is (strongly) rigid.*

Proof. By 4.6, 4.13, we may assume A is semi-minimal. Further we may assume A is Zariski dense in $\underline{G}$. If A is locally modular, we are done by 4.11. Otherwise by 5.6 there exist an algebraic group $\underline{H}$ defined over k, isogenous to $\underline{G}$, and a surjective map $h : A \to \underline{H}(k)$ with finite kernel. $\underline{H}$ is also a semi-Abelian variety, so $\underline{H}(k)$ is (strongly) rigid. By 4.14, A is (strongly) rigid. □

Proof of 5.1. The proof is now a sequence of reductions, leading to 5.6.

We may assume X has finite stabilizer; otherwise we may quotient out the connected component of the stabilizer. Let A be the maximal semi-pluriminimal connected subgroup of Γ. Then 4.3 (ii) holds. By 5.7, 4.3 (i) holds also. By assumption $X \cap \Gamma$ is Zariski dense in X. Choose a definable $Y \subseteq X \cap \Gamma$, such that Y is Zariski dense in X, and of least possible dimension and multiplicity. Since X is irreducible, whenever Y is written as a finite union $\bigcup_i Y_i$, one of the sets Y_i must be Zariski dense in X. Hence Y has multiplicity one. Moreover if Y' is a subset of Y of the same dimension, then Y' is Zariski dense in X, since the complement cannot be.

Observe that if translation by an element a stabilizes Y in the sense that $\dim(Y \cap (Y+a)) = \dim(Y)$, then the Zariski closure of $Y \cap (Y+a)$ must be X, so (as $X \cap (X+a)$ is Zariski closed and contains $Y \cap (Y+a)$) the element a stabilizes X as a set. Thus the dimension-theoretic stabilizer of Y is contained in the set-stabilizer of X; so it is finite.

Thus 4.3 (iii) is also true of Y. So by 4.3, a subset Y' of Y of the same dimension is contained in a single coset $c+A$ of A. In particular $(c+A) \cap X$ is Zariski dense in X. Replacing X by $X-c$, and Γ by A, we may assume $\Gamma = A$ is semi-pluriminimal.

Write A as a sum of orthogonal subgroups A_i, with A_i semi-minimal. Let B be the sum of all nonlocally modular A_i, and C the sum of the rest. By 5.4, if A_i, A_j are nonlocally modular, then they are nonorthogonal to k, and hence to each other, so $i = j$. Thus B is semi-minimal. By 4.15, Y is contained in a single coset of B. Translating, we may assume Y is contained in B.

Let $\underline{G}$ be the Zariski closure of B. Then $\underline{G}$ is a group subvariety of $\underline{S}$, containing X (the Zariski closure of $Y \subseteq B$). By 5.6, there exist an algebraic group $\underline{S}_0$ defined over k and a bijective rational homomorphism $\underline{h} : \underline{G} \to \underline{S}_0$ defined over K; and $\underline{h}(B) = \underline{S}_0(k)$. So S_0 is semi-Abelian. Let X_0 be the Zariski closure of $\underline{h}(Y)$. Since $\underline{h}(Y) \subseteq \underline{h}(B) \subseteq \underline{S}_0(k)$, X_0 is defined over k. Clearly $\underline{h}^{-1}(X_0)$ contains X. Since $\underline{h}$ is bijective, $\underline{h}^{-1}(X_0) = X$.

Remark 5.8. We observe that Theorem 5.1 for saturated K implies the same statement for arbitrary models K. The assumptions that Γ is of finite dimension and that $\Gamma \cap X$ is Zariski dense in X should both be understood in the universal domain however. There are two points to observe here.

(i) Suppose S is defined over L. The domain and kernel of h, being algebraic subgroups of S, are defined over a separable extension of L ($= L^s$). Hence up to a p-rational isomorphism, S_0 is defined over L. Let L^* be a saturated elementary extension of L, and let $k^* = \bigcap_n L^{*p^n}$ (in characteristic $p > 0$) or k^*=constants of L^* (in the differential case). In either case k^*, L are linearly disjoint over k. (In char $p > 0$, it is because L^{*p^n} is linearly disjoint from L over L^{p^n} for each n.) Theorem 5.1 states that S_0 is defined over k^*. It follows that a p-birational copy of S_0 exists over the algebraically closed field k. More precisely, there exist S_0' defined over k, a surjective map $h' : \mathrm{dom}(h) \to S_0'$ defined over L, and a p-birational map $h'' : S_0' \to S_0$ defined over k^*, such that $h = h''h'$. It follows similarly that $h''^{-1}X_0 = h(X + t)$ is defined over $k^* \cap L = k$. (The t in S such that $h(X+t)$ is defined over k^* may be chosen in L.)

(ii) Note that if $\Gamma = \bigcap_n \Gamma_n$, where Γ_n is a definable group, then $\Gamma \cap X$ is Zariski dense in X iff, for each n, $\Gamma_n \cap X$ is Zariski dense in X. This is because $\Gamma \cap X$ is Zariski dense in X if and only if, for each m, there exist $a_1, \ldots, a_m$ in $\Gamma \cap X$ such that $(a_1, \ldots, a_m)$ is a generic point of X^m in the sense of algebraically closed fields. (And the compactness theorem of model theory applies.) Similarly one can deal with the case where for each n, $C_n \cap X$ is Zariski dense in X, where C_n is some coset of Γ_n. In this case in the saturated extension there will be a coset of $\bigcap_n \Gamma_n$ meeting X in a Zariski dense set.

Thus we have proved the following restatement of Theorem 5.1.

Theorem 5.9. *Let K be either a separably closed field of characteristic $p > 0$, with a finite p-basis fixed, or a differentially closed field of* char 0. *Let $k = \bigcap_n K^{p^n}$ if $p > 0$, and $k = \{x : Dx = 0\}$ if the characteristic is 0. Let S be a semi-Abelian variety defined over K. Let Γ_n $(n = 1, 2, \ldots)$ be a descending sequence of definable subgroups of S, such that $\bigcap_n \Gamma_n$ has finite dimension. Assume that for each n, for some coset C_n of Γ_n, $X \cap C_n$ is Zariski dense in X. Then there exist a semi-Abelian variety S_0 defined over k, a subvariety X_0 of S_0 defined over k, and a rational homomorphism h from a group subvariety of S into S_0, such that $X = h^{-1}(X_0) + c$ for some c.*

6. A QUESTION OF VOLOCH'S

We prove here a refinement of Theorem 1.1, conjectured by Voloch (Theorem 6.4). In characteristic 0, a somewhat weaker version was proved in [BV93]. It is related to a conjecture of Lang concerning integral points on an open affine subset of an Abelian variety (see [BV93] and [La91]). We also observe some uniformities that arise from our method of proof. In this section we use only basic model theory, really only the compactness theorem, in the style of A. Robinson.

For simplicity, as in 1.2, we will assume away the homomorphisms that occur in the conclusion of 1.1. Thus throughout this section, let k be an algebraically closed field, K an extension field, $\underline{A}$ an Abelian variety defined over K with no nonzero homomorphic images defined over k, $\underline{X}$ a subvariety of $\underline{A}$, and Γ a p'-finitely generated subgroup of A. (There is no difficulty in working with semi-Abelian varieties and special subvarieties as in 1.1; we restrict to the Abelian case for simplicity only.)

Recall:

Corollary 1.2. *$\underline{X}(K) \cap \Gamma = \underline{Y}(K) \cap \Gamma$, where $\underline{Y}$ is a (reducible) subvariety of $\underline{X}$, equal to a finite union of translates of Abelian subvarieties of $\underline{A}$.*

We wish to exploit the method of proof to observe a uniformity in the finite number involved.

Theorem 6.1. *Let $k, K, \underline{A}, \underline{X}$ be as above. There exists a finite number of Abelian subvarieties $\underline{B}_1, \ldots, \underline{B}_m$ with the following property. For any extension field K' of K and any p'-finitely-generated subgroup Γ of $\underline{A}(K')$ of rank g, there exists a union $\underline{Y}$ of a finite number l of cosets of the $\underline{B}_i$, such that $\underline{X}(K') \cap \Gamma = \underline{Y}(K') \cap \Gamma$.*

Moreover, l depends only on $\underline{A}, \underline{X}$, and g but not on the actual choice of Γ. In characteristic p, we have: $l \leq mp^{rg}$ for a certain fixed r depending on $\underline{A}, \underline{X}$ alone.

Proof. We give the proof in characteristic p, the characteristic 0 case being analogous, using a derivation on K over k. In 6.1 we may first replace K by a finitely generated field, and then by the separable closure of that field. Then $k = \bigcap_n K^{p^n}$. We now have the following lemma:

Lemma 6.2. *Let K be a separably closed field, and let $\underline{A}$ be an Abelian variety defined over K, with no nonzero homomorphic images defined over $k = \bigcap_n K^{p^n}$. Then there exist integers r, m, and Abelian subvarieties $\underline{B}_i$ of $\underline{A}$ (at most m of them) with the following property: For any coset C of $p^r A(K)$, there exists a subvariety $\underline{Y}$ of $\underline{X}$ with $C \cap \underline{X}(K) = C \cap \underline{Y}(K)$; and $\underline{Y}$ is the union of at most m cosets of some of the $\underline{B}_i$.*

Proof. Note that for a given choice of r, m and $\underline{B}* = \{\underline{B}_i\}_{i=1,\ldots,m}$, there exists a first order formula $\psi(y) = \psi(r, m, \underline{B}*)(y)$ such that: $\psi(c)$ holds in K iff for every union $\underline{Y}$ of at most m cosets of some of the $\underline{B}_i$, $(c+p^r A(K)) \cap \underline{X}(K) \neq (c+p^r A(K)) \cap \underline{Y}(K)$. (The formula quantifies universally over the possible cosets of the $\underline{B}_i$.)

Note that as r and m grow bigger, the formula $\psi(c)$ grows stronger. Suppose for contradiction that there are no r, m and Abelian subvarieties $\underline{B}_i$ as asserted in the theorem. For any choice of $r, m, \underline{B}*$ there exists c with $\psi(r, m, \underline{B}*)(y)$. Hence by compactness, there exists an element c in some elementary extension K^* of K such that all the formulas $\psi(r, m, \underline{B}*)$ hold of c. K^* may be chosen countably saturated. Let $C = c + p^\infty A$. Then by Corollary 1.2, $C \cap \underline{X}(K^*) = C \cap \underline{Y}(K^*)$ for some finite union $\underline{Y}$ of cosets of m Abelian subvarieties $\underline{B}_i$ of $\underline{A}$. In particular $C \cap \underline{X}(K^*) \subseteq \underline{Y}(K^*)$. By compactness, for some r, $(c + p^r A) \cap \underline{X}(K^*) \subseteq \underline{Y}$. So $(c+p^r A) \cap \underline{X}(K^*) = (c+p^r A) \cap \underline{Y}(K^*)$. Thus $\neg\psi(c)$ holds with $\psi = \psi(r, m, \{\underline{B}_i\}_i)$, a contradiction. □

We now finish the proof of 6.1. We simply observe that as Γ is p'-generated by g elements, so is $(\Gamma + p^r A(K))/p^r A(K)$, hence it has size at most p^{rg}. So Γ is contained in p^{rg} cosets of $p^r A(K)$, and the theorem follows. □

Remark 6.3. Assume the situation of 6.1, but let $\underline{A}$, $\underline{X}$ vary within an algebraic family of Abelian varieties and subvarieties. Then the same proof shows that k, m do not depend on the actual choice of $\underline{A}, \underline{X}$, and that the $\underline{B}_i$ also vary within an algebraic family.

We now assume given a discrete valuation v of K/k. (In characteristic 0 the assumption of discreteness of v is not needed; it can probably be eliminated in characteristic p too.) Then one can define a v-adic distance $d_v(a, \underline{X})$ between a point a of A and a subvariety $\underline{X}$ of $\underline{A}$. We will define below a quantity $\lambda_{\underline{X}}(a)$, which should be considered as $-\log d_v(a, \underline{X})$. Thus 6.1 states that for $a \in \Gamma$, if $d_v(a, \underline{X}) = 0$ then also $d_v(a, \underline{Y}) = 0$, or equivalently that if $\lambda_{\underline{X}}(a)$ is infinite then $\lambda_{\underline{Y}}(a)$ is infinite. We prove a continuous version of this, conjectured by Voloch.

Theorem 6.4. *With the assumptions above, there exist a finite union $\underline{Y}$ of translates of Abelian subvarieties of $\underline{A}$, and a constant c, such that: for all $a \in \Gamma$, $\lambda_{\underline{X}}(a) \leq c \cdot \lambda_{\underline{Y}}(a)$.*

Remark 6.5. This statement naturally carries over to points of the v-adic closure of Γ in $\underline{A}(K_v)$, where K_v is the v-adic completion of K.

Definition. To define λ, we consider $\underline{A}$ as embedded in projective space $\mathbf{P}^m$. Let $R = \{t \in K : v(t) \geq 0\}$ be the valuation ring of v. Any point x of $\mathbf{P}^m(K)$ can be written in projective coordinates as $x = (x_0 : \cdots : x_m)$, with $v(x_i) \geq 0$ and $v(x_i) = 0$ for some i. We let

$$\lambda'_{\underline{X}}(x) = \inf\{v(f(x_0 : \cdots : x_m)) : f \text{ a homogeneous polynomial in } R[X] \text{ vanishing on } X\}.$$

Note that if $\{f_j\}$ are homogeneous polynomials generating the ideal of $\underline{X}$ in $R[X]$, then $\lambda'_{\underline{X}}(x) = \min\{v(f_j(x_0 : \cdots : x_m)) : j\}$. We are interested in small distances from $\underline{X}$, hence large values of λ', so for convenience we let $\lambda_{\underline{X}}(x) = \max(1, \lambda'_{\underline{X}}(x))$.

Lemma 6.6. *Let K have characteristic $p > 0$, $\underline{A}$ a group variety over K. Let $\underline{X}, \underline{Y}$ be subvarieties of $\underline{A}$ defined over K. Suppose $\underline{X}(K^s) \cap p^r A(\underline{K}^s) = \underline{Y}(K^s) \cap p^r \underline{A}(K^s)$. Then for some integer c and for all $a \in p^r \underline{A}(K)$, $\lambda_{\underline{X}}(a) \leq c \cdot \lambda_{\underline{Y}}(a)$.*

Proof. Every separable extension of K embeds over K into some elementary extension of K^s. Hence the hypothesis implies that $\underline{X}(K') \cap p^r \underline{A}(K') = \underline{Y}(K') \cap p^r \underline{A}(K')$ for all separable extensions K' of K.

Suppose for contradiction that there is no integer c as required. Consider the language describing an extension field K^* of K, a valuation v^* of K^* extending v on K, with value group Z^* (extending the value group $\mathbf{Z}$ of v), and an element a^* of $\underline{A}(K^*)$. We define λ^* from v^* as λ is defined from v. The following statements are first-order:

(i) a^* is in $p^r \underline{A}(K^*)$.

(ii) v^* extends v ($v^*(\alpha) = v(\alpha)$ for $\alpha \in K$). Z^* is an ordered group with a least positive element (hence containing the integers $\mathbf{Z}$ as a convex subgroup).

(iii) $\lambda^*_{\underline{X}}(a^*) > c\lambda^*_{\underline{Y}}(a^*)$ ($c = 1, 2, \ldots$).

By assumption any finite number of the above axioms can be satisfied (with $K^* = K, v^* = v$). Hence by compactness there exist K^*, v^*, a^* with these properties. Let R^* be the valuation ring of v^*, and let

$$M' = \{x \in R^* : v^*(x) > n\lambda^*_{\underline{Y}}(a^*), \text{ all } n\}.$$

This is a prime ideal of R^*. $v^*(x) = v(x) \in \mathbf{Z}$ for x in $R^* \cap K$, so M' meets $R^* \cap K$ trivially; thus K embeds into $K' = R^*/M'$.

K' is a separable extension of K. For suppose it is not. Let $t \in K$, $v(t) = 1$. Then (since K/k has transcendence degree 1) there is a p-th root of t in K'; i.e. there is s in R^* with $s^p - t$ in M', i.e. $v^*(s^p - t)$ large. But either $v^*(s) = 0$ and $v^*(s^p - t) = 0$, or $v^*(s) \geq 1$ so $v^*(s^p) \geq p$ and $v^*(s^p - t) = 1$; a contradiction.

Let a' be the image of a^* in K'. By (iii) we have that $\lambda_{\underline{X}}(a^*)$ is in M', so a' is in $X(K')$. Since $a^* = p^r b^*$ for some b^*, and b^* may be written with projective coordinates from R^*, we see that $a' = p^r b'$ for some b' in $\underline{A}(K')$. Thus by the first paragraph, a' is in $\underline{Y}(K')$. But $\lambda_{\underline{Y}}(a^*)$ is not in M', so a' is not in $\underline{Y}(K')$. This contradiction proves the lemma. □

Lemma 6.7. *Let there be given a derivation D of K over k compatible with the valuation v, in the sense that for some constant b in the value field, if $v(x) > b$, then $v(Dx) > v(x) - b$. (We can take $b = 1$.) Let $\underline{A}$ be a group variety over K, $\underline{X}, \underline{Y}$ be subvarieties of $\underline{A}$ defined over K, and B be a subgroup of $\underline{A}(K)$ defined by a differential equation. Suppose $\underline{X}(K^d) \cap B = \underline{Y}(K^d) \cap B$ holds in some differential closure K^d of K. Then for some integer c and for all $a \in B(K)$, $\lambda_{\underline{X}}(a) \leq c \cdot \lambda_{\underline{Y}}(a)$.*

Proof. Entirely analogous to 6.6 (and indeed we could have used 6.7 to prove 6.6). We need only note that in an elementary extension (K^*, D^*, v^*) of (K, D, v), the derivation D^* continues to satisfy that $v^*(x) > b$ implies $v^*(D^*x) > v^*(x) - b$, hence is continuous, and hence induces a derivation of R^*/M' of the proof of 6.6. □

Proof of 6.4. Again we limit ourselves to giving the proof in characteristic $p > 0$. We have $\Gamma \subseteq \Xi + p^r A(K)$ for some finite set Ξ. By 6.2, for some finite union $\underline{Y}$ of translates of Abelian subvarieties of $\underline{A}$,

$$\underline{X} \cap (\Xi + p^r A(K^s)) = \underline{Y} \cap (\Xi + p^r A(K^s)).$$

Thus by 6.6, for some c and all $a \in (\Xi + p^r A(K^s))$, $\lambda_{\underline{X}}(a) \leq c \cdot \lambda_{\underline{Y}}(a)$. (Actually we require here a version of 6.4 valid for $(\Xi + p^r A(K^s))$ in place of $p^r A(K^s)$; this can be proved in the same way.) □

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ABSTRACT. We give a proof of the geometric Mordell-Lang conjecture, in any characteristic. Our method involves a model-theoretic analysis of the kernel of Manin's homomorphism and of a certain analog in characteristic p.

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MODEL-THEORETIC INVARIANTS: APPLICATIONS TO RECURSIVE AND HYPERARITHMETIC OPERATIONS

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Introduction. Both in informal mathematical reasoning and in foundations one sometimes *identifies* objects a and a' in two structures $\mathfrak{A}$, $\mathfrak{A}'$ (of the same similarity type) whatever the 'nature' of the a, a' themselves may be. Thus, a mathematician does not distinguish between the rationals in one commutative field of characteristic 0 and any other. Or, if Φ is an axiom system of arithmetic with the individual constant 0 and the function constant S, and $\Phi \vdash \forall y(\mathrm{S}y \neq 0) \wedge \forall x \forall y(\mathrm{S}x = \mathrm{S}y \rightarrow x = y)$, then $\mathrm{S}^n 0$, also denoted by $\bar{\mathrm{n}}$, is taken to denote the *same* object in each model $\mathfrak{A} = \langle A, 0, S \rangle$ of Φ. In contrast, the term $\iota_x[(x = 0 \wedge \alpha) \vee (x = \mathrm{S}0 \wedge \neg \alpha)]$ would not if α is a formally undecided sentence since it denotes 0 in $\mathfrak{A}$ if $\models_{\mathfrak{A}} \alpha$ and $S0$ if $\models_{\mathfrak{A}} \neg \alpha$ (and $S0 \neq 0$). In terms of the notion: a in A and a' in A' *correspond*, one defines in an obvious way which elements are common to all structures in a class $\mathscr{A}$.

Our main interest is to see what ideas are implicit in this informal notion of correspondence. In Section 1 we give two analyses in model-theoretic terms, applicable to wide classes of structures, and coinciding with the informal notion in familiar cases. The principal general result establishes (for the case of general models) sharp explicit definability for each element of the common part. Since one of the main properties brought out in our analyses is that the common part of $\mathscr{A}$ does not admit automorphisms, we call it the *hard core* of $\mathscr{A}$.

The most interesting applications come about through the following link with recursion theory. The class of recursive sets of natural numbers made its first appearance in the literature in Gödel [31], not in connection with Church's problematic thesis, but under the name of 'entscheidungs-definite Eigenschaften'. In the spirit of the times this notion was formulated proof-theoretically, but it has a clear model-theoretic content. One considers Φ (containing $\bar{0}, \bar{1}, \ldots$) and calls $\alpha(x)$ *entscheidungsdefinit* provided, for all $\bar{\mathrm{n}}$, $\Phi \vdash \alpha(\bar{\mathrm{n}})$ or $\Phi \vdash \neg \alpha(\bar{\mathrm{n}})$. Thus, granted that $\bar{\mathrm{n}}$ denotes the *same* object in each model $\mathfrak{A}$ of Φ so that $\mathfrak{N}$ is the hard core of $\{\mathfrak{A}: \models_{\mathfrak{A}} \Phi\}$, the set $\mathfrak{N} \cap \{x: \models_{\mathfrak{A}} \alpha(x)\}$ is *independent* of $\mathfrak{A}$, and α is a uniform invariant definition of it; more precisely, invariant *on* $\mathfrak{N}$, since for (absolute) invariance, $\{x: \models_{\mathfrak{A}} \alpha(x)\}$ itself would have to be independent of $\mathfrak{A}$. (It turns out that the latter sets are *finite* sets of natural numbers.)

If $\mathfrak{C}$ is the hard core of a class $\mathscr{A}$ of structures there is then an unambiguous extension of the notions of invariance, of invariance *on* $\mathfrak{C}$, and of several related notions set out in Section 2. The extensions are meaningful for arbitrary classes of structures, but, of course, dependent on the *language* used in the definitions, e.g., whether finite or infinite formulae α are considered.

The principle involved in the applications is simply that familiar from ordinary mathematical analyses of concrete situations in abstract terms. Thus, we have

(a) Generalization of known results about some familiar cases; given results about general models (e.g., the finiteness theorem), or about recursive and recursively enumerable sets of natural numbers (e.g., maximal simple sets of Friedberg [58]), one reformulates them by replacing some or all occurrences of 'finite' by 'invariantly definable in $\mathscr{A}$', and 'recursive' by 'invariantly definable *on* $\mathfrak{C}$ in $\mathscr{A}$', and tries to prove the results for unfamiliar $\mathscr{A}$.

(b) Analyses of the familiar results. Since in any particular application distinct general notions may coincide, one has to analyse which general notions are involved in a special case. [Thus, for systems of arithmetic, $\mathfrak{C} = \mathfrak{N}$ and the collection of finite subsets of $\mathfrak{C}$ has the following properties: it has cardinal $\aleph_0$ $(=\overline{\overline{\mathfrak{C}}})$; it can be mapped into (and onto) $\mathfrak{C}$ by a function invariantly definable on $\mathfrak{C}$; each element of the collection has cardinal $<\aleph_0$ (and so $<\overline{\overline{\mathfrak{C}}}$), is invariantly definable, bounded, i.e., included in an invariantly definable set, *and* conversely.] In particular, different proofs of the same theorem in the familiar case may generalize to proofs of different theorems. This is beautifully illustrated in the case of various maximal simple set constructions; cf. Kreisel-Sacks [a].

Our principal illustrations concern ω-models of countable sets of axioms, i.e., typically, models satisfying the infinite formula $(\forall x \in \omega)\,(x=\bar{0} \vee x=\bar{1} \vee \ldots)$, and recursion theory on the recursive ordinals, though extensions to other segments of the ordinals are unambiguous. Not unexpectedly, the theory becomes particularly neat if one uses languages with suitable infinitely long expressions, *provided one restricts* *the syntax not by simple-minded purely external conditions* (e.g., cardinality of the expressions), *but by conditions on* the satisfaction relation $\models_{\mathfrak{A}} \alpha$ *between the pair of objects* $\mathfrak{A}$ *and* α (cf. Discussion, Section 2.2).

While this general model-theoretic approach is implicit in Gödel [31], the discovery of close relations between ω-models and the concept of hyperarithmeticity is due to Mostowski, cf. Grzegorczyk-Mostowski-Ryll-Nardzewski [58] and the polished exposition of Mostowski [62]. Our basic new point is that in the earlier theory one did not analyze just why the numerals played a special role in general models for formal systems of arithmetic, and simply compared subsets of $\mathfrak{N}$ which are invariantly definable on $\mathfrak{N}$ in general and ω-models, respectively. In terms of the present terminology, this neglects the fact that $\mathfrak{N}$ itself is (absolutely) invariantly definable in ω-models. but not in general models; so, sets definable *on* $\mathfrak{N}$ in ω-models are automatically (absolutely) invariantly definable, but not in general models. In short, in the old theory the notions of hard core and absolute invariance were neglected. Some modifications needed for a smooth theory were first introduced in Kreisel [61]. (The partial success of the old theory is explained in 2.1, Lemma 2, below: some elementary results hold for sets definable *on* an *arbitrary part* of the hard core.)

In connection with relative recursiveness and invariant definability from the diagram of Y, $Y \subset \mathfrak{C}$ (relative representability in a class of models of Φ, of Mostowski [62]) there is an important distinction between two types of models of Φ which collapses for general models but not for ω-models (Definition 2(c), Lemma 2, Proposition C below). This point enters into our revision of Spector's generalization [55] of Post's problem; cf. Section 3.2.

1. Basic Notions. In 1.1 the notion of common part is formulated abstractly and applied in 1.2 to a series of important classes $\mathscr{A}$. For these also a simpler formulation of the informal notion is meaningful. Suppose we have some kind of 'standard' maximum (concrete) structure $\mathfrak{A}_0$ and consider substructures $\mathfrak{A} \in \mathscr{A}$ ($\mathfrak{A}_0$ need not $\in \mathscr{A}$). Then the common part may literally be defined to be the set-theoretic intersection of all the universes of those $\mathfrak{A}$. This simpler idea is sufficient for many applications, where it coincides with the general notion.

1.1. Let $\mathscr{A}$ be a class of structures satisfying Φ, possibly containing individual constants (which the structure must specify).

Definition. $\mathfrak{A}_1$, *of the similarity type of* $\mathfrak{A}$, *is* rigidly *contained in*

$\mathfrak{A}$, *iff there is a* unique *function* F *from* A_1 *into* A, *such that the restriction of* $\mathfrak{A}$ *to the range* $\tilde{A}$ *of* F *is isomorphic to* $\mathfrak{A}_1$. (*We continue to use* $\tilde{A}$ *in this sense below;* $\tilde{A}$ *is closed under the operations of* $\mathfrak{A}$ *and contains the objects of* A *denoted by the constants of* Φ.)

Corollary. *Let* $\mathfrak{C}$ *denote the set of constants of* Φ *and let* $\mathfrak{C}_1$ *be an arbitrary labeling of* $A_1 - \mathfrak{C}$. *Then* $\mathfrak{A}_1$ *is rigidly contained in all* $\mathfrak{A} \in \mathscr{A}$ *if and only if (i) in each* $\mathfrak{A}$ *the diagram* Δ_1 *of* $\mathfrak{A}_1$ *is satisfiable, (ii) if* $\mathfrak{C}_1'$ *is a congruent labeling to* $\mathfrak{C}_1$, *i.e., obtained from* $\mathfrak{C}_1$ *by replacing* $c \in \mathfrak{C}_1$ *by* $c^* \in \mathfrak{C}_1'$ (* *not in the language of* Φ) $\Delta_1, \Delta_1^* \models_{\mathfrak{A}} c = c^*$.

(i) expresses that $\mathfrak{A}_1$ is contained in $\mathfrak{A}$; (ii) expresses rigidity.

Theorem. *Let* $\mathscr{A}$ *be the class of general models of* Φ. *If* $\mathfrak{A}_1$ *is rigidly contained in* $\mathscr{A}$, *let* $c_{\mathfrak{A}}$ *be the element of* $\mathfrak{A}$ *corresponding to* c *in* A_1. *Then, for each* $c \in A_1$, *there is a quantifier-free formula* $\Delta_c(x, \vec{y})$ ($\vec{y}$ *standing for a finite sequence), such that* (i) $c_{\mathfrak{A}}$ *is the only object* $\models_{\mathfrak{A}} \exists \vec{y} \Delta_c(x, \vec{y})$, *i.e., uniformity,* (ii) $\models_{\mathfrak{A}} (\forall x)[(\exists \vec{y})\Delta_c(x, \vec{y}) \leftrightarrow (\exists \vec{y} \in \tilde{A}^{\omega})\Delta_c(x, \vec{y})]$ (*i.e.,* $\tilde{A}$ *is a* basis *for the existential formula*). (iii) *Further, all basic operations and relations are* decided *by* Φ *for these explicit definitions as arguments.*

Proof by compactness, using rigidity; decidability follows because the full diagram is satisfied in each $\mathfrak{A}$.

Definition. $\mathscr{A}$ *possesses a hard core iff there is a maximal structure rigidly contained in each* $\mathfrak{A} \in \mathscr{A}$.

1.2. To analyse the informal notion of *common part* (c.p.) of $\mathscr{A}$ we use the following evident properties:

A1. The hard core of $\mathscr{A}$ is (isomorphic to) a substructure of c.p.,

A2. C.p. is a substructure of each $\mathfrak{A} \in \mathscr{A}$ (though not necessarily *rigidly* contained),

A3. C.p. is a monotone decreasing function of $\mathscr{A}$.

By (A3), c.p. is a substructure of each $\mathfrak{A}$: $\mathfrak{A} \subset \mathfrak{A}_0 \wedge \mathfrak{A} \in \mathscr{A}$, i.e. of the *set theoretic intersection* considered in 1 above. Below are some examples of intrinsic interest, for which hard cores and intersections are determined for suitable $\mathfrak{A}_0$. In this way it is decided whether A1–A3 determine c.p.

(a) The concrete structure ('general directed magnitude'): the set of *complex numbers* with *addition* and *multiplication*. (i) Commutative fields of characteristic 0; intersection: real rationals. (ii) Real closed fields;

intersection: real algebraic numbers. (iii) Algebraically closed fields of transcendence degree ≥ 1; intersection: algebraic complex numbers. Note that in (i), (ii) the intersections satisfy the axioms mentioned, and are isomorphic to the hard cores of (the general models of) (i), (ii) resp., but not in (iii).

Going over to arbitrary models, the common parts of (i), (ii) are determined by (A1–3) to be isomorphic to the respective hard cores, but not in (iii). Intuitively in the case of two algebraically closed fields, one would not identify $\sqrt{2}+i$ in $\mathfrak{A}$ and $\sqrt{2}+i$ in $\mathfrak{A}'$ since there is nothing to distinguish $\sqrt{2}+i$ and $\pm\sqrt{2}\pm i$: only *sets* of solutions of an irreducible equation are identified. Evidently automorphisms are to be avoided. The hard core of (iii) consists of the real rational numbers and thus agrees with the informal notion of common part.

(b) The concrete structure [set in the sense of the cumulative theory of types built up from the empty set: a model is standard if it consists of (well-founded) sets and is transitive]: *sets with the* $\in$ *relation*, $\emptyset$, and the *operations* $a \cup b$ and $\{a\}$. The following axioms are specially suited to the present problems. (i) Extensionality, the usual 'definitions' of the constants, the (predicatively) restricted replacement axiom, i.e.,

$$(\forall a)[(\forall x \in a)\exists! y\alpha(x, y) \rightarrow (\exists z)(\forall y)[y \in z \leftrightarrow \exists x(x \in a \wedge \alpha(x, y))]],$$

where α may contain parameters, but any quantifiers in $\alpha(x, y)$ are restricted to be elements of a, x, y or the parameters. (ii) In addition to (i), the axiom of union, and of infinity (asserting the existence of the set of all finite von Neumann ordinals): let $\pi(b)$ express closure of b under predecessor, i.e., $\forall x(x \cup \{x\} \in b \rightarrow x \in b)$, then:

$$(\exists \omega)(\emptyset \in \omega \wedge \forall a[a \in \omega \leftrightarrow (\forall b)\{[\pi(b) \wedge a \in b] \rightarrow \emptyset \in b\}]).$$

The intersection of (i) consists of just the hereditarily finite sets (also for the full replacement axiom); if the structure only contained $\in$ ($\emptyset$, $\{\}$, $\cup$ defined) the intersection would be empty because the hereditarily finite sets built up from any set would also be a model. The intersection of (ii) consists of the hereditarily hyperarithmetic sets of rank $<\omega_1$ (the first non-recursive ordinal). (Essentially Theorem 2 (i) of GKT [1] and Kreisel [61], but proved more simply by the methods of GMR [1] or Section 2 below.)

Going over to general models, by extensionality, each model is iso-

[1] GKT = Gandy-Kreisel-Tait [60]; GMR = Grzegorczyk-Mostowski-Ryll-Nardzewski [61].

morphic to a structure of (not necessarily well-founded) sets where $\in$, $\emptyset$, $\cup$, $\{\}$ have standard meaning. Considering only those sets that *are* well-founded, even (ii) has as intersection only the hereditarily finite sets. But without this condition, *each general model of* (ii) *contains a subset which is isomorphic to the union of* ω, *the set* ϱ_1 *of all recursive subsets of* ω, *the set* ϱ_2 *of all subsets of* ϱ_1 *with partial recursive characteristic function, etc., and closed under union* [2]. For $X \in \varrho_1$, if α is any invariant definition of X *on* ω, $\forall y[y \in x \leftrightarrow y \in \omega \wedge \alpha(y)]$ is an explicit definition of an element $X_{\mathfrak{A}}$ of $\mathfrak{A}$ corresponding to X. Note: $X_{\mathfrak{A}}$ contains in general 'non-standard' elements of $\omega_{\mathfrak{A}}$: $\omega \cap X_{\mathfrak{A}} = X$, but $X_{\mathfrak{A}} \neq X$; also, for given X and $\mathfrak{A}$, $X_{\mathfrak{A}}$ is not uniquely determined by X; finally, the definition is not *uniform* because for given α, $X_{\mathfrak{A}} \neq X_{\mathfrak{A}'}$. Thus the hard core is a proper part of this common substructure [cf. the notion of *prime model*; but the structure described is not a model of (ii)].

Dana Scott has shown (by use of Theorem 1.1) for *arbitrary* consistent extensions of (i): if the extension is consistent with Zermelo-Fraenkel's axioms *plus* regularity the hard core consists precisely of the hereditarily finite sets, but not, e.g., if $\exists! x(x = \{x\})$ is a theorem of the extension.

An ω-model of (ii) [or (i)] is, by definition, a model in which the denotation of ω consists precisely of the hereditarily finite sets built up from the empty set [3]. Now the hard core of all ω-models of (ii) consists precisely of the hereditarily hyperarithmetic sets of rank $< \omega_1$, and the same remains true if one adds to (ii) any Π_1^1 set of axioms in the notation (which have an ω-model at all), by Theorem 2 (ii) of GKT, for those elements of the hard core which *are* of rank $< \omega_1$. Furthermore any such element is explicitly definable uniformly for all ω-models.

(c) The concrete structure: *the ordinals, with the ordering relation* $<$, and *binary relations* P_n (defined by transfinite induction), 0 and *successor*

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[2] Compare the notion of GKT: the set $X^2_{\mathfrak{A}}$ represents in $\mathfrak{A}$ the subset X^2 of $\mathfrak{P}(\omega)$ (and for higher types). Here strong uniqueness conditions are required. To each $X' \subseteq \omega$ there corresponds in $\mathfrak{A}$ the (unique) class $\{X : X \in \mathfrak{A} \wedge X \cap \omega = X'\}$; for representable X', and all $\mathfrak{A} \in \mathscr{A}$, this is not empty. Then, for representable X', either $X^2_{\mathfrak{A}}$ contains the whole class or is disjoint from it. These conditions imply continuity of the characteristic functions of representable sets, and thus, e.g., (non-empty) finite sets of finite subsets of ω are not representable; cf. Feferman's review of GKT in J. Symb. Logic.

[3] This does not mean that *all* sets in an ω-model of (ii) are well-founded though, of course, ω is. Models satisfying this stronger condition are called β-models ('β' for bon-ordre) by Mostowski. The Shepherdson-Cohen minimal models make up the hard core of the β-models for ordinary set theory.

S; a model is standard if it is an initial segment of the ordinals. The following axioms are useful. (i) Total order, successor axioms, schema of proof by transfinite induction for arbitrary α: $\forall x[(\forall y<x)\alpha(y)\cdot\rightarrow\alpha(x)]\rightarrow\rightarrow\forall x\alpha(x)$; definition by transfinite recursion: suppose all quantifiers in α_n are bounded by x or y, and α_n does not contain P_n; then $\forall x\forall y[P_n(x, y) \leftrightarrow \alpha_n^{[x]}(P_n, x, y)]$ is an axiom where $\alpha_n^{[x]}$ is obtained from α_n by replacing $P(t, z)$ by $x\leqslant t \vee P(t, z)$, i.e., $\lambda y P_n(x, y)$ is defined from $\lambda z<x\lambda y P_n(z, y)$; the supremum: if all quantifiers in $\alpha(x, y)$ are bounded, $(\forall a)[(\forall x<a)\exists y\alpha(x, y)\cdot\rightarrow \exists z(\forall x<a)(\exists y<z)\alpha(x, y)]$. (ii) In addition to (i) the axiom of infinity $(\exists\omega)(\forall x<\omega)\{Sx \neq \omega \wedge [x=0 \vee (\exists y<x)(Sy=x)]\}$.

The intersection of (i) consists of the finite ordinals, and that of (ii) the recursive ordinals.

Going over to general models, every model is isomorphic to one in which each element whose set of predecessors has ordinal σ, *is* the (abstract) ordinal σ. (Other elements can in general *not* be replaced by the abstract order type of their predecessors because a non-well-ordering may permit automorphisms and so different elements may define the same order type.) Considering only the ordinals, the intersection of (ii) is still only ω because we can consistently add the set of axioms $a_1<\omega$, $a_2<a_1$, ..., $a_{n+1}<a_n$, ...

The *hard core* of all consistent recursively enumerable extensions of (i) is also ω, by use of provably disjoint recursively enumerable sets which are not recursively separable, and decidability of the P_n on the hard core.

An ω-model of (ii) is one in which ω *is* ω. The hard core of all ω-models consists of the recursive ordinals, with P_n determined uniquely. The same remains true on addition of an arbitrary Π_1^1 set of axioms (in the present notation) provided they have an ω-model. Each recursive ordinal α can be explicitly defined, e.g., by defining first an ordering of the finite ordinals with initial segment of type ω_1, as in Gandy [60], and then a functional relation $P_n(x, y) \wedge y<\omega$, $\forall x\exists! y P_n(x,y)$, mapping the ordinals $<\alpha<\omega_1$ into the segment and preserving order [4].

2. Invariants. The theorem of 1.1 gives results on uniform definability of singletons. Now we are concerned with *subsets* of the hard core which can be defined (in one sense or another) in each structure $\mathfrak{A} \in \mathscr{A}$. Let Q

[4] To study the hard core of a (transfinite) type theory in which variables are explicitly supplied with (ordinal) types, one has to combine the considerations of (b) and (c).

be a set of constants intended to define elements of the hard core; but the results obtained will use only the weaker condition: for distinct $c, c' \in \mathbb{C}$, $\Phi \models \mathrm{c} \neq \mathrm{c}'$. One may therefore suppose that the symbols c denote themselves, and so the letters C and $\mathbb{C}$ may be interchanged.

Throughout, $X \subseteq \mathbb{C}$, $C_1 \subseteq \mathbb{C}$, $C_2 \subseteq \mathbb{C}$ and c denotes an element of $\mathbb{C}$, φ a formula with one free variable. (The extension to $X \subseteq \mathbb{C}^n$ is immediate, by use of formulae φ with n free variables. If needed, $\vec{c}$ denotes an element of C^n.)

Definitions. *X is* invariantly *or* uniformly (a) *defined*, (b) *defined on C_1, by φ. in $\mathscr{A}$ iff*:

(a) $(\forall \mathfrak{A} \in \mathscr{A})[X = \{a : a \models_{\mathfrak{A}} \varphi\}]$, (b) $(\forall \mathfrak{A} \in \mathscr{A})[X = C_1 \cap \{a : a \models_{\mathfrak{A}} \varphi\}]$.

X is (a′) *definable*, (b′) *definable on C_1, in $\mathscr{A}$ iff*:

(a′) $(\forall \mathfrak{A} \in \mathscr{A})\exists\varphi[X = \{a : a \models_{\mathfrak{A}} \varphi\}]$, (b′) $(\forall \mathfrak{A} \in \mathscr{A})(\exists\varphi)[X = C_1 \cap \{a : a \models_{\mathfrak{A}} \varphi\}$, *where, as usual, φ may contain constants for arbitrary elements of A.*

For $Y \subseteq C_2$, X is invariantly defined from Y on C_1 *by φ, containing a predicate letter* Y *not in the language of Φ, iff*:

(c) $(\forall \mathfrak{A} \in \mathscr{A})(\forall Y' \subseteq A)[Y' \cap C_2 = Y \rightarrow X = C_1 \cap \{a : a, Y' \models_{\mathfrak{A}} \varphi\}]$.

(Evidently, there are notions corresponding to, e.g., (b′), non-uniform definability, etc.)

Remark. If Φ is a set theory, it is *not* assumed that Y' or Y is necessarily an object in the universe of $\mathfrak{A}$. Thus, if Φ contains the comprehension axiom, the letter Y does not appear in it. In short, Y is regarded as a *property*, not a *set*. More formally: $a, Y' \models_{\mathfrak{A}} \varphi$ should be written: $a, Y' \models_{\mathfrak{A}'} \varphi$ where $\mathfrak{A}'$ is the structure obtained from $\mathfrak{A}$ by adding an additional relation type.

Correspondingly, in 1.2 (b) and (c): if Y is regarded as a set, Y may appear in α in the replacement and supremum axioms resp., but not if Y is regarded as a property. Note that in the former case the hard core may be altered, but not in the latter.

Some notions of *implicit* definability corresponding to the above explicit ones are these:

(ai) Let the set Ψ contain the predicate symbol X, and possibly others Y, not contained in Φ. Ψ is a *uniform implicit definition* of X in $\mathscr{A}$ if and only if:

$$(\forall \mathfrak{A} \in \mathscr{A})(\forall X' \subseteq A)[X = X' \leftrightarrow (\exists \vec{Y})(X', \vec{Y} \models_{\mathfrak{A}} \Psi)].$$

(a′i) X is implicitly definable in (each structure of) $\mathscr{A}$ iff:

$$(\forall \mathfrak{A} \in \mathscr{A})(\exists \psi)(\forall X' \subseteq A)[X = X' \leftrightarrow \exists \vec{Y}(X', \vec{Y} \models_{\mathfrak{A}} \psi)]$$

where ψ is a *single* formula.

(bi) ψ is a *uniform implicit definition* of X *on* C_1 iff:

$$(\forall \mathfrak{A} \in \mathscr{A})(\exists X', \vec{Y})(X', \vec{Y} \models_{\mathfrak{A}} \psi) \wedge \forall X'[(\exists \vec{Y'})(X', \vec{Y} \models_{\mathfrak{A}} \psi) \cdot \rightarrow X = X' \cap C_1].$$

Note that in contrast to (ai), X itself need not satisfy $(\exists \vec{Y})(X, \vec{Y} \models_{\mathfrak{A}} \psi)$.

Terminology. For $\mathbf{C} = \omega$, (b) is usually called: reckonable, strongly representable, (strongly) definable, or binumerable. (a) is not generally considered. (bi) is (a slight generalization of) Fraïssé's [61a] relative recursiveness when Φ is the diagram of the basic relations (his: $R_1, \ldots, R_p$ on p. 320) on $\mathbf{C}$ (his: E). In his Section 5, Fraïssé's intended meaning is almost certainly better expressed in terms of models whose universe is E itself (p. 323, l. 5), than by his formal definition.

2.1. *Technical lemmas on general models.* Most of the results below are easy, known, and/or given in Kreisel-Krivine [a], Chapter V. Φ is an arbitrary (possibly uncountable) consistent set of sentences in ordinary first-order language, and $\mathscr{A}$ its class of general models.

Lemma 1. *If X is definable in $\mathscr{A}$, uniformly or not, explicitly or implicitly, X is a* finite *subset of* $\mathbf{C}$.

Lemma 2. For $\mathbf{C}_1 \subset \mathbf{C}$, *if X is definable* on $\mathbf{C}_1$ *in $\mathscr{A}$, there is a φ and a set Φ_1, of cardinality $< \aleph_0 + \Phi$, such that for all c (of course: not all a) and $\mathfrak{A} \in \mathscr{A}$,*

$$c \in X \leftrightarrow [\Phi, \Phi_1 \vdash \varphi(c)] \text{ and } c \notin X \leftrightarrow [\Phi, \Phi_1 \vdash \neg \varphi(c)].$$

Thus, (i) for countable Φ, Φ_1 is finite. (ii) If (the set of Gödel numbers of) Φ is recursively enumerable in some set $\mathfrak{N}_0$ ($\subseteq \mathfrak{N}$), X is recursive in $\mathfrak{N}_0$ *on* $\mathbf{C}_1$, say $\subseteq \mathfrak{N}$; here as usual, X is called: recursive *on* $\mathbf{C}_1$, if there are two recursively enumerable sets Y_1, Y_2 which are not necessarily complementary on $\mathfrak{N}$ itself, such that $\mathbf{C}_1 \cap Y_1 = \mathbf{C}_1 \cap \complement Y_2 = X$. (iii) An example of Keisler shows that, in general, Φ_1 cannot be dropped even for countable Φ, and one of Dana Scott that Φ may have to be infinite for uncountable Φ; but in these examples the hard core is empty. Lemma 2 shows that a definable set is almost invariantly definable, and an analogous argument establishes the analogous result for implicit definability. (iv) If Φ is a Rosser theory, in the sense of Smullyan [61], p. 135, extended to Φ that are not recursively enumerable, a definable set is invariantly definable.

(ii) extends immediately to relative invariant definability from Y. The collapse of the distinction made after Definition 2(c), for the case of *general* models, follows from:

Lemma 3. *Let Φ contain at least second-order type theory in which every finite subset of C exists, i.e., $\Phi \models_{\mathfrak{A}} (\exists S)\forall x[S(x) \leftrightarrow (x = \mathsf{c}_{n_1} \vee \ldots \vee x = \mathsf{c}_{n_p})]$. Suppose Y does not occur in Φ, $Y \subseteq C$ and $\Psi = \{\mathsf{Y}(\mathsf{c}) : c \in Y\} \cup \{\neg \mathsf{Y}(\mathsf{c}) : c \notin Y\}$. Then the addition of γ: $\exists S \forall x[S(x) \leftrightarrow \mathsf{Y}(x)]$ to $\Phi \cup \Psi$ is conservative.*

If Φ, Ψ, $\gamma \vdash \alpha$ then also Φ, Ψ_1, $\gamma \vdash \alpha$ for some finite subset Ψ_1 of Ψ. Replace $\mathsf{Y}(t)$ throughout by the disjunction of $t = \mathsf{c}$, $\mathsf{Y}(\mathsf{c}) \in \Psi_1$ to form ψ_1', γ'. Then $\Phi \vdash \psi_1' \wedge \gamma'$, and hence, if α does not contain Y, $\Phi \vdash \alpha$.

2.2. *Technical lemmas on ω-models.* We consider ω-models only for systems of set theory and of ordinals (cf. Section 1.2) and only *countable* Φ, the latter because the ω-rule is not complete for uncountable Φ. For more general definitions and counterexamples, cf. Kreisel-Krivine [a].

We first consider sets of finite formulae and then examine for what infinite formulae the arguments are valid. Since finite ordinals are used in the arithmetization of the syntax of finitary language, we are first concerned with sets of integers: for $X \subseteq \omega$, *X is (uniformly) definable on ω if and only if X is (uniformly) definable*, since, if $\alpha(x)$ defines X *on* ω, $\alpha(x) \wedge x \in \omega$, resp. $\alpha(x) \wedge x < \omega$ defines X in all ω-models considered.

Lemma 2 of 2.1 holds for ω-models of *countable* Φ by GMR. Consequently, either directly by Kreisel [61], p. 113, or by the completeness of the ω-rule:

If Φ is Π_1^1 in (= inductively defined from) a set $X_0 \subseteq \omega$, and $X \subseteq \omega$ is definable in all ω-models of Φ, then X is hyperarithmetic in X_0.

The following propositions show that, *in contrast to general models, the class of invariantly definable sets* (of integers) *depends on* Φ, and similarly for related notions.

Proposition A. (i) *If $\Phi \in \Pi_1^1$, the invariantly definable sets are hyperarithmetic, but* (ii) *there are $\Phi \in \Sigma_1^1$ such that* all *Π_1^1 subsets of ω are invariantly definable* (*in contrast to Lemma* 1).

Proof: (i) by GMR, (ii), e.g., by Kreisel [61], p. 119 (a).

Proposition B. *Let Φ be ordinary axiomatic second-order arithmetic not containing the letters* X, X' *and let X be non-hyperarithmetic, Φ_1 the diagram of X for* X, *Φ_1' its diagram for* X'. *Then (i)* $\Phi \cup \Phi_1 \cup \Phi_1' \cup \cup \{\forall S \exists x[S(x) \leftrightarrow \neg \mathsf{X}(x)]\}$ *has an ω-model, (ii) its extension by* $(\exists S)\forall x [S(x) \leftrightarrow \mathsf{X}'(x)]$ *does not.*

Proof: (i) by the basis lemma of GKT, (ii) obvious.

This contrasts with Lemma 3; but though of course (i) assures the existence of all *finite* or even *hyperarithmetic* subsets of ω (the comprehension axiom is provable for them), *it does not assure the existence of all sets $\subseteq \omega$ which are invariantly definable in* (i). A simple computation shows this about invariant definability *from X*: let X be O (the set of ordinal notations), and Φ as in Proposition B; then the class of sets invariantly definable in $\Phi \cup \Phi_1$ from (the property) O can be enumerated by a function hyperarithmetic in O, but all functions hyperarithmetic in O are invariantly definable in $\Phi \cup \Phi_1 \cup \{\exists S \forall x[S(x) \leftrightarrow \mathsf{X}(x)]\}$ from (the set) O by Mostowski [62; II, 5, 6]. Thus for ω-models, the distinction (cf. the remark in the first part of Section 2) between definability from properties and sets is real.

Proposition C. *If Φ is Π_1^1 in X_0 ($X_0 \subseteq \omega$) then the ω-rule applied to Φ closes off after $\leqslant \omega_1^{X_0}$ steps, where $\omega_1^{X_0}$ is the least ordinal not invariantly definable from X_0 (in all ω-models.)*

Proof as in Theorem 3 of Spector [61], where it is stated for absolutely Π_1^1 sets Φ. It is nearly stated in Mostowski [62], p. 41, II. 5.3 with $\omega_1^{|\Phi|}$ ($|\Phi|$ the characteristic function of Φ) instead of the generally smaller $\omega_1^{X_0}$. The improvement expresses that a rule uses only the occurrence of a formula in a set of axioms, not its absence.

Recall that the ordinary logical rules (for the consequence relation of general models) close off after ω steps, and note that, in the case of general models, for all X_0, ω is the least ordinal not invariantly definable from X_0.

Discussion. To extend, e.g., Lemma 1, whose proof depends on the finiteness theorem for general models, it is necessary to extend the latter to ω-models. Propositions A–C (are set out to) suggest that the relevant property of finiteness is invariant definability in general models and this turns out to be true (cf. next section). Part of Lemma 1 concerns the step from (invariant, possibly implicit) definability to *uniform* explicit definability, and therefore involves essentially a choice of language. Inspection of, e.g., the proof of Proposition A in Kreisel [61], shows: all that is needed of the formulae Φ is that the satisfaction relation ('the structure described by F satisfies φ') be Σ_1^1 in F and (the Gödel number of) φ and X_0. By use of countable models, F may be taken to be number-theoretic functions. This is satisfied not only by finite formulae, but also infinite formulae φ, e.g., infinite disjunctions and finite strings of quantifiers, provided, regarded as syntactic objects, they are hyperarithmetic in X_0; the collection of such φ is countable, and thus the φ may be given *finite* Gödel numbers. (NB: One cannot rely on simple-minded syntactic conditions, e.g., $\forall x_1 \ldots \forall x_n \ldots [\neg R(x_1,x_2) \vee \neg R(x_2,x_3) \vee \ldots]$ expresses that R is well-founded, hence the satisfaction relation is Π_1^1, but the syntactic object is a simple $\omega+\omega$ sequence. Dana Scott [*], in a related context, introduced the restriction to finite sequences of quantifiers for purely pragmatic reasons; but there too the analysis in terms of the satisfaction relation seems to be the correct theoretical foundation.) Going back to (countable) general models of first-order formulae Φ, a model F (on ω) may be taken as given by a *set* F of number-theoretic functions, namely the characteristic functions of the relations in Φ, and of the Skolem functions of Φ; then the relation 'F satisfies Φ' ($\leftrightarrow$ F satisfies a certain infinite set of propositional formulae) is Π_1^0 in F and Φ. From this point of view, Mostowski's proof of the finiteness theorem in [61] generalizes naturally to ω-models (though in 3.1 we use a different method); only the final step of his proof needs elaboration because, in the case of general models, one can, and he does, take advantage of special compactness properties of continuous mappings from 2^ω into a discrete space.

3. Applications. Only some points of principle are considered.

3.1. *Theory of ω-models.* Theorem. *Suppose Φ is Π_1^1 in X_0 and every $\Phi_1 \subseteq \Phi$, which is hyperarithmetic in X_0, has an ω-model. Then Φ itself has an ω-model.*

Corollary. *If $\Phi \vdash_\omega \varphi$, then there is a $\Phi_1 \subseteq \Phi$, Φ_1 hyperarithmetic in X_0, such that $\Phi_1 \vdash_\omega \varphi$.*

(It seems to be open whether relative hyperarithmeticity, i.e. definability from the *set* X_0, can be replaced by definability from the *property* X_0.)

Proof. Since Φ is Π_1^1 in X_0 there is an arithmetic functional K from $\mathfrak{P}(\omega)$ to $\mathfrak{P}(\omega)$, $\forall X[K(X) \supset X]$ and Φ is the least class $\supset X_0$: $K(\Phi) = \Phi$.

If, for all ordinals α, $X^0 = X_0$, $\alpha \neq 0$: $X^\alpha = \bigcup_{\beta<\alpha} K(X^\beta)$, $X^\alpha = \Phi$ for $\alpha \geqslant \omega_1^{X_0}$ by Spector's [61] Theorem 3. Now let Ω be the (arithmetic) functional mapping sets X of formulae into the set of all logical consequences of X together with a *single* application of the ω-rule. Again $\Omega(X) \supset X$; let the closure be $Cn_\omega(X)$. Consider the sequence of *pairs* of sets of formulae:

$$\langle \Phi^0, X^0 \rangle = \langle X_0, X_0 \rangle; \text{ for } \alpha \neq 0,\ \langle \Phi^\alpha, X^\alpha \rangle = \langle X^\alpha \cup \bigcup_{\beta<\alpha} \Omega(\Phi^\beta), \bigcup_{\beta<\alpha} K(X^\beta) \rangle.$$

The closure rule is still arithmetic; by above, for $\alpha \geqslant \omega_1^{X_0}$, $X^\alpha = \Phi$ and so $\Phi^\alpha \supset \Phi$. By completeness of the ω-rule, $\Phi^\alpha \supset Cn_\omega(\Phi)$, for all $\alpha \geqslant \omega_1^{X_0}$. So, if $\Phi \vdash_\omega \varphi$, $\varphi \in \Phi^\alpha$ for some $\alpha < \omega_1^{X_0}$. Since, for $\alpha < \omega_1^{X_0}$, $\Phi^\alpha \subset Cn_\omega(X^\alpha)$, and Φ^α is hyperarithmetic in X_0, the theorem is proved.

In view of this generalization we prefer to speak of the finiteness theorem for general models, and not of: compactness theorem. For, though compactness is used in most proofs (and shortens them), a generalization to non-compact cases is possible, as above.

The use of the theorem simplifies and improves a good deal of GKT. Also it brings out more clearly the relation between (i) my construction, in Kreisel [61], p. 120, (ii), of a Π_1^1 set Φ (extension of ordinary second-order arithmetic) which is not ω-equivalent to any $\Pi_1^1 \cap \Sigma_1^1$ (sub)set and (ii) non-finite axiomatizability of ordinary arithmetic. (Of course, if no special conditions are imposed on Φ, one can take any non-hyperarithmetic Π_1^1 set X and put $\Phi = \{\mathsf{P}(\bar{n}) : n \in X\}$ imitating the construction of infinite sets not equivalent to finite ones for general models.)

3.2. *Recursion theory on the recursive ordinals.* The following two languages are suitable for isolating important areas of classical recursion theory.

(a) *Elementary theory of recursively enumerable sets.* The only constants are $\emptyset$, $=$, $\subseteq$ and one kind of variable. In the interpretations, $\emptyset$, $=$, $\subseteq$ have the usual meaning. We consider three interpretations (for the range of the variables):

(i) Recursively enumerable subsets of ω. For this interpretation, $R_0(x) \underset{\text{def}}{=} (\exists y)(x = \complement y)$ defines the recursive sets, and (Lacombe): $\text{Fin}(x) \underset{\text{def}}{=} (\forall y)[y \subseteq x \to R_0(y)]$ the finite ones. Non-trivial parts of recursion theory can be formulated in this language such as decomposition of recursively enumerable sets or existence of a maximal simple set, Friedberg [58], but cf. (b) below.

In terms of model-theoretic invariants (for *general* models of the theory of ordinals), recursive functions are those invariantly definable on ω; a recursively enumerable set is defined as the *range* of such a function. We consider ranges included (ii) in the hard core ($=\omega$ for general models), (iii) in ω, and go over to ω-models (of Π_1^1 sets of axioms).

(ii) Π_1^1 subsets of ω_1 [5]. Now $R_0(x)$ defines the sets hyperarithmetic *on* ω_1, and Fin(x) the finite ones.

(iii) Π_1^1 subsets of ω. Now, $R_0(x)$ defines the hyperarithmetic subsets of ω, and Fin(x) the finite ones. This coincides with Mostowski [62].

It is not impossible that all three interpretations are equivalent for this language. H. Rogers suggested during the symposium that the three classes of sets may be isomorphic with respect to inclusion, and R. L. Vaught that the theory might be decidable.

(b) *Elementary theory of monadic recursive functions.* The language contains two kinds of variables ($x, y, \ldots$ for individuals and $f, g, \ldots$ for monadic functions), constants 0, $=$ and application $[f(x)]$.

(i) Universe of individuals$=\omega$, of functions: number-theoretic recursive functions. (a) (i) above can be interpreted here by defining $f \subseteq g: \forall x \exists y [f(x)=g(y)]$, and so the definitions of R_0 and Fin carry over. Alternative definitions here are: $R_0^+(f) \underset{\text{def}}{=} (\exists g)(\forall x)\{g(x)=0 \leftrightarrow (\exists y)[f(y)=x]\}$, $F(f) \underset{\text{def}}{=} (\forall g)$ [the range of the function g restricted to the range of f is R_0^+]. $B(f) \underset{\text{def}}{=} \exists g [f \subseteq g \wedge F(g)]$ here also defines the finite ($=$bounded recursively enumerable) sets.

By use of inverse pairing functions the theory of n-ary recursive functions can be interpreted in the present language.

Model-theoretically, we again regard the universe of individuals as (ii) the hard core, (iii) ω, the functions as those (ii) invariantly definable on the hard core, (iii) on ω, and go over to ω-models (of Π_1^1 sets of axioms).

(ii) Universe of individuals$=\omega_1$, of functions: those invariantly definable *on* $\omega_1 \cdot \forall f [R_0^+(f) \leftrightarrow R_0(f)]$, but $\neg \forall f [F(f) \leftrightarrow \text{Fin}(f)]$ since $F(f)$ defines the absolutely hyperarithmetic subsets of ω_1, and $\neg \forall f [B(f) \rightarrow F(f)]$.

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[5] What is meant is: subsets of ω_1, which are the range of some function which is invariantly definable *on* ω_1, in the sense of the present paper. The notation is chosen because any such set X can also be defined as follows: If W is the initial segment of a recursive ordering of ω of type ω_1, e.g., as in Gandy [60], and $|n|$ the ordinal of the initial segment determined by n (in the field of W), then $X=\{|n|: n \propto X'\}$ where X' is a Π_1^1 subset of ω.

Thus by using F, Fin, B, respectively, in the three places where 'finite' occurs in the maximal simple set theorem, one gets 27 formally distinct generalizations! For details, cf. Kreisel-Sacks [a].

(iii) Universe of *individuals* $=\omega$, of *functions*: those (whose graph is) invariantly definable *on* ω, i.e., the (absolutely) hyperarithmetic number-theoretic functions. The range of every such function is hyperarithmetic and thus $\forall f F(f)$ (and not all Π_1^1 subsets of ω can be enumerated). So the theory of Π_1^1 sets cannot be interpreted here by the definition a(iii) above.

Relative recursiveness (between recursively enumerable sets X and Y). For general models of recursively enumerable axiom systems Φ, containing a minimum Φ_0 of arithmetic, the following relations are equivalent: (1) X is invariantly definable from Y *on* the common part of all models of Φ, Y being regarded as a set, (2) with Y being regarded as a property, (3) if German lower case letters denote invariantly definable subsets of of the common part [i.e. enumerated by f: $F(f)$], there is g: $R_0(g)$, $n \in X \leftrightarrow (\exists \mathfrak{a} \subset Y)(\exists \mathfrak{a}' \subset \complement Y)[g(n, \mathfrak{a}, \mathfrak{a}')=1]$, $n \notin X \leftrightarrow (\exists \mathfrak{a} \subset Y)(\exists \mathfrak{a}' \subset \complement Y)$ $[g(n, \mathfrak{a}, \mathfrak{a}')=2]$, (4) as (3) with $n \in X$ replaced by $\mathfrak{n} \subset X$, $n \notin X$ by $\mathfrak{n} \subset \complement X$, $g(n, \mathfrak{a}, \mathfrak{a}')$ by $\mathfrak{g}(\mathfrak{n}, \mathfrak{a}, \mathfrak{a}')$.

Note that for recursively enumerable sets X, Y the relations (3), (4) can be expressed in the language (b) above for the interpretations b(i), b(iii). – The existence of Φ_0 shows that the relations (1) and (2) are invariant invariants! i.e., the same relation is defined for each $\Phi \supset \Phi_0$ considered. – The common part of all models of Φ *together* with the diagram of Y (on ω) is still ω, even if Y is considered as a set.

For ω-models and Π_1^1 sets Φ. By Mostowski [62], (1) is an invariant invariant, but the argument does not apply to (2) for all Y (on ω_1). Also, the use of infinite formulae does not alter (1), but, in general, increases (2). By definition the common part of all ω-models of Φ is not altered if Y is regarded as a property, but in general is increased if Y is regarded as a set. – Trivially (1) $\supset$ (2) $\supset$ (3) $\supset$ (4); the inclusions are proper. (1) $\supset$ (2) by p. 187, (2) $\supset$ (3) by recent work of Driscoll since (2) is transitive and (3) is not, (3) $\supset$ (4) because a simple computation shows that (4) is transitive.

For each of the relations (1)–(4) Post's problem generalizes to this [for the interpretation b(ii)]. To find Π_1^1 sets of recursive ordinals X, Y which are incomparable for the relation considered. – Friedberg's proof [58] generalises directly for (3) and hence for (4), Kreisel-Sacks [a]. It is open for (2). A satisfactory solution for (1) requires X, Y with the property that the common part of all ω-models of Φ_0 *together* with the diagrams on ω_1 of X, Y resp., be still ω_1. (Cf. *generic* sets of Cohen [63, 64].) (*Added in proof*. The existence of such X and Y has been established by Sacks, in his *Metarecursively enumerable sets and admissible ordinals*, forthcoming in Bull. Amer. Math. Soc.)

The generalization above conflicts with Spector's [55] on two counts. First he uses Π_1^1 subsets of ω, instead of ω_1, second the relation (1), which is equivalent to relative hyperarithmeticity for all Π_1^1 systems Φ containing Φ_0.

RECURSIVE FUNCTIONALS AND QUANTIFIERS OF FINITE TYPES I

BY
S. C. KLEENE

This is the third of a series of papers in these Transactions on hierarchies obtained by quantifying variables of recursive predicates. In *Recursive predicates and quantifiers* [10] variables for natural numbers (type 0) were quantified, and in *Arithmetical predicates and function quantifiers* [14] also variables for one-place number-theoretic functions (type 1). In the present paper we shall quantify also variables for functions (i.e. functionals) of higher finite types (types 2, 3, 4, $\cdots$). A theory of recursive functions and predicates of variables of these types has not previously been developed. For the hierarchy results in the form that new predicates are definable by increasing the highest type of variable quantified(1), or the number of alternations of the quantifiers of the highest type, it would suffice to extend the notion of primitive recursiveness to the higher types of variables. This is how we began the investigation in 1952(2). However there are situations where the question "What becomes of this theory for higher types?" calls for an extension of general and partial recursiveness. For example, before we can extend Post's notion of degree of unsolvability [25; 19], which proved fruitful in the study of hierarchies of number-theoretic predicates [5; 14; 16; 26], to predicates with type-1 variables, we must have a notion of relative general recursiveness for such predicates, which when uniform amounts to having general recursiveness for functions with type-2 variables. Accordingly we now give an extension of general and partial recursiveness. The treatment entails incidentally a somewhat new treatment of general and partial recursive functions of variables of types 0, 1. As will appear, many of the known results for types 0, 1 extend to the higher types, but not all. In this Part I we leave undiscussed various aspects of the subject on which work is in progress or completed which we hope to report in a Part II.

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(1) This result was obtained by Tarski [28], using set variables. Also cf. [29].

(2) Cf. XXXIX below. This paper incorporates (with a new method of proof) the results alluded to in the last sentence of the abstract of [14] as presented to the Association for Symbolic Logic on December 29, 1952 (J. Symb. Logic vol. 18 (1953) p. 190), and is the paper referred to in [14, p. 312] and [16, p. 212] as to be written under the title *Analytic predicates and function quantifiers of higher finite types*. The term "analytic" was applied in [14] to the predicates obtained by quantifying variables of types ≤ 1, and we have meanwhile decided that it would invite confusion to extend the use of "analytic" to include quantification of higher-type variables; hence the present change of title.

1. Primitive recursive functions. 1.1. We shall cite our *Introduction to metamathematics* [13] simply as IM, and follow it in terminology and notation (cf. bottom p. 538) except as otherwise specified(3).

1.2. The objects of *type* 0 are the natural numbers $0, 1, 2, \cdots$. As type-0 variables, i.e. variables ranging over the natural numbers, we use $a, b, c, \cdots, a_1, a_2, \cdots$ or $\alpha^0, \beta^0, \gamma^0, \cdots, \alpha_1^0, \alpha_2^0, \cdots$, etc.

For each $j > 0$, the objects of *type* j are the one-place functions from type-$j-1$ objects to natural numbers(4). As type-j variables, i.e. variables ranging over the type-j objects, we use $\alpha^j, \beta^j, \gamma^j, \cdots, \alpha_1^j, \alpha_2^j, \cdots$, etc. (For $j > 1$, they are "functional" variables.)

In illustrations, we may use simply $\alpha, \beta, \gamma, \cdots, \alpha_1, \alpha_2, \cdots$ for type-1 variables (as in [12; 14; 16]); and $\mathbf{F}, \mathbf{G}, \mathbf{H}, \cdots$ for type-2 variables.

Sometimes we omit the type index on the letters $\alpha^j, \beta^j, \gamma^j, \cdots$ used as type-j variables in contexts where the type should be clear (e.g. $\alpha^5(\beta^4)$ can be written $\alpha^5(\beta)$ or $\alpha(\beta^4)$ without ambiguity). One must then not confuse $\alpha, \beta, \gamma, \cdots$ for variables of types shown by the context with $\alpha, \beta, \gamma, \cdots$ for type-1 variables.

Letters like $\phi, \psi, \chi, \cdots$ will be used (as in [12; 14] with types 0, 1) for functions of a given finite number of variables, each variable being of a specified one of our types, taking a natural number as value. (These functions are "functionals," when any of their variables are of type > 0.)

1.3. In the theory of general and partial recursive functions of variables of types 0, 1 a useful role was played from the beginning by the primitive recursive functions (cf. e.g. [8], IM). They constitute a subclass of the general recursive functions, each function of which subclass can be calculated for given arguments by steps which can be proved by known reasoning to terminate; and they can be enumerated effectively. Subsequent research showed that smaller classes of functions can play the same role, e.g. the elementary functions of Csillag and Kalmár (cf. IM, p. 285). Such results are of interest, e.g. for type 0 as a possible line of approach toward a negative solution of Hilbert's tenth problem (1900). However for our purposes the largeness of the class of primitive recursive functions has seemed rather convenient. Accordingly we shall begin the theory of recursive functions of variables of types $0, 1, 2, \cdots$ with them(5), leaving open the question what smaller classes of functions could play the same role.

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(3) Several notations from papers subsequent to IM are: $\bar{\alpha}(x)$ for $\prod_{i<x} p_i^{\alpha(i)+1}$ (cf. [12; 14, Footnote 2]), Seq(w) for $w \neq 0$ & $(i)_{i<\text{lh}(w)}[(w)_i \neq 0]$ (cf. [15, p. 416]), Ext(w, u) for Seq(w) & $(Ex)_{x \leq \text{lh}(w)}[u = \prod_{i<x} p_i^{(w)_i}]$ (cf. Spector [27, p. 588]).

(4) I.e., arbitrary such functions, not merely recursive ones. For the type-1 case, cf. [12, p. 683].

(5) Our extension of primitive recursiveness to allow higher types of variables does not alter the notion for functions of variables of types 0, 1 (by VIII below). What Péter calls a "primitive recursion on the second level (II-te Stufe)" XXIV below can be used to define number-theoretic functions that are not primitive recursive (cf. [24, pp. 247–248, 252, 256] and [23, pp. 68, 97–99]).

For each application of the following schemata S1–S8, $\mathfrak{b}$ is any list (possibly empty) of variables, distinct from one another and from the other variables of the schema, and each of a specified one of our types. Furthermore, a is a number variable, α is a function variable of type 1, q is a given natural number, ψ and χ are given functions of the indicated variables, and ϕ is the function being defined. For S6, $\mathfrak{a}_1$ is a list of distinct variables, containing at least $k+1$ of type j, from which $\mathfrak{a}$ results by moving the $(k+1)$st type-j variable to the front of the list. The expressions shown at the right will be explained in §3.

S1	$\phi(a, \mathfrak{b}) = a' = a+1$	$\langle 1, \langle n_0, \cdots, n_r\rangle\rangle$.
S2	$\phi(\mathfrak{b}) = q$	$\langle 2, \langle n_0, \cdots, n_r\rangle, q\rangle$.
S3	$\phi(a, \mathfrak{b}) = a$	$\langle 3, \langle n_0, \cdots, n_r\rangle\rangle$.
S4	$\phi(\mathfrak{b}) = \psi(\chi(\mathfrak{b}), \mathfrak{b})$	$\langle 4, \langle n_0, \cdots, n_r\rangle, g, h\rangle$.
S5	$\begin{cases} \phi(0, \mathfrak{b}) = \psi(\mathfrak{b}), \\ \phi(a', \mathfrak{b}) = \chi(a, \phi(a, \mathfrak{b}), \mathfrak{b}) \end{cases}$	$\langle 5, \langle n_0, \cdots, n_r\rangle, g, h\rangle$.
S6	$\phi(\mathfrak{a}) = \psi(\mathfrak{a}_1)$	$\langle 6, \langle n_0, \cdots, n_r\rangle, j, k, g\rangle$.
S7	$\phi(\alpha, a, \mathfrak{b}) = \alpha(a)$	$\langle 7, \langle n_0, \cdots, n_r\rangle\rangle$.
S8	$\phi(\alpha^j, \mathfrak{b}) = \alpha^j(\lambda\alpha^{j-2}\, \chi(\alpha^j, \alpha^{j-2}, \mathfrak{b}))$	$\langle 8, \langle n_0, \cdots, n_r\rangle, j, h\rangle$.

S6, S7, S8 may also be designated as S6.j ($j \geqq 0$), S7.1, S8.j ($j \geqq 2$).

The schemata will be used under the convention that only the order of listing the variables within each type is material. For example, $\phi(a, b, \alpha) = a+1$, $\phi(a, \alpha, b) = a+1$, $\phi(\alpha, a, b) = a+1$ are all admissible as applications of S1, but not $\phi(b, a, \alpha) = a+1$ (though this function can be introduced by an application of S1 followed by one of S6).

A function $\phi(\mathfrak{a})$ is *primitive recursive*, if there is a *primitive recursive description* (analogous to IM, p. 220) of it in terms of S1–S8 used under the stated convention.

1.4. We call a primitive recursive description *irredundant*, if in it each function except the last is used as the ψ or χ of a later schema application. By the *maximum type* of $\mathfrak{a}$ we mean the greatest of the types of the variables $\mathfrak{a}$ if $\mathfrak{a}$ is nonempty (0 if $\mathfrak{a}$ is empty).

I. *In an irredundant primitive recursive description of a function $\phi(\mathfrak{a})$ with r the maximum type of $\mathfrak{a}$, each function has the same maximum type r of its variables and for $r \geqq 2$ ($r = 1$) the same variables of types r, $r-1$ (type r), and hence* S6.j, S7.j, S8.j *can be used only for $j \leqq r$.*

Proof. In an application of S4, S5, S6 or S8, the ϕ has no variables of types >0 which are not variables of the ψ and χ, and lacks no such variables of the ψ or χ other than the variable α^{j-2} of the χ in S8.

II. *If $\phi(\mathfrak{a})$ is primitive recursive (with a given description), and $\mathfrak{a}'$ comes from $\mathfrak{a}$ by a permutation (without repetitions or omissions), then $\phi'(\mathfrak{a}') = \phi(\mathfrak{a})$*

is also primitive recursive (with a description obtainable from the given one by suffixing applications of S6).

Proof. Inversely to S6, k successive applications of S6 move a variable from the first to the $(k+1)$st position among the type-j variables. By S6 with this inverse (for different k's), any two type-j variables can be interchanged.

III. *If* $\phi(\mathfrak{a})$ *is primitive recursive (with a given description), then* $\phi(\mathfrak{a}, \mathfrak{c}) = \phi(\mathfrak{a})$, *where* $\mathfrak{a}$, $\mathfrak{c}$ *are distinct variables, is primitive recursive (with a description consisting of applications of the same respective schemata as the given one).*

Proof. In the given description of $\phi(\mathfrak{a})$ any variables not in $\mathfrak{a}$ but in $\mathfrak{c}$ can be changed to new distinct variables. Then the variables $\mathfrak{c}$ can be introduced at each application of S1–S3, S7 and can be carried through each application of S4–S6, S8.

1.5. Although our schemata as stated introduce functions $\phi(\mathfrak{a})$ with values of type 0, we can get functions with values of any higher type j by introducing $\phi(\mathfrak{a}, \alpha^{j-1})$ and considering $\lambda\alpha^{j-1}\,\phi(\mathfrak{a}, \alpha^{j-1})$ as a function of $\mathfrak{a}$. The latter we call *primitive recursive* when $\phi(\mathfrak{a}, \alpha^{j-1})$ is(6).

IV. *For each* $n \geqq 1$, *if* $\phi(\mathfrak{a}, \sigma^n, \mathfrak{b})$ *and* $\theta(\mathfrak{a}, \mathfrak{b}, \tau^{n-1})$ *are primitive recursive, so is* $\phi(\mathfrak{a}, \mathfrak{b}) = \phi(\mathfrak{a}, \lambda\tau^{n-1}\,\theta(\mathfrak{a}, \mathfrak{b}, \tau^{n-1}), \mathfrak{b})$.

Proof. First replace in IV $\theta(\mathfrak{a}, \mathfrak{b}, \tau^{n-1})$, $\phi(\mathfrak{a}, \mathfrak{b})$ by $\theta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \tau^{n-1})$, $\phi(\mathfrak{a}, \mathfrak{b}, \mathfrak{c})$. The resulting proposition (of which IV is the case for $\mathfrak{c}$ empty) we prove by induction on n, and, within that, induction on the length l of a given primitive recursive description of $\phi(\mathfrak{a}, \sigma^n, \mathfrak{b})$. Cases are numbered according to the schema last applied in that description. We treat in detail the two cases requiring the most care, and summarize the others.

CASE 5. $\phi(\mathfrak{a}, \sigma, \mathfrak{b})$ is introduced by S5 thus, writing $\mathfrak{a} = (a, \mathfrak{d})$,

$$\begin{cases} \phi(0, \mathfrak{d}, \sigma, \mathfrak{b}) = \psi(\mathfrak{d}, \sigma, \mathfrak{b}), \\ \phi(a', \mathfrak{d}, \sigma, \mathfrak{b}) = \chi(a, \phi(a, \mathfrak{d}, \sigma, \mathfrak{b}), \mathfrak{d}, \sigma, \mathfrak{b}), \end{cases}$$

where ψ, χ are previously introduced primitive recursive functions. We need to show that $\phi(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}) = \phi(a, \mathfrak{d}, \lambda\tau\,\theta(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau), \mathfrak{b})$, where $\theta(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau)$ is primitive recursive, is primitive recursive. Using II and III, so are

$$\theta_1(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u, \tau) = \theta(u, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau),$$
$$\theta(a, b, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u, \tau) = \theta(u, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau).$$

By the hypothesis of the induction on l,

$$\psi(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u) = \psi(\mathfrak{d}, \lambda\tau\,\theta_1(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u, \tau), \mathfrak{b}),$$
$$\chi(a, b, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u) = \chi(a, b, \mathfrak{d}, \lambda\tau\,\theta(a, b, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u, \tau), \mathfrak{b})$$

are primitive recursive. As a new application of S5, let

(6) More generally, a function of an arbitrary one of the "finite types" considered in [18, §5] shall be *primitive recursive*, if the function of "special type" correlated to it there is primitive recursive under 1.3 here.

$$\begin{cases} \phi(0, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u) = \psi(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u), \\ \phi(a', \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u) = \chi(a, \phi(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u), \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u). \end{cases}$$

Using S6, S3, S4, let

$$\phi_1(u, a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}) = \phi(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u),$$

$$\eta(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}) = a,$$

$$\phi(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}) = \phi_1(\eta(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}), a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}) = \phi(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, a).$$

To see that then $\phi(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}) = \phi(a, \mathfrak{d}, \lambda\tau\, \theta(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau), \mathfrak{b})$ as required, we prove by induction on a that, for any fixed $\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u$,

$$(1) \qquad \phi(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u) = \phi(a, \mathfrak{d}, \lambda\tau\, \theta(u, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau), \mathfrak{b}).$$

BASIS. $\phi(0, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u) = \psi(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u) = \psi(\mathfrak{d}, \lambda\tau\, \theta_1(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u, \tau), \mathfrak{b})$
$= \psi(\mathfrak{d}, \lambda\tau\, \theta(u, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau), \mathfrak{b}) = \phi(0, \mathfrak{d}, \lambda\tau\, \theta(u, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau), \mathfrak{b})$.

IND. STEP. $\phi(a', \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u) = \chi(a, \phi(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u), \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u)$
$= \chi(a, \phi(a, \mathfrak{d}, \lambda\tau\, \theta(u, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau), \mathfrak{b}), \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u)$ (by hyp. ind. on a)
$= \chi(a, \phi(a, \mathfrak{d}, \lambda\tau\, \theta(u, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau), \mathfrak{b}), \mathfrak{d}, \lambda\tau\, \theta(u, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau), \mathfrak{b})$
$= \phi(a', \mathfrak{d}, \lambda\tau\, \theta(u, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau), \mathfrak{b})$.

CASE 8. SUBCASE 1. σ is α^j. Then $n = j$. We have

$$\phi(\sigma, \mathfrak{b}) = \sigma(\lambda\alpha^{j-2}\, \chi(\sigma, \alpha^{j-2}, \mathfrak{b}))$$

with χ primitive recursive, and need to obtain primitive recursively

$$\phi(\mathfrak{b}, \mathfrak{c}) = \{\lambda\tau\, \theta(\mathfrak{b}, \mathfrak{c}, \tau)\}(\lambda\alpha^{j-2}\, \chi(\lambda\tau\, \theta(\mathfrak{b}, \mathfrak{c}, \tau), \alpha^{j-2}, \mathfrak{b}))$$
$$= \theta(\mathfrak{b}, \mathfrak{c}, \lambda\alpha^{j-2}\, \chi(\lambda\tau\, \theta(\mathfrak{b}, \mathfrak{c}, \tau), \alpha^{j-2}, \mathfrak{b})).$$

By III and S6, we can express $\theta(\mathfrak{b}, \mathfrak{c}, \tau)$ as $\theta(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c}, \tau)$. By the hyp. ind. on l, $\chi(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c}) = \chi(\lambda\tau\, \theta(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c}, \tau), \alpha^{j-2}, \mathfrak{b}) = \chi(\lambda\tau\, \theta(\mathfrak{b}, \mathfrak{c}, \tau), \alpha^{j-2}, \mathfrak{b})$ is primitive recursive, and thence using II, so is $\chi_1(\mathfrak{b}, \mathfrak{c}, \alpha^{j-2}) = \chi(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c})$. Finally, by the hyp. ind. on n,

$$\theta(\mathfrak{b}, \mathfrak{c}, \lambda\alpha^{j-2}\, \chi_1(\mathfrak{b}, \mathfrak{c}, \alpha^{j-2})) = \theta(\mathfrak{b}, \mathfrak{c}, \lambda\alpha^{j-2}\chi(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c}))$$
$$= \theta(\mathfrak{b}, \mathfrak{c}, \lambda\alpha^{j-2}\, \chi(\lambda\tau\, \theta(\mathfrak{b}, \mathfrak{c}, \tau), \alpha^{j-2}, \mathfrak{b})) = \phi(\mathfrak{b}, \mathfrak{c})$$

is primitive recursive. SUBCASE 2. σ is not α^j. Writing $\mathfrak{a} = (\alpha^j, \mathfrak{d})$, we express $\theta(\alpha^j, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau)$ as $\theta(\alpha^j, \alpha^{j-2}, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau)$, apply the hyp. ind. on l to introduce $\chi(\alpha^j, \alpha^{j-2}, \mathfrak{d}, \mathfrak{b}, \mathfrak{c})$, and make a new application of S8 to introduce $\phi(\alpha^j, \mathfrak{d}, \mathfrak{b}, \mathfrak{c})$.

CASE 7. SUBCASE 1. σ is α. Writing $\mathfrak{b} = (a, \mathfrak{d})$, we need to get primitive recursively $\phi(a, \mathfrak{d}, \mathfrak{c}) = \{\lambda\tau\, \theta(a, \mathfrak{d}, \mathfrak{c}, \tau)\}(a) = \theta(a, \mathfrak{d}, \mathfrak{c}, a)$. But from $\theta(a, \mathfrak{d}, \mathfrak{c}, \tau)$, since τ is of type 0, we can obtain $\theta(a, \mathfrak{d}, \mathfrak{c}, a)$ by S6, S3, S4 (cf. Case 5).

CASES 1, 2, 3 and CASE 7 SUBCASE 2. Simply use a new application of the same schema omitting σ and adding $\mathfrak{c}$ as variables $\mathfrak{b}$.

CASE 6. SUBCASE 1. σ is the $(k+1)$st type-j variable for ψ. Then $\phi(\sigma, \mathfrak{b}) = \psi(\mathfrak{b}_1, \sigma, \mathfrak{b}_2)$ where $\mathfrak{b} = (\mathfrak{b}_1, \mathfrak{b}_2)$. By hyp. ind. on l, $\psi(\mathfrak{b}, \mathfrak{c}) = \psi(\mathfrak{b}_1, \lambda\tau\, \theta(\mathfrak{b}, \mathfrak{c}, \tau), \mathfrak{b}_2)$

is primitive recursive, and this is $\phi(\mathfrak{b}, \mathfrak{c})$. SUBCASE 2. otherwise. Then $\phi(\mathfrak{a}, \sigma, \mathfrak{b}) = \psi(\mathfrak{a}_1, \sigma, \mathfrak{b}_1)$ where $(\mathfrak{a}_1, \mathfrak{b}_1)$ is a permutation of $(\mathfrak{a}, \mathfrak{b})$. By II, $\theta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \tau)$ can be expressed as $\theta_1(\mathfrak{a}_1, \mathfrak{b}_1, \mathfrak{c}, \tau)$. Apply the hyp. ind. on l to ψ, and use S6.

CASE 4. Express $\theta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \tau)$ as $\theta(b, \mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \tau)$, apply the hyp. ind. on l to ψ and χ, and use S4.

1.6. By a *full substitution* we mean one in which, for each one β^i of the m variables of a function $\psi(\mathfrak{b})$, there is substituted a function of the same list $\mathfrak{a}$ of variables (formed when $j > 0$ by use of the λ-operator). For example, a full substitution into $\psi(a, \alpha, \mathbf{F})$ gives $\phi(\mathfrak{a}) = \psi(\chi_1(\mathfrak{a}), \lambda t\ \chi_2(\mathfrak{a}, t), \lambda\sigma\ \chi_3(\mathfrak{a}, \sigma))$.

V. *The class of primitive recursive functions is closed under full substitution.*

Proof. If $\mathfrak{b}$ includes variables from the list $\mathfrak{a}$, first change them to new variables throughout a given primitive recursive description of ψ. Now we illustrate using the above example (supposing $\mathfrak{a}$, $\mathfrak{b}$ distinct). Using III and II, we can express $\psi(a, \alpha, \mathbf{F})$, $\chi_1(\mathfrak{a})$, $\chi_2(\mathfrak{a}, t)$ as $\psi(a, \alpha, \mathbf{F}, \mathfrak{a})$, $\chi_1(\alpha, \mathbf{F}, \mathfrak{a})$, $\chi_2(\mathbf{F}, \mathfrak{a}, t)$. Now, by one application of S4 and two of IV, we obtain successively

$$\psi_1(\alpha, \mathbf{F}, \mathfrak{a}) = \psi(\chi_1(\alpha, \mathbf{F}, \mathfrak{a}), \alpha, \mathbf{F}, \mathfrak{a}) = \psi(\chi_1(\mathfrak{a}), \alpha, \mathbf{F}),$$
$$\psi_2(\mathbf{F}, \mathfrak{a}) = \psi_1(\lambda t\ \chi_2(\mathbf{F}, \mathfrak{a}, t), \mathbf{F}, \mathfrak{a}) = \psi(\chi_1(\mathfrak{a}), \lambda t\ \chi_2(\mathfrak{a}, t), \mathbf{F}),$$
$$\phi(\mathfrak{a}) = \psi_2(\lambda\sigma\ \chi_3(\mathfrak{a}, \sigma), \mathfrak{a}) = \psi(\chi_1(\mathfrak{a}), \lambda t\ \chi_2(\mathfrak{a}, t), \lambda\sigma\ \chi_3(\mathfrak{a}, \sigma)).$$

VI. *For each $j \geqq 1$, the function $\phi(\alpha^j, \alpha^{j-1}, \mathfrak{b}) = \alpha^j(\alpha^{j-1})$ is primitive recursive.*

Proof, by induction on j. BASIS ($j = 1$). Use S7. IND STEP. By hyp. ind., III and II, $\phi(\alpha^{j+1}, \alpha^{j-1}, \alpha^j, \mathfrak{b}) = \alpha^j(\alpha^{j-1})$ is primitive recursive. Using S8.$j+1$, so is $\phi(\alpha^{j+1}, \alpha^j, \mathfrak{b}) = \alpha^{j+1}(\lambda\alpha^{j-1}\ \phi(\alpha^{j+1}, \alpha^{j-1}, \alpha^j, \mathfrak{b})) = \alpha^{j+1}(\lambda\alpha^{j-1}\ \alpha^j(\alpha^{j-1})) = \alpha^{j+1}(\alpha^j)$.

REMARK 1. Consider the following schemata, for $j \geqq 1$:

$$\text{S4.}j\ \phi(\mathfrak{b}) = \psi(\lambda\alpha^{j-1}\ \chi(\mathfrak{b}, \alpha^{j-1}), \mathfrak{b}).\ \text{S7.}j\ \phi(\alpha^j, \alpha^{j-1}, \mathfrak{b}) = \alpha^j(\alpha^{j-1}).$$

(S4 = S4.0, S7 = S7.1.) Using IV and VI, (S1-S7, S8.2-S8.r) = (S1-S7, S4.1-S4.$r-1$, S7.2-S7.r), and S4.r is derivable from either list.

VII. *The class of primitive recursive functions is closed under explicit definition, using (besides given functions, constant natural numbers, and variables) the λ-operator to form terms for substitution for function variables.*

Proof. We have full substitution by V. Also we have all constant and identity functions, using (besides II) S2, S3 and VI. The lemma follows as in IM, p. 221.

1.7. We now reconcile the present notion of primitive recursiveness with the notions in the literature. For functions $\phi(\mathfrak{a})$ of number variables $\mathfrak{a}$ only, cf. e.g. IM, p. 220 (and for $\mathfrak{a}$ empty, Remark 1, p. 223). In e.g. [14, p. 313], we further called a function $\phi(\mathfrak{a})$ with $\mathfrak{a}$ of types 0, 1 "primitive recursive," if as a function of its type-0 variables it is primitive recursive uniformly in its type-1 variables (IM, p. 234 and Remark 1, p. 238).

VIII. *A function* $\phi(\mathfrak{a})$ *of variables* $\mathfrak{a}$ *of types* <2 *is primitive recursive in the present sense, if and only if it is primitive recursive in the former sense.*

Proof. For "if," use VII. For "only if," use I and IM, pp. 220–221.

1.8. Primitive recursiveness can be relativized to assumed functions $\psi_1, \cdots, \psi_l$ (briefly, Ψ) by using *primitive recursive derivations from* Ψ (analogous to IM, p. 224) instead of primitive recursive descriptions. In a primitive recursive derivation from Ψ, besides using S1-S8, we may introduce any one $\psi_i(\mathfrak{c})$ of Ψ by the schema for a full substitution (cf. 1.6), for only its function variables, of functions of variables $\mathfrak{b}$ distinct from its number variables; e.g., if ψ_i is $\psi_i(c, \gamma_2^1, \gamma_3^2)$:

$$\text{S0.}i \qquad \phi(c, \mathfrak{b}) = \psi_i(c, \lambda t\, \chi_2(t, \mathfrak{b}), \lambda\sigma\, \chi_3(\sigma, \mathfrak{b})) \qquad \langle 0, \langle n_0, \cdots, n_r\rangle, i, h_2, h_3\rangle.$$

When $\mathfrak{c}$ is number variables only, S0.i is $\phi(\mathfrak{c}, \mathfrak{b}) = \psi_i(\mathfrak{c})$. (Alternatively, if S8.$j$ ($j \geqq 2$) is replaced by S4.j ($j \geqq 1$), S7.j ($j \geqq 2$) of Remark 1, each ψ_i can be introduced simply by $\phi(\mathfrak{c}, \mathfrak{b}) = \psi_i(\mathfrak{c})$.)

Now relativized versions I*–VIII* of I–VIII can be proved for functions primitive recursive on Ψ. As I*, in an irredundant primitive recursive derivation of a function $\phi(\mathfrak{a})$ with $\mathfrak{a}$ of types $\leqq r$ from functions Ψ of variables of types $\leqq s$ (introduced by S0.i ($1 \leqq i \leqq l$)), each function has all the variables in $\mathfrak{a}$ of types > 0 and only variables of types $\leqq \max(r, s-1)$. As VIII*, when Ψ are number-theoretic functions, and ϕ a function of variables of types $\leqq 1$, the present notion of relative primitive recursiveness coincides with the former one (IM, p. 224, [14, p. 313]).

1.9. When Ψ vary and Θ are fixed, ϕ is *primitive recursive in* Ψ, Θ, *uniformly in* Ψ, if there is a primitive recursive derivation of ϕ from Ψ, Θ with a fixed analysis (analogous to IM, p. 234). Then when Ψ are one-place functions, $\phi(\mathfrak{a})$ is primitive recursive in Ψ, Θ uniformly in Ψ, exactly if ϕ regarded as a function $\phi(\mathfrak{a}, \Psi)$ is primitive recursive in Θ. For on reconstruing Ψ as variables for the function ϕ, the applications of S0.i to introduce ψ_i become applications of S7.1 or S8.j (with applications of S6); and inversely. It follows that, for fixed one-place functions Ψ_0, we can consider any function $\phi(\mathfrak{a})$ primitive recursive in Ψ_0, Θ as the value for the fixed Ψ_0 of a function $\phi(\mathfrak{a}, \Psi)$ primitive recursive in Θ.

1.10. As usual, the notions of primitive recursiveness, absolute and relative, extend via representing functions (IM, p. 227) to cases when some or all of ϕ, Ψ, Θ are replaced by predicates.

2. **Alterations of quantifiers. 2.1.** Let $\langle a_0, \cdots, a_n\rangle = p_0^{a_0} \cdot \cdots \cdot p_n^{a_n}$ ($=1$ in the case $n = -1$, i.e. $\langle\ \rangle = 1$). For each $j \geqq 1$ (7), let $\langle \alpha_0^j, \cdots, \alpha_n^j\rangle = \lambda\tau^{j-1} \langle \alpha_0^j(\tau^{j-1}), \cdots, \alpha_n^j(\tau_n^{j-1})\rangle$ and $(\alpha^j)_i = \lambda\tau^{j-1} (\alpha^j(\tau^{j-1}))_i$. Then, for each j ($j \geqq 0$),

(7) This makes $\langle \alpha_0^j, \cdots, \alpha_n^j\rangle = \lambda\tau^{j-1}\, 1$ for $n = -1$ & $j > 0$ ($= 1$ for $n = -1$ & $j = 0$). We can write "$\langle \alpha_0^j, \cdots, \alpha_n^j\rangle$" (with "$j$" shown) without ambiguity even when -1 is a possible value of n. But "$\langle\ \rangle$" shall mean 1, unless we show the type j as by "$\alpha^j = \langle\ \rangle$" or "$\langle\ \rangle^j$".

$$(2) \qquad (\langle \overset{j}{\alpha}_0, \cdots, \overset{j}{\alpha}_n \rangle)_i = \overset{j}{\alpha}_i \qquad (0 \leqq i \leqq n).$$

Now we have, generalizing IM, p. 285 (17) and [14, p. 315, (1) and (5)] to any $j \geqq 0$,

$$(3) \quad (E\overset{j}{\alpha}_0) \cdots (E\overset{j}{\alpha}_n) A(\overset{j}{\alpha}_0, \cdots, \overset{j}{\alpha}_n) \equiv (E\overset{j}{\alpha}) A((\overset{j}{\alpha})_0, \cdots, (\overset{j}{\alpha})_n),$$

$$(4) \qquad (\overset{j}{\alpha})(E\overset{j+1}{\beta}) A(\overset{j}{\alpha}, \overset{j+1}{\beta}) \equiv (E\overset{j+1}{\beta})(\overset{j}{\alpha}) A(\overset{j}{\alpha}, \lambda\overset{j}{\sigma}\, \overset{j+1}{\beta}(\langle \overset{j}{\alpha}, \overset{j}{\sigma} \rangle)),$$

and their duals $(\bar{3})$, $(\bar{4})$.

Proofs. (3) Given $\alpha_0^j, \cdots, \alpha_n^j$ for the left, $\alpha^j = \langle \alpha_0^j, \cdots, \alpha_n^j \rangle$ is an α^j for the right, by (2). The converse implication is immediate. (4) $(\alpha^j)(E\beta^{j+1})A(\alpha^j, \beta^{j+1}) \equiv (E\beta)(\alpha^j)A(\alpha^j, \lambda\sigma^j\, \beta(\alpha^j, \sigma^j)) \equiv (E\beta^{j+1})(\alpha^j)A(\alpha^j, \lambda\sigma^j\, \beta^{j+1}(\langle \alpha^j, \sigma^j \rangle))$, since by (2), given any two-place β (not an object of one of our types 1.2) for the middle expression, $\beta^{j+1} = \lambda\tau^j\, \beta((\tau^j)_0, (\tau^j)_1)$ is one for the right.

2.2. Using induction on j, we define two primitive recursive functions mp_j and pm_j, for each $j \geqq 1$, as follows:

$$\mathrm{mp}_1(\alpha^1, \beta^1) = \alpha^1(\beta^1(0)),$$
$$\mathrm{pm}_1(\alpha^2, \beta^0) = \alpha^2(\lambda x\ \beta^0),$$
$$\mathrm{mp}_{j+1}(\alpha^{j+1}, \beta^{j+1}) = \alpha^{j+1}(\lambda\sigma^{j-1}\, \mathrm{pm}_j(\beta^{j+1}, \sigma^{j-1})),$$
$$\mathrm{pm}_{j+1}(\alpha^{j+2}, \beta^j) = \alpha^{j+2}(\lambda\sigma^j\, \mathrm{mp}_j(\beta^j, \sigma^j)).$$

For $j \geqq 1$, let $\mathrm{mp}_j(\alpha^j) = \lambda\beta^j\, \mathrm{mp}_j(\alpha^j, \beta^j)$ and $\mathrm{pm}_j(\alpha^{j+1}) = \lambda\beta^{j-1}\, \mathrm{pm}_j(\alpha^{j+1}, \beta^{j-1})$; and let $\mathrm{mp}_0(\alpha^0) = \lambda x\ \alpha^0$ and $\mathrm{pm}_0(\alpha^1) = \alpha^1(0)$. The objects α^j of type j are mapped into objects $\alpha^{j+1} = \mathrm{mp}_j(\alpha^j)$ of type $j+1$, and $\alpha^j = \mathrm{pm}_j(\alpha^{j+1})$ is the inverse mapping, as the following proposition states.

IX. *For each* $j \geqq 0$,

$$(5) \qquad \mathrm{pm}_j(\mathrm{mp}_j(\alpha^j)) = \alpha^j.$$

The proof, using for $j \geqq 1$ induction on j, is straightforward.

Now we have, generalizing [14, (3)] to any $j \geqq 0$,

$$(6) \qquad (E\alpha^j)A(\alpha^j) \equiv (E\alpha^{j+1})A(\mathrm{pm}_j(\alpha^{j+1})),$$

and its dual $(\bar{6})$. For given an α^j for the left, $\mathrm{mp}_j(\alpha^j)$ is an α^{j+1} for the right, by (5).

2.3. It will be useful to have formulas for raising the type j by any number $m-j \geqq 0$. Using induction on m for $m \geqq j \geqq 1$, let

$$\mathrm{mp}_j^j(\overset{j}{\alpha}, \overset{j-1}{\beta}) = \overset{j}{\alpha}(\overset{j-1}{\beta}),$$
$$\mathrm{mp}_j^{m+1}(\overset{j}{\alpha}, \overset{m}{\beta}) = \mathrm{mp}_m(\lambda\overset{m-1}{\beta}\ \mathrm{mp}_j^m(\overset{j}{\alpha}, \overset{m-1}{\beta}), \overset{m}{\beta}),$$
$$\mathrm{pm}_j^j(\overset{j}{\alpha}, \overset{j-1}{\beta}) = \overset{j}{\alpha}(\overset{j-1}{\beta}),$$
$$\mathrm{pm}_j^{m+1}(\overset{m+1}{\alpha}, \overset{j-1}{\beta}) = \mathrm{pm}_j^m(\lambda\overset{m-1}{\beta}\, \mathrm{pm}_m(\overset{m+1}{\alpha}, \overset{m-1}{\beta}), \overset{j-1}{\beta}).$$

Also let

$$\mathrm{mp}_0^0(\alpha^0) = \alpha^0, \qquad \mathrm{mp}_0^m(\alpha^0, \beta^{m-1}) = \alpha^0 \qquad (m \geqq 1),$$

$$\mathrm{pm}_0^0(\alpha^0) = \alpha^0, \qquad \mathrm{pm}_0^1(\alpha^1) = \alpha^1(0), \qquad \mathrm{pm}_0^m(\alpha^m) = \alpha^m(\lambda\beta^{m-2}\, 0) \qquad (m \geqq 2).$$

Finally, for $m \geqq j \geqq 1$ or $m > j = 0$, let $\mathrm{mp}_j^m(\alpha^j) = \lambda\beta^{m-1}\, \mathrm{mp}_j^m(\alpha^j, \beta^{m-1})$; and for $m \geqq j \geqq 1$, let $\mathrm{pm}_j^m(\alpha^m) = \lambda\beta^{j-1}\, \mathrm{pm}_j^m(\alpha^m, \beta^{j-1})$.

X. *For each* $m \geqq j \geqq 0$,

$$\mathrm{pm}_j^m(\mathrm{mp}_j^m(\alpha^j)) = \alpha^j, \tag{7}$$

$$(E\alpha^j) A(\alpha^j) \equiv (E\alpha^m) A(\mathrm{pm}_j^m(\alpha^m)) \qquad (m \geqq j \geqq 0). \tag{8}$$

2.4. When $j_0 \leqq j_1 \leqq \cdots \leqq j_n \leqq m$, let $\langle \alpha_0^{j_0}, \cdots, \alpha_n^{j_n} \rangle^m = \langle \mathrm{mp}_{j_0}^m(\alpha_0^{j_0}), \cdots, \mathrm{mp}_{j_n}^m(\alpha_n^{j_n}) \rangle$; and for $j \leqq m$, let $(\alpha^m)_i^j = \mathrm{pm}_j^m((\alpha^m)_i)$. Then using (2) and (7),

$$(\langle \alpha_0^{j_0}, \cdots, \alpha_n^{j_n} \rangle^m)_i^{j_i} = \alpha_i^{j_i} \qquad (0 \leqq i \leqq n). \tag{9}$$

Consistently with 2.1, we may write $\langle \alpha_0^{j_0}, \cdots, \alpha_n^{j_n} \rangle^m$ as simply $\langle \alpha_0^{j_0}, \cdots, \alpha_n^{j_n} \rangle$ when $m = j_n$, and $(\alpha^m)_i^m$ as $(\alpha^m)_i$. Finally, by $\langle \mathfrak{a} \rangle$ let us mean $\langle \alpha_0^{j_0}, \cdots, \alpha_n^{j_n} \rangle$ where $\alpha_0^{j_0}, \cdots, \alpha_n^{j_n}$ is the result of arranging $\mathfrak{a}$ in order of nondecreasing type preserving the given order within each type[7].

2.5. A consecutive set of $n+1$ quantifiers of like kind and assorted types $\leqq m$ can be contracted into one type-m quantifier of the same kind, by using first (8) and then (3) (or ($\bar{8}$), ($\bar{3}$)). The operation can be performed in one step using (9). For example, when $p \leqq m$,

$$(a)(\alpha_1^p)(\alpha_2^m) A(a, \alpha_1^p, \alpha_2^m) \equiv (\alpha^m) A((\alpha^m)_0^0, (\alpha^m)_1^p, (\alpha^m)_2).$$

The prefix $(\alpha^j)(E\beta^k)$ for any $j \leqq m$ and $k \leqq m+1$ can be changed to $(E\beta^{m+1})(\alpha^m)$ (or dually) by use of (8), ($\bar{8}$) and then (4) (or ($\bar{4}$)). It is more expeditious, except when (4) or ($\bar{4}$) is immediately applicable, to proceed in the manner used in proving (4), using (9) to effect the contraction required when $k > 0$. For example,

$$(\alpha^2)(Ex) A(\alpha^2, x) \equiv (E\alpha^3)(\alpha^2) A(\alpha^2, \alpha^3(\alpha^2)),$$

$$(\alpha^4)(E\beta^3) A(\alpha^4, \beta^3) \equiv (E\beta)(\alpha^4) A(\alpha^4, \lambda\sigma^2\, \beta(\alpha^4, \sigma^2))$$

$$\equiv (E\beta^5)(\alpha^4) A(\alpha^4, \lambda\sigma^2\, \beta^5(\langle \alpha^4, \sigma^2 \rangle)).$$

The same technique can be used in advancing and contracting several quantifiers simultaneously (cf. [14, Footnote 10]); e.g.

$$(x)(E\alpha^3)(\beta^3)(E\gamma^1) A(x, \alpha^3, \beta^3, \gamma^1) \equiv$$

$$(E\alpha)(E\gamma)(x)(\beta^3) A(x, \lambda\sigma^2\, \alpha(x, \sigma^2), \beta^3, \lambda t\, \gamma(x, \beta^3, t)) \equiv$$

$$(E\alpha^3)(E\gamma^4)(x)(\beta^3) A(x, \lambda\sigma^2\, \alpha^3(\langle x, \sigma^2 \rangle), \beta^3, \lambda t\, \gamma^4(\langle x, \beta^3, t \rangle)) \equiv$$

$$(E\alpha^4)(\beta^3) A((\beta^3)_0^0, \lambda\sigma^2\, \{(\alpha^4)_0^3\}(\langle (\beta^3)_0^0, \sigma^2 \rangle), (\beta^3)_1, \lambda t\, \{(\alpha^4)_1\}(\langle (\beta^3)_0^0, (\beta^3)_1, t \rangle)).$$

2.6. These alterations of quantifiers are effected by substitutions of primitive recursive functions in the scope. As another application of the contraction formulas:

XI. *Given any function* $\psi(\mathfrak{a})$ *of variables* $\mathfrak{a}$ *of types* $\leqq m$, *there is a function* $\psi'(\alpha^m)$ *such that each of* $\psi(\mathfrak{a})$ *and* $\psi'(\alpha^m)$ *comes from the other by substitution of primitive recursive functions.*

Proof. For example, given $\psi(a, \alpha_1^p, \alpha_2^m)$ $(p \leqq m)$, take $\psi'(\alpha^m) = \psi((\alpha^m)_0^0, (\alpha^m)_1^p, (\alpha^m)_2)$. Then $\psi(a, \alpha_1^p, \alpha_2^m) = \psi'(\langle a, \alpha_1^p, \alpha_2^m \rangle)$.

2.7. We can use XI in connection with 1.9 when Ψ include more-than-one-place functions. For example (with $p \leqq m$), if $\phi(\mathfrak{a})$ is primitive recursive in $\psi(a, \alpha_1^p, \alpha_2^m)$, Θ, then using $\psi(a, \alpha_1^p, \alpha_2^m) = \psi'(\langle a, \alpha_1^p, \alpha_2^m \rangle)$ we have $\phi(\mathfrak{a})$ primitive recursive in ψ', Θ for $\psi' = \lambda\alpha^m \psi((\alpha^m)_0^0, (\alpha^m)_1^p, (\alpha^m)_2)$. Thence by 1.9 we obtain a function $\phi(\mathfrak{a}, \alpha^{m+1})$ primitive recursive in Θ such that $\phi(\mathfrak{a}) = \phi(\mathfrak{a}, \psi') = \phi(\mathfrak{a}, \lambda\alpha^m \psi((\alpha^m)_0^0, (\alpha^m)_1^p, (\alpha^m)_2)$.

3. **Partial and general recursive functions. 3.1.** By Church's thesis for number-theoretic functions [4] and our extension of it to functions with variables of type 1 as well [12] and their converses, the general recursive functions of variables of types 0, 1 comprise exactly the functions of such variables which are "effectively calculable" (cf. IM, §62 and end §61). The effective calculability is relative when there are type-1 variables. Thus, for a function $\phi(\alpha)$ of a type-1 variable α, we imagine after Turing [30, pp. 172–173, 200] an "oracle" for α who at any step in the computation can be called upon to supply the value of $\alpha(y)$ for a computed y [3]. The use of the word "oracle" emphasizes that in the steps from a y to the value of $\alpha(y)$ the otherwise mechanical character of the computation is transcended. The computation is carried out by a preassigned procedure, using steps all of them mechanical except those from y to $\alpha(y)$, which we "assume" provided upon demand.

For a function $\phi(\mathbf{F})$ of a type-2 variable $\mathbf{F}$, it seems natural to image similarly an "oracle" for $\mathbf{F}$ who at any step in the computation can be called upon to supply the value of $\mathbf{F}(\beta)$ when a procedure for computing $\beta(y)$ for any y has arisen. Quite as before, we mean by this that $\phi(\mathbf{F})$ can be computed by a preassigned procedure, using steps all of them mechanical except for steps from a β to the value of $\mathbf{F}(\beta)$. Such steps we "assume" provided upon demand, under circumstances in which we could in the same sense, i.e. with the help of the oracle for $\mathbf{F}$, compute $\beta(y)$ for any y $(y = 0, 1, 2, \cdots)$. When the oracle for $\mathbf{F}$ is used, if the computations of $\beta(y)$ for each $y = 0, 1, 2, \cdots$ are considered to be parts of the computation of $\phi(\mathbf{F})$, that computation for the given $\mathbf{F}$ will not be a finite object, unlike the computation of $\phi(\alpha)$ for a given α. It cannot be otherwise when $\mathbf{F}(\beta)$ depends on all the values of β.

(3) Turing had in mind only particular α's (primarily, in our terms, the representing function of $(x)\overline{T}_1(y, y, x)$), while here (as in [12] and IM) we are adding the idea of uniformity in α (the procedure for computing not varying with α, but only the answers the oracle gives) to obtain the effective calculability of a function of α.

So, it seems, this is the direction the generalization of effective calculability to type 2 must take, if we are to allow arbitrary type-2 objects **F** as arguments. The resulting notion, though it employs a very potent oracle, is still an immense restriction on the notion of an arbitrary function $\phi(\mathbf{F})$, and should be of mathematical interest. That the computation of $\phi(\mathbf{F})$ is not in general a finite object is already illustrated by the primitive recursive functions $\phi(\mathbf{F})$. Our objective now is to characterize what, we believe, is the class of all functions $\phi(\mathbf{F})$ which are effectively calculable (and more generally, of all such functions $\phi(\mathfrak{a})$ for $\mathfrak{a}$ of assorted types).

Besides developing the theory for arbitrary **F**'s as we are proposing here, we can consider specializations and alternatives under restrictions on the **F**'s. Our basic idea of the effective calculability of a function ϕ admitting nonconstructive objects as arguments is that the calculation should be performable by preassigned rules, constructively except for the use of the oracles necessary to put in those nonconstructive objects. Basically, each of those nonconstructive objects can be a function (so ϕ is a functional), and the role of the oracle for it is to give its value when supplied its argument(s). Different notions of "effectively calculable" or "constructive" functional may be related on this basis (9).

3.2. For functions of variables of types 0, 1, there are a number of ways of describing the class of the general recursive functions, or in other words there are a number of notions equivalent to general recursiveness as usually formulated (cf. IM, §62). In setting out to give an extension to higher types of variables that shall preserve Church's thesis, one should not assume a

(9) For example, in [18] we confine **F** to functions (called *countable functionals* of type 2) each of whose values $\mathbf{F}(\beta)$ is determined by finitely many values of $\beta(y)$. Then we employ an oracle to which we give a different role in relation to **F** than above. We may ask this oracle a question of the form, "Does $\mathbf{F}(\beta)$ have the same value for all β's such that $\beta(0) = b_0, \cdots, \beta(x-1) = b_{x-1}$, and if so what is that value?" (The question can be asked by supplying to the oracle a value of $\bar{\beta}(x)$ (3).) The oracle may reply either "No comment" or "Yes, and that value is —"; but, for a given β, there must be some x for which he gives the second kind of answer. In [18, §1] we note that this amounts to having an oracle in the basic sense for a certain kind of type-1 function $\mathbf{F}^{(\,)}$ (or $\alpha^{(2)}$ if $\mathbf{F} = \alpha^2$) called an *associate* of **F**. Under the thesis that the effectively calculable functions on a given domain are those which are general recursive on that domain (indeed, the case of it for a type-1 variable already entertained in [12] and IM end §61), the functions $\phi(\mathbf{F})$ "effectively calculable" with this new kind of oracle are those with ϕ_1 partial recursive, where $\phi_1(\mathbf{F}^{(\,)}) = \phi(\mathbf{F})$ for every associate $\mathbf{F}^{(\,)}$ of **F**; these $\phi(\mathbf{F})$ we call *recursively countable* (cf. [18, Theorem 2]). All functions $\phi(\mathbf{F})$ general recursive (as defined below) for **F** countable are recursively countable [18, Theorem 4]. Whether the converse is true is an open problem. This does not mean that the class of the general recursive functions $\phi(\mathbf{F})$ may be too narrow to represent effective calculability when **F** is restricted to be countable (i.e. that our thesis may fail for such **F**). It only means (under the thesis) that there may be functions $\phi(\mathbf{F})$ for which $\phi_1(\mathbf{F}^{(\,)})$ is effectively calculable, though $\phi(\mathbf{F})$ itself is not, in the basic sense. That is, the new oracle may be more potent. (We failed in an attempt to determine whether there is an effective way to perform the job the new oracle does by using the basic oracle for **F**.) There are purposes for which one may be more interested in using the new oracle (i.e. in having merely ϕ_1 rather than ϕ effectively calculable in the basic sense).

priori that it is immaterial which of the existing notions for types 0, 1 he extends. We shall in fact start with a new formulation, which provides the essentials for Church's thesis quite directly, and later (in §8 and Part II) we shall examine extensions of the several existing notions.

3.3. One essential for Church's thesis to hold is that there be available a modicum of elementary operations of computation. This is provided by the schemata for primitive recursive functions.

3.4. The second essential is that there be the means for reflecting upon computation procedures already set up as objects and computing further computation procedures from them.

This second essential entails that there should be no means for deciding in general whether a computation procedure terminates. For otherwise by diagonalization we could effectively get outside our class of computation procedures, so Church's thesis could not apply.

A function ϕ, taking a natural number as value, which is defined over some subset proper or improper of a set D we call a *partial function over* D; we then sometimes think of ϕ as a function from D to $\{0, 1, 2, \cdots; \mathfrak{u}\}$ where $\mathfrak{u}$ stands for "undefined" (cf. IM, pp. 325–326). In the following, we deal with partial functions $\phi(\mathfrak{a})$ for $\mathfrak{a}$ a given list of variables of the types introduced in 1.2, so D is the set of all n-tuples of objects of the respective types of $\mathfrak{a}$.

When $\phi(\mathfrak{a})$ and $\psi(\mathfrak{a})$ are partial functions, we write $\phi(\mathfrak{a}) \simeq \psi(\mathfrak{a})$ to express that both $\phi(\mathfrak{a})$ and $\psi(\mathfrak{a})$ are defined with the same value or both are undefined (IM, p. 327).

3.5. In order to allow computation procedures, or what comes to the same thing recursive (or "computable") functions, to be treated as objects in computing further computation procedures, we shall assign numbers called "indices" to the recursive functions. (The indices take over the role of the Gödel numbers in our previous theory of general and partial recursive functions; cf. IM, pp. 292, 330, 340 ff., and [17, §3].)

We begin with an assignment of indices to the primitive recursive functions. Consider such a function $\phi(\mathfrak{a})$. Let r be the maximum type of $\mathfrak{a}$ (1.4), and let $n_0, \cdots, n_r$ be the numbers of variables in $\mathfrak{a}$ of types $0, \cdots, r$, respectively (thus $n_r > 0$, except for $\mathfrak{a}$ empty). We assign to $\phi(\mathfrak{a})$ an index determined by a given primitive recursive description of $\phi(\mathfrak{a})$. For $\phi(\mathfrak{a})$ introduced by a given one of the schemata S1–S8, the *index* is the number shown at the right opposite the schema (cf. 1.3, 2.1), where (in the cases of S4, S5, S6 and S8) g and h are the *indices* already determined for the ψ and χ by the descriptions of them as part of the given description of ϕ.

Under this method of indexing, inversely an index of $\phi(\mathfrak{a})$ *determines* an irredundant description of $\phi(\mathfrak{a})$, say that one in which at each application of S4 or S5 the entire descriptions of the ψ and the χ, and nothing else, each appear separately and successively preceding the application of S4 or S5, and similarly at each application of S6 or S8.

3.6. Now for example we can define a function $\phi(\mathfrak{a})$ whose value for arguments $\mathfrak{a}$ is $\phi_{\mathfrak{a}}(\mathfrak{a})$ where, for each $\mathfrak{b}$, $\phi_{\mathfrak{b}}(\mathfrak{a})$ is the function with index $\chi(\mathfrak{b})$. If $\{z\}(\mathfrak{a})$ is the function of $\mathfrak{a}$ with index z, the function $\phi(\mathfrak{a})$ can be written $\phi(\mathfrak{a}) = \{\chi(\mathfrak{a})\}(\mathfrak{a})$. To introduce this $\phi(\mathfrak{a})$ it would suffice to use S4 with $\chi(\mathfrak{a})$ and the function $\psi(a, \mathfrak{b}) = \{a\}(\mathfrak{b})$ as given functions. This suggests the definition of "partial recursive function" we now adopt.

3.7. To obtain the partial recursive functions, we use the schemata S1–S8, now with $\simeq$ instead of $=$ and interpreting $\alpha^j(\lambda\alpha^{j-2}\chi(\alpha^j, \alpha^{j-2}, \mathfrak{b}))$ for S8 to be undefined when $\lambda\alpha^{j-2}\chi(\alpha^j, \alpha^{j-2}, \mathfrak{b})$ is incompletely defined([10]), and one further schema

S9 $\quad \phi(a, \mathfrak{b}, \mathfrak{c}) \simeq \{a\}(\mathfrak{b}) \qquad \langle 9, \langle n_0, \cdots, n_r\rangle, \langle m_0, \cdots, m_s\rangle\rangle,$

where s is the maximum type of $\mathfrak{b}$, and $m_0, \cdots, m_s$ are the numbers of variables in $\mathfrak{b}$ of types $0, \cdots, s$, respectively. A function $\phi(\mathfrak{a})$ is *partial recursive*, if there is a *partial recursive description* of it in terms of these schemata. Each partial recursive description of $\phi(\mathfrak{a})$ determines an *index* and inversely, in the same manner as before (3.5). Finally, for each z which is an index of a partial recursive function $\phi(\mathfrak{a})$ and each $\mathfrak{a}$, $\{z\}(\mathfrak{a})$ is the value if defined of $\phi(\mathfrak{a})$; otherwise $\{z\}(\mathfrak{a})$ is undefined. (For number variables $\mathfrak{a}$, $\{z\}(\mathfrak{a})$ is not the function so written in IM, p. 340, but the analog of it here.)

A partial recursive function $\phi(\mathfrak{a})$ which is defined for all $\mathfrak{a}$ is *general recursive*. A predicate is *partial* (*general*) *recursive*, if its representing function is partial (general) recursive.

3.8. The definition of $\{z\}(\mathfrak{a})$ via an interaction of schemata and indexing can be elaborated as follows. First, $\{z\}(\mathfrak{a})$ is defined only when z is an index of a function $\phi(\mathfrak{a})$. Second, the schemata S1–S9 can be written as equations to be satisfied by $\{z\}(\mathfrak{a})$ as a partial function of z and a varying list $\mathfrak{a}$ of variables; e.g.:

S1 $\quad \{\langle 1, \langle n_0, \cdots, n_r\rangle\rangle\}(a, \mathfrak{b}) \simeq a'.$

S5a $\{\langle 5, \langle n_0, \cdots, n_r\rangle, g, h\rangle\}(0, \mathfrak{b}) \simeq \{g\}(\mathfrak{b}).$

S5b $\{\langle 5, \langle n_0, \cdots, n_r\rangle, g, h\rangle\}(a', \mathfrak{b}) \simeq \{h\}(a, \{\langle 5, \langle n_0, \cdots, n_r\rangle, g, h\rangle\}(a, \mathfrak{b}), \mathfrak{b}).$

S8 $\{\langle 8, \langle n_0, \cdots, n_r\rangle, j, h\rangle\}(\alpha^j, \mathfrak{b}) \simeq \alpha^j(\lambda\alpha^{j-2}\{h\}(\alpha^j, \alpha^{j-2}, \mathfrak{b})).$

S9 $\{\langle 9, \langle n_0, \cdots, n_r\rangle, \langle m_0, \cdots, m_s\rangle\rangle\}(a, \mathfrak{b}, \mathfrak{c}) \simeq \{a\}(\mathfrak{b}).$

Third, these equations can be construed as the direct clauses of a transfinite inductive definition (cf. [11, §7], IM §53) of the predicate $\{z\}(\mathfrak{a}) \simeq w$ (each applicable however only for z an index of a function of $\mathfrak{a}$). Thus S1 is a basic clause, and S5a, S5b, S8, S9 can be read as inductive clauses; e.g.:

([10]) Likewise $\psi(\chi(\mathfrak{b}), \mathfrak{b})$ for S4 ($\chi(a, \phi(a, \mathfrak{b}), \mathfrak{b})$ for S5) shall be undefined when $\chi(\mathfrak{b})$ ($\phi(a, \mathfrak{b})$) is undefined, i.e. it shall have the "weak sense" IM, p. 327.

S5b If $\{\langle 5, \langle n_0, \cdots, n_r\rangle, g, h\rangle\}(a, \mathfrak{b}) \simeq u$ and $\{h\}(a, u, \mathfrak{b}) \simeq w$,

then $\{\langle 5, \langle n_0, \cdots, n_r\rangle, g, h\}\rangle(a', \mathfrak{b}) \simeq w$.

S8 If for each α^{j-2}, $\{h\}(\alpha^j, \alpha^{j-2}, \mathfrak{b}) \simeq \gamma^{j-1}(\alpha^{j-2})$,

then $\{\langle 8, \langle n_0, \cdots, n_r\rangle, j, h\rangle\}(\alpha^j, \mathfrak{b}) \simeq \alpha^j(\gamma^{j-1})$.

S9 If $\{a\}(\mathfrak{b}) \simeq w$, then $\{\langle 9, \langle n_0, \cdots, n_r\rangle, \langle m_0, \cdots, m_s\rangle\rangle\}(a, \mathfrak{b}, \mathfrak{c}) \simeq w$.

As the extremal clause of the inductive definition, $\{z\}(\mathfrak{a}) \simeq w$ is to hold only as required by the direct clauses.

Now we can prove by mathematical induction, in the form corresponding to this inductive definition, that, for given values of z and $\mathfrak{a}$, the proposition $\{z\}(\mathfrak{a}) \simeq w$ holds for at most one natural number w. The proof is afforded by the fact that, for given z and $\mathfrak{a}$, at most one of the equations has $\{z\}(\mathfrak{a})$ as left member.

Thereby $\{z\}(\mathfrak{a})$ is defined as the partial function whose value, for given z and $\mathfrak{a}$, is the unique w (if any) such that $\{z\}(\mathfrak{a}) \simeq w$ holds by the inductive definition.

Mathematical induction in the aforesaid form can be used to establish properties of $\{z\}(\mathfrak{a})$ when defined. We shall refer to such an induction as *induction on* $\{z\}(-)$.

3.9. We consider an example to illustrate how the inductive definition of $\{z\}(\mathfrak{a}) \simeq w$ provides a computation process for $\{z\}(\mathfrak{a})$ (but one which in general makes the computation an infinite object, cf. 3.1).

Given z, $\mathbf{F}$, where z is an index of a function of $\mathbf{F}$, how do we compute $\{z\}(\mathbf{F})$? Say e.g. $z = \langle 4, \langle 0, 0, 1\rangle, g, h\rangle$. Then the value of $\{z\}(\mathbf{F})$ is that of $\{g\}(\{h\}(\mathbf{F}), \mathbf{F})$, if the latter is defined. Consider first $\{h\}(\mathbf{F})$. Say e.g. $h = \langle 8, \langle 0, 0, 1\rangle, 2, k\rangle$. Then the value of $\{h\}(\mathbf{F})$ is that of $\mathbf{F}(\beta)$ as supplied by the "oracle" for $\mathbf{F}$, provided that, for each y, $\{k\}(\mathbf{F}, y)$ is defined with value $\beta(y)$. That is, if, for each y, by proceeding similarly to compute $\{k\}(\mathbf{F}, y)$ a value would be obtained, call it $\beta(y)$, a value of $\{h\}(\mathbf{F})$ results, call it u ($u = \mathbf{F}(\beta)$). Say $g = \langle 9, \langle 1, 0, 1\rangle, \langle 0, 0, 1\rangle\rangle$. Then the value of $\{g\}(u, \mathbf{F})$ is that of $\{u\}(\mathbf{F})$ if defined. Say $u = \langle 2, \langle 0, 0, 1\rangle, 5\rangle$. Then the value of $\{u\}(\mathbf{F})$ is 5; this is the value of $\{g\}(u, \mathbf{F})$, of $\{g\}(\{h\}(\mathbf{F}), \mathbf{F})$, and of $\{z\}(\mathbf{F})$, which we sought.

3.10. By our definition, the class of the partial recursive functions is closed under applications of the primitive recursive schemata S1–S8 ([11]). In particular, primitive recursive functions are partial recursive; and the class of partial recursive functions is closed under primitive recursion, and under

([11]) In the presence of S9, primitive recursion S5 can clearly be replaced by some selection of initial functions, as was the case for types $\leqq 1$ in the presence of the author's μ-schema (cf. [9] and Grzegorczyk [6]). However it adds little to the work below to keep S5, and we prefer doing so to being drawn here into the investigation of what initial functions constitute a good choice for this purpose. Besides, some of the theory for partial recursive functions includes conveniently theory for primitive recursive functions by S5 being kept.

substitution so long at least as substitution for a function variable takes place only directly for the argument of a function variable (for while we have S8, we do not immediately have IV). Clearly I, II, III, VI extend to partial recursive functions.

As to IV, the former proof gives it to us for the case $\phi(\mathfrak{a}, \sigma^n, \mathfrak{b})$ is primitive recursive, $\theta(\mathfrak{a}, \mathfrak{b}, \tau^{n-1})$ is general recursive, and $n=1$ (so that Case 8 Subcase 1 of the proof will not arise). The extension of IV more generally (and thence of V, VII) will be dealt with in §4 and in Part II.

The extension of VIII will be given by XVII and XXXI below.

3.11. XII. *$\{z\}(\mathfrak{a})$ is a partial recursive function of z, $\mathfrak{a}$ such that $\{0\}(\mathfrak{a})$, $\{1\}(\mathfrak{a})$, $\{2\}(\mathfrak{a})$, $\cdots$ is an enumeration with repetitions of the partial recursive functions of* $\mathfrak{a}$. (Enumeration theorem for partial recursive functions.)

Proof. $\{z\}(\mathfrak{a})$ is partial recursive, because it is introduced by S9 with $\mathfrak{c}$ empty. Thence, for each fixed z, using S4 to substitute for z the constant function $\chi(\mathfrak{a}) = z$ given by S2 with z as the q, $\{z\}(\mathfrak{a})$ is a partial recursive function of $\mathfrak{a}$. Any partial recursive function $\phi(\mathfrak{a})$ has an index z by 3.7, and $\phi(\mathfrak{a}) \simeq \{z\}(\mathfrak{a})$.

REMARK 2. XII is the analog of IM, Theorem XXII, p. 341, but our use of indices gives it to us at the very beginning of the theory. By XII, we get all partial recursive functions using only S9 with $\mathfrak{c}$ empty and substitution of constants z, but of course S1–S8 enter into our definition of the function $\{a\}(\mathfrak{b})$ introduced by S9.

3.12. XIII. *For each $m \geqq 1$: There is a primitive recursive function $S^m(z, y_1, \cdots, y_m)$ such that, if $\phi(y_1, \cdots, y_m, \mathfrak{b})$ is a partial recursive function of the variables $y_1, \cdots, y_m$, $\mathfrak{b}$ with index z, then for each fixed $y_1, \cdots, y_m$, $S^m(z, y_1, \cdots, y_m)$ is an index of $\phi(y_1, \cdots, y_m, \mathfrak{b})$ as a function of* $\mathfrak{b}$ (analogously to IM, Theorem XXIII, p. 342).

Proof, by induction on m. BASIS ($m = 1$). We have $\phi_y(\mathfrak{b}) \simeq \phi(y, \mathfrak{b}) \simeq \phi(\chi(\mathfrak{b}), \mathfrak{b})$ by S4 where $\chi(\mathfrak{b}) = y$ by S2. So take $S^1(z, y) = \langle 4, [(z)_1/2], z, \langle 2, [(z)_1/2], y\rangle\rangle$, noting that, if $(z)_1 = \langle n_0, n_1, \cdots, n_r\rangle$ with $n_0 > 0$, $[(z)_1/2] = \langle n_0-1, n_1, \cdots, n_r\rangle$. IND. STEP. Take $S^{m+1}(z, y_1, \cdots, y_m, y_{m+1}) = S^1(S^m(z, y_1, \cdots, y_m), y_{m+1})$.

XIV. *Given any partial recursive function $\psi(z, \mathfrak{b})$, an index e of $\psi(e, \mathfrak{b})$ can be found; thus we can solve for z the equation*

$$\{z\}(\mathfrak{b}) \simeq \psi(z, \mathfrak{b}).$$

(The recursion theorem, analogous to IM, Theorem XXVII, pp. 352–353.)

Proof. By the same method as before; i.e. let f be an index of $\psi(S^1(y, y), \mathfrak{b})$ as a function of y, $\mathfrak{b}$, and take $e = S^1(f, f)$.

3.13. XV. *If $\psi_0(\mathfrak{a})$, $\psi_1(\mathfrak{a})$ are partial recursive functions, and $Q(\mathfrak{a})$ is a partial recursive predicate, the function*

$$\phi(\mathfrak{a}) \simeq \begin{cases} \psi_0(\mathfrak{a}) & \text{if } Q(\mathfrak{a}), \\ \psi_1(\mathfrak{a}) & \text{if } \overline{Q}(\mathfrak{a}) \end{cases}$$

(undefined exactly if $Q(\mathfrak{a})$ is undefined, or $Q(\mathfrak{a})$ is true and $\psi_0(\mathfrak{a})$ is undefined, or $Q(\mathfrak{a})$ is false and $\psi_1(\mathfrak{a})$ is undefined) is partial recursive.

Proof. Let $\chi(\mathfrak{a})$ be the representing function of $Q(\mathfrak{a})$, and let e_i be an index of $\psi_i(\mathfrak{a})$ $(i = 0, 1)$. Define a primitive recursive function π by

$$\pi(i) = \begin{cases} e_0 \text{ if } i = 0, \\ e_1 \text{ if } i \neq 0. \end{cases}$$

Then $\phi(\mathfrak{a}) \simeq \{\pi(\chi(\mathfrak{a}))\}(\mathfrak{a})$.

XVI. *If $\chi(\mathfrak{b}, y)$ is partial recursive, so is $\mu y[\chi(\mathfrak{b}, y)=0]$.* (Cf. IM, pp. 279, 329–330.)

Proof. We have $\mu y[\chi(\mathfrak{b}, y)=0] \simeq \phi(\mathfrak{b}, 0)$ where

$$\phi(\mathfrak{b}, y) \simeq \begin{cases} 0 \text{ if } \chi(\mathfrak{b}, y) = 0, \\ \phi(\mathfrak{b}, y')' \text{ if } \chi(\mathfrak{b}, y) \neq 0. \end{cases}$$

Note that this does make $\phi(\mathfrak{b}, 0)$ undefined not only when, as y increases, a y for which $\chi(\mathfrak{b}, y)$ is undefined occurs before one for which $\chi(\mathfrak{b}, y) = 0$, but also (as then $\phi(\mathfrak{b}, 0) \simeq 1+\phi(\mathfrak{b}, 1) \simeq 2+\phi(\mathfrak{b}, 2) \simeq \cdots$) when all $\chi(\mathfrak{b}, y)$ are false. To get $\phi(\mathfrak{b}, y)$ as a partial recursive function, let

$$\psi(z, \mathfrak{b}, y) \simeq \begin{cases} 0 \text{ if } \chi(\mathfrak{b}, y) = 0, \\ \{z\}(\mathfrak{b}, y')' \text{ if } \chi(\mathfrak{b}, y) \neq 0 \end{cases}$$

(using XV), and set $\phi(\mathfrak{b}, y) \simeq \psi(e, \mathfrak{b}, y)$ where e is index of $\psi(e, \mathfrak{b}, y)$ given by XIV.

XVII. *Each function $\phi(\mathfrak{a})$ of variables $\mathfrak{a}$ of type $\leqq 1$ partial (general) recursive in the former sense* (e.g. IM, Chapters XI, XII) *is partial (general) recursive in the present sense.*

Proof. By the former normal form theorem (IM, Theorem XIX, p. 330) with XVI and VIII. (We are not yet considering here any but completely defined functions as values of our type-1 variables, so IM, Theorem XIX is available to us. The situation when incompletely defined functions are allowed as type-1 arguments will be considered in Part II.)

XVIII. *If $\psi_0(\mathfrak{a}), \cdots, \psi_m(\mathfrak{a}), Q_0(\mathfrak{a}), \cdots, Q_m(\mathfrak{a})$ are partial recursive, so is the function*

$$\phi(\mathfrak{a}) \simeq \begin{cases} \psi_0(\mathfrak{a}) \textit{ if } Q_0(\mathfrak{a}), \\ \textit{otherwise } \psi_1(\mathfrak{a}) \textit{ if } Q_1(\mathfrak{a}), \\ \cdots\cdots\cdots \\ \textit{otherwise } \psi_m(\mathfrak{a}) \textit{ if } Q_m(\mathfrak{a}) \end{cases}$$

(undefined exactly if there is no i $(0 \leqq i \leqq m)$ such that $Q_0(\mathfrak{a}), \cdots, Q_{i-1}(\mathfrak{a})$ are false, $Q_i(\mathfrak{a})$ is true and $\psi_i(\mathfrak{a})$ is defined). (Definition by cases.)

Proof. Let $e_0, \cdots, e_m$ $(q_0, \cdots, q_m)$ be indices of $\psi_0(\mathfrak{a}), \cdots, \psi_m(\mathfrak{a})$ (of the representing functions of $Q_0(\mathfrak{a}), \cdots, Q_m(\mathfrak{a})$), and let

$$\rho(i) = \begin{cases} e_0 & \text{if } i=0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ e_{m-1} & \text{if } i=m-1, \\ e_m & \text{if } i \geqq m, \end{cases}$$

$$\sigma(i) = \begin{cases} q_0 & \text{if } i=0, \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \\ q_{m-1} & \text{if } i=m-1, \\ q_m & \text{if } i \geqq m. \end{cases}$$

Then using XVI, let

$$\phi(\mathfrak{a}) \simeq \{\rho(\mu i[\{\sigma(i)\}(\mathfrak{a})=0])\}(\mathfrak{a}).$$

We shall compare XVIII with the previous result for types 0, 1 (IM, Theorem XX(c), p. 337) in Part II.

3.14. Relative partial and general recursiveness are obtained by allowing assumed functions $\psi_1, \cdots, \psi_l$ (briefly, Ψ), introduced by applications of S0.i (1.8, excluding the alternative there). Indices become *indices from* Ψ; $\{z\}(\mathfrak{a})$ becomes $\{z\}^{\Psi}(\mathfrak{a})$. For uniformity, cf. 1.9; for predicates, cf. 1.10.

In an application of S0.i it may now happen that some $\lambda\alpha^{j-2}\, \chi_k(\alpha^{j-2}, \mathfrak{b})$ is an incompletely defined function and thus not an object from the range of γ_k^{j-1} as established in 1.2. Here we shall take the result of the substitution into ψ_i to be undefined in this case, just as in 3.7 for S8.j we took $\alpha^j(\lambda\alpha^{j-2}\, \chi(\alpha^j, \alpha^{j-2}, \mathfrak{b}))$ to be undefined when $\lambda\alpha^{j-2}\, \chi(\alpha^j, \alpha^{j-2}, \mathfrak{b})$ is incompletely defined.

On the other hand, there is no reason here for not allowing $\psi_1, \cdots, \psi_l$ to be partial functions.

3.15. When $\psi_1, \cdots, \psi_l$ are completely defined functions, the transformations of 1.9 (extended to include S9) and 2.7 provide an alternative (for most purposes) to developing the theory in relativized form.

4. **Construction of indices. 4.1.** We now write Ix(z) (PRI(z)), and say z is an ***index*** (***primitive recursive index***), when z is an index of a partial (primitive) recursive function $\phi(\mathfrak{a})$ as assigned in 3.7 (in 3.5)([12]). In this case we also say z is an ***index for*** $\mathfrak{a}$. Then if $\mathfrak{a}$ is a list of variables with r and $n_0, \cdots, n_r$ as in 3.5, $\langle n_0, \cdots, n_r\rangle = (z)_1$ and $r = \text{tp}(z)$ where tp is the primitive recursive function defined thus:

$$\text{tp}(z) = \mu i_{i<z}(j)_{i<j<z}\overline{p_j \mid (z)_1}.$$

XIX. *The predicates* Ix(z) *and* PRI(z) *are primitive recursive.*

Proof for Ix(z). Ix(z) satisfies the course-of-values recursion

([12]) It is a trifling departure from the analogy to Gödel numbers (IM, pp. 292, 330, 340) that we do not call any z such that $(\mathfrak{a})[\phi(\mathfrak{a}) \simeq \{z\}(\mathfrak{a})]$ an "index" of $\phi(\mathfrak{a})$. By 3.7, such a z can fail to be an index only when $\phi(\mathfrak{a})$ is the completely undefined function of $\mathfrak{a}$.

$$\begin{aligned}\mathrm{Ix}(z) \equiv &\Big[((z)_0=1 \vee (z)_0=3) \;\&\; z=\prod_{i<2} p_i^{(z)_i} \;\&\; (z)_{1,0}>0\Big]\\
&\vee \Big[(z)_0=2 \;\&\; z=\prod_{i<3} p_i^{(z)_i} \;\&\; (z)_1>0\Big]\\
&\vee \Big[(z)_0=4 \;\&\; z=\prod_{i<4} p_i^{(z)_i} \;\&\; \mathrm{Ix}((z)_2) \;\&\; (z)_{2,1}=2\cdot(z)_1 \;\&\; \mathrm{Ix}((z)_3) \;\&\; (z)_{3,1}=(z)_1\Big]\\
&\vee \Big[(z)_0=5 \;\&\; z=\prod_{i<4} p_i^{(z)_i} \;\&\; \mathrm{Ix}((z)_2) \;\&\; 2\cdot(z)_{2,1}=(z)_1 \;\&\; \mathrm{Ix}((z)_3) \;\&\; (z)_{3,1}=2\cdot(z)_1\Big]\\
&\vee \Big[(z)_0=6 \;\&\; z=\prod_{i<5} p_i^{(z)_i} \;\&\; (z)_3<(z)_{1,(z)_2} \;\&\; \mathrm{Ix}((z)_4) \;\&\; (z)_{4,1}=(z)_1\Big]\\
&\vee \Big[(z)_0=7 \;\&\; z=\prod_{i<2} p_i^{(z)_i} \;\&\; (z)_{1,0}>0 \;\&\; (z)_{1,1}>0\Big]\\
&\vee \Big[(z)_0=8 \;\&\; z=\prod_{i<4} p_i^{(z)_i} \;\&\; (z)_2>1 \;\&\; (z)_{1,(z)_2}>0 \;\&\; \mathrm{Ix}((z)_3)\\
&\qquad\qquad \& \; (z)_{3,1}=p_{(z)_2 \dot{-} 2}\cdot(z)_1\Big]\\
&\vee \Big[(z)_0=9 \;\&\; z=\prod_{i<3} p_i^{(z)_i} \;\&\; (z)_{1,0}>(z)_{2,0} \;\&\; (i)_{0<i\leq z}((z)_{1,i}\geq(z)_{2,i})\Big].\end{aligned}$$

4.2. XX. *There is a primitive recursive function* $\pi(j, k, g)$ *such that, for* $\mathfrak{a}$, $\mathfrak{a}_1$ *as in Schema* S6, *if* g *is an index of* $\psi(\mathfrak{a})$, *then* $\pi(j, k, g)$ *is one of* $\phi(\mathfrak{a}_1)$.

Proof. Let $\pi(j, k, g, 0) = g$, $\pi(j, k, g, l+1) = \langle 6, (g)_1, j, k, \pi(j, k, g, l)\rangle$, $\pi(j, k, g) = \pi(j, k, g, k)$ (cf. the proof of II).

XXI. *There is a primitive recursive function* $\iota(z, m)$ *such that, if* z *is an index of* $\phi(\mathfrak{a})$, *and* $\mathfrak{c}$ *are variables distinct from* $\mathfrak{a}$ *and consisting of* $m_0, \cdots, m_s$ *of types* $0, \cdots, s$, *then* $\iota(z, \langle m_0, \cdots, m_s\rangle)$ *is an index of* $\phi(\mathfrak{a}, \mathfrak{c}) = \phi(\mathfrak{a})$ *with* $\lambda(\iota(z, \langle m_0, \cdots, m_s\rangle)) = \lambda(z)$ (cf. III).

Proof. We define $\iota(z, m)$ by the course-of-values recursion

$$\iota(z, m) = \begin{cases} \langle (z)_0, m\cdot(z)_1, (z)_2\rangle & \text{if } (z)_0=2 \vee (z)_0=9,\\ \langle (z)_0, m\cdot(z)_1, \iota((z)_2, m), \iota((z)_3, m)\rangle & \text{if } (z)_0=4 \vee (z)_0=5,\\ \langle 6, m\cdot(z)_1, (z)_2, (z)_3, \iota((z)_4, m)\rangle & \text{if } (z)_0=6,\\ \langle 8, m\cdot(z)_1, (z)_2, \iota((z)_3, m)\rangle & \text{if } (z)_0=8,\\ \langle (z)_0, m\cdot(z)_1\rangle & \text{otherwise.}\end{cases}$$

XXII. *For each* $n \geq 1$, *there is a primitive recursive function* $\gamma_n(z, w, p)$ *with the following property. Suppose* $\mathfrak{a}$ *contains exactly* p *type-*n *variables; and let* z, w *be indices of* $\phi(\mathfrak{a}, \sigma^n, \mathfrak{b})$, $\theta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \tau^{n-1})$, *respectively. Then, for values of*

$\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$ *such that* $\lambda\tau^{n-1}\,\theta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \tau^{n-1})$ *is completely defined and* $\phi(\mathfrak{a}, \sigma^n, \mathfrak{b})$ *is defined when* $\sigma^n = \lambda\tau^{n-1}\,\theta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \tau^{n-1})$,

$$(10) \qquad \phi(\mathfrak{a}, \lambda\tau^{n-1}\,\theta(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \tau^{n-1}), \mathfrak{b}) = \{\gamma_n(z, w, p)\}(\mathfrak{a}, \mathfrak{b}, \mathfrak{c}).$$

Proof, by induction on n. Take any $n \geqq 1$. We shall take $\gamma_n(z, w, p) = 0$, except for $\mathrm{Ix}(z)$ & $(z)_{1,n} > p$ & $\mathrm{Ix}(w)$ & $p_{n-1}\cdot(z)_1 | p_n\cdot(w)_1$, when the appropriate following case shall apply. The first eight cases correspond to those in the proof of IV. We use induction on $\{z\}(-)$ (cf. end 3.8) in proving (10) from the specifications for $\gamma_n(z, w, p)$ worked out in the cases. In conclusion we shall show that a primitive recursive γ_n exists that meets all the specifications.

CASE 5. $(z)_0 = 5$. First, starting from z, w as indices of $\phi(a, \mathfrak{d}, \sigma, \mathfrak{b})$, $\theta(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau)$, we construct indices (right column) for the functions (left column) used in the treatment of Case 5 for IV.

$\theta_1(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u, \tau)$:	$A = \pi(0, (w)_{1,0} \dot{-} (1+(2\dot{-}n)), w)$.
$\theta(a, b, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u, \tau)$:	$B = \langle 6, 4\cdot(w)_1, 0, (w)_{1,0}+1, \langle 6, 4\cdot(w)_1, 0, (w)_{1,0}+1, \iota(A, 4)\rangle\rangle$.
$\psi(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u)$:	$C = \gamma_n((z)_2, A, p)$.
$\chi(a, b, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u)$:	$D = \gamma_n((z)_3, B, p)$.
$\phi(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u)$:	$E = \langle 5, 2\cdot(C)_1, C, D\rangle$.
$\phi_1(u, a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c})$:	$F = \langle 6, (E)_1, 0, (E)_{1,0}\dot{-}1, E\rangle$.
$\eta(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c})$:	$G = \langle 3, (C)_1\rangle$.
$\phi(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c})$:	$H = \langle 4, (C)_1, F, G\rangle$.

Let $\gamma_n(z, w, p) = H$. Consider any particular $(a_0, \mathfrak{d}_0, \mathfrak{b}_0, \mathfrak{c}_0)$ such that $\lambda\tau\,\theta(a_0, \mathfrak{d}_0, \mathfrak{b}_0, \mathfrak{c}_0, \tau)$ is completely defined and $\phi(a_0, \mathfrak{d}_0, \sigma_0, \mathfrak{b}_0)$ is defined for $\sigma_0 = \lambda\tau\,\theta(a_0, \mathfrak{d}_0, \mathfrak{b}_0, \mathfrak{c}_0, \tau)$. Then $\psi(\mathfrak{d}_0, \sigma_0, \mathfrak{b}_0)$ is defined and $\chi(a, b, \mathfrak{d}_0, \sigma_0, \mathfrak{b}_0)$ is defined for $(a, b) = (0, \phi(0, \mathfrak{d}_0, \sigma_0, \mathfrak{b}_0)), \cdots, (a_0-1, \phi(a_0-1, \mathfrak{d}_0, \sigma_0, \mathfrak{b}_0))$. This information suffices for the applications of the hypothesis of the induction on $\{z\}(-)$ to conclude, by the method of proof under IV (taking $(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, u) = (\mathfrak{d}_0, \mathfrak{b}_0, \mathfrak{c}_0, a_0)$, and considering only $a \leqq a_0$ for the induction on a at the end), that taking $\phi(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}) \simeq \{H\}(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c})$ will make $\phi(a_0, \mathfrak{d}_0, \mathfrak{b}_0, \mathfrak{c}_0) = \phi(a_0, \mathfrak{d}_0, \lambda\tau\,\theta(a_0, \mathfrak{d}_0, \mathfrak{b}_0, \mathfrak{c}_0, \tau), \mathfrak{b}_0)$.

CASE 8. $(z)_0 = 8$. SUBCASE 1. $n = (z)_2$ & $p=0$ (only for $n > 1$).

$\theta(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c}, \tau)$:	$A = \langle 6, p_{n-2}\cdot(w)_1, n-2, (w)_{1,n-2}, \iota(w, p_{n-2})\rangle$.
$\chi(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c})$:	$B = \gamma_n((z)_3, A, 0)$.
$\chi_1(\mathfrak{b}, \mathfrak{c}, \alpha^{j-2})$:	$C = \pi(n-2, (B)_{1,n-2}\dot{-}1, B)$.
$\phi(\mathfrak{b}, \mathfrak{c})$:	$D = \gamma_{n-1}(w, C, (w)_{1,n-1}\dot{-}1)$.

Let $\gamma_n(z, w, p) = D$. Consider any $\mathfrak{b}_0$, $\mathfrak{c}_0$ such that $\lambda\tau\,\theta(\mathfrak{b}_0, \mathfrak{c}_0, \tau)$ is completely defined and $\phi(\sigma_0, \mathfrak{b}_0)$ is defined for $\sigma_0 = \lambda\tau\,\theta(\mathfrak{b}_0, \mathfrak{c}_0, \tau)$. Then $\chi(\sigma_0, \alpha^{j-2}, \mathfrak{b}_0)$ is defined for each α^{j-2}, and so, by hyp. ind. on $\{z\}(-)$, letting $\chi(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c})$

$\simeq \{B\}(\alpha^{j-2}, \mathfrak{b}, \mathfrak{c})$ does make $\chi(\alpha^{j-2}, \mathfrak{b}_0, \mathfrak{c}_0) = \chi(\sigma_0, \alpha^{j-2}, \mathfrak{b}_0)$. But putting $\alpha_0^{j-1} = \lambda\alpha^{j-2}\, \chi(\sigma_0, \alpha^{j-2}, \mathfrak{b}_0)$, $\phi(\sigma_0, \mathfrak{b}_0) = \theta(\mathfrak{b}_0, \mathfrak{c}_0, \alpha_0^{j-1})$, so by hyp. ind. on n letting $\phi(\mathfrak{b}, \mathfrak{c}) \simeq \{D\}(\mathfrak{b}, \mathfrak{c})$ does make $\phi(\mathfrak{b}_0, \mathfrak{c}_0) = \phi(\sigma_0, \mathfrak{b}_0)$.

Subcase 2. $n \neq (z)_2 \vee p \neq 0$.

$$\theta(\alpha^j, \alpha^{j-2}, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau)\colon\ A = \langle 6, p_{(z)_2 \dot{-} 2}\cdot(w_1), (z)_2 \dot{-} 2, (w)_{1,(z)_2 \dot{-} 2}, \iota(w, p_{(z)_2 \dot{-} 2})\rangle.$$

$$\chi(\alpha^j, \alpha^{j-2}, \mathfrak{d}, \mathfrak{b}, \mathfrak{c})\colon\ B = \gamma_n((z)_3, A, p + \overline{\mathrm{sg}}\,|\, n \dot{-} ((z)_2 \dot{-} 2)\,|\,).$$

$$\phi(\alpha^j, \mathfrak{d}, \mathfrak{b}, \mathfrak{c})\colon\qquad C = \langle 8, [(B)_1/p_{(z)_2 \dot{-} 2}], (z)_2, B\rangle.$$

Case 9. $(z)_0 = 9$. Subcase 1. $p \geqq (z)_{2,n}$ (so σ^n is one of the variables $\mathfrak{c}$ for the schema S9). Let $\gamma_n(z, w, p) = \langle 9, [(w)_1/p_{n-1}], (z)_2\rangle$.

Subcase 2. $p < (z)_{2,n}$. Write $\mathfrak{a} = (a, \mathfrak{d})$, $\mathfrak{b} = (\mathfrak{e}, \mathfrak{f})$ with $(\mathfrak{d}, \sigma, \mathfrak{e})$ the $\mathfrak{b}$, and $\mathfrak{f}$ the $\mathfrak{c}$, for the schema. Now $\phi(a, \mathfrak{d}, \sigma, \mathfrak{b}) \simeq \{a\}(\mathfrak{d}, \sigma, \mathfrak{e})$ and $\theta(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau) \simeq \{S^1(w, a)\}(\mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau)$ by XIII. So for values of a, $\mathfrak{d}$, $\mathfrak{b}$, $\mathfrak{c}$ which make $\sigma_0 = \lambda\tau\, \theta(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau)$ completely defined and $\{a\}(\mathfrak{d}, \sigma_0, \mathfrak{e})$ defined, the hyp. ind. on $\{z\}(-)$ gives that $\phi(a, \mathfrak{d}, \lambda\tau\, \theta(a, \mathfrak{d}, \mathfrak{b}, \mathfrak{c}, \tau), \mathfrak{b}) = \{\gamma_n(a, S^1(w, a), p)\}(\mathfrak{d}, \mathfrak{b}, \mathfrak{c})$. So using XIII it will suffice to take $\gamma_n(z, w, p) = S^3(D, g_n, w, p)$ where D is an index of $\lambda gwpa\mathfrak{d}\mathfrak{b}\mathfrak{c}\, \{\{g\}(a, S^1(w, a), p)\}(\mathfrak{d}, \mathfrak{b}, \mathfrak{c})$ constructed as follows and g_n is an index of γ_n. Let e be an index of $\lambda gwpa\, \{g\}(a, S^1(w,a), p)$. We construct further indices.

$$\lambda gwpa\mathfrak{d}\mathfrak{b}\mathfrak{c}\, \{g\}(a, S^1(w, a), p)\colon\ A = \iota(e, [(w)_1/2p_{n-1}]).$$

$$\lambda b\mathfrak{d}\mathfrak{b}\mathfrak{c}gwpa\, \{b\}(\mathfrak{d}, \mathfrak{b}, \mathfrak{c})\colon\qquad B_0 = \langle 9, 2^4\cdot[(w)_1/p_{n-1}], [(w)_1/2p_{n-1}]\rangle.$$

$$B_{i+1} = \langle 6, (B_0)_1, 0, (B_0)_{1,0} \dot{-} 1, B_i\rangle.$$

$$\lambda bgwpa\mathfrak{d}\mathfrak{b}\mathfrak{c}\, \{b\}(\mathfrak{d}, \mathfrak{b}, \mathfrak{c})\colon\qquad C = \langle 6, (B_0)_1, 0, 4, B_4\rangle.$$

$$\lambda gwpa\mathfrak{d}\mathfrak{b}\mathfrak{c}\, \{\{g\}(a, S^1(w, a), p\}(\mathfrak{d}, \mathfrak{b}, \mathfrak{c})\colon\ D = \langle 4, (A)_1, C, A\rangle.$$

Conclusion. Bringing together the definitions of $\gamma_n(z, w, p)$ proposed in the cases, we can by the recursion theorem XIV pick an index g_n of γ_n (which g_n is called for in Case 9) so that the resulting recursion is satisfied. But with this g_n fixed, we can see as follows from the form of the recursion for $\gamma_n(z, w, p)$ that γ_n is primitive recursive, using for $n > 1$ the hyp. ind. on n by which γ_{n-1} (which occurs in Case 8, Subcase 1) is primitive recursive. The right side is built up primitive recursively from z, w, p and parts $\gamma_n(Z, W, P)$ with $Z < z$ (under the hypothesis of the case in which the part occurs) and Z, W, P primitive recursive functions of z, w, p. Thus the recursion is a course-of-values recursion on z with "nesting" or "substitution for the parameters" w, p, and hence by Péter [22, §§1, 2, especially Nr. 20, p. 632] or [23, §§3, 5] defines a primitive recursive function γ_n.

4.3. Now we recapitulate XXII, taking $\mathfrak{c}$ empty and putting $\phi(\mathfrak{a}, \mathfrak{b}) \simeq \{\gamma_n(z, w, p)\}(\mathfrak{a}, \mathfrak{b})$. This gives us a version of IV extended to partial and general recursive functions.

XXIII. *For* $n \geqq 1$: *If* $\phi(\mathfrak{a}, \sigma^n, \mathfrak{b})$ *and* $\theta(\mathfrak{a}, \mathfrak{b}, \tau^{n-1})$ *are partial recursive, there is a partial recursive function* $\phi(\mathfrak{a}, \mathfrak{b})$ *such that*

$$\phi(\mathfrak{a}, \mathfrak{b}) = \phi(\mathfrak{a}, \lambda\tau^{n-1}\, \theta(\mathfrak{a}, \mathfrak{b}, \tau^{n-1}), \mathfrak{b}) \tag{11}$$

for values of $\mathfrak{a}, \mathfrak{b}$ *for which* $\lambda\tau^{n-1}\, \theta(\mathfrak{a}, \mathfrak{b}, \tau^{n-1})$ *is completely defined and* $\phi(\mathfrak{a}, \lambda\tau^{n-1}\, \theta(\mathfrak{a}, \mathfrak{b}, \tau^{n-1}), \mathfrak{b})$ *is defined.* Thus: *If* $\phi(\mathfrak{a}, \sigma^n, \mathfrak{b})$ *and* $\theta(\mathfrak{a}, \mathfrak{b}, \tau^{n-1})$ *are general recursive, so is* $\phi(\mathfrak{a}, \mathfrak{b}) = \phi(\mathfrak{a}, \lambda\tau^{n-1}\, \theta(\mathfrak{a}, \mathfrak{b}, \tau^{n-1}), \mathfrak{b})$.

4.4. Using partial or general recursiveness in functions Ψ, and indices from Ψ (3.14), we obtain a relativized version XXIII* of XXIII.

4.5. XXIV. *For any* $j \geqq 0$: *If* $\psi(\alpha^j, \mathfrak{b})$ *and* $\chi(a, \alpha^j, \alpha^{j+1}, \mathfrak{b})$ *are general recursive, so is the function* $\phi(a, \alpha^j, \mathfrak{b})$ *defined by*

$$\begin{cases} \phi(0, \alpha^j, \mathfrak{b}) = \psi(\alpha^j, \mathfrak{b}), \\ \phi(a', \alpha^j, \mathfrak{b}) = \chi(a, \alpha^j, \lambda\alpha^j\, \phi(a, \alpha^j, \mathfrak{b}), \mathfrak{b}). \end{cases}$$

(Functional recursion; for $j = 0$, Péter's "primitive Rekursion der II-ten Stufe"(5).)

Proof. Let $\chi(z, a, \alpha^j, \mathfrak{b})$ be the partial recursive function given by XXIII such that

$$\chi(z, a, \alpha^j, \mathfrak{b}) = \chi(a \dot{-} 1, \alpha^j, \lambda\alpha^j\, \{z\}(a \dot{-} 1, \alpha^j, \mathfrak{b}), \mathfrak{b}) \tag{12}$$

for values of $z, a, \mathfrak{b}$ which make $\lambda\alpha^j\, \{z\}(a \dot{-} 1, \alpha^j, \mathfrak{b})$ completely defined. Using XV, let

$$\psi(z, a, \alpha^j, \mathfrak{b}) \simeq \begin{cases} \psi(\alpha^j, \mathfrak{b}) & \text{if } a=0, \\ \chi(z, a, \alpha^j, \mathfrak{b}) & \text{if } a \neq 0. \end{cases}$$

Using the recursion theorem XIV, pick an index e of $\psi(e, a, \alpha^j, \mathfrak{b})$, and let $\phi(a, \alpha^j, \mathfrak{b}) \simeq \{e\}(a, \alpha^j, \mathfrak{b}) \simeq \psi(e, a, \alpha^j, \mathfrak{b})$. Now, for any given $\mathfrak{b}$, we prove by induction on a that $\phi(a, \alpha^j, \mathfrak{b})$ is defined for every α^j and satisfies the recursion in XXIV. Basis. $\phi(0, \alpha^j, \mathfrak{b}) \simeq \psi(e, 0, \alpha^j, \mathfrak{b}) \simeq \psi(\alpha^j, \mathfrak{b})$, which is defined since ψ is general recursive. Ind. step. $\phi(a', \alpha^j, \mathfrak{b}) \simeq \psi(e, a', \alpha^j, \mathfrak{b}) \simeq \chi(e, a', \alpha^j, \mathfrak{b}) \simeq \chi(a' \dot{-} 1, \alpha^j, \lambda\alpha^j\, \{e\}(a' \dot{-} 1, \alpha^j, \mathfrak{b}), \mathfrak{b})$ (by (12), since by hyp. ind. $\lambda\alpha^j\, \{e\}(a' \dot{-} 1, \alpha^j, \mathfrak{b}) = \lambda\alpha^j\, \phi(a, \alpha^j, \mathfrak{b})$ is completely defined) $\simeq \chi(a, \alpha^j, \lambda\alpha^j\, \phi(a, \alpha^j, \mathfrak{b}), \mathfrak{b})$.

4.6. XXV. *There are primitive recursive functions* $\nu_0, \nu_1, \nu_2, \cdots$ *such that* $\nu_0(n)$ ($\nu_j(n)$ *for a* $j > 0$) *is a primitive recursive index of* $\lambda a_1 \cdots a_n \langle a_1, \cdots, a_n\rangle$ (*of* $\lambda\alpha_1^j \cdots \alpha_n^j\tau^{j-1} \langle \alpha_1^j(\tau^{j-1}), \cdots, \alpha_n^j(\tau^{j-1})\rangle$).

Proof for $j > 0$, using induction on n. Basis. Let $\nu_j(0) = \langle 2, p_{j-1}, 1\rangle$. Ind. step. Assume $\nu_j(n)$ is an index of $\lambda\alpha_1^j \cdots \alpha_n^j\tau^{j-1} \langle \alpha_1^j(\tau^{j-1}), \cdots, \alpha_n^j(\tau^{j-1})\rangle$. Let h be an index of $\lambda n b \alpha^j(\tau^{j-1})\, b \cdot p_n^{\alpha^j(\tau^{j-1})}$. We construct further indices.

$$\lambda b \alpha_1^j \cdots \alpha_n^j \alpha^j \tau^{j-1}\, b \cdot p_n^{\alpha^j(\tau^{j-1})} : \mathrm{A} = \pi(j, n, \iota(S^1(h, n), p_j^n)).$$

$$\lambda \alpha_1^j \cdots \alpha_n^j \alpha^j \tau^{j-1} \langle \alpha_1^j(\tau^{j-1}), \cdots, \alpha_n^j(\tau^{j-1})\rangle : \mathrm{B} = \iota(\nu_j(n), p_j).$$

$$\lambda \alpha_1^j \cdots \alpha_{n+1}^j \tau^{j-1} \langle \alpha_1^j(\tau^{j-1}), \cdots, \alpha_{n+1}^j(\tau^{j-1})\rangle : \nu_j(n+1) = \langle 4, (\mathrm{B})_1, \mathrm{A}, \mathrm{B}\rangle.$$

5. Reduction of the inductive definition of $\{z\}(\mathfrak{a})\simeq w$ to an explicit definition. **5.1.** In computing $\{z\}(\mathfrak{a})$ the numbers $n_0, \cdots, n_r$ in general vary, but r does not increase (for by I with 3.10 r remains fixed throughout the partial recursive description determined by z, while applications of S9 start new partial recursive descriptions determined by the a with the same or smaller r, etc.). In analyzing the inductive definition of $\{z\}(\mathfrak{a}) \simeq w$ it will help to contract all the variables of each type $j \leqq r$ into one.

So for each $r \geqq 0$ we define (writing $n_j = (z)_{1,j}$; cf. 2.1, 4.1)

$$\{z\}[a, \alpha^1, \cdots, \alpha^r]$$
$$\simeq \{z\}((a)_0, \cdots, (a)_{n_0-1}, (\alpha^1)_0, \cdots, (\alpha^1)_{n_1-1}, \cdots, (\alpha^r)_0, \cdots, (\alpha^r)_{n_r-1})$$

if $\text{Ix}(z)$ and $\text{tp}(z) \leqq r$; otherwise $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ shall be undefined. Then by (2), when z is an index for $a_1, \cdots, a_{n_0}, \overset{1}{\alpha}_1, \cdots, \overset{1}{\alpha}_{n_1}, \cdots, \overset{r}{\alpha}_1, \cdots, \overset{r}{\alpha}_{n_r}$,

$$\{z\}(a_1, \cdots, a_{n_0}, \overset{1}{\alpha}_1, \cdots, \overset{1}{\alpha}_{n_1}, \cdots, \overset{r}{\alpha}_1, \cdots, \overset{r}{\alpha}_{n_r})$$
$$\simeq \{z\}[\langle a_1, \cdots, a_{n_0}\rangle, \langle \overset{1}{\alpha}_1, \cdots, \overset{1}{\alpha}_{n_1}\rangle, \cdots, \langle \overset{r}{\alpha}_1, \cdots, \overset{r}{\alpha}_{n_r}\rangle]. \tag{13}$$

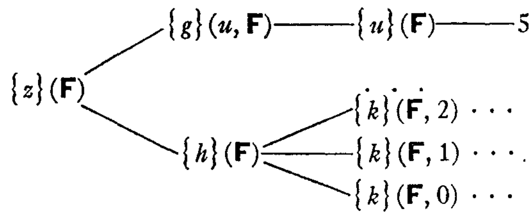
5.2. XXVI. *For each $r \geqq 2$, there are primitive recursive predicates I and J such that, when $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ is defined,*

$$\{z\}[a, \alpha^1, \cdots, \alpha^r] = w \equiv (\beta^{r-1})(E\xi^{r-2})I(z, a, \alpha^1, \cdots, \alpha^r, w, \beta^{r-1}, \xi^{r-2}), \tag{14}$$

$$\{z\}[a, \alpha^1, \cdots, \alpha^r] = w \equiv (E\beta^{r-1})(\xi^{r-2})J(z, a, \alpha^1, \cdots, \alpha^r, w, \beta^{r-1}, \xi^{r-2}). \tag{15}$$

The proof occupies 5.3–5.9. We assume $r \geqq 2$, though afterwards (in 5.21–5.22) we shall use parts of the material for $r = 0, 1$. For 5.3–5.6, though of course z can take any value in the recursions, we assume for the discussion that $\text{Ix}(z)$ & $\text{tp}(z) \leqq r$.

5.3. The "stages" in the computation of $\{z\}(\mathbf{F})$ in the example of 3.9 can be arranged in a "tree," thus:



Similarly (assuming $\text{Ix}(z)$ & $\text{tp}(z) \leqq r$) we can consider the stages in the computation of $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ arranged in a tree. However to avoid syntactical considerations here, we consider the "positions" in the tree as being occupied, not by the expressions beginning with "$\{z\}[a, \alpha^1, \cdots, \alpha^r]$" at the 0- (or leftmost) position, but instead by the $r+2$-tuples of objects which occur successively as arguments of $\lambda z a \alpha^1 \cdots \alpha^r \{z\}[a, \alpha^1, \cdots, \alpha^r]$ in carrying out the computation. Thus let $(z_0, a_0, \alpha_0^1, \cdots, \alpha_0^r)$

$= (z, a, \alpha^1, \cdots, \alpha^r)$; this is the $r+2$-tuple at the 0-position. Say a branch has been generated as far as the $r+2$-tuple $(z_n, a_n, \alpha_n^1, \cdots, \alpha_n^r)$ at its n-position. Then there are one, two or infinitely many choices of the $r+2$-tuple $(z_{n+1}, a_{n+1}, \alpha_{n+1}^1, \cdots, \alpha_{n+1}^r)$ at the $n+1$-position.

Here we exclude the possibility of zero choices, so each branch of the tree extends ad infinitum, by letting

$$(z_{n+1}, a_{n+1}, \overset{1}{\alpha}_{n+1}, \cdots, \overset{r}{\alpha}_{n+1}) = (z_n, a_n, \overset{1}{\alpha}_n, \cdots, \overset{r}{\alpha}_n)$$

in the cases (S1–S3, S7) when the computation terminates by the schema giving the value of $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r)$ outright. (We should accordingly redraw the tree shown above to replace the "$-$ 5" by "$-$ $\{u\}(\mathbf{F})$ $-$ $\{u\}(\mathbf{F})$ $\cdots$ ".) Likewise, in the case for S9, which says $\{z\}(a, \mathfrak{b}, \mathfrak{c}) \simeq \{a\}(\mathfrak{b})$, when the a is not an index for the $\mathfrak{b}$, we shall not go on from the $r+2$-tuple for $\{z\}(a, \mathfrak{b}, \mathfrak{c})$ to that for $\{a\}(\mathfrak{b})$, but repeat the former ad infinitum. It is a consequence of $\text{Ix}(z)$ & $\text{tp}(z) \leqq r$ and these provisions, by induction on n, that along every branch for every n, $\text{Ix}(z_n)$ & $\text{tp}(z_n) \leqq r$.

Which $r+2$-tuple we choose in passing from an n-position already reached to an $n+1$-position we describe by a choice of the $(n+1)$st value $\lambda\tau^{r-3}\rho(n, \tau^{r-3})$ of a function $\lambda n\,\lambda\tau^{r-3}\rho(n, \tau^{r-3})$, where in case $r = 2$ the τ^{r-3} is to be omitted so it is the $(n+1)$st value $\rho(n)$ of $\lambda n\,\rho(n)$ that is chosen. In the case of a branching at an application of S4, the choice of the lower $r+2$-tuple (which starts the computation of the $\chi(\mathfrak{b})$) is described by taking $\lambda\tau\,\rho(n, \tau) = \lambda\tau\,0$, and of the upper $r+2$-tuple (for the $\psi(\chi(\mathfrak{b}), \mathfrak{b})$) by taking $\lambda\tau\,\rho(n, \tau) = \lambda\tau\,1$. Similarly at an application of the second equation for S5, the choice of the lower $r+2$-tuple (for the $\phi(a, \mathfrak{b})$) is described by $\lambda\tau\,\rho(n, \tau) = \lambda\tau\,0$ and of the upper (for the $\chi(a, \phi(a, \mathfrak{b}), \mathfrak{b})$) by $\lambda\tau\,\rho(n, \tau) = \lambda\tau\,1$. At an application of S8.2, the choice of the $r+2$-tuple which starts the computation of the $\chi(\alpha^2, x, \mathfrak{b})$ for a given x is described by taking $\lambda\tau\,\rho(n, \tau) = \lambda\tau\,x = \lambda\tau\,\text{mp}_0^{r-2}(x, \tau)$ (cf. 2.3); and more generally, at an application of S8.j ($2 \leqq j \leqq r$), the choice of the $r+2$-tuple which starts the computation of the $\chi(\alpha^j, \alpha^{j-2}, \mathfrak{b})$ for a given α^{j-2} is described by $\lambda\tau\,\rho(n, \tau) = \lambda\tau\,\text{mp}_{j-2}^{r-2}(\alpha^{j-2}, \tau)$. In all other cases, $\lambda\tau\,\rho(n, \tau) = \lambda\tau\,0$.

An *n-position* in the tree is described by the first n choices, which are given by $\lambda\tau\,\bar{\rho}(n; \tau)$ where $\bar{\rho}(n; \tau) = \prod_{i<n} p_i^{\rho(i,\tau)+1}$. A *branch* in its entirety is described by the function $\lambda n\,\lambda\tau\,\rho(n, \tau)$ or simply ρ.

We would like to express $(z_n, a_n, \alpha_n^1, \cdots, \alpha_n^r)$ for the given $z, a, \alpha^1, \cdots, \alpha^r$ as a function of the position $\gamma^{r-2} = \lambda\tau\,\bar{\rho}(n; \tau)$ by primitive recursion on n. However we cannot do just this. For, at a branching corresponding to an application of S4, the a_{n+1} for the upper $n+1$-position is not immediately forthcoming from the $n+1$-position $\gamma^{r-2} = \lambda\tau\,\bar{\rho}(n+1; \tau)$ and the $r+2$-tuple $(z_n, a_n, \alpha_n^1, \cdots, \alpha_n^r)$ at the n-position, but depends (through the $\chi(\mathfrak{b})$ of $\psi(\chi(\mathfrak{b}), \mathfrak{b})$ which is contracted into that a_{n+1}) on the outcome of the entire computation (of $\chi(\mathfrak{b})$) which starts with the lower $n+1$-position; and similarly at a branching corresponding to an application of the second equation of S5. The n-position from which two $n+1$-positions thus issue (via S4 or S5b) we call a *node*.

To get around this difficulty, we now alter the above definition of the $r+2$-tuples $(z_n, a_n, \alpha_n^1, \cdots, \alpha_n^r)$ throughout the tree, by assuming a function $\eta^{r-1}(\gamma^{r-2})$ of position γ^{r-2} in the tree, and expressing $(z_n, a_n, \alpha_n^1, \cdots, \alpha_n^r)$ for the given $z, a, \alpha^1, \cdots, \alpha^r$ as a function of the position $\gamma = \lambda\tau\, \bar{\rho}(n; \tau)$ and this η by primitive recursion on n, in the manner that would be correct as described above if at each position γ in the tree $\eta(\gamma)$ were the value of the $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r]$ there. The use of this η is the key to XXVI–XXVIII in the present treatment.

5.4. We begin with the recursion for z_n and a_n. As basis,

$$z_0 = z, \qquad a_0 = a. \tag{16}$$

The definitions of z_{n+1} and a_{n+1} from z_n, a_n are given by the following table, where $x = \mathrm{pm}_0^{r-2}(\lambda\tau\, \rho(n, \tau))$ (cf. X), $u = \eta(\lambda\tau\, \bar{\rho}(n; \tau)*2)$, and in Case 6 $k = (z_n)_3$. For the computation of $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r]$, the nonvanishing exponents of a_n after the $((z_n)_{1,0})$th exponent (if any) are immaterial, but it is simplest to keep them in defining a_{n+1}.

Case	Case hypothesis	z_{n+1}	a_{n+1}
1	$(z_n)_0=4\ \&\ x=0$	$(z_n)_3$	a_n
2	$(z_n)_0=4\ \&\ x=1$	$(z_n)_2$	$2^u\cdot \prod_{i<a_n} p_{i+1}^{(a_n)_i}$
3	$(z_n)_0=5\ \&\ (a_n)_0=0$	$(z_n)_2$	$\prod_{i<a_n} p_i^{(a_n)_{i+1}}$
4	$(z_n)_0=5\ \&\ (a_n)_0>0\ \&\ x=0$	z_n	$[a_n/2]$
5	$(z_n)_0=5\ \&\ (a_n)_0>0\ \&\ x=1$	$(z_n)_3$	$2^{(a_n)_0\dot{-}1}\cdot 3^u\cdot \prod_{i<a_n} p_{i+2}^{(a_n)_{i+1}}$
6	$(z_n)_0=6\ \&\ (z_n)_2=0$	$(z_n)_4$	$2^{(a_n)_k}\cdot \prod_{i<k} p_{i+1}^{(a_n)_i}\cdot \prod_{k<i<a_n} p_i^{(a_n)_i}$
7	$(z_n)_0=6\ \&\ (z_n)_2>0$	$(z_n)_4$	a_n
8	$(z_n)_0=8\ \&\ (z_n)_2=2$	$(z_n)_3$	$2^x\cdot \prod_{i<a_n} p_{i+1}^{(a_n)_i}$
9	$(z_n)_0=8\ \&\ (z_n)_2>2$	$(z_n)_3$	a_n
10	$(z_n)_0=9\ \&\ \mathrm{Ix}((a_n)_0)$ $\&\ (a_n)_{0,1}=(z_n)_2$	$(a_n)_0$	$\prod_{i<a_n} p_i^{(a_n)_{i+1}}$
11	otherwise	z_n	a_n

To analyze this recursion, we have, for $m > n$, $\lambda\tau\, \rho(n, \tau) = \lambda\sigma\, (\bar{\rho}(m;\sigma))_n \dot{-} 1$ and $\lambda\tau\, \bar{\rho}(n;\tau)*2 = \lambda\sigma\, (\prod_{i<n} p_i^{(\bar{\rho}(m;\sigma))_i}) \cdot p_n$, where $\bar{\rho}(m;\sigma) = \{\lambda\tau\, \bar{\rho}(m;\tau)\}(\sigma)$. For each n, let $b_n = \langle z_n, a_n\rangle$, so that

$$z_n = (b_n)_0, \qquad a_n = (b_n)_1. \tag{17}$$

Using these relations, the simultaneous recursion for z_n and a_n gives

$$\begin{aligned} b_0 &= \langle z, a\rangle, \\ b_{n+1} &= \chi(n, b_n, \lambda\tau\, \bar{\rho}(m;\tau), \eta) \qquad (m > n) \end{aligned} \tag{18}$$

with a primitive recursive χ. Now define β by the primitive recursion

$$\begin{cases} \beta(0, z, a, \gamma^{r-2}, \eta) = \langle z, a\rangle, \\ \beta(n+1, z, a, \gamma^{r-2}, \eta) = \chi(n, \beta(n, z, a, \gamma^{r-2}, \eta), \gamma^{r-2}, \eta). \end{cases}$$

Then, by induction on n,

$$b_n = \beta(n, z, a, \lambda\tau\, \bar{\rho}(m;\tau), \eta) \qquad (m \geqq n). \tag{19}$$

Taking $m = n = \text{lh}(\{\lambda\tau\, \bar{\rho}(n;\tau)\}(\lambda\sigma^{r-4}\, 0))$ (omit σ^{r-4} if $r \leqq 3$),

$$b_n = \beta(z, a, \lambda\tau\, \bar{\rho}(n;\tau), \eta) \tag{20}$$

with a new primitive recursive β. This with (17) gives z_n, a_n by primitive recursive functions of z, a, η and the position $\gamma = \lambda\tau\, \bar{\rho}(n;\tau)$; and more generally, (19) with (17) gives z_n, a_n by primitive recursive functions of z, a, η, n and any position $\delta = \lambda\tau\, \bar{\rho}(m;\tau)$ as far or further out on any branch through the position γ in question.

It remains to deal similarly with α_n^j for $j = 1, \cdots, r$. First $\alpha_0^j(v^{j-1}) = \alpha^j(v^{j-1})$. Furthermore $\alpha_{n+1}^j(v) = \alpha_n^j(v)$, except in the case $(z_n)_0 = 6$ & $(z_n)_2 = j$, when (writing $k = (z_n)_3$)

$$\alpha_{n+1}^j(v) = 2^{(\alpha_n^j(v))_k} \cdot \prod_{i<k} p_{i+1}^{(\alpha_n^j(v))_i} \cdot \prod_{k<i<\alpha_n^j(v)} p_i^{(\alpha_n^j(v))_i},$$

and in the case $(z_n)_0 = 8$ & $(z_n)_2 = j+2$, when

$$\alpha_{n+1}^j(v) = 2^{\sigma^j(v)} \cdot \prod_{i<\alpha_n^j(v)} p_{i+1}^{(\alpha_n^j(v))_i}$$

where $\sigma^j(v) = \text{pm}_j^{r-2}(\lambda\tau\, \rho(n, \tau), v)$ (cf. X). Using (17) and (19), thus

$$\begin{aligned} \alpha_0^j(v) &= \alpha^j(v), \\ \alpha_{n+1}^j(v) &= \chi_j(n, z, a, \alpha_n^j(v), \lambda\tau\, \bar{\rho}(m;\tau), \eta, v) \qquad (m > n) \end{aligned} \tag{21}$$

with a primitive recursive χ_j. Continuing from (21) as before from (18), we obtain primitive recursive β_j's such that

$$\overset{j}{\alpha}_n(v) = \beta_j(n, z, a, \overset{j}{\alpha}(v), \lambda\tau\, \bar{\rho}(m;\tau), \eta, v) \qquad (m \geqq n), \tag{22}$$

$$\overset{j}{\alpha}_n(v) = \beta_j(z, a, \overset{j}{\alpha}(v), \lambda\tau\, \bar{\rho}(n;\tau), \eta, v). \tag{23}$$

5.5. A type-$r-2$ object δ^{r-2} will describe a position in the tree under 5.3 exactly if (a) δ^{r-2} is of the form $\lambda\tau\, \bar{\rho}(m;\tau)$ for some number m and function ρ and (b) the value of $\lambda\tau\, \rho(n,\tau)$ obeys certain restrictions for $n=0, \cdots, m-1$. Now (a) is expressed by[3]

$$(24)\quad \begin{array}{ll} (\tau)[\mathrm{Seq}(\delta^{r-2}(\tau))\ \&\ \mathrm{lh}(\delta^{r-2}(\tau)) = \mathrm{lh}(\delta^{r-2}(\lambda\sigma^{r-4}\, 0))] & (r > 2), \\ \mathrm{Seq}(\delta) & (r = 2). \end{array}$$

As to (b), suppose (a) is satisfied and the restrictions are obeyed for $0, \cdots, n-1$ where $n < m$. Then $\gamma^{r-2} = \lambda\tau\, \bar{\rho}(n;\tau)$ is a position, at which the z_n and a_n are given by (19) with (17) from z, a, η, n and $\delta^{r-2} = \lambda\tau\, \bar{\rho}(m;\tau)$. The restrictions for n are expressed by

$$(25)\quad \begin{array}{ll} (\tau)[\rho(n,\tau)=0] \vee \{[(z_n)_0=4 \vee ((z_n)_0=5\ \&\ (a_n)_0>0)]\ \&\ (\tau)[\rho(n,\tau)=1]\} & \\ \quad \vee \{(z_n)_0=8\ \&\ (z_n)_2=2\ \&\ (\tau)[\rho(n,\tau)=\mathrm{pm}_0^{r-2}(\lambda\tau\, \rho(n,\tau))]\} & \\ \quad \vee (Ej)_{3\leq j\leq r}\{(z_n)_0=8\ \&\ (z_n)_2=j & \\ \quad \&\ (\tau)[\rho(n,\tau)=\mathrm{mp}_{j-2}^{r-2}(\lambda \overset{j-3}{v}\ \mathrm{pm}_{j-2}^{r-2}(\lambda\tau\, \rho(n,\tau), \overset{j-3}{v}), \tau)]\} & (r > 2), \\ \rho(n)=0 \vee \{[(z_n)_0=4 \vee ((z_n)_0=5\ \&\ (a_n)_0>0)]\ \&\ \rho(n)=1\} & \\ \quad \vee \{(z_n)_0 = 8\ \&\ (z_n)_2 = 2\} & (r = 2). \end{array}$$

A formula expressing (b) is obtained by prefixing $(n)_{n<m}$ to (25), and replacing m by $\mathrm{lh}(\delta^{r-2}(\lambda\sigma^{r-4}\, 0))$, $\rho(n,\tau)$ by $(\delta^{r-2}(\tau))_n \dot{-} 1$, z_n by $(\beta(n, z, a, \delta^{r-2}, \eta))_0$ and a_n by $(\beta(n, z, a, \delta^{r-2}, \eta))_1$. Forming the conjunction of this with (24), and for $r > 2$ advancing the quantifiers (τ^{r-3}) by predicate calculus and contracting them by $(\overline{3})$, we obtain a primitive recursive predicate P such that (omitting τ^{r-3} for $r = 2$)

$$\{\delta^{r-2} \text{ is a position}\} \equiv (\tau^{r-3})P(z, a, \delta^{r-2}, \eta^{r-1}, \tau^{r-3}). \tag{26}$$

5.6. The tree described in 5.3 for given $(z, a, \alpha^1, \cdots, \alpha^r)$ and η does correspond to the computation of $\{z\}[a, \alpha^1, \cdots, \alpha^r]$, if $\eta(\gamma)$ at each position γ is the value of $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r]$ there. (Only if $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ is defined are there such η's.) However we must study the tree for η's in general.

For any η, let us say η is *locally correct* at a given position γ, if $\eta(\gamma)$ has the right value in case γ corresponds to a schema application (of S1–S3, S7) that gives a value outright, and otherwise if $\eta(\gamma)$ and the numbers $\eta(\delta)$ for each of the one or more $n+1$-positions issuing from the n-position γ have the relationship required by the schema application to which γ corresponds.

We analyze this notion, assuming γ is a position. In the case of a node, only the relationship of $\eta(\gamma)$ to the $\eta(\delta)$ at the upper $n+1$-position is in question, since the relationship to the $\eta(\delta)$ at the lower $n+1$-position has already been taken care of in the tree construction by incorporating that $\eta(\delta)$ into the a_{n+1} at the upper $n+1$-position. The local correctness of η at γ is expressed by

$$\{(z_n)_0=1 \;\&\; \eta(\gamma)=(a_n)_0+1\} \vee \{(z_n)_0=2 \;\&\; \eta(\gamma)=(z_n)_2\} \vee \{(z_n)_0=3 \;\&\; \eta(\gamma)=(a_n)_0\}$$
$$\vee \{[(z_n)_0=4 \vee ((z_n)_0=5 \;\&\; (a_n)_0>0)] \;\&\; \eta(\gamma)=\eta(\lambda\tau\, \gamma(\tau) * 2^2)\}$$
$$\vee \{[((z_n)_0=5 \;\&\; (a_n)_0=0) \vee (z_n)_0=6 \vee (z_n)_0=9] \;\&\; \eta(\gamma)=\eta(\lambda\tau\, \gamma(\tau) * 2)\}$$
$$\vee \{(z_n)_0=7 \;\&\; \eta(\gamma)=(\overset{1}{\alpha_n}((a_n)_0))_0\} \vee (Ej)_{2\le j\le r}\{(z_n)_0=8 \;\&\; (z_n)_2=j \;\&$$
$$\eta(\gamma)=(\overset{j}{\alpha_n}(\lambda\overset{j-2}{\sigma}\; \eta(\lambda\tau\, \gamma(\tau) * [2 \exp 1+\mathrm{mp}_{j-2}^{r-2}(\overset{j-2}{\sigma}, \tau)])))_0\}.$$

Eliminating $z_n, a_n, \alpha_n^1, \cdots, \alpha_n^r$ by (17), (20) and (23) with $\gamma = \lambda\tau\, \bar{\rho}(n;\tau)$, we obtain a primitive recursive predicate C such that, for γ a position,

(27) $\{\eta^{r-1}$ is locally correct at $\gamma^{r-2}\} \equiv C(z, a, \alpha^1, \cdots, \alpha^r, \gamma^{r-2}, \eta^{r-1})$.

5.7. Consider, as we have been doing, the tree for $(z, a, \alpha^1, \cdots, \alpha^r)$ based on a given η, and at any position γ let $(z_n, a_n, \alpha_n^1, \cdots, \alpha_n^r)$ be the $r+2$-tuple which occupies it. *If* $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ *is defined, and* η *is locally correct at every position* γ, *then at every position* γ $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r]$ *is defined and* $= \eta(\gamma)$, *i.e.* η *gives the correct values.*

Proof. Assume $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ is defined. Then $\mathrm{Ix}(z) \;\&\; \mathrm{tp}(z) \leqq r$ by 5.1, and we can use induction on $\{z\}(-)$ (end 3.8). Assume η is locally correct. We give two of the cases for the induction.

CASE 1. $(z)_0 = 1$, i.e. z is an index $\langle 1, \langle n_0, \cdots, n_r\rangle\rangle$ for an application of S1. So $\{z\}[a, \alpha^1, \cdots, \alpha^r] = (a)_0+1$. By 5.3, the tree is unbranched with $(z_n, a_n, \alpha_n^1, \cdots, \alpha_n^r) = (z, a, \alpha^1, \cdots, \alpha^r)$ for every n. So at every position γ, $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r] = (a)_0+1 = \eta(\gamma)$, since η is locally correct at γ.

CASE 4. $(z)_0 = 4$, i.e. z is an index $\langle 4, \langle n_0, \cdots, n_r\rangle, g, h\rangle$ for an application of S4. So $\{z\}[a, \alpha^1, \cdots, \alpha^r] = \{z\}(\mathfrak{b}) = \{g\}(\{h\}(\mathfrak{b}), \mathfrak{b})$, $\{h\}[a, \alpha^1, \cdots, \alpha^r] = \{h\}(\mathfrak{b})$ and $\{g\}[a_1, \alpha^1, \cdots, \alpha^r] = \{g\}(\{h\}(\mathfrak{b}), \mathfrak{b})$, where $g = (z)_2$, $h = (z)_3$, $a_1 = 2^{\{h\}(\mathfrak{b})} \cdot \prod_{i<a} p_{i+1}^{(a)_i}$. Each of these three expressions is defined, since the first is[10]. In the tree for $(z, a, \alpha^1, \cdots, \alpha^r)$ (at the 0-position) and the given η, the next $r+2$-tuples are $(h, a, \alpha^1, \cdots, \alpha^r)$ at the lower 1-position, and at the upper 1-position $(g, a_1^*, \alpha^1, \cdots, \alpha^r)$ where $a_1^* = 2^{\eta(\gamma)} \cdot \prod_{i<a} p_{i+1}^{(a)_i}$ for $\gamma = \lambda\tau\, 2$. In the construction of any tree, for any $r+2$-tuple $(z_n, a_n, \alpha_n^1, \cdots, \alpha_n^r)$, the set of the next $r+2$-tuples (i.e. how many and what they are) is completely determined by $(z_n, a_n, \alpha_n^1, \cdots, \alpha_n^r)$, except in the case of a node when for the upper next $r+2$-tuple the value of η at the position γ of the lower next $r+2$-tuple is also used. It follows that the trees for $(h, a, \alpha^1, \cdots, \alpha^r)$ and $(g, a_1^*, \alpha^1, \cdots, \alpha^r)$ are exactly the lower and upper subtrees which remain from the tree for $(z, a, \alpha^1, \cdots, \alpha^r)$ upon omitting the initial $r+2$-tuple of that, when the functions η_0 and η_1 used for the subtrees are respectively the functions which correlate to each position the

number correlated to the corresponding position of the whole tree, i.e. $\eta_0(\gamma) = \eta(\lambda\tau\, 2*\gamma(\tau))$ and $\eta_1(\gamma) = \eta(\lambda\tau\, 2^2*\gamma(\tau))$. So the local correctness of η throughout the whole tree implies that of η_0 and η_1 throughout the respective subtrees. Applying the hypothesis of the induction to the lower subtree, at each γ $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r]$ is defined and $= \eta_0(\gamma)$, i.e. η_0 gives the correct values throughout the lower subtree. Hence in particular, $\eta_0(\lambda\tau\, 1) = \{h\}(\mathfrak{b}) = \eta(\lambda\tau\, 2)$; so $a_1^* = a_1$. So $\{g\}[a_1^*, \alpha^1, \cdots, \alpha^r]$ is defined, since it is $\{g\}[a_1, \alpha^1, \cdots, \alpha^r]$; and by applying the hyp. ind. to the upper subtree, we can conclude that at each position γ $\{z_n\}[a_n, \alpha^1, \cdots, \alpha^r]$ is defined and $= \eta_1(\gamma)$, i.e. η_1 gives correct values throughout that. These correct values under η_0 and η_1 in the two subtrees become correct values under η at the corresponding positions of the whole tree. Finally, by the local correctness of η at the 0-position $\lambda\tau\, 1$ of the whole tree, the value under η is correct also there.

5.8. *If $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ is defined, then there is a function η such that, in the tree for $(z, a, \alpha^1, \cdots, \alpha^r)$ based on η, the function η is locally correct at every position γ.*

The proof is similar to that in 5.7, using the result of 5.7 to identify a_1^* and a_1 in Case 4 (and similarly in Case 5 for $(a)_0 > 0$).

5.9. To complete the proof of XXVI, assume that $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ is defined. Then by 5.8 there is an η which is locally correct throughout the tree for $(z, a, \alpha^1, \cdots, \alpha^r)$ based on that η. By 5.7, for each such η, the values are defined and given correctly throughout the tree, in particular at the 0-position, so $\eta(\lambda\tau\, 1) = \{z\}[a, \alpha^1, \cdots, \alpha^r]$. Thus

$$\begin{aligned} &\{z\}[a, \alpha^1, \cdots, \alpha^r] = w \\ (28) \quad &\equiv (\eta)\{(\gamma)[(\gamma \text{ is a position}) \rightarrow (\eta \text{ is locally correct at } \gamma)] \rightarrow \eta(\lambda\tau\, 1) = w\} \\ &\equiv (E\eta)\{(\gamma)[(\gamma \text{ is a position}) \rightarrow (\eta \text{ is locally correct at } \gamma)] \;\&\; \eta(\lambda\tau\, 1) = w\}. \end{aligned}$$

Using (26) and (27) in the two expressions at the right (omitting τ^{r-3} for $r=2$), advancing the quantifiers, and for $r > 2$ simplifying the resulting prefixes $(\eta^{r-1})(E\gamma^{r-2})(\tau^{r-3})$ and $(E\eta^{r-1})(\gamma^{r-2})(E\tau^{r-3})$ by use of $(\bar{8})$, $(\bar{4})$, $(\bar{3})$ and (8), (4), (3), we obtain the forms in (14) and (15).

5.10. XXVII. *For each $r \geqq 2$ there is a primitive recursive predicate K, and for each $r > 2$ a primitive recursive predicate L, such that*

$$(29) \quad (\{z\}[a, \alpha^1, \cdots, \alpha^r] \text{ is defined}) \equiv (\beta^{r-1})(E\xi^{r-2})K(z, a, \alpha^1, \cdots, \alpha^r, \beta^{r-1}, \xi^{r-2}) \qquad (r \geqq 2),$$

$$(30) \quad (\{z\}[a, \alpha^1, \cdots, \alpha^r] \text{ is defined}) \equiv (E\beta^{r-1})(\xi^{r-2})L(z, a, \alpha^1, \cdots, \alpha^r, \beta^{r-1}, \xi^{r-2}) \qquad (r > 2).$$

The proof, continuing from 5.3–5.8, occupies 5.11–5.18. We assume $r \geqq 2$, and for the discussion in 5.11–5.12 $\text{Ix}(z)$ & $\text{tp}(z) \leqq r$. (Cf. LI below.)

5.11. Let us say γ^{r-2} is *below* a position δ^{r-2}, if δ^{r-2} is the upper $n+1$-position issuing from a node (the node being an n-position), and γ^{r-2} is an m-position with $m \geqq n+1$ on a branch through the lower $n+1$-position issuing from the node.

We analyze this notion, assuming δ a position. First, for γ to be below δ, the position δ must be an $n+1$-position, i.e.

$$\text{(31)} \qquad \mathrm{lh}(\delta(\lambda\sigma^{r-4}\, 0)) > 0.$$

In this case, $n = \mathrm{lh}(\delta(\lambda\sigma^{r-4}\, 0)) \dot{-} 1$, and the z_n, a_n at the n-position are given from $\delta = \lambda\tau\, \bar{p}(n+1; \tau)$ by (17) and (19) with $m = n+1$. Next, this n-position must be a node with δ the upper $n+1$-position, i.e.

$$\text{(32)} \qquad [(z_n)_0=4 \vee ((z_n)_0=5 \;\&\; (a_n)_0>0)] \;\&\; \mathrm{pm}_0^{r-2}(\lambda\tau\ (\delta(\tau))_n \dot{-} 1)=1.$$

In this case the lower $n+1$-position is $\lambda\tau\ [\delta(\tau)/p_n]$. Finally, γ must be a position at least as far out as $\lambda\tau\ [\delta(\tau)/p_n]$ on a branch through the latter, i.e. (using (26))(3)

$$\text{(33)} \qquad (\tau)P(z, a, \gamma, \eta, \tau) \;\&\; (\tau)\mathrm{Ext}(\gamma(\tau), [\delta(\tau)/p_n]).$$

Forming the conjunction of (31)–(33), eliminating z_n, a_n and n as indicated, and for $r > 2$ advancing and contracting the quantifiers, we obtain a primitive recursive predicate B such that, for δ a position (and omitting τ^{r-3} for $r = 2$),

$$\text{(34)} \qquad \{\gamma^{r-2} \text{ is below } \delta^{r-2}\} \equiv (\tau^{r-3})B(z, a, \gamma^{r-2}, \delta^{r-2}, \eta^{r-1}, \tau^{r-3}).$$

We say γ is *below* a branch of the tree, if γ is below some position on the branch.

5.12. The computation *terminates* at a given position $\gamma = \lambda\tau\, \bar{p}(n; \tau)$ if S1–S3 or S7 applies there, i.e. if $(z_n)_0=1 \vee (z_n)_0=2 \vee (z_n)_0=3 \vee (z_n)_0=7$. Using (20) with (17), we obtain a primitive recursive T such that, for ρ a branch,

$$\text{(35)} \quad \{\text{the computation terminates at the } n\text{-position on } \rho\} \equiv T(z, a, n, \rho, \eta^{r-1}).$$

5.13. Let $D(z, a, \alpha^1, \cdots, \alpha^r) \equiv \{\mathrm{Ix}(z) \;\&\; \mathrm{tp}(z) \leqq r$, and for every η, in the tree constructed for $(z, a, \alpha^1, \cdots, \alpha^r)$ on the basis of η, the computation terminates along each branch ρ below which η is locally correct$\}$.

To analyze this, note that "ρ is a branch" is expressed using (26) by $(n)(\tau^{r-3})P(z, a, \lambda\tau\, \bar{p}(n; \tau), \eta^{r-1}, \tau^{r-3})$. Using also (34), (27) and (35),

$$\text{(36)} \quad \begin{aligned} &D(z, a, \alpha^1, \cdots, \alpha^r) \equiv \mathrm{Ix}(z) \;\&\; \mathrm{tp}(z) \leqq r \;\&\; \\ &(\eta^{r-1})(\rho)\{(n)(\tau^{r-3})P(z, a, \lambda\tau\, \bar{p}(n; \tau), \eta^{r-1}, \tau^{r-3}) \\ &\qquad \& \; (n)(\gamma^{r-2})[(\tau^{r-3})B(z, a, \gamma^{r-2}, \lambda\tau\, \bar{p}(n; \tau), \eta^{r-1}, \tau^{r-3}) \\ &\qquad \rightarrow C(z, a, \alpha^1, \cdots, \alpha^r, \gamma^{r-2}, \eta^{r-1})] \rightarrow (En)T(z, a, n, \rho, \eta^{r-1})\}. \end{aligned}$$

For $r > 2$, and any two-place function $\lambda n\tau^{r-3}\rho(n, \tau^{r-3})$, let $\rho^{r-2}(\sigma^{r-3}) = \rho((\sigma^{r-3})_0^0, (\sigma^{r-3})_1)$ (cf. 2.4); then by (9), $\rho(n, \tau^{r-3}) = \rho^{r-2}(\langle n, \tau^{r-3}\rangle)$. Using this for $r > 2$ to replace the quantification of the two-place ρ by quantification of ρ^{r-2} (for $r = 2$, ρ is a one-place function ρ^1), then advancing and contracting quantifiers on the right (cf. 2.5), we obtain a primitive recursive predicate K such that

$$(37)\quad D(z, a, \alpha^1, \cdots, \alpha^r) \equiv (\beta^{r-1})(E\xi^{r-2})K(z, a, \alpha^1, \cdots, \alpha^r, \beta^{r-1}, \xi^{r-2}).$$

5.14. *If* $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ *is defined, then* $D(z, a, \alpha^1, \cdots, \alpha^r)$.

Proof. Assume $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ is defined. Then $\mathrm{Ix}(z)$ & $\mathrm{tp}(z) \leqq r$, and we use induction on $\{z\}(-)$ to show that the other conjunctive member of $D(z, a, \alpha^1, \cdots, \alpha^r)$ holds.

Case 1. $(z)_0 = 1$. Then $(z_0)_0 = 1$, which gives $T(z, a, 0, \rho, \eta)$ and thence $(En)T(z, a, n, \rho, \eta)$, for any η and the only branch ρ.

Case 4. $(z)_0 = 4$. Then the tree for $(z, a, \alpha^1, \cdots, \alpha^r)$ based on any given η begins with a node, with $(h, a, \alpha^1, \cdots, \alpha^r)$ at the lower 1-position and $(g, a_1^*, \alpha^1, \cdots, \alpha^r)$ at the upper (cf. Case 4 in 5.7). Consider any given branch ρ below which the η is locally correct.

Subcase 0. The 1-position on ρ is the lower 1-position issuing from the node, i.e. is occupied by $(h, a, \alpha^1, \cdots, \alpha^r)$. Then the branch ρ minus its 0-position corresponds to a branch ρ_0 in the lower subtree, i.e. in tree for $(h, a, \alpha^1, \cdots, \alpha^r)$ based on η_0, below which η_0 is locally correct. So, since $\{h\}[a, \alpha^1, \cdots, \alpha^r]$ is defined, by the hyp. ind. the computation terminates along ρ_0 in the lower subtree, i.e. $(En)T(h, a, n, \rho_0, \eta_0)$, and hence along ρ in the whole tree (with n one greater), i.e. $(En)T(z, a, n, \rho, \eta)$.

Subcase 1. The 1-position on ρ is the upper one, i.e. is occupied by $(g, a_1^*, \alpha^1, \cdots, \alpha^r)$. Then the local correctness of η below ρ in the whole tree entails the local correctness of η_0 in the lower subtree. Hence by 5.7 η_0 gives the correct values in the lower subtree, so $a_1^* = a_1$. So $\{g\}[a_1^*, \alpha^1, \cdots, \alpha^r]$ is defined, and we can apply the hyp. ind. to $\{g\}[a_1^*, \alpha^1, \cdots, \alpha^r]$ (in the same manner as to $\{h\}(a, \alpha^1, \cdots, \alpha^r)$ in Subcase 0) to conclude that $(En)T(g, a_1^*, n, \rho_1, \eta_1)$ and hence $(En)T(z, a, n, \rho, \eta)$.

5.15. *If* $D(z, a, \alpha^1, \cdots, \alpha^r)$, *then* $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ *is defined.*

Proof. Assume $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ is undefined. To conclude that then $\bar{D}(z, a, \alpha^1, \cdots, \alpha^r)$, assume further that $\mathrm{Ix}(z)$ & $\mathrm{tp}(z) \leqq r$. We shall "construct" a function η, and a branch ρ of the tree for $(z, a, \alpha^1, \cdots, \alpha^r)$ based on η, such that at each n-position along ρ $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r]$ is undefined, and η is locally correct below ρ. This will contradict the second conjunctive member of $D(z, a, \alpha^1, \cdots, \alpha^r)$; for wherever $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r]$ is undefined, the computation is unterminated.

The construction will proceed by stages for $n = 0, 1, 2, \cdots$. At Stage n, the first $n+1$ $r+2$-tuples $(z_0, a_0, \alpha_0^1, \cdots, \alpha_0^r), \cdots, (z_n, a_n, \alpha_n^1, \cdots, \alpha_n^r)$ along ρ will have been picked, for each of these the function value $\{z_i\}$

$[a_i, \alpha_i^1, \cdots, \alpha_i^r]$ will be undefined, $\eta(\delta)$ will have been picked at all positions δ below the positions of the $n+1$ $r+2$-tuples already picked, η will be locally correct at each δ for which $\eta(\delta)$ has thus been picked, and the further $r+2$-tuples along ρ and values of η will not yet have been picked.

At Stage 0 we necessarily have $(z_0, a_0, \alpha_0^1, \cdots, \alpha_0^r) = (z, a, \alpha^1, \cdots, \alpha^r)$, the function value $\{z_0\}[a_0, \alpha_0^1, \cdots, \alpha_0^r]$ is undefined by our assumption that $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ is undefined, and no values of η have been picked.

Now we give some cases for the step from Stage n to Stage $n+1$.

CASE 1. $(z_n)_0 = 1$. This case can be excluded, since under it $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r]$ would be defined (with value $(a_n)_0+1$).

CASE 4. $(z_n)_0 = 4$. Then $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r] \simeq \{g\}(\{h\}(\mathfrak{b}), \mathfrak{b})$ (cf. Case 4 in 5.7).

SUBCASE 0. $\{h\}(\mathfrak{b})$ is undefined. Then we take its $r+2$-tuple $(h, a_n, \alpha_n^1, \cdots, \alpha_n^r)$ as $(z_{n+1}, a_{n+1}, \alpha_{n+1}^1, \cdots, \alpha_{n+1}^r)$ and pick no further values of η. Clearly the requirements are met.

SUBCASE 1. $\{h\}(\mathfrak{b})$ is defined. By 5.8, there is a function η_0 such that throughout the tree for $(h, a_n, \alpha_n^1, \cdots, \alpha_n^r)$ based on η_0 the function η_0 is locally correct. By 5.7, the values which this η_0 gives are correct; in particular, $\eta_0(\lambda\tau\, 1) = \{h\}(\mathfrak{b})$. So if as our $(z_{n+1}, a_{n+1}, \alpha_{n+1}^1, \cdots, \alpha_{n+1}^r)$ we take $(g, a_{n,1}, \alpha_n^1, \cdots, \alpha_n^r)$ where $a_{n,1} = 2^{\{h\}(\mathfrak{b})} \cdot \prod_{i<a_n} p_{i+1}^{(a_n)_i}$, and if at the same time we extend the selection of values of η by employing at positions below the $n+1$-position just filled the values given by η_0 at the corresponding positions of the tree for $(h, a_n, \alpha_n^1, \cdots, \alpha_n^r)$ based on η_0, then $(z_{n+1}, a_{n+1}, \alpha_{n+1}^1, \cdots, \alpha_{n+1}^r)$ will meet the condition for occupying an $n+1$-position based on η. Now $\{z_{n+1}\}[a_{n+1}, \alpha_{n+1}^1, \cdots, \alpha_{n+1}^r] \simeq \{g\}(\{h\}(\mathfrak{b}), \mathfrak{b}) \simeq \{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r]$, which is undefined; and η is locally correct at all points below the segment of ρ thus far chosen.

CASE 8.j. $(z_n)_0 = 8$ & $(z_n)_2 = j$. Then $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r] \simeq \{z_n\}((\alpha_n^j)_0, \mathfrak{b})$, $\simeq (\alpha_n^j(\lambda\sigma^{j-2}\chi((\alpha_n^j)_0, \sigma^{j-2}, \mathfrak{b})))_0$. Since $\{z_n\}[a_n, \alpha_n^1, \cdots, \alpha_n^r]$ is undefined, $\chi((\alpha_n^j)_0, \sigma^{j-2}, \mathfrak{b})$ is undefined for some σ^{j-2}. Choosing such a σ^{j-2}, we take as our $(z_{n+1}, a_{n+1}, \alpha_{n+1}^1, \cdots, \alpha_{n+1}^r)$ the $r+2$-tuple for $\chi((\alpha_n^j)_0, \sigma^{j-2}, \mathfrak{b})$, i.e. we make the choice described by $\lambda\tau\, \rho(n, \tau) = \lambda\tau\, \mathrm{mp}_{j-2}^{r-2}(\sigma^{j-2}, \tau)$, and do not choose further values of η.

When $(z_n, a_n, \alpha_n^1, \cdots, \alpha_n^r)$ has been thus picked for every n, the branch ρ will have been constructed, and the value of $\eta(\delta)$ will have been picked and will be locally correct, at exactly each position δ below ρ. We complete the construction of η by taking $\eta(\delta) = 0$ for all other δ.

5.16. Let $E(z, a, \alpha^1, \cdots, \alpha^r) \equiv \{\mathrm{Ix}(z)$ & $\mathrm{tp}(z) \leqq r$, and there is an η such that, in the tree constructed for $(z, a, \alpha^1, \cdots, \alpha^r)$ on the basis of η, the function η is locally correct at every position, and the computation terminates along every branch$\}$.

Using (26), (27) and (35) (omitting τ^{r-3} for $r = 2$),

$$E(z, a, \alpha^1, \cdots, \alpha^r) \equiv \mathrm{Ix}(z) \,\&\, \mathrm{tp}(z) \leq r \\ \&\, (E\eta^{r-1})\{(\gamma^{r-2})[(\tau^{r-3})P(z, a, \gamma^{r-2}, \eta^{r-1}, \tau^{r-3}) \\ \rightarrow C(z, a, \alpha^1, \cdots, \alpha^r, \gamma^{r-2}, \eta^{r-1})] \\ \&\, (\rho)[(n)(\tau^{r-3})P(z, a, \lambda\tau\, \bar{\rho}(n; \tau), \eta^{r-1}, \tau^{r-3}) \rightarrow (En)T(z, a, n, \rho, \eta^{r-1})]\}. \tag{38}$$

For $r > 2$, proceeding as in 5.13 we obtain a primitive recursive L such that

$$E(z, a, \alpha^1, \cdots, \alpha^r) \\ \equiv (E\beta^{r-1})(\xi^{r-2})L(z, a, \alpha^1, \cdots, \alpha^r, \beta^{r-1}, \xi^{r-2}) \qquad (r > 2) \tag{39}$$

For $r = 2$, the quantifier (ρ) is of type $1 = r-1$ instead of (after contraction) $r-2$, so $(\gamma^{r-2})(\rho)$ can only be contracted into a (ξ^{r-1}). A counterexample to (39), (30) and (41) for $r = 2$ will be given in 8.9 below.

5.17. *If* $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ *is defined, then* $E(z, a, \alpha^1, \cdots, \alpha^r)$. For assume $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ is defined. Then by 5.8 there exists an η which is locally correct throughout the tree for $(z, a, \alpha^1, \cdots, \alpha^r)$ based on η. Also by 5.14, for any η, the computation terminates along every branch below which η is locally correct; and so, with this η, along every branch.

5.18. *If* $E(z, a, \alpha^1, \cdots, \alpha^r)$, *then* $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ *is defined.* For assume $E(z, a, \alpha^1, \cdots, \alpha^r)$, and pick such an η (cf. 5.16). Then, assuming $\{z\}[a, \alpha^1, \cdots, \alpha^r]$ is undefined, we reach a contradiction by constructing a branch ρ along which the computation does not terminate, as in 5.15 but using the η already picked here (everywhere locally correct) instead of constructing one by stages as there.

5.19. XXVIII. *For each* $r \geqq 2$ *there is a primitive recursive predicate* M, *and for each* $r > 2$ *a primitive recursive predicate* N, *such that*

$$\{z\}[a, \alpha^1, \cdots, \alpha^r] \simeq w \\ \equiv (\beta^{r-1})(E\xi^{r-2})M(z, a, \alpha^1, \cdots, \alpha^r, w, \beta^{r-1}, \xi^{r-2}) \qquad (r \geqq 2), \tag{40}$$

$$\{z\}[a, \alpha^1, \cdots, \alpha^r] \simeq w \\ \equiv (E\beta^{r-1})(\xi^{r-2})N(z, a, \alpha^1, \cdots, \alpha^r, w, \beta^{r-1}, \xi^{r-2}) \qquad (r > 2). \tag{41}$$

Proof of (40). Using (29) and (14),

$$\{z\}[a, \alpha^1, \cdots, \alpha^r] \simeq w \equiv (\beta^{r-1})(E\xi^{r-2})K(z, a, \alpha^1, \cdots, \alpha^r, \beta^{r-1}, \xi^{r-2}) \\ \&\, (\beta^{r-1})(E\xi^{r-2})I(z, a, \alpha^1, \cdots, \alpha^r, w, \beta^{r-1}, \xi^{r-2}). \tag{42}$$

5.20. To get results for $r = 0, 1$ similar to those above for $r \geqq 2$, and analogous to known theorems in the former theory of partial and general recursive functions for $r = 0, 1$, various known techniques are available (e.g. IM, p. 322 (D1) or (D2), or [11, §8] with [15, §18 and p. 424]). We elect here to adapt the foregoing treatment.

5.21. XXIX. *There are primitive recursive predicates* $T(s, z, a)$ *and* $T'(z, a, y)$ *such that*

(43) $(\{z\}[a, \alpha] \text{ is defined}) \equiv (Ey)T(\bar{\alpha}(y), z, a)$ ($r = 1$ case),

(44) $(\{z\}[a] \text{ is defined}) \equiv (Ey)T'(z, a, y)$ ($r = 0$ case).

Proof, for $r = 1$. Assume $\{z\}[a, \alpha]$ defined. Then by 5.17, $E(z, a, \alpha)$ (cf. 5.16). Since $\text{tp}(z) \leqq r < 2$, S8 cannot be used (cf. I, 3.10); so in the tree for an η given by $E(z, a, \alpha)$ the only branching will be at applications of S4 or the second equation of S5 (i.e. at nodes) with two $n+1$-positions issuing from the n-position. Hence by Brouwer's fan theorem ([1, Theorem 2; 2, Theorem 2; 3], König [20]), there is an $m > 0$ such that along each branch the computation terminates at an n-position with $n < m$. Now we confine our attention to what remains from the tree when all its branches are pruned off beginning with their m-positions; call it the m-tree. In the m-tree, the positions are described by numbers $g = \bar{\rho}(n) < \prod_{i<m} p_i^2$. Let $y = \bar{\eta}(u)$ with u chosen so that $u = \text{lh}(y) \geqq \prod_{i<m} p_i^2$ (then $y > u > m$). For operations that involve only values of η only at positions in the m-tree, η is replaceable by $\lambda t\, (y)_t \dot{-} 1$.

In particular, adapting 5.5 (with 5.4), we have

(45)
$$\{d \text{ is a position in the } m\text{-tree}\} \equiv \text{Seq}(d) \;\&\; \text{lh}(d) < m$$
$$\& \; (n)_{n<\text{lh}(d)}[(d)_n = 1 \vee \{[(z_n)_0 = 4 \vee ((z_n)_0 = 5 \;\&\; (a_n)_0 > 0)] \;\&\; (d)_n = 2\}]$$

where $z_n = (\beta(n, z, a, d, \lambda t\, (y)_t \dot{-} 1))_0$, $a_n = (\beta(n, z, a, d, \lambda t\, (y)_t \dot{-} 1))_1$, which is of the form $P(z, a, d, m, y)$ with a primitive recursive P.

To adapt 5.6 (with 5.4), let us further assume for the choice of u that $y = \bar{\eta}(u) > a_n$ at each position g in the m-tree ($n = \text{lh}(g)$). This condition is expressed, writing $\text{M} = \prod_{i<m} p_i^2$, by

(46) $$(g)_{g<\text{M}}[P(z, a, g, m, y) \rightarrow y > (\beta(z, a, g, \lambda t\, (y)_t \dot{-} 1))_1],$$

which is of the form $G(z, a, m, y)$ with a primitive recursive G. Also $\alpha_n(v) = \beta_1(z, a, (\bar{\alpha}(y))_v \dot{-} 1, g, \lambda t\, (y)_t \dot{-} 1, v)$ for $v < y$. Now for g a position in the m-tree,

(47) $\{\eta$ is locally correct at g, or g is an $m-1$-position at which the computation is unterminated$\} \equiv C(z, a, \bar{\alpha}(y), g, y)$

with a primitive recursive C.

The part of a branch ρ of the whole tree that belongs to the m-tree can be represented by its $m-1$-position d. Thus branches of the m-tree are represented by the numbers d such that $P(z, a, d, m, y) \;\&\; \text{lh}(d) = m \dot{-} 1$. Adapting 5.12, for such d, and $n < m$,

(48) $\{$the computation terminates at the n-position on $d\} \equiv T(z, a, n, d, y)$

with a primitive recursive T.

Combining these remarks,

$$(\{z\}[a, \alpha] \text{ is defined}) \rightarrow (Ey)\{\mathrm{Ix}(z) \ \& \ \mathrm{tp}(z) \leqq 1 \ \& \ (Em)_{0<m<y}\{\mathrm{Seq}(y) \ \& \ \mathrm{lh}(y) \geqq \prod_{i<m} p_i^2 \ \& \ G(z, a, m, y) \ \& \ (g)_{g<y}[P(z, a, g, m, y) \rightarrow C(z, a, \bar{\alpha}(y), g, y)] \ \& \ (d)_{d<y}[P(z, a, d, m, y) \ \& \ \mathrm{lh}(d) = m \dot{-} 1 \rightarrow (En)_{n<m}T(z, a, n, d, y)]\}\}. \tag{49}$$

Upon replacing y except in $\bar{\alpha}(y)$ by $\mathrm{lh}(\bar{\alpha}(y))$, the scope of (Ey) in (49) assumes the form $T(\bar{\alpha}(y), z, a)$ with a primitive recursive T. Thus

$$(\{z\}[a, \alpha] \text{ is defined}) \rightarrow (Ey)T(\bar{\alpha}(y), z, a). \tag{50}$$

Conversely, assume $(Ey)T(\bar{\alpha}(y), z, a)$, and pick such a y. Then y is of the form $\bar{\eta}(u)$ for some function η, with $u = \mathrm{lh}(y) \geqq \prod_{i<m} p_i^2$ for some $m > 0$, and $G(z, a, m, y)$. The rest of $T(\bar{\alpha}(y), z, a)$ then gives everything stated in $E(z, a, \alpha)$ for the tree based on this η, except the local correctness at n-positions for $n \geqq m$, since the computation terminates along every branch at an n-position with $n < m$. We can by altering values of η at most at n-positions with $n \geqq m$ obtain the local correctness there also (without spoiling it for $n < m$, or altering the m-tree or the termination). Then 5.18 applies. Thus

$$(Ey)T(\bar{\alpha}(y), z, a) \rightarrow (\{z\}[a, \alpha] \text{ is defined}). \tag{51}$$

5.22. XXX. *There is a primitive recursive function* $U(y)$ *such that*

$$\{z\}[a, \alpha] \simeq U(\mu y T(\bar{\alpha}(y), z, a)), \tag{52}$$

$$T(\bar{\alpha}(y), z, a) \rightarrow U(y) = \{z\}[a, \alpha] \qquad (r = 1 \text{ case}), \tag{53}$$

$$\{z\}[a] \simeq U(\mu y T'(z, a, y)), \tag{54}$$

$$T'(z, a, y) \rightarrow U(y) = \{z\}[a] \qquad (r = 0 \text{ case}). \tag{55}$$

Proof, for $r = 1$. By (43) it will suffice for (52) to pick U to satisfy (53). So assume $T(\bar{\alpha}(y), z, a)$ (then $\{z\}[a, \alpha]$ is defined). By the proof of (51), $y = \bar{\eta}(u)$ for some η, u and m with $u > m > 0$; and, by altering values of η at most for some arguments $\bar{\rho}(n)$ with $n \geqq m$, η becomes locally correct everywhere in the tree for (z, a, α) based on it. So by 5.7, $\{z\}[a, \alpha] = \eta(1) = (\bar{\eta}(u))_1 \dot{-} 1 = (y)_1 \dot{-} 1$ (since $\eta(1)$ is unaltered). So take $U(y) = (y)_1 \dot{-} 1$.

5.23. XXXI. *Each function* $\phi(\mathfrak{a})$ *of variables* $\mathfrak{a}$ *of types* $\leqq 1$ *partial* (*general*) *recursive in the present sense is partial* (*general*) *recursive in the former sense* (e.g. IM, Chapters XI, XII).

Proof. By (52) or (54) with (13) and IM, Theorem XVIII, p. 330.

5.24. For each $n \geqq 1$, putting $\alpha = \langle \alpha_1, \cdots, \alpha_n \rangle$,

$$\bar{\alpha}(x) = \prod_{i<\mathrm{lh}(\bar{\alpha}_1(x))} p_i \exp 1 + \langle (\bar{\alpha}_1(x))_i \dot{-} 1, \cdots, (\bar{\alpha}_n(x))_i \dot{-} 1 \rangle. \tag{56}$$

This enables us by substitution into XXX to obtain the usual forms of the normal form theorem (IM, p. 292, [14, Footnote 2]) but now with indices.

For example, let

$$T_2^{1,1}(u, v, z, a, b) = T\left(\prod_{i<\mathrm{lh}(u)} p_i \exp 1+\langle (u)_i \dot{-} 1, (v)_i \dot{-} 1\rangle, z, \langle a, b\rangle\right).$$

XXXII. *If $\phi(a, b, \alpha, \beta)$ is partial recursive with index z,*

(57) $$\phi(a, b, \alpha, \beta) \simeq U(\mu y T_2^{1,1}(\bar{\alpha}(y), \bar{\beta}(y), z, a, b)),$$

(58) $$T_2^{1,1}(\bar{\alpha}(y), \bar{\beta}(y), z, a, b) \rightarrow U(y) = \phi(a, b, \alpha, \beta).$$

5.25. To get the enumeration theorem under type-0 quantification (IM, Theorems IV, IV*, pp. 281, 292) but now with indices, take e.g. a general recursive predicate $R(a, b, \alpha, \beta, y)$; let its representing function be χ with index h, so

$$R(a, b, \alpha, \beta, y) \equiv \chi(a, b, \alpha, \beta, y)=0 \equiv \{h\}(a, b, \alpha, \beta, y)=0.$$

By S9 and XVI, $\lambda zab\alpha\beta\, \mu y[\{z\}(a, b, \alpha, \beta, y)=0]$ is a partial recursive function, say with index e. By XIII, the partial recursive function $\lambda ab\alpha\beta\, \mu y[\{h\}(a, b, \alpha, \beta, y)=0]$, i.e. $\lambda ab\alpha\beta\, \mu y R(a, b, \alpha, \beta, y)$, has the index $f = S^1(e, h)$. In the following equivalence (59), the left side expresses the condition of definition of this function for particular a, b, α, β as given directly by its definition by the μ-operator, and the right the condition as given by (57) (or (43) via (13) and (56)) from f being an index of it.

XXXIII. *To each general recursive predicate $R(a, b, \alpha, \beta, y)$, there are numbers f, g such that*

(59) $$(Ey)R(a, b, \alpha, \beta, y) \equiv (Ey)T_2^{1,1}(\bar{\alpha}(y), \bar{\beta}(y), f, a, b),$$

(60) $$(y)R(a, b, \alpha, \beta, y) \equiv (y)\overline{T}_2^{1,1}(\bar{\alpha}(y), \bar{\beta}(y), g, a, b).$$

5.26. Other results for the present T predicates with indices follow from the foregoing in the same manner as in IM for the T predicates with Gödel numbers.

6. **Reduction in type of a quantifier. 6.1.** The following theorem and its dual correspond for $r \geqq 2$ to [14, (7) and (8), p. 316] for $r = 1$. (Cf. XLII and XLIII below.)

XXXIV. *Suppose $r \geqq 2$. Let $\mathfrak{a}^r$ be variables of types $\leqq r$, $\mathfrak{b}^{r-2}$ variables of types $\leqq r-2$, $(Q\mathfrak{b}^{r-2})$ quantifiers on the variables $\mathfrak{b}^{r-2}$, and $P(\mathfrak{a}^r, \sigma^r, \mathfrak{b}^{r-2})$ a general recursive predicate. Then there is a primitive recursive predicate $R(\mathfrak{a}^r, \eta^{r-1}, \xi^{r-2})$ such that*

(61) $$(E\sigma^r)(Q\mathfrak{b}^{r-2})P(\mathfrak{a}^r, \sigma^r, \mathfrak{b}^{r-2}) \equiv (E\eta^{r-1})(\xi^{r-2})R(\mathfrak{a}^r, \eta^{r-1}, \xi^{r-2}).$$

Similarly if P is merely partial recursive, with (61) *holding for those values of $\mathfrak{a}^r$ such that $P(\mathfrak{a}^r, \sigma^r, \mathfrak{b}^{r-2})$ is defined for all σ^r, $\mathfrak{b}^{r-2}$.*

The proof occupies 6.2–6.9.

6.2. Assume for P merely partial recursive that the values of $\mathfrak{a}^r$ under consideration make $\lambda\sigma^r\mathfrak{b}^{r-2}\,P(\mathfrak{a}^r, \sigma^r, \mathfrak{b}^{r-2})$ completely defined. Say z_0 is an index of the representing function of P. Then $P(\mathfrak{a}^r, \sigma^r, \mathfrak{b}^{r-2}) \equiv \{z_0\}(\mathfrak{a}^r, \sigma^r, \mathfrak{b}^{r-2}) = 0$. Using this in the left side of (61), applying (13) and (15), advancing the quantifier $(E\beta^{r-1})$ and contracting adjacent quantifiers of like kind by (4), (3), $(\bar{3})$ with (8), $(\bar{8})$, writing $\mathfrak{a}^r = (\mathfrak{c}^{r-2}, \alpha_1^{r-1}, \cdots, \alpha_{m_{r-1}}^{r-1}, \alpha_1^r, \cdots, \alpha_{m_r-1}^r)$ where $\mathfrak{c}^{r-2}$ are of types $\leqq r-2$, $\alpha_0^{r-1} = \langle\alpha_1^{r-1}, \cdots, \alpha_{m_{r-1}}^{r-1}\rangle$ and $\alpha_0^r = \langle\alpha_1^r, \cdots, \alpha_{m_r-1}^r\rangle$, and using (2), we obtain

$$(62)\qquad (E\sigma^r)(Q\mathfrak{b}^{r-2})P(\mathfrak{a}^r, \sigma^r, \mathfrak{b}^{r-2}) \equiv (E\sigma^r)(\xi^{r-2})P(\mathfrak{c}^{r-2}, \alpha_0^{r-1}, \alpha_0^r, \sigma^r, \xi^{r-2})$$

with a primitive recursive P on the right. Thus the general case of XXXIV is reduced to the case of it for a primitive recursive predicate $P(\mathfrak{c}^{r-2}, \alpha^{r-1}, \alpha^r, \sigma^r, \xi^{r-2})$ of variables of the types shown.

6.3. Let z_1 be a primitive recursive index (cf. 4.1) of the representing function of such a $P(\mathfrak{c}^{r-2}, \alpha^{r-1}, \alpha^r, \sigma^r, \xi^{r-2})$. Then

$$(63)\qquad P(\mathfrak{c}^{r-2}, \alpha^{r-1}, \alpha^r, \sigma^r, \xi^{r-2}) \equiv \{z_1\}(\mathfrak{c}^{r-2}, \xi^{r-2}, \alpha^{r-1}, \alpha^r, \sigma^r) = 0.$$

By I, in computing $\{z_1\}(\mathfrak{c}^{r-2}, \xi^{r-2}, \alpha^{r-1}, \alpha^r, \sigma^r)$, applications of S8.$r$ with σ^r as the α^j of S8.r will always be to introduce $\sigma^r(\lambda\tau^{r-2}\,\{h\}(\tau^{r-2}, \mathfrak{d}^{r-2}, \alpha^{r-1}, \sigma^r, \alpha^r))$ with $\mathfrak{d}^{r-2}$ of types $\leqq r-2$; i.e. the $\alpha^j(\lambda\alpha^{j-2}\,\chi(\alpha^j, \alpha^{j-2}, \mathfrak{b}))$ will always be of this form with only $(h, \mathfrak{d}^{r-2})$ varying from one application to the next. The idea of the proof of XXXIV is that hence we can in the computation replace the type-r function σ^r by a type-$r-1$ function η^{r-1} chosen so that (cf. 2.4) $\eta^{r-1}(\langle h, \mathfrak{e}\rangle) = \sigma^r(\lambda\tau^{r-2}\,\{h\}(\tau^{r-2}, \mathfrak{d}, \alpha^{r-1}, \sigma^r, \alpha^r))$ when $\mathfrak{e} = (\epsilon^0, \cdots, \epsilon^{r-2})$ and $\mathfrak{d} = \mathfrak{e}^{(h)} = ((\epsilon^0)_0, \cdots, (\epsilon^0)_{n_0-1}, \cdots, (\epsilon^{r-2})_0, \cdots, (\epsilon^{r-2})_{n_{r-2}-1})$ for $n_j = (h)_{1,j} \dot{-} \overline{\text{sg}}(r-2-j)$. In particular, if $\mathfrak{d} = (\delta_1^0, \cdots, \delta_{n_0}^0, \cdots, \delta_1^{r-2}, \cdots, \delta_{n_{r-2}}^{r-2})$ for such n_j, then $\mathfrak{d}^* = (\delta^0, \cdots, \delta^{r-2})$ for $\delta^j = \langle\delta_1^j, \cdots, \delta_{n_j}^j\rangle$ is such an $\mathfrak{e}$, i.e. $\mathfrak{d}^{*(h)} = \mathfrak{d}$.

6.4. To carry out this idea, we introduce two primitive recursive functions $\zeta_0(z)$ and $\zeta_1(z)$ which will give indices of $\{z\}(\mathfrak{d}, \alpha^{r-1}, \alpha^r, \sigma^r)$ and $\{z\}(\mathfrak{d}, \alpha^{r-1}, \sigma^r, \alpha^r)$, respectively, as functions of $\mathfrak{d}, \alpha^{r-1}, \eta^{r-1}, \alpha^r$ (cf. 6.5).

The definitions of ζ_0 and ζ_1 are by simultaneous course-of-values recursion. Let $\zeta_i(z) = 0$ $(i = 0, 1)$, except for $\text{PRI}(z)$ & $\text{tp}(z) = r$ & $(z)_{1,r-1} = 1$ & $(z)_{1,r} = 2$, when the appropriate following case shall apply.

CASE 1. $(z)_0 = 1$. Let $\zeta_i(z) = \langle 1, p_{r-1}\cdot[(z)_1/p_r]\rangle$, so e.g., when z is an index by S1 of $\lambda a\mathfrak{d}_1\alpha^{r-1}\alpha^r\sigma^r\,a+1$, then $\zeta_0(z)$ is one of $\lambda a\mathfrak{d}_1\alpha^{r-1}\eta^{r-1}\alpha^r\,a+1$.

CASE 4. $(z)_0 = 4$. Let $\zeta_i(z) = \langle 4, p_{r-1}\cdot[(z)_1/p_r], \zeta_i((z)_2), \zeta_i((z)_3)\rangle$.

CASE 6.r. $(z)_0 = 6$ & $(z)_2 = r$. Let $\zeta_0(z) = \zeta_1((z)_4)$ and $\zeta_1(z) = \zeta_0((z)_4)$, if $(z)_3 = 1$; otherwise, $\zeta_i(z) = \zeta_i((z)_4)$.

CASE 8.r. $(z)_0 = 8$ & $(z)_2 = r$. Let $\zeta_0(z) = \langle 8, p_{r-1}\cdot[(z)_1/p_r], r, \zeta_0((z)_3)\rangle$. We want $\zeta_1(z)$ to be a primitive recursive index $\theta(z)$ of $\lambda\mathfrak{d}\alpha^{r-1}\eta^{r-1}\alpha^r\,\eta^{r-1}(\langle(z)_3, \mathfrak{d}^*\rangle)$, where $\mathfrak{d}$ is a list of variables such that z is an index for $\mathfrak{d}, \alpha^{r-1}, \sigma^r, \alpha^r$, and θ is primitive recursive. To construct this index, let e be a primitive recursive index of $\lambda c \epsilon^0 \cdots \epsilon^{r-2}\alpha^{r-1}\eta^{r-1}\alpha^r\, \eta^{r-1}(\langle c, \epsilon^0, \cdots, \epsilon^{r-2}\rangle)$. Suppose e.g. $r > 2$, and write $n_j = (z)_{1,j}$. We construct further indices as follows, using XIII, XXI, XXV, XXII.

$$\lambda \overset{0}{\epsilon} \cdots \overset{r-2}{\epsilon}\, \overset{r-2}{\delta_1} \cdots \overset{r-2}{\delta}_{n_{r-2}} \overset{r-1}{\alpha}\, \overset{r-1}{\eta}\, \overset{r}{\alpha}\, \overset{r-1}{\eta}(\langle (z)_3, \overset{0}{\epsilon}, \cdots, \overset{r-2}{\epsilon}\rangle):$$

$$\mathrm{A} = \iota(\overset{1}{S}(e, (z)_3), \overset{n_{r-2}}{p_{r-2}}).$$

$$\lambda \overset{0}{\epsilon} \cdots \overset{r-3}{\epsilon}\, \overset{r-2}{\delta_1} \cdots \overset{r-2}{\delta}_{n_{r-2}} \overset{r-1}{\alpha}\, \overset{r-1}{\eta}\, \overset{r}{\alpha}\, \overset{r-3}{\tau} \langle \overset{r-2}{\delta_1}(\overset{r-3}{\tau}), \cdots, \overset{r-2}{\delta}_{n_{r-2}}(\overset{r-3}{\tau})\rangle:$$

$$\mathrm{C} = \langle 6, (\mathrm{B})_1, r-3, 1, \mathrm{B}\rangle \text{ where } \mathrm{B} = \iota(\nu_{r-2}(n_{r-2}), p_0 \cdot \cdots \cdot p_{r-3} \cdot \overset{2}{p}_{r-1} \cdot p_r).$$

$$\lambda \overset{0}{\epsilon} \cdots \overset{r-3}{\epsilon}\, \overset{r-2}{\delta_1} \cdots \overset{r-2}{\delta}_{n_{r-2}} \overset{r-1}{\alpha}\, \overset{r-1}{\eta}\, \overset{r}{\alpha}\, \overset{r-1}{\eta}(\langle (z)_3, \overset{0}{\epsilon}, \cdots, \overset{r-3}{\epsilon}, \overset{r-2}{\delta}\rangle):$$

$$\mathrm{D} = \gamma_{r-2}(\mathrm{A}, \mathrm{C}, 0).$$

Here $\delta^{r-2} = \langle \delta_1^{r-2}, \cdots, \delta_{n_{r-2}}^{r-2}\rangle$ (end 6.3). Continuing in this manner to substitute successively $\delta^{r-3}, \cdots, \delta^1$ for $\epsilon^{r-3}, \cdots, \epsilon^1$, and, using S4, δ^0 for ϵ^0, we obtain an index $\theta(z)$ of $\lambda \mathfrak{b} \alpha^{r-1}\eta^{r-1}\alpha^r\, \eta^{r-1}(\langle (z)_3, \mathfrak{b}^*\rangle)$ with θ primitive recursive. But $\theta(z)$ is a primitive recursive index, since e, $\nu_{r-2}(n_{r-2})$ are primitive recursive indices, and the operations used produce primitive recursive indices when applied to primitive recursive indices.

6.5. *Choose any* α^{r-1}, σ^r, α^r, *and let* η^{r-1} *be defined thence by*

$$\text{(A)} \quad \eta^{r-1}(\gamma^{r-2}) = \begin{cases} \sigma^r(\lambda\tau \{h\}(\tau, \mathfrak{e}^{(h)}, \alpha^{r-1}, \sigma^r, \alpha^r)) \textit{ if } \gamma^{r-2} = \langle h, \mathfrak{e}\rangle \textit{ with} \\ h \textit{ a primitive recursive index for } (\tau, \mathfrak{e}^{(h)}, \alpha^{r-1}, \sigma^r, \alpha^r), \\ 0 \textit{ otherwise.} \end{cases}$$

If $\mathrm{PRI}(z)$ & $\mathrm{tp}(z) = r$ & $(z)_{1,r-1} = 1$ & $(z)_{1,r} = 2$, *then*

$$\{\zeta_0(z)\}(\mathfrak{b}, \alpha^{r-1}, \eta^{r-1}, \alpha^r) = \{z\}(\mathfrak{b}, \alpha^{r-1}, \alpha^r, \sigma^r), \tag{64}$$

$$\{\zeta_1(z)\}(\mathfrak{b}, \alpha^{r-1}, \eta^{r-1}, \alpha^r) = \{z\}(\mathfrak{b}, \alpha^{r-1}, \sigma^r, \alpha^r) \tag{65}$$

when $\mathfrak{b}$ *are variables such that* z *is an index for* $(\mathfrak{b}, \alpha^{r-1}, \sigma^r, \alpha^r)$.

Proof, by induction on z.

CASE 6.r for $\zeta_0(z)$ with $(z)_3 = 1$. By the theorem and case hypotheses, $\{z\}(\mathfrak{b}, \alpha^{r-1}, \alpha^r, \sigma^r) = \{(z)_4\}(\mathfrak{b}, \alpha^{r-1}, \sigma^r, \alpha^r)$ where $\mathrm{PRI}((z)_4)$ & $\mathrm{tp}((z)_4) = r$ & $((z)_4)_{1,r-1} = 1$ & $((z)_4)_{1,r} = 2$. By hyp. ind., $\{\zeta_1((z))_4\}(\mathfrak{b}, \alpha^{r-1}, \eta^{r-1}, \alpha^r) = \{(z)_4\}(\mathfrak{b}, \alpha^{r-1}, \sigma^r, \alpha^r)$. Thence (64) follows by the definition $\zeta_0(z) = \zeta_1((z)_4)$.

CASE 8.r for $\zeta_1(z)$. We have $\{z\}(\mathfrak{b}, \alpha^{r-1}, \sigma^r, \alpha^r) = \sigma^r(\lambda\tau \{(z)_3\}(\tau, \mathfrak{b}, \alpha^{r-1}, \sigma^r, \alpha^r))$ with $(z)_3$ a primitive recursive index for $(\tau, \mathfrak{b}, \alpha^{r-1}, \sigma^r, \alpha^r)$. So by (A), $\eta^{r-1}(\langle (z)_3, \mathfrak{b}^*\rangle) = \{z\}(\mathfrak{b}, \alpha^{r-1}, \sigma^r, \alpha^r)$; and by definition, $\{\zeta_1(z)\}(\mathfrak{b}, \alpha^{r-1}, \eta^{r-1}, \alpha^r) = \eta^{r-1}(\langle (z)_3, \mathfrak{b}^*\rangle)$.

6.6. Now we formulate a property $F(\alpha^{r-1}, \eta^{r-1}, \alpha^r)$ of a type-$r-1$ variable η^{r-1} which, when (A) holds (cf. 6.7), expresses that η^{r-1} takes the same value for any two arguments $\gamma_0^{r-2} = \langle h_0, \mathfrak{e}_0\rangle$ and $\gamma_1^{r-2} = \langle h_1, \mathfrak{e}_1\rangle$ which represent the same function of τ, i.e. such that $\lambda\tau \{h_0\}(\tau, \mathfrak{e}_0^{(h_0)}, \alpha^{r-1}, \sigma^r, \alpha^r) = \lambda\tau \{h_1\}$

$(\tau, \mathfrak{e}_1^{(h_1)}, \alpha^{r-1}, \sigma^r, \alpha^r)$. However we use (65) to state it without using σ^r. Thus let

$$\begin{aligned} F(\alpha^{r-1}, \eta^{r-1}, \alpha^r) \equiv\ & (h_0)(\mathfrak{e}_0)(h_1)(\mathfrak{e}_1)\{(i)_{i<2}[\mathrm{PRI}(h_i) \ \&\ \mathrm{tp}(h_i)=r \\ & \&\ (h_i)_{1,r-2}>0 \ \&\ (h_i)_{1,r-1}=1 \ \&\ (h_i)_{1,r}=2] \\ & \&\ (\overset{r-2}{\tau})(Ew)(i)_{i<2}[\{\zeta_1(h_i)\}(\overset{r-2}{\tau}, \overset{(h_i)}{\mathfrak{e}_i}, \overset{r-1}{\alpha}, \overset{r-1}{\eta}, \overset{r}{\alpha}) = w] \\ & \to \eta^{r-1}(\langle h_0, \mathfrak{e}_0\rangle) = \eta^{r-1}(\langle h_1, \mathfrak{e}_1\rangle)\}. \end{aligned}$$

But (omitting σ^{r-3} if $r=2$) for h_i an index for $(\tau, \mathfrak{e}^{(h_i)}, \alpha^{r-1}, \sigma^r, \alpha^r)$,

$$\begin{aligned} & \{\zeta_1(h_i)\}(\overset{r-2}{\tau}, \overset{(h_i)}{\mathfrak{e}_i}, \overset{r-1}{\alpha}, \overset{r-1}{\eta}, \overset{r}{\alpha}) = \{\zeta_1(h_i)\}[\overset{0}{\epsilon_i}, \cdots, \overset{r-3}{\epsilon_i}, \\ (66) \qquad & \lambda \overset{r-3}{\sigma} 2^{\tau^{r-2}(\sigma^{r-3})} \cdot \prod_{s<n_{r-2,i}} p_{s+1} \exp(\overset{r-2}{\epsilon_i}(\overset{r-3}{\sigma}))_s, \langle \overset{r-1}{\alpha}, \overset{r-1}{\eta}\rangle, \langle \overset{r}{\alpha}\rangle]. \end{aligned}$$

Using (66) and (14) in the expression for $F(\alpha^{r-1}, \eta^{r-1}, \alpha^r)$, and advancing and contracting quantifiers, we obtain a primitive recursive G such that

$$(67) \qquad F(\alpha^{r-1}, \eta^{r-1}, \alpha^r) \equiv (E\beta^{r-1})(\xi^{r-2})G(\alpha^{r-1}, \eta^{r-1}, \alpha^r, \beta^{r-1}, \xi^{r-2}).$$

6.7. *For any given $\alpha^{r-1}, \sigma^r, \alpha^r$, if η^{r-1} is defined by* (A), *then* $F(\alpha^{r-1}, \eta^{r-1}, \alpha^r)$. For suppose the antecedent of the implication in $F(\alpha^{r-1}, \eta^{r-1}, \alpha^r)$ is satisfied by given $h_0, \mathfrak{e}_0, h_1, \mathfrak{e}_1$. By the first part of this antecedent with (65), then $\{\zeta_1(h_i)\}(\tau, \mathfrak{e}_i^{(h_i)}, \alpha^{r-1}, \eta^{r-1}, \alpha^r) = \{h_i\}(\tau, \mathfrak{e}_i^{(h_i)}, \alpha^{r-1}, \sigma^r, \alpha^r)$. This with the second part gives $\lambda\tau\ \{h_0\}(\tau, \mathfrak{e}_0^{(h_0)}, \alpha^{r-1}, \sigma^r, \alpha^r) = \lambda\tau\ \{h_1\}(\tau, \mathfrak{e}_1^{(h_1)}, \alpha^{r-1}, \sigma^r, \alpha^r)$. Hence by (A), $\eta^{r-1}(\langle h_0, \mathfrak{e}_0\rangle) = \eta^{r-1}(\langle h_1, \mathfrak{e}_1\rangle)$.

6.8. *Choose any $\alpha^{r-1}, \eta^{r-1}, \alpha^r$ such that $F(\alpha^{r-1}, \eta^{r-1}, \alpha^r)$, and let σ^r be defined thence by*

$$(\mathrm{B}) \qquad \sigma^r(\gamma^{r-1}) = \begin{cases} \eta^{r-1}(\langle h, \mathfrak{d}^*\rangle) \text{ if } \gamma^{r-1} = \lambda\tau\ \{\zeta_1(h)\}(\tau, \mathfrak{d}, \alpha^{r-1}, \eta^{r-1}, \alpha^r) \text{ with} \\ h \text{ a primitive recursive index for } (\tau, \mathfrak{d}, \alpha^{r-1}, \sigma^r, \alpha^r), \\ 0 \text{ otherwise.} \end{cases}$$

If $\mathrm{PRI}(z)$ & $\mathrm{tp}(z)=r$ & $(z)_{1,r-1}=1$ & $(z)_{1,r}=2$, *then* (64) *and* (65) *hold for variables* $\mathfrak{d}$ *such that z is an index for* $(\mathfrak{d}, \alpha^{r-1}, \sigma^r, \alpha^r)$. (By $F(\alpha^{r-1}, \eta^{r-1}, \alpha^r)$, the value of $\sigma^r(\gamma^{r-1})$ in the first case under (B) is independent of the choice of the h, $\mathfrak{d}$.)

Proof. As before (6.5), except for one case.

CASE 8.r for $\zeta_1(z)$. We have $\{z\}(\mathfrak{d}, \alpha^{r-1}, \sigma^r, \alpha^r) = \sigma^r(\lambda\tau\ \{(z)_3\}(\tau, \mathfrak{d}, \alpha^{r-1}, \sigma^r, \alpha^r))$ with $(z)_3$ a primitive recursive index for $(\tau, \mathfrak{d}, \alpha^{r-1}, \sigma^r, \alpha^r)$. By hyp. ind., $(\tau)[\{\zeta_1((z)_3)\}(\tau, \mathfrak{d}, \alpha^{r-1}, \eta^{r-1}, \alpha^r) = \{(z)_3\}(\tau, \mathfrak{d}, \alpha^{r-1}, \sigma^r, \alpha^r)]$. So by (B), $\sigma^r(\lambda\tau\ \{(z)_3\}(\tau, \mathfrak{d}, \alpha^{r-1}, \sigma^r, \alpha^r)) = \eta^{r-1}(\langle (z)_3, \mathfrak{d}^*\rangle)$; and by definition (6.4), $\{\zeta_1(z)\}(\mathfrak{d}, \alpha^{r-1}, \eta^{r-1}, \alpha^r) = \eta^{r-1}(\langle (z)_3, \mathfrak{d}^*\rangle)$.

6.9. Combining 6.7 and 6.5,

$$\begin{aligned} (68) \qquad & (\alpha^{r-1})(\sigma^r)(\alpha^r)(E\eta^{r-1})\{F(\alpha^{r-1}, \eta^{r-1}, \alpha^r) \ \&\ (z)\{\mathrm{PRI}(z) \ \&\ \mathrm{tp}(z) = r \ \&\ \\ & (z)_{1,r-1}=1 \ \&\ (z)_{1,r}=2 \to (\mathfrak{d})[\{\zeta_0(z)\}(\mathfrak{d}, \alpha^{r-1}, \eta^{r-1}, \alpha^r) = \{z\}(\mathfrak{d}, \alpha^{r-1}, \alpha^r, \sigma^r)]\}\} \end{aligned}$$

where, for each such z, the $\mathfrak{d}$ are variables such that z is an index for $(\mathfrak{d}, \alpha^{r-1}, \sigma^r, \alpha^r)$. By 6.8,

$$(69)\quad \begin{aligned}&(\alpha^{r-1})(\eta^{r-1})(\alpha^r)\{F(\alpha^{r-1}, \eta^{r-1}, \alpha^r) \rightarrow (E\sigma^r)(z)\{\text{PRI}(z) \,\&\, \text{tp}(z) = r \,\&\\ &(z)_{1,r-1}=1 \,\&\, (z)_{1,r}=2 \rightarrow (\mathfrak{d})[\{\zeta_0(z)\}(\mathfrak{d}, \alpha^{r-1}, \eta^{r-1}, \alpha^r) = \{z\}(\mathfrak{d}, \alpha^{r-1}, \alpha^r, \sigma^r)]\}\}.\end{aligned}$$

Applying (68) and (69) (cf. 6.3),

$$(70)\quad \begin{aligned}&(E\sigma^r)(\xi^{r-2})[\{z_1\}(\mathfrak{c}^{r-2}, \xi^{r-2}, \alpha^{r-1}, \alpha^r, \sigma^r)=0]\\ &\equiv (E\eta^{r-1})\{F(\alpha^{r-1}, \eta^{r-1}, \alpha^r) \,\&\, (\xi^{r-2})[\{\zeta_0(z_1)\}(\mathfrak{c}^{r-2}, \xi^{r-2}, \alpha^{r-1}, \eta^{r-1}, \alpha^r) = 0]\}.\end{aligned}$$

Thence, using (63) and (67), and advancing and contracting quantifiers,

$$(71)\quad \begin{aligned}&(E\sigma^r)(\xi^{r-2})P(\mathfrak{c}^{r-2}, \alpha^{r-1}, \alpha^r, \sigma^r, \xi^{r-2})\\ &\qquad\qquad \equiv (E\eta^{r-1})(\xi^{r-2})R(\mathfrak{c}^{r-2}, \alpha^{r-1}, \alpha^r, \eta^{r-1}, \xi^{r-2})\end{aligned}$$

with a primitive recursive R.

7. **Predicates of order r. 7.1.** Consider a predicate $P(\mathfrak{a})$, where $\mathfrak{a}$ is a list of variables of our types 0, 1, 2, $\cdots$ (1.2). We call $P(\mathfrak{a})$ *r-expressible* in certain predicates and functions (the *primitives*), if there is a syntactically-constructed expression (an *r-expression*) for $P(\mathfrak{a})$ in terms of variables of our types, the primitives (only applied to arguments), and the symbols of the predicate calculus with quantification only of variables of types $< r$. If P is r-expressible in predicates general recursive in (completely defined) functions Ψ, we say P is of *order r* in Ψ. The notion extends to the case P is replaced by a function ϕ via the representing predicate of ϕ (IM, p. 199), and to the case any of Ψ are replaced by predicates via their representing functions (IM, p. 227), and for Ψ varying is *uniform* if the same r-expression can serve for all values of Ψ and the recursiveness in Ψ of its primitives is uniform.

7.2. As in [14, 2.2], the class of the primitives for predicates of order r in Ψ can be enlarged to include the functions general recursive in Ψ.

7.3. Clearly, P is general recursive in Ψ exactly if P is of order 0 in Ψ. We say P is *arithmetical* (*analytic*) in Ψ, if P is of order 1 (2) in Ψ. By XVII and XXXI (and, for Ψ nonempty, 3.15), this agrees with previous usage in the case then considered that the variables of P are of types $\leqq 1$ and of Ψ of type 0 (cf. IM, pp. 239, 284–285, 291–292, [14, pp. 313–314]).

In that case, a smaller class of primitives than the predicates and functions general recursive in Ψ suffices. Indeed, for predicates arithmetical in Ψ, by IM, Theorem VII* (b), pp. 285, 292 the primitives $=$, $+$, $\cdot$, Ψ suffice[13]. For predicates analytic in Ψ, by [14, 2.3 and Footnote 6] the primitives $=$, $+1$, Ψ suffice; this result is included in the $r = 1$ case of XXXVI below.

[13] The constant natural numbers are expressible in 0, 1, $+$ (or 0, $+1$); then 0, 1 are replaceable using number quantification by $w=0$, $w=1$ (e.g. IM, p. 411 with $n=0$); and the latter are 1-expressible in $=$, $+$ (or $+1$), $\cdot$, thus: $w=0 \equiv (x)[x+w=x] \equiv (x)[x+1\neq w]$, $w=1 \equiv (x)[x\cdot w=x]$.

7.4. XXXV. *For each* $r \geqq 1$: *If a function* $\phi(\mathfrak{a})$ *of variables* $\mathfrak{a}$ *of types* $\leqq r+1$ *is general recursive in functions* Ψ *of variables of types* $\leqq r$, *then* $\phi(\mathfrak{a})=w$ *is* $r+1$*-expressible in* $=, +, \cdot, \Psi$, *with a prenex* $r+1$*-expression in which all the type-r quantifiers are universal, and also one in which all are existential.*

Proof. PART (a). $\phi(\mathfrak{a})$ is primitive recursive in Ψ. We use induction on the length of a primitive recursive derivation of $\phi(\mathfrak{a})$ from Ψ by our schemata, as for IM, Theorem I, I*, pp. 241, 292(13). Cases 1–7 are essentially as before, using in Case 5 ideas of Dedekind and Gödel, and introducing only number quantifiers (type 0). Using in the applications of the hyp. ind. prenex forms with their type-r quantifiers only universal (existential), the resulting prenex forms have the same property.

CASE 8. $\phi(\alpha^j, \mathfrak{b}) = \alpha^j(\lambda\alpha^{j-2}\, \chi(\alpha^j, \alpha^{j-2}, \mathfrak{b}))$ where χ comes earlier in the derivation. Now

$$(72)\quad \begin{aligned} \phi(\alpha^j, \mathfrak{b})=w &\equiv (\gamma^{j-1})\{(\alpha^{j-2})[\chi(\alpha^j, \alpha^{j-2}, \mathfrak{b})=\gamma^{j-1}(\alpha^{j-2})] \rightarrow \alpha^j(\gamma^{j-1})=w\} \\ &\equiv (E\gamma^{j-1})\{(\alpha^{j-2})[\chi(\alpha^j, \alpha^{j-2}, \mathfrak{b})=\gamma^{j-1}(\alpha^{j-2})] \,\&\, \alpha^j(\gamma^{j-1})=w\}. \end{aligned}$$

The quantifier (γ^{j-1}) or $(E\gamma^{j-1})$ is of type $\leqq r$, since $j \leqq r+1$. By hyp. ind., $\chi(\alpha^j, \alpha^{j-2}, \mathfrak{b})=w$, and thence by substitution $\chi(\alpha^j, \alpha^{j-2}, \mathfrak{b})=\gamma^{j-1}(\alpha^{j-2})$, is $r+1$-expressible in $=, +, \cdot, \Psi$. For $j-1 = r$, use of a prenex form for $\chi(\alpha^j, \alpha^{j-2}, \mathfrak{b})= w$ with its type-r quantifiers only existential leads from the two forms of (72) to prenex forms of $\phi(\alpha^j, \mathfrak{b})=w$ with their type-r quantifiers all universal and all existential, respectively. CASE 0 (cf. 1.8) is similar, since each variable γ^{j-1} of each of Ψ is of type $\leqq r$.

PART (b). Otherwise. For Ψ nonempty, $\phi(\mathfrak{a})=w$ can be transformed by 3.15, 2.7, 1.9 into $\phi(\mathfrak{a}, \Psi')=w$, where $\phi(\mathfrak{a}, \mathfrak{b})$ is partial recursive absolutely, and Ψ' are functions primitive recursive in Ψ. Using (13) to express $\phi(\mathfrak{a}, \mathfrak{b}) \simeq w$ in the form $\{z\}[a, \alpha^1, \cdots, \alpha^{r+1}] \simeq w$, applying XXVI to obtain expressions equivalent to the latter when $\phi(\mathfrak{a}, \mathfrak{b})$ is defined, and substituting Ψ' for $\mathfrak{b}$ by IV* (1.8, 1.5), we obtain $\phi(\mathfrak{a})=w \equiv (\beta^r)(E\xi^{r-1})R(\mathfrak{a}, w, \beta^r, \xi^{r-1}) \equiv (E\beta^r)(\xi^{r-1})S(\mathfrak{a}, w, \beta^r, \xi^{r-1})$ with R, S primitive recursive in Ψ. The conclusion follows by applying the result of Part (a) to the representing functions of R, S as its ϕ.

7.5. XXXVI. *For each* $r \geqq 1$: *If a predicate P of variables of types* $\leqq r+1$ *is of order* $r+1$ *in functions* Ψ *of variables of types* $\leqq r$, *then P is* $r+1$*-expressible in* $=, +1, \Psi$.

Proof. In an $r+1$-expression for $P(\mathfrak{a})$ under 7.1, consider each prime part $Q(\mathfrak{b})$ where Q is general recursive in Ψ. We can write $Q(\mathfrak{b}) \equiv \phi(\mathfrak{b})=0$ where ϕ is the representing function of Q, and apply XXXV. Finally, $+$ and $\cdot$ can be replaced by their representing predicates (e.g. by IM, p. 411), which by Dedekind's method (cf. IM, p. 242) can be 2-expressed in $=, 0, +1$(13).

7.6. REMARK 3. In XXXV, XXXVI, by $=$ we mean of course the predicate $\alpha^0=\beta^0$. For $r > 0$, $\alpha^r=\beta^r \equiv (\tau^{r-1})[\alpha^r(\tau^{r-1})=\beta^r(\tau^{r-1})]$. Thus the predicate $\alpha^r = \beta^r$ is of order r (but not less, as will be shown in Remark 7 end 7.11).

REMARK 4. Although in 7.1 we did not list the λ-operator in the vocabulary for r-expressions, for expressing predicates of order r under 7.1 and 7.2 we can use it via IV, IV*, XXIII or XXIII* (4.4).

REMARK 5. The restriction on the types of the variables in XXXV and XXXVI is necessary in general. For example, $\alpha^{r+2}(\lambda\beta^r\,\chi(\beta^r)) = w$ with a primitive recursive χ is not $r+1$-expressible in $=, +, \cdot$ as there would be no way the α^{r+2} could be used; and likewise $\psi(\lambda\beta^r\,\chi(a, \beta^r)) = w$ is not $r+1$-expressible in $=, +, \cdot, \psi$ (unless $\psi(\lambda\beta^r\,\chi(a, \beta^r))$ is constant).

7.7. In an $r+1$-expression, (a) any function $\psi(\mathfrak{b})$ can be replaced as primitive by its representing predicate $\psi(\mathfrak{b}) = w$ (e.g. as in IM, p. 411), and (b) any function $\psi(\mathfrak{b})$ which is the representing function of a predicate can be replaced as primitive by that prediate $Q(\mathfrak{b})$ (from (a) similarly to [14, 2.5]). Only type-0 quantifiers are introduced by these replacements.

7.8. Applying XXXVI with 7.7, similarly to [14, 2.6], if a predicate P of variables of types $\leqq r+1$ is of order $r+1$ in predicates and functions Ψ of variables of types $\leqq r$ of order $r+1$ in Θ, then P is of order $r+1$ in Θ.

7.9. We turn now to reductions which minimize the use of quantifiers rather than of primitives. For brevity we state the theorems for Ψ empty, but via 3.15 as in the proof of XXXV Part (b) they have relativized forms. The $r = 1$ case of XXXVII for $\mathfrak{a}$ of types $\leqq 1$ is [14, Theorem 1].

XXXVIIa. *For each $r \geqq 1$: Each predicate $P(\mathfrak{a})$ of order $r+1$ is expressible in one of the following forms where $B(\mathfrak{a})$ is of order r and each R is general recursive*:

$$(\mathrm{c}_1)\quad B(\mathfrak{a})\quad \begin{matrix} (\alpha^r)(E\xi^{r-1})R(\mathfrak{a}, \alpha^r, \xi^{r-1}) & (E\alpha^r)(\beta^r)(E\xi^{r-1})R(\mathfrak{a}, \alpha^r, \beta^r, \xi^{r-1}) \cdots \\ (E\alpha^r)(\xi^{r-1})R(\mathfrak{a}, \alpha^r, \xi^{r-1}) & (\alpha^r)(E\beta^r)(\xi^{r-1})R(\mathfrak{a}, \alpha^r, \beta^r, \xi^{r-1}) \cdots . \end{matrix}$$

Proof, for variables $\mathfrak{a}$ of types $\leqq r$. Essentially as before ([14, 3.6]). In detail: Consider some $r+1$-expression for $P(\mathfrak{a})$ in general recursive predicates (7.1). If this $r+1$-expression contains no quantifiers of type r, it is of the first form $B(\mathfrak{a})$. Otherwise bring it to prenex form. Now apply the following steps in order.

STEP 1. Contract each sequence of several adjacent quantifiers of like kind to one quantifier of the same kind and the maximum m of their types, by 2.5. (The scope remains general recursive, using for $m > 0$ XXIII.)

STEP 2. If the rightmost quantifier of type r has no type-$r-1$ quantifier to the right of it, reduce it to type $r-1$ by XXXIV (61) or its dual ($\overline{61}$) if $r \geqq 2$ (by [14, (7) or (8)] if $r = 1$). If then no type-r quantifier remains, we have the first form $B(\mathfrak{a})$.

STEP 3. If more than one quantifier of types $< r$ stand to the right of the rightmost type-r quantifier, remove all but one of them, by advancing the rightmost of them and performing contractions (or using the technique illustrated end 2.5).

STEP 4. Remove each group of quantifiers of types $< r$ included between two type-r quantifiers, or to the left of the leftmost type-r quantifier, by advancing the type-r quantifier immediately to their right and performing contractions, and if type-r quantifiers of the same kind are thereby brought together contract them.

REMARK 6. As before [14, Remark p. 317], the described precedure is best in the sense formulated in XLI below.

Proof, without restriction on $\mathfrak{a}$. In this case we cannot use Step 2 in general. In lieu of it we may be obliged to introduce a redundant $(E\xi^{r-1})$ or (ξ^{r-1}) at the right (whereupon XLI will no longer apply).

XXXVIIb. *Equivalently the forms can be written as follows where the B for each is of order r (i.e. the same predicates are expressible in a given form of (c_1) as in the respective form of (c_2)):*

$$(c_2) \qquad B(\mathfrak{a}) \qquad \begin{matrix} (\alpha^r)B(\mathfrak{a}, \alpha^r) & (E\alpha^r)(\beta^r)B(\mathfrak{a}, \alpha^r, \beta^r) & \cdots \\ (E\alpha^r)B(\mathfrak{a}, \alpha^r) & (\alpha^r)(E\beta^r)B(\mathfrak{a}, \alpha^r, \beta^r) & \cdots . \end{matrix}$$

A third part of this theorem, using set variables (cf. [14, p. 317]), is intended for Part II of the paper.

7.10. XXXVIII. *For each $r \geqq 1$: To each of the forms* (c) *of* XXXVII *after the first, when $\mathfrak{a}$ are variables of types $\leqq r+1$, there is an enumerating predicate of that form with primitive recursive scope for the predicates of that form. For example, to each fixed list $\mathfrak{a}$ of such variables, there is a primitive recursive predicate $T(z, \mathfrak{a}, \alpha^r, \xi^{r-1})$ such that, to any general recursive predicate $R(\mathfrak{a}, \alpha^r, \xi^{r-1})$,*

$$(73) \qquad (\alpha^r)(E\xi^{r-1})R(\mathfrak{a}, \alpha^r, \xi^{r-1}) \equiv (\alpha^r)(E\xi^{r-1})T(f, \mathfrak{a}, \alpha^r, \xi^{r-1}),$$

$$(74) \qquad (E\alpha^r)(\xi^{r-1})R(\mathfrak{a}, \alpha^r, \xi^{r-1}) \equiv (E\alpha^r)(\xi^{r-1})\overline{T}(g, \mathfrak{a}, \alpha^r, \xi^{r-1})$$

when f, g are indices of $R(\mathfrak{a}, \alpha^r, \xi^{r-1})$, $\overline{R}(\mathfrak{a}, \alpha^r, \xi^{r-1})$, respectively(14).

Proof. Say e.g. $\mathfrak{a}$ consists of exactly one variable γ^i of each type $\leqq r+1$. By (14), when z is an index of a general recursive function of $(\mathfrak{a}, \alpha^r, \xi^{r-1})$,

$$(75) \qquad \begin{aligned} &(\alpha^r)(E\xi^{r-1})\{z\}[\langle\gamma^0\rangle, \cdots, \langle\gamma^{r-2}\rangle, \langle\gamma^{r-1}, \xi^{r-1}\rangle, \langle\gamma^r, \alpha^r\rangle, \langle\gamma^{r+1}\rangle] = 0 \\ &\equiv (\alpha^r)(E\xi^{r-1})(\beta^r)(E\eta^{r-1})I(z, \langle\gamma^0\rangle, \cdots, \\ &\qquad\qquad \langle\gamma^{r-2}\rangle, \langle\gamma^{r-1}, \xi^{r-1}\rangle, \langle\gamma^r, \alpha^r\rangle, \langle\gamma^{r+1}\rangle, 0, \beta^r, \eta^{r-1}), \end{aligned}$$

I being primitive recursive. The expression on the right comes to the form

(14) The T's here are different from those of 5.21–5.26. For $r = 1$ with $\mathfrak{a}$ of types $\leqq 1$, there are also enumerating predicates using those; e.g. for $\mathfrak{a} = (a, b, \alpha)$, quantifying β in XXXIII and writing $T_2^{\alpha,\beta}(z, a, b, y) \equiv T_2^{1,1}(\bar{\alpha}(y), \bar{\beta}(y), z, a, b)$: (73a) $(\beta)(Ey)R(a, b, \alpha, \beta, y) \equiv (\beta)(Ey)T_2^{\alpha,\beta}(f, a, b, y)$, (74a) $(E\beta)(y)R(a, b, \alpha, \beta, y) \equiv (E\beta)(y)\overline{T}_2^{\alpha,\beta}(g, a, b, y)$ when f, g are indices of $\lambda ab\alpha\beta\, \mu yR(a, b, \alpha, \beta, y)$, $\lambda ab\alpha\beta\, \mu y\overline{R}(a, b, \alpha, \beta, y)$, respectively.—Cf. XXXVIII with L below.

$(\alpha^r)(E\xi^{r-1})T(z, \mathfrak{a}, \beta^r, \xi^{r-1})$ with primitive recursive T by Step 4. For f, g as stated, (73) and (74) follow by using (13).

XXXIX. *For each* $r \geqq 1$: *The class of the predicates expressible in a given one of the forms* (c_1) *after the first, when* $\mathfrak{a}$ *are variables of types* $\leqq r+1$, *is the same whether a general recursive or only a primitive recursive* R *be allowed.*

7.11. XL. *For each* $r \geqq 1$: *To each of the forms* (c) *after the first, when* $\mathfrak{a}$ *is a nonempty list of variables of types* $\leqq r+1$, *there is a predicate expressible in that form but not in the dual form (a fortiori, not in any of the forms with fewer type-r quantifiers).*

Proof. Say $\mathfrak{a} = (\gamma^m, \mathfrak{b})$. Consider e.g. the first upper form, and the predicate $(\alpha^r)(E\xi^{r-1})T((\gamma^m)_0^0, \gamma^m, \mathfrak{b}, \alpha^r, \xi^{r-1})$ of the form (cf. 2.4). For any general recursive R, using (74) and (9), $(E\alpha^r)(\xi^{r-1})R(\gamma^m, \mathfrak{b}, \alpha^r, \xi^{r-1}) \equiv (E\alpha^r)(\xi^{r-1}) \overline{T}((\langle g\rangle^m)_0^0, \gamma^m, \mathfrak{b}, \alpha^r, \xi^{r-1}) \not\equiv (\alpha^r)(E\xi^{r-1})T((\langle g\rangle^m)_0^0, \gamma^m, \mathfrak{b}, \alpha^r, \xi^{r-1})$ for a certain number g. This shows that $\langle g\rangle^m$ is a value of γ^m for which $(E\alpha^r)(\xi^{r-1}) R(\gamma^m, \mathfrak{b}, \alpha^r, \xi^{r-1})$ is inequivalent to our predicate.

Second proof for $m = 0$. Similarly using $(\alpha^r)(E\xi^{r-1})T(a, a, \mathfrak{b}, \alpha^r, \xi^{r-1})$.

XLI. *To any prenex form with quantifiers of types* $\leqq r$ *and recursive scope and a nonempty list* $\mathfrak{a}$ *of free variables of types* $\leqq r$, *there is a predicate of that form which is expressible in no others of the forms* (c_1) *than the one to which Steps* 1–4 *reduce it except forms with more quantifiers.*

Proofs. As before [14, Corollary Theorem 3, p. 319].

XLII. *When* $\mathfrak{a}$ *includes variables of type* > 1, *there is a predicate* $(\alpha)R(\mathfrak{a}, \alpha)$ *with* R *recursive which is not expressible in the form* $(x)R(\mathfrak{a}, x)$ *with* R *recursive, and dually* (in contrast to [14, (7) and (8), p. 316], which can be stated for any list $\mathfrak{a}$ of free variables of types $\leqq 1$).

Proof. For example, with $(\mathfrak{a}, \alpha) = (a, \alpha^2, \xi^1)$, were $(\xi^1)\overline{T}(a, a, \alpha^2, \xi^1) \equiv (x)R(a, \alpha^2, x)$ with a recursive R, then XXXIV would bring $(E\alpha^2)(\xi^1) \overline{T}(a, a, \alpha^2, \xi^1)$ to the form $(E\eta^1)(x)R(a, \eta^1, x)$ with a recursive R, contradicting XL.

XLIII. *The conclusion of* XXXIV *does not hold in general when the free variables* $\mathfrak{a}$ *include ones of type* $> r$.

Proof. Were $(E\xi^r)T(a, a, \alpha^{r+1}, \xi^r) \equiv (E\eta^{r-1})(\xi^{r-2})R(a, \alpha^{r+1}, \eta^{r-1}, \xi^{r-2})$ with a recursive R, then by a (correct) application of XXXIV we would have $(\alpha^{r+1})(E\xi^r)T(a, a, \alpha^{r+1}, \xi^r) \equiv (\alpha^r)(E\xi^{r-1})R(a, \alpha^r, \xi^{r-1})$ with a recursive R, contradicting XL.

REMARK 7. For $r \geqq 1$, were $\alpha^{r+1} = \beta^{r+1}$ (cf. Remark 3) of order r, we would have

(a) $$\alpha^{r+1} = \beta^{r+1} \equiv (Q\mathfrak{b}^{r-1})R(\alpha^{r+1}, \beta^{r+1}, \mathfrak{b}^{r-1})$$

with R general recursive and $(Q\mathfrak{b}^{r-1})$ quantifiers on variables of types $\leqq r-1$. Letting $\psi(a, \alpha^r)$ be the representing function of $(E\xi^{r-1})T(a, a, \alpha^r, \xi^{r-1})$, we would then have

(b) $$(\alpha^r)(E\xi^{r-1})T(a, a, \alpha^r, \xi^{r-1}) \equiv (Q\mathfrak{b}^{r-1})R(\lambda\alpha^r\, \psi(a, \alpha^r), \lambda\alpha^r\, 0, \mathfrak{b}^{r-1}).$$

Since by XXIII* $R(\lambda\alpha^r \psi(a, \alpha^r), \lambda\alpha^r 0, \mathfrak{b}^{r-1})$ is general recursive in ψ, by XXXV (applied to its representing function) it is $r+1$-expressible in $=, +, \cdot$, ψ, with a prenex $r+1$-expression in which all the type-r quantifiers are existential; in this expression, $\psi(a, \alpha^r)$ can be replaced (IM, p. 411) by the represented predicate $(E\xi^{r-1})T(a, a, \alpha^r, \xi^{r-1})$. Using the result in the right member of (b), and advancing and contracting quantifiers, we would obtain for $(\alpha^r)(E\xi^{r-1})T(a, a, \alpha^r, \xi^{r-1})$ an expression of the form $(E\alpha^r)(\xi^{r-1})R(a, \alpha^r, \xi^{r-1})$ with R recursive, contradicting XL. Similarly, were $\alpha^1 = \beta^1$ of order 0, we would have $(x)\overline{T}_1(a, a, x)$ recursive, contradicting IM, Theorem V Part I, p. 283.

7.12. XLIV. *For each $r \geqq 1$ and $m \leqq r+1$: To each of the forms* (c) *after the first, there is a predicate $C(\gamma^m)$ of the form such that any predicate of the form with its free variables $\mathfrak{a}$ of types $\leqq m$ is expressible by substitution of a primitive recursive function of $\mathfrak{a}$ for γ^m in $C(\gamma^m)$.*

Proof. The case of a general $\mathfrak{a}$ is reduced to $\mathfrak{a} = \gamma^m$ by XI. We use the same predicate as for XL. By XXIII and (73), for any general recursive R, $(\alpha^r)(E\xi^{r-1})R((\gamma^m)_1, \alpha^r, \xi^{r-1}) \equiv (\alpha^r)(E\xi^{r-1})T(f, \gamma^m, \alpha^r, \xi^{r-1})$ for a suitable number f. Now $(\alpha^r)(E\xi^{r-1})R(\gamma^m, \alpha^r, \xi^{r-1}) \equiv (\alpha^r)(E\xi^{r-1})R((\langle f, \gamma^m\rangle)_1, \alpha^r, \xi^{r-1})$ $\equiv (\alpha^r)(E\xi^{r-1})T((\langle f, \gamma^m\rangle)_0^0, \langle f, \gamma^m\rangle, \alpha^r, \xi^{r-1})$.

Second proof for $m = 0$. Again we use the same predicate. Let e be an index of $\lambda ab\alpha^r\xi^{r-1} R(a, \alpha^r, \xi^{r-1})$. Then by XIII, $S^1(e, a)$ is one of $\lambda b\alpha^r\xi^{r-1} R(a, \alpha^r, \xi^{r-1})$. So by (73) (substituting $S^1(e, a)$ for b), $(\alpha^r)(E\xi^{r-1})R(a, \alpha^r, \xi^{r-1})$ $\equiv (\alpha^r)(E\xi^{r-1})T(S^1(e, a), S^1(e, a), \alpha^r, \xi^{r-1})$.

7.13. XLV. *For each $r \geqq 1$ and $k \geqq 0$: Let P be a predicate of variables of types $\leqq r+1$, and Ψ predicates of variables of types $\leqq r$. If P is of order r in Ψ, and Ψ are expressible in both the $k+1$-(type-r)-quantifier forms of* (c), *then P is expressible in both the $k+1$-(type-r)-quantifier forms of* (c).

Proof. Consider, in a given r-expression for P under 7.1, each prime occurrence $R(\mathfrak{a})$ ($\equiv \phi(\mathfrak{a}) = 0$) of a predicate general recursive in Ψ. According as this occurrence is positive or negative (cf. [14, p. 321]), apply XXXV with 7.7 (b) to replace it by a prenex $r+1$-expression with the type-r quantifiers only universal or only existential and with $=, +, \cdot, \Psi$ as the primitives. Now consider in this prenex form of each $R(\mathfrak{a})$ each prime part $Q(\mathfrak{b})$ with one of Ψ as its predicate symbol Q. According as this part is a positive or negative occurrence in the expression for P, replace it by the $k+1$-quantifier form for Q with (α^r) first or $(E\alpha^r)$ first. Now the quantifiers can be advanced and contracted so that a $k+1$-quantifier expression for P with (α^r) first is obtained (cf. [14, pp. 321–322]). Reversing the above choices, a $k+1$-quantifier expression for P with $(E\alpha^r)$ first is obtained similarly.

8. **μ-recursiveness versus general recursiveness. 8.1.** We say a function ϕ is *partial μ-recursive*, if it is describable by a succession of applications of the primitive recursive schemata S1–S8 (written with $\simeq$ instead of $=$, and

taking $\alpha^j(\lambda\alpha^{j-2}\,\chi(\alpha^j, \alpha^{j-2}, \mathfrak{b}))$ to be undefined when $\lambda\alpha^{j-2}\,\chi(\alpha^j, \alpha^{j-2}, \mathfrak{b})$ is incompletely defined(10)) and one further schema

S10 $$\phi(\mathfrak{b}) \simeq \mu y[\chi(\mathfrak{b}, y) = 0].$$

A function is *μ-recursive*, if it is partial μ-recursive and completely defined. Also cf. 3.14, 1.8–1.10, 2.7, 3.15.

By XVI the μ-recursive (partial μ-recursive) functions constitute a subset of the general (partial) recursive functions, which by the normal form theorem XXX, XXXII (or [8], IM) is the whole set in the case of functions of variables of types ≤ 1.

8.2. In comparing μ- and general recursiveness for functions of variables of types > 1, we employ the particular type-2 object $\mathbf{E}$ defined thus:

$$\mathbf{E}(\alpha) = \begin{cases} 0 & \text{if } (Et)[\alpha(t) = 0], \\ 1 & \text{otherwise.} \end{cases}$$

This is a simple example of a functional depending on infinitely many values of its function arguments. (Another is the representing function $\psi(\alpha^1, \beta^1)$ of $\alpha^1 = \beta^1$; cf. Remark 3 in 7.6. Note that $\psi(\alpha, \beta) = \overline{\mathrm{sg}}(\mathbf{E}(\lambda t\, \overline{\mathrm{sg}}|\alpha(t) - \beta(t)|))$ and $\mathbf{E}(\alpha) = \overline{\mathrm{sg}}(\psi(\lambda t\, \mathrm{sg}(\alpha(t)), \lambda t\, 1))$.)

8.3. In the description of a partial μ-recursive function $\phi(\mathbf{F}, a_1, \cdots, a_n, \beta)$ with a single type-1 variable β and a single type-2 variable $\mathbf{F}$, only S1–S5, S6.0, S7, S8.2, S10 can be used (if we exclude identical uses of S6.1 and S6.2).

By the ***F**-height* of such a description we shall mean the greatest number of applications of S8.2 in any branch when the description is written in tree form (cf. IM, pp. 106–107, taking into account the analogy between descriptions and proofs, derivations and deductions, pp. 220, 224). Thus the **F**-height after an application of S4 is the maximum of the **F**-heights for the ψ and the χ; after an application of S8.2, the **F**-height for the χ increased by one.

A partial function $\phi_1(\mathfrak{a})$ is an *extension* of a partial function $\phi(\mathfrak{a})$, if $\phi_1(\mathfrak{a})$ is defined and $= \phi(\mathfrak{a})$ for each $\mathfrak{a}$ for which $\phi(\mathfrak{a})$ is defined.

XLVI. *If $\phi(\mathbf{F}, a_1, \cdots, a_n, \beta)$ is partial μ-recursive with a description of **F**-height h, then there is a function $\phi_1^\beta(a_1, \cdots, a_n)$ partial recursive in L_h^β uniformly in β such that, for each β (and the fixed **E** of* 8.2), *$\phi_1^\beta(a_1, \cdots, a_n)$ is an extension of $\phi(\mathbf{E}, a_1, \cdots, a_n, \beta)$. Similarly without the β, i.e. for a $\phi(\mathbf{F}, a_1, \cdots, a_n)$ using L_h simply*(15).

Proof with the β, by induction on the length of a description of ϕ by S1–S8, S10.

(15) For L_h^β, cf. [19, p. 400] taking $A(a) \equiv \beta((a)_0) = (a)_1$, or [16, p. 198] with $Q(a) \equiv \beta((a)_0) = (a)_1$. For this §8, it is immaterial whether we rework the definitions and theory of L_h^β, L_h, and H_y ([14, §6 ff.] or [16, p. 200]) to use the T_1^P bases on indices as in 5.21–5.26 above (which reworking does not alter the degrees), or agree that in this section the T_1^P is to be the one based on Gödel numbers (as in IM, [19; 14; 16]).

CASES 1, 2, 3, 7. $\phi(\mathbf{F}, a_1, \cdots, a_n, \beta)$ is introduced by one of S1–S3, S7. Then the $\mathbf{F}$-height is 0, and $\phi_1^\beta(a_1, \cdots, a_n) = \phi(\mathbf{E}, a_1, \cdots, a_n, \beta)$ is primitive recursive uniformly in β, and hence general recursive in L_0^β uniformly in β.

CASE 4. $\phi(\mathbf{F}, a_1, \cdots, a_n, \beta) = \psi(\mathbf{F}, \chi(\mathbf{F}, a_1, \cdots, a_n, \beta), a_1, \cdots, a_n, \beta)$ by S4. Let the $\mathbf{F}$-heights of ψ, χ be h_1, h_2 (so $h = \max(h_1, h_2)$). By hyp. ind. there are $\psi_1^\beta, \chi_1^\beta$, partial recursive in $L_{h_1}^\beta, L_{h_2}^\beta$ uniformly in β, hence (since $L_k^\beta(a) \equiv L_{k+1}^\beta(\theta(a))$ with a primitive recursive θ, by IM, p. 343 or [14, Lemma 1, p. 325]) in L_h^β uniformly in β, such that $\psi_1^\beta(b, a_1, \cdots, a_n), \chi_1^\beta(a_1, \cdots, a_n)$ are extensions of $\psi(\mathbf{E}, b, a_1, \cdots, a_n, \beta)$, $\chi(\mathbf{E}, a_1, \cdots, a_n, \beta)$, respectively. Then $\phi_1^\beta(a_1, \cdots, a_n) = \psi_1^\beta(\chi_1^\beta(a_1, \cdots, a_n), a_1, \cdots, a_n)$ is an extension of $\phi(\mathbf{E}, a_1, \cdots, a_n, \beta)$.

CASE 8. $\phi(\mathbf{F}, a_1, \cdots, a_n, \beta) = \mathbf{F}(\lambda x\, \chi(\mathbf{F}, x, a_1, \cdots, a_n, \beta))$ by S8.2. The $\mathbf{F}$-height of χ is $h-1$, and by hyp. ind. there is a function χ_1^β partial recursive in L_{h-1}^β uniformly in β such that $\chi_1^\beta(x, a_1, \cdots, a_n)$ is an extension of $\chi(\mathbf{E}, x, a_1, \cdots, a_n, \beta)$. Then the function ϕ_0^β defined by

$$\phi_0^\beta(a_1, \cdots, a_n) = \mathbf{E}(\lambda x\, \chi_1^\beta(x, a_1, \cdots, a_n)) \tag{76}$$
$$= \begin{cases} 0 & \text{if } (Ex)[\chi_1^\beta(x, a_1, \cdots, a_n) = 0], \\ 1 & \text{otherwise,} \end{cases}$$

for the fixed $\mathbf{E}$ and any β, is an extension of $\phi(\mathbf{E}, a_1, \cdots, a_n, \beta)$; here, to correspond to our interpretation of S8 for χ a partial function, $\phi_0^\beta(a_1, \cdots, a_n)$ is to be undefined for given $\beta, a_1, \cdots, a_n$ when $\lambda x\, \chi_1^\beta(x, a_1, \cdots, a_n)$ is incompletely defined. But since χ_1^β is partial recursive in L_{h-1}^β uniformly in β, there is by [14, Lemma 1] a primitive recursive θ such that

$$(Ex)[\chi_1^\beta(x, a_1, \cdots, a_n) = 0]$$
$$\equiv (Ex)T_1^{L_{h-1}^\beta}(\theta(a_1, \cdots, a_n), \theta(a_1, \cdots, a_n), x) \equiv L_h^\beta(\theta(a_1, \cdots, a_n)) \tag{77}$$

for values of $\beta, a_1, \cdots, a_n$ which make $\lambda x\, \chi_1^\beta(x, a_1, \cdots, a_n)$ completely defined, i.e. which make $\phi_0^\beta(a_1, \cdots, a_n)$ defined. So by replacing $(Ex)[\chi_1^\beta(x, a_1, \cdots, a_n) = 0]$ by $L_h^\beta(\theta(a_1, \cdots, a_n))$ in (76), we obtain a function

$$\phi_1^\beta(a_1, \cdots, a_n) = \begin{cases} 0 & \text{if } L_h^\beta(\theta(a_1, \cdots, a_n)), \\ 1 & \text{otherwise,} \end{cases} \tag{78}$$

which is an extension of ϕ_0^β and thence of $\phi(\mathbf{E}, a_1, \cdots, a_n, \beta)$, and is clearly partial (indeed, primitive) recursive in L_h^β uniformly in β.

8.4. Now we give an argument by which any person who accepts the primitive recursive functions as effectively calculable, and who allows such a type-2 object as $\mathbf{E}$, must admit that the μ-recursive functions are not all the effectively calculable functions.

Since $T_1^\alpha(a, a, t)$ is primitive recursive as a predicate of α, a, t (IM, p. 292, or §5 above), the function

$$\tau(t, \alpha, a) = \begin{cases} 0 \text{ if } T_1^{\alpha}(a, a, t), \\ 1 \text{ otherwise} \end{cases}$$

is primitive recursive. Hence using III and S8, so is

$$\tau(\mathbf{F}, \alpha, a) = \mathbf{F}(\lambda t\, \tau(t, \alpha, a)).$$

Using induction on k (and in the induction step, III and IV), we define a succession of primitive recursive functions $\lambda_k(\mathbf{F}, a, \beta)$ $(k = 0, 1, 2, \cdots)$:

$$\lambda_0(\mathbf{F}, a, \beta) = \text{sg}\,|\,\beta((a)_0) - (a)_1\,|\,,$$
$$\lambda_{k+1}(\mathbf{F}, a, \beta) = \tau(\mathbf{F}, \lambda t\, \lambda_k(\mathbf{F}, t, \beta), a).$$

Upon giving $\mathbf{F}$ the fixed value $\mathbf{E}$, we have

$$(79) \qquad \tau(\mathbf{E}, \alpha, a) = \begin{Bmatrix} 0 \text{ if } (Et)[\tau(t, \alpha, a) = 0] \\ 1 \text{ otherwise} \end{Bmatrix} = \begin{cases} 0 \text{ if } (Et)T_1^{\alpha}(a, a, t), \\ 1 \text{ otherwise}, \end{cases}$$

and thence by induction on k, for each β,

$$(80) \qquad \lambda_k(\mathbf{E}, a, \beta) = \{\text{the representing function of } L_k^{\beta}(a)\}.$$

Similarly, omitting the β and taking $\lambda_0(\mathbf{F}, a) = 0$, we obtain for $k = 0, 1, 2, \cdots$ a primitive recursive function $\lambda_k(\mathbf{F}, a)$ such that

$$(81) \qquad \lambda_k(\mathbf{E}, a) = \{\text{the representing function of } L_k(a)\}.$$

Now consider the function

$$\lambda(k, \mathbf{F}, a) = \lambda_k(\mathbf{F}, a).$$

This must be accepted as being effectively calculable (accepting that the primitive recursive functions of variables of types 0, 1, 2 are). For, given k, $\mathbf{F}$, a, we can via the induction on k find effectively a primitive recursive description of λ_k, and then "compute" $\lambda_k(\mathbf{F}, a)$.

But $\lambda(k, \mathbf{F}, a)$ is not μ-recursive. For by (81) and the definition of $\lambda(k, \mathbf{F}, a)$, $\lambda(k, \mathbf{E}, a)$ is the representing function of $L(k, a) \equiv L_k(a)$, which is of degree of recursive unsolvability $\mathbf{0}^{(\omega)}$ [19, p. 401; 16, p. 198]). But by XLVI, were $\lambda(k, \mathbf{F}, a)$ μ-recursive, then, for some h, $\lambda(k, \mathbf{E}, a)$ would be recursive in L_h, and thus would be of degree $\leq \mathbf{0}^{(h)} < \mathbf{0}^{(\omega)}$[16].

[16] As μ-recursiveness is one of the simplest of the equivalent notions of "effective calculability" or "computability" for functions of variables of types 0, 1 (cf. 3.2), we considered it as a possible definition for the higher types at the beginning of our study of the subject in 1952, but rejected it for the reason given now. In a 1955 paper Grzegorczyk [7, p. 170] seems to be proposing this definition; his formulation is equivalent to μ-recursiveness as defined here, when our schema S8.j for $j \geq 2$ (or S4.$j-1$ and S7.j of 1.6, Remark 1) is added to the schemata he explicitly mentions (and indeed without some such addition the higher-type arguments of the functions could not be utilized) and the (maybe inessential) restriction is imposed on our μ-schema S10 that at each application the $\phi(\mathfrak{b})$ be completely defined.

8.5. The function $\lambda(k, \mathsf{F}, a)$ is general recursive, as we see by writing the equations for $\lambda_k(\mathsf{F}, a)$ with k as argument, and applying XXIV (or by showing with the help of XXI and XXII that $\lambda_k(\mathsf{F}, a)$ has a primitive recursive index $\xi(k)$ with ξ primitive recursive, and putting $\lambda(k, \mathsf{F}, a) = \{\xi(k)\}(\mathsf{F}, a)$).

We can go much further. Let $u_b \simeq U(\mu v T_1(u, b_O, v))$ where $0_O = 1$, $(b+1)_O = 2^{b_O}$ (cf. [14, p. 325] or [16, pp. 199–200]). Consider the recursion

$$\text{(82)} \qquad \{z\}(y, \mathsf{F}, a) \simeq \begin{cases} 0 \text{ if } y=1, \\ \chi(z, y, \mathsf{F}, a) \text{ if } y=2^{(y)_0} \neq 1, \\ \{z\}([(y)_2]_{(a)_1}, \mathsf{F}, (a)_0) \text{ if } y=3\cdot 5^{(y)_2}, \end{cases}$$

where χ is a partial recursive function given by XXIII such that

$$\text{(83)} \qquad \chi(z, y, \mathsf{F}, a) = \tau(\mathsf{F}, \lambda t\, \{z\}((y)_0, \mathsf{F}, t), a)$$

whenever $\lambda t\, \{z\}((y)_0, \mathsf{F}, t)$ is completely defined. By XVIII, the right side of (82) is of the form $\psi(z, y, \mathsf{F}, a)$ with ψ partial recursive; so by the recursion theorem XIV, we can find a solution e of (82) for z. Let $\kappa(y, \mathsf{F}, a) \simeq \{e\}(y, \mathsf{F}, a)$. Now by induction on y over the class O of ordinal notations (loc. cit.),

$$\text{(84)} \qquad y \in O \rightarrow \{\kappa(y, \mathsf{E}, a) \text{ is the representing function of } H_y(a)\}.$$

8.6. We collect these results in a pair of contrasting theorems. Instead of speaking of completely defined functions $\phi(\mathsf{E}, a_1, \cdots, a_n)$ ($\phi(\mathsf{E}, a_1, \cdots, a_n, \beta)$) for $\phi(\mathsf{F}, a_1, \cdots, a_n)$ ($\phi(\mathsf{F}, a_1, \cdots, a_n, \beta)$) partial μ-, or partial, recursive, we can equivalently (by 1.9 extended to include S10 or S9) speak of functions $\phi(a_1, \cdots, a_n)$ ($\phi(a_1, \cdots, a_n, \beta)$) μ-, or general, recursive in E. For the notion "arithmetical," cf. 7.3, IM, pp. 239, 284–285, 291–292, [14, 2.1]. For "hyperarithmetical," cf. [16, p. 210]; a function is *hyperarithmetical* if its representing predicate is such.

XLVII. *The functions* $\phi(a_1, \cdots, a_n)$ ($\phi(a_1, \cdots, a_n, \beta)$) *$\mu$-recursive in* E *are exactly the arithmetical functions.*

Proof, with β present. By XLVI, for each $\phi(a_1, \cdots, a_n, \beta)$ μ-recursive in E, $\lambda a_1 \cdots a_n\, \phi(a_1, \cdots, a_n, \beta)$ is general recursive, a fortiori arithmetical, in L_h^β for some h uniformly in β, and hence is arithmetical uniformly in β (using 7.8 or [14, 2.6 with uniformity], and induction on h), i.e. $\phi(a_1, \cdots, a_n, \beta)$ is arithmetical. Conversely, for any arithmetical function $\phi(a_1, \cdots, a_n, \beta)$, $\lambda a_1 \cdots a_n\, \phi(a_1, \cdots, a_n, \beta)$ is μ-recursive in L_k^β for some k uniformly in β (using $\phi(a_1, \cdots, a_n, \beta) = \mu w[\phi(a_1, \cdots, a_n, \beta) = w]$, [16, IV*, VII*, p. 197 with uniformity], and [14, 9.8 (31) and (32), and Lemma 16* 9.7]). But by (80), $L_k^\beta(a) \equiv \lambda_k(\mathsf{E}, a, \beta) = 0$; and $\lambda_k(\mathsf{F}, a, \beta)$ is primitive, a fortiori μ-, recursive.

XLVIII. *The functions* $\phi(a_1, \cdots, a_n)$ *general recursive in* E *are exactly the hyperarithmetical functions.*

Proof. Suppose $\phi(a_1, \cdots, a_n)$ is general recursive in E. Equivalently, $\phi(a_1, \cdots, a_n)$ is completely defined and $= \phi(\mathsf{E}, a_1, \cdots, a_n)$ where

$\phi(\mathbf{F}, a_1, \cdots, a_n)$ is partial recursive, say with index z. Using XXVI with (13), $\phi(\mathbf{E}, a_1, \cdots, a_n)=w$ is expressible in both the forms $(\beta)(Ex)R(\mathbf{E}, a_1, \cdots, a_n, w, \beta, x)$ and $(E\beta)(x)S(\mathbf{E}, a_1, \cdots, a_n, w, \beta, x)$ with R and S primitive recursive. But by XLVII, $R(\mathbf{E}, a_1, \cdots, a_n, w, \beta, x)$ and $S(\mathbf{E}, a_1, \cdots, a_n, w, \beta, x)$ are arithmetical, hence (XLV or [14, Corollary p. 322 with a free function variable β]) expressible in both 1-function-quantifier forms; and hence so is $\phi(\mathbf{E}, a_1, \cdots, a_n)=w$, i.e. $\phi(\mathbf{E}, a_1, \cdots, a_n)=w$ is hyperarithmetical (by the second definition [16, p. 210]).

Conversely, suppose $\phi(a_1, \cdots, a_n)$ is hyperarithmetical, i.e. $\phi(a_1, \cdots, a_n)=w$ is hyperarithmetical. Then by [16, XXIV, p. 204, or the first definition p. 210], for some $y \in O$, $\phi(a_1, \cdots, a_n)=w$ and hence $\phi(a_1, \cdots, a_n)$ $(=\mu w[\phi(a_1, \cdots, a_n)=w])$, is general recursive in $H_y(a)$. But by (84), $H_y(a) \equiv \kappa(y, \mathbf{E}, a)=0$; and $\kappa(y, \mathbf{F}, a)$ is partial recursive.

8.7. XLIX. (a) *A predicate $P(\mathfrak{a})$ expressible in both the forms $(Ex)R(\mathfrak{a}, x)$ and $(x)S(\mathfrak{a}, x)$ with R, S general recursive is general recursive.* (Converse holds trivially.) (b) *When $\mathfrak{a}$ includes variables of types > 1, a general recursive predicate $P(\mathfrak{a})$ may fail to be expressible by any (finite) number of number-quantifiers prefixed to a μ- (or primitive) recursive scope* (in contrast to IM, Theorem VI* (a), pp. 284, 292, and Corollary Theorem IV*, pp. 282, 292, for types 0, 1).

Proof. (a) By the former proof, IM, Theorem VI (b), p. 284. (b) By 8.5, $\lambda(k, \mathbf{F}, a)=0$ is general recursive. But it is not expressible e.g. in the form $(Ex)(y)R(k, \mathbf{F}, a, x, y)$ with a μ-recursive R. For by 8.4, $\lambda(k, \mathbf{E}, a)=0 \equiv L(k, a)$, which is not arithmetical; but using XLVII, for a given μ-recursive R, $(Ex)(y)R(k, \mathbf{E}, a, x, y)$ is arithmetical.

8.8. L. *With variables of types > 1, there is no enumeration theorem for the predicates definable by a given succession of number quantifiers applied to a general (μ-, or primitive) recursive scope. For example, there is no general recursive predicate $S(z, \mathbf{F}, a, x)$ with the property that, to each general recursive $R(\mathbf{F}, a, x)$, there is a number e such that $(Ex)R(\mathbf{F}, a, x) \equiv (Ex)S(e, \mathbf{F}, a, x)$* (in contrast to XXXIII or IM, Theorem IV*, pp. 281, 292 for types 0, 1).

Proof. (We can use XLVIII, but the following is basically simpler.) Take any fixed general recursive $S(z, \mathbf{F}, a, x)$. We construct as follows a general recursive $R(\mathbf{F}, a, x)$ such that $(Ex)R(\mathbf{F}, a, x) \equiv (Ex)S(e, \mathbf{F}, a, x)$ for no e. Let σ be the representing function of S. Using 8.4 and IV via 3.10, define $\rho(\mathbf{F}, a, x) = \tau(x, \lambda y\, \tau(\mathbf{F}, \lambda t\, \sigma((t)_0, \mathbf{F}, (t)_1, (t)_2), y), a)$, and take $R(\mathbf{F}, a, x) \equiv \rho(\mathbf{F}, a, x)=0$. Let $\mathbf{s}$ be the degree of $S(z, \mathbf{E}, a, x)$ ($=$ the degree of $S((t)_0, \mathbf{E}, (t)_1, (t)_2)$). Then $(Ex)S(z, \mathbf{E}, a, x)$ is of degree $\leqq \mathbf{s}'$, while $(Ex)R(\mathbf{E}, a, x)$ is of degree $\mathbf{s}'' > \mathbf{s}'$ (cf. [19, 1.4]).

8.9. LI. *The condition of definition of a partial recursive function $\phi(\mathfrak{a})$ of variables of types $\leqq 2$ is not in general expressible in the form $(E\beta)(x)R(\mathfrak{a}, \beta, x)$ with recursive R* (dual to XXVII (29)), *a fortiori not in the form $(Ex)R(\mathfrak{a}, x)$ with recursive R* (in contrast to XXIX or IM, Theorem XIX, p. 330 for types 0, 1). Thus (30), (39) and (41) do not hold for $r = 2$.

Proof. Suppose (cf. 8.5)

(a) $\{\kappa(y, \mathbf{F}, a) \text{ is defined}\} \equiv (E\beta)(x)R(y, \mathbf{F}, a, \beta, x)$

with a primitive recursive R (cf. XXXIX). Using (29) and (13), for z an index of κ,

(85) $\{\kappa(y, \mathbf{F}, a) \text{ is defined}\} \equiv (\beta)(Ex)K(z, \langle y, a\rangle, \langle\ \rangle^1, \langle\mathbf{F}\rangle, \beta, x).$

But then, using XLVII etc. as in the first part of the proof of XLVIII, $\{\kappa(y, \mathbf{E}, a) \text{ is defined}\}$ would be hyperarithmetical; so by the converse part of XLVIII, there would be a partial recursive $R(y, \mathbf{F}, a)$ such that

(b) $\{\kappa(y, \mathbf{E}, a) \text{ is defined}\} \equiv R(y, \mathbf{E}, a).$

Using XV, we could define a partial recursive $\kappa'(y, \mathbf{F}, a)$ by

$$\kappa'(y, \mathbf{F}, a) = \begin{cases} \kappa(y, \mathbf{F}, a) & \text{if } R(y, \mathbf{F}, a), \\ 0 & \text{otherwise.} \end{cases}$$

Then $\kappa'(y, \mathbf{E}, a)$ would be a completion of $\kappa(y, \mathbf{E}, a)$. This is absurd. For by XLVIII and [16, XXIV or p. 210], $\kappa'(y, \mathbf{E}, a)$ would be recursive in H_u for some $u \in O$, i.e. of degree $\leq 0^{(|u|)}$; but any completion of $\kappa(y, \mathbf{E}, a)$ must be of degree $\geq 0^{(|u|+1)} > 0^{(|u|)}$, since by (84) $H_{2^u}(a) \equiv \kappa(2^u, \mathbf{E}, a) = 0$.

8.10. LII. *There is a primitive recursive* $R(\alpha^2, \beta^1)$ *such that* $(\alpha^2)(E\beta^1)R(\alpha^2, \beta^1)$ *is true, but* $(\alpha^2)R(\alpha^2, \lambda x\, \chi(\alpha^2, x))$ *is false for every general recursive* $\chi(\alpha^2, x)$ (in contrast to XVI or IM, Theorem III, p. 279 for $R(\mathfrak{a}, \beta^0)$)[17].

Proof. By [16, XXVI, p. 208] there is a primitive recursive $R(\beta^1, x)$ such that

(86a) $(E\beta^1)(x)R(\beta^1, x)$, (86b) $\overline{(E\beta^1)_{\beta^1\,\mathrm{hyp}}(x)R(\beta^1, x)}$

where $(E\beta^1)_{\beta^1\,\mathrm{hyp}}$ is an existential quantifier over the 1-place hyperarithmetical functions. Advancing the x in (86a) (cf. 2.5) and bringing (86b) to prenex form,

(87a) $(\alpha^2)(E\beta^1)R(\beta^1, \alpha^2(\beta^1))$, (87b) $(\beta^1)_{\beta^1\,\mathrm{hyp}}(Ex)\overline{R}(\beta^1, x)$.

Taking $R(\alpha^2, \beta^1) \equiv R(\beta^1, \alpha^2(\beta^1))$, (87a) gives one part of LII. Let

$$\alpha_0^2(\beta^1) = \begin{cases} \mu x\overline{R}(\beta^1, x) & \text{if } (Ex)\overline{R}(\beta^1, x), \\ 0 & \text{otherwise,} \end{cases}$$

so by (87b)

(17) In March 1957, Kreisel raised the questions (a) whether there is any such $R(\alpha^2, \beta^1)$ and (b) whether the $R(\alpha^2, \beta^1)$ obtained by advancing quantifiers in $(x)\{(Ez)T_1(x, x, z) \vee (z)\overline{T}_1(x, x, z)\}$ (cf. [10, p. 71]) is such, and we answered (a) as here. Subsequently Kreisel found two simple examples of the like with $R(\alpha^3, \beta^2)$ instead of $R(\alpha^2, \beta^1)$ (cf. [21], and also [18, Theorem 3]). Finally in June 1957, Kreisel gave an argument which, supported by a lemma we provided, answers (b) affirmatively; this will appear in Part II.

$$(88) \qquad (\overset{1}{\beta})_{\beta^1\ \mathrm{hyp}}\overline{R}(\overset{1}{\beta}, \overset{2}{\alpha_0}(\overset{1}{\beta})).$$

But $(Ex)\overline{R}(\beta^1, x) \equiv L_1^{\beta^1}(g)$ for some number g (by [16, VII*, p. 197 with uniformity and $a = 0$] or [14, Lemma 1, p. 325 with $n = 0$]). Using XV, XVI and 8.4,

$$\alpha_0(\mathbf{F}, \beta^1) = \begin{cases} \mu x\overline{R}(\beta^1, x) & \text{if } \lambda_1(\mathbf{F}, g, \beta^1) = 0, \\ 0 & \text{otherwise} \end{cases}$$

is a general recursive function such that, by (80),

$$(89) \qquad \overset{2}{\alpha_0}(\overset{1}{\beta}) = \alpha_0(\mathbf{E}, \overset{1}{\beta}).$$

Now suppose there were a general recursive $\chi(\alpha^2, x)$ such that $(\alpha^2)R(\alpha^2, \lambda x\, \chi(\alpha^2, x))$, i.e. $(\alpha^2)R(\lambda x\, \chi(\alpha^2, x), \alpha^2(\lambda x\, \chi(\alpha^2, x)))$, whence $R(\lambda x\, \chi(\overset{2}{\alpha_0}, x), \overset{2}{\alpha_0}(\lambda x\, \chi(\overset{2}{\alpha_0}, x)))$. This would contradict (88), since by (89), XXIII and XLVIII, $\lambda x\, \chi(\overset{2}{\alpha_0}, x) = \lambda x\, \chi(\lambda\beta^1\, \alpha_0(\mathbf{E}, \beta^1), x)$ would be hyperarithmetical.

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A RECURSIVELY ENUMERABLE DEGREE WHICH WILL NOT SPLIT OVER ALL LESSER ONES

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0. Introduction

Two of the early theorems about recursively enumerable (r.e.) degrees were the "splitting theorem"

$$\bigwedge a \bigvee b_0 \bigvee b_1 (0 < a \to . \ a = b_0 \cup b_1 \wedge b_0 | b_1)$$

and the "density theorem"

$$\bigwedge a \bigwedge c \bigvee b (c < a \to c < b < a) .$$

The first says that any nonzero r.e. degree can be written as the join of two incomparable r.e. degrees. Let $(\boldsymbol{c}, \boldsymbol{a})$ be any interval of r.e. degrees where the notation is to always imply that $\boldsymbol{c}, \boldsymbol{a}$ are r.e. degrees and $\boldsymbol{c} < \boldsymbol{a}$. The second theorem says that $(\boldsymbol{c}, \boldsymbol{a}) \neq \emptyset$. Both were proved by Sacks, the former in [4, p. 217, Corollary 1] and the latter in [5]. It was tempting to conjecture that a common generalization of these two theorems might be true:

$$(*) \qquad \bigwedge a \bigwedge c \bigvee b_0 \bigvee b_1 (c < a \to . \ a = b_0 \cup b_1 \wedge c < b_0 \wedge c < b_1 \wedge b_0 | b_1).$$

That is, for every interval $(\boldsymbol{c}, \boldsymbol{a})$ it might be possible to write $\boldsymbol{a}$ as the join of two members of $(\boldsymbol{c}, \boldsymbol{a})$. This questions was specifically raised by Robinson in [1, p. 313] where some generalizations of the density theorem may be found. The purpose of this paper is to show that (*) is not true.

The only progress that has been made in the other direction, i.e., towards determining when one r.e. degree will split over another, is the following unpublished theorem of Robinson:

$$\bigwedge a \bigwedge c \bigvee b_0 \bigvee b_1 (c < a \wedge c' = 0' .\rightarrow. a = b_0 \cup b_1 \wedge c < b_0 \wedge c < b_1 \wedge b_0 | b_1).$$

The plan of the paper is as follows. In § 1 we formulate the problem in a way appropriate to our solution. In § 2 we present the construction of a counterexample to (∗) and discuss the intuition behind the construction. In § 3 we list some properties of the construction and prove the two main propositions while assuming the rest. In § 4 we discuss the verification of the properties of the construction taken for granted in § 3. This last section presents peculiar difficulties in that we have not yet found any reasonable way to carry out the verification. The proofs in this part of the paper do not provide new insight. They merely serve to demonstrate that the construction, which is intuitively plausible, really does serve the purpose for which it is intended.

For general information on the theory of recursive functions the reader should consult [3]. A good introduction to degree theory is [7].

1. Formulation of the problem

For $A, C \subset \omega$ let $A + C$ denote $\{2x : x \in A\} \cup \{2x + 1 : x \in C\}$ so that $\deg(A + C) = \deg(A) \cup \deg(C)$. We shall construct r.e. sets A and C such that $\deg(A) \nleq \deg(C)$, and for any r.e. sets B^0, B^1

$$\deg(A + C) = \deg(B^0 + C) \cup \deg(B^1 + C) \Rightarrow \deg(A) \leqslant \deg(B^0 + C) \vee \deg(A) \leqslant \deg(B^1 + C) .$$

This is clearly sufficient to refute (∗), because we can take $a = \deg(A + C)$ and $c = \deg(C)$. The sets A and C will be enumerated in an infinite number of stages numbered 0, 1, 2, ... at most one number being enumerated in $A \cup C$ at any given stage.

It is customary in recursion theory to consider partial functions whose arguments and values range through ω. Here we find it convenient to turn partial functions into total ones by introducing ω as a new element of the codomain. Thus instead of a partial function φ in $\cup\{\omega^A : A \subset \omega\}$ we shall consider $\varphi' \in (\omega \cup \{\omega\})^\omega$, where $\varphi'(x) = \varphi(x)$ if $\varphi(x)$ is defined and $\varphi'(x) = \omega$ otherwise. For the enumeration of A and C we suppose given binary functions Ξ_i, ξ_i, θ_i, θ_i, Ψ_i^0, ψ_i^0, Ψ_i^1, ψ_i^1 and finite sets of natural numbers $B_i^0(s)$, $B_i^1(s)$ for each i and $s < \omega$. More precisely $\lambda x\, \Xi_i(x, s)$,

$\lambda x\, \xi_i(x, s)$, $\lambda x\, \theta_i(x, s)$, $\lambda x\, \theta_i(x, s)$, $\lambda x\, \Psi_i^0(x, s)$, $\lambda x\, \psi_i^0(x, s)$, $\lambda x\, \Psi_i^1(x, s)$, $\lambda x\, \psi_i^1(x, s)$, $B_i^0(s)$, and $B_i^1(s)$ are assumed to be given immediately after stage s. For the reader unfamiliar with Church's λ-notation we recall that $\lambda x\, \tau(x, s)$ denotes the unary function f such that $f(x) = \tau(x, s)$ for each x. The sequences $\langle B_i^0(s) : s < \omega\rangle$ and $\langle B_i^1(s) : s < \omega\rangle$ are to be increasing with respect to $\subset$ but not necessarily strictly increasing. We say that *n is enumerated in* $\boldsymbol{B}_i^0$ *at stage* s if either $s = 0$ and $n \in B_i^0(0)$, or $s > 0$ and $n \in B_i^0(s) - B_i^0(s-1)$. Similarly for $\boldsymbol{B}_i^1$. We shall suppose further that for all i, s and t the following seven conditions are satisfied, for convenience we drop the subscript i:

(1) $x \leqslant \xi(x, s) \leqslant \xi(x+1, s)$, and similarly for θ, ψ^0 and ψ^1;

(2) if $\xi(x, s), \xi(x, t) < \omega$ and $s < t$ then $\xi(x, s) \leqslant \xi(x, t)$, and similarly for θ, ψ^0, and ψ^1;

(3) $\Xi(x, s) = \omega$ if and only if $\xi(x, s) = \omega$, and similarly for $\langle \Theta, \theta\rangle$, $\langle \Psi^0, \psi^0\rangle$, and $\langle \Psi^1, \psi^1\rangle$;

(4) if $\Xi(x, s+1) \neq \Xi(x, s) < \omega$ or $\xi(x, s) < \xi(x, s+1)$ then some number $\leqslant \xi(x, s)$ is enumerated in $\boldsymbol{C}$ at stage $s+1$;

(5) if $\Theta(x, s+1) \neq \Theta(x, s) < \omega$ or $\theta(x, s) < \theta(x, s+1)$ then some number $\leqslant \theta(x, s)$ is enumerated by $\boldsymbol{B}^0 \cup \boldsymbol{B}^1 \cup \boldsymbol{C}$ at stage $s+1$;

(6) if $\Psi^0(x, s+1) \neq \Psi^0(x, s) < \omega$ or $\psi^0(x, s) < \psi^0(x, s+1)$ then some number $\leqslant \psi^0(x, s)$ is enumerated in $\boldsymbol{A} \cup \boldsymbol{C}$ at stage $s+1$, and similarly for $\langle \Psi^1, \psi^1\rangle$;

(7) if x is enumerated in $\boldsymbol{B}^0$ at stage $s+1$ then $\Psi^0(x, s) = 1$, and similarly for $\langle \boldsymbol{B}^1, \Psi^1\rangle$.

For $i < \omega$ and $j < 2$ let $\boldsymbol{B}_i^j$ denote $\cup\{B_i^j(s) : s < \omega\}$. For a sequence φ in $(\omega \cup \{\omega\})^\omega$ let $\lim_s \varphi(s)$ be n if $\varphi(s) = n$ for all sufficiently large s, and be ω otherwise. Let $\boldsymbol{\xi}_i$ denote the unary function defined by $\boldsymbol{\xi}_i(x) = \lim_s \xi_i(x, s)$. Let $\boldsymbol{\theta}_i$, $\boldsymbol{\psi}_i^0$, and $\boldsymbol{\psi}_i^1$ be defined similarly. Let $\boldsymbol{\Xi}_i$ be the unary function defined by $\boldsymbol{\Xi}_i(x) = \lim_s \Xi_i(x, s)$ if $\boldsymbol{\xi}_i(x) < \omega$, and $\boldsymbol{\Xi}_i(x) = \omega$ otherwise. Let $\boldsymbol{\Theta}_i$, $\boldsymbol{\Psi}_i^0$, and $\boldsymbol{\Psi}_i^1$ be defined similarly. In the latter sections of the paper we shall show how effectively to enumerate $\boldsymbol{A}$ and $\boldsymbol{C}$, from the given functions and sets, so that, for all i.

(8) $\boldsymbol{\Xi}_i \neq \boldsymbol{A}$,

(9) $\boldsymbol{A} = \boldsymbol{\Theta}_i \wedge \boldsymbol{B}_i^0 = \boldsymbol{\Psi}_i^0 \wedge \boldsymbol{B}_i^1 = \boldsymbol{\Psi}_i^1 .\!\rightarrow\!. \boldsymbol{A} \leqslant_T \boldsymbol{B}_i^0 + \boldsymbol{C} \vee \boldsymbol{A} \leqslant_T \boldsymbol{B}_i^1 + \boldsymbol{C}$.

This program is sufficient for ous needs provided that for all i, B_i^0, B_i^1, Ξ_i, ξ_i, etc. can be effectively generated in such a way that for every r.e. set $\boldsymbol{D}$ recursive in $\boldsymbol{C}$ there exists i such that $\boldsymbol{\Xi}_i = \boldsymbol{D}$, and for every pair of r.e. sets $\langle \boldsymbol{D}^0, \boldsymbol{D}^1\rangle$ which satisfies

(10) $\boldsymbol{D}^0 \leqslant_T \boldsymbol{A} + \boldsymbol{C} \wedge \boldsymbol{D}^1 \leqslant_T \boldsymbol{A} + \boldsymbol{C} \wedge \boldsymbol{A} \leqslant_T \boldsymbol{D}^0 + \boldsymbol{D}^1 + \boldsymbol{C}$,

there exists an i such that

(11) $A = \boldsymbol{\theta}_i \wedge D^0 = B_i^0 = \psi_i^0 \wedge D^1 = B_i^1 = \psi_i^1$.

To complete this section we indicate briefly how effectively to generate the given functions Ξ_i, ξ_i, etc., and B_i^0, B_i^1 in a suitable way. Let

$$\langle\langle '\Xi_i, '\boldsymbol{\theta}_i, '\psi_i^0, '\psi_i^1, 'B_i^0, 'B_i^1\rangle : i < \omega\rangle ,$$

be an effective enumeration of all sextuples $\langle '\Xi, '\boldsymbol{\theta}, '\psi^0, '\psi^1, 'B^0, 'B^1\rangle$ such that $'B^0$, $'B^1$ are recursively enumerable subsets of ω, and $'\Xi$, $'\Theta$, $'\Psi^0$, $'\Psi^1$ are partial recursive functionals mapping 2^ω, $2^\omega \times 2^\omega \times 2^\omega$, $2^\omega \times 2^\omega$ and $2^\omega \times 2^\omega$, respectively, into $(2 \cup \{\omega\})^\omega$. Let $A(s)$ and $C(s)$ denote the finite sets of numbers which have been enumerated in A and C respectively by the end of stage s. If x, $i < s$ and $'\Xi_i(C(s))(x)$ is computed in $<s$ steps, let $'\xi_i(x, s)$ be the number of steps in its computation. Otherwise let $'\xi_i(x, s)$ be ω. Define

$$\xi_i(x, s) = \sup(\{'\xi_i(x, s)\} \cup \{\xi_i(y, s) : y < x\} \cup \{\xi_i(x, t) : \xi_i(x, t) < \omega \wedge t < s\}) .$$

Let $\Xi_i(x, s) = {'\Xi_i}(C(s))(x)$ if $\xi_i(x, s) < \omega$, and let $\Xi_i(x, s) = \omega$ otherwise. The pair $\langle \Theta_i, \theta_i\rangle$ is defined similarly except that the function arguments are now $B_i^0(s)$, $B_i^1(s)$, $C(s)$. Similarly for the definition of each of the pairs $\langle \Psi_i^0, \psi_i^0\rangle$ and $\langle \Psi_i^1, \psi_i^1\rangle$ the function arguments are $A(s)$, $C(s)$. For $j < 2$ let $'B_i^j(s)$ be the finite set of numbers enumerated in $'B_i^j$ by the end of stage s. For $j < 2$ let $B_i^j(s)$ be $\emptyset$ if $s \leqslant i$, and be

$$B_i^j(s-1) \cup \{x : x < s \wedge x \in {'B_i^j}(s) \wedge \Psi_i^j(x, s) = 1\}$$

otherwise. It is easy to check that the conditions (1)–(7) are all satisfied and that if $\langle \boldsymbol{D}, \boldsymbol{D}^0, \boldsymbol{D}^1\rangle$ is any triple of r.e. sets satisfying $\boldsymbol{D} \leqslant_T \boldsymbol{C}$ and (10), then there exists i such that $\Xi_i = \boldsymbol{D}$ and (11).

2. The priority tree and the construction

In this section we shall state a method of effectively enumerating A and C so that the requisite conditions

$$\mathfrak{C}_i \qquad A = \boldsymbol{\theta}_i \wedge B_i^0 = \psi_i^0 \wedge B_i^1 = \psi_i^1 .\!\rightarrow\!. A \leqslant_T B_i^0 + C \vee A \leqslant_T B_i^1 + C$$

and $\mathfrak{C}_i' : \Xi_i \neq A$, are satisfied for all i. As is often the case in the construction of r.e. sets we shall have to weave together many different strategies to allow for all possible contingencies. We first define a tree structure which will provide us with a way of classifying the atomic parts into which our construction naturally falls. Let a_0, a_1, a_2, w, d_0, d_1 and f be primitive objects. A finite sequence $\boldsymbol{a} = \langle \boldsymbol{a}_0, ..., \boldsymbol{a}_l \rangle$ is a *characteristic sequence* or *characteristic* for short if either $\boldsymbol{a} = \emptyset$, or

$$\boldsymbol{a}_l \in \{a_0, a_1, a_2, w, d_0, d_1\} \cup (\{f\} \times \omega),$$

$$\Lambda i < l\, (\boldsymbol{a}_i \in \{w, d_0, d_1\} \cup (\{f\} \times \omega)).$$

Let the set of characteristics $\boldsymbol{a}$ such that $\boldsymbol{a}_l \notin \{a_0, a_1, a_2\}$ be denoted by $\boldsymbol{G}$, and let the set of all characteristics be denoted by $\boldsymbol{G}^*$. By $\boldsymbol{a} \frown a_0$ we mean $\langle \boldsymbol{a}_0, ..., \boldsymbol{a}_l, a_0 \rangle$, by $\boldsymbol{a} \frown (f, j)$ we mean $\langle \boldsymbol{a}_0, ..., \boldsymbol{a}_l, (f, j) \rangle$,and so on. The *order of* $\boldsymbol{a}$ denoted $\mathrm{O}(\boldsymbol{a})$ is defined inductively for all $\boldsymbol{a} \in \boldsymbol{G}$ by $\mathrm{O}(\emptyset) = 0$, $\mathrm{O}(\boldsymbol{b} \frown d_0) = \mathrm{O}(\boldsymbol{b})$, $\mathrm{O}(\boldsymbol{b} \frown w) = \mathrm{O}(\boldsymbol{b} \frown d_1) = \mathrm{O}(\boldsymbol{b}) + 1$, and $\mathrm{O}(\boldsymbol{b} \frown (f, n)) = n + 1$. Let $<^*$ be the unique strict linear ordering of $\{a_0, a_1, a_2, w, d_0, d_1\} \cup (\{f\} \times \omega)$ such that

$$w <^* a_2 <^* (f, i) <^* (f, j) <^* a_1 <^* d_1 <^* a_0 <^* d_0$$

for all $i, j \in \omega$ such that $i < j$. For characteristic sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ we define $\boldsymbol{a} < \boldsymbol{b}$ to hold if there exists i such that $\boldsymbol{a}_i$ and $\boldsymbol{b}_i$ are both defined and different, and $\boldsymbol{a}_i <^* \boldsymbol{b}_i$ for the least such i.

For each $\boldsymbol{a} \in \boldsymbol{G}$, where $\boldsymbol{a} = \langle \boldsymbol{a}_0, ..., \boldsymbol{a}_l \rangle$ we define

$$D_0(\boldsymbol{a}) = \{i : \mathrm{V} j (j \leqslant l \wedge \boldsymbol{a}_j = d_0 \wedge \mathrm{O}(\boldsymbol{a} \upharpoonright j) = i)\},$$

$$F(\boldsymbol{a}) = \{i : \mathrm{V} j (j \leqslant l \wedge \boldsymbol{a}_j = (f, i) \wedge \Lambda k (j < k \leqslant l \rightarrow \boldsymbol{a}_k \notin$$
$$\notin \{(f, 0), (f, 1), ..., (f, i-1)\}))\}.$$

Certain $\boldsymbol{a} \in \boldsymbol{G}^*$ will have no significance for us. The reader will easily verify after the statement of the construction that if $\boldsymbol{a}$ is the characteristic of one of the stages, then:

L1. $\quad \mathrm{O}(\boldsymbol{b}) = n \in D_0(\boldsymbol{b}) \wedge \boldsymbol{b} \subset \boldsymbol{a} .\rightarrow . \boldsymbol{b} \frown d_0 \not\subset \boldsymbol{a} \wedge \boldsymbol{b} \frown a_0 \neq \boldsymbol{a}$,

L2. $\quad \mathrm{O}(\boldsymbol{b}) = n \wedge \boldsymbol{b} \frown (f, j) \subset \boldsymbol{a} .\rightarrow . j \notin F(\boldsymbol{b}) \cup D_0(\boldsymbol{b}) \wedge j \leqslant n$.

Call $\boldsymbol{a} \in \boldsymbol{G}^*$ *legitimate* it is satisfies L1 and L2.

The construction. As well as enumerating A and C we simultaneously construct auxiliary functions: φ with domain $G \times \omega \times \omega$, and ϵ, μ, ν, and $\boldsymbol{a}$ all with domain $G \times \omega$. The codomain of $\boldsymbol{a}$ is $G \cup \{\omega\}$; the others have codomain $\omega \cup \{\omega\}$. In stage s we shall specify $\varphi(\boldsymbol{b}, n, s)$, $\epsilon(\boldsymbol{b}, s)$, $\mu(\boldsymbol{b}, s)$, $\nu(\boldsymbol{b}, s)$, and $\boldsymbol{a}(\boldsymbol{b}, s)$ for all $\boldsymbol{b} \in G$ and $n \in \omega$. To prevent unwanted clashes between the values of the auxiliary functions we choose a recursive partition of ω into infinite sets, one member of the partition being associated in an effective manner with each member of the set

$$\{\boldsymbol{c} : \boldsymbol{c} \in G\} \cup \{(\boldsymbol{c}, n) : \boldsymbol{c} \in G \wedge n \in \omega\}\,.$$

The numbers in the member of the partition associated with $\boldsymbol{c}$ are called *$\boldsymbol{c}$-numbers*, those in the member associated with $(\boldsymbol{c}, n)$ are called *$(\boldsymbol{c}, n)$-numbers*. If $\epsilon(\boldsymbol{b}, s) \neq \omega$ then $\epsilon(\boldsymbol{b}, s)$ will be a $\boldsymbol{b}$-number, and if $\varphi(\boldsymbol{b}, n, s) \neq \omega$ it will be a $(\boldsymbol{b}, n)$-number.

The characteristic functions of $A(s)$ and $B_i(s)$ will be denoted by $\lambda x\, A(x, s)$ and $\lambda x\, B_i(x, s)$. In stage 0 we let $\varphi(\boldsymbol{b}, n, 0) = \epsilon(\boldsymbol{b}, 0) = \mu(\boldsymbol{b}, 0) = \nu(\boldsymbol{b}, 0) = \boldsymbol{a}(\boldsymbol{b}, 0) = \omega$ for all $\boldsymbol{b} \in G$ and $n \in \omega$. Each stage s has a characteristic associated with it which we denote by $\boldsymbol{a}(s)$. We let $\boldsymbol{a}(0) = \langle a_0 \rangle$. Stage $s + 1$ will consist of substages numbered $(s + 1, 0)$, $(s + 1, 1)$, ... and so on. Before stage $(s + 1, i)$ we shall have defined $\boldsymbol{a}(s + 1)_j$ for each $j < i$. We shall also have specified a certain interval $I(i, s)$ of ω, where $I(i, 0) = \omega$. Let $\boldsymbol{b}$ denote $\langle \boldsymbol{a}(s + 1)_0, ..., \boldsymbol{a}(s + 1)_{i-1} \rangle$ and $\mathrm{O}(\boldsymbol{b}) = n$. The various cases that can arise in Stage $(s + 1, i)$ are as follows. In each one we are tacitly assuming that no earlier case holds.

C0. $I(i, s) = \emptyset$. Let $\boldsymbol{a}(s + 1) = \boldsymbol{b} \frown a_0$ and stop the whole construction. (It will be shown that C0 never occurs.)

C1. There exists a $\boldsymbol{c}$ such that $\boldsymbol{b} \subset \boldsymbol{a}(\boldsymbol{c}, s)$, $\epsilon(\boldsymbol{c}, s) < \omega$, $\epsilon(\boldsymbol{c}, s) \in A(s)$, and $\nu(\boldsymbol{c}, s) = n \notin D_0(\boldsymbol{b})$. Choose such a $\boldsymbol{c}$ minimal with respect to $<$ and then minimal with respect to $\subset$. Act according to the first of the following subcases which holds. Stage $(s + 1, i)$ is said to *pertain to* $\epsilon(\boldsymbol{c}, s)$.

C1.1. $n \notin F(\boldsymbol{a}(\boldsymbol{c}, s))$. Enumerate $\varphi(\boldsymbol{a}(\boldsymbol{c}, s), n, s)$ in C. Let $\boldsymbol{a}(\boldsymbol{c}, s + 1) = \nu(\boldsymbol{c}, s + 1) = \omega$. Let $\boldsymbol{a}(s + 1)_i$ be a_0.

C1.2. $n \in F(\boldsymbol{a}(\boldsymbol{c}, s))$ and some number $\leqslant \theta_n(\epsilon(\boldsymbol{c}, s), t)$ has been enumerated in B_n^0 since stage t where stage $t + 1$ was the one in which $\epsilon(\boldsymbol{c}, s)$ was enumerated in A. Let $\boldsymbol{d}$ be the greatest initial segment of $\boldsymbol{a}(\boldsymbol{c}, s)$ such that $\boldsymbol{d} \frown (f, n) \subset \boldsymbol{a}(\boldsymbol{c}, s)$. If there exists m, $n < m \leqslant \mathrm{O}(\boldsymbol{d})$ and $m \notin D_0(\boldsymbol{d})$, let $\boldsymbol{a}(\boldsymbol{c}, s + 1) = \boldsymbol{d}$ and $\nu(\boldsymbol{c}, s + 1)$ be the least such m. Otherwise let $\boldsymbol{a}(\boldsymbol{c}, s + 1) = \nu(\boldsymbol{c}, s + 1) = \omega$. Let $\varphi(\boldsymbol{d}, n, s + 1) = \omega$ and $\boldsymbol{a}(s + 1)_i$ be a_0. We say $\varphi(\boldsymbol{d}, n, s)$ is *destroyed through* C1.2 if $\varphi(\boldsymbol{d}, n, s) < \omega$.

C1.3. $n \in F(\boldsymbol{a}(\boldsymbol{c}, s))$ and some number $\leqslant \theta_n(\epsilon(\boldsymbol{c}, s), t)$ has been enumerated in $\boldsymbol{B}_n^1$ since stage t where stage $t+1$ was the one in which $\epsilon(\boldsymbol{c}, s)$ was enumerated in $\boldsymbol{A}$. Let $\boldsymbol{d}$ be defined as in C1.2. Enumerate $\varphi(\boldsymbol{d}, n, s)$ in $\boldsymbol{C}$ if $\varphi(\boldsymbol{d}, n, s) < \omega$. If there exists an m, $n < m \leqslant O(\boldsymbol{a}(\boldsymbol{c}, s))$ and $m \notin D_0(\boldsymbol{a}(\boldsymbol{c}, s))$, let $\boldsymbol{a}(\boldsymbol{c}, s+1) = \boldsymbol{a}(\boldsymbol{c}, s)$ and $\nu(\boldsymbol{c}, s+1)$ be the least such m. Otherwise let $\boldsymbol{a}(\boldsymbol{c}, s+1) = \nu(\boldsymbol{c}, s+1) = \omega$. Let $\varphi(\boldsymbol{a}(\boldsymbol{c}, s), n, s+1) = \omega$ and $\boldsymbol{a}(s+1)_i = a_0$. We say that $\varphi(\boldsymbol{a}(\boldsymbol{c}, s), n, s)$ is *destroyed through* C1.3. If one of C1.1, 2, 3 occurs, we say stage $s+1$ *pertains to* $\epsilon(\boldsymbol{c}, s)$.

C1.4. *Otherwise.* Let $\boldsymbol{a}(s+1)_i = d_0$. Let $I(i+1, s)$ be obtained by subtracting from $I(i, s)$ each interval $[0, \mu(\boldsymbol{d}, s)]$ such that $\boldsymbol{d} \cap a_0 < \boldsymbol{b} \cap \boldsymbol{d}_0$ and $\epsilon(\boldsymbol{d}, s) < \omega$.

C2. There exists $\boldsymbol{c} \supset \boldsymbol{b}$ such that $\epsilon(\boldsymbol{c}, s) < \omega$, $n \notin D_0(\boldsymbol{c})$, $F(\boldsymbol{c}) - F(\boldsymbol{b})$ contains no number $< n$ and $\varphi(\boldsymbol{c}, n, s) = \omega$. Choose such $\boldsymbol{c}$ minimal with respect to $<$ and then maximal with respect to $\subset$. We say that stage $(s+1, i)$ is associated with $\boldsymbol{c}$. There are two subcases.

C2.1. There exists a $(\boldsymbol{c}, n)$-number p in $I(i, s)$ such that: $p > \mu(\boldsymbol{c}, s)$; $p > \varphi(\boldsymbol{c}, n, t)$ for each $t < s$ such that $\varphi(\boldsymbol{c}, n, t) < \omega$; $p > \mu(\boldsymbol{d}, s)$ for each $\boldsymbol{d}$ such that $\epsilon(\boldsymbol{d}, s) < \omega$, and either $\boldsymbol{c} \cap (f, n) \subset \boldsymbol{d}$ or $\boldsymbol{d} < \boldsymbol{c} \cap (f, n)$;

$$A(x, s) = \Theta_n(x, s) \qquad \text{for all } x \leqslant \mu(\boldsymbol{c}, s),$$

$$B_n^i(x, s) = \Psi_n^i(x, s) \qquad \text{for all } x \leqslant \Theta_n(\mu(\boldsymbol{c}, s), s) \quad i \leqslant 1\,,$$

$$\sup(\{\theta_n(\mu(\boldsymbol{c}, s), s)\} \cup \{\psi_n^i(x, s) : x \leqslant \theta_n(\mu(\boldsymbol{c}, s), s), i \leqslant 1\}) < p\,.$$

Let $\varphi(\boldsymbol{c}, n, s+1)$ be the least such $(\boldsymbol{c}, n)$-number. Let $\mu(\boldsymbol{c}, s+1) = \varphi(\boldsymbol{c}, n, s+1)$ and $\boldsymbol{a}(s+1)_i = a_0$. In this case we say that stage $s+1$ *is associated with* $\boldsymbol{c}$.

C2.2. *Otherwise.* Let $\boldsymbol{a}(s+1)_i = d_0$. Let $I(i+1, s)$ be obtained by subtracting from $I(i, s)$ each interval $[0, \mu(\boldsymbol{d}, s)]$ such that $\boldsymbol{d} \cap a_0 < b \cap \boldsymbol{d}_0$ and $\epsilon(\boldsymbol{d}, s) < \omega$. Go to stage $(s+1, i+1)$.

C3. $\epsilon(\boldsymbol{b}, s) = \omega$. Enumerate $\sup I(i, s)$ in $\boldsymbol{C}$ if $\sup I(i, s) < \omega$. Let $\epsilon(\boldsymbol{b}, s+1)$ and $\mu(\boldsymbol{b}, s+1)$ be the least $\boldsymbol{b}$-number which is $>$ every number used at a stage $\leqslant s$. Let $\boldsymbol{a}(s+1)_i = a_1$.

C4. $\xi_n(\epsilon(\boldsymbol{b}, s), s) \geqslant \sup I(i, s)$. Let $\boldsymbol{a}(s+1)_i = d_1$. Let $I(i+1, s)$ be obtained by subtracting from $I(i, s)$ each interval $[0, \mu(\boldsymbol{d}, s)]$ such that $\boldsymbol{d} \cap a_1 < \boldsymbol{b} \cap d_1$ and $\epsilon(\boldsymbol{d}, s) < \omega$. Go to stage $(s+1, i+1)$.

C5. $\xi_n(\epsilon(\boldsymbol{b}, s), s) < \varphi(\boldsymbol{b}, j, s)$ for each $j \leqslant n$ such that $j \notin F(\boldsymbol{b})$. There are three subcases:

C5.1. $\mu(\boldsymbol{b}, s) < \xi_n(\epsilon(\boldsymbol{b}, s), s)$. Let $\boldsymbol{a}(s+1)_i = a_2$ and

$$\mu(\boldsymbol{b}, s+1) = \sup\{\mu(\boldsymbol{b}, s), \xi_n(\epsilon(\boldsymbol{b}, s), s)\}\ .$$

C5.2. $\Xi_n(\epsilon(\boldsymbol{b}, s), s) = 1 - A(\epsilon(\boldsymbol{b}, s), s)$. Let $\boldsymbol{a}(s+1)_i = w$, and $I(i+1, s)$ be obtained by subtracting from $I(i, s)$ each interval $[0, \mu(\boldsymbol{d}, s)]$ such that $\boldsymbol{d} \leqslant \boldsymbol{b}$ and $\epsilon(\boldsymbol{d}, s) < \omega$. Go to stage $(s+1, i+1)$.

C5.3. Otherwise. Enumerate $\epsilon(\boldsymbol{b}, s)$ in A provided that $A(\epsilon(\boldsymbol{b}, s), s) = 0$. Let $\boldsymbol{a}(s+1)_i = a_2$. If there exists $m \leqslant n$, $m \notin D_0(\boldsymbol{b})$, let $\boldsymbol{a}(\boldsymbol{b}, s+1) = \boldsymbol{b}$, and $\nu(\boldsymbol{b}, s+1)$ be the least such m. Otherwise let $\boldsymbol{a}(\boldsymbol{b}, s+1) = \nu(\boldsymbol{b}, s+1) = \omega$.

C6. There exists $j \leqslant n$, $j \notin F(\boldsymbol{b})$, such that $\varphi(\boldsymbol{b}, j, s) \leqslant \xi_n(\epsilon(\boldsymbol{b}, s), s)$. Choose the greatest such j. Let $\boldsymbol{a}(i, s+1) = (f, j)$, and let $I(i+1, s)$ be obtained by subtracting from $I(i, s)$ each interval $[0, \mu(\boldsymbol{d}, s)]$ such that $\boldsymbol{d} < \boldsymbol{b} \frown (f, j)$ and $\epsilon(\boldsymbol{d}, s) < \omega$, and by deleting from $I(i, s)$ all numbers which are either not greater than

$$\sup(\{\epsilon(\boldsymbol{b}, s)\} \cup \{\varphi(\boldsymbol{b}, k, s) : k < j \text{ and } \varphi(\boldsymbol{b}, k, s) < \omega\})$$

or $> \varphi(\boldsymbol{b}, j, s)$. Let $\mu(\boldsymbol{b}, s+1) = \sup\{\mu(\boldsymbol{b}, s), \xi_n(\epsilon(\boldsymbol{b}, s), s)\}$. Go to stage $(s+1, i+1)$.

This completes the list of cases. Unless stage $(s+1, i+1)$ is specified, stage $(s+1, i)$ is the last substage of stage $s+1$. Only some values of the auxiliary functions for argument $s+1$ have been given explicitly by the statement above. The remaining values are specified as follows. If $\boldsymbol{a}(s+1) < \boldsymbol{d}$ or some number $\leqslant \epsilon(\boldsymbol{d}, s)$ is enumerated in $\boldsymbol{C}$ at stage $s+1$ then $\epsilon(\boldsymbol{d}, s+1) = \omega$. Otherwise $\epsilon(\boldsymbol{d}, s+1) = \epsilon(\boldsymbol{d}, s)$. In like manner $\varphi(\boldsymbol{d}, n, s+1)$ is defined from $\varphi(\boldsymbol{d}, n, s)$. If $\epsilon(\boldsymbol{d}, s+1) = \omega$ then $\mu(\boldsymbol{d}, s+1) = \boldsymbol{a}(\boldsymbol{d}, s+1) = \nu(\boldsymbol{d}, s+1) = \omega$; otherwise $\mu(\boldsymbol{d}, s+1) = \mu(\boldsymbol{d}, s)$, $\boldsymbol{a}(\boldsymbol{d}, s+1) = \boldsymbol{a}(\boldsymbol{d}, s)$, and $\nu(\boldsymbol{d}, s+1) = \nu(\boldsymbol{d}, s)$. We say that $\epsilon(\boldsymbol{d}, s)$ is assigned at stage $t+1$ if $\epsilon(\boldsymbol{d}, s) < \omega$ and t is the least number such that $\epsilon(\boldsymbol{d}, t+1) = \epsilon(\boldsymbol{d}, s)$. Similarly for the other auxiliary functions. We say that $\epsilon(\boldsymbol{d}, s)$ is destroyed at stage $t+1$ if $s \leqslant t$ and $\epsilon(\boldsymbol{d}, s) = \epsilon(\boldsymbol{d}, t) < \omega = \epsilon(\boldsymbol{d}, t+1)$. Similarly for the other auxiliary functions; $\varphi(\boldsymbol{d}, n, s)$ is destroyed through C1 if it is destroyed either through C1.2 or through C1.3.

We now state some trivial properties of the auxiliary functions in the construction, properties which may be verified by straightforward induction. These properties should be borne in mind for when we have to establish the chain of propositions which witnesses the validity of the construction.

t1. $\epsilon(c, s) < \omega$ if and only if $\mu(c, s) < \omega$, and $\epsilon(c, s) \leqslant \mu(c, s)$.

t2. If $\varphi(c, n, s) < \omega$, then $n \leqslant O(c)$, $n \notin D_0(c)$, and

$$\epsilon(c, s) < \varphi(c, n, s) \leqslant \mu(c, s) < \omega .$$

t3. $a(c, s) = \omega$ if and only if $\nu(c, s) = \omega$.

t4. If $a(c, s) \neq \omega$, then $\epsilon(c, s) < \omega$, $\epsilon(c, s)$ is enumerated in A at a stage $\leqslant s$, $a(c, s) \subset c$, $\nu(c, s) \leqslant O(a(c, s))$, and $\nu(c, s) \notin D_0(a(c, s))$.

t5. If $s < t$ and $a(c, u) \neq \omega$ for all u in $s \leqslant u \leqslant t$, then $a(c, t) \subset a(c, s)$ and $\nu(c, t) \geqslant \nu(c, s)$. Further, if $a(c, t) \neq a(c, s)$ then $\nu(c, t) \neq \nu(c, s)$.

t6. If $a(c, s) \neq \omega$ and $a(c, s) \neq c$ then for some $m < \nu(c, s)$, $a(c, s)^\frown (f, m) \subset c$.

t7. If $\epsilon(c, t) \leqslant \epsilon(b, t) < \omega$ and $c < b$ or $c \subset b$, and $\epsilon(c, t)$ is destroyed at stage $s + 1$ then $\epsilon(b, t)$ is destroyed at a stage $\leqslant s + 1$. Similarly with $\varphi(b, n, t)$ in place of $\epsilon(b, t)$.

t8. If $s \leqslant t$ and $\epsilon(b, s)$, $\epsilon(b, t)$ are both $< \omega$ then $\epsilon(b, s) \leqslant \epsilon(b, t)$. Similarly for $\langle \mu(b, s), \mu(b, t)\rangle$ and $\langle \phi(b, n, s), \phi(b, n, t)\rangle$.

t9. If $i \leqslant j$ and there is a stage $(s + 1, j)$ then $I(j, s) \subset I(i, s)$.

Motivation of the construction. With each $d \in G$ will be associated a finite number of strategies. If $d < e \in G$ then the strategies associated with d will have priority over those associated with e. Let $d \in G$ and $O(d) = m$. We can think of d as representing the hypothesis $H(d)$: "there are infinitely many s such that $a(s) \supset d$, and there are at most a finite number of s such that $a(s) < d$". A stage $(s + 1, i)$ such that $a(s + 1) \upharpoonright i \supset d$ is based on the hypothesis $H(d)$. With d is associated a strategy $S(d)$ whose aim is the satisfaction of $\mathfrak{C}'_m$; $S(d)$ is pursued in each stage $(s + 1, i)$ such that $a(s + 1) \upharpoonright i = d$.

Of course $S(d)$ must respect the need to satisfy $\mathfrak{C}_i$ for $i \leqslant m$. If $i \in F(d)$ then under $H(d)$ the strategy for satisfying $\mathfrak{C}_i$ is associated with the greatest $\boldsymbol{e} \subset \boldsymbol{d}$ such that $\boldsymbol{e} \frown (f, i) \subset \boldsymbol{d}$. (Such $\boldsymbol{e}$ exists from the definition of $F(\boldsymbol{d})$.) If $i \in D_0(\boldsymbol{d})$ then under $H(\boldsymbol{d})$

$$A = \theta_i \wedge B_i^0 = \psi_i^0 \wedge B_i^1 = \psi_i^1$$

is false, and so no action need be taken to satisfy $\mathfrak{C}_i$. Finally if $i \leqslant m$ and $i \notin F(\boldsymbol{d}) \cup D_0(\boldsymbol{d})$ then the strategy for $\mathfrak{C}_i$ is associated with the greatest $\boldsymbol{e} \subset \boldsymbol{d}$ such that $O(\boldsymbol{e}) = i$.

Let $\boldsymbol{a}(s+1) \upharpoonright i = \boldsymbol{d}$ then in stage $(s+1, i)$, $I(i, s)$ is the "universe" in which $S(\boldsymbol{d})$ must be pursued. We digress for a moment to compute the endpoints of the interval $I(i, s)$. The right-hand endpoint of $I(i, s)$ if any is just the least number of the form $\varphi(\boldsymbol{b}, j, s)$ such that $\boldsymbol{b} \frown (f, j) \subset \boldsymbol{d}$. This follows, because from the statement of the construction $\sup I(i+1, s) = \sup I(i, s)$ except in C6, and in C6 $\sup I(i+1, s) = \inf\{\sup I(i, s), \varphi(\boldsymbol{b}, j, s)\}$. The left-hand endpoint of $I(i, s)$ is the least number exceeding all those of the forms:

(i) $\epsilon(\boldsymbol{b}, s)$ where $\boldsymbol{b} \frown (f, j) \subset \boldsymbol{d}$ for some j,

(ii) $\varphi(\boldsymbol{b}, k, s)$ where $\varphi(\boldsymbol{b}, k, s) < \omega$ and $\boldsymbol{b} \frown (f, j) \subset \boldsymbol{d}$ for some $j > k$,

(iii) $\mu(\boldsymbol{c}, s)$ where $\mu(\boldsymbol{c}, s) < \omega$ and either $\boldsymbol{c} < \boldsymbol{d}$ or $\boldsymbol{c} \frown d_0 \subset \boldsymbol{d}$ or $\boldsymbol{c} \frown d_1 \subset \boldsymbol{d}$ or $\boldsymbol{c} \frown w \subset \boldsymbol{d}$.

This can be checked by looking through those cases of stage $(s+1, i)$ which do not terminate stage $s+1$, namely C1.4, C2.2, C4, C5.2, and C6. In looking through these cases one should bear in mind the following points. Firstly for all $\boldsymbol{c}$, $\epsilon(\boldsymbol{c}, s) < \omega$ if and only if $\mu(\boldsymbol{c}, s) < \omega$. (This can easily be verified by induction on s.) Secondly $\boldsymbol{e} \frown a_0 < \boldsymbol{c} \frown d_0$ if and only if either $\boldsymbol{e} < \boldsymbol{c} \frown d_0$ or $\boldsymbol{e} = \boldsymbol{c}$. Similarly for C4, $\boldsymbol{e} \frown a_1 < \boldsymbol{c} \frown d_1$ if and only if $\boldsymbol{e} < \boldsymbol{c} \frown d_1$ or $\boldsymbol{e} = \boldsymbol{c}$. One of the facts we shall prove later is that C0 never occurs, i.e. $I(i, s)$ is never $\emptyset$.

We now return to the discussion of $S(\boldsymbol{d})$. Let $H(\boldsymbol{d})$ be true, and $\{s_j : j < \omega\}$ be a strictly increasing sequence such that $\boldsymbol{a}(s_j + 1) \upharpoonright i = \boldsymbol{d}$. Suppose further that no stage $> s_0$ has characteristic $< \boldsymbol{d}$. In the limit $I(i, s_j)$ is a final segement of ω, i.e. for some p in ω

$$\lim_j I(i, s_j) = \{x : x \in \omega \wedge x > p\} .$$

In pursuing $S(\boldsymbol{d})$ we first give $\epsilon(\boldsymbol{d}, s)$ a value $< \omega$ larger than any other number yet used. This is the purpose of C3 in the construction. In C3 we enumerate $\sup I(i, s)$ in C because by the rules for determining $\varphi(\boldsymbol{e}, j, s+1)$ all the values of $\lambda \boldsymbol{x}\, \lambda y\, \varphi(\boldsymbol{x}, y, s)$ which bound $I(i, s)$ on the right will thereby be destroyed. Suppose $\epsilon(\boldsymbol{d}, s) = \omega > \epsilon(\boldsymbol{d}, s+1)$, $s' > s$, and

$a(s'+1) \upharpoonright i = \boldsymbol{d}$, then $\epsilon(\boldsymbol{d}, s')$ will be $\epsilon(\boldsymbol{d}, s+1)$ provided there is no stage $>s$ and $\leqslant s'$ with characteristic $<\boldsymbol{d}$. Further $\epsilon(\boldsymbol{d}, s')$ will be in $I(i, s')$. Recall that $O(\boldsymbol{d}) = m$. Our intention in $S(\boldsymbol{d})$ is to make Ξ_m and A differ at argument $\epsilon(\boldsymbol{d}, s)$. Associated with $\epsilon(\boldsymbol{d}, s)$ when s in one of the numbers s_h we shall have a number $\varphi(\boldsymbol{d}, j, s) \in I(i, s)$ for each $j < m$, $j \notin D_0(\boldsymbol{d})$. Also in a stage $(s_h + 1, i)$ one of our aims, after having made $\epsilon(\boldsymbol{d}, s) < \omega$, is to make $\varphi(\boldsymbol{d}, m, s) < \omega$ if $m \notin D_0(\boldsymbol{d})$. For $j \leqslant m$ and $j \notin D_0(\boldsymbol{d})$ the numbers $\varphi(\boldsymbol{d}, j, s)$ will be increasing with j, and if $k > j$, $\varphi(\boldsymbol{d}, k, s)$ will be chosen after $\varphi(\boldsymbol{d}, j, s)$. When $\varphi(\boldsymbol{d}, j, s)$ is chosen, i.e. given a value $<\omega$, the choice is based on the assumption that

$$A = \boldsymbol{\theta}_j \wedge B_j^0 = \boldsymbol{\psi}_j^0 \wedge B_j^1 = \boldsymbol{\psi}_j^1 .$$

Given this assumption we can choose $\varphi(\boldsymbol{d}, j, s)$ such that for all $p \leqslant \mu(\boldsymbol{d}, s)$ if p is subsequently enumerated in A then some number $\leqslant \varphi(\boldsymbol{d}, j, s)$ will be forced into B_j^0 or B_j^1. For if we change a value of A and restrain numbers from C, the only way $\boldsymbol{\theta}_j$ can change to restore the equality of A and $\boldsymbol{\theta}_j$ is through a sufficiently small number being enumerated in B_j^0 or B_j^1. The number $\mu(\boldsymbol{d}, s)$ bounds all the numbers associated with $\epsilon(\boldsymbol{d}, s)$ at the end of stage s.

Now we come to the heart of the strategy $S(\boldsymbol{d})$. If

$$\epsilon(\boldsymbol{d}, s) < \omega,$$

$$\Xi_m(\epsilon(\boldsymbol{d}, s), s) = A(\epsilon(\boldsymbol{d}, s), s) = 0 ,$$

we wish to enumerate $\epsilon(\boldsymbol{d}, s)$ in A and henceforth restrain numbers $\leqslant \xi_m(\epsilon(\boldsymbol{d}, s), s)$ from C, so as to force the inequality $\Xi_m \neq A$, thus satisfying $\mathfrak{C}'_m$. (This is what happens if C5.3 occurs in a stage $(s+1, i)$ with $a(s+1) \upharpoonright i = \boldsymbol{d}$.) However, for each $j \leqslant m$, $j \notin D_0(\boldsymbol{d}) \cup F(\boldsymbol{d})$, we are implicitly constructing an effective reduction of A to $B_j^0 + C$. These reductions have higher priority than the satisfaction of $\mathfrak{C}'_m$. For this reason we are only free to enumerate $\epsilon(\boldsymbol{d}, s)$ in A if $\xi_m(\epsilon(\boldsymbol{d}, s), s) < \varphi(\boldsymbol{d}, j, s)$ for each $j \leqslant m$, $j \notin D_0(\boldsymbol{d}) \cup F(\boldsymbol{d})$. (We can delete $D_0(\boldsymbol{d})$ if we like because for $j \notin D_0(\boldsymbol{d})$, $\varphi(\boldsymbol{d}, j, s) = \omega$.) The intuition here is as follows. If $\xi_m(\epsilon(\boldsymbol{d}, s), s) <$ $< \varphi(\boldsymbol{d}, j, s)$ and $\epsilon(\boldsymbol{d}, s)$ is enumerated in A, we can correct the reduction of A to $B_j^0 + C$ by enumerating $\varphi(\boldsymbol{d}, j, s)$ in C while at the same time restraining numbers $\leqslant \xi_m(\epsilon(\boldsymbol{d}, s), s)$ from C. If there exists an s such that $a(s+1) \upharpoonright i = \boldsymbol{d}$, $\epsilon(\boldsymbol{d}, s) < \omega$, and the conditions for enumerating $\epsilon(\boldsymbol{d}, s)$ in A are satisfied, then by C5.3 $\epsilon(\boldsymbol{d}, s)$ will be enumerated in A if and only if $\Xi_m(\epsilon(\boldsymbol{d}, s), s) = A(\epsilon(\boldsymbol{d}, s), s) = 0$. Subsequently, provided no stage of characteristic $<\boldsymbol{d}$ intervenes, if $\boldsymbol{a}(s+1) \upharpoonright i = \boldsymbol{d}$ and none of C0, C1, and C2 occurs at stage $(s+1, i)$, we shall have $\boldsymbol{a}(s+1) \upharpoonright (i+1) = \boldsymbol{d}^\frown w$. Here w signifies that $S(\boldsymbol{d})$ has been satisfactorily completed. As mentioned above C0 will actually never occur, and the roles of C1 and C2 will be explained later.

If $\boldsymbol{a}(s+1) \upharpoonright i = \boldsymbol{d}$ and $\epsilon(\boldsymbol{d}, s) < \omega$ then ignoring C1 and C2 for the moment, $\boldsymbol{a}(s+1) \supset \boldsymbol{d}^\frown d_1$ if and only if $\xi_m(\epsilon(\boldsymbol{d}, s), s) \geqslant \sup I(i, s)$. Intuitively d_1 signifies that we are waiting for $\xi_m(\epsilon(\boldsymbol{d}, s), s)$ to be within the universe $I(i, s)$ in which $S(\boldsymbol{d})$ must be pursued. This is the content of C4. If there are infinitely many such s, then $\Xi_m(\epsilon(\boldsymbol{d})) = \omega$ because $\sup I(i, s)$ takes arbitrarily large values. Thus the aim of $S(\boldsymbol{d})$ is satisfied in this case.

Now we consider the eventuality that, for infinitely many s such that $\boldsymbol{a}(s+1) \upharpoonright i = \boldsymbol{d}$, neither C1 nor C2 occurs at stage $(s+1, i)$, that $\epsilon(\boldsymbol{d}, s) < \omega$, whence C3 does not occur, that $\xi_m(\epsilon(\boldsymbol{d}, s), s) < \sup I(i, s)$ whence C4 does not occur, and that $S(\boldsymbol{d})$ neither has been nor can be completed by C5. There will be a least j such that, for infinitely many such s, j is the greatest number for which $j \leqslant m$, $j \notin F(\boldsymbol{d})$, and $\varphi(\boldsymbol{d}, j, s) \leqslant \xi_m(\epsilon(\boldsymbol{d}, s), s) < \omega$. For each such s we shall have $\boldsymbol{a}(s+1) \upharpoonright (i+1) = \boldsymbol{d}^\frown (f, j)$ and there will be only a finite number of stages with characteristic $<\boldsymbol{d}^\frown (f, j)$. Now we pursue all strategies associated with characteristics $\boldsymbol{e} \supset \boldsymbol{d}^\frown (f, j)$ to the left of $\varphi(\boldsymbol{d}, j, s)$, and to the right of

$$\sup(\{\epsilon(\boldsymbol{d}, s)\} \cup \{\varphi(\boldsymbol{d}, k, s) : k < j \text{ and } \varphi(\boldsymbol{d}, k, s) < \omega\}).$$

This is part of the definition of $I(i+1, s)$ in C6.

We now associate with $\boldsymbol{d}$ a reduction of $\boldsymbol{A}$ to $\boldsymbol{B}_j^1 + \boldsymbol{C}$ generated in the following way. A number p in $I(i+1, s)$ will only be enumerated in $\boldsymbol{A}$ when we are sure that putting p in $\boldsymbol{A}$ will force a number $q \leqslant \varphi(\boldsymbol{d}, j, s)$ into either $\boldsymbol{B}_j^0$ or $\boldsymbol{B}_j^1$. (We are assuming here that $\boldsymbol{A} = \boldsymbol{\theta}_j \wedge \boldsymbol{B}_j^0 = \boldsymbol{\psi}_j \wedge \boldsymbol{B}_j^1 = \boldsymbol{\psi}_j^1$ and that numbers $\leqslant \varphi(\boldsymbol{d}, j, s)$ are restrained from $\boldsymbol{C}$ while we wait for q to be enumerated in either $\boldsymbol{B}_j^0$ or $\boldsymbol{B}_j^1$.) The method of reducing $\boldsymbol{A}$ to $\boldsymbol{B}_j^1 + \boldsymbol{C}$ is based on q always being enumerated in $\boldsymbol{B}_j^1$ rather than $\boldsymbol{B}_j^0$. If it happens that in response to the enumeration of p in $\boldsymbol{A}$ we have $q < \varphi(\boldsymbol{d}, j, s)$ enumerated in $\boldsymbol{B}_j^0$ rather than $\boldsymbol{B}_j^1$ then we may change the value of $\varphi(\boldsymbol{d}, j, s)$ without putting a number $\leqslant \varphi(\boldsymbol{d}, j, s)$ in $\boldsymbol{C}$. This is because in the strategy for reducing $\boldsymbol{A}$ to $\boldsymbol{B}_j^0 + \boldsymbol{C}$, which is associated with the greatest $\boldsymbol{e} \subset \boldsymbol{d}$ for which $\mathrm{O}(\boldsymbol{e}) = j$, for each p in $I(i+1, s)$ we relate $\boldsymbol{A}(p)$ to the values of $\boldsymbol{B}_j^0$ and $\boldsymbol{C}$ for arguments $\leqslant \varphi(\boldsymbol{d}, j, s)$. When $\boldsymbol{B}_j^0(q)$ changes for some $q \leqslant \varphi(\boldsymbol{d}, j, s)$ we may give $\varphi(\boldsymbol{d}, j, s)$ a new value $>\xi_j(\epsilon(\boldsymbol{d}, s), s)$. Now we consider the next stage s at which $\boldsymbol{a}(s+1) \upharpoonright (i+1) = \boldsymbol{d}^\frown (f, j)$. If $\varphi(\boldsymbol{d}, j, s) > \xi_j(\epsilon(\boldsymbol{d}, s), s)$ is still true we have a contradiction, thus some stage of characteristic $< \boldsymbol{d}^\frown (f, j)$ must intervene. Since j was chosen as small as possible there can only be a finite number of stage of characteristic $< \boldsymbol{d}^\frown (f, j)$, and hence only a finite number of occasions on which q is enumerated in B_j^0 rather than B_j^1. Thus the reduction of A to $B_j^1 + C$ will be sound.

If there are infinitely many s with $\boldsymbol{a}(s+1) \upharpoonright (i+1) = \boldsymbol{d}^\frown (f, j)$ then in the course of the construction $\varphi(\boldsymbol{d}, j, s)$ takes arbitrarily large finite values and is non-decreasing if we ignore those s such that $\varphi(\boldsymbol{d}, j, s) = \omega$. Since $\varphi(\boldsymbol{d}, j, s) \leqslant \xi_m(\epsilon(\boldsymbol{d}, s), s)$, $\xi_m\, \varepsilon(\boldsymbol{d}) = \omega$ where $\varepsilon(\boldsymbol{d})$ is $\lim_s \epsilon(\boldsymbol{d}, s)$. Thus the aim of $S(\boldsymbol{d})$ is also satisfied in this case.

As has been indicated in the discussion above the requirement $\mathfrak{C}_j$ is satisfied by first attempting to construct a reduction of A to $B_j^0 + C$, and then constructing a reduction of A to $B_j^1 + C$ if necessary. This is an oversimplification in the following sense. In the strategies corresponding to members $\boldsymbol{e}$ of G where $\boldsymbol{e} \subset \boldsymbol{d}^\frown (f, n)$ we have to begin again attacking the conditions $\mathfrak{C}_i$ for $i > n$. This is the reason why $\mathrm{O}(\boldsymbol{d}^\frown (f, n)) = n + 1$. The reductions of the second kind bring considerable complexity to the construction, because "waiting" is involved. Thus when $\epsilon(\boldsymbol{d}, s)$ is enumerated in A we must initially let $\boldsymbol{a}(\boldsymbol{d}, s) = \boldsymbol{d}$ and $\nu(\boldsymbol{d}, s)$ be the least $k \leqslant m$, $k \notin D_0(\boldsymbol{d})$. Intuitively $\nu(\boldsymbol{d}, s)$ is the least j such that

$$A = \boldsymbol{\theta}_j \wedge B_j^0 = \psi_j^0 \wedge B_j^1 = \psi_j^1$$

is part of the hypothesis $H(\boldsymbol{d})$.

At an arbitrary stage after $\epsilon(\boldsymbol{d}, s)$ has been enumerated in A, if $\boldsymbol{a}(\boldsymbol{d}, s)$ and $\nu(\boldsymbol{d}, s)$ are still both $\neq \omega$, it means that instead of the original hypothesis only the weaker hypothesis $H(\boldsymbol{a}(\boldsymbol{d}, s))$ remains, and we are currently repairing the injury to the strategies concerned with satisfying $\mathfrak{C}_{\nu(\boldsymbol{d},s)}$. If $\nu(\boldsymbol{d}, s) \notin F(\boldsymbol{a}(\boldsymbol{d}, s))$ then we repair the injury to the reduction of A to $B^0_{\nu(\boldsymbol{d},s)} + C$, caused by enumerating $\epsilon(\boldsymbol{d}, s)$ in A, by enumerating $\varphi(\boldsymbol{a}(\boldsymbol{d}, s), \nu(\boldsymbol{d}, s), s)$ in C. This is what happens in C1.1. In this case the number enumerated in C is small enough that not only is the injury to the satisfaction of $\mathfrak{C}_{\nu(\boldsymbol{d},s)}$ repaired but also the injury to each $\mathfrak{C}_j$, $\nu(\boldsymbol{d}, s) < j \leqslant m$. For this reason, in C1.1 $\boldsymbol{a}(\boldsymbol{d}, s+1) = \nu(\boldsymbol{d}, s+1) = \omega$.

If $\nu(\boldsymbol{d}, s) \in F(\boldsymbol{a}(\boldsymbol{d}, s))$ then we must wait for some number $\leqslant \theta_n(\epsilon(\boldsymbol{d}, s), t)$ to be enumerated in either $B^0_{\nu(\boldsymbol{d},s)}$ or $B^1_{\nu(\boldsymbol{d},s)}$ in response to the enumeration of $\epsilon(\boldsymbol{d}, s)$ in A at stage $t+1$. This gives rise to the cases C1.2, C1.3, and C1.4. In C1.2 where there is a response in $B^0_{\nu(\boldsymbol{d},s)}$, we make

$\varphi(e, \nu(d, s), s+1) = \omega$ where e is the greatest initial segment of $a(d, s)$ such that $e \frown (f, \nu(d, s) \subset a(d, s)$. Our goal here is to bring about a stage of characteristic $< e \frown (f, \nu(d, s)$. Intuitively at this point the hypothesis $H(d)$ under which $\epsilon(d, s)$ was enumerated in A has been discredited to the extent that only $H(e)$ remains. This is why we define $a(d, s+1) = e$ provided there exists j with $j \notin \mathrm{O}(e)$ and $\nu(d, s) < j \leqslant \mathrm{O}(e)$, and why $a(d, s+1) = \omega$ otherwise.

If a number $\leqslant \theta_n(\epsilon(d, s), t)$ is enumerated in $B^1_{\nu(d,s)}$ before $B^0_{\nu(d,s)}$ we have C1.3. This is the case we expect under $H(a(d, s))$. Thus here $a(d, s+1) = a(d, s)$ provided there exists a j with $j \notin \mathrm{O}(a(d, s))$ and $\nu(d, s) < j \leqslant \mathrm{O}(a(d, s))$. Otherwise $a(d, s+1) = \nu(d, s+1) = \omega$. We enumerate $\varphi(e, \nu(d, s), s)$ in C in order to repair the reduction of A to $B^0_{\nu(d,s)} + C$ which is associated with the greatest $e' \subset e$ satisfying $\mathrm{O}(e') = \nu(d, s)$. We make $\varphi(a(d, s), \nu(d, s), s+1) = \omega$ in order to repair the reduction of A to $B^1_{\nu(d,s)} + C$ associated with e.

Finally while we are waiting for a response in $B^0_{\nu(d,s)} \cup B^1_{\nu(d,s)}$ to the enumeration of $\epsilon(d, s)$ in A, in a stage $(s+1, i)$ where we should be continuing to repair the harm done by $\epsilon(d, s)$ we let $a(s+1)_i = d_0$ by C1.4. This signifies the apparent falsity of $A = \boldsymbol{\theta}_j \wedge B^0_j = \psi^0_j \wedge B^1_j = \psi^1_j$ for $j = \nu(d, s)$. For such a stage $(s+1, i)$ we shall have $a(s+1) \upharpoonright i = b$ where b is the least initial segment of d such that $\mathrm{O}(b) = \nu(d, s)$.

We can sum up the role of C1 as follows. When a number is enumerated in A to attack $\mathfrak{C}'_m$, the attacks on $\mathfrak{C}_0, \mathfrak{C}_1, ..., \mathfrak{C}_m$ are injured. The role of C1 is to permit the attack on $\mathfrak{C}_0$ to be repaired, then the attack on $\mathfrak{C}_1$ to be repaired, and so on. At a particular stage after $\epsilon(d, s)$ has been put into A, $\mathfrak{C}_{\nu(d,s)}$ is the condition we are currently repairing and $H(a(d, s))$ is that part of the hypothesis $H(d)$ which is still good.

The only part of the construction not yet touched on is C2. This is concerned with giving $\varphi(d, j, s)$ a suitable value $< \omega$ once $\epsilon(d, s)$ has been given a value $< \omega$, where j runs through all numbers $\leqslant \mathrm{O}(d)$ which are not in $D_0(d)$. The selection of $\varphi(d, j, s)$ is associated with e where e is the greatest initial segment of d such that $\mathrm{O}(e) = j$. The purpose of $\varphi(d, j, s)$ is to bound the computations of $\boldsymbol{\theta}_j$, ψ^0_j, and ψ^1_j on sufficiently large initial segments of ω. We want the computations to be good ones in the sense that $\boldsymbol{\theta}_j = A$, $\psi^0_j = B^0_j$, and $\psi^1_j = B^1_j$ all appear true at the particular stage and for the particular initial segments. The choice of $\varphi(d, j, s)$ is associated with e because the attempt to reduce A to $B^0_j + C$, which is relevant to $\epsilon(d, s)$, is associated with e. In C2.1 it may seem odd that we choose d, for which we wish to make $\varphi(d, j, s) < \omega$, minimal with respect to $<$ and then *maximal* with respect to $\subset$. The reason for this is as follows. Suppose that in a certain substage $(s+1, i)$ we have $d_0 \subsetneq d_1$, $\varphi(d_0, j, s) = \varphi(d_1, j, s) = \omega$, $\epsilon(d_0, s)$ and $\epsilon(d_1, s) < \omega$, $a(s+1) \restriction i = e$, and for $k \in \{0, 1\}$, e is the greatest initial segment of d_k such that $O(e) = j$. Suppose further that C1 does not occur at stage $(s+1, i)$ then we shall have C2. In this situation it turns out that $d_0 {}^\frown (f, j) \subset d_1$. We want $\varphi(d_1, j, s') < \varphi(d_0, j, s') < \omega$ for some $s' > s$ so that $\varphi(d_1, j, s')$ will be in the "universe" of $S(d_0 {}^\frown (f, j))$. By making $\varphi(d_1, j, s') < \omega$ first, and then making $\varphi(d_0, j, s') < \omega$ at a later stage it is easy to obtain the desired inequality. If in stage $s+1$ we are trying to make $\varphi(d, j, s+1) < \omega$, and if the computations related to $\mathfrak{C}_j$ we wish to bound cannot be bounded in $I(i, s)$, then we have $a(s+1) \restriction (i+1) = e {}^\frown d_0$. This is C2.2. Just as in C1.4 d_0, signifies that seemingly $A = \theta_j \wedge B_j^0 = \psi_j^0 \wedge B_j^1 = \psi_j^1$ is not true.

In the next section we shall show that there is a sequence $\langle b_i : i < \omega \rangle$ in G such that, for every i, b_i has length i, $b_i \subset a(s+1)$ for infinitely many s, and $a(s+1) < b_i$ for at most a finite number of s. The members of $\langle b_i : i < \omega \rangle$ are called *preferred*. The strategies corresponding to these characteristics are the ones which make the construction succeed because for each such strategy only a finite number of stages have higher priority.

As we have mentioned above if $O(b_i) = m$, $j \leqslant m$, and $j \notin F(b_i)$, then there is a strategy for reducing A to $B_j^1 + C$ associated with b_i. Since this strategy is only pursued when $a(s+1) \supset b_i {}^\frown (f, j)$ it is only important if $b_{i+1} = b_i {}^\frown (f, j)$. This completes our discussion of some of the intuitive ideas behind the construction.

3. Properties of the construction

In this section we shall list all the properties of the construction that we need. We prove the two most important propositions while assuming the rest. The proofs of the other propositions are deferred until §4. The propositions will be numbered P1 – P12. The principal propositions are P11 and P12; from them we shall deduce that all the requirements $\mathfrak{C}_m$ and $\mathfrak{C}'_m$ are satisfied.

P1. *Let $s > t+1$ and $\epsilon(c, s) = \epsilon(c, t)$ be enumerated in A at stage $t+1$. Let $a(c, s)$ be defined and $\nu(c, s) = n \leqslant O(a(c, s))$. Let e be the least initial segment of $a(c, s)$ such that $O(e) = n$. Then*

(1) *if a stage* $>t+1$ *and* $\leqslant s$ *has characteristic* $\supset \boldsymbol{e}$ *it has characteristic* $\supset \boldsymbol{e} \frown d_0$

(2) *if* $\boldsymbol{a}(s+1) \supset \boldsymbol{e}$, *then either* $\boldsymbol{a}(s+1) = \boldsymbol{e} \frown a_0$ or $\boldsymbol{a}(s+1) \supset \boldsymbol{e} \frown d_0$. (*Notice that if* $\boldsymbol{e} \neq \boldsymbol{a}(\boldsymbol{c}, s)$, *then* $\boldsymbol{a}(\boldsymbol{c}, s) < \boldsymbol{e} \frown a_0$. *This is because* $n \notin D_0(\boldsymbol{a}(\boldsymbol{c}, s))$ *by the way in which* $\nu(\boldsymbol{c}, s)$ *is defined, whence* $\boldsymbol{e} \frown d_0 \not\subset \boldsymbol{c}$.)

P2. (1) *If* $\epsilon(\boldsymbol{b}, s), \epsilon(\boldsymbol{c}, s) < \omega$, *and either* $\boldsymbol{c} < \boldsymbol{b}$ *or* $\boldsymbol{c} \frown d_0 \subset \boldsymbol{b}$ *or* $\boldsymbol{c} \frown d_1 \subset \boldsymbol{b}$ or $\boldsymbol{c} \frown w \subset \boldsymbol{b}$, *then* $\mu(\boldsymbol{c}, s) < \epsilon(\boldsymbol{b}, s)$.

(2) *If* $\boldsymbol{c} \frown (f, j) \subset \boldsymbol{b}$, $\epsilon(\boldsymbol{b}, s)$ *and* $\epsilon(\boldsymbol{c}, s)$ *are both* $<\omega$, $\boldsymbol{a}(s+1) \supset \boldsymbol{c}$, *and* $\boldsymbol{a}(s+1) < \boldsymbol{c} \frown a_0$, *then*

$$\epsilon(\boldsymbol{c}, s) < \epsilon(\boldsymbol{b}, s) \leqslant \mu(\boldsymbol{b}, s) < \varphi(\boldsymbol{c}, j, s)\,.$$

Further if $\varphi(\boldsymbol{c}, k, s) < \omega$ *and* $k < j$, *then*

$$\epsilon(\boldsymbol{c}, s) < \varphi(\boldsymbol{c}, k, s) < \epsilon(\boldsymbol{b}, s) \leqslant \mu(\boldsymbol{b}, s) < \varphi(\boldsymbol{c}, j, s)\,.$$

P.3. (1) *With the notation of the statement of the construction, if* $\epsilon(\boldsymbol{b}, s) < \omega$, *then* $\epsilon(\boldsymbol{b}, s) \geqslant \inf I(i, s)$ *and* $\mu(\boldsymbol{b}, s) < \sup I(i, s)$.

(2) *If there is a stage* $(s+1, i+1)$, *then* $I(i+1, s) \neq \emptyset$, i.e. C0 *never occurs.*

P4. *Let* $\boldsymbol{b}$ *be a characteristic sequence of length i such that* $\mathrm{O}(\boldsymbol{b}) = n$. *Let* $\boldsymbol{b} \subset \boldsymbol{a}(s+1)$, $\boldsymbol{a}(s+1) < \boldsymbol{b} \frown a_0$, *and* $\epsilon(\boldsymbol{b}, s) < \omega$. *Then*

(1) *for* $j \leqslant n$, $\varphi(\boldsymbol{b}, j, s) < \omega$ *if and only if* $j \notin D_0(\boldsymbol{b})$,

(2) *for* $j_1 < j_2 \leqslant n$ *and* $\{j_1, j_2\} \cap D_0(\boldsymbol{b}) = \emptyset$, *we have* $\epsilon(\boldsymbol{b}, s) < \varphi(\boldsymbol{b}, j_1, s) < \varphi(\boldsymbol{b}, j_2, s)$ *and* $\varphi(\boldsymbol{b}, j_1, s)$ *is assigned before* $\varphi(\boldsymbol{b}, j_2, s)$.

(3) *If* $\varphi(\boldsymbol{b}, j, s) < \omega$ *and is assigned at stage* $v+1$ *then no number* $\leqslant \varphi(\boldsymbol{b}, j, s)$ *is enumerated in* $A \cup C$ *at a stage* $>v$ and $\leqslant s$.

P5. *Let* $\epsilon(\boldsymbol{c}, t)$ *be enumerated in A at stage* $t+1$. *Let u be the least number* $\geqslant t$ *such that* $\boldsymbol{a}(\boldsymbol{c}, u+1) = \omega$. *If stage* $u+1$ *pertains to* $\epsilon(\boldsymbol{c}, t)$ *and* C1.2 *occurs let* $\boldsymbol{d}$ *be the greatest initial segment of* $\boldsymbol{a}(\boldsymbol{c}, u)$ *such that* $\boldsymbol{d} \frown (f, \nu(\boldsymbol{c}, u)) \subset \boldsymbol{a}(\boldsymbol{c}, u)$. *If* $u = t$ *let* $\boldsymbol{d} = \boldsymbol{c}$. *Otherwise, let* $\boldsymbol{d} = \boldsymbol{a}(\boldsymbol{c}, u)$. *Let s be the least number* $>u$ *such that* $\boldsymbol{a}(s+1) \supset \boldsymbol{d}$ *and* $\boldsymbol{a}(s+1) < \boldsymbol{d} \frown a_0$, *and suppose that no stage* $>u$ *and* $\leqslant s$ *has characteristic* $< \boldsymbol{d}$ If $\boldsymbol{d} = \boldsymbol{c}$, *then* $\boldsymbol{c} \frown w \subset \boldsymbol{a}(s+1)$. *If* $\boldsymbol{d} \neq \boldsymbol{c}$, *then* $\boldsymbol{d} \frown (f, m) \subset \boldsymbol{c}$ *for some m and* $\boldsymbol{a}(s+1) < \boldsymbol{d} \frown (f, m)$.

Remark. From P1(1), $s+1$ is the least number $>t+1$ such that $\boldsymbol{a}(s+1) \supset \boldsymbol{d}$ and $\boldsymbol{a}(s+1) < \boldsymbol{d} \frown a_0$. Also no stage $>t$ and $\leqslant s+1$ has characteristic $< \boldsymbol{d}$: by hypothesis for stages $>u$, and otherwise by the existence of $\boldsymbol{a}(c, u)$.)

P6. (1) *Let* $\boldsymbol{b} \frown d_0 \subset \boldsymbol{a}(t+1)$, $\epsilon(\boldsymbol{b}, t) < \omega$, $t \leqslant s$, *and* $\boldsymbol{a}(u) \nless \boldsymbol{b} \frown d_0$ *for all* u *in* $t < u \leqslant s+1$. *Then* $\epsilon(\boldsymbol{b}, s+1) = \epsilon(\boldsymbol{b}, t)$, $\mu(\boldsymbol{b}, s+1) = \mu(\boldsymbol{b}, t)$, *and no number* $\leqslant \mu(\boldsymbol{b}, t)$ *is enumerated in* $A \cup C$ *at a stage* $>t$ *and* $\leqslant s+1$. *Similarly for* d_1 *in place of* d_0.

(2) *Let* $\boldsymbol{b} \frown a_1 = \boldsymbol{a}(t+1)$, $t \leqslant s$, *and* $\boldsymbol{a}(u) \nless \boldsymbol{b}$ *for all* u *in* $t < u \leqslant s+1$. *Then* $\epsilon(\boldsymbol{b}, s+1) = \epsilon(\boldsymbol{b}, t+1)$.

(3) *Let* $\boldsymbol{b} \frown w \subset \boldsymbol{a}(t+1)$, $t \leqslant s$, *and* $\boldsymbol{a}(u) \nless \boldsymbol{b}$ *for all* u *in* $t < u \leqslant s+1$. *Then* $\epsilon(\boldsymbol{b}, s+1) = \epsilon(\boldsymbol{b}, t)$, $\mu(\boldsymbol{b}, s+1) = \mu(\boldsymbol{b}, t)$, *and no number* $\leqslant \mu(\boldsymbol{b}, t)$ *is enumerated in* $A \cup C$ *at a stage* $>t$ *and* $\leqslant s+1$. *Further, if* $\boldsymbol{a}(s+1) \supset \boldsymbol{b}$ *and* $\boldsymbol{a}(s+1) < \boldsymbol{b} \frown a_0$ *then* $\boldsymbol{a}(s+1) \supset \boldsymbol{b} \frown w$.

(4) *Let* $\boldsymbol{b} \frown a_2 = \boldsymbol{a}(t+1)$, $t < s$, $\boldsymbol{a}(u) \nless \boldsymbol{b} \frown a_2$ *for all* u *in* $t < u \leqslant s+1$, *and* $\epsilon(\boldsymbol{b}, t)$ *not be enumerated in* A *at a stage* $>t$. *Then*

$$\epsilon(\boldsymbol{b}, s+1) = \epsilon(\boldsymbol{b}, t), \qquad \mu(\boldsymbol{b}, s+1) = \mu(\boldsymbol{b}, t+1),$$

no number $\leqslant \mu(\boldsymbol{b}, t+1)$ *is enumerated in* $A \cup C$ *at a stage* $>t$ *and* $\leqslant s+1$, *and* $\boldsymbol{a}(s+1) \neq \boldsymbol{b} \frown a_2$.

(5) *Let* $\boldsymbol{b} \frown (f, j) \subset \boldsymbol{a}(t+1)$, $t \leqslant s$, *and* $\boldsymbol{a}(u) \nless \boldsymbol{b} \frown (f, j)$ *for all* u *in* $t < u \leqslant s+1$. *Then* $\epsilon(\boldsymbol{b}, s+1) = \epsilon(\boldsymbol{b}, t)$ *and* $\varphi(\boldsymbol{b}, k, s+1) = \varphi(\boldsymbol{b}, k, t)$ *for all* $k < j$ *such that* $k \notin D_0(\boldsymbol{b})$. *Further no number* $\leqslant \epsilon(\boldsymbol{b}, t)$, *or* $\leqslant \varphi(\boldsymbol{b}, k, t)$ *for some* $k < j$ *and* $k \notin D_0(\boldsymbol{b})$, *is enumerated in* $A \cup C$ *at a stage* $>t$ *and* $\leqslant s+1$.

(6) *Let* $\boldsymbol{b} \frown a_0 = \boldsymbol{a}(t+1)$, $\boldsymbol{b} \subset \boldsymbol{c}$, $\boldsymbol{b} \frown d_0 \notin \boldsymbol{c}$, $t \leqslant s$, $\epsilon(\boldsymbol{c}, s) < \omega$, *and* $\boldsymbol{a}(u) \nless \boldsymbol{b} \frown a_0$ *for all* u *in* $t < u \leqslant s+1$. *Suppose there is no stage* $u+1 > t$ *at which* C1 *occurs and which pertains to* $\epsilon(\boldsymbol{d}, u)$ *where* $\boldsymbol{d} \subset \boldsymbol{b}$ *or* $\boldsymbol{d} < \boldsymbol{b} \frown a_0$. *Then at stage* $s+1$ *no number* $\leqslant \mu(\boldsymbol{c}, s)$ *is enumerated in* C.

P7. *Let* $\boldsymbol{b} \frown (f, j) \subset \boldsymbol{a}(s+1)$ *for infinitely many* s, *and* $\boldsymbol{a}(s+1) < \boldsymbol{b} \frown (f, j)$ *for at most a finite number of* s. *There exists* s *such that* $\boldsymbol{b} \frown (f, j) \subset \boldsymbol{a}(s+1)$ *and* $\varphi(\boldsymbol{b}, j, s)$ *is arbitrarily large and* $< \omega$.

P8. *Let* $\epsilon(\boldsymbol{c}, t)$ *be enumerated in* A *at stage* $t+1$, $s > t$, $\epsilon(\boldsymbol{c}, s) = \epsilon(\boldsymbol{c}, t)$, $O(\boldsymbol{b}) = n$, $\boldsymbol{b} \subset \boldsymbol{c}$, $\boldsymbol{b}$ *have length* i, $\boldsymbol{a}(s+1) \supset \boldsymbol{b}$, C1.4 *occur at stage* $(s+1, i)$, *stage* $(s+1, i)$ *pertain to* $\epsilon(\boldsymbol{c}, t)$, *and* $\boldsymbol{d}$ *be the greatest initial segment of* $\boldsymbol{a}(\boldsymbol{c}, s)$ *such that* $\boldsymbol{d} \frown (f, n) \subset \boldsymbol{a}(\boldsymbol{c}, s)$. *Then* $\varphi(\boldsymbol{d}, n, s) = \varphi(\boldsymbol{d}, n, t)$.

P9. (1) *Let* b, c, *and* t *satisfy*: $O(b) = n \notin D_0(b)$, $b \subset c$, $\epsilon(c, t)$ *is enumerated in* A *at stage* $t + 1$, b *has length* i, *and, for infinitely many* s, *stage* $(s + 1, i)$ *pertains to* $\epsilon(c, t)$. *Then* $A \neq \theta_n$.

(2) *Let* $O(b) = n$, b *have length* i, *and let there be at most a finite number of stages with characteristic* $< b \frown d_0$. *Let* c *be such that, for infinitely many* s, $a(s + 1) \supset b \frown d_0$ *and stage* $(s + 1, i)$ *is associated with* c. *Then either* $A \neq \theta_n$, *or* $B_n^0 \neq \psi_n^0$, *or* $B_n^1 \neq \psi_n^1$.

P10. *Let* $\epsilon(c, t)$ *be enumerated in* A *at stage* $t + 1$. *Let* $n \notin D_0(b)$ *and* $n \leqslant O(b)$. *Let* $b \frown (f, m) \subset c$ where $m \leqslant n$, *or* $b = c$. *Then* $\theta_n(\epsilon(c, t), t) \leqslant \varphi(b, n, t) < \omega$.

The ten propositions we have listed above are the main-links in the chain of reasoning which shows the success of the construction. Before we go any further we must single out those characteristic sequences which play a vital role in the enumeration of A and C. First define $\varepsilon(d)$ to be $\epsilon(d, s_0)$ if $\epsilon(d, s) = \epsilon(d, s_0)$ for all $s \geqslant s_0$, and to be ω otherwise. Let $\mu(d)$ and $\varphi(d, j)$ be defined similarly. A characteristic sequence b is called *preferred* provided that the following conditions p1 – p8 are satisfied.

p1. $b \in G$.

p2. $b \subset a(s + 1)$ for infinitely many s and $a(s + 1) \nless b$ for all sufficiently large s.

p3. $\varepsilon(c) < \omega$ for each $c \subset b$ such that $c \neq b$ and $c \frown d_0 \not\subset b$.

p4. If $b = c \frown w$, where $O(c) = m$, then $\mu(c) < \omega$, $\xi_m \varepsilon(c) < \omega$, $\xi_m \varepsilon(c) \leqslant \mu(c)$, and $\Xi_m \varepsilon(c) \neq A\, \varepsilon(c)$. Further for all but a finite number of s, $c \subset a(s + 1)$ and $a(s + 1) < c \frown a_0$ imply $b \subset a(s + 1)$.

p5. If $b = c \frown d_0$ where $O(c) = m$, then either $\mu(c) < \omega$ or $\mu(c, s) = \omega$ for all sufficiently large s. Further, either $\theta_m \neq A$ or $\psi_m^0 \neq B_m^0$ or $\psi_m^1 \neq B_m^1$.

p6. If $b = c \frown d_1$, then $\mu(c) < \omega$. Further, if $O(c) = m$, then $\xi_m \varepsilon(c) = \omega$.

p7. If $b = c \frown (f, j)$, then $j \notin D_0(c) \cup F(c)$, $\varphi(c, k) < \omega$ exists for each $k < j$ such that $k \notin D_0(c)$. Further, if $O(c) = m$, then $\xi_m \varepsilon(c) = \omega$. Also $\varphi(c, j, s)$ takes arbitrarily large values.

p8. Each proper initial segment of $\boldsymbol{b}$ is preferred.

Because of p8 the above definition is to be regarded as being by induction on the length of $\boldsymbol{b}$. The empty sequence $\emptyset$ is trivially a preferred sequence. We now state and prove the two main propositions.

P11. *Let $\boldsymbol{b}$ be a preferred sequence. Then some proper extension of $\boldsymbol{b}$ is preferred.*

P12. *For each $m < w$, either $A \leqslant_T C + B_m^0$ or $A \leqslant_T C + B_m^1$ or $\boldsymbol{\theta}_m \neq A$ or $\boldsymbol{\psi}_m^0 \neq B_m^0$ or $\boldsymbol{\psi}_m^1 \neq B_m^1$.*

A proof of the theorem from P11 and P12 will be found at the end of this section.

Proof of P11. Let $\boldsymbol{b}$ be a preferred sequence and $O(\boldsymbol{b}) = n$. Let s_0 be the least number such that $\boldsymbol{a}(s) \nless \boldsymbol{b}$ for all $s \geqslant s_0$. Suppose that $\boldsymbol{b} \frown w \subset \boldsymbol{a}(s+1)$ for infinitely many s, and let s_1 be the least number $\geqslant s_0$ such that $\boldsymbol{b} \frown w \subset \boldsymbol{a}(s_1 + 1)$. From P6 (3), for all $s \geqslant s_1$, we have $\epsilon(\boldsymbol{b}, s) = \epsilon(\boldsymbol{b}, s_1)$, $\mu(\boldsymbol{b}, s) = \mu(\boldsymbol{b}, s_1)$, and no number $\leqslant \mu(\boldsymbol{b}, s_1)$ is enumerated in $A \cup C$ at stage $s + 1$. From the hypothesis of C5.2 we see that $\Xi_n(\epsilon(\boldsymbol{b}, s_1), s_1) = 1 - A(\epsilon(\boldsymbol{b}, s_1), s_1)$ and $\xi_n(\epsilon(\boldsymbol{b}, s_1), s_1) \leqslant \mu(\boldsymbol{b}, s_1)$. It follows that $\boldsymbol{\varepsilon}(\boldsymbol{b}) = \epsilon(\boldsymbol{b}, s_1)$, $\boldsymbol{\mu}(\boldsymbol{b}) = \mu(\boldsymbol{b}, s_1)$, $\xi_n \boldsymbol{\varepsilon}(\boldsymbol{b}) = \xi_n(\epsilon(\boldsymbol{b}, s_1), s_1) \leqslant \boldsymbol{\mu}(\boldsymbol{b})$, and $\Xi_n \boldsymbol{\varepsilon}(\boldsymbol{b}) = 1 - A\boldsymbol{\varepsilon}(\boldsymbol{b})$. Further, if $s > s_1$, $\boldsymbol{b} \subset \boldsymbol{a}(s+1)$, and $\boldsymbol{a}(s+1) < \boldsymbol{b} \frown a_0$, then again by P6(3) we have $\boldsymbol{a}(s+1) \supset \boldsymbol{b} \frown w$. Thus $\boldsymbol{b} \frown w$ is preferred.

For the rest suppose that $\boldsymbol{b} \frown w \not\subset \boldsymbol{a}(s+1)$ for all $s \geqslant s_0$. Let $t \geqslant s_0$ and $\boldsymbol{a}(t+1) = \boldsymbol{b} \frown a_2$ and $\epsilon(\boldsymbol{b}, t)$ be enumerated in A at stage $t + 1$. Let $\boldsymbol{d}$ be defined as in the statement of P5 but from $\boldsymbol{b}$ instead of $\boldsymbol{c}$. If $\boldsymbol{d}$ does not exist then $\boldsymbol{a}(\boldsymbol{b}, u+1) \neq \omega$ for all $u \geqslant t$. From P1 it follows easily that for all $u > t$, if $\boldsymbol{a}(u+1) \supset \boldsymbol{b}$, then either $\boldsymbol{a}(u+1) \supset \boldsymbol{b} \frown d_0$ or $\boldsymbol{a}(u+1) = \boldsymbol{b} \frown a_0$. For this application of P1 we let $\boldsymbol{c} = \boldsymbol{b}$. Let $\boldsymbol{e}$ be defined as in the statement of P1, then $\boldsymbol{e} \subset \boldsymbol{b}$ and if $\boldsymbol{e} \neq \boldsymbol{b}$ then $\boldsymbol{b} < \boldsymbol{e} \frown a_0$ from the way $\boldsymbol{a}(\boldsymbol{b}, s)$ and $\nu(\boldsymbol{b}, s)$ are defined. If $\boldsymbol{d}$ does exist, consider the least $s > t$ if any such that $\boldsymbol{a}(s+1) \supset \boldsymbol{d}$ and $\boldsymbol{a}(s+1) < \boldsymbol{d} \frown a_0$. If s exists then from the conclusion of P5 either $\boldsymbol{a}(s+1) < \boldsymbol{d} \frown (f, m) \subset \boldsymbol{b}$ or $\boldsymbol{a}(s+1) \supset \boldsymbol{b} \frown w$. This is impossible, hence s does not exist. Also from the definition of $\boldsymbol{d}$, $\boldsymbol{d} \subset \boldsymbol{b}$ and $\boldsymbol{d} \frown d_0 \not\subset \boldsymbol{b}$. Thus since $\boldsymbol{b} \subset \boldsymbol{a}(s+1)$ for infinitely many s, we have $\boldsymbol{d} = \boldsymbol{b}$ and $\boldsymbol{a}(s+1) \nless \boldsymbol{b} \frown a_0$ for all $s > t$. Let us now consider $t \geqslant s_0$ such that $\boldsymbol{a}(t+1) = \boldsymbol{b} \frown a_2$ and $\boldsymbol{\varepsilon}(\boldsymbol{b}, t)$ is not enumerated in A at any stage $>t$. From P6(4) if $s > t$ and $\boldsymbol{a}(s+1) \leqslant \boldsymbol{b} \frown a_2$, then $\boldsymbol{a}(u) < \boldsymbol{b} \frown a_2$ for some u in $t < u \leqslant s$ which is impossible. Thus in all these cases for all sufficiently large s, say $s \geqslant s_1$, $\boldsymbol{a}(s+1) \not\leqslant \boldsymbol{b} \frown a_2$.

Let j be the least number, if any, such that $\boldsymbol{b} \frown (f, j) \subset \boldsymbol{a}(s+1)$ for infinitely many s. Choose s_1 such that $\boldsymbol{b} \frown (f, j) \subset \boldsymbol{a}(s_1+1)$ and $\boldsymbol{a}(u) \not< \boldsymbol{b} \frown (f, j)$ for all $u > s_1$. From P6(5) $\boldsymbol{\varepsilon}(\boldsymbol{b}) = \epsilon(\boldsymbol{b}, s_1)$ and $\boldsymbol{\phi}(\boldsymbol{b}, k) = \phi(\boldsymbol{b}, k, s_1)$ for each $k < j$ such that $k \notin D_0(\boldsymbol{b})$. From P7, $\phi(\boldsymbol{b}, j, s)$ takes arbitrarily large values. For each $s \geqslant s_1$ such that $\boldsymbol{b} \frown (f, j) \subset \boldsymbol{a}(s+1)$, from C6 at stage $(s+1, \mathrm{lh}(\boldsymbol{b}))$ we have $j \notin D_0(\boldsymbol{b}) \cup F(\boldsymbol{b})$ and $\phi(\boldsymbol{b}, j, s) \leqslant \leqslant \xi_n(\boldsymbol{\varepsilon}(\boldsymbol{b}), s)$. It follows that $\xi_n \boldsymbol{\varepsilon}(\boldsymbol{b}) = \omega$. Thus $\boldsymbol{b} \frown (f, j)$ is preferred.

For the rest suppose that $\boldsymbol{a}(s+1) < \boldsymbol{b} \frown a_1$ for at most a finite number of s. Choose s_1 such that $\boldsymbol{a}(s+1) \not< \boldsymbol{b} \frown a_1$ for all $s \geqslant s_1$ and $\boldsymbol{a}(s_1+1) = \boldsymbol{b} \frown a_1$ if possible. From P6(2) $\boldsymbol{\varepsilon}(\boldsymbol{b}) = \epsilon(\boldsymbol{b}, s_1)$ whence $\boldsymbol{a}(s+1) \neq \boldsymbol{b} \frown a_1$ for all $s > s_1$. Suppose $\boldsymbol{b} \frown d_1 \subset \boldsymbol{a}(s+1)$ for infinitely many s and in particular for $s = s_2 > s_1$. Then $\epsilon(\boldsymbol{b}, s_2) < \omega$. From P6(1) with d_1 in place of d_0, $\epsilon(\boldsymbol{b}) = \epsilon(\boldsymbol{b}, s_2)$ and $\mu(\boldsymbol{b}) = \mu(\boldsymbol{b}, s_2)$. Further for each $s > s_2$ such that $\boldsymbol{a}(s+1) \supset \boldsymbol{b} \frown d_1$, C4 occurs at stage $(s+1, \mathrm{lh}(\boldsymbol{b}))$ whence $\xi_n(\boldsymbol{\varepsilon}(\boldsymbol{b}), s) \leqslant \sup I(i, s)$. Recall that $\sup I(i, s)$ is either ω or the least of the numbers $\phi(\boldsymbol{c}, j, s)$ such that $\boldsymbol{c} \frown (f, j) \subset \boldsymbol{b}$. From P7 for each such $\boldsymbol{c}$, $\phi(\boldsymbol{c}, j, s)$ takes arbitrarily large values. Also $\phi(\boldsymbol{c}, j, s)$ is increasing with s if we ignore s such that $\phi(\boldsymbol{c}, j, s) = \omega$. Hence there exists an s such that $\boldsymbol{a}(s+1) \supset \boldsymbol{b} \frown d_1$ and $\sup I(i, s)$ is arbitrarily large. Thus $\xi_n \boldsymbol{\varepsilon}(\boldsymbol{b}) = \omega$ and hence in this case $\boldsymbol{b} \frown d_1$ is preferred.

For the rest suppose $\boldsymbol{a}(s+1) < \boldsymbol{b} \frown a_0$ for at most a finite number of s. There can be at most a finite number of pairs $\langle \boldsymbol{d}, u \rangle$ such that either $\boldsymbol{d} \subset \boldsymbol{b}$ or $\boldsymbol{d} < \boldsymbol{b} \frown d_0$, and $\epsilon(\boldsymbol{d}, n)$ is enumerated in A at stage $u+1$. Otherwise for some $\boldsymbol{d} \subset \boldsymbol{b}$ we should have $\boldsymbol{a}(u+1) = \boldsymbol{d} \frown a_2$ for infinitely many u which is incompatible with our case hypothesis if $\boldsymbol{d} = \boldsymbol{b}$, and with $\boldsymbol{b}$ being preferred if $\boldsymbol{d} \neq \boldsymbol{b}$. For each such pair $\langle \boldsymbol{d}, u \rangle$ at most a finite number of stages pertain to $\epsilon(\boldsymbol{d}, u)$. Choose s_1 greater than all these stages such that $\boldsymbol{a}(s+1) \not< \boldsymbol{b} \frown a_0$ for all $s \geqslant s_1$, and $\boldsymbol{a}(s_1+1) = \boldsymbol{b} \frown a_0$ if possible. From P6.6 we see that if $s \geqslant s_1$, $\boldsymbol{b} \subset \boldsymbol{c}$, and $\boldsymbol{b} \frown d_0 \not\subset \boldsymbol{c}$, then $\phi(\boldsymbol{c}, n, s)$ will not be destroyed at stage $s+1$ through some number $\leqslant \phi(\boldsymbol{c}, n, s)$ being enumerated in $\boldsymbol{c}$. By choice of s_1, for such s and $\boldsymbol{c}$, $\phi(\boldsymbol{c}, n, s)$ will be destroyed at stage $s+1$ neither through $\boldsymbol{a}(s+1)$ being $< \boldsymbol{c}$, nor through C1. Thus $\phi(\boldsymbol{c}, n, s)$ will not be destroyed at all. We now claim that, although $\boldsymbol{a}(s+1) = \boldsymbol{b} \frown a_0$ can occur for $s \geqslant s_1$ through C2.1 at stage $s+1$, there can be at most a finite number of $s \geqslant s_1$ such that $\boldsymbol{a}(s+1) = \boldsymbol{b} \frown a_0$. For any such s there exists $\boldsymbol{c} \supset \boldsymbol{b}$ such that $\boldsymbol{c} \not\supset \boldsymbol{b} \frown d_0$, $\epsilon(\boldsymbol{c}, s) < \omega$, and

$\phi(c, n, s+1) < \phi(c, n, s) = \omega$. Further $\epsilon(c, s)$ must have been assigned at a stage $\leqslant s_1$, whence c can take only a finite number of values. This justifies the claim.

Thus for all sufficiently large $s \geqslant s_2$, $a(s+1) \supset b$ implies $a(s+1) \supset b \cap d_0$. Choose the least $t \geqslant s_2$, if any, such that $\epsilon(b, t) < \omega$ and $a(t+1) \supset b \cap d_0$. Then from P6(1), $\epsilon(b) = \epsilon(b, t)$ and $\mu(b) = \mu(b, t) < \omega$. Also, if t does not exist, $\epsilon(b, s)$ and $\mu(b, s)$ are both ω for all sufficiently large s, because there are only a finite number of s such that $a(s+1) = b \cap a_1$, i.e. there are only a finite number of s such that $\epsilon(b, s+1) < \epsilon(b, s) = \omega$. Suppose that there are infinitely many s such that $a(s+1) \supset b \cap d_0$ and C1.4 occurs at stage $(s+1, \mathrm{lh}(b))$. For each such s, stage $(s+1, \mathrm{lh}(b))$ pertains to some $\epsilon(c, s)$ where $c \supset b$, $c \not\supset b \cap d_0$, and there exists a $t < s$ such that $\epsilon(c, t) = \epsilon(c, s)$ was enumerated in A at stage $t+1$. Since $a(t+1) < b \cap d_0$ there are only a finite number of possibilities for the pair $\langle c, t\rangle$. Thus for infinitely many s, $a(s+1) \supset b \cap d_0$ and stage $(s+1, \mathrm{lh}(b))$ pertains to the same $\epsilon(c, t)$. It now follows immediately from P9(1) that $A \neq \theta_n$.

There remains the case in which, for infinitely many s, $a(s+1) \supset b \cap d_0$ and C2.2 occurs at stage $(s+1, i)$. From the hypothesis of C2 for each such s, stage $(s+1, i)$ is associated with some $c \supset b$ such that $\epsilon(c, s) < \omega$, $c \not\supset b \cap d_0$, $\phi(c, n, s) = \omega$, and $F(c) - F(b)$ contains no number $< n$. Since $c = b$ or $c < b \cap d_0$ there are only a finite number of possibilities for c. From P9(2) it follows that either $A \neq \theta_n$, or $B_n^0 = \psi_n^0$, or $B_n^1 \neq \psi_n^1$. This completes the proof that some proper extension of b is preferred.

P12. *For each $m < \omega$, either $A \leqslant_T C + B_m^0$ or $A \leqslant_T C + B_m^1$ or $\theta_m \neq A$ or $\psi_m^0 \neq B_m^1$ or $\psi_m^1 \neq B_m^1$.*

Proof. Suppose that $\theta_m = A$, $\psi_m^0 = B_m^0$, and $\psi_m^1 = B_m^1$. A number can only be enumerated in A at stage $s+1$ if it has the form $\epsilon(b, s)$. If $\epsilon(b, s)$ is assigned at stage t then $\epsilon(b, s) > \epsilon(b', s')$ for all b' and all $s' \leqslant t$ such that $\epsilon(b', s') < \omega$. This follows immediately from the provisions of C3. Thus to compute A it is sufficient to have an algorithm for telling, given b and t such that $\epsilon(b, t) < \omega$, whether $\epsilon(b, t)$ is in A or not. Let a be the longest preferred sequence such that $\mathrm{O}(a) = m+1$. We shall confine our attention to stages sufficiently large that all subsequent stages have characteristics either $\supset a$ or $> a$. We shall consider only numbers $\epsilon(b, t)$ such that $b \supset a$, because for b neither $\supset a$ nor $> a$ there can be at most a finite number of values of $\epsilon(b, t)$, and if $b > a$ then $\epsilon(b, t)$ will eventually be destroyed. If $m \in D_0(a)$ then $c \cap d_0 \subset a$ for some c such that $\mathrm{O}(c) = m$.

Being an initial segment of $\boldsymbol{a}$, $\boldsymbol{c} \frown d_0$ is preferred which by p5 contradicts our supposition that $\theta_m = \boldsymbol{A}$, $\psi_m^0 = \boldsymbol{B}_m^1$, and $\psi_m^1 = \boldsymbol{B}_m^1$. Thus $m \notin D_0(\boldsymbol{a})$.

Suppose that (f, m) is the last member of $\boldsymbol{a}$. We shall show that with the aid of oracles for $\boldsymbol{C}$ and $\boldsymbol{B}_m^1$ we can effectively tell whether $\epsilon(\boldsymbol{b}, t)$ is in $\boldsymbol{A}$ or not. Let $\boldsymbol{b}_m$ be the least initial segment of $\boldsymbol{b}$ such that $\boldsymbol{a} \subset \boldsymbol{b}_m$ and either $\boldsymbol{b}_m = \boldsymbol{b}$ or $\boldsymbol{b}_m \frown (f, n) \subset \boldsymbol{b}$ where $n < m$. If $\epsilon(\boldsymbol{b}, s) < \omega$ then $\epsilon(\boldsymbol{b}_m, s) \leqslant \epsilon(\boldsymbol{b}, s) < \omega$ also. This is obvious when $\boldsymbol{b} = \boldsymbol{b}_m$ and can easily be proved by induction on s when $\boldsymbol{b} \neq \boldsymbol{b}_m$. If $\epsilon(\boldsymbol{b}_m, s) < \omega$ and $\boldsymbol{a} \subset \boldsymbol{a}(s+1)$ then $\phi(\boldsymbol{b}_m, m, s) < \omega$. Otherwise we should have $\boldsymbol{a} < \boldsymbol{a}(s+1)$ through C2. Let $\boldsymbol{b}_m^*$ be the unique preferred sequence which has the same length as $\boldsymbol{b}_m$. If $\boldsymbol{b}_m^* < \boldsymbol{b}_m$ then $\epsilon(\boldsymbol{b}, t)$ will eventually be destroyed.

If $\boldsymbol{b}_m^* = \boldsymbol{b}_m$ then one of $\boldsymbol{b}_m \frown d_0$, $\boldsymbol{b}_m \frown d_1$, and $\boldsymbol{b}_m \frown w$ is preferred, or $\boldsymbol{b}_m \frown (f, n')$ is preferred for some $n' > m$. Otherwise the choice of $\boldsymbol{a}$ would be refuted. From p4–p7 if $\epsilon(\boldsymbol{b}, t)$ is never destroyed then $\boldsymbol{\mu}(\boldsymbol{b}_m) < \omega$ for $\boldsymbol{\phi}(\boldsymbol{b}_m, m) < \omega$. Thus in any case $\boldsymbol{\phi}(\boldsymbol{b}_m, m) < \omega$. Note that $m \notin D_0(\boldsymbol{b}_m)$ because $m \notin D_0(\boldsymbol{a})$ and $\mathrm{O}(\boldsymbol{c}) > m$ for each $\boldsymbol{c}$ such that $\boldsymbol{a} \subset \boldsymbol{c} \subset \boldsymbol{b}_m$.

Now let $\boldsymbol{b}_m < \boldsymbol{b}_m^*$. Either $\boldsymbol{\varepsilon}(\boldsymbol{b}_m^*) < \omega$ or $\boldsymbol{\varepsilon}(\boldsymbol{b}_m^* \frown d_0) < \omega$, because $\boldsymbol{\varepsilon}(\boldsymbol{c}) < \omega$ whenever $\boldsymbol{c}$ is preferred and $\boldsymbol{c} \frown d_0$ is not. From P2.1 for all sufficiently large s such that $\phi(\boldsymbol{b}_m, m, s) < \omega$ we have

$$\phi(\boldsymbol{b}_m, m, s) \leqslant \mu(\boldsymbol{b}_m, s) < \min\{\boldsymbol{\varepsilon}(\boldsymbol{b}_m^*), \boldsymbol{\varepsilon}(\boldsymbol{b}_m^* \frown d_0)\} < \omega .$$

From above if $\epsilon(\boldsymbol{b}, t)$ is never destroyed then there are infinitely many s such that $\phi(\boldsymbol{b}_m, m, s) < \omega$. Hence if $\epsilon(\boldsymbol{b}, t)$ is never destroyed then $\boldsymbol{\phi}(\boldsymbol{b}_m, m) < \omega$.

Suppose $\phi(\boldsymbol{b}_m, m, s)$ is destroyed at stage $s+1$ and that $\epsilon(\boldsymbol{b}, s) < \omega$ is not destroyed at stage $s+1$. Then either some number $\leqslant \phi(\boldsymbol{b}_m, m, s)$ is enumerated in $\boldsymbol{C}$ or $\phi(\boldsymbol{b}_m, m, s)$ is destroyed through C1.2 or C1.3 at stage $s+1$. If $\phi(\boldsymbol{b}_m, m, s)$ is destroyed through C1.2 at stage $s+1$ and stage $s+1$ pertains to $\epsilon(\boldsymbol{c}, s)$, then $\boldsymbol{b}_m \frown (f, m) \subset \boldsymbol{c}$. But $\boldsymbol{b}_m \frown (f, m)$ is not legitimate, thus C1.2 is impossible. If $\phi(\boldsymbol{b}_m, m, s)$ is destroyed through C1.3 at stage $s+1$, let stage $s+1$ pertain to $\epsilon(\boldsymbol{c}, s)$ where $\epsilon(\boldsymbol{c}, s)$ was enumerated in $\boldsymbol{A}$ at stage $u+1$. Then $\boldsymbol{a}(\boldsymbol{c}, s) = \boldsymbol{b}_m$ and $\nu(\boldsymbol{c}, s) = m$. Since C1.3 occurs some number $\leqslant \theta_m(\epsilon(\boldsymbol{c}, s), u)$ is enumerated in $\boldsymbol{B}_m^1$ at a stage $>u$ and $\leqslant s$. From P10 $\theta_m(\epsilon(\boldsymbol{c}, s), u) \leqslant \phi(\boldsymbol{b}_m, m, u)$. From P1 no stage $>u+1$ and $\leqslant s+1$ has characteristic $\supset \boldsymbol{a}$, because $\boldsymbol{a} \subset \boldsymbol{b}_m = \boldsymbol{a}(\boldsymbol{c}, s)$ and $\mathrm{O}(\boldsymbol{a}) > m = \nu(\boldsymbol{c}, s)$. Thus if $\phi(\boldsymbol{b}_m, m, s)$ is destroyed at stage $s+1$, $\epsilon(\boldsymbol{b}, s) = \epsilon(\boldsymbol{b}, s+1) < \omega$, and s' is the greatest number $\leqslant s$ such that $\boldsymbol{a}(s'+1) \supset \boldsymbol{a}$, then some number $\leqslant \phi(\boldsymbol{b}_m, m, s')$ is enumerated in $\boldsymbol{C} \cup \boldsymbol{B}_m^1$ at a stage $>s'$ and $\leqslant s+1$.

Suppose that $\epsilon(\boldsymbol{b}, t)$ is enumerated in A at stage $t + 1$ and that no number $\leqslant \epsilon(\boldsymbol{b}, t)$ is enumerated in C at a stage $>t$. Let v be the least number $>t$ such that stage $v + 1$ pertains to $\epsilon(\boldsymbol{b}, t) = \epsilon(\boldsymbol{b}, v)$ and one of the following five possibilities holds:

(i) C1.1 occurs and $\boldsymbol{b}_m \subset \boldsymbol{a}(\boldsymbol{b}, v)$,

(ii) C1.2 occurs and there exists $\boldsymbol{d} \subsetneqq \boldsymbol{a}$ such that $\boldsymbol{d}$ is the greatest initial segment of $\boldsymbol{a}(\boldsymbol{b}, v)$ for which $\boldsymbol{d} \cap (f, \nu(\boldsymbol{b}, v)) \subset \boldsymbol{a}(\boldsymbol{b}, v)$,

(iii) C1.3 occurs and there exists $\boldsymbol{d} \supset \boldsymbol{a}$ such that $\boldsymbol{d}$ is the greatest initial segment of $\boldsymbol{a}(\boldsymbol{b}, v)$ for which $\boldsymbol{d} \cap (f, \nu(\boldsymbol{b}, v)) \subset \boldsymbol{a}(\boldsymbol{b}, v)$,

(iv) $\boldsymbol{a}(\boldsymbol{b}, v) = \boldsymbol{b}_m$ and $\nu(\boldsymbol{b}, v) = m$,

(v) $\nu(\boldsymbol{b}, v + 1) > m$.

Note that $\nu(\boldsymbol{b}, t + 1) \leqslant m$ because some number $\leqslant m$ is in $F(\boldsymbol{b}) - D_0(\boldsymbol{b})$ namely the least number k such that for some $\boldsymbol{c}$, $\boldsymbol{c} \cap (f, k) \subset \boldsymbol{b}$. Note also from P1 that if $u > t$, $\nu(\boldsymbol{b}, u) \leqslant m$, and $\boldsymbol{a}(u + 1) < \boldsymbol{b}$, then $\boldsymbol{a}(u + 1) < \boldsymbol{a}$. If v does not exist then either $\epsilon(\boldsymbol{b}, t)$ is destroyed at some stage with characteristic $<\boldsymbol{a}$, or $\epsilon(\boldsymbol{b}, t)$ is never destroyed and $\nu(\boldsymbol{b}, v) \leqslant m$ for all $v > t$, which yields $\boldsymbol{a} \not\subset \boldsymbol{a}(v)$ for all $v > t$. But the only stages considered have characteristics $\supset \boldsymbol{a}$ or $>\boldsymbol{a}$, and infinitely many have characteristics $\supset \boldsymbol{a}$. Thus v exists.

By induction on z, if $t < z \leqslant v$ then $\boldsymbol{b}_m \subset \boldsymbol{a}(\boldsymbol{b}, z)$ and $\nu(b, z) \leqslant m$. Let $n' < m$ and $\boldsymbol{a} \subset \boldsymbol{b}' \cap (f, n') \subset \boldsymbol{b}$. Again by induction on z, if $t < z \leqslant v$ and $\boldsymbol{b}' \cap (f, n') \subset \boldsymbol{a}(\boldsymbol{b}, z)$ then $\nu(\boldsymbol{b}, z) \leqslant n'$. These two inductions are similar. We treat the first one in detail. Suppose $z = t + 1 \leqslant v$ then $\boldsymbol{a}(\boldsymbol{b}, z) = \boldsymbol{b} \supset \boldsymbol{b}$ through C5.2 and $\nu(\boldsymbol{b}, z) \leqslant m$ since $t < v$. Now for induction let $t + 1 < z \leqslant v$, $\boldsymbol{a}(\boldsymbol{b}, z - 1) \supset \boldsymbol{b}_m$, and $\nu(\boldsymbol{b}, z - 1) \leqslant m$. Clearly $\nu(\boldsymbol{b}, z) \leqslant m$, otherwise $z = v + 1$. If $\boldsymbol{a}(\boldsymbol{b}, z) \neq \boldsymbol{a}(\boldsymbol{b}, z - 1)$ then stage z pertains to $\epsilon(\boldsymbol{b}, t)$ and C1.2 occurs since $\nu(\boldsymbol{b}, z) \neq \omega$. Let $\boldsymbol{d}$ be the greatest initial segment of $\boldsymbol{a}(\boldsymbol{b}, z - 1)$ such that $\boldsymbol{d} \cap (f, \nu(\boldsymbol{b}, z - 1)) \subset \boldsymbol{a}(\boldsymbol{b}, z - 1)$. Suppose C1.2 occurs at stage z. Then $\boldsymbol{d} \supset \boldsymbol{a}$, otherwise $z = v + 1$ through (ii). Hence $\nu(\boldsymbol{b}, z - 1) < m$, and $\boldsymbol{b}_m \subset \boldsymbol{d}$ by definition of $\boldsymbol{b}_m$ because $\boldsymbol{d} \cap (f, \nu(\boldsymbol{b}, z - 1)) \subset \boldsymbol{a}(\boldsymbol{b}, z - 1) \subset \boldsymbol{b}$. Also $\boldsymbol{a}(\boldsymbol{b}, z) \supset \boldsymbol{b}_m$ since $\boldsymbol{a}(\boldsymbol{b}, z) = \boldsymbol{d}$. This completes the induction step. We now return to the main thread of our argument in which we aim to show that, if $\epsilon(\boldsymbol{b}, t)$ is enumerated in A at stage $t + 1$ and no number $\leqslant \epsilon(\boldsymbol{b}, t)$ is enumerated in C at a stage $>t$, then there exists $s > t$ such that $\epsilon(\boldsymbol{b}, s + 1) = \epsilon(\boldsymbol{b}, t)$, $\phi(\boldsymbol{b}_m, m, s) = \phi(\boldsymbol{b}_m, m, t)$, and $\phi(\boldsymbol{b}_m, m, s + 1) = \omega$.

If $\boldsymbol{a}(\boldsymbol{b}, v) = \boldsymbol{b}_m$ and $\nu(\boldsymbol{b}, v) < m$ then either (i) or (ii) obtains. Otherwise C1.3 occurs at stage $v + 1$, and (iii) is ruled out by the nonexistence of suitable $\boldsymbol{d}$ while (v) is ruled out by the eligibility of m as a value of

$\nu(\boldsymbol{b}, v+1)$. Now consider the case in which $\boldsymbol{a}(\boldsymbol{b}, v) \neq \boldsymbol{b}_m$. Let $\boldsymbol{b}_m = \boldsymbol{b}$ then $\boldsymbol{a}(\boldsymbol{b}, t+1) = \boldsymbol{b}_m$. Further, let z be the least number $>t$ such that $\boldsymbol{a}(\boldsymbol{b}, z+1) \neq \boldsymbol{b}_m$, then $z < v$ and $\nu(\boldsymbol{b}, z) \leqslant m$. Since $\boldsymbol{a}(\boldsymbol{b}, z+1) \neq \boldsymbol{a}(\boldsymbol{b}, z)$, C1.2 occurs at stage $z+1$ and

$$\boldsymbol{a}(\boldsymbol{b}, z+1) \frown (f, \nu(\boldsymbol{b}, z)) \subset \boldsymbol{b}_m = \boldsymbol{b} .$$

Now $\boldsymbol{a}(\boldsymbol{b}, z+1)$ is a proper initial segment of $\boldsymbol{a}$ by definition of $\boldsymbol{b}_m$. Hence $v = z$ contradiction. Thus $\boldsymbol{b}_m \neq \boldsymbol{b}$ which means that $\boldsymbol{b}_m \frown (f, n) \subset \boldsymbol{b}$ for some $n < m$. From the inductive result about n' and $\boldsymbol{b}'$ stated above we get $\boldsymbol{a}(\boldsymbol{b}, v) \supset \boldsymbol{b}_m \frown (f, n)$ and $\nu(\boldsymbol{b}, v) \leqslant n$ by taking n' and $\boldsymbol{b}'$ to be n and $\boldsymbol{b}_m$ respectively. Let n_0 be the least number $\geqslant \nu(\boldsymbol{b}, v)$ such that for some $\boldsymbol{b}_0 \supset \boldsymbol{a}$ we have $\boldsymbol{b}_0 \frown (f, n_0) \subset \boldsymbol{a}(\boldsymbol{b}, v)$. If $\nu(\boldsymbol{b}, v) < n_0$ either (i) or (ii) holds, because if C1.3 occurs no $\boldsymbol{d}$ appropriate to (iii) exists and $n_0 \leqslant n$ is eligible as a value of $\nu(\boldsymbol{b}, v+1)$. If $n_0 \notin F(\boldsymbol{a}(\boldsymbol{b}, v))$ then there exist $n_1 < n_0$ and $\boldsymbol{b}_1$ such that $\boldsymbol{b}_0 \subset \boldsymbol{b}_1 \frown (f, n_1) \subset \boldsymbol{a}(\boldsymbol{b}, v)$. Using the same inductive result as before, $\nu(\boldsymbol{b}, v) \leqslant n_1$ which contradicts the choice of n_0. Thus $n_0 \in F(\boldsymbol{a}(\boldsymbol{b}, v))$. Hence if $\nu(\boldsymbol{b}, v) = n_0$ C1.3 occurs at stage $v+1$, whence (iii) is satisfied because $\boldsymbol{b}_0 \supset \boldsymbol{a}$. We conclude that whether $\boldsymbol{a}(\boldsymbol{b}, v) = \boldsymbol{b}_m$ or not, (v) may be ignored.

If (ii) holds then by P5 some stage $>v+1$ has characteristic $<\boldsymbol{a}$. The point to notice here is that $\boldsymbol{d}$ in the statement of P5 is a proper initial segment of $\boldsymbol{a}$. Thus (ii) may also be ignored.

We now show that if (i), (iii), or (iv) holds then $\phi(\boldsymbol{b}_m, m, t)$ is destroyed at some stage $>t+1$ and $\leqslant v+1$. If (i) holds then $\phi(\boldsymbol{a}(\boldsymbol{b}, v), \nu(\boldsymbol{b}, v), v)$ is enumerated in C at stage $v+1$. If $\boldsymbol{b}_m \frown (f, n) \subset \boldsymbol{a}(\boldsymbol{b}, v)$ then from P2.2 and P4.2.

$$\phi(\boldsymbol{a}(\boldsymbol{b}, v), \nu(\boldsymbol{b}, v), t) < \phi(\boldsymbol{b}_m, n, t) < \phi(\boldsymbol{b}_m, m, t) .$$

If $\boldsymbol{b}_m \frown (f, n) \not\subset \boldsymbol{a}(\boldsymbol{b}, v)$ then $\boldsymbol{b}_m = \boldsymbol{a}(\boldsymbol{b}, v)$ because we are assuming (i). Since $m \in \mathrm{F}(\boldsymbol{b}_m)$, in this case $\nu(\boldsymbol{b}, v) < m$ whence by P4.2

$$\phi(\boldsymbol{a}(\boldsymbol{b}, v), \nu(\boldsymbol{b}, v), t) < \phi(\boldsymbol{b}_m, m, t) .$$

Note that the values of ϕ appearing in these last two inequalities are $<\omega$ by P4.1. It is now clear that if (i) holds, $\phi(\boldsymbol{b}_m, m, t)$ will certainly be destroyed at a stage $>t+1$ and $\leqslant v+1$ unless $\phi(\boldsymbol{a}(\boldsymbol{b}, v), \nu(\boldsymbol{b}, v), t)$ is destroyed through C1 at a stage $z+1$, $t < z < v$. Note that $\phi(\boldsymbol{a}(\boldsymbol{b}, v), \nu(\boldsymbol{b}, v), t)$ is not destroyed through $\boldsymbol{a}(z+1) < \boldsymbol{a}(\boldsymbol{b}, v)$, because this would mean that $\epsilon(\boldsymbol{b}, t)$ was destroyed at stage $z+1$ contrary to the choice of v.

For proof by contradiction suppose that $\phi(\boldsymbol{a}(\boldsymbol{b}, v), \nu(\boldsymbol{b}, v), t)$ is destroyed through C1 at stage $z + 1$. Let stage $z + 1$ pertain to $\epsilon(\boldsymbol{b}', v)$ where $\epsilon(\boldsymbol{b}', v) = \epsilon(\boldsymbol{b}', t')$ was enumerated in A at stage $t' + 1$. Then $\boldsymbol{a}(\boldsymbol{b}, v) \subset \boldsymbol{b}'$ and $\nu(\boldsymbol{b}, v) = \nu(\boldsymbol{b}', z) \neq \nu(\boldsymbol{b}', z + 1)$. Thus $\boldsymbol{b}' \neq \boldsymbol{b}$. If $t' < t$, let $\boldsymbol{b}'_0$ be the least initial segment of $\boldsymbol{b}'$ such that $\mathrm{O}(\boldsymbol{b}'_0) = \nu(\boldsymbol{b}', t) \leqslant \nu(\boldsymbol{b}', z)$, then $\boldsymbol{b}'_0 \subset \boldsymbol{a}(\boldsymbol{b}, v) \subset \boldsymbol{b}$. Now $\boldsymbol{a}(t + 1) = \boldsymbol{b} \frown a_2$, but since $\boldsymbol{a}(t + 1) \supset \boldsymbol{b}'_0$ we have $\boldsymbol{a}(t + 1) \supset \boldsymbol{b}'_0 \frown d_0$ through C1 and $\epsilon(\boldsymbol{b}', t')$. Thus $\boldsymbol{b}'_0 \frown d_0 \subset \boldsymbol{b}$. But $\boldsymbol{b}'_0 \frown d_0 \not\subset \boldsymbol{b}'$ since $\nu(\boldsymbol{b}', t) \notin \mathrm{D}_0(\boldsymbol{a}(\boldsymbol{b}', t))$, and $\boldsymbol{a}(\boldsymbol{b}, v) \frown d_0 \not\subset \boldsymbol{b}$ by definition of $\boldsymbol{a}(\boldsymbol{b}, v)$. It follows that $\boldsymbol{b}'_0$ is a proper initial segment of $\boldsymbol{a}(\boldsymbol{b}, v)$, whence $\boldsymbol{b}'_0 \frown d_0 \subset \boldsymbol{a}(\boldsymbol{b}, v)$ since $\boldsymbol{a}(\boldsymbol{b}, v) \subset \boldsymbol{b}$. Since $\boldsymbol{a}(\boldsymbol{b}, v) \subset \boldsymbol{b}', \boldsymbol{b}'_0 \frown d_0 \subset \boldsymbol{b}'$ which contradicts our finding above.

Now consider the case in which $t < t'$, let $\boldsymbol{b}_0$ be the least initial segment of $\boldsymbol{b}$ such that $\mathrm{O}(\boldsymbol{b}_0) = \nu(\boldsymbol{b}, t') \leqslant \nu(\boldsymbol{b}, v)$. Since $\phi(\boldsymbol{a}(\boldsymbol{b}, v), \nu(\boldsymbol{b}, v), t)$ is destroyed through C1.2 or C1.3 at stage $z + 1$ pertaining to $\epsilon(\boldsymbol{b}', t')$, $\boldsymbol{a}(\boldsymbol{b}, v) \subset \boldsymbol{b}'$ and $\boldsymbol{a}(\boldsymbol{b}, v) \frown d_0 \not\subset \boldsymbol{b}'$. Since $\nu(\boldsymbol{b}, v) \geqslant \mathrm{O}(\boldsymbol{b}_0)$, we have $\boldsymbol{b}_0 \subset \boldsymbol{a}(\boldsymbol{b}, v) \subset \boldsymbol{b}'$. Considering stage $(t' + 1, \mathrm{lh}(\boldsymbol{b}_0))$ we see from P1 that $\boldsymbol{b}' \supset \boldsymbol{b}_0 \frown d_0$ because $\boldsymbol{a}(t' + 1) = \boldsymbol{b}' \frown a_2$. Hence $\boldsymbol{b}_0 \frown a_0 \subset \boldsymbol{a}(\boldsymbol{b}, v) \subset \boldsymbol{a}(\boldsymbol{b}, t')$, and $\nu(\boldsymbol{b}, t') \in \mathrm{D}_0(\boldsymbol{a}(\boldsymbol{b}, t'))$ which is a contradiction of the way $\nu(\boldsymbol{b}, t)$ is defined. This completes the proof that if (i) holds then $\phi(\boldsymbol{b}_m, m, t)$ will be destroyed at a stage $>t + 1$ and $\leqslant v + 1$.

We now consider the case in which (iii) holds. Here $\phi(\boldsymbol{d}, \nu(\boldsymbol{b}, v), v)$ is enumerated in $\boldsymbol{C}$ at stage $v + 1$. Notice that $\boldsymbol{b}_m \subset \boldsymbol{d}$ because $\boldsymbol{a} \subset \boldsymbol{d}$, $\boldsymbol{d} \frown (f, \nu(\boldsymbol{b}, v)) \subset \boldsymbol{b}$, $m \in F(\boldsymbol{a})$, and $\nu(\boldsymbol{b}, v) \leqslant m$. From P2.2 and P4.2 we get $\phi(\boldsymbol{d}, \nu(\boldsymbol{b}, v), t) < \phi(\boldsymbol{b}_m, m, t)$ either because $\boldsymbol{d} = \boldsymbol{b}_m$ and $\nu(\boldsymbol{b}, v) = n < m$, or because $\boldsymbol{b}_m \frown (f, n) \subset \boldsymbol{d}$ and $n < m$. Thus $\phi(\boldsymbol{b}_m, m, t)$ is certainly destroyed at some stage $>t + 1$ and $\leqslant v + 1$ unless $\phi(\boldsymbol{d}, \nu(\boldsymbol{b}, v), t)$ is destroyed through C1 at a stage $z + 1, t < z < v$. But such destruction of $\phi(\boldsymbol{d}, \nu(\boldsymbol{b}, v), t)$ through C1 can be ruled out in the same way as for $\phi(\boldsymbol{a}(\boldsymbol{b}, v), \nu(\boldsymbol{b}, v), t)$ in the treatment of (i). This completes our consideration of (iii).

Finally suppose that (iv) holds. Then C1.3 occurs at stage $v + 1$, because if C1.2 occurs from P5 there will be a stage $>v + 1$ with characteristic $< \boldsymbol{a}$. In this case $\phi(\boldsymbol{b}_m, m, t)$ is destroyed at stage $v + 1$ if not before.

We can summarize our findings as follows. If $\epsilon(\boldsymbol{b}, s) < \omega$ and $\boldsymbol{a} \subset \boldsymbol{a}(s + 1)$ then $\phi(\boldsymbol{b}_m, m, s) < \omega$. If $\epsilon(\boldsymbol{b}) < \omega$ then $\Phi(\boldsymbol{b}_m, m) < \omega$. Let $\phi(\boldsymbol{b}_m, m, s)$ be destroyed at stage $s + 1$, $\epsilon(\boldsymbol{b}, s) = \epsilon(\boldsymbol{b}, s + 1) < \omega$, and s' be the greatest number $\leqslant s$ such that $\boldsymbol{a}(s' + 1) \supset \boldsymbol{a}$, then some number $\leqslant \phi(\boldsymbol{b}_m, m, s')$ is enumerated in $\boldsymbol{C} \cup \boldsymbol{B}^1_m$ at a stage $>s'$ and $\leqslant s + 1$. If $\epsilon(\boldsymbol{b}, t)$ is enumerated in $\boldsymbol{A}$ at stage $t + 1$ and no number $\leqslant \epsilon(\boldsymbol{b}, t)$ is enumerated in $\boldsymbol{C}$ at a stage $>t$ then $\phi(\boldsymbol{b}_m, m, t)$ is eventually destroyed before $\epsilon(\boldsymbol{b}, t)$ is destroyed. Thus, if $\epsilon(\boldsymbol{b}, s) < \omega$, $\epsilon(\boldsymbol{b}, s) \notin A$ if and only if either $\epsilon(\boldsymbol{b}, s)$ is destroyed before being enumerated in A or there exists v such that $A(\epsilon(\boldsymbol{b}, s), v) = 0$, $\epsilon(\boldsymbol{b}, v) = \epsilon(\boldsymbol{b}, s)$, $\boldsymbol{a} \subset \boldsymbol{a}(v+1)$, $\phi(\boldsymbol{b}_m, m, v) < \omega$, and no number $\leqslant \phi(\boldsymbol{b}_m, m, v)$ is enumerated in $\boldsymbol{B}_m^1 \cup \boldsymbol{C}$ at a stage $>v$. This shows that $\boldsymbol{A}$ is recursive in $\boldsymbol{C} + \boldsymbol{B}_m^1$ if $\boldsymbol{a}$ has last member (f, m).

Now suppose that (f, m) is not the last member of $\boldsymbol{a}$. We shall show that with the aid of oracles for $\boldsymbol{C}$ and $\boldsymbol{B}_m^0$ we can effectively tell whether $\epsilon(\boldsymbol{b}, t) \in A$ or not. The argument is very like the one we have given above. Let $\boldsymbol{b}_m$ be the least initial segment of $\boldsymbol{b}$ such that $\boldsymbol{a} \subset \boldsymbol{b}_m$ and either $\boldsymbol{b}_m = \boldsymbol{b}$ or $\boldsymbol{b}_m \frown (f, n) \subset \boldsymbol{b}$ where $n \leqslant m$. If $\epsilon(\boldsymbol{b}, s) < \omega$ and $\boldsymbol{a} \subset \boldsymbol{a}(s+1)$ then $\phi(\boldsymbol{b}_m, m, s) < \omega$. Otherwise we should have $\boldsymbol{a} < \boldsymbol{a}(s+1)$ by C2. Just as before we can argue that if $\boldsymbol{\varepsilon}(\boldsymbol{b}) < \omega$ then $\boldsymbol{\phi}(\boldsymbol{b}_m, m) < \omega$. If $\phi(\boldsymbol{b}_m, m, s)$ is destroyed at stage $s+1$ and $\epsilon(\boldsymbol{b}, s) = \epsilon(\boldsymbol{b}, s+1) < \omega$ then either some number $\leqslant \phi(\boldsymbol{b}_m, m, s)$ is enumerated in $\boldsymbol{C}$ at stage $s+1$, or there exists $t < s$ such that no stage $>t+1$ and $\leqslant s+1$ has characteristic $\supset \boldsymbol{a}$ and some number $\leqslant \phi(\boldsymbol{b}_m, m, t)$ is enumerated in $\boldsymbol{B}_m^0$ at a stage $>t$ and $\leqslant s$. The difference from the case in which (f, m) is the last member of $\boldsymbol{a}$ is that now $m \notin F(\boldsymbol{b}_m)$. Thus if $\phi(\boldsymbol{b}_m, m, s)$ is destroyed through C1, it must now be through C1.2. To complete the proof it has to be shown that if $\epsilon(\boldsymbol{b}, t)$ is enumerated in $\boldsymbol{A}$ at stage $t+1$ and no number $\leqslant \epsilon(\boldsymbol{b}, t)$ is enumerated in $\boldsymbol{C}$ at a stage $>t$, then $\phi(\boldsymbol{b}_n, m, t)$ is destroyed before $\epsilon(\boldsymbol{b}, t)$. We define v as before except that (iv) now becomes:

(iv) $\qquad \boldsymbol{b}_m \frown (f, m) \subset \boldsymbol{a}(\boldsymbol{b}, v)$ and $\nu(\boldsymbol{b}, v) = m$.

As before the possibilities (ii) and (v) may be ignored. For (i), (iii), and (iv) we follow the same line of argument as before with some minor changes. We conclude that in each case $\phi(\boldsymbol{b}_m, m, t)$ is destroyed at a stage $\leqslant v+1$ and $\epsilon(\boldsymbol{b}, v+1) < \omega$. It follows easily that $\boldsymbol{A}$ is recursive in $\boldsymbol{C} + \boldsymbol{B}_m^0$.

We conclude this part by showing that Propositions 11 and 12 ensure the success of the construction. For each m there exists a preferred characteristic sequence $\boldsymbol{c}$ such that $\mathrm{O}(\boldsymbol{c}) = m$ and $\boldsymbol{c} \frown d_0$ is not preferred. To see this consider arbitrary legitimate $\boldsymbol{b} \in \boldsymbol{G}$. By induction on i it follows from L2 that (f, i) occurs at most 2^i times in $\boldsymbol{b}$. Therefore if $\boldsymbol{b}$ is long enough either (f, i) occurs in $\boldsymbol{b}$ for some $i \geqslant n$ or there exists a consecutive part of $\boldsymbol{b}$ of length $2n$ having no member of the form (f, i). From L1 two consecutive members of $\boldsymbol{b}$ cannot both be d_0. Further, if $\boldsymbol{b}_0 \subset \boldsymbol{b}_1 \subset \boldsymbol{b}$ and $\mathrm{lh}(\boldsymbol{b}_1) = \mathrm{lh}(\boldsymbol{b}_0 + 1)$, then $\mathrm{O}(\boldsymbol{b}_1) \leqslant \mathrm{O}(\boldsymbol{b}_0) + 1$, and

$O(\boldsymbol{b}_1) = O(\boldsymbol{b}_0) + 1$ unless either $\boldsymbol{b}_1 = \boldsymbol{b}_0 \frown (f, i)$ for some i or $\boldsymbol{b}_1 = \boldsymbol{b}_0 \frown d_0$. If $\boldsymbol{b}_1 = \boldsymbol{b}_0 \frown d_0$, then $O(\boldsymbol{b}_1) = O(\boldsymbol{b}_0)$. It follows that if $\boldsymbol{b}$ is long enough then $O(\boldsymbol{b}_0) = n$ for some $\boldsymbol{b}_0 \subset \boldsymbol{b}$ such that $\boldsymbol{b}_0 \frown d_0 \not\subset \boldsymbol{b}$. From P11 there exist preferred characteristic sequences of arbotrary length, whence the desired $\boldsymbol{c}$ exists. Since either $\boldsymbol{c} \frown d_1$, or $\boldsymbol{c} \frown w$, or $\boldsymbol{c} \frown (f, i)$ is preferred for some i, $\Xi_m \neq \boldsymbol{A}$ by p6, p4, or p7 respectively. Thus the construction certainly satisfies $\mathfrak{C}'_m$. From P12 the construction also satisfies $\mathfrak{C}_m$.

4. Verification of Propositions 1 to 10

This part of the paper is very complex. The complexity stems from the way in which the construction was discovered. From very crude beginnings the final format of the construction was achieved only after a series of modifications each designed to eliminate a flaw found in the previous attempt. This process of evolution yielded only a cloudy intuition as to why the construction should work.

Proposition P1 will be proved outright. Propositions P2–6 are proved by simultaneous course-of-values induction. Each part of each of these propositions is an assertion about the first $s + 1$ stages of the construction. In each case there is the tacit assumption that there are $s + 1$ stages. Further in the induction step the various parts of P2–6 are to be regarded as being proved in the order:

P2(1), P4(1), P5, P2(2), P3(1), P4(2), P4(3), P6(3), P3(2) .

The remaining parts of P6 are to be placed after P3(2). Each of P7–10 is proved in the normal way assuming only the truth of earlier propositions.

Proof of P1. Suppose the hypothesis is true. To prove P1(1) consider v such that $t + 1 < v + 1 \leqslant s$ and $a(v + 1) \supset e$. Then $\nu(\boldsymbol{c}, v) \leqslant \nu(\boldsymbol{c}, s) = n$. If $\nu(\boldsymbol{c}, v) < n$ let $\boldsymbol{q}$ be the least initial segment of $\boldsymbol{a}(\boldsymbol{c}, v)$ such that $O(\boldsymbol{q}) = \nu(\boldsymbol{c}, v)$. Then $\boldsymbol{q} \subsetneqq \boldsymbol{e} \subset \boldsymbol{a}(\boldsymbol{c}, s) \subset \boldsymbol{c}$. From the definition of ν, $\nu(\boldsymbol{c}, v) \notin D_0(\boldsymbol{a}(\boldsymbol{c}, v))$ whence $\nu(\boldsymbol{c}, v) \notin D_0(\boldsymbol{a}(\boldsymbol{c}, s))$ because $\boldsymbol{a}(\boldsymbol{c}, s) \subset \boldsymbol{a}(\boldsymbol{c}, v)$. Hence $\boldsymbol{q} \frown d_0 \not\subset \boldsymbol{e}$. Therefore $\boldsymbol{e} \not\subset \boldsymbol{a}(v + 1)$ because otherwise through C1 $\epsilon(\boldsymbol{c}, v)$ will ensure $\boldsymbol{a}(v + 1) \supset \boldsymbol{q} \frown d_0$ or $\boldsymbol{a}(v + 1) = \boldsymbol{q} \frown a_0$ if $\boldsymbol{a}(v + 1) \supset q$. Thus $\nu(\boldsymbol{c}, v) = n$ and $\boldsymbol{a}(\boldsymbol{c}, v) = \boldsymbol{a}(\boldsymbol{c}, s)$. Reasoning as above we see that $\boldsymbol{a}(v + 1) \supset \boldsymbol{e} \frown d_0$ or $\boldsymbol{a}(v + 1) = \boldsymbol{e} \frown a_0$. If $\boldsymbol{a}(v + 1) = \boldsymbol{e} \frown a_0$ then $\nu(\boldsymbol{c}, v + 1) = \nu(\boldsymbol{c}, v)$, whence stage $v + 1$ pertains to $\epsilon(\boldsymbol{c}', v)$ where $\boldsymbol{e} \subset \boldsymbol{c}'$,

$c' \neq c$, $\nu(c', v) = n$, and $\epsilon(c', v)$ is enumerated in A at stage $t' + 1 < v + 1$. Now $e \frown d_0 \not\subset a(c, v)$ since $n \notin D_0(a(c, v))$. Also $a(c, v) \frown d_0 \not\subset c$ by the definition of $a(c, v)$. Thus $e \frown d_0 \not\subset c$. Similarly $e \frown d_0 \not\subset c'$. By induction on v it is legitimate to use P1(1) to draw conclusions about stages $\leqslant v$. Thus $t < t' < v$ makes $a(t' + 1) = e \frown a_0$ or $a(t' + 1) \supset e \frown d_0$ since $a(t' + 1) \supset e$. Similarly $t' < t < v$ makes $a(t + 1) \supset e \frown d_0$ or $a(t + 1) = e \frown a_0$ since $a(t + 1) \supset e$. Thus $t < t'$ and $t' < t$ both lead to contradiction which completes the proof. P1(2) is also evident from the above proof.

Proof of P2(1). Let $\epsilon(b, s)$, $\epsilon(c, s) < \omega$, and either $c < b$ or $c \frown d_0 \subset b$ or $c \frown d_1 \subset b$ or $c \frown w \subset b$. Let $\epsilon(b, s)$ be assigned at stage $t + 1$. If $c < b$ or $c \frown d_0 \subset b$, $\epsilon(c, s)$ cannot be assigned at a stage $>t + 1$ because $c \frown a_1 < b$. However, if $c \subset b$ and $c \frown d_0 \not\subset b$, then $\epsilon(c, t) < \omega$, for otherwise $a(t + 1)$ could not be $b \frown a_1$. Thus in any case $\epsilon(c, t) < \omega$. Let c_0 be the greatest common initial segment of c and b and let c_0 have length l_0. Then by inspection of C1.4, C2.2, C4, C5.1 and C6 $\mu(c, t) < \inf I(l_0 + 1, t)$. Since $I(\mathrm{lh}(b), t) \subset I(l_0 + 1, t)$, it follows that

$$\epsilon(c, t) = \epsilon(c, t + 1) \leqslant \mu(c, t) < \epsilon(b, t + 1) = \epsilon(b, s)$$

and that the number enumerated in C at stage $t + 1$ is $>\mu(c, t)$. We can now observe that $\epsilon(c, t) = \epsilon(c, s)$ because if the value $\epsilon(c, t)$ is destroyed at a stage $\leqslant s$, $\epsilon(b, t + 1)$ will also be destroyed.

For proof by contradiction let u be the least number, $t \leqslant u < s$, such that $\mu(c, u + 1) \neq \mu(c, u)$. There are three possibilities for the case which produces this inequality at stage $u + 1$: C2.1, C5.1 or C6. If C5 or C6 occurs then $c \frown w \subset b$ otherwise stage $u + 1$ would have characteristic $<b$. From stage $t + 1$ we have $\mu(c, t) \geqslant \xi_l(\epsilon(c, t), t)$ where $l = \mathrm{O}(c)$. Since $\mu(c, u + 1) \neq \mu(c, u)$, $\xi_l(\epsilon(c, u), u) > \xi_l(\epsilon(c, t), t)$. Thus some number $\leqslant \xi_l(\epsilon(c, t), t)$ must be enumerated in C at a stage $>t + 1$ and $\leqslant u$. This is impossible because it would destroy $\epsilon(b, t + 1)$. We now turn to the other case in which C2.1 occurs at stage $u + 1$. Here there must exist m such that $\phi(c, m, u + 1) < \omega = \phi(c, m, u)$. Then $a(u + 1) = c_1 \frown a_0$ where c_1 is the greatest initial segment of c with $\mathrm{O}(c_1) = m$. By choice of t, $c_1 \subset b$, whence $\phi(c, m, t) < \omega$. For $\phi(c, m, t) = \omega$ implies $c_1 \frown d_0 \subset a(t+1) = b \frown a_1$ which implies $a(u + 1) < b$, contradiction. Let v be the least number $\geqslant t$ such that $\phi(c, m, v + 1) = \omega$, then $t < v < u$. Since $\epsilon(b, v+1) < \omega$, $\phi(c, m, v)$ is destroyed through C1 at stage $v + 1$. Let stage $v + 1$ pertain to $\epsilon(d, v)$. Then $c \subset a(c, v) \subset d$, $\nu(d, v) = m \in F(a(d, v))$, and either $c = a(d, v)$ or c is the greatest initial segment of $a(d, v)$ such that

$c^\frown (f, m) \subset a(c, v)$. If $c \neq d$ then $c^\frown (f, k) \subset d$ for some $k \leqslant m$. Therefore unless $c^\frown w \subset b$ the stage $z + 1$ at which $\epsilon(d, v)$ is enumerated in A has characteristic $d^\frown a_2 < b$, in which case $z < t$. If $c^\frown w \subset b$ and $z > t$, then by P6(3), $a(z + 1) \supset c^\frown w$, contradiction. Thus $z < t$ in every case. Now let c_2 be the least initial segment of c such that $O(c_2) = m$, then $c_2 \subset c_1 \subset b$. Hence $a(t + 1) \supset c_2$. By P1(1) applied to $\epsilon(d, v)$ we get $b^\frown a_1 = a(t + 1) \supset c_2{}^\frown d_0$ whence $b \supset c_2{}^\frown d_0$. Also $c_2 \subset c_1$, and $c_2{}^\frown d_0 \not\subset c_1$ since $m = \nu(c, v) \notin D_0(a(d, v))$. Therefore $a(u + 1) = c_1{}^\frown a_0 < c_2{}^\frown d_0$, whence $a(u + 1) < b$ contrary to the choice of t. Since u does not exist the proposition is proved.

Proof of P2(2). Let $c^\frown (f, j) \subset b$, $\epsilon(b, s)$ and $\epsilon(c, s)$ both be $< \omega$, $a(s + 1) \supset c$, and $a(s + 1) < c^\frown a_0$. We have to show that

$$(*) \qquad \epsilon(c, s) < \epsilon(b, s) \leqslant \mu(b, s) < \phi(c, j, s)\,.$$

Consider the greatest $t < s$ such that $\epsilon(b, t) = \omega$ then $\epsilon(b, t + 1)$ is defined through C3. Let c have length l then in stage $(t + 1, l)$ C6 occurs whence $I(l + 1, t)$ is a subinterval of $(\epsilon(c, t), \phi(c, j, t))$. Thus we get

$$\epsilon(c, t + 1) = \epsilon(c, t) < \epsilon(b, t + 1) = \mu(b, t + 1) < \phi(c, j, t + 1) = \omega\,.$$

Without loss of generality suppose $\phi(c, j, s) < \omega$. Consider the greatest $u < s$ such that $\phi(c, j, u) = \omega$. Then $u > t$, and $\phi(c, j, u + 1) < \omega$ through C2.1, whence

$$\epsilon(b, u + 1) \leqslant \mu(b, u + 1) \leqslant \phi(c, j, u + 1) = \phi(c, j, s)\,.$$

We have $\epsilon(c, s) = \epsilon(c, t)$ because if $\epsilon(c, t)$ is destroyed before stage $s + 1$ so will be $\epsilon(b, t + 1)$.

Let v be the least number if any such that $u < v < s$ and $\phi(c, j, s) \leqslant \mu(b, v + 1)$. To complete the proof of (*) it is sufficient to show that v does not exist. Now the inequality $\mu(b, v) < \mu(b, v + 1)$ must arise from C2.1, C5 or C6 at stage $v + 1$. Suppose C5 or C6 occurs then $\mu(b, v + 1) = \sup\{\mu(b, v), \xi_n(\epsilon(b, v), v)\}$. Since C4 does not occur at stage $(v + 1, \mathrm{lh}(b))$ we have

$$\xi_n(\epsilon(b, v), v) < \sup I(\mathrm{lh}(b), v) \leqslant \phi(c, j, v)$$

which contradicts the choice of v.

Thus $\mu(b, v) < \mu(b, v + 1)$ arises from C2.1, and $\phi(b, m, v + 1) < \phi(b, m, v) = \omega$ for some m. If $m >$ the least member of $F(b) - F(c)$ then

$\boldsymbol{a}(v+1) \supset \boldsymbol{c}^\frown (f, j)$ whence

$$\epsilon(\boldsymbol{b}, v+1) \leqslant \mu(\boldsymbol{b}, v+1) \leqslant \phi(\boldsymbol{c}, j, v)$$

which contradicts the choice of v. Thus $m \leqslant$ the least member of $F(\boldsymbol{b}) - F(\boldsymbol{c})$ and in particular $m \leqslant j$. Let $\boldsymbol{d}$ be the greatest initial segment of $\boldsymbol{c}$ with $\mathrm{O}(\boldsymbol{d}) = m$. Then stage $v+1$ has characteristic $\boldsymbol{d}^\frown a_0$. Now $\phi(\boldsymbol{b}, m, u) = \phi(\boldsymbol{b}, m, u+1) < \omega$ either because $m < j$ or because $m = j$ and $\boldsymbol{c} \subset \boldsymbol{b}$. Hence there exists a least z, $u < z < v$, such that $\phi(\boldsymbol{b}, m, z+1) = \omega$. By choice of u and t, $\boldsymbol{a}(z+1) \nless \boldsymbol{b}$ and no number $\leqslant \phi(\boldsymbol{b}, m, z) = \phi(\boldsymbol{b}, m, u)$ is enumerated in $\boldsymbol{C}$ at stage $z+1$. Thus $\phi(\boldsymbol{b}, m, z)$ is destroyed through C1 at stage $z+1$. Let stage $z+1$ pertain to $\epsilon(\boldsymbol{e}, z)$ where $\epsilon(\boldsymbol{e}, z)$ is enumerated in $\boldsymbol{A}$ at stage $x+1 < z+1$. Then $\boldsymbol{b} \subset \boldsymbol{e}$, $\nu(\boldsymbol{e}, z) = m$, and $\boldsymbol{a}(\boldsymbol{e}, z+1)$ is either $\boldsymbol{b}$ or ω. Note that $\phi(\boldsymbol{c}, j, x) < \omega$ from P4(1) and $\epsilon(\boldsymbol{e}, x) < \phi(\boldsymbol{c}, j, x)$ from P2.2 at stage $x+1$. Since $\phi(\boldsymbol{c}, j, x) \leqslant \phi(\boldsymbol{c}, j, u+1)$ at no stage $>u$ and $\leqslant s$ is a number $\leqslant \epsilon(\boldsymbol{e}, x)$ enumerated in $\boldsymbol{C}$. Let $z \leqslant y < s$, $\boldsymbol{a}(\boldsymbol{e}, y) \neq \omega$, and $\epsilon(\boldsymbol{e}, x)$ be destroyed at stage $y+1$. Then $z < y$ and $\boldsymbol{a}(y+1) < \boldsymbol{e}$. Further $\boldsymbol{a}(y+1) < \boldsymbol{g}$ where $\boldsymbol{g}$ is the least initial segment of $\boldsymbol{e}$ such that $\mathrm{O}(\boldsymbol{g}) = \nu(\boldsymbol{e}, y)$. Otherwise through C1 we have either $\boldsymbol{a}(y+1) > \boldsymbol{g}$, $\boldsymbol{a}(y+1) = \boldsymbol{g}^\frown a_0$, or $\boldsymbol{a}(y+1) \supset \boldsymbol{g}^\frown d_0$. Each of these possibilities is incompatible with $\boldsymbol{a}(y+1) < \boldsymbol{e}$. Since $\boldsymbol{a}(\boldsymbol{e}, y) \subset \boldsymbol{a}(\boldsymbol{e}, z+1) = \boldsymbol{b}$ we have $\boldsymbol{g} \subset \boldsymbol{b}$ whence $\epsilon(\boldsymbol{b}, y+1) = \omega$, contradiction. Thus if $z \leqslant y < s$, $\epsilon(\boldsymbol{e}, x) = \epsilon(\boldsymbol{e}, y)$, $\boldsymbol{a}(\boldsymbol{e}, y) \neq \omega$, and $\boldsymbol{a}(\boldsymbol{e}, y+1) = \omega$, then stage $y+1$ pertains to $\epsilon(\boldsymbol{e}, y)$.

If $j = m$ then j is the last member of $F(\boldsymbol{b}) - F(\boldsymbol{c})$ whence $\boldsymbol{b}^\frown (f, m)$ is illegitimate. In this case $\boldsymbol{a}(\boldsymbol{e}, z) = \boldsymbol{b}$, C1.3 occurs at stage $z+1$ and $\phi(\boldsymbol{c}, j, z)$ is enumerated in $\boldsymbol{C}$ at stage $z+1$. This contradicts the choice of u since $u < z < s$. Hence $m < j$.

The key to the remainder of the proof that v does not exist is the following claim.

There exists y such that $z \leqslant y < s$, stage $y+1$ pertains to $\epsilon(\boldsymbol{e}, z)$ and either $\boldsymbol{f} \subset \boldsymbol{c}$ and C1.2 *occurs at stage $y+1$ or $\boldsymbol{c} \subset \boldsymbol{f}$ and* C1.1 *or* C1.3 *occurs at stage $y+1$ where $\boldsymbol{f}$ is the greatest initial segment of $\boldsymbol{a}(\boldsymbol{e}, y)$ such that $\boldsymbol{f}^\frown (f, \nu(\boldsymbol{e}, y)) \subset \boldsymbol{a}(\boldsymbol{e}, y)$ if $\nu(\boldsymbol{e}, y) \in F(\boldsymbol{a}(\boldsymbol{e}, y))$ and $\boldsymbol{f} = \boldsymbol{a}(\boldsymbol{e}, y)$ otherwise.* For proof by contradiction suppose no such y exists. Let j_0 be the least number $\leqslant j$ such that for some $\boldsymbol{c}_0$ (to be chosen of greatest possible length once j_0 is found) $\boldsymbol{c} \subset \boldsymbol{c}_0{}^\frown (f, j_0) \subset \boldsymbol{b}$ and there exists no y_0 standing in the same relation to $\boldsymbol{c}_0$ as the proposed y to $\boldsymbol{c}$. It is clear that j_0 exists because $\boldsymbol{c}^\frown (f, j) \subset \boldsymbol{b}$ whence j has the properties required of j_0. Let z_0 be the least number $\geqslant z$ such that stage $z_0 + 1$ pertains to

$\epsilon(e, z)$ and $\nu(e, z_0 + 1) \geqslant j_0$, then $z_0 < s$. Otherwise by P1(2) either $a(s + 1) \not\supset c$ or $a(s + 1) \not< c \frown a_0$. From the choice of c_0 there is no y_0, $z \leqslant y_0 < s$, such that $f_0 \subset c_0$, stage $y_0 + 1$ pertains to $\epsilon(e, z)$ and C1.2 occurs, where f_0 is the greatest initial segment of $a(e, y_0)$ such that $f_0 \frown (f, \nu(e, y_0)) \subset a(e, y_0)$. Hence $c_0 \frown (f, j_0) \subset a(e, z_0)$, because $a(e, y_0 + 1) \neq a(e, y_0)$ only through C1.2 and then $a(e, y_0 + 1)$ is f_0. Let $\nu(e, z_0) = j_1$ then $j_1 < j_0$.

Let C1.1 occur at stage $z_0 + 1$ then z_0 serves for y_0; contradiction. Let C1.2 occur at stage $z_0 + 1$ then there exists a greatest c_1 such that $c_0 \subset c_1 \frown (f, j_1) \subset a(e, z_0)$. We claim that $a(e, z_0 + 1) = c_1$ and $\nu(e, z_0 + 1) = j_0$. If not, there exists j_2 and c_2 such that $c_0 \frown (f, j_0) \subset c_2 \frown (f, j_2) \subset c_1$. It can now be seen that j_2 has the defining property of j_0 with c_2 playing the role of c_0. But $j_2 < j_0$, contradicting the choice of j_0. Let C1.3 occur at stage $z_0 + 1$ and $c_1 = a(e, z_0)$. By the same argument as for C1.2 $a(e, z_0 + 1) = c_1$ and $\nu(e, z_0 + 1) = j_0$. Above we remarked that $z_0 < s$. By the same token there exists a least z_1, $z_0 < z_1 < s$, such that stage $z_1 + 1$ pertains to $\epsilon(e, z)$. Now c_0 is the greatest initial segment of c_1 such that $c_0 \frown (f, j_0) \subset c_1$. Otherwise there exist c_2 and j_2 such that $c_0 \frown (f, j_0) \subset c_2 \frown (f, j_2) \subset c_1$ and $j_2 < j_0$ which would contradict the choice of j_0 as above. Clearly $a(e, z_1) = c_1$ and $\nu(e, z_1) = j_0$. At stage $z_1 + 1$ the defining property of c_0 is contradicted. This completes the proof that y exists.

As above let f be the greatest initial segment of $a(e, y)$ such that $f \frown (f, \nu(e, y)) \subset a(e, y)$ if $\nu(e, y) \in F(a(e, y))$, and let f be $a(e, y)$ otherwise. Recall that u is the greatest number $<s$ such that $\phi(c, j, u) = \omega$ and that $u < z < s$. We now consider the various possibilities for f. If $f = c$ and C1.2 or C1.3 occurs at stage $y + 1$ then $\phi(c, j, y)$ is destroyed at stage $y + 1$ contrary to the definition of u. If $f = c$ and C1.1 occurs at stage $y + 1$ then y can be replaced by a lesser number such that C1.2 occurs. If f is a proper initial segment of c then by P5 some stage $>y$ and $\leqslant s$ has characteristic $<c$, contradiction. Here one should note that $f \not\supset c$, whence C1.2 occurs at stage $y + 1$, whence $a(e, y + 1) = f$ or $a(e, y + 1) = \omega$; thus the "d" of P5 for $\epsilon(e, z)$ is a proper initial segment of c.

The case which remains is that in which $f \supset c \frown (f, j)$ and either C1.1 or C1.3 occurs at stage $y + 1$. Let $\nu(e, y) = p$; then $\phi(f, p, y)$ is enumerated in C at stage $y + 1$ and $p \notin D_0(f)$. Recall that $\epsilon(e, z)$ is enumerated in A at stage $x + 1$, $\nu(e, z) = m < j$. Let d_0 be the greatest initial segment of c such that $O(d_0) = j$; then $a(u + 1) = d_0 \frown a_0$ by choice of u. By choice of z, $u < z$. Thus if $x < u$, then $\nu(e, u) \leqslant m$ and P1.2 applied to $\epsilon(e, x)$

and stage $u + 1$ yields: $a(u + 1) \not\supset d_0$. From this contradiction we infer that $u < x$. Applying P2(2) and P4(1) to stage $x + 1$, $\phi(f, p, x) < \phi(c, j, x)$. Also $\phi(c, j, x) = \phi(c, j, y)$ since both x and y are $>u$ and $\leqslant s$. Thus if $\phi(f, p, x) = \phi(f, p, y)$, $\phi(c, j, y)$ is destroyed at stage $y + 1$ contradicting the choice of u. Hence there exists a least z', $x < z' < y$, such that $\phi(f, p, z' + 1) = \omega$. Now $\phi(f, p, z')$ cannot be destroyed either by stage $z' + 1$ having characteristic $<f$ which would destroy $\epsilon(b, z')$ also, or by the enumeration in C of some number $\leqslant \phi(f, p, z')$ which would destroy $\phi(c, j, z')$, contradicting the choice of u. Thus C1.2 or C1.3 occurs at stage $z' + 1$ with respect to some $\epsilon(e', z')$ such that $f \subset e'$ and $\nu(e', z') = p$.

Let $\epsilon(e', z')$ be enumerated in A at stage $x' + 1 < z' + 1$. Suppose $x < x'$, then $x < x' < y$. Let f' be the least initial segment of e such that $O(f') = \nu(e, y)$ then $f' \subset f$ because $O(f) \geqslant \nu(e, y)$. Further, applying P1(1) to $\epsilon(e, x) = \epsilon(e, y)$ we have $a(x' + 1) \supset f' \,^\frown\, d_0$ since by choice of x' we have $a(x' + 1) = e' \,^\frown\, a_2 \supset f \supset f'$. But $f' \,^\frown\, d_0 \not\subset f$ since $f \subset a(e, y)$ and $\nu(e, y) \notin D_0(a(e, y))$, and $f \,^\frown\, d_0 \not\subset e'$ because $\phi(f, p, z')$ is destroyed through C1 at a stage pertaining to $\epsilon(e', z')$. This is a contradiction. Similarly $x' < x$ leads to a contradiction, because $x' < x < z'$, $e \supset f$, and $O(f) \geqslant \nu(e', z') = p$. If $x = x'$ then $e = e'$ and so $\nu(e, z') = \nu(e, y)$. But either C1.2 of C1.3 occurs at stage $z' + 1$ whence $\nu(e', z') < \nu(e', z' + 1)$. Since $z' < y$ we have a contradiction in this case also. This completes the proof that v does not exist and of the first part of P2(2).

Now suppose that $k < j$ and $k \notin D_0(c)$. As before let t be the greatest number $<s$ such that $\epsilon(b, t) = \omega$, and let $\mathrm{lh}(c) = l$. From stage $(t + 1, l)$ $I(l + 1, t)$ has least member $>\phi(c, k, t)$ since $\phi(c, k, t) < \phi(c, j, t)$ by P4(2). Thus we have from C3 at stage $t + 1$

$$\phi(c, k, t) = \phi(c, k, t + 1) < \epsilon(b, t + 1) = \epsilon(b, s)\,.$$

Let u be the greatest number, if any, such that $t < u < s$ and $\phi(c, k, u + 1) = \omega$. To complete the proof it is enough to show that u does not exist. If $a(u + 1) < c$ or some number $\leqslant \phi(c, k, u)$ is enumerated in C at stage $u + 1$ then $\epsilon(b, u + 1) = \omega$ which is a contradiction. Thus $\phi(c, k, u)$ is destroyed through C1 at stage $u + 1$. Let stage $u + 1$ pertain to $\epsilon(d, u)$ where $\epsilon(d, u) = \epsilon(d, v)$ was enumerated in A at stage $v + 1 < u + 1$. Then $\nu(d, u) = k$. Since there exists $q \notin D_0(c)$, $k < q \leqslant O(c)$, j for instance, we have $a(d, u + 1) = c$. There are two possibilities: $d = c$ or $d \supset c \,^\frown\, (f, m)$ for some $m \leqslant k$. In either of these cases $v \geqslant t$ yields $\epsilon(b, v + 1) = \omega$, a contradiction. If $v < t$, then $v < t < u$. Also $\nu(d, u) = k$. Now $a(t + 1) \supset b \supset g$ where g is the least initial segment of d such that $O(g) = k$. Hence by P1(1)

$a(t+1) \supset g \frown d_0$. But $g \subsetneq c$ since $k < j$, and $g \frown d_0 \not\subset c$ since $\nu(d, u) = k$. Thus $a(t+1) = b \frown a_1 \supset g \frown d_0$ is a contradiction. Hence u does not exist and $\phi(c, k, t) = \phi(c, k, s)$, which completes the proof of P2(2).

Proof of P3(1). By examining the cases in the construction which lead to another substage, namely C1.4, C2.2, C4, C5.2 and C6, we can easily see that $I(i, s)$ is the intersection of all the intervals of the following three kinds:

(i) $(\mu(c, s), \omega)$ where $\epsilon(c, s) < \omega$ and either $c < b$, or $c \frown d_0 < b$, or $c \frown d_1 \subset b$, or $c \frown w \subset b$,

(ii) $(\epsilon(c, s), \phi(c, j, s))$ where $\epsilon(c, s) < \omega$ and $c \frown (f, j) \subset b$,

(iii) $(\phi(c, k, s), \phi(c, j, s))$ where $\epsilon(c, s) < \omega$, $c \frown (f, j) \subset b$, $k < j$, and $\phi(c, k, s) < \omega$.

With the notation of the statement of the construction suppose $\epsilon(b, s) < \omega$. From P2(1) and (2) it is immediate that $\epsilon(b, s) \geqslant \inf I(i, s)$ and $\mu(b, s) < \sup I(i, s)$. This completes the proof of P3(1).

Proof of P3(2). Suppose stage $(s+1, i+1)$ exists, we must show that $I(i+1, s) \neq \emptyset$. We continue the line of argument begun in the proof of P3(1). There are several cases according as to which of C1.4, C2.2, C4, C5.2 and C6 occurs in stage $(s+1, i)$. Suppose C1.4 occurs then $I(i+1, s)$ is formed from $I(i, s)$ by subtracting each interval $[0, \mu(d, s)]$ such that $d \frown a_0 < b \frown d_0$ and $\epsilon(d, s) < \omega$. Consider particular such d. If $d < b$ we have already seen that $\mu(d, s) < \inf I(i, s)$. Otherwise $b \subset d$. For any c with $\epsilon(c, s) < \omega$ and $c \frown (f, j) \subset b$ we have $\phi(c, j, s) > \mu(d, s)$ by P2(2). Thus $\sup I(i, s) \in I(i+1, s)$ which completes this case. All the other cases except C6 may be treated similarly. Let C6 occur in stage $(s+1, i)$. As a preliminary for this case we show that if $c \supset b \frown w$, then $\epsilon(c, s) = \omega$. For proof by contradiction suppose $c \supset b \frown w$ and $\epsilon(c, s) < \omega$. Let $\epsilon(c, s)$ be assigned at stage $t+1$ then $a(t+1) \supset b \frown w$. Now $a(u) \not< b \frown w$ for all u in $t < u \leqslant s$, otherwise $\epsilon(c, t+1)$ would be destroyed. By P6(3) $a(s+1) \supset b \frown w$ contrary to the hypothesis that C6 occurs at stage $(s+1, i)$. It follows that $I(i+1, s)$ may be formed from $I(i, s)$ by subtracting $[0, \epsilon(b, s)]$, $(\phi(b, j, s), \omega)$, $[0, \phi(b, k, s)]$ for each $k < j$ such that $\phi(b, k, s) < \omega$, and each interval $[0, \mu(d, s)]$ such that $\epsilon(d, s) < \omega$, $d \supset b$, $d \not\supset b \frown w$, and $d < b \frown (f, j)$. (Recall that for $d < b$ we already have $\mu(d, s) < \inf I(i, s)$.) Consider such d then since d must be legitimate we have $d \supset b \frown (f, k)$ where $k \notin D_0(b) \cup F(b)$ and $k < j$. By P4.1 and

P4.2

$$\epsilon(\boldsymbol{b}, s) < \phi(\boldsymbol{b}, k, s) < \phi(\boldsymbol{b}, j, s) < \omega .$$

Also $\mu(\boldsymbol{d}, s) < \phi(\boldsymbol{b}, k, s)$ from P2(2). Now $\epsilon(\boldsymbol{b}, s) < \phi(\boldsymbol{b}, j, s) \leqslant \mu(\boldsymbol{b}, s)$ whence $\phi(\boldsymbol{b}, j, s) \in I(i, s)$ from P3(1). It follows that $\phi(\boldsymbol{b}, j, s) \in I(i+1, s)$ which completes the proof.

Proof of P4. Recall the hypothesis: $\mathrm{lh}(\boldsymbol{b}) = i$, $\mathrm{O}(\boldsymbol{b}) = n$, $\boldsymbol{b} \subset \boldsymbol{a}(s+1)$, $\boldsymbol{a}(s+1) < \boldsymbol{b} \frown a_0$, and $\epsilon(\boldsymbol{b}, s) < \omega$. For P4(1) note that C2.1 cannot occur unless $n \notin D_0(\boldsymbol{c})$. Thus $\phi(\boldsymbol{b}, j, s) < \omega$ certainly implies $j \notin D_0(\boldsymbol{b})$. For the rest of P4(1) suppose for proof by contradiction that $j \leqslant n$, $j \notin D_0(\boldsymbol{b})$, and $\phi(\boldsymbol{b}, j, s) = \omega$. Let $\boldsymbol{c}$ be the greatest initial segment of $\boldsymbol{b}$ such that $\mathrm{O}(\boldsymbol{c}) = j$. By C2 either $\boldsymbol{a}(s+1) \supset \boldsymbol{c} \frown d_0$ or $\boldsymbol{a}(s+1) = \boldsymbol{c} \frown a_0$. In the former case $j \in D_0(\boldsymbol{b})$ and in the latter $\boldsymbol{c} = \boldsymbol{b}$ and $\boldsymbol{a}(s+1) = \boldsymbol{b} \frown a_0$. In either event we have the desired contradiction. This completes P4(1).

For P4(2) let $j_1 < j_2 \leqslant n$ and neither j_1 nor j_2 be in $D_0(\boldsymbol{b})$. We must show that $\epsilon(\boldsymbol{b}, s) < \phi(\boldsymbol{b}, j_1, s) < \phi(\boldsymbol{b}, j_2, s)$ and that $\phi(\boldsymbol{b}, j_1, s)$ is assigned before $\phi(\boldsymbol{b}, j_2, s)$. For $i = 1, 2$, let $\boldsymbol{c}_i$ be the greatest initial segment of $\boldsymbol{b}$ such that $\mathrm{O}(\boldsymbol{b}) = j_i$. Then $\boldsymbol{c} \subsetneqq \boldsymbol{c}_1$ since $j_1 < j_2$. By C2.1 $\phi(\boldsymbol{b}, j_2, s)$ is assigned at a stage $t+1$ with characteristic $\boldsymbol{c}_2 \frown a_0$. If $\phi(\boldsymbol{b}, j_1, t) = \omega$, then by C2 stage $t+1$ would have characteristic either $\boldsymbol{c}_1 \frown d_0$ or $\boldsymbol{c}_1 \frown a_0$. Since $j_1 \notin D_0(\boldsymbol{b})$, $\phi(\boldsymbol{b}, j_1, t) < \omega$. Also by C2.1 at stage $t+1$, $\mu(\boldsymbol{b}, t) < \phi(\boldsymbol{b}, j_2, t+1)$ whence $\phi(\boldsymbol{b}, j_1, t) < \phi(\boldsymbol{b}, j_2, t+1)$. Choose the least v if any such that $t < v < s$ and $\phi(\boldsymbol{b}, j_1, v+1) = \omega$. If $\phi(\boldsymbol{b}, j_1, v)$ is destroyed other than C1, $\phi(\boldsymbol{b}, j_2, t+1)$ is also destroyed at stage $v+1$ which is impossible. Thus stage $v+1$ pertains to some $\epsilon(\boldsymbol{c}, v)$ such that $\nu(\boldsymbol{c}, v) = j_1$ and $\boldsymbol{a}(\boldsymbol{c}, v+1) = \boldsymbol{b}$. Note that $\boldsymbol{a}(\boldsymbol{c}, v+1) \neq \omega$ since $j_1 < j_2 \notin D_0(\boldsymbol{b})$ and $j_2 \leqslant \mathrm{O}(\boldsymbol{b})$. To complete the proof of the proposition it is clearly sufficient to show that v does not exist.

Suppose $\boldsymbol{a}(\boldsymbol{c}, u) \neq \omega$ for all u in $v < u \leqslant s$. Then $\boldsymbol{a}(\boldsymbol{c}, s) \subset \boldsymbol{a}(\boldsymbol{c}, v+1) = \boldsymbol{b}$ and $\nu(\boldsymbol{c}, s) \leqslant \mathrm{O}(\boldsymbol{a}(\boldsymbol{c}, s))$. Let $\boldsymbol{e}$ be the least initial segment of $\boldsymbol{a}(\boldsymbol{c}, s)$ with $\mathrm{O}(\boldsymbol{e}) = \nu(\boldsymbol{c}, s)$ then from P1(2) $\boldsymbol{a}(s+1) = \boldsymbol{e} \frown a_0$ or $\boldsymbol{a}(s+1) \supset \boldsymbol{e} \frown d_0$ since $\boldsymbol{a}(s+1) \supset \boldsymbol{b}$. This contradicts $\boldsymbol{a}(s+1) < \boldsymbol{b} \frown a_0$ since $\boldsymbol{e} \subset \boldsymbol{b}$ and $\boldsymbol{e} \frown d_0 \not\subset \boldsymbol{b}$. Thus there exists a least number u such that $v < u < s$ and $\boldsymbol{a}(\boldsymbol{c}, u+1) = \omega$. Let $\boldsymbol{a}(u+1) < \boldsymbol{c}$ then $\boldsymbol{a}(u+1) \supset \boldsymbol{b} \supset \boldsymbol{a}(\boldsymbol{c}, u)$. Otherwise $\phi(\boldsymbol{b}, j_2, u)$ is destroyed at stage $u+1$. Let $\boldsymbol{e}$ now be the least initial segment of $\boldsymbol{a}(\boldsymbol{c}, u)$ with $\mathrm{O}(\boldsymbol{e}) = \nu(\boldsymbol{c}, u)$. From P1(2) $\boldsymbol{a}(u+1) = \boldsymbol{e} \frown a_0$ or $\boldsymbol{a}(u+1) \supset \boldsymbol{e} \frown d_0$, which contradicts $\boldsymbol{a}(u+1) < \boldsymbol{c}$. Hence $\boldsymbol{a}(u+1) \nless \boldsymbol{c}$.

For proof by contradiction assume that some number $\leqslant \epsilon(\boldsymbol{c}, u)$ is enumerated in C at stage $u + 1$. Since $a(c, v + 1) = b$ and $\nu(c, v) = j_1$ notice that either $c = b$ or $c \supset b ^\frown (f, j)$ for some $j \leqslant j_1$. Let $\epsilon(c, v)$ be enumerated in A at stage $z + 1$. From P2(2) $\epsilon(c, z) < \phi(b, j, z)$ if $c \supset b ^\frown (f, j)$. From P4(1) and (2), $\phi(b, j, z) \leqslant \phi(b, j_2, z) < \omega$. Thus in any case, i.e. even if $c = b$, we have $\epsilon(c, z) < \phi(b, j_2, z) < \omega$. Also $\phi(b, j_2, z) \leqslant \phi(b, j_2, u)$ and so $\epsilon(c, u) < \phi(b, j_2, u)$. Hence $\phi(b, j_2, u)$ is destroyed at stage $u + 1$, contradiction. Thus no number $\leqslant \epsilon(c, u)$ is enumerated in C at stage $u + 1$, whence stage $u + 1$ pertains to $\epsilon(c, u)$.

Let d be defined as in P5. If d is a proper initial segment of b then by P5 some stage $> u$ and $\leqslant s + 1$ has characteristic $< b$ contradicting either the choice of t or $a(s + 1) \supset b$. Hence $d = b$, $a(c, u) = b$, and either C1.1 or C1.3 occurs at stage $u + 1$. If $\nu(c, u) \geqslant j_2$, then, since $\nu(c, v) = j_1$, for some x we have $v < x \leqslant u$, $\nu(c, x) = j_2$, and $\phi(b, j_2, x + 1) = \omega$, contradicting the choice of t. Therefore $\nu(c, u) = y < j_2$ and C1.1 occurs at stage $u + 1$. (If C1.3 occurred, $a(c, u + 1)$ would be $\neq \omega$ since $j_2 \notin D_0(b)$.) In this case $\phi(b, y, u)$ is enumerated in C at stage $u + 1$. From P4(1) and (2), $\phi(b, y, z) < \phi(b, j_2, z) < \omega$. Suppose $z < t$ then $z < t < u$. Let c_3 be the least initial segment of c such that $\mathrm{O}(c_3) = y = \nu(c, u)$. Then $c_3 {}^\frown d_0 \not\subset c_2$ since $y \notin D_0(b)$, and yet $c_3 \subset c_2$ because $y \leqslant j_2$. By choice of t, $a(t + 1) = c_2 {}^\frown a_0 \supset c_3$. But from P1(1) applied to $\epsilon(c, z)$, we now get $a(t + 1) \supset c_3 {}^\frown d_0$; contradiction. Thus $t < z$ whence $\phi(b, j_2, u) = \phi(b, j_2, z)$. Since $\phi(b, j_2, z)$ is not destroyed at stage $u + 1$, $\phi(b, y, z)$ is destroyed at some stage $w + 1 \leqslant u$. Now $\phi(b, y, z)$ must be destroyed through C1 at stage $w + 1$, otherwise $\phi(b, j_2, t + 1)$ would also be destroyed. Let stage $w + 1$ pertain to $\epsilon(e, w)$ where $\epsilon(e, w)$ was enumerated in A at stage $r + 1 < w + 1$. Suppose $r < z$ then $r < z < w$. Now $e \supset b$ and $\nu(e, w) = y$, whence $a(z + 1) \supset c_3 {}^\frown d_0$ by applying P1(1) to $\epsilon(e, r)$. This contradicts $a(z + 1) = c ^\frown a_2$ which follows from the choice of z. Similarly if $z < r$, then $z < r < u$. Applying P1(1) to $\epsilon(c, z)$ we get $a(r + 1) \supset c_3 {}^\frown d_0$, which contradicts $a(r + 1) = e ^\frown a_2$. Thus $z = r$ and $c = e$. But now $\nu(c, w + 1) > \nu(e, z) = y$ which contradicts $\nu(c, u) = y$. This contradiction completes the proof that v cannot exist and hence the proof of P4(2).

For P4(3) suppose $\phi(b, j, s) < \omega$ and that $\phi(b, j, s)$ is assigned at stage $v + 1$. We must show that no number $\leqslant \phi(b, j, s)$ is enumerated in $A \cup C$ at a stage $> v$ and $\leqslant s$. If a number $\leqslant \phi(b, j, s)$ is enumerated in C at a stage $> v$ and $\leqslant s$, then $\phi(b, j, v + 1)$ would be destroyed too early. For the rest, assume for proof by contradiction that $\epsilon(c, u)$ is enumerated in

$\boldsymbol{A}$ at stage $u+1$, $v<u<s$ and that $\epsilon(\boldsymbol{c}, u) < \phi(\boldsymbol{b}, j, u)$. Then $\boldsymbol{b} \not\subset \boldsymbol{c}$ and none of $\boldsymbol{b}^\frown d_0$, $\boldsymbol{b}^\frown d_1$ and $\boldsymbol{b}^\frown w$ are initial segments of $\boldsymbol{c}$ by P2(1) applied to stage $u+1$. Also $\boldsymbol{c}^\frown a_2 \not\subset \boldsymbol{b}$, otherwise $\phi(\boldsymbol{b}, j, u)$ would be destroyed at stage $u+1$. There are now three possibilities:

(i) $\boldsymbol{c}^\frown w \subset \boldsymbol{b}$,

(ii) $\boldsymbol{c} = \boldsymbol{b}$,

(iii) $\boldsymbol{b}^\frown (f, k) \subset \boldsymbol{c}$ for some k.

Let $\epsilon(\boldsymbol{b}, s)$ be assigned at stage $t+1 < v+1$ and $\mathrm{O}(\boldsymbol{c}) = l$.

Suppose (i) holds. Then $\boldsymbol{c}^\frown w \subset \boldsymbol{a}(t+1)$, $t \leqslant u$ and $\boldsymbol{a}(r) \not\subset \boldsymbol{c}$ for all r in $t < r \leqslant u+1$. By P6(3) $\boldsymbol{a}(u+1) \supset \boldsymbol{c}^\frown w$ since $\boldsymbol{a}(u+1) \supset \boldsymbol{c}$ and $\boldsymbol{a}(u+1) = \boldsymbol{c}^\frown a_2 < \boldsymbol{c}^\frown a_0$. This is a contradiction.

Suppose (ii) holds. Let w be the least number $\geqslant u$ such that $\boldsymbol{a}(\boldsymbol{c}, w+1) = \omega$. Applying P1(1) to $\epsilon(\boldsymbol{c}, u)$ we can see that w exists and $w \leqslant s$, because $\boldsymbol{a}(s+1) \supset \boldsymbol{b} = \boldsymbol{c}$ and $\boldsymbol{a}(s+1) < \boldsymbol{b}^\frown a_0$. Also by P1(2), $w \neq s$ whence $w < s$. Let $\boldsymbol{d}$ be defined from $\boldsymbol{c}$ and u, as $\boldsymbol{d}$ is defined from $\boldsymbol{c}$ and t in P5. Let x be the least number $\geqslant w$ such that $\boldsymbol{a}(x+1) \supset \boldsymbol{d}$ and $\boldsymbol{a}(x+1) < \boldsymbol{d}^\frown a_0$, then $x \leqslant s$ because either $\boldsymbol{d} = \boldsymbol{c}$ or $\boldsymbol{d}^\frown (f, m) \subset \boldsymbol{c}$ for some m. By P5, if $\boldsymbol{d} \neq \boldsymbol{c}$, some stage $> w$ and $\leqslant s+1$ has characteristic $< \boldsymbol{c} = \boldsymbol{b}$, contradiction. Hence $\boldsymbol{d} = \boldsymbol{c} = \boldsymbol{b}$. If $\nu(\boldsymbol{c}, w) \geqslant j$ then there exists y, $u < y \leqslant w$, such that $\nu(\boldsymbol{c}, y) = j$ and stage $y+1$ pertains to $\epsilon(\boldsymbol{c}, u)$. This is because $\boldsymbol{a}(\boldsymbol{c}, k) = \boldsymbol{c}$ for all k in $u+1 \leqslant k \leqslant w$ and $j \notin D_0(\boldsymbol{c}) = D_0(\boldsymbol{b})$. But from stage $y+1$, $\phi(\boldsymbol{c}, j, y+1) = \omega$. This contradicts $v \leqslant y < s$, whence $\nu(\boldsymbol{c}, w) = m$ for some $m < j$. Since $\boldsymbol{a}(\boldsymbol{c}, w+1) = \omega$ C1.1 occurs at stage $w+1$. (Note that either C1.1 or C1.3 occurs because $\boldsymbol{d} = \boldsymbol{a}(\boldsymbol{c}, w)$ and that if C1.3 occurs then j is eligible as a value of $\nu(\boldsymbol{c}, w+1)$.) Thus $\phi(\boldsymbol{c}, m, w)$ is enumerated in $\boldsymbol{C}$ at stage $w+1$. Applying P4(2) to stage $u+1$ we have $\phi(\boldsymbol{c}, m, u) < \phi(\boldsymbol{c}, j, u) = \phi(\boldsymbol{c}, j, s) < \omega$ whence $\phi(\boldsymbol{c}, m, y+1) = \omega$ for some y, $u < y < w$. Otherwise $\phi(\boldsymbol{b}, j, v+1)$ would be destroyed at stage $w+1$. Consider the least such y: then $\phi(\boldsymbol{c}, m, y)$ is destroyed through C1 at stage $y+1$. Let stage $y+1$ pertain to $\epsilon(\boldsymbol{e}, y)$ and $z+1 < y+1$ be the stage at which $\epsilon(\boldsymbol{e}, y)$ is enumerated in $\boldsymbol{A}$. Then $\boldsymbol{e} \supset \boldsymbol{c}$ and $\nu(\boldsymbol{e}, y) = m \neq \nu(\boldsymbol{e}, y+1)$. Since $y < w$ and $\nu(\boldsymbol{c}, w) = m$ we have $\boldsymbol{e} \neq \boldsymbol{c}$ and thus $u \neq z$. Suppose $u < z$; then $u < z < w$, $\nu(\boldsymbol{c}, w) = m < j$ and $\boldsymbol{a}(z+1) \supset \boldsymbol{c}$. Applying P1(1) to $\epsilon(\boldsymbol{c}, u)$ we get a contradiction. Similarly if $z < u$, then $z < u < y$, $\nu(\boldsymbol{e}, y) = m < j$ and $\boldsymbol{a}(u+1) \supset \boldsymbol{c}$. Applying P1(1) to $\epsilon(\boldsymbol{e}, z)$ we get a contradiction. This means that (ii) cannot hold.

Thus (iii) holds, i.e. $\boldsymbol{c} \supset \boldsymbol{b}^\frown (f, k)$ for some k. Since $\boldsymbol{a}(u+1) = \boldsymbol{c}^\frown a_2$ from P2(2) we have $\phi(\boldsymbol{b}, j, u) < \epsilon(\boldsymbol{c}, u)$ if $j < k$. Hence $j \geqslant k$. From P4(2) $\phi(\boldsymbol{b}, j, u) \geqslant \phi(\boldsymbol{b}, k, u)$. Let w be the least number if any such that $w > u$, stage $w + 1$ pertains to $\epsilon(\mathbf{c}, w)$ and either

$$\mathbf{d} \subsetneqq \mathbf{b} \text{ and C1.2 occurs at stage } w + 1$$

or

$$\mathbf{d} \supset \mathbf{b} \text{ and either C1.1 or C1.3 occurs },$$

where $\mathbf{d}$ is the greatest initial segment of $\mathbf{a}(\mathbf{c}, w)$ such that $\mathbf{d} \frown (f, \nu(\mathbf{c}, w)) \subset \mathbf{a}(\mathbf{c}, w)$ if $\nu(\mathbf{c}, w) \in F(\mathbf{a}(\mathbf{c}, w))$ and where $\mathbf{d}$ is $\mathbf{a}(\mathbf{c}, w)$ otherwise. Let $\nu(\mathbf{c}, w) = l$. Suppose that $w \geqslant s$ or does not exist and that $\mathbf{a}(\mathbf{c}, q) \neq \omega$ for all q, $u < q \leqslant s$. Then $\mathbf{b} \subset \mathbf{a}(\mathbf{c}, s)$, otherwise the least y such that $\mathbf{b} \not\subset \mathbf{a}(\mathbf{c}, y + 1)$ would be a candidate for w.

Let m be the least number if any such that $m < \nu(\mathbf{c}, s)$ and for some $\mathbf{e} \supset \mathbf{b}$, $\mathbf{e} \frown (f, m) \subset \mathbf{c}$ and $\mathbf{a}(\mathbf{c}, s) \not\subset \mathbf{e}$. Let z be the least number $\geqslant u$ such that $\nu(\mathbf{c}, z + 1) \geqslant m$. Then $z < s$ and $\mathbf{e} \frown (f, m) \subset \mathbf{a}(\mathbf{c}, z + 1)$ because $\mathbf{a}(\mathbf{c}, s) \subset \mathbf{a}(\mathbf{c}, z + 1) \subset \mathbf{c}$. Note that $\mathbf{e}$ is the greatest initial segment of $\mathbf{a}(\mathbf{c}, z + 1)$ such that $\mathbf{e} \frown (f, m) \subset \mathbf{a}(\mathbf{c}, z + 1)$. Otherwise there exist $\mathbf{e}'$ and $m' < m$ such that $\mathbf{e}' \frown (f, m') \subset \mathbf{a}(\mathbf{c}, z + 1)$ and $\mathbf{e}' \supset \mathbf{b}$, which contradicts the choice of m. Suppose $\nu(\mathbf{c}, z + 1) = m$, then because $m < \nu(\mathbf{c}, s)$ there exists a least r, $z < r < s$, such that stage $r + 1$ pertains to $\epsilon(\mathbf{c}, z)$. Either C1.1 or C1.3 occurs at stage $r + 1$, or $\mathbf{a}(\mathbf{c}, r + 1) = \mathbf{e}$. In the former case, since $\mathbf{a}(\mathbf{c}, r) \supset \mathbf{a}(\mathbf{c}, s) \supset \mathbf{b}$, r is eligible as w contradicting $w \geqslant s$; and in the latter the choice of $\mathbf{e}$ is contradicted since $\mathbf{a}(\mathbf{c}, s) \subset \mathbf{a}(\mathbf{c}, r + 1)$. Thus $\nu(\mathbf{c}, z + 1) > m$, whence $m \in D_0(\mathbf{a}(\mathbf{c}, z + 1))$, which means that for some $m' < m$ and $\mathbf{e}' \supset \mathbf{e}$ we have $\mathbf{a}(\mathbf{c}, z + 1) \supset \mathbf{e}' \frown (f, m')$. It is easy to see that $\mathbf{a}(\mathbf{c}, s) \not\subset \mathbf{e}'$, whence m' has the defining properties of m and the minimality of m is contradicted. We conclude that m does not exist whence $\mathbf{a}(\mathbf{c}, s) = \mathbf{b}$ or $\nu(\mathbf{c}, s) \leqslant k \leqslant \mathrm{O}(\mathbf{b})$. Applying P1(2) to $\epsilon(\mathbf{c}, u)$ we get a contradiction of the hypotheses that $\mathbf{a}(s + 1) \supset \mathbf{b}$ and $\mathbf{a}(s + 1) < \mathbf{b} \frown a_0$. It follows that either $w < s$ or $\mathbf{a}(\mathbf{c}, q) = \omega$ for some q in $u < q \leqslant s$.

Suppose w does not exist or $w \geqslant s$. Consider the least x such that $u < x < s$ and $\mathbf{a}(\mathbf{c}, x + 1) = \omega$. Note that $\mathbf{a}(\mathbf{c}, u + 1) \neq \omega$ because $\mathbf{b} \frown (f, k) \subset \mathbf{c}$. Stage $x + 1$ pertains to $\epsilon(\mathbf{c}, u)$ otherwise $\phi(\mathbf{b}, j, v + 1)$ would be destroyed at stage $x + 1$. We can now repeat the argument made above about m but reading x for s. We can conclude that either $\mathbf{a}(\mathbf{c}, x) = \mathbf{b}$ or $\mathbf{a}(\mathbf{c}, x) \supset \mathbf{b} \frown (f, k)$ and $F(\mathbf{a}(\mathbf{c}, x)) - F(\mathbf{b})$ has no member $< \nu(\mathbf{c}, x)$. We now argue by cases to show that in fact x is a candidate for w. Let $\nu(\mathbf{c}, x) = l' \in F(\mathbf{a}(\mathbf{c}, x))$ and let $\mathbf{d}'$ be the greatest initial segment of $\mathbf{a}(\mathbf{c}, x)$ such that $\mathbf{d}' \frown (f, l') \subset \mathbf{a}(\mathbf{c}, x)$. If $l' < k$ then $\mathbf{d}' \supset \mathbf{b}$ implies that either C1.3 occurs at stage $x + 1$ or C1.2 occurs and $\mathbf{a}(\mathbf{c}, x + 1) = \mathbf{d}'$ because some number $\leqslant k$ and $> l'$ is in $F(\mathbf{d}')$ and hence

$\notin D_0(\boldsymbol{d}')$. If $l' < k$ and $\boldsymbol{d}' \supsetneq \boldsymbol{b}$, then either C1.2 occurs at stage $x + 1$ or C1.3 occurs and $\boldsymbol{a}(\boldsymbol{c}, x + 1) = \boldsymbol{a}(\boldsymbol{c}, x)$ because some number $>l'$ and $\leqslant k$ is in $F(\boldsymbol{a}(\boldsymbol{c}, x))$. If $l' = k$ then $\boldsymbol{d}' = \boldsymbol{b}$ and either C1.3 occurs or C1.2 occurs at stage $x + 1$ and $\boldsymbol{a}(\boldsymbol{c}, x + 1) = \boldsymbol{b}$ because some number $>k$ and $\leqslant O(\boldsymbol{b})$, j for instance, is not in $D_0(\boldsymbol{b})$. If $l' > k$ then $\boldsymbol{a}(\boldsymbol{c}, x) = \boldsymbol{b}$, otherwise $F(\boldsymbol{a}(\boldsymbol{c}, x)) - F(\boldsymbol{b})$ contains a number $< l' = \nu(\boldsymbol{c}, x)$. Either C1.2 occurs at stage $x + 1$ or C1.3 occurs and $\nu(\boldsymbol{c}, x) \geqslant j$ since $\boldsymbol{a}(\boldsymbol{c}, x + 1) = \omega$. The latter is impossible because it implies that for some y, $u < y \leqslant x$, $\boldsymbol{a}(\boldsymbol{c}, y) = \boldsymbol{b}$, $\nu(\boldsymbol{c}, y) = j$ and $\phi(\boldsymbol{b}, j, y + 1) = \omega$. This completes every case except that in which $\nu(\boldsymbol{c}, x) \notin F(\boldsymbol{a}(\boldsymbol{c}, x))$, but then C1.1 occurs at stage $x + 1$ and $\boldsymbol{d} = \boldsymbol{a}(\boldsymbol{c}, x) \supset \boldsymbol{b}$. We have finally shown that w exists and $w < s$.

We complete the proof by obtaining a contradiction. If $\boldsymbol{d} \supsetneq \boldsymbol{b}$ then C1.2 occurs at stage $w + 1$ and by P5 some stage $>w$ and $\leqslant s + 1$ has characteristic $< \boldsymbol{b}$, contradiction. Suppose $\boldsymbol{b} = \boldsymbol{d}$ and C1.1 occurs at stage $w + 1$ and $l = \nu(\boldsymbol{c}, w) \geqslant j$. Then $\boldsymbol{a}(\boldsymbol{c}, w) = \boldsymbol{b}$ and $\phi(\boldsymbol{b}, j, y + 1) = \omega$ for some y in $u < y \leqslant w$, contradicting the choice of v. Notice that if $\boldsymbol{d} = \boldsymbol{b}$ and C1.3 occurs at stage $w + 1$ then $l = k$. Thus $\boldsymbol{b} \subset \boldsymbol{d}$, $\phi(\boldsymbol{d}, l, w)$ is enumerated in $\boldsymbol{C}$ at stage $w + 1$ through C1.1 or C1.3 and if $\boldsymbol{d} = \boldsymbol{b}$ then $l \leqslant j$. If $\boldsymbol{d} \supset \boldsymbol{b}^\frown (f, k)$ we get $\phi(\boldsymbol{d}, l, u) \leqslant \mu(\boldsymbol{d}, u) < \phi(\boldsymbol{b}, k, u)$ from P2(2) and $\phi(\boldsymbol{b}, k, u) \leqslant \phi(\boldsymbol{b}, j, u)$ from P4(2). If $\boldsymbol{d} = \boldsymbol{b}$, then $\phi(\boldsymbol{d}, l, u) \leqslant \phi(\boldsymbol{b}, j, u)$ from P4(2). Thus in any case $\phi(\boldsymbol{d}, l, u) \leqslant \phi(\boldsymbol{b}, j, u) = \phi(\boldsymbol{b}, j, w)$. Hence there is a least y such that $u < y < w$ and $\phi(\boldsymbol{d}, l, y + 1) = \omega$; otherwise $\phi(\boldsymbol{b}, j, u)$ would be destroyed at stage $w + 1$. Further, $\phi(\boldsymbol{d}, l, y)$ must be destroyed through C1; otherwise either $\epsilon(\boldsymbol{c}, y + 1) = \omega$ or $\phi(\boldsymbol{b}, j, t + 1) = \omega$ both of which are impossible. Let stage $y + 1$ pertain to $\epsilon(\boldsymbol{e}, y)$ where $\epsilon(\boldsymbol{e}, y)$ was enumerated in $\boldsymbol{A}$ at stage $z + 1 < y + 1$, then $\boldsymbol{b} \subset \boldsymbol{d} \subset \boldsymbol{e}$, $\boldsymbol{d}^\frown d_0 \not\subset \boldsymbol{e}$ and $\nu(\boldsymbol{e}, y) = l$. If $z < u$, then $z < u < y$. In this case applying P1.1 to $\epsilon(\boldsymbol{e}, z)$ we can refute $\boldsymbol{a}(u + 1) = \boldsymbol{c}^\frown a_2$. By repeating the argument that m cannot exist, with w playing the role of s, we see that either $\boldsymbol{a}(\boldsymbol{c}, w) = \boldsymbol{b}$ or $\nu(\boldsymbol{c}, w) \leqslant k$. If $u < z$ then $u < z < w$. In this case applying P1(1) to $\epsilon(\boldsymbol{c}, u)$ we can refute $\boldsymbol{a}(z + 1) = \boldsymbol{e}^\frown a_2$. Finally, if $u = z$ then $\boldsymbol{e} = \boldsymbol{c}$ and $\nu(\boldsymbol{c}, y) = l = \nu(\boldsymbol{c}, w)$. This is impossible because C1 pertains to $\epsilon(\boldsymbol{c}, y) = \epsilon(\boldsymbol{c}, w)$ at stage $y + 1$, whence $\nu(\boldsymbol{c}, y) < \nu(\boldsymbol{c}, y + 1) \leqslant \nu(\boldsymbol{c}, w)$. This last contradiction shows that (iii) cannot hold and completes the proof of P4(3).

Proof of P5. We recall the hypotheses: $\epsilon(\boldsymbol{c}, t)$ is enumerated in $\boldsymbol{A}$ at stage $t + 1$, u is the least number $\geqslant t$ such that $\boldsymbol{a}(\boldsymbol{c}, u + 1) = \omega$, s is the least number $>u$ such that $\boldsymbol{a}(s + 1) \supset \boldsymbol{d}$ and $\boldsymbol{a}(s + 1) < \boldsymbol{d}^\frown a_0$ and no stage

$>u$ and $\leqslant s$ has characteristic $< \boldsymbol{d}$. If stage $u + 1$ pertains to $\epsilon(\boldsymbol{c}, t)$ and C1.2 occurs, $\boldsymbol{d}$ is the greatest initial segment of $\boldsymbol{a}(\boldsymbol{c}, u)$ such that $\boldsymbol{d} ^\frown (f, \nu(\boldsymbol{c}, u)) \subset \boldsymbol{a}(\boldsymbol{c}, u)$. If $u = t$, $\boldsymbol{d} = \boldsymbol{c}$. Otherwise $\boldsymbol{d} = \boldsymbol{a}(\boldsymbol{c}, u)$. We have to show that if $\boldsymbol{d} = \boldsymbol{c}$ then $\boldsymbol{c} ^\frown w \subset \boldsymbol{a}(s + 1)$ and that if $\boldsymbol{d} \neq \boldsymbol{c}$ then $\boldsymbol{d} ^\frown (f, m) \subset \boldsymbol{c}$ for some m and $\boldsymbol{a}(s + 1) < \boldsymbol{d} ^\frown (f, m)$.

Let $\mathrm{O}(\boldsymbol{d}) = l$. From the definition of $\boldsymbol{a}(\boldsymbol{c}, s + 1)$ in C1 and C5.2 we see that either $\boldsymbol{d} = \boldsymbol{c}$ or $\boldsymbol{d} ^\frown (f, m) \subset \boldsymbol{c}$ for some m. It follows that $\epsilon(\boldsymbol{d}, t) < \omega$ and that $\xi_l(\epsilon(\boldsymbol{d}, t), t) \leqslant \mu(\boldsymbol{d}, t + 1) < \omega$. We claim that no number $\leqslant \xi_l(\epsilon(\boldsymbol{d}, t), t)$ is enumerated in $\boldsymbol{C}$ at a stage $>t$ and $\leqslant s$. For proof by contradiction let v be the least number, $t < v < s$, such that some number $\leqslant \xi_l(\epsilon(\boldsymbol{d}, t), t)$ is enumerated in $\boldsymbol{C}$ at stage $v + 1$. Then $\boldsymbol{a}(v + 1) \nless \boldsymbol{d}$ because either $v < u$ and $\boldsymbol{a}(\boldsymbol{c}, v + 1)$ is defined, or $u < v + 1 \leqslant s$.

Suppose that C3 occurs at stage $v + 1$ and that $\boldsymbol{a}(v + 1) = \boldsymbol{e} ^\frown a_1$. Then $\boldsymbol{e} ^\frown a_1 \nless \boldsymbol{d}$. Let $\boldsymbol{e} \subsetneq \boldsymbol{d}$. Since $\boldsymbol{e} ^\frown d_0 \subset \boldsymbol{d}$ implies $\boldsymbol{a}(v + 1) < \boldsymbol{d}$, we have $\boldsymbol{e} ^\frown d_0 \not\subset \boldsymbol{d}$ and hence $\epsilon(\boldsymbol{e}, t) < \omega$. By P2(1) or P2(2) as appropriate $\epsilon(\boldsymbol{e}, t) < \epsilon(\boldsymbol{d}, t)$. If $\epsilon(\boldsymbol{e}, t)$ is destroyed at stage $w + 1$, $t \leqslant w < v$, then either $\boldsymbol{a}(w + 1) < \boldsymbol{e}$ or some number $\leqslant \epsilon(\boldsymbol{e}, t)$ is enumerated in $\boldsymbol{C}$ at stage $w + 1$. Now $\boldsymbol{a}(w + 1) \nless \boldsymbol{e}$ since by hypothesis $\boldsymbol{a}(w + 1) \nless \boldsymbol{d}$ and if a number $\leqslant \epsilon(\boldsymbol{e}, t)$ is enumerated in $\boldsymbol{C}$ at stage $w + 1$ the choice of v is refuted. Remember that $\epsilon(\boldsymbol{d}, t) \leqslant \xi_l(\epsilon(\boldsymbol{d}, t), t)$ from §1. Hence $\boldsymbol{e}$ is not a proper initial segment of $\boldsymbol{d}$.

If $\boldsymbol{d} \subset \boldsymbol{e}$ then $\boldsymbol{d} ^\frown d_0 \subset \boldsymbol{e}$. Otherwise we should have either a contradiction by P1(1) if $v < u$ or a contradiction of $\boldsymbol{a}(v + 1) \nless \boldsymbol{d} ^\frown a_0$ if $u \leqslant v < s$. Thus either $\boldsymbol{d} < \boldsymbol{e}$ or $\boldsymbol{d} ^\frown d_0 \subset \boldsymbol{e}$. Let $\boldsymbol{d}_0$ be the greatest common initial segment of $\boldsymbol{d}$ and $\boldsymbol{e}$ and let $\boldsymbol{d}_0$ have length l_0. From stage $(v + 1, l_0)$ the least member of $I(l_0 + 1, v)$ is $>\mu(\boldsymbol{d}, v)$. But $\mu(\boldsymbol{d}, v) \geqslant \mu(\boldsymbol{d}, t + 1) \geqslant \xi_l(\epsilon(\boldsymbol{d}, t), t)$ and so the choice of v is refuted, because in stage $v + 1$ the number enumerated in $\boldsymbol{C}$ is $\sup I(\mathrm{lh}(\boldsymbol{e}), v) \geqslant \inf I(l_0 + 1, v)$.

It follows that one of C1.1 and C1.3 occurs at stage $v + 1$. Let stage $v + 1$ pertain to $\epsilon(\boldsymbol{e}, v)$ where $\epsilon(\boldsymbol{e}, v) = \epsilon(\boldsymbol{e}, w)$ is enumerated in $\boldsymbol{A}$ at stage $w + 1 < v + 1$ and where $\nu(\boldsymbol{e}, v) = k$. Let $\boldsymbol{f} = \boldsymbol{a}(\boldsymbol{e}, v)$ if C1.1 occurs at stage $v + 1$ and otherwise let $\boldsymbol{f}$ be the greatest initial segment of $\boldsymbol{a}(\boldsymbol{e}, v)$ such that $\boldsymbol{f} ^\frown (f, k) \subset \boldsymbol{a}(\boldsymbol{e}, v)$. Then $\phi(\boldsymbol{f}, k, v)$ is enumerated in $\boldsymbol{C}$ at stage $v + 1$.

Suppose $\boldsymbol{e} < \boldsymbol{d}$ then $w < t$ because no stage $>t$ and $\leqslant s$ has characteristic $>\boldsymbol{d}$. Let $\boldsymbol{e}_0$ be the least initial segment of $\boldsymbol{e}$ such that $\mathrm{O}(\boldsymbol{e}_0) = \nu(\boldsymbol{e}, t)$. Let t_0 be the least stage $>t$ pertaining to $\epsilon(\boldsymbol{e}, w)$ then $t_0 \leqslant v + 1$ and $\boldsymbol{e}_0 ^\frown a_0 = \boldsymbol{a}(t_0)$. From stage $t + 1$ either $\boldsymbol{e}_0 < \boldsymbol{d}$ or $\boldsymbol{e}_0 ^\frown d_0 \subset \boldsymbol{c}$ which makes $\boldsymbol{a}(t_0) < \boldsymbol{d}$, contradiction. The same argument shows that if $w < t$, then $\boldsymbol{e} \not\subset \boldsymbol{d}$.

Suppose $\boldsymbol{e} \subset \boldsymbol{d}$ and $t < w$; then $\boldsymbol{a}(w+1) = \boldsymbol{e} ^\frown a_2 \nless \boldsymbol{d}$. In this case $\boldsymbol{e} ^\frown w \subset \boldsymbol{d}$, because $\boldsymbol{e} = \boldsymbol{d}$ would contradict the choice of s. From P6(3), since $\boldsymbol{a}(t+1) \supset \boldsymbol{e} ^\frown w$ and $\boldsymbol{a}(z) \nless \boldsymbol{e}$ for all z in $t < z \leqslant w+1$, we have $\boldsymbol{a}(w+1) \supset \boldsymbol{e} ^\frown w$; contradiction. Suppose $\boldsymbol{d} \subset \boldsymbol{e}$ and $t < w$; then $\boldsymbol{d} ^\frown d_0 \subset \boldsymbol{e}$. Otherwise $\boldsymbol{a}(w+1) \supset \boldsymbol{d}$ and $\boldsymbol{a}(w+1) < \boldsymbol{d} ^\frown a_0$ contradicting the choice of s. If $\boldsymbol{d} \subset \boldsymbol{e}$ and $w < t$, let $\boldsymbol{d}_1$ be the least initial segment of $\boldsymbol{e}$ such that $\mathrm{O}(\boldsymbol{d}_1) = \nu(\boldsymbol{e}, t)$; then $\boldsymbol{d}_1 \not\subset \boldsymbol{d}$. Otherwise by P1(2) $\boldsymbol{a}(t+1) \supset \boldsymbol{d}_1 ^\frown d_0$ or $\boldsymbol{a}(t+1) = \boldsymbol{d}_1 ^\frown a_0$, contradiction. Further $\boldsymbol{d} ^\frown d_0 \not\subset \boldsymbol{d}_1$, because otherwise $\boldsymbol{a}(t+1) < \boldsymbol{e}$ and $\epsilon(\boldsymbol{e}, w)$ would be destroyed at stage $t+1$. Hence $\boldsymbol{d} \subset \boldsymbol{e}$ and $w < t$ imply $\boldsymbol{a}(v+1) \supset \boldsymbol{d}_1 \supsetneq \boldsymbol{d}$ and $\boldsymbol{a}(v+1) < \boldsymbol{d} ^\frown a_0$, which contradicts the choice of s. Notice also that if $\boldsymbol{d} < \boldsymbol{e}$, then certainly $t < w$, *otherwise* $\epsilon(\boldsymbol{e}, w)$ *is destroyed at stage* $t+1$ *making it impossible for stage* $v+1$ to pertain to $\epsilon(\boldsymbol{e}, w)$.

There now remain only two possibilities regarding $\boldsymbol{d}$, t, $\boldsymbol{e}$ and w:

(i) $t < w$ and $\boldsymbol{d} < \boldsymbol{e}$ or $\boldsymbol{d} ^\frown d_0 \subset \boldsymbol{e}$;

(ii) $t = w$ and $\boldsymbol{c} = \boldsymbol{e}$.

From the definition of $\boldsymbol{f}$ we know that either $\boldsymbol{f} = \boldsymbol{e}$ or $\boldsymbol{f} ^\frown (f, k_0) \subset \boldsymbol{e}$ for some $k_0 \leqslant k$. Let (i) hold; then either $\boldsymbol{d} ^\frown d_0 \subset \boldsymbol{f}$ or $\boldsymbol{d} < \boldsymbol{f}$ or $\boldsymbol{f} ^\frown (f, k_0) \subset \boldsymbol{e}$ and $\boldsymbol{f} ^\frown (f, k_1) \subset \boldsymbol{d}$ for some $k_1 \leqslant k_0$ or $\boldsymbol{f} ^\frown w \subset \boldsymbol{d}$. Suppose $\boldsymbol{f} ^\frown w \subset \boldsymbol{d}$; then $\boldsymbol{a}(t+1) \supset \boldsymbol{f} ^\frown w$. Applying P6(3) we have $\boldsymbol{a}(w+1) \supset \boldsymbol{f} ^\frown w$, because $w > t$, $\boldsymbol{a}(u) \nless \boldsymbol{f}$ for all u in $t < u \leqslant w$, $\boldsymbol{a}(w+1) \supset \boldsymbol{f}$ and $\boldsymbol{a}(w+1) < \boldsymbol{f} ^\frown a_0$. This contradicts $\boldsymbol{a}(w+1) = \boldsymbol{e} ^\frown a_2$. Thus either $\boldsymbol{d} ^\frown d_0 \subset \boldsymbol{f}$ or $\boldsymbol{d} < \boldsymbol{f}$ or $\boldsymbol{f} ^\frown (f, k_1) \subset \boldsymbol{d}$ where $k_1 \leqslant k$. In each of these cases it is easy to see using P2 and P4 that $\mu(\boldsymbol{d}, w) < \phi(\boldsymbol{f}, k, w) < \omega$, because $\boldsymbol{a}(w+1) \supset \boldsymbol{f}$ and $\boldsymbol{a}(w+1) < \boldsymbol{f} ^\frown a_0$. Note that $\epsilon(\boldsymbol{d}, w) \neq \omega$ for the following reasons: $\epsilon(\boldsymbol{d}, t) \leqslant \xi_l(\epsilon(\boldsymbol{d}, t), t) < \omega$ and if a number $\leqslant \epsilon(\boldsymbol{d}, t)$ is enumerated in $\boldsymbol{C}$ at a stage $> t$ and $\leqslant w$ then the choice of v is contradicted and no stage $> t$ and $\leqslant w$ has characteristic $< \boldsymbol{d}$ by hypothesis. Since $w < v$, $\phi(\boldsymbol{f}, k, w) \leqslant \phi(\boldsymbol{f}, k, v)$ and so the number $\phi(\boldsymbol{f}, k, v)$ enumerated in $\boldsymbol{C}$ at stage $v+1$ is $> \mu(\boldsymbol{d}, t+1)$ and hence $\nleqslant \xi_l(\epsilon(\boldsymbol{d}, t), t)$. This contradicts the choice of v.

Now let (ii) hold and suppose firstly that C1.1 occurs at stage $v+1$, then $\boldsymbol{f} = \boldsymbol{d}$ because $\boldsymbol{a}(\boldsymbol{e}, v+1) = \boldsymbol{a}(\boldsymbol{c}, v+1) = \omega$ in this case. If $\boldsymbol{f} = \boldsymbol{d} = \boldsymbol{c}$ then from the occurrence of C5 at stage $t+1$, $\xi_l(\epsilon(\boldsymbol{d}, t), t) < \phi(\boldsymbol{f}, k, t)$ because $k \notin F(\boldsymbol{f})$. If $\boldsymbol{f} = \boldsymbol{d} \neq \boldsymbol{c}$, then $\boldsymbol{c} \supset \boldsymbol{f} ^\frown (f, k_0)$ for some $k_0 < k$ and $\xi_l(\epsilon(\boldsymbol{d}, t), t) < \phi(\boldsymbol{f}, k, t)$ since $\boldsymbol{a}(t+1) \supset \boldsymbol{f} ^\frown (f, k_0)$ and $k \notin F(\boldsymbol{f})$. Secondly suppose C1.3 occurs at stage $v+1$; then $\boldsymbol{f} ^\frown (f, k) \subset \boldsymbol{a}(\boldsymbol{c}, v) \subset \boldsymbol{c}$. By induction on z, $\boldsymbol{a}(\boldsymbol{c}, z) \supset \boldsymbol{f} ^\frown (f, k)$ for all z in $v \leqslant z \leqslant u$, whence $\boldsymbol{f} ^\frown (f, k) \subset \boldsymbol{d}$. Now since C5 or C6 occurs at stage $(t+1, \mathrm{lh}(\boldsymbol{d}))$ rather than C4 we have $\xi_l(\epsilon(\boldsymbol{d}, t), t) \leqslant \sup I(\mathrm{lh}(\boldsymbol{d}), t)$. But $\sup I(\mathrm{lh}(\boldsymbol{d}), t) \leqslant \phi(\boldsymbol{f}, k, t)$

because $f^\frown (f, k) \subset d$. Thus again $\xi_l(\epsilon(d, t), t) < \phi(f, k, t)$. Now in every case $\phi(f, k, t) < \omega$ from P4(1), whence $\xi_l(\epsilon(d, t), t) < \phi(f, k, v)$ because $\phi(f, k, t) \leqslant \phi(f, k, v)$. This contradicts the choice of v and completes the proof of the claim that no number $\leqslant \xi_l(\epsilon(d, t), t)$ is enumerated in C at a stage $>t$ and $\leqslant s$.

The remainder of the proof of P5 is straightforward. Suppose $d \neq c$ then $d^\frown (f, m) \subset c$ for some m. Let u_0 be the least number $>t$ such that either $a(c, u_0 + 1) = d$ or $u_0 = u$. Stage $u_0 + 1$ pertains to $\epsilon(c, t)$ and C1.2 occurs, whence $\phi(d, m, u_0 + 1) = \omega$. Hence $\phi(d, m, s)$ is assigned at some stage $u_1 + 1 \leqslant s$ where $u_1 > u_0$. Now $\epsilon(d, t) = \epsilon(d, s)$ whence

$$\xi_l(\epsilon(d, t), t) \leqslant \phi(d, t + 1) \leqslant \mu(d, u_1) < \phi(d, m, s) .$$

For any m_0 such that $m < m_0 \leqslant l$ and $m_0 \notin D_0(d) \cup F(d)$ we have $\xi_l(\epsilon(d, t), t) < \phi(d, m_0, t) < \omega$ since $a(t + 1) \supset d^\frown (f, m)$. (Of course, for $m_0 > l$ or $m_0 \in D_0(d)$, $\phi(d, m_0, s) = \omega$.) Let p be the length of d. We have shown above that $\xi_l(\epsilon(d, s), s) = \xi_l(\epsilon(d, t), t)$, whence C4 does not occur in stage $(s + 1, p)$ because $\sup I(p, t) \leqslant \sup I(p, s)$. This last inequality is east to verify when one notices from the description of stage $(s + 1, i)$ that if C6 occurs in stage $(s + 1, i)$ then $\sup I(i + 1, s) = \inf\{\sup I(i, s), \phi(b, j, s)\}$, and that otherwise $\sup I(i + 1, s) = \sup I(i, s)$. From the hypotheses of P5 neither C1 nor C2 occurs at stage $(s + 1, p)$. Further since $\epsilon(d, s) < \omega$, nor does C3 occur. Thus either $a(s + 1) \supset d^\frown w$ or $a(s + 1) = d^\frown a_2$ or $a(s + 1) \supset d^\frown (f, m')$ where m' is the greatest $j \leqslant l$ such that $j \notin F(d)$ and $\phi(d, j, s) \leqslant \xi_l(\epsilon(d, s), s)$. But above we have shown that $\xi_l(\epsilon(d, s), s) = \xi_l(\epsilon(d, t), t) < \phi(d, m_0, t) \leqslant \phi(d, m_0, s)$ for all m_0, $m \leqslant m_0 \leqslant l$. Hence $m' < m$. This completes the proof of P5 when $d \neq c$. If $d = c$ then from the occurrence of C5.3 at stage $(t + 1, p)$ we have

$$\Xi_l(\epsilon(c, t), t + 1) = 1 - A(\epsilon(c, t), t + 1) .$$

It follows easily that $a(s + 1) \supset d^\frown w$ using the same kind of argument as for the case in which $d \neq c$.

Proof of P6(1). Let $b^\frown d_0 \subset a(t + 1)$, $\epsilon(b, t) < \omega$, $t \leqslant s$ and $a(u) \not\subset b^\frown d_0$ for all u in $t < u \leqslant s + 1$. For proof by contradiction fix t and choose the least $s \geqslant t$ such that either $\epsilon(b, s + 1) \neq \epsilon(b, t)$ or $\mu(b, s + 1) \neq \mu(b, t)$ or some number $\leqslant \mu(b, t)$ is enumerated in $A \cup C$ at a stage $>t$ and $\leqslant s + 1$. If $\epsilon(b, s + 1) \neq \epsilon(b, s)$ then some number $\leqslant \epsilon(b, s)$ and hence $\leqslant \mu(b, s)$ is enumerated in C at stage $s + 1$, since $a(s + 1) \not\subset b$ by hypothesis. If

$\mu(\boldsymbol{b}, s+1) \neq \mu(\boldsymbol{b}, t)$ then either some number $\leqslant \epsilon(\boldsymbol{b}, s)$ is enumerated in $\boldsymbol{C}$ at stage $s+1$ or $\mu(\boldsymbol{b}, s) < \mu(\boldsymbol{b}, s+1) < \omega$ through the occurrence of C2.1, C5.1 or C6 at stage $s+1$. If C5.1 or C6 occurs then $\boldsymbol{a}(s+1) < \boldsymbol{b} \frown d_0$ contrary to hypothesis.

Let C2.1 occur at stage $s+1$ then for some $n \notin D_0(\boldsymbol{b})$, $\phi(\boldsymbol{b}, n, s+1) < \phi(\boldsymbol{b}, n, s) = \omega$. Let $\boldsymbol{c}$ be the greatest initial segment of $\boldsymbol{b}$ for which $\mathrm{O}(\boldsymbol{c}) = n$. Then $\boldsymbol{a}(s+1) = \boldsymbol{c} \frown a_0$ whence $\boldsymbol{c} \frown d_0 \not\subset \boldsymbol{b} \frown d_0$. Thus $\boldsymbol{c}$ is a proper initial segment of $\boldsymbol{b}$. Also $\phi(\boldsymbol{b}, n, t) < \omega$; otherwise $\boldsymbol{a}(t+1)$ would be either $\supset \boldsymbol{c} \frown d_0$ or $= \boldsymbol{c} \frown a_0$. (When d_0 is replaced by d_1 we may have $\boldsymbol{c} = \boldsymbol{b}$ in which case $\phi(\boldsymbol{b}, n, t) < \omega$ because $\boldsymbol{b} \frown d_1 \subset \boldsymbol{a}(t+1)$.) Since $\phi(\boldsymbol{b}, n, s) = \omega$ either some number $\leqslant \phi(\boldsymbol{b}, n, t)$ is enumerated in $\boldsymbol{C}$ at a stage $\geqslant t+1$ and $\leqslant s$ or we have $\phi(\boldsymbol{b}, n, u+1) = \omega$ through C1 for some u, $t \leqslant u < s$. In the former case recall that $\phi(\boldsymbol{b}, n, t) \leqslant \mu(\boldsymbol{b}, t)$. In the latter case let stage $u+1$ pertain to $\epsilon(\boldsymbol{d}, u)$ then $\boldsymbol{b} \subset \boldsymbol{a}(\boldsymbol{d}, u) \subset \boldsymbol{d}$ and $\nu(\boldsymbol{d}, u) = n$. Further, either $\boldsymbol{b} = \boldsymbol{d}$ or $\boldsymbol{b} \frown (f, m) \subset \boldsymbol{d}$ for some m.

Thus $\epsilon(\boldsymbol{d}, u)$ cannot be enumerated in A at a stage $\geqslant t+1$ whence $\epsilon(\boldsymbol{d}, t) = \epsilon(\boldsymbol{d}, u)$ and both $\nu(\boldsymbol{d}, t)$ and $\boldsymbol{a}(\boldsymbol{d}, t)$ are $\neq \omega$. From the way in which $\boldsymbol{a}(\boldsymbol{d}, t)$ and $\nu(\boldsymbol{d}, t)$ are defined, $\nu(\boldsymbol{d}, t) = n' \leqslant n$ and $\boldsymbol{a}(\boldsymbol{d}, u) \subset \boldsymbol{a}(\boldsymbol{d}, t)$ where $n' \notin D_0(\boldsymbol{a}(\boldsymbol{d}, t))$. Let $\boldsymbol{c}'$ be the least initial segment of $\boldsymbol{b}$ such that $\mathrm{O}(\boldsymbol{c}') = n'$ then $\boldsymbol{c}' \subset \boldsymbol{c}$. Since $n' \notin D_0(\boldsymbol{a}(\boldsymbol{d}, t)$ we have $\boldsymbol{c}' \frown d_0 \not\subset \boldsymbol{b}$. Also $\boldsymbol{c}'$ is a proper initial segment of $\boldsymbol{b}$ because $\boldsymbol{c}$ is. Since $\boldsymbol{a}(t+1) \supset \boldsymbol{c}'$ we should have $\boldsymbol{a}(t+1) \supset \boldsymbol{c}' \frown d_0$ or $\boldsymbol{a}(t+1) = \boldsymbol{c}' \frown a_0$ through C1 and $\epsilon(\boldsymbol{d}, t)$. This contradicts $\boldsymbol{a}(t+1) \supset \boldsymbol{b}$. (When d_0 is replaced by d_1 we may have $\boldsymbol{c}' = \boldsymbol{c} = \boldsymbol{b}$. We still have a contradiction because $\boldsymbol{a}(t+1) \supset \boldsymbol{b} \frown d_1$.)

At this point we can infer that some number $\leqslant \mu(\boldsymbol{b}, s) = \mu(\boldsymbol{b}, t)$ is enumerated in $A \cup C$ at stage $s+1$. Suppose $\epsilon(\boldsymbol{c}, s)$ is enumerated in A; then $\boldsymbol{b} \not< \boldsymbol{c}$ by P2(1). Also $\boldsymbol{a}(s+1) = \boldsymbol{c} \frown a_2 \not< \boldsymbol{b} \frown d_0$ by hypothesis. Thus either $\boldsymbol{c} \frown w \subset \boldsymbol{b}$ or $\boldsymbol{b} \frown d_0 \subset \boldsymbol{c}$ is also ruled out by P2(1) we have $\boldsymbol{c} \frown w \subset \boldsymbol{b}$. Since $\boldsymbol{c} \frown w \subset \boldsymbol{a}(t+1)$, $\boldsymbol{a}(s+1) \supset \boldsymbol{c}$ and $\boldsymbol{a}(s+1) < \boldsymbol{c} \frown a_0$, we have $\boldsymbol{a}(s+1) \supset \boldsymbol{c} \frown w$ by P6(3). This contradicts $\boldsymbol{a}(s+1) = \boldsymbol{c} \frown a_2$.

There remains the case in which some number $\leqslant \mu(\boldsymbol{b}, t) = \mu(\boldsymbol{b}, s)$ is enumerated in $\boldsymbol{C}$ at stage $s+1$. Then one of the three cases C1.1, C1.3 or C3 occurs at stage $s+1$. If C3 occurs, then sup $I(i, s)$ is enumerated in $\boldsymbol{C}$ and from the way in which $I(i, s)$ is defined sup $I(i, s)$ has the form $\phi(\boldsymbol{c}, j, s)$, where $\boldsymbol{c} \frown (f, j) \subset \boldsymbol{a}(s+1)$. If $\boldsymbol{c} \subset \boldsymbol{b}$, then either $\boldsymbol{c} \frown w \subset \boldsymbol{b}$ or $\boldsymbol{c} \frown (f, k) \subset \boldsymbol{b}$, where $k \leqslant j$. In the former case we have $\boldsymbol{a}(s+1) \supset \boldsymbol{c} \frown w$ by P6(3) as above, which contradicts $\boldsymbol{a}(s+1) \supset \boldsymbol{c} \frown (f, j)$. In the latter case $\mu(\boldsymbol{b}, s) < \phi(\boldsymbol{c}, j, s)$ by P2(2) and P4(2) which is contrary to assumption. Since $\boldsymbol{c} \not\subset \boldsymbol{b}$ either $\boldsymbol{b} \frown d_0 \subset \boldsymbol{c}$ or $\boldsymbol{b} < \boldsymbol{c}$. (When d_0 is replaced by d_1

we have the additional possibility that $b ^\frown d_1 \subset c$.) By P2(1) $\mu(b, s) < \epsilon(c, s)$ which again contradicts the assumption that $\phi(c, j, s) \leqslant \mu(b, s)$.

Thus either C1.1 of C1.3 occurs at stage $s + 1$. Again the number enumerated in C has the form $\phi(c, j, s)$. Also $a(s + 1) = c_0 ^\frown a_0$ where c_0 is the least initial segment of c such that $O(c_0) = j$. If $b ^\frown d_0 \subset c_0$ or $b < c_0$, then $\mu(b, s) < \epsilon(c, s)$ by P2(1), contrary to $\phi(c, j, s) \leqslant \mu(b, s)$. (If d_0 is replaced by d_1, again there is the case in which $b ^\frown d_1 \subset c$.) Thus $c_0 \subset b$. Let stage $s + 1$ pertain to $\epsilon(d, s) = j$. Also $c = a(d, s)$ if C1.1 occurs while c is the greatest initial segment of $a(d, s)$ such that $c ^\frown (f, j) \subset a(d, s)$ if C1.3 occurs. Let $\epsilon(d, s)$ be enumerated in A at a stage $\leqslant t$; then $a(d, s) \subset a(d, t)$ and $\nu(d, t) \leqslant \nu(d, s)$. Let c_1 be the least initial segment of b such that $O(c_1) = \nu(d, t)$. Then $c_1 \subset c_0$. Since $c_1 \subset b$ and $b ^\frown d_0 \subset a(t + 1)$ we have $c_1 ^\frown d_0 \subset a(t + 1)$ through C1.4 at stage $(t + 1, \mathrm{lh}(c_1))$. Thus there exists a u, $t < u \leqslant s$, such that $c_1 ^\frown a_0 = a(u + 1)$, which contradicts the assumption that $a(u + 1) \not< b ^\frown d_0$. (When d_0 is replaced by d_1 we have $a(t + 1) \supset b ^\frown d_1$ whence $c_1 ^\frown d_0 \subset b$.) There remains the case in which $\epsilon(d, s)$ is enumerated in A at a stage $u + 1 > t$. Since $a(d, s) = c$ when C1.1 occurs at stage $s + 1$ and since $\nu(d, s) = j$, either $c = d$ or $c ^\frown (f, i) \subset d$ where $i \leqslant j$. If $b < c$, then $\mu(b, s) < \epsilon(c, s)$ by P2(1) which contradicts $\phi(c, j, s) \leqslant \mu(b, s)$. If $b \subset c$, then $b ^\frown d_0 \subset c$ since otherwise we should have $a(u + 1) < b ^\frown d_0$. Again P2(1) yields a contradiction. If $c < b$, then $a(u + 1) < b$ contrary to hypothesis. Thus $c \subsetneqq b$. If $c ^\frown w \subset b$, then since $a(u + 1) \supset c$ and $a(t + 1) \supset c ^\frown w$, it is easy to show by P6(3) that $a(u + 1) \supset c ^\frown w$. But $a(u + 1) \supset c ^\frown w$ contradicts $a(u + 1) = d ^\frown a_2$. Since $c ^\frown w \not\subset b$ we have $c \neq d$; otherwise $a(u + 1) = c ^\frown a_2 < b$. As noted above $c ^\frown (f, i) \subset d$ for some $i \leqslant j$. Since $a(u + 1) = d ^\frown a_2 \not< b$, $c ^\frown (f, k) \subset b$ for some $k \leqslant i$. From P2(2), $\mu(b, t) < \phi(c, k, t)$. Since $k \leqslant j$ we have $\phi(c, k, t) \leqslant \phi(c, j, t) < \omega$ by P4(1) and (2). It follows that $\mu(b, s) = \mu(b, t) < \phi(c, j, s)$, contradiction. This completes the proof of P6(1).

Proof of P6(2). Let $b ^\frown a_1 = a(t + 1)$, $t < s$ and $a(u) \not< b ^\frown a_1$ for all u in $t < u \leqslant s + 1$. For proof by contradiction fix t and choose the least $s > t$ such that $\epsilon(b, s + 1) \neq \epsilon(b, t + 1)$. Then some number $\leqslant \epsilon(b, t + 1)$ is enumerated in C at stage $s + 1$ since $a(s + 1) \not< b$ by hypothesis. One of three cases C1.1, C1.3 and C3 occurs at stage $s + 1$.

If C3 occurs, then sup $I(i, s)$ is enumerated in C. From the definition of $I(i, s)$, sup $I(i, s)$ has the form $\phi(c, j, s)$, where $c ^\frown (f, j) \subset a(s + 1)$. If $c \subset b$, then either $c ^\frown w \subset b$ or $c ^\frown (f, k) \subset b$ where $k \leqslant j$. In the former case we easily obtain a contradiction via P6(3). In the latter $\mu(\boldsymbol{b}, s) < \phi(\boldsymbol{c}, j, s)$ by P2(2) and P4(2), contradicting $\phi(\boldsymbol{c}, j, s) \leqslant \epsilon(\boldsymbol{b}, t+1) = \epsilon(\boldsymbol{b}, s)$. If $\boldsymbol{c} \not\subset \boldsymbol{b}$ then either $\boldsymbol{b} < \boldsymbol{c}$ or $\boldsymbol{b} \frown d_0 \subset \boldsymbol{c}$ or $\boldsymbol{b} \frown d_1 \subset \boldsymbol{c}$. In each of these cases $\mu(\boldsymbol{b}, s) < \epsilon(\boldsymbol{c}, s)$ by P2(1), yielding the same contradiction.

Therefore either C1.1 or C1.3 occurs at stage $s+1$. Again the number enumerated in $\boldsymbol{C}$ has the form $\phi(\boldsymbol{c}, j, s)$. Also $\boldsymbol{a}(s+1) = \boldsymbol{c}_0 \frown a_0$ where $\boldsymbol{c}_0$ is the least initial segment of $\boldsymbol{c}$ such that $\mathrm{O}(\boldsymbol{c}_0) = j$. If $\boldsymbol{b} \frown d_0 \subset \boldsymbol{c}_0$, $\boldsymbol{b} \frown d_1 \subset \boldsymbol{c}_0$ or $\boldsymbol{b} < \boldsymbol{c}_0$, then $\mu(\boldsymbol{b}, s) < \epsilon(\boldsymbol{c}, s)$ by P2(1); contradiction. Thus $\boldsymbol{c}_0 \subset \boldsymbol{b}$. Let stage $s+1$ pertain to $\epsilon(\boldsymbol{d}, s)$. Then $\nu(\boldsymbol{d}, s) = j$. Further $\boldsymbol{c} = \boldsymbol{a}(\boldsymbol{d}, s)$ if C1.1 occurs, while $\boldsymbol{c}$ is the greatest initial segment of $\boldsymbol{a}(\boldsymbol{d}, s)$ such that $\boldsymbol{c} \frown (f, j) \subset \boldsymbol{a}(\boldsymbol{d}, s)$ if C1.3 occurs. If $\epsilon(\boldsymbol{d}, s)$ is enumerated in $\boldsymbol{A}$ at a stage $\leqslant t$, then $\boldsymbol{a}(\boldsymbol{d}, t)$ and $\nu(\boldsymbol{d}, t)$ are both $\neq \omega$, $\boldsymbol{a}(\boldsymbol{d}, s) \subset \boldsymbol{a}(\boldsymbol{d}, t)$ and $\nu(\boldsymbol{d}, t) \leqslant \nu(\boldsymbol{d}, s)$. Let $\boldsymbol{c}_1$ be the least initial segment of $\boldsymbol{b}$ such that $\mathrm{O}(\boldsymbol{c}_1) = \nu(\boldsymbol{d}, t)$. Then $\boldsymbol{c}_1 \subset \boldsymbol{c}_0$. Since $\boldsymbol{c}_1 \subset \boldsymbol{b}$ and $\boldsymbol{a}(t+1) = \boldsymbol{b} \frown a_1$, we have $\boldsymbol{c}_1 \frown d_0 \subset \boldsymbol{a}(t+1)$ through C1.4 at stage $(t+1, \mathrm{lh}(\boldsymbol{c}_1))$. Thus there exists u, $t < u \leqslant s$, such that $\boldsymbol{c}_1 \frown a_0 = \boldsymbol{a}(u+1)$. This contradicts the hypothesis that $\boldsymbol{a}(u+1) \not< \boldsymbol{b} \frown a_1$. There remains the case in which $\epsilon(\boldsymbol{d}, s)$ is enumerated in $\boldsymbol{A}$ at a stage $u+1 > t$. Notice that either $\boldsymbol{c} = \boldsymbol{d}$ or $\boldsymbol{c} \frown (f, i) \subset \boldsymbol{d}$ where $i \leqslant j$, because $\nu(\boldsymbol{d}, s) = j$ and either $\boldsymbol{c}(\boldsymbol{a}(\boldsymbol{d}, s)$ or $\boldsymbol{c} \frown (f, j) \subset \boldsymbol{a}(\boldsymbol{d}, s)$. If $\boldsymbol{b} < \boldsymbol{c}$, then $\mu(\boldsymbol{b}, s) < \epsilon(\boldsymbol{c}, s)$ by P2(1) which contradicts $\phi(\boldsymbol{c}, j, s) \leqslant \epsilon(\boldsymbol{b}, t+1) = \epsilon(\boldsymbol{b}, s)$. If $\boldsymbol{b} \subset \boldsymbol{c}$ then either $\boldsymbol{b} \frown d_0 \subset \boldsymbol{c}$ or $\boldsymbol{b} \frown d_1 \subset \boldsymbol{c}$ since otherwise we should have $\boldsymbol{a}(u+1) < \boldsymbol{b} \frown a_1$. Again P2(1) yields a contradiction. If $\boldsymbol{c} < \boldsymbol{b}$ we have $\boldsymbol{a}(u+1) < \boldsymbol{b}$, again contrary to assumption. Thus $\boldsymbol{c} \subsetneqq \boldsymbol{b}$. If $\boldsymbol{c} \frown w \subset \boldsymbol{b}$ then by P6(3) we get $\boldsymbol{c} \frown w \subset \boldsymbol{a}(u+1)$, since $\boldsymbol{a}(u+1) \supset \boldsymbol{c}$ and $\boldsymbol{a}(u+1) < \boldsymbol{c} \frown a_0$. This contradicts $\boldsymbol{a}(u+1) = \boldsymbol{d} \frown a_2$. Since $\boldsymbol{c} \frown w \not\subset \boldsymbol{b}$ we have $\boldsymbol{c} \neq \boldsymbol{d}$; otherwise $\boldsymbol{a}(u+1) = \boldsymbol{c} \frown a_2 < \boldsymbol{b}$. As noted above $\boldsymbol{c} \frown (f, i) \subset \boldsymbol{d}$ for some $i \leqslant j$. Since $\boldsymbol{a}(u+1) = \boldsymbol{d} \frown a_2 \not< \boldsymbol{b}$, $\boldsymbol{c} \frown (f, k) \subset \boldsymbol{b}$ for some $k \leqslant i$. Through C3 at stage $t+1$ some number $\leqslant \phi(\boldsymbol{c}, k, t)$ is enumerated in $\boldsymbol{C}$ at stage $t+1$. By P4(2), $\phi(\boldsymbol{c}, k, t) \leqslant \phi(\boldsymbol{c}, j, t)$. Hence $\phi(\boldsymbol{c}, j, t+1) = \omega$, and, assuming without loss of generality that $\phi(\boldsymbol{c}, j, s) < \omega$, let v be the greatest number $< s$ such that $\phi(\boldsymbol{c}, j, v) = \omega$. Clearly $v > t$ and $\phi(\boldsymbol{c}, j, s)$ is assigned through C2.1 at stage $v+1$ whence $\epsilon(\boldsymbol{b}, t+1) = \epsilon(\boldsymbol{b}, v) < \phi(\boldsymbol{c}, j, v+1) = \phi(\boldsymbol{c}, j, s)$. This contradiction completes the proof of P6(2).

Proof of P6(3). Let $\boldsymbol{b} \frown w \subset \boldsymbol{a}(t+1)$, $t \leqslant s$ and $\boldsymbol{a}(u) \not< \boldsymbol{b}$ for all u in $t < u \leqslant s+1$. For proof by contradiction fix t and choose the least $s \geqslant t$ such that either $\epsilon(\boldsymbol{b}, s+1) \neq \epsilon(\boldsymbol{b}, t)$, or $\mu(\boldsymbol{b}, s+1) \neq \mu(\boldsymbol{b}, t)$, or some number $\leqslant \mu(\boldsymbol{b}, t)$ is enumerated in $\boldsymbol{A} \cup \boldsymbol{C}$ at stage $s+1$, or $\boldsymbol{a}(s+1) \supset \boldsymbol{b}$

and $a(s+1) < b^\frown a_0$ and $a(s+1) \not\supset b^\frown w$. We may suppose that P6(3) holds for c instead of b at stages $\leq s+1$ when c is a proper initial segment of b.

Suppose that $\mu(b, s) < \mu(b, s+1) < \omega$ through the occurrence of C2.1 at stage $s+1$; then $\phi(b, j, s+1) < \omega = \phi(b, j, s)$ for some $j \notin D_0(b)$. By P4(1), $\phi(b, j, t) < \omega$. Hence there exists u, $t \leq u < s$, such that $\phi(b, j, t)$ is destroyed at stage $u+1$. By hypothesis $a(u+1) \nless b$. If some number $\leq \phi(b, j, t)$ is enumerated in C at stage $u+1$, the choice of s is contradicted. Thus $\phi(b, j, t)$ is destroyed by C1. Let stage $u+1$ pertain to $\epsilon(c, u)$; then $\nu(c, u) = j$. If C1.2 occurs then b is the greatest initial segment of $a(c, u)$ such that $b^\frown (f, j) \subset a(c, u)$, and if C1.3 occurs then b is $a(c, u)$. Thus either $b^\frown (f, k) \subset c$ for some k or $b = c$. Let $\epsilon(c, u)$ be enumerated in A at stage $v+1$. If $v < t$ let e be the least initial segment of b such that $O(e) = \nu(c, t)$. Note that e exists because $\nu(c, t) \leq \nu(c, u) \leq O(b)$. Applying P1(2) to $\epsilon(c, v)$, we have $a(t+1) = e^\frown a_0$ or $a(t+1) \supset e^\frown d_0$ because $a(t+1) \supset b \supset e$. But $e^\frown d_0 \not\subset b$ and so we have a contradiction. Thus $v \geq t$. Since $a(v+1) < b^\frown a_0$, by choice of s we have $a(v+1) \supset b^\frown w$. But $a(v+1) = c^\frown a_2$ whence $c \supset b^\frown w$ which contradicts our findings above. Thus $\mu(b, s) < \mu(b, s+1) < \omega$ cannot occur through C2.1.

Suppose that $\mu(b, s) < \mu(b, s+1) < \omega$ through C5.1 or C6. Then $a(s+1) = b^\frown a_2$ or $a(s+1) \supset b^\frown (f, k)$ for some k. But $a(s+1) \supset b$ and $a(s+1) < b^\frown a_0$ imply $a(s+1) \supset b^\frown w$. This can be seen as follows. We have $\mu(b, s) = \mu(b, t)$, and no number $\leq \mu(b, t)$ is enumerated in C at a stage $>t$ and $\leq s$. Let $O(b) = n$. From stage $(t+1, \mathrm{lh}(b))$, $\xi_n(\epsilon(b, t), t) \leq$ $\leq \mu(b, t)$ because we have C5.2 rather than C5.1. It follows easily that $\xi_n(\epsilon(b, s), s) = \xi_n(\epsilon(b, t), t)$. Also from the last paragraph it is clear that $\phi(b, j, s) = \phi(b, j, t)$ for all $j \leq n$. Let $\mathrm{lh}(b) = i$ as in the statement of the construction then $\sup I(i, s) \geq \sup I(i, t)$ because the left hand side is in the inf of certain values $\phi(c, j, s)$ and the right hand side is the inf of the corresponding numbers $\phi(c, j, t)$. (By P4(1) the values concerned are $< \omega$ whence $\phi(c, j, s) \geq \phi(c, j, t)$.) It is now clear that $a(s+1) \supset b^\frown w$ because $a(s+1) \supset b$ and because the inequalities making $a(t+1) \supset b^\frown w$ are still true at stage $s+1$. This is a contradiction whence neither C5.1 nor C6 obtains at stage $s+1$.

Since $a(s+1) \nless b$, if $\epsilon(b, s+1) \neq \epsilon(b, t)$ or $\mu(b, s+1) \neq \mu(b, t)$, then some number $\leq \mu(b, t)$ is enumerated in C at stage $s+1$. It only remains to consider the cases in which some number $\leq \mu(b, t)$ is enumerated in $A \cup C$ at stage $s+1$.

Suppose that $\epsilon(c, s) \leq \mu(b, s) = \mu(b, t)$ is enumerated in A at stage $s+1$.

By P2(1) if $\boldsymbol{b} < \boldsymbol{c}$, $\boldsymbol{b} \frown d_0 \subset \boldsymbol{c}$ or $\boldsymbol{b} \frown w \subset \boldsymbol{c}$, then $\mu(\boldsymbol{b}, s) < \epsilon(\boldsymbol{c}, s)$ which contradicts the choice of s. Also as shown above, if $\boldsymbol{a}(s+1) \supset \boldsymbol{b}$ and $\boldsymbol{a}(s+1) < \boldsymbol{b} \frown a_0$, then $\boldsymbol{a}(s+1) \supset \boldsymbol{b} \frown w$. Hence $\boldsymbol{c} \not\supset \boldsymbol{b}$. If $\boldsymbol{c} \subsetneqq \boldsymbol{b}$, then $\boldsymbol{c} \frown w \subset \boldsymbol{b}$, otherwise $\boldsymbol{a}(s+1)$ would be $<\boldsymbol{b}$. But now $\boldsymbol{a}(s+1) \supset \boldsymbol{c} \frown w$ because $\boldsymbol{a}(s+1) \supset \boldsymbol{c}$ and $\boldsymbol{a}(s+1) < \boldsymbol{c} \frown a_0$. This contradicts $\boldsymbol{a}(s+1) = \boldsymbol{c} \frown a_2$, whence $\boldsymbol{c} \not\subset \boldsymbol{b}$. The only remaining possibility is $\boldsymbol{c} < \boldsymbol{b}$ which also contradicts the hypothesis.

Finally suppose that some number $\leqslant \mu(\boldsymbol{b}, s) = \mu(\boldsymbol{b}, t)$ is enumerated in C at stage $s+1$. Let C3 occur at stage $s+1$, then sup $I(i, s)$ is enumerated in $\boldsymbol{C}$ at stage $s+1$ for some i. From the way $I(i, s)$ is defined sup $I(i, s)$ has the form $\phi(\boldsymbol{c}, u, s)$ and $\boldsymbol{a}(s+1) \supset \boldsymbol{c} \frown (f, j)$. Reasoning just as in the last paragraph we can rule out the possibilities $\boldsymbol{b} < \boldsymbol{c}$, $\boldsymbol{c} < \boldsymbol{b}$ and $\boldsymbol{b} \subset \boldsymbol{c}$. Thus $\boldsymbol{c} \subsetneqq \boldsymbol{b}$. If $\boldsymbol{c} \frown w \subset \boldsymbol{b}$, then $\boldsymbol{a}(s+1) \supset \boldsymbol{c} \frown w$, because $\boldsymbol{a}(s+1) \supset \boldsymbol{c}$ and $\boldsymbol{a}(s+1) < \boldsymbol{c} \frown a_0$. This contradicts $\boldsymbol{a}(s+1) \supset \boldsymbol{c} \frown (f, j)$. Since $\boldsymbol{a}(s+1) \nless \boldsymbol{b}$ we have $\boldsymbol{c} \frown (f, k) \subset \boldsymbol{b}$ for some $k \leqslant j$. From P2(2) and P4(2) we have

$$\mu(\boldsymbol{b}, s) < \phi(\boldsymbol{c}, k, s) \leqslant \phi(\boldsymbol{c}, j, s)$$

which contradicts the hypothesis. Thus C3 does not occur at stage $s+1$.

Suppose that C1.1 or C1.3 occurs at stage $s+1$ and that stage $s+1$ pertains to $\epsilon(\boldsymbol{c}, s)$. Then $\phi(\boldsymbol{d}, j, s)$ is enumerated in $\boldsymbol{C}$ at stage $s+1$ where $j = \nu(\boldsymbol{c}, s)$. If C1.1 occurs then $\boldsymbol{d} = \boldsymbol{a}(\boldsymbol{c}, s)$ and if C1.3 occurs then $\boldsymbol{d}$ is the greatest initial segment of $\boldsymbol{a}(\boldsymbol{c}, s)$ such that $\boldsymbol{d} \frown (f, j) \subset \boldsymbol{a}(\boldsymbol{c}, s)$. By P2(1) we cannot have $\boldsymbol{b} < \boldsymbol{d}$ or $\boldsymbol{b} \frown d_0 \subset \boldsymbol{d}$ or $\boldsymbol{b} \frown w \subset \boldsymbol{d}$. Let $\epsilon(\boldsymbol{c}, s)$ be enumerated in A at stage $u+1$. Suppose $u < t$ and let $\nu(\boldsymbol{d}, t) = j' \leqslant j$. Let $\boldsymbol{e}$ be the least initial segment of $\boldsymbol{c}$ such that $\mathrm{O}(\boldsymbol{e}) = j'$; then $\boldsymbol{e} \subset \boldsymbol{d}$. Applying P1(2) to $\epsilon(\boldsymbol{c}, u)$ at stage $t+1$ we see that $\boldsymbol{e} \not\subset \boldsymbol{b}$ unless $\boldsymbol{e} \frown d_0 \subset \boldsymbol{b}$, because $\boldsymbol{a}(t+1) \supset \boldsymbol{b} \frown w$. Now $\boldsymbol{b} \nless \boldsymbol{e}$ otherwise $\epsilon(\boldsymbol{c}, u)$ would be destroyed at stage $t+1$, contradiction. If either $\boldsymbol{e} < \boldsymbol{b}$ or $\boldsymbol{e} \frown d_0 \subset \boldsymbol{b}$ let v be the least number $>t$ and $\leqslant s$ such that stage $v+1$ pertains to $\epsilon(\boldsymbol{c}, u)$. Then $\boldsymbol{a}(v+1) < \boldsymbol{b}$ contradiction. Thus $\boldsymbol{b} \subsetneqq \boldsymbol{e}$, whence $\boldsymbol{b} \frown w \subset \boldsymbol{e}$ since otherwise $\epsilon(\boldsymbol{c}, u)$ would be destroyed at stage $t+1$. But above we showed $\boldsymbol{b} \frown w \not\subset \boldsymbol{d}$ and $\boldsymbol{e} \subset \boldsymbol{d}$, contradiction.

Since $u < t$ leads to a contradiction, $u \geqslant t$. If $\boldsymbol{a}(u+1) \supset \boldsymbol{b}$ and $\boldsymbol{a}(u+1) < \boldsymbol{b} \frown d_0$, then as before $\boldsymbol{a}(u+1) \supset \boldsymbol{b} \frown w$. (Note that by choice of u, $\boldsymbol{a}(u+1) \neq \boldsymbol{b} \frown a_0$.) Hence if $\boldsymbol{d} \supset \boldsymbol{b}$ then either $\boldsymbol{d} \supset \boldsymbol{b} \frown d_0$ or $\boldsymbol{d} \supset \boldsymbol{b} \frown w$, because $\boldsymbol{a}(u+1) = \boldsymbol{c} \frown a_2$, $\boldsymbol{c} \supset \boldsymbol{d}$ and $\boldsymbol{d} \frown d_0 \not\subset \boldsymbol{c}$. But above we showed that $\boldsymbol{b} \nless \boldsymbol{d}$, $\boldsymbol{b} \frown d_0 \not\subset \boldsymbol{d}$ and $\boldsymbol{b} \frown w \not\subset \boldsymbol{d}$. Thus $\boldsymbol{d} \not\supset \boldsymbol{b}$ and $\boldsymbol{b} \nless \boldsymbol{d}$. Since $\boldsymbol{a}(u+1) \nless \boldsymbol{b}$ by hypothesis, $\boldsymbol{d} \nless \boldsymbol{b}$. It follows that $\boldsymbol{d} \subsetneqq \boldsymbol{b}$. If C1.3 occurs at stage $s+1$ then $\boldsymbol{d} \frown (f, j) \subset \boldsymbol{c}$. If C1.1 occurs then either

$d^\frown (f, k) \subset c$ for some $k < j$ or $d = c$. If $d = c$ then $a(u + 1) = d^\frown a_2$ and so $d^\frown w \subset b$ since $a(u + 1) \not< b$ by hypothesis. Since $a(t + 1) \supset d^\frown w$, $a(u + 1) \supset d$ and $a(u + 1) < d^\frown a_0$, applying the proposition to d we get $a(u + 1) \supset d^\frown w$, contradicting $a(u + 1) = c^\frown a_2$. Thus $d \neq c$, and the argument also shows that $d^\frown w \not\subset b$. In this case $d^\frown (f, k) \subset c$ for some $k \leqslant j$. Since $a(u + 1) = c^\frown a_2 \not< b$, $d^\frown (f, m) \subset b$ for some $m \leqslant k$. From P2(2) and P4, $\mu(b, u) < \phi(d, j, u) \leqslant \phi(d, j, u) < \omega$. Hence $\mu(b, t) < \phi(d, j, s)$. But $\phi(d, j, s)$ is meant to be the number $\leqslant \mu(b, t)$ enumerated in C at stage $s + 1$. This contradiction completes the proof.

Proof of P6(4). Let $b^\frown a_2 = a(t + 1)$, $t < s$, $a(u) \not< b^\frown a_2$ for all u in $t < u \leqslant s + 1$ and $\epsilon(b, t)$ not be enumerated in A at a stage $>t$. For proof by contradiction choose the least $s > t$ such that $\epsilon(b, s + 1) \neq \epsilon(b, t)$, or $\mu(b, s + 1) \neq \mu(b, t + 1)$, or some number $\leqslant \mu(b, t + 1)$ is enumerated in $A \cup C$ at a stage $>t$ and $\leqslant s + 1$, or $a(s + 1) = b^\frown a_2$.

Let $O(b) = n$. Observe that

$$\epsilon(b, s) = \epsilon(b, t), \qquad \mu(b, s) = \mu(b, t + 1), \qquad \mu(b, t + 1) \geqslant \xi_n(\epsilon(b, t), t).$$

Now $\xi_n(\epsilon(b, s), s) = \xi_n(\epsilon(b, t), t) < \omega$ because no number $\leqslant \mu(b, t + 1)$ is enumerated in C at a stage $>t$ and $\leqslant s$. Thus $\Xi_n(\epsilon(b, s), s) = \Xi_n(\epsilon(b, t), t)$. Since $\epsilon(b, t)$ is not enumerated in A at a stage $>t$, we also have $A(\epsilon(b, s), s) = A(\epsilon(b, t), t)$.

Now suppose that $a(s + 1) \supset b$ and $a(s + 1) < b^\frown a_0$. From P4(1) $\phi(b, j, s) < \omega$ if and only if $\phi(b, j, t) < \omega$. Further, if $\phi(b, j, s) \neq \phi(b, j, t)$ for some j, then $\phi(b, j, s)$ is assigned at some stage $>t + 1$, whence $\mu(b, s) \neq \mu(b, t + 1)$, contradiction. Let $\mathrm{lh}(b) = i$ as in the statement of the construction then $\sup I(i, s) \geqslant \sup I(i, t)$ because the left hand side is the inf of the members $\phi(c, j, t)$ such that $c^\frown (f, j) \subset b$, while the right hand side is the inf of the corresponding numbers $\phi(c, j, s)$. By P4(1) the values concerned are $< \omega$ whence $\phi(c, j, s) \geqslant \phi(c, j, t)$. Now C5 will occur at stage $(s + 1, i)$ because C5 occurs at stage $(t + 1, i)$.

Suppose there is a stage $u + 1 \leqslant t$ at which $\epsilon(b, t)$ is enumerated in A then by P5 there exists a v, $u < v < t$, such that $a(v + 1) \supset b^\frown w$. The alternative, namely that $a(v + 1) < b$, would destroy $\epsilon(b, t)$ before stage $t + 1$, contradiction. From P6(3) we get $a(t + 1) \supset b^\frown w$, contradiction. Thus $A(\epsilon(b, t), t) = 0$. It follows that C5.1 occurs at stage $(t + 1, i)$ since $a(t + 1) = b^\frown a_2$ and $\epsilon(b, t)$ is not enumerated in A at stage $t + 1$. From above $\mu(b, s) \geqslant \xi_n(\epsilon(b, s), s)$ whence C5.1 cannot occur at stage $(s + 1, i)$ if $a(s + 1) \supset b$. Since $A(\epsilon(b, s), s) = 0$ and $\epsilon(b, s)$ is not enumerated in A

at any stage $>t$, neither can C5.3 occur at stage $(s+1, i)$. Thus $\mathbf{a}(s+1) \neq \mathbf{b} \frown a_2$. Indeed we have the stronger result that if $\mathbf{a}(s+1) \supset \mathbf{b}$, then $\mathbf{a}(s+1) = \mathbf{b} \frown a_0$ or $\mathbf{a}(s+1) \supset \mathbf{b} \frown d_0$. For the rest of the proof we can follow the same line as for P6(3).

Proof of P6(5). Let $\mathbf{b} \frown (f, j) \subset \mathbf{a}(t+1)$, $t \leqslant s$ and $\mathbf{a}(u) \not< \mathbf{b} \frown (f, j)$ for all u in $t < u \leqslant s+1$. For proof by contradiction fix t and choose the least $s \geqslant t$ such that the conclusion fails.

Suppose that $\phi(\mathbf{b}, k, s+1) \neq \phi(\mathbf{b}, k, t)$ for some $k < j$, $k \notin D_0(\mathbf{b})$. Then $\phi(\mathbf{b}, k, s+1) = \omega$. Suppose that $\phi(\mathbf{b}, k, t)$ is destroyed by C1 at stage $s+1$. Let stage $s+1$ pertain to $\epsilon(\mathbf{c}, s)$; then $\nu(\mathbf{c}, s) = k$. If C1.2 occurs, $\mathbf{b}$ is the greatest initial segment of $\mathbf{a}(\mathbf{c}, s)$ such that $\mathbf{b} \frown (f, k) \subset \mathbf{a}(\mathbf{c}, s)$ and if C1.3 occurs, then $\mathbf{b}$ is $\mathbf{a}(\mathbf{c}, s)$. Thus either $\mathbf{b} \frown (f, m) \subset \mathbf{c}$ for some $m \leqslant k$ or $\mathbf{b} = \mathbf{c}$. Let $\epsilon(\mathbf{c}, s)$ be enumerated in $\mathbf{A}$ at stage $v+1$. If $v < t$, then by P1 $\mathbf{a}(t+1) \not\supset \mathbf{b} \frown (f, j)$, contradiction. Thus $v \geqslant t$ which means that either $\mathbf{a}(v+1) \supset \mathbf{b} \frown (f, m) < \mathbf{b} \frown (f, j)$ or $\mathbf{a}(v+1) = \mathbf{b} \frown a_2 < \mathbf{b} \frown (f, j)$, again a contradiction. Thus rather than $\phi(\mathbf{b}, k, t)$ being destroyed by C1 at stage $s+1$ some number $\leqslant \phi(\mathbf{b}, k, t)$ is enumerated in $\mathbf{C}$ at stage $s+1$.

Suppose that $\epsilon(\mathbf{c}, s)$ is enumerated in $\mathbf{A}$ at stage $s+1$ and that $\epsilon(\mathbf{c}, s) \leqslant \phi(\mathbf{b}, k, t)$ where $k < j$ and $k \notin D_0(\mathbf{b})$. By P2(1) if $\mathbf{b} < \mathbf{c}$, $\mathbf{b} \frown d_0 \subset \mathbf{c}$, or $\mathbf{b} \frown d_1 \subset \mathbf{c}$, then $\mu(\mathbf{b}, s) < \epsilon(\mathbf{c}, s)$ which contradicts $\epsilon(\mathbf{c}, s) \leqslant \phi(\mathbf{b}, k, t)$. Thus if $\mathbf{b} \subset \mathbf{c}$ then $\mathbf{b} \frown (f, l) \subset \mathbf{c}$ for some $l \geqslant j$ because $\mathbf{a}(s+1) = \mathbf{c} \frown a_2 \not< \mathbf{b} \frown (f, j)$. Now $\phi(\mathbf{b}, k, s) < \epsilon(\mathbf{c}, s)$ from P2(2) since $k < l$, again a contradiction. Therefore $\mathbf{b} \not\subset \mathbf{c}$. If $\mathbf{c} \subsetneq \mathbf{b}$, then $\mathbf{c} \frown w \subset \mathbf{b}$, otherwise $\mathbf{a}(s+1) < \mathbf{b}$. By P6(3) $\mathbf{a}(s+1) \supset \mathbf{c} \frown w$ because $\mathbf{a}(s+1) \supset \mathbf{c}$ and $\mathbf{a}(s+1) < \mathbf{c} \frown a_0$. This contradicts $\mathbf{a}(s+1) = \mathbf{c} \frown a_2$ whence $\mathbf{c} \not\subset \mathbf{b}$. The only remaining possibility is $\mathbf{c} < \mathbf{b}$ which also contradicts the hypothesis. We conclude that in every case some number is enumerated in $\mathbf{C}$ at stage $s+1$, that number being either $\leqslant \epsilon(\mathbf{b}, t)$ or $\leqslant \phi(\mathbf{b}, k, t)$ for some $k < j$ and $k \notin D_0(\mathbf{b})$.

Let C3 occur at stage $s+1$. Then sup $I(i, s)$ is enumerated in $\mathbf{C}$ at stage $s+1$ for some i. From the way $I(i, s)$ is defined sup $I(i, s)$ has the form $\phi(\mathbf{c}, m, s)$ and $\mathbf{a}(s+1) \supset \mathbf{c} \frown (f, m)$. As in the last paragraph we get $\mathbf{b} \not< \mathbf{c}$, $\mathbf{b} \not\subset \mathbf{c}$ and $\mathbf{c} \frown w \not\subset \mathbf{b}$. Since $\mathbf{c} \frown (f, m) \subset \mathbf{a}(s+1) \not< \mathbf{b}$ we have $\mathbf{c} \frown (f, l) \subset \mathbf{b}$ for some $l \leqslant m$. Now from P2(2) and P4(2) we have $\mu(\mathbf{b}, s) < \phi(\mathbf{c}, l, s) \leqslant \phi(\mathbf{c}, m, s)$ which contradicts the choice of s. Thus C3 is impossible.

Suppose that C1.1 or C1.3 occurs at stage $s+1$ and that stage $s+1$ pertains to $\epsilon(\mathbf{c}, s)$. Then $\phi(\mathbf{d}, m, s)$ is enumerated in $\mathbf{C}$ at stage $s+1$ where $m = \nu(\mathbf{c}, s)$. If C1.1 occurs, then $\mathbf{d} = \mathbf{a}(\mathbf{c}, s)$ and if C1.3 occurs, then $\mathbf{d}$ is the greatest initial segment of $a(c, s)$ such that $d \frown (f, m) \subset a(c, s)$. From P2(1), each of

$$b < d, \qquad b \frown d_0 \subset d, b \frown d_1 \subset d, \qquad b \frown w \subset d,$$

leads to a contradiction. Let $\epsilon(c, s)$ be enumerated in A at stage $u + 1$. Suppose $u < t$ and let $\nu(c, t) = m' \leqslant m$. Let e be the least initial segment of c such that $O(e) = m'$ then $e \subset d$. From P1(2) applied to stage $t + 1$ if $b \supset e$ then $b \supset e \frown d_0$. Since $b \nless d$ from above, $b \nless e$. Also the first stage $v + 1 > t$ which pertains to $\epsilon(c, u)$ is $\leqslant s + 1$ and $a(v + 1) = e \frown a_0$. Hence $e < b$ and $e \frown d_0 \subset b$ are both impossible. Thus $b \supsetneq e$. Since $b \frown d_0 \subset e$, $b \frown d_1 \subset e$ and $b \frown w \subset e$ are all ruled out above, we have $b \frown (f, l) \subset e$ for some l. If $l < j$ then $a(v + 1) < b \frown (f, j)$, contradiction. If $l > j$ then $\epsilon(c, u)$ is destroyed at stage $t + 1$, contradiction.

Now $b \frown (f, j) \subset d$ is impossible because by P2(2) it would follow that

$$\epsilon(b, t) \leqslant \phi(b, k, t) < \epsilon(d, t) < \phi(d, m, s)$$

for each $k < j$, $k \notin D_0(b)$. (It does not matter if there is no such k.) Hence $u < t$ is impossible.

Suppose $u \geqslant t$. If C1.3 occurs at stage $s + 1$ then $d \frown (f, m) \subset c$. If C1.1 occurs then either $d \frown (f, l) \subset c$ for some $l < m$ or $d = c$. If $d = b$, then $j \leqslant m$, otherwise $a(u + 1) < b \frown (f, j)$. In this case from P4(2)

$$\epsilon(b, t) < \phi(b, k, t) < \phi(b, m, t) \leqslant \phi(d, m, s)$$

for each $k < j$, $k \notin D_0(b)$. This contradicts the choice of s, whence $d \neq b$. If $d \supsetneq b$, then $d \supset b \frown (f, l)$ for some l, since $b \frown d_0 \subset d$, $b \frown d_1 \subset d$ and $b \frown w \subset d$ are all ruled out above. Now $j \leqslant l$ since $a(u + 1) \nless b \frown (f, j)$. From P2(2)

$$\epsilon(b, u) < \phi(b, k, u) < \epsilon(d, u) < \phi(d, m, s)$$

for each $k < j$, $k \notin D_0(b)$. Again this is a contradiction whence $d \not\supset b$. From above $b \nless d$ and $d \nless b$ since $a(u + 1) \nless b$. Hence $d \subsetneq b$. If $d = c$, then $a(u + 1) = d \frown a_2$ and so $d \frown w \subset b$ since $a(u + 1) \nless b$ by hypothesis. Now $a(u + 1) \supset d \frown w$ by P6(3) since $a(t + 1) \supset d \frown w$; contradiction. Thus $d \neq c$ and $d \frown w \not\subset b$. Since $a(u + 1) \nless b$, the only remaining possibility is that $d \frown (f, k) \subset b$ for some $k \leqslant m$. From P2(2) and P4

$$\mu(b, u) < \phi(d, k, u) \leqslant \phi(d, m, u) < \omega .$$

This contradicts the choice of s and completes the proof of P6(5).

Proof of P6(6). Let $\boldsymbol{b}^\frown a_0 = \boldsymbol{a}(t+1)$, $\boldsymbol{b} \subset \boldsymbol{c}$, $\boldsymbol{b}^\frown d_0 \not\subset \boldsymbol{c}$, $t \leqslant s$, $\epsilon(\boldsymbol{c}, s) < \omega$ and $\boldsymbol{a}(u) \not< \boldsymbol{b}^\frown a_0$ for all u in $t < u \leqslant s+1$. Suppose there is no stage $u+1 > t$ at which C1 occurs and which pertains to $\epsilon(\boldsymbol{d}, u)$ where $\boldsymbol{d} \subset \boldsymbol{b}$ or $\boldsymbol{d} < \boldsymbol{b}^\frown a_0$. We have to show that at stage $s+1$ no number $\leqslant \mu(\boldsymbol{c}, s)$ is enumerated in $\boldsymbol{C}$. For proof by contradiction suppose that some number $\leqslant \mu(\boldsymbol{c}, s)$ is enumerated in $\boldsymbol{C}$ at stage $s+1$, and that stage $s+1$ is the first at which P6(6) if false.

Suppose C3 occurs at stage $s+1$. Then some number $\phi(\boldsymbol{d}, j, s)$ is enumerated in $\boldsymbol{C}$ at stage $s+1$ where $\boldsymbol{d}^\frown (f, j) \subset \boldsymbol{a}(s+1)$. Since $\boldsymbol{a}(s+1) \not< \boldsymbol{b}^\frown a_0$, $\boldsymbol{d} \not< \boldsymbol{b}$ and also $\boldsymbol{d} \supset \boldsymbol{b}$ implies $\boldsymbol{d} \supset \boldsymbol{b}^\frown d_0$. If $\boldsymbol{b} < \boldsymbol{d}$ then $\boldsymbol{c} < \boldsymbol{d}$, whence $\mu(\boldsymbol{c}, s) \leqslant \epsilon(\boldsymbol{d}, s) < \phi(\boldsymbol{d}, j, s)$, contradiction. Thus either $\boldsymbol{d}^\frown w \subset \boldsymbol{b}$, $\boldsymbol{d}^\frown (f, i) \subset \boldsymbol{b}$ for some $i \leqslant j$ or $\boldsymbol{b}^\frown d_0 \subset \boldsymbol{d}$. Suppose $\boldsymbol{d}^\frown w \subset \boldsymbol{b}$ then from P6(3) $\boldsymbol{a}(s+1) \supset \boldsymbol{d}^\frown w$, because $\boldsymbol{a}(t+1) \supset \boldsymbol{d}^\frown w$, $\boldsymbol{a}(s+1) \supset \boldsymbol{d}$ and $\boldsymbol{a}(s+1) < \boldsymbol{d}^\frown a_0$. Thus $\boldsymbol{d}^\frown w \not\subset \boldsymbol{b}$. Suppose $\boldsymbol{d}^\frown (f, i) \subset \boldsymbol{b}$ where $i \leqslant j$, then $\mu(\boldsymbol{c}, s) < \phi(\boldsymbol{d}, i, s)$ by P2(2) and $\phi(\boldsymbol{d}, i, s) \leqslant \phi(\boldsymbol{d}, j, s)$ by P4(2). Thus $\mu(\boldsymbol{c}, s) < \phi(\boldsymbol{d}, j, s)$ which is also a contradiction. Also $\boldsymbol{b}^\frown d_0 \subset \boldsymbol{d}$ is impossible, because by P2(1) $\mu(\boldsymbol{c}, s) < \epsilon(\boldsymbol{d}, s)$. Thus either C1.1 or C1.3 occurs at stage $s+1$.

Let stage $s+1$ pertain to $\epsilon(\boldsymbol{d}, s)$; then by hypothesis either $\boldsymbol{b} < \boldsymbol{d}$ or $\boldsymbol{b}^\frown d_0 \subset \boldsymbol{d}$. The number enumerated in $\boldsymbol{C}$ at stage $s+1$ has the form $\phi(\boldsymbol{e}, j, s)$ where either $\boldsymbol{e} = \boldsymbol{d}$ or $\boldsymbol{e}^\frown (f, i) \subset \boldsymbol{d}$ for some $i \leqslant j$. Let $u+1$ be the stage $\leqslant s$ where $\epsilon(\boldsymbol{d}, u) = \epsilon(\boldsymbol{d}, s)$ is enumerated in $\boldsymbol{A}$. Now $u > t$, otherwise $\epsilon(\boldsymbol{d}, u)$ would be destroyed at stage $t+1$. If $\boldsymbol{e}^\frown w \subset \boldsymbol{b}$ then $\boldsymbol{e}^\frown w \subset \boldsymbol{a}(u+1)$ by P6(3). This contradicts $\boldsymbol{a}(u+1) = \boldsymbol{d}^\frown a_2$ since either $\boldsymbol{e} = \boldsymbol{d}$ or $\boldsymbol{e}^\frown (f, i) \subset \boldsymbol{d}$. Hence $\boldsymbol{e}^\frown w \not\subset \boldsymbol{b}$. If $\boldsymbol{b} < \boldsymbol{e}$ or $\boldsymbol{b}^\frown d_0 \subset \boldsymbol{e}$, then $\mu(\boldsymbol{c}, s) < \epsilon(\boldsymbol{e}, s)$ by P2(1), whence $\mu(\boldsymbol{c}, s) < \phi(\boldsymbol{e}, j, s)$. Since $\boldsymbol{d}^\frown a_2 = \boldsymbol{a}(u+1) \not< \boldsymbol{b}^\frown a_0$, the only other possibility is that $\boldsymbol{e}^\frown (f, i) \subset \boldsymbol{b}$ where $i \leqslant j$. In this case $\mu(\boldsymbol{c}, u) < \phi(\boldsymbol{e}, i, u) \leqslant \phi(\boldsymbol{e}, j, u)$ by P2(2) and P4(2). Thus again $\mu(\boldsymbol{c}, s) < \phi(\boldsymbol{e}, j, s)$ unless $\mu(\boldsymbol{c}, u) < \mu(\boldsymbol{c}, s)$.

Let v be the least number if any such that $u \leqslant v < s$ and $\mu(\boldsymbol{c}, v) < \mu(\boldsymbol{c}, v+1)$. Since $\boldsymbol{b} \subset \boldsymbol{c}$, $\boldsymbol{b}^\frown d_0 \not\subset \boldsymbol{c}$ and $\boldsymbol{a}(v+1) \not< \boldsymbol{b}^\frown a_0$, C2.1 occurs at stage $v+1$. Thus $\boldsymbol{a}(v+1) = \boldsymbol{c}_0{}^\frown a_0$ for some $\boldsymbol{c}_0 \subset \boldsymbol{b}$. Let $\boldsymbol{c}_1$ be the least initial segment of $\boldsymbol{d}$ such that $\mathrm{O}(\boldsymbol{c}_1) = j$. Since $\phi(\boldsymbol{e}, j, s)$ is enumerated in $\boldsymbol{C}$ at stage $s+1$, $\nu(\boldsymbol{d}, s) = j$. Also $\boldsymbol{c}_1 \subset \boldsymbol{e} \subset \boldsymbol{b}$ since $\mathrm{O}(\boldsymbol{e}) \geqslant j$. From P1(1) if $\boldsymbol{a}(v+1) \supset \boldsymbol{c}_1$, then $\boldsymbol{a}(v+1) \supset \boldsymbol{c}_1{}^\frown d_0$. Now $\boldsymbol{c}_0$ and $\boldsymbol{c}_1$ are comparable since both are $\subset \boldsymbol{b}$. If $\boldsymbol{c}_1 \subsetneqq \boldsymbol{e}$, then $\boldsymbol{c}_1{}^\frown d_0 \not\subset \boldsymbol{e}$ since $\nu(\boldsymbol{d}, s) = j \notin D_0(\boldsymbol{a}(\boldsymbol{d}, s))$. Also from above $\boldsymbol{e}^\frown d_0 \not\subset \boldsymbol{b}$. Since $\boldsymbol{c}_1 \subset \boldsymbol{e} \subset \boldsymbol{b}$ it follows that $\boldsymbol{c}_1{}^\frown d_0 \not\subset \boldsymbol{b}$. But if $\boldsymbol{c}_1 \subset \boldsymbol{c}_0$ then $\boldsymbol{a}(v+1) = \boldsymbol{c}_0{}^\frown a_0 \supset \boldsymbol{c}_1$ whence $\boldsymbol{b} \supset \boldsymbol{c}_0 \supset \boldsymbol{c}_1{}^\frown d_0$; contradiction. Thus $\boldsymbol{c}_0 \subsetneqq \boldsymbol{c}_1$.

Let $O(\boldsymbol{c}_0) = j_0$; then $\phi(\boldsymbol{c}, j_0, v) = \omega$. But $\boldsymbol{c}_0 {}^\frown \boldsymbol{d}_0 \not\subset \boldsymbol{b}$, whence $\boldsymbol{c}_0 {}^\frown \boldsymbol{d}_0 \not\subset \boldsymbol{d}$, whence $\phi(\boldsymbol{c}, j_0, u) < \omega$. Let w be the least number such that $u \leqslant w < v$ and $\phi(\boldsymbol{c}, j_0, w+1) = \omega$. Such w exists because stage $v+1$ must be concerned with assigning a value $< \omega$ to $\phi(\boldsymbol{c}, j_0, v+1)$, whence $\phi(\boldsymbol{c}, j_0, v) = \omega$. Since $\phi(\boldsymbol{c}, j_0, w+1) = \omega$ either $\phi(\boldsymbol{c}, j_0, w)$ is destroyed through C1 or some number $\leqslant \phi(\boldsymbol{c}, j_0, w)$ is enumerated in $\boldsymbol{C}$ at stage $w+1$. The former contradicts the hypothesis of the proposition, the latter contradicts the choice of s. Therefore w does not exist, whence v does not exist.

It follows that if C1 holds at stage $s+1$ then $\mu(\boldsymbol{c}, s) < \phi(\boldsymbol{e}, j, s)$. But some number $\leqslant \mu(\boldsymbol{c}, s)$ is supposed to be enumerated in $\boldsymbol{C}$ at stage $s+1$ whence $\phi(\boldsymbol{e}, j, s) \leqslant \mu(\boldsymbol{c}, s)$. This concludes the proof.

Proof of P7. Let $\boldsymbol{b} {}^\frown (f, j) \subset \boldsymbol{a}(s+1)$ for infinitely many s and $\boldsymbol{a}(s+1) < \boldsymbol{b} {}^\frown (f, j)$ for at most a finite number of s. We must show that there exists s such that $\boldsymbol{b} {}^\frown (f, j) \subset \boldsymbol{a}(s+1)$ and $\phi(\boldsymbol{b}, j, s)$ is arbitrarily large. Let $\boldsymbol{S} = \{s: \boldsymbol{a}(s+1) \supset \boldsymbol{b} {}^\frown (f, j)\}$. For proof by contradiction assume that $\phi(\boldsymbol{b}, j, s)$ is bounded for $s \in \boldsymbol{S}$. There exist at most finitely many $s \in \boldsymbol{S}$ such that C3 occurs at stage $s+1$, because at each such stage $\phi(\boldsymbol{b}, j, s)$ is destroyed and its next finite value is strictly greater. It follows that the set

$$\mathfrak{C} = \{\langle \boldsymbol{c}, t\rangle : \boldsymbol{c} \supset \boldsymbol{b} {}^\frown (f, j), \epsilon(\boldsymbol{c}, t) < \omega, \text{ and } \epsilon(\boldsymbol{c}, t-1) = \omega\}$$

is finite. C1 occurs at stage $s+1$ for at most finitely many s, because for each such s there exists $\langle \boldsymbol{c}, t\rangle \in \mathfrak{C}$ such that stage $s+1$ pertains to $\epsilon(\boldsymbol{c}, t)$ and at most a finite number of stages pertain to each $\epsilon(\boldsymbol{c}, t)$. There are at most a finite number of $s \in \boldsymbol{S}$ such that C2 occurs at stage $s+1$. For each such s there exists $\langle \boldsymbol{c}, t\rangle \in \mathfrak{C}$ and n such that $\epsilon(\boldsymbol{c}, s) = \epsilon(\boldsymbol{c}, t)$, $n \leqslant O(\boldsymbol{c})$, and

$$\sup\{\phi(\boldsymbol{c}, n, u) : \phi(\boldsymbol{c}, n, u) < \omega, u \leqslant s\} < \phi(\boldsymbol{c}, n, s+1) < \phi(\boldsymbol{b}, j, s) .$$

Since $\phi(\boldsymbol{b}, j, s)$ is bounded as s runs through $\boldsymbol{S}$ at most a finite number of $s \in \boldsymbol{S}$ pertain to each pair $\langle \boldsymbol{c}, t\rangle$. If C5.1 occurs at stage $s+1$, $s \in \boldsymbol{S}$, then for some $\langle \boldsymbol{c}, t\rangle \in \mathfrak{C}$, $\epsilon(\boldsymbol{c}, s) = \epsilon(\boldsymbol{c}, t)$ and $\mu(\boldsymbol{c}, s) < \mu(\boldsymbol{c}, s+1) < \phi(\boldsymbol{b}, j, s)$. If C5.2 occurs at stage $s+1$, $s \in \boldsymbol{S}$, then for some $\langle \boldsymbol{c}, t\rangle \in \mathfrak{C}$, $\epsilon(\boldsymbol{c}, t)$ is enumerated in $\boldsymbol{A}$ at stage $s+1$. Thus there are at most a finite number of $s \in \boldsymbol{S}$ such that C5 occurs at stage $s+1$. Since one of C1, C2, C3 and C5 occurs at every stage, and $\boldsymbol{S}$ is infinite, we have the desired contradiction.

Proof of P8. Let $\epsilon(c, t)$ be enumerated in A at stage $t + 1, s > t$, $\epsilon(c, s) = \epsilon(c, t)$, $O(b) = n$, $b \subset c$, $\mathrm{lh}(b) = i$, $a(s + 1) \supset b$, C1.4 occur at stage $(s + 1, i)$, stage $(s + 1, i)$ pertain to $\epsilon(c, t)$, and d be the greatest initial segment of $a(c, s)$ such that $d ^\frown (f, n) \subset a(c, s)$. We must show that $\phi(d, n, s) = \phi(d, n, t)$. For proof by contradiction suppose that $\phi(d, n, s) \neq \phi(d, n, t)$. Let u be the least number $\geqslant t$ such that $\phi(d, n, u+1) \neq \phi(d, n, t)$; then $u + 1 \leqslant s$ and $\phi(d, n, t)$ is destroyed at stage $u + 1$. Since stage $(s + 1, i)$ pertains to $\epsilon(c, t)$ we have $b \subset d \subset c$. Since $\epsilon(c, u + 1) = \epsilon(c, t)$, $a(u + 1) \not< d$. Therefore either $\phi(d, n, t)$ is destroyed through C1 or some number $\leqslant \phi(d, n, t)$ is enumerated in C at stage $u + 1$.

Suppose that $\phi(d, n, t)$ is destroyed through C1. Let stage $u + 1$ pertain to $\epsilon(c', u)$ where $\epsilon(c', u)$ was enumerated in A at stage $t' + 1$. Then $\nu(c', u) = n$, $c' \supset d \supset b$ and $a(u + 1) = b ^\frown a_0$. From the hypothesis it is clear that $\nu(c, s) = n$, whence $c \neq c'$ and $t' \neq t$ because $\nu(c', u + 1) > n$. Suppose $t' < t$; then $a(t + 1) \supset b ^\frown d_0$ by applying P1(1) to $\epsilon(c', u)$. This contradicts $a(t + 1) = c ^\frown a_2$ because $n \notin D_0(c)$. A similar contradiction arises from supposing $t < t'$. Thus some number $\leqslant \phi(d, n, t)$ is enumerated in C at stage $u + 1$.

Suppose that C3 occurs at stage $u + 1$ then the number enumerated in C at stage $u + 1$ has the form $\phi(e, j, u)$ where $e ^\frown (f, j) \subset a(u + 1)$. Now $e ^\frown (f, j) \not< b$ since $\epsilon(c, t) = \epsilon(c, u + 1)$. By P1(1) if $b \subset e$, then $b ^\frown d_0 \subset e$. If $e ^\frown w \subset b$, then $e ^\frown w \subset a(u + 1)$ by P6(3); contradiction. If $e ^\frown (f, k) \subset b$ for some $k \leqslant j$, then $\mu(d, t) < \phi(e, k, t) \leqslant \phi(e, j, t) < \omega$ by P2(2) and P4, whence $\phi(e, j, u) \nleqslant \phi(d, n, t)$. If $b ^\frown d_0 \subset e$ or $b < e$, then $\mu(d, u) < \epsilon(e, u)$ by P2(1), whence again $\phi(e, j, u) \nleqslant \phi(d, n, t)$. Since this exhausts all possible relationships between e and b, C3 does not occur at stage $u + 1$.

It is clear that either C1.1 or C1.3 occurs at stage $u + 1$. Let $a(u + 1) = b' ^\frown a_0$ and stage $u + 1$ pertain to $\epsilon(c', u)$ where $\epsilon(c', u) = \epsilon(c', t')$ was enumerated in A at stage $t' + 1$. Let $\nu(c', u) = n'$ and if $n' \in F(a(c', u))$, let d' be the greatest initial segment of $a(c', u)$ such that $d' ^\frown (f, n') \subset a(c', u)$. Clearly $b' \subset d'$.

Suppose C1.1 occurs at stage $u + 1$; then the number enumerated in C is $\phi(a(c', u), n', u)$. Let $t' = t$; then $c = c'$ whence $a(c, u + 1) = \omega$. Thus stage $(s + 1, i)$ cannot pertain to $\epsilon(c, t)$, contradiction. Let $t' < t$ then $c \not< b'$ since $\epsilon(c', t + 1) = \epsilon(c', t)$, and if $c \supset b'$, then $c \supset b' ^\frown d_0$ by P1(1). Now $c \supset b' ^\frown d_0$ is impossible because it implies $a(u + 1) < c$ and hence $\epsilon(c, u + 1) \neq \epsilon(c, u)$. Similarly $b' \not< c$. Thus $c \subset b'$ and $c \neq b'$. Since $\epsilon(c', t + 1) < \omega$ we have $a(t + 1) \not< b'$, whence $c ^\frown w \subset b' \subset a(t' + 1)$. By

P6(3) we get $c \cap w \subset a(t+1)$, contradiction. Let $t' > t$, then $b' \not< b$ since $\epsilon(b, t'+1) < \omega$ and $c' \supset b$ implies $c' \supset b \cap d_0$ by P1(1). Thus $b' \supset b$ implies $b' \supset b \cap d_0$, because either $b' = c'$, or $b' \subset c'$ and $b' \cap d_0 \not\subset c'$. If $b < b'$ or $b \cap d_0 \subset b'$, then either $d < a(c', u)$ or $d \cap d_0 \subset a(c', u)$, since $d \supset b$, $a(c', u) \supset b'$ and $b \cap d_0 \not\subset d$. In this case, by P2(1) and P4(1)

$$\phi(d, n, t) = \phi(d, n, t') < \phi(a(c', u), n', t') < \omega .$$

It follows that $\phi(d, n, t) < \phi(a(c', u), n', u)$.

There remains the case in which $b' \supsetneq b$. Recall that either $b = d \subset c$ or $b \subset d \subset c$ and $d < b \cap d_0$. Notice also that if b and $a(c', u)$ are incomparable with respect to $\subset$, then $a(c', u) > b$. Otherwise $a(t'+1) = c' \cap a_2 < c$ and $\epsilon(c, t)$ is destroyed at stage $t'+1$, contradiction. Recall that if $c' \supset b$, then $c' \supset b \cap d_0$. Thus either $a(c', u) \subset b$ or $a(c', u) \supset b \cap d_0$ or $a(c', u) > b$, whence either $a(c', u) \subset b$ or $a(c', u) \supset d \cap d_0$ or $a(c', u) > d$. In the second and third cases $\phi(d, n, t) < \phi(a(c', u), n', u)$ as above. Thus suppose $a(c', u) \subset b$. If $a(c', u) = c' \subset b$, then $c' \neq b$ from above, and $c' \cap w \subset b$ since $c' \cap a_2 = a(t'+1) \not< b$. By P6(3) $c' \cap w \subset b$ implies $c' \cap w \subset a(t'+1)$, impossible. Therefore $a(c', u) \cap (f, j') \subset c'$ for some $j' < n'$. If $a(c', u) = b$, then $a(t'+1) \supset b$ and $a(t'+1) \not\supset b \cap d_0$, contrary to P1(1). Also $a(c', u) \cap w \subset b$ can be ruled out just as $c' \cap w \subset b$ was ruled out above. Since $a(c', u) \cap (f, j') \subset c' < b$ it must be the case that $a(c', u) \cap (f, j) \subset b$ for some $j \leqslant j'$. From P2(2) and P4 we have $\phi(d, n, t') < \phi(a(c', u), n', t') < \omega$, whence $\phi(d, n, t) = \phi(d, n, t') < < \phi(a(c', u), n', u)$. Again this is a contradiction.

Thus C1.3 must occur at stage $u+1$. Let $t' = t$; then $c' = c$, $\nu(c, u) = n' < n$ and $b \subset a(c, s) \subset a(c, u)$ by comparing stages $u+1$ and $s+1$. From C1.3 $a(c, u+1) = a(c, u)$ and $n' < \nu(c, u+1) \leqslant n$. By induction on v, if $u \leqslant v$ and $a(c, w) \neq \omega$ for all w in $u \leqslant w \leqslant v$, then $d' \subset a(c, v)$. Hence $d' \subset a(c, s)$. Since C1.4 occurs at stage $(s+1, i)$, $n \in F(a(c, s))$ and so $d' \subset d$, Since $n' < n$, $d' \cap (f, n') \subset d$. From P2(2) and P4(1) we have $\phi(d, n, t) < \phi(d', n', t) < \omega$. Thus the number $\phi(d', n', u)$ enumerated in C at stage $u+1$ is $> \phi(d, n, t)$, contradiction. The cases in which $t' < t$ and $t' > t$ are treated in the same way as when C1.1 occurs at stage $u+1$ except that when $t' > t$, d' and n' now play the roles which $a(c', u)$ and j' played before.

Proof of P9(1). Let b, c and t satisfy: $O(b) = n \notin D_0(b)$, $b \subset c$, $\epsilon(c, t)$ is enumerated in A at stage $t+1$, $\mathrm{lh}(b) = i$, and for infinitely many s stage

$(s+1, i)$ pertains to $\epsilon(\mathbf{c}, t)$. We must show that $\mathbf{A} \neq \boldsymbol{\theta}_n$. Let $\mathbf{S}$ denote the set of all s such that C1.4 occurs at stage $(s+1, i)$ and stage $(s+1, i)$ pertains to $\epsilon(\mathbf{c}, t)$. Since $\mathbf{S}$ is infinite $\boldsymbol{\varepsilon}(\mathbf{c}) = \epsilon(\mathbf{c}, t)$. For all $s \in \mathbf{S}$, $\nu(\mathbf{c}, s) = n$ and $\mathbf{a}(\mathbf{c}, s)$ takes a fixed value which will be denoted by $\mathbf{e}$. From the hypotheses of C1, $\mathbf{b} \subset \mathbf{e} \subset \mathbf{c}$ and $n \in F(\mathbf{e})$. Let $\mathbf{d}$ be the greatest initial segment of $\mathbf{e}$ such that $\mathbf{d} \frown (f, n) \subset \mathbf{e}$; then $\mathbf{d} \supset \mathbf{b}$. Recall from the definition of $\mathbf{a}(\mathbf{c}, s)$ that either $\mathbf{e} = \mathbf{c}$ or $\mathbf{e} \frown (f, m) \subset \mathbf{c}$ for some $m < n$. From P4(1) $\phi(\mathbf{e}, n, t) < \omega$, because $n \in F(\mathbf{e})$ whence $n \notin D_0(\mathbf{e})$. Let $\phi(\mathbf{e}, m\ t)$ be assigned at stage $u+1$. Now $\epsilon(\mathbf{c}) < \phi(\mathbf{e}, n, t)$ immediately when $\mathbf{e} = \mathbf{c}$, and from P2(2) and P4 otherwise. It follows that $\epsilon(\mathbf{c}, u) = \boldsymbol{\varepsilon}(\mathbf{c})$, because when $\epsilon(\mathbf{c}, u)$ is assigned it is larger than any number yet used.

Suppose for the moment that $\mathbf{e} \frown (f, m) \subset \mathbf{c}$ where $m < n$. Let $\mathbf{b}^m, \mathbf{b}^n$, be the greatest initial segments of $\mathbf{e}$ such that $\mathrm{O}(\mathbf{b}^m) = m$, $\mathrm{O}(\mathbf{b}^n) = n$ respectively. Then $\mathbf{b}^m \subsetneqq \mathbf{b}^n$ since $m < n$ and $n \leqslant \mathrm{O}(\mathbf{e})$. Also $\mathbf{b}^m \frown d_0 \not\subset \mathbf{b}^n$ since $m \notin D_0(\mathbf{e})$. Now $\mathbf{a}(u+1) = \mathbf{b}^n \frown a_0$ whence $\phi(\mathbf{e}, m, u) < \omega$. Further $\phi(\mathbf{e}, m, u)$ must have been assigned after $\boldsymbol{\varepsilon}(\mathbf{c})$ since otherwise $\phi(\mathbf{e}, m, u)$ would have been destroyed when $\boldsymbol{\varepsilon}(\mathbf{c})$ was assigned. Thus $\boldsymbol{\varepsilon}(\mathbf{c}) < \phi(\mathbf{e}, m, u)$. We conclude that, whether $\mathbf{e} \frown (f, m) \subset \mathbf{c}$ or not, $\boldsymbol{\varepsilon}(\mathbf{c}) = \epsilon(\mathbf{c}, u) \leqslant \mu(\mathbf{e}, u)$.

It follows from the occurrence of C2.1 at stage $u+1$ that

$$A(\boldsymbol{\varepsilon}(\mathbf{c}), u) = \theta_n(\boldsymbol{\varepsilon}(\mathbf{c}), u)$$

$$B_n^i(x, u) = \Psi_n^i(x, u) \quad \text{for all } x \leqslant \theta_n(\epsilon(\mathbf{c}), u),\ i \leqslant 1$$

$$\phi(\mathbf{e}, n, u+1) > \sup(\{\theta_n(\epsilon(\mathbf{c}), u)\} \cup \{\psi_n^i(x, u) \colon x \leqslant \theta_n(\epsilon(\mathbf{c}), u), i \geqslant 1\}).$$

From P4(3), no number $\leqslant \phi(\mathbf{e}, n, u+1)$ is enumerated in $\mathbf{A} \cup \mathbf{C}$ at a stage $> u$ and $\leqslant t$. It follows that no number $\leqslant \theta_n(\epsilon(\mathbf{c}), u)$ is enumerated in $\mathbf{B}_n^0 \cup \mathbf{B}_n^1 \cup \mathbf{C}$ at a stage $> u$ and $\leqslant t$ and hence that $\theta_n(\boldsymbol{\varepsilon}(\mathbf{c}), u) = \theta_n(\boldsymbol{\varepsilon}(\mathbf{c}), t)$ and $\theta_n(\boldsymbol{\varepsilon}(\mathbf{c}), u) = \theta_n(\boldsymbol{\varepsilon}(\mathbf{c}), t)$. Now

$$\phi(\mathbf{e}, n, t) \leqslant \phi(\mathbf{d}, n, t) < \omega$$

from P2(2) and P4 since $\mathbf{e} \supset \mathbf{d} \frown (f, n)$. From P8, $\boldsymbol{\phi}(\mathbf{d}, n) = \phi(\mathbf{d}, n, t)$ whence no number $\leqslant \phi(\mathbf{d}, n, t)$ is enumerated in $\mathbf{C}$ after stage t. Since C1.4 occurs at stage $(s+1, i)$ for infinitely many s in $\mathbf{S}$, no number $\leqslant \theta_n(\boldsymbol{\varepsilon}(\mathbf{c}), t)$ is enumerated in $\mathbf{B}_n^0 \cup \mathbf{B}_n^1$ after stage t. It follows that $\boldsymbol{\theta}_n \boldsymbol{\varepsilon}(\mathbf{c}) = \theta_n(\boldsymbol{\varepsilon}(\mathbf{c}), t)$ and $\boldsymbol{\theta}_n \boldsymbol{\varepsilon}(\mathbf{c}) = \theta_n(\boldsymbol{\varepsilon}(\mathbf{c}), t)$. But since $\boldsymbol{\varepsilon}(\mathbf{c})$ is enumerated in $\mathbf{A}$ at stage $t+1$, $\mathbf{A}\,\boldsymbol{\varepsilon}(\mathbf{c}) = 1$ and $\theta_n(\boldsymbol{\varepsilon}(\mathbf{c}), t) = 0$. Thus $\mathbf{A}\,\boldsymbol{\varepsilon}(\mathbf{c}) \neq \boldsymbol{\theta}_n \boldsymbol{\varepsilon}(\mathbf{c})$ which completes the proof.

Proof of P9(2). Let $O(\boldsymbol{b}) = n$, $\mathrm{lh}(\boldsymbol{b}) = i$, and let there be at most a finite number of stages with characteristic $< \boldsymbol{b} \frown d_0$. Let $\boldsymbol{c}$ be such that, for infinitely many s, $\boldsymbol{a}(s+1) \supset \boldsymbol{b} \frown d_0$ and stage $(s+1, i)$ is associated with $\boldsymbol{c}$. We must show that either $A \neq \theta_n$, or $B_n^0 \neq \psi_n^0$, or $B_n^1 \neq \psi_n^1$. We have $\varepsilon(\boldsymbol{c}) < \omega$ and $\phi(\boldsymbol{c}, n, s) = \omega$ for all sufficiently large s, because otherwise $\phi(\boldsymbol{c}, n, s+1) < \omega = \phi(\boldsymbol{c}, n, s)$ for infinitely many s which would make $\boldsymbol{a}(s+1) = \boldsymbol{b} \frown a_0 < \boldsymbol{b} \frown d_0$ for infinitely many s. We claim that if $\boldsymbol{d} < \boldsymbol{b} \frown d_0$ or $\boldsymbol{d} = \boldsymbol{b}$, then either $\mu(\boldsymbol{d}) < \omega$ or $\mu(\boldsymbol{d}, s) = \omega$ for all sufficiently large s. For proof by contradiction fix $\boldsymbol{d}$ which refutes the claim. Since at most a finite number of stage have characteristic $< \boldsymbol{b} \frown d_0$, $\epsilon(\boldsymbol{d}, s)$ and $\mu(\boldsymbol{d}, s)$ are $< \omega$ for all sufficiently large s. For the same reason there are at most a finite number of s such that C1 occurs at stage $s+1$ and stage $s+1$ pertains to $\epsilon(\boldsymbol{e}, s)$ where $\boldsymbol{e} \frown a_2 < \boldsymbol{b} \frown d_0$. Remember that to each number $\epsilon(\boldsymbol{e}, t)$ at most a finite number of stages pertain. Further using P6(3) if $\boldsymbol{e} \frown w \subset \boldsymbol{b}$ then $\boldsymbol{a}(s+1) = \boldsymbol{e} \frown a_2$ for at most a finite number of s. Thus at most a finite number of stages $s+1$ pertain to $\epsilon(\boldsymbol{e}, s)$ where $\boldsymbol{e} < \boldsymbol{b} \frown d_0$ or $\boldsymbol{e} \subset \boldsymbol{b}$.

If possible fix m such that $\phi(\boldsymbol{d}, m, s+1) = \omega > \phi(\boldsymbol{d}, m, s)$ for infinitely many s. Let $\boldsymbol{S}$ consist of all s such that $\phi(\boldsymbol{d}, m, s+1) = \omega > \phi(\boldsymbol{d}, m, s)$, $\boldsymbol{a}(s+1) \nless \boldsymbol{b} \frown d_0$, and if stage $s+1$ pertains to $\epsilon(\boldsymbol{e}, s)$, then $\boldsymbol{e} \nless \boldsymbol{b} \frown d_0$ and $\boldsymbol{e} \not\subset \boldsymbol{b}$. From our remarks above $\boldsymbol{S}$ is infinite. Consider $s \in \boldsymbol{S}$. If $\phi(\boldsymbol{d}, m, s)$ were destroyed through C1 then stage $s+1$ would pertain to $\epsilon(\boldsymbol{e}, s)$ such that $\boldsymbol{e} = \boldsymbol{d}$ or $\boldsymbol{d} \frown (f, j) \subset \boldsymbol{e}$. But $\boldsymbol{d} = \boldsymbol{b}$ or $\boldsymbol{d} < \boldsymbol{b} \frown d_0$, whence $\boldsymbol{e} = \boldsymbol{b}$ or $\boldsymbol{e} < \boldsymbol{b} \frown d_0$, contradiction. Thus $\phi(\boldsymbol{d}, m, s)$ must be destroyed by a number $\leqslant \phi(\boldsymbol{d}, m, s)$ being enumerated in $\boldsymbol{C}$ at stage $s+1$.

Suppose C3 occurs at stage $s+1$ then $\phi(\boldsymbol{e}, j, s)$ is enumerated in $\boldsymbol{C}$ where $\boldsymbol{e} \frown (f, j) \subset \boldsymbol{a}(s+1)$. If $\epsilon(\boldsymbol{d}, s)$ was assigned at stage $u+1$ then no stage $> u$ and $\leqslant s+1$ has characteristic $< \boldsymbol{d}$. Thus by P6(3) if $\boldsymbol{e} \frown w \subset \boldsymbol{d}$ then $\boldsymbol{a}(s+1) \supset \boldsymbol{e} \frown w$; contradiction. Recall that either $\boldsymbol{d} = \boldsymbol{b}$ or $\boldsymbol{d} < \boldsymbol{b} \frown d_0$ and $\boldsymbol{a}(s+1) \nless \boldsymbol{b} \frown d_0$. Examining cases we find that either $\boldsymbol{d} < \boldsymbol{e}$, or $\boldsymbol{d} \frown d_0 \subset \boldsymbol{e}$, or $\boldsymbol{e} \frown (f, k) \subset \boldsymbol{d}$ for some $k \leqslant j$. From P2 and P4 we now have $\mu(\boldsymbol{d}, s) < \phi(\boldsymbol{e}, j, s)$, contradiction.

Suppose C1.1 or C1.3 occurs at stage $s+1$ and that stage $s+1$ pertains to $\boldsymbol{e}$, then $\phi(\boldsymbol{f}, j, s)$ is enumerated in $\boldsymbol{C}$ at stage $s+1$ for some $\boldsymbol{f}$ such that either $\boldsymbol{f} = \boldsymbol{e}$ or $\boldsymbol{e} \supset \boldsymbol{f} \frown (f, k)$ for some $k \leqslant j$. By definition of $\boldsymbol{S}$, $\boldsymbol{e} \nless \boldsymbol{b} \frown d_0$ and $\boldsymbol{e} \not\subset \boldsymbol{b}$. Since either $\boldsymbol{d} = \boldsymbol{b}$ or $\boldsymbol{d} < \boldsymbol{b} \frown d_0$, we have either $\boldsymbol{d} < \boldsymbol{e}$ or $\boldsymbol{d} \frown d_0 \subset \boldsymbol{e}$. By P2(1) $\mu(\boldsymbol{d}, s) < \epsilon(\boldsymbol{e}, s)$. If $\boldsymbol{f} = \boldsymbol{e}$ we have $\epsilon(\boldsymbol{e}, s) < \phi(\boldsymbol{f}, j, s)$ immediately. If $\boldsymbol{f} \frown (f, k) \subset \boldsymbol{e}$ for some $k \leqslant j$, let $\epsilon(\boldsymbol{e}, s)$ be enumerated in A at stage $t+1$. Then from P2 and P4

$$\epsilon(\boldsymbol{e}, s) = \epsilon(\boldsymbol{e}, t) < \phi(\boldsymbol{f}, k, t) \leqslant \phi(\boldsymbol{f}, j, t) < \omega .$$

Thus in this case we also have $\epsilon(\boldsymbol{e}, s) < \phi(\boldsymbol{f}, j, s)$. If follows that

$$\phi(\boldsymbol{d}, m, s) \leqslant \mu(\boldsymbol{d}, s) < \epsilon(\boldsymbol{e}, s) < \phi(\boldsymbol{f}, j, s) .$$

This is a contradiction since the number enumerated in $\boldsymbol{C}$ at stage $s + 1$ should be $\leqslant \phi(\boldsymbol{d}, m, s)$. Therefore m does not exist.

It follows that for all sufficiently large s, $\mu(\boldsymbol{d}, s) < \mu(\boldsymbol{d}, s + 1) < \omega$ cannot arise through C2.1. The only other way $\mu(\boldsymbol{d}, s) < \mu(\boldsymbol{d}, s + 1) < \omega$ can arise is through C5.1 or C6 which require $\boldsymbol{a}(s + 1) < \boldsymbol{b} \frown d_0$. Again this is impossible for all sufficiently large s. This completes the proof of the claim.

Let $\boldsymbol{S}'$ consist of all s such that $\boldsymbol{a}(s + 1) \supset \boldsymbol{b} \frown d_0$, stage $(s + 1, i)$ is associated with $\boldsymbol{c}$, no stage $>s$ has characteristic $< \boldsymbol{b} \frown d_0$, and for all $\boldsymbol{d}$ and $\boldsymbol{t}$

$$(\boldsymbol{d} = \boldsymbol{b} \text{ or } \boldsymbol{d} < \boldsymbol{b} \frown d_0 \;.\wedge.\; t > s) \to \mu(\boldsymbol{d}, t) = \boldsymbol{\mu}(\boldsymbol{d}) .$$

From the claim and the hypothesis of the proposition, $\boldsymbol{S}'$ is infinite. We next claim that for s_0 and s_1 in $\boldsymbol{S}'$, inf $I(i, s_0) =$ inf $I(i, s_1)$. Recall from the construction that inf $I(i, s)$ is the least number exceeding certain numbers p which arise in the stages $(s + 1, j), j < i$. When $\boldsymbol{a}(s + 1) \supset \boldsymbol{b}$ some of the numbers p have the form $\mu(\boldsymbol{d}, s)$, $\boldsymbol{d} < \boldsymbol{b}$, which are the same for $s = s_0$ as for $s = s_1$ by choice of $\boldsymbol{S}'$. The other numbers p derive from the proper initial segments of $\boldsymbol{b}$ and are the same for $s = s_0$ as for $s = s_1$ by P6. For example suppose $\boldsymbol{d} \frown (f, j) \subset \boldsymbol{b}$ then by C6 inf $I(i, s)$ must exceed $\epsilon(\boldsymbol{d}, s)$ and also $\phi(\boldsymbol{d}, k, s)$ for each $k < j$ such that $\phi(\boldsymbol{d}, k, s) < \omega$. By P6.5, $\epsilon(\boldsymbol{d}, s_0) = \epsilon(\boldsymbol{d}, s_1)$ and $\phi(\boldsymbol{d}, k, s_0) = \phi(\boldsymbol{d}, k, s_1)$ for each $k < j$, $k \notin D_0(\boldsymbol{d})$. This establishes the claim.

On the other hand from P7 it follows that sup $I(i, s)$ is unbounded for $s \in \boldsymbol{S}'$. Now recall the conditions that p must satisfy from C2.1. Note from the first claim that $\mu(\boldsymbol{c}, s) = \boldsymbol{\mu}(\boldsymbol{c}) < \omega$ for all sufficiently large s. Note, also from our first claim, that

$$\sup \{\mu(\boldsymbol{d}, s)\colon \epsilon(\boldsymbol{d}, s) < \omega \;.\wedge.\; \boldsymbol{c} \frown (f, n) \subset \boldsymbol{d} \text{ or } \boldsymbol{d} < \boldsymbol{c} \frown (f, n)\}$$

is fixed for all sufficiently large s. Since $\phi(\boldsymbol{c}, n, s + 1) = \omega$ for each $s \in \boldsymbol{S}'$, either $A(x) \neq \boldsymbol{\theta}_n(x)$ for some $x \leqslant \boldsymbol{\mu}(\boldsymbol{c})$, or $\boldsymbol{\theta}_n \boldsymbol{\mu}(\boldsymbol{c}) = \omega$, or $B^i_n(x) \neq \psi^i_n(x)$ for some $x \leqslant \boldsymbol{\theta}_n \boldsymbol{\mu}(\boldsymbol{c})$ and some $i \leqslant 1$, or $\psi^i_n(x) = \omega$ for some $x \leqslant \boldsymbol{\theta}_n \boldsymbol{\mu}(\boldsymbol{c})$ and some $i \leqslant 1$. In each of these cases either $A \neq \boldsymbol{\theta}_n$, or $B^0_n \neq \psi^0_n$, or $B^1_n \neq \psi^1_n$.

Proof of P10. Let $\epsilon(\boldsymbol{c}, t)$ be enumerated in A at stage $t+1$. Let $n \leqslant O(\boldsymbol{b})$ and $n \in D_0(\boldsymbol{b})$. Let $\boldsymbol{b} \frown (f, m) \subset \boldsymbol{c}$ where $m \leqslant n$, or $\boldsymbol{b} = \boldsymbol{c}$. We have to show that $\theta_n(\epsilon(\boldsymbol{c}, t), t) \leqslant \phi(\boldsymbol{b}, n, t) < \omega$. Suppose $\boldsymbol{b} \frown (f, n) \subset \boldsymbol{c}$. Then from P4(1) and P2(2) either $n \leqslant O(\boldsymbol{c})$, $n \notin D_0(\boldsymbol{c})$ and $\phi(\boldsymbol{c}, n, t) < \phi(\boldsymbol{b}, n, t) < \omega$, or there exist $\boldsymbol{b}'$ and $n' < n$ such that $n' \leqslant O(\boldsymbol{b}')$, $n' \notin D_0(\boldsymbol{b}')$ and $\boldsymbol{b} \frown (f, n) \subset \boldsymbol{b}' \frown (f, n') \subset \boldsymbol{c}$. From P4(1) and P2(2), $\phi(\boldsymbol{b}', n', t) < \phi(\boldsymbol{b}, n, t) < \omega$. Thus we can ignore the case in which $\boldsymbol{b} \frown (f, n) \subset \boldsymbol{c}$. We treat the case in which $\boldsymbol{b} \frown (f, m) \subset \boldsymbol{c}$ where $m < n$, and indicate in parentheses the changes necessary for the case in which $\boldsymbol{b} = \boldsymbol{c}$. From P4(1) $\phi(\boldsymbol{b}, n, t) < \omega$. Let $\phi(\boldsymbol{b}, n, t)$ be assigned at stage $u+1$. Then $\epsilon(\boldsymbol{c}, t) = \epsilon(\boldsymbol{c}, u)$, otherwise $\phi(\boldsymbol{b}, n, u+1)$ would be destroyed at the stage where $\epsilon(\boldsymbol{c}, t)$ is assigned (where $\epsilon(\boldsymbol{c}, u)$ is destroyed). Note that $\phi(\boldsymbol{b}, m, u) < \omega$ because the definition of $\phi(\boldsymbol{b}, m, u+1)$ has priority over that of $\phi(\boldsymbol{b}, n, u+1)$. Also $\epsilon(\boldsymbol{c}, t) = \epsilon(\boldsymbol{c}, u) < \phi(\boldsymbol{b}, m, u) \leqslant \mu(\boldsymbol{b}, u)$ because $\phi(\boldsymbol{b}, m, u)$ must have been assigned after $\epsilon(\boldsymbol{c}, u)$. From the application of C2.1 at stage $u+1$ we see that $\theta_n(\epsilon(\boldsymbol{c}, u), u) < \phi(\boldsymbol{b}, n, t)$ and for each $z \leqslant \theta_n(\epsilon(\boldsymbol{c}, u), u)$

$$B_n^i(z, u) = \Psi_n^i(z, u), \qquad \Psi_n^i(z, u) < \phi(\boldsymbol{b}, n, u+1) = \phi(\boldsymbol{b}, n, t) \quad \text{for } i = 0, 1.$$

The desired conclusion follows provided that no number $\leqslant \phi(\boldsymbol{b}, n, t)$ is enumerated in $\boldsymbol{A} \cup \boldsymbol{C}$ at a stage $> u$ and $\leqslant t$.

Suppose for proof by contradiction that some number $\leqslant \phi(\boldsymbol{b}, n, t)$ is enumerated in $\boldsymbol{A} \cup \boldsymbol{C}$ at stage $v+1$, $u < v < t$, then the number must be enumerated in $\boldsymbol{A}$. Otherwise $\phi(\boldsymbol{b}, n, u+1)$ would be destroyed at stage $v+1$ contrary to the choice of u. Let $\epsilon(\boldsymbol{d}, v)$ be enumerated in A at stage $v+1$. Since $\epsilon(\boldsymbol{d}, v) \leqslant \phi(\boldsymbol{b}, n, v)$, from P2 $\boldsymbol{b} \frown (f, n) \not< \boldsymbol{d}$. From P6(3) if $\boldsymbol{b} \frown w \subset \boldsymbol{d}$, then $\boldsymbol{b} \frown w \subset \boldsymbol{a}(t+1)$, whence $\boldsymbol{b} \frown w \not\subset \boldsymbol{d}$. Also $\boldsymbol{d} \not< \boldsymbol{b}$ since $\epsilon(\boldsymbol{c}, v+1) < \omega$. For the same reason, if $\boldsymbol{d} \subset \boldsymbol{b}$ then $\boldsymbol{d} \frown w \subset \boldsymbol{c}$. (If $\boldsymbol{b} = \boldsymbol{c}$, $\boldsymbol{d} = \boldsymbol{b}$ is impossible because $\epsilon(\boldsymbol{d}, v)$ cannot be enumerated in A twice.) But if $\boldsymbol{d} \frown w \subset \boldsymbol{c}$, comparing the stage where $\epsilon(\boldsymbol{c}, v)$ is assigned with stage $v+1$, from P6(3) we deduce that $\boldsymbol{d} \frown w \subset \boldsymbol{a}(v+1)$, contradiction. The only remaining possibility is that $\boldsymbol{b} \frown (f, l) \subset \boldsymbol{d}$ for some $l \leqslant n$. Let x be the least number $> v$ such that $\boldsymbol{a}(\boldsymbol{d}, x+1) = \omega$. Let $\boldsymbol{f}$ and y be defined from $\boldsymbol{d}$, v and x as $\boldsymbol{d}$ and s are defined from $\boldsymbol{c}$, t and u in the statement of P5. If $\boldsymbol{f} \subsetneqq \boldsymbol{b}$ then $y \leqslant t$. Further no stage $s+1$, $x \leqslant s < y$, has characteristic $< \boldsymbol{f}$ since $\epsilon(\boldsymbol{c}, y) = \epsilon(\boldsymbol{c}, u)$. Now from P5 $\boldsymbol{a}(y+1) < \boldsymbol{b}$ whence $\epsilon(\boldsymbol{c}, u)$ is destroyed before stage $t+1$, contradiction. Thus $\boldsymbol{b} \subset \boldsymbol{f}$.

We claim that there exists z, $v < z \leqslant x$, such that stage $z+1$ pertains to $\epsilon(\boldsymbol{d}, v)$ and one of the following possibilities holds

(i) C1.1 occurs, $\boldsymbol{b} \frown (f, l) \subset \boldsymbol{a}(\boldsymbol{d}, z)$ and $\nu(\boldsymbol{d}, z) \leqslant l$,

(ii) C1.3 occurs and there exists b' such that $b \subsetneqq b' \frown (f, \nu(d, z)) \subset$ $\subset a(d, z)$ and $\nu(d, z) \leqslant l$,

(iii) C1.2 occurs, $l = n = \nu(d, z)$ and b is the greatest $b' \subset a(d, z)$ such that $b' \frown (f, l) \subset a(d, z)$,

(iv) C1.1 occurs, $a(d, z) = b$ and $\nu(d, z) \leqslant n$.

To prove the claim choose b' and $l' \leqslant l$ if possible such that $b \subset b'$ and $b' \frown (f, l') \subset a(d, x)$. Minimize l' and then maximize the length of b'. Let z_0 be the least number $> v$ such that $\nu(d, z_0 + 1) > l'$.

Suppose $\nu(d, z_0) < l'$ and $\nu(d, z_0 + 1) < \omega$; then either $l' \in D_0(a(d, z_0+1))$ or $l' \nleqslant O(a(d, z_0 + 1))$ since $b' \frown (f, l') \subset a(d, z_0 + 1)$. In either case there exists $l'' < l'$ and b'' such that $b'' \frown (f, l'') \subset a(d, x)$. But this contradicts the choice of l', whence $\nu(d, z_0) < l'$ implies $\nu(d, z_0 + 1) = \omega$ and $z_0 = x$. If $\nu(d, z_0) < l'$ and C1.1 occurs at stage $z_0 + 1$, then we can take $z = x$ and (i) holds. If $\nu(d, z_0) < l'$ and either C1.2 or C1.3 occurs at stage $z_0 + 1$, then $\nu(d, z_0)$ has the defining properties of l', contradiction. (In the case of C1.2 recall that $f \supset b$ whence $b \subsetneqq f \frown (f, \nu(d, z_0)) \subset a(d, z_0)$.)

Now suppose $\nu(d, z_0) = l'$. Recall that

$$b \subsetneqq b' \frown (f, l') \subset a(d, x) \subset a(d, z_0) .$$

If C1.3 occurs at stage $z_0 + 1$, (ii) holds with $z = z_0$. If C1.1 occurs at stage $z_0 + 1$, (i) holds with $z = z_0$. Now let C1.2 occur at stage $z_0 + 1$ and consider cases as follows. If $l' < l$ and $z_0 < x$ then $b' \frown (f, l') \subsetneqq a(d, z_0 + 1) \frown$ $\frown (f, l')$. This implies that $b' \subsetneqq b'' \frown (f, l'') \subset a(d, z_0 + 1)$ for some $l'' < l'$ and some b'', contradiction. If $l' < l$ and $z_0 = x$, then $f = b'$ and $b \frown (f, l)$ $\subset f \frown (f, l') \subset a(d, x)$. Since $a(d, x + 1) = \omega$

$$x > l' \wedge x \leqslant O(f) \;.\!\!\rightarrow\!. \; x \in D_0(f) .$$

Again the choice of l' is contradicted. If $l' = l$ and $b \subsetneqq b'$, then the choice of l' can be refuted as before. If $l' = l$, $b = b'$ and $l < n$, then $z_0 = x$ since otherwise $a(d, z_0 + 1) = b'$. We have a contradiction because n is eligible as a value for $\nu(d, z_0 + 1)$. Finally, if $l' = l$, $b = b'$ and $l = n$, then we can take $z = x$ and (iii) holds.

We conclude that if b' and l' exist then z exists. Suppose now that there are no such b' and l' then clearly $a(d, x) = b$ and so $f = b$ also. Stage $x + 1$ pertains to $\epsilon(d, v)$, otherwise $\phi(b, n, v)$ would be destroyed at stage $x + 1$ through $a(x + 1)$ being $< b$ or some number $\leqslant \epsilon(d, v)$ being enumerated in C. Since $a(d, x) = f$ either C1.1 or C1.3 occurs at stage $x + 1$. Since $a(d, x + 1) = \omega$ either C1.1 occurs or $\nu(d, x) \geqslant n$. Now consider cases. Let $l = n$ then $\phi(b, n, u + 1)$ is destroyed at stage $x' + 1$

through C1.2 where x' is the least number $>v$ such that $\boldsymbol{a}(\boldsymbol{d}, x'+1) = \boldsymbol{b}$. Since $\nu(\boldsymbol{d}, x') = l \leqslant O(\boldsymbol{b})$ we have $x' < t$ which contradicts $\phi(\boldsymbol{b}, n, u+1) = \phi(\boldsymbol{b}, n, t)$. Thus $l < n$. Suppose $\nu(\boldsymbol{d}, x) \leqslant n$ and C1.1 occurs at stage $x + 1$ then (iv) holds with $z = x$. If $\boldsymbol{a}(\boldsymbol{d}, x) > n$ then for some $x'', x' < x'' < x$, $\nu(\boldsymbol{d}, x'') = n$ and $\phi(\boldsymbol{b}, n, u+1)$ is destroyed through C1.3 at stage $x'' + 1 \leqslant t$, contradiction. We have shown that z exists whether $\boldsymbol{b}'$ and l' exist or not.

Now $z < t$ because $\nu(\boldsymbol{d}, z) \leqslant O(\boldsymbol{b})$. If (iii) holds then $\phi(\boldsymbol{b}, n, u+1)$ is destroyed at stage $z + 1 \leqslant t$, contradiction. In each of the other cases there exists $\boldsymbol{e}$ and $j = \nu(\boldsymbol{d}, z) \leqslant n$ such that $j \leqslant O(\boldsymbol{e})$, $j \notin D_0(\boldsymbol{e})$, $\phi(\boldsymbol{e}, j, z)$ is enumerated in $\boldsymbol{C}$ at stage $z + 1$ and either $\boldsymbol{e} = \boldsymbol{b}$ or $\boldsymbol{e} \supset \boldsymbol{b}^\frown(f, l)$. From P4 and P2(2)

$$\phi(\boldsymbol{e}, j, v) \leqslant \phi(\boldsymbol{b}, n, v) = \phi(\boldsymbol{b}, n, u+1) .$$

Since $\phi(\boldsymbol{b}, n, t) = \phi(\boldsymbol{b}, n, u+1)$ and $z < t$, $\phi(\boldsymbol{e}, j, z) \neq \phi(\boldsymbol{e}, j, v)$. Let z' be the least number $>v$ such that $\phi(\boldsymbol{e}, j, z'+1) = \omega$. Now $\phi(\boldsymbol{e}, j, v)$ is destroyed through C1 at stage $z' + 1$. Otherwise $\phi(\boldsymbol{b}, n, u+1)$ would also be destroyed at stage $z' + 1$. Let stage $z' + 1$ pertain to $\epsilon(\boldsymbol{d}', z')$; then $\nu(\boldsymbol{d}, z) = j = \nu(\boldsymbol{d}', z')$ and $z' < z$. Therefore $\boldsymbol{d} \neq \boldsymbol{d}'$. Since $\boldsymbol{d}' \supset \boldsymbol{b}$ and $\boldsymbol{d}' \not\supset \boldsymbol{b}^\frown d_0$, $\epsilon(\boldsymbol{d}', z')$ cannot be enumerated in A after stage $v + 1$. Also $\epsilon(\boldsymbol{d}', z')$ cannot be enumerated in A before stage $v + 1$, because this would be inconsistent with $\boldsymbol{a}(v+1) \supset \boldsymbol{b}^\frown(f, l)$. This contradiction completes the proof of P10.

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Measurable cardinals and analytic games

by

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Introduction. A subset P of ω^ω is *determinate* if, in the sense of [5] the game $G_\omega(P)$ is determined. The assumption that every projective set is determinate implies that every projective set is Lebesgue measurable (see [6]) and leads to a complete solution to the problem of reduction and separation principles in the classical and effective projective hierarchies [1], [4]. Because if these and other consequences it would be pleasant to have a proof that every projective set is determinate. The best available result is that every $F_{\sigma\delta}$ is determinate [2]. It is not provable in Zermelo–Fraenkel set theory that every analytic (Σ_1^1) set is determinate [5]. (1)

We assume the existence of a measurable cardinal and prove that every analytic set is determinate. Our proof is fairly simple and makes a very direct use of the large cardinal assumption (we present it in terms of a Ramsey cardinal) and the fact that open games are determined. We believe that larger cardinals will yield a generalization of our proof to all projective sets. The assumption that measurable cardinals exist is known not to imply even that all Δ_2^1 sets are determinate. (This follows from [1], [4] and work of Silver.)

§ 1. Definitions. (For more information on the analytical hierarchy see [7], [8]; on infinite games see [5]; on large cardinals see [10], [11].)

Let ω be the set of all natural numbers. If $f\colon \omega \to A$, the function $\bar{f}$ is defined by setting $\bar{f}(n)$ equal to the sequence $\langle f(0), f(1), \ldots, f(n-1)\rangle$ of the first n values of f. Let Seq be the set of all finite sequences of natural numbers. Let $n \to k_n$ be some enumeration of Seq with the property that k_n has length $\leqslant n$. The *Kleene–Brouwer ordering* of Seq is defined by

$$\bar{f}(m) < \bar{g}(n) \leftrightarrow \begin{cases} \bar{f}(m) \text{ is a proper extension of } \bar{g}(n)\,, \\ \text{or at the least } p \text{ for which } f(p) \neq g(p)\,, \\ f(p) < g(p)\,. \end{cases}$$

(1) Harvey Friedman (unpublished) has shown that the determinateness of Borel sets cannot be proved in Zermelo set theory. Whether it can be proved in Zermelo-Fraenkel set theory remains open.

Let $R(i, j, k)$ be a relation in Seq^3 and let f and g map ω into ω. A sequence $\bar{k}(n)$ is *secured* with respect to f, g, and R if

$$(\exists m \leqslant n) R(\bar{f}(m), \bar{g}(m), \bar{k}(m)) .$$

A fundamental fact is that $(h)(\exists n) R(\bar{f}(n), \bar{g}(n), \bar{h}(n))$ holds if and only if the Kleene–Brouwer ordering of the sequences unsecured with respect to f, g, and R is a well-ordering.

Let $\varkappa$ be an uncountable cardinal number. Let $\varkappa^{[n]}$ be the set of all subsets of $\varkappa$ of cardinality n. Let $\mathcal{F}$ be a set such that, if $F \in \mathcal{F}$, there is an $n < \omega$ such that $F\colon \varkappa^{[n]} \to \omega$. $X \subseteq \varkappa$ is a *homogeneous set* for $\mathcal{F}$ if, for each $F \in \mathcal{F}$ with $F\colon \varkappa^{[n]} \to \omega$ and elements a and b of $\varkappa^{[n]}$ which are subsets of X, $F(a) = F(b)$. If α is an ordinal number, $\varkappa \to (\alpha)^{<\omega}$ means that, for every countable $\mathcal{F}$, there is a homogeneous set for $\mathcal{F}$ of order type α. $\varkappa$ is a *Ramsey cardinal* if $\varkappa \to (\varkappa)^{<\omega}$. Every measurable cardinal is Ramsey.

Let A and B be sets and let $C \subseteq A^\omega \times B^\omega$. The (Gale–Stewart) infinite game defined by A, B, and C is given as follows: Players I and II move alternately, choosing elements of A and B respectively at each turn. In this way functions $f\colon \omega \to A$ and $g\colon \omega \to B$ are produced. I wins if $\langle f, g\rangle \in C$. I has a winning strategy if there is a function which, given the first n plays of II, gives an $(n+1)$st play for I in such a manner that I wins whatever II plays. A game is *determined* if either I or II has a winning strategy. Let $A = B = \omega$. The game defined by A, B, and C is *analytic* (Σ_1^1) if there is a relation $R(i, j, k)$ in Seq^3 such that

$$C = \{\langle f, g\rangle\colon (\exists h)(n) R(\bar{f}(n), \bar{g}(n), \bar{h}(n))\}$$

(i.e., if C is the projection of a closed set in $(\omega^\omega)^3$ under the product topology.) The game is *Borel* if some R satisfies the condition above and furthermore there is a countable ordinal α such that for no f and g is the Kleene–Brouwer ordering of sequences unsecured with respect to f, g, and $\mathrm{Seq}^3 - R$ a well-ordering of order type $> \alpha$.

§ 2. The determinateness of analytic sets.

THEOREM. (a) *If* $(\exists \varkappa)[\varkappa \to (\omega_1)^{<\omega}]$ *every analytic game is determined.*

(b) *If* $(\exists \varkappa)(\alpha < \omega_1)[\varkappa \to (\alpha)^{<\omega}]$ *every Borel game is determined.*

Proof. Only (a) will be proved, as the prove of (b) is similar. Let $\varkappa \to (\omega_1)^{<\omega}$. Let $R \subseteq \mathrm{Seq}^3$. I and II, moving alternately, produce functions $f\colon \omega \to \omega$ and $g\colon \omega \to \omega$. II wins if

$$(h)(\exists n) R(\bar{f}(n), \bar{g}(n), \bar{h}(n)) .$$

Call this Game 1. We consider a second game (Game 2). I picks $f\colon \omega \to \omega$ and II picks not only $g\colon \omega \to \omega$ but also $G\colon \omega \to \varkappa$. (At stage n, II selects the ordered pair $\langle g(n), G(n)\rangle$.) Via the enumeration k_n, G induces a map

G^*: Seq$\to\varkappa$. II wins Game 2 if $G^*(k) = 0$ for all k secured with respect to f, g, and R and G^* preserves the Kleene–Brouwer ordering on the unsecured sequences.

LEMMA 1. *Game 2 is determined.*

Proof. This is the Gale–Stewart result for open games [3]. If I has no winning strategy, II makes the least plays at each move such that I still has no winning strategy. Since II wins provided that he has not lost by some finite stage, this strategy wins for II. (There is a concealed use of the Axiom of Choice in this proof which can be eliminated.)

LEMMA 2. *If* II *has a winning strategy for Game* 2, II *has a winning strategy for Game* 1.

Proof. If II wins Game 2, the Kleene–Brouwer ordering on the unsecured sequences is a well-ordering. (The converse of Lemma 2 can be proved without assuming $\varkappa\to(\omega_1)^{<\omega}$ but only that $\varkappa$ is uncountable.)

LEMMA 3. *If* I *has a winning strategy for Game* 2, I *has a winning strategy for Game* 1.

Proof. Let $f(n) = f^{**}(\bar{g}(n), \bar{G}(n))$ be a winning strategy for I for Game 2. Let $\bar{f}(n)$ and $\bar{g}(n)$ be any finite sequences. Let $k_{i_1}, \ldots, k_{i_m}$ for $i_j \leqslant n$ be the sequences unsecured with respect to f', g', and R for any f', g' agreeing with $\bar{f}(n)$, $\bar{g}(n)$ respectively. (Since k_j has length $\leqslant j$, its being secured depends only on $\bar{f}(j)$ and $\bar{g}(j)$.) Let $Q \in \varkappa^{[m]}$. There is a unique sequence $\bar{G}(n)$ such that $G(p) = 0$ if k_p is secured and G^* maps $\{k_{i_1}, \ldots, k_{i_m}\}$ into Q so as to preserve the Kleene–Brouwer ordering. We define

$$F_{\bar{f}(n),\bar{g}(n)}\colon \varkappa^{[m]}\to\omega \text{ by } F_{\bar{f}(n),\bar{g}(n)}(Q) = f^{**}(\bar{g}(n), \bar{G}(n))$$

where $\bar{G}(n)$ is the sequence defined above.

Let $\mathcal{F} = \{F_{\bar{f}(n),\bar{g}(n)}\colon \bar{f}(n), \bar{g}(n) \in \text{Seq}\}$. Let X be a homogeneous set for $\mathcal{F}$ order type ω_1. We define a strategy f^* for I for Game 1 inductively by

$$f^*(\bar{g}(n)) = F_{\bar{f}(n),\bar{g}(n)}(Q)$$

where $\bar{f}(n)$ is the result of applying f^* to the first n plays $g(p)$ and $Q \in \varkappa^{[m]}$ is any subset of X. If f^* is not a winning strategy, there is a play g such that, for the play f given by f^*, the Kleene–Brouwer ordering of the sequences unsecured with respect to f, g, and R is a well-ordering. Let G be such that $G(n) = 0$ for k_n secured and G^* maps the unsecured sequences in an order preserving manner into X. f is then the play according to f^{**} against g and G, and so we have a contradiction.

§ 3. Further results. Combining our argument with the methods of [11], we can prove that, if $(\exists\varkappa)[\varkappa\to(\omega_1)^{<\omega}]$, then every Σ^1_1 game has a Δ^1_3 winning strategy. (We owe this observation to Solovay.)

For sets A and B, give $A^\omega \times B^\omega \times \omega^\omega$ the product topology, where A, B, and ω are given the discrete topology. Let $C \subseteq A^\omega \times B^\omega$ be the projection of a closed set in $A^\omega \times B^\omega \times \omega^\omega$. Our argument can be used to show that the game defined by A, B, and C is determined, on the assumption that a Ramsey cardinal larger than the cardinals of A and B exists.

In [5], using a theorem of Davis [2], Mycielski shows that, if every analytic game is determined, then every uncountable $CA(\Pi^1_1)$ set has a perfect subset. Our theorem thus gives a new and very different proof of a result of Solovay [12] and Mansfield: If $(\exists\varkappa)[\varkappa \to (\omega_1)^{<\omega}]$ then every uncountable CA set has a perfect subset.

By a theorem of Shoenfield [8], the assertion that all Borel games are determined relativizes to L, the universe of constructible sets. Hence we have that, if $(\exists\varkappa)(\alpha < \omega_1)[\varkappa \to (\alpha)^{<\omega}]$, then "all Borel games are determined" holds in L. We should note that Silver has shown that $(\exists\varkappa)(\alpha < \omega_1)[\varkappa \to (\alpha)^{<\omega}]$ relativizes to L.

§ 4. What games are determined? We believe that the best way of approaching this problem is to see what games one can prove to be determined using plausible large cardinal assumptions. Nevertheless, some guesses may prove useful.

Addison and Moschovakis [1] suggest that *definability* may be the crucial property which guarantees determinateness. Their "Axiom of Definable Determinateness" asserts that, if $A = B = \omega$ and C is ordinal definable from a member of ω^ω, then the game defined by A, B, and C is determined. A problem with this axiom is that there is little hope at present of proving it from large cardinal assumptions or of using anything like its full strength in deducing consequences (in both cases, because of the unmanageable "ordinal definable"). A weaker proposition suggested by Takeuti and by Solovay, which may not have these defects, is that the Axiom of Determinateness holds in the smallest transitive class containing all ordinals and all subsets of ω and satisfying the axioms of set theory.

Another approach is to consider games of arbitrary length. For any ordinal α, sets A, B, and $C \subseteq A^\alpha \times B^\alpha$ define a game of length 2α in the obvious way. (See § 7 of [5].) The proposition that all games of length 2 are determined is equivalent to the Axiom of Choice [5]. Give A and B the discrete topology and $A^\alpha \times B^\alpha$ the product topology. If C is open, the game determined by A, B, and C is an *open game of length* 2α. The proposition that all open games of every length are determined is inconsistent. Even if $A = B = \omega$, the Axiom of Choice can be used to construct an undetermined open game of length ω_1. However, it apparently is possible that every open game of length $< \omega_1$ with $A = B = \omega$ is determined. Let P_n be the proposition that all open games of length $\omega \cdot n$ where $A = B = \omega$ are determined. It is easy to show that, for $n < \omega$, P_{n+1} is equivalent to the assertion that all Σ^1_n sets are determinate. This suggests that length may well be a sufficient compensation for what we lose when we restrict ourselves to open games.

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ENUMERABLE SETS ARE DIOPHANTINE

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Hilbert's tenth problem has the following formulation (cf. [1]):

Specify a procedure which in a finite number of steps enables one to determine whether or not a given diophantine equation with an arbitrary number of indeterminates and with rational integer coefficients has a solution in rational integers.

When this problem was formulated, one could only speak of a positive solution to the problem since a precise concept of algorithm had not yet been developed. The appearance of this concept has made algorithmic undecidability proofs possible.

Undecidable problems were first found in mathematical logic, then in algebra and number theory. In particular, M. Davis, H. Putnam, and Julia Robinson [2] proved that there is no algorithm which allows one to determine the existence of integer solutions to so-called *exponential diophantine* equations, i. e. equations which can be constructed from natural numbers and variables by means of addition, multiplication, and exponentiation. Using this result and the work of Robinson [3], we will show that Hilbert's tenth problem is also algorithmically undecidable.

1. Lower-case Latin letters except for i and j will be used throughout as variables ranging over positive integers; i and j will vary over the nonnegative integers.

We will say that a predicate** $\mathcal{P}(u, v)$ *has exponential growth* if $\mathcal{P}(u, v)$ implies the inequality $v \leq u^u$ and for each k there exist u, v such that $\mathcal{P}(u, v)$ and $u^k < v$.

A predicate $\mathcal{S}(x_1, \cdots, x_n)$ is called *diophantine* if one can find a polynomial*** M such that $\mathcal{S}(x_1, \cdots, x_n)$ holds if, and only if there exist integers $y_1, \cdots, y_j$ such that $M(x_1, \cdots, x_n, y_1, \cdots, y_j) = 0$.

From the work of Davis, Putnam, and Robinson cited above, it follows that if even one diophantine predicate had exponential growth, then every enumerable**** predicate would be diophantine.

The predicate "v is the $2u$th Fibonacci number" has exponential growth. We will show that it is diophantine. This will prove the following assertion:

Every enumerable predicate is diophantine.

Moreover, *for each n one can find an $(n+1)$-place diophantine predicate $\mathfrak{U}_n(x_1, \cdots, x_n, s)$, such that any enumerable n-place predicate can be obtained from $\mathfrak{U}_n$ by fixing a value for s.*

Since there exist enumerable but algorithmically undecidable predicates (cf. [4, 5]), the following assertion holds:

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* *Editor's note.* The present translation incorporates suggestions made by the author.

** We consider predicates to be properties or relations which are representable by formulas in the language of formal arithmetic.

*** Without loss of generality we may assume that the degree of the polynomial M is not greater than 4.

**** A predicate is *enumerable* if one can specify an effectively computable sequence of n-tuples of numbers containing all and only those n-tuples for which the predicate holds.

*There is no algorithm which enables one to determine whether or not an arbitrary diophantine equation has a solution.**

Combining our result with the results of [6] we obtain the following corollaries:

1) *One can specify a fifth-degree polynomial* $Q(y_1, \cdots, y_k, z)$ *with integer coefficients such that any enumerable set* $\mathfrak{M}$ *of natural numbers* (for example, the set of prime numbers) *coincides with the set of natural values of the polynomial* $Q(y_1, \cdots, y_k, a_{\mathcal{M}})$, *where* $a_{\mathcal{M}}$ *is a certain number effectively constructed for the set* $\mathfrak{M}$.

2) *One can specify polynomials* $R(y_1, \cdots, y_k, z)$ *and* $S(y_1, \cdots, y_k, z)$ *with integer coefficients such that any enumerable set* $\mathfrak{M}$ *of integers coincides with the set of integral values of the ratio*

$$\frac{S(y_1, \ldots, y_k, a_{\mathcal{M}})}{R(y_1, \ldots, y_k, a_{\mathcal{M}})},$$

where $a_{\mathcal{M}}$ *is a certain number effectively constructed for the set* $\mathfrak{M}$.

3) *One can specify a fifth-degree polynomial* $D(y_1, \cdots, y_k)$ *with integer coefficients such that there is no algorithm which enables one to determine for a given number n whether or not the equation* $D(y_1, \cdots, y_k) = n$ *has a solution.*

2. Definition 1. $\phi_0 = 0$, $\phi_1 = 1$, $\phi_{n+1} = \phi_n + \phi_{n-1}$. ϕ_j is called the jth *Fibonacci number.*

Lemma 1. $\phi_{2(n+1)} = 3\phi_{2n} - \phi_{2(n-1)}$.

Corollary. $\phi_{2(n-1)} = 3\phi_{2n} - \phi_{2(n+1)}$.

Lemma 2. $\phi_{2(k+j)} \equiv -\phi_{2(k+1-j)} \pmod{\phi_{2k} + \phi_{2k+2}}$, $0 \leq j \leq k+1$ (induction on j; induction step by Lemma 1 and its corollary).

Lemma 3. $\phi_{2(2k+1+j)} \equiv \phi_{2j} \pmod{\phi_{2k} + \phi_{2k+2}}$ (induction on j; basis by Lemmas 2 and 1, induction step by Lemma 1).

Lemma 4. $\phi_{2((2k+1)i+j)} \equiv \phi_{2j} \pmod{\phi_{2k} + \phi_{2k+2}}$ (induction on i; induction step by Lemma 3).

Corollary to Lemmas 4 and 2.

$$\varphi_{2((2k+1)i+j)} \equiv \begin{cases} \varphi_{2j} & \text{for } 0 \leqslant j \leqslant k, \\ \varphi_{2k} + \varphi_{2k+2} - \varphi_{2(2k+1-j)} & \text{for } k+1 \leqslant j \leqslant 2k \end{cases}$$

mod $(\phi_{2k} + \phi_{2k+2})$.

Definition 2. For each $m \geq 2$, $\psi_{m,0} = 0$, $\psi_{m,1} = 1$, $\psi_{m,n+1} = m\psi_{m,n} - \psi_{m,n-1}$.

Lemma 5. *If* $m \geq 2$, $d \mid (m-3)$, *then* $\psi_{m,j} \equiv \phi_{2j} \pmod{d}$ (induction on j; induction step by Lemma 1).

Lemma 6. *If the numbers* k, m, n, v *are such that* $m \geq 2$, $v < \phi_{2k+1}$, $(\phi_{2k} + \phi_{2k+2}) \mid (m-3)$, $\psi_{m,n} \equiv v \pmod{\phi_{2k} + \phi_{2k+2}}$, *then there exist numbers* i, j *such that* $v = \phi_{2j}$, $n = (2k+1)i + j$ (by Lemma 5 and the corollary to Lemmas 4 and 2).

Lemma 7. *If* $m \geq 2$, $l \mid (m-2)$, *then* $\psi_{m,j} \equiv j \pmod{l}$ (induction on j).

Lemma 8. $\phi_{i+1}^2 - \phi_i\phi_{i+1} - \phi_i^2 = (-1)^i$ (induction on i).

Lemma 9. *If the numbers* j, k *are such that* $(k^2 - jk - j^2)^2 = 1$, *then there exists a number* i *such that* $j = \phi_i$, $k = \phi_{i+1}$ (complete induction on $j + k$: if $j > 0$, then $j \leq k$; set $j_1 = k - j$, $k_1 = j$, then $(k_1^2 - j_1k_1 - j_1^2)^2 = 1$, $j_1 + k_1 < j + k$).

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*It does not matter here whether we ask for integer, positive integer, or nonnegative integer solutions since it is known that these three decision problems are equivalent (cf. [4,5]).

Lemma 10. *For each* $m \geq 2$, $\psi^2_{m,i+1} - m\psi_{m,i}\psi_{m,i+1} + \psi^2_{m,i} = 1$ (induction on i).

Lemma 11. *If the numbers* j, k, m *are such that* $m \geq 2$, $j \leq k$, $k^2 - mjk + j^2 = 1$, *then there exists a number* i *such that* $j = \psi_{m,i}$, $k = \psi_{m,i+1}$ (analogous to Lemma 9).

Lemma 12. $\gcd(\phi_i, \phi_j) = \phi_{\gcd(i,j)}$ (proved in [7]).

Corollary. $\phi_n \mid \phi_{jn}$.

Lemma 13. *If* p *and* q *are prime numbers,* $p \mid \phi_n$, $q \neq p$, *then* $p\phi_n \nmid \phi_{qn}$. *If* p *is a prime,* $p \neq 2$, $p \mid \phi_n$, *then* $p\phi_n \mid \phi_{pn}$, *but* $p^2\phi_n \nmid \phi_{pn}$.

If p *is a prime,* $p \neq 2$, $p \mid \phi_n$, *then* $p\phi_n \mid \phi_{pn}$, *but* $p^2\phi_n \nmid \phi_{pn}$.

If $2 \mid \phi_n$, $4 \nmid \phi_n$, *then* $4\phi_n \mid \phi_{2n}$, *but* $8\phi_n \nmid \phi_{2n}$.

If $4 \mid \phi_n$, *then* $2\phi_n \mid \phi_{2n}$, *but* $4\phi_n \nmid \phi_{2n}$.

The lemma is proved in [7].

Lemma 14. *If* p *is a prime,* $p \mid \phi_n$, $p \nmid r$, *then* $p\phi_n \nmid \phi_{rn}$ (induction on the number of prime factors in the number r; induction step by the corollary to Lemma 12 and by Lemma 13).

Lemma 15. *If* p *is prime,* $p \neq 2$, $p \mid \phi_n$, *then* $p^i\phi_n \mid \phi_{p^i n}$, *but* $p^{i+1}\phi_n \nmid \phi_{p^i n}$ (induction on i; induction step by the corollary to Lemma 12 and by Lemma 13).

Lemma 16. *If* $4 \mid \phi_n$, *then* $2^i\phi_n \mid \phi_{2^i n}$, *but* $2^{i+1}\phi_n \nmid \phi_{2^i n}$ (analogous to Lemma 15).

Lemma 17. $\phi_s^2 \mid \phi_{rs}$ *if and only if* $\phi_s \mid r$ (follows from Lemmas 13–16).

Corollary to Lemmas 12 and 17. *If* $\phi_s^2 \mid \phi_t$, *then* $\phi_s \mid t$.

Lemma 18. $2\phi_{2n} < \phi_{2(n+1)} \leq 3\phi_{2n}$ (follows from Lemma 1).

Lemma 19. $n \leq 2^{n-1} \leq \phi_{2n} < 3^n$ (induction on n; induction step by Lemma 18).

3. Theorem. *In order for* v *to be the* $2u$*th Fibonacci number, it is necessary and sufficient that there exist numbers* g, h, l, m, x, y, z *such that:*

$$u \leqslant v < l, \tag{1}$$
$$l^2 - lz - z^2 = 1, \tag{2}$$
$$g^2 - gh - h^2 = 1, \tag{3}$$
$$l^2 \mid g, \tag{4}$$
$$l \mid m - 2, \tag{5}$$
$$(2h + g) \mid (m - 3), \tag{6}$$
$$x^2 - mxy + y^2 = 1, \tag{7}$$
$$l \mid (x - u), \tag{8}$$
$$(2h + g) \mid (x - v). \tag{9}$$

Sufficiency. Suppose that the numbers $u, v, g, h, l, m, x, y, z$ satisfy conditions (1)–(9). By Lemma 9 and (2) it follows that there is a number s such that

$$l = \phi_s. \tag{10}$$

By Lemmas 9 and 8 it follows from (3) that there exists a number k such that

$$h = \phi_{2k}, \quad g = \phi_{2k+1}. \tag{11}$$

From this we have

$$2h + g = \phi_{2k} + \phi_{2k+2}. \tag{12}$$

The corollary to Lemmas 12 and 17 together with (10)–(11), (4) imply

$$l \mid (2k+1). \tag{13}$$

By (1), (4), (11), (5) it follows that

$$2 \le l < \phi_{2k+1}, \quad m \ge 2. \tag{14}$$

By Lemma 11, it follows from (14), (7) that there exists a number n such that

$$x = \psi_{m,n}. \tag{15}$$

By Lemma 6, it follows from (14), (1), (12), (6), (9) that there are numbers i, j such that

$$v = \phi_{2j}, \quad n = (2k+1)\,i + j. \tag{16}$$

By Lemma 7, it follows from (14), (5), (15) that $x \equiv n \pmod{l}$. From this and from (8), (16), (13) it follows that

$$u \equiv j \pmod{l}. \tag{17}$$

By Lemma 19, it follows from (16) that $j \le v$. This together with (1), (17) implies that $u = j$ and, by (16), $v = \phi_{2u}$. Thus, sufficiency is established.

Necessity. Suppose that $v = \phi_{2u}$. By Lemma 19 the first inequality in (1) is fulfilled. Set $l = \phi_{6s+1}$, $z = \phi_{6s}$, where s is large enough so that the second inequality in (1) is also satisfied. By Lemma 8, condition (2) holds. Put $g = \phi_{l(6s+1)}$, $h = \phi_{l(6s+1)-1}$. By Lemma 17, condition (4) is satisfied. By Lemma 12, l is odd since $2 = \phi_3$. Therefore, by Lemma 8, condition (3) is also satisfied. By Lemma 12, $\gcd(h, g) = 1$ and, since l is odd and divides g, we have $\gcd(2h+g, l) = 1$. Therefore, by the Chinese Remainder Theorem, we can find a number m such that conditions (5)–(6) are fulfilled. Set $x = \psi_{m,u}$, $y = \psi_{m,u+1}$. By Lemma 10, condition (7) is satisfied; by Lemma 7, condition (8) holds; by Lemma 5, condition (9) is satisfied. This proves necessity.

Conditions (1)–(9) can easily be replaced by a single diophantine predicate (cf. [4, 5]). Thus, the predicate "v is the $2u$th Fibonacci number" is diophantine. Lemma 19 implies that it has exponential growth.

4. Certain constructions in this paper use the methods of Julia Robinson presented in [8].

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ADDENDUM*

Lemma 17 has a simpler proof.

Lemma A. $\phi_{i+j} = \phi_{i-1}\phi_j + \phi_i\phi_{j+1}$ (induction on j; also proved in [7]).

Lemma B. $\phi_{in+1} \equiv \phi^i_{n+1} \pmod{\phi_n}$ (induction on i; inductive transition by Lemma A).

Lemma C. $\phi_{mn} \equiv m\phi_n\phi^{m-1}_{n+1} \pmod{\phi^2_n}$ (induction on m; inductive transition by Lemmas A and B).

Lemma 17 is an immediate consequence of Lemmas 12 and C.

* *Added in translation.*

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CATEGORICITY IN POWER[1]

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Introduction. A theory, Σ, (formalized in the first order predicate calculus) is *categorical in power* κ if it has exactly one isomorphism type of models of power κ. This notion was introduced by Łoś [9] and Vaught [16] in 1954. At that time they pointed out that a theory (e.g., the theory of dense linearly ordered sets without end points) may be categorical in power $\aleph_0$ and fail to be categorical in any higher power. Conversely, a theory may be categorical in every uncountable power and fail to be categorical in power $\aleph_0$ (e.g., the theory of algebraically closed fields of characteristic 0). Łoś then raised the following question.

Is a theory categorical in one uncountable power necessarily categorical in every uncountable power?

The principal result of this paper is an affirmative answer to that question. We actually prove a stronger result, namely: If a theory is categorical in some uncountable power then every uncountable model of that theory is *saturated.* (Terminology used in the Introduction will be defined in the body of the paper; roughly speaking, a model is saturated, or *universal-homogeneous,* if it contains an element of every possible elementary type relative to its subsystems of strictly smaller power.) It is known[2] that a theory can have (up to isomorphism) at most one saturated model in each power. It is interesting to note that our results depend essentially on an analogue of the usual analysis of topological spaces in terms of their derived spaces and the Cantor-Bendixson theorem.

The paper is divided into five sections.

In §1 terminology and some meta-mathematical results are summarized. In particular, for each theory, Σ, there is described a theory, Σ^*, which has essentially the same models as Σ but is "neater" to work with.

In §2 is defined a topological space, $S(A)$, corresponding to each subsystem, A, of a model of a theory, Σ; the points of $S(A)$ being the "isomorphism types" of elements with respect to A. With each monomorphism (= isomorphic imbedding), $f: A \to B$, is associated a "dual" continuous map, $f^*: S(B) \to S(A)$. Then there is defined for each $S(A)$ a decreasing sequence $\{S^\alpha(A)\}$ of subspaces which is analogous to (but different from)

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(1) Except for minor emendations this paper is identical with the author's doctoral dissertation submitted to the University of Chicago in August 1962.

(2) Cf. [10] where the result was shown to follow from the more general result of [5].

the usual sequence of derived spaces in a topological space([3]). The basic difference is that for us the definition of "derived space" will involve not only $S(A)$ but all of its inverse images under maps of the type, $f^*: S(B) \to S(A)$; that is, not only A but every system A can be imbedded into. It is well known that those topological spaces whose αth derived space vanishes at some ordinal α have particularly simple properties. Similarly, those theories, Σ, such that for some ordinal α, $S^\alpha(A)$ vanishes for every A which is a subsystem of a model of Σ have particularly simple properties. We have chosen to call such theories totally transcendental. Theorem 2.8, which is an analogue of the Cantor-Bendixson theorem, states that totally transcendental theories are characterized by a certain countability condition.

§3 gathers together some results depending on Ramsey's theorem. In particular, Theorem 3.8 states that any theory categorical in an uncountable power is totally transcendental. Much of §3 is related to the results of Ehrenfeucht and Mostowski [3] and Ehrenfeucht [1] and [2].

Some properties of models of totally transcendental theories are established in §4. These have to do with the existence of prime models and the existence of sets of *indiscernible* elements.

Finally §5 applies the results of the preceding sections to solve the problem of Łoś.

This paper was written while the author was at the University of California at Berkeley. It is a pleasant duty to acknowledge the more than usual debt he owes for the advice and encouragement of Professor S. MacLane of the University of Chicago and Professor R. L. Vaught of the University of California.

1. **Preliminaries.** Ordinals are defined so that each ordinal is equal to the set of smaller ordinals. Cardinals are those ordinals not set-theoretically equivalent to any preceding ordinal. We use the Greek letters $\alpha, \beta, \gamma, \cdots$ to denote ordinals, reserving δ for limit ordinals; λ and κ will always denote cardinals and m and n non-negative integers. κ^+ denotes the least cardinal $> \kappa$. The cardinality of a set X is denoted by $\kappa(X)$. An infinite cardinal κ is *regular* if for every $\beta < \kappa$ and every well-ordered set $[\lambda_\alpha; \alpha < \beta]$ of cardinals with each $\lambda_\alpha < \kappa, \sum_{\alpha<\beta} \lambda_\alpha < \kappa$. In much that follows finite cardinals will present anomalous cases; therefore, we shall use the notation $\kappa = \kappa'$ (modulo $\aleph_0$) to mean $\kappa + \aleph_0 = \kappa' + \aleph_0$.

A relational system, $A = \langle |A|, R_i^A \rangle_{i \in I}$ is a set $|A|$ together with an indexed set $\{R_i^A\}_{i \in I}$ of finitary relations on $|A|$. Then $|A|$ is the *universe* of $A, \kappa(A) = \kappa(|A|)$, the power of A, R_i^A the ith relation of A, and I the *index* set of A. If $\tau \in \omega^I$ and each R_i^A is a $\tau(i)$-ary relation, then τ is the *similarity type* of A. Suppose A and B are systems of similarity type τ. Then a map $f: |A| \to |B|$ is a *monomorphism* if f is one-one, and, for each

([3]) As defined, for example, in [7, pp. 126-134].

$i \in I$ and $a_1, \cdots, a_{\tau(i)} \in A$, $R_i^A a_1, \cdots, a_{\tau(i)}$ if and only if $R_i^B f(a_1), \cdots, f(a_{\tau(i)})$. If a monomorphism maps A *onto* B it is an *isomorphism* and A is isomorphic to $B(A \cong B)$. If $|A| \subseteq |B|$ and the identity map is a monomorphism of A into B then A is a *subsystem* of $B(A \subseteq B)$. Corresponding to each $X \subseteq |A|$, there is a unique subsystem of A with universe X, denoted by $A|X$.

In certain auxiliary constructions it is convenient to consider *generalized relation systems* which have in addition to finitary relations, a set of *distinguished elements* and a set of finitary *operations*. The preceding concepts may be extended to generalized relation systems in an obvious fashion. In particular, a subsystem will always contain all the distinguished elements and be closed under all the operations.

Corresponding to each similarity type τ is a first order (with identity) language, L_τ. The symbols of L_τ are the usual logical connectives: $\sim$, $\vee, \wedge, \rightarrow, \leftrightarrow$; quantifiers: $\exists, \forall$; an equality sign: $=$; a denumerable set of variables: $v_0, v_1, \cdots$; and a $\tau(i)$-ary relation symbol, R_i, for each $i \in I$. (Corresponding to generalized relation systems we have *generalized languages* which have, in addition to the preceding symbols, individual constants and operation symbols.) The language, L_τ is *countable* if it has only a countable number of symbols. The reader is assumed familiar with the notion of *term* and *formula* in such a language. An *open* formula is a formula containing no quantifiers. A *sentence* is a formula with no free variables. A *universal* sentence is a sentence in prenex form containing no existential quantifiers. If ψ is a formula of L_τ with no free variables other than $v_0, \cdots, v_{n-1}$, A is a system of type τ, and $a_0, \cdots, a_{n-1} \in A$; then $\vdash_A \psi(a_0, \cdots, a_{n-1})$ means that $a_0, \cdots, a_{n-1}$ satisfies ψ in A (in the usual sense) when v_m denotes a_m. If $t(v_1, \cdots, v_n)$ is a term of L_τ and $\vdash_A a_0 = t(a_1, \cdots, a_n)$, then we say a_0 is the value of the term t when v_m denotes $a_m (m \leqq n)$ and write $a_0 = t^A(a_1, \cdots, a_n)$. A consistent set, Σ, of sentences of L is a theory of L_τ. A system, A (of similarity type τ), is a *model* of Σ if for every $\sigma \in \Sigma$, $\vdash_A \sigma$. If ψ is a sentence of L_τ, $\vdash_\Sigma \psi$ means that for every model A of Σ, $\vdash_A \psi$. The theory Σ is *complete* if for every sentence ψ of L, either $\vdash_\Sigma \psi$ or $\vdash_\Sigma \sim \psi$. If Σ is a theory having an infinite model and κ is an infinite cardinal then Σ is *categorical* in power κ (κ-categorical) if all models of Σ of power κ are isomorphic. By a result of Vaught [16] and Łoś [9], if Σ is κ-categorical and has no finite models then Σ is complete.

If A is a system of type τ and $X \subseteq |A|$ we may form a new system $(A, a)_{a \in X}$ by taking each element of X as a distinguished element. We denote by $L(A)$ the language corresponding to the similarity type of $(A, a)_{a \in |A|}$. (The symbols of $L(A)$ are the symbols of L together with a new individual constant $\bar{a}$ for each $a \in A$ (4).) The *diagram* of $A, \mathscr{D}(A)$, is the

(4) To avoid all ambiguities one should write $\bar{a}^A$ rather than $\bar{a}$; however, in our uses the A will always be clear from context.

set of all open sentences (i.e., formulas without variables) of $L(A)$ which are valid in $(A, a)_{a \in |A|}$. If A and B are systems of type τ then A is *elementary equivalent* to B if A and B are models of the same complete theory of L_τ $(A \equiv B)$. If $X \subseteq |A|$ and f is a mapping of X into $|B|$ then f is an *elementary monomorphism* $((A, x)_{x \in X} \equiv (B, f(x))_{x \in X})$ if for every $x_0, \cdots, x_n \in X$ and every formula, ψ, of L_τ, $\vdash_A \psi(x_0, \cdots, x_n)$ implies $\vdash_B \psi(f(x_0), \cdots, f(x_n))$.

Suppose $A = \langle A, R_i^A \rangle_{i \in I}$ is a relation system of type τ. For each formula ψ of L_τ, if m is the smallest number such that the free variables of ψ are among $v_0, \cdots, v_{m-1}$, then we denote by ψ^A the m-ary relation on $|A|$ such that $\psi^A a_0, \cdots, a_{m-1}$ if and only if $\vdash_A \psi(a_0, \cdots, a_{m-1})$. Then define

$$A^* = (A, \psi^A)_{\psi \in \text{ formulas of } L_\tau}.$$

Let τ^* be similarity type of A^*. If Σ is a theory in L_τ define Σ^* as those sentences ψ of L_{ρ^*} such that $\vdash_{A^*} \psi$ for every model A of Σ. The next lemma follows easily from these definitions.

Lemma 1.1. (a) *A' is a model of Σ^* if and only if there is a model A of Σ such that $A^* = A'$.*

(b) *$A \cong B$ if and only if $A^* \cong B^*$.*

(c) *If A and B are models of Σ, $X \subseteq |A|$, and f a map of X into B, then $(A, x)_{x \in X} \equiv (B, f(x))_{x \in X}$ if and only if the map $f: A^* | X \to B^*$ is a monomorphism.*

(d) *Σ is κ-categorical if and only if Σ^* is κ-categorical.*

(e) *If Σ is a theory in L_τ and ψ is a formula in L_{τ^*} having no free variables other than $v_0, \cdots, v_{n-1}$, then there is a relation symbol R of degree n in L_{τ^*} such that $\vdash_{\Sigma^*} \psi(v_0, \cdots, v_{n-1}) \leftrightarrow R(v_0, \cdots, v_{n-1})$.*

For the case that Σ is a complete theory the following results were established in [10].

Lemma 1.2. *Suppose Σ is a complete theory in L_τ. Denote by $\mathscr{N}(\Sigma^*)$ the class of subsystems of models of Σ^*.*

(a) *Σ^* is a complete theory in L_{τ^*}.*

(b) *If $\{A_\alpha; \alpha < \delta\}$ is an increasing chain of members of $\mathscr{N}(\Sigma^*)$ then $\bigcup_{\alpha<\delta} A_\alpha \in \mathscr{N}(\Sigma^*)$. If each A_α is a model of Σ^* then the union is a model of Σ^*.*

(c) *If $A_1, A_2 \in \mathscr{N}(\Sigma^*)$ then there is an $A_3 \in \mathscr{N}(\Sigma^*)$ and monomorphisms $f_1: A_1 \to A_3$ and $f_2: A_2 \to A_3$.*

(d) *If $A_0, A_1, A_2 \in \mathscr{N}(\Sigma^*)$ and $g_1: A_0 \to A_1$ and $g_2: A_0 \to A_2$ are monomorphisms, then there is an $A_3 \in \mathscr{N}(\Sigma^*)$ and monomorphisms $f_1: A_1 \to A_3$ and $f_2: A_2 \to A_3$ such that $f_1 g_1 = f_2 g_2$(5).*

(5) In [5] Jónsson considered classes of relation systems satisfying certain condition which he numbered I-VI. In order to apply Jónsson's result to an arbitrary complete theory, [10] devised the Σ^* theory and showed that $\mathscr{N}(\Sigma^*)$ satisfied Jósson's conditions. In Theorem 1.2, (b), (c) and (d) assert respectively that $\mathscr{N}(\Sigma^*)$ satisfies Jónsson's conditions V, III, and IV.

A notion that we shall find convenient is that of a category of maps. If $\mathscr{S}$ is a class of mathematical objects(6) then a *category* which object class $\mathscr{S}$ is a class, $\mathscr{C}$, of triples called maps (denoted by $f: A \to B$) where $A, B \in \mathscr{S}$, f is a function of $|A|$ into $|B|$, and such that (i) (identity: $A \to A) \in \mathscr{C}$ for each $A \in \mathscr{S}$ and (ii) if $(f: A \to B)$ and $(g: B \to C) \in \mathscr{C}$ then $(gf: A \to C) \in \mathscr{C}$. A is the *domain* and B the *co-domain* of $f: A \to B$(7).

2. **Transcendence in rank.** We shall be interested in elementary monomorphisms among subsystems of a complete theory Σ. By 1.1(c) it is therefore convenient to consider Σ^* instead of Σ. *Throughout the rest of this paper we shall adopt the following convention*: T will always denote a complete theory in a countable language, L, T has an infinite model, and there is a theory Σ such that $T = \Sigma^*$. We will denote the class of subsystems of models of T by $\mathscr{M}(T)$.

If $A \in \mathscr{M}(T)$ it follows from 1.1(e) that $T(A) = T \cup \mathscr{D}(A)$ is a complete theory in $L(A)$(8). We denote by $F(A)$ the set of formulas of $L(A)$ which have no free variable other than v_0. If the formulas of $F(A)$ which are equivalent in the theory $T(A)$ are identified (i.e., ψ is identified with ψ' if $\vdash_{T(A)} (\forall v_0)\psi \leftrightarrow \psi')$(9) then $F(A)$ may be considered as a Boolean algebra with $\wedge$, $\vee$, and $\sim$ as $\cap$, $\cup$, and complementation respectively (10). A maximally consistent set of formulas in $F(A)$ will be a dual prime ideal (ultrafilter) in $F(A)$ considered as a Boolean algebra. The set of such dual prime ideals is the Stone space of $F(A)$ and will be denoted by $S(A)$. $S(A)$ is a Boolean space with a basis consisting of the sets.

$$U_\psi = \{p \in S(A); \psi \in p\} \qquad (\psi \in F(A)).$$

It follows that $S(A)$ has a basis of power $= \kappa(A)$ (modulo $\aleph_0$).

The space $S(A)$ may be thought of as the ways of extending $T(A)$ to a complete theory in a language having one more individual constant than $L(A)$ has. Suppose $A, B \in \mathscr{M}(T), B \supseteq A, b \in B$, and $\bar{b}$ is the constant in the $L(B)$ corresponding to b. We denote by $p_{b,B}$ the unique $p \in S(B)$ con-

(6) A "mathematical object," A, is a set $|A|$ with some associated structure. In every case in this paper an object is either a relational system or a topological space.

(7) It is more usual to abstract the composition properties of the maps and define a category as a class of elements with a binary operation defined for some pairs of elements and which satisfies certain axioms. Since we are interested not in categories, *per se*, but in certain instances of them, the definition we have given is more convenient.

(8) We could have chosen to present this entire section "syntactically" by considering, instead of the class $\mathscr{M}(T)$, the class of all complete extensions of T in languages which are extensions of L by the addition of new individual constants.

(9) It follows from 1.1(e) that we would get the same Boolean algebra if we assumed that $F(A)$ contained only open formulas. Notice that for open formulas $\vdash_A$ is equivalent to $\vdash_{T(A)}$, but for formulas in general the two are not equivalent unless A is a model of T.

(10) The close relationship between the properties of the various Boolean algebras of formulas of the language L and the model-theoretic properties of T has been observed by several authors. See especially [13] and [18].

taining the formula: $v_0 = \overline{b}$. Let $p_{b,B,A} = p_{b,B} \cap F(A)$. If $q \in S(A)$ we say b *realizes* q in B if $q = p_{b,B,A}$. Clearly, every $b \in B$ realizes some point of $S(A)$. By the Completeness Theorem every $p \in S(A)$ is realized in some extension of A. Suppose $B_1, B_2 \in \mathscr{N}(T), B_1 B_2 \supseteq A$ and $b_1 \in B_1, b_2 \in B_2$; then the map: $A \cup \{b_1\} \to A \cup \{b_2\}$ which is the identity on A and maps b_1 to b_2 is a monomorphism if and only if b_1 and b_2 realize the same point in $S(A)$. Thus, $S(A)$ is the set of "isomorphism types of elements with respect to A."

LEMMA 2.1. *If $A \in \mathscr{N}(T)$ then there is a model of $T, B, B \supseteq A$ such that each $p \in S(A)$ is realized in B.*

Proof. Let $\{p_\alpha; \alpha < \gamma\}$ be a well-ordered list of the points of $S(A)$. We assert there exists an increasing chain $\{B_\alpha; \alpha < \gamma\}$ of models of T such that each $B_\alpha \supseteq A$ and each p_β with $\beta < \alpha$ is realized in B_α. The proof is by induction on α. Assume the sequence exists for all $\beta < \alpha$. If α is a limit ordinal let $B_\alpha = \bigcup_{\beta<\alpha} B_\beta$ and the result follows from 1.2(b). Suppose $\alpha = \beta + 1$. By the Completeness Theorem there is a model of $T, C, C \supseteq A$ such that p_β is realized in C. By 1.2(d) there is a model of T, D, and monomorphisms $f_1 : C \to D$ and $f_2 : B_\beta \to D$ such that $f_1 = f_2$ on A. If we identify B_β with $f_2(B_\beta)$ then D may be taken as B_α. B_γ is the B satisfying the theorem.

Suppose that $A, B \in \mathscr{N}(T)$ and $f : A \to B$ is a monomorphism. Then f induces a monomorphism $\tilde{f} : F(A) \to F(B)$ defined by: $\tilde{f}(\psi)$ is the formula obtained by substituting (for each $a \in A$) $\overline{f(a)}$ for each occurrence of $\overline{a}$ in ψ. In turn, $\tilde{f}$ induces a map $f^* : S(B) \to S(A)$ defined by $f^*(p) = \tilde{f}^{-1}(p)$. The map f^* is continuous (cf. [14]), indeed $f^{*-1}(U_\psi) = U_{\tilde{f}(\psi)}$; the map f^* is onto $S(A)$, for if $q \in S(A)$ there is some $p \in S(B)$ with $p \supseteq \tilde{f}(q)$. If, in particular, $B \supseteq A$ and $i_{AB} : A \to B$ is the identity map[11] and $p \in S(B)$ then $i^*_{AB}(p) = p \cap F(A)$.

Let $\mathscr{S}(T) = \{S(A); A \in \mathscr{N}(T)\}$ and $\mathscr{C}(T) = \{(f^* : S(B) \to S(A)); A, B \in \mathscr{N}(T)$ and $f : A \to B$ a monomorphism$\}$. Then $\mathscr{C}(T)$ is a category of continuous onto maps with object class $\mathscr{S}(T)$. It is "dual" to the category of monomorphisms between members of $\mathscr{N}(T)$. Therefore, corresponding to each of 1.2(b), (c) and (d) there is a dual statement which holds in the category $\mathscr{C}(T)$. It should be especially noted that since a formula, ψ, involves only a finite number of individual constants, for each U_ψ in the basis of $S(A)$ there is some finite $B \subseteq A$ such that U_ψ is the inverse image under i^*_{BA} of a member of the basis of $S(B)$.

The next definition is a generalization of the usual definition of derived spaces to a definition involving a class of spaces and a category of maps between them. Though we shall deal explicitly only with the category

(11) Henceforth, whenever $A \subseteq B$ the identity map of A into B will be denoted by i_{AB}.

$\mathscr{C}(T)$, it will be obvious that Definition 2.2 and many of the following results and proofs remain valid in many other categories of continuous onto maps betweem compact spaces.

DEFINITION 2.2. For each ordinal α and each $S(A) \in \mathscr{S}(T)$, subspaces $S^\alpha(A)$ and $\mathrm{Tr}^\alpha(A)$ are defined inductively by:

(1) $S^\alpha(A) = S(A) - \bigcup_{\beta<\alpha} \mathrm{Tr}^\beta(A)$

(2) $p \in \mathrm{Tr}^\alpha(A)$ if (i) $p \in S^\alpha(A)$ and (ii) for every map $(f^* : S(B) \to S(A)) \in \mathscr{C}(T), f^{*-1}(p) \cap S^\alpha(B)$ is a set of isolated points in $S^\alpha(B)$.

$p \in S(A)$ is *algebraic* if $p \in \mathrm{Tr}^0(A)$; p *is transcendental in rank* α if $p \in \mathrm{Tr}^\alpha(A)$ [12].

THEOREM 2.3. (a) *$S^\alpha(A)$ is a closed and hence compact subspace of $S(A)$.*

(b) *If $(f^* : S(B) \to S(A)) \in \mathscr{C}(T)$ then* (i) *$f^*(S^\alpha(B)) = S^\alpha(A)$, and* (ii) *if $p \in S^\alpha(A)$ then $p \in \mathrm{Tr}^\alpha(A)$ if and only if $f^{*-1}(p) \cap S^\alpha(B) \subseteq \mathrm{Tr}^\alpha(B)$.*

Proof. (a) The proof is by induction on α. Suppose $\alpha = \beta + 1$. Then $S^\alpha(A) = S^\beta(A) - \mathrm{Tr}^\beta(A)$. $\mathrm{Tr}^\beta(A)$ is a set of isolated points in $S^\beta(A)$ and is therefore open in $S^\beta(A)$. So $S^\alpha(A)$ is closed.

Suppose $\alpha = \delta$. Then $S^\delta(A) = \bigcap_{\beta<\delta} S^\beta(A)$ and is closed since it is the intersection of closed sets.

(b) Notice first that, since $\mathrm{Tr}^\alpha(A) = S^\alpha(A) - S^{\alpha+1}(A)$, (b)(ii) will follow immediately from (b)(i). We shall use the following topological result.

PROPOSITION. *Suppose G is a compact space, H a Hausdorff space, f: $G \to H$ a continuous onto map, and $p \in H$, a limit point of H; then $f^{-1}(p)$ contains a limit point of G.*

Proof of proposition. If $f^{-1}(p)$ contained only isolated points then $G - f^{-1}(p)$ would be closed and hence compact. Then $f(G - f^{-1}(p)) = H - \{p\}$ would be compact and hence closed, so p would not be a limit point of H.

The proof of (b)(i) is by induction on α. Assume result for all $\beta < \alpha$. We first show that $f^*(S^\alpha(B)) \subseteq S^\alpha(A)$; that is, we show for each $\beta < \alpha$ that if $q \in S^\alpha(B)$ then $f^*(q) = p \notin \mathrm{Tr}^\beta(A)$. Since $q \notin \mathrm{Tr}^\beta(B)$ there is some $(g^* : S(C) \to S(B)) \in \mathscr{C}(T)$ such that $g^{*-1}(q) \cap S^\beta(C)$ contains a limit point, say r. Then $(f^*g^* : S(C) \to S(A)) \in \mathscr{C}(T)$ and $r \in (f^*g^*)^{-1}(p)$ so $p \notin \mathrm{Tr}^\beta(A)$.

Finally, to prove that $f^*(S^\alpha(\beta)) \supseteq S^\alpha(A)$ we must show for each $p \in S^\alpha(A)$ that $f^{*-1}(p) \cap S^\alpha(B) \neq \emptyset$. Suppose the contrary for some $p \in S^\alpha(A)$. Since f^* is onto, $f^{*-1}(p)$ is closed and compact and therefore there is a largest β (necessarily $< \alpha$) such that $f^{*-1}(p) \cap S^\beta(B) \neq \emptyset$. Then $f^{*-1}(p)$

[12] The terminology *algebraic* and *transcendental* are suggested by the theory of algebraically closed fields of characteristic 0, see Example I below. Our notion of algebraic is also related to a generalized notion of algebraic extension considered by Jónsson [6].

$\cap S^\beta(B) \subseteq \mathrm{Tr}^\beta(B)$. Since $p \notin \mathrm{Tr}^\beta(A)$ there is some $(g^*: S(C) \to S(A)) \in \mathscr{L}(T)$ such that $g^{*-1}(p) \cap S^\beta(C)$ contains a limit point of $S^\beta(C)$, say r. By 1.2(d) there is a $D \in \mathscr{N}(T)$ and monomorphisms $h_1: B \to D$ and $h_2: C \to D$ such that $h_1 f = h_2 g$. By the induction assumption, h_2^* maps $S^\beta(D)$ onto $S^\beta(C)$. By the proposition above, $h_2^{*-1}(r)$ contains a limit point of $S^\beta(D)$ say s. Then $h_1^*(s) \in S^\beta(B) \cap f^{*-1}(p)$ but $h_1^*(s) \notin \mathrm{Tr}^\beta(B)$ from 2.2. This contradicts $f^{*-1}(p) \cap S^\beta(B) \subseteq \mathrm{Tr}^\beta(B)$ and the result is established.

COROLLARY 2.4. *If $p \notin \mathrm{Tr}^\alpha(A)$ then there is a finite $F \subseteq A$ such that $i_{FA}^*(p) \in \mathrm{Tr}^\alpha(F)$.*

Proof. $S(A)$ has a neighborhood U such that $S^\alpha(A) \cap U = \{p\}$. As remarked earlier, since U is determined by some formula there is some finite $F \subseteq A$ such that $S(F)$ has a neighborhood V with $U = i_{FA}^{*-1}(V)$. By 2.3 (b)(i), $i_{FA}^*(p) = i_{FA}^*(U \cap S^\alpha(A)) = V \cap S^\alpha(F)$. Therefore, $i_{FA}^*(p) \in \mathrm{Tr}^\alpha(F)$.

THEOREM 2.5. (a) *If $p \in \mathrm{Tr}^\alpha(A)$ there is an integer n such that for every $(f^*: S(B) \to S(A)) \in \mathscr{L}(T)$ the set $f^{*-1}(p) \cap S^\alpha(B)$ has power $\leq n$. The least such integer will be called the degree of p*[13]

(b) *If $p \in \mathrm{Tr}^\alpha(A)$ and $(f^*: S(B) \to S(A)) \in \mathscr{L}(T)$ then degree $p = \sum_q$ degree $q(q \in f^{*-1}(p) \cap \mathrm{Tr}^\alpha(B))$.*

Proof. (a) Suppose the opposite for some $p \in \mathrm{Tr}^\alpha(A)$. Then there would be, for each $n \in \omega$, a $B_n \in \mathscr{N}(T)$ and monomorphisms $f_n: A \to B_n$ such that $f_n^{*-1}(p) \cap S^\alpha(B_n)$ has power $> n$. By iterative applications of 1.2(d) to these B_n's there is a sequence $A \subseteq A_1 \subseteq A_2 \cdots$ such that $i_{AA_n'}^{-1}(p) \cap S^\alpha(A_n)$ has power greater than n. Let $A' = \bigcup_{n \in \omega} A_n$. Then $i_{AA'}^{*-1}(p) \cap S^\alpha(A')$ is infinite and since it is compact, has a limit point. So $p \notin \mathrm{Tr}^\alpha(A)$ contradicting the assumption.

(b) For each $q \in \mathrm{Tr}^\alpha(B) \cap f^{*-1}(p)$ there is some $C_q \in \mathscr{N}(T)$ and a monomorphism $g_q: B \to C_q$ such that $g_q^{*-1}(q) \cap S^\alpha(C_q)$ has power degree q. Similarly there is some $C_p \in \mathscr{N}(T)$ and a monomorphism $g_p: A \to C_p$ such that $g_p^{*-1}(p) \cap S^\alpha(C_p)$ has power degree p. By repeated applications of 1.2(d) there is a $C \in \mathscr{N}(T)$ and a monomorphism $g: B \to C$ such that $g^{*-1}(q) \cap S^\alpha(C)$ has power degree q (for each $q \in f^{*-1}(p) \cap S^\alpha(B)$) and $(f^*g^*)^{-1}(p) \cap S^\alpha(C)$ has power degree p. But

$$(f^*g^*)^{-1}(p) \cap S^\alpha(C) = \bigcup_q g^{*-1}(q) \cap S^\alpha(C) \quad (q \in f^{*-1}(p) \cap S^\alpha(B))$$

and the result follows.

Lemma 2.6. (a) *There is an ordinal $\alpha_T < (2^{\aleph_0})^+$ which is the least ordinal*

[13] It is possible to combine the rank and degree into a single new rank by varying the Definition 2.2 slightly. To do so replace in 2.2(2) the words "set of isolated points" by "a single point."

such that for all $A \in \mathscr{A}(T)$ *and all* $\beta > \alpha_T$, $S^{\alpha_T}(A) = S^{\beta}(A)$.

(b) *If* $S^{\alpha_T}(A) = \emptyset$ *for some* $A \in \mathscr{A}(T)$, *then* α_T *is not a limit ordinal and for every* $B \in \mathscr{A}(T)$, $S^{\alpha_T}(B) = \emptyset$ *and* $S^{\beta}(B) = \emptyset$ *for any* $\beta < \alpha_T$.

Proof. (a) From 2.4 it follows that $\mathrm{Tr}^{\alpha}(A)$ is empty for every $A \in \mathscr{A}(T)$ if it is empty for every finite $A \in \mathscr{A}(T)$. There are at most $2^{\aleph_0}$ isomorphism types of finite systems $\in \mathscr{A}(T)$ and for each such finite system $\kappa(S(A)) \leqq 2^{\aleph_0}$.

(b) Suppose $A, B \in \mathscr{A}(T)$ and $S^{\beta}(A) = \emptyset$ Then by 1.2(c) and 2.3(b) $S^{\beta}(B) = \emptyset$. That the least ordinal at which this occurs cannot be a limit ordinal follows from 2.3(a) and the compactness of $S(A)$.

We say T is *totally transcendental* if $S^{\alpha_T}(A) = \emptyset$ for some (and hence every) $A \in \mathscr{A}(T)$.

THEOREM 2.7. *If* T *is totally transcendental then* $\kappa(S(A)) = \kappa(A)$ (*modulo* $\aleph_0$) *for every* $A \in \mathscr{A}(T)$.

Proof. For every $p \in \mathrm{Tr}^{\alpha}(A)$ we may choose a member $U(p)$ of the basis of $S(A)$ such that $U(p) \cap S^{\alpha}(A) = \{p\}$. Clearly if $p \neq p'$ then $U(p) \neq U(p')$. Since T is totally transcendental, every $p \in S(A)$ is transcendental in some rank. Thus the correspondence of p to $U(p)$ is a one-one correspondence between $S(A)$ and a subset of the basis of $S(A)$. So $\kappa(S(A)) \leqq \kappa(A) + \aleph_0$. On the other hand, the formula: $v_0 = \bar{a}$, determines for each $a \in A$ a unique element of $S(A)$; so $\kappa(S(A)) \geqq \kappa(A)$.

The next theorem is an analogue of the Cantor-Bendixson theorem and the proof is similar to proofs of that theorem.

THEOREM 2.8. T *is totally transcendental if and only if* $S(A)$ *is countable for every countable* $A \in \mathscr{A}(T)$.

Proof. If T is totally transcendental then $S(A)$ is countable for countable A by Theorem 2.7.

Conversely, suppose T is not totally transcendental. Then for every $A \in \mathscr{A}(T), S^{\alpha_T}(A) \neq \emptyset$. There is some $A \in \mathscr{A}(T)$ such that $S^{\alpha_T}(A)$ has more than one point; for otherwise, every $p \in S^{\alpha_T}(A)$ would be transcendental in rank α_T, and by definition, there are no points transcendental in rank α_T. Thus, there is some $A_1 \in \mathscr{A}(T)$ such that $S^{\alpha_T}(A_1)$ may be divided into two disjoint nonempty components (closed-open sets), say U_0 and U_1. As remarked earlier, U_0 and U_1 are determined by finite subsets of A_1. Hence, without loss of generality we take A_1 to be finite. There must be some $B \in \mathscr{A}(T), B \supseteq A_1$, such that $i^{*-1}_{A_1B}(U_0) \cap S^{\alpha_T}(B)$ has more than one point; for otherwise, each $p \in U_0$ would be transcendental in rank α_T Similarly for U_1. By 1.2(d) we may find an $A_2 \supseteq A_1$ such that $i^{*-1}_{A_1A_2}(U_0) \cap S^{\alpha_T}(A_2)$ and $i^{*-1}_{A_1A_2}(U_1) \cap S^{\alpha_T}(A_2)$ both have more than one point. Thus we may decompose $S^{\alpha_T}(A_2)$ into four disjoint nonempty components, U_{00},

U_{01}, U_{10}, U_{11} such that $i^{*-1}_{A_1A_2}(U_j) \cap S^{\alpha T}(A_2) = U_{j0} \cup U_{j1}$ $(j = 0, 1)$. As before we may take A_2 to be finite. We proceed inductively to find an increasing chain of systems $\{A_n; n < \omega\}$ such that each $A_n \in \mathscr{N}(T)$, is finite, each $S^{\alpha T}(A_n)$ may be decomposed into 2^n disjoint nonempty components $U_{j_0 \cdots j_{n-1}}$ $(j_k = 0, 1)$ and

$$i^{*-1}_{A_nA_{n+1}}(U_{j_0 \cdots j_{n-1}}) \cap S^{\alpha T}(A_{n+1}) = U_{j_0 \cdots j_{n-1}0} \cup U_{j_0 \cdots j_{n-1}1}.$$

Let $A = \bigcup_n A_n$. For each $t \in 2^\omega$, Let $V_t = \bigcap_n i^{*-1}_{A_nA}(U_{t(0) \cdots t(n-1)}) \cap S^{\alpha T}(A)$. Then $V_t \neq \emptyset$ since it is the intersection of closed nonempty sets. Obviously, $t_1 \neq t_2$ implies $V_{t_1} \cap V_{t_2} = \emptyset$. Thus $S^{\alpha T}(A)$ has power $2^{\aleph_0}$ though A is countable.

We shall conclude this section with three examples. In each case we shall describe the theory Σ such that $T = \Sigma^*$. We shall then describe $S(A)$ for each $A \in \mathscr{N}(T)$. To do this it is convenient to know when a consistent set of formulas of $F(A)$ is contained in a unique $p \in S(A)$. We give the following sufficient condition:

A consistent set of formulas, $Q \subseteq F(A)$, is contained in a unique $p \in S(A)$ if whenever B is a model of $T, B \supseteq A$, and $b, b' \in B$ satisfy every formula of Q, then there is an automorphism of B carrying b to b' and leaving each element of A fixed([14]).

For suppose p and p' were points of $S(A)$ which contain Q. By 2.1 there is a model of $T, B, B \supseteq A$, and with $b, b' \in B$ realizing p and p' respectively. Our condition then asserts that there is an automorphism of B having A fixed and carrying b to b'. Therefore b and b' realize the same point of $S(A)$, that is $p = p'$.

EXAMPLE I. Let Σ be the theory of algebraically closed fields of characteristic 0([15]). As mentioned earlier this theory is categorical in very uncountable power but not in power $\aleph_0$. Suppose $A \in \mathscr{N}(T)$, let $\Delta(A)$ be the field generated by A. Suppose $Q(v_0)$ is a polynomial with coefficients in $\Delta(A)$ and irreducible over $\Delta(A)$. By the condition above the formula: $Q(v_0) = 0$ determines a unique point of $S(A)$. Since this point is determined by a single formula it is an isolated point of $S(A)$. Let P be the set of all formulas: $Q(v_0) \neq 0$ where $Q(v_0)$ is a polynomial with coefficients in $\Delta(A)$. Then all the formulas of P are satisfied precisely by those elements transcendental (in the usual field-theoretical sense) over $\Delta(A)$. Therefore, by our condition and the Steinitz theorems P is included in a unique $p \in S(A)$. Obviously, the above are all the points of $S(A)$. Since $S(A)$ is infinite and compact it must have a limit point which can only be the point deter-

([14]) If we weaken this condition to assert that there is a model of $T, C \supseteq B$, such that C has an automorphism carrying b to b' and leaving each element of A fixed, then this condition is also necessary; cf. [10].

([15]) For a more detailed discussion of this case see Abraham Robinson, *Complete theories*, North-Holland, Amsterdam, 1956.

mined by P. Thus $S(A)$ consists of: (1) isolated points corresponding to the distinct elements of $\Delta(A)$ and to the algebraic extensions of $\Delta(A)$, and (2) a single limit point corresponding to the transcendental extensions of $\Delta(A)$. If $B \supseteq A$ then any element algebraic over $\Delta(A)$ is *a fortiori* algebraic over $\Delta(B)$. So if $p \in S(A)$ is an isolated point, then $i_{AB}^{*-1}(p)$ is a set of isolated points; hence $p \in \mathrm{Tr}^0(A)$. For each $A \in \mathscr{N}(T), S^1(A)$ is then a single point so $S^1(A) = \mathrm{Tr}^1(A)$. Thus T is totally transcendental and $\alpha_T = 2$.

EXAMPLE II. Suppose there are two relation symbols: R_0, a one-ary relation symbol, and R_1, an $(n+1)$-ary relation symbol and let the formulas of Σ assert that in any model of Σ $A = \langle |A|, R_0^A, R_1^A \rangle$:

(1) $|A|$ is infinite, and

(2) the set of pairs $(a_0, (a_1, \cdots, a_n))$ such that $R_1^A a_0, a_1, \cdots, a_n$ is a one-one correspondence between $|A| - R_0^A$ and the n-tuples of distinct elements of R_0^A.

This theory is obviously categorical in every infinite power.

For each n-tuple of distinct elements of $R_0^A, a_1, \cdots, a_n$, let $\langle a_1, \cdots, a_n \rangle$ denote the unique a_0 such that $R_1^A, a_0, a_1, \cdots, a_n$. Suppose $B \in \mathscr{N}(T)$. By 2.1 there is a model of $T, A \supseteq B$, such that every $p \in S(B)$ is realized in A. Denote by $\hat{B}$ the closure of B in A; more precisely, $\hat{B}$ is the smallest subsystem such that $B \subseteq \hat{B} \subseteq A$, and $\langle a_1, \cdots, a_n \rangle \in \hat{B}$ if and only if $a_1, \cdots, a_n \in \hat{B}$. It is easy to see that every $a \in \hat{B}$ is characterized by a unique formula of $F(B)$, and so each $a \in \hat{B}$ realizes an isolated point of $S(B)$.

Notice that every one-one map of $R_0^A - \hat{B}$ onto itself induces an automorphism of A which leaves $\hat{B}$ fixed. So every element of $R_0^A - \hat{B}$ realizes the same point of $S(B)$. Similarly, two elements $\langle a_1, \cdots, a_n \rangle$ and $\langle a_1', \cdots, a_n' \rangle$ realizes the same point of $S(B)$ if and only if for all $1 \leqq i \leqq n, a_i = a_i'$ whenever a_i or $a_i' \in \hat{B}$. Call a point $p \in S(B)$ of type m if it is realized by an element $\langle a_1, \cdots, a_n \rangle$ and exactly m of the a_i's $\in \hat{B}$;

Suppose $C \in \mathscr{N}(T)$, $A \supseteq C \supseteq B$, $\langle a_1, \cdots, a_n \rangle \in A, m$ of the a_i's $\in \hat{B}$, and $m+1$ of the a_i's $\in \hat{C}$. Then $\langle a_1, \cdots, a_n \rangle$ realizes a point of type m in $S(B)$ and of type $m+1$ in $S(C)$. Thus, for every $B \in \mathscr{N}(T)$ we can find a $C \supseteq B$ such that for every $p \in S(B)$ of type $m < n, i_{BC}^{*-1}(p)$ contains an infinite set of points of type $m+1$.

From the above considerations it is easy to show that: (1) the points of $S(B)$ realized by elements of $\hat{B} \in \mathrm{Tr}^0(B)$, (2) the point of $S(B)$ realized by the elements of $R_0^A - \hat{B}$ is transcendental in rank 1, and (3) the points of $S(B)$ of type m are transcendental in rank $n - m$. Therefore, T is totally transcendental and $\alpha_T = n + 1$.

EXAMPLE III. Consider the Cantor set, i.e., 2^ω with the product topology. Let Y be a closed nonempty subset of 2^ω. There will be a denumerable set R_n $(n \in \omega)$ of singulary relation symbols and the theory Σ will assert that for any model of Σ, A, and any two finite sets $K_0, K_1 \subseteq \omega, \bigcap_{n\in K_1} R_n^A \cap \bigcap_{n\in K_0}(|A| - R_n^A)$ is empty or infinite depending on whether

$$\left\{t \in Y;\ \bigwedge_{n\in K_1} t(n) = 1 \wedge \bigwedge_{n\in K_0} t(n) = 0\right\}$$

is empty or nonempty. Thus the points of Y correspond to the isomorphism types of single element subsystems of models of Σ. If $A \in \mathscr{N}(T)$, then the points of $S(A)$ realized by elements of A are isolated, indeed algebraic points; while the points realized by elements not in A form a space homeomorphic to Y. None of the latter points can be algebraic since each one could be realized by an infinite set of elements in some $B \supseteq A$. So $S^1(A)$ is homeomorphic to Y, and, if $B \supseteq A, i^*_{AB}$ maps $S^1(B)$ homeomorphically onto $S^1(A)$. A point $p \in S^1(A)$ will be in $S^{1+\alpha}(A)$ if and only if the corresponding point of Y is in $Y^{(\alpha)}$, the αth derived set of Y. The theory is totally transcendental if and only if Y has a vanishing perfect kernel, that is, if Y is countable. If α_Y is the least ordinal such that $Y^{(\alpha)} = Y^{(\alpha+1)}$ then $\alpha_T = 1 + \alpha_Y$.

3. **Results depending on Ramsey's theorem.** In this section we have gathered together some results depending on the following theorem of Ramsey [12].

THEOREM 3.1 (RAMSEY). *Suppose Y is an infinite set and $Y^{(n)}$ the set of subsets of Y having exactly n elements. If $Y^{(n)} = C_1 \cup \cdots \cup C_m$ is a partition of $Y^{(n)}$ into a finite number of mutually disjoint sets, then there is a $j \leqq m$ and an infinite set $Y_1 \subseteq Y$ such that $Y_1^{(n)} \subseteq C_j$.*

Much of this section is related to results of Ehrenfeucht and Mostowski [3] and Ehrenfeucht [1], [2]. In particular, Theorems 3.2, 3.4 and 3.5 below are only slight variants of the results of [3].

THEOREM 3.2. *Suppose Σ is a (generalized) theory in a language L, Σ has an infinite model, and $(X, <)$ is an arbitary linearly ordered set. Then there is a model of Σ, A, such that $|A| \supseteq X$ and whenever $n \in \omega, x_0 < \cdots < x_{n-1}$ and $x_0' < \cdots < x_{n-1}'$ are contained in X and ψ is a formula of L with no free variables other than $v_0, \cdots, v_{n-1}$, then $\vdash_A \psi(x_0, \cdots, x_{n-1}) \leftrightarrow \psi(x_0', \cdots, x_{n-1}')$.*

Proof. Suppose there is added to L a new constant, $\overline{x}$, for each $x \in X$, and there is added to Σ the sentence

(I) $$\text{“}\overline{x_1} \neq \overline{x_2}\text{”}$$

for each pair x_1, x_2 of distinct elements of X. Suppose further that whenever $n \in \omega, x_0 < \cdots < x_{n-1}$ and $x_0' < \cdots < x_{n-1}'$ and ψ is a formula of L with free variables among $v_0, \cdots, v_{n-1}$, there is added to Σ the formula

(II) $$\text{“}\psi(\overline{x_0}, \cdots, \overline{x_{n-1}}) \leftrightarrow \psi(\overline{x_0'}, \cdots, \overline{x_{n-1}'})\text{.”}$$

The result is a set of sentences, say $\bar{\Sigma}$, extending Σ.

To prove the theorem it is sufficient to prove $\bar{\Sigma}$ consistent. Suppose $\bar{\Sigma}$ is inconsistent. Then there is an inconsistent $\Sigma_1 \subseteq \bar{\Sigma}$ such that $\Sigma_1 = \Sigma$ together with a finite number of sentences of type (I) and (II). Let the sentences of type (II) appearing in Σ_1 be:

$$\text{``}\psi_1[\bar{x}]_1 \leftrightarrow \psi_1[\bar{x}]_1'\text{''}, \cdots, \text{``}\psi_m[\bar{x}]_m \leftrightarrow \psi_m[\bar{x}]_m'\text{''}$$

where the notation $[\bar{x}]$ is an abbreviation for a sequence of constants $\bar{x}_0, \cdots, \bar{x}_{n-1}$.

Consider first the case where each $[\bar{x}]_j$ has the same number of elements, say n. Let A be an infinite model of Σ and "$<*$" a linear ordering of $|A|$ (in general, having nothing to do with any of the original relations of A). If $[a]$ and $[a]'$ are n-tuples of $|A|$ which are properly ordered by $<*$ then we say

$$[a] \approx [a]' \quad \text{if} \quad \vdash_A \bigwedge_{j \leq m} \psi_j[a] \leftrightarrow \psi_j[a]'.$$

This equivalence $\rightarrow$ partitions $|A|^{(n)}$ into (at most) 2^m equivalence classes. Applying Ramsey's theorem, we may find some infinite subset $Y \subseteq |A|$ such that $Y^{(n)}$ lies entirely within one equivalence class. That is, if $[a]$ and $[a]'$ are properly ordered n-tuples of Y then $[a] \approx [a]'$. Since Σ contains only a finite number of sentences of type (I) and (II), it contains of the new constants added to L, only those corresponding to some finite subset of X, say X_1. We may now pick in Y a finite subset, Y_1, which is order-isomorphic to X_1. Then $(A, a)_{a \in Y_1}$ is a model of Σ_1, contradicting its inconsistency.

Consider the general case where all the $[x]_j$'s do not necessarily have the same number of elements. Notice that it is sufficient to prove the theorem for X, a linearly ordered set without maximal elements, since any linearly ordered set can be imbedded in such a one. Now, let N be the maximum number of elements in any $[x]_j$ $(j \leq m)$. Then a properly ordered $[x] = (x_0, \cdots, x_{n-1})$ may be imbedded in a properly ordered set $(x_0, \cdots, x_{n-1}, x_n, \cdots, x_{N-1})$. The general result then follows from the first considered case.

The next theorem expresses the well-known fact that one can eliminate existential quantifiers by the use of operation symbols. A proof may be found in the first chapter of [4].

THEOREM 3.3. *Suppose Σ is a theory in a countable language, L; then there is a countable generalized language, $L^\# \supseteq L$ and a theory $\Sigma^\#$ of $L^\#$ such that:*

(i) *every $\sigma \in \Sigma^\#$ is a universal sentence, and*

(ii) *for every sentence, ψ of L, $\vdash_{\Sigma^\#} \psi$ if and only if $\vdash_\Sigma \psi$.*

Thus, if A is a model of $\Sigma^{\#}$, then $A \upharpoonright L$ (A restricted to the relations corresponding to symbols of L) is a model of Σ.

Suppose A is a model of $\Sigma^{\#}$ and $X \subseteq |A|$. The set of elements $a \in A$ such that there is a term, $t(v_1, \cdots, v_n)$ in $L^{\#}$ and $x_1, \cdots, x_n \in X$ with $a = t^A(x_1, \cdots, x_n)$ is by 3.3(i) the universe of a model of $\Sigma^{\#}$, denoted by $M(X, A)$.

THEOREM 3.4. *If Σ is a theory of L, has an infinite model, and $(X, <)$ is an arbitrary linearly ordered set; then there is a model of $\Sigma^{\#}, A$, with $X \subseteq |A|$ such that if:* (i) *$t_0(v_0, \cdots, v_{n_0}), \cdots, t_m(v_0, \cdots, v_{n_m})$ are terms in $L^{\#}$,* (ii) *x_{jk} and x'_{jk} ($j \leq m, k \leq n_j$) are elements of X and the mapping of x_{jk} to x'_{jk} is an order isomorphism between them, and* (iii) *ψ is a formula of $L^{\#}$ with free variables among $v_0, \cdots, v_m$; then*

$$\vdash_A \psi(t_0^A(x_{00}, \cdots, x_{0n_0}), \cdots, t_m^A(x_{m0}, \cdots, x_{mn_m}))$$
$$\leftrightarrow \psi(t_0^A(x'_{00}, \cdots, x'_{0n_0}), \cdots, t_0^A(x'_{m0}, \cdots, x'_{mn_m})).$$

Proof. Apply Theorem 3.2 to $\Sigma^{\#}$.

THEOREM 3.5. *Suppose Σ is a theory with an infinite model and $(X, <)$ is an arbitrary linearly ordered set. Then there is a model of $\Sigma, B, |B| \supseteq X$, such that any order endomorphism (automorphism) of X may be extended to an endomorphism (automorphism) of B.*

Proof. Extend Σ to $\Sigma^{\#}$ and apply Theorem 3.4 to get a model of $\Sigma^{\#}$ containing X. Take $B = M(X, A) \upharpoonright L$. If $f: X \to X$ is an order endomorphism, define $\tilde{f}: M(X, A) \to M(X, A)$ by $\tilde{f}(t^A(x_0, \cdots, x_n)) = t^A(f(x_0), \cdots, f(x_n))$. By 3.4 $\tilde{f}$ is well defined and is a monomorphism; it is obviously onto if f is onto.

The preceding two theorems may be strengthened by extending Σ to $\Sigma^{*\#}$ rather than $\Sigma^{\#}$. Using 1.1(c) this will then prove:

THEOREM 3.6. (a) *For formulas, ψ, of L, Theorem 3.4 remains valid if in the last line $\vdash_A$ is replaced by $\vdash_{M(X,A)}$.*

(b) *Under the hypothesis of Theorem 3.5 there is a model of $\Sigma, B, |B| \supseteq X$, such that any order endomorphism (automorphism) of X may be extended to an elementary endomorphism (automorphism) of B.*

Suppose A is a model of $\Sigma, X \subseteq |A|$, and $a, a' \in A$. We say *a is elementarily equivalent over X with respect of A to a'* if the map $: X \cup \{a\} \to X \cup \{a'\}$ which is the identity on X and maps a to a' is an elementary monomorphism.

THEOREM 3.7. *Suppose Σ is a theory in a countable language, L, and Σ has an infinite model. Then for every infinite κ there is a model of $\Sigma, A, \kappa(A) = \kappa$, such that for every countable $X \subseteq |A|, A$ contains only a countable number of elementary equivalence classes over X.*

Proof(16). Let $(X_\kappa, <)$ be a linearly ordered set having the order type of initial ordinal κ. Apply Theorem 3.4 to $\Sigma^{*\#}$ to get B and let $A = M(X_\kappa, B) \upharpoonright L$. Suppose Y is a countable subset of $|A|$. Then $Y \subseteq M(X_0, A)$ for some countable subset $X_0 \subseteq X_\kappa$. For each $a \in A$ there is some term $t(v_0, \cdots, v_n)$ in $L^{*\#}$ and elements $x_0, \cdots, x_n \in X_\kappa$ such that $a = t^A(x_0, \cdots, x_n)$. By 3.6(a) the elementary equivalence class of a over Y is determined by $t(v_0, \cdots, v_n)$ and the ordering relations between $x_0, \cdots, x_n$ and X_0. $L^{*\#}$ is a countable language and has only a countable number of distinct terms. X_0 is a countable well-ordered set and so there are only a countable number of ways of interpolating a finite set into it. Therefore A has only a countable number of equivalence classes over Y.

THEOREM 3.8. *If T is categorical in some power $\kappa > \aleph_0$ then T is totally transcendental.*

Proof(17). Suppose T were not totally transcendental. Then by 2.8 there would be a countable $C \in \mathscr{M}(T)$ with $\kappa(S(C)) > \aleph_0$. So we could certainly have a model of T, B, such that $\kappa(B) = \kappa, B \supseteq C$, and an uncountable number of points of $S(C)$ are realized in B. This B is clearly not isomorphic to the model of power κ proven to exist in Theorem 3.7.

A theory T may be categorical in power $\aleph_0$ and not be totally transcendental. For example, consider the theory of dense linearly ordered sets without end points. Let A be a linearly ordered set having the order type of the rationals. It can be shown that distinct Dedekind cuts in A correspond to distinct points in $S(A)$ so $\kappa(S(A)) = 2^{\aleph_0}$. By 2.8 the theory cannot be totally transcendental. Theorem 3.9, below, is proved by a generalization of this argument.

Suppose A is a model of T, R a relation of degree n of $A, X \subseteq |A|$, and S_n the permutation group on $(0, \cdots, n-1)$. Following Ehrenfeucht [1] we define R to be *connected over* X if for every sequence of n distinct elements $x_0, \cdots, x_{n-1}$ of X there is an $s \in S_n$ such that $\vdash_A R(x_{s(0)}, \cdots, x_{s(n-1)})$. R is *anti-symmetric over* X if for every sequence of n distinct elements $x_0, \cdots, x_{n-1}$ of X there is an $s \in S_n$ such that $\vdash_A \sim R(x_{s(0)}, \cdots, x_{s(n-1)})$.

THEOREM 3.9. *If T is totally transcendental and A a model of T, then no relation of A is connected and anti-symmetric over any infinite $X \subseteq |A|$* (18).

(16) For the case $X = \emptyset$, this result was obtained by Ehrenfeucht [2]. Indeed, he showed that if equivalence of two elements of A is defined to mean that there is an automorphism of A mapping one to the other, there is still a model of Σ of power κ which has only a countable number of equivalence classes. The proof is similar to that of 3.7 but X must be taken as a somewhat more complicated linear ordering.

(17) The crux of this proof, that $S(C)$ is countable for every countable $C \in \mathscr{M}(T)$, was established by Vaught [10] for the case where T is categorical in power $\kappa = \kappa^{\aleph_0}$.

(18) For the case where T is categorical in power 2^κ, this result was obtained by Ehrenfeucht [1]. Dana Scott (unpublished), by a different and simpler proof, extended the result to theories categorical in power $\kappa^{\aleph_0}$.

Proof. Suppose some relation of degree n of A, say R, were connected and anti-symmetric over an infinite set $X \subseteq |A|$. Impose an arbitrary linear order on X and say that two properly ordered n-tuples of X are equivalent, $(x_0, \cdots, x_{n-1}) \approx (x'_0, \cdots, x'_{n-1})$ if

$$\vdash_A \bigwedge_{s \in S_n} R(x_{s(0)}, \cdots, x_{s(n-1)}) \leftrightarrow R(x'_{s(0)}, \cdots, x'_{s(n-1)}).$$

Then "$\approx$" partitions the properly ordered n-tuples of X into a finite number of equivalence classes. By Ramsey's theorem we may find an infinite $Y \subseteq X$ such that every properly ordered n-tuple of Y is in the same equivalence class. That is, S_n may be decomposed into two sets S_n^+ and S_n^- such that for any $y_0 < \cdots < y_{n-1} \in Y$,

$$\text{(I)} \qquad \vdash_A \bigwedge_{s \in S_n^+} R(y_{s(0)}, \cdots, y_{s(n-1)}) \wedge \bigwedge_{s \in S_n^-} \sim R(y_{s(0)}, \cdots, y_{s(n-1)}).$$

R is connected and anti-symmetric on Y so neither S_n^+ nor S_n^- is empty. Hence there exists an $s_1 \in S_n^+$, $s_2 \in S_n^-$, and a cycle $(m-1, m)$ such that $s_1 = s_2 \cdot (m-1, m)$.

Using the Completeness Theorem one easily shows that the existence of Y implies that for any arbitrary order type, γ, there is a model of T, B, containing an ordered set, Y, of type γ and such that any $y_0 < \cdots < y_{n-1} \in Y$ satisfies (I). In particular, let Y have the order type of the real numbers and let $Z \subseteq Y$ be a countable dense subset. We assert that distinct elements in Y realize distinct points in $S(Z)$. For suppose $y < y' \in Y$. Pick $n-1$ elements of $Z, z_0, \cdots, z_{m-1}, z_{m+1}, \cdots, z_{n-1}$ such that

$$z_0 < \cdots < z_{m-2} < y < z_{m-1} < y' < z_{m+1} < \cdots < z_{n-1}.$$

Then $(z_0, \cdots, z_{m-1}, y', z_{m+1}, \cdots, z_{n-1})$ will, after permutation by s_1, satisfy R. But $(z_0, \cdots, z_{m-1}, y, z_{m+1}, \cdots, z_{n-1})$ will, after permutation by s_1, not satisfy R, since its proper order will now be permuted by s_2. So $\kappa(S(Z)) = 2^{\aleph_0}$ and T cannot be totally transcendental.

4. **Models of totally transcendental theories.** A neat characterization of models of a theory T is given by the following lemma.

LEMMA 4.1. *$A \in \mathscr{N}(T)$ is a model of T if and only if the points of $S(A)$ which are realized in A form a dense subset of $S(A)$.*

Proof. By 2.1 there is a $B \supseteq A$ such that B is a model of T which realizes every point in $S(A)$. By 1.1(c) a necessary condition for A to be a model of T is that i_{AB} be an elementary monomorphism. Trivially, this is also a sufficient condition. By a theorem of Tarski[19] a necessary and sufficient condition that i_{AB} be an elementary monomorphism is that every formula of $F(A)$ which is satisfied by some $b \in B$ be also satisfied by some $a \in A$.

[19] Theorem 1.10 of [15].

But every $\psi \in F(A)$ and consistent with $T(A)$ (i.e., $\neq 0$ in the Boolean algebra $F(A)$) is satisfied in B. Hence, a necessary and sufficient condition that A be a model of T is that every $\psi \neq 0$ in $F(A)$ be satisfied in A, which is equivalent to the condition that some point in each of the sets $U_\psi = \{p \in S(A); \psi \in p\}$ is realized in A. But the sets U_ψ $(\psi \in F(A))$ form a basis for $S(A)$, and the lemma is proved.

LEMMA 4.2. *If T is totally transcendental then for every $A \in \mathscr{N}(T)$ the isolated points are dense in $S(A)$; indeed if U is an open set of $S(A)$ and $p \in U$ is a point of the minimal transcendental rank of the points of U, then p is an isolated point in $S(A)$.*

Proof. Suppose $p \in U$ is of the minimal transcendental rank, say α, of the points of U. By definition there is a neighborhood V of p such that $V \cap S^\alpha(A) = \{p\}$. But $U \cap S^\alpha(A) = U$. So $V \cap S^\alpha(A) \cap U = V \cap U = \{p\}$, and p is isolated.

Suppose $A, B \in \mathscr{N}(T), B \supseteq A$, and B is a model of T. B is *prime* over A if for every model of T, B', and monomorphism $f: A \to B'$, there is a monomorphism $g: B \to B'$ with $f = g$ on A.

THEOREM 4.3. *Suppose T is such that for every $A \in \mathscr{N}(T)$ the isolated points are dense in $S(A)$, then every $A \in \mathscr{N}(T)$ has a model of T prime over it*[20].

Proof. Let $A \in \mathscr{N}(T)$ and $\kappa = \kappa(A) + \aleph_0$. Then $S(A)$ has at most κ isolated points. Let $\{p_\alpha; \alpha < \kappa\}$ be a listing (possibly with repetitions) of the isolated points of $S(A)$. Choose some increasing chain $\{A_\alpha; \alpha < \kappa\}$ of members of $\mathscr{N}(T)$ such that: (1) $A_0 = A$, (2) $A_\delta = \bigcup_{\beta<\delta} A_b$, (3) $A_{\alpha+1} = A_\alpha$ if p_α is realized in A_α, and (4) if p_α is not realized in A_α, then $A_{\alpha+1} - A_\alpha$ has a single element, a_α, which realizes some isolated point q in $S(A_\alpha)$ such that $q \supseteq p_\alpha$.

If C is a model of T and $f_0: A \to C$ is a monomorphism then there is a sequence of monomorphisms $\{(f_\alpha: A_\alpha \to C); \alpha < \kappa\}$ such that for $\alpha' > \alpha$, $f_{\alpha'}$ extends f_α. This is proved by induction on α. The induction is trivial in cases (1), (2), and (3) above. In case (4) suppose $f_\alpha: A_\alpha \to C$ is a monomorphism and $a_\alpha \in A_{\alpha+1} - A_\alpha$ satisfies the isolated point q in $S(A_\alpha)$. Then $f_\alpha^{*-1}(q)$ is an open set in $S(C)$ and by hypothesis contains an isolated point, say q'. By 4.1 there is a $c \in C$ realizing q'. Let $f_{\alpha+1}(a_\alpha) = c$ and the monomorphism is extended.

$A_\kappa = \bigcup_{\alpha<\kappa} A_\alpha$ then realizes every isolated point in $S(A)$ and every monomorphism of A into a model of T can be extended to a monomorphism of A_κ. We may now list the isolated points of $S(A_\kappa)$ and repeat the above process to get an $A_{\kappa\cdot 2}$ realizing every isolated point in $S(A_\kappa)$ and

[20] For the case where A is countable the existence of a prime model over A was proved under a somewhat weaker hypothesis in [18].

such that every monomorphism of A into a model of T may be extended to a monomorphism of $A_{\kappa\cdot 2}$. Iterating ω times we obtain

$$A_{\kappa\cdot\omega} = \bigcup_{n\in\omega} A_{\kappa\cdot n}$$

such that any monomorphism of A into a model of T can be extended to a monomorphism of $A_{\kappa\cdot\omega}$, and $A_{\kappa\cdot\omega}$ realizes every isolated point in $S(A_{\kappa\cdot n})$ for each $n \in \omega$. But the topology on $S(A_{\kappa\cdot\omega})$ is that induced by the $S(A_{\kappa\cdot n})$'s; for each formula $\psi \in F(A_{\kappa\cdot\omega})$ must be already in some $F(A_{\kappa\cdot n})$, hence the neighborhood U_ψ of $S(A_{\kappa\cdot\omega})$ is the inverse image of the corresponding neighborhood in $S(A_{\kappa\cdot n})$. So, $A_{\kappa\cdot\omega}$ realizes every isolated point in $S(A_{\kappa\cdot\omega})$ and is by 4.1 a model of T.

For the next theorem we shall need some results about increasing sequences of systems and the corresponding sequence of Boolean spaces. We summarize these in the next lemma.

LEMMA 4.4. *Suppose T is totally transcendental.* (a) *If $\{A_\alpha; \alpha < \gamma\}$ is an increasing sequence of members of $\mathscr{N}(T)$, $A = \bigcup_{\alpha<\gamma} A_\alpha$, and $\{p_\alpha; \alpha < \gamma\}$ a sequence such that $p_\alpha \in S(A_\alpha)$ and $i^*_{A_\alpha A_\beta}(p_\beta) = p_\alpha$ $(\alpha \leqq \beta < \gamma)$ then:*

(i) *there is an $\alpha_0 < \gamma$ such that for all α, if $\alpha_0 \leqq \alpha < \gamma$ then transcendental rank and degree of p_α equal the transcendental rank and degree of p_{α_0}, and*

(ii) *there is a unique $p \in S(A)$ such that*

$$p \in \bigcap_{\alpha<\gamma} i^{*-1}_{A_\alpha A}(p_\alpha).$$

This point will have transcendental rank and degree equal to that of the p_{α_0} defined in (i).

(b) *If $\{A_\alpha; \alpha < \gamma\}$ is an increasing sequence of members of $\mathscr{N}(T)$ and p is an isolated point in $S(A_0)$, then there is a sequence $\{p_\alpha; \alpha < \gamma\}$ of points such that $p_\alpha \in S(A_\alpha)$ $(\alpha < \gamma)$, $p_0 = p$, $i_{A_\alpha A_\beta}(p_\beta) = p_\alpha$ $(\alpha < \beta < \gamma)$ and each p_α is isolated in $S(A_\alpha)$.*

Proof. (a) If $\gamma = \beta + 1$ then $A = A_\beta$ and the result is trivial. Suppose $\gamma =$ a limit ordinal δ. By 2.3, $\beta \geqq \alpha$ implies transcendental rank $p_\beta \leqq$ transcendental rank p_α. Since there can be no infinite decreasing sequence of ordinal numbers, the transcendental rank must remain constant from some α on. By a similar argument (now using 2.5), the transcendental rank and degree must remain constant from some α_0 on. Let p_{α_0} have transcendental rank ν and degree n. By 2.3, $\bigcap_{\alpha<\delta} i^{*-1}_{A_\alpha A}(p_\alpha)$ can have no point of rank $> \nu$, and $i_{A_0 A}(p_{\alpha_0}) \cap S^\nu(A)$ is not empty; but by 2.5(b) $i^{*-1}_{A_{\alpha_0} A_\alpha}(p_{\alpha_0}) \cap S^\nu(A_\alpha) = \{p_\alpha\}$ (for $\alpha \geqq \alpha_0$) and so

$$\bigcap_{\alpha<\delta} i^{*-1}_{A_\alpha A}(p_\alpha) \cap S^\nu(A) = i^{*-1}_{A_{\alpha_0} A}(p_{\alpha_0}) \cap S^\nu(A).$$

We assert that $\bigcap_{\alpha<\delta} i^{*-1}_{A_\alpha A}(p_\alpha)$ can contain only one point. For suppose it contained distinct points p_1 and p_2. Then there would be a formula $\psi \in F(A)$ such that $\psi \in p_1$ and $\neg\psi \in p_2$. There is some $\alpha < \delta$ such that $\psi \in F(A_\alpha)$ and so $\psi \in p_\alpha$ and $\neg\psi \in p_\alpha$ which is impossible. (Topologically, this argument amounts to the statement that $S(A)$ is a Hausdorff space with its topology determined by that of the $S(A_\alpha)$'s.) By 2.5 this unique $p \in S(A)$ must have degree equal to the degree of p_{α_0} and (a) is proved.

(b) There is no loss of generality in assuming that for limit ordinals $\delta, A_\delta = \bigcup_{\alpha<\delta} A_\alpha$; for whenever it is not so for some δ we may interpolate $\bigcup_{\alpha<\delta} A_\alpha$ into the sequence. Consider a sequence $\{p_\alpha; \alpha < \gamma\}$ such that for each $\alpha, p_\alpha \in S(A_\alpha)$ and p_α is a point of minimal transcendental rank in $\bigcap_{\beta<\alpha} i^{*-1}_{A_\beta A_\alpha}(p_\beta)$. We show inductively that such a sequence exists and that it satisfies (b). Assume a sequence defined and satisfying (b) for $\beta < \alpha$. If $\alpha = \beta + 1$, then by 4.2 any point of minimal transcendental rank in $i^{*-1}_{A_\beta A_\alpha}(p_\beta)$ is isolated.

If $\alpha = \delta$ then by (a) and its proof

$$\bigcap_{\beta<\delta} i^{*-1}_{A_\beta A_\delta}(p_\beta) = i^{*-1}_{A_{\alpha_0} A_\delta}(p_{\alpha_0}) \cap S^\nu(A_\delta)$$

and is a single point, say p_δ. The point p_{α_0} is isolated in $S(A_{\alpha_0})$ so $i_{A_{\alpha_0} A_\delta}(p_{\alpha_0})$ is an open set in $S(A_\delta)$. Thus to prove p_δ isolated in $S(A_\delta)$ it will suffice to show that $i^{*-1}_{A_{\alpha_0} A_\delta}(p_{\alpha_0}) \cap S^\nu(A_\delta) = i^{*-1}_{A_{\alpha_0} A_\delta}(p_{\alpha_0})$. Suppose this equality did not hold. Then there would be a $p' \in i^{*-1}_{A_{\alpha_0} A_\delta}(p_{\alpha_0})$ with transcendental rank of $p' < \nu$. By the argument used in the proof of (a) there would be a $\beta, \alpha_0 \leqq \beta < \delta$, such that $i^*_{A_\beta A_\delta}(p') \neq i_{A_\beta A_\delta}(p_\delta) = p_\beta$. Since p_β has transcendental rank and degree the same as p_{α_0} and $i^*_{A_\beta A_\delta}(p') \in i^{*-1}_{A_{\alpha_0} A_\beta}(p_{\alpha_0})$, by 2.5 transcendental rank of $i^*_{A_\beta A_\delta}(p') <$ transcendental rank $(p_{\alpha_0}) =$ transcendental rank (p_β). This contradicts the assumption that p_β is of minimal rank in $\bigcap_{\beta'<\beta} i^{*-1}_{A_{\beta'} A_\beta}(p_{\beta'})$

THEOREM 4.5. *Suppose T is totally transcendental and $\{A_\alpha; \alpha < \gamma\}$ is an increasing chain of members of $\mathscr{M}(T)$ such that for each limit ordinal $\delta < \gamma$, $A_\delta = \bigcup_{\alpha<\delta} A_\alpha$. Then there is an increasing chain $\{B_\alpha; \alpha < \gamma\}$ of models of T such that B_α is prime over A_α (for each $\alpha < \gamma$) and for each limit ordinal $\delta < \gamma, B_\delta = \bigcup_{\alpha<\delta} B_\alpha$*[21].

Proof. Let $A = \bigcup_{\alpha<\gamma} A_\alpha$. We shall show inductively that there exists an increasing sequence of systems $\{C_\alpha; \alpha < \gamma\}$ and of models of $T, \{B_\alpha; \alpha < \gamma\}$ such that (i) $C_\alpha = A \cup B_\alpha$, (ii) $B_\alpha \supseteq A_\alpha$, (iii) $B_\delta = \bigcup_{\beta<\delta} B_\beta$ (for limit ordinals $\delta < \gamma$), and (iv) if D is a model of T, $\alpha = \beta + 1$, $\alpha' \geqq \alpha$ and $f: A_{\alpha'} \cup B_\beta \to D$ is a monomorphism then there is a monomorphism $g: A_{\alpha'} \cup B_\alpha \to D$ with $g \supseteq f$. The sequence $\{B_\alpha; \alpha < \gamma\}$ will then satisfy the theorem.

(21) It may be shown by example that the assumption that T is totally transcendental is stronger than the assumption that the isolated points are dense in $S(A)$ for every $A \in \mathscr{M}(T)$. Theorem 4.3 was proved under the weaker assumption but we have been unable to do the same for Theorem 4.5.

Assume the sequence $\{C_\beta; \beta < \alpha\}$ satisfying (i)-(iv). If $\alpha = \delta$ let $C_\delta = \bigcup_{\beta<\delta} C_\beta$ and $B_\delta = \bigcup_{\beta<\delta} B_\beta$.

If $\alpha = \beta + 1$ we proceed as in the proof of Theorem 4.3. Let $\{p_\nu; \nu < \kappa\}$ be a list of the isolated points of $S(A_\alpha \cup B_\beta)$. By 4.4 we may find a sequence of points $\{p_{0,\eta}; \alpha \leqq \eta < \gamma\}$ such that $p_{0,\alpha} = p_0, p_{0,\eta}$ is an isolated point of $S(A_\eta \cup B_\beta)$ and $\eta' > \eta$ implies $p_{0,\eta'} \supseteq p_{0,\eta}$. Let $q_0 = \bigcup_{\eta<\gamma} p_{0,\eta}$. If there is an element of C_β realizing q_0 denote it by a_0; otherwise add an element satisfying q_0 to C_β and denote it by a_0. By the method of the proof of 4.3 we may iterate this process $\kappa \cdot \omega$ times and find a sequence $\{a_\nu; \nu < \kappa \cdot \omega\}$ such that $A_\alpha \cup B_\beta \cup \{a_\nu; \nu < \kappa \cdot \omega\}$ is a model of T, and for each α' $(\alpha \leqq \alpha' < \gamma)$ a_ν realizes an isolated point in $S(A_{\alpha'} \cup B_\beta \cup \{a_{\nu'}; \nu' < \nu\})$. This latter condition implies (by the same argument used in the proof of 4.3) that (iv) holds for α.

Let $C_\alpha = C_\beta \cup \{a_\nu; \nu < \kappa \cdot \omega\}$ and $B_\alpha = B_\beta \cup \{a_\nu; \nu < \kappa \cdot \omega\}$.

Using condition (iv), above, a simple induction shows that any monomorphism of A_α into a model of T may be extended to a monomorphism of B_α into the same model, i.e., B_α is prime over A_α. Theorem 4.5 is proved.

Suppose $A, B \in \mathscr{N}(T), A \subseteq B$, and $X \subseteq |B| - |A|$. X is a set of elements *indiscernible* over A if every one-one map of

$$|A| \cup X \to |A| \cup X$$

which is the identity on $|A|$ is a monomorphism. That is, for any open formula, ψ, of L, any $a_1, \cdots, a_m \in A$, and any two sets of distinct elements $x_1, \cdots, x_n$ and $x_1', \cdots, x_n' \in X$; $\psi(a_1, \cdots, a_m, x_1, \cdots, x_n)$ if and only if $\psi(a_1, \cdots, a_m, x_1', \cdots, x_n')$.

THEOREM 4.6. *Suppose T is totally transcendental, $A, B \in \mathscr{N}(T), A \subseteq B$, and $\kappa(A) < \kappa(B) = \kappa$. Then* (i) *if κ is a regular uncountable cardinal, there is an $X \subseteq |B| - |A|$ such that $\kappa(X) = \kappa$ and X is a set of elements indiscernible over A*; (ii) *if κ is uncountable but not regular there is still for each $\lambda < \kappa$ a set $X \subseteq |B| - |A|$ such that $\kappa(X) > \lambda$ and X is a set of elements indiscernible over A.*

Proof. Since for every infinite λ, λ^+ is regular, (ii) will follow immediately from (i) by choosing some $C, A \subseteq C \subseteq B$ and $\kappa(C)$ regular.

So assume κ regular. Suppose $C \in \mathscr{N}(T), \kappa(C) < \kappa$ and $A \subseteq C \subseteq B$. By 2.7 $\kappa(S(C)) < \kappa$, and from the regularity of κ it follows that there is some $p \in S(C)$ which is realized by κ distinct elements of B. From the set of all pairs (C, p) satisfying the above conditions we pick one, say (C_0, p_0), such that transcendental rank of p_0 is the minimum, say ν, and the degree of p_0 is the minimum, say n, among those having rank ν.

Suppose $C' \in \mathscr{N}(T)$, $\kappa(C') < \kappa$ and $C_0 \subseteq C' \subseteq B$. Then $i^{*-1}_{C_0C'}(p_0)$ has power $< \kappa$ and hence must contain some point, p', which is realized by κ elements of B. Since ν is the minimal transcendental rank of such points, transcendental rank $(p') \geq \nu$. But by 2.3 transcendental rank $(p') \leq$ transcendental rank $(p_0) = \nu$. A similar argument may be made for degree, so transcendental rank $(p_0) =$ transcendental rank (p') and degree (p_0) $=$ degree(p'). By 2.5 there is only one such point in $i^{*-1}_{C_0C'}(p_0)$. Thus, $i^{*-1}_{C_0C'}(p_0)$ has exactly one point realized by κ elements and that point has transcendental rank ν and degree n.

We show that inductively that there exists a set of κ distinct elements $\{x_\alpha; \alpha < \kappa\} \subseteq |B| - |C_0|$ such that, letting $C_\alpha = C_0 \cup \{x_\beta; \beta < \alpha\}$ and $p_\alpha \in S(C_\alpha)$ the point realized by x_α, p_α is the unique point of rank ν and degree n in $i^{*-1}_{C_0C_\alpha}(p_0)$. For if $\{x_\beta; \beta < \alpha\}$ is defined, then by the discussion of the preceding paragraph there are κ elements of B which realize p_α and we pick x_α to be one of these. Notice that $\beta < \alpha$ implies $i^{*}_{C_\beta C_\alpha}(p_\alpha) = p_\beta$ and hence x_α realizes p_β for all $\beta \leq \alpha$.

Suppose $\beta_1 < \cdots < \beta_m$ and $\beta'_1 < \cdots < \beta'_m$. Denote by D_m and D'_m the systems having universe $|C_0| \cup \{x_{\beta_1}, \cdots, x_{\beta_m}\}$ and $|C_0| \cup \{x_{\beta'_1}, \cdots, x_{\beta'_m}\}$ respectively. We assert that the map $f_m: D_m \to D'_m$ which is the identity on C_0 and carries x_{β_i} to $x_{\beta'_i}$ $(i \leq m)$ is an isomorphism. Then proof is by induction on m. Assume $f_{m-1}: D_{m-1} \to D'_{m-1}$ is an isomorphism. Let q be the point of $S(D_{m-1})$ realized by x_{β_m} and q' the point of $S(D'_{m-1})$ realized by $x_{\beta'_m}$. To prove f_m to be an isomorphism it is sufficient to show that $f^*_{m-1}(q') = q$. Since x_{β_m} realizes a point (namely p_0) of transcendental rank ν and degree n in $S(C_0)$ and a point (namely p_{β_m}) of transcendental rank ν and degree n in $S(C_{\beta_m})$ and $C_0 \subseteq D_{m-1} \subseteq C_{\beta_m}$, it follows from 2.3 and 2.5 that q is of transcendental rank ν and degree n. As proved above, there is a unique point of transcendental rank ν and degree n in $i^{*-1}_{C_0D_{m-1}}(p_0)$, and q must be this point. Similary, q' must be the unique point of rank ν and degree n in $i^{*-1}_{C_0D'_{m-1}}(p_0)$. Since f_{m-1} is the identity on C_0,

$$f^*_{m-1}(i^{*}_{C_0D'_{m-1}}(p_0)) = i^{*-1}_{C_0D_{m-1}}(p_0).$$

Therefore $f^*_{m-1}(q') = q$ and f_m is an isomorphism.

Finally, we assert that X is indiscernible over A, indeed over C_0. Consider an open formula ψ of $L, a_1, \cdots, a_m \in C_0$, and sequences of distinct elements $(x_{\beta_1}, \cdots, x_{\beta_r})$ and $(x_{\beta'_1}, \cdots, x_{\beta'_r})$ in X. We must show that $\psi(a_1, \cdots, a_m, x_{\beta_1}, \cdots, x_{\beta_r})$ if and only if $\psi(a_1, \cdots, a_m, x_{\beta'_1}, \cdots, x_{\beta'_r})$. We have already shown this in the case when $\beta_1 < \cdots < \beta_r$ and $\beta'_1 < \cdots < \beta'_r$. But by 3.9, $\psi(a_1, \cdots, a_m, x_{\beta_1}, \cdots, x_{\beta_r})$ cannot depend on the order of the β_i's. (We actually apply 3.9 to the theory $T(\{a_1, \cdots, a_m\})$ which extends T by adding $a_1, \cdots, a_m$ as "distinguished elements" but by 2.8, $T(\{a_1, \cdots, a_m\})$ is totally transcendental if T is.) Theorem 4.6 is now proved.

5. **Saturated models and categoricity in power.** Suppose B is an infinite system $\in \mathcal{N}(T)$. B is *saturated* if for every $A \subseteq B$ with $\kappa(A) < \kappa(B)$, every point of $S(A)$ is realized in B.

From 4.1 we see that if $B \in \mathscr{N}(T)$ is saturated, then B is a model of T. Saturated systems were considered in [10], and the following result was established([22]).

THEOREM 5.1. *If A and B are saturated models of T of the same power, then A is isomorphic to B.*

Thus a sufficient condition for T to be categorical in power κ is that every model of T of power κ be saturated([23]).

Suppose $B \in \mathscr{N}(T)$ is an uncountable system. B is *saturated over countable subsystems* if for every countable $A \subseteq B$, B realizes every point of $S(A)$. By 4.1, every $B \in \mathscr{N}(T)$ which is saturated over countable subsystems is a model of T.

THEOREM 5.2. *If T is totally transcendental and $\kappa > \aleph_0$, then there is a model of T of power κ which is saturated over countable subsystems*([24]).

Proof. Let B_0 be an arbitrary model of T of pwer κ. Then $S(B_0) = \kappa$ by 2.7. Therefore, there is a model of T, $B_1 \supseteq B_0$ such that $\kappa(B_1) = \kappa$ and every point of $S(B_0)$ is realized in B_1. Proceeding inductively, we see that there is an increasing chain of models of T of power κ, $\{B_\alpha; \alpha < \omega_1\}$ such that every point of $S(B_\alpha)$ is realized in $B_{\alpha+1}$ (for all $\alpha < \omega_1$). Then $B = \bigcup_{\alpha<\omega_1} B_\alpha$ is a model of T of power κ which is saturated over countable subsystems. For if A is a countable subsystem of B, then there is an $\alpha < \omega_1$ such that $A \subseteq B_\alpha$; then every $p \in S(A)$ is realized in $B_{\alpha+1}$ and, *a fortiori*, in B.

LEMMA 5.3. *Suppose T is totally transcendental and B is an uncountable model of T which is not saturated. Then there is a countable model of T, $A \subseteq B$, with a subsystem $A' \subseteq A$ such that* (i) *there is an infinite set $Y \subseteq |A| - |A'|$ of elements indiscernible over A', and* (ii) *there is a $q \in S(A')$ which is not realized in A.*

([22]) In [10] *universal homogeneous* systems are considered. This is a terminology of Jónsson [5]. If K is a class of similar relational systems and $A \in K$ then: (1) A is universal for K if A contains an isomorphic image of every $B \in K$ with $\kappa(B) \leqq \kappa(A)$, (2) A is homogeneous in K if whenever $B_1, B_2 \in K$, $B_1, B_2 \subseteq A$, $\kappa(B_i) < \kappa(A)$, and $f: B_1 \rightarrow B_2$ is an isomorphism, then f may be extended to an automorphism of A. Jónsson showed that under certain simple conditions on K that any two universal homogenous systems of the same power are isomorphic. In the case that $K = \mathscr{N}(T)$, universal-homogeneous is equivalent to saturated. This was shown in the countable case by Vaught [18] and in the uncountable case by Keisler (Theorem A2 of [8]).

([23]) That the problem of categoricity in power could be approached this way was noticed by Vaught. He proved [10; 17] (assuming the generalized continuum hypothesis) that if T is categorical in an increasing sequence of powers then it is categorical in the limit power.

([24]) In the case $\kappa = \kappa^{\aleph_0}$, this result was proved in [10] without the assumption that T is totally transcendental. However, it is possible to give an example of a theory T which is not totally transcendental and a cardinal $\kappa > \aleph_0$ with $\kappa^{\aleph_0} \neq \kappa$ such that no model of T of power κ is countably saturated.

Proof. Since B is not saturated there is some $C \subseteq B, \kappa(C) < \kappa(B)$, and a $p \in S(C)$ which is not realized in B. By 4.6 there is a countable infinite set, Y, of elements indiscernible over C contained in $|B| - |C|$. By the Löwenheim-Skolem theorem there is a countable submodel of B, A_0, such that $A_0 \supseteq Y$. For each $a \in A_0$ let p_a be the point of $S(C)$ realized by a. Then no $p_a = p$ since no element of B realizes p. Hence, there is for each $a \in A_0$ a formula $\psi_a \in F(C)$ such that $\psi_a \in p_a$ and $\sim \psi_a \in p$. Since ψ_a involves only a finite number of sysmbols we may find for each some finite $C_a \subseteq C$ such that $\psi_a \in F(C_a)$. Let $A'_1 = \bigcup_{a \in A_0} C_a$. Then no $a \in A_0$ realizes $i^*_{A'_1C}(p)$ in $S(A'_1)$. Let A_1 be a countable submodel of B such that $A_1 \supseteq A_0 \cup A'_1$. By iteration we may find a sequence of countable models, $A_0 \subseteq \cdots A_n \subseteq \cdots$, and a sequence of systems, $A'_1 \subseteq \cdots A'_n \subseteq \cdots$, such that $A'_n \subseteq A_n \cap C$ and no $a \in A_n$ realizes $i^*_{A'_{n+1}C}(p)$ in $S(A'_{n+1})$. Let $A = \bigcup_{n \in \omega} A_n$ and $A' = \bigcup_{n \in \omega} A'_n$. Then $Y \subseteq |A| - |A'|$ is a set of elements indiscernible over A' and no $a \in A$ realizes $i^*_{A'C}(p)$ in $S(A')$.

THEOREM 5.4. *Suppose T is totally transcendental and has an uncountable model which is not saturated. Then for each $\kappa > \aleph_0$, T has a model of power κ which is not saturated over countable subsystems.*

Proof. Let A, A', and Y be as in Lemma 5.3 and $q \in S(A')$ be not realized in A. By the completeness theorem there is an $A_\kappa \in \mathcal{M}(T)$ such that $A_\kappa \supseteq A' \cup Y$ and $A_\kappa - A'$ is a set of κ elements indiscernible over A'. (For we can assert the existence of such an A_κ by a set Σ (of power κ) of sentences, and the existence of $A' \cup Y$ shows that every finite subset of Σ, and therefore Σ, is consistent.) Let $\{y_\alpha; \alpha < \kappa\}$ be a well-ordering of $A_\kappa - A'$, and $A_\alpha = A' \cup \{y_\beta; \beta < \alpha\}$. Apply Theorem 4.5 to get an increasing chain of models of T, $\{\beta_\alpha; \alpha < \kappa\}$, with B_α prime over A_α and for each limit ordinal $\delta < \kappa$, $B_\delta = \bigcup_{\alpha < \delta} B_\alpha$.

We assert that q is not realized in any B_α. The proof is by induction on α. For $\alpha < \omega$, the existence of the model $A \supseteq A' \cup Y$ and not realizing q, implies B_α does not realize q. If $\alpha = \delta$, the induction hypothesis implies no B_β $(\beta < \delta)$ realizes q and, hence, $B_\delta = \bigcup_{\beta < \delta} B_\beta$ does not realize q. Finally, if $\alpha = \beta + 1 > \omega$, then by the indiscernibility of $A_\kappa - A'$ over A', there is an isomorphism of A_α onto A_β which is the identity on A'. So there is a monomorphism of B_α into B_β which is the identity on A'. By the induction hypothesis B_β does not realize q, therefore B_α does not realize q.

$B_\kappa = \bigcup_{\alpha < \kappa} B_\alpha$ is of power κ and does not realize q.

THEOREM 5.5. *If T is categorical in some power $\kappa > \aleph_0$, then every uncountable model of T is saturated.*

Proof. By 3.8, T is totally transcendental. By 5.2, there is a model of T of power κ which is saturated over countable subsystems. If T had an uncountable model which was not saturated, then by 5.4 it would have a model of power κ which was not saturated over countable subsystems, and T would not be categorical in power κ.

THEOREM 5.6. *If T is categorical in one uncountable power then T is categorical in every uncountable power.*

Proof. The proof is immediate from 5.1 and 5.5.

We shall conclude by mentioning some open questions[25]. The first two questions are about theories categorical in uncountable powers but not in power $\aleph_0$.

(1) Does every such theory have exactly $\aleph_0$ isomorphism types of countable models?

(2) Is any such theory finitely axiomatizable?

The next two questions concern theories in languages with an uncountable number of symbols.

(3) If $\kappa > \aleph_0$, Σ is a theory in a language having $\leqq \kappa$ symbols, and Σ is categorical in some power $> \kappa$, is Σ necessarily categorical in every power $> \kappa$?

(4) If $\kappa > \aleph_0$ and every model of Σ has power $\geqq \kappa$ can Σ be categorical in power κ?

We return to theories in countable languages. From 4.3 and 4.6 it follows that if T is totally transcendental and $\kappa > \aleph_0$ we may find a model of T, A, and a set $X \subseteq |A|$ with $\kappa(X) = \kappa(A) = \kappa$ such that any one-one map of X into itself may be extended to an endomorphism of A. This raises the following question.

(5) If T is totally transcendental and $\kappa \geqq \aleph_0$, is there always a model of T, A, with a set $X \subseteq |A|$ such that $\kappa(A) = \kappa(X) = \kappa$ and any one-one map of X onto itself may be extended to an automorphism of A?

Notice this would follow from 3.5 and 3.9 if whenever T were totally transcendental we could find a $T^{\#}$ which was totally transcendental. In [1], Theorem 2 asserts the affirmative of this question for theories categorical in power 2^{κ}, but Vaught has pointed out a fallacy in the proof given.

Finally, we consider some questions about the ordinal α_T defined in 2.6. In 2.6 we showed that $\alpha_T < (2^{\aleph_0})^+$. The first question is:

(6) Is α_T ever uncountable?

We can answer this question in one case.

THEOREM 5.7. *If T is totally transcendental, $\alpha_T < \omega_1$.*

Proof. By 2.4 if $p \in \mathrm{Tr}^{\alpha}(A)$ there is a finite $B \subseteq A$ such that $i^*_{BA}(p) \in \mathrm{Tr}^{\alpha}(B)$. By 2.7 $S(B)$ is countable for every finite $B \in \mathcal{M}(T)$. Thus we

(25) Problems (1) through (4) below are not due to the author; they seem to have been considered by several people. Problem (5) has recently been answered affirmatively by Jack Silver.

need only to show that there are only a countable number of isomorphism types of finite members of $\mathcal{N}(T)$. We prove inductively for each $n \in \omega$ that there are only a countable number of isomorphism types of members of $\mathcal{N}(T)$ of power n. For $n = 0$ there is obviously only one. (Strictly, the empty set is not a subsystem. But since we can define $F(\emptyset)$, there is no harm in treating it as a member of $\mathcal{N}(T)$.) Assume only a countable number of isomorphism types of systems of power m. By 2.7 there are only a countable number of ways of adding an element to each system of power m, so there are only a countable number of isomorphism types of members of $\mathcal{N}(T)$ of power $m + 1$.

Another question is:

(7) What model-theoretical conditions on T imply that α_T is finite? Plausible possiblilities are T being categorical in some power, or $T = \Sigma^*$ with Σ finitely axiomatizable.

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HYPERANALYTIC PREDICATES

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Consider the two operations on number-theoretic predicates which allow us to (1) pass from a predicate $P(x_1, \ldots, x_n)$ to a predicate $Q(y_1, \ldots, y_m)$ recursive in $P(x_1, \ldots, x_n)$, and (2) pass from a predicate $P(y, x_1, \ldots, x_n)$ to the predicate $(Ey)P(y, x_1, \ldots, x_n)$. If we start with recursive predicates and apply these operations any finite number of times, we obtain the arithmetical hierarchy. If we now extend application of these operations into the transfinite (letting the predicates that we are defining determine, in any one of several equivalent ways, "how many times" these operations can be iterated) we obtain the hyperarithmetic hierarchy.

From this point of view the hyperarithmetic predicates are just those which can be defined "constructively" from recursive predicates, except for use of the existential number quantifier. This is made precise in Kleene's XLVIII of RFI(2), which asserts that a number-theoretic predicate is hyperarithmetic if and only if it is recursive in the type-2 object $^2\mathbf{E}$ that embodies number quantification,

$$\begin{aligned} {}^2\mathbf{E}(\alpha) &= 0 \quad \text{if } (Et)[\alpha(t) = 0], \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

Kleene extends this approach in RFII to higher types, by calling a predicate $P(\alpha_1, \ldots, \alpha_n, x_1, \ldots, x_m)$ of number and function variables *hyperanalytic*, if it is recursive in the type-3 object $^3\mathbf{E}$ which embodies function quantification,

$$\begin{aligned} {}^3\mathbf{E}(\mathbf{F}) &= 0 \quad \text{if } (E\alpha)[\mathbf{F}(\alpha) = 0], \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

He then defines a hierarchy $H_a^2(\alpha)$ $(a \in O^2)$ of hyperanalytic predicates, very similar to the hierarchy $H_a(x)$ $(a \in O)$ of hyperarithmetic predicates, and asks if every hyperanalytic predicate is recursive in some $H_a^2(\alpha)$. Clarke studies this hierarchy in some detail in [1] and conjectures that it does not exhaust the hyperanalytic predicates.

In §1 we prove Clarke's conjecture. What seems to go wrong in the definition of O^2 and $H_a^2(\alpha)$, is that only countable ordinals (which are order-types of hyperanalytic well-orderings) are used, and in an effective manner. Since $^3\mathbf{E}$ is powerful

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(2) References to the bibliography are by number, except for Kleene's [3], [6] and [7] which we call IM, RFI, and RFII respectively.

enough to allow us to refer to the totality of countable ordinals, we are able to diagonalize through O^2 and define a hyperanalytic predicate, recursive in no $H_a^2(\alpha)$.

In this paper we define and study a hierarchy which exhausts the class of hyperanalytic predicates. Our definition is a generalized induction (like those of O in [4] and $\{z\}(\mathfrak{c})\simeq w$ in RFI) whose classical interpretation as a transfinite induction employs uncountable ordinals. (Since there are only countably many hyperanalytic predicates, our definition yields a hierarchy indexed by countable ordinals, in fact, exactly those which are order-types of hyperanalytic well-orderings; but this is "after the fact," and is just noneffective enough to prohibit diagonalization.)

A number-theoretic predicate is hyperarithmetic if and only if both it and its negation are Π_1^1, i.e., expressible in the form $(\alpha)A(\alpha, x_1, \ldots, x_n)$, with A arithmetical. The correct analog of Π_1^1 in type 3 seems to be *recursively enumerable in* $^3\mathbf{E}$, i.e., expressible in the form "$\mathrm{f}(\alpha_1, \ldots, \alpha_n, x_1, \ldots, x_m)$ is defined," with f partial recursive in $^3\mathbf{E}$. In §§4, 5 we prove some elementary properties of predicates r.e. in $^3\mathbf{E}$, including the analog of the statement above, several "choice axioms" and the fact that this class of predicates is closed under number quantification. These results are inspired by Gandy's [2], where similar theorems are used to study the theory of hyperarithmetic predicates relativized to an arbitrary type-2 object.

In §6 we give a slight improvement of Kleene's representation theorems for the predicate $\{z\}(\mathfrak{c})\simeq w$ (for type-3), and we prove that the class of predicates r.e. in $^3\mathbf{E}$ is not closed under existential function quantification. This leads us in §7 to a study of a hierarchy that extends the hyperanalytic predicates and is similar in many respects to the hierarchy of Δ_2^1 predicates.

We use the notation and terminology of Kleene's RFI and RFII, with very few exceptions. However the only results about recursive functionals that are essential are those in §§1–4 of RFI. We quote here a version of Kleene's XXII in RFI, which we shall find very useful.

KLEENE'S SUBSTITUTION LEMMA. *There are primitive recursive functions* $\gamma_1(z, w)$ *and* $\gamma_2(z, w)$ *such that for a list of variables* $\mathfrak{c}$ *of types* $\leqq 3$:

(1) $\{\gamma_1(e, a)\}(\mathfrak{c}) = \{e\}(\mathfrak{c}, \lambda t\{a\}(\mathfrak{c}, t))$, *when* $\lambda t\{a\}(\mathfrak{c}, t)$ *is completely defined and* $\{e\}(\mathfrak{c}, \lambda t\{a\}(\mathfrak{c}, t))$ *is defined.*

(2) $\{\gamma_2(e, a)\}(\mathfrak{c}) = \{e\}(\mathfrak{c}, \lambda\alpha\{a\}(\mathfrak{c}, \alpha))$, *when* $\lambda\alpha\{a\}(\mathfrak{c}, \alpha)$ *is completely defined and* $\{e\}(\mathfrak{c}, \lambda\alpha\{a\}(\mathfrak{c}, \alpha))$ *is defined.*

We thank the referee for discovering a host of minor errors (the remaining host is our own responsibility) as well as a more serious error in the proof of Theorem 6.

This paper was presented to the Association of Symbolic Logic at the April 4, 1966 meeting in New York. The abstract from that talk (to appear in the Journal of Symbolic Logic) gives a concise summary of the most interesting results in the paper and can serve as a more technical introduction than our preceding remarks.

1. **A hyperanalytic function not recursive in any** $H_a^2(\alpha)$. The set O^2 and the predicates $H_a^2(\alpha)$ ($a \in O^2$) are defined in §11 of RFII.

LEMMA 1 (IMPLICIT IN RFII). *There is a primitive recursive function* g(a), *such that if* $a \in O^2$, *then* $\lambda\alpha\{g(a)\}(^3\mathbf{E}, \alpha)$ *is completely defined and*

$$H_a^2(\alpha) \equiv \{g(a)\}(^3\mathbf{E}, \alpha) = 0.$$

For an arbitrary function ξ, set

$$D_\xi = \{x : \xi(\langle 1, x \rangle) = 0\},$$

$$x \leqq_\xi y \equiv \xi(\langle x, y \rangle) = 0\text{(}^3\text{)}.$$

We say that ξ *is a well-ordering*, $\xi \in$ WO, if the following three conditions are satisfied:

(1) $\leqq_\xi$ is a well-ordering on D_ξ with minimum 1.

(2) For each $x \in D_\xi$, 2^x is the successor of x in D_ξ.

(3) For each $x \in D_\xi$, x is a limit point of $\leqq_\xi$ if and only if $x = 3^{(x)_1}$, $(x)_1 \neq 0$.

It is clear that the predicate $\xi \in$ WO is analytic, hence recursive in $^3\mathbf{E}$.

LEMMA 2. *Let*

$P(\xi, \alpha, x) \equiv \xi \in$ WO & $x \in D_\xi$ & [α *is a similarity mapping from the initial segment of* $\leqq_\xi$ *up to and including* x *onto some initial segment of the partial ordering* $<_O^2$].

Then $P(\xi, \alpha, x)$ *is hyperanalytic, i.e. recursive in* $^3\mathbf{E}$.

Proof. We shall define the characteristic function f(ξ, α, x) of $P(\xi, \alpha, x)$ using the Recursion Theorem XIV of RFI. Thus, we shall define a partial recursive function h(f, $^3\mathbf{E}$, ξ, α, x), then seek a number f such that

$$h(f, {}^3\mathbf{E}, \xi, \alpha, x) \simeq \{f\}(^3\mathbf{E}, \xi, \alpha, x)$$

and set

$$f(\xi, \alpha, x) \simeq \{f\}(^3\mathbf{E}, \xi, \alpha, x).$$

To simplify notation, here and in similar cases in the future, we avoid the symbols "h(f, $^3\mathbf{E}$, ξ, α, x)" and "$\{f\}(^3\mathbf{E}, \xi, \alpha, x)$" and simply use "f($\xi, \alpha, x$)" as synonymous with these, even before we complete the definition of h(f, $^3\mathbf{E}$, ξ, α, x) and the application of the recursion theorem.

(3) We follow the notation of IM, RFI and RFII, with only one addition: for a partial function f(c), f(c)$\downarrow$ $\equiv$ f(c) is defined.

The primitive recursive functions $(a)_i$, lh(a) and $a*b$ are defined in §45 of IM. Other notations that we often use are:

Seq $(w) \equiv w \neq 0$ & $(i)_{i<\text{lh}(w)}[(w)_i \neq 0]$.

$\langle x_0, \ldots, x_m \rangle = p_0^{x_0}, \ldots, p_m^{x_m}$; $\langle \alpha_0, \ldots, \alpha_n \rangle = \lambda t \langle \alpha_0(t), \ldots, \alpha_n(t) \rangle$. (If $m = -1$, then $\langle\ \rangle = 1$; if $n = -1$, then $\langle\ \rangle = \lambda t 1$.)

$[x_0, \ldots, x_m] = \langle x_0 + 1, \ldots, x_m + 1 \rangle$.

After $f(\xi, \alpha, x)$ has been defined as a partial function, we shall prove that it is completely defined and that it is the characteristic function of $P(\xi, \alpha, x)$.

The definition of $f(\xi, \alpha, x)$ from its index f is by cases, using XVIII of RFI. As in several later constructions, the partial recursive predicates which determine the cases are quite complicated; we believe that producing explicit formulas for them would only conceal our motivation. Instead, we give intuitive instructions for the computations to be performed, from which the explicit formulas can be derived by routine applications of the results in §§1–4 of RFI.

Since the cases split into subcases and these further into sub-subcases etc., we use a numbering which follows this "tree structure." Thus, if there are three subcases of case 121 we call them 1211, 1212 and 1213. After the number of a case we identify in brackets the case hypothesis.

In some instances we ask that the value of a certain partial predicate or function be computed, and then the main computation splits into subcases according to the value obtained (if any). We label these branching points in the computation "questions" and prefix their numbers with a "?," e.g. ?122 below, to be read "question 122." Sometimes, when this reflects our motivation, we phrase the instructions at these branching points as questions to the oracle $^3\mathbf{E}$. Thus ?1 below "ask $^3\mathbf{E}$ if $\xi \in \mathrm{WO}$ & $x \in D_\xi$" means "go to 12 if $\xi \in \mathrm{WO}$ & $x \in D_\xi$, go to 11 if not ($\xi \in \mathrm{WO}$ & $x \in D_\xi$)," where the predicate $\xi \in \mathrm{WO}$ & $x \in D_\xi$ is recursive in $^3\mathbf{E}$.

Instructions for the computation of $f(\xi, \alpha, x)$:

?1. Ask $^3\mathbf{E}$ if $\xi \in \mathrm{WO}$ & $x \in D_\xi$.

11 [No to ?1]. Give output 1.

12 [Yes to ?1]. Go to 121, 122 or 123 according as $x = 1$, $[x = 2^{(x)_0}$ & $(x)_0 \neq 0]$ or $x = 3^{(x)_1}$ $((x)_1 \neq 0)$. If none of these cases applies, give output 1.

121 $[x = 1]$. If $\alpha(1) = 1$, give output 0. If $\alpha(1) \neq 1$, give output 1.

?122 $[x = 2^y$, where $y = (x)_0 \neq 0]$. Compute $f(\xi, \alpha, y)$.

1221 $[f(\xi, \alpha, y) \simeq 0$ in ?122]. If $\alpha(x) = 2^{\alpha(y)}$, give output 0, otherwise give output 1.

1222 $[f(\xi, \alpha, y) \simeq 1$ in ?122]. Give output 1.

?123 $[x = 3^w$, where $w = (x)_1 \neq 0]$. Ask $^3\mathbf{E}$ if $\alpha(x) = 3.5^{(\alpha(x))_2}.7^{(\alpha(x))_3}$ & $(t)[t <_\xi x \rightarrow f(\xi, \alpha, t) \simeq 0]$ & $(Et)\,[t <_\xi x$ & $\alpha(t) = (\alpha(x))_3]$.

1231 [No to ?123]. Give output 1.

?1232 [Yes to ?123; let $y = (\alpha(x))_2$, $u = (\alpha(x))_3$]. Ask $^3\mathbf{E}$ if $[y$ defines y_n general recursively from $\lambda\alpha\{g(u)\}\,(^3\mathbf{E}, \alpha)$ as a function of $n_O]$ & $y_0 = u$ & $(n)(Es)(Et)[s <_\xi t$ & $t <_\xi x$ & $\alpha(s) = y_n$ & $\alpha(t) = y_{n+1}]$.

12321 [No to ?1232]. Give output 1.

12322 [Yes to ?1232]. Give output 0.

We must now verify that each of the questions above is *legitimate*, i.e., that we are indeed asking for the computation of a predicate or function which is partial recursive in $^3\mathbf{E}$.

In ?1 this is obvious, and in ?122 we ask for a computation which can be performed from the index f of $f(\xi, \alpha, x)$.

In ?123 we have a tacit application of the Subsitution Lemma. The predicate

$$Q(\xi, \alpha, x, \beta) \equiv \alpha(x) = 3 \cdot 5^{(\alpha(x))_2} \cdot 7^{(\alpha(x))_3}$$
$$\& \ (t)[\beta(t) = 0]$$
$$\& \ (Et)[t <_\xi x \ \& \ \alpha(t) = (\alpha(x))_3]$$

is clearly recursive in ${}^3\mathbf{E}$. We wish to evaluate $Q(\xi, \alpha, x, \beta)$ for

$$\beta(t) \simeq 0 \quad \text{if } \overline{t <_\xi x},$$
$$\simeq 0 \quad \text{if } t <_\xi x \ \& \ \mathrm{f}(\xi, \alpha, x) \simeq 0,$$
$$\simeq 1 \quad \text{if } t <_\xi x \ \& \ \mathrm{f}(\xi, \alpha, x) \simeq 1.$$

By the Substitution Lemma, there is a partial recursive $Q'({}^3\mathbf{E}, \xi\ \alpha, x)$ which agrees with $Q(\xi, \alpha, x, \beta)$ for this β, if β is completely defined. It is this $Q'({}^3\mathbf{E}, \xi, \alpha, x)$ that we are instructed to evaluate in ?123. (Notice that as before we are suppressing the dependence of $Q'({}^3\mathbf{E}, \xi, \alpha, x)$ on f.)

A similar tacit application of the Substitution Lemma occurs in ?1232, where the substitution is for a type-2 object. The clause [y defines y_n general recursively from $\mathbf{F}$ as a function of n_O] is recursive in ${}^3\mathbf{E}$, as a predicate of y and a type-2 object $\mathbf{F}$. Since the other clauses are clearly recursive in ${}^3\mathbf{E}$, we interpret ?1232 as instructions to evaluate a certain recursive $R({}^3\mathbf{E}, \xi, \alpha, x, \mathbf{F})$, for $\mathbf{F} = \lambda\alpha\{\mathrm{g}(u)\}({}^3\mathbf{E}, \alpha)$. The Substitution Lemma gives us a partial recursive $R'({}^3\mathbf{E}, \xi, \alpha, x)$ which agrees with $R({}^3\mathbf{E}, \xi, \alpha, x, \mathbf{F})$ for $\mathbf{F} = \lambda\alpha\{\mathrm{g}(u)\}({}^3\mathbf{E}, \alpha)$, if $\lambda\alpha\{\mathrm{g}(u)\}({}^3\mathbf{E}, \alpha)$ is completely defined, and it is this $R'({}^3\mathbf{E}, \xi, \alpha, x)$ that we evaluate.

In the more complicated computation instructions of §4 we shall let the reader supply these tacit references to the Substitution Lemma. We hope that the detailed analyses of these two cases above are sufficiently clear to make the process routine.

If $\xi \in \mathrm{WO}\ \&\ x \in D_\xi$ is false, then $\mathrm{f}(\xi, \alpha, x) \simeq 1$ by case 11.

If $\xi \in \mathrm{WO}\ \&\ x \in D_\xi$ let $|x|_\xi$ be the order type of the initial segment $[1, x]_\xi$ of $\leqq_\xi$ up to and including x. We prove that $\mathrm{f}(\xi, \alpha, x)$ is defined and agrees with the characteristic function of $P(\xi, \alpha, x)$ by transfinite induction on $|x|_\xi$.

If $x = 1$, $\mathrm{f}(\xi, \alpha, x)$ is defined and has the correct value by case 121.

If $x = 2^y$ ($y = (x)_0$), we may assume by the induction hypothesis that for each $t \leqq_\xi y$, $\mathrm{f}(\xi, \alpha, t)$ is defined and has the correct value. In particular ?122 is answered and the computation shifts to either 1221 or 1222, whence the correct value is obtained.

If $x = 3^w$ ($w = (x)_1$), we may assume the induction hypothesis for each $t <_\xi x$. Thus ?123 with the interpretation given it above is answered, since β is completely defined. If $\alpha(x)$ is not of the proper form, or if α does not map each $[1, t]_\xi$ ($t < x$) onto some initial segment of $<_0^2$, we give the correct output 1 by 1231. If these conditions are satisfied, we further ask ?1232. Since by induction hypothesis $u = \alpha(t) \in O^2$, $\lambda\alpha\{\mathrm{g}(u)\}({}^3\mathbf{E}, \alpha)$ is completely defined and hence ?1232 is answered correctly in the way it was interpreted, whence we are led to either 12321 or 12322 and the correct output.

THEOREM 1. *There exists a hyperanalytic function* $\mathfrak{f}(x)$ *which is not recursive in any* $H_a^2(\alpha)$ $(a \in O^2)$.

Proof. Since

$$a \in O^2 \equiv (E\xi)(E\alpha)(Ex)[P(\xi, \alpha, x) \;\&\; \alpha(x) = a],$$

the set O^2 is hyperanalytic. Then the predicate

$$\begin{aligned} Q(\alpha) &\equiv \alpha(0) \in O^2 \;\&\; H^2_{\alpha(0)}(\lambda t \alpha(t+1)) \\ &\equiv \alpha(0) \in O^2 \;\&\; \{g(\alpha(0))\}({}^3\mathbf{E}, \lambda t \alpha(t+1)) = 0 \end{aligned}$$

is hyperanalytic. Now each $H_a^2(\alpha)$ is recursive in $Q(\alpha)$, hence $Q(\alpha)$ cannot be recursive in any $H_b^2(\alpha)$; because then $H^2_{2^b}(\alpha)$ would be recursive in $H_b^2(\alpha)$, which is impossible.

Put

$$\begin{aligned} f(e, \mathbf{F}, t) &\simeq 0 && \text{if } \{e\}(\mathbf{F}, t) \text{ is not defined,} \\ &\simeq \{e\}(\mathbf{F}, t)+1 && \text{if } \{e\}(\mathbf{F}, t) \text{ is defined,} \end{aligned}$$

and let

$$\mathfrak{f}(t) = \mathfrak{f}(t, \lambda\alpha \mathfrak{q}(\alpha), t),$$

where $\mathfrak{q}(\alpha)$ is the characteristic function of $Q(\alpha)$. It is easy to verify that $\mathfrak{f}(t)$ is hyperanalytic but not recursive in $Q(\alpha)$, hence not recursive in any $H_a^2(\alpha)$.

2. **Functions recursive in a type-3 object.** Let τ^2 and τ^3 be given objects of types 2 and 3 respectively. For each number-theoretic function τ we shall define a set of numbers $N(\tau^3, \tau^2, \tau) = N(\tau)$, and for each $z \in N(\tau)$ a number-theoretic function $\mathfrak{f}_z^{\tau^3,\tau^2,\tau}(t) = \mathfrak{f}_z^{\tau}(t)$.

The definition will be by induction and will be given simultaneously for all τ. Thus it is most properly viewed as an inductive definition of a predicate $\lambda z \tau z \in N(\tau)$ and a partial functional $\lambda z \tau t \mathfrak{f}_z^{\tau}(t)$, defined when $z \in N(\tau)$.

For typographical reasons we often write $\mathfrak{f}(\tau, z; t)$ or $\mathfrak{f}^{\tau}(z; t)$ for $\mathfrak{f}_z^{\tau}(t)$.

DEFINITION 1. (1) For each q, $\langle 1, q\rangle \in N(\tau)$ and $\mathfrak{f}^{\tau}_{\langle 1,q\rangle}(t) = q$. (*Introduction of constants.*)

(2) $\langle 2, 0\rangle \in N(\tau)$ and $\mathfrak{f}^{\tau}_{\langle 2,0\rangle}(t) = \tau(t)$. (*Introduction of* τ.)

(3) If $w \in N(\tau)$, then $\langle 3, w\rangle \in N(\tau)$ and $\mathfrak{f}^{\tau}_{\langle 3,w\rangle}(t) = \tau^2(\lambda u \mathfrak{f}^{\tau}_w(u))$. (*Introduction of* τ^2.)

(4) For $w \in N(\tau)$ and e any number, put $e(\tau, w, t) \simeq \{e\}^{\alpha}(t)$, where $\alpha = \lambda u \mathfrak{f}^{\tau}_w(u)$.

If $w \in N(\tau)$ and if for each t, $e(\tau, w, t)\downarrow$ and $e(\tau, w, t) \in N(\tau)$ ([3]), then $\langle 4, w, e\rangle \in N(t)$ and $\mathfrak{f}^{\tau}_{\langle 4,w,e\rangle}(t) = \mathfrak{f}^{\tau}_{e(\tau,w,t)}(\mathrm{t})$. (*Diagonalization.*)

(5) Suppose that for each α and each t, $\{e\}^{\alpha,\tau}(t)\downarrow$; we say that e defines a τ-recursive functional $\{e\}(\alpha, \tau) = \lambda t \{e\}^{\alpha,\tau}(t)$.

If e defines a τ-recursive functional $\{e\}(\alpha, \tau)$ and if for each α, $w \in N(\{e\}(\alpha, \tau))$, then $\langle 5, w, e\rangle \in N(\tau)$ and $\mathfrak{f}^{\tau}_{\langle 5,w,e\rangle}(t) = \tau^3(\lambda\alpha \mathfrak{f}(\{e\}(\alpha, \tau), w; t))$. (*Introduction of* τ^3.)

(6) Suppose that for each t, $\{e\}^\tau(t)\downarrow$, put $\alpha=\lambda t\{e\}^\tau(t)$. If $w\in N(\alpha)$, then $\langle 6, w, e\rangle \in N(\tau)$ and $f^\tau_{\langle 6,w,e\rangle}(t)=f^\alpha_w(t)$. (*Transfer.*)

(7) *Recursion clause*: $z\in N(\tau)$ *and* $f^\tau_z(t)=u$ *only by* (1)–(6).

The main result of this section is that the class of functions $\lambda t f^\tau_z(t)$ $(z\in N(\tau))$ coincides with the class of functions recursive in τ^3, τ^2, τ.

Classically one interprets inductions of this type as definitions by transfinite induction over an initial segment of the ordinals. Clause (5) raises the possibility that in this case the induction extends into the third number class; in any case, we only use ordinals which are limits of fundamental sequences of cardinality not exceeding that of the continuum. We assign to each z, τ such that $z\in N(\tau)$ the ordinal $|z|^{\tau^3,\tau^2,\tau}=|z|^\tau$ at which we verify that $z\in N(\tau)$. The definition is by induction following the inductive definition of $z\in N(\tau)$.

Definition 2. (1) $|\langle 1, q\rangle|^\tau=0$.

(2) $|\langle 2, 0\rangle|^\tau=0$.

(3) $|\langle 3, w\rangle|^\tau=|w|^\tau+1$.

(4) $|\langle 4, w, e\rangle|^\tau=supremum_t\{|w|^\tau+1, |e(\tau, w, t)|^\tau+1\}$.

(5) $|\langle 5, w, e\rangle|^\tau=supremum_\alpha\{|w|^{\{e\}(\alpha,\tau)}+1\}$.

(6) $|\langle 6, w, e\rangle|^\tau=|w|^\alpha+1$, *where* $\alpha=\lambda t\{e\}^\tau(t)$.

Whenever possible we prefer to give proofs or definitions by induction on $z\in N(\tau)$ rather than by transfinite induction on $|z|^\tau$. One can, of course, interpret all such arguments as classical transfinite inductions.

Theorem 2. *There is a primitive recursive function* $p(z)$, *such that if* $z\in N(\tau)$, *then*

$$f^\tau_z(t)\simeq\{p(z)\}(\tau^3, \tau^2, \tau, t).$$

Thus each $f^\tau_z(t)$ *is recursive in* τ^3, τ^2, τ.

Proof. We first define $p(z)$ as a partial recursive function from a Gödel number p using the recursion theorem, and then observe that it is totally defined and indeed primitive recursive. The proof that $p(z)$ has the desired property will be by induction on $z\in N(\tau)$. We treat the most interesting cases right in the definition.

Case (1). $z=\langle 1, q\rangle$ *for some* q. Put $p(z)=\langle 2, \langle 1, 1, 1, 1\rangle, q\rangle$.

Case (2). $z=\langle 2, 0\rangle$. Put $p(z)=\langle 7, \langle 1, 1, 1, 1\rangle\rangle$.

Case (3). $z=\langle 3, w\rangle$ *for some* w. Choose a primitive recursive $g_1(e)$ such that for each τ, u, t, $\{g_1(e)\}(\tau^3, \tau^2, \tau, u, t)\simeq\{e\}(\tau^3, \tau^2, \tau, u)$ and put

$$p(z)=\langle 8, \langle 1, 1, 1, 1\rangle, 2, g_1(p(w))\rangle.$$

If the induction hypothesis is satisfied, we know that for each u,

$$\{p(w)\}(\tau^3, \tau^2, \tau, u)\simeq f^\tau_w(u).$$

Thus

$$\{p(z)\}(\tau^3, \tau^2, \tau, t)\simeq\tau^2(\lambda u\{g_1(p(w))\}(\tau^3, \tau^2, \tau, u, t))$$
$$\simeq\tau^2(\lambda u\{p(w)\}(\tau^3, \tau^2, \tau, u))\simeq\tau^2(\lambda u f^\tau_w(u))\simeq f^\tau_z(t).$$

Case (4). $z = \langle 4, w, e\rangle$ *for some* w, e. The functional

$$\phi(\tau^3, \tau^2, \tau, \alpha, p, e, t) \simeq \{\mathrm{p}(\{e\}^\alpha(\mathrm{t}))\}(\tau^3, \tau^2, \tau, t)$$

is evidently partial recursive. Using the Substitution Lemma, choose a primitive recursive $\mathrm{g}_2(p, w, e)$ such that

$$\{\mathrm{g}_2(p, w, e)\}(\tau^3, \tau^2, \tau, t) = \phi(\tau^3, \tau^2, \tau, \alpha, p, e, t),$$

when $\alpha = \lambda u\{\mathrm{p}(w)\}(\tau^3, \tau^2, \tau, u)$ is completely defined and $\phi(\tau^3, \tau^2, \tau, \alpha, p, e, t)$ is defined. Put $\mathrm{p}(z) = \mathrm{g}_2(p, w, e)$.

If the induction hypothesis is satisfied, then α is completely defined, and for each t, $\{e\}^\alpha(t)\downarrow$ & $\{\mathrm{p}(\{e\}^\alpha(t))\}(\tau^3, \tau^2, \tau, t) \simeq \mathrm{f}_z^\tau(t)$, so that $\mathrm{p}(z)$ has the required property.

Case (5). $z = \langle 5, w, e\rangle$ *for some* w, e. Using the Substitution Lemma, choose a primitive recursive $\mathrm{g}_3(p, w, e)$ such that

$$\{\mathrm{g}_3(p, w, e)\}(\tau^3, \tau^2, \alpha, \tau, t) = \{\mathrm{p}(w)\}(\tau^3, \tau^2, \{e\}(\alpha, \tau), t),$$

when $\{e\}(\alpha, \tau) = \lambda u\{e\}^{\alpha,\tau}(u)$ is completely defined and the right-hand side of the equation is defined. Put $\mathrm{p}(z) = \langle 8, \langle 1, 1, 1, 1\rangle, 3, \mathrm{g}_3(p, w, e)\rangle$.

If the induction hypothesis is satisfied, then for each α, $\{e\}(\alpha, \tau)$ is completely defined and $\{\mathrm{p}(w)\}(\tau^3, \tau^2, \{e\}(\alpha, \tau), t) = \mathrm{f}(\{e\}(\alpha, \tau), w; t)$, so that

$$\{\mathrm{p}(z)\}(\tau^3, \tau^2, \tau, t) \simeq \tau^3(\lambda\alpha \mathrm{f}(\{e\}(\alpha, \tau), w; t)).$$

Case (6). $z = \langle 6, w, e\rangle$ *for some* w, e. Using the Substitution Lemma, choose a primitive recursive $\mathrm{g}_4(p, w, e)$ such that

$$\{\mathrm{g}_4(p, w, e)\}(\tau^3, \tau^2, \tau, t) = \{\mathrm{p}(w)\}\{\tau^3, \tau^2, \lambda u\{e\}^\tau(u), t),$$

when $\lambda u\{e\}^\tau(u)$ is completely defined and the right-hand side is defined. Put $\mathrm{p}(z) = \mathrm{g}_4(p, w, e)$.

Case (7). *Otherwise*. Put $\mathrm{p}(z) = 0$.

To simplify notation for the next theorem we introduce the conventions

$$\mathbf{x} = x_1, \ldots, x_m; \qquad \boldsymbol{\tau} = \tau_1, \ldots, \tau_n.$$

(Now $[\mathbf{x}] = [x_1, \ldots, x_m]$ and $\langle\boldsymbol{\tau}\rangle = \langle\tau_1, \ldots, \tau_n\rangle$ are defined in ([3]), including the case for empty $\mathbf{x}$ or $\boldsymbol{\tau}$.)

THEOREM 3. *There is a primitive recursive function* $\mathrm{q}(e, x)$ *such that*:

(1) $\{e\}(\tau^3, \tau^2, \boldsymbol{\tau}, \mathbf{x})\downarrow \equiv \mathrm{q}(e, [\mathbf{x}]) \in N(\langle\boldsymbol{\tau}\rangle)$.

(2) $\{e\}(\tau^3, \tau^2, \boldsymbol{\tau}, \mathbf{x}) \simeq w \to (t)[\mathrm{f}(\langle\boldsymbol{\tau}\rangle, \mathrm{q}(e, [\mathbf{x}]); t) = w]$.

Proof. For this proof only, let us introduce the additional notation conventions:

$$\{e\}(\boldsymbol{\tau}, \mathbf{x}) \simeq \{e\}(\tau^3, \tau^2, \boldsymbol{\tau}, \mathbf{x}).$$

$$x = [\mathbf{x}] = [x_1, \ldots, x_m]; \qquad \tau = \langle\boldsymbol{\tau}\rangle = \langle\tau_1, \ldots, \tau_n\rangle.$$

We shall define $\mathrm{q}(e, x)$ from its Gödel number q using the recursion theorem in the usual fashion. The definition will be by cases on e being an index of the proper kind for x. We number the cases S1, S2, etc. and give as case hypothesis the definition of $\{e\}(\boldsymbol{\tau}, \boldsymbol{x})$ rather than the number-theoretic condition that e and x must satisfy. For example, Case S1 below, $\{e\}(\boldsymbol{\tau}, \boldsymbol{x}) = x_1'$, should be described by

$$\mathrm{Seq}\,(x)\ \&\ (En)[e = \langle 1, \langle \mathrm{lh}(x), n, 1, 1\rangle\rangle].$$

The implication from left to right in (1) together with (2) of the theorem are proved by induction on $\{e\}(\boldsymbol{\tau}, \boldsymbol{x}) \simeq w$. We treat the most interesting cases right in the definition. Proof of the implication from right to left in (1) is by induction on $\mathrm{q}(e, x) \in N(\tau)$ and will be outlined after the definition is completed.

Case S1. $\{e\}(\boldsymbol{\tau}, \boldsymbol{x}) = x_1'$. Choose a primitive recursive $\mathrm{r}_1(x)$ such that for each $\alpha, x, t, \{\mathrm{r}_1(x)\}^{\alpha}(t) = \langle 1, (x)_0\rangle$, and put $\mathrm{q}(e, x) = \langle 4, \langle 2, 0\rangle, \mathrm{r}_1(x)\rangle$. Clearly $\mathrm{q}(e, x) \in N(\tau)$ and $\mathrm{f}(\tau, \mathrm{q}(e, x); t) = \mathrm{f}(\tau, \langle 1, (x)_0\rangle; t) = x_1'$.

Case S2. $\{e\}(\boldsymbol{\tau}, \boldsymbol{x}) = k$. Put $\mathrm{q}(e, x) = \langle 1, k\rangle$.

Case S3. $\{e\}(\boldsymbol{\tau}, \boldsymbol{x}) = x_1$. Treat similarly to Case S1.

Case S4. $\{e\}(\boldsymbol{\tau}, \boldsymbol{x}) \simeq \{g\}(\{h\}(\boldsymbol{\tau}, \boldsymbol{x}), \boldsymbol{\tau}, \boldsymbol{x})$. Choose a primitive recursive $\mathrm{r}_2(q, g, x)$ such that for each α, t, $\{\mathrm{r}_2(q, g, x)\}^{\alpha}(t) \simeq \{q\}(g, [\alpha(0)]*x)$. Put

$$\mathrm{q}(e, x) = \langle 4, \mathrm{q}(h, x), \mathrm{r}_2(q, g, x)\rangle.$$

Assume that $\{h\}(\boldsymbol{\tau}, \boldsymbol{x}) \simeq u\ \&\ \{g\}(u, \boldsymbol{\tau}, \boldsymbol{x}) \simeq w$. By induction hypothesis, $\mathrm{q}(h, x) \in N(\tau)$, $\mathrm{f}^{\tau}(\mathrm{q}(h, x); t) = u$ (all t), $\mathrm{q}(g, [u]*x) \in N(\tau)$ and $\mathrm{f}^{\tau}(\mathrm{q}(g, [u]*x); t) = w$ (all t). Thus $\mathrm{q}(e, x) \in N(\tau)$, since $\mathrm{q}(h, x) \in N(\tau)$ and for $\alpha = \lambda u \mathrm{f}^{\tau}(\mathrm{q}(h, x); u)$ and each t,

$$\{\mathrm{r}_2(q, g, x)\}^{\alpha}(t) \simeq \{q\}(g, [\alpha(0)]*x) \simeq \mathrm{q}(g, [u]*x) \in N(\tau).$$

Moreover, $\mathrm{f}^{\tau}(\mathrm{q}(e, x); t) = \mathrm{f}^{\tau}(\mathrm{q}(g, [u]*x); t) = w$.

Case S5. *Subcase* (a). $\{e\}(\boldsymbol{\tau}, 0, \boldsymbol{x}) \simeq \{g\}(\boldsymbol{\tau}, \boldsymbol{x})$. Choose a primitive recursive $\mathrm{r}_3(q, g, x)$ such that for each α, t, $\{\mathrm{r}_3(q, g, x)\}^{\alpha}(t) \simeq \{q\}(g, x)$ and put $\mathrm{q}(e, [0]*x) = \langle 4, \langle 2, 0\rangle, \mathrm{r}_3(q, g, x)\rangle$.

Assume that $\{g\}(\boldsymbol{\tau}, \boldsymbol{x}) \simeq w$. By induction hypothesis, $\mathrm{q}(g, x) \in N(\tau)$ and for each t, $\mathrm{f}^{\tau}(\mathrm{q}(g, x); t) = w$. Since $\langle 2, 0\rangle \in N(\tau)$ and since for each α,

$$\{\mathrm{r}_3(q, g, x)\}^{\alpha}(t) \simeq \mathrm{q}(g, x) \in N(\tau),$$

we have $\mathrm{q}(e, [0]*x) \in N(\tau)$ and $\mathrm{f}^{\tau}(\mathrm{q}(e, [0]*x); t) = \mathrm{f}^{\tau}(\mathrm{q}(g, x); t) = w$.

REMARK. The simpler definition $\mathrm{q}(e, [0]*x) = \mathrm{q}(g, x)$ would serve equally well for this part of the theorem. In proving the implication from right to left in (1) however, this definition would not immediately give us a usable induction hypothesis. With the definition we chose, if $\mathrm{q}(e, [0]*x) \in N(\tau)$, then $\mathrm{q}(g, x) \in N(\tau)$ and $\mathrm{q}(g, x)$ is a *predecessor* of $\mathrm{q}(e, [0]*x)$ in the definition of $N(\tau)$ (in particular $|\mathrm{q}(g, x)|^{\tau} < |\mathrm{q}(e, [0]*x)|^{\tau}$). Thus in treating this case of the inductive proof of the right-to-left implication in (1), the induction hypothesis assures us that $\{g\}(\boldsymbol{\tau}, \boldsymbol{x}) \downarrow$, which implies immediately that $\{e\}(\boldsymbol{\tau}, 0, \boldsymbol{x}) \downarrow$.

Case S5. *Subcase* (b). $\{e\}(\boldsymbol{\tau}, y+1, \boldsymbol{x}) \simeq \{h\}(y, \{e\}(\boldsymbol{\tau}, y, \boldsymbol{x}), \boldsymbol{\tau}, \boldsymbol{x})$. Choose a primitive recursive $\mathrm{r}_4(q, h, y, x)$ such that for each α, t, $\{\mathrm{r}_4(q, h, y, x)\}^{\alpha}(t) \simeq \{q\}(h, [y]*([\alpha(0)*x))$ and put $\mathrm{q}(e, [y+1]*x) = \langle 4, \mathrm{q}(e, [y]*x), \mathrm{r}_4(q, h, y, x)\rangle$.

Proof is as in Case S4.

Case S6. *Subcase* (a). $\{e\}(\boldsymbol{\tau}, x) \simeq \{g\}(\boldsymbol{\tau}, x_{k+1}, x_1, \ldots, x_k, x_{k+2}, \ldots, x_m)$. As in *Subcase* (a) of Case S5, put $\mathrm{q}(e, x) = \langle 4, \langle 2, 0\rangle, \mathrm{r}_5(q, g, e, x)\rangle$, where $\mathrm{r}_5(q, g, e, x)$ is primitive recursive and such that for each α, t,

$$\{\mathrm{r}_5(q, g, e, x)\}^{\alpha}(t) \simeq \{q\}(g, [x_{k+1}, x_1, \ldots, x_k, x_{k+2}, \ldots, x_m]).$$

Case S6. *Subcase* (b). $\{e\}(\boldsymbol{\tau}, x) \simeq \{g\}(\tau_{k+1}, \tau_1, \ldots, \tau_k, \tau_{k+2}, \ldots, \tau_n, x)$. Choose a primitive recursive $\mathrm{r}_6(e)$ such that for each τ, t,

$$\{\mathrm{r}_6(e)\}^{\tau}(t) = \langle \tau_{k+1}, \tau_1, \ldots, \tau_k, \tau_{k+2}, \ldots, \tau_n\rangle(t)$$

and put $\mathrm{q}(e, x) = \langle 6, \mathrm{q}(g, x), \mathrm{r}_6(e)\rangle$.

Case S7. $\{e\}(\boldsymbol{\tau}, x) = \tau_1(x_1)$. Choose a primitive recursive $\mathrm{r}_7(x)$ such that for each α, t, $\{\mathrm{r}_7(x)\}^{\alpha}(t) = \langle 1, (\alpha((x)_0 - 1))_0\rangle$ and put $\mathrm{q}(e, x) = \langle 4, \langle 2, 0\rangle, \mathrm{r}_7(x)\rangle$. Then $\mathrm{f}^{\boldsymbol{\tau}}(\mathrm{q}(e, x); t) = \mathrm{f}^{\boldsymbol{\tau}}(\langle 1, (\tau(x_1))_0\rangle; t) = (\tau(x_1))_0 = \tau_1(x_1)$.

Case S8. *Subcase* (a). $\{e\}(\boldsymbol{\tau}, x) \simeq \tau^2(\lambda u\{g\}(\boldsymbol{\tau}, u, x))$. Choose a primitive recursive $\mathrm{r}_8(q, g, x)$ such that for each α, u, $\{\mathrm{r}_8(q, g, x)\}^{\alpha}(u) \simeq \{q\}(g, [u]*x)$ and put $\mathrm{q}(e, x) = \langle 3, \langle 4, \langle 2, 0\rangle, \mathrm{r}_8(q, g, x)\rangle\rangle$.

If $\{e\}(\boldsymbol{\tau}, x) \simeq w$, then by induction hypothesis $\mathrm{q}(g, [u]*x) \in N(\tau)$ for each u, hence $\langle 4, \langle 2, 0\rangle, \mathrm{r}_8(q, g, x)\rangle \in N(\tau)$ and $\mathrm{f}^{\boldsymbol{\tau}}(\langle 4, \langle 2, 0\rangle, \mathrm{r}_8(q, g, x)\rangle; u) = \{g\}(\boldsymbol{\tau}, u, x)$. Thus $\mathrm{q}(e, x) \in N(\tau)$ and $\mathrm{f}^{\boldsymbol{\tau}}(\mathrm{q}(e, x); t) = \tau^2(\lambda u \mathrm{f}^{\boldsymbol{\tau}}(\langle 4, \langle 2, 0\rangle, \mathrm{r}_8(q, g, x)\rangle; u)) = \tau^2(\lambda u\{g\}(\boldsymbol{\tau}, u, x))$.

Case S8. *Subcase* (b). $\{e\}(\boldsymbol{\tau}, x) \simeq \tau^3(\lambda\alpha\{g\}(\alpha, \boldsymbol{\tau}, x))$. Choose a primitive recursive $\mathrm{r}_9(e)$ such that for each α, τ, t, $\{\mathrm{r}_9(e)\}^{\alpha,\tau}(t) \simeq \langle\alpha(t)\rangle * \tau(t)$ and put

$$\mathrm{q}(e, x) = \langle 5, \mathrm{q}(g, x), \mathrm{r}_9(e)\rangle.$$

If $\{e\}(\boldsymbol{\tau}, x) \simeq w$, then by induction hypothesis, for each α,

$$\mathrm{q}(g, x) \in N(\langle\alpha, \tau_1, \ldots, \tau_n\rangle) = N(\{\mathrm{r}_9(e)\}(\alpha, \tau));$$

hence $\mathrm{q}(e, x) \in N(\tau)$ and

$$\mathrm{f}^{\boldsymbol{\tau}}(\mathrm{q}(e, x); t) = \tau^3(\lambda\alpha \mathrm{f}(\{\mathrm{r}_9(e)\}(\alpha, \tau), \mathrm{q}(g, x); t)) = \tau^3(\lambda\alpha\{g\}(\alpha, \boldsymbol{\tau}, x)).$$

Case S9. $\{e\}(\boldsymbol{\tau}, \beta_1, \ldots, \beta_k, a, x, y_1, \ldots, y_j) \simeq \{a\}(\boldsymbol{\tau}, x)$. Choose a primitive recursive $\mathrm{r}_{10}(e)$ such that for each α, t, $\{\mathrm{r}_{10}(e)\}^{\alpha}(t) = \langle(\alpha(t))_0, (\alpha(t))_1, \ldots, (\alpha(t))_{n-1}\rangle$ and put $\mathrm{q}(e, [x_1, \ldots, x_m, a, y_1, \ldots, y_j]) = \langle 6, \mathrm{q}(a, x), \mathrm{r}_{10}(e)\rangle$.

If $\{e\}(\boldsymbol{\tau}, \beta_1, \ldots, \beta_k, a, x, y_1, \ldots, y_j) \simeq w$, then by induction hypothesis $\mathrm{q}(a, x) \in N(\tau)$ and for each t, $\mathrm{f}^{\boldsymbol{\tau}}(\mathrm{q}(a, x); t) = w$. Hence

$$\mathrm{q}(e, [x_1, \ldots, x_m, a, y_1, \ldots, y_j]) \in N(\langle\tau_1, \ldots, \tau_n, \beta_1, \ldots, \beta_k\rangle)$$

and

$$\mathrm{f}(\langle\tau_1, \ldots, \tau_n, \beta_1, \ldots, \beta_k\rangle, \; \mathrm{q}(e, [x_1, \ldots, x_m, a, y_1, \ldots, y_j]); t) = \mathrm{f}(\mathrm{q}(a, x); t) = w.$$

Case S10. *Otherwise*. Put $\mathrm{q}(e, x) = 0$.

The recursion theorem gives us a partial recursive $\mathrm{q}(e, x)$ with Gödel number q that satisfies all the above clauses. It is obvious that $\mathrm{q}(e, x)$ is completely defined; that it is primitive recursive can be inferred from the nature of the clauses, which (once q is chosen) give a nested course-of-values recursion for $\mathrm{q}(e, x)$.

We have already indicated in each of the clauses how to prove by induction on the definition of $\{e\}(\boldsymbol{\tau}, \boldsymbol{x}) \simeq w$ the implication from left to right in (1), and (2).

To prove the implication from right to left in (1) by induction on the definition of $q(e, \boldsymbol{x}) \in N(\tau)$, we first remark that $0 \notin N(\tau)$; thus if $q(e, \boldsymbol{x}) \in N(\tau)$, then e and $\boldsymbol{x}$ must satisfy one of the case hypotheses for cases S1–S9. It is a routine exercise to verify in each case (using the induction hypothesis) that $\{e\}(\boldsymbol{\tau}, \boldsymbol{x})$ must be defined. (Notice the remark after *Subcase* (a) of Case S5.)

COROLLARY 3.1: *If* $g(t)$ *is recursive in* $\tau^3, \tau^2, \boldsymbol{\tau}$, *then there is a* $z \in N(\tau)$ *such that for all* t, $g(t) = f_z^{\tau}(t)$.

Proof. Let g be an index of $g(t)$, i.e., for all t, $g(t) = \{g\}(\tau^3, \tau^2, \boldsymbol{\tau}, t)$; then for each t, $q(g, [t]) \in N(\langle\tau\rangle)$ and $f(\langle\tau\rangle, q(g, [t]); t) = g(t)$. Choose e so that for each α, t, $\{e\}^{\alpha}(t) = q(g, [t])$ and let $w = \langle 4, \langle 2, 0\rangle, e\rangle$; then $w \in N(\langle\tau\rangle)$ and $f(\langle\tau\rangle, w; t) = g(t)$. Choose m so that for each τ, $[m]^{\tau}(t) = \langle\tau(t)\rangle$ and let $z = \langle 6, w, m\rangle$; then $z \in N(\tau)$ and $f_z^{\tau}(t) = f_w^{\langle\tau\rangle}(t) = g(t)$.

3. **The hyperanalytic hierarchy.** Let $\mathfrak{b}$ be a list of objects of types 2 and 3([4]), let τ^i ($i = 2, 3$) be the contraction of all objects of type i in $\mathfrak{b}$ ($\tau^i = \lambda\alpha^{i-1}\,1$, if $\mathfrak{b}$ contains no objects of type i). Using LXVII of RFII we obtain primitive recursive functions $c(e)$ and $c^{-1}(e)$ so that:

$$\{e\}(\mathfrak{b}, \boldsymbol{\tau}, \boldsymbol{x}) \simeq \{c(e)\}(\tau^3, \tau^2, \boldsymbol{\tau}, \boldsymbol{x}),$$
$$\{e\}(\tau^3, \tau^2, \boldsymbol{\tau}, \boldsymbol{x}) \simeq \{c^{-1}(e)\}(\mathfrak{b}, \boldsymbol{\tau}, \boldsymbol{x}).$$

DEFINITION 3. Let $\mathfrak{b}$, τ^3, τ^2 be related as above, let $\tau = \tau_1, \ldots, \tau_n$, put

$$\mathfrak{a} = \mathfrak{b}, \boldsymbol{\tau}.$$

(1) $N(\mathfrak{a}) = N(\tau^3, \tau^2, \langle\boldsymbol{\tau}\rangle)$.
(2) For $z \in N(\mathfrak{a})$, $f_z^{\mathfrak{a}}(t) = f^{\mathfrak{a}}(z; t) = f(\mathfrak{a}, z; t) = f(\tau^3, \tau^2, \langle\boldsymbol{\tau}\rangle, z; t)$.
(3) $r(z) = c^{-1}(p(z))$, where $p(z)$ is the function of Theorem 2.
(4) $s(e, \boldsymbol{x}) = q(c(e), [\boldsymbol{x}])$, where $q(e, \boldsymbol{x})$ is the function of Theorem 3.
(5) For $z \in N(\mathfrak{a})$, $|z|^{\mathfrak{a}} = |z|(\mathfrak{a}) = |z|^{\langle\boldsymbol{\tau}\rangle}$.
(6) $\kappa(\mathfrak{a}) = \textit{supremum}\ \{|z|^{\mathfrak{a}} : z \in N(\mathfrak{a})\}$.

(Recall that if $\boldsymbol{\tau}$ is empty, then $\langle\boldsymbol{\tau}\rangle = \langle\ \rangle = \lambda t\,1$ and if $\boldsymbol{x}$ is empty, $[\boldsymbol{x}] = [\ ] = 1$; the clauses of Definition 3 apply to the cases of empty $\boldsymbol{\tau}$ or $\boldsymbol{x}$ with these conventions.)

([4]) We collect here several notation conventions to which we adhere throughout the paper:
$\boldsymbol{x} = x_1, \ldots, x_m$;
$\boldsymbol{\tau} = \tau_1, \ldots, \tau_n$; $\boldsymbol{\alpha} = \alpha_1, \ldots, \alpha_k$; $\boldsymbol{\beta} = \beta_1, \ldots, \beta_l$;
$\mathfrak{b}$ = a list of objects of types 2 and 3;
$\mathfrak{a}$ = a list of objects of types 1, 2 and 3 (usually $\mathfrak{a} = \mathfrak{b}, \boldsymbol{\tau}$);
$\mathfrak{c}$ = a list of objects of types 0, 1, 2 and 3 (usually $\mathfrak{c} = \mathfrak{a}, \boldsymbol{x}$);
$\mathfrak{b}^* = {}^3\mathbf{E}, \mathfrak{b}$; $\mathfrak{a}^* = {}^3\mathbf{E}, \mathfrak{a}$; $\mathfrak{c}^* = {}^3\mathbf{E}, \mathfrak{c}$.

Using Theorems 2 and 3 we can easily verify the following results:

(A) $\{e\}(\mathfrak{a}, \mathfrak{x})\downarrow \equiv s(e, \mathfrak{x}) \in N(\mathfrak{a})$.

(B) *If* $z \in N(\mathfrak{a})$, *then* $\lambda t\{r(z)\}(\mathfrak{a}, t)$ *is completely defined and* $f_z^{\mathfrak{a}}(t) = \{r(z)\}(\mathfrak{a}, t)$.

(C) *If* $g(t)$ *is recursive in* $\mathfrak{a}$, *then there exists a* $z \in N(\mathfrak{a})$ *such that* $g(t) = f_z^{\mathfrak{a}}(t)$.

(D) *If* $(\alpha)[z \in N(\mathfrak{a}, \alpha)]$, *then* $\lambda\alpha t\{r(z)\}(\mathfrak{a}, \alpha, t)$ *is completely defined and*

$$f(\mathfrak{a}, \alpha, z; t) = \{r(z)\}(\mathfrak{a}, \alpha, t).$$

(E) *If* $g(a, t)$ *is recursive in* $\mathfrak{a}$, *then there exists a* z *such* $(\alpha)[z \in N(\mathfrak{a}, \alpha)]$ *and* $g(\alpha, t) = f(\mathfrak{a}, \alpha, z; t)$.

This "construction" of all functions and functionals recursive in $\mathfrak{a}$ by (transfinite) induction naturally defines a certain "hierarchy" on the one- and two-sections of $\mathfrak{a}$. However, if no assumptions are imposed on $\mathfrak{a}$, then this hierarchy may be trivial; e.g., if $\mathfrak{a}$ consists of recursive functions and functionals, then each function recursive in $\mathfrak{a}$ is recursive, and hence equal to some $f_z^{\mathfrak{a}}(t)$ with $|z|^{\mathfrak{a}} = 1$.

Put

$$\mathfrak{b}^* = {}^3\mathbf{E}, \mathfrak{b}, \qquad \mathfrak{a}^* = {}^3\mathbf{E}, \mathfrak{a} = {}^3\mathbf{E}, \mathfrak{b}, \boldsymbol{\tau}.$$

Relativizing Kleene's definition, we call a predicate, set or function *hyperanalytic in* $\mathfrak{a}$ if it is recursive in $\mathfrak{a}^*$.

In this section we define two transfinite sequences of predicates, ${}_1\mathbf{G}_z(u, v)$ and ${}_2\mathbf{G}_z(u, \alpha, v, \beta)$ $(z \in N(\mathfrak{a}^*))$ such that:

(1) ${}_i\mathbf{G}_z$ *is hyperanalytic in* $\mathfrak{a}$, *uniformly for* $z \in N(\mathfrak{a}^*)$.

(2) *If* $|z|^{\mathfrak{a}^*} \leqq |w|^{\mathfrak{a}^*}$, *then* ${}_i\mathbf{G}_z$ *is recursive in* ${}_i\mathbf{G}_w$, *uniformly for* $z, w \in N(\mathfrak{a}^*)$.

(3) *A function (functional) is hyperanalytic in* $\mathfrak{a}$ *if and only if it is recursive in some* ${}_1\mathbf{G}_z({}_2\mathbf{G}_z)$.

We also prove that the denumerable set of ordinals $|z|^{\mathfrak{a}^*}$ $(z \in N(\mathfrak{a}^*))$ has order-type $\omega_1(\mathfrak{a}^*)$ = least denumerable ordinal which cannot be realized by an ordering hyperanalytic in $\mathfrak{a}$.

LEMMA 3. *Let* $\mathfrak{b}$ *be a list of objects of types* 2 *and* 3, *let* $\mathfrak{b}^* = {}^3\mathbf{E}, b$, *let*

$$\boldsymbol{\alpha} = \alpha_1, \ldots, \alpha_k, \qquad \boldsymbol{\beta} = \beta_1, \ldots, \beta_j$$

be two lists of function variables. There is a primitive recursive $i(k, j, z)$ *such that, if* $z \in N(\mathfrak{b}^*, \boldsymbol{\alpha})$, *then* $\lambda w\boldsymbol{\beta}\{i(k, j, z)\}(\mathfrak{b}^*, \boldsymbol{\alpha}, \boldsymbol{\beta}, w)$ *is completely defined and*

$$|w|(\mathfrak{b}^*, \boldsymbol{\beta}) \leqq |z|(\mathfrak{b}^*, \boldsymbol{\alpha}) \equiv \{i(k, j, z)\}(\mathfrak{b}^*, \boldsymbol{\alpha}, \boldsymbol{\beta}, w) = 0.$$

(*I.e. the initial segment of ordinals* $|w|(\mathfrak{b}^*, \boldsymbol{\beta})$ *which are* $\leqq |z|(\mathfrak{b}^*, \boldsymbol{\alpha})$ *is hyperanalytic in* $\mathfrak{b}$, $\boldsymbol{\alpha}$, *uniformly for* $z \in N(\mathfrak{b}^*, \boldsymbol{\alpha})$.)

We do not prove this lemma here, since it will be an immediate consequence of Theorem 6. However one can easily construct an elementary proof similar to that of Theorem 2.

LEMMA 4. *Let* $\mathfrak{a}=\mathfrak{b}, \boldsymbol{\tau}$ *be a list of objects of types* 1, 2, 3. *The predicate* $\lambda z\, z \in N(\mathfrak{a})$ *is not recursive in* $\mathfrak{a}$.

Proof. Suppose it were; put

$$\mathfrak{g}(t) = 0 \qquad \text{if } t \notin N(\mathfrak{a}),$$
$$= \mathfrak{f}_t^{\mathfrak{a}}(t)+1 \qquad \text{if } t \in N(\mathfrak{a}).$$

Now (**B**) implies that $\mathfrak{g}(t)$ is recursive in $\mathfrak{a}$, though it is different from each $\mathfrak{f}_z^{\mathfrak{a}}(t)$, contradicting (**C**).

DEFINITION 4. Let $\mathfrak{c}$ be a list of variables of types $\leqq 3$. A predicate $P(\mathfrak{c})$ is *recursively enumerable in* $\mathfrak{a}$ (*r.e. in* $\mathfrak{a}$), if for some e,

$$P(\mathfrak{c}) \equiv \{e\}(\mathfrak{a}, \mathfrak{c})\downarrow .$$

LEMMA 5 (BOUNDEDNESS). *If a predicate* $P(\boldsymbol{\alpha}, \boldsymbol{x})$ (*with variables of types* $\leqq 1$) *is r.e. in* $\mathfrak{a}^*$, *then there exists a primitive recursive* $\mathfrak{g}(\boldsymbol{x})$ *such that*

$$P(\boldsymbol{\alpha}, \boldsymbol{x}) \equiv \mathfrak{g}(\boldsymbol{x}) \in N(\mathfrak{a}^*, \boldsymbol{\alpha}). \tag{1}$$

Moreover, if (1) *holds with some* $\mathfrak{g}(\boldsymbol{x})$, *hyperanalytic in* $\mathfrak{a}$, *then* $P(\boldsymbol{\alpha}, \boldsymbol{x})$ *is hyperanalytic in* $\mathfrak{a}$ *if and only if*

$$\textit{supremum}\ \{|\mathfrak{g}(\boldsymbol{x})|(\mathfrak{a}^*, \boldsymbol{\alpha}) : P(\mathfrak{a}^*, \boldsymbol{\alpha})\} < \kappa(\mathfrak{a}^*). \tag{2}$$

REMARK. We prove in Corollary 7.1 that the predicate $\lambda z\boldsymbol{\alpha} z \in N(\mathfrak{a}^*, \boldsymbol{\alpha})$ is r.e. in $\mathfrak{a}^*$.

Proof. For the first assertion, choose e so that

$$P(\boldsymbol{\alpha}, \boldsymbol{x}) \equiv \{e\}(\mathfrak{a}^*, \boldsymbol{\alpha}, \boldsymbol{x})\downarrow$$

and put $\mathfrak{g}(\boldsymbol{x})=\mathfrak{s}(e, \boldsymbol{x})$.

Assume that (1) and (2) are satisfied with a $\mathfrak{g}(\boldsymbol{x})$ hypernalytic in $\mathfrak{a}$; then for some $z \in N(\mathfrak{a}^*)$ we have

$$\text{supremum}\ \{|\mathfrak{g}(\boldsymbol{x})|(\mathfrak{a}^*, \boldsymbol{\alpha}) : P(\boldsymbol{\alpha}, \boldsymbol{x})\} \leqq |z|(\mathfrak{a}^*) < \kappa(\mathfrak{a}^*).$$

Letting $\mathfrak{a}^*=\mathfrak{b}^*, \boldsymbol{\tau}$ as usual(4), we then have

$$P(\boldsymbol{\alpha}, \boldsymbol{x}) \equiv |\mathfrak{g}(\boldsymbol{x})|(\mathfrak{a}^*, \boldsymbol{\alpha}) \leqq |z|(\mathfrak{a}^*)$$
$$\equiv \{\mathfrak{i}(n, n+k, z)\}(\mathfrak{b}^*, \boldsymbol{\tau}, \boldsymbol{\tau}, \boldsymbol{\alpha}, \mathfrak{g}(\boldsymbol{x})) = 0,$$

so $P(\boldsymbol{\alpha}, \boldsymbol{x})$ is hypernalytic in $\mathfrak{a}$.

Now assume that (1) is satisfied with a $\mathfrak{g}(\boldsymbol{x})$ which is hyperanalytic in $\mathfrak{a}$, but

$$\text{supremum}\ \{|\mathfrak{g}(\boldsymbol{x})|(\mathfrak{a}^*, \boldsymbol{\alpha}) : P(\boldsymbol{\alpha}, \boldsymbol{x})\} \geqq \kappa(\mathfrak{a}^*).$$

Then

$$z \in N(\mathfrak{a}^*) \equiv (E\boldsymbol{\alpha})(E\boldsymbol{x})[P(\boldsymbol{\alpha}, \boldsymbol{x})\ \&\ |z|(\mathfrak{a}^*) \leqq |\mathfrak{g}(\boldsymbol{x})|(\mathfrak{a}^*, \boldsymbol{\alpha})]$$
$$\equiv (E\boldsymbol{\alpha})(E\boldsymbol{x})[P(\boldsymbol{\alpha}, \boldsymbol{x})\ \&\ \{\mathfrak{i}(n+k, n, \mathfrak{g}(\boldsymbol{x}))\}(\mathfrak{b}^*, \boldsymbol{\tau}\, \boldsymbol{\alpha}, \boldsymbol{\tau}, z) = 0].$$

If the predicate $P(\boldsymbol{\alpha}, \boldsymbol{x})$ is hyperanalytic in $\mathfrak{a}$, the predicate in brackets is hyperanalytic in $\mathfrak{a}$; since the class of predicates hyperanalytic in $\mathfrak{a}$ is closed under number and function quantification, $z \in N(\mathfrak{a}^*)$ is hypernalytic in $\mathfrak{a}$, contradicting Lemma 4.

DEFINITION 5. For each $z \in N(\mathfrak{a}^*)$(4), put

$$ {}_1\mathbf{G}_z(u, v) \equiv |v|^{\mathfrak{a}^*} \leqq |z|^{\mathfrak{a}^*} \mathbin{\&} |u|^{\mathfrak{a}^*} \leqq |v|^{\mathfrak{a}^*}, $$

$$ {}_2\mathbf{G}_z(u, \alpha, v, \beta) \equiv |v|^{\mathfrak{a}^*,\beta} \leqq |z|^{\mathfrak{a}^*} \mathbin{\&} |u|^{\mathfrak{a}^*,\alpha} \leqq |v|^{\mathfrak{a}^*,\beta}. $$

THEOREM 4. (1) *Each* ${}_i\mathbf{G}_z$ *is hyperanalytic in* $\mathfrak{a}$, *uniformly for* $z \in N(\mathfrak{a}^*)$ $(i=1, 2)$.
(2) *If* $|z|^{\mathfrak{a}^*} = |w|^{\mathfrak{a}^*}$, *then* ${}_i\mathbf{G}_z \equiv {}_i\mathbf{G}_w$ $(i=1, 2)$.
(3) *If* $|z|^{\mathfrak{a}^*} \leqq |w|^{\mathfrak{a}^*}$, *then* ${}_i\mathbf{G}_z$ *is recursive in* ${}_i\mathbf{G}_w$, *uniformly for* $z, w \in N(\mathfrak{a}^*)$.
$(4)_1$ *If* $P(\mathfrak{x})$ *is hyperanalytic in* $\mathfrak{a}$, *then* $P(\mathfrak{x})$ *is recursive in some* ${}_1\mathbf{G}_z$.
$(4)_2$ *If* $P(\boldsymbol{\alpha}, \mathfrak{x})$ *is hyperanalytic in* $\mathfrak{a}$, *then* $P(\boldsymbol{\alpha}, \mathfrak{x})$ *is recursive in some* ${}_2\mathbf{G}_z$.

Proof. (1) is an immediate consequence of Lemma 3 and (2) is obvious. To prove (3), notice that if $|z|^{\mathfrak{a}^*} \leqq |w|^{\mathfrak{a}^*}$, then

$$ {}_1\mathbf{G}_z(u, v) \equiv {}_1\mathbf{G}_w(v, z) \mathbin{\&} {}_1\mathbf{G}_w(u, v), $$

and similarly for ${}_2\mathbf{G}_z$, ${}_2\mathbf{G}_w$.

To prove $(4)_2$, choose a predicate $Q(\alpha, \mathfrak{x})$ of one function variable with the same degree as $P(\boldsymbol{\alpha}, \mathfrak{x})$, e.g., $Q(\alpha, \mathfrak{x}) \equiv P((\alpha)_1, \ldots, (\alpha)_k, \mathfrak{x})$. Now take e so that

$$ Q(\alpha, \mathfrak{x}) \equiv \{e\}(\mathfrak{a}^*, \alpha, \mathfrak{x}) \downarrow \equiv s(e, \mathfrak{x}) \in N(\mathfrak{a}^*, \alpha). $$

Since $Q(\alpha, \mathfrak{x})$ is hyperanalytic, the Boundedness Lemma implies that for some z,

$$ \text{supremum}\ \{|s(e, \mathfrak{x})|(\mathfrak{a}^*, \alpha) : Q(\alpha, \mathfrak{x})\} \leqq |z|(\mathfrak{a}^*). $$

Thus $Q(\alpha, \mathfrak{x}) \equiv |s(e, \mathfrak{x})|(\mathfrak{a}^*, \alpha) \leqq |z|(\mathfrak{a}^*) \equiv {}_2\mathbf{G}_z(s(e, \mathfrak{x}), \alpha, s(e, \mathfrak{x}), \alpha)$.

Proof of $(4)_1$ is similar.

DEFINITION 6. For each $\mathfrak{a} = \mathfrak{b}, \boldsymbol{\tau}$, $\{|z|^{\mathfrak{a}} : z \in N(\mathfrak{a})\}$ is a countable set of ordinals; let ψ be the unique order-preserving function which maps $\{|z|^{\mathfrak{a}} : z \in N(\mathfrak{a})\}$ onto an initial segment of the countable ordinals.

(1) $|z|_{\mathrm{c}}^{\mathfrak{a}} = |z|_{\mathrm{c}}(\mathfrak{a}) = \psi(|z|^{\mathfrak{a}})$ $(z \in N(\mathfrak{a}))$.

(2) $\omega_1(\mathfrak{a}) = supremum$ $\{|z|_{\mathrm{c}}^{\mathfrak{a}} : z \in N(\mathfrak{a})\}$.

(3) A countable ordinal η *is recursive in* $\mathfrak{a}$, if there exists a function $g(x, y)$, recursive in $\mathfrak{a}$, such that the relation $\lambda xy g(x, y) = 0$ is a well-ordering of order-type η.

REMARK. For $z, w \in N(\mathfrak{a})$, $|z|^{\mathfrak{a}} \leqq |w|^{\mathfrak{a}} \equiv |z|_{\mathrm{c}}^{\mathfrak{a}} \leqq |w|_{\mathrm{c}}^{\mathfrak{a}}$.

THEOREM 5. $\omega_1(\mathfrak{a}^*)$ *is the smallest countable ordinal which is not recursive in* $\mathfrak{a}^*$(4).

Proof. We first show that each $|z|_{\mathrm{c}}^{\mathfrak{a}^*}$ is recursive in $\mathfrak{a}^*$. Put

$$ \begin{aligned} u \in C_z \equiv |u|^{\mathfrak{a}^*} &\leqq |z|^{\mathfrak{a}^*} \mathbin{\&} (v)[|v|^{\mathfrak{a}^*} = |u|^{\mathfrak{a}^*} \to u \leqq v], \\ g(x, y) &= 0 \quad \text{if } x \in C_z \mathbin{\&} y \in C_z \mathbin{\&} |x|^{\mathfrak{a}^*} \leqq |y|^{\mathfrak{a}^*}, \\ &= 1 \quad \text{otherwise.} \end{aligned} $$

Lemma 3 implies that $g(x, y)$ is recursive in $\mathfrak{a}^*$, and it is obvious that $\lambda xy g(x, y) = 0$ is a well-ordering with order-type $|z|_{\mathrm{c}}^{\mathfrak{a}^*}$.

To prove the converse, let $\preceq$ be a well-ordering recursive in $\mathfrak{a}^*$, with order-type η, put

$$D = \{\langle n, x\rangle : x \preceq x\} \cup \{5\},$$

$$\begin{aligned} g(x, y) &= 0 \quad \text{if } x \in D \,\&\, y \in D \\ &\qquad \& \,[y = 5 \vee [(x)_1 \prec (y)_1] \vee [(x)_1 = (y)_1 \,\&\, (x)_0 \leqq (y)_0]], \\ &= 1 \quad \text{otherwise.} \end{aligned}$$

It is clear that $g(x, y)$ is recursive in $\mathfrak{a}^*$ and that $\lambda xy g(x, y)=0$ is a well-ordering with order-type $\eta\omega+1$. If x_0 is the minimum of the ordering $\preceq$, then the ordering on D has the following properties:

(1) $\langle 0, x_0\rangle = 3^{x_0}$ is the minimum.

(2) 5 is the maximum.

(3) The successor of any element $y \in D$ $(y \neq 5)$ is $2y$.

(4) An element of D is a limit point if and only if it is 5 or of the form 3^x $(x \neq x_0)$.

Let $z \in N(\mathfrak{a}^*)$ be such that

$$f_z^{\mathfrak{a}^*}(x) = g((x)_0, (x)_1),$$

and choose a primitive recursive $m(e, x)$ such that for each α,

$$\begin{aligned} \{m(e, x)\}^{\alpha}(t) &\simeq \langle 2, 0\rangle \quad \text{if } \alpha(\langle t, x\rangle) \neq 0, \\ &\simeq \{e\}(t) \quad \text{if } \alpha(\langle t, x\rangle) = 0. \end{aligned}$$

We define a primitive recursive function $h(u)$ from a Gödel number h using the recursion theorem by:

$$\begin{aligned} h(u) &\simeq \langle 2, 0\rangle && \text{if } u = \langle 0, x_0\rangle, \\ &\simeq \langle 3, h(x)\rangle && \text{if } u = 2x \text{ for some } x, \\ &\simeq \langle 4, z, m(h, u)\rangle && \text{if } u = 3^x \text{ for some } x \neq x_0, \text{ or } u = 5, \\ &\simeq 0 && \text{otherwise.} \end{aligned}$$

It is now easy to verify by transfinite induction on the ordering on D that

$$u \in D \to h(u) \in N(\mathfrak{a}^*), \tag{5}$$

$$\begin{aligned} u, v \in D \,\&\, g(u, v) = 0 &\to |h(u)|^{\mathfrak{a}^*} \leqq |h(v)|^{\mathfrak{a}^*} \\ &\to |h(u)|_c^{\mathfrak{a}^*} \leqq |h(v)|_c^{\mathfrak{a}^*}. \end{aligned} \tag{6}$$

Thus $h(5) \in N(\mathfrak{a}^*)$ and $\eta < \eta\omega + 1 \leqq |h(5)|_c^{\mathfrak{a}^*} < \omega_1(\mathfrak{a}^*)$.

REMARK. Theorem 5 allows us to think of $N(\mathfrak{a}^*)$ as a notation system for the ordinals recursive in $\mathfrak{a}^*$, much as Kleene's O is a notation system for the recursive ordinals.

For $i=1, 2$, the predicates ${}_i\mathbf{G}_z$ $(z \in N(\mathfrak{a}^*))$ provide a nondecreasing, transfinite sequence of degrees, indexed by the notations in $N(\mathfrak{a}^*)$, which exhausts the degrees in the i-section of $\mathfrak{a}^*$. One can easily see that this sequence need not be strictly increasing; however we shall prove in Corollary 8.1 that the sequence of degrees of the predicates ${}_i\mathbf{G}_z$ has order-type $\omega_1(\mathfrak{a}^*)$.

REMARK. At the suggestion of the referee we outline a proof that for each a, $\kappa(a^*) > \aleph_1$. Using the notation of §2, consider the functional

$$\begin{aligned} \mathrm{h}(\xi, x, z, \tau, t) &= \mathrm{f}_z^\tau(t)+1 \quad \textit{if } \xi \in \mathrm{WO} \ \&\ x \in D_\xi \ \&\ |z|^\tau \leqq |x|_\xi, \\ &= 0 \qquad\qquad\quad \textit{otherwise}, \end{aligned}$$

where for $x \in D_\xi$, $|x|_\xi$ is the order-type of the initial segment of $\leqq_\xi$ up to and including x. It is easy to show by the methods of §1 that the functional h is hyperanalytic (in the fixed objects τ^3, τ^2). Now if each $|z|^\tau$ were countable, we would have

$$z \in N(\tau) \equiv (E\xi)(Ex)[\mathrm{h}(\xi, x, z, \tau, 0) > 0].$$

In the notation of §3 this means that $z \in N(a^*)$ would be recursive in a^*, contradicting Lemma 4.

4. **Minimum functions.** In this section we prove two theorems on ordinal notations which are basic for the theory of predicates r.e. in a^*. The results are inspired by [2], where Gandy announces similar results for type 2.

We revert to the notation of §2, where $N(\tau)$, $\mathrm{f}_z^\tau(t)$ and $|z|^\tau$ ($z \in N(\tau)$) are defined, relative to arbitrary but fixed parameters τ^2, τ^3.

If $z \notin N(\tau)$, put $|z|^\tau = \text{supremum } \{|w|^\alpha : w \in N(\alpha)\}$.

THEOREM 6. *There is a partial recursive functional* $\phi({}^3\mathbf{E}, \tau^3, \tau^2, z, \alpha, w, \beta) \simeq \phi(z, \alpha, w, \beta)$ *such that*:

(1) $[z \in N(\alpha)\ \&\ |z|^\alpha \leqq |w|^\beta] \rightarrow \phi(z, \alpha, w, \beta) \simeq 0$.

(2) $|w|^\beta < |z|^\alpha \rightarrow \phi(z, \alpha, w, \beta) \simeq 1$.

REMARK. Using Theorem 4, one can easily define a functional $\phi_1(z, \alpha, w, \beta)$ which will have properties (1) and (2) whenever $z \in N(\alpha)$, or a functional $\phi_2(z, \alpha, w, \beta)$ which will have properties (1) and (2) whenever $w \in N(\alpha)$. The nontrivial applications of Theorem 6 involve computing $\phi(z, \alpha, w, \beta)$ when we know $z \in N(\alpha) \vee w \in N(\alpha)$, but do not know which of the disjuncts is true; in such cases $\phi(z, \alpha, w, \beta)$ gives us this information.

Proof. We define $\phi(z, \alpha, w, \beta)$ from its index $\hat{\phi}$ using the recursion theorem.

For each number u we consider seven cases 1–7, according as u is in one of the forms $\langle 1, q\rangle$, $\langle 2, 0\rangle$, $\langle 3, x_1\rangle$, $\langle 4, x_1, e\rangle$, $\langle 5, x_1, e\rangle$, $\langle 6, x_1, e\rangle$ (cases 1–6), or in none of these forms (case 7). In the computation we distinguish $7^2 = 49$ cases labeled ij ($1 \leqq i \leqq 7$, $1 \leqq j \leqq 7$), where in case ij we assume that z is in case i and w is in case j. (The computation is trivial, except when $3 \leqq i \leqq 6$, $3 \leqq j \leqq 6$.) For each of these cases we give informal instructions for computing $\phi(z, \alpha, w, \beta)$ in the manner of the proof of Theorem 1.

We shall be referring to the function $\mathrm{p}(z)$ of Theorem 2; recall that when $z \in N(\tau)$, then $\{\mathrm{p}(z)\}(\tau^3, \tau^2, \tau, t) \simeq \mathrm{f}_z^\tau(t)$.

Several times below we use the expression "ask ${}^2\mathbf{E}$." In these instances we wish to emphasize that the predicate whose value we are determining only involves number quantification; surely questions to ${}^2\mathbf{E}$ can be rephrased as questions to ${}^3\mathbf{E}$.

A. *Trivial cases.*

A1. $1 \leqq i \leqq 2 \vee j=7$. Give (output) 0.

A2. $[i=7 \,\&\, j<7] \vee [i \geqq 3 \,\&\, 1 \leqq j \leqq 2]$. Give 1.

B. *Cases ij* with $3 \leqq i \leqq 5$, $3 \leqq j \leqq 5$.

Case 33. $[z=\langle 3, z_1\rangle \,\&\, w=\langle 3, w_1\rangle]$. Give output $\phi(z_1, \alpha, w_1, \beta)$.

Case 34. $[z=\langle 3, z_1\rangle \,\&\, w=\langle 4, w_1, m\rangle]$.

?1. Compute $\phi(z_1, \alpha, w_1, \beta)$.

11 $[\phi(z_1, \alpha, w_1, \beta) \simeq 0$ in ?1]. Give 0.

?12 $[\phi(z_1, \alpha, w_1, \beta) \simeq 1$ in ?1]. Let $\delta=\lambda u\{p(w_1)\}(\tau^3, \tau^2, \beta, u)$; ask ${}^2\mathbf{E}$ if $\lambda s\{m\}^{\delta}(s)$ is completely defined.

121 [No to ?12]. Give 0.

?122 [Yes to ?12]. Ask ${}^2\mathbf{E}$ if $(Es)[\phi(z_1, \alpha, \{m\}^{\delta}(s), \beta)=0]$.

1221 [Yes to ?122]. Give 0.

1222 [No to ?122]. Give 1.

Case 43. $[z=\langle 4, z_1, e\rangle \,\&\, w=\langle 3, w_1\rangle]$. This case is symmetric to Case 34. The computation is that of Case 34, except that z and w, z_1 and w_1, α and β and e and m are interchanged; however we do not change the order of the arguments in ϕ. We give this symmetric computation here so we can omit it in some other cases later.

?1. Compute $\phi(z_1, \alpha, w_1, \beta)$.

11 $[\phi(z_1, \alpha, w_1, \beta) \simeq 1$ in ?1]. Give 1.

?12 $[\phi(z_1, \alpha, w_1, \beta) \simeq 0$ in ?1]. Let $\gamma=\lambda u\{\mathrm{p}(z_1)\}(\tau^3, \tau^2, \alpha, u)$; ask ${}^2\mathbf{E}$ if $\lambda t\{e\}^{\gamma}(t)$ is completely defined.

121 [No to ?12]. Give 1.

?122 [Yes to ?12]. Ask ${}^2\mathbf{E}$ if $(Et)[\phi(\{e\}^{\gamma}(t), \alpha, w_1, \beta)=1]$.

1221 [Yes to ?122]. Give 1.

1222 [No to ?122]. Give 0.

Case 35. $[z=\langle 3, z_1\rangle \,\&\, w=\langle 5, w_1, m\rangle]$.

?1. Ask ${}^3\mathbf{E}$ if m defines a β-recursive functional $\{m\}(\delta, \beta)$.

11 [No to ?1]. Give 0.

?12 [Yes to ?1]. Ask ${}^3\mathbf{E}$ if $(E\delta)[\phi(z_1, \alpha, w_1, \{m\}(\delta, \beta))=0]$.

121 [Yes to ?12]. Give 0.

122 [No to ?12]. Give 1.

Case 53. $[z=\langle 5, z_1, e\rangle \,\&\, w=\langle 3, w_1\rangle]$. Symmetric to Case 35.

Case 44. $[z=\langle 4, z_1, e\rangle \,\&\, w=\langle 4, w_1, m\rangle]$.

?1. Compute $\phi(z_1, \alpha, w_1, \beta)$.

?11 $[\phi(z_1, \alpha, w_1, \beta) \simeq 0$ in ?1]. Let $\gamma=\lambda u\{\mathrm{p}(z_1)\}(\tau^3, \tau^2, \alpha, u)$; ask ${}^2\mathbf{E}$ if $\lambda t\{e\}^{\gamma}(t)$ is completely defined.

111 [No to ?11]. Give 1.

?112 [Yes to ?11]. Ask ${}^2\mathbf{E}$ if $(t)[\phi(\{e\}^{\gamma}(t), \alpha, w_1, \beta)=0]$.

1121 [Yes to ?112]. Give 0.

?1122 [No to ?112]. Let $\delta = \lambda u\{p(w_1)\}(\tau^3, \tau^2, \beta, u)$; ask ${}^2\mathbf{E}$ if $\lambda s\{m\}^\delta(s)$ is completely defined.

11221 [No to ?1122]. Give 0.

?11222 [Yes to ?1122]. Ask ${}^2\mathbf{E}$ if

$$(t)[\phi(\{e\}^\gamma(t), \alpha, w_1, \beta) = 0 \vee (Es)[\phi(\{e\}^\gamma(t), \alpha, \{m\}^\delta(s), \beta) = 0]].$$

112221 [Yes to ?1122]. Give 0.

112222 [No to ?1122]. Give 1.

?12 [$\phi(z_1, \alpha, w_1, \beta) \simeq 1$ in ?1]. (This subcase is symmetric to ?11, and the computation is symmetric to ?11–112222.) Let $\delta = \lambda u\{p(w_1)\}(\tau^3, \tau^2, \beta, u)$; ask ${}^2\mathbf{E}$ if $\lambda s\{m\}^\delta(s)$ is completely defined.

121 [No to ?12]. Give 0.

?122 [Yes to ?12]. Ask ${}^2\mathbf{E}$ if $(s)[\phi(z_1, \alpha, \{m\}^\delta(s), \beta) = 1]$.

1221 [Yes to ?122]. Give 1.

?1222 [No to ?122]. Let $\gamma = \lambda u\{p(z_1)\}(\tau^3, \tau^2, \alpha, u)$; ask ${}^2\mathbf{E}$ if $\lambda t\{e\}^\gamma(t)$ is completely defined.

12221 [No to ?1222]. Give 1.

?12222 [Yes to ?1222]. Ask ${}^2\mathbf{E}$ if

$$(s)[\phi(z_1, \alpha, \{m\}^\delta(s), \beta) = 1 \vee (Et)[\phi(\{e\}^\gamma(t), \alpha, \{m\}^\delta(s), \beta) = 1]].$$

122221 [Yes to ?12222]. Give 1.

122222 [No to ?12222]. Give 0.

Case 45. [$z = \langle 4, z_1, e\rangle$ & $w = \langle 5, w_1, m\rangle$].

?1. Ask ${}^3\mathbf{E}$ if m defines a β-recursive functional $\{m\}(\delta, \beta)$.

11 [No to ?1]. Give 0.

?12 [Yes to ?1]. Ask ${}^3\mathbf{E}$ if $(E\delta)[\phi(z_1, \alpha, w_1, \{m\}(\delta, \beta)) = 0]$.

121 [No to ?12]. Give 1.

?122 [Yes to ?12]. Let $\gamma = \lambda u\{p(z_1)\}(\tau^3, \tau^2, \alpha, u)$; ask ${}^2\mathbf{E}$ if $\lambda t\{e\}^\gamma(t)$ is completely defined.

1221 [No to ?122]. Give 1.

1222 [Yes to ?122]. Ask ${}^3\mathbf{E}$ if $(t)(E\delta)[\phi(\{e\}^\gamma(t), \alpha, w_1, \{m\}(\delta, \beta)) = 0]$.

12221 [Yes to ?1222]. Give 0.

12222 [No to ?1222]. Give 1.

Case 54. [$z = \langle 5, z_1, e\rangle$ & $w = \langle 4, w_1, m\rangle$]. Symmetric to Case 45.

Case 55. [$z = \langle 5, z_1, e\rangle$ & $w = \langle 5, w_1, m\rangle$].

?1. Ask ${}^3\mathbf{E}$ if e defines an α-recursive functional $\{e\}(\gamma, \alpha)$.

11 [No to ?1]. Give 1.

?12 [Yes to ?1]. Ask ${}^3\mathbf{E}$ if m defines a β-recursive functional $\{m\}(\delta, \beta)$.

121 [No to ?12]. Give 0.

?122 [Yes to ?12]. Ask ${}^3\mathbf{E}$ if $(\gamma)(E\delta)[\phi(z_1, \{e\}(\gamma, \alpha), w_1, \{m\}(\delta, \beta)) = 0]$.

1221 [Yes to ?122]. Give 0.

1222 [No to ?122]. Give 1.

C. *Cases* ij with $i \geqq 3$ & $j \geqq 3$ & $[i = 6 \vee j = 6]$.

A number $w = \langle 6, w_1, m\rangle$ is a member of $N(\beta)$ if and only if $\lambda s\{m\}^\beta(s) = \delta$ is completely defined and $w_1 \in N(\delta)$; and in that case $|w|^\beta = |w_1|^\delta + 1$. Thus, once we verify that $\lambda s\{m\}^\beta(s)$ is completely defined, case *i*6 is treated very much like case *i*3, the other case where $|w|^\beta$ is always a successor ordinal (if $w \in N(\beta)$). We give one of these cases and leave the rest for the reader.

Case 36. $[z = \langle 3, z_1\rangle \ \&\ w = \langle 6, w_1, m\rangle]$.

?1. Ask $^2\mathbf{E}$ if $\lambda s\{m\}^\beta(s) = \delta$ is completely defined.

11 [No to ?1]. Give 0.

?12 [Yes to ?1]. Compute $\phi(z_1, \alpha, w_1, \delta)$.

121 $[\phi(z_1, \alpha, w_1, \delta) \simeq 0]$. Give 0.

122 $[\phi(z_1, \alpha, w_1, \delta) \simeq 1]$. Give 1.

To prove that this functional $\phi(z, \alpha, w, \beta)$ satisfies the conclusion of the theorem, we first show by induction on $z \in N(\alpha)$ that, if $|z|^\alpha \leqq |w|^\beta$, then $\phi(z, \alpha, w, \beta) \simeq 0$ and then by induction on $w \in N(\beta)$ that if $|w|^\beta < |z|^\alpha$, then $\phi(z, \alpha, w, \beta) \simeq 1$. The induction on $z \in N(\alpha)$ breaks down into 42 cases (since z must be in one of cases 1–6 while w is arbitrary) of which only 16 are nontrivial. Similarly the induction on $w \in N(\beta)$ has but 16 nontrivial cases.

Let us say that 0 *is the proper value* (for z, α, w, β) if $z \in N(\alpha)$ & $|z|^\alpha \leqq |w|^\beta$ and that 1 is the proper value (for z, α, w, β) if $|w|^\beta < |z|^\alpha$ (which implies $w \in N(\beta)$). We must verify in each of the 16 nontrivial cases, and under each of the two assumptions, that 0 or 1 is the proper value, that the computation yields that proper value as output. Each time we may assume as induction hypothesis that the computation leads to the proper value for arguments that are "predecessors" of z or w as members of $N(\alpha)$ or $N(\beta)$.

In 33 the verification is immediate, since the ind. hyp. guarantees that the proper value for z_1, α, w_1, β is the same as the proper value for z, α, w, β.

Consider 34 and assume that the proper value is 0. We claim that the computation will be along one of the branches of Diagram A below and will reach the end of some branch, thus giving output 0.

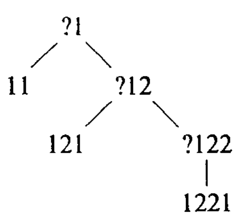

Diagram A.

Case 34; proper value $=0$.

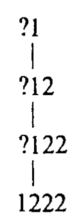

Diagram B.

Case 34; proper value $=1$.

To substantiate the claim we must verify that each time a question is asked in that computation, the partial predicate involved is defined and the answer leads the computation along one of the branches of the diagram. The ind. hyp. guarantees that $\phi(z_1, \alpha, w_1, \beta)$ is defined, hence ?1 is answered. If $\phi(z_1, \alpha, w_1, \beta) \simeq 1$ and we go to ?12, then by ind. hyp. we know that $|w_1|^\beta < |z_1|^\alpha$, hence $w_1 \in N(\beta)$. Thus $\delta = \lambda u\{p(w_1)\}(\tau^3, \tau^2, \beta, u)$ is completely defined and the Substitution Lemma (which is tacitly appealed to in ?12) guarantees that ?12 is answered. If the answer to ?12 leads us to ?122, then we know that $\lambda s\{m\}^\delta(s)$ is completely defined, and by ind. hyp. $\phi(z_1, \alpha, \{m\}^\delta(s), \beta)$ is defined for each s, so again ?122 is well posed for the Substitution Lemma. Now the ind. hyp. implies that the answer to ?122 must be "yes," since the assumption that it is "no" together with the ind. hyp. easily leads to the conclusion $|w|^\beta < |z|^\alpha$.

A similar verification is needed for each of the two assumptions, that 0 or 1 is the proper value, in each of the nontrivial cases. One simple way to obtain these verifications is to draw diagrams of all the possible computation paths and check that the proper value is reached along each allowable branch. We draw one more of these diagrams and omit the details.

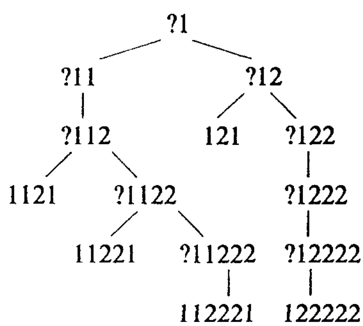

Diagram C.

Case 44; proper value = 0.

THEOREM 7. *There is a partial recursive* $\psi({}^3\mathbf{E}, \tau^3, \tau^2, \tau, h) \simeq \psi(\tau, h)$ *such that*

$$\lambda t\{h\}(t) \text{ is completely defined } \& \ (Et)[\{h\}(t) \in N(\tau)] \rightarrow (t)|(\{h\}(\psi(\tau, h))|^\tau \leqq |\{h\}(t)|^\tau.$$

(In particular,

$$\lambda t\{h\}(t) \text{ comp. defined } \& \ (Et)[\{h\}(t) \in N(\tau)] \rightarrow \{h\}(\psi(\tau, h)) \in N(\tau).)$$

Proof. We shall define $\psi(\tau, h) \simeq \psi({}^3\mathbf{E}, \tau^3, \tau^2, \tau, h)$ from an index $\tilde{\psi}$ by the recursion theorem in the usual way. We assume throughout that $\lambda t\{h\}(t)$ is completely defined.

If $(Et)[\{h\}(t) \in N(\tau)]$, put

$$\text{rank}(h) = \text{least } s\ (t)[|\{h\}(s)|^\tau \leqq |\{h\}(t)|^\tau].$$

Let h' be a primitive recursive function of h such that

$$\{h'\}(t) \simeq \{h\}(t+1).$$

We notice that if rank $(h) > 0$, then rank $(h') = $ rank $(h) - 1$.

Recall that by Theorem 3:

$$\psi(\tau, h)\downarrow \; \equiv \mathrm{q}(\bar{\psi}, [h]) \in N(\tau).$$

In the instructions for the computation of $\psi(\tau, h)$ from $\bar{\psi}$ we write $\mathrm{h}(t)$ for $\{h\}(t)$.

?1. Compute $\phi(\mathrm{h}(0), \tau, \mathrm{q}(\bar{\psi}, [h']), \tau)$.

?11 $[\phi(\mathrm{h}(0), \tau, \mathrm{q}(\bar{\psi}, [h']), \tau)\simeq 0$ in ?1]. Ask ${}^2\mathbf{E}$ if $(t)[\phi(\mathrm{h}(0), \tau, \mathrm{h}(t), \tau)=0]$.

111 [Yes to ?11]. Give 0.

112 [No to ?11]. Give $\psi(\tau, h')+1$.

?12 $[\phi(\mathrm{h}(0), \tau, \mathrm{q}(\bar{\psi}, [h']), \tau)\simeq 1$ in ?1]. Compute $\phi(\mathrm{h}(0), \tau, \mathrm{h}(\psi(\tau, h')+1), \tau)$.

121 $[\phi(\mathrm{h}(0), \tau, \mathrm{h}(\psi(\tau, h')+1), \tau)\simeq 0$ in ?12]. Follow the instructions of ?11.

122 $[\phi(\mathrm{h}(0), \tau, \mathrm{h}(\psi(\tau, h')+1), \tau)\simeq 1$ in ?12]. Give $\psi(\tau, h')+1$.

We prove that if $(Et)[\mathrm{h}(t) \in N(\tau)]$, then $\psi(\tau, h)$ gives the correct output rank (h), by induction on rank (h).

BASIS. Rank $(h)=0$, i.e., $(t)[|\mathrm{h}(0)|^{\tau} \leqq |\mathrm{h}(t)|^{\tau}]$. Since $\mathrm{h}(0) \in N(\tau)$,

$$\phi(\mathrm{h}(0), \;\tau, \;\mathrm{q}(\bar{\psi}, [h']), \tau)$$

is defined and ?1 is answered. If the answer sends us to ?11, the answer to this must be "yes," so by 111 the output is correctly 0. If the answer to ?1 sends us to ?12, then $|\mathrm{q}(\bar{\psi}, [h'])|^{\tau} < |\mathrm{h}(0)|^{\tau}$, so $\mathrm{q}(\bar{\psi}, [h']) \in N(\tau)$, i.e., $\psi(\tau, [h'])$ is defined, so ?12 is answered; now the answer must lead us to 121 and thence to ?11 and the correct output, since by assumption $|\mathrm{h}(0)|^{\tau} \leqq |\mathrm{h}(\psi(\tau, h')+1)|^{\tau}$.

INDUCTION STEP. Rank $(h)>0$. Now we may assume that $\psi(\tau, h')\simeq$ rank (h'); we must show that $\psi(\tau, h)\simeq\psi(\tau, h')+1$. Since $\psi(\tau, h')$ is defined, $\mathrm{q}(\bar{\psi}, [h']) \in N(\tau)$, so ?1 is answered. If the answer to ?1 leads us to ?11, the answer to this must be "no," so we are led to 112 and the correct output. If the answer to ?1 leads us to ?12, then from the answer to that we either go to ?11 and thence to the correct output as before or directly to the correct output through 122.

5. **Predicates r.e. in ${}^3\mathbf{E}$.**

DEFINITION 7. Let $\mathfrak{b}$, τ^3, τ^2, $\boldsymbol{\tau}$ and $\boldsymbol{x}$ be as in Definition 3, let $\boldsymbol{\alpha}=\alpha_1, \ldots, \alpha_k$, $\boldsymbol{\beta}=\beta_1, \ldots, \beta_j$.

(1) $\phi(\mathfrak{b}, z, \boldsymbol{\alpha}, w, \boldsymbol{\beta})\simeq\phi({}^3\mathbf{E}, \tau^3, \tau^2, z, \langle\boldsymbol{\alpha}\rangle, w, \langle\boldsymbol{\beta}\rangle)$.

(2) $\psi(\mathfrak{b}, \boldsymbol{\tau}, h)\simeq\psi({}^3\mathbf{E}, \tau^3, \tau^2, \langle\boldsymbol{\tau}\rangle, h)$.

(3) *Let* $\mathrm{h}(e, x)$ *be primitive recursive and such that*

$$\{\mathrm{h}(e, \boldsymbol{x})\}(t) = \mathrm{s}(e, \boldsymbol{x}, t),$$

put

$$\nu(\mathfrak{b}, \boldsymbol{\tau}, \boldsymbol{x}, e) \simeq \psi(\mathfrak{b}, \boldsymbol{\tau}, \mathrm{h}(e, \boldsymbol{x})).$$

The functionals ϕ, ψ and ν are clearly partial recursive in ${}^3\mathbf{E}$. In particular, if $\mathfrak{b}^*={}^3\mathbf{E}, \mathfrak{b}$, then $\lambda z\boldsymbol{\alpha}w\boldsymbol{\beta}\phi(\mathfrak{b}^*, z, \boldsymbol{\alpha}, w, \boldsymbol{\beta})$ is partial recursive in $\mathfrak{b}^*$, and similarly with ψ and ν.

For any list $\mathfrak{a}=\mathfrak{b}, \tau(^4)$, if $z \notin N(\mathfrak{a})$, put $|z|^{\mathfrak{a}}=\text{supremum } \{|z|^{\mathfrak{b},\alpha} : w \in N(\mathfrak{b}, \alpha)\}$. Using Theorems 6 and 7 we can easily verify the following results:

(F) $z \in N(\mathfrak{b}, \alpha)$ & $|z|(\mathfrak{b}, \alpha) \leqq |w|(\mathfrak{b}, \beta) \rightarrow \phi(\mathfrak{b}, z, \alpha, w, \beta) \simeq 0$.

(G) $|w|(\mathfrak{b}, \beta) < |z|(\mathfrak{b}, \alpha) \rightarrow \phi(\mathfrak{b}, z, \alpha, w, \beta) \simeq 1$.

(H) *Let* $\mathfrak{a}=\mathfrak{b}, \tau, \mathrm{h}(t) \simeq \{h\}(t)$; *then*

$$\lambda t\,\mathrm{h}(t) \textit{ comp. defined } \& \ (Et)\{\mathrm{h}(t) \in N(\mathfrak{a})] \rightarrow \psi(\mathfrak{a}, h)\downarrow \ \& \ (t)[|\mathrm{h}(\psi(\mathfrak{a}, h))|^{\mathfrak{a}} \leqq |\mathrm{h}(t)|^{\mathfrak{a}}].$$

(I) *Let* $\mathfrak{c}=\mathfrak{b}, \tau, x$; *then*

$$(Et)[\{e\}(\mathfrak{c}, t)\downarrow] \rightarrow \nu(\mathfrak{c}, e)\downarrow \ \& \ \{e\}(\mathfrak{c}, \nu(\mathfrak{c}, e))\downarrow.$$

From (F), with $\mathfrak{b}^*$ for $\mathfrak{b}$, we can easily prove Lemma 3; (I) is Gandy's main result in [2], for type 3.

These results allow us to establish several elementary properties of the class of predicates r.e. in $^3\mathbf{E}$. (We leave two basic normal form theorems for §6.) Contrary to the situation in §3, where we had to restrict ourselves to predicates with variables of types $\leqq 1$, we study here predicates $P(\mathfrak{c})$, r.e. in $^3\mathbf{E}$, where $\mathfrak{c}$ is a list of variables of types $\leqq 3$ (Definition 4). Relativized versions of Theorems 7 and 8 are obtained by substituting specific functionals for some of the variables in the list $\mathfrak{c}$.

(*Added November* 1966. We thank Abraham Robinson for showing to us in May, 1966 a mimeographed copy of R. Platek, *Foundations of recursion theory*, Ph.D. Thesis, Stanford University, 1966. This mimeographed copy (dated January, 1966) contains a proof of (I) (with a different ν) as well as proofs of the immediate corollaries of (I) in Theorem 7 and Theorem 8 below. Platek's independent proofs of these results utilize an analysis of hyperanalytic computations that is based on a λ-calculus instead of our analysis through the induction in Definition 1.)

THEOREM 7. (1) *If* $P(\mathfrak{c}, y)$ *is r.e. in* $^3\mathbf{E}$, *then* $(Ey)P(\mathfrak{c}, y)$ *is also r.e. in* $^3\mathbf{E}$.

(2) *If* $P(\mathfrak{c})$ *and* $Q(\mathfrak{c})$ *are r.e in* $^3\mathbf{E}$, *then* $P(\mathfrak{c}) \vee Q(\mathfrak{c})$ *is also r.e. in* $^3\mathbf{E}$.

(3) *A predicate* $P(\mathfrak{c})$ *is recursive in* $^3\mathbf{E}$ (*hyperanalytic*) *if and only if both* $P(\mathfrak{c})$ *and* $\bar{P}(\mathfrak{c})$ *are r.e. in* $^3\mathbf{E}$.

(4) *Let* $P(\mathfrak{c})$ *and* $Q(\mathfrak{c})$ *be r.e. in* $^3\mathbf{E}$. *There exist predicates* $P_1(\mathfrak{c})$ *and* $Q_1(\mathfrak{c})$, *r.e. in* $^3\mathbf{E}$. *such that*:

(a) $P_1(\mathfrak{c}) \rightarrow P(\mathfrak{c})$; $Q_1(\mathfrak{c}) \rightarrow Q(\mathfrak{c})$.

(b) $P(\mathfrak{c}) \vee Q(\mathfrak{c}) \rightarrow P_1(\mathfrak{c}) \vee Q_1(\mathfrak{c})$.

(c) *It is impossible that* $P_1(\mathfrak{c})$ & $Q_1(\mathfrak{c})$.

Proof. (1) Put $\mathfrak{c}^*={}^3\mathbf{E}, \mathfrak{c}$ and choose e so that $P(\mathfrak{c}, y) \equiv \{e\}(\mathfrak{c}^*, y)\downarrow$; now **(I)** implies that $(Ey)P(\mathfrak{c}, y) \equiv \{e\}(\mathfrak{c}^*, \nu(\mathfrak{c}^*, e))\downarrow$.

(2) follows immediately from (1).

(3) The "only if" part is trivial. To prove the "if" part, write $\mathfrak{c}=\mathfrak{b}, \tau, x(^4)$, put $\mathfrak{b}^*={}^3\mathbf{E}, \mathfrak{b}$, and choose e, m so that

$$P(\mathfrak{c}) \equiv \{e\}(\mathfrak{b}^*, \tau, x)\downarrow \ \equiv \mathrm{s}(e, x) \in N(\mathfrak{b}^*, \tau),$$

$$\bar{P}(\mathfrak{c}) \equiv \{m\}(\mathfrak{b}^*, \tau, x)\downarrow \ \equiv \mathrm{s}(m, x) \in N(\mathfrak{b}^*, \tau).$$

Now for each $\mathfrak{c}$, exactly one of $\mathrm{s}(e, x)$, $\mathrm{s}(m, x)$ is a member of $N(\mathfrak{b}^*, \tau)$; hence $\lambda\mathfrak{c}\phi(\mathfrak{b}^*, \mathrm{s}(e, x), \tau, \mathrm{s}(m, x), \tau)$ is completely defined and

$$P(\mathfrak{c}) \equiv \phi(\mathfrak{b}^*, \mathrm{s}(e, x), \tau, \mathrm{s}(m, x), \tau) = 0.$$

(4) As in (3), pick e, m so that

$$P(\mathfrak{c}) \equiv \mathrm{s}(e, x) \in N(\mathfrak{b}^*, \tau),$$
$$Q(\mathfrak{c}) \equiv \mathrm{s}(m, x) \in N(\mathfrak{b}^*, \tau).$$

Put

$$P_1(\mathfrak{c}) \equiv P(\mathfrak{c}) \,\&\, \phi(\mathfrak{b}^*, \mathrm{s}(e, x), \tau, \mathrm{s}(m, x), \tau) \simeq 0,$$
$$Q_1(\mathfrak{c}) \equiv Q(\mathfrak{c}) \,\&\, \phi(\mathfrak{b}^*, \mathrm{s}(e, x), \tau, \mathrm{s}(m, x), \tau) \simeq 1.$$

The verification that P_1 and Q_1 have the desired properties is easy.

COROLLARY 7.1. *Let* $\mathfrak{a} = \mathfrak{b}, \tau$, $\mathfrak{a}^* = {}^3\mathbf{E}, \mathfrak{a}$. *The predicate* $\lambda z\mathfrak{a}\ z \in N(\mathfrak{a}^*)$ *is r.e. in* ${}^3\mathbf{E}$.

Proof. By (A),

$$\{e\}(\mathfrak{a}^*, x)\downarrow \ \equiv \mathrm{s}(e, x) \in N(\mathfrak{a}^*).$$

We claim that for each $\mathfrak{a}$,

$$\text{supremum } \{|\mathrm{s}(e, x)|^{\mathfrak{a}^*} : \{e\}(\mathfrak{a}^*, x)\downarrow\} = \kappa(\mathfrak{a}^*);$$

because if this supremum were less than $\kappa(\mathfrak{a}^*)$, then Lemma 5 would imply that $\lambda ex\ \mathrm{s}(e, x) \in N(\mathfrak{a}^*)$ is recursive in $\mathfrak{a}^*$ which is easily seen to be absurd. Thus

$$\begin{aligned} z \in N(\mathfrak{a}^*) &\equiv (Ee)(Ex)[\{e\}(\mathfrak{a}^*, x)\downarrow \ \&\ |z|^{\mathfrak{a}^*} \leqq |\mathrm{s}(e, x)|^{\mathfrak{a}^*}] \\ &\equiv (Ee)(Ex)[\{e\}(\mathfrak{a}^*, x)\downarrow \ \&\ \phi(\mathfrak{b}^*, z, \tau, \mathrm{s}(e, x), \tau) \simeq 0], \end{aligned}$$

which is r.e. in ${}^3\mathbf{E}$ by (1) of the theorem.

REMARK. Theorem 9 in §6 implies that $\lambda z\mathfrak{a}\ z \in N(\mathfrak{a})$ is also r.e. in ${}^3\mathbf{E}$.

The most interesting consequences of Theorems 5 and 6 are several *choice axioms* which hold for predicates r.e. in ${}^3\mathbf{E}$. Again, these results are inspired by Gandy's similar theorems for type 2 in [2].

In the equivalence of Theorem 8 we use the restricted quantifiers $(E\alpha)_{\mathfrak{c}}$, $(E\mathrm{g})_{\mathfrak{c}}$; these are to be read "there exists an α recursive in $\mathfrak{c}$," "there exists a g recursive in $\mathfrak{c}$." We use "g" as a variable for functions or functionals with arguments of types $\leqq 3$.

THEOREM 8. *Let* $\mathfrak{c}$, $\mathfrak{d}$ *be lists of variables of types* $\leqq 3$, *for each of the equivalences below let P be a predicate in the indicated variables which is r.e. in* ${}^3\mathbf{E}$.

(1) $(\mathfrak{d})(Ey)P(\mathfrak{c}, \mathfrak{d}, y) \equiv (E\mathrm{g})_{\mathfrak{c}^*}(\mathfrak{d})P(\mathfrak{c}, \mathfrak{d}, \mathrm{g}(\mathfrak{d}))$.

(2) $(\mathfrak{d})(E\alpha)_{\mathfrak{c}^*,\mathfrak{d}}P(\mathfrak{c}, \mathfrak{d}, \alpha) \equiv (E\mathrm{g})_{\mathfrak{c}^*}(\mathfrak{d})P(\mathfrak{c}, \mathfrak{d}, \lambda t\mathrm{g}(\mathfrak{d}, t))$.

(*Particular cases of* (1) *and* (2) *are*:

(3) $(x)(Ey)P(\mathfrak{c}, x, y) \equiv (E\alpha)_{\mathfrak{c}^*}(x)P(\mathfrak{c}, x, \alpha(x))$,

(4) $(x)(E\alpha)_{\mathfrak{c}^*}P(\mathfrak{c}, x, \alpha) \equiv (E\alpha)_{\mathfrak{c}^*}(x)P(\mathfrak{c}, x, \lambda t\alpha(\langle x, t\rangle))$.)

Proof. The implications from right to left are trivial. To prove (1) and (2) from left to right, let $\mathfrak{c}=\mathfrak{a}, x$ as usual, and pick e in each case so that

$$P(\mathfrak{c}, \mathfrak{b}, -) \equiv \{e\}(\mathfrak{c}^*, \mathfrak{b}, -)\downarrow.$$

(1) Put $\mathrm{g}(\mathfrak{b}) \simeq \nu(\mathfrak{c}, \mathfrak{b}, e)$; now (I) implies that $\mathrm{g}(\mathfrak{b})$ is completely defined, recursive in $\mathfrak{c}^*$, and satisfies the right-hand side of (1).

(2) We compute,

$$(\mathfrak{b})(E\alpha)_{\mathfrak{c}^*,\mathfrak{b}}P(\mathfrak{c}, \mathfrak{b}, \alpha) \to (\mathfrak{b})(Ee)[(t)[\{e\}(\mathfrak{c}^*, \mathfrak{b}, t)\downarrow] \,\&\, P(\mathfrak{c}, \mathfrak{b}, \lambda t\{e\}(\mathfrak{c}^*, \mathfrak{b}, t))]$$
$$\to (\mathfrak{b})(Ee)[(t)[\{e\}(\mathfrak{c}^*, \mathfrak{b}, t)\downarrow] \,\&\, (\beta)[\beta = \lambda t\{e\}(\mathfrak{c}^*, \mathfrak{b}, t) \to P(\mathfrak{c}, \mathfrak{b}, \beta)]].$$

We can easily show that the predicate in brackets is r.e. in ${}^3\mathbf{E}$, using the Substitution Lemma and the fact that the class of predicates r.e. in ${}^3\mathbf{E}$ is closed under universal (number or function) quantification and conjunction. Thus (1) applies, and there exists a functional $\mathrm{g}(\mathfrak{b})$, recursive in $\mathfrak{c}^*$, such that

$$(\mathfrak{b})[(t)[\{\mathrm{g}(\mathfrak{b})\}(\mathfrak{c}^*, \mathfrak{b}, t)\downarrow] \,\&\, P(\mathfrak{c}, \mathfrak{b}, \lambda t\{\mathrm{g}(\mathfrak{b})\}(\mathfrak{c}^*, \mathfrak{b}, t))].$$

We obtain then a functional $\mathrm{g}(\mathfrak{b}, t)$ that satisfies the right-hand side of (2) by putting

$$\mathrm{g}(\mathfrak{b}, t) \simeq \{\mathrm{g}(\mathfrak{b})\}(\mathfrak{c}^*, \mathfrak{b}, t).$$

COROLLARY 8.1. *For $i=1, 2$, put $z \in N_i(\mathfrak{a}^*) \equiv z \in N(\mathfrak{a}^*) \,\&\, (w)[|w|^{\mathfrak{a}^*} < |z|^{\mathfrak{a}^*} \to {}_i\mathbf{G}_z]$ is not recursive in ${}_i\mathbf{G}_w$ & $(w)[|w|^{\mathfrak{a}^*} = |z|^{\mathfrak{a}^*} \to z \leq w]$.*

(1) *The predicate $\lambda z\mathfrak{a}\; z \in N_i(\mathfrak{a}^*)$ is r.e. in ${}^3\mathbf{E}$.*

(2) *The set $N_i(\mathfrak{a}^*)$ is linearly ordered by $\lambda zw\; |z|_{\mathfrak{c}}^{\mathfrak{a}^*} \leq |w|_{\mathfrak{c}}^{\mathfrak{a}^*}$ with order-type $\omega_1(\mathfrak{a}^*)$.*

Proof. (1) follows easily from Corollary 7.1, Lemma 3 and the remark that the predicate "$\mathbf{F}$ is recursive in $\mathbf{G}$" is hyperanalytic.

To prove (2) by contradiction, assume that the ordering on $N_i(\mathfrak{a}^*)$ is similar to an initial segment of $\omega_1(\mathfrak{a}^*)$, say $\{|w|_{\mathfrak{c}}^{\mathfrak{a}^*} : |w|_{\mathfrak{c}}^{\mathfrak{a}^*} < |z|_{\mathfrak{c}}^{\mathfrak{a}^*}\}$. Put

$$P(\mathfrak{a}, x, y) \equiv \{not\,(|x|_{\mathfrak{c}}^{\mathfrak{a}^*} < |z|_{\mathfrak{c}}^{\mathfrak{a}^*}) \,\&\, y = 0\}$$
$$\vee\; [|x|_{\mathfrak{c}}^{\mathfrak{a}^*} < |z|_{\mathfrak{c}}^{\mathfrak{a}^*} \,\&\, y \in N_i(\mathfrak{a}^*)$$
$$\&\, (\alpha)[\alpha \textit{ is a mapping from } \{u : |u|_{\mathfrak{c}}^{\mathfrak{a}^*} \leq |x|_{\mathfrak{c}}^{\mathfrak{a}^*}\}$$
$$\textit{into } \{v : v \in N_i(\mathfrak{a}^*) \,\&\, |v|_{\mathfrak{c}}^{\mathfrak{a}^*} \leq |y|_{\mathfrak{c}}^{\mathfrak{a}^*}\} \textit{ which}$$
$$\textit{preserves the ordinal ordering} \to \alpha(x) = y]].$$

It is easy to verify that $\{v : v \in N_i(\mathfrak{a}^*) \,\&\, |v|_{\mathfrak{c}}^{\mathfrak{a}^*} \leq |y|_{\mathfrak{c}}^{\mathfrak{a}^*}\}$ is hyperanalytic in $\mathfrak{a}$, uniformly for $y \in N_i(\mathfrak{a}^*)$, and hence that $P(\mathfrak{a}, x, y)$ is r.e. in ${}^3\mathbf{E}$. Since, obviously, $(x)(Ey)P(\mathfrak{a}, x, y)$, Theorem 8 implies that for some α, hyperanalytic in $\mathfrak{a}$,

$$(x)P(\mathfrak{a}, x, \alpha(x)).$$

But then

$$y \in N_i(\mathfrak{a}^*) \equiv (Ex)[|x|_{\mathfrak{c}}^{\mathfrak{a}^*} < |z|_{\mathfrak{c}}^{\mathfrak{a}^*} \,\&\, \alpha(x) = y],$$

so that $N_i(\mathfrak{a}^*)$ is hyperanalytic in $\mathfrak{a}$, which is easily seen to be absurd.

REMARK. The sequence of predicates ${}_i\mathbf{G}_z\,(z \in N_i(\mathfrak{a}^*))$ provides a strictly increasing sequence of degrees of length $\omega_1(\mathfrak{a}^*)$, which is indexed by a set r.e. in $\mathfrak{a}^*$ and exhausts the i-section of $\mathfrak{a}^*$.

6. **Normal forms for predicates r.e. in $^3\mathbf{E}$.** One of the main results of RFI is the Representation Theorem XXVIII, which asserts (for type 3) that there exist primitive recursive predicates $L(e, \mathfrak{c}, w, \alpha, \mathbf{F})$ and $M(e, \mathfrak{c}, w, \alpha, \mathbf{F})$([4]), such that

$$\{e\}(\mathfrak{c}) \simeq w \equiv (E\mathbf{F})(\alpha)L(e, \mathfrak{c}, w, \alpha, \mathbf{F}),$$

$$\{e\}(\mathfrak{c}) \simeq w \equiv (\mathbf{F})(E\alpha)M(e, \mathfrak{c}, w, \alpha, \mathbf{F}).$$

In this section we shall outline proofs of strengthened versions of these theorems, in which we specify *bases* (in the sense of [5]) for the quantifiers $(E\mathbf{F})$, $(\mathbf{F})$ above (our L and M however will be analytic). As corollaries we show that the class of predicates r.e. in $^3\mathbf{E}$ is closed under restricted functional quantification $(E\mathbf{F})_{\mathfrak{c}^*}$, but not closed under unrestricted function quantification $(E\alpha)$.

THEOREM 9. (1) *If $R(\mathfrak{c}, \mathbf{F})$ is hyperanalytic, then $(E\mathbf{F})_{\mathfrak{c}^*}R(\mathfrak{c}, \mathbf{F})$ is r.e. in $^3\mathbf{E}$.*

(2) *If $P(\mathfrak{c})$ is r.e. in $^3\mathbf{E}$, then there exists an analytic $R(\mathfrak{c}, \mathbf{F})$ such that*

$$P(\mathfrak{c}) \equiv (E\mathbf{F})R(\mathfrak{c}, \mathbf{F}) \equiv (E\mathbf{F})_{\mathfrak{c}^*}R(\mathfrak{c}, \mathbf{F}).$$

(Thus $P(\mathfrak{c})$ is r.e. in $^3\mathbf{E}$, if and only if it is expressible in the form $(E\mathbf{F})_{\mathfrak{c}^*}R(\mathfrak{c}, \mathbf{F})$, with analytic R.)

Proof. (1) We compute

$$(E\mathbf{F})_{\mathfrak{c}^*}R(\mathfrak{c}, \mathbf{F}) \equiv (Ee)[(\alpha)[\{e\}(\mathfrak{c}^*, \alpha)\downarrow] \;\&\; R(\mathfrak{c}, \lambda\alpha\{e\}(\mathfrak{c}^*, \alpha))];$$

now the Substitution Lemma implies that the predicate in brackets is r.e. in $^3\mathbf{E}$, hence $(E\mathbf{F})_{\mathfrak{c}^*}R(\mathfrak{c}, \mathbf{F})$ is r.e. in $^3\mathbf{E}$ by (1) of Theorem 7.

To prove (2), we shall define an analytic predicate $L(\tau^3, \tau^2, \tau, z, \mathbf{F})$, such that in the notation of §2,

(i) $z \in N(\tau) \equiv (E\mathbf{F})L(\tau^3, \tau^2, \tau, z, \mathbf{F})$

$$\equiv (E\mathbf{F})[\mathbf{F} \text{ is recursive in } {}^3\mathbf{E}, \tau^3, \tau^2, \tau \;\&\; L(\tau^3, \tau^2, \tau, z, \mathbf{F})];$$

the result will then follow by (A).

The construction is a lengthy but straightforward analysis of the transfinite induction for $z \in N(\tau)$, so we shall omit many of the details.

(ii) We define a predicate $P(u, \alpha, v, \beta, \delta)$ by the *disjunction* of the following seven clauses. ($P(u, \alpha, v, \beta, \delta)$ will assert that, if $v \in N(\beta)$, then $u \in N(\alpha)$ and u in $N(\alpha)$ is an immediate predecessor of v in $N(\beta)$, relative to δ.) (Notation: $q = w = (v)_1$, $e = (v)_2$.)

(1) $v = \langle 1, q\rangle \;\&\; u = \langle 1, q\rangle$.

(2) $v = \langle 2, 0\rangle \;\&\; u = \langle 2, 0\rangle$.

(3) $v = \langle 3, w\rangle \;\&\; u = w \;\&\; a = \beta$.

(4) $v = \langle 4, w, e\rangle \;\&\; \alpha = \beta \;\&\; [(a) \vee (b) \vee (c)]$, where

(a) $u = w$.

(b) $[\lambda t\{e\}^{\delta}(t)$ is not completely defined$] \;\&\; u = 0$.

(c) $[\lambda t\{e\}^{\delta}(t)$ is completely defined$] \;\&\; (Et)[u = \{e\}^{\delta}(t)]$.

(5) $v = \langle 5, w, e\rangle$ & [(d) ∨ (e)], where:

(d) [e does not define a β-recursive functional] & $u=0$.

(e) [e defines a β-recursive functional $\{e\}(\gamma, \beta)$] & $u=w$ & $(E\gamma)[\alpha=\{e\}(\gamma,\beta)]$.

(6) $v=\langle 6, w, e\rangle$ & [(f) ∨ (g)], where:

(f) [$\lambda t\{e\}^{\beta}(t)$ is not completely defined] & $u=0$.

(g) [$\lambda t\{e\}^{\beta}(t)=\gamma$ is completely defined] & $u=w$ & $\alpha=\gamma$.

(7) [v does not satisfy any of the case hypotheses for (1)–(6)] & $u=0$.

(iii) Let $\mathbf{F}(z, \tau)$ and $\mathbf{G}(z, \tau, t)$ be variables for functionals of the indicated list of variables. We define a predicate $S(\mathbf{F}, \mathbf{G})$ by the *conjunction* of the following eight clauses. (We suppress in the notation the dependence of $S(\mathbf{F}, \mathbf{G})$ on τ^3, τ^2.)

(1) $\mathbf{F}(\langle 1, q\rangle, \tau) = 0$ & $\mathbf{G}(\langle 1, q\rangle, \tau, t)=q$.

(2) $\mathbf{F}(\langle 2, 0\rangle, \tau)=0$ & $\mathbf{G}(\langle 2, 0\rangle, \tau, t)=\tau(t)$.

(3) $\mathbf{F}(\langle 3, w\rangle, \tau)=0 \rightarrow [\mathbf{F}(w, \tau)=0$ & $\mathbf{G}(\langle 3, w\rangle, \tau, t)=\tau^2(\lambda u\mathbf{G}(w, \tau, u))]$.

(4) $\mathbf{F}(\langle 4, w, e\rangle, \tau)=0 \rightarrow [\mathbf{F}(w, \tau)=0$ & $[\delta=\lambda u\mathbf{G}(w,\tau,u) \rightarrow [\lambda t\{e\}^{\delta}(t)$ is completely defined] & $(t)\mathbf{F}(\{e\}^{\delta}(t), \tau)=0$ & $\mathbf{G}(\langle 4, w, e\rangle, \tau, t)=\mathbf{G}(\{e\}^{\delta}(t), \tau, t)]]$.

(5) $\mathbf{F}(\langle 5, w, e\rangle, \tau)=0 \rightarrow$ [e defines a τ-recursive functional $\{e\}(\gamma, \tau)$] & $(\gamma)[\mathbf{F}(w, \{e\}(\gamma, \tau)) = 0]$ & $\mathbf{G}(\langle 5, w, e\rangle, \tau, t)=\tau^3(\lambda\gamma\mathbf{G}(w, \{e\}(\gamma, \tau), t))]$.

(6) $\mathbf{F}(\langle 6, w, e\rangle, \tau)=0 \rightarrow [\gamma=\lambda t\{e\}^{\tau}(t)$ is completely defined & $\mathbf{F}(w, \gamma)=0$ & $\mathbf{G}(\langle 6, w, e\rangle, \tau, t)=\mathbf{G}(w, \gamma, t)]$.

(7) If x is not in any of the forms $\langle 2, 0\rangle$, $\langle 1, q\rangle$, $\langle 3, w\rangle$, $\langle 4, w, e\rangle$, $\langle 5, w, e\rangle$, $\langle 6, w, e\rangle$, then $\mathbf{F}(x, \tau)\neq 0$.

(8) (*Well-ordering clause.*) For an arbitrary function β, put

$$\beta_n = \lambda t\beta(\langle n, t\rangle).$$

$$(\alpha)(\beta)[(x)[\mathbf{F}(\alpha(x), \beta_x) = 0 \;\&\; P(\alpha(x+1), \beta_{x+1}, \alpha(x), \beta_x, \lambda t\mathbf{G}((\alpha(x))_1, \beta_{x+1}, t))] \rightarrow (Ex)[\alpha(x) = \langle 2, 0\rangle \vee \alpha(x) = \langle 1, (\alpha(x))_1\rangle]].$$

We now claim:

(iv) $z \in N(\tau) \rightarrow (E\mathbf{F})(E\mathbf{G})[\mathbf{F}, \mathbf{G}$ are recursive in ${}^3\mathbf{E}, \tau^3, \tau^2, \tau$, & $S(\mathbf{F}, \mathbf{G})$ & $\mathbf{F}(z, \tau)=0]$.

(v) $(E\mathbf{F})(E\mathbf{G})[S(\mathbf{F}, \mathbf{G})$ & $\mathbf{F}(z, \tau)=0] \rightarrow z \in N(\tau)$.

From these two equivalences we can obtain (i) and then complete the proof of the theorem by routine contractions of variables.

Proof of (iv). If $z \in N(\tau)$, put

$$\mathbf{F}(x, \gamma) = 0 \quad \text{if } |x|^{\gamma} \leqq |z|^{\tau},$$
$$= 1 \quad \text{otherwise},$$
$$\mathbf{G}(x, \gamma, t) = f_x^{\gamma}(t) \quad \text{if } |x|^{\gamma} \leqq |z|^{\tau}$$
$$= 0 \quad \text{otherwise}.$$

Lemma 3 and Theorem 2 imply that $\mathbf{F}$ and $\mathbf{G}$ are recursive in ${}^3\mathbf{E}$, τ^3, τ^2, τ, and of course $\mathbf{F}(z, \tau)=0$.

The verification of the first seven clauses of $S(\mathbf{F}, \mathbf{G})$ is immediate. To verify clause (8), we prove by cases on the definition of $P(u, \alpha, v, \beta, \delta)$, that for each u, v, α, β,

$$[P(u, \alpha, v, \beta, \lambda t\mathbf{G}((v)_1, \alpha, t)) \,\&\, \mathbf{F}(v, \beta) = 0] \rightarrow [v = \langle 2, 0\rangle \vee v = \langle 1, (v)_1\rangle] \vee [|u|^{\alpha} < |v|^{\beta}].$$

(For example, if $v = \langle 4, w, e\rangle$, then $\alpha = \beta$ and u must satisfy one of (a), (b), (c) with $\delta = \lambda t\mathbf{G}(w, \alpha, t) = \lambda t f_w^{\alpha}(t)$. Since $\mathbf{F}(v, \beta) = 0$, $v \in N(\beta)$, hence $w \in N(\alpha)$ and (b) cannot be true; it is then evident that whether (a) or (c) holds, $|u|^{\alpha} < |v|^{\beta}$.) Clause (8) then follows, since we cannot have an infinite decreasing sequence of ordinals

$$|\alpha(0)|^{\beta_0} > |\alpha(1)|^{\beta_1} > \cdots.$$

Proof of (v). Assume that $S(\mathbf{F}, \mathbf{G}) \,\&\, \mathbf{F}(z, \tau) = 0$. Consider the set $\mathcal{T}$ of all finite sequences $\{(u_0, \beta_0), \ldots, (u_x, \beta_x)\}$ of pairs of numbers and functions such that

$$u_0 = z \,\&\, \beta_0 = \tau \,\&\, (s)_{s<x}P(u_{s+1}, \beta_{s+1}, u_s, \beta_s, \lambda t\mathbf{G}((u_s)_1, \beta_{s+1}, t)) \,\&\, (s)_{s\leqq x}\mathbf{F}(u_s, \beta_s) = 0.$$

We can think of $\mathcal{T}$, partially ordered under extension of finite sequences, as a *tree*, where at each node $\{(u_0, \beta_0), \ldots, (u_x, \beta_x)\}$ there may be one, countably many or uncountably many choices for (u_{x+1}, β_{x+1}), depending on the prime number expansion of u_x.

Consider the predicate of sequences of $\mathcal{T}$,

$$\text{(vi)} \qquad u_x \in N(\beta_x) \,\&\, \mathbf{G}(u_x, \beta_x, t) = f(\beta_x, u_x; t).$$

Clause (8) of the definition of $S(\mathbf{F}, \mathbf{G})$ asserts that this predicate has the *well-founded property* on $\mathcal{T}$, i.e., if we follow any branch along the tree, we must meet a node where (vi) holds. We shall prove that (vi) holds for all sequences in $\mathcal{T}$ (and hence $z \in N(\tau)$) by *bar induction* on $\mathcal{T}$ (as in [10]). (One may easily rephrase the proof as a classical proof by contradiction.)

Basis of bar induction:

$$(\alpha)(\beta)[(x)\{(\alpha(0), \beta_0), \ldots, (\alpha(x), \beta_x)\} \in \mathcal{T} \rightarrow (Ek)(x)_{x\geqq k}[\alpha(x) \in N(\beta_x) \,\&\, \mathbf{G}(\alpha(x), \beta_x, t) = f(\beta_x, \alpha(x); t)]].$$

This is immediate from clause (8) of definition (ii) and clauses (1) and (2) of definitions (ii) and (iii).

Ind. step of bar induction: Let $\{(u_0, \beta_0), \ldots, (u_x, \beta_x)\}$ be a sequence in $\mathcal{T}$. We must show $u_x \in N(\beta_x) \,\&\, \mathbf{G}(u_x, \beta_x, t) = f(\beta_x, u_x; t)$, utilizing the following:

Ind. hypothesis: *if* (u_{x+1}, β_{x+1}) *is any pair such that*

$$\{(u_0, \beta_0), \ldots, (u_x, \beta_x), (u_{x+1}, \beta_{x+1})\} \in \mathcal{T},$$

then $u_{x+1} \in N(\beta_{x+1}) \,\&\, \mathbf{G}(u_{x+1}, \beta_{x+1}, t) = f(\beta_{x+1}, u_{x+1}; t)$. Since $\mathbf{F}(u_x, \beta_x) = 0$, we know that u_x must be in one of the forms $\langle 2, 0\rangle$, $\langle 1, q\rangle$, etc., and hence in the definition of $\mathbf{F}(u_x, \beta_x)$ one of the first six clauses must apply. We only treat one of the cases, the others being similar.

Clause (4). $u_x = \langle 4, w, e\rangle$. First choose $u_{x+1} = w$, $\beta_{x+1} = \beta_x$; clearly

$$\mathbf{F}(w, \beta_x) = 0 \;\&\; P(w, \beta_x, u_x, \beta_x, \lambda t\mathbf{G}(w, \beta_x, t)),$$

so the sequence $\{(u_0, \beta_0), \ldots, (u_x, \beta_x), (w, \beta_x)\} \in \mathscr{T}$ and by ind. hyp.

$$w \in N(\beta_x) \;\&\; \mathbf{G}(w, \beta_x, \mathrm{t}) = \mathrm{f}(\beta_x, w; t).$$

Moreover, if $\delta = \lambda t\mathbf{G}(w, \beta_x, t)$, then $\lambda t\{e\}^\delta(t)$ is comp. defined and for each t, $\mathbf{F}(\{e\}^\delta(t), \beta_x) = 0$. Thus for each t, if we choose $u_{x+1} = \{e\}^\delta(t)$, $\beta_{x+1} = \beta_x$, we have $\{(u_0, \beta_0), \ldots, (u_x, \beta_x), (\{e\}^\delta(t), \beta_x)\} \in \mathscr{T}$, so by ind. hyp.

$$\{e\}^\delta(t) \in N(\beta_x) \;\&\; \mathbf{G}(\{e\}^\delta(t), \beta_x, t) = \mathrm{f}(\beta_x, \{e\}^\delta(t); t).$$

This implies $\langle 4, w, e\rangle \in N(\beta_x)$, and clause (4) of definition (iii) implies that

$$\mathbf{G}(\langle 4, w, e\rangle, \beta_x, t) = \mathrm{f}(\beta_x, \langle 4, w, e\rangle; t).$$

Remark. One half of Kleene's Representation Theorem follows from (2) of Theorem 9 if we remark that $\{e\}(\mathfrak{c}) \simeq w$ is r.e. in ${}^3\mathbf{E}$ and absorb in the usual way all but one of the function quantifiers in the prefix of the analytic predicate $R(\mathfrak{c}, \mathbf{F})$ in the quantifier $(E\mathbf{F})$.

Corollary 9.1. (1) *If* $P(\mathfrak{c}, \alpha)$ *and* $Q(\mathfrak{c}, \mathbf{F})$ *are r.e. in* ${}^3\mathbf{E}$, *then so are* $(E\alpha)_{\mathfrak{c}^*}P(\mathfrak{c}, \alpha)$ *and* $(E\mathbf{F})_{\mathfrak{c}^*}P(\mathfrak{c}, \mathbf{F})$.

(2) *In the notation of Theorem* 8, *if* $P(\mathfrak{c}, \mathfrak{d}, \mathbf{F})$ *is r.e. in* ${}^3\mathbf{E}$, *then*

$$(\mathfrak{d})(E\mathbf{F})_{\mathfrak{c}^*,\mathfrak{d}}P(\mathfrak{d}, \mathfrak{c}, \mathbf{F}) \equiv (Eg)_{\mathfrak{c}^*}(\mathfrak{d})P(\mathfrak{c}, \mathfrak{d}, \lambda\alpha\mathrm{g}(\mathfrak{c}, \mathfrak{d}, \alpha)).$$

Proof. (1) By (2) of Theorem 9, there exists an analytic predicate $R(\mathfrak{c}, \mathbf{F}, \mathbf{G})$, such that

$$Q(\mathfrak{c}, \mathbf{F}) \equiv (E\mathbf{G})_{\mathfrak{c}^*,\mathbf{F}}R(\mathfrak{c}, \mathbf{F}, \mathbf{G});$$

hence

$$(E\mathbf{F})_{\mathfrak{c}^*}Q(\mathfrak{c}, \mathbf{F}) \equiv (E\mathbf{F})_{\mathfrak{c}^*}(E\mathbf{G})_{\mathfrak{c}^*,\mathbf{F}}R(\mathfrak{c}, \mathbf{F}, \mathbf{G}) \equiv (E\mathbf{F})_{\mathfrak{c}^*}R(\mathfrak{c}, (\mathbf{F})_0, (\mathbf{F})_1),$$

which is r.e. in ${}^3\mathbf{E}$ by (1) of Theorem 9.

(2) $(\mathfrak{d})(E\mathbf{F})_{\mathfrak{c}^*,\mathfrak{d}}P(\mathfrak{d}, \mathfrak{c}, \mathbf{F}) \rightarrow (\mathfrak{d})(Ee)[(E\mathbf{F})_{\mathfrak{c}^*\mathfrak{d}}[(\alpha)[\{e\}(\mathfrak{c}^*, \mathfrak{d}, \alpha) \simeq \mathbf{F}(\alpha)] \;\&\; P(\mathfrak{c}, \mathfrak{d}, \mathbf{F})]]$.

The predicate in brackets is r.e. in ${}^3\mathbf{E}$, by (1) of Theorem 9, hence by Theorem 8,

$$(Eg)_{\mathfrak{c}^*}(\mathfrak{d})P(\mathfrak{c}, \mathfrak{d}, \lambda\alpha\{\mathrm{g}(\mathfrak{d})\}(\mathfrak{c}^*, \mathfrak{d}, \alpha)).$$

Put then

$$\mathrm{g}(\mathfrak{c}, \mathfrak{d}, \alpha) \simeq \{\mathrm{g}(\mathfrak{d})\}(\mathfrak{c}^*, \mathfrak{d}, \alpha).$$

Corollary 9.2. *A predicate* $P(\mathfrak{c})$ *is hyperanalytic if and only if it can be expressed in both forms* $(E\mathbf{F})_{\mathfrak{c}^*}R(\mathfrak{c}, \mathbf{F})$, $(\mathbf{F})_{\mathfrak{c}^*}Q(\mathfrak{c}, \mathbf{F})$, *with analytic* R, Q.

Proof. Immediate from Theorems 9 and 7.

THEOREM 10. *If* $P(\mathfrak{c})$ *is r.e. in* ${}^3\mathbf{E}$, *then there exists a predicate* $Q(\mathfrak{c}, \alpha)$, *r.e. in* ${}^3\mathbf{E}$, *such that*

$$P(\mathfrak{c}) \equiv (\alpha)\bar{Q}(\mathfrak{c}, \alpha).$$

Proof. We shall define a predicate $Q(\tau^3, \tau^2, \tau, \alpha, z)$, r.e. in ${}^3\mathbf{E}$, and prove

$$\text{(i)} \qquad z \in N(\tau) \equiv (\alpha)\bar{Q}(\tau^3, \tau^2, \tau, \alpha, z);$$

from this the theorem follows by (A).

Let $P(u, \alpha, v, \beta, \delta)$ be the analytic predicate defined in the proof of Theorem 9, let $\mathrm{p}(z)$ be the function of Theorem 2. By the Substitution Lemma, there exists a predicate $R_1(\tau^3, \tau^2, \tau, u, \alpha, v, \beta)$, partial recursive in ${}^3\mathbf{E}$, such that

$$P(u, \alpha, v, \beta, \delta) \equiv R_1(\tau^3, \tau^2, \tau, u, \alpha, v, \beta)$$

for

$$\delta = \lambda t\{\mathrm{p}((v)_1)\}(\tau^3, \tau^2, \tau, t),$$

when δ is completely defined. Put

$$Q(u, \alpha, v, \beta) \equiv [\delta \text{ is comp. defined}] \;\&\; R_1(\tau^3, \tau^2, \tau, u, \alpha, v, \beta),$$

where we are suppressing the dependence of Q on τ^3, τ^2, τ. Evidently Q is r.e. in ${}^3\mathbf{E}$. We claim

$$\text{(ii)} \quad z \in N(\tau) \to (\alpha)(\beta)[\alpha(0) = z \;\&\; \beta_0 = \tau \;\&\; (x)Q(\alpha(x+1), \beta_{x+1}, \alpha(x), \beta_x) \to (Ex)[\alpha(x) = \langle 2, 0\rangle \vee \alpha(x) = \langle 1, (\alpha(x))_1\rangle]],$$

$$\text{(iii)} \quad (\alpha)(\beta)[\alpha(0) = z \;\&\; \beta_0 = \tau \;\&\; (x)Q(\alpha(x+1), \beta_{x+1}, \alpha(x), \beta_x) \to (Ex)[\alpha(x) = \langle 2, 0\rangle \vee \alpha(x) = \langle 1, (\alpha(x))_1\rangle]] \to z \in N(\tau).$$

Now (i) follows easily from (ii) and (iii), since the predicate in brackets is clearly the negation of a predicate r.e. in ${}^3\mathbf{E}$.

Proof of (ii) (by induction on $z \in N(\tau)$). Given α, β such that

$$\alpha(0) = z \;\&\; \beta_0 = \tau \;\&\; (x)Q(\alpha(x+1), \beta_{x+1}, \alpha(x), \beta_x),$$

choose α', β' such that

$$\alpha'(x) = \alpha(x+1), \qquad \beta'_x = \beta_{x+1}.$$

Clearly $(x)Q(\alpha'(x+1), \beta'_{x+1}, \alpha'(x), \beta'_x)$. Now we verify that since $Q(\alpha(1), \beta_1, \alpha(0), \beta_0)$, we must have $\alpha(1) \in N(\beta_1)$, i.e., $\alpha'(0) \in N(\beta'_0)$; hence by induction hypothesis, $(Ex)[\alpha'(x) = \langle 2, 0\rangle \vee \alpha'(x) = \langle 1, (\alpha'(x))_1\rangle]$ which easily implies

$$(Ex)[\alpha(x) = \langle 2, 0\rangle \vee \alpha(x) = \langle 1, (\alpha(x))_1\rangle].$$

Proof of (iii). Assume the left-hand side of (iii). Consider the tree $\mathscr{T}$ of all finite sequences $\{(u_0, \beta_0), \ldots, (u_x, \beta_x)\}$ such that

$$u_0 = z \;\&\; \beta_0 = \tau \;\&\; (s)_{s<x}Q(u_{s+1}, \beta_{s+1}, u_s, \beta_s).$$

We prove by *bar induction* on $\mathscr{T}$, that for every sequence $\{(u_0, \beta_0), \ldots, (u_x, \beta_x)\} \in \mathscr{T}$,

(iv) $$u_x \in N(\beta_x);$$

in particular then $z \in N(\tau)$.

Basis of bar induction: This is immediate from the left-hand side of (iii).

Ind. step of bar induction: We must show that $u_x \in N(\beta_x)$, under the ind. hyp. that $u_{x+1} \in N(\beta_{x+1})$, whenever $\{(u_0, \beta_0), \ldots, (u_x, \beta_x), (u_{x+1}, \beta_{x+1})\} \in \mathscr{T}$, i.e., whenever $Q(u_{x+1}, \beta_{x+1}, u_x, \beta_x)$. Here we take cases on the definition of

$$Q(u_{x+1}, \beta_{x+1}, u_x, \beta_x),$$

and again we omit all cases save one.

Clause (4). $u_x = \langle 4, w, e\rangle$. First take $u_{x+1} = w$, $\beta_{x+1} = \beta_x$; as in the proof of Theorem 9, it follows that $w \in N(\beta_x)$, and hence $\delta = \lambda t\{\mathrm{p}(w)\}(\tau^3, \tau^2, \beta_x, t)$ is completely defined. If $\lambda t\{e\}^\delta(t)$ is not completely defined, put

$$\begin{aligned} \alpha'(s) &= u_s \quad \text{if } s \leqq x, & \beta'_s &= \beta_s \quad \text{if } s \leqq x, \\ &= 0 \quad \text{if } s > x, & &\beta_x \quad \text{if } s > x, \end{aligned}$$

and check that α', β' satisfy the antecedent of the left-hand side of (iii), but not the consequent. Hence $\lambda t\{e\}^\delta(t)$ is completely defined, and for each t we can take $u_{x+1} = \{e\}^\delta(t)$, $\beta_{x+1} = \beta_x$ and verify that $\{(u_0, \beta_0), \ldots, (u_x, \beta_x), (u_{x+1}, \beta_{x+1})\} \in \mathscr{T}$. Then by ind. hyp. $\{e\}^\delta(t) \in N(\beta_x)$, so $\langle 4, w, e\rangle \in N(\beta_x)$.

COROLLARY 10.1. *If* $P(\mathfrak{c})$ *is r.e. in* $^3\mathbf{E}$, *then there exists an analytic predicate* $R(\mathfrak{c}, \alpha, \mathbf{F})$, *such that*

$$P(\mathfrak{c}) \equiv (\alpha)(\mathbf{F})R(\mathfrak{c}, \alpha, \mathbf{F}) \equiv (\alpha)(\mathbf{F})_{\mathfrak{c}^*,\alpha}R(\mathfrak{c}, \alpha, \mathbf{F}).$$

Proof. Immediate from Theorems 9 and 10.

REMARK. The second half of Kleene's Representation Theorem follows from this corollary.

COROLLARY 10.2. *There exists a predicate* $P(z, \alpha)$, *r.e. in* $^3\mathbf{E}$, *such that* $(E\alpha)P(z, \alpha)$ *is not r.e. in* $^3\mathbf{E}$.

Proof. The predicate $z \in N(\mathfrak{a}^*, \alpha)$(4) is r.e. in $^3\mathbf{E}$, by Corollary 7.1; we show that $(E\alpha)[z \in N(\mathfrak{a}^*, \alpha)]$ cannot be r.e. in $^3\mathbf{E}$.

Let $P(\mathfrak{c})$ be any predicate r.e. in $^3\mathbf{E}$. By the theorem,

$$P(\mathfrak{c}) \equiv (\alpha)\bar{Q}(\mathfrak{c}, \alpha)$$

for a suitable Q, r.e. in $^3\mathbf{E}$, and by (A)

$$Q(\mathfrak{c}, \alpha) \equiv \mathrm{s}(e, \mathfrak{x}) \in N(\mathfrak{a}^*, \alpha)$$

for a suitable e. Thus

$$\bar{P}(\mathfrak{c}) \equiv (E\alpha)[\mathrm{s}(e, \mathfrak{x}) \in N(\mathfrak{a}^*, \alpha)],$$

and if $(E\alpha)[z \in N(\mathfrak{a}^*, \alpha)]$ were r.e. in $^3\mathbf{E}$, $\bar{P}(\mathfrak{c})$ would be r.e. in $^3\mathbf{E}$. Since $P(\mathfrak{c})$ was an arbitrary predicate r.e. in $^3\mathbf{E}$, this is absurd.

7. **An extension of the hyperanalytic hierarchy.** In this section we outline briefly the theory of predicates $P(\mathfrak{c})$ of the form $(E\alpha)R(\mathfrak{c}, \alpha)$, with R r.e. in ${}^3\mathbf{E}$. We are mostly interested in the case when $\mathfrak{c}$ contains variables of types ≤ 1, when this class resembles in many ways the class of Σ_2^1 number-theoretic predicates. (A hierarchy of Δ_2^1 number-theoretic predicates very similar to the hierarchy $\mathscr{D}(\mathfrak{a}^*)$ constructed below, is given in §16.6 of [8].)

Definition. (1) Let $\mathfrak{a}$ be a list of objects of types 2 and 3, $\mathfrak{c}$ a list of variables of types ≤ 3([4]). Let $\mathscr{S}(\mathfrak{a})$ be the class of predicates $P(\mathfrak{c})$ such that for some $R(\mathfrak{c}, \alpha)$, r.e. in $\mathfrak{a}$,

$$P(\mathfrak{c}) \equiv (E\alpha)R(\mathfrak{c}, \alpha).$$

Let $\mathscr{P}(\mathfrak{a})$ be the class of predicates $P(\mathfrak{c})$ such that $\bar{P}(\mathfrak{c}) \in \mathscr{S}(\mathfrak{a})$, let

$$\mathscr{D}(\mathfrak{a}) = \mathscr{S}(\mathfrak{a}) \cap \mathscr{P}(\mathfrak{a}).$$

A set or function is in $\mathscr{S}(\mathfrak{a})$, $\mathscr{P}(\mathfrak{a})$ or $\mathscr{D}(\mathfrak{a})$ if its representing predicate is in $\mathscr{S}(\mathfrak{a})$, $\mathscr{P}(\mathfrak{a})$ or $\mathscr{D}(\mathfrak{a})$ respectively.

(2) $z \in S(\mathfrak{a}) \equiv (E\alpha)[z \in N(\mathfrak{a}, \alpha)]$.

(3) For $z \in S(\mathfrak{a})$, put

$$\|z\|^{\mathfrak{a}} = \|z\|(\mathfrak{a}) = \textit{infinum}\ \{|z|^{\mathfrak{a},\alpha} : z \in N(\mathfrak{a}, \alpha)\}.$$

Put

$$\lambda(\mathfrak{a}) = \textit{supremum}\ \{\|z\|^{\mathfrak{a}} : z \in S(\mathfrak{a})\}.$$

In the list of results (i)–(x) below we follow the conventions of ([4]), so that in particular $\mathfrak{a}^* = {}^3\mathbf{E}, \mathfrak{a}$.

(i) *$\mathscr{S}(\mathfrak{a}^*)$ is closed under the operations &, $\vee$, (Ex), (x), $(E\alpha)$.*

(ii) *$\mathscr{D}(\mathfrak{a}^*)$ is closed under the operations &, $\vee$, $\neg$, (Ex), (x), $(E\alpha)$, (α) and contains all predicates r.e. in $\mathfrak{a}^*$.*

Proof. By Theorem 10.

(iii) *The predicate $z \in S(\mathfrak{a}, \tau)$ is complete for predicates $P(\tau, x)$ in $\mathscr{S}(\mathfrak{a})$. I.e., for each $P(\tau, x)$ in $\mathscr{S}(\mathfrak{a})$, there exists a primitive recursive $g(x)$ so that*

$$P(\tau, x) \equiv g(x) \in S(\mathfrak{a}, \tau).$$

Proof. By (A).

(iv) *$\mathscr{S}(\mathfrak{a}^*)$ does not contain the predicate $z \notin S(\mathfrak{a}^*, \tau)$, and hence $\mathscr{S}(\mathfrak{a}^*)$ is not closed under the operations $\neg$, (α).*

Proof. Assume that $\mathscr{S}(\mathfrak{a}^*)$ contains the predicate $z \notin S(\mathfrak{a}^*, \tau)$. Then $\mathscr{S}(\mathfrak{a}^*) = \mathscr{D}(\mathfrak{a}^*)$, so by **(iii)** $\mathscr{D}(\mathfrak{a}^*)$ has a complete predicate. However the closure properties **(ii)** easily imply that $\mathscr{D}(\mathfrak{a}^*)$ cannot have a complete predicate.

By **(ii)** and Corollary 7.1, the predicate $z \notin N(\mathfrak{a}^*, \alpha)$ is in $\mathscr{S}(\mathfrak{a}^*)$; if $\mathscr{S}(\mathfrak{a}^*)$ were closed under (α), then the predicate $(\alpha)[z \notin N(\mathfrak{a}^*, \alpha)] \equiv z \notin S(\mathfrak{a}^*)$ would be in $\mathscr{S}(\mathfrak{a}^*)$, contradicting the first part of **(iv)**.

(v) *Let* $\mathfrak{b}$ *be a list of objects of types* 2 *and* 3, *let* $\boldsymbol{\alpha}=\alpha_1,\ldots,\alpha_k$, $\boldsymbol{\beta}=\beta_1,\ldots,\beta_j$ *be two lists of function variables*(4). *There exist predicates* $L(k, j, z, \boldsymbol{\alpha}, w, \boldsymbol{\beta})$ *and* $M(k, j, z, \boldsymbol{\alpha}, w, \boldsymbol{\beta})$ *in* $\mathscr{S}(\mathfrak{b}^*)$, *such that if* $z \in S(\mathfrak{b}^*, \boldsymbol{\alpha})$, *then*

$$\|w\|(\mathfrak{b}^*, \boldsymbol{\beta}) \leqq \|z\|(\mathfrak{b}^*, \boldsymbol{\alpha}) \equiv L(k, j, z, \boldsymbol{\alpha}, w, \boldsymbol{\beta}),$$

$$\neg(\|w\|(\mathfrak{b}^*, \boldsymbol{\beta}) \leqq \|z\|(\mathfrak{b}^*, \boldsymbol{\alpha})) \equiv M(k, j, z, \boldsymbol{\alpha}, w, \boldsymbol{\beta}).$$

(*I.e., the initial segment of ordinals* $\|w\|(\mathfrak{b}^*, \boldsymbol{\beta})$ *which are* $\leqq \|z\|(\mathfrak{b}^*, \boldsymbol{\alpha})$ *is in* $\mathscr{D}(\mathfrak{b}^*, \boldsymbol{\alpha})$, *uniformly for* $z \in S(\mathfrak{b}^*, \boldsymbol{\alpha})$.)

Proof. If $z \in S(\mathfrak{b}^*, \boldsymbol{\alpha})$, then the following equivalences hold:

$$\|w\|(\mathfrak{b}^*, \boldsymbol{\beta}) \leqq \|z\|(\mathfrak{b}^*, \boldsymbol{\alpha})$$
$$\equiv (E\gamma)[w \in N(\mathfrak{b}^*, \boldsymbol{\beta}, \gamma) \;\&\; (\delta)\neg[|z|(\mathfrak{b}^*, \boldsymbol{\alpha}, \delta) < |w|(\mathfrak{b}^*, \boldsymbol{\alpha}, \gamma)]].$$
$$\neg[\|w\|(\mathfrak{b}^*, \boldsymbol{\beta}) \leqq \|z\|(\mathfrak{b}^*, \boldsymbol{\alpha})]$$
$$\equiv (E\delta)[z \in N(\mathfrak{b}^*, \boldsymbol{\beta}, \delta) \;\&\; (\gamma)\neg[|w|(\mathfrak{b}^*, \boldsymbol{\beta}, \gamma) < |z|(\mathfrak{b}^*, \boldsymbol{\alpha}, \delta)]].$$

Lemma 3 and Corollary 7.4 imply that in each case the predicate in brackets on the right is r.e. in $\mathfrak{b}^*$, so the right-hand side is in $S(\mathfrak{b}^*)$.

(vi) (*Boundedness.*) *Let* $P(\boldsymbol{\alpha}, x)$ *be in* $\mathscr{S}(\mathfrak{a}^*)$, *let* $\mathfrak{g}(x)$ *be a function in* $\mathscr{D}(\mathfrak{a}^*)$ *such that*

(1) $$P(\boldsymbol{\alpha}, x) \equiv \mathfrak{g}(x) \in S(\mathfrak{a}^*, \boldsymbol{\alpha}).$$

Then $P(\boldsymbol{\alpha}, x)$ *is in* $\mathscr{D}(\mathfrak{a}^*)$ *if and only if*

(2) $$\textit{supremum}\ \{\|\mathfrak{g}(x)\|(\mathfrak{a}^*, \boldsymbol{\alpha}) : P(\boldsymbol{\alpha}, x)\} < \lambda(\mathfrak{a}^*).$$

The proof is similar to that of Lemma 5, using (v) instead of Lemma 3.

(vii) *For each* $z \in S(\mathfrak{a}^*)$, *put*

$$\mathbf{S}_z(u, \alpha, v, \beta) \equiv \|v\|(\mathfrak{a}^*, \beta) \leqq \|z\|(\mathfrak{a}^*) \;\&\; \|u\|(\mathfrak{a}^*, \alpha) \leqq \|v\|(\mathfrak{a}^*, \beta).$$

Then

(1) *Each* $\mathbf{S}_z$ *is in* $\mathscr{D}(\mathfrak{a}^*)$, *uniformly for* $z \in S(\mathfrak{a}^*)$.

(2) *If* $\|z\|^{\mathfrak{a}^*} = \|w\|^{\mathfrak{a}^*}$, *then* $\mathbf{S}_z \equiv \mathbf{S}_w$.

(3) *If* $\|z\|^{\mathfrak{a}^*} < \|w\|^{\mathfrak{a}^*}$, *then* $\mathbf{S}_z$ *is recursive in* $\mathbf{S}_w$.

(4) *If* $P(\boldsymbol{\alpha}, x)$ *is in* $\mathscr{S}(\mathfrak{a}^*)$, *then* $P(\boldsymbol{\alpha}, x)$ *is recursive in some* $\mathbf{S}_z$.

The proof is similar to that of Theorem 4, using (vi) instead of Lemma 5.

(viii) (*Choice.*) *Let* $P(x, y)$ *be in* $\mathscr{S}(\mathfrak{a}^*)$, *assume that* $(x)(Ey)P(x, y)$; *then there exists a function* $\mathfrak{g}(x)$ *in* $\mathscr{D}(\mathfrak{a}^*)$ *such that* $(x)P(x, \mathfrak{g}(x))$.

Proof. Using (iii), choose a primitive recursive $\mathfrak{f}(x, y)$ such that

$$P(x, y) \equiv \mathfrak{f}(x, y) \in S(\mathfrak{a}^*),$$

and put

$$\mathfrak{g}(x) = y \equiv \mathfrak{f}(x, y) \in S(\mathfrak{a}^*)$$
$$\& (u)[\|\mathfrak{f}(x, u)\|^{\mathfrak{a}^*} \leqq \|\mathfrak{f}(x, y)\|^{\mathfrak{a}^*} \rightarrow [\|\mathfrak{f}(x, u)\|^{\mathfrak{a}^*} = \|\mathfrak{f}(x, y)\|^{\mathfrak{a}^*} \;\&\; y \leqq u]].$$

Now (v) easily implies that $\mathfrak{g}(x) = y$ is in $\mathscr{S}(\mathfrak{a}^*)$; since $\mathfrak{g}(x) \neq y \equiv (Ez)[z \neq y \& \mathfrak{g}(x) = z]$, the predicate $\mathfrak{g}(x) = y$ (and hence the function $\mathfrak{g}(x)$) is in $\mathscr{D}(\mathfrak{a}^*)$.

(ix) *For each* $\mathfrak{a}$, $\{\|z\|^{\mathfrak{a}} : z \in S(\mathfrak{a})\}$ *is a countable set of ordinals; let* ψ *be the unique order-preserving function which maps* $\{\|z\|^{\mathfrak{a}} : z \in S(\mathfrak{a})\}$ *onto an initial segment of the countable ordinals. Put*

(1) $\|z\|^{\mathfrak{a}}_{c} = \psi(\|z\|^{\mathfrak{a}})$ $(z \in S(\mathfrak{a}))$;

(2) $\omega^{\mathfrak{a}}_{2} =$ *supremum* $\{\|z\|^{\mathfrak{a}}_{c} : z \in S(\mathfrak{a})\}$;

(3) *a countable ordinal* η *is a* $\mathscr{D}(\mathfrak{a})$*-ordinal if there exists a predicate in* $\mathscr{D}(\mathfrak{a})$ *which is a well-ordering with order-type* η. *Then* $\omega^{\mathfrak{a}^*}_{2}$ *is the supremum of all* $\mathscr{D}(\mathfrak{a}^*)$*-ordinals.*

Proof. That each $\|z\|^{\mathfrak{a}^*}_{c}$ is a $\mathscr{D}(\mathfrak{a}^*)$-ordinal is easily proved from (**v**) by the method of proof of the first part of Theorem 5.

We prove that $\omega^{\mathfrak{a}^*}_{2}$ is not a $\mathscr{D}(\mathfrak{a}^*)$-ordinal by contradiction; assume that $Q(x, y)$ is in $\mathscr{D}(\mathfrak{a}^*)$ and defines a well-ordering with field D and order-type $\omega^{\mathfrak{a}^*}_{2}$. Put

$$P(x, z) \equiv [x \notin D \;\&\; z = 0] \vee [x \in D \;\&\; z \in S(\mathfrak{a}^*) \;\&\; (E\alpha)\,[\alpha \text{ maps the initial segment of the ordering defined by } Q \text{ up to and including } x \text{ into } \{w : \|w\|^{\mathfrak{a}^*} \leq \|z\|^{\mathfrak{a}^*}\} \text{ in an order-preserving fashion}]].$$

It is easy to verify that $P(x, z)$ is in $\mathscr{S}(\mathfrak{a}^*)$, and clearly $(x)(Ez)P(x, z)$; thus by (**viii**) there is a $g(x)$ in $\mathscr{D}(\mathfrak{a}^*)$, so that $(x)P(x, g(x))$. Hence

$$z \in S(\mathfrak{a}^*) \equiv (Ex)[x \in D \;\&\; \|z\|^{\mathfrak{a}^*} \leq \|g(x)\|^{\mathfrak{a}^*}]$$

which implies that $z \in S(\mathfrak{a}^*)$ is in $\mathscr{D}(\mathfrak{a}^*)$, contradicting (**iv**).

(**x**) *Put*

$$z \in S_2(\mathfrak{a}^*) \equiv z \in S(\mathfrak{a}^*) \;\&\; (w)[\|w\|^{\mathfrak{a}^*} < \|z\|^{\mathfrak{a}^*} \rightarrow \mathbf{S}_z \text{ is not hyperanalytic in } \mathbf{S}_w] \;\&\; (w)[\|w\|^{\mathfrak{a}^*} = \|z\|^{\mathfrak{a}^*} \rightarrow z \leq w].$$

Then the predicate $z \in S_2(\mathfrak{a}^*)$ *is in* $\mathscr{S}(\mathfrak{a}^*)$ *and the set of ordinals* $\{\|z\|^{\mathfrak{a}^*}_{c} : z \in S_2(\mathfrak{a}^*)\}$ *has order-type* $\omega^{\mathfrak{a}^*}_{2}$.

Proof. To prove the first assertion we simply verify that the predicate "**F** is hyperanalytic in **G**" is r.e. in $^3\mathbf{E}$, and hence in $\mathscr{D}(\mathfrak{a}^*)$. The second assertion is proved by the method used in (**ix**).

8. **Comments on results for types other than** 3. Suppose that we alter our basic Definition 1 by deleting clauses (5) and (6). It can be verified that the set of functions $f^{\tau}_{z}(t)$ $(z \in N(\tau))$ coincides then with the set of functions recursive in τ^2, τ. In fact all the subsequent theorems of this paper hold with this modification, and the proofs are somewhat simpler, since there are two fewer cases to worry about. However we obtain nothing in this way (other than the basic characterization) which is not either explicit or implicit in Gandy's [2] and Shoenfield's [9]. (We do think though that our methods are simpler.)

On the other hand one may attempt to extend Definition 1 by adding clauses like (5) and (6) which introduce objects of types > 3. We believe that in this way one can obtain natural hierarchies for the *hyper-(order-r) predicates* with variables of types $< r$ (RFII 11.26) for each r, but we have not carried out the relevant computations.

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SOLUTION OF POST'S REDUCTION PROBLEM AND SOME OTHER PROBLEMS OF THE THEORY OF ALGORITHMS. I.[1)]

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CONTENTS

Introduction

In mathematics there have often arisen problems consisting of an infinite number of individual questions. Problems of this kind are: the problem of finding the greatest common divisor of two whole numbers; the problem of solving systems of algebraic equations with integral coefficients (Hilbert's Tenth Problem); identity problems in semigroups and groups; and many others. To solve such a problem means to indicate a general method, an algorithm, for solving all the individual questions of which the problem consists. For example, Euclid's algorithm enables us to find the greatest common divisor of any pair of whole numbers.

To the intuitive presentation of algorithms as finite systems of rules according to which one transforms constructive objects, words in some alphabet, natural numbers, geometric figures, there corresponds a rigorous mathematical definition given in the papers of Church, Kleene, Markov, and others [5]. The task of constructing an algorithm satisfying certain requirements is called an algorithmic problem.

While many concrete algorithms have been known for a long time, proofs of the non-existence of algorithms for solving problems became possible only after

1) The basic contents of this paper were presented at a meeting of the Moscow Mathematical Society on October 16, 1956.

the notion of algorithms was made precise. Such proofs have, in fact, appeared. Some important algorithmic problems have turned out to be undecidable. Among them are the identity problems in semigroups and groups, the problem of homotopy of paths on polyhedrons, the decision problem of the restricted predicate calculus, and others [3]. The algorithmic problems studied up to recent times have been, in most cases, decision problems of recursively enumerable sets. This means the following. Let M be some set of constructive objects, and let E be a subset of M. The decision problem of the set E consists in constructing an algorithm which enables us to find out, for any object $y \in M$, whether y belongs to E or not.

A set E is called recursively enumerable if there exists an effective process which enables us to construct, one after the other, all the elements of E. If M is the set of pairs of words in a group G given by a finite number of generators and defining equations, and if S is the set of pairs of equal words, then S is a recursively enumerable set, since the process of deriving equations of words from the defining equations according to the rules of inference for groups enables us to obtain every pair of equal words. The decision problem of the set S is the identity problem in the group G [3].

From an arbitrary algorithmic problem one can pass to an arithmetic problem by establishing an effective one-one correspondence between the constructive objects figuring in the initial problem and the natural numbers (Gödelization) [4]. In this way, the notion of algorithm reduces to that of partial recursive function (p. r. f.) [5]. In the language of arithmetic the decision problem of the subset E of the natural number sequence N consists in the computability of the function $\psi_E(n)$ which is equal to 1 when $n \in E$ and to 0 when $n \overline{\in} E$. The function $\psi_E(n)$ is called the decision or characteristic function (c.f.) of the set E. If $\psi_E(n)$ is a general recursive function (g. r. f.), then the set E is called recursive (r. s.). The set F of values of a g. r. f. $\phi(t)$ is called recursively enumerable (r. e.). The process of constructing the elements of F consists in the computation of the values of the function $\phi(t)$.

Every recursive set is recursively enumerable. The decision problems of logic and mathematics the undecidability of which has been proved in a series of papers (cf. [3]) lead (after arithmetization) to recursively enumerable, but not recursive, sets. The first such sets were constructed by Post [1]. The undecidability of algorithmic problems has been proved either directly (the creative set in [1]) or by reducing the problem to another one the undecidability of which has been proved earlier (the identity problems in semigroups and groups [3]).

However the possibility of such a reduction, for arbitrary undecidable problems, of the decidability of recursively enumerable sets occasioned some doubt and was formulated precisely by Post in 1944 under the name of the reduction problem [1].

If Post's Problem were to be answered in the positive sense, then the undecidability of a large number of algorithmic problems, namely the decision problems of r. e. sets, could be proved by reducing to them problems which are already known to be undecidable, using the latter as "standards of undecidability". In the negative case, it would be necessary to study the possible degrees of undecidability of decision problems of r. e. sets.

Investigating the reduction problem, Post constructed r. e. non-recursive sets of three types: creative, simple, and hypersimple (special case of simple) sets [1]. Other types of r. e. non-recursive sets have been considered by Dekker, V. A. Uspenskiĭ and A. A. Mučnik [6]. In [1] Post considered a special case of reducibility, reducibility by means of tables, and proved that a creative set is not reducible by tables to a hypersimple set.

B. A. Trachtenbrot [7] generalized reducibility by tables, introducing the idea of a general recursive operator, and proved that a creative set is not reducible to a hypersimple set by means of any general recursive operator. But the surprising result of Dekker [8] showed that, for any r. e. non-recursive set G, there exists a hypersimple set H such that G and H are reducible to each other. The significance of Dekker's result also consists in the fact that it gives a criterion for decidability of r. e. sets.

After this, B. A. Trachtenbrot and A. V. Kuznecov undertook a study of partial recursive operators.

Almost at the same time as Dekker's paper a paper of Kleene and Post appeared on degrees of unsolvability of arithmetic sets. They established that in the class of sets reducible to r. e. sets (this class also contains non-recursively enumberable sets) there are sets which are not reducible to each other [9]. The reduction problem of r. e. sets remained open.

Along with the decision problems in the theory of algorithms and its applications one also studied the problem of separability of r. e. sets [10].

Algorithmic problems of the most general form, considered by Ju. T. Medvedev, appear as problems of computing a function[1] (constructing an algorithm) fulfilling certain conditions. Such problems we shall call M-problem [11], [12]. To every M-problem corresponds the class of functions fulfilling the conditions of the given problem. These conditions are usually connected with sets of numbers.[2] Medvedev investigated M-problems depending on arbitrary sets of numbers. However, for the theory of algorithms and its applications, the greatest

1) We consider functions of natural numbers, assuming natural numbers as values.
2) By numbers we always mean natural numbers, among which we agree to include zero.

interest is presented by decision problems and problems of separability and enumerability of so-called arithmetic or recursively projective sets (r. p. sets) [13]; above all, decision problems of r. e. sets. The present paper is devoted to the study of these classes of problems. In the first part we introduce the necessary ideas and give a solution of Post's Problem.[1)]

CHAPTER I.

Functional Representation of Partial Recursive Operators

§1. Sequences and Quasisequences

By a sequence we mean a finite or infinite sequence of numbers. Infinite sequences can be interpreted as functions, and we shall identify them with functions (of one argument).

Let us introduce a symbol λ. By a quasisequence we mean a finite or infinite sequence of numbers and the symbol λ. The numbers and the symbol λ occurring in a quasisequence are called its components. The length $M\{l\}$ of a quasisequence l is the number of its components ($1 \leq M\{l\} \leq \infty$). To an infinite quasisequence l there corresponds a unique partial function (p. f.) $l(n)$ ($n \geq 1$), undefined for $n = m$ if the mth component of l is the symbol λ; in the contrary case, $l(m)$ is equal to the mth component of the quasisequence l. We shall denote the nth component of the quasisequence l by $l(n)$ ($n \geq 1$).

We call the numbers and the symbol λ elements. We shall say that elements a and b are mutually consistent, $a \sim b$, if $a = b \vee a = \lambda \vee b = \lambda$. A system of elements $a_1, a_2, \cdots, a_s, \cdots$ is called consistent if its elements are pairwise mutually consistent. The cover $\bigcup a_s = a_1 \cup a_2 \cup \cdots \cup a_s \cup \cdots$ of a consistent system of elements $\{a_s\}$ is defined to be any element a_s which is a number, and, if all $a_s = \lambda$, then it is the symbol λ.

Quasisequences l_1 and l_2 are mutually consistent, $l_1 \sim l_2$, if $l_1(n) \sim l_2(n)$ for all $n \leq \min(M\{l_i\})$. A system of quasisequences $f_1, f_2, \cdots, f_s, \cdots$ is called consistent if its quasisequences are pairwise mutually consistent. The cover $\bigcup_s f_s = f_1 \cup f_2 \cup \cdots \cup f_s \cup \cdots$ of the quasisequences of a consistent system $\{f_s\}$ is defined to be a quasisequence f such that $f(i) = \bigcup_s f_s(i)$ if $f_s(i)$ is defined for some s; $M\{f\} = \sup_s(M\{f_s\})$.

For example, the cover of the quasisequences $(2, \lambda, 0, \lambda)$ and $(\lambda, 4, \lambda, \lambda, 7, \lambda)$ is the quasisequence $(2, 4, 0, \lambda, 7, \lambda)$.

The cover operation is associative and commutative.

---

1) After the printing of the present paper the author learned of the paper of R. Friedberg in which a proof is outlined of Theorem 1 of Chapter II on the existence of incomparable r. e. sets.

Let us introduce a relation $\gtrsim$ for elements and quasisequences. Let a_1 and a_2 be elements and let f_1 and f_2 be quasisequences.

$a_1 \gtrsim a_2$[1] means that either $a_1 = a_2$ or $a_2 = \lambda$.

$f_1 \gtrsim f_2$[1] means that $f_1(n) \gtrsim f_2(n)$ for all $n \leq \min(M\{f_i\})$.

For example, $(2, \lambda, 4, 7) \gtrsim (\lambda, \lambda, 4, \lambda, 5)$.

Obviously,

$$\bigcup f_s \gtrsim f_i. \tag{1.1}$$

Theorem 1. *A system of quasisequences* $\{f_s\}$ *is consistent when and only when there exists a quasisequence* g *such that*

$$(s)\,(g \gtrsim f_s)\ \&\ M\{g\} \gtrsim M\{f_s\}. \tag{1.2}$$

Let us define, finally, a relation $\eqsim$ on quasisequences. Quasisequences e_1 and e_2 are compatible, $e_1 \eqsim e_2$, if $e_1(n) = e_2(n)$ for all $n \leq \min(M\{e_i\})$.

The relation $\eqsim$ is connected with the relation $\gtrsim$ by the formula

$$f \eqsim h \longleftrightarrow f \gtrsim h\ \&\ h \gtrsim f. \tag{1.3}$$

The relations $\gtrsim$ and $\eqsim$ are not transitive, since quasisequences may have different lengths. However, the following weak transitivity holds:

$$e_1 \gtrsim g\ \&\ g \gtrsim e_2\ \&\ M\{g\} \geq \min(M\{e_i\}) \rightarrow e_1 \gtrsim e_2, \tag{1.4}$$

$$e_1 \eqsim g\ \&\ g \eqsim e_2\ \&\ M\{g\} \geq \min(M\{e_i\}) \rightarrow e_1 \eqsim e_2. \tag{1.5}$$

If $e \sim f$ and e is a sequence, then $e \gtrsim f$; if e and f are sequences, then $e \sim f \longleftrightarrow e \eqsim f$.

If $e \eqsim f$ and $M\{e\} \leq M\{f\}$, then e is called a segment of f, and f is an extension of e.

A sequence of 0's and 1's is called a predicate. A quasisequence of 0's, 1's, and the symbol λ is called a quasipredicate.

A U-sequence is a quasisequence consisting of 1's and the symbol λ. If f is a predicate, then by ${}^{\lambda}f$ we denote the U-sequence obtained from f by substituting λ for 0. If e is a U-sequence, then by ${}^{0}e$ we denote the predicate obtained from e by substituting 0 for all λ's. Both transformations preserve the compatibility relation $\eqsim$.

§ 2. Functional Representation of Operators

Let us call a partial function $\delta(\omega)$ orderly if it is defined on a segment of the natural number sequence N $(\omega \leq L)$ or on all of N. Let there be established an effective one-one correspondence between the set of ordered pairs of finite quasisequences and the set of natural numbers. Let us call the number δ

1) $a_1 \gtrsim a_2$ $(f_1 \gtrsim f_2)$ is read: a_1 covers a_2 $(f_1$ covers $f_2)$.

corresponding to the pair of quasisequences $[d, d']$ the index of this pair.

To every orderly function $\delta(\omega)$ there corresponds some sequence of pairs of quasisequences $[d(\omega), d'(\omega)]$ $(\omega = 0, 1, 2, \cdots)$, where the index of the pair $[d(\omega), d'(\omega)]$ is $\delta(\omega)$.

We shall say that the function $\delta(\omega)$ satisfies the consistency condition if $(\omega_1)(\omega_2)(d(\omega_1)) \sim d(\omega_2) \rightarrow d'(\omega_1) \sim d'(\omega_2)$.

Every such function $\delta(\omega)$ determines a mapping (operator) Δ of infinite quasisequences into quasisequences (p. f. into p. f.): $\Delta(f) = g$, where

$$g = \bigcup_{f \gtrsim d(\omega)} d'(\omega).$$

We shall say that the function $\delta(\omega)$ realizes a quasifunctional representation (q. f. r.) of the operator Δ on p. f.

Just as the concept of partial recursive function (p. r. f.) is a precise form of the notion of a computable function, so also the concept of a partial recursive operator (p. r. o.) is a precise form of the idea of an effective mapping of systems of functions into functions [7], [9]. We shall start from the definition of p. r. o. given in [7].

Theorem 2. *The q. f. r. of any p. r. o.* T $(T(f) = g)$ *is realized by some primitive recursive function (prim. r. f.)* $\pi(v)$.

The p. r. o. T is given by a finite system of equations of terms Ω, from which, according to the rules of inference and using the values of the function $f(n)$, we obtain the equation $g(m) = k$. By means of a Gödelian arithmetization of the derivations of equations from the system Ω, we arrive at a primitive recursive sequence $\{a_v\}$ of Gödel numbers of derivations of equations $g(m(v)) = k(v)$, where $m(v)$ and $k(v)$ are primitive recursive functions.

Assume that, in the derivation with number a_v, we used the values $f(n_1)$, $f(n_2), \cdots, f(n_{i_v})$ of the function $f(n)$, $n_1 < n_2 < \cdots < n_{i_v}$. Let us denote by $p(v)$ the quasisequence of length n_{i_v} the n_ith component of which is equal to $f(n_i)$ $(1 \leq i \leq i_v)$, and the other components of which are λ. By $p'(v)$ we denote the quasisequence of length $m(v)$, all components of which, except the last, are equal to λ, and the last component of which is $k(v)$. The index of the pair $[p(v), p'(v)]$ is denoted by $\pi(v)$. It is easily seen that $\pi(v)$ is a primitive recursive function realizing the q. f. r. of the operator T.

The q. f. r. is a generalization of the functional representation (f. r.) of operators first considered by Ju. T. Medvedev [11].

Let us limit ourselves to the set of infinite sequences (i.e., functions) as the domain of definition of operators. Let us effectively enumerate the ordered pairs of finite sequences. Every orderly function (ord. f.) $\delta(\omega)$ determines a sequence of pairs of sequences $\{[d(\omega), d'(\omega)]\}$, where $\delta(\omega)$ is the index of the pair $[d(\omega), d'(\omega)]$. $\delta(\omega)$ will satisfy the consistency condition if

$$(\omega_1)(\omega_2)(d(\omega_1)) \approx d(\omega_2) \rightarrow d'(\omega_1) \approx d'(\omega_2). \tag{1.6}$$

An orderly function $\delta(\omega)$ satisfying this condition determines a mapping Δ of infinite sequences f into sequences g (functions into orderly functions),

$$\Delta(f) = g = \bigcup_{d(\omega) \approx f} d'(\omega).$$

Yu. T. Medvedev proved a theorem on the functional representation (f.r.) of p.r.o.:

The f.r. of any p.r.o. T is realized by some prim. r. f.

The proof of this theorem is similar to the proof of Theorem 2.

Some strong converse theorems are true.

Theorem 3. *Every orderly p.r.f. $\delta(\omega)$ satisfying the consistency condition realizes a q.f.r. of some p.r.o. Δ.*

Theorem 4. *Every orderly p.r.f. $\delta(\omega)$ satisfying the consistency condition* (1.6) *realizes a f.r. of some p.r.o. Δ.*

§3. A Universal Partial Recursive Operator

Theorem 5. *There exists a partial recursive function $\theta(x, \omega)$ universal for orderly partial recursive functions $\delta(\omega)$.*

Let us take a partial recursive function $r(x, \omega)$ universal for partial recursive functions $r_x(\omega)$. The function $\theta_x(\omega) = \theta(x, \omega)$ is assumed to be defined at a point (x_0, ω_0) and equal to $r(x_0, \omega_0)$ if $r(x_0, \omega)$ is defined for all $\omega \leq \omega_0$. In the opposite case, $\theta(x_0, \omega_0)$ is undefined.

If $r_{x0}(\omega) = r(x_0, \omega)$ is an orderly partial recursive function, then

$$\theta(x_0, \omega) \equiv r(x_0, \omega).$$

On the other hand, $\theta_{x0}(\omega)$ is an orderly p.r.f. From this the assertion of Theorem 5 follows.

There exists a p.r.o. R transforming every orderly function $\epsilon(\omega)$ into an orderly function $\delta(\omega)$ satisfying condition (1.6), where $\delta(\omega) \equiv \epsilon(\omega)$ if $\epsilon(\omega)$ satisfies (1.6).

Let $\epsilon(\omega)$ be the index of a pair of sequences $[l(\omega), l'(\omega)]$.

The transformation R consists in the following:

$$\delta(0) = \epsilon(0).$$

Let the values $\delta(0), \cdots, \delta(\omega)$ be defined. We denote the pair of sequences with index $\delta(v)$ by $[d(v), d'(v)]$. Then

$$\delta(\omega+1)=\begin{cases}\epsilon(\omega+1), \text{ if } (v)_{v\leq\omega}\,[d(v)\approx l(\omega+1)\longrightarrow d'(v)\approx l'(\omega+1)],\\ \delta(\omega) \text{ otherwise.}\end{cases}$$

Let us now apply the transformation R to the orderly p.r.f. $\theta_x(\omega)$:

$$\phi_x(\omega)=R[\theta_x(\omega)],$$

where $\phi(x, \omega)=\phi_x(\omega)$ is a p.r.f. universal for orderly p.r.f., satisfying condition (1.6).

According to Medvedev's theorem, the f.r. of any p.r.o. T is realized by the orderly p.r.f. $\phi_x(\omega)$ for some x. Let us denote by T_x the p.r.o. the f.r. of which is realized by the function $\phi_x(\omega)$. The p.r.o. T_x (depending on the parameter x) is said to be universal.

If in the definition of f.r. we take predicates instead of sequences, then we arrive at the definition of the functional representation of operators transforming predicates into predicates. Following B. A. Trachtenbrot [7], we call such operators considered only on the set of predicates Π-operators (Π-o).

It is easy to check that Theorems 2–4 and Medvedev's theorem are true for f.r. of Π-operators.

Not changing notation, we formulate the following theorem.

Theorem 6. *There exists a p.r.f.* $\phi(x, \omega)=\phi_x(\omega)$ *universal for orderly p.r.f., satisfying the consistency condition* (1.6) (*with respect to pairs of predicates*); *moreover, the f.r. of any partial recursive* Π-o P *is realized by some orderly function* $\phi_x(\omega)$. *We denote this partial recursive* Π-o *by* P_x *and call it a universal partial recursive* Π-o.

§4. The Calculus of M-problems

The basic notions of the calculus of M-problems is presented in [11], [12].[1)] By an M-problem (problem A) we mean the task of constructing a function $f(n)$ possessing a given property $\mathfrak{A}_A$. All the functions possessing the given property constitute a class K_A completely characterizing the problem A. Every function $f(n)\in K_A$ is called a resolving function of the problem A ($f\in A$).

Let us consider a series of examples of M-problems.

The decision problem A_E of a set E was already defined as the task of computing (constructing) the c.f. $\psi_E(n)$:

$$K_{A_E}=K_E=\{\psi_E(n)\}.$$

The separability problem for non-intersecting sets E_0, E_1 consists in the computation of any function $f(n)$ possessing the property $\mathfrak{A}_{E_0E_1}$:

1) In [11, 12] these problems occur under the name of mass problems.

$$f(n) = \begin{cases} 0 \text{ when } n \in E_0, \\ 1 \text{ when } n \in E_1, \\ 0 \text{ or } 1 \text{ when } n \overline{\in} E_0 \cup E_1. \end{cases}$$

Let us denote this problem by $B_{E_0 E_1}$. The class of resolving functions K_{E_0, E_1} is infinite if $N \setminus E_0 \cup E_1$ is infinite.

The enumerability problem C_G of a set G is defined as the class of functions $K_{C(G)} = \{u(t)\}$ satisfying the following condition:

$$(t)\,(u(t) \in G)\ \&\ (z)\,[z \in G \rightarrow (Et)\,(u(t) = z)].$$

An M-problem is said to be solvable if there is at least one general recursive *function among the resolving functions.*

The solvability of the separability problem $B_{E_0 E_1}$ means that the sets E_0, E_1 are recursively separable [10], and the solvability of the enumerability problem of a set G means that G is r. e.

An M-problem B is reducible to an M-problem A, $A \geq B$, if there exists a p. r. o. T mapping any function $f \in A$ into some (depending on f) function $g \in B$.

The problems A and B are equivalent, $A \approx B$, if they are reducible to each other.

The set of problems equivalent to the problem A is called the degree of difficulty of the problem $A - |A|$. The degrees of difficulty form a partially ordered set Ω, where $|A| \leq |B|$ if the problem A is reducible to the problem B. We also call the elements of Ω problems.

Medvedev has proved that Ω is a distributive lattice.

By $A \wedge B$ and $A \vee B$ we denote, respectively, the lattice union (conjunction) and intersection (disjunction) of the problems A and B. We also define the recursive conjunction and disjunction of a sequence of problems $\{A_i\}$ (which, generally speaking, does not coincide with the countable union and intersection of the A_i) as elements of the lattice Ω. We select from each class K_{A_i} a function $f_i(n) \in A_i$.

We enumerate pairs of numbers (i, n) by some fixed g. r. f.'s $i = \alpha_1(m)$, $n = \alpha_2(m)$

$$g(m) = f_{\alpha_1(m)}\,(\alpha_2(m)). \tag{1.7}$$

By the resolving functions of the problem $\bigwedge_{i=1}^{\infty} A_i$ we mean the functions $g(m)$ defined in (1.7) for arbitrarily chosen $f_i \in A_i$; the degree of difficulty $|\bigwedge_{i=1}^{\infty} A_i|$ does not depend on the choice of the g. r. f.'s $\alpha_1(m)$, $\alpha_2(m)$. This completely defines the problem $\bigwedge_{i=1}^{\infty} A_i$.

Let us set

$$h_i(0) = i; \quad h_i(n+1) = f_i(n). \tag{1.8}$$

By the resolving functions of the problem $\bigvee_{i=1}^{\infty} A_i$ we mean all functions $h_i(n)$ defined by (1.8) for arbitrarily chosen $f_i \in A_i$.

Theorem 7 (Medvedev). *For any A and B, there exists a least element C in the set of those $\tilde{C} \in \Omega$ such that $A \wedge \tilde{C} \geq B$.*

We call the problem C the reducibility problem (implication) of the problem B to the problem A and we denote it by $A \supset B$.

The idea of the proof of Theorem 7 consists in defining the class $K_{A \supset B}$ to be all those functions which are everywhere defined and which realize f.r.'s of operators S such that

$$f \in A \rightarrow S(f) \in B. \tag{1.9}$$

We shall say that an operator S with property (1.9) reduces problem B to problem A.

CHAPTER II.

Decision Problems of Recursively Enumerable Sets

§1. The Semilattice $\mathfrak{A}(P)$

As we mentioned in the Introduction, there exist recursively enumerable, nonrecursive sets. The conjunction of two decision problems of r.e. sets (as well as recursive conjunctions) is, as is well known, also a decision problem of some r.e. set. Hence, decision problems of r.e. sets form an upper semilattice [9]. In this semilattice there is a largest element—the degree of undecidability of the universal (creative) set U to which every r.e. set is reducible [1]; and a least element—the decision problem of recursive sets $|A_0|$. Let us denote this semilattice by $\mathfrak{A}(P)$.

§2. Post's Reduction Problem

Do there exist in $\mathfrak{A}(P)$ elements different from $|A_U|$ and $|A_0|$?

It will be shown that there exist a "great many" (in the sense indicated below) such elements.

According to the general definition of the reducibility of M-problems, the decision problem of a set D is reducible to the decision problem of a set E (more briefly, the set D is reducible to the set E) if there exists a g.r.o. T mapping the c.f. of the set E into the c.f. of the set D. The g.r.o. T can be assumed to be a Π-operator [7].

In what follows we shall need a series of new concepts and definitions.

The complement of the element $a = 1$ is defined to be the element $ca = \lambda$, and conversely, the complement of the element $b = \lambda$ is defined to be the element $cb = 1$.

By the complement of a U-sequence f we mean the U-sequence $g = cf$ of length $M\{f\}$ such that $g(n) = c(f(n))$.

By the coupling of quasisequences h_1 and h_2 we mean the quasisequence $e = h_1 \circ h_2$ defined by the conditions:

a) h_1 is a segment of e;

b) for $M\{h_2\} \geq n \geq M\{h_1\}$, $e(n) = h_2(n)$;

c) $M\{e\} = \max_{i=1,2} M\{h_i\}$.

The coupling operation is associative, i.e., the coupling $h_1 \circ h_2 \circ \cdots \circ h_k$ of k quasisequences is defined.

We note that $h_1 \circ h_2 \circ \cdots \circ h_i \mathrel{\overline{\sim}} h_1 \circ h_2 \circ \cdots \circ h_k$ $(i \leq k)$.

By the coupling of the quasisequences $h_1, h_2, \cdots, h_k, \cdots$ we mean the cover $\bigcup_{i=1}^{\infty} e_i$ where $e_i = \mathop{\circ}_{k=1}^{i} h_k = h_1 \circ h_2 \circ \cdots \circ h_k$. By $(h \circ a)$ (where a is an element) we denote the quasisequence $h \circ e_a$, where e_a is the infinite quasisequence all components of which are equal to a. We define the quasisequence $[h]_\alpha$(where h is a quasisequence and α is a number) as the segment of $(h \circ \lambda)$ of length α.

By the convolution of e with f relative to α (where e and f are U-sequences and α is a number) we mean the U-sequence $(e, f)_\alpha = e \cup ([e]_\alpha \circ cf)$.

Lemma 1. *Let e and f be U-sequences, and α and n numbers. If $M\{f\} \geq n > \alpha$ and at least one of the two elements $(e \circ \lambda)(n)$ and $f(n)$ is equal to λ, then*

$$h = (e, f)_\alpha \times f.^{1)} \tag{2.1}$$

Let us denote $[e]_\alpha \circ cf$ by g; $h = e \cup g$ and $g(n) = cf(n)$ since $n > \alpha$.

If $(e \circ \lambda)(n) = \lambda$, then $h(n) = (e \circ \lambda)(n) \cup g(n) = \lambda \cup cf(n) = cf(n)$, i.e., $h(n) \neq f(n)$ and $h \times f$.

If $f(n) = \lambda$, then $g(n) = cf(n) = 1$ and $h(n) = 1$.

Again $f(n) \neq h(n)$ and $f \times h$. Q.E.D.

Lemma 2. $[e]_\alpha \mathrel{\overline{\sim}} (e, f)_\alpha$.

In fact, $[e]_\alpha \mathrel{\overline{\sim}} e$ and $[e]_\alpha \mathrel{\overline{\sim}} ([e]_\alpha \circ cf)$. Hence, $[e]_\alpha \mathrel{\overline{\sim}} e \cup ([e]_\alpha \circ cf) = (e, f)_\alpha$.

In Chapter I, § 3, we introduced the f.r. of the universal p.r. Π-o P_x, realized by the orderly p.r.f. $\phi_x(\omega) = \phi(x, \omega)$.

Let us denote by $[d_x \mid \omega \mid, d_x' \mid \omega \mid]$ the pair of finite predicates with index $\phi_x(\omega)$.

We introduce now a parametric representation for p.r.f.:

1) $\times$ is the symbol for incompatibility of sequences.

$$z = \phi(x, \omega):\ x = x(t),\ \omega = \omega(t),\ \ z = z(t),$$

where $x(t)$, $\omega(t)$, $z(t)$ are primitive recursive functions, and where $x(0) = \omega(0) = 0$ and

$$t_1 > t_2 \,\&\, x(t_1) = x(t_2) \to \omega(t_1) > \omega(t_2). \tag{2.2}$$

The predicates $d_{x(t)}|\omega(t)|$ and $d'_{x(t)}|\omega(t)|$ we denote by $\bar{f}|t|$[1] and $\bar{f}'|t|$, respectively.

Let us set ${}^{\lambda}\bar{f}|t| = f|t|$, ${}^{\lambda}\bar{f}'|t| = f'|t|$ (cf. the end of §1, Chapter I);

$$M\{f|t|\} = m(t);\ \ M\{f'|t|\} = m'(t).$$

We call a number t a characteristic value of the operator P_{x_0} if $x(t) = x_0$.

Now we shall formulate a theorem from which the solution of Post's Reduction Problem follows.

Theorem 1. *There exist r. e. non-recursive sets E and G such that, for all x, $P_x(\bar{e}) \neq \bar{g}$ and $P_x(\bar{g}) \neq \bar{e}$, where $\bar{e}$ and $\bar{g}$ are the characteristic functions of E and G respectively, i.e., the r. e. sets E and G are not reducible to each other (cf. Theorem 6 of Chapter* I).

We construct a recursive sequence of numbers $\{t_i\}$ and U-sequences $\{e_i\}$, $\{g_i\}$, where

$$t_0 = 0,\ \ e_0 = f|0|,\ \ g_0 = cf'|0|.$$

Let the numbers $t_0, \cdots, t_{2k}$ and the predicates $e_0, e_1, \cdots, e_{2k}, g_0, \cdots, g_{2k}$ already be defined.

Let us put $t_{2k+1} = \mu t$,[2] satisfying the conditions 1a)–1d):

1a) $f|t| \gtrsim g_{2k}$;

1b) $f|t| \not\approx [g_{2k}]_{\alpha(t,\,2k)}$,

where

$$\left.\begin{aligned} &\alpha(t, q) = \max\{\bar{\alpha}(t, q), \bar{\alpha}'(t, q)\},\\ &\bar{\alpha}(t, q) = \max_{i \le j(t, q)} (m(t_i)) + 1;\quad \bar{\alpha}'(t, q) = \max_{i \le j(t, q)} (m'(t_i)) + 1,\\ &j(t, q) = \max_{\substack{i \le q\\ x(t_i) \le x(t)}} (i);\end{aligned}\right\} \tag{2.3}$$

1c) $(i)_{i<k}(x(t_{2i+1}) = x(t) \,\&\, f|t_{2i+1}| \not\approx g_{2k} \to f|t_{2i+1}| \not\approx e_{2k})$;

1d) $f'|t| \times (e_{2k}, f'|t|)_{\alpha(t,\,2k)}$.

Let us also set $e_{2k+1} = (e_{2k}, f'|t_{2k}|)_{\alpha(t_{2k+1},\,2k)}$; $g_{2k+1} = g_{2k} \cup f|t_{2k+1}|$ and $t_{2k+2} = \mu t$ determined by 2a)–2d):

2a) $f|t| \gtrsim e_{2k+1}$;

2b) $f|t| \not\approx [e_{2k+1}]_{\alpha(t,\,2k+1)}$;

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1) Do not confuse the predicate $\bar{f}|t|$ with $\bar{f}(t)$—the tth component of the predicate $\bar{f}$.
2) μt signifies the least t.

2c) $(i)_{i\leq k}(x(t_{2i}) = x(t)\ \&\ f\mid t_{2i}\mid \approx e_{2k+1} \rightarrow f'\mid t_{2i}\mid \approx g_{2k+1})$;

2d) $f'\mid t\mid \times (g_{2k+1}, f'\mid t\mid)_{\alpha(t,\,2k+1)}$,

$e_{2k+2} = e_{2k+1} \cup f\mid t_{2k+2}\mid$,

$g_{2k+2} = (g_{2k+1}, f'\mid t_{2k+2}\mid)_{\alpha(t_{2k+2},\,2k+1)}$.

Let us put $e^* = \bigcup_i e_i$, $g^* = \bigcup_i g_i$, $e = (e^* \circ \lambda)$, $g = (g^* \circ \lambda)$; e and g are U-sequences.

The predicates (c. f.) $\bar{e}$ and $\bar{g}$ are obtained from e and g by the transformation defined at the end of § 1 of Chapter I,

$$\bar{e} = {}^0e,\ \bar{g} = {}^0g.$$

Let us note that the predicates $\bar{e}(n)$ and $\bar{g}(n)$ are defined for $n \geq 1$.

Let us assume that $0 \in E$ and $0 \in G$. This, of course, does not affect the reducibility of the sets E and G.

It is easily seen that the sequences $\{t_i\}$, $\{e_i\}$, $\{g_i\}$ defined by 1a)–1d), 2a)–2d) are computable (recursive), and from this it follows that E and G are r. e.

We call the function $\alpha(t, q)$ defined in (2.3) a control function with respect to the sequences $\{t_i\}$, $\{e_i\}$, $\{g_i\}$ satisfying properties 1a)–1d) and 2a)–2d). Regulating functions play an important role in the proofs of the basic theorems of this paper.

First of all we note that $\alpha(t, q)$ is a g. r. f. A number s is said to be:

a) minimal with respect to x_0 if

$$r > s \rightarrow x(t_r) \geq x_0 \geq x(t_s); \tag{2.4}$$

b) strongly minimal with respect to x_0 if

$$r > s \rightarrow x(t_r) > x_0 \geq x(t_s); \tag{2.5}$$

c) minimal if

$$r > s \rightarrow x(t_r) \geq x(t_s);$$

d) strongly minimal if

$$r > s \rightarrow x(t_r) > x(t_s).$$

Lemma 3. *If the number s is minimal with respect to x_0, then, for $r > s$,*

1°. $\alpha(t_r, r-1) \geq \alpha(t_{s+1}, s) \geq \max(M\{e_s\}, M\{d_s\}) + 1$,

2°. $[e_s]_{M\{e_s\}+1} \approx [e_s]_{\alpha(t_{s+1},\,s)} \approx \begin{cases} e^* \approx e, \\ e_r, \end{cases}$

$$[g_s]_{M\{e_s\}+1} \approx [g_s]_{\alpha(t_{s+1},\,s)} \approx \begin{cases} g^* \approx g, \\ g_r, \end{cases}$$

3°. *$f\mid t_s\mid$ is a segment of e_r; $f'\mid t_s\mid = g_r$, for even s;*

$f \mid t_s \mid$ is a segment of g_r; $f' \mid t_s \mid = e_r$, for odd s.

Lemma 4. *Let a number s be strongly minimal with respect to x_0. Then $\alpha(t_{s+1}, s) = \alpha(t, r)$ for all t and r such that $x(t) = x_0$ and $r \geq s$.*

Let us prove both lemmas. If s is minimal with respect to x_0, then

$$j(t_{s+1}, s) = \max_{\substack{i \leq s \\ x(t_i) \leq x(t_{s+1})}} \{i\} = s.$$

Since $s \leq r - 1$ and $x(t_s) \leq x(t_r)$, then

$$j(t_{s+1}, s) = s \leq j(t_r, r-1) = \max_{\substack{i \leq r-1, \\ x(t_i) \leq x(t_r)}} \{i\},$$

$$\alpha(t_{s+1}, s) = \max_{i \leq s} \{m(t_i), m'(t_i)\} + 1,$$

$$\left.\begin{matrix} M\{e_s\} \\ M\{g_s\} \end{matrix}\right\} \leq \max_{i \leq s} \{m(t_i), m'(t_i)\},$$

which proves assertion 1° of Lemma 3.

From the proposition 1° it follows that $[e_s]_{M\{e_s\}+1} \approx [e_s]_{\alpha(t_{s+1}, s)}$.

Let us prove assertion 2° by induction on r.

For $r = s$, we have $[e_s]_{\alpha(t_{s+1}, s)} \approx e_s = e_r$.

Let $[e_s]_{\alpha(t_{s+1}, s)} \approx e_{2k}$, where $2k \geq s$. Then $[e_s]_{\alpha(t_{s+1}, s)} = [e_{2k}]_{\alpha(t_{s+1}, s)}$. By definition, $e_{2k+1} = (e_{2k}, f' \mid t_{2k+1} \mid)_{\alpha(t_{2k+1}, 2k)}$. According to Lemma 2, $[e_{2k}]_{\alpha(t_{2k+1}, 2k)} \approx e_{2k+1}$, and, since $\alpha(t_{2k+1}, 2k) \geq \alpha(t_{s+1}, s)$, then $[e_{2k}]_{\alpha(t_{s+1}, s)} \approx e_{2k+1}$ and $[e_s]_{\alpha(t_{s+1}, s)} \approx e_{2k+1}$.

Now let us put $[e_s]_{\alpha(t_{s+1}, s)} \approx e_{2k+1}$, where $2k+1 \geq s$. Then $[e_s]_{\alpha(t_{s+1}, s)} = [e_{2k+1}]_{\alpha(t_{s+1}, s)}$; $e_{2k+2} = e_{2k+1} \cup f \mid t_{2k+2} \mid$; moreover, by condition 2b), $f \mid t_{2k+2} \mid \approx [e_{2k+1}]_{\alpha(t_{2k+2}, 2k+1)}$, and, by virtue of $2k+1 \geq s$, we have $f \mid t_{2k+2} \mid \approx [e_{2k+1}]_{\alpha(t_{s+1}, s)}$. Hence

$$e_{2k+2} \approx [e_{2k+1}]_{\alpha(t_{s+1}, s)} = [e_s]_{\alpha(t_{s+1}, s)}.$$

Thus, for $r > s$,

$$[e_s]_{\alpha(t_{s+1}, s)} \approx e_r, \tag{2.6}$$

and, since $M\{e_s\} \leq M\{e_r\}$, then e_s is a segment of e_r. From the definition of e_{i+1}, we have $e_{i+1} = e_i \cup d_i$, where d_i is some U-sequence. Hence, $e^* = \bigcup_i e_i = \bigcup_{r>s} e_r$, if at least one U-sequence is defined for $r > s$. From (2.6) follows:

$$[e_s]_{\alpha(t_{s+1}, s)} \approx e^* \approx (e^* \circ \lambda) = e.$$

Thus, the first part of assertion 2° of Lemma 3 has been proved. The second part is proved in an analogous way.

Let us prove assertion 3°. For even s

$$e_s = e_{s-1} \cup f \mid t_s \mid \text{ and } f \mid t_s \mid \gtrsim e_{s-1} \text{ (condition 2a)).}$$

Therefore $f \mid t_s \mid$ is a segment of e_s and, so, also a segment of e_r.

According to 2d), $f'(t_s) \times g_s$; g_s is a segment of g_r for $r > s$. Hence $f'(t_s) \times g_r$.

The second part of proposition 3° of Lemma 3 can be proved in the same way.

Proceeding to the proof of Lemma 4, let us note that, for t and r such that $x(t) = x_0$ & $r > s$,

$$j(t, r) = j(t_{s+1}, s) = s. \tag{2.7}$$

In fact, $j(t, r) = \max\limits_{\substack{i \leq r, \\ x(t_i) \leq x(t) = x_0}} \{i\}$.

Comparing expressions (2.3) for $\alpha(t, r)$ and $\alpha(t_{s+1}, s)$, we see that $\alpha(t, r) = \alpha(t_{s+1}, s)$. Q.E.D.

(In general, from the equation $j(t, q) = j(t', q')$ it follows that $\alpha(t, q) = \alpha(t', q')$.)

The proof of Theorem 1 is based on the following lemma.

Lemma 5 (Finiteness Principle for the Set of Characteristic Values). *For any x, the operator P_x has not more than a finite set of characteristic values t_i.*

Let us prove Lemma 5 by induction on x.

a) $x = 0$. Let $x(t_0) = x(t_{2i+1}) = x = 0$.

The numbers $t_0 = 0$ and t_{2i+1} are minimal with respect to 0. Therefore $f \mid 0 \mid \eqsim e_r$, and $f' \mid 0 \mid \times g_r$ $(r > 0)$. According to condition 2c), the operator P_0 has no even characteristic values t_{2k} other than $t_0 = 0$.

We also have $f \mid t_{2i+1} \mid \eqsim g_r$ and $f' \mid t_{2i+1} \mid \times e_r$ $(r > 2i + 1)$. By condition 1c), the operator P_0 cannot have characteristic values t_{2k+1}, where $k > i$.

b) Assume the lemma proved for $x = 0, 1, \cdots, x_0$. Let us prove it for $x = x_0 + 1$.

Let s be a strongly minimal number with respect to x_0. Such an s exists, since $x(t_0) = 0$ and we have assumed that, for all $x \leq x_0$, the equation $x(t_i) = x$ has not more than a finite set of solutions. Let t_s be the largest of them. We denote by t_{2i_1} and t_{2i_2+1} any characteristic values of T_{x_0+1} such that $2i_1 > s$ and $2i_2 + 1 > s$.

The numbers $2i_1$ and $2i_2 + 1$ are minimal with respect to $x_0 + 1$. Hence,

$$f\,|\,t_{2i}\,| \approx e_r \text{ and } f'\,|\,t_{2i_1}\,| \times g_r \ (r > 2i_1),$$
$$f\,|\,t_{2i_2+1}\,| \approx g_r \text{ and } f'\,|\,t_{2i_2+1}\,| \approx e_r \ (r > 2i_2 + 1).$$

According to conditions 1c) and 2c), there cannot be any characteristic values t_r of the operator $T_{x_0} + 1$ for $r > \max(2i_1, 2i_2 + 1)$.

Lemma 5 has been proved.

Lemma 6. *In the U-sequences e and g there are infinitely many components* λ.

If the U-sequence $e^* = \bigcup_i e_i$ is finite, then the assertion of the lemma is obvious for e, since $e = (e^* \circ \lambda)$.

Let e^* be an infinite U-sequence. Then $e^* = e$ and the sequences $\{t_i\}$, $\{e_i\}$, $\{g_i\}$ are infinite.

In this case there is an infinite sequence of strongly minimal numbers: $s_1 < s_2 < \cdots < s_p < s_{p+1} < \cdots$.

As s_1 we take a strongly minimal number with respect to $x = 0$: $x(t_{s_1}) = 0$.

Let $s_1 < s_2 < \cdots < s_p$ be defined, with $x(t_{s_p}) = x_p$.

Let us put $x'_p = x(t_{s_p+1}) > x_p$.

s_{p+1} is a strongly minimal number with respect to x'_p, and therefore, it is a strongly minimal number in general, since $x'_p \geq x_{p+1} = x(t_{s_p+1}) > x_p$. In addition, $s_{p+1} \geq s_p + 1$. By virtue of the fact that $e_{i+1} = e_i \cup d_i$,

$$e^* = \bigcup_{i=1}^{\infty} e_i = \bigcup_{p=1}^{\infty} e_{s_p} \text{ and } M\{e_{s_p}\} \longrightarrow \infty \text{ as } p \longrightarrow \infty. \tag{2.8}$$

According to assertion 2° of Lemma 3, $[e_{s_p}]_{M\{e_{s_p}\}+1} \approx e$. Therefore the component of e with index $M\{e_{s_p}\} + 1$ is equal to λ. From (2.8) we obtain the statement of the lemma for the U-sequence e. The proof for the U-sequence g is carried through in an analogous way.

Now we can begin directly on the proof of Theorem 1.

Let us assume that $P_{x_0}(\overline{e}) = \overline{g}$ for some x_0, and whenever the number t_{2i} is defined then t_{2i+1} also exists. The case when t_{2i} is defined but the number t_{2i+1} does not exist it will be necessary to treat separately.

Let us take a number s strongly minimal with respect to x_0. By our assumptions, the number t_{2k+1} and U-sequences e_{2k+1}, g_{2k+1} are defined for some $2k + 1 \geq s$. Then one can find a characteristic value t of the operator P_{x_0}, satisfying conditions 2a')–2d') for any $2k + 1 \geq s$, for which t_{2k+1}, e_{2k+1}, g_{2k+1} are defined:

2a') $f\,|\,t\,| \gtrsim e_{2k+1}$;

2b') $f' | t | \approx [e_{2k+1}]_{\alpha(t, 2k+1)}$;

2c') $f' | t | \times (g_{2k+1}, f' | t |)_{\alpha(t, 2k+1)}$;

2d') $(i)_{i \leq k} (x(t_{2i}) = x(t) = x_0 \,\&\, f | t_{2i} | \approx e_{2k+1} \rightarrow f' | t_{2i} | \approx g_{2k+1})$.

The condition 2d') is fulfilled for all $2k + 1 \geq s$, since $x(t_{2i}) = x_0 \rightarrow 2i \leq s$, $e_s \approx e_{2k+1}$; and since $m(t_{2i}) \leq M\{e_s\} \leq M\{e_{2k+1}\}$, then $f | t_{2i} | \approx e_{2k+1}$ means that $f | t_{2i} |$ is a segment of e_s, $e_s = e$. Hence $f | t_{2i} |$ is a segment of e.

Since we have assumed that $P_{x_0}(\bar{e}) = \bar{g}$, and $x(t_{2i}) = x_0$, then $\bar{f} | t_{2i} | \approx e \rightarrow \bar{f}' | t_{2i} | \approx \bar{g}$, but the transformation of U-sequences into predicates and the inverse transformation preserve the relation $\approx$ (§1 of Chapter I). Thus, we have $f | t_{2i} | \approx e \rightarrow f' | t_{2i} | \approx g$ and

$$f | t_{2i} | \approx e_{2k+1} \rightarrow f' | t_{2i} | \approx g. \tag{2.9}$$

From the strong minimality of s with respect to x_0 ($2i \leq s$) it follows, according to Lemma 3, that

$$f' | t_{2i} | \approx g \rightarrow f' | t_{2i} | \approx g_{2k+1}. \tag{2.10}$$

Comparing (2.9) and (2.10) we see that condition 2d') is fulfilled.

Conditions 2a') and 2b') also hold if $\bar{f} | t | \approx \bar{e}$, since, then, $f | t | \approx e$, $e \gtrsim e_{2k+1}$, and 2a') is fulfilled. According to Lemma 4, $\alpha(t, 2k+1) = \alpha(t_{s+1}, s)$.

In the proof of Lemma 3 we saw that $[e_{2k+1}]_{\alpha(t_{s+1}, s)} = [e_s]_{\alpha(t_{s+1}, s)} \approx e$, from which 2b') follows for $f | t | \approx e$.

Let $n > \alpha(t_{s+1}, s)$ and $g(n) = \lambda$ (according to Lemma 6 such an n exists). From $P_{x_0}(\bar{e}) = \bar{g}$, the existence of t follows such that $x(t) = x_0$, $\bar{f} | t | \approx e$, $\bar{f}' | t | \approx \bar{g}$ and $m'(t) \geq n > \alpha(t_{s+1}, s)$.

For such t we have $f' | t | \approx g$. Hence $f' | t | (n) = g(n) = \lambda$. In addition, $m'(t) \geq n > \alpha(t_{s+1}, s)$.

Applying Lemma 1 and taking into account that $\alpha(t, 2k+1) = \alpha(t_{s+1}, s)$, we obtain $f' | t | \times (g_{2k+1}, f' | t |_{\alpha(t, 2k+1)})$, i.e., 2c') holds.

Let us put $t' = \mu t$ satisfying $x(t) = x_0$ and conditions 2a')–2d') for any $2k + 1 \geq s$; and let $i = \max_{t_i \leq t'} \{i\}$. Let us take $2k + 1$ such that $2k + 1 \geq \max(i', s)$ and t_{2k+1}, e_{2k+1}, g_{2k+1} are defined. Such a number $2k + 1$ exists by virtue of the assumptions we have made. Then $t' = \mu t$ satisfying conditions 2a)–2d), i.e., $t' = t_{2k+2}$, $x(t_{2k+2}) = x_0$, which contradicts the strong minimality of s with respect to x_0.

If the number t_{2i} *is defined but* t_{2i+1} *does not exist, then* $e^* = \bigcup_{j=1}^{2i} e_j$, $g^* = \bigcup_{j=1}^{2i} g_j$, and E and G are finite, and therefore, recursive. Then $P_{x_1}(\bar{g}) = \bar{e}$ for some x_1. In this case, the proof proceeds just as before but with the replacement of e by g, g by e, and the conditions 2a')–2d') by corresponding conditions 1a')–1d'). In a similar way one proves that $P_x(\bar{g}) \neq \bar{e}$.

We have proved that, for all x, $P_x(\bar{e}) \neq \bar{g}$ and $P_x(\bar{g}) \neq \bar{e}$. From this it follows that the sets E and G are not recursive and are not reducible to each other. Such sets (problems) are called incomparable.

One can strengthen Theorem 1.

Theorem 2. *There exists a recursive sequence of Gödel numbers of r. e. sets* $\{E_n\}$ *such that the decision problem of each r. e. set* E_n *is not reducible to the recursive conjunction* $\bigwedge_{m \neq n} A_{E_m}$ (A_{E_m} *is the decision problem for* E_m).

Following Kleene and Post, we call such sets $E_1, E_2, \cdots, E_n, \cdots$ recursively independent.

The proof of Theorem 2 is carried out by the same methods as in the proof of Theorem 1, but it is much more complicated. Therefore we have omitted it.

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RECURSIVELY ENUMERABLE SETS OF POSITIVE INTEGERS AND THEIR DECISION PROBLEMS

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Introduction. Recent developments of symbolic logic have considerable importance for mathematics both with respect to its philosophy and practice. That mathematicians generally are oblivious to the importance of this work of Gödel, Church, Turing, Kleene, Rosser and others as it affects the subject of their own interest is in part due to the forbidding, diverse and alien formalisms in which this work is embodied. Yet, without such formalism, this pioneering work would lose most of its cogency. But apart from the question of importance, these formalisms bring to mathematics a new and precise mathematical concept, that of the general recursive function of Herbrand-Gödel-Kleene, or its proved equivalents in the developments of Church and Turing.[1] It is the purpose of this lecture to demonstrate by example that this concept admits of development into a mathematical theory much as the group concept has been developed into a theory of groups. Moreover, that stripped of its formalism, such a theory admits of an intuitive development which can be followed, if not indeed pursued, by a mathematician, layman though he be in this formal field. It is this intuitive development of a very limited portion of a sub-theory of the hoped for general theory that we present in this lecture. We must emphasize that, with a few exceptions explicitly so noted, we have obtained formal proofs of all the consequently mathematical theorems here developed informally. Yet the real mathematics involved must lie in the informal development. For in every instance the informal "proof" was first obtained; and once gotten, transforming it into the formal proof turned out to be a routine chore.[2]

We shall not here reproduce the formal definition of *recursive function of positive integers*. A simple example of such a function is an

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[1] For "general recursive function" see [9] ([8] a prerequisite), [12] and [11]; for Church's "λ-definability," [1] and [6]; for Turing's "computability," [24] and the writer's related [18]. To this may be added the writer's method of "canonical systems and normal sets" [19]. See pp. 39–42 and bibliography of [6] for a survey of the literature and further references. Numbers in brackets refer to the bibliography at the end of the paper.

[2] Our present formal proofs, while complete, will require drastic systematization and condensation prior to publication.

arbitrary polynomial $P(x_1, x_2, \cdots, x_n)$, with say non-negative integral coefficients, and not identically zero. If the x's are assigned arbitrary positive integral values expressed, for example, in the arabic notation, the algorithms for addition and multiplication in that notation enable us to calculate the corresponding positive integral value of the polynomial. That is, $P(x_1, x_2, \cdots, x_n)$ is an *effectively calculable function of positive integers*. The importance of the technical concept recursive function derives from the overwhelming evidence that it is coextensive with the intuitive concept effectively calculable function.[3]

A set of positive integers is said to be *recursively enumerable* if there is a recursive function $f(x)$ of one positive integral variable whose values, for positive integral values of x, constitute the given set. The sequence $f(1), f(2), f(3), \cdots$ is then said to be a *recursive enumeration* of the set. The corresponding intuitive concept is that of an *effectively enumerable* set of positive integers. To prepare us in part for our intuitive approach, consider the following three examples of recursively enumerable sets of positive integers.

(a): $1^2, 2^2, 3^2, \cdots.$

(b): $1, 2, 2^{1+2}, 2^{1+2+2^{1+2}}, \cdots.$

(c):
$$\begin{matrix} 1^2, & 2^2, & 3^2, & \cdots \\ 1^3, & 2^3, & 3^3, & \cdots \\ 1^4, & 2^4, & 3^4, & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots. \end{matrix}$$

In the first example, the set is given by a recursive enumeration thereof via the recursive function x^2. In the second example, the set is generated in a linear sequence, each new element being effectively obtained from the elements previously generated, in this case by raising 2 to the power the sum of the preceding elements. The set is effectively enumerable, since the nth element of the sequence can be found, given n, by regenerating the sequence through its first n elements. In the third example, we rather imagine the positive integers $1, 2, 3, \cdots$ generated in their natural order, and, as each positive integer n is generated, a corresponding process set up which generates $n^2, n^3, n^4, \cdots$, all these to be in the set. Actually, the standard method for proving that an enumerable set of enumerable sets is enumerable yields an effective enumeration of the set.

[3] See Kleene [13, footnote 2]. In the present paper, "recursive function" means "general recursive function."

Several more examples would have to be given to convey the writer's concept of a *generated set*, in the present instance of positive integers. Suffice it to say that each element of the set is at some time written down, and earmarked as belonging to the set, as a result of predetermined effective processes. It is understood that once an element is placed in the set, it stays there. The writer elsewhere has referred to a generalization which may be restated *every generated set of positive integers is recursively enumerable.*[4] For comparison purposes this may be resolved into the two statements: every generated set is effectively enumerable, every effectively enumerable set of positive integers is recursively enumerable. The first of these statements is applicable to generated sets of arbitrary symbolic expressions; their converses are immediately seen to be true. We shall find the above concept and generalization very useful in our intuitive development. But while we shall frequently say, explicitly or implicitly, "set so and so of positive integers is a generated, and hence recursively enumerable set," as far as the present enterprise is concerned that is merely to mean "the set has intuitively been shown to be a generated set; it can indeed be proved to be recursively enumerable." Likewise for other identifications of informal concepts with corresponding mathematically defined formal concepts.

At a few points in our informal development we have to lean upon the formal development. The latter is actually yet another formalism, due to the writer [19] but proved completely equivalent to that of general recursive function. It will suffice to give the equivalent of "recursively enumerable set of positive integers" in this development.

A positive integer n is represented in the most primitive fashion by a succession $11 \cdots 1$ of n strokes. For working purposes, we introduce the letter b, and consider "strings" of 1's and b's such as $11b1bb1$. An operation on such strings such as "*b1bP produces P1bb1*" we term a normal operation. This particular normal operation is applicable only to strings starting with $b1b$, and the derived string is then obtained from the given string by first removing the initial $b1b$, and then tacking on $1bb1$ at the end. Thus $b1bb$ becomes $b1bb1$. "*gP produces Pg′*" is the form of an arbitrary normal operation. A system in normal form, or normal system, is given by an initial string A of 1's and b's, and a finite set of normal operations "*g_iP produces Pg_i'*," $i=1, 2, \cdots, \mu$. The derived strings of the system are A and all strings obtainable from A be repeated applications of the μ normal

[4] See [19, p. 201 and footnote 18]. In this connection note Kleene's use of the word "Thesis" in [14, p. 60]. We still feel that, ultimately, "Law" will best describe the situation [18].

operations. Each normal system uniquely defines a set, possibly null, of positive integers, namely the integers represented by those derived strings which are strings of 1's only. It can then be proved that every recursively enumerable set of positive integers is the set of positive integers defined by some normal system, and conversely.[5] *We here, as below, arbitrarily extend the concept recursively enumerable set to include the null set.*

By the *basis B* of a normal system, and of the recursively enumerable set of positive integers it defines, we mean the string of letters and symbols here represented by

$$A\,;\ g_1P \textit{ produces } Pg_1',\ \cdots,\ g_\mu P \textit{ produces } Pg_\mu'.$$

When meaningfully interpreted, B determines the normal system, and recursively enumerable set of positive integers, in question. Each basis is but a finite sequence of the symbols 1, b, P, the comma, semicolon and the letters of the word "*produces*." The set of bases is therefore enumerably infinite, and can indeed be effectively generated in a sequence of distinct elements

$$O: \quad B_1, B_2, B_3, \cdots.$$

Since each B_i defines a unique recursively enumerable set of positive integers and each such set is defined by at least one B_i, O is also an ordering of all recursively enumerable sets of positive integers, though each set will indeed recur an infinite number of times in O. We may then say, in classical terms, that whereas there are $2^{\aleph_0}$ arbitrary sets of positive integers, there are but $\aleph_0$ recursively enumerable sets.

By the *decision problem* of a given set of positive integers we mean the problem of effectively determining for an arbitrarily given positive integer whether it is, or is not, in the set. While, in a certain sense, the theory of recursively enumerable sets of positive integers is potentially as wide as the theory of general recursive functions, the decision problems for such sets constitute a very special class of decision problems. Nevertheless they are important, as is shown by the following special and general examples.

One of the problems posed by Hilbert in his Paris address of 1900 [**10**, problem 10] is the problem of determining for an arbitrary diophantine equation with rational integral coefficients whether it has, or has not, a solution in rational integers. If the variables in a

[5] We have thus restricted the normal operations and normal systems of [**19**] because of the following result. If in the initial string and in the normal operations of a normal system with primitive letters 1, a_1', $\cdots$, a_μ', each a_i', $i=1, \cdots, \mu'$, is replaced by $b1 \cdots 1b$ with i 1's, a normal system with primitive letters 1, b results, defining the same set of strings on 1 only as the original normal system.

diophantine equation be chosen from a given enumerably infinite set of variables, it is clear that the set of diophantine equations is enumerably infinite. Indeed they can be effectively put into one-one correspondence with the set of positive integers. Since for any one diophantine equation, and assignment of rational integral values to its variables, it can be effectively determined whether or no the equation is satisfied by those values, the set of diophantine equations having rational integral solutions can be generated. The corresponding integers under the above one-one correspondence can then also be generated, and, indeed, constitute a recursively enumerable set of positive integers.[6] And under that correspondence, Hilbert's problem is transformed into the decision problem of that recursively enumerable set.

The assertions of an arbitrary symbolic logic[7] constitute a generated set A of what may be called symbol-complexes or formulas. We assume that A is a subset of an infinite generated set E of symbol-complexes, which in one case may be the set of meaningful enunciations of the logic, in another the set of all symbol-complexes of a given mode of symbolization. The decision problem of the logic, more precisely its deducibility problem [**3**], is then the problem of determining of an arbitrary member of E whether it is, or is not, in A. Granting that every generated set is effectively enumerable, the members of E can be effectively set in one-one correspondence with the set of positive integers. The positive integers corresponding to the members of A then constitute a generated, and hence, under our generalization, a recursively enumerable set of positive integers. And under that correspondence the decision problem of the symbolic logic is transformed into the decision problem of this recursively enumerable set of positive integers.

Closely related to the technical concept recursively enumerable set of positive integers is that of a *recursive* set of positive integers. This is a set for which there is a recursive function $f(x)$ such that $f(x)$ is say 2 when x is a positive integer in the set, 1 when x is a positive integer not in the set. We may also make this the definition of the decision problem of the set being *recursively solvable*. For 2 and 1 may be regarded as the two possible truth-values, true, false, of the proposition "positive integer x is in the set," and the definition of recursive set is equivalent to this truth-value being recursively calculable for all positive integers x. If then recursive function is coextensive with

[6] In view of [**17**] we inadvertantly carried through our formal verification with "rational integral solution" replaced by "positive integral solution."

[7] See Church [5, p. 225] for our omitting the qualifying "finitary."

effective calculability, recursive solvability is coextensive with solvability in the intuitive sense. In particular, the decision problem of a recursively enumerable set would be solvable or unsolvable according as the set is, or is not, recursive. More generally than in our two illustrations, through the more precise mechanism of Gödel representations [8], a wide variety of decision and other problems are transformed into problems about positive integers; and whether those problems are, or are not, solvable in the intuitive sense would be equivalent to their being, or not being, recursively solvable in the precise technical sense.

Gödel's classic theorem on the incompleteness and extendibility of symbolic logics [8] in all but wording led him to the recursive unsolvability of a generalization of the above problem of Hilbert [8, 9, 22]. Church explicitly formulated the concept of recursive unsolvability, and arrived at the unsolvability of a number of problems; certainly he proved them recursively unsolvable [1–4]. The above problem of Hilbert begs for an unsolvability proof (see [17]). Like the classic unsolvability proofs, these proofs are of unsolvability by means of given instruments. What is new is that in the present case these instruments, in effect, seem to be the only instruments at man's disposal.

Related to the question of solvability or unsolvability of problems is that of the reducibility or non-reducibility of one problem to another. Thus, if problem P_1 has been reduced to problem P_2, a solution of P_2 immediately yields a solution of P_1, while if P_1 is proved to be unsolvable, P_2 must also be unsolvable. For unsolvable problems the concept of reducibility leads to the concept of *degree of unsolvability*, two unsolvable problems being of the same degree of unsolvability if each is reducible to the other, one of lower degree of unsolvability than another if it is reducible to the other, but that other is not reducible to it, of incomparable degrees of unsolvability if neither is reducible to the other. A primary problem in the theory of recursively enumerable sets is the problem of determining the degrees of unsolvability of the unsolvable decision problems thereof. We shall early see that for such problems there is certainly a highest degree of unsolvability. Our whole development largely centers on the single question of whether there is, among these problems, a lower degree of unsolvability than that, or whether they are all of the same degree of unsolvability. Now in his paper on *ordinal logics* [26, section 4], Turing presents as a side issue a formulation which can immediately be restated as the general formulation of the "recursive reducibility" of one problem to another, and proves a result which immediately generalizes to the result that for any "recursively given" unsolvable problem there is another of higher degree of unsolvability.[8] While his theorem does not help us in our search for that lower degree of unsolvability, his formulation makes our problem precise. It remains a problem at the end of this paper. But on the way we do obtain a number of special results, and towards the end obtain some idea of the difficulties of the general problem.

1. **Recursive versus recursively enumerable sets.** The relationship between these two concepts is revealed by the following

THEOREM. *A set of positive integers is recursive when and only when both it and its complement with respect to the set of all positive integers are recursively enumerable.*[9]

For simplicity, we assume both the set S and its complement $\bar{S}$ to be infinite. If, then, S is recursive, there is an effective method for telling of any positive integer n whether it is, or is not, in S. Generate the positive integers $1, 2, 3, \cdots$ in their natural order, and, as a positive integer is generated, test its being or not being in S. Each time a positive integer is thus found to be in S, write it down as belonging to S. Thus, an effective process is set up for effectively enumerating the elements of S. Hence, S is recursively enumerable. Likewise $\bar{S}$ can be shown to be recursively enumerable.

Conversely, let both S and $\bar{S}$ be recursively enumerable, and let $n_1, n_2, n_3, \cdots$ be a recursive enumeration of S; $m_1, m_2, m_3, \cdots$, of $\bar{S}$. Given a positive integer n, generate in order $n_1, m_1, n_2, m_2, n_3, m_3$, and so on, comparing each with n. Since n must be either in S or in $\bar{S}$, in a finite number of steps we shall thus come across an n_i or m_j identical with n, and accordingly discover n to be in S, or $\bar{S}$. An effective method is thus set up for determining of any positive integer n whether it is, or is not, in S. Hence, S is recursive.

COROLLARY. *The decision problem of a recursively enumerable set is recursively solvable when and only when its complement is recursively enumerable.*

For then and only then is the recursively enumerable set recursive. It is readily proved that the logical sum and logical product of two

[8] Both our generalization of his formulation and of his theorem have been carried through, rather hastily, by the formalism of [19], without, as yet, an actual equivalence proof. It may be that Tarski's Theorem 9.1 [23] can be transformed into a like absolute theorem.

[9] The only portion of this theorem we can find in the literature is Rosser's Corollary II [20, p. 88].

recursively enumerable sets are recursively enumerable, the complement of a recursive set, and the logical sum, and hence logical product, of two recursive sets are recursive.

Clearly, *any finite set of positive integers is recursive.* For if $n_1, n_2, \cdots, n_\nu$ are the integers in question, we can test n being, or not being, in the set by directly comparing it with $n_1, n_2, \cdots, n_\nu$.[10] Likewise for a set whose complement is finite. For arbitrary infinite sets we have the following result of Kleene [12]. *An infinite set of positive integers is recursive when and only when it admits of a recursive enumeration without repetitions in order of magnitude.* Indeed, if $n_1, n_2, n_3, \cdots$ is a recursive enumeration of S without repetitions in order of magnitude, all n_i's beyond the nth must exceed n. Hence we can test n being, or not being, in S by generating the first n members of the given recursive enumeration of S, and seeing whether n is, or is not, one of them. Conversely, if infinite S is recursive, the recursive enumeration thereof we set up in the proof of our first theorem is of the elements of S without repetition, and in order of magnitude.

A direct consequence of the first half of the last result is the following

THEOREM. *Every infinite recursively enumerable set contains an infinite recursive set.*

For, if $n_1, n_2, n_3, \cdots$ is a recursive enumeration of an infinite set S, for each n_i there must be, in this sequence, a later $n_j > n_i$. Hence, generate the elements $n_1, n_2, n_3, \cdots$ in order, and let $m_1 = n_1$, $m_2 = n_{i_2}$, the first n_i greater than n_1, $m_3 = n_{i_3}$, the first n_i beyond n_{i_2} greater than n_{i_2}, and so on. The sequence $m_1, m_2, m_3, \cdots$ is then a recursive enumeration of a subset of S without repetitions in order of magnitude. That subset is therefore infinite, and recursive.

Basic to the entire theory is the following result we must credit to Church, Rosser, Kleene, jointly [1, 20, 12].

THEOREM. *There exists a recursively enumerable set of positive integers which is not recursive.*[11]

By our first theorem this is equivalent to the existence of a recursively enumerable set of positive integers whose complement is

[10] The mere existence of a general recursive function defining the finite set is in question. Whether, given some definition of the set, we can actually discover what the members thereof are, is a question for a theory of proof rather than for the present theory of finite processes. For sets of finite sets the situation is otherwise, as seen in §11.

[11] In each of our existence theorems we show how to set up the basis of the set in question—at least, the corresponding formal proof does exactly that.

not recursively enumerable. Generate in order the distinct bases $B_1, B_2, B_3, \cdots$ of all recursively enumerable sets of positive integers as mentioned in the introduction, and keep track of these bases as the first, second, third, and so on, in this enumeration O. As the nth basis B_n is generated, with $n=1, 2, 3, \cdots$, set going the processes whereby the corresponding recursively enumerable set is generated, and whenever n is thus generated by B_n, place n in a set U. Being a generated set of positive integers, U is recursively enumerable. A positive integer n, then, is, or is not, in U according as it is, or is not, in the nth recursively enumerable set in O considered as an ordering of all recursively enumerable sets. Hence, n is, or is not, in $\overline{U}$, the complement of U, according as it is not, or is, in the nth set in O. We thus see that $\overline{U}$ differs from each recursively enumerable set in the presence or absence of at least one positive integer. Hence $\overline{U}$ is not recursively enumerable.

COROLLARY. *There exists a recursively enumerable set of positive integers whose decision problem is recursively unsolvable.*

Taken singly, finite sets, or sets whose complements are finite, are rather trivial examples of recursive sets. On the other hand, if we define two sets of positive integers to be *abstractly* the same if one can be transformed into the other by a recursive one-one transformation of the set of all positive integers into itself, then all infinite recursive sets with infinite complements are abstractly the same. Our theory being essentially an abstract theory of recursively enumerable sets, our interest therefore centers in recursively enumerable sets that are not recursive. Such sets, as well as their complements, are always infinite. We do not further pursue the question of two sets being abstractly the same, for that is but a special case of each set being one-one reducible to the other (§4).

2. **A form of Gödel's theorem.** Given any basis B, and positive integer n, the couple (B, n) may be used to represent the proposition, true or false, "n is in the set generated by B." By interlacing the process for generating the distinct bases in the sequence $B_1, B_2, B_3, \cdots$ and the process for generating the positive integers in the sequence $1, 2, 3, \cdots$ by the addition of 1's, we can effectively generate the distinct couples (B, n) in the single infinite sequence

$$O': \quad (B_1, 1), (B_2, 1), (B_1, 2), (B_3, 1), (B_2, 2), (B_1, 3), \cdots .$$

On the one hand, the set of all couples (B, n) is thus a generated set of expressions which we shall call E. On the other hand, O' leads to an effective 1-1 correspondence between the members of E and the set of positive integers, (B, n) corresponding to m if (B, n) is the mth member of O'. We may call m the Gödel representation[12] of (B, n). Given a generated subset of E, the Gödel representations of its members will constitute a generated set of positive integers, and conversely. Thus, in the former case we can generate the members of the subset of E, and, as a couple (B, n) is generated, find its Gödel representation m by regenerating O'. The set of these m's is thus a generated set. Likewise for the converse. If, therefore, we formally define a subset of E to be recursively enumerable if the set of Gödel representations of its members is recursively enumerable,[13] we can conclude that every generated subset of E is recursively enumerable, and, of course, conversely. Similarly for a like formal definition of a recursive subset of E.

While E is just the set of couples (B, n), it may be interpreted as the set of enunciations "n is in the set generated by B." The subset T of E consisting of those couples (B, n) for which n *is* in the set generated by B may then be interpreted as the set of true propositions in E, while $\overline{T}$, the complement of T with respect to E, consists of the false propositions in E.

Actually, T itself can be generated as follows. Generate B_1, B_2, $B_3, \cdots$ in order. As a B is generated, set up the process for generating the set of positive integers determined by B, and, whenever a positive integer n is thus generated, write down the couple (B, n). Each (B, n) for which n is in the set generated by B will thus be written down, and conversely. This generated set of (B, n)'s is then T. We therefore conclude that *T is recursively enumerable.*

Now let F be any recursively enumerable subset of $\overline{T}$. If (B, n) is in F, it is in $\overline{T}$, and hence n is certainly not in the set generated by B. Now generate the members of F, and if (B, n) is thus generated, find the nth member B_n of $O: B_1, B_2, B_3, \cdots$, and if B_n is B, place n in a set of positive integers S_0. Since S_0 is thus a generated set of positive integers, it is recursively enumerable. It will therefore be determined by some basis B. Let this basis be in the νth in O, that is, let the basis be B_ν, and form the couple (B_ν, ν). Now by construction, S_0 consists of those members of F of the form (B_n, n). Suppose that (B_ν, ν) is in F. Then, on the one hand, proposition (B_ν, ν) being false,

[12] Rather is the Gödel representation in [8] not just an effectively corresponding positive integer, but one which, when expressed according to a specific algorithm, is "formally similar," in the sense of Ducasse [7, p. 51], to the symbolic expression represented.

[13] In our own development [19], "recursively enumerable subset of E" is defined directly as a normal subset of E, or rather of the set of symbolic representations of the members of E.

ν would not be in the set generated by B_ν, that is (1): ν would not be in S_0. But (B_ν, ν) being of the form (B_n, n), (2): ν would be in S_0. Our assumption thus leading to a contradiction, it follows that (B_ν, ν) *is not in* F. But ν can only be in S_0 by (B_ν, ν) being in F. Hence, ν *is not in* S_0. Finally, (B_ν, ν) as proposition says that ν is in S_0. The proposition (B_ν, ν) is therefore false, that is (B_ν, ν) *is in* $\overline{T}$.

For any recursively enumerable subset F of $\overline{T}$ there is then always this couple (B_ν, ν) in $\overline{T}$, but not in F. On the one hand, then, $\overline{T}$ can never be F. Hence, $\overline{T}$ *is not recursively enumerable.* By the definitions of this section, and the first theorem of the last, it follows that T, *while recursively enumerable, is not recursive.* By the decision problem of T we mean the problem of determining for an arbitrarily given member of E whether it is, or is not, in T. But that can be interpreted as the decision problem for the class of recursively enumerable sets of positive integers, that is, the problem of determining for any arbitrarily given recursively enumerable set, that is, arbitrarily given basis B of such a set, and arbitrary positive integer n whether n is, or is not, in the set generated by B. We may therefore say that the *decision problem for the class of all recursively enumerable sets of positive integers is recursively unsolvable*, and hence, in all probability, unsolvable in the intuitive sense.

On the other hand, since (B_ν, ν) of $\overline{T}$ is not in F, T and F together can never exhaust E. Now T, or any recursively enumerable subset T' of T, in conjunction with F may be called a recursively generated logic relative to the class of enunciations E. For the appearance of (B, n) in T' assures us of the truth of the proposition "n is in the set generated by B," while its presence in F would guarantee its falseness. We can then say *that no recursively generated logic relative to E is complete*, since F alone will lead to the (B_ν, ν) which is neither in T' nor in F. That is, (B_ν, ν) is undecidable in this logic. Moreover, if, with a given "basis" for F, the above argument is carried through formally,[14] the recursively enumerable S_0 obtained above will actually be given by a specific basis B which can be constructed by that formal argument. Having found this B, we can then regenerate $O: B_1, B_2, B_3, \cdots$, until B is reached, and thus determine the ν such that $B = B_\nu$. That is, given the basis of F, the (B_ν, ν) in $\overline{T}$ and not in F can actually be found. If then we add this (B_ν, ν) to F, a wider recursively enumerable subset F' of $\overline{T}$ results. We may then say that *every recursively generated logic relative to E can be extended.* Outwardly, these two results, when formally developed, seem to be

[14] Here, the basis of F may be taken to be the basis of the recursively enumerable set of Gödel representations of the members of F. But see the preceding footnote.

Gödel's theorem in miniature. But in view of the generality of the technical concept general recursive function, they implicitly, in all probability, justify the generalization that every symbolic logic is incomplete and extendible relative to the class of propositions constituting E.[15] The conclusion is unescapable that even for such a fixed, well defined body of mathematical propositions, *mathematical thinking is, and must remain, essentially creative.* To the writer's mind, this conclusion must inevitably result in at least a partial reversal of the entire axiomatic trend of the late nineteenth and early twentieth centuries, with a return to meaning and truth as being of the essence of mathematics.

3. **The complete set K; creative sets.** Return now to the effective 1-1 correspondence between the set E of distinct (B, n)'s and the set of positive integers obtained via the effective enumeration O' of E. Since T is a recursively enumerable subset of E, the positive integers corresponding to the elements of T constitute a recursively enumerable set of positive integers, K. We shall call K the *complete set*.[16] Since $\bar{T}$ is not recursively enumerable, $\bar{K}$, which consists of the positive integers corresponding to the elements of $\bar{T}$, is not recursively enumerable. Now let B be the basis of a recursively enumerable subset α of $\bar{K}$. The elements of E corresponding to the members of α constitute, then, a recursively enumerable subset F of $\bar{T}$. Find then the (B_ν, ν) of $\bar{T}$ not in F, and, via O', the positive integer n corresponding to (B_ν, ν). This n will then be an element of $\bar{K}$ not in α.

Actually, we have no general method of telling when a basis B defines a recursively enumerable subset of $\bar{K}$. Indeed, the above method will yield a unique positive integer n for any basis B of a recursively enumerable set α of positive integers. However, when α is a subset of $\bar{K}$, n will also be in $\bar{K}$, but not in α.

Furthermore, even the formal proof of this result merely gives an effective method for finding n, given B. But this method itself can be formalized, so that, as a result, n is given as a "recursive function of B." This can mean that a recursive function $f(m)$ can be set up such that $n=f(m)$ where $B=B_m$. We now isolate this property of K by setting up the

DEFINITION. *A creative set C is a recursively enumerable set of positive integers for which there exists a recursive function giving a unique*

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[15] See Kleene's Theorem XIII in [12] for a mathematically stateable theorem approximating the generality of our informal generalization.

[16] "A complete set" might be better. Just how to abstract from K the property of completeness is not, at the moment, clear. By contrast, see "creative set" below.

positive integer n for each basis B of a recursively enumerable set of positive integers α such that whenever α is a subset of $\bar{C}$, n is also in $\bar{C}$, but not in α.

THEOREM. *There exists a creative set; to wit, the complete set K.*

Actually, the class of creative sets is infinite, and very rich indeed as shown by the following easily proved results.[17] If C is a creative set, and E a recursively enumerable set of positive integers, then if E contains $\bar{C}$, CE is creative, if $\bar{C}$ contains E, $C+E$ is creative. Results of §1 enable us actually to construct creative sets according to the first method by using E's which are the complements of recursive subsets of C. Results of the rest of this section lead to constructions using the second method.

It is convenient to talk as if the n in the definition of a creative set were determined by the α thereof instead of by the basis B of α. Clearly *every creative set C is a recursively enumerable set which is not recursive.* For were $\bar{C}$ recursively enumerable, there could be no n in $\bar{C}$ not in the recursively enumerable subset $\bar{C}$ of $\bar{C}$. *The decision problem of each creative set is therefore recursively unsolvable.* On the other hand, the *complement $\bar{C}$ of any creative set C contains an infinite recursively enumerable set.* Recall that every finite set is recursive, and hence recursively enumerable. With, then, α of the definition of creative set as the null set, find the $n=n_1$ of $\bar{C}$ "not in α." With α the unit set having n_1 as sole member, $n=n_2$ will be in $\bar{C}$, and distinct from n_1. With α consisting of n_1 and n_2, $n=n_3$ will be in $\bar{C}$, and distinct from n_1 and n_2, and so on. The set of positive integers n_1, n_2, n_3, $\cdots$ is then an infinite generated, and hence recursively enumerable, subset of $\bar{C}$.

Actually, with this subset of $\bar{C}$ as α, a new element n_ω of $\bar{C}$ is obtained, and so on into the constructive transfinite. But this process is essentially creative. For any mechanical process could only yield n's forming a generated, and hence recursively enumerable, subset α of $\bar{C}$, and hence could be transcended by finding that n of $\bar{C}$ not in α.

4. **One-one reducibility, to K; many-one reducibility.** Let S_1 and S_2 be any two sets of positive integers. One of the simplest ways in which the decision problem of S_1 would be reduced to the decision problem of S_2 would arise if we had an effective method which would determine for each positive integer n a positive integer m such that n is, or is not, in S_1 according as m is, or is not, in S_2. For if we could

[17] Of course, all sets abstractly the same as a given creative set, in the sense of §1, are creative. Likewise for our later simple and hyper-simple sets.

somehow determine whether m is, or is not, in S_2, we would determine n to be, or not be, in S_1 correspondingly. If "effective method" be replaced by "recursive method," we shall say, briefly, that S_1 is then *many-one reducible* to S_2. If, furthermore, different n's always lead to different m's, we shall say that S_1 is *one-one reducible* to S_2.[18] "Recursive method" here can mean that $m=f(n)$, where $f(n)$ is a recursive function.

THEOREM. *The decision problem of every recursively enumerable set of positive integers is one-one reducible to the decision problem of the complete set K.*

For let B' be a basis of any one recursively enumerable set S'. The effective one-one correspondence between all (B, n)'s and all positive integers yielded by the effective enumeration O' of E, the set of all (B, n)'s, then yields a unique positive integer m for each (B', n), B' fixed, and thus a unique m for each n, different n's yielding different m's. Now n is, or is not, in S' according as (B', n) is in T, or $\overline{T}$, and hence according as m is in K, or $\overline{K}$, whence our result.

Since K itself is recursively enumerable, we may say that for recursively enumerable sets of positive integers with recursively unsolvable decision problems there is a *highest degree of unsolvability relative to one-one reducibility*, namely, that of K. Actually, one-one reducibility is a special case of all the more general types of reducibility later introduced, and, though the proof of this is still in the informal stage, these latter are special cases of general recursive, that is, Turing reducibility. The same result then obtains relative to these special types of reducibility and, more significantly, for reducibility in the general sense.[19]

We have thus far explicitly obtained two recursively enumerable sets with recursively unsolvable decision problems, the U of our first section, and K. We may note that a certain necessary and sufficient condition for the many-one reducibility of K to a recursively enumerable set, the proof of which is still in the informal stage, has as an immediate consequence that K is many-one reducible to U. It would then follow that K and U are of the same degree of unsolvability relative to many-one reducibility.

[18] The resulting one-to-one correspondence is then between $S_1+\overline{S}_1$ and a subset, recursively enumerable indeed, of $S_2+\overline{S}_2$. Of course, both $S_1+\overline{S}_1$ and $S_2+\overline{S}_2$ constitute the set of all positive integers.

[19] It seems rather obvious that K and the problem of Church [1] are each at least many-one reducible to the other; likewise for the problem of [1] and of [2, 3]. Had we verified this in detail, we would have called this highest degree of unsolvability of decisions problems of recursively enumerable sets the *Church degree of unsolvability*.

5. **Simple sets.** It is readily proved that the necessary and sufficient condition that every recursive set be one-one reducible to a given recursively enumerable set of positive integers S is that S is infinite, and $\bar{S}$ contains an infinite recursively enumerable set. We are thus led to ask if there exist sets satisfying the following

DEFINITION. *A simple set is a recursively enumerable set of positive integers whose complement, though infinite, contains no infinite recursively enumerable set.*

We now prove the

THEOREM. *There exists a simple set.*

Recall the set T of all couples (B, n) such that positive integer n is in the recursively enumerable set of positive integers determined by basis B. Since T is recursively enumerable, we can set up an effective enumeration

$$O'': \quad (B_{i_1}, n_1), (B_{i_2}, n_2), (B_{i_3}, n_3), \cdots$$

of its members. The subscript of each B is its subscript in the effective enumeration $O: B_1, B_2, B_3, \cdots$ of all distinct B's. Now the complement of a set containing no infinite recursively enumerable set is equivalent to the set itself having an element in common with each infinite recursively enumerable set. Generate then the distinct bases $B_1, B_2, B_3, \cdots$, and as a B_i is generated, regenerate the sequence O'' of (B, n)'s in T, and the first time, if ever, B is B_i, and *n is greater than $2i$*, place n in a set S. The resulting set S is then a generated, and hence recursively enumerable, set of positive integers. We proceed to prove it simple.

If S' is an infinite recursively enumerable set of positive integers, it will be determined by some basis B_i, and will have some element m greater than $2i$. Since (B_i, m), being then in T, will appear in O'', our construction will place m in S, if some earlier (B_i, n) of O'' has not already contributed an element of S' to S. That is, S has an element in common with each infinite recursively enumerable S'. As for $\bar{S}$ being infinite, note that each B_i contributes at most one element to S. The first n B's in O therefore contribute at most n elements to S. Each B_i with $i \geqq n+1$ can only contribute to S an element greater than $2n+2$. Of the first $2n+2$ positive integers, at most n are therefore in S, and hence at least $n+2$ are in the consequently infinite $\bar{S}$.[20]

[20] $n > i$ can replace $n > 2i$ in the above construction, but the proof will then depend on there being an infinite number of bases defining the null set.

Having one simple set, the method of our succeeding §8 can be modified to yield a rich infinite class of simple sets. Clearly, *every simple set S is a recursively enumerable set that is not recursive.* For were S recursive, $\bar{S}$ would be an infinite recursively enumerable subset of $\bar{S}$. *The decision problem of each simple set is therefore recursively unsolvable.* We thus have obtained two infinite mutually exclusive classes of recursively enumerable sets with recursively unsolvable decision problems, the class of creative sets, and the class of simple sets. They are poles apart in that the complements of creative sets have a creative infinity of infinite recursively enumerable subsets, those of simple sets, not one.

In passing, we may note that every recursively enumerable set of positive integers S with recursively unsolvable decision problem leads to an incompleteness theorem for symbolic logics relative to the class of propositions $n \in S$, n an arbitrary positive integer. Creative sets S are then exactly those recursively enumerable sets of this type each of which admits a universal extendibility theorem as well, simple sets S those for which, given S, each logic can prove the falsity of but a finite number of the infinite set of false propositions $n \in S$.

It is readily seen that no creative set C can be one-one reducible to a simple set S. For under such a reduction, each infinite recursively enumerable subset of $\bar{C}$, proved above to exist, would be transformed into an infinite recursively enumerable subset of $\bar{S}$, contradicting the simplicity of S. Simple sets thus offer themselves as *candidates* for recursively enumerable sets with decision problems of lower degree of unsolvability than that of the complete set K. Even for many-one reducibility the situation is no longer immediately obvious; for an infinite recursively enumerable subset of $\bar{C}$ could thus be transformed into a finite subset of $\bar{S}$, the complement of simple S, without contradiction. However we can actually go much further than that.

6. Reducibility by truth-tables. If S_1 is many-one reducible to S_2, positive integer n being, or not being, in S_1 may be said to be determined by its correspondent m being, or not being, in S_2 in accordance with the truth-table

(S_2) m	n (S_1)
$+$	$+$
$-$	$-$

Here, the two signs $+$, $-$ under m represent the two possibilities m is in S_2, m is not in S_2, respectively. And by the sign under n in the same horizontal row as the corresponding sign under m the table in the same language tells whether n correspondingly is $(+)$, or is not $(-)$, in S_1. The table then says that when m is in S_2, n is in S_1, when m is not in S_2, n is not in S_1, as required by many-one reducibility. Now there are altogether four ways in which n being, or not being, in S_1 can be made to depend solely on m being, or not being, in S_2, the signs under n being $+, -$ as above; or $+, +$; $-, -$; $-, +$. If then we have an effective method which for each positive integer n will not only determine a unique corresponding positive integer m, but also one of these four "first order" truth-tables, and if in each case the table is such that for the correct statement of membership or non-membership of m in S_2, it gives the correct statement of membership or non-membership of n in S_1, then the decision problem of S_1 will thus be reduced to the decision problem of S_2. For here also, given n, if we could somehow determine whether m is, or is not, in S_2, we could thereby determine which row of the corresponding table correctly describes the membership or non-membership of m in S_2, and from that row correctly determine whether n is, or is not, in S_1.

More generally, let there be an effective method which for each positive integer n determines a finite sequence of positive integers $m_1, m_2, \cdots, m_\nu$, ν as well as the m's depending on n. Let that method correspondingly determine for each n a "νth order" truth-table of the form

(S_2)	m_1	$m_2 \cdots m_\nu$	n	(S_1)
	$+$	$+ \cdots +$	$-$	
	$+$	$+ \cdots -$	$+$	
	.	. . . . .	.	
	.	. . . . .	.	
	.	. . . . .	.	
	$-$	$- \cdots -$	$-$	

Each horizontal row, to the left of the vertical bar, specifies one of the 2^ν possible ways in which the ν m_i's may, or may not, be in S_2, to the right of the bar correspondingly commits itself to one of the statements n is in S_1, n is not in S_1. If then for each n that row of the corresponding table which gives the correct statements for the m's being or not being in S_2 also gives the correct statement regarding the membership or non-membership of n in S_1, the decision problem of S_1 is again thereby reduced to the decision problem of S_2.

If such a situation obtains with "effective method" replaced by "recursive method," we shall say that *S_1 is reducible to S_2 by truth-tables*. "Recursive method" here can mean that a suitable Gödel representation of the couple consisting of the sequence $m_1, m_2, \cdots,$

m_ν and the truth-table of order ν is a recursive function of n. If the orders of the truth-tables arising in such a reduction are bounded, we shall say that S_1 *is reducible to* S_2 *by bounded truth-tables*. Since there are 2^{2^ν} distinct truth-tables of order ν, reducibility by bounded truth-tables is equivalent to reducibility by truth-tables in which but a finite number of distinct tables arise.

7. **Non-reducibility of creative sets to simple sets by bounded truth-tables.** Let us suppose that creative set C is reducible to simple set S by bounded truth-tables. Let $T_1, T_2, \cdots, T_\kappa$ be the finite set of distinct truth-tables entering into such a reduction. That reduction then effectively determines for each positive integer n a finite sequence of positive integers $m_1, m_2, \cdots, m_\nu$, and a unique T_i, $1 \leqq i \leqq \kappa$.

The gist of our reductio-ad-absurdam proof consists in showing that under the assumed reduction we can obtain for each natural number p a sequence of m's at least p of which are in S. We then immediately have our desired contradiction. For in each case $p \leqq \nu$. The finite set of ν's, the orders of the T_i's, being bounded, p cannot then be arbitrarily large as stated.

More precisely we prove by mathematical induction that under the assumed reduction the following would be true. *For each natural number p an effective process Π_p can be set up which will determine for each recursively enumerable subset α of $\overline{C}$ an element n of $\overline{C}$ not in α, and which for the corresponding $m_1, m_2, \cdots, m_\nu$ and T_i yielded by the assumed reduction will correctly designate p of these m's as belonging to S.* The mode of designation may be assumed to be by specifying the sequence of subscripts, $i_1, i_2, \cdots, i_p$, of the m's to be designated, with say $i_1 < i_2 < \cdots < i_p$. With the assumed reduction adjoined to this process, Π_p then determines for each α in question the quadruplet (n, M, T_i, I), M being the sequence of m's, I the sequence of subscripts of the p designated m's.

For $p=0$, Π_p is immediately given by the creative character of C. For that immediately gives us for each recursively enumerable subset α of $\overline{C}$ a definite element n of $\overline{C}$ not in α. The assumed reduction yields the corresponding M and T_i; and with no members of M *designated* as being in S, I is the null sequence.

Inductively, assume that we have the process Π_p for $p=k$. Let α be any given recursively enumerable subset of $\overline{C}$, and let $(n', M', T_{i'}, I')$ be the corresponding quadruplet yielded by Π_k. Now suppose n is a positive integer for which the assumed reduction yields the same table $T_{i'}$ as it did for n', and a sequence of m's, M, consequently of the same length as M', having the following property. For each undesignated element of M', the correspondingly placed element of M is identical with that of M'; for each element of M' designated as being in S, the corresponding element of M is also in S. Such an n must then be in $\bar{C}$ along with n'. For that row of $T_{i'}$ which correctly tells of the m's of M' whether they are, or are not, in S will also be the correct row for M. And since in the former case that row must say that n' is in $\bar{C}$, in the latter case it will say that n is in $\bar{C}$, and correctly so. We proceed to show how all such n's may be generated.

We first show how to generate all M's obtainable from M' by replacing the designated elements of M' by arbitrary elements of S. For any one such M, the replacing elements, being finite in number, will be among the first N elements, for some positive integer N, of a given recursive enumeration of S. Generate then the positive integers $1, 2, 3, \cdots$, and as a positive integer N is generated, generate the first N elements of the given recursive enumeration of S. For each N place in a set β the at most N^k sequences M that can be obtained from M' by replacing the designated elements of M' by elements chosen from the first N elements of S. The generated set of sequences β then consists of all M's obtainable from M' by replacing the designated elements of M' by arbitrary elements of S.

The n's we wish to generate are then those positive integers for which the assumed reduction yields the table $T_{i'}$ and a sequence of m's, M, such that M is a member of β. Generate then the elements of β. As an element M of β is generated, generate the positive integers $1, 2, 3, \cdots$, and as a positive integer n is generated, find the corresponding sequence of m's and table yielded by the reduction of C to S. If then that sequence of m's is M, and the table is $T_{i'}$, *add n to the given set α*. As seen above, each such n will be in $\bar{C}$. Hence the resulting generated, and hence recursively enumerable, set α' is a subset of $\bar{C}$ containing α. Our reason for thus adding the desired n's to α instead of just forming the class thereof is that the iterative process we are about to set up requires a cumulative effect.

As a result of our hypothesis and construction we thus have a derived process Π_k' which for every recursively enumerable subset α of $\bar{C}$ yields a definite recursively enumerable subset α' of $\bar{C}$ containing α. Starting with α, we may then iterate the process Π_k' to obtain the infinite sequence $A:\alpha_1, \alpha_2, \alpha_3, \cdots$, where $\alpha_1=\alpha$, $\alpha_{n+1}=(\alpha_n)'$. Each member of A is thus a recursively enumerable subset of $\bar{C}$, and contained in the next member of A. By applying the original process Π_k to the members of A we correspondingly obtain the infinite sequence $\Sigma:\sigma_1, \sigma_2, \sigma_3, \cdots$, where σ_j is the quadruplet $(n^{(j)}, M^{(j)}, T_i^{(j)}, I^{(j)})$ yielded by Π_k for α_j. We then observe the following. If for $j_1 \neq j_2$

the T's of σ_{j_1} and σ_{j_2} are the same, and the I's are the same, then the sequences obtained from the M's by deleting the designated m's cannot be identical. For if they also were identical, then, with say $j_1 < j_2$, $n^{(j_2)}$ would have been assigned to $\alpha^{(j_1+1)}$, whereas it actually is outside of $\alpha^{(j_2)}$ which contains $\alpha^{(j_1+1)}$. Hence, the infinite sequence Σ', obtained from Σ by deleting from each σ_j the integer $n^{(j)}$ and the designated m's of $M^{(j)}$, itself consists of distinct elements.

It follows that *there are an infinite number of distinct undesignated m's appearing in Σ*. Indeed, the distinct $T_i^{(j)}$'s of Σ are at most κ in number. With $T_i^{(j)}$ fixed, the order $\nu^{(j)}$ of $T_i^{(j)}$ is fixed; and since $1 \leqq i_1^{(j)} < i_2^{(j)} < \cdots < i_k^{(j)} \leqq \nu^{(j)}$, the number of distinct I's is finite. Finally, with T and I fixed, were the total number of distinct undesignated m's finite, the number of distinct ways in which those $\nu^{(j)} - k$ undesignated m's could assume values would be finite. Hence Σ' would be finite, not infinite.

Now were each of this infinite set of undesignated m's in $\bar{S}$, we could regenerate the elements of Σ, and as an element σ_j thereof is generated, place all of its undesignated m's in a set γ, and thus obtain an infinite generated, and hence recursively enumerable, subset of $\bar{S}$. As this contradicts the simplicity of S, it follows that *at least one undesignated m arising in Σ is in S.*

We can then find a unique such m, as well as a σ in which it occurs, as follows. With $N = 1, 2, 3, \cdots$, generate the first N elements of the given recursive enumeration of S, and the first N elements of Σ, and test the latter in order to see if any undesignated m is among those first N elements of S. If a particular undesignated m of Σ in S, proved above to exist, is the Lth member of S, and in the Kth member σ_K of Σ, then an affirmative answer to the above test will certainly be obtained for $N = \max (L, K)$. Find then the first N for which an affirmative answer is obtained, and let (m, M, T_i, I) be the first σ to yield the affirmative answer for this N, $m_{i'}$ the first undesignated m of M thus found to be in S. We can then add $m_{i'}$ to the designated m's of M, thus obtaining a quadruplet (n, M, T_i, I_1), where I_1 designates $(k+1)$ of the m's of M as being in S, and where n is certainly a member of $\bar{C}$ not in the originally given α. But the whole process leading up to (n, M, T_i, I_1) is determined by that α. It is therefore the desired process Π_p for $p = k+1$.

Under the assumed reduction of C to S, Π_p would therefore exist for every natural number p. With α say the null set, we would thus obtain for every natural number p a quadruplet (n_p, M_p, T_{i_p}, I_p) such that p of the members of the sequence M_p are in S. Yet the total length of M_p is the order of T_{i_p}, and hence bounded. Hence the

THEOREM. *No creative set is reducible to a simple set by bounded truth-tables.*

We recall that every recursively enumerable set of positive integers is one-one reducible to the creative set K, the complete set. Hence the

COROLLARY. *Every simple set is of lower degree of unsolvability than the complete set K relative to reducibility by bounded truth-tables.*

8. **Counter-example for unbounded truth-tables.** We recall that for the particular simple set S constructed in §5, of the first $2m+2$ positive integers at most m were in S, m being any positive integer. Hence, of the $m+1$ integers $m+2, m+3, \cdots, 2m+2$, at least one is in $\bar{S}$. By setting $m=2^n-1$, with $n=1, 2, 3, \cdots$, we can effectively generate the infinite sequence of mutually exclusive finite sequences

$$\sigma: \qquad (3, 4), (5, 6, 7, 8), \cdots, (2^n+1, 2^n+2, \cdots, 2^{n+1}), \cdots$$

such that each sequence in σ has at least one member thereof in $\bar{S}$. An effective one-one correspondence between the positive integers 1, 2, 3, $\cdots$ and the elements of σ is then obtained by making the positive integer n correspond to the sequence $(2^n+1, 2^n+2, \cdots, 2^{n+1})$ constituting the nth element of σ.

Given a creative set C, regenerate the elements of S, placing each in a set S_1. Furthermore, regenerate the elements of C, and as an element n thereof is generated, place all of the positive integers in the nth sequence of σ in S_1. The resulting set S_1 is a generated, and hence recursively enumerable, set of positive integers. Since S_1 contains S, $\bar{S}$ contains $\bar{S}_1$. As S is simple, $\bar{S}$, and hence $\bar{S}_1$, does not have an infinite recursively enumerable subset. Moreover, $\bar{S}_1$ is also infinite. For $\bar{C}$ is infinite. And, for each element of $\bar{C}$, the corresponding sequence in σ has only those of its members that are already in S also in S_1, and hence at least one element in $\bar{S}_1$. Hence, S_1 is simple.

Likewise we see that a positive integer n is in C, or $\bar{C}$, according as all of the integers in the nth sequence of σ are in S_1, or at least one is in $\bar{S}_1$. If then we make correspond to each positive integer n the sequence of 2^n positive integers $(2^n+1, 2^n+2, \cdots, 2^{n+1})$, and the truth-table of order 2^n in which the sign under n is $+$ in that row in which the signs under the 2^n "m's" are all $+$, and in every other row the sign under n is $-$, we have a reduction of C to S_1 by truth-tables. Hence the

THEOREM. *For each creative set C a simple set S can be constructed such that C is reducible to S by unbounded truth-tables.*

COROLLARY. *A simple set S can be constructed which is of the same degree of unsolvability as the complete set K relative to reducibility by truth-tables unrestricted.*

Simple sets as such do not therefore give us the absolutely lower degree of unsolvability than that of K we are seeking.

9. **Hyper-simple sets.** The counter-example of the last section suggests that we seek a set satisfying the following

DEFINITION. *A hyper-simple set H is a recursively enumerable set of positive integers whose complement $\bar{H}$ is infinite, while there is no infinite recursively enumerable set of mutually exclusive finite sequences of positive integers such that each sequence has at least one member thereof in $\bar{H}$.*[21]

In this definition we may use the original Gödel method for representing a finite sequence of positive integers $m_1, m_2, \cdots, m_\nu$ by the single positive integer $2^{m_1} 3^{m_2} \cdots p_\nu^{m_\nu}$, where $2, 3, \cdots, p_\nu$ are the first ν primes in order of magnitude. A set of finite sequences of positive integers is then recursively enumerable if the set of Gödel representations of those sequences is recursively enumerable.

THEOREM. *A hyper-simple set exists.*

Our intuitive argument must again draw upon the formal development to the effect that each recursively enumerable set of finite sequences of positive integers will be determined by a "basis" B^*, and that all such bases can be generated in a single infinite sequence of distinct bases

$$O^*: \quad B_1^*, B_2^*, B_3^*, \cdots .$$

As in §2, generate the elements of O^*, and as an element B^* is generated, set up the process for generating the set of sequences determined by B^*, and as a sequence s is thus generated, write down the couple (B^*, s). The resulting set of couples is then a generated set, and can indeed be effectively ordered in a sequence of distinct couples

$$O_1^*: \quad (B_{i_1}^*, s_1), (B_{i_2}^*, s_2), (B_{i_3}^*, s_3), \cdots .$$

[21] Mutually exclusive sequences here mean no element of one sequence is an element of another. Curry suggests that "hyper-simple" is linguistically objectionable, and should be replaced by "super-simple." But we would not then know what to use in place of the letter H.

O_1^* then consists of all distinct couples (B^*, s) such that finite sequence s is a member of the recursively enumerable set of finite sequences of positive integers determined by basis B^*.

Now the condition that no infinite recursively enumerable set of mutually exclusive finite sequences of positive integers has the property that each sequence has at least one positive integer thereof in $\bar{H}$ is equivalent to each such set of sequences having at least one sequence all of whose members are in H. Our method of constructing the desired hyper-simple set H will then consist in placing in H for certain B^*'s in O^* all of the positive integers in a sequence in the set of sequences determined by B^*. For purposes of presentation we shall call each such basis B^* a *contributing basis*, while every B^* determining an infinite recursively enumerable set of mutually exclusive sequences will be called a *relevant basis*. Set H, if recursively enumerable, will then be hyper-simple if each relevant basis is a contributing basis, and $\bar{H}$ is infinite.

If B^* is a relevant basis, then among the infinite number of mutually exclusive sequences generated by B^* there must be a sequence each of whose elements exceeds an arbitrarily given positive integer N. For did every sequence generated by B^* have as element one of the integers $1, 2, \cdots, N$, for any $N+1$ of these sequences at least two would have one of these integers in common. We shall then generate H by regenerating sequence O_1^*, and, as an element $(B_{i_n}^*, s_n)$ thereof is generated, we shall place all the elements of s_n in H if $B_{i_n}^*$ has not thus been made a contributing basis earlier in the process, while the elements of s_n are all greater than a certain positive integer N_n, about to be determined; otherwise none. Inductively, assume N_m to have been determined for $1 \leqq m < n$, and thus the entire process up to the time $(B_{i_n}^*, s_n)$ was brought up for consideration. Let $B_{j_1}^*, B_{j_2}^*, \cdots, B_{j_\nu}^*$ be the bases that have thus far contributed to H, and in the order in which they became contributing bases. These bases are then distinct, and hence their subscripts, which give their position in the sequence O^* of all distinct bases, are distinct. Let $k_1, k_2, \cdots, k_\nu$ be the largest integer placed in H by the first contributing basis, by the first two, $\cdots$, by the first ν. The result being cumulative, $k_1 \leqq k_2 \leqq \cdots \leqq k_\nu$. The crux of our construction is to make N_n depend not on the history of all these ν contributions to H, but only on that part of that history up to and including the last contribution, if any, made by a B^* preceding $B_{i_n}^*$ in O^*. Specifically, if $B_{j_\mu}^*$ is the last of the above ν contributing bases preceding $B_{i_n}^*$ in O^*, that is, with $j_\mu < i_n$, N_n is to be one more than the largest integer present in H as a result of all the contributions made up to and including the contribution made by $B_{j_\mu}^*$. That is, $N_n = k_\mu + 1$. Actually, if none of the ν contributing bases precede $B_{i_n}^*$ in O^*, no condition is to be placed on s_n, and all of its elements are placed in H so long as $B_{i_n}^*$ is distinct from the ν contributing bases obtained thus far.

Furthermore, in our induction assume that we have been able to keep a record of the sequence $B_{j_1}^*, B_{j_2}^*, \cdots, B_{j_\nu}^*$, of $k_1, k_2, \cdots, k_\nu$, and also of $j_1, j_2, \cdots, j_\nu$ up to the time $(B_{i_n}^*, s_n)$ was about to be generated. We then generate $(B_{i_n}^*, s_n)$, and by regenerating O^* find the place of $B_{i_n}^*$ in O^* thus determining the subscript i_n. Our criterion for determining whether, or no, the elements of s_n are to be placed in H then becomes effective. In the latter case, the record is unchanged as we generate $(B_{i_{n+1}}^*, s_{n+1})$. In the former, $B_{i_n}^*$ is written into the record as $B_{j_{\nu+1}}^*$, i_n as $j_{\nu+1}$ while we can write in for $k_{\nu+1}$ the maximum of k_ν and the largest integer in s_n. The entire process is thus effective at each stage, and H is thus a generated, and hence recursively enumerable, set of positive integers. We proceed to prove it hyper-simple.

Let B^* be any relevant basis. Of the finite number of bases preceding B^* in O^*, but a finite number can be contributing bases. Let $B_{j_\mu}^*$ be the last of these contributing bases, if any, appearing in the sequence $B_{j_1}^*, B_{j_2}^*, B_{j_3}^*, \cdots$ of distinct contributing bases determined by the above generation of H. There will then be a sequence s generated by B^* each of whose elements is greater than $k_\mu + 1$. When then (B^*, s), a definite element of O_1^*, is generated in the course of generating H, B^* will contribute each element of s to H unless it became a contributing basis earlier in the process. Hence, every relevant basis is a contributing basis.

It also follows, or is easily seen directly, that the number of contributing bases is infinite. Consider then the infinite sequence of contributing bases $B_{j_1}^*, B_{j_2}^*, B_{j_3}^*, \cdots$, the corresponding infinite sequence of subscripts $j_1, j_2, j_3, \cdots$, and the associated infinite sequence $k_1, k_2, k_3, \cdots$. Since the contributing bases are distinct, so are their subscripts. Hence, for each j_m, among the infinite set of j's following j_m there is a unique least j, $j_{m'}$. Consider then the resulting infinite sequence $j_{\lambda_1}, j_{\lambda_2}, j_{\lambda_3}, \cdots$, where j_{λ_1} is the least j in the whole infinite sequence of j's, while $\lambda_2 = (\lambda_1)'$, $\lambda_3 = (\lambda_2)'$, $\cdots$. Now k_{λ_n} is the largest integer contributed to H through the contributing basis with subscript j_{λ_n}. Since j_{λ_n} is the smallest j following $j_{\lambda_{n-1}}$ it is less than all succeeding j's. Hence B^* with subscript j_{λ_n} precedes in O^* all bases following that B^* in the above infinite sequence of contributing bases. Hence, each element added to H by contributing bases thus following B^* with subscript j_{λ_n} must exceed $k_{\lambda_n} + 1$. It follows, on the one hand, that for each positive integer n, $k_{\lambda_n}+1$ is in $\bar{H}$. On the other hand, $k_{\lambda_{n+1}}$ itself exceeds $k_{\lambda_n}+1$ so that $k_{\lambda_{n+1}}+1>k_{\lambda_n}+1$. These members of $\bar{H}$ therefore constitute an infinite subset of the consequently infinite $\bar{H}$. Hence, H is hyper-simple.

Clearly, *every hyper-simple set H is simple.* For an infinite recursively enumerable subset of $\bar{H}$, as set of unit sequences, would contradict H being hyper-simple. Our construction of §6, in view of §8, gives us, however, *a simple set which is not hyper-simple.* Hyper-simple sets thus constitute a third class of recursively enumerable sets with recursively unsolvable decision problems—a class which is a proper subclass of the class of simple sets.

10. Non-reducibility of creative sets to hyper-simple sets by truth-tables unrestricted. Let creative set C be reducible by truth-tables to a recursively enumerable set of positive integers H. The given reduction will again determine for each positive integer n a finite sequence of positive integers $m_1, m_2, \cdots, m_\nu$, and a truth-table T of order ν such that that row of the table which correctly tells of the m's whether they are, or are not, in H will correctly tell of n whether it is, or is not, in C. Of course ν and T as well as the m's depend on n, and the set of distinct T's now entering into our reduction may be infinite, and hence the set of distinct ν's unbounded.

Let $l_1, l_2, \cdots, l_\mu$ be any given finite sequence of distinct positive integers. A particular hypothesis on the l's being, or not being, in H may then be symbolized by a sequence of μ signs, each $+$ or $-$, such as $+-\cdots+$, such that the ith sign is $+$, or $-$, according as the hypothesis says that l_i is in H, or $\bar{H}$, respectively. We shall speak of such a sequence of signs as a *truth-assignment* for the l's, the ith sign in that sequence as the *sign of l_i* in that truth-assignment. Of the 2^μ possible truth-assignments for the l's, constituting a set V_1, one and only one correctly tells of each l_i whether it is, or is not, in H. Every set V of truth-assignments for the l's is then a subset of V_1, and will be called a *possible set* of truth-assignments if it includes this *correct* truth-assignment.

Let then V be any given possible set of truth-assignments for the l's. Let n be a positive integer with corresponding $m_1, m_2, \cdots, m_\nu$, T yielded by the given reduction of C to H such that *each m not an l is in H.* The correct row of table T must then have the following two properties. First, the sign under each m not an l must be $+$. Second, the signs under those m's which are l's must be the same as the signs of those integers in some one and the same truth-assignment for the l's in V, in fact, as in the correct truth assignment for the l's. Any row of T having these two properties, given the l's, m's and V, will be called a *relevant row* of T. Since for our n the correct row of T is thus a relevant row, it follows that n will surely be in $\bar{C}$ if for each relevant row of T the sign under n is $-$.

Generate then the positive integers 1, 2, 3, $\cdots$, and as a positive integer N is generated, generate the first N members of a given recursive enumeration of H, and for each n, with $1 \leqq n \leqq N$, find the corresponding $m_1, m_2, \cdots, m_\nu, T$ yielded by the given reduction of C to H. Of those m's, if any, which are not l's, see if each is one of those first N members of H. If they all are, see if for each relevant row of T the sign under n is $-$. If that also is true, place n in a set α_V. Since each such n must be in $\bar{C}$, as seen above, α_V is a subset of $\bar{C}$. And being a generated set, α_V is therefore a recursively enumerable subset of $\bar{C}$.

C being creative, we can therefore find a definite positive integer n' in $\bar{C}$ but not in α_V, and, by the given reduction, the corresponding $m'_1, m'_2, \cdots, m'_{\nu'}, T'$. Let $p_1, p_2, \cdots, p_\lambda$ be those m''s, if any, which are not l's. Now suppose that each p is in H. Then for at least one relevant row of T' the sign under n' must be $+$. For otherwise, if p_i is say the k_ith element in the given recursive enumeration of H, n' would have been placed in α_V in the above generation thereof for $N = \max(k_1, k_2, \cdots, k_\lambda, n')$. Since n' is in $\bar{C}$, such a relevant row cannot be the correct row. But, with each p in H, the signs in that row under m's that are not l's are correctly $+$. Hence the sign under at least one m' that is an l must be incorrect. But, by our definition of a relevant row, the signs under all such m''s are the same as the signs of those integers in at least one truth-assignment in V. Such a truth-assignment in V cannot therefore be the correct truth-assignment for the l's, and hence may be deleted from V. Perform this deletion for all such truth-assignments in V, and for all such relevant rows of T', to obtain the set of truth-assignments V'. Under our hypothesis that each p is in H, V' will then be a proper subset of V, and yet a possible set of truth-assignments for the l's.

Actually, let V be any given set of truth-assignments for the l's, possible or not. Each step of the above construction can then still be carried out, though the constructed entities need not now have all the properties they otherwise possess.[22] In particular, the set of integers, possibly null, $p_1, p_2, \cdots, p_\lambda$ can be found, all different from any l. Likewise, whether the p's are, or are not, all in H, the subset V' of V can be found. What we can say is that if V is a possible set,

[22] Recall that in the definition of creative set, §3, each B determines an n, whether the α determined by B is, or is not, a subset of $\bar{C}$.

and if furthermore each p is in H, then V' is a proper subset of V, and itself is also a possible set of truth-assignments for the l's.

For the given sequence of l's, start then with $V=V_1$, the possible set of all 2^μ truth-assignments for the l's, obtain the corresponding p's, $p_1', p_2', \cdots, p_{\lambda'}'$ and corresponding V', $V_2=(V_1)'$. With $V=V_2$, likewise find $p_1'', p_2'', \cdots, p_{\lambda''}''$, and $V_3=(V_2)'$, and so on. Now each V_{i+1} is a subset of V_i, while V_1 is but a finite set of 2^μ members. Hence in at most 2^μ steps we shall come across a V_κ such that either $V_{\kappa+1}$ is identical with V_κ, or is null. But if all the p''s, p'''s, $\cdots$, $p^{(\kappa)}$'s were in H, V_1 being a possible set, $V_2, \cdots, V_\kappa$ as well as $V_{\kappa+1}$ would all be possible sets, each a proper subset of the preceding. $V_{\kappa+1}$ could not then either be identical with V_κ, or null. It follows that at least one of the $p_i^{(j)}$'s with $1 \leqq j \leqq \kappa$ is in $\overline{H}$. Each $p_i^{(j)}$ is an integer that is not one of the l's. If then we take this finite set of $p_i^{(j)}$'s and arrange them in a sequence of distinct elements in say order of magnitude, we obtain for our arbitrarily given sequence of distinct positive integers $l_1, l_2, \cdots, l_\mu$ a sequence of distinct positive integers $k_1, k_2, \cdots, k_\nu$ having no element in common with the former sequence, and having at least one element in $\overline{H}$.

Starting with the null sequence as the sequence of l's, we can thus find the sequence of k's, $(k_1', k_2', \cdots, k_{\nu'}')$ of distinct positive integers at least one of which is in $\overline{H}$. Inductively, let us have thus generated the sequences $(k_1', k_2', \cdots, k_{\nu'}'), \cdots, (k_1^{(\mu)}, k_2^{(\mu)}, \cdots, k_{\nu^{(\mu)}}^{(\mu)})$, mutually exclusive, of distinct positive integers, each having at least one element in $\overline{H}$. With the single sequence $k_1', \cdots, k_{\nu^{(\mu)}}^{(\mu)}$ as the sequence of l's, we can find the corresponding sequence of k's, $(k_1^{(\mu+1)}, k_2^{(\mu+1)}, \cdots, k_{\nu^{(\mu+1)}}^{(\mu+1)})$ of distinct positive integers with no element in common with any of the preceding sequences, and having at least one element in $\overline{H}$.

With creative C reducible to recursively enumerable H by truth-tables we can thus obtain an infinite generated, and hence recursively enumerable, set of mutually exclusive finite sequences of positive integers each having an element in $\overline{H}$. The set H is therefore not hyper-simple. Hence the

THEOREM. *No creative set is reducible to a hyper-simple set by truth-tables.*

COROLLARY. *Every hyper-simple set is of lower degree of unsolvability than the complete set K relative to reducibility by truth-tables.*

Despite this result, the brief discussion of Turing reducibility, still in the informal stage, entered into in the next section makes it dubious that hyper-simple sets as such will give us the desired absolutely lower degree of unsolvability than that of K. But, in the absence of a counter-example, they remain candidates for this position.

11. **General (Turing) reducibility.** The process envisaged in our concept of a generated set may be said to be *polygenic*. In a *monogenic* process act succeeds act in one time sequence. The intuitive picture is that of a machine grinding out act after act (Turing [24]) or a set of rules directing act after act (Post [18]). The actual formulations are in terms of "atomic acts," the first leading to a development proved by Turing [25] equivalent to those arising from general recursive function or λ-definability, and hence of the same degree of generality. In our intuitive discussion the acts may be "molecular."

An effective solution of the decision problem for a recursively enumerable set S_1 of positive integers may therefore be thought of as a machine, or set of rules, which, given any positive integer n, will set up a monogenic process terminating in the correct answer, "yes" or "no," to the question "is n in S_1." Now suppose instead, says Turing [26] in effect, this situation obtains with the following modification. That at certain times the otherwise machine determined process raises the question is a certain positive integer in a given recursively enumerable set S_2 of positive integers, and that the machine is so constructed that were the correct answer to this question supplied on every occasion that arises, the process would automatically continue to its eventual correct conclusion.[23] We could then say that the machine effectively reduces the decision problem of S_1 to the decision problem of S_2. Intuitively this should correspond to the most general concept of the reducibility of S_1 to S_2. For the very concept of the decision problem of S_2 merely involves the answering for an arbitrarily given single positive integer m of the question is m in S_2; and in finite time but a finite number of such questions can be asked. A corresponding formulation of "Turing reducibility" should then be the same degree of generality for effective reducibility as say general recursive function is for effective calculability.[24]

We may note that whereas in reducibility by truth-tables the posi-

[23] Turing picturesquely suggests access to an "oracle" which would supply the correct answer in each case. The "if" of mathematics is however more conducive to the development of a theory.

[24] A reading of McKinsey [16] suggested generalizing the reducibility of a recursively enumerable set S to a recursively enumerable set S' to the reducibility of S to a finite set of recursively enumerable sets $S_1, S_2, \cdots, S_n$. However, no absolute gain in generality is thus achieved, as a single recursively enumerable set S' can be constructed such that reducing S to $(S_1, S_2, \cdots, S_n)$ is equivalent to reducing S to S'. Points of interest, however, do arise.

tive integers m of which we ask the questions "is m in S_2" are effectively determined, for given n, by the reducibility process, in Turing reducibility, except for the first such m, the very identity of the m's for which this question is to be asked depends, in general, on the correct answers having been given to these questions for all preceding m's. The mode of this dependence is, however, effective, hence we still have effective reducibility in the intuitive sense.

Let now creative set C be Turing reducible to a recursively enumerable set S of positive integers. We shall talk as if our intuitive discussion has already been formalized. Generate the positive integers $1, 2, 3, \cdots$, and as a positive integer N is generated, for each n with $1 \leq n \leq N$ proceed as follows. Set going the reducibility process of C to S for n. Each time a question of the form "is m in S" is met, see if m is among the first N integers in a given recursive enumeration of S. If it is, supply the answer "yes," thus enabling the reducibility process to continue. Finally, if under these circumstances the process terminates in a "no" for the initial question of n being in C, place n in a set α_0. This α_0 is then a recursively enumerable subset of $\overline{C}$ consisting of all members thereof for which the given Turing reduction of C to S leads only to questions of the form "is m in S" whose answer is "yes."

Find then n_0 of $\overline{C}$ not in α_0, and set the reducibility process going for n_0. Now if at any time a wrong answer is supplied to a question "is m in S," we can nevertheless expect our machine for reducing C to S either to effectively pick up the wrong answer and operate on it to give a next step in the process, or to cease operating. Generate then the positive integers $1, 2, 3, \cdots$, and as a positive integer N is generated, generate the first N members of the given recursive enumeration of S, and make the reducibility process for n_0 *effective* though perhaps *incorrect* as follows. Each time a question of the form "is m in S" is reached, see if m is among the first N members of S. If it is, answer the question "yes," and correctly so; if not, answer the question "no," whether that answer be correct or no. If now this *pseudo-reduction* terminates in a "no," place the finite number of m's thus arising in a set β_{n_0}. Note that β_{n_0} consists of all such m's for all such pseudo-reductions for the given n_0. Being a generated set of positive integers, *β_{n_0} is recursively enumerable.*

Now let the correct, though possibly non-effective, reducibility process for n_0 involve the μ questions "is m_i in S," $i=1, 2, \cdots, \mu$. Let $m_{i_1}, m_{i_2}, \cdots, m_{i_\nu}$ be those of these m's actually in S, and let them be the n_1st, n_2nd, $\cdots$, n_νth members of the given recursive enumeration of S. If then $N \geq M = \max(n_1, n_2, \cdots, n_\nu)$, or $M=1$ if

$\nu=0$, the corresponding psuedo-reduction for n becomes the correct reduction. For, inductively, if that be so through the time a question "is m in S" is raised, m will be $m_1, m_2, \cdots$, or m_μ, hence will, or will not, be in S according as it is, or is not, one of the first N members of S. The answer is then correctly given by that pseudo-reduction, which therefore continues to be correct through the raising of the next question. Finally, since n_0 is in $\bar{C}$, the correct reduction, now the pseudo-reduction, must terminate with a "no."

It follows that all N's with $N>M$ merely repeat the contribution to β_{n_0} made by $N=M$, that is, of the integers $m_1, m_2, \cdots, m_\mu$. Since but a finite number of m's are contributed by N's with $N<M$, it follows that *β_{n_0} is a finite set.* Finally, were each of the integers $m_1, m_2, \cdots, m_\mu$ in S, n_0 would be in α_0. Hence, *at least one member of β_{n_0} is in $\bar{S}$.*

Formally, we would thus obtain a basis for a finite recursively enumerable set of positive integers at least one of whose members is in $\bar{S}$. Instead of recursively enumerable sets of finite sequences of positive integers, we would thus be led to consider recursively enumerable sets of bases for finite recursively enumerable sets of positive integers. Though, in the last analysis, each sequence in the former case must be generated atom by atom, there will come a time for each sequence when the process will say "this sequence is completed." In the latter case, in general, we cannot have an effective method which, for each basis, will give a point in the ensuing process at which it can say all members of the finite set in question have already been obtained, even though, with the process made monogenic, there always is such a stage in the process.

This suggests, then, that we strengthen the condition of hypersimplicity still further by replacing "infinitive recursively enumerable set of mutually exclusive finite sequences of positive integers" in the definition of §9 by "infinite recursively enumerable set of bases defining mutually exclusive finite recursively enumerable sets of positive integers." Whether such a "hyper-hyper-simple" set exists, or whether, if it exists, it will lead to a stronger non-reducibility result than that of the last section we do not know.

On the other hand, an equivalent definition of hyper-simple set is obtained if, for example, we replace the quoted phrase by "recursively enumerable set of finite sequences of positive integers having for each positive integer n a member each of whose elements exceeds n." We now can say that with this as the definition of a hyper-simple set, the corresponding extension to a hyper-hyper-simple set cannot be made. For we prove the

THEOREM. *For any recursively enumerable set of positive integers S, with infinite $\bar{S}$, there exists a recursively enumerable set of bases defining finite recursively enumerable sets of positive integers, each set having at least one element in $\bar{S}$, and at least one set having each of its elements greater than an arbitrarily given positive integer n.*

Briefly, with n given, for each positive integer N, and each positive integer m, place all of the integers $n+1, n+2, \cdots, n+m$ in a set α_n if all, or all but the last, are among the first N members of a given recursive enumeration of S. It is readily seen that α_n is a generated, and hence recursively enumerable, set of positive integers. A corresponding basis $B^{(n)}$ can actually be found, and the set of $B^{(n)}$'s, $n=1, 2, 3, \cdots$, being a generated set, is therefore recursively enumerable. Moreover, if ν_n is the smallest integer in the infinite $\bar{S}$ greater than n, α_n will consist of exactly the integers $n+1, n+2, \cdots, \nu_n$, and hence will be finite, with indeed ν_n as the only element in $\bar{S}$, and with each element greater than n.

As a result we are left completely on the fence as to whether there exists a recursively enumerable set of positive integers of absolutely lower degree of unsolvability than the complete set K, or whether, indeed, all recursively enumerable sets of positive integers with recursively unsolvable decision problems are absolutely of the same degree of unsolvability. On the other hand, if this question can be answered, that answer would seem to be not far off, if not in time, then in the number of special results to be gotten on the way.[25]

Such then is the portion of "Recursive theory" we have thus far developed. In fixing our gaze in the one direction of answering the lower degree of unsolvability question, we have left unanswered many questions that stud even the short path we have traversed. Moreover, both our special, and the general Turing, definitions of reducibility are applicable to arbitrary decision problems whose questions in symbolic form are recursively enumerable, and indeed to problems with recursively enumerable set of questions whose answers belong to a recursively enumerable set. Thus, only partly leaving the field of decisions problems of recursively enumerable sets, work of Turing [26] suggests the question is the problem of determining of an arbitrary basis B whether it generates a finite, or infinite, set of positive

[25] This is a matter of practical concern as well as of theoretical interest. For according as the second or first of the above alternatives holds will the method of reducing new decision problems to problems previously proved unsolvable be, or not be, the general method for proving the unsolvability of decision problem either of recursively enumerable sets of positive integers or of problems equivalent thereto.

integers of absolutely higher degree of unsolvability than K. And if so, what is its relationship to that decision problem of absolutely higher degree of unsolvability than K yielded by Turing's theorem.

Actually, the theory of recursive reducibility can be but one chapter in the theory of recursive unsolvability, and that, but one volume of the theory and applications of general recursive functions. Indeed, if general recursive function is the formal equivalent of effective calculability, its formulation may play a rôle in the history of combinatory mathematics second only to that of the formulation of the concept of natural number.

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MATHEMATICS

NON-STANDARD ANALYSIS

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1. *Introduction.*

It has been known since the publication of a classical paper by SKOLEM [9] that there exist proper extensions of the system of natural numbers $N(n \geqslant 0)$ which possess all properties of N that are formulated in the Lower predicate calculus in terms of some given set of number-theoretic relations or functions, e.g. addition, multiplication, and equality. Such an extension of the natural numbers is known as a (strong) non-standard model of arithmetic.

Now let R_0 be the set of all real numbers. Let K_0 be the set of sentences formulated in the Lower predicate calculus in terms of (individual constants for) all elements of R_0 and in terms of (distinct symbols for) all relations that are definable in R_0, including singulary relations. In a well-defined sense all elementary statements about *functions* in R_0 can be expressed within K_0. Thus if the real-valued function $f(x)$ is defined on the subset of S of R_0, and not elsewhere, then there exists a binary relation $F(x, y)$ in the vocabulary of K_0 such that $F(a, b)$ holds in R_0 if and only if $a \in S$ and $b = f(a)$. The fact that $F(x, y)$ denotes a function with domain of definition S is expressed by the sentence

$$[(x)(y)[F(x, y) \supset T(x)]] \wedge [(x)[T(x) \supset [(\exists y)(z)[F(x, y) \wedge [F(x, z) \supset E(y, z)]]]]],$$

where $T(x)$ is the singulary relation which defines S and $E(y, z)$ stands for $y = z$. Note that a different relation, $F'(x, y)$ corresponds to "the same" $f(x)$ if it is obtained the restricting the domain of definition of $f(x)$ to a proper subset of S. However, for ease of understanding we shall in the sequel use a less formal notation and include expressions like $x = y$, $y = f(x)$, $xy = z$ among our sentences. It is not difficult to translate these sentences into the strict formalism of K_0.

Let R^* be a model of K_0 which is a proper extension of R_0 with respect to all the relations and individual constants contained in R_0. R^* will be called a *non-standard model of analysis*. The existence of non-standard models of analysis follows from a familiar application of the extended completeness theorem of the Lower predicate calculus (*e.g.* [7]). Such models may also be constructed in the form of ultra-powers (*e.g.* [6]). The latter method affords us an insight into the structure of non-standard models of analysis and enables us to discuss the question to what extent we can single out certain distinguished models of this kind. Considerable progress can be made in this direction, but for the work of the present paper any one non-standard model of analysis will do as well as another.

It is our main purpose to show that these models provide a natural approach to the age old problem of producing a calculus involving infinitesimal (infinitely small) and infinitely large quantities. As is well known, the use of infinitesimals, strongly advocated by Leibnitz and unhesitatingly accepted by Euler fell into disrepute after the advent of Cauchy's methods which put Mathematical Analysis on a firm foundation. Accepting Cauchy's standards of rigor, later workers in the domain of non-archimedean quantities concerned themselves only with fragments of the edifice of Mathematical Analysis. We mention only DU BOIS-REYMOND's Calculus of infinities [2] and HAHN's work on non-archimedean fields [4] which in turn were followed by the theories of ARTIN–SCHREIER [1] and, returning to analysis, of HEWITT [5] and ERDÖS, GILLMAN, and HENRIKSEN [3]. Finally, a recent and rather successful effort of developing a calculus of infinitesimals is due to SCHMIEDEN and LAUGWITZ [8] whose number system consists of infinite sequences of rational numbers. The drawback of this system is that it includes zero-divisors and that it is only partially ordered. In consequence, many classical results of the Differential and Integral calculus have to be modified to meet the changed circumstances.

Our present approach yields a proper extension of classical Analysis. That is to say, the standard properties of specific functions (e.g. the trigonometric functions, the Bessel functions) and relations, in a sense made precise within the framework of the Lower predicate calculus, still hold in the wider system. However, the new system contains also infinitely small and infinitely large quantities and so we may reformulate the classical definitions of the Infinitesimal calculus within a Calculus of infinitesimals and at the same time add certain new notions and results.

There are various non-trivial interconnections between the theories mentioned in this introduction. For example (as is not generally realized) the ultra-power construction coincides, in certain special cases which are relevant here, with the construction of residue fields in the theory of rings of continuous functions. Similarly, there are various connections between these theories and the work of the present paper. We have no space to deal with them here. Details and proofs of the results described in the present paper will be given in the volume "Introduction to Model theory and to the Metamathematics of Algebra" which is due to be published in the series "Studies in Logic and the Foundations of Mathematics".

2. *Non-standard analysis and non-archimedean fields.*

Let R^* be any non-standard model of analysis. Then $R^* \supseteq R_0$ but $R \neq R_0$. It follows that R^* is non-archimedean. The elements of $R_0 \subseteq R^*$

will be referred to as the *standard* elements of R^* or standard numbers. Since R^* is non-archimedean the following two subsets of R^* are not empty.

M_0, the set of all $a \in R^*$ such that $|a| < r$ for some $r \in R_0$. M_0 is a ring. The elements of M_0 will be said to be *finite*.

M_1, the set of all $a \in R^*$ such that $|a| < r$ for all positive $r \in R_0$. The elements of M_1 will be said to be *infinitesimal*. M_1 is a prime ideal in M_0 and M_0/M_1 is isomorphic to R_0.

Let $S = (a_1, a_2, \dots)$ be any subset of R^*. We write $O(a_1, a_2, \dots)$ for the module $M_0a_1 + M_0a_2 + \dots \subseteq R^*$ (weak sum) and we write $o(a_1, a_2, \dots)$ for the module $M_1a_1 + M_1a_2 + \dots \subseteq R^*$. In particular $M_0 = O(1)$ and $M_1 = o(1)$. We write $a =_1 b$ if $a - b \in M_1$, i.e. if the difference between a and b is infinitesimal and we say in that case that a is *infinitely close* to b. Every finite number (i.e. every element of M_0) is infinitely close to some standard number. We write

$$a = b \text{ mod. } O(a_1, a_2, \dots)$$

if $a - b \in O(a_1, a_2, \dots)$, with a similar notation for o.

So far we have formulated only a number of obvious, and in part well-known, notions and facts concerning all ordered fields which are extensions of the field of real numbers. We now make use of the fact that R^* is a non-standard model of analysis. Let $N'(x)$ be the singulary relation which defines the natural numbers in R_0. Then $N'(x)$ defines a set N^* in R^*. N^* turns out to be a non-standard model of the natural numbers with respect to all relations definable in N. Similarly, we obtain a non-standard model of the rational numbers, R_1^* as a subset of R^*. It can be shown that every standard transcendental number is infinitely close to some element of R_1^*.

Syntactically, or linguistically, our method depends on the fact that we may enrich our vocabulary by the introduction of new relations, such as $R_0'(x)$, $M_0'(x)$, $M_1'(x)$ which define R_0, M_0, M_1, in R^*. (Note that the singulary relations just mentioned are, provably, not definable in terms of the vocabulary of K_0). We are therefore in a position to reformulate the notions and procedures of classical analysis in non-archimedean language. Since all the "standard" results of analysis still hold we may make use of them as much as we please and we may therefore carry out our reformulation either at the level of the fundamental definitions (of a limit, of an integral, ...) or at the level of the proof or, finally, by introducing non-standard notions into a result obtained by classical methods.

We consider in the first instance functions, relations, sets, etc. which are defined already in R_0, so that appropriate symbols are available for them in the original vocabulary. Such concepts will be called *standard* (functions, relations, sets, etc.). For example the interval $a < x < b$, will be called a standard interval in R^* if a and b are standard numbers (elements of R_0). The interval $\eta < x < b$, where b is standard and η in-

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finitesimal positive, is not a standard interval. The function $\sin x$ is a standard function. Strictly speaking we ought to refer here also to the interval of definition of the function but, as in ordinary analysis, this will frequently be taken for granted.

At a more advanced stage it turns out to be nexessary to go beyond standard functions, etc. Consider a standard function of $n+1$ variables, $y=f(x_1, \dots, x_n, t)$, $n \geqslant 1$. Regard t as a parameter and define $g(x_1, \dots, x_n)$ by $g(x_1, \dots, x_n)=f(x_1, \dots, x_n, \tau)$ where τ is not (necessarily) standard. Then $g(x_1, \dots, x_n)$ will be called a *quasi-standard* function. (Note that need not be a continuous function of its arguments.) For example, the function $f(x, n)=\sqrt{n/\pi}\, e^{-nx^2}$ is a standard function of two variables. The function $g(x)=f(f, \omega)=\sqrt{\omega/\pi}\, e^{-\omega x^2}$, where ω is an infinite (non-finite) positive number, is quasi-standard.

Quasi-standard relations, etc. are introduced in a similar way. For example, the interval $\eta<x<b$ mentioned above is quasi-standard.

3. *Examples in non-standard Analysis.*

Let s_n be a standard sequence, i.e. a function defined in the first instance on the natural numbers N and taking values in R_0. Then the definition of s_n extends automatically to the elements of N^* (the non-standard positive integers). Let s be a standard number.. We define –

3.1. s is called the *limit* of s_n iff s_ω is infinitely close to s for all infinitely large positive integers ω. In symbols –

$$(\omega)[N'(\omega) \wedge \sim R_0(\omega) \supset |s-s_\omega| \in M_1].$$

This compares with the classical definition (s is the limit of s_n if for every $\delta>0$ there exists a positive integer ω_0 such that, etc.). It can be shown that the two definitions are equivalent under the stated conditions (s_n and s standard). That is to say $\lim_{n\to\infty} s_n=s$ in R_0 if and only if 3.1 holds in R^*. The proof involves the formalization of the classical definition as a sentence in K_0. Similarly, the following is an equivalent definition of the concept of a limit point (accumulation point) of a sequence for standard s_n and s.

3.2. s is a limit point of s_n iff $s_\omega={}_1s$ for some infinite positive integer ω.

Similarly –

3.3. A standard sequence s_n is bounded if and only if s_n is finite for all infinitely large n.

The theorem of Bolzano–Weierstrass for standard bounded sequences can of course still be proved by classical methods. Using non-standard analysis we obtain an alternative proof along the following lines.

Let $\langle a, b\rangle$ be a closed interval containing all elements of the standard sequence s_n. Then there exists a sentence X of K_0 which states that for every positive integer m, the partition of $\langle a, b\rangle$ into m sub-intervals of equal length $(b-a)/m$ yields at least one sub-interval I that contains an unbounded number of elements of s_n. X holds also in R^* and so, taking m infinite, we obtain a sub-interval I^* of $\langle a, b\rangle$, of infinitesimal length, that contains an unbounded number of elements of s_n. Both end points of I^* are infinitely close to a single standard number s. This is the required limit point.

We have the following version of Cauchy's necessary and sufficient condition for convergence, which may either be proved directly or transcribed from the standard version.

3.4. The standard sequence s_n converges iff $s_\omega =_1 s_{\omega'}$ for all infinitely large ω and ω'.

The theory of infinite series may be reduced to that of infinite sequences in the usual way.

Coming next to functions of a real variable, suppose that the standard function $f(x)$ is defined in a standard open interval $a<x<b$. Then the standard number l is the limit of $f(x)$ as x tends to be from the left iff $f(b-\eta) =_1 l$ for all positive infinitesimal η. $f(x)$ is continuous at the standard point x_0, $a<x_0<b$ iff $f(x_0+\eta) =_1 f(x_0)$ for all infinitesimal η. Again these conditions are equivalent to the classical definitions. Accordingly, $f(x)$ is continuous in (a, b) if $f(x_0+\eta) =_1 f(x_0)$ for all *standard* x_0 in the open interval and for all infinitesimal η. The natural question now arises what non-standard condition corresponds to *uniform* continuity in the interval $a<x<b$. The answer is

3.5. $f(x)$ is uniformly continuous in $(a, b)-a, b$, and $f(x)$ standard— if $f(x_0+\eta) =_1 f(x_0)$ for all infinitesimal η and for *all* x_0 in the open interval (a, b).

In a similar way, we may distinguish between ordinary continuity and uniform continuity of a standard sequence of functions $s_n(x)$.

3.5. $f(x)$ has the derivative f_0 at the standard point x_0 (f_0 a standard number) if for all infinitesimal η

$$\frac{f(x_0+\eta) - f(x_0)}{\eta} =_1 f_0$$

a formula which may be used in practice. Various "standard" results of the Differential calculus, including Rolle's theorem can in fact be established readily by means of non-standard Analysis.

We touch only briefly upon integration and remark that, up to infinitesimal quantities, Cauchy's integral and Riemann's integral can be defined by means of a partition of a given standard interval into an infinite number of subintervals of infinitesimal length combined with the formation of the usual sums such as $\Sigma(x_n - x_{n-1})y_n$. The non-standard definition of the Lebesgue integral appears to be more intricate and has not been carried out in detail so far.

Next, we discuss differential notation in connection with functions of several variables. We do not select a specific element of M_1 as *the* differential but regard any infinitesimal increments as differentials of the independent variables. Thus, the theorem on the existence of the total differential may be stated as follows.

3.6. Let (x_0, y_0) be a standard point (a point with standard numbers as coordinates), and let S be the standard plane set given by

$$(x-x_0)^2+(y-y_0)^2<r^2$$

where r is a standard positive number. Suppose that the standard function $f(x, y)$ possesses continuous first derivatives in S. Let dx, dy be any pair of infinitesimal numbers and let $df=f(x_0+dx, y_0+dy)-f(x_0, y_0)$. Then

3.7. $$df = \frac{\partial f}{\partial x_0}\,dx + \frac{\partial f}{\partial y_0}\,dy \quad \text{mod. } o(dx, dy).$$

Although this formula does not yield ordinary equality between the left and right hand side it can be applied without difficulty, e.g. to the calculation of the derivative of a function defined implicitly by $f(x, y)=0$.

More generally, it is true that much of the classical work in Differential Geometry has been done in terms of a vague notion of infinitesimals, and the same applies to Analytical Mechanics. It is a matter of general belief that all this work could, if necessary, be rewritten to conform to the rigor of contemporary Mathematics but nobody would think of carrying out this task. It is therefore not without interest that we may now justify the use of infinitesimals in all these problems directly. As an example, the case of the osculating plane of a skew curve has been considered in detail. Thus, let C be a standard skew curve in three dimensions (the equations for C are expressed in terms of standard functions) and let P be a standard point on C. Let Π be the set of all planes drawn through P and any two neighboring points P_1, P_2 such that P, P_1, P_2, are not collinear. Then the osculating plane of C at P may be defined as a standard plane p through P such that p is *infinitely near* to all elements of Π in a sense which can be made precise without difficulty. This definition leads to the usual equation for the osculating plane.

Going on to an example which is of greater contemporary interest, let G be a *standard* n-parametric Lie group, n finite. Thus G is defined by analytic functions in R_0, but the passage to R^* extends it automatically to a wider group G^*. Let $(e_1, \ldots, e_n)$ be the set of parameters for the identity in both G and G^*. Then the "infinitely small" transformations in G^* are given by the sets $(e_1+\eta_1, \ldots, e_n+\eta_n)$, where $\eta_1, \ldots, \eta_n$ are infinitesimal. These transformations now constitute a genuine subgroup G' of G^*, which may be analyzed further.

We pass on to the consideration of quasi-standard functions. Let $f(x, t)$ be a standard function defined in a standard set S_1, and let $g(x)=f(x, \omega)$, where ω is non-standard, e.g. infinite or infinitesimal.

Then it can be shown that all functionals or operators which apply to the standard functions $g_t(x)=f(x, t)$, t standard, are extended in a natural and unique way to the function $g(x)$. For example, suppose that the integrals

$$\int_a^b g_t(x)dx = \int_a^b f(x, t)dx = F(t)$$

are defined, in R_0, in some definite sense (e.g. as Riemann integrals) for the range of t under consideration. Then $F(t)$ is a standard function, and we shall regard $F(\omega)$ as the value of the integral $\int_a^b g(x)dx$. It is not difficult to see that if the function $g(x)$ is obtained from different families $f_1(x, t)$, $f_2(x, t)$, so that $g(x)=f_1(x, \omega_1)=f_2(x, \omega_2)$ then the use of either of these families leads to the same value for the integral $\int_a^b g(x)dx$. Moreover, the definition preserves the properties of an integral to the extent to which they can be expressed in the Lower predicate calculus, e.g. approximation by sums of the form $\Sigma(x_n-x_{n-1})y_n$.

The same argument applies to functional operators. Thus, if the derivatives $\dfrac{\partial f(x, t)}{\partial x}=h(x, t)$ exist then we define $\dfrac{dg}{dt}=h(x, \omega)$. It may be mentioned that all these definitions take on a rather more concrete form it we consider non-standard models which are in the form of ultraproducts.

Quasi-standard functions yield a natural realization of generalized functions. Thus, a Dirac delta-function on an interval I may be defined as a quasi-standard function $\delta(x)$ such that for a given standard x_0 in I, $\delta(x)$ is infinitesimal for all standard $x\neq x_0$ in I, and $\int_I \delta(x)dx=1$. For instance, $\sqrt{\omega/\pi}\, e^{-\omega(x-x_0)^2}$, where ω is an infinite natural number, is a delta function for $I=R^*$ and $(1+\cos(x-x_0))^\omega / \int_{-\pi}^{\pi} (1+\cos t)^\omega\, dt$ is a delta function in the interval $(-\pi, \pi)$. For given I and $x_0 \in I$, there are many delta functions, as opposed to the situation in the theory of distributions. Quasi-functions can be added, subtracted, multiplied, and divided by one another provided the divisor does not vanish. It is natural to consider such functions in connection with a concrete application.

For any finite number a we write $b=st\{a\}$ (read 'b is the standard part of a') for the standard number b which is infinitely close to a. Consider now a standard function $\phi=\phi(x, y, z)$ which is harmonic in a region V bounded by a standard surface S. Let $P=(x, y, z)$ be a standard point in the interior of V, so that (Green's formula)

$$\phi(x, y, z) = \frac{1}{4\pi}\int_S \left(\frac{1}{r}\frac{\partial\phi}{\partial n} - \frac{\partial}{\partial n}\left(\frac{1}{r}\right)\phi\right) dS.$$

The formula is usually obtained by applying Green's identity to the pair of functions ϕ and $\psi = 1/r$. The singularity of ψ at P is taken into account by a familiar procedure.

Instead of taking $\psi = 1/r$ we may introduce the potential ψ_ϱ of a homogeneous sphere of infinitesimal radius ϱ round P. In this case there is no singularity at P and after first checking that Green's identity applies, we obtain the formula

$$\phi(x, y, z) = \frac{1}{4\pi}\, st\left\{ \int_S \left(\psi_\varrho \frac{\partial \phi}{\partial n} - \frac{\partial \psi_\varrho}{\partial n} \phi \right) dS \right\}.$$

An interesting result is obtained if we apply similar considerations to Volterra's formula for the solution of the two-dimensional wave equation.

Coming next to classical Applied Mathematics, it would of course be natural to reword the usual statements about particles of fluid and about infinitesimal surfaces and volumes in terms of the present theory. However, we pass over this possibility and consider instead a particular point in Fluid mechanics that gives rise to certain conceptual difficulties.

It is the assumption of boundary layer theory, e.g. for flow round a body or through a pipe, that the equations of inviscid flow are valid everywhere except in a narrow layer along the wall. It is found that the thickness of the layer, δ, is proportional to $R^{-1/2}$ where R is the Reynolds number, supposed large. Within this boundary layer, the flow is determined by means of the boundary layer equations which are obtained by simplifying the Navier–Stokes equations of viscous flow. However, when solving these equations, it proves natural to suppose that the boundary layer is *infinitely thick*. For example, for the case of a straight wall along the x-axis, the boundary layer equations are solved for boundary conditions at $y = 0$ and $y \to \infty$ a procedure which is clearly incompatible with the previous assumption on the smallness of δ. This conceptual difficulty can be resolved by supposing that the inviscid fluid equations hold for all positive standard values $y > 0$ while the influence of viscosity is confined to values of y that belong to $O(\delta)$, δ infinitesimal (so that the Reynolds number R is infinite). Introducing $y' = y/\delta$ we may then derive and solve the boundary layer equations for $0 < y' < \infty$ which is a region in which the flow has not been defined previously. There are other problems in continuous media mechanics that should be amenable to a similar analysis.

In reality it is of course not true that the region in which viscosity is effective may be regarded as infinitely thin. It can in fact be seen with the naked eye both in certain laboratory experiments and in every day life. Thus, the above model is intended only as a conceptually clear picture within which it should be easier to discuss some of the more intricate theoretical questions of the subject such as the conditions near the edge of the layer.

For phenomena on a different scale, such as are considered in Modern

Physics, the dimensions of a particular body or process may not be observable directly. Accordingly. the question whether or not a scale of non-standard analysis is appropriate to the physical world really amounts to asking whether or not such a system provides a better explanation of certain observable phenomena than the standard system of real numbers. The possibility that this is the case should be borne in mind.

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The Recursively Enumerable Degrees are Dense*

By GERALD E. SACKS

Our main result is: if **b** and **c** are recursively enumerable degrees such that $\mathbf{b} < \mathbf{c}$, then there exists a recursively enumerable degree **d** such that $\mathbf{b} < \mathbf{d} < \mathbf{c}$. By degree we mean degree of recursive unsolvability as defined by Kleene and Post [2]. A degree is called recursively enumerable if it is the degree of a recursively enumerable set. The upper semi-lattice of recursively enumerable degrees has a least member, **0**, the degree of all recursive sets, and a greatest member, **0**′, the degree of all complete sets. Post [4] asked: does there exist a recursively enumerable degree **d** such that $\mathbf{0} < \mathbf{d} < \mathbf{0}'$? Friedberg [1] and Muchnik [3] answered Post's question in the affirmative. Sacks [5] showed that every countable, partially ordered set is imbeddable in the upper semi-lattice of recursively enumerable degrees. Muchnik [3] announced that if **c** is a non-zero, recursively enumerable degree, then there exists a recursively enumerable degree **d** such that $\mathbf{0} < \mathbf{d} < \mathbf{c}$.

The arguments of Friedberg [1], Muchnik [3] and Sacks [5] have a great deal in common. All three authors make use of a method which may be roughly described as follows (a precise abstract description is given in Sacks [5]): One or more sets are to be recursively enumerated. Certain requirements are to be met. A typical requirement is: the set B is not recursive in the set A with Gödel number e. Unfortunately, the requirements tend to conflict. Thus we may wish to add n to B to insure that $B \neq \{e\}^A$, but the addition of n to B may ruin what we did earlier in the enumeration to insure $A \neq \{f\}^B$. We resolve all conflicts by appealing to a system of priorities assigned to the requirements before the enumeration begins. Then it can be shown that each requirement is "ruined" only finitely often, and that, consequently, each requirement is eventually met.

In the argument below we assign priorities to requirements, but we are forced to permit each requirement to be "ruined" infinitely often. Nonetheless, we are still able to meet each requirement. Thus there is an important combinatorial difference between the method of Friedberg [1] and Muchnik [3] and the method we use below. The difference is a consequence of the fact that Friedberg and Muchnik deal only with finite, initial segments of functions while we are forced to deal indirectly with entire functions. Almost all

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of the complications we encounter below arise from the non-recursiveness of B. We are given a non-recursive, recursively enumerable set B, and we wish to find a recursively enumerable set D such that B is recursive in D but D is not recursive in B. Since B is to be recursive in D, we are forced to add many members to D in complete disregard of all priorities. Thus even the requirement of highest priority may be "ruined" infinitely often. If B were recursive, we could put all of B in D in one step, and then the arguments of [1] and [3] would suffice.

The basic combinatorial principle of this paper appears in primitive form in Sacks [6]. Our basic source of notation is [2]. We conclude with intuitive remarks and an open problem.

THEOREM. *If* **b** *and* **c** *are recursively enumerable degrees such that* $\mathbf{b} < \mathbf{c}$, *then there exists a recursively enumerable degree* **d** *such that* $\mathbf{b} < \mathbf{d} < \mathbf{c}$.

PROOF. Let f, g respectively, be a one-one recursive function whose range is a set B, C respectively, of degree **b**, **c** respectively. We define:

$$b(s, n) = \begin{cases} 0 & \text{if } (Et)_{t<s}(f(t) = n) \\ 1 & \text{otherwise ;} \end{cases}$$

$$c(s, n) = \begin{cases} 2 & \text{if } (Et)_{t<s}(g(t) = n) \\ 1 & \text{otherwise .} \end{cases}$$

Then $\lim_s b(s, n)$ exists for each n; in addition, $\lim_s b(s, n)$ is the representing function of B. Similarly, $2 - \lim_s c(s, n)$ is the representing function of C. Let $k(s)$ be a recursive function such that, for each s, $s < k(s) < k(s + 1)$. We define three recursive functions:

$$y_b(s, n, e) = \begin{cases} \mu y T_1^1(\prod_{j<y} p_j^{b(s,j)}, e, n, y) & \text{if } (Ey)_{y<s} T_1^1(\prod_{j<y} p_j^{b(s,j)}, e, n, y) \\ k(s) & \text{otherwise ;} \end{cases}$$

$$h(0, n, e) = 0 ;$$

$$h(s + 1, n, e) = h(s, n, e) + sg(|\, y_b(s + 1, n, e) - y_b(s, n, e) \,|) ;$$

$$t(s, i, e) = \sum_{j \leq i} h(s, j, e) .$$

We define four recursive functions, $y(s, n, e)$, $m(s, e)$, $r(s, n, e)$ and $d(s, n)$, simultaneously by induction on s. The desired degree **d** will be the degree of the predicate $(Es)(d(s, n) = 0)$.

Stage $s = 0$. We set $y(0, n, e) = m(0, e) = 0$ and $r(0, n, e) = d(0, n) = 1$ for all n and e, except that $d(0, p(f(0), 1)) = 0$.

Stage $s > 0$. We set

$$y(s, n, e) = \begin{cases} \mu y T_1^1(\prod_{j<y} p_j^{d(s-1,j)}, e, n, y) & \text{if } (Ey)_{y<s} T_1^1(\prod_{j<y} p_j^{d(s-1,j)}, e, n, y) \\ 0 & \text{otherwise .} \end{cases}$$

The definition of $m(s, e)$ has two cases:

Case m1. There exists an $n < m(s-1, e)$ such that

$$c(s, n) \neq \mathrm{U}(y(s, n, e)) \;\&\; y(s, n, e) \neq y(s-1, n, e)\,.$$

We set $m(s, e)$ equal to the least such n.

Case m2. Otherwise.

We set $m(s, e)$ equal to the least n such that

$$\begin{aligned} & m(s-1, e) \leqq n < 2m(s-1, e) + s \\ & \& \; (Et)_{t \leqq n}(c(s, t)) \neq \mathrm{U}(y(s, t, e))\,. \end{aligned}$$

If no such n exists, then $m(s, e) = m(s-1, e) + s$; this last is in accord with the definition of the bounded, least number operator.

Let $p(i, m) = p_i^{2+m}$ and $p(e, i, m) = p(e, p(i, m))$. We define $r(s, n, e)$ and $d(s, p(e, i, m))$ for all e, n, i and m by a simultaneous induction on e. First we set $d(s, p(f(s), 1)) = 0$.

$$r(s, n, e) = \begin{cases} 0 & \text{if } (Ei)_{i<e}(Et)_{t \leqq n}(Em)[p(i, m) < y(s, t, e) \\ & \quad \& \; d(s, p(i, m)) \neq d(s-1, p(i, m))]\,, \\ 0 & \text{if } (Et)_{t \leqq n}[p(f(s), 1) < y(s, t, e)]\,, \\ 1 & \text{otherwise}\,. \end{cases}$$

The definition of $d(s, p(e, i, m))$ has three cases.

Case d1. $t(s, i, e) > m$. We set

$$d(s, p(e, i, m)) = \begin{cases} d(s-1, p(e, i, m)) & \text{if } (Eu)_{u \leqq e}(Ev)_{v < m(s,u)}[r(s, v, u) = 1 \\ & \qquad \& \; p(e, i, m) < y(s, v, u)]\,, \\ 0 & \text{otherwise}\,. \end{cases}$$

Case d2. $t(s, i, e) = m$. We set

$$d(s, p(e, i, m)) = \begin{cases} 0 & \text{if } c(s, i) = 2 \\ & \quad \& \; (j)_{j<i}[d(s-1, j) = \mathrm{U}(y_b(s, j, e))] \\ & \quad \& \sim (Eu)_{u \leqq e}(Ev)_{v < m(s,u)}[r(s, v, u) = 1 \\ & \qquad \& \; p(e, i, m) < y(s, v, u)]\,, \\ d(s-1, p(e, i, m)) & \text{otherwise}\,. \end{cases}$$

Case d3. $t(s, i, e) < m$. We set $d(s, p(e, i, m)) = d(s-1, p(e, i, m))$.

The construction is concluded by setting $d(s, w) = d(s-1, w)$ for all w not equal to $p(f(s), 1)$ or $p(e, i, m)$ for some e, i and m. Let $d(n) = \lim_s d(s, n)$ for each n. Let D be the set whose representing function is d; D is recursively enumerable since $n \in D$ if and only if $(Es)(d(s, n) = 0)$. We list two notes which will be needed later:

(N1) $(s)(n)(e)(r(s, n, e) = 0 \rightarrow r(s, n+1, e) = 0)$;

(N2) $(s)(n)(e)(y(s, n, e) = 0 \rightarrow n \geqq m(s, e))$.

We prove (N2) by induction on s. Certainly,

$$(n)(e)(y(0, n, e) = 0 \rightarrow n \geqq m(0, e)) .$$

Fix $s > 0$, n and e. Suppose $y(s, n, e) = 0$, and

$$(n)(e)(y(s-1, n, e) = 0 \rightarrow n \geqq m(s-1, e)) .$$

Suppose *Case* m1 of the definition of $m(s, e)$ holds. If $y(s-1, n, e) = 0$, then $n \geqq m(s-1, e) > m(s, e)$. Suppose $y(s-1, n, e) > 0$. Then

$$y(s, n, e) \neq y(s-1, n, e)$$
$$\& \ 0 = \mathrm{U}(y(s, n, e)) \neq c(s, n) > 0 ;$$

Consequently, $n \geqq m(s, e)$. Now suppose *Case* m2 of the definition of $m(s, e)$ holds. Once again, $\mathrm{U}(y(s, n, e)) \neq c(s, n)$; it follows $n \geqq m(s, e)$.

LEMMA 1. *Let $y(s, n, e) > 0$, $m(s, e) > n$ and $p(f(s), 1) \geqq y(s, t, e)$ for all $t \leqq n$. Let $d(s, p(i, m)) = d(s-1, p(i, m))$ for all i, t and m such that $i < e$, $t \leqq n$ and $p(i, m) < y(s, t, e)$. Then $y(s, n, e) = y(s+1, n, e)$ and $m(s+1, e) > n$.*

PROOF. Since $y(s, n, e) > 0$, we have

$$y(s, n, e) = \mu y T_1^1(\textstyle\prod_{j<y} p_j^{d(s-1, j)}, e, n, y) .$$

If $d(s, j) = d(s-1, j)$ for all $j < y(s, n, e)$, then $y(s+1, n, e) = y(s, n, e)$. The hypothesis of our lemma tells us that $r(s, n, e) = 1$. But then it follows from the definition of $d(s, j)$ that $d(s, j) = d(s-1, j)$ for all $j < y(s, n, e)$, since $m(s, e) > n$.

Note (N1) makes clear that the above argument also works for any $t < n$. Thus we have

$$(t)_{t \leqq n}(y(s, t, e) = y(s+1, t, e)) .$$

Suppose $m(s+1, e) \leqq n$. Then $m(s+1, e) < m(s, e)$, and *Case* m1 of the definition of $m(s+1, e)$ holds. But then we have a t (namely, $m(s+1, e)$) such that $t \leqq n$, and $y(s, t, e) \neq y(s+1, t, e)$.

For each e, we say e is stable if $\lim_s y(s, n, e)$ exists and is positive for all n. Since there are infinitely many e which are not Gödel numbers of systems of equations, there are infinitely many non-stable e. Let $\{e_0 < e_1 < e_2 < \cdots\}$ be the set of all non-stable e. For each j, let n_j be the least n such that $\lim_s y(s, n, e_j)$ either does not exist or is zero. Lemma 2 is our basic combinatorial principle (cf. [6, Lemma 4]).

LEMMA 2. *For each k and v, there is an $s \geqq v$ such that*

$$(j)_{j<k}[m(s, e_j) \leqq n_j \vee r(s, n_j, e_j) = 0 \vee y(s, n_j, e_j) = 0] .$$

PROOF. Fix k and v. We suppose there is no s with the desired properties and then define an infinite descending sequence of natural numbers.

We propose the following system of equations as a means of defining two functions, $S(t)$ and $M(t)$, simultaneously by induction:

$$S(0) = v\,;$$
$$M(t) = \mu j_{j<k}[n_j < m(S(t), e_j) \;\&\; r(S(t), n_j, e_j) = 1 \;\&\; y(S(t), n_j, e_j) > 0]\,;$$
$$S(t+1) = \mu s(Em)[s \geqq S(t) \;\&\; m < y(S(t), n_{M(t)}, e_{M(t)}) \;\&\; d(s, m) \neq d(S(t)-1, m)]\,.$$

Clearly $S(0) \geqq v$. Suppose $t \geqq 0$, $S(t)$ is well-defined and $S(t) \geqq v$. Then $M(t) < k$, since we have supposed the lemma to be false. Thus

$$y(S(t), n_{M(t)}, e_{M(t)}) > 0\,,$$

and $\lim_s y(s, n_{M(t)}, e_{M(t)})$ does not exist or does equal 0. Then there must be an $s > S(t)$ such that

$$m < y(S(t), n_{M(t)}, e_{M(t)}) \;\&\; d(s-1, m) \neq d(S(t)-1, m)\,.$$

But then $S(t+1)$ is well-defined and $S(t+1) \geqq v$.

For each $t \geqq 0$, let

$$u(t+1) = \mu m[d(S(t+1), m) \neq d(S(t)-1, m)]\,.$$

Fix $t > 0$. We show $u(t+1) < u(t)$. Since we know

$$u(t+1) < y(S(t), n_{M(t)}, e_{M(t)})\,,$$

it suffices to show

$$y(S(t), n_{M(t)}, e_{M(t)}) \leqq u(t)\,.$$

It follows from the definition of S that

$$d(w, m) = d(S(t-1)-1, m)$$

whenever $S(t) > w \geqq S(t-1)$ and $m < y(S(t-1), n_{M(t-1)}, e_{M(t-1)})$. Consequently,

$$d(S(t), u(t)) \neq d(S(t)-1, u(t))\,.$$

First suppose $u(t) = p(f(S(t)), 1)$. Then $y(S(t), n_{M(t)}, e_{M(t)}) \leqq u(t)$, since $r(S(t), n_{M(t)}, e_{M(t)}) = 1$. Now suppose $u(t) = p(i, f, m)$ for some i, f and m. Let $s = S(t)$, $n = n_{M(t)}$ and $e = e_{M(t)}$. If $i < e$, then the definition of r tells us that $y(s, n, e) \leqq u(t)$, since $r(s, n, e) = 1$. If $i \geqq e$, then the definition of d tells us that $y(s, n, e) \leqq u(t)$, since $n < m(s, e)$, $r(s, n, e) = 1$ and $d(s, u(t)) \neq d(s-1, u(t))$.

We introduce two predicates:

$A(e)$: if e is stable, then the set $\{m(s, e) \mid s \geqq 0\}$ is finite.

$B(e)$: the set $\{m \mid d(p_e^m) = 0\}$ is recursive in B.

We will prove $(e)A(e)$ and $(e)B(e)$ by means of a simultaneous induction on e. From $(e)A(e)$ it will follow that $\mathbf{c} \not\leqq \mathbf{d}$. From $(e)B(e)$ it will follow that $\mathbf{d} \not\leqq \mathbf{b}$.

LEMMA 3. $(i)_{i<e}B(i) \rightarrow A(e)$.

PROOF. We know $\mathbf{c} \not\leqq \mathbf{b}$. We suppose that $B(i)$ is true for each $i < e$ and that $A(e)$ is false, and we show $\mathbf{c} \leqq \mathbf{b}$. Thus the set $\{m(s, e) \mid s \geqq 0\}$ is infinite, and for each n, $\lim_s y(s, n, e)$ exists and is positive. Let $R(n, s)$ denote the predicate

$$\begin{aligned} & m(s, e) > n \\ & \& \ (i)(m)(t)[p(i, m) < y(s, t, e) \ \& \ i < e \ \& \ t \leqq n \\ & \qquad\qquad \rightarrow d(s-1, p(i, m)) = d(p(i, m))] \\ & \& \ (z)(t)[z \in B \ \& \ p(z, 1) < y(s, t, e) \ \& \ t \leqq n \\ & \qquad\qquad \rightarrow d(s-1, p(z, 1)) = 0] \, . \end{aligned}$$

(Recall that $z \in B$ if and only if $d(p(z, 1)) = 0$.) Since $B(i)$ is true for all $i < e$, it follows that R is recursive in B. Since the set $\{m(s, e) \mid s \geqq 0\}$ is infinite, and since $\lim_s y(s, n, e)$ exists and is positive for each n, it follows that $(n)(Es)R(n, s)$. Let

$$w(n) = \mu s R(n, s) \, .$$

Then w is recursive in B, and for each n, $w(n+1) \geqq w(n)$.

Fix n. We show $\lim_s y(s, n, e) = y(w(n), n, e)$. Let s be such that $s \geqq w(n)$ and

$$y(w(n), n, e) = y(s, n, e) \ \& \ R(n, s) \, .$$

Since $m(s, e) > n$, it follows from *note* (N2) that $y(s, t, e) > 0$ for all $t \leqq n$. We know f enumerates B without repetitions. It follows from the definition of R that $p(f(s), 1) \geqq y(s, t, e)$ for all $t \leqq n$. But then Lemma 1 tells us that

$$y(s+1, t, e) = y(s, t, e)$$

for all $t \leqq n$, and that $m(s+1, e) > n$. Thus

$$y(w(n), n, e) = y(s+1, n, e) \quad \text{and} \quad R(n, s+1) \, .$$

It follows that $\lim_s y(s, n, e) = y(w(n), n, e)$.

Finally, we show

$$\lim_s c(s, n) = \mathrm{U}(y(w(n), n, e))$$

for all n. If this last is true, then C is recursive in B, since $2 - \lim_s c(s, n)$ is the representing function of C; and since w is recursive in B. Fix n and suppose $\lim_s c(s, n) \neq \text{U}(y(w(n), n, e))$. There must be an s^* such that for all $s \geqq s^*$,

$$c(s, n) = \lim_s c(s, n) \neq \text{U}\big(y(w(n), n, e)\big) = \text{U}\big(y(s, n, e)\big) .$$

Fix $s > s^*$ and suppose $m(s-1, e) \leqq m(s^*, e) + n$. If *Case* m1 of the definition of $m(s, e)$ holds, then $m(s, e) \leqq m(s^*, e) + n$. If *Case* m2 holds and $n < 2m(s-1, e) + s$, then $m(s, e) \leqq m(s^*, e) + n$, since $c(s, n) \neq \text{U}(y(s, n, e))$. If $n \geqq 2m(s-1, e) + s$, then $m(s, e) \leqq n$. Thus

$$m(s, e) \leqq m(s^*, e) + n$$

for all $s \geqq s^*$. This last is impossible, since the set $\{m(s, e) \mid s \geqq 0\}$ is infinite.

Suppose $\{e\}^b(n)$ is defined for all n. Then $\lim_s y_b(s, n, e)$ exists for each n, and $\lim_s t(s, i, e)$ exists for each i. In addition, $\lim_s t(s, i, e)$ (regarded as a function of i) is recursive in B. All this is clear from the definition of y_b and t.

LEMMA 4. *If* $\{e\}^b(n)$ *is defined for all* n *and* $(u)_{u \leqq e} A(u)$, *then* $d(p(e, i, \lim_s t(s, i, e))) = 0$ *for only finitely many* i.

PROOF. We suppose the lemma is false and show C is recursive in B. Our first claim is:

$$d = \{e\}^b .$$

Fix j. Let s be so large that

$$d(j) = d(w-1, j) \;\&\; \text{U}\big(y_b(w, j, e)\big) = \{e\}^b(j)$$

for all $w \geqq s$. Let $t(i) = \lim_s t(s, i, e)$ for all i. Let w and i be such that $j < i$, $w \geqq s$, $c(w, i) = 2$,

$$0 = d\big(w, p\big(e, i, t(i)\big)\big) \neq d\big(w-1, p\big(e, i, t(i)\big)\big) ,$$

and $t(i) = t(w, i, e)$. It follows from *Case* d2 of the definition of d that

$$d(w-1, j) = \text{U}\big(y_b(s, j, e)\big) ,$$

since $j < i$. But then $d(j) = \{e\}^b(j)$, since $w \geqq s$, and our first claim is proved.

Our second claim is: for all sufficiently large i,

$$d\big(p\big(e, i, t(i)\big)\big) = \lim_s c(s, i) .$$

Our first and second claims, together with the fact that t is recursive in B, imply that C is recursive in B. Our second claim is a consequence of Lemma 2 and $(u)_{u \leqq e} A(u)$. If $u \leqq e$ and u is stable, then $A(u)$ tells us that the set $\{m(s, u) \mid s \geqq 0\}$ is finite. If $u \leqq e$ and u is stable, let $m(u)$ be the greatest member of $\{m(s, u) \mid s \geqq 0\}$. If $u \leqq e$ and u is non-stable, then $u = e_i$ for some i; let $m(u) = n_i$. Recall that n_i is the least witness to the fact that e_i is non-stable. Thus $\lim_s y(s, v, u)$ exists if $u \leqq e$ and $v < m(u)$. Let y be so large that $y(s, v, u) \leqq y$ if $s \geqq 0$, $u \leqq e$ and $v < m(u)$. Fix $i \geqq y$. We show

$d(p(e, i, t(i))) = \lim_s c(s, i)$. If $\lim_s c(s, i) = 1$, then $c(s, i) = 1$ for all s, and $d(s, p(e, i, t(i))) = 1$ for all s. Suppose $\lim_s c(s, i) = 2$. Let w be so large that $c(s, i) = 2$, $t(s, i, e) = t(i)$, and

$$(j)_{j<i}[d(s-1, j) = \text{U}(y_b(s, j, e))]$$

for all $s \geqq w$. The existence of w follows from our first claim. By Lemma 2, there is an $s \geqq w$ such that for all $u \leqq e$, if u is non-stable (hence equal to e_i for some i), then

$$m(s, u) \leqq n_i \vee r(s, n_i, u) = 0 \vee y(s, n_i, u) = 0 .$$

We show $d(s, p(e, i, t(i))) = 0$. *Case* d2 is such that we need only show

$$\sim(Eu)_{u \leqq e}(Ev)_{v<m(s,u)}[r(s, v, u) = 1 \;\&\; p(e, i, t(i)) < y(s, v, u)] .$$

Fix $u \leqq e$ and $v < m(s, u)$. Suppose u is stable. Then $y(s, v, u) \leqq y \leqq i \leqq p(e, i, t(i))$. Suppose u is non-stable. Let $u = e_i$. If $v < n_i$, then $v < m(u)$ and $y(s, v, u) \leqq y \leqq p(e, i, t(i))$. Suppose $v \geqq n_i$. Then $m(s, u) > n_i$, and consequently,

$$r(s, n_i, u) = 0 \vee y(s, n_i, u) = 0 .$$

If $r(s, n_i, u) = 0$, then $r(s, v, u) = 0$ by *Note* (N1), since $n_i \leqq v$. Suppose $y(s, n_i, u) = 0$. Then $n_i \geqq m(s, u)$ by *Note* (N2). But $m(s, u) > n_i$ since we have supposed $v \geqq n_i$.

LEMMA 5. $(u)_{u \leqq e}A(u) \rightarrow B(e)$.

PROOF. Our first claim is (L).

(L) $\quad (Ey)(m)(x)(i)[m < t(x, i, e) \;\&\; p(e, i, m) \geqq y \rightarrow d(p(e, i, m)) = 0] .$

The proof of our first claim is virtually identical with the proof of the second claim of Lemma 4. For each $u \leqq e$, we define $m(u)$ as in Lemma 4. Then $\lim_s y(s, v, e)$ exists if $u \leqq e$ and $v < m(u)$. Define y as in Lemma 4. Fix m, x and i so that $m < t(x, i, e)$ and $p(e, i, m) \geqq y$. We show $d(p(e, i, m)) = 0$. By Lemma 2 there is an $s \geqq x$ such that for all $u \leqq e$, if u is non-stable (hence equal to e_i for some i), then

$$m(s, e_i) \leqq n_i \vee r(s, n_i, e_i) = 0 \vee y(s, n_i, e_i) = 0 .$$

We show $d(s, p(e, i, m)) = 0$. Since $s \geqq x$, we have

$$m < t(x, i, e) \leqq t(s, i, e) .$$

Thus the definition of $d(s, p(e, i, m))$ is given by *Case* d1. We need only show

$$\sim(Eu)_{u \leqq e}(Ev)_{v<m(s,u)}[r(s, v, u) = 1 \;\&\; p(e, i, m) < y(s, v, u)] .$$

Fix $u \leqq e$ and $v < m(s, u)$. If u is stable, then $y(s, v, u) \leqq y \leqq p(e, i, m)$. Suppose u is non-stable. Then we repeat the argument at the end of Lemma 4 to show $r(s, v, u) = 0$ or $p(e, i, m) \geqq y(s, v, u)$.

We are ready to prove $B(e)$. We will use only Lemma 4 and (L). First we suppose that $\{e\}^b(n)$ is undefined for some n; let N be the least such n. It follows from the definition of y_b that $\lim_s t(s, i, e) = \infty$ for all $i \geqq N$. Let y have the property described by (L). Then

$$(i)(m)[i \geqq N \ \& \ p(e, i, m) \geqq y \rightarrow d(p(e, i, m)) = 0] .$$

If $i < N$, then $\lim_s t(s, i, e)$ is finite; consequently, the set

$$\{p(e, i, m) \mid d(p(e, i, m)) = 0 \ \& \ i < N\}$$

is finite by *Case* d3 of the definition of d. Now $d(p_e^n) = 0$ only if $p_e^n = p(e, i, m)$ for some i and m or $p_e^n = p(f(s), 1)$. It follows that the set $\{n \mid d(p_e^n) = 0\}$ is recursive, hence recursive in B.

Now suppose $\{e\}^b(n)$ is defined for all n. Then $\lim_s t(s, i, e)$ is finite for all i. Let $\lim_s t(s, i, e) = t(i)$. Recall that $t(s + 1, i, e) \geqq t(s, i, e)$ for all s and i. It follows from (L) that

$$(i)(m)[m < t(i) \ \& \ p(e, i, m) \geqq y \rightarrow d(p(e, i, m)) = 0] .$$

By Lemma 4, the set

$$\{p(e, i, t(i)) \mid d(p(e, i, t(i))) = 0\}$$

is finite. By *Case* d3, $d(p(e, i, m)) = 1$ if $m > t(i)$. Since $t(i)$ is recursive in B, it follows that the set $\{n \mid d(p_e^n) = 0\}$ is recursive in B.

Lemmas 3 and 5 constitute a proof of $(e)A(e)$ and $(e)B(e)$.

LEMMA 6. *C is not recursive in D.*

PROOF. Suppose $\lim_s c(s, n) = \{e\}^a(n)$ for all n. Then e is stable, and by $A(e)$, the set $\{m(s, e) \mid s \geqq 0\}$ has a greatest member; call it M. Let w be so large that

$$\lim_s c(s, n) = c(s, n) = \mathrm{U}(y(s, n, e))$$

when $s \geqq w$ and $n \leqq M$. If $s \geqq w$, then either

(a) $c(s, n) \neq \mathrm{U}(y(s, n, e))$ for some $n \leqq m(s, e)$, or

(b) $m(s, e) = m(s - 1, e) + s$.

If (a) holds, then $M < m(s, e)$. Thus (b) must hold for infinitely many s. But then $M < m(s, e)$ for some s.

LEMMA 7. *D is not recursive in B.*

PROOF. Suppose $d(n) = \{e\}^b(n)$ for all n, where $b(n) = \lim_s b(s, n)$ for all n. Then $\lim_s t(s, i, e)$ is finite for each i. Let $t(i) = \lim_s t(s, i, e)$. By Lemma 4, there is an n such that $d(p(e, i, t(i))) = 1$ for each $i \geqq n$. We shall find an i such that $i \geqq n$ and $d(p(e, i, t(i))) = 0$. We proceed as in Lemma 4. Define y

as in Lemma 4. Fix $i \geqq y + n$. Define w and s as in Lemma 4. Then the final argument of Lemma 4 tells us that $d(s, p(e, i, t(i))) = 0$.

Let **d** be the degree of D. It follows from Lemmas 6 and 7 that $\mathbf{c} \nleqq \mathbf{d}$ and $\mathbf{d} \nleqq \mathbf{b}$. We have $\mathbf{b} \leqq \mathbf{d}$, since $d(p(m, 1)) = 0$ if and only if $m \in B$. It remains only to show that D is recursive in C. We give an intuitive description of a procedure E for computing D from C. Our description is such that the translation of E into a system of equations which define D recursively in C is not difficult. We need a recursive predicate $Q(u, v, s, e, i, m)$:

$$u \leqq e \;\&\; v < m(s, u) \;\&\; r(s, v, u) = 1 \;\&\; p(e, i, m) < y(s, v, u) .$$

We also need a predicate $R(s, e, n)$:

$$\begin{aligned} &(u)(i)(m)(t)[u < e \;\&\; t \leqq n \;\&\; p(u, i, m) < y(s, t, e) \\ &\qquad \rightarrow d(s-1, p(u, i, m)) = d(p(u, i, m))] \\ &\& \;(i)(t)[i \in B \;\&\; t \leqq n \;\&\; p(i, 1) < y(s, t, e) \\ &\qquad \rightarrow d(s-1, p(i, 1)) = d(p(i, 1))] . \end{aligned}$$

We need the next two lemmas to describe E.

LEMMA 8. $Q(u, v, s, e, i, m) \;\&\; R(s, u, v) \rightarrow (w)_{w \geqq s} Q(u, v, w, e, i, m)$.

PROOF. We proceed with an induction on $w \geqq s$. Fix $w \geqq s$ and suppose $Q(u, v, w, e, i, m)$ and $R(w, u, v)$ hold. Recall that f, the recursive function whose range is B, is one-one. It follows from Lemma 1 that $v < m(w+1, u)$ and that

$$y(w, t, u) = y(w+1, t, u)$$

for all $t \leqq v$. But then $R(w+1, u, v)$ holds and $r(w+1, v, u) = 1$, and consequently, $Q(u, v, w+1, e, i, m)$.

LEMMA 9. $(w)(Es)_{s \geqq w}(u)(v)[Q(u, v, s, e, i, m) \rightarrow R(s, u, v)]$.

PROOF. Fix w, e, i and m. For each $u \leqq e$, define $m(u)$ as in the second half of the proof of Lemma 4. Define y as in Lemma 4. Thus $y(s, v, u) \leqq y$ if $s \geqq 0$, $u \leqq e$ and $v < m(u)$. It is safe to suppose that w is so large that

$$(s)(v)(s > w \;\&\; v \leqq y \rightarrow d(s-1, v) = d(v)) .$$

By Lemma 1 there is an $s > w$ such that for any non-stable $u \leqq e$, we have

$$m(s, u) \leqq n_j \vee r(s, n_j, u) = 0 \vee y(s, n_j, u) = 0 ,$$

where $u = e_j$; recall that if u is non-stable, then $m(u) = n_j$.

Fix u and v and suppose $Q(u, v, s, e, i, m)$ holds. We show $R(s, u, v)$. First we suppose u is stable. Then $v < m(s, u) \leqq m(u)$, since $Q(u, v, s, e, i, m)$ holds; consequently,

$$(n)(t)[t \leqq v \;\&\; n < y(s, t, u) \rightarrow d(s-1, n) = d(n)] ,$$

since $s > w$. But then $R(s, u, v)$ holds. Now we suppose u is non-stable. Let $u = e_j$. Then $m(u) = n(j)$. If $m(s, u) \leqq m(u)$, then $v < m(u)$, and as above, $R(s, u, v)$ holds. Thus

$$r(s, m(u), u) = 0 \vee y(s, m(u), u) = 0 .$$

If $r(s, m(u), u) = 0$, then $v < m(u)$, since $r(s, v, u) = 1$; this last follows from (N1). Suppose $y(s, m(u), u) = 0$. Then it follows from (N2) that $m(u) \geqq m(s, u)$. But then $v < m(u)$.

We return to the description of procedure E. Fix z. We compute $d(z)$ with the help of C. If z is not of the form $p(b, 1)$ or $p(e, i, m)$, then $d(z) = 1$. If $z = p(b, 1)$, then $d(z) = 0$ if and only if $b \in B$. Suppose $z = p(e, i, m)$. The predicate $(Es)(t(s, i, e) = j)$ is recursive in B, since

$$t(s, i, e) \leqq t(s + 1, i, e)$$

for all s. We consider three cases:

(a) $(s)(t(s, i, e) < m)$;

(b) $(Es)(t(s, i, e) > m)$;

(c) $\lim_s t(s, i, e) = m$.

With the help of B, we can decide which one of (a), (b) and (c) holds; note that the monotonicity of $t(s, i, e)$, regarded as a function of s, is vital. If (a) holds, then *Case* d3 tells us that $d(z) = 1$. Suppose (b) holds. Let w^* be such that $t(s, i, e) > m$ for all $s \geqq w^*$. We can determine w^* with the help of B. Suppose $d(s, z) = 1$ for all $s < w^*$. Then $d(z) = 0$ if and only if

$$(Es)(u)(v)[s \geqq w^* \ \& \ \sim Q(u, v, s, e, i, m)] ,$$

since *Case* d1 applies when $s \geqq w^*$. By Lemma 9, there is an $s \geqq w^*$ such that

$$(u)(v)[Q(u, v, s, e, i, m) \rightarrow R(s, u, v)] .$$

We can find s if we know the value of $d(x)$ for some of the following x: $x = p(u, j, k)$, where $u < e$; $x = p(j, 1)$, where $j \in B$. (Recall that $d(p(j, 1)) = 0$ if and only if $j \in B$.) If $\sim Q(u, v, s, e, i, m)$ holds for all u and v, then $d(z) = 0$; we only have to check $u \leqq e$ and $v < m(s, u)$. Suppose for some u and v, $Q(u, v, s, e, i, m)$ holds. Then we have $R(s, u, v)$ by definition of s. By Lemma 8, it follows that

$$(w)_{w \geqq s} Q(u, v, w, e, i, m) .$$

But then $d(z) = 0$ if and only if $d(w, z) = 0$ for some $w < s$.

Finally, we suppose (c) holds. Let w' be such that $t(s, i, e) = m$ for all $s \geqq w'$. We can determine w' with the help of B. Suppose $d(s, z) = 1$ for all $s < w'$. Then $d(z) = 0$ if and only if

$$(Es)[s \geqq w' \;\&\; c(s, i) = 2 \;\&\; (u)(v) \sim Q(u, v, s, e, i, m) \;\&\; (j)_{j<i}(d(s-1, j) = U(y_b(s, j, e)))] ,$$

since *Case* d2 applies when $s \geqq w'$. If $j < i$, then

$$y_b(s, j, e) = \lim_s y_b(s, j, e)$$

for all $s \geqq w'$, since $t(w', i, e) = \lim_s t(s, i, e) = m$. Let $v^* \geqq w'$ be such that for all $s \geqq v^*$,

$$c(s, i) = \lim_s c(s, i) \;\&\; (j)_{j<i}[d(s-1, j) = d(j)] .$$

We can determine v^* with the help of C and the values of $d(x)$ for $x < z$, since $j < i$ implies that $j < p(e, i, m)$. If $\lim_s c(s, i) = 1$ or $d(j) \neq \text{U}(\lim_s y_b(s, j, e))$ for some $j < i$, then $d(z) = 1$. Suppose this last hypothesis is false. Suppose also that $d(s, z) = 1$ for all $s < v^*$. Then $d(z) = 0$ if and only if

$$(Es)(u)(v)[s \geqq v^* \;\&\; \sim Q(u, v, s, e, i, m)] ,$$

since *Case* d2 applies when $s \geqq v^*$. We now continue as in *Case* (*b*). With the help of Lemmas 8 and 9, $d(z)$ is easily determined.

That completes our computation of $d(z)$ from C. To find $d(z)$, we had to know the value of $d(x)$ for finitely many $x : x < z$; $x = p(u, j, k)$, where $u < e$; $x = p(j, 1)$, where $j \in B$. We had to know that $d(p(j, 1)) = 0$ if and only if $j \in B$. We used very heavily the fact that B is recursive in C.

The basic combinatorial principle of this paper is contained in Lemma 2; this same principle appeared in simpler form in [6]. We combined this principle with a further combinatorial principle expressed by Lemma 4 in order to show the recursively enumerable degrees are dense. Lemma 4 was needed to show D is not recursive in B. The prethinking which inspired Lemma 4 may be described as follows. C is not recursive in B. So let us keep planting members of C in D until D looks enough like C to guarantee that D is not recursive in B. But let us not plant members of C in D with utter abandon, because we wish to have C not recursive in D. At the same time let us plant B in D with utter abandon so that B will be recursive in D. We plant a member of C in D when we set $d(s, p(e, i, t(s, i, e))) = 0$; this happens only if $c(s, i) = 2$. (Recall $c(s, i) = 2$ only if $i \in C$.) In order to prevent us from planting too much of C in D, we must have a method of unplanting members of C already planted in D. Suppose we have planted $i \in C$ in D; that is, we set $d(s, p(e, i, t(s, i, e))) = 0$. If for some $w > s$, $t(w, i, e) > t(s, i, e)$, then *Case* d1 or d2 may give us a chance of setting $d(w, p(e, i, 1 + t(s, i, e))) = 0$; if this last happens, we have unplanted i. We have the opportunity to unplant i if and only if $\lim_s t(s, i, e) > t(s, i, e)$. This happens if and only if $\{e\}^b(i)$ is undefined or unequal to $y_b(s, i, e)$. If for some i, $\{e\}^b(i)$ is undefined, there is no need to plant any member of C in D. If $\{e\}^b(i)$ is defined for all i, then Lemma 4 tells us that we do not permanently plant infinitely much of C in D. Of course the proof of Lemma 4 turns on the fact that C is not recursive in B.

There are still many open questions concerning the recursively enumerable degrees; however, it may be possible to say something about the elementary theory of recursively enumerable degrees if the following question is answered: if $\mathbf{b}$ and $\mathbf{c}$ are non-zero, recursively enumerable degrees such that $\mathbf{b} < \mathbf{c}$, does there exist a recursively enumerable degree $\mathbf{d}$ such that $\mathbf{d} < \mathbf{c}$ and $\mathbf{d} \cup \mathbf{b} = \mathbf{c}$? In a talk at the Berkeley Model Theory Symposium (1963), J. R. Shoenfield conjectured that the answer is yes. We agree with him, but we are unable to prove it.

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MATHEMATICS

Measurable Cardinals and Constructible Sets

by

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A cardinal number $\mathfrak{m}$ will be called measurable if and only if there is a set X of cardinality $\mathfrak{m}$ and a non-trivial, real-valued, countably additive measure μ defined on all subsets of X. (The term non-trivial can be taken to mean that $\mu(X) = 1$ and $\mu(\{x\}) = 0$, for all $x \in X$). If $2^{\aleph_0} = \aleph_1$, Banach and Kuratowski [1] proved that $\aleph_1$ is not measurable. Ulam [12] proved that if there is a measurable cardinal, then either $2^{\aleph_0}$ is measurable or there exists a 2-valued measurable cardinal (2-valued in the sense that the measure μ can be assumed to take on only the values 0 and 1). Ulam and Tarski showed that no cardinal less than the first strongly inaccessible cardinal beyond $\aleph_0$ can be 2-valued measurable (cf. [12], esp. footnote 1, p. 146). Last year, using some new results of Hanf, Tarski proved [11] that many inaccessibles, in particular the first beyond $\aleph_0$, are not 2-valued measurable (for other proofs cf. [6] and [2]). Even though the least 2-valued measurable cardinal, if it exists at all, now appears to be incredibly large since Tarski's results apply to a seemingly inexhaustible number of inaccessible cardinals, it still seems plausible to many people including the author to assume that such cardinals do exist. However, this assumption has some surprising consequences, for, as shall be outlined below, we can show that the existence of measurable cardinals contradicts Gödel's axiom of constructibility.

We shall work within the system of [4] but shall not follow the notation of [4] too closely. The axiom $V = L$ is assumed in the form of the following statement:

(*) *If M is a class such that*

(i) $M \subseteq PM \subseteq \bigcup_{x \in M} Px$;

(ii) $x - y, \bigcup x, \breve{x}, x|x, E \upharpoonright x \in M$, *for all* $x, y \in M$;
then $V = M$.

(Above, the symbol P denotes the power set operation so that PM is the class of all subsets of the class M, and $\bigcup$ the union operation; of course, $\bigcup x = \bigcup_{y \in x} y$). The terms $\breve{x}$ and $x \mid y$ denote, respectively, the operations of forming the converse of the relational part of the set x and of forming the relative product of the relational parts of the sets x and y. The class E is the membership relation between sets; hence, $E \restriction x = \{\langle u, v\rangle : u \in v \in x\}$). That the statement (*) is equivalent to $V = L$ follows essentially from the lemma given by Hajnal ([5], p. 133) and the theorem of Shepherdson ([9], p. 186). The possibility of using the specific operations mentioned in condition (ii) of (*) follows from some unpublished results of Tarski.

Let us now assume that measurable cardinals exist. Since the axiom of choice follows from $V = L$, we can identify cardinals with initial ordinals. Let $\omega_\varkappa$, then, be the least measurable cardinal. Since $2^{\aleph_0} = \aleph_1$ follows from $V = L$, we can use the arguments of [12] to conclude that $\omega_\varkappa$ must be the least 2-valued measurable cardinal and that $\omega_\varkappa$ is a strongly inaccessible number; hence, $\omega_\varkappa = \varkappa$. Let $\mu \in \{0, 1\}^{P\varkappa}$ be 2-valued, non-trivial, countably additive measure defined on all subsets of $\varkappa$. (In general if A is a class and b is a set, then A^b denotes the class of all functions with domain b and range included in A). We now employ the measure μ to define certain relations Q_μ and E_μ over the class $V^\varkappa$ as in the theory of the reduced products (ultra products) of relational systems (cf. [3] and [6]).

Definition 1.

(i) $Q_\mu = \{\langle f, g\rangle : f, g \in V^\varkappa \wedge \mu(\{\xi < \varkappa : f(\xi) = g(\xi)\}) = 1\}$;

(ii) $E_\mu = \{\langle f, g\rangle : f, g \in V^\varkappa \wedge \mu(\{\xi < \varkappa : f(\xi) \in g(\xi)\}) = 1\}$.

Lemma 1. *Q_μ is a congruence relation for E_μ over $V^\varkappa$.*

The proof is very easy and uses only the finite additivity of the measure μ. Our main interest will lie in the structure of the equivalence classes f/Q_μ under the quotient relation E_μ/Q_μ. However, the equivalence classes are not sets and the quotient relation does not really exist. The next lemma gives some facts about relation E_μ which will allow us to replace the equivalence classes by sets thus overcoming this difficulty.

Lemma 2. (i) *If $\{h \in V^\varkappa : hE_\mu f\} = \{h \in V^\varkappa : hE_\mu g\}$, then $fQ_\mu g$;*

(ii) $\{h \in V^\varkappa : hE_\mu f\} = \{h \in V^\varkappa : \exists k\, [k \in (\bigcup_{\xi<\varkappa} f(\xi) \cup \{0\})^\varkappa \wedge kE_\mu f \wedge hQ_\mu k]\}$;

(iii) $\sim \exists f\, [f \in (V^\varkappa)^\omega \wedge \forall \nu\, [\nu \in \omega \rightarrow f(\nu+1)\, E_\mu f(\nu)]]$.

Statement (i) shows that the equivalence class of f is determined by

$$\{h \in V^\varkappa : hE_\mu f\},$$

This is best proved by contradiction and requires the axiom of choice to find a function h which distinguishes f from g.

Statement (ii) implies that the number of equivalence classes included in the class $\{h \in V^\varkappa : hE_\mu f\}$ is bounded by the cardinality of the set $(\bigcup_{\xi<\varkappa} f(\xi) \cup \{0\})^\varkappa$.

Statement (iii) implies that the relation E_μ is well founded. The proof of (iii) is the first place where the countable additivity of μ is needed in the lemmas. The countable additivity at once reduces (iii) to the corresponding statement for the membership relation E, which follows easily from the axiom of foundation.

Using Lemma 2 we can now prove a statement which shows that $V^\varkappa$ can be mapped onto a class in such a way that the image of Q_μ is the identity relation and the image of E_μ is the membership relation. The method of proof is essentially that of [8] (Theorem 3, p. 147) or of [9] (Theorem 1.5, p. 171); see also [7].

LEMMA 3. *There is a (unique) function σ with domain $V^\varkappa$ such that for $f, g \in V^\varkappa$,*

(i) $\sigma(f) = \{\sigma(h) : h \in V^\varkappa_\cup \wedge hE_\mu f\}$;

(ii) $\sigma(f) = \sigma(g)$ *if and only if* $fQ_\mu g$;

(iii) $\sigma(f) \in \sigma(g)$ *if and only if* $fE_\mu g$.

DEFINITION 2. $M = \{\sigma(f) : f \in V^\varkappa\}$.

In other words, the class M is the range of the function σ; it is the class to which we shall apply the hypothesis of (∗). We note first:

LEMMA 4. $M \subseteq PM \subseteq \bigcup_{x \in M} Px$.

The first inclusion follows at once from 3 (i) and Def. 2 To prove the second, let $y \in PM$. Using the axiom of choice find $z \in P(V^\varkappa)$ such that $y = \{\sigma(g) : g \in z\}$. Let $f \in V^\varkappa$ be defined so that for $\xi < \varkappa, f(\xi) = \{g(\xi) : g \in z\}$. Then $y \in \sigma(f)$. Before we can check the second hypothesis of (∗), we need to prove a more general fact about M that can be used in many different ways. In the following $\Phi(v_0, ..., v_{k-1})$ will stand for any formula of set theory with free variables $v_0, ..., v_{k-1}$ and with *all* quantifiers restricted to V (that is, no bound class variables). Further, $\Phi^{(M)}(v_0, ..., v_{k-1})$ is the result of relativising all the quantifiers of $\Phi(v_0, ..., v_{k-1})$ to the class M.

LEMMA 5. *If* $f_0, ..., f_{k-1} \in V^\varkappa$, *then* $\Phi^{(M)}(\sigma(f_0), ..., \sigma(f_{k-1}))$ *if and only if* $\mu(\{\xi < \varkappa : \Phi(f_0(\xi), ..., f_{k-1}(\xi))\}) = 1$.

The proof proceeds by induction on the number of logical symbols in the formula, and is exactly the same proof as that for reduced products (cf. [3], sec. 2). Now by using the proper formulas and Lemma 4 one can easily prove that M satisfies hypothesis (ii) of (∗); hence, we have:

COROLLARY 5.1. $V = M$.

To obtain other corollaries, it is useful to have a short notation for the images of the constant functions in $V^\varkappa$ under the mapping σ.

DEFINITION 3. $x^* = \sigma(\{\langle \xi, x \rangle : \xi < \varkappa\})$.

COROLLARY 5.2. *If* $x_0, ..., x_{k-1} \in V$, *then* $\Phi^{(M)}(x_0^*, ..., x_{k-1}^*)$ *if and only if* $\Phi(x_0, ..., x_{k-1})$.

Corollary 5.2 is a direct consequence of Lemma 5 obtained by substituting the constant functions for the $f_0, ..., f_{k-1}$. Next, if we combine 5.1 with 5.2 using the formula $\Phi(\varkappa)$ that expresses in formal terms that $\varkappa$ is the least 2-valued measurable cardinal, we prove at once:

COROLLARY 5.3. $\varkappa = \varkappa^*$.

To show how a contradiction is reached, we introduce next a special ordinal number that does not correspond to a constant function but is the image of the identity function.

DEFINITION 4. $\delta = \sigma(\{\langle\xi, \xi\rangle : \xi < \varkappa\})$.

LEMMA 6. *If* $\lambda < \varkappa$, *then* $\lambda^* < \delta < \varkappa^*$.

Recalling that less than between ordinals is the same as membership, we see that the inequality $\delta < \varkappa^*$ follows from 3 (iii) and Definitions 3 and 4. The proof of the inequality $\lambda^* < \delta$ reduces simply to the equation $\mu(\{\xi < \varkappa : \lambda \leqslant \xi\}) = 1$, which follows from the fact that $\varkappa$ is the least 2-valued measurable cardinal.

Notice that from 5.2 it follows at once that the mapping from sets x to sets x^* is one-one; hence, the set $\{\lambda^* : \lambda < \varkappa\}$ must have cardinality $\varkappa$. From 6 it follows that δ must have cardinality at least that of $\varkappa$. On the other hand 5.3 and 6 together imply that $\delta < \varkappa$, which contradicts the choice of $\varkappa$ as an initial ordinal.

In case one does not wish to assume that $V = L$, the above method of proof can be used for the following definite statement: If $\varkappa$ is the least 2-valued measurable cardinal, then $\boldsymbol{PP}\varkappa \in L$.

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STABLE THEORIES

BY

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ABSTRACT

We study $K_T(\lambda) = \sup \{|S(A)| : |A| \leqq \lambda\}$ and extend some results for totally transcendental theroies to the case of stable theories. We then investigate categoricity of elementary and pseudo-elementary classes.

0. **Introduction** In this article we shall generalize Morley's theorems in [2] to more general languages.

In Section 1 we define our notations.

In Theorems 2.1, 2.2. we in essence prove the following theorem: every first-order theory T of arbitrary infinite cardinality satisfies one of the possibilities:

1) for all χ, $|A| = \chi \Rightarrow |S(A)| \leqq \chi + 2^{|T|}$, (where $S(A)$ is the set of complete consistent types over a subset A of a model of T).

2) for all χ, $|A| = \chi \Rightarrow |S(A)| \leqq \chi^{|T|}$, and there exists A such that $|A| = \chi$, $|S(A)| \geqq \chi^{\aleph_0}$.

3) for all χ there exists A, such that $|A| = \chi$, $|S(A)| > |A|$.

Theories which satisfy 1 or 2 are called *stable* and are similar in some respects to totally transcendental theories. In the rest of Section 2 we define a generalization of Morley's *rank of transcendence*, and prove some theorems about it. Theorems whose proofs are similar to the proofs of the analogous theorems in Morley [2], are not proven here, and instead the number of the analogous theorem in Morley [2] is mentioned.

In Section 3, theorems about the existence of sets of *indiscernibles* and *prime models* on sets are proved.

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In Section 4, a two-cardinal Skolem-Löwenhiem theorem is given without proof, and is followed by some theorems about *categorical* elementary and pseudo-elementary classes.

Among them appear:

THEOREM. *If T is categorical in λ, $\lambda > |T| + \aleph_0$, $\lambda \neq \inf\{\mu: \mu^{\aleph_0} > \mu + |T|\}$ then T is categorical in every cardinal $\geqq \lambda$, and in some cardinal $< \mu(|T|) < \beth((2^{|T|})^+)$.*

THEOREM. *If the class of reducts of models of T to the language L is categorical in λ, $\lambda > |T|$, $\beth_\gamma > |T|$ and the ordinal γ is divided by $(2^{|T|})^+$, then the class of reducts of models of T to the language L is categorical in $\beth_\gamma$.*

Some of the results of this article appear in my notices [8], [7].

After proving the theorems in this article, an unpublished article of J. P. Ressayre [5] came to my attention. It deals with categorical theories and includes results previously obtained by F. Rowbottom. Among the results in Ressayre's article are a weaker version of Theorems 2.1 and 2.2, a partial version of 3.5, and a somewhat weaker version of 4.6.

1. **Notations.** M will denote a model, $|M|$ is the set of its elements, $|A|$ is the cardinality of A, and $\|M\|$ is the cardinality of the model M. We shall write $a \in M$ instead $a \in |M|$. $\alpha, \beta, \gamma, i, j, k, l$, will denote ordinale, δ a limit ordinal and n, m natural numbers.

λ, χ, μ will denote infinite cardinals. λ^+ is the first cardinal greater than λ. $\beth(\chi,\alpha)$ is defined by induction: $\beth(\chi,0) = \chi, \beth(\chi,\alpha + 1) = 2^{\beth(\chi,\alpha)}$, and $\beth(\chi, \delta) = \bigcup_{\alpha<\delta}\beth(\chi,\alpha)$; $\beth(\alpha) = \beth_\alpha = \beth(\aleph_0,\alpha)$. If $\chi = \aleph_\alpha$ then $\aleph(\chi,\beta) = \alpha_{\alpha+\beta}$, where $\aleph_\alpha = \omega_\alpha$ is the α'th infinite cardinal.

T will denote a fixed first-order theory with equality. If $\psi(x)$ is a formula in the language of T with one variable., $\psi(M)$ is the set of elements satisfying ψ. $M \models \psi[a]$ if $\psi[a]$ is satisfied in M. Without loss of generality we assume that for every formula $\psi(x_1, \cdots, x_n)$ there is a predicate $R(x_1, \cdots, x_n)$ such that $(\forall \bar{x})(\psi(x_1, \cdots, x_n) \equiv R(x_1, \cdots, x_n)) \in T$ and that there are no function symbols in the language. Morley [2] explains why there is no loss of generality here. The language of T will be denoted by $L(T)$. The predicates in $L(T)$ will be $\{R_i: i < |T|\}$. T is complete unless stated otherwise. Usually x, y, z will be individual variables, $\bar{x}, \bar{y}, \bar{z}$— finite sequences of variables, a, b, c will denote elements of models, and $\bar{a}, \bar{b}, \bar{c}$ will denote finite sequences of elements of models. It is implicitly assumed that different sequences of variables contain no common variables. $\langle\,\rangle$ will be the empty sequence. $\bar{a}_i$ or $\bar{a}(i)$ will be the i'th element of the sequence $\bar{a}$. Instead of writing $(\forall n<\omega)(\bar{a}_n \in A)$ we shall write $\bar{a}_n \in A$ or $\bar{a} \in A$. A, B, C will denote substructures of T-models, and when we speak about a set A, or define A, we speak about its relations as well. That is we do not distinguish between the substructure A and the set A. By $A \subset M$ we mean that $A \subset |M|$, and the relations on A are the relations on M restricted to A. $T(A)$ is the theory T together with all the true sentences $R[\bar{a}]$, $\bar{a} \in A$, and $T(A)$ is a complete theory. When writing $R[\bar{a}]$ we assure implicitly that the length of the sequence $\bar{a}$ is equal to the number of places in the predicate R.

We define p to be a *type* on A iff p is a set whose elements are of the form $\psi(\bar{x}, \bar{a})$, where $\bar{a} \in A$, and ψ is an arbitrary formula in L. q, r will also denote types. If for every ψ, $\bar{a} \in A \rightarrow \psi(\bar{x}, \bar{a}) \in p$ or $\psi(\bar{a}, \bar{a}) \in p$, p is called a *complete type* on A. If A is not mentioned, then it is assumed A is the empty set. When we speak about a type we implicitly assume that $T(A) \cup p$ is a consistent set. We define $p | A = \{\psi(\bar{x}, \bar{a}) \in p : \bar{a} \in A\}$. If not otherwise assumed $\bar{x} = x$ in p.

$S^T(A)$ is the set of complete types on A. As T is fixed we write $S(A)$. If I is a set of predicates then $p | I = \{\psi \in p : \psi = R(x, \bar{a})$ or $\psi = \neg R(x,, \bar{a})$ and $R \in I\}$, $S_I(A) = \{p | I : p \in S(A)\}$, $p | R = p | \{R\}$, and $S_R(A) = S_{\{R\}}(A)$. By our notations we can distinguish easily between $p | I$ and $p | A$. On $S(A)$ $(S_I(A))$ a compact topology is defined by the sub-base which has the following sets as elements: for every $\phi = \psi(x, \bar{a})$, $V_\phi = \{p : \psi(x, \bar{a}) \in p\}$. *$M$ realizes* a type p on $|M|$, if there is an element b of M such that for every $\psi(x, \bar{a}) \in p$ $M \models \psi[b, \bar{a}]$ (that is : $\psi(b, \bar{a})$ is satisfied in M). *M omits p* if it does not realize p. M is called *λ-saturated* if every type on A with $A \subset M$, $|p| < \lambda$, is realized in M. If M is $\|M\|$-saturated it is called *saturated*.

$\mu(\chi)$ is the smallest cardinal such that if T with $|T| = \chi$, has a model omitting a type p in every cardinal smaller than $\mu(\chi)$ and not smaller than $|T|$, then it has such a model in every cardinal $\geqq |T|$. In Vaught [9] the following results are mentioned:

$$\mu(\chi) < \beth_\gamma \text{ where } \gamma = (2^\chi)^+;\ \mu(\aleph_0) = \beth_{\omega_1};\ \mu(\beth_\delta) = \beth(\beth_{\delta+1}) \text{ when } \operatorname{cf} \delta = \omega.$$

T is *categorical* in λ if all models of T of cardinality λ are isomorphic. $pc\ (T_1, T)$ is the class of reducts of models of T_1 to $L(T)$. (We assume implicitly that

$T = T_1 \cap L(T)$.) $pc(T_1, T)$ is *categorical* in λ if all models in $pc(T_1, T)$ of cardinality λ are isomorphic.

2. On possible cardinalities of $S(A)$

DEFINITION 2.1: $K_T(\lambda) = \sup\{|S(A)| : |A| \leqq \lambda\} = \inf\{\mu : |A| \leqq \lambda \Rightarrow |S(A)| < \mu\}$.

NOTATIONS: η, τ will denote *ordinal sequences* of zeroes and ones. For $0 \leqq i < l(\eta)$ η_i is the i'th element of the sequence, where $l(\eta)$ is the length of the sequence. $\psi^{\eta(i)}$ will denote ψ if $\eta(i) = 0$, and $\neg\psi$ if $\eta(i) = 1$. $\eta \mid \alpha$ is the sequence of the first α elements of η.

THEOREM 2.1. 1) *If there exists* A, $|A|^{|T|} = |A|$, $|S(A)| > |A|$. *Then for every* λ, $K_T(\lambda) \geqq \inf\{(2^\chi)^+ : 2^\chi > \lambda\}$.

2) There exists A as mentioned in 1, iff tthere exists a predicate R such that: $\Gamma_R = \{(\exists x)(\bigwedge_{0 \leqq i < l(\eta)} R(x, \bar{y}^{\eta|i})^{\eta(i)}) : l(\eta) < \omega\} \cup T$ is consistent.

REMARKS. The same argument will show that if there exists an A such that $|A|^{|T|} < |S(A)|$, then Γ_R is consistent.

Proof. Let us assume that A satisfies $|A|^{|T|} = |A|$, $|S(A)| > |A|$. Then we shall show that there exists a consistent Γ_R as mtntioned in 2, and that the consistency of Γ_R implies the conclusion of 1. This will prove the theorem.

Now for every R, we define $p_1 \sim p_2 \pmod R$ iff $p_1 \mid R = p_2 \mid R$. This is an equivalence relation on $S(A)$, which divides it into $|S_R(A)|$ equivalence classes. Since, for every $p_1, p_2 \in S(A)$, $p_1 \neq p_2$, there is an R such that $P_1 \nsim p_2 \pmod R$, $|S(A)| \leqq |\prod_R S_R(A)| = \prod_R |S_R(A)|$. If for every R, $|S_R(A)| \leqq |A|$, then $|S(A)| \leqq |A|^{|T|} = |A|$, a contradiction. Hence, there esxits an R such that $|S_R(A)| > |A| \geqq \aleph_0$. We shall prove that Γ_R is consistent.

For every $\bar{a}$ such that $\bar{a} \in A$, $R(x, \bar{a})$ divides $S_R(A)$ into two sets: the types p such that $R(x, \bar{a}) \in p$, and the types p such that $\neg R(x, \bar{a}) \in p$. If in every such division one of the sets is of cardinality $\leqq |A|$, for example the set $\{p \in S_R(A) : R(x, \bar{a}^{\tau(\bar{a})}) \in p\}$ then,

$$|S_R(A)| = |\bigcup_{\bar{a}} \{p \in S_R(A) : R(x, \bar{a})^{\tau(\bar{a})} \in p\} \cup \{p \in S_R(A) : \text{for all } \bar{a}$$

$$R(x, \bar{a})^{\tau(\bar{a})} \notin p\}| \leqq \sum_{\bar{a}} |\{p \in S_R(A) : R(x, \bar{a})^{\tau(\bar{a})} \in p\}| + 1 = |A|, \quad \text{a contradiction.}$$

So there exists $\bar{a} = \bar{a}^{\langle\rangle}$ such that $R(x, \bar{a}^{\langle\rangle})$ divides $S_R(A)$ into two sets of cardinality $> |A|$. For every one of them we can repeat the above discussion and find. $\bar{a}^{\langle 0\rangle}$, $\bar{a}^{\langle 1\rangle}$ such that there exists $>|A|$ types p with either $R(x,\bar{a}^{\langle\rangle})$, $R(x,\bar{a}^{\langle 0\rangle})\in p$; $R(x,\bar{a}^{\langle\rangle})$, $\neg R(x,\bar{a}^{\langle 0\rangle})\in p$; $\neg R(x,\bar{a}^{\langle\rangle})$, $R(x,\bar{a}^{\langle 1\rangle})\in p$; or $\neg R(x,\bar{a}^{\langle\rangle}), \neg R(x,\bar{a}^{\langle 1\rangle})\in p$. We can continue defining $\bar{a}^{\eta}$, and proving by it the consistency of Γ_R. And so we have shown one direction.

Let $\chi=\inf\{\mu: 2^{\mu}>\lambda\}$. We define

$$\Gamma=\{R(\chi_\eta y^{\eta|\gamma})^{\eta(\gamma)}: l(\eta)=\chi, \gamma<\chi\}\cup T.$$

It is easy to see that if Γ is not consistent then Γ_R is not consistent. Let M be a model of Γ, and A_1 the set of elements which realize the variables $\{(\bar{y}^{\eta|\gamma})_n: l(\eta)=\chi, \gamma<\chi, \eta<l(\bar{y}^{\eta|\gamma})\}$. The cardinality of A_1 is $\leq\sum_{\gamma<\chi}2^{|\gamma|}\leq\lambda$, and in M 2^{χ} different complete types on A_1 are realized. (The types realized by elements which realizes the variables χ_η, $l(\eta)=\chi$). So $|A_1|\leq\lambda$. $|S(A_1)|\geq 2^{\chi}>\lambda$, and so $K_T(\lambda)\geq(2^{\chi})^+>\lambda^+$.

Definition 2.2. If in T there is no predicate R such that Γ_R is consistent, T is called *stable.*

Definition 2.3. If for every λ, $K_T(\lambda)\leq\lambda^+ + (2^{|T|})^+$ then T is called *super stable.*

Theorem 2.2. 1) *If T is stable and there exists A, $|A|\geq 2^{|T|}$ such that $S(A)|>|A|$, then for every λ, $K_T(\lambda)>\lambda^{\chi_0}$. So there exists arbitrarily large powers for which $K_T(\lambda)>\lambda^+ + (2^{|T|})^+$.*

2) *There exists A as mentioned in 1 iff there exists a sequence of ω predicates $\langle R^n: n<\omega\rangle$ such that*

$$\Gamma\langle R^n: n\,\omega<\rangle=\{R^m(x^f,\bar{y}^{g,h})\equiv\neg R^m(x^{f'},\bar{y}^{g,h})\text{: for all}$$

$$f=\langle i_0,\cdots,i_{m-1},i_m,\cdots,i_l\cdots: l<\omega\rangle, f'=\langle i_0,\cdots,i_{m-1},i_m,\cdots',i_l,\cdots': l<\omega\rangle,$$

$$i'_m\neq i_m, g=\langle i_0,\cdots,i_{m-1}\rangle, h=\{i_m,i'_m\}\text{ and}$$

$$i_l,i'_l<\omega\text{ for all } l<\omega\}$$

is consistent.

3) *If T is super stable and there exists A with $|S(A)|>|A|,|T|$ and if $\lambda>|A|+|T|$, $\lambda\leq S(A)$ is regular then there exists $B\subset A$, $|B|=|T|$* such that $|S(B)|\geq\lambda$. We can conclude that, for super stable T, if $K_T(\lambda)>\lambda^+>|T|$ then $K_T(|T|)>|T|^+$.

Proof. The way we prove 1 and 2 will be similar to that of Theorem 2.1. First, we shall prove from the assumption of 1 that there exists $\langle R^n : n<\omega\rangle$ such that $\Gamma\langle R^n : n<\omega\rangle$ is consistent, and then that if $\Gamma\langle R^n : n<\omega\rangle$ is consistent then for every λ there exists A, such that $|A| = \lambda$, $|S(A)| \geqq \lambda^{\aleph_0}$. Then choosing such an A for $\lambda = \aleph(2^{|T|}, \omega)$, we close the circle.

Let A be as in the assumption of 1.

LEMMA 2.3. *There exists R^0, a predicate of $L(T)$, such that the partition of $S(A)$ by the equivalence relation (mod R^0) contains at least $|T|^+$ classes of cardinality $> |A|$.*

Proof of the lemma. If not $-|S(A)| \leqq \sum_R |S_R(A)| + |T|^{|T|} = |A|$, a contradiction.

For every one of the $|T|^+$ classes there exists R_i that divides it in a similar manner. But there are only $|T|$ predicates. So there exists R^1 such that there are $|T|^+$ classes (mod R^0) such that in each of their partitions by R^1 there are $|T|^+$ classes of cardinality $> |A|$. It is easy to see that we can continue to define R_n for $n < \omega$.

Now $\langle R^n : n<\omega\rangle$ is defined. By the construction just mentioned there exists for every n $\{p(j; i_0, \cdots, i_{m-1}) : j < |T|^+,\ i_l < |T|^+, m < n\}$ such that the following three conditions are satisfied:

$p(j; i_0, \cdots, i_{m-1}) \in S_R m(A)$; if $j \neq j'$ then $p(j : i_0, \cdots, i_{m-1})$ and $p(j'; i_0, \cdots, i_{m-1})$ are contradictory; and $p(i_1) \cup P(i_2; i_1) \cup p(i_3, i_1, i_2) \cup \cdots \cup p(i_m;\ i_0, \cdots, i_{m-1})$ is consistent.

From this it can be easily seen that $\Gamma\langle R^n : n<\omega\rangle$ is consistent. Now we shall prove that if $\Gamma\langle R^n : n\langle\omega\rangle$ is consistent, then for every λ there exists an A such that $|A| = \lambda$, $|S(A)| \geqq \lambda^{\aleph_0}$. Let $\Gamma = T \cup \{R^m(x^f, \bar{y}^{g,h}) \equiv R^m(x^{f'}, \bar{y}^{g,h})$: for, all $m < \omega$, $f = \langle i_0, \cdots, i_{m-1}, i_m, \cdots, i_l, \cdots : l<\omega\rangle$, $g = \langle i_0, \cdots, i_m\rangle$, $h = \{i'_m, i_m\}$, and $f' = \langle i_0, \cdots, i_{m-1}, i'_m, \cdots, i'_l \cdots : l<\omega\rangle$ such that $(\forall j < \omega)\ (i_j < \lambda \wedge i'j < \lambda)\}$.

If Γ is inconsistent, then a finite subset of Γ is inconsistent and so $\Gamma < R^n : n<\omega\rangle$ is inconsistent, a contradiction. Therefore Γ has a model. Let A be the set of elements realizing the variables appearing in $\bar{y}^{g,h}$. Then elements realizing different variables from $\{x^f : f = < i_0, \cdots, i_l, \cdots : l<\omega\rangle, i_l<\lambda\}$ realizes different types on A. So $|A| \leqq \sum_{m<\omega} \lambda^m = \lambda$, $|S(A)| \geqq \lambda^{\aleph_0}$.

Now it remains to prove part 3. We can try again to build the construction that appears in the beginning of the proof replacing "more than $|A|$" by "at least λ", As that attempt must fail by our assumption, we get a set S of $\geqq \lambda$ types in $S(A)$. such that for every R there are no more than $|T|$ equivalence classes of power $\geqq \lambda$,

$\{S_i(R): i < j_R \leqq |T|\}$. Now $|S - \bigcup_i S_i(R)| < \lambda$ and $|S - \cap_R \bigcup_i S_i(R)| \leqq \sum_R |S - \bigcup_i S_i(R)| < \lambda$ and this implies that $|\cap_R \bigcup_i S_i(R)| \geqq \lambda > |A|$. If $p_1, p_2 \in \cap_R \bigcup_i S_i(R)$, $p_1 \neq p_2$ there is an R such that $p_1 | R \neq p_2 | R$; but $p_1 | R$ is one of $|T|$ elements of $\{p | R : p \in \bigcup_i S_i(R)\}$ (by the definition of $S_i(R)$), and so there is $A(R) \subset A$, $|A(R)| = |T|$ such that for every $p_1, p_2 \in \cap_R \bigcup_i S_i(R)$ if $p_{1R} | \neq p_2 | R$ then $p_1 | A(R) \neq p_2 | A(R)$. It follows that $|S(\bigcup_R A(R)| \geqq |\cap_R \bigcup_i S_i(R) \geqq \lambda$, and $|\bigcup_R A(R)| \leqq |T|$.

Remark. By a more refined proof we can replace $\Gamma\langle R^n : n < \omega\rangle$ by the more elegant set

$$\Gamma'\langle R^n : n<\omega\rangle = T \bigcup \{(\exists x) \bigwedge_{j=0}^{m} [R^j(x, \bar{y}^g) \wedge \bigwedge_{h=0}^{i_{j-1}} \cdots R^j(x, \bar{y}^f)] : m < \omega,$$

$$g = \langle i_0, \cdots, i_j\rangle,\ f = \langle i_0, \cdots, i_{j-1}, h\rangle,\ i_0, \cdots, i_m < \omega\}$$

Definition 2.4. We shall define $S_I^\alpha(A)$ and $TR_I^\alpha(A)$ by induction on α, where I is a set of predicates in $L(T)$. $S_I^0(A) = S_I(A)$. $TR_I^\alpha(A)$ will be the set of types in $S_I^\alpha(A)$, which have, in every extension B of A, at most one extension which is an element of $S_I^\alpha(B)$. $S_I^\alpha(A) = S_I(A) - \bigcup_{i<\alpha} TR_I^i(A)$.

Remark. An analogous definition appears in Morley [1], 2.2 and footnote 13.

Theorem 2.4. *If R is a predicate of $L(T)$, Γ_R is consistent iff $S_R^\alpha(A) \neq 0$ for every α and A. If for some α and A $S_R^\alpha(A) = 0$, then there exists $\beta < \omega_1$ such that for every A, $S_R^\beta(A) = 0$.*

Proof. As in Morley [1], 2.7, 2.8.

Remark. In fact, $\beta < \omega$.

Definition 2.5. 1) If Γ_R is not consistent, then to every type $p \in S(A)$, we define $\mathrm{Rank}(R, p)$ as the first α such that $p | R \in TR_R^\alpha(A)$.

2) If T is stable then $\mathrm{Rank}(p) = \langle \mathrm{Rank}(R_i, p) : i < |T|\rangle$.

Lemma 2.5. *It is possible to define a lexicographic order on Rank(p), such that there is no monotonically decreasing sequence of type $|T|^+$.*

Proof. Immediate.

Theorem 2.6. 1) *If $B \subset A$, and $p \in S(A)$, then* $\mathrm{Rank}(R, p) \leqq \mathrm{Rank}(R, p)| B)$ *and* $\mathrm{Rank}(p) \leqq \mathrm{Rank}(p | B)$, *and there is no more than one extension q of $p | B$, $q \in S(A)$, such that* $\mathrm{Rank}(q) = \mathrm{Rank}(p | B)$.

2) *For all A, and $p \in S(A)$, and for every R, there exists a finite set $B \subset A$, such that* $\mathrm{Rank}(R,p) = \mathrm{Rank}(R, p \mid B)$.

Proof. See Morley [2] 2.4, 2.6. Notice the difference in terminology. (Rank here is rank and degree there.)

3. On some properties of stable theories.

THEOREM. 3,1. *If M is a model of a stable theory T, $|T| < \lambda = |A| < \|M\|$, $K_T(\lambda) = \lambda^+$ and A a substructure of M, then there exists a set Y in M, $|Y| = \lambda^+$, which is indiscernible on A (that is, for all $y_1, \cdots, y_h$; $z_1, \cdots, z_n \in Y$. $a_1, \cdots, a_m \in A$, $M \models R(y_1, \cdots, y_n, a_1, \cdots, a_m) \equiv R(z_1, \cdots, z_n, a_1, \cdots, a_m)$ if for every $i \neq j$, $y_i \neq y_j$ and $z_i \neq z_j$).*

REMARK 1. A similar theorem, for totally transcendental theories appears in Morley [2] 4.6. Rowbottom has a weaker unpublished theorem.

REMARK 2. In fact we can prove more: in every $B \subset M$, $|B| > \lambda$, and for every regular $\chi \leqq |B|$, $\chi > \lambda$, there is such a Y, provided $|B| < \chi \Rightarrow |\{p \in S(A)$: p is realized in $M\}| < \chi$.

Proof. In $S(A)$ there are λ types, and so at least one of them, p, is realized at least $|A|^+$ times. Let the set of elements of M realizing p be B.

LEMMA 3.2. *There exists A_1, $|A_1| = \lambda$ $A \subset A_1 \subset M$, and $p_1 \in S(A_1)$ $p_1 \supset p$, such that, if $M \supset B_1 \supset A_1$, $|B_1| = \lambda$, p_1 has one and only one extension of the same rank in $S(B_1)$ and the extension is realized $\geqq \lambda^+$ times in M.*

Proof. of 3.2. Let us assume the lemma is not correct. We shall define by induction C_i which fulfills the following conditions:

1) $C_i = \{\langle A(k,j), p(k,j)\rangle : j;\ k \leqq i\}$ where $p(i,j) \in S(A(i,j))$, $A(i,j) \supset A$, $A(ij)| = \lambda$.

2) If $p(i,j) \subsetneqq p(i',j')$ then $i < i'$ and there exists $p(i+1,j'')$ such that $p(i,j) \subsetneqq p(i+1,j'') \subseteq p(i',j')$.

3) If $p(i,j) \subsetneqq p(i+1,j')$ then $\mathrm{Rank}(p(i,j)) < \mathrm{Rank}(p(i+1,j'))$ or $|B(i,j)| > \lambda \geqq |B(i+1,j')|$, where $B(i,j)$ is the set of elements of M realizing $p(i,j)$.

4) For every i,j,i',j', $p(i,j) \subset p(i',j')$ or $p(i,j) \supset p(i',j')$ or they are contradictory (that is, $T \cup p(i,j) \cup p(i',j')$ is inconsistent);

5) $C_i \subset C_j$ for $i < j$.

We shall not prove the conditions explicitly as they are obvious from the construction.

Let $C_0 = \{\langle A, p\rangle\} = \{\langle A(0,0), p(0,0)\rangle\}$.

Let us define C_{i+1}. If $|B(i,j)| > \lambda$ then by our assumption there exists $A_1 \subset M$, $A(ij) \subset A_1$, $|A_1| = \lambda$ such that every extension of $p(i,j)$ to A_1 has a smaller rank or is realized at most λ times. Then we add to C_i $\langle A(i+1,k), p(i+1,k)\rangle$ (where $A(i+1,k) = A_1$) for every extension of $p(i,j)$, $p(i+1,k)$, which belongs to $S(A(i+1,k))$ and is realized in M. Their number is $\leqq \lambda$ as $|A_1| = \lambda$ implies $|S(A_1)| \leqq \lambda$. We do so to every $\langle A(i,j), p(i,j)\rangle \in C_i$, and we get C_{i+1}. (We have enough indices so that there will be no confusion.) It is easily seen that $|C_{i+1}| \leqq |C_i| + \lambda |C_i|$, and for every j, $|A(i+1,j)| \leqq |A(i,j')| + \lambda \leqq \lambda$ for some j'.

Now we define C_δ. Let $\langle A^1, p^1\rangle < \langle A^2, p^2\rangle$ if $A^1 \subset A^2$ and $p^1 \subset p^2$. I $\langle A^i, p^i\rangle$ $i < j$ is an increasing sequence, then $\bigcup_{i<j}\langle A^i, p^i\rangle = \langle \bigcup_{i<j} A^i, \bigcup_{i<j} p^i\rangle$. The elements of C_δ will be the elements of $\bigcup_{i<\delta} C_i$, and unions of increasing sequences in $\bigcup_{i<\delta'} C_i$, $\langle A^1, p^1\rangle$, such that p^1 is realized in M.

It will now be proved that $C_{|T|^+} = C_{|T|^+ +1} = C_{|T|_+ +2} = \cdots$. It is sufficient to show that $\bigcup_i\{C_i : i < |T|^+\} = C_{|T|^+}$. That comes from the construction, for if it is not correct, there is an increasing sequence $\langle\langle A^i, p^i\rangle : i < |T|^+\rangle$. Then Rank$(p_i)$ is decreasing sequence, and by Lemma 2.4 that sequence cannot be strictly decreasing, so there exists an i such that Rank(p_i) = Rank$(p_{i+2}) = \cdots$. By condition 3 $|\{a \in M\colon$ a realizes $p_{i+1}\}| \leqq \lambda$ (as Rank(p_i) = Rank(p_{i+1}) and $p_i \subsetneqq p_{i+1}$) and similarly $|\{a \in M\colon$ a realizes $p_{i+1}\}| > \lambda$ (as Rank(p_{i+1}) = Rank$(_{pi+2})$ and $p_{i+1} \subsetneqq p_{i+2}$), a contradiction.

We shall now show that $|C_i| \leqq \lambda$ and $|A(i,j)| \leqq \lambda$ for $i \leqq |T|^+$. If not, let k be the first ordinal that contradicts our assertion. If $k = i+1$ then $|C_k| \leqq |C_i| + \lambda |C_i| \leqq \lambda$ and for every j, for some j' $|A(k,j)| \leqq |A(i,j')| + \lambda = \lambda$ (as remarked in the definition of C_{i+1}), so that k has to be a limit ordinal, and $k \leqq |T|^+$. Let $A^i = \bigcup\{A(l,j) : j; l \leqq i\}$. Now it can be seen easily that $|A^i| \leqq |C_i| \cdot \max_j |A(i,j)| \leqq \lambda$ for $i < k$, and from that, and the construction, it can be easily seen that $|A^k| \leqq \lambda$, and therefore $|S(A_k)| = \lambda$. Now the $\{B(k,j) : j\}$ are disjoint sets, and every one of them is the union of sets realizing some complete types on A_k, and by the construction $B(k,j) \neq 0$, and so the number of $B(k,j)$ is no more than λ. Thus $|C_k - \bigcup_{i<k} C_i| \leqq \lambda$. We can conclude that $|C_k| \leqq |C_k - \bigcup_{i<k} C_i| + \sum_{i<k} |C_i| \leqq \lambda + k\ \lambda = \lambda$, a contradiction; and so $|C_{|T|^+}| \leqq \lambda$, $|A^{|T|^+}| \leqq \lambda$.

For every b with $b \in B(0,0)$, the set of $\langle A(i,j), p(i,j)\rangle$ in $C_{|T|^+}$ such that $b \in B(i,j)$ is an increasing sequence in $C_{|T|^+}$. The union of the sequence is also in $C_{|T|^+}$, and so there is a last such element in $C_{|T|^+}$, $\langle A^b, p^b\rangle$. The set of elements of M realizing

p^b will be denoted by B^b. Now if there is an element of $C_{|T|+}$ greater than $\langle A^b, p^b\rangle$, then by the construction of C_i there is such an element $\langle A', p'\rangle$ such that b realizes p', in contradiction to the definition of $\langle A^b, p^b\rangle$. Therefore $|B^b| \leqq \lambda$. Now $B = B(0,0) \subset \bigcup\{B_b : b \in B\} = \bigcup\{B(i,j) : | B(i,j) | \leqq \lambda; j; i < |T|^+\}$ $\lambda < |B| \leqq |C_{|T|+}| \cdot \lambda = \lambda$ — contradiction.

So we have proved Lemma 3.2.

It follows that without loss of generality we can assume that for every C, such that $A \subset C \subset M$ and $|C| = \lambda$, p has one and only one extension in $S(C)$ which is of the same rank, and that extension is realized at least λ^+ times. Let the set of elements of M realizing p be B.

We define by induction the sequence $\{y_i : i < \lambda^+\}$. y_0 is an arbitrary element of B. If we define y_i for every $i < j < \lambda^+$, then $y_j = y(j)$ will be an element of M that realizes the only extension of p to a type q in $S(A \bigcup \{y_i : i < j\})$ such that $\mathrm{Rank}(p) = \mathrm{Rank}(q)$. By the definitions of B and p, there is such a y_j.

LEMMA 3.3. *If* $i_1 < i_2 < \cdots < i_n < \lambda^+$, $j_1 < \cdots < j_n < \lambda^+$ *then for every predicate* R *in* T *and every* $\bar{a}$, $\bar{a} \in A$,

$$M \models R[y(i_1), \cdots, y(i_n), \bar{a}] \equiv R[y(j_1), \cdots, y(j_n), \bar{a}].$$

Proof of Lemma 3.3. Without loss of generality $i_k = k$.

Now, in the construction of the y_i, in every stage in $S(A\bigcup\{y_i : i < j\})$ there is only one extension p_j of p such that $\mathrm{Rank}(p_1) = \mathrm{Rank}(p)$, so the type which y_j realizes on $A\bigcup\{y_i : i < j\}$ is independent of the choice of y_j. If $\{z_i : i < j\}$ satisfies: for every i, z_i realizes a type q_i on $A\bigcup\{z_k : k < i\}$ such that $q_i \supset p$, $\mathrm{Rank}(q_i) = \mathrm{Rank}(p)$, then it can be easily proved by induction that $\langle y_{i_1}, \cdots, y_{in}\rangle$ satisfies the same type on A as $\langle z_{i_1}, \cdots, \cdots, z_{i_n}\rangle$. Now, if we choose y_{j_1} as the first y, and y_{j_2} as the second, etc., they will satisfy the same formulaes as $y_1, \cdots, y_n$. It remains to prove that after choosing $y_{j_1}, \cdots, y_{j_k}$ as the first k y's we can choose $y_{j_{k+1}}$ as the next y. That is, perhaps $y_{j_{k+1}}$ realized a type p on $A\bigcup\{y_{j_1}, \cdots, y_{j_k}\}$, such that $\mathrm{Rank}(\bar{p}) < \mathrm{Rank}(p)$. But if q is the type of $y_{j_{k+1}}$ on $A\bigcup\{y_1, \cdots, y_l\}$ $(l = j_{k+1} - 1)$ then $\mathrm{Rank}(p) = \mathrm{Rank}(q) \leqq \mathrm{Rank}(\bar{p}) < \mathrm{Rank}(p)$, contradiction. So Lemma 3.3 is proved.

LEMMA 3.4. *Y is indiscernible on A.*

Proof. The proof is the same as in Morley [2] 4.6, since in every cardinal χ there is an ordered set that has more than χ Dedekind cuts.

So Theorem 3.1 is proved.

DEFINITION 3.1. Let K be a class of models, A a substructure of such models, $M \in K$ is called K-prime on A, if for every $M_1 \supset A$, $M_1 \in K$, there exists an isomorphism from M into M_1 that is the identity on A.

THEOREM 3.5. *If T is a stable theory, and $|A| \leqq \sum_{\chi<\lambda} 2^{\chi} \Rightarrow |S(A)| < 2^{\lambda}$, or $\lambda \geqq |T|^{+}$, then among the λ-saturated models of T, there is a prime model on every substructure A of a model of T.*

REMARK. An analogous theorem appears in Morley [2] 4.3.

DEFINITION 3.2. $p \in S(A)$ is called λ-isolated if there is a type $p_1 \subset p$, $|p_1| < \lambda$, such that p is the only element in $S(A)$ that includes p_1.

Proof of 3.5. In order that the model we will build on A be λ-saturated, we should realize every type of cardinality $< \lambda$, and in order that it be a prime we should realize only types which are realized in every λ-saturated model including A. So it is sufficient to show that if p is type on a set A, $|p| < \lambda$, then there exists an extension p_1 of p, $p_1 \in S(A)$, and p_1 is λ-isolated. For if it is right, we can add an element to A for every λ-isolated type. And if we continue adding such elements for every type p, $|p| < \lambda$ (by adding an element which realizes a λ-isolated complete type containing it) we shall get the wanted prime model.

Now let $\lambda \geqq |T|^{+}$ and $|p| < \lambda$ where p is a type on A. Among the elements of $S(A)$ containing p, there is a q with minimal $\text{Rank}(R_0, q)$, so there are a finite number of formulaes which define the type completely with regard to R_0 (among the extension of p). We adjoin these formulaes to p, and continue with $R_1, R_2, \cdots$. Because of the compactness theorem, this operation does not lead to a contradiction at the limit. So after $|T|$ steps we get the required type — a type of power $\leqq |p| + |T| < \lambda$, which has only one extension in $S(A)$.

It remains to deal with the case $|A| \leqq \sum_{\chi<\lambda} 2^{\chi} \Rightarrow |S(A)| < 2^{\lambda}$. Let p be a type on A, $|p| < \lambda$, which contradicts our conjecture. Let $p = p_{\langle\rangle}$. If $p_{\langle\rangle}$ has more than one extension to a type in $S(A)$, then there is a formula $R(x, \bar{a})$, such that $p_{\langle 0\rangle} = p \bigcup \{R(x, \bar{a})\}$, and $p_{\langle 1\rangle} = p \bigcup \{-R(x, \bar{a})\}$ are consistent. We continue with $p_{\langle 1\rangle}$ and $p_{\langle 0\rangle}$ as with $p_{\langle\rangle}$ and can define p_η for every sequence η of ones and zeroes, $l(\eta) \leqq \lambda$, $|p_\eta| \leqq \lambda$, such that:

1) if η_1 is not is not an initial segment of η_2 or conversely, then $p_{\eta_1} \cup p_{\eta_2}$ is not consistent;

2) if η_1 is an initial segment of η_2, then $p_{\eta_1} \subset p_{\eta_2}$; and

3) if $l(\eta)$ is a limit ordinal then $p_\eta = \bigcup_{i<l(\eta)} p_{\eta|i}$. Then $\{p_\eta : l(\eta) \leqq \lambda\}$ are 2^λ contradictory types on a set of cardinality $\leqq \lambda + \sum_{\chi<\lambda} 2^\chi = \sum_{<\lambda} 2^\chi < 2^\lambda$, a contradiction.

4. **On categorical elementary and pseudo elementary classes.**

THEOREM 4.1. *Let M be a model of a not necessarily complete theory T, Q predicate in $L(T)$, p a type. Let $(2^{|T|})^+ = \gamma$.*

1) *If M omits the type p, and $\beth(|Q(M)|,\gamma) \leqq \|M\|$, then in every cardinal $\geqq |T|$, there is a model M_1 of T which omits p and such that $|Q(M_1)| \leqq |T|$.*

2) *If M omits the type p, and $\beth(|Q(M)|,\gamma) \leqq \|M\|$, $|Q(M)| \geqq \beth_\gamma$ then for all cardinals $\chi \geqq \lambda \geqq |T|$, there is a model M_1 of T which omits p and such that $|Q(M_1)| = \lambda$, $\|M_1\|_1 = \chi$.*

Proof. The proof is by the methods of Morley [3] and is not given here. (Also see Vaught [9].)

REMARKS. The theorem can be slightly improved as don by Morley [2], in analogous theorems.

THEOREM 4.2. *If $p \subset (T_1, T)$ is categorical in a cardinal $\lambda > |T_1|$, then for every χ, $|T_1| \leqq \chi < \lambda$, $K_T(\chi) = \chi^+$, and so T is stable.*

Proof. By Morley [1], 3.7 (the proof for the non-denumerable case in the same) there exists a model M of T_1, $\|M\| = \lambda$, such that for every $A \subset M$, at most $|A| + |T_1|$ types on A are realized in M, and it follows from this that the same holds for the reduct of M to $L(T)$. If $K_T(\chi) = \chi^+$, $|T_1| \leqq \chi < \lambda$, there is a reduct to $L(T)$ of a model of T_1 of cardinality λ, for which there exists $A \subset M$ satisfying $|A| = \chi$, and $> \chi$ types of $S(A)$ are realized on A in the model. This contradicts the categoricity.

THEOREM 4.3. *If $p \subset (T_1, T)$ is not categorical in $\lambda_1 = \beth(\gamma \cdot \alpha) > |T_1|$* (where $\gamma = (2^{|T|})^+$, $\alpha > 0$), *then it has a non-$|T|^+$-saturated model in every cardinality. This is also true if we replace the assumption by: "$p \subset T_1, T)$ has a non-saturated model in λ_1".*

Proof. As any two saturated models of the same cardinality > [illegible] are isomorphic (see Morley and Vaught [1]), the second assumption follows from the first.

Let M be a non-saturated model such that $\| M \| = \lambda_1$ and M is the reduct to $L(T)$ of M_1. Then there exists $A \subset M$ $|A| < \| M \|$, and $p \in S(A)$, such that p is omitted in M. When we adjoin to M_1 the relations $Q(M) = A$ and to every predicate R of $L(T)$ $\psi_R(M_2) = \{\bar{a}: R(x, \bar{a}) \in p\}$, we get a model M_2. Now $| Q(M_2)| < \| M_2 \|$ and M_2 omits $p_1 = \{(\forall \bar{y})(\bigwedge_i Q(\bar{y}_i) \rightarrow R(x, \bar{y}) \equiv \psi_R(\bar{y})): R$ a predicate in $L(T)\}$.

By 4.1 in every cardinality there is a model M_3 of the theory of M_2 such that $|Q(M_3)| \leq |T_1|$, and M_3 omits the type $\{R(x, \bar{a}): \bar{a}_i \in Q(M_3), M_3 \models \psi_R[a]$, R a predicate of $L(T)\}$ which is a type on a set of cardinality $\leqq |T_1|$ (its consistency follows from the theory of M_2), and this proves the theorem.

Lemma 4.4. 1) *If* $|T_1| \leqq \lambda$, λ *is regular, and* $|A| < \lambda \Rightarrow |S^T(A)| \leqq \lambda$, *then* $p \subset T_1, T)$ *has a saturated model in* λ.

2) *If* $|T_1| \leqq \lambda$, $\mu \leqq \lambda$, μ *is regular and* $|A| \leqq \lambda \Rightarrow |S^T(A)| \leqq \lambda$, *then* $p \subset (T_2, T)$ *has a* λ*-saturated model in* λ.

Proof. Since the proofs are essentially similar, we prove only 1). Let $T_1 = \bigcup_{i < |T_1|} T_1^i$ where $T_1^i \subset T_1^j$ if $i < j$ and $T_1^i = T_1$ for $i \geqq |T_1|$ and $|T_1^i| < \lambda$. By the conditions in 1 we can easily define a sequence $\langle M^i: i \leqq \lambda \rangle$ such that: $|i| \leqq \| M^i \| < \lambda$; M^i as a model of T_1^i; if $i < j$ then the reduct of M^j to $L(T_1^i)$ is an elementary extension of M^i; the $L(T)$-types on M^i are $< p_j^i : j < j_0 \leqq \lambda \rangle$ and p_j^i is realized in M^j; $M^\delta = \bigcup_{i < \delta} M^i$. M_λ is the required model.

Corollary. 1) *If* T *is not stable then* $p \subset (T_1, T)$ *has a saturated model in a regular cardinal* $\lambda \geqq |T_1|$ *iff* $\chi < \lambda = 2^\chi \leqq \lambda$.

Proof. Suppose there exists $\chi_1 < \lambda < 2^{\chi_1}$. Let $\chi = \inf\{\chi: 2^\chi > \lambda\}$. As T is not stable, by Theorem 2.1, there exists A, $|A| \leqq \sum_{\mu < \chi} 2^\mu \leqq \lambda$ such that there exists $2^\chi > \lambda$ contradicting types of power $\chi < \lambda$ on A. If T has a saturated model M of power λ, then there is $A' \subset M$, with A' isomorphic to A. Thus in M more than $\| M \|$ contradicting types have to be realized, a contradiction. The opposite direction in trivial by 4.4.1, since always, $S(A) \leqq 2^{|A|| + |T|}$.

Theorem 4.5. 1) *If* $|T_1| = \aleph_\alpha$ *and* T *is not stable then the number of isomorphism types of* $p \subset (T_1, T)$ *in* $\aleph_\beta$ *is at least* $|\beta - \alpha|$.

2) *If* $|T_1| = \aleph_\alpha$ *and* T *is not super stable, then the number of isomrophism types of* $p \subset T_1, (T)$ *in* $\aleph_\beta$ *is at least* $|(\beta - \alpha)/\omega|$.

3) *If* $p \subset (T_1, T)$ *is categorical in a cardinal* $> |T_1|$, *different from* $\inf\{\chi: \chi \geqq T, \chi^{\aleph_0} > \chi + |T|\}$, *then a) T is superstable, b)* $K_T(\lambda) = \lambda^+$ *for* $\lambda \geqq |T|$. c) $p \subset (T_1, T)$ *is categorical in a cardinal* $> |T_1|$ *iff all models in it are saturated, and* d) $p \subset (T_1, T)$ *is caterogical in* $\beth(\gamma \cdot \alpha)$ *for* $\gamma = (2^{|T|})^+$, $\alpha > 0$, $\beth(\gamma.\alpha) > |T_1|$.

Proof. 1, 2) If $K_T(\lambda) > \lambda^+$, then for every $\chi \geqq \lambda^+$ there is a model M in $p\subset (T_1, T)$ such that there exists a set A with more than $|A|$ types realized on it in M, $|A| = \lambda$, and there is no such set of greater cardinality. (The existence is proved as in 4.2.)

3) By 4.2, for every χ with $|T_1| \leqq \chi < \lambda$, $K_T(\chi) = \chi^+$. So, if λ is regular, then there is a model in $p \subset (T_1, T)$ of cardinality λ which is saturated by Lemma 4.1.2. If λ is singular, then $\lambda > \chi = \inf\{\chi: \chi \geqq |T|, \chi^{\aleph_0} > \chi\}$, and so $K_T(\chi) = \chi^+$, and as $\chi^{\aleph_0} > \chi$, this implies that T is super stable. As $K_T(|T_1|) = |T_1|^+$, by 2.2 $K_T(\lambda) = \lambda^+$, and so by Lemma 4.4.1 $p \subset (T_1, T)$ has a $|T_1|^+$-saturated model in λ. Therefore by 4.3, $p \subset (T_1, T)$ is categorical in $\beth(\gamma \cdot \alpha)$ ($\gamma = (2^{|T|})^+$, $\alpha > 0$), and so $K_T(\mu) = \mu^+$ for every $\mu \geqq |T_1|$. That implies by Lemma 4.4, that in every power $\mu > |T_1|$ and regular $\chi \leqq \mu$, there exists a model of power μ in $p\subset (T_1, T)$, which is χ-saturated. So if $p\subset (T_1\ T)$ is categorical in μ, its only model in μ is saturated. It is clear that if $p\subset (T_1, T)$ has only saturated models in $\mu > |T|$, then it is categorical in μ.

Remark. In 4.3 and 4.5.3 we apply a two-cardinal theorem to a categoricity theorem. In fact, a more general connection exists among the following conditions on χ, λ, μ ($\chi \leqq \lambda, \mu$):

1) If $|T| < \chi$ and T has a model which omits a type p and such that $\|M\| = \lambda$, $|Q(M)| < \lambda$, then T has a model M' which omits p such that $\mu = \|M'\| > |Q(M')|$.

2) If $|T_1| < \chi$ and $p \subset (T_1\ T)$ is categorical in μ, then it is categorical in λ.

3) If $|T_1| < \chi$ and every model of power μ in $p \subset (T_1, T)$ is homogeneous, then the same holds for λ.

1 implies 3. (Keisler proves this in [1].) 1 implies 2 if $\mu \neq \inf\{\lambda_1 : \lambda_1^+ \geqq \chi, \lambda_1^{\aleph_0} \geqq \lambda_1\}$ or if $\chi_1 < \chi \Rightarrow \aleph(\chi_1, \omega) < \chi$. 3 implies 1 if χ is not greater than the first measurable cardinal, and there is no weakly compact χ_1 such that $\chi_1 < \chi \leqq (2^{\chi_1})^+$. 2 implies 1 if in addition $\mu \neq \inf\{\lambda_1 : \lambda_1^+ \geqq \chi, \lambda_1^{\aleph_0} \geqq \lambda\}$ or $\chi_1 < \chi \Rightarrow \aleph(\chi_1, \omega) < \chi$.

Theorem 4.6. *If T is categorical in a power* λ, $\lambda > |T|$, $\lambda \neq \inf\{\chi: \chi^{\aleph_0} > \chi + |T|\}$, *then there exists a cardinal* λ_0, *such that T is categorical in every cardinal*

$\geqq \lambda_0$, *and is not categorical in any power* χ, $|T| < \chi < \lambda_0$. *Furthermore* λ_0 *is such that* $\lambda_0 < \mu(|T|) < \beth((2^{|T|})^+)$.

Proof. If for every $\chi < \mu(|T|)$ T has a model M which is not $|T|^+$-saturated, $\|M\| \geqq \chi$, then it has such a model in every cardinal $> |T|$ a contradiction by 4.4. (For if $A \subset M$, $|A| \leqq |T|$, $p \in S(A)$, and p is omitted, we adjoin to M the constants $\{c_i : i < |T|\}$ a names for the element of A, and relations as in 4.3, and the result follows by the definition of $\mu(|T|)$.) Now if M is a $|T|^+$-saturated but not saturated model of T, then there exists $A \subset M$, $|A| < \|M\|$, $|A| > |T|$, $p \in S(A)$, such that p is omitted. As $K_T(|A|) = |A|^+$ and $\|M\| > |A| > |T| \Rightarrow \|M\| > |T|^+$, there exists an indiscernible set Y over A, $|Y| = |A|^+$, by Theorem 3.1. If $Y = \{y_i : i < |A|^+\}$, let $B = A \cup \{y_i : i < \chi\}\}$, where $\{y_i : i < \chi\}$ is indiscernible over A, and M_1 be a prime model over B among the $|T|^+$-saturated models, which exists by 3.2. Now it will be proved that p is not realized in M_1. In the construction of M_1, we adjoined to B the elements of $\{c_i : i < |B|\}$ one after another, such that c_j realizes a $|T|^+$-isolated type on $B \cup_i \{c_i : i < j\}$, defined by p_j, $|p_j| < |T|^+$. If c_k realizes p, let $B_1 = \{c_k\}$, and $B_{i+1} = B_i \cup \{b : b$ is mentioned in p_l, and $c_l \in B_i\}$. Now $|\bigcup_i B_i| \leqq |T|$, and it can be easily seen that in a prime model over $A \cup (\{y_i : i < \chi\} \cap \bigcup_{i<\omega} B_i)$, p is realized, and so it is realized in M a contradiction, so p is not realized in M_1. As we can take $\chi = \beth_\gamma, (2^{|T|})^+ | \gamma, \chi > |A|$, it follows that T has a non-saturated model in χ, in contradiction to 4.3, 4.4.3. So every $|T|^+$-saturated model is saturated. If T is not categorical in λ_1, then it has a non-$|T|^+$-saturated model of cardinality λ_1, and so T is not categorical in any cardinal λ_2. $|T| < \lambda_2 \leqq \lambda_1$. As we have shown that there exits a cardinal $\lambda < \mu(|T|)$ in which every model of T is $|T|^+$-saturated, the theorem follows.

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THE PROBLEM OF PREDICATIVITY

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One of the principal objectives of the foundations of mathematics is to "explain", as far as possible, the basic concepts of mathematics. In the simplest case, concepts are explained in terms of other concepts. Thus the concept of rational number is explained in terms of the simpler concepts of integer and ordered pair. The object is not to reduce all problems about rational numbers to problems about integers and ordered pairs, but to be able to say that we understand the notion of a rational number as fully as we understand the notions of integer and ordered pair.

In the case of such fundamental concepts as natural number or set, such a procedure is impossible; we cannot hope to explain these concepts fully in terms of other concepts.(1) However, we can hope to explain the concept of set in stages. Thus we might start with certain concepts which we regard as being thoroughly understood; these might include some particularly simple sets which we feel offer no problem to the understanding. In terms of these, we may explain certain further sets. We then explain still further sets in terms of these, and continue the process indefinitely (perhaps into the transfinite). The sets obtained in this way are said to be predicative,(2) or, more precisely, predicative in terms of the concepts with which we started.

It seems unlikely that the predicative sets include all sets, or even all sets required to exist by the axioms of set theory. For these axioms require the existence of sets with impredicative definitions, i.e. defini-

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(1) The Frege-Russell definition of natural number in terms of sets, although of great interest for the study of axiomatics, is hardly such an explanation; for the notion of an arbitrary set is surely much more complicated than that of natural number.

(2) This seems to be in rough agreement with Russell's notion of predicativity. Of course, even when the basic concepts are established, our definition is quite imprecise. The study of the various possible precise notions of predicativity and their relations to one another is the chief problem of predicativity.

tions containing quantifiers over a class of sets containing the set being defined. Of course, such sets may have other definitions which are not impredicative; but it is doubtful that this is always the case.

Several authors have investigated the possibility of doing mathematics with only predicative sets. Although much of interest has been discovered in this way, it seems quite unlikely that this program can be fully realized. But even if it were, this would not be a final answer to the problem of predicativity. For the notion of an impredicative set is of considerable interest, and should be investigated as far as possible.

We shall discuss only the simple case of sets of natural numbers.(3) We consider two notions which may be applied to the study of predicativity, namely, Kleene's analytical hierarchy and Gödel's constructible sets,(4) and obtain relations between them.

We refer to [6] for information on the analytical hierarchy; we shall follow the notation used there. We also use the notation of [1] for the positions in the hierarchy. Thus a set which is in the form of the analytical hierarchy having n function quantifiers beginning with an existential (universal) quantifier will be referred to as a Σ_n^1 predicate (Π_n^1 predicate). A predicate which is both a Σ_n^1 predicate and a Π_n^1 predicate will be called a Δ_n^1 predicate. We generally omit the superscripts, since we are not interested in other hierarchies. Finally, we agree that, when not otherwise specified, "number" shall mean "natural number"; "set" shall mean "set of numbers"; and "function" shall mean "function whose arguments and values are numbers".

Now an analytical set is one which may be defined by means of variables for numbers and functions; quantifiers over these variables; propositional connectives; symbols for the individual numbers; and symbols for primitive recursive predicates (of numbers and functions). If we regard the individual numbers, the set of all numbers, the propositional connectives, and the primitive recursive predicates as basic notions, then the number of function quantifiers in such a definition is a measure of the impredicativity of the definition.(5) Hence the analytical

(3) This seems the simplest case which one can study; for finite sets present no problems, and the set of natural numbers is, conceptually, the simplest infinite set. However, the principal difficulties concerning predicativity already arise in this simplest case.

(4) It is interesting to note that both of these notions were originally developed to deal with problems quite other than those considered here.

(5) Of course, one should not assume the functions as basic notions, since a function has the same order of complexity as a set.

hierarchy becomes an arrangement of the analytical sets in order of increasing impredicativity.

From this point of view the arithmetical sets, which are just the analytical sets whose definitions use no function quantifiers, are predicative. If we are willing to expand our class of basic notions a little, we can say much more. For Kleene [7] has shown that every Π_1 set is of the form $\hat{x}(\alpha(x) \in O)$, where α is a primitive recursive function and O is the set of Church-Kleene notations for constructive ordinals. Since the definition of O is of a predicative nature, it seems reasonable to add O to our list of basic concepts. It then follows that analytical sets whose definitions require only one function quantifier are predicative.

To study the case of more quantifiers, we use the notion of constructible set [5]. The constructible sets may be regarded as the sets which are predicative when the notion of an arbitrary ordinal is added to the list of basic concepts above.(6) This notion can hardly be regarded as predicative; so constructible sets are predicative in only a weak sense. This is modified somewhat by the fact that only countable ordinals need be used; for it is proved in [5] that the order of any constructible set (of numbers) is a countable ordinal.

We shall show that analytical sets whose definitions require only two function quantifiers are constructible, and hence predicative in a weak sense. This will be a consequence of the following result.

THEOREM. *If $A(\alpha,\beta)$ is a Π_1 predicate, β_0 is a constructible function, and the class of α such that $A(\alpha,\beta_0)$ is non-empty, then this class contains a constructible function.*

This result may be expressed in another way. We shall use a subscript L on a function quantifier to indicate that the quantifier is over all constructible functions.

COROLLARY 1. If $A(\alpha,\beta,\gamma)$ is arithmetical and γ_0 is constructible, then

$$(1)\qquad (\alpha)(E\beta)\,A(\alpha,\beta,\gamma_0) \equiv (\alpha)_L(E\beta)_L A(\alpha,\beta,\gamma_0),$$

$$(2)\qquad (E\alpha)(\beta)\,A(\alpha,\beta,\gamma_0) \equiv (E\alpha)_L(\beta)_L A(\alpha,\beta,\gamma_0).$$

(6) This fact is not too apparent from the definition in [5], but is quite clear from the original definition of [4].

PROOF.(7) Using the theorem,

$$(E\alpha)(\beta)A(\alpha,\beta,\gamma_0) \rightarrow (E\alpha)_L(\beta)A(\alpha,\beta,\gamma_0) \rightarrow E\alpha)_L(\beta)_L A(\alpha,\beta,\gamma_0).$$

Again using the theorem, we have for α constructible

$$(E\beta)\bar{A}(\alpha,\beta,\gamma_0) \rightarrow (E\beta)_L\bar{A}(\alpha,\beta,\gamma_0).$$

Contraposing and adding a quantifier $(E\alpha)_L$, we get

$$(E\alpha)_L(\beta)_L A(\alpha,\beta,\gamma_0) \rightarrow (E\alpha)_L(\beta)A(\alpha,\beta,\gamma_0) \rightarrow (E\alpha)(\beta)A(\alpha,\beta,\gamma_0).$$

This proves (2), and (1) follows by duality.

From (2) we readily obtain

$$(3) \qquad (\alpha)(E\beta)(\gamma)A(\alpha,\beta,\gamma) \rightarrow (\alpha)_L(E\beta)_L(\gamma)_L A(\alpha,\beta,\gamma).$$

It follows from (2) that if $A(\alpha,\beta,x)$ is arithmetical, then

$$\hat{x}(E\alpha)(\beta)A(\alpha,\beta,x) = \hat{x}(E\alpha)_L(\beta)_L A(\alpha,\beta,x).$$

Now, as we shall note later, every arithmetical predicate is absolute (in the sense of [5]). Hence the relativization of the theorem of set theory which asserts that $\hat{x}(E\alpha)(\beta)A(\alpha,\beta,x)$ is a set asserts that $\hat{x}(E\alpha)_L(\beta)_L A(\alpha,\beta,x)$ is a constructible set. Since the complement of a constructible set is constructible, we have proved:

COROLLARY 2. Every Σ_2 or Π_2 set of numbers is constructible.

Can these results be extended to more quantifiers? A difficulty in answering this question is that we do not know if every set is constructible, or even if the existence of a non-constructible set is consistent with the axioms of set theory. Of course, all the above results become trivial (for any number of quantifiers) if every set is constructible.

(7) The implication in one direction of (1) and (2) was proved by Mostowski [8].

If we assume the existence of a non-constructible set, we can show that Corollary 1 does not extend to more quantifiers, i.e., that the converse of (3) does not hold in general. According to Addison [2], there is an arithmetical predicate A such that

$$\alpha \text{ is constructible} \equiv (E\beta)(\gamma)\,A(\alpha,\beta,\gamma).$$

If the converse of (3) held, we would have

$$\begin{aligned}(E\alpha)(\alpha \text{ is not constructible}) &\rightarrow (E\alpha)(\beta)(E\gamma)\bar{A}(\alpha,\beta,\gamma)\\ &\rightarrow (E\alpha)_L(\beta)_L(E\gamma)_L\bar{A}(\alpha,\beta,\gamma)\\ &\rightarrow (E\alpha)_L(\beta)(E\gamma)\bar{A}(\alpha,\beta,\gamma)\\ &\rightarrow (E\alpha)_L\,(\alpha \text{ is not constructible})\end{aligned}$$

(where we have used Corollary 1). But the right hand side is clearly a contradiction. We do not know if the existence of a non-constructible set implies the existence of a non-constructible set which is a Σ_3 or a Π_3 set; but this seems quite likely.

The Δ_2 sets have proved to be of particular interest in both logical and topological investigations. Our results show that each Δ_2 set is predicative in terms of the concept of countable ordinal. Actually, a much smaller set of ordinals will do. Let us call σ a Δ_2 ordinal if it is the ordinal of a Δ_2 well-ordering of a set of numbers. If a set C is constructible and its order is a Δ_2 ordinal, then the methods of [2] show that C is a Δ_2 set. Conversely, let C be a Δ_2 set. Then there is an α satisfying the following condition: α is the characteristic function of a well-ordering of a set of ordinals whose ordinal is the order of C. But the methods of [2] show that this is a Δ_2 predicate of α. Hence by the Kondo-Addison theorem [3], α may be chosen recursive in Δ_2 predicates. It follows that the order of C is a Δ_2 ordinal. We have thus proved the following theorem.

THEOREM. *The Δ_2 sets of numbers coincide with the constructible sets of numbers whose orders are Δ_2 ordinals.*

Thus the Δ_2 sets are predicative in terms of the notion of Δ_2 ordinals. It seems doubtful if the notion of Δ_2 ordinal can be explained in terms of any essentially simpler notion.[8]

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(8) By contrast, Spector [9] has proved that the Δ_1 ordinals are just the constructive ordinals.

We now turn to the proof of our main theorem. We shall use the notation of [5], except that we shall use α, β, and γ as special variables for functions and σ as a special variable for ordinals.

We first recall that an arithmetic predicate can be expressed in terms of free variables for numbers and functions; quantifiers over number variables; equality; propositional connectives; and addition and multiplication of numbers. All of these are either proved absolute in [5], or can easily be proved absolute by the methods used there. (In the case of addition and multiplication, we make use of Peano's inductive definition of these operations.) It follows that every arithmetical predicate is absolute.

LEMMA. If $(Ey)\mathfrak{A}(x,y) \equiv (y)\mathfrak{B}(x,y) \equiv \mathfrak{C}(x)$, where $\mathfrak{A}$ and $\mathfrak{B}$ are absolute notions, then $\mathfrak{C}(x)$ is absolute.

PROOF. Relativizing the given equivalence,

$$(E\bar{y})\mathfrak{A}(\bar{x},\bar{y}) \equiv (\bar{y})\mathfrak{B}(\bar{x},\bar{y}) \equiv \mathfrak{C}_l(\bar{x}).$$

Hence

$$\begin{aligned} \mathfrak{C}(\bar{x}) &\equiv (y)\ \mathfrak{B}(\bar{x},y) \\ &\supset (\bar{y})\ \mathfrak{B}(\bar{x},\bar{y}) \\ &\equiv \mathfrak{C}_l\ (\bar{x}) \\ &\equiv (E\bar{y})\ \mathfrak{A}(\bar{x},\bar{y}) \\ &\supset (Ey)\ \mathfrak{A}(\bar{x},y) \\ &\equiv \mathfrak{C}(\bar{x}). \end{aligned}$$

COROLLARY. $x\,\mathfrak{We}\,y$ is absolute.

PROOF. We have

$$\begin{aligned} x\,\mathfrak{We}\,y &\equiv (u)[y\,\mathfrak{Con}\,x\,[(u \neq 0 \wedge u \subseteq x) \supset (Ev)(v \in u \wedge u \cdot y``\{v\} = 0)]] \\ &\equiv (Eu)(E\sigma)[y \subseteq x^2 \wedge u\,\mathfrak{Fn}\,x \wedge \mathfrak{Un}\,(u^{-1}) \wedge \mathfrak{W}(u) \subseteq \sigma \wedge \\ &\qquad (w)(z)(\langle w,z\rangle \in y \equiv u`w \in u`z)]. \end{aligned}$$

The parts in brackets are proved absolute by the usual methods.

Now let $A(\alpha, \beta)$ be a Π_1 predicate. By [7, § 26], there is a number e such that

$$(4) \qquad A(\alpha,\beta) \equiv R^{\alpha,\beta} \text{ is a well-ordering,}$$

where $R^{\alpha,\beta}$ is the predicate which is recursive in α,β with Gödel number e. Using the absoluteness of arithmetical predicates, we see that $R^{\alpha,\beta}$ is absolute. Hence by the above corollary and (4), $A(\alpha,\beta)$ is absolute.

We now show that $(E\alpha)\,A(\alpha,\beta)$ is absolute. Using (4), this is equivalent to

$$(E\alpha)(Eu)(E\sigma)[u\,\mathfrak{Fn}\,\omega \wedge \mathfrak{Un}(u^{-1}) \wedge \mathfrak{W}(u) \subset \sigma \wedge (i)(j)(\langle i,j\rangle \in R^{\alpha,\beta} \equiv u`i \in u`j)].$$

Using the absoluteness of $R^{\alpha,\beta}$, this can be rewritten

$$(E\alpha)(Eu)(E\sigma)[u\,\mathfrak{Fn}\,\omega \wedge \mathfrak{W}(u) \subseteq \sigma \wedge (i)\,\mathfrak{A}(\alpha \restriction i, u \restriction i, \beta)],$$

where $\mathfrak{A}$ is absolute. By the usual methods for contraction of functions, this is equivalent to

$$(5) \qquad (Eu)(E\sigma)(u\,\mathfrak{Fn}\,\omega \wedge \mathfrak{W}(u) \subseteq \sigma \times \omega \wedge (i)\,\mathfrak{B}(u \restriction i, \beta))$$

with B absolute.

We now show, by methods similar to that used in proving (4), that (5) is equivalent to

$$(E\sigma)(\mathfrak{C}(\beta,\sigma) \text{ is not a well-ordering})$$

where $\mathfrak{C}$ is an absolute operation; in view of the above corollary and the absoluteness of ordinals, this will imply that $(E\alpha)\,A(\alpha,\beta)$ is absolute. Since the proof of the equivalence is the same as that in [7], we shall confine ourselves to describing $\mathfrak{C}(\beta,\sigma)$. We think of $\sigma \times \omega$ as well-ordered by the usual well-ordering of pairs of ordinals [5, 7.81]. The field of $\mathfrak{C}(\beta,\sigma)$ consists of all u such that for some j,

$$u\,\mathfrak{Fn}\,j \wedge \mathfrak{W}(u) \subseteq \sigma \times \omega \wedge (i)\,(i \in j \supset \sim \mathfrak{B}(u \restriction i, \beta)).$$

If u and v are in the field of $\mathfrak{C}(\beta, \sigma)$, then u precedes v in the ordering $\mathfrak{C}(\beta, \sigma)$ if either $v = u \upharpoonright i$, where $i \in \mathfrak{D}(u)$; or there is an $i \in \mathfrak{D}(u) \cdot \mathfrak{D}(v)$ such that $u \upharpoonright i = v \upharpoonright i$ and $u\text{‘}i$ precedes $v\text{‘}i$ in the ordering of $\sigma \times \omega$. The proof that $\mathfrak{C}(\beta, \sigma)$ is absolute is standard.

Now since $\mathrm{A}(\alpha, \beta)$ and $(E\alpha)\,\mathrm{A}(\alpha, \beta)$ are absolute, we have for β_0 constructible

$$\begin{aligned}(E\alpha)\,\mathrm{A}(\alpha, \beta_0) &\equiv (E\alpha)_L\,\mathrm{A}_e(\alpha, \beta_0)\\ &\equiv (E\alpha)_L\,\mathrm{A}(\alpha, \beta_0),\end{aligned}$$

which is the desired result.

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On the Singular Cardinals Problem*

Jack Silver

In this paper we show, for example, that if the GCH holds for every cardinal less than κ, a singular cardinal of uncountable cofinality, then the GCH holds at κ itself. This result is contrary to the previous expectations of nearly all set-theorists, including myself. Another consequence of Theorem 1.1 is that if the GCH holds for every singular cardinal cofinal with ω, then it holds for every singular cardinal.

The immediate stimulus for this result was some work of Kanamori and Magidor[1] concerning nonregular uniform ultrafilters over ω_1. The other principal influences were a result of Scott concerning the GCH at measurable cardinals, some work of Keisler on ultrapowers of the sort defined in 1.3, the two-cardinal theory developed by several model-theorists, some work of Prikry and Silver on indecomposable ultrafilters [3], [4], as well as Cohen's methods and work on nonstandard models of set theory [2].

Our terminology is mostly standard. If κ is a cardinal, S is called a *stationary subset of* κ if it intersects every closed cofinal subset of κ. A function $h: \lambda \to \kappa$ is *continuous* if, for every limit ordinal $\alpha \in \lambda$, $h(\alpha)$ is the least upper bound of $\{h(\beta): \beta \in \alpha\}$. If κ is a cardinal, κ^+ is the least cardinal greater than κ. Also, $\kappa^{(\beta)}$ is the βth cardinal greater than κ. Thus $\kappa^{(0)} = \kappa$, $\kappa^{(1)} = \kappa^+$, etc. The cofinality of κ is λ iff λ is the least cardinal such that κ can be written as a union of λ sets, each of cardinality $< \kappa$. κ is singular iff its cofinality is $< \kappa$.

1. Model-theoretic preliminaries. Suppose $\langle A, E \rangle$ is a model of ZFC, i.e., A is the universe of sets and E the membership relation for the model. If $a \in E$, let a_E be the

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[1]The result of Magidor states, in particular: If there is a regular, nonuniform ultrafilter over ω_1 and $2^{\aleph_\alpha} = \aleph_{\alpha+1}$ for all $\alpha < \omega$, then $2^{\aleph_{\omega_1}} = \aleph_{\omega_1+1}$.

E-extension of a, i.e., $\{b \in A : b\ E\ a\}$. We sketch the proofs of two well-known lemmas, the first of which establishes a relation between the cardinalities of a_E and b_E where a is a cardinal in the sense of $\langle A, E\rangle$, and b is the successor cardinal in the sense of $\langle A, E\rangle$. Note that only Lemma 1.1 is needed for the GCH form of the main result.

LEMMA 1.1. *Suppose $\langle A, E\rangle$ is a model of ZFC, and $a, b \in A$ are such that $\langle A, E\rangle \models$ a is a cardinal, and $\langle A, E\rangle \models b = a^+$ (i.e., $\langle A, E\rangle \models b$ is the successor cardinal of a). Then* card $b_E \leqq (\text{card } a_E)^+$.

PROOF. Let $\mu = \text{card } a_E$. We claim that E totally orders b_E in such a way that every member of b_E has at most μ predecessors. This will be sufficient since any ordered set whose every element has at most μ predecessors must itself have cardinality at most μ^+.

Clearly E totally orders b_E since $\langle A, E\rangle \models b$ is an ordinal, and E is the membership relation. It only remains to see that if $c \in b_E$, then c has at most μ E predecessors. Since $\langle A, E\rangle \models b = a^+ \wedge c \in b$, we have $\langle A, E\rangle \models \text{card } c \leqq b$, so there exists an element $g \in A$ such that $\langle A, E\rangle \models g$ is a 1-1 mapping of c into a. One easily verifies that $\{\langle u, v\rangle : \langle A, E\rangle \models g(u) = v\}$ is really a 1-1 mapping of c_E into a_E, whence card $c_E \leqq \text{card } a_E = \mu$. But c_E is just the set of E predecessors of c.

LEMMA 1.2. *Suppose $\langle A, E\rangle$ is a model of ZFC, and a, b, $d \in A$ are such that $\langle A, E\rangle \models a$ is a cardinal, $\langle A, E\rangle \models b = a^{(d)}$, and d_E has order type δ, an ordinal, under E. Then* card $b_E \leqq (\text{card } a_E)^{(\delta)}$.

PROOF. Lemma 1.1 enables one to carry out an easy induction on δ.

We now sketch some methods used by Keisler in his first proof of the two-cardinal transfer theorems for ω-logic.

DEFINITION 1.3. Suppose M is a transitive model of ZFC and, for some ordinal Υ, D is an ultrafilter in $P\Upsilon \cap M$. We define Ult(M, D) and the canonical injection.

Let $S = \{f \in M : f : \Upsilon \to M\}$. Define an equivalence relation $\sim_D$ on S by $f \sim_D g$ if $\{i \in \Upsilon : f(i) = g(i)\} \in D$. If $f \in S$, let

$$\begin{aligned} f/D &= \text{the equivalence class of } f \text{ with respect to } \sim_D \\ &= \{g \in S : g \sim_D f, \text{ and nothing in } S \text{ of rank smaller than that of } g \text{ is } \sim_D f\}. \end{aligned}$$

Finally, Ult(M, D) is that structure $\langle A, E\rangle$ where $A = \{f/D : f \in S\}$ and $(f/D)E(g/D)$ iff $\{i \in \Upsilon : f(i) \in g(i)\} \in D$. The canonical injection j of M into Ult(M,D) is defined by $j(x) = c_x/D$ where $c_x : \Upsilon \to \{x\}$ is the constant function x.

LEMMA 1.4. *If M, D, and S are as in Definition* 1.3, *and $f_1, \cdots, f_n \in S$, then*

$$\text{Ult}(M, D) \models \varphi(f_1/D, \cdots, f_n/D) \textit{ iff } \{i \in \Upsilon : \langle M, \varepsilon\rangle \models \varphi(f_1(i), \cdots, f_n(i))\} \in D,$$

φ any first-order formula. Hence the canonical injection j is an elementary monomorphism.

PROOF. One proceeds as usual by induction on formulas. In handling the existential quantifier step (the only nontrivial step), one uses the fact that the axiom of choice holds in M.

To avoid metamathematical complications, we systematically ignore the fact that satisfaction cannot be defined for the structure M and $\mathrm{Ult}(M, D)$ we will be using. There are well-known devices for handling this technical difficulty.

2. The main theorems.

THEOREM 2.1. *If κ is a singular cardinal of uncountable cofinality and $\{\nu < \kappa : 2^\nu = \nu^+\}$ is a stationary subset of κ, then $2^\kappa = \kappa^+$.*

PROOF. Let $T = \{\nu < \kappa : 2^\nu = \nu^+\}$ and let λ be the cofinality of κ. Suppose that h is a continuous, strictly-increasing map of λ onto a cofinal subset of κ. One easily shows that $\{\alpha < \lambda : h(\alpha) \in T\}$, which we call X, is a stationary subset of λ. Thus X is a stationary subset of λ such that, for all $\alpha \in X$, $2^{h(\alpha)} = h(\alpha)^+$.

Let $\mu = 2^\lambda$. Using either the method of Cohen or the method of Boolean-valued models, we can form an extension of the original universe in which μ is countable, but such that all cardinals greater than μ are preserved. Henceforth we work in that extension and call the original universe M. Thus, if ν is a cardinal of M and ν exceeds μ, then ν is really a cardinal. Moreover, if $U = {}^\lambda\lambda \cap M$ = set of functions from λ into λ which are members of M, then U is countable since it is in 1-1 correspondence with μ. It is our objective to show that, in M, $2^\kappa = \kappa^+$ holds.

A function $f : \lambda \to \lambda$ is called *regressive* if, for all $\alpha \neq 0, f(\alpha) < \alpha$. Since U is countable, there is an ultrafilter D in $P\lambda \cap M$ such that $X \in D$ and every regressive member of $U = {}^\lambda\lambda \cap M$ is constant on some member of D. To see this, let $\{f_i : i \in \omega, i > 0\}$ be the set of regressive members of U. Form a sequence $X_0 \supseteq X_1 \supseteq X_2 \supseteq \cdots$ of subsets of λ, each in M and stationary subsets of λ in the sense of M, such that $X_0 = X$ and f_i is constant on X_i. This is possible by a theorem of Fodor [1], which says that if X_i is stationary and f_{i+1} regressive, then there is a stationary subset of X_i on which f_{i+1} is constant (the regularity of λ is also used). Finally, let $D = \{B \in M : B \subseteq \lambda, B \text{ includes some } X_i\}$.

Form $\mathrm{Ult}(M, D) = \langle A, E \rangle$ and let j be the canonical injection of M into $\mathrm{Ult}(M, D)$. Let e be the element of A represented by the identity function from λ into λ. The basic property of D implies that the set of E predecessors of e is precisely $\{j(\alpha) : \alpha < \lambda\}$. Since h is continuous, $j(h)$ is continuous in the sense of $\langle A, E \rangle$. Therefore, if $d = j(h)(e)$, then every predecessor of d is a predecessor of some $j(h)(j(\alpha)) = j(h(\alpha)), \alpha < \lambda$. But $j(h(\alpha))$ has fewer than κ predecessors, for each such predecessor is represented by some member of M which maps λ into $h(\alpha)$, and, κ being a strong limit cardinal in M, there are fewer than κ such functions. Hence d has exactly κ E predecessors.

Since $\{\alpha < \lambda : 2^{h(\alpha)} = h(\alpha)^+\} \in D$, Lemma 1.4 assures us that in $\mathrm{Ult}(M, D)$, $2^{j(h)(e)} = j(h)(e)^+$, i.e., $\mathrm{Ult}(M, D) \models 2^d = d^+$. Let b be such that $\mathrm{Ult}(M, D) \models d^+ = b$. Let $Q = \{Z : \mathrm{Ult}(M, D) \models Z \subseteq d\}$. Since $\mathrm{Ult}(M, D) \models 2^d = b$, there is a 1-1 map of Q into b_E. By Lemma 1.1, there is a 1-1 map of b_E into κ^+. Hence there is a 1-1 map of Q into κ^+.

We complete the argument by showing that, if $2^\kappa = \kappa^+$ fails in M, then there is a 1-1 map of κ^{++} into Q, contradicting the preceding paragraph. By preservation of cardinals $> \mu$, $(\kappa^{++})^M = \kappa^{++}$. Hence, if $2^\kappa = \kappa^+$ fails in M, there is a 1-1 sequence $\langle C_\alpha\colon \alpha < \kappa^{++}\rangle$ of subsets of κ, each $C_\alpha \in M$. Set $k(\alpha) =$ that B such that $\mathrm{Ult}(M, D) \models B = d \cap j(C_\alpha)$. k is 1-1, for if $\gamma \in C_\alpha - C_\beta$, then $j(\gamma)Ek(\alpha)$ while not $j(\gamma)Ek(\beta)$.

THEOREM 2.2. *If κ is a singular cardinal of uncountable cofinality λ and β is an ordinal $< \lambda$ such that $\{\nu < \kappa\colon 2^\nu \leqq \nu^{(\beta)}\}$ is a stationary subset of κ, then $2^\kappa \leqq \kappa^{(\beta)}$.*

PROOF. One proceeds much as in the proof of Theorem 2.1, using Lemma 1.2 instead of 1.1.

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Automorphisms of the lattice of recursively enumerable sets Part I: Maximal sets

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TABLE OF CONTENTS

Introduction

Let $\mathfrak{E}$ denote the lattice of recursively enumerable (r.e.) sets under inclusion, and let $\mathfrak{E}^*$ denote the quotient lattice of $\mathfrak{E}$ modulo the ideal $\mathcal{F}$ of finite sets. For $A \in \mathfrak{E}$ let A^* denote the equivalence class in $\mathfrak{E}^*$ which contains A. A r.e. set A is *maximal* if A^* is a coatom (maximal element) of $\mathfrak{E}^*$. Let Aut $\mathfrak{E}$ (Aut $\mathfrak{E}^*$) denote the group of automorphisms of $\mathfrak{E}$ ($\mathfrak{E}^*$). Our principal result is that for any two maximal sets A and B, there exists $\Phi \in$ Aut $\mathfrak{E}$ such that $\Phi(A) = B$. It easily follows that for each $k \geqq 1$ the group Aut $\mathfrak{E}^*$ is k-ply transitive on the coatoms of $\mathfrak{E}^*$.

This answers a question of D. A. Martin, and implies that there are no nontrivial elementary classes of maximal sets, thereby answering a question of A. H. Lachlan [1, p. 36]. Furthermore, it demonstrates greater uniformity of structure of $\mathfrak{E}$ than was supposed, and indirectly lends further evidence for the decidability of the elementary theory of $\mathfrak{E}$, a major open question studied by Lachlan in [2] and [4]. Further results and open questions on automorphisms are discussed in §7.

In contrast, we show that the above automorphism Φ cannot always be chosen to be recursively *presented*, in the sense that for some recursive

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permutation h of N, $\Phi(W_n^*) = W_{h(n)}^*$ for all n, where $\{W_n\}_{n \in N}$ is a fixed acceptable numbering [10, p. 41] of the r. e. sets. We also give some background information about Aut $\mathfrak{E}$ including results by Lachlan, Martin, and the author which positively answer the following questions posed by Rogers [10, pp. 228–229] of whether: (1) there exists $\Phi \in \text{Aut}\, \mathfrak{E}^*$ not induced by a recursive permutation; (2) every $\Phi \in \text{Aut}\, \mathfrak{E}^*$ is induced by some $\Psi \in \text{Aut}\, \mathfrak{E}$; (3) the properties of hypersimplicity, creativeness, and Turing degree are non-invariant under Aut $\mathfrak{E}^*$.

The proof of the main theorem involves a priority argument like those introduced by Sacks [11], [12], and [13], where opposing requirements interfere with one another infinitely often, but here an altogether different combinatorial device is needed to resolve this conflict. The usual "infinite-injury" priority argument (which has heretofore been used almost exclusively to study r.e. *degrees* rather than the structure of r.e. *sets*) has the following feature. At a given stage s, a negative requirement "restrains" certain elements from entering a particular r.e. set A in an attempt to preserve some computation $\{e\}_s^{A_s}(x)$. Sacks' resolution of conflicts depends upon the fact that a negative requirement cannot later restrain new elements, without first ceasing to restrain the old, since the computation must be destroyed and later reestablished.

In constructing an automorphism Φ, we specify a permutation h of N such that $\Phi(W_n^*) = W_{h(n)}^*$ for all $n \in N$. Positive requirements cause us to enumerate elements in $W_{h(n)}$ as W_n is enumerated, while an element once enumerated in say $W_{h(n+1)}$ may be restrained from entering $W_{h(0)}$ unless the cardinality of $W_0 \cap W_{n+1}$ is sufficiently large. Such negative requirements may restrain many new elements from $W_{h(0)}$ without unrestraining any of the old. However, infinitely many $n \in N$ may then combine to permanently restrain infinitely many elements from $W_{h(0)}$, thereby threatening the positive requirement corresponding to W_0. This conflict is resolved by a different method from those on r.e. degrees, and involves roughly "playing off" sets of lower priority against one another.

The major difficulty in the case of maximal sets A and B arises from the fact that we may be forced to enumerate elements in certain r.e. sets being matched, and that later these elements may be enumerated in A or B, disrupting the correspondence. Our essential tool is what we call the *Extension Theorem* (Theorem 2.2) which asserts roughly that if we can satisfy certain minimal hypotheses on those elements which arrive in A or B after being enumerated in certain r.e. sets being matched, then that matching may be extended to an automorphism. In §2 we motivate the

Extension Theorem, and in §§4, 5, and 6 we prove it. In §3 we verify the hypotheses of the Extension Theorem for the maximal set case. The Extension Theorem also plays a crucial role in other automorphism results discussed in §7.

Unexplained notation and conventions can be found in Rogers [10]. Let $A =^* B$ denote that the symmetric difference of A and B is finite. Hence, $A =^* B$ just if $A^* = B^*$. Let $A \subseteq^* B$ denote that $A \cap \bar{B} =^* \emptyset$. For any class $\mathcal{C} \subseteq \mathcal{E}$ let $\mathcal{C}^* = \{A^*: A \in \mathcal{C}\}$, and all $\mathcal{C}$ a *skeleton* if $\mathcal{C}^* = \mathcal{E}^*$. For $A, B \in \mathcal{E}$ we write $A \equiv B$ if there is a recursive permutation p (of N) such that $p(A) = B$, and we write $A \equiv_{\mathcal{E}} B$ ($A^* \equiv_{\mathcal{E}^*} B^*$) if there exists $\Phi \in \operatorname{Aut} \mathcal{E}$ ($\operatorname{Aut} \mathcal{E}^*$) such that $\Phi(A) = B$ ($\Phi(A^*) = B^*$). A *recursive array* is a recursive sequence of r.e. sets. A *simultaneous enumeration* of a given recursive array $\{U_n\}_{n \in N}$ is a 1:1 recursive function g with range $\{\langle m, n\rangle : m \in U_n\}$. Thus, at each stage s, $g(s) = \langle m, n\rangle$ causes *one* element m to be enumerated in *one* r.e. set U_n. Fixing g, let $U_{n,s}$ denote those elements enumerated in U_n by stage s, and

$$U_n \backslash U_m = \{x: (\exists s)[x \in U_{n,s} - U_{m,s}]\} ,$$

those elements appearing in U_n before U_m. (The notation $X \backslash Y$ should not be confused with $X - Y$ which denotes $X \cap \bar{Y}$.) We let $X \searrow Y$ denote $(X \backslash Y) \cap Y$. We use these notations only when we have in mind a *particular* simultaneous enumeration.

A *standard enumeration* (of the r.e. sets) is a simultaneous enumeration of $\{U_n\}_{n \in N}$, where the latter is some acceptable numbering of the r.e. sets. From now on we fix a standard enumeration of our acceptable numbering $\{W_n\}_{n \in N}$, thus yielding the double array $\{W_{n,s}: n, s \in N\}$ of finite approximations. It will be convenient later to introduce other recursive arrays $\{U_n\}_{n \in N}$ with simultaneous enumerations. With respect to such $\{U_{n,s}: n, s \in N\}$, we define the *e-state of an element x at stage s*,

$$\sigma(s, e, x) = \{i: i \leqq e \text{ and } x \in U_{i,s}\} .$$

The e-states, being finite sets, are identified with their characteristic functions which are ordered lexicographically as usual (see Rogers [10, p. 235]). We say e-state σ is *higher* than e-state τ if its characteristic function precedes that of τ.

1. Background material

In Chapter 12 of [10] Rogers presents much material on $\mathcal{E}$ and $\mathcal{E}^*$, and raises several questions concerning automorphisms of $\mathcal{E}$ and $\mathcal{E}^*$, and lattice-invariant properties. A property is *lattice-invariant* or *lattice-theoretic in*

$\mathfrak{E}(\mathfrak{E}^*)$ if it is preserved by every automorphism of $\mathfrak{E}$ (respectively $\mathfrak{E}^*$). We say that a permutation p (of N) *induces* an automorphism Φ of $\mathfrak{E}(\mathfrak{E}^*)$ if for all n, $p(W_n) = \Phi(W_n)$ (respectively $(p(W_n))^* = \Phi(W_n^*)$).

It is easy to see that any recursive permutation induces an automorphism of $\mathfrak{E}$, and that every automorphism of $\mathfrak{E}$ induces an automorphism of $\mathfrak{E}^*$. Kent [10, p. 233] uses a cohesive set to show that there are $2^{\aleph_0}$ automorphisms of $\mathfrak{E}$, but the induced automorphisms of $\mathfrak{E}^*$ are all trivial. Rogers [10, p. 229] asks whether:

(i) every automorphism of $\mathfrak{E}^*$ is induced by a recursive permutation;

(ii) every automorphism of $\mathfrak{E}^*$ is induced by an automorphism of $\mathfrak{E}$.

Lachlan (unpublished) has answered (i) by showing that there are $2^{\aleph_0}$ automorphisms of $\mathfrak{E}^*$, and we answer (ii) by proving that any $\Phi \in \operatorname{Aut} \mathfrak{E}^*$ is induced by a permutation (of N), and hence by some $\Psi \in \operatorname{Aut} \mathfrak{E}$.

However, in Lachlan's method, $\Phi \in \operatorname{Aut} \mathfrak{E}^*$ is induced by a permutation p which is obtained by "piecing together" recursive permutations in a nonrecursive fashion, in such a way that for any A and B, if $p(A) = B$ then $A \equiv B$. Thus, the method produces no new elements in the $\mathfrak{E}^*$*-orbit of* A^*, i. e., the set $\{B^*: A^* \equiv_{\mathfrak{E}^*} B^*\}$. To prove the noninvariance of hypersimplicity, creativeness, and Turing degree, one must use a different device such as Martin's method to generate new elements in the $\mathfrak{E}^*$-orbit of a nonrecursive $A \in \mathfrak{E}$. In fact we show that the $\mathfrak{E}^*$-orbit of *every* nonrecursive set $A \in \mathfrak{E}$ contains some $B \not\equiv A$.

THEOREM 1.1 (Lachlan). *There are* $2^{\aleph_0}$ *automorphisms of* $\mathfrak{E}^*$.

Proof. For each n, we define a recursive set R_n such that $R_n \subseteq R_{n-1}$, R_n is infinite, and $R_{n-1} - R_n$ is infinite. For convenience, let $R_{-1} = N$. Choose $R_n \subseteq R_{n-1} - W_n$ if $R_{n-1} \cap W_n$ is finite, and $R_n \subseteq R_{n-1} \cap W_n$ otherwise. In either case, make R_n recursive, R_n infinite, and $R_{n-1} - R_n$ infinite. Note that $R_n \subseteq W_n$ or $R_n \subseteq \bar{W}_n$, for all n.

For all $n \in N$ and $j \in \{0, 1\}$ choose partial recursive functions ψ_j^n which are permutations of $R_{n-1} - R_n$, such that ψ_0^n is the identity, and ψ_1^n carries S_n into some $T \neq^* S_n$ for some infinite r.e. $S_n \subseteq R_{n-1} - R_n$. For any function $f: N \to \{0, 1\}$, define the permutation p_f of N, such that p_f restricted to $R_{n-1} - R_n$ is $\psi_{f(n)}^n$. Clearly p_f is a permutation of N because each $m \in \bar{R}_n$ (and hence $m \in \operatorname{domain} p_f$), where $W_n = \{m\}$. Since $\psi_0^n(S_n) \neq^* \psi_1^n(S_n)$, p_f and p_g cannot induce the same automorphism on $\mathfrak{E}^*$ if $f \neq g$.

It suffices to show that $p_f(W_n)$ and $p_f^{-1}(W_n)$ are r.e. for each n. Clearly, $\bigcup \{\psi_{f(i)}^i: i \leq n\}$ is a partial recursive function with domain $\bar{R}_n$. Also $p_f(\bar{R}_n \cap W_n) = \bigcup \{\psi_{f(i)}^i(\bar{R}_n \cap W_n): i \leq n\}$ which is therefore r.e. Furthermore

$p_f(R_n \cap W_n) = R_n \cap W_n$ since either $R_n \subseteq W_n$, or R_n and W_n are disjoint. Likewise $p_f^{-1}(W_n)$ is r.e. ☒

COROLLARY 1.2. *Not every automorphism of* $\mathcal{E}^*$ *is induced by a recursive permutation of* N.

On the other hand, we shall show that every automorphism of $\mathcal{E}^*$ is induced by *some* permutation of N. First we need a convenient way to describe an arbitrary $\Phi \in \operatorname{Aut} \mathcal{E}^*$ relative to our fixed indexing $\{W_n\}_{n \in N}$.

Definition. $\Phi \in \operatorname{Aut} \mathcal{E}^*$ is *presented by* h, a permutation of N, if $\Phi(W_n^*) = W_{h(n)}^*$ for all n.

THEOREM 1.3. *Every* $\Phi \in \operatorname{Aut} \mathcal{E}^*$ *is induced by some permutation* p *of* N. *Furthermore, if* Φ *is presented by* h, p *can be chosen recursive in* h join $\varnothing''$.

Proof. Fix $\Phi \in \operatorname{Aut} \mathcal{E}^*$, and let Φ be presented by h. For any $x \in N$, define e-states,

$$\sigma(e, x) = \{i : i \leqq e \text{ and } x \in W_i\} ,$$
$$\hat{\sigma}(e, x) = \{i : i \leqq e \text{ and } x \in W_{h(i)}\} .$$

Note that the sets $\sigma(e, x)$ are uniformly recursive in $\varnothing'$, and $\hat{\sigma}(e, x)$ in h join $\varnothing'$. Let $p = \bigcup_n p_n$, where finite functions p_n are defined as follows. For convenience, let $p_{-1} = \varnothing$, and $\sigma(-1, x) = \hat{\sigma}(-1, x) = \varnothing$ for all x. For n even, let

$$x_n = (\mu x)[x \notin \text{domain } p_{n-1}] ,$$
$$e_n = \max \{e : -1 \leqq e \leqq n \text{ and } (\exists y)[y \notin \text{range } p_{n-1} \text{ and } \sigma(e, x_n) = \hat{\sigma}(e, y)]\} ,$$
$$y_n = (\mu y)[y \notin \text{range } p_{n-1} \text{ and } \sigma(e_n, x_n) = \hat{\sigma}(e_n, y)] .$$

Define $p_n(x_n) = y_n$. Note that e_n and hence y_n may be found recursively in h join $\varnothing''$. For n odd, we let $y_n = (\mu y)[y \notin \text{range } p_{n-1}]$, and proceed as above with x_n and y_n interchanged, and σ, $\hat{\sigma}$ interchanged. Clearly, $p = \bigcup_n p_n$ is a permutation and $p \leqq_T h$ join $\varnothing''$. Since $\Phi \in \operatorname{Aut} \mathcal{E}^*$ it easily follows by induction on n that $p(W_n) =^* W_{h(n)}$ for all n. ☒

COROLLARY 1.4. *Every* $\Phi \in \operatorname{Aut} \mathcal{E}^*$ *is induced by some* $\Psi \in \operatorname{Aut} \mathcal{E}$.

Proof. The above p induces $\Psi \in \operatorname{Aut} \mathcal{E}$ which induces Φ. ☒

COROLLARY 1.5. *If* $A, B \in \mathcal{E}$ *are infinite and coinfinite then* $A \equiv_{\mathcal{E}} B$ *if and only if* $A^* \equiv_{\mathcal{E}^*} B^*$.

Proof. If $A \equiv_{\mathcal{E}} B$ then clearly $A^* \equiv_{\mathcal{E}^*} B^*$. Conversely, if $A^* \equiv_{\mathcal{E}^*} B^*$ via Φ, then in the proof of Theorem 1.3, set $W_0 = A$ and $W_{h(0)} = B$, so that the permutation p obtained satisfies $P(A) = B$. ☒

COROLLARY 1.6. *Any property of* r.e. *sets which is well-defined on* $\mathcal{E}^*$ *is invariant under* Aut $\mathcal{E}$ *just if it is invariant under* Aut $\mathcal{E}^*$.

Note that the proof of Theorem 1.3 establishes the following fact which will be useful in §6.

COROLLARY 1.7. *Given any permutation* h *of* N *define* $\sigma(e, x)$ *and* $\hat{\sigma}(e, x)$ *as in the proof of Theorem* 1.3. *Then* h *presents some* $\Phi \in \text{Aut}\,\mathcal{E}^*$ *if and only if for every* e *and* e*-state* σ_0,

$$[\{x: \sigma(e, x) = \sigma_0\} \text{ is infinite}] \Leftrightarrow [\{x: \hat{\sigma}(e, x) = \sigma_0\} \text{ is infinite}] .$$

(Let $\mathcal{L}$ be any countable sublattice of the lattice $\mathfrak{N} = \langle 2^N; \subseteqq \rangle$ such that $\mathcal{L}$ is closed under finite differences and contains ϕ and N. The above theorem and corollaries clearly hold with $\mathcal{L}$ and $\mathcal{L}^*$ $(= \mathcal{L}/\mathcal{F})$ in place of $\mathcal{E}$ and $\mathcal{E}^*$ if $\emptyset'$ is replaced by join $\{A: A \in \mathcal{L}\}$, namely $\{\langle m, n\rangle: m \in A_n\}$ where $\{A_n\}_{n \in N}$ are the members of $\mathcal{L}$. The method of Theorem 1.3 can thus be used to give an alternate (but similar) proof of Lachlan's result [1, Lemma 14] that for such sublattices $\mathcal{L}_1$ and $\mathcal{L}_2$, $\mathcal{L}_1 \cong \mathcal{L}_2$ just if $\mathcal{L}_1^* \cong \mathcal{L}_2^*$.)

To present some $\Phi \in \text{Aut}\,\mathcal{E}^*$ it is clearly equivalent (and usually easier) to given function f and g satisfying

$$(1.1) \qquad (\forall n)\ [\Phi(W_n^*) = W_{f(n)}^* \text{ and } \Phi^{-1}(W_n^*) = W_{g(n)}^*] ,$$

in which case we say that Φ is *presented by the pair* $\langle f, g\rangle$. (Given f and g, one constructs a corresponding presentation h recursive in f join g by the well-known "padding" of indices [10, p. 83], and conversely given h, let $f = h$ and $g = h^{-1}$.)

All known constructions of some automorphism Φ consist of building a permutation p of N which induces the proposed Φ, and simultaneously giving functions f and g (not necessarily recursive) satisfying (1.1). The permutation p guarantees that the proposed Φ preserves the inclusion ordering (and is thus a 1:1 homomorphism from $\mathcal{E}$ into $\mathfrak{N}$), while the functions f and g insure that p and p^{-1} map $\mathcal{E}$ into $\mathcal{E}$.

Definition. An automorphism Φ of $\mathcal{E}^*$ is *effective* if Φ is presented by some recursive permutation h (or equivalently by a pair of recursive functions $\langle f, g\rangle$).

Clearly, any recursive permutation p induces an effective automorphism, but not conversely. For example, Martin's method always produces effective automorphisms, and yet suffices to prove noninvariance of hypersimplicity. Nevertheless, since effective automorphisms are not sufficient for the maximal set result, one must examine nonrecursive presentations h. For example, one might expect to get noneffective automorphisms corresponding to presentations $h \leqq_T \emptyset'$ but this is false.

THEOREM 1.8 (Jockusch). *If* $\Phi \in \text{Aut}\ \mathcal{E}^*$ *is presented by some* $h \leqq_T \emptyset'$, *the* Φ *is effective.*

Proof. Let Φ be presented by h where $h = \lim_s h_s$, and h_s is a recursive permutation for all s. Then Φ is effectively presented by the pair of recursive functions $\langle f, g\rangle$, where $W_{f(x)} = \bigcup_s W_{h_s(x),s}$, and $W_{g(x)} = \bigcup_s W_{h_s^{-1}(x),s}$. ⊠

THEOREM 1.9 (Martin). *There exists a hypersimple set* H *and* $\Phi \in \text{Aut}\ \mathcal{E}$ *such that* $\Phi(H)$ *is not hypersimple.*

Proof. We define effectively a sequence of recursive permutations p_s and an increasing sequence of finite sets H_s with $H = \bigcup_s H_s$. Let S_s denote $p_s(H_s)$. We shall have $S_s \subseteq S_{s+1}$ for all s. Let $\{D_n^e\}_{n \in N}$ be the e^{th} (candidate for a) strong array, arranged so that D_n^e is a finite set, and either D_n^e is defined for all n, or else is undefined for all $n >$ some m.

A number e is *satisfied at stage* s if either some D_n^e and D_m^e are defined by stage s and $D_n^e \cap D_m^e \neq \emptyset$, with $n \neq m$, or else D_n^e is defined and $\subseteq H_s$. The *complete e-state of* x *at stage* s is the pair of e-states $\langle \sigma, \tau\rangle$ where

$$\sigma = \{i : i \leqq e \text{ and } x \in W_{i,s}\}$$
$$\tau = \{i : i \leqq e \text{ and } p_s(x) \in W_{i,s}\}\ .$$

stage $s = 0$. Let p_0 be the identity, and $H_0 = \emptyset$.

stage $s + 1$. Look for a number e such that e is not satisfied at stage s, and such that there is an element D of the e^{th} strong array and a number $x \in D - H_s$ such that for some $y \notin H_s$:

(i) x and y are in the same complete e-state at s,

(ii) $\min(x, y, p_s(x), p_s(y)) \geqq e$, and

(iii) $p_s(y) \geqq 2x$.

If e does not exist, set $p_{s+1} = p_s$, and $H_{s+1} = H_s$. Otherwise, let e_s be the least such e, and x_s and y_s the least x and y corresponding to e_s. Let q be the permutation of N which transposes x and y and is the identity off $\{x, y\}$. Let $p_{s+1} = p_s q$, and $H_{s+1} = H_s \cup \{x_s\}$.

LEMMA 1. $S = \bigcup_s S_s$ *is coinfinite and not hypersimple.*

Proof. If $z \in S$, $z = p_{s+1}(x_0) \neq p_s(x_0)$ for some s, x_0. For each x_0, there can be only one such z. Since $z = p_s(y_s)$, condition (iii) insures that for every n there are $\leqq n$ members of S smaller than $2n$.

LEMMA 2. *If* $H = \bigcup_s H_s$ *is coinfinite,* H *is hypersimple.*

Proof. Assume to the contrary that $\{D_n^e\}_{n \in N}$ witnesses the nonhypersimplicity of H, with e minimal. If $i < e$, by the minimality of e the i^{th}

candidate for a strong array is finite or else i is eventually satisfied. Let s_0 be a stage such that for all $s \geqq s_0$, $e_s \geqq e$.

Since the complete e-state of x "increases" with s, owing to condition (i), each x goes to a final complete e-state as s goes to infinity. A complete e-state is *well-resided* if infinitely many members of $\bar{H}$ finally settle in it. Since all but finitely many members of $\bar{H}$ lie in well-resided complete e-states, there is an n such that either $D_n^e \subseteq H$ or $D_n^e - H \neq \emptyset$, and all $x \in D_n^e - H$ have $x \geqq e$, $p(x) \geqq e$ and are in well-resided complete e-states. In the latter case, if x_0 is the least such x, then there exists a stage $s \geqq s_0$, when some $y \geqq e$, with $p_s(y) \geqq 2x_0$ and $y \notin H_{s+1}$, is in the same complete e-state as x_0. Then $x_0 \in H_{s+1}$, a contradiction.

LEMMA 3. *$p = \lim_s p_s$ exists and is a permutation of N.*

Proof. If $e = e_s$, and either $p_s(x) \neq p_{s+1}(x)$ or $p_s^{-1}(x) \neq p_{s+1}^{-1}(x)$ then by condition (ii), $e \leqq x$. By Lemma 2, each e can be e_s only finitely often and thus for each x, $\lim_s p_s(x)$ and $\lim_s p_s^{-1}(x)$ exist.

LEMMA 4. *H is hypersimple.*

Proof. $H = p^{-1}(S)$, so H is coinfinite.

LEMMA 5. *p induces an automorphism of $\mathcal{E}$, and in fact an <u>effective</u> automorphism of $\mathcal{E}^*$.*

Proof. Recursively in $\emptyset'$ define the function $k(e)$ to be the least stage t such that $e_s \geqq e$ for all stages $s \geqq t$. Define functions f and g by

$$W_{f(n)} = \bigcup_{s \geqq k(n)} p_s(W_{n,s}), \text{ and } W_{g(n)} = \bigcup_{s \geqq k(n)} p_s^{-1}(W_{n,s}) .$$

Clearly, $p(W_n) = W_{f(n)}$ and $p^{-1}(W_n) = W_{g(n)}$ for all n, because after stage $k(n)$, p only interchanges elements in the same complete e-state by condition (i). Both f and g (and hence p) are $\leqq_T \emptyset'$. To see that p induces an *effective* $\Phi \in \mathrm{Aut}\, \mathcal{E}^*$, apply Theorem 1.3 or replace $k(n)$ above by 0 to produce *recursive* functions f and g. ⊠

(Note that in the above proof, the e^{th} positive requirement, which asserts that $D_j^e \subset H$ for some j, once met is never injured although it may require attention finitely often before being fully met. The e^{th} negative requirement, which asserts that only elements in the same complete e-state may be interchanged, may be injured by the i^{th} positive requirement only if $i < e$.)

Martin's method can easily be adapted to construct $A \in \mathcal{E}$ and $\Phi \in \mathrm{Aut}\, \mathcal{E}$ such that A and $\Phi(A)$ have different Turing degree. (The noninvariance of Turing degree also follows from the subsequent theorem on maximal sets.)

2. Automorphisms and maximal sets

Given maximal sets A and B such that $A^* \neq B^*$ let us consider various attempts to find $\Phi \in \operatorname{Aut} \mathcal{E}^*$ such that $\Phi(A^*) = B^*$. Kent observed [10, p. 233] that if a permutation p of N is the identity except on a cohesive set, then p induces an automorphism of $\mathcal{E}^*$. We might first try to extend Kent's method by searching for a permutation p such that: (i) p is the identity on $A \cap B$; (ii) $p(\bar{A}) =^* \bar{B}$; and (iii) p induces an automorphism of $\mathcal{E}^*$. Unfortunately, a permutation satisfying (i) and (ii) for A nonrecursive and $\bar{A} \cap \bar{B} =^* \emptyset$ never satisfies (iii). To see this let C be a major subset of A, as constructed by Lachlan [1, p. 29], so that $A - C$ is infinite but for all $W \in \mathcal{E}$ if $\bar{A} \subsetneqq^* W$ then $\bar{C} \subsetneqq^* W$. Thus, $\bar{C} \subsetneqq^* B$ because $\bar{A} \subsetneqq^* B$. Hence, $\bar{B} \subsetneqq^* C$. Now if p satisfies (i) and (ii) then

$$p(C) = p(C \cap \bar{B}) \cup p(C \cap B) =^* \bar{A} \cup (C \cap B) = D\ ,$$

and $\bar{A} \subsetneqq^* D$ but $\bar{C} \not\subseteq^* D$, so that D cannot be r.e.

As a second attempt we might try to construct Φ using Martin's method of the preceding section, but this will also fail in general because any such Φ is always effective. In this section we first construct maximal sets A and B such that $\Phi(A^*) \neq B^*$ for every *effective* $\Phi \in \operatorname{Aut} \mathcal{E}^*$. We then motivate the main theorem that $A \equiv_{\mathcal{E}} B$ for any two maximal sets A and B, and derive its corollaries. We reduce the proof to two parts which we handle in subsequent sections.

We assume familiarity with the Yates' construction of a maximal set A [10, p. 235] with respect to any standard enumeration (of the r.e. sets) $\{U_n\}_{n \in N}$. Briefly, a marker Λ_e^A seeks an element x in the highest possible e-state with respect to $\{U_n\}_{n \in N}$, and moves (monotonically) to settle on the e^{th} member of $\bar{A}$. It is well-known that we may not achieve $\bar{A} \subsetneqq U_0$ even if U_0 is infinite (since almost all x may appear in U_0 only *after* $x \in A$). However, clearly $\bar{A} \subsetneqq U_0$ just if infinitely many x appear in U_0 *before* A (i.e., if $U_0 \backslash A$ is infinite), since each marker Λ_e^A will not rest until it has reached some $x \in U_0$. This fact is the crux of the following proof.

THEOREM 2.1. *There exist maximal sets A and B such that for every effective $\Phi \in \operatorname{Aut} \mathcal{E}^*$, $\Phi(A^*) \neq B^*$.*

Proof. First let us consider how to handle a single requirement R_e, which asserts that $A^* \not\equiv_{\mathcal{E}^*} B^*$ via Φ with presentation φ_e. We give standard enumerations $\{U_n\}_{n \in N}$ and $\{V_n\}_{n \in N}$ such that maximal sets A and B are constructed simultaneously in the above way with respect to $\{U_n\}_{n \in N}$ and $\{V_n\}_{n \in N}$ respectively. We let $U_{n+2} = V_{n+2} = W_n$ for all n, and define U_0, U_1, V_0, and V_1 so as to meet R_e.

Let $U_1 = W_{i_1}$ = the even numbers, enumerated so that $U_1\backslash A$ is infinite. Let $V_0 = W_{\varphi_e(i_1)}$ if $\varphi_e(i_1)$ is defined, and $= \emptyset$ otherwise. Define U_0 = the odd numbers so that $U_0\backslash A$ is infinite just if $V_0\backslash B$ is infinite. Let $W_{i_0} = U_0$, and define $V_1 = W_{\varphi_e(i_0)}$ if $\varphi_e(i_0)$ is defined, and $= \emptyset$ otherwise.

Clearly A and B will be maximal sets. Now suppose that $A^* \equiv_{\mathcal{E}^*} B^*$ via Φ with presentation φ_e. Then $\varphi_e(i_0)$ and $\varphi_e(i_1)$ are defined, $\Phi(U_0^*) = V_1^*$, and $\Phi(U_1^*) = V_0^*$. Now $V_0\backslash B$ is infinite because if $U_0\backslash A$ is finite then $\bar{A} \subseteq U_1$ by construction of A, and hence $\bar{B} \subseteq^* V_0$ (because $\Phi(A^*) = B^*$), which forces $V_0 - B$ and thus $V_0\backslash B$ to be infinite. Now $V_0\backslash B$ infinite insures that $U_0\backslash A$ is infinite by definition of U_0. Since $U_0\backslash A$ and $V_0\backslash B$ are infinite, $\bar{A} \subseteq U_0$ and $\bar{B} \subseteq V_0$ by the remarks immediately preceding the statement of the theorem. But $\bar{A} \subseteq U_0$ and $\Phi(U_0^*) = V_1^*$ imply that $\bar{B} \subseteq^* V_1$. Thus, $\bar{B} \subseteq^* V_0 \cap V_1 =^* \emptyset$ contradicting the maximality of B.

This method may be modified as follows to handle simultaneously all requirements. The arrays $\{U_y\}$, $\{V_y\}$, $y \in N$, are listed in finite collections called "e-blocks." The 0-blocks are respectively U_0, U_1, W_0, and V_0, V_1, W_0, where U_0, U_1, V_0, and V_1 are defined as above with $e = 0$. Assume that we have defined both arrays through the e-blocks, and let $\{U_i\}$, $\{V_i\}$, for $i \leq$ some j, denote the resulting arrays. Let $\sigma^n(j, x)$ denote the j-state at stage n of element x with respect to $\{U_i\}_{i\leq j}$. The $(e + 1)$-blocks of $\{U_n\}$ and $\{V_n\}$ respectively will be $X_1, Y_1, X_2, Y_2, \cdots, X_m, Y_m, W_{e+1}$, and $\hat{X}_1, \hat{Y}_1, \cdots, \hat{X}_m, \hat{Y}_m, W_{e+1}$. The pairs $\langle X_k, Y_k\rangle$ and $\langle \hat{X}_k, \hat{Y}_k\rangle$ play the role of $\langle U_0, U_1\rangle$ and $\langle V_0, V_1\rangle$ respectively of the earlier proof, correspond to a fixed j-state σ_k, and are arranged in decreasing order according to the order of σ_k. For fixed j-state σ_k, $0 \leq k \leq 2^{j+1}$, define the r.e. set,

$$C_k = \{x: x \in (\bigcap \{W_i: i \in \sigma_k\})\backslash A\} .$$

Given an enumeration $\{c_n^k\}$ of $C_k\backslash A$, define

$$C_k^{\text{even}} = \{c_{2n}^k\}_{n\in N}, \text{ and } C_k^{\text{odd}} = \{c_{2n+1}^k\}_{n\in N} ,$$

which play the role of the evens and odds in the earlier proof.

Let $D_k = W_{\varphi_e(i')}$, for some $W_{i'} = C_k$, if $\varphi_e(i')$ is defined, and $= \emptyset$ otherwise. Note that if σ_k is the true j-state of (cofinitely members of) $\bar{A}$ then $\bar{A} \subseteq C_k$. Thus, if $A^* \equiv_{\mathcal{E}^*} B^*$ with presentation φ_e, then $\bar{B} \subseteq D_k$, so $D_k\backslash B$ is infinite. Define $X_k \subseteq C_k^{\text{odd}}$, $Y_k \subseteq C_k^{\text{even}}$, $\hat{X}_k = (W_{\varphi_e(i_1)} \cap D_k)$, and $\hat{Y}_k \subseteq W_{\varphi_e(i_0)} \cap D_k$, for $W_{i_0} = X_k$, and $W_{i_1} = Y_k$ as before, such that $X_k\backslash A$ is infinite just if $\hat{X}_k\backslash B$ is infinite.

If $\sigma_{k'}$ is the true j-state of $\bar{A}$ then for all $k < k'$, C_k and hence X_k and Y_k will be finite and will cease interfering with $X_{k'}$ and $Y_{k'}$, which will witness $A^* \not\equiv_{\mathcal{E}^*} B^*$ with presentation φ_e as before. ☒

(We discovered the above proof after Lachlan suggested a similar idea for proving the weaker fact that there is no uniform method for finding an effective Φ between maximal sets A and B given their indices.)

Let us return to the problem of proving that $A \equiv_{\mathcal{E}} B$ for given maximal sets A and B. Rather than working directly with $\{W_n\}_{n\in N}$, it will be convenient to define recursive arrays of r.e. sets, $\{U_n\}_{n\in N}$ and $\{V_n\}_{n\in N}$ which depend upon A and B respectively, and which are skeletons, i.e., for which there exist functions F and G, such that

$$(2.1) \qquad (\forall n)\,[W_n =^* U_{F(n)} \text{ and } W_n =^* V_{G(n)}] \,.$$

(Note that even if the arrays include *all* r.e. sets, they may not be acceptable numberings since the functions F and G need not be recursive.)

To specify the automorphism Φ we shall specify a permutation p of N, and functions f and g such that $p(A) = B$, and

$$(\forall n)\,[p(U_n) =^* W_{f(n)} \text{ and } p^{-1}(V_n) =^* W_{g(n)}] \,.$$

Thus Φ is presented (with respect to $\{W_n\}_{n\in N}$) by the pair $\langle f\circ F, g\circ G\rangle$. (The permutation p serves only to aid in defining f and g and to witness that Φ preserves inclusion. For the present case of maximal sets A and B, the noneffectiveness of Φ follows entirely from the nonrecursiveness of F and G, which will be $\leq_T \varnothing'''$, since f and g will be recursive.)

Let us assume that A and B have been given and the corresponding arrays $\{U_n\}_{n\in N}$, $\{V_n\}_{n\in N}$ have been somehow determined (as we will do in §3). How do we define f and g? From now on we let $\hat{U}_n$ denote $W_{f(n)}$ and $\hat{V}_n$ denote $W_{g(n)}$ so that our intended correspondence under Φ will be $\Phi(U_n^*) = \hat{U}_n^*$ and $\hat{V}_n^* = \Phi^{-1}(V_n^*)$. (Picture two copies of N with A, U_n, $\hat{V}_n$ on the left-hand side and B, $\hat{U}_n$, V_n on the right-hand side.) To produce $\hat{U}_n$ and $\hat{V}_n$ we give a simultaneous enumeration of a recursive sequence of r.e. sets including all of A, B, U_n, V_n, $\hat{U}_n$, and $\hat{V}_n$ for all $n \in N$. (This simultaneous enumeration determines $X\backslash Y$ as well as $X\searrow Y = (X\backslash Y) \cap Y$ for any X and Y in the array.)

The problem may be split into two parts corresponding to $p(\bar{A})$ and $p(A)$ respectively. We must insure that both

$$(2.2) \qquad p(\bar{A} \cap U_n) =^* (\bar{B} \cap \hat{U}_n) \text{ and } p^{-1}(\bar{B} \cap V_n) =^* (\bar{A} \cap \hat{V}_n) \qquad \text{for all } n\,,$$

and

$$(2.3) \qquad p(A \cap U_n) =^* (B \cap \hat{U}_n) \text{ and } p^{-1}(B \cap V_n) =^* (A \cap \hat{V}_n) \qquad \text{for all } n\,.$$

Decompose $\hat{U}_n$ into the two r.e. sets $\hat{U}_n^+ = \hat{U}_n\backslash B$, and $\hat{U}_n^- = B\searrow \hat{U}_n$, and $\hat{V}_n = \hat{V}_n^+ \cup \hat{V}_n^-$ with A in place of B. Condition (2.2) causes us to enumerate many elements in $\hat{U}_n^+$ ($\hat{V}_n^+$), but these enumerations must be sufficiently restrained so that the r.e. sets $\hat{U}_n^+ \searrow B$ ($\hat{V}_n^+ \searrow A$) do not *already* prevent extensions by $\hat{U}_n^-$ ($\hat{V}_n^-$) to $\hat{U}_n$ ($\hat{V}_n$) satisfying condition (2.3). For example, if there are infinitely many $y \in \hat{U}_0^+ \searrow B$ then to meet condition (2.3) we need infinitely many $x \in U_0 \cap A$, which usually means producing infinitely many $x \in U_0 \searrow A$ because in general we may have $A \searrow U_0 = \emptyset$. Now if the above elements y enter $\hat{U}_0^+ \searrow B$ after $y \in V_0$, and if we can find corresponding elements $x \in U_0 \searrow A$ then it is easy to enumerate $x \in \hat{V}_0^-$ if necessary. However, if all the above y enter $U_0^+ \searrow B$ *while* $y \notin V_0$ then we must have infinitely many x entering $U_0 \searrow A$ while $x \notin \hat{V}_0^+$. In general, for each n such that $y \in \hat{U}_n^+$ ($\notin V_n$), the corresponding x must be $\in U_n$ ($\notin \hat{V}_n^+$), and x is otherwise restricted.

A *full e-state* of an element x is a triple $\nu = \langle e, \sigma, \tau\rangle$ where σ and τ are e-states of x with respect to certain arrays. Given $\nu = \langle e, \sigma, \tau\rangle$ and $\nu' = \langle e', \sigma', \tau'\rangle$ let $\nu \leqq \nu'$ denote that $e = e'$, $\sigma \subseteqq \sigma'$ and $\tau \supseteqq \tau'$. Replacing single r.e. sets in the above example by full e-states yields an obvious necessary condition (2.4) below which asserts that "A covers B". The dual condition (2.5) asserts that "B dual covers A". (This terminology will be further explained and amplified in §4.) The following "Extension Theorem" guarantees that these necessary conditions are also *sufficient* for an arbitrary pair of arrays $\{\hat{U}_n^+\}_{n\in N}$, $\{\hat{V}_n^+\}_{n\in N}$ to be extended to $\{\hat{U}_n\}_{n\in N}$, $\{\hat{V}_n\}_{n\in N}$ satisfying (2.3). The rather complicated proof of the Extension Theorem is deferred to §§ 4, 5, and 6.

THEOREM 2.2 (Extension Theorem). *Let A and B be infinite* r.e. *sets, and $\{U_n\}_{n\in N}$, $\{V_n\}_{n\in N}$, $\{\hat{U}_n^+\}_{n\in N}$, and $\{\hat{V}_n^+\}_{n\in N}$ be recursive arrays of* r.e. *sets. Suppose there is a simultaneous enumeration of a recursive array including all above such that $B \searrow \hat{U}_n^+ = \emptyset = A \searrow \hat{V}_n^+$, for all n. For each full e-state $\nu = \langle e, \sigma, \tau\rangle$ define the* r.e. *set*

$$D_\nu^A = \{x\colon x \in A_{s+1} - A_s \text{ for some } s \text{ such that } (\forall n)_{\leqq e}[[n \in \sigma \Leftrightarrow x \in U_{n,s}] \text{ and } [n \in \tau \Leftrightarrow x \in \hat{V}_{n,s}^+]]\}\ .$$

Similarly, define D_ν^B with B, $\hat{U}_n^+$, V_n in place of A, U_n, $\hat{V}_n^+$ respectively. Furthermore, suppose that

$$(2.4) \qquad (\forall\nu)\ [D_\nu^B \text{ infinite} \Rightarrow (\exists\nu')[\nu \leqq \nu' \text{ and } D_{\nu'}^A \text{ infinite}]]$$

and

$$(2.5) \qquad (\forall\nu)\ [D_\nu^A \text{ infinite} \Rightarrow (\exists\nu')[\nu' \leqq \nu \text{ and } D_{\nu'}^B \text{ infinite}]]\ .$$

Then there exists a 1: 1 *map p from A onto B and recursive arrays of* r.e. *sets $\{\hat{U}_n^-\}_{n\in N}$, $\{\hat{V}_n^-\}_{n\in N}$ such that the extensions $\hat{U}_n = \hat{U}_n^+ \cup \hat{U}_n^-$ and $\hat{V}_n = \hat{V}_n^+ \cup \hat{V}_n^-$ satisfy for all n,*

$$[p(A \cap U_n) =^* (B \cap \hat{U}_n)] \text{ and } [p^{-1}(B \cap V_n) =^* (A \cap \hat{V}_n)]\,.$$

THEOREM 2.3. *Given any two maximal sets A and B there is an automorphism Φ of $\mathcal{E}$ such that $\Phi(A) = B$.*

Proof. Fix A and B. In § 3 (Theorem 3.1) we define recursive arrays $\{U_n\}_{n \in N}$, $\{V_n\}_{n \in N}$ satisfying (2.1). We then define (Theorem 3.2) recursive arrays $\{\hat{U}_n^+\}_{n \in N}$, $\{\hat{V}_n^+\}_{n \in N}$ satisfying (2.2), and a simultaneous enumeration of all these r.e. sets satisfying the hypotheses (2.4) and (2.5) of the Extension Theorem. By the Extension Theorem (to be proved in §§4, 5 and 6) we can extend the latter two arrays to $\{\hat{U}_n\}_{n \in N}$ and $\{\hat{V}_n\}_{n \in N}$ satisfying (2.3), and hence yielding an automorphism $\Psi \in \text{Aut}\,\mathcal{E}^*$ mapping A^* to B^*. By Corollary 1.5, there exists $\Phi \in \text{Aut}\,\mathcal{E}$ such that $\Phi(A) = B$. ⊠

COROLLARY 2.4 (Martin). *Turing degree is not lattice invariant.*

Proof. By Yates [18] there exists a complete maximal set, but by Sacks [14] there exists an incomplete maximal set. ⊠

COROLLARY 2.5. *For any $k > 0$, let $A_1, A_2, \cdots, A_k$ and $B_1, B_2, \cdots, B_k$ be two sequences each containing exactly k maximal sets whose complements are pairwise disjoint. Then there exists an automorphism Φ of $\mathcal{E}$ such that $\Phi(A_i) = B_i$ for all $i \leq k$.*

Proof. Find recursive sets R_i, $i \leq k$, such that $N = \bigcup_{i \leq k} R_i$, $R_i \cap R_j = \emptyset$ for $i \neq j$, and $R_i \supset \bar{A}_i$ for all $i \leq k$. Likewise find recursive sets $\hat{R}_i$, $i \leq k$, with B_i in place of A_i. Apply Theorem 2.3 to find a 1:1 function p_i from R_i onto $\hat{R}_i$ such that $p_i(\bar{A}_i) = \bar{B}_i$ and p_i induces an isomorphism from $\{W_n \cap R_i\}_{n \in N}$ to $\{W_n \cap \hat{R}_i\}_{n \in N}$. Then $p = \bigcup_{i \leq k} p_i$ is a permutation of N which induces the desired Φ. ⊠

Definition. A set A is *quasimaximal of rank n*, if A is the intersection of n maximal sets whose complements are pairwise disjoint.

COROLLARY 2.6. *Let A and B be any two quasimaximal sets of the same rank. Then there exists $\Phi \in \text{Aut}\,\mathcal{E}$, such that $\Phi(A) = B$.*

COROLLARY 2.7. *For every $k \geq 1$ the group $\text{Aut}\,\mathcal{E}^*$ is k-ply transitive on the coatoms of $\mathcal{E}^*$.*

Proof. Apply Corollary 2.6 and the fact that if $A_1^*, A_2^*, \cdots, A_k^*$ are distinct coatoms in $\mathcal{E}^*$ then there exist maximal sets $A_1', A_2', \cdots, A_k'$ whose complements are pairwise disjoint and such that $A_i' \in A_i^*$ for all $i \leq k$. ⊠

3. Satisfying condition (2.2) and the hypotheses of the Extension Theorem

Fix maximal sets A and B from now on. For any r.e. set C, define,

$$\mathcal{C}(C) = \{W_e\colon W_e \cup C = N \text{ or } W_e \subseteq^* C\} .$$

Note that if C is coinfinite, then $\mathcal{C}^*(C) = \mathcal{E}^*$ just if C is maximal. We shall define a recursive array $\{U_n\}_{n\in N}$, with $U_0 = A$, which contains exactly those sets in $\mathcal{C}(A)$. In addition we give a simultaneous enumeration of $\{U_n\}_{n\in N}$, which is "order-preserving" in the sense that

$$(\forall n)(\forall m)_{>n}(\text{a.a. } x)\ [x \in (U_n\backslash A) \cap (U_m\backslash A) \Rightarrow x \in (U_n\backslash U_m)] ,$$

where "a.a. x" denotes "for almost all (cofinitely many) x". (Roughly, the order in which elements x are enumerated in U_n, U_m while in $\bar{A}$ coincides with the rank of the indices n, m for almost all x.) Similarly, $\{V_n\}_{n\in N}$ is defined with B in place of A. This order-preserving property explains our use of $\{U_n\}_{n\in N}$ and $\{V_n\}_{n\in N}$ in place of $\{W_n\}_{n\in N}$, since it then becomes easy to define $\{\hat{U}_n^+\}_{n\in N}$ and $\{\hat{V}_n^+\}_{n\in N}$ meeting the hypotheses of the Extension Theorem as well as condition (2.2).

The maximality of A and B is used only to insure that $\mathcal{C}^*(A) = \mathcal{C}^*(B) = \mathcal{E}^*$ so that there exist functions F and G satisfying (2.1). If A and B are replaced by any nonrecursive r.e. sets C and D, our proof shows that there is a permutation p of N such that $p(C) = D$, and p induces an isomorphism from $\mathcal{C}^*(C)$ onto $\mathcal{C}^*(D)$, rather than an automorphism of $\mathcal{E}^*$.

(This fact is obvious for $\mathcal{C}^*(C)$, $\mathcal{C}^*(D)$ replaced by either the principal ideals, $\mathcal{I}^*(C)$, $\mathcal{I}^*(D)$, or the principal filters, $\mathcal{F}^*(\bar{C})$, $\mathcal{F}^*(\bar{D})$, but these two trivial proofs are so different that we can see no way to combine them for $\mathcal{C}^*(C) = \mathcal{I}^*(C) \cup \mathcal{F}^*(\bar{C})$, $\mathcal{C}^*(D) = \mathcal{I}^*(D) \cup \mathcal{F}^*(\bar{D})$. As usual define, $\mathcal{I}(X) = \{W_e\colon W_e \subseteq X\}$, and $\mathcal{F}(\bar{X}) = \{W_e\colon W_e \supseteq \bar{X}\}$.)

For any r.e. set C, $\mathcal{C}(C)$ can be given by a recursive array $\{Y_n\}_{n\in N}$, with $Y_0 = C$, namely

$$Y_{2n+1} = W_n \cap C, \text{ and}$$
$$Y_{2n+2} = \{x\colon x \in W_n \text{ and } (\forall y)_{\le x}[y \in C \cup W_n]\} .$$

Note that if $W_n \supseteq \bar{C}$ then $Y_{2n+2} = W_n$. Otherwise, $Y_{2n+2} =^* \varnothing$. (The same device is used by Lachlan in the major subset construction [1, p. 30], and elsewhere.) If infinite, $\bar{C}$ is cohesive with respect to $\mathcal{C}(C)$, and hence we may define the e-state of $\bar{C}$ to be the e-state of infinitely many elements of $\bar{C}$ with respect to $\{Y_n\}_{n\in N}$.

THEOREM 3.1 (Order-Preserving Enumeration Theorem). *Given any coinfinite* r.e. *set C, there is a recursive array $\{Z_n\}_{n\in N}$, with $Z_0 = C$, containing exactly the* r.e. *sets in $\mathcal{C}(C)$, and a simultaneous enumeration of $\{Z_n\}_{n\in N}$, such that*

1) $(\forall n)$ *[$Z_n\backslash C$ is infinite $\Rightarrow Z_n - C$ infinite]; and*

2) $(\forall n)(\forall m)_{>n}(\text{a.a. } x)[x \in (Z_n \backslash C) \cap (Z_m \backslash C) \Rightarrow x \in (Z_n \backslash Z_m)]$,
where "a.a. x" denotes "for almost all x."

Proof. Fix C and a simultaneous enumeration of the array $\{Y_n\}_{n \in N}$ defined above such that

(3.1) $$(\forall n)\ [Y_n \backslash C \text{ is infinite} \Leftrightarrow Y_n - C \text{ is infinite}]\ ,$$

where $C_s = Y_{0,s}$. Define $Z_{0,s} = C_s$ for all s. (It will be obvious that (3.1) holds with Z_n in place of Y_n.)

For each $e > 0$ we need some $F(e)$ such that $Z_{F(e)} = Y_e$. Our choice of $F(e)$ will depend upon Y_e as well as the $(e-1)$-state of $\bar{C}$ with respect to $\{Y_i\}_{i \in N}$. For each $e > 0$ and $\sigma \subseteqq \{0, 1, \cdots, e-1\}$ we have a marker $\Lambda(e, \sigma)$ which comes to rest on an index $F(e)$, for which $Z_{F(e)} = Y_e$, in case the $(e-1)$-state of $\bar{C}$ is σ. An integer j is *fresh* at some stage if it has never been occupied by a marker. Define $Z_{e,s}$ for all $e > 0$ as follows.

Stage $s = 0$. Let $Z_{e,0} = \varnothing$ for all $e > 0$.

Stage $s + 1$. Enumerate one new pair $\langle x_0, y_0 \rangle$ where $x_0 \in Y_{y_0}$. If $y_0 = 0$, go to stage $s + 2$. Otherwise,

Step 1. If $x_0 \in C_{s+1}$ leave all markers fixed. Otherwise, move those markers $\Lambda(e, \sigma)$, for which $e > y_0$ and $y_0 \notin \sigma$, to the first fresh elements, maintaining the present order of the markers.

Step 2. For each $\sigma \subseteqq \{0, 1, \cdots, s\}$ assign a marker $\Lambda(s + 1, \sigma)$ to the first fresh element in order of the s-state ranking of σ.

Step 3. For each e, $0 < e \leqq s + 1$, and each $\sigma \subseteqq \{0, 1, \cdots, e - 1\}$, let $F(e, \sigma, s + 1)$ denote the present position of $\Lambda(e, \sigma)$. Let (x', e', σ') be the first triple (x, e, σ) with the following properties P1–P3. If there exists such (x', e', σ'), enumerate x' in $Z_{F(e',\sigma',s+1),s+1}$. Otherwise, go to stage $s + 2$.

P1. $x \in Y_{e,s+1}$ and $x \notin Z_{F(e,\sigma,s+1),s}$.

P2. $(\forall y)_{<x}[y \in C_{s+1} \cup Z_{F(e,\sigma,s+1),s}]$.

P3. Either $x \in C_{s+1}$, or $\sigma = \varnothing$, or the elements of σ are $a_1 < a_2 < \cdots < a_k$, and

$$(\forall j)_{\leqq k}[x \in Z_{F(a_j,\sigma_j,s+1),s}]\ ,$$

where $\sigma_j = \{a_1, a_2, \cdots, a_{j-1}\}$ if $j > 1$, and $= \varnothing$ if $j = 1$.

LEMMA 1. *For each e and σ, marker $\Lambda(e, \sigma)$ moves infinitely often if and only if $Y_y \supseteqq \bar{C}$ for some $y < e$, $y \notin \sigma$.*

Proof. By Step 1 of the construction and (3.1).

LEMMA 2. *For all n, if $(Z_n \backslash C)$ is infinite, then some marker $\Lambda(e, \sigma)$ comes to rest on n, such that*

$$(\forall y)[y \in \sigma \cup \{e\} \Rightarrow Y_y \supseteq \bar{C}] .$$

Proof. By Step 3 of the construction and (3.1).

Now let $e_1 < e_2 < e_3 < \cdots$ be a listing of all e such that $Y_e \supseteq \bar{C}$. For each e, let $\rho_e = \{e_j : e_j < e\}$. By Lemma 1, $\Lambda(e, \rho_e)$ comes to rest on some element denoted $F(e)$, and $Z_{F(e)} = Y_e$ by Step 3. The markers $\Lambda(e, \rho_e)$ once appointed remain in order of e (by Steps 1 and 2), and hence for all i and e, $i < e$ if and only if $F(i) < F(e)$. By Lemmas 1 and 2, for all $i \neq F(e_j)$, for some j, $Z_i \backslash C$ is finite. The last clause of Step 3 insures that for all x if $x \in (Z_{F(e_i)} \backslash C) \cap (Z_{F(e_j)} \backslash C)$, where $i < j$, then $x \in Z_{F(e_i)} \backslash Z_{F(e_j)}$. ⊠

THEOREM 3.2. *Given maximal sets A and B, let $\{U_n\}_{n \in N}$, and $\{V_n\}_{n \in N}$ be the recursive arrays with simultaneous enumerations given by Theorem 3.1 for $C = A$ and B respectively. Then there exist recursive arrays $\{\hat{U}_n^+\}_{n \in N}$, $\{\hat{V}_n^+\}_{n \in N}$ together with a simultaneous enumeration of a recursive array including all of $\{U_n\}_{n \in N}$, $\{V_n\}_{n \in N}$, $\{\hat{U}_n^+\}_{n \in N}$, and $\{\hat{V}_n^+\}_{n \in N}$ which satisfies the hypotheses of the Extension Theorem, and condition* (2.2), *which here asserts that for all n, $(\hat{U}_n^+)^* \in \mathcal{C}^*(B)$, $(\hat{V}_n^+)^* \in \mathcal{C}^*(A)$, and*

$$[\bar{A} \subseteq U_n \Leftrightarrow \bar{B} \subseteq^* \hat{U}_n^+] \text{ and } [\bar{B} \subseteq V_n \Leftrightarrow \bar{A} \subseteq^* \hat{V}_n^+] .$$

Proof. We define A_s and B_s below. Given A_s and B_s let ψ_s be the partial 1 : 1 function which maps $\bar{A}_s$ onto $\bar{B}_s$ such that $\psi_s(x) < \psi_s(y)$ just if $x < y$. Let ψ_{-1} be the identity permutation on N. Let $\psi = \lim_s \psi_s$.

Stage $s = 4t$. Enumerate $x \in U_{y,s}$ where x was enumerated in U_y at stage t in the given simultaneous enumeration of $\{U_n\}_{n \in N}$. Let A_s denote $U_{0,s}$.

Stage $s = 4t + 1$. Enumerate $x \in V_{y,s}$ where x was enumerated in V_y at stage t in the given simultaneous enumeration of $\{V_n\}_{n \in N}$. Let B_s denote $V_{0,s}$.

Stage $s = 4t + 2$. Let (x', e') be the first pair (x, e) with the following properties P1–P4. If there exists (x', e'), enumerate $\psi_{s-1}(x') \in \hat{U}_{e',s}^+$. Otherwise go to stage $s + 1$.

P1. $x \in (U_{e,s-1} - A_{s-1})$ and $\psi_{s-1}(x) \notin \hat{U}_{e,s-1}^+$.
P2. $(\forall i)_{<e}[x \in U_{i,s-1} \Leftrightarrow \psi_{s-1}(x) \in \hat{U}_{i,s-1}^+]$.
P3. $(\forall i)_{<e}[x \in \hat{V}_{i,s-1}^+ \Leftrightarrow \psi_{s-1}(x) \in V_{i,s-1}]$.

Define,

$$\sigma_0 = \{i : i \leq e \text{ and } \psi_{s-1}(x) \in \hat{U}_{i,s-1}^+\} ,$$
$$\tau_0 = \{i : i \leq x \text{ and } \psi_{s-1}(x) \in V_{i,s-1}\} ,$$
$$u_x = (\text{the unique } u)_{<s-1}[x \in U_{e,u+1} - U_{e,u}] .$$

P4. $(\exists v)(\exists y)(\exists \sigma_1)(\exists \tau_1)[u_x < v < s - 1$ and $y \in A_{v+1} - A_v$ and $\sigma_1 \supseteqq \{e\} \cup \sigma_0$ and $\tau_1 \subseteqq \tau_0$ and $\sigma_1 = \{i\colon i < e$ and $y \in U_{i,v}\}$ and $\tau_1 = \{i\colon i \leqq x$ and $y \in \hat{V}^+_{i,v}\}]$. (Property P4 asserts that *after* x appeared in U_e there appeared in A some y whose pair of states $\langle \sigma_1, \tau_1 \rangle$ at that time "covers" the pair $\langle \sigma_0 \cup \{e\}, \tau_1 \rangle$ now proposed for $\psi_{s-1}(x)$, in that $\sigma_1 \supseteqq \sigma_0 \cup \{e\}$, $\tau_1 \subseteqq \tau_0$. The crucial point is that we measure state σ_0 only with respect to $i < e$ not $i \leqq x$ (as for τ_0), which would impose more restraint on enumerating $\psi_{s-1}(x) \in \hat{U}^+_e$, since in general $x > e$. By the order-preserving enumeration property, this will still suffice for Lemma 1.)

Stage $s = 4t + 3$. Similarly attempt to enumerate some element $\psi^{-1}_{s-1}(x')$ in $\hat{V}^+_{e',s}$, where A, U_n, $\hat{U}^+_n$, V_n, $\hat{V}^+_n$, and $\psi_{s-1}(x)$ in the preceding stage are replaced by B, V_n, $\hat{V}^+_n$, U_n, $\hat{V}^+_n$, and $\psi^{-1}_{s-1}(x)$. Also in P3 (which now refers to U_i and $\hat{U}^+_i$) replace "(for all $j)_{<e}$" by "(for all $j)_{\leqq e}$" reflecting the fact that the matching $U_e \leftrightarrow \hat{U}_e$ has higher priority than $\hat{V}_e \leftrightarrow V_e$.

LEMMA 1. *The hypotheses* (2.4) *and* (2.5) *of the Extension Theorem are met by the above arrays and simultaneous enumeration.*

Proof. Property P1 of the construction guarantees that $B \searrow \hat{U}^+_n = \emptyset = A \searrow \hat{V}^+_n$, for all n. By trivially "speeding up" the enumeration of A if necessary we may assume that for infinitely many x, x appears in A before x has appeared in *any* U_i, $i > 0$, or $\hat{V}^+_j$, $j \geqq 0$, and similarly for B, V_i, and $\hat{U}^+_j$. Thus, we need verify (2.4) only for $\sigma \neq \emptyset$. Verification of (2.5) is similar.

Fix i and full i-state $\nu_0 = \langle i, \sigma_0, \tau_0 \rangle$ with $\sigma_0 \neq \emptyset$. Assume that $D^A_{\nu_1}$ is finite for all $\nu_1 \geqq \nu_0$. We must show that for each $\tau_1 \subseteqq \tau_0$ only finitely many x whose i-state with respect to $\{V_n\}_{n \in N}$ is τ_1 are allowed to enter $\hat{U}^+_{\sigma_0} = \bigcap \{\hat{U}^+_n\colon n \in \sigma_0\}$. Since i-states increase with time it follows that $D^B_{\nu_0}$ must be finite also.

Let $e = \max \sigma_0$. (Note that $e \leqq i$.) It follows by P2 of the construction and the order-preserving enumeration of $\{U_n\}_{n \in N}$ that $\hat{U}^+_e \searrow \hat{U}^+_n$ is finite for all $n < e$. Hence, if there are infinitely many $x \in \hat{U}^+_{\sigma_0}$, almost all such x enter $\hat{U}^+_e$ only *after* $x \in \hat{U}^+_n$, for all $n \in \sigma_0 - \{e\}$. But by P4 of the construction and our assumption that $D^A_{\nu_1}$ is finite for all $\nu_1 \geqq \nu_0$, at most finitely many x already in $\bigcap \{\hat{U}^+_n\colon n \in \sigma_0 - \{e\}\}$ will be allowed to enter $\hat{U}^+_e$ while the i-state of x with respect to $\{V_n\}_{n \in N}$ is some $\tau_1 \subseteqq \tau_0$.

LEMMA 2. $(\forall e)[\psi(U_e \cap \bar{A} =^* \hat{U}^+_e \cap \bar{B}$ *and* $\psi^{-1}(V_e \cap \bar{B}) =^* (\hat{V}^+_e \cap \bar{A})]$.

Proof. Fix e, and by induction assume the above for all $i < e$. If $U_e \not\supset \bar{A}$ then $U_e \backslash A$ is finite by (3.1), and hence $\hat{U}^+_e \backslash B$ and $\hat{U}^+_e - B$ are finite. Assume that $U_e \supset \bar{A}$. Choose y_0 such that for all $x \in \bar{A}$, $x \geqq y_0$ implies

$$(\forall i)_{<e}[[x \in U_i \Leftrightarrow \psi(x) \in \hat{U}_i^+] \text{ and } [x \in \hat{V}_i^+ \Leftrightarrow \psi(x) \in V_i]] .$$

We claim that for any $x_0 \in \bar{A}$, $x_0 \geqq y_0$ implies $\psi(x_0) \in \hat{U}_e^+$. Let $m = \max\{e, x_0\}$. Choose s_0 such that for all $s \geqq s_0$, $\psi_s(x) = \psi_{s_0}(x)$, and the m-state of x_0 $(\psi(x_0))$ at stage s with respect to each of $\{U_n\}_{n \in N}$, $\{\hat{U}_n^+\}_{n \in N}$, $\{V_n\}_{n \in N}$, $\{\hat{V}_n^+\}_{n \in N}$ equals the m-state of x_0 $(\psi(x_0))$ at stage s_0. To see that $\psi(x_0)$ must be in $\hat{U}_{e,s_0}^+$ first define

$$U_{\sigma_0} = \bigcap \{U_i : i \leqq e \text{ and } U_i \supseteqq \bar{A}\} .$$

Now $U_{\sigma_0} \supseteqq \bar{A}$ and hence $U_{\sigma_0} \searrow A$ must be infinite by the nonrecursiveness of A. Define

$$\tau_0 = \{i : i \leqq x_0 \text{ and } \psi(x_0) \in V_i\} .$$

Now if $i \leqq x_0$ and $i \notin \tau_0$, then $V_i - B$ is finite and hence $V_i \backslash B$ is finite by (3.1). But for any i, $V_i \backslash B$ finite implies $\hat{V}_i^+ \backslash A$ finite. Hence, there exist $\sigma_1 \supseteqq \sigma_0$ and $\tau_1 \subseteqq \tau_0$ such that

$$(\exists t)(\exists v)_{>t}(\exists y)[y \in A_{v+1} - A_v \text{ and } \sigma_1 = \{i : i \leqq e \text{ and } y \in U_{i,v}\}$$
$$\text{and } \tau_1 = \{i : i \leqq x_0 \text{ and } y \in \hat{V}_{i,v}\}] .$$

But then by P4, $\psi(x)$ is eventually enumerated in $\hat{U}_e^+$. The case of $\psi^{-1}(V_e) =^* \hat{V}_e^+$ is handled similarly. ☒

4. Proof of the Extension Theorem

Part I: Motivation

From now on we fix r.e. sets A and B and recursive arrays $\{U_n\}_{n \in N}$, $\{V_n\}_{n \in N}$, $\{\hat{U}_n^+\}_{n \in N}$, $\{\hat{V}_n^+\}_{n \in N}$, and a simultaneous enumeration g_0 of all the above which satisfies the hypotheses of the Extension Theorem (Theorem 2.2). We shall define recursive arrays $\{\hat{U}_n^-\}_{n \in N}$, $\{\hat{V}_n^-\}_{n \in N}$ by stages. At a given stage we may:

(1) enumerate one new element in one of the given r.e. sets in order according to g_0;

(2) enumerate an element x in one or more of the new r.e. sets $\hat{U}_n^-$ $(\hat{V}_n^-)$ where x has already been enumerated in A (respectively B); or

(3) enumerate no new element.

By (1) above, our new enumeration restricted to the given r.e. sets also satisfies the hypotheses of the Extension Theorem. If X is any of the above r.e. sets, let X_s denote those elements enumerated in X by the end of stage s of our construction. Define

$$\hat{U}_{n,s} = \hat{U}_{n,s}^+ \cup \hat{U}_{n,s}^-, \quad \hat{V}_{n,s} = \hat{V}_{n,s}^+ \cup \hat{V}_{n,s}^-, \quad \hat{U}_n = \bigcup_s \hat{U}_{n,s}, \text{ and } \hat{V}_n = \bigcup_s \hat{V}_{n,s} .$$

To prove the Extension Theorem we must give a 1 : 1 map p from A

onto B and construct $\hat{U}_n^-$, $\hat{V}_n^-$ in such a way that the extensions $\hat{U}_n$, $\hat{V}_n$ satisfy for all n,

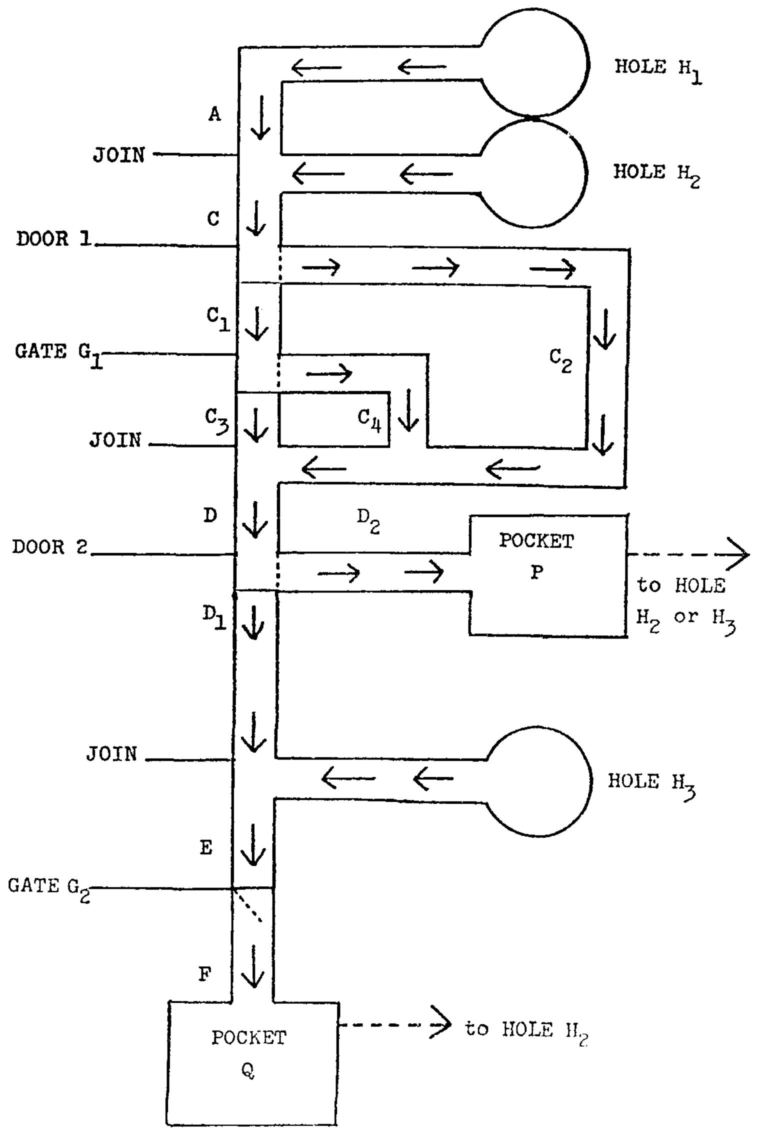
(4.1) $$p(A \cap U_n) =^* (B \cap \hat{U}_n) \text{ and } p^{-1}(B \cap V_n) =^* (A \cap \hat{V}_n) .$$

We present the construction in the informal style of Lerman's "pinball machines" [6]. Like Rogers' "movable markers," Lerman's pinball machines are designed to give the reader a clear picture of the dynamics of the construction rather than simultaneous definitions of a long list of recursive functions and recursive predicates, which the reader must then decode to obtain the desired mental picture. We believe that Lerman's device, which is especially well-suited to our particular construction, will become a standard vehicle like movable markers for presenting more difficult priority constructions, especially those of the "infinite injury" type. We use Lerman's terminology but unlike Lerman, we have decided not to separate the recursion theory from the combinatorics. We have attempted to break down the construction and proofs into a few digestible pieces which we present in §§ 4, 5, and 6.

Our construction involves two *pinball machines*, denoted M and $\hat{M}$. Machine M is shown in the diagram, and $\hat{M}$ is identical except that each symbol X is replaced by $\hat{X}$. There are two copies of N, whose elements $\{n\}_{n \in N}$, and $\{\hat{n}\}_{n \in N}$ act as "balls" in M and $\hat{M}$ respectively. Each machine consists of *tracks*, *gates*, *holes*, *doors*, *pockets*, and *joins*. The *surface of the machine* is that portion covered by the solid arrows. A *track* is a section of the surface of the machine between any two of the following: door, gate, pocket, or join. The surface of the machine is decomposed into tracks some of which in this diagram are labeled A, C, D, E, F sometimes with subscripts.

An element x ($\hat{x}$) initially *enters* the surface of machine M ($\hat{M}$) via hole H_1 ($\hat{H}_1$) when x is enumerated in A (B). Later x ($\hat{x}$) may *re-enter* the surface of M ($\hat{M}$) via hole H_2 or H_3 ($\hat{H}_2$ or $\hat{H}_3$). Having entered or re-entered the surface of the machine, x ($\hat{x}$) proceeds via a sequence of consecutive *moves* in the direction of the arrows which we call *down*, until x ($\hat{x}$) lands in one of the two *pockets* P or Q ($\hat{P}$ or $\hat{Q}$). Such a sequence of moves beginning when x ($\hat{x}$) enters or re-enters the surface of M ($\hat{M}$) and ending when x ($\hat{x}$) reaches a pocket is called a *play*.

On a single move (which will involve one stage of our construction) x ($\hat{x}$) proceeds down along a certain track until it reaches the next door, gate, or pocket. Which track x ($\hat{x}$) takes for its next move after reaching door i will be determined by *Rule* R_i ($\hat{R}_i$), $1 \leq i \leq 2$. When x ($\hat{x}$) reaches gate G_1 or G_2 ($\hat{G}_1$ or $\hat{G}_2$) it may be enumerated in $\hat{V}_n^-$ ($\hat{U}_n^-$) for certain n according to



rules R_3 or R_4 ($\hat{R}_3$ or $\hat{R}_4$), before being placed on the next track. This is the only change of any e-state of x ($\hat{x}$) allowed during a play until x ($\hat{x}$) enters a pocket.

An element x having entered M and having reached a pocket never leaves M although x may later be removed from pocket Q (P) and be placed above hole H_2 (hole H_2 or H_3) later to re-enter M. This motion is indicated in the diagram by the dotted arrows and is governed by rules R_i, $8 \leqq i \leqq 12$. Machine $\hat{M}$ is similar.

For each $x \in A$ (B) the element x $(\hat{x})$ enters M $(\hat{M})$ when x is enumerated in A (B) and thereafter re-enters M $(\hat{M})$ at most finitely often, eventually resting forever in one of the two pockets. Let M_s, P_s, and Q_s consist of those elements in machine M and in pockets P and Q respectively at the end of stage s of the construction. Define sets $P_\omega = \lim_s P_s$, and $Q_\omega = \lim_s Q_s$. Similarly, define $\hat{M}_s, \hat{P}_s, \hat{Q}_s, \hat{P}_\omega$, and $\hat{Q}_\omega$ for machine $\hat{M}$. Since A (B) will be the disjoint union of P_ω and Q_ω, $(\hat{P}_\omega$ and $\hat{Q}_\omega)$, we will achieve (4.1) by giving two partial functions p_1 and p_2 whose domains (dom) are respectively P_ω and Q_ω, and whose ranges (ran) are respectively, $\hat{Q}_\omega$ and $\hat{P}_\omega$, and such that for all $i \in \{1, 2\}$ and all n,

$$\begin{aligned} &p_i(\text{dom } p_i \cap U_n) =^* (\text{ran } p_i \cap \hat{U}_n) \text{ and} \\ &p_i^{-1}(\text{ran } p_i \cap V_n) =^* (\text{dom } p_i \cap \hat{V}_n)\ . \end{aligned} \tag{4.2}$$

We can then let $p = \bigcup \{p_i : i \leq 2\}$. (The antisymmetry of notation, $p_1(P_\omega) = \hat{Q}_\omega$ not $\hat{P}_\omega$, is due to the antisymmetry of the construction and a desire to avoid a relabeling of pockets of $\hat{M}$ which would require a separate diagram.)

The *full e-state at stage s*, denoted $\nu(s, e, x)$, of an element $x \in M_s$, is the triple $\langle e, \sigma(s, e, x), \tau(s, e, x)\rangle$ where $\sigma(s, e, x)$ and $\tau(s, e, x)$ denote the e-states of x at stage s measured with respect to $\{U_{n,s}\}_{n,s \in N}$ and $\{\hat{V}_{n,s}\}_{n,s \in N}$ respectively. Similarly, the full e-state at stage s of $\hat{x} \in \hat{M}_s$, denoted $\nu(s, e, \hat{x})$, is the triple $\langle e, \sigma(s, e, \hat{x}), \tau(s, e, \hat{x})\rangle$, where e-states $\sigma(s, e, \hat{x})$ and $\tau(s, e, \hat{x})$ are measured with respect to $\{\hat{U}_{n,s}\}_{n,s \in N}$ and $\{V_{n,s}\}_{n,s \in N}$ respectively. (Since we will always use the notation $\hat{x}$ to denote the integer x considered as an element of $\hat{M}$, there will be no confusion in the definition of σ, τ, or ν, which from now on will be used only in the above context.) Recall that given $\nu = \langle e, \sigma, \tau\rangle$ and $\nu' = \langle e', \sigma', \tau'\rangle$, $\nu \leq \nu'$ means that $e = e'$, $\sigma \subseteq \sigma'$, and $\tau \supseteq \tau'$.

Intuitively, the role of the various pockets is the following. Most of the elements $\hat{y}$ entering the surface of $\hat{M}$ will fall into pocket $\hat{Q}$ where they await a stage s at which there is a certain x in M (specifically an x on track D) such that $\nu(s, d, x) = \nu(s, d, \hat{y})$, where $d = d(s, \hat{y})$ is a certain recursive function. We then place x in pocket P, and regard x as a proper "mate" for $\hat{y}$. Several (but at most finitely many) different $\hat{y} \in \hat{Q}$ may be permanently matched to the same $x \in P$. Using Corollary 1.7 it will be shown at the end of § 6 that this suffices. The function $d(s, \hat{y})$ which *applies* to elements $\hat{y} \in \hat{M}$ will be *determined* by elements $x \in M$ (specifically $x \in C, D$) so that as more potential mates x in some ν_0 arrive in D the function $d(s, \hat{y})$ for most $\hat{y}$ in ν_0 has large value thereby increasing the maximum e for which $\nu(s, e, x)$ and $\nu(s, e, \hat{y})$ must agree. On the other hand when few x in ν_0 appear in D, then $d(s, \hat{y})$, for $\hat{y}$ in ν_0, tends to have a small value, making it easier to choose a mate x from D as above. The heart of the proof is Lemma 6.1 which uses this principle.

From now on fix the following notation concerning full e-states. Let $\nu(e, x)$ and $\nu(e, \hat{x})$ denote $\lim_s \nu(s, e, x)$ and $\lim_s \nu(s, e, \hat{x})$ which of course exist. Given an e-state ρ and any $i \leqq e$, let $[\rho]_i$ denote $\rho \cap \{0, 1, \cdots, i\}$. Given a full e-state ν, and any $i \leqq e$, let $[\nu]_i$ denote the full i-state $\langle i, [\sigma]_i, [\tau]_i \rangle$. For technical convenience let $[\rho]_{-1} = \emptyset$, and $\nu_{-1} = \langle -1, \emptyset, \emptyset \rangle$. Given $\nu = \langle e, \sigma, \tau \rangle$ and $\nu' = \langle e', \sigma', \tau' \rangle$, we say that ν' *extends* ν, denoted $\nu \prec \nu'$, if $e \leqq e'$, $[\sigma']_e = \sigma$, and $[\tau']_e = \tau$. (Note that $\nu_{-1} \prec \nu$ for all ν.) Given $\nu_0 = \langle e_0, \sigma_0, \tau_0 \rangle$ and $\nu_1 = \langle e_1, \sigma_1, \tau_1 \rangle$, let $\nu_0 \leqq^\tau \nu_1$ ($\nu_0 \leqq^\sigma \nu_1$) denote that $\nu_0 \leqq \nu_1$ and $\tau_0 = \tau_1$ ($\sigma_0 = \sigma_1$). (As usual $\nu_0 \prec \nu_1$ ($\nu_0 < \nu_1$) denotes that $\nu_0 \prec \nu_1$ ($\nu_0 \leqq \nu_1$) and $\nu_0 \neq \nu_1$.) Given any set $\mathcal{S}$ of full states and any ν_0 define $\mathcal{S}[\nu_0] = \{\nu : \nu \in \mathcal{S} \text{ and } \nu_0 \prec \nu\}$. For any $\nu = \langle e, \sigma, \tau \rangle$, we define the *length* of ν, denoted $|\nu|$, to be e.

For each track X of M we have a r.e. sequence of elements $x_1, x_2, \cdots$, such that x_n is added to the sequence at stage s just if x_n enters track X at stage s. We define a corresponding r.e. sequence $\mathcal{S}(X)$ of full e-states as follows. If element x enters track X at stage s, let $\mathcal{S}_s(X)$ denote the finite sequence $\{\nu(s, e, x) : e \leqq x\}$, and $\mathcal{S}_s(X) = \emptyset$ otherwise. (If x exists it will be unique.) Let $\mathcal{S}(X)$ denote the sequence which is the concatenation of $\{\mathcal{S}_s(X) : s \in N\}$. The resulting pair of sequences in referred to as *stream* X. It will follow from Lemma 4.2 that each element x enters a track at most finitely often, although a given ν may appear on $\mathcal{S}(X)$ infinitely often, which we denote by "$\nu \in \mathcal{S}(X)$ i.o." For a track $\hat{X}$ of $\hat{M}$ the sequence $\mathcal{S}(\hat{X})$ and stream $\hat{X}$ are defined similarly, with $\nu(s, e, x)$ replaced by $\nu(s, e, \hat{x})$.

Given stream X and full e-state ν, we say that X *covers* (*exactly covers*, *τ-exactly covers*, *dual covers*) ν if some $\nu' \in \mathcal{S}(X)$ i.o., where $\nu \leqq \nu'$ ($\nu = \nu'$, $\nu \leqq^\tau \nu'$, $\nu \geqq \nu'$). Given streams X and Y, we say that X *covers* (*exactly covers*, *τ-exactly covers*, *dual covers*) Y if X covers (exactly covers, τ-exactly covers, dual covers) every ν such that $\nu \in \mathcal{S}(Y)$ i.o. These definitions are extended in the obvious way in case one or both are streams in $\hat{M}$. Since x ($\hat{x}$) enters track A ($\hat{A}$) just when x is enumerated in A (B) we will identify the set A (B) with stream A ($\hat{A}$).

Remark. The hypotheses (2.4) and (2.5) of the Extension Theorem assert respectively that A covers B and B dual covers A.

We say that streams X and Y are *equivalent* if X exactly covers Y, and Y exactly covers X. Rule R_1 uses a slight variation of the well-known Friedberg decomposition method [10, p. 230] to decompose stream C into streams C_1 and C_2, each equivalent to C. We let $\mathcal{R}_s$ denote a certain sequence defined by induction on s and containing (exactly once) each pair $\langle \nu, j \rangle$ for all $j \in \{1, 2\}$ and all full e-states ν, for $e < s$. Let $\mathcal{R}_0 = \{\langle \nu_{-1}, 1\rangle, \langle \nu_{-1}, 2\rangle\}$.

RULE R_1. *Suppose that sequence $\mathcal{R}_s$ is given. If an element x enters track C at stage s, then at stage $s + 1$ it enters either track C_1 or C_2 (with $\nu(s + 1, x, x) = \nu(s, x, x)$) as follows. Let $\langle \nu', i' \rangle$ be the first pair $\langle \nu, i \rangle$ on the sequence $\mathcal{R}_s$ such that $\nu \prec \nu(s, x, x)$. Remove $\langle \nu', i' \rangle$ from its present position on $\mathcal{R}_s$, place it at the end of the sequence, and place x on track $C_{i'}$. In this case we say that $\langle \nu', i' \rangle$ is reset at stage $s + 1$. Finally, whether an element x entered track C_1 or not, add $\langle \nu, i \rangle$ at the end of the sequence (in any fixed effective order) for each $i \in \{1, 2\}$ and each full s-state ν. Let $\mathcal{R}_{s+1}$ denote the resulting sequence.*

RULE $\hat{R}_1$. *Like R_1 but with $\hat{C}$, $\hat{C}_1$, $\hat{C}_2$ in place of C, C_1, C_2.*

LEMMA 4.1. *Streams C_1 and C_2 ($\hat{C}_1$ and $\hat{C}_2$) are each equivalent to stream C ($\hat{C}$).*

Proof. Clearly a given pair $\langle \nu, i \rangle$ is reset infinitely often just if $\nu \in \mathbb{S}(C)$ i.o., in which case Rule R_1 guarantees that $\nu \in \mathbb{S}(C_1)$ i.o., and $\nu \in \mathbb{S}(C_2)$ i.o. Conversely, $\nu \in \mathbb{S}(C_1)$ i.o. obviously implies that $\nu \in \mathbb{S}(C)$ i.o. ⊠

Although streams C and D are not equivalent, D exactly covers C (because D exactly covers C_2 which exactly covers C), and C covers D (because if x enters track D at stage s, then x entered track C at some stage $t < s$, such that

$$\nu(s, x, x) \leqq^{\sigma} \nu(t, x, x)$$

by rules R_3, R_6, and R_7.)

The next rules depend upon certain key recursive functions $d(s, \hat{x})$ ($d'(s, x)$) to be defined in §5, which will depend only upon $\mathbb{S}(C)$ and $\mathbb{S}(D)$ ($\mathbb{S}(\hat{C})$ and $\mathbb{S}(\hat{D})$) and which will satisfy

$$(\forall x)(\exists s)(\forall t)_{\geq s}[d(t + 1, \hat{x}) \leqq d(t, \hat{x})]\ , \text{ and} \tag{4.3}$$

$$(\forall x)(\exists s)(\forall t)_{\geq s}[d'(t + 1, x) \leqq d'(t, x)]\ . \tag{4.4}$$

Now (4.3) and (4.4) obviously imply

$$(\forall x)[\lim_s d(s, \hat{x}) \text{ and } \lim_s d'(s, x) \text{ exist}]\ , \tag{4.5}$$

which will be assumed from now on. When functions d and d' are defined, (4.3) and (4.4) will be verified without circularity.

Rule R_2 will be precisely stated in §6. Briefly, if x enters track D at stage s, then at stage $s + 1$, it enters either pocket P or track D_1 according to which elements are in P_s and $\hat{Q}_{s+1}$. (Very roughly, x enters pocket P just if some $\hat{y} \in \hat{Q}_{s+1}$ needs x as a mate.) If $x \in P_{s+1} - P_s$, there may be several $y \in P_s - P_{s+1}$. These are handled by Rule R_{12}. Rule $\hat{R}_2$ is similar.

In §5, we shall give Rules R_3 and R_4 which govern the enumeration of x in $\hat{V}^-_{n,s+1}$ as x passes gate G_1 or G_2, and Rule R_5 whereby $x \in Q_s \cap Q_{s+1}$ may sometimes be enumerated in $\hat{V}^-_{n,s+1}$. Rules $\hat{R}_3$, $\hat{R}_4$ and $\hat{R}_5$ for $\hat{U}^-_n$ are similar.

Rule R_6. *Element* $x \in \hat{V}^-_{n,s+1} - \hat{V}^-_{n,s}$ *only if at stage* $s + 1$, *Rule* R_3, R_4, *or* R_5 *applies to* x.

Rule R_7. *If* $x \in U_{n,s+1} - U_{n,s}$ *and* $x \in A_s$, *then* x *is in some pocket* M *at the end of stage* s.

In particular, Rules R_6 and R_7 imply that the only change in full x-state of x allowed during a play (until x reaches a pocket) is under R_3 or R_4. Rules $\hat{R}_6$ and $\hat{R}_7$ for $\hat{M}$ are similar. In addition to the above rules for a play, we need rules concerning the pockets.

Rule R_8. *If* $x \in Q_s$, *and if either* $x \in U_{n,s+1} - U_{n,s}$ *for some* $n \leqq x$, *or if* $d'(s + 1, x) \neq d'(s, x)$, *remove* x *from pocket* Q *at stage* $s + 1$, *and place* x *above hole* H_2.

Rule $\hat{R}_8$. *If* $\hat{x} \in \hat{Q}_s$, *and if either* $x \in V_{n,s+1} - V_{n,s}$ *for some* $n \leqq x$ *or if* $d(s + 1, \hat{x}) \neq d(s, \hat{x})$, *remove* $\hat{x}$ *from pocket* $\hat{Q}$ *at stage* $s + 1$, *and place* x *above hole* $\hat{H}_2$.

Rule R_9. *If* $x \in Q_s$, *then* $x \in Q_{s+1}$, *unless* x *is removed at stage* $s + 1$ *under* R_8.

In §6, we shall give Rules R_{10} and R_{11} which tell when $x \in P_s$ may be removed from pocket P at stage $s + 1$ in case $x \in U_{n,s+1} - U_{n,s}$ or $\hat{Q}_s \neq \hat{Q}_{s+1}$. Rules $\hat{R}_9$, $\hat{R}_{10}$, $\hat{R}_{11}$, and $\hat{R}_{12}$ for pockets $\hat{Q}$ and $\hat{P}$ are similar.

Rule R_{12} *If* $x \in P_s$, *then* $x \in P_{s+1}$ *unless* x *is removed under Rule* R_2, R_{10}, *or* R_{11}. *Furthermore, if* $x \in P_s - P_{s+1}$, *and* x *last entered pocket* P *at stage* $t \leqq s$, *then at stage* $s + 1$, x *is placed above hole* H_3 *if* $\nu(s + 1, x, x) = \nu(t, x, x)$, *and above hole* H_2 *otherwise*.

The Construction. At stage $s = 0$ do nothing. Assume inductively that we are at a stage $s + 1$ such that each $x \in M_s$ and $\hat{x} \in \hat{M}_s$ is either in a pocket or above a hole.

Case 1. At the end of stage s some $x \in M_s$ or $\hat{x} \in \hat{M}_s$ is above a hole. Choose x_1 to be the least such $x \in M_s$ if such exists, and otherwise choose $\hat{x}_1$ to be the least such $\hat{x} \in \hat{M}_s$. Release x_1 ($\hat{x}_1$) from its hole onto the surface of

M ($\hat{M}$) at stage $s + 1$, and continue processing x_1 ($\hat{x}_1$) on successive moves according to the rules until x_1 ($\hat{x}_1$) enters a pocket.

(During this play a move for x_1 at stage t may cause some $d(t, \hat{y}) \neq d(t - 1, \hat{y})$ if x_1 enters track C or D, or may cause a change in the membership of $\hat{Q}_t$ or P_t if Rule R_2 applies to x_1, or the membership of Q_t if x_1 finally enters pocket Q, thereby causing other elements to be removed from pockets P, Q, $\hat{P}$, or $\hat{Q}$ and placed above holes H_2, H_3, $\hat{H}_2$, or $\hat{H}_3$. These elements are not processed until after x_1 ($\hat{x}_1$) has reached a pocket.)

It will follow from Lemma 4.2 that each x ($\hat{x}$) can re-enter M ($\hat{M}$) at most finitely often. Since M_s ($\hat{M}_s$) is finite, after finitely many applications of Case 1 we shall be in Case 2.

Case 2. Each $x \in M_s$ and $\hat{x} \in \hat{M}_s$ is in a pocket at the end of stage s. Consider the next pair $\langle x, Y \rangle$ such that element x is enumerated in r.e. set Y according to our given simultaneous enumeration g_0.

Subcase 2A. $Y = A$ (B). Enumerate x in A_{s+1} (B_{s+1}), and release x ($\hat{x}$) from hole H_1 ($\hat{H}_1$) onto track A ($\hat{A}$). Continue processing x ($\hat{x}$) on successive moves according to the rules until x ($\hat{x}$) reaches a pocket. (By modifying g_0 if necessary we may assume that if $x \in A_s$ or B_s, then $x < s$.)

Subcase 2B. $Y = U_n$ (V_n). Enumerate x in $U_{n,s+1}$ ($V_{n,s+1}$). If $x \notin A_s$ (B_s) go to stage $s + 2$. Otherwise, x ($\hat{x}$) must be in a pocket of M ($\hat{M}$) and is processed according to either Rule R_8 or R_{10} ($\hat{R}_8$ or $\hat{R}_{10}$) according to which pocket contains x ($\hat{x}$).

Subcase 2C. $Y = \hat{U}_n^+$ ($\hat{V}_n^+$). Enumerate x in $\hat{U}_{n,s+1}^+$ ($\hat{V}_{n,s+1}^+$), and go to stage $s + 2$. (By hypothesis on g_0, in this case $x \notin A_s$ (B_s), and thus no processing is necessary.)

LEMMA 4.2. *Each element x ($\hat{x}$) re-enters the surface of M ($\hat{M}$) at most finitely often.*

Proof. By the construction, if x is placed above hole H_2 at stage $s + 1$ then $x \in P_s$ or $x \in Q_s$. But then, by Rules R_8 and R_9, if $x \in Q_s - Q_{s+1}$ then either $\nu(s + 1, x, x) \neq \nu(s, x, x)$ or $d'(s + 1, x) \neq d'(s, x)$. By Rule R_{12} if $x \in P_s - P_{s+1}$ and x is placed above hole H_2 at stage $s + 1$ then $\nu(s + 1, x, x) \neq \nu(t, x, x)$, where x last entered pocket P at stage $t < s$, because otherwise x is placed above hole H_3. Clearly, $\lim_s \nu(s, x, x)$ exists, and by (4.5) $\lim_s d'(s, x)$ exists. Hence, x is placed above (and hence re-enters from) hole H_2 at most finitely often. But then clearly x is placed above hole H_3 finitely often, because after x re-enters from H_3 it (eventually) goes to pocket Q, from which it can re-enter only via H_2. The case of $\hat{x}$ in $\hat{M}$ is similar. ⊠

5. Proof of the Extension Theorem

Part II: Covering

In this section we define the previously mentioned recursive functions $d(s, \hat{x})$ and $d'(s, x)$ and give Rules R_3, R_4, R_5 ($\hat{R}_3$, $\hat{R}_4$, and $\hat{R}_5$) under which elements are enumerated in the r.e. sets $\hat{V}_n^-$ ($\hat{U}_n^-$). By Rule R_6 ($\hat{R}_6$) x is enumerated in $\hat{V}_n^-$ ($\hat{U}_n^-$) only if R_3, R_4, or R_5 ($\hat{R}_3$, $\hat{R}_4$, or $\hat{R}_5$) applies. In § 6 we define certain mappings to verify that this enumeration satisfies the conclusion of the Extension Theorem.

These rules are defined so that using the Extension Theorem hypothesis,

$$A \text{ covers } B, \text{ and } B \text{ dual covers } A\ , \tag{5.1}$$

we can prove that for any streams X of M and $\hat{X}$ of $\hat{M}$,

$$C \text{ covers } \hat{X}, \text{ and } \hat{C} \text{ dual covers } X\ , \tag{5.2}$$

Associated with pocket $\hat{Q}$ there is a r.e. sequence of full e-states $\mathcal{S}(\hat{Q})$, which is the concatenation of the finite sequences $\mathcal{S}_s(\hat{Q})$, $s \in N$, defined as follows. If $\hat{x} \in \hat{Q}_{s+1} - \hat{Q}_s$, or if $x \in \hat{Q}_s \cap \hat{Q}_{s+1}$ and $\sigma(s+1, x, \hat{x}) \neq \sigma(s, x, \hat{x})$ (necessarily because of Rule $\hat{R}_5$), then $\mathcal{S}_{s+1}(\hat{Q})$ is the sequence $\{\nu(s+1, e, \hat{x})$: $e \leqq x\}$, and $\mathcal{S}_{s+1}(\hat{Q}) = \varnothing$ otherwise. (By Lemma 4.2, each $\hat{x}$ causes at most finitely many ν to be added to $\mathcal{S}(\hat{Q})$.) Using $\mathcal{S}(\hat{Q})$ we will consider $\hat{Q}$ as a stream of $\hat{M}$.

One of the main purposes of Rules R_3, $\hat{R}_4$, and $\hat{R}_5$ is to achieve

$$D \text{ exactly covers } \hat{Q}\ , \tag{5.3}$$

and similarly for $\hat{R}_3$, R_4, R_5, $\hat{D}$ and Q (although in fact we shall need somewhat more).

Before defining Rules R_3, R_4, R_5 and the recursive function $d(s, \hat{x})$, we recursively define uniformly in s sequences $\mathcal{K}_s$, $\mathcal{M}_s$, and $\mathcal{P}_s$ of full e-states which will be determined by streams C and D, and which play a key role from now on. Furthermore, $\mathcal{K}_s$ will contain (exactly once) each full j-state, for all $j < s$.

Notation. Given $\nu_1, \nu_2 \in \mathcal{K}_s$ let $\nu_1 \leqq_s^* \nu_2$ denote that ν_1 precedes ν_2 on the sequence $\mathcal{K}_s$. (Thus, $\leqq_s^*$ is a linear ordering of all full j-states, for all $j < s$, which is recursive uniformly in s.)

Let $\mathcal{K}_0 = \{\nu_{-1}\}$. Given $\mathcal{K}_s$, define

$$\mathcal{K}_{s+1}^2 = \{\nu : \nu \in \mathcal{K}_s \text{ and } (\exists \nu')[\nu <^r \nu' \text{ and } [\nu' \in \mathcal{S}_{s+1}(C) \text{ or } \nu' \in \mathcal{S}_{s+1}(D)]]\}\ .$$

Define $\mathcal{K}_{s+1}^1 = \mathcal{K}_s - \mathcal{K}_{s+1}^2$. Consider $\mathcal{K}_{s+1}^1$ and $\mathcal{K}_{s+1}^2$ each as sequences with the ordering induced by $\leqq_s^*$. Let $\mathcal{K}_{s+1}^3$ be the sequence of all full s-states arranged in some effective order (uniformly in s) such that $\nu = \langle e', \sigma', \tau' \rangle$ precedes $\nu = \langle e, \sigma, \tau \rangle$ if $\tau' \subsetneqq \tau$ or if $\tau = \tau'$ and $\sigma' \supsetneqq \sigma$. Let $\mathcal{K}_{s+1}$ denote the sequence which is the concatenation of the sequences $\mathcal{K}^1_{s+1}$, $\mathcal{K}^2_{s+1}$, and $\mathcal{K}^3_{s+1}$ in that order.

LEMMA 5.1. *For all e-states* σ, σ', τ, *and all* $s > e$,

$$\sigma' \supsetneqq \sigma \Longrightarrow [\langle e, \sigma', \tau \rangle \leqq^{*}_{s} \langle e, \sigma, \tau \rangle] .$$

Proof. Fix e, σ, τ, and $\sigma' \supsetneqq \sigma$. The assertion is clearly true for $s = e + 1$ by the ordering of $\mathcal{K}^3_{e+1}$. Assume it is true for some $s > e$. If $\nu' = \langle e, \sigma', \tau \rangle \in \mathcal{K}^2_{s+1}$ via some ν'' where $\nu' <^{\tau} \nu''$, then $\langle e, \sigma, \tau \rangle = \nu \in \mathcal{K}^2_{s+1}$ also via ν'' because $\nu \leqq^{\tau} \nu' \leqq^{\tau} \nu''$. Otherwise, $\nu' \in \mathcal{K}^1_{s+1}$. In either case, clearly $\nu' \leqq^{*}_{s+1} \nu$. ⊠

If $\nu \in \mathcal{K}^2_{s+1}$, we say that ν is *reset at stage* $s + 1$. Let $\mathcal{K}_\omega$ denote the set of those ν which are reset at most finitely often. From the definitions of $\mathcal{K}_{s+1}$ and $\mathcal{K}_\omega$ immediately follows the statement

$$(5.4) \qquad (\forall \nu)[\nu \in \mathcal{K}_\omega \Longrightarrow (\exists s)(\forall t)_{\geqq s}(\forall \nu')[\nu \in \mathcal{K}^2_t \Longrightarrow \nu <^{*}_{t} \nu']] .$$

Given any sequence $\mathcal{S}$ of full e-states, and any ν_0, we say ν_0 is *maximal with respect to* $\mathcal{S}$ just if

$$(\forall \nu)[[\nu \in \mathcal{S} \text{ i.o. and } \nu_0 \leqq^{\tau} \nu] \Longrightarrow \nu = \nu_0] .$$

LEMMA 5.2. *For all* ν, $\nu \in \mathcal{K}_\omega$ *just if* ν *is maximal with respect to* $\mathcal{S}(D)$.

Proof. By the definition of $\mathcal{K}^2_{s+1}$, ν is reset finitely often just if ν is maximal with respect to *both* $\mathcal{S}(D)$ and $\mathcal{S}(C)$. However, any ν which is maximal with respect to $\mathcal{S}(D)$ is maximal with respect to $\mathcal{S}(C)$ also, because D exactly covers C_2 and hence C. ⊠

By induction on s, we now define $\mathcal{M}_s$ and $\mathcal{P}_s$ such that $\mathcal{M}_s \subseteqq \mathcal{P}_s \subseteqq \mathcal{K}_s$. It will suffice to define $\mathcal{M}_s$ because we let

$$\mathcal{P}_s = \{\nu \colon (\exists \nu')[\nu' \in \mathcal{M}_s \text{ and } \nu \leqq^{\tau} \nu']\} .$$

Let $\mathcal{M}_0 = \{\nu_{-1}\}$. Given $\mathcal{K}_{s+1}$ and $\mathcal{M}_s$, define $\mathcal{M}_{s+1}$ as follows. Fix $\nu = \langle e, \sigma, \tau \rangle$ and assume that we have already determined whether $\nu' \in \mathcal{M}_{s+1}$ for all $\nu' = \langle e', \sigma', \tau' \rangle$ such that $e' < e$. We say that ν is *excluded from* $\mathcal{M}_{s+1}$ if one of the following two clauses holds:

Clause 1. $(\exists \nu')[\nu' \leqq^{*}_{s} \nu$ and $\nu' \in \mathcal{K}^2_{s+1}]$.

Clause 2. $(\exists \nu')(\exists \hat{X})[\nu' \notin \mathcal{P}_s$ and $\nu' \in \mathcal{S}_{s+1}(\hat{X})$ and $\nu' = \langle e', \sigma', \tau' \rangle$ where $e' < e]$.

Define $\nu \in \mathcal{M}_{s+1}$ just if ν is not excluded from $\mathcal{M}_{s+1}$ and either $\nu \in \mathcal{M}_s$ or $\nu \in \mathcal{S}_{s+1}(D)$. Define

$$\mathcal{M}_\omega = \{\nu \colon \nu \in \mathcal{M}_s \text{ for almost all } s\} ,$$

$$\mathcal{P}_\omega = \{\nu \colon \nu \in \mathcal{P}_s \text{ for almost all } s\} .$$

Note that $\mathfrak{M}_\omega \subseteq \mathcal{K}_\omega$ by Clause 1 above, although of course $\mathcal{P}_\omega \nsubseteq \mathcal{K}_\omega$. Clause 2 above is included only to facilitate proving the crucial Lemma 5.11 below from which it easily follows that C covers any stream $\hat{X}$ of $\hat{M}$, and furthermore that,

$$D \ \tau\text{-exactly covers any stream } \hat{X} \text{ of } \hat{M}\,. \tag{5.5}$$

To give the reader some intuition about $\mathfrak{M}_\omega$ and $\mathcal{P}_\omega$ we first derive their crucial properties using the hypothesis (5.5), which can later be dropped after Lemma 5.16. Since Lemma 5.11 will be proved by induction on $e = |\nu|$, we now define for any stream X of M (or $\hat{X}$ of $\hat{M}$),

$$\mathcal{S}^e(X) = \mathcal{S}(X) \cap \{\nu : |\nu| = e\}\,.$$

LEMMA 5.3. *Assuming* (5.5),

i) $(\forall\nu)[\nu \in \mathcal{K}_\omega$ *and* $\nu \in \mathcal{S}(D)$ i.o. $\Rightarrow \nu \in \mathfrak{M}_\omega]$,

ii) $(\forall\nu)[\nu \in \mathcal{P}_s$ *for infinitely many* $s \Rightarrow \nu \in \mathcal{P}_\omega]$.

Proof. Fix e and assume that i) and ii) hold for all ν' such that $|\nu'| = e' < e$. For any such ν' and any stream $\hat{X}$ of $\hat{M}$,

$$\nu' \in \mathcal{S}_{s+1}(\hat{X}) - \mathcal{P}_s \text{ for at most finitely many } s, \tag{5.6}$$

because of inductive hypothesis ii) for ν' and because D τ-exactly covers $\mathcal{S}^{e'}(\hat{X})$ by (5.5). Thus, by (5.6) any $\nu \in \mathcal{S}^e(D)$ is excluded from $\mathfrak{M}_{s+1}$ under Clause 2 at most finitely often.

Now for i) fix $\nu = \langle e, \sigma, \tau\rangle$ such that $\nu \in \mathcal{K}_\omega$ and $\nu \in \mathcal{S}(D)$ i.o. Since $\nu \in \mathcal{K}_\omega$, ν is excluded from $\mathfrak{M}_{s+1}$ under Clause 1 at most finitely often. Thus, $\nu \in \mathfrak{M}_\omega$ because $\nu \in \mathcal{S}(D)$ i.o.

For ii) fix $\nu_1 = \langle e, \sigma, \tau\rangle \in \mathcal{P}_s$ for infinitely many s. Then either $\nu_1 \in \mathcal{P}_\omega$ immediately or else $\nu_1 \in \mathcal{P}_{s+1} - \mathcal{P}_s$ for infinitely many s. For such an s, some $\nu_2 \in \mathfrak{M}_{s+1} - \mathfrak{M}_s$, and hence $\nu_2 \in \mathcal{S}_{s+1}(D)$, where $\nu_1 \leqq^\tau \nu_2$. Choose any $\nu_3 \in \mathcal{S}(D)$ i.o. where $\nu_1 \leqq^\tau \nu_3$, and ν_3 is maximal with respect to $\mathcal{S}(D)$. By part i), $\nu_3 \in \mathfrak{M}_\omega$ and hence $\nu_1 \in \mathcal{P}_\omega$, thereby proving Lemma 5.3. ⊠

We turn now briefly to the dual part of the machines. Sequences $\mathcal{K}'_s$, $\mathfrak{M}'_s$, $\mathcal{P}'_s$ are defined similarly to $\mathcal{K}_s$, $\mathfrak{M}_s$, and $\mathcal{P}_s$ with $\mathcal{S}(C)$ and $\mathcal{S}(D)$ replaced by $\mathcal{S}(\hat{C})$ and $\mathcal{S}(\hat{D})$, with the roles of σ and τ interchanged, and with all the obvious changes. All the obvious dual lemmas hold.

Rule R_3 involves a certain r.e. sequence $\mathcal{H}$ of full e-states, which is the concatenation of the finite sequences $\mathcal{H}_s$, $s \in N$, defined as follows. $\mathcal{H}_s = \emptyset$ unless there is some track $\hat{X}$ of $\hat{M}$, some element $\hat{y}_1$ on track $\hat{X}$ at stage s, and some full e_1-state ν_1 such that

$$\nu_1 \in \mathcal{S}_s(\hat{X}) - \mathcal{P}_{s-1}$$

via $\hat{y}_1$, in which case $\mathcal{H}_s$ consists of the following full e_1-states (in some effective order uniformly in s):

$$\{\nu\colon (\exists t)_{\leqq s}[\hat{y}_1 \in \hat{M}_t \text{ and } \nu(e_1, t, \hat{y}_1) \leqq^{\tau} \nu]\} .$$

Once added to $\mathcal{H}$, a given ν is never removed from $\mathcal{H}$ or altered in position, although it may later be *checked* during an application of Rule R_3. Let $\mathcal{H}_{\leqq s}$ denote the sequence of elements added to $\mathcal{H}$ by the end of stage s.

RULE R_3. *Suppose an element x enters track C_1 at stage s. Let $\nu_0 = \langle x, \sigma_0, \tau_0\rangle$ denote $\nu(s, x, x)$. Let $\nu_1 = \langle e_1, \sigma_1, \tau_1\rangle$ be the first member $\nu = \langle e, \sigma, \tau\rangle \in \mathcal{H}_{\leqq s}$ such that:*

i) *ν has not been checked by the end of stage s;*

ii) *$e \leqq x$; and*

iii) *$\nu \leqq^{\sigma} [\nu_0]_e$ (i.e., $\sigma = [\sigma_0]_e$, and $\tau \supseteq [\tau_0]_e$).*

If ν_1 exists, then at stage $s + 1$ check ν_1, enumerate x in $\hat{V}^-_{n,s+1}$ for each $n \leqq e_1$ such that $n \in \tau_1 - \tau_0$ (so that $\nu(s + 1, e_1, x) = \nu_1$), and place x on track C_3. If ν_1 fails to exist, then at stage $s + 1$ place x on track C_4, and let $\nu(s + 1, x, x) = \nu(s, x, x)$.

We shall later prove that each ν may be added to $\mathcal{H}$ at most finitely often. Thus, the following lemma will show that the enumeration under Rule R_3 above has been sufficiently restrained so that $\hat{C}$ dual covers C_3.

LEMMA 5.4. *Fix e. If each ν such that $|\nu| < e$ is added to $\mathcal{H}$ at most finitely often then $\hat{C}$ dual covers $\mathcal{S}^e(C_3)$.*

Proof. Fix e. Assume that each ν of length $< e$ is added to $\mathcal{H}$ at most finitely often. Since each ν on the sequence $\mathcal{H}$ is checked at most once, there is a stage, say s_0, after which no ν of length $< e$ is ever checked on $\mathcal{H}$. Fix $\nu_0 \in \mathcal{S}^e(C_3)$ i.o. Now there exist infinitely many stages $t_j \geqq s_0$, $j \in N$, such that for each $j \in N$ at stage t_j ($t_j + 1$) some element $x_j \geqq e$ enters track C_1 (C_3), where $\nu(t_j + 1, e, x_j) = \nu_0$. But by Rule R_3, at stage $t_j + 1$, x_j enters track C_3 only if some $\nu_j \in \mathcal{H}$ is checked, where $\nu_j \prec \nu(t_j + 1, x_j, x_j)$. Furthermore, $\nu_0 \prec \nu_j$ because $t_{j+1} \geqq s_0$. Thus, infinitely often some ν_0' is added to $\mathcal{H}$ such that $\nu_0 \prec \nu_0'$.

However, each element $\hat{y}$ of $\hat{M}$ causes at most finitely many ν to be added to $\mathcal{H}$, because of Lemma 4.2. Thus, for some track $\hat{X}$ of $\hat{M}$ and some $\nu_1 \leqq^{\tau} \nu_0$, $\nu_1 \in \mathcal{S}(\hat{X})$ i.o. But then $\nu_2 \in \mathcal{S}(\hat{C})$ i.o. for some $\nu_2 \leqq^{\tau} \nu_1$, and hence $\hat{C}$ dual covers ν_0. ⊠

RULE $\hat{R}_3$. *Same as Rule R_3, but with C_1, C_3, C_4, $\hat{C}$, $\hat{V}^-_n$, $\mathcal{H}$ replaced by $\hat{C}_1$, $\hat{C}_3$, $\hat{C}_4$, C, $\hat{U}^-_n$, and $\mathcal{H}'$ (which is defined in the analogous way using $\mathcal{P}'_{s-1}$ and $\mathcal{S}_s(X)$), and with the roles of σ and τ interchanged.*

LEMMA 5.5. *Fix e. If each* ν *such that* $|\nu| < e$ *is added to* $\mathcal{H}'$ *at most finitely often then C covers* $\mathcal{S}^e(\hat{C}_3)$.

Proof. Same as Lemma 5.4, using $\hat{R}_3$. ⊠

The purpose of Rule R_3 is to allow sufficient enumeration at Gate G_1 to yield

LEMMA 5.6. *If* $\hat{X}$ *is any stream of* $\hat{M}$, *and C covers* $\hat{X}$ *then D* τ*-exactly covers* $\hat{X}$.

Proof. Fix a stream $\hat{X}$ of $\hat{M}$ such that C covers $\hat{X}$. Assume for a contradiction that some $\nu_1 \in \mathcal{S}(\hat{X})$ i.o., but no $\nu \in \mathcal{S}(D)$ i.o., if $\nu_1 \leqq^\tau \nu$. Then there exists s' such that no ν is added to $\mathcal{S}(D)$ at any stage $s \geqq s'$ if $\nu_1 \leqq^\tau [\nu]_e$. Replacing ν_1 by some extension if necessary we may assume that $\nu_1 \in \mathcal{S}(\hat{X})$ i.o., and

$$(5.7) \qquad (\forall \nu)[\nu_1 \leqq^\tau [\nu]_e \Longrightarrow \nu \notin \mathcal{S}(D)] .$$

By (5.7), $\nu_1 \notin \mathcal{P}_s$ for any s. Thus, by the definition of $\mathcal{H}$, each $\nu \in \mathcal{H}$ i.o. if $\nu_1 \leqq^\tau \nu$.

Let $\nu_1 = \langle e, \sigma_1, \tau_1 \rangle$. Since C covers $\hat{X}$, some $\nu_2 = \langle e, \sigma_2, \tau_0 \rangle \in \mathcal{S}(C)$ i.o. (and hence $\nu_2 \in \mathcal{S}(C_1)$ i.o.) where $\sigma_2 \supseteqq \sigma_1$ and $\tau_0 \subseteqq \tau_1$. Furthermore, $\nu_1 \in \mathcal{S}_{s+1}(\hat{X}) - \mathcal{P}_s$ for infinitely many s implies that $\nu_3 = \langle e, \sigma_2, \tau_1 \rangle \in \mathcal{H}$ i.o.

Now $\nu_1 \leqq^\tau \nu_3$, and thus by (5.7) $\nu_3 \notin \mathcal{S}(D)$. Hence, $\nu_3 \notin \mathcal{S}(C_3)$ and ν_3 once added to $\mathcal{H}$ is never checked under Rule R_3. Choose s_0 such that no ν preceding ν_3 on the sequence $\mathcal{H}$ is checked at any stage $s \geqq s_0$. Choose $s_1 \geqq s_0$ such that some $x \geqq e$ enters track C_1 at stage s_1, where $\nu(s_1, e, x) = \nu_2 = \langle e, \sigma_2, \tau_0 \rangle$. But $\tau_0 \subseteqq \tau_1$, and hence at stage $s + 1$ by Rule R_3, $\nu_3 = \langle e, \sigma_2, \tau_1 \rangle$ is checked, and x is placed on track C_3 with $\nu(s_1 + 1, e, x) = \nu_3$, contrary to (5.7). ⊠

LEMMA 5.7. *D* τ*-exactly covers* $\hat{A}$.

Proof. By Lemma 5.6 and hypothesis (5.1), which asserts that A (and hence C) covers $\hat{A}$ $(= B)$. ⊠

Using $\mathcal{P}_s$ we define the recursive function $d(s, \hat{x})$ as follows. Let $v_x = \mu s[x \in B_{s+1} - B_s]$ if s exists, and be undefined otherwise. If v_x is defined and $s > v_x$, define

$$\begin{aligned} d(s, \hat{x}) = \max \{e\colon & [\nu(s, x, \hat{x})]_e \in \mathcal{P}_s \text{ and} \\ & (\forall t)[v_x < t < s \Longrightarrow e \leqq d(t, , \hat{x})]\} , \end{aligned}$$

and $d(s, \hat{x}) = -1$ otherwise. Note that $d(s, \hat{x})$ is a recursive function since $\mathcal{P}_s$ is recursive uniformly in s. Note also that the second clause guarantees that d satisfies (4.3) without any assumptions on $\mathcal{P}_s$, thereby justifying our use of (4.5) in Lemma 4.3.

Notation. From now on $\nu^*(s, \hat{x})$ will denote $[\nu(s, x, \hat{x})]_{d(s,\hat{x})}$, and $\nu^*(\hat{x})$ denotes $\lim_s \nu^*(s, \hat{x})$ which exists by (4.3).

RULE $\hat{\mathrm{R}}_4$. *Suppose an element $\hat{x}$ enters track $\hat{E}$ at stage s. Then at stage $s+1$, place $\hat{x}$ on track $\hat{F}$ after performing the following enumeration. Let $\mathcal{S}_{\leqq t}(D)$ denote the sequence which is the concatenation of the sequences $\{\mathcal{S}_u(D): u \leqq t\}$. Let $\nu'(s+1) = \langle e, \sigma', \tau' \rangle$ denote $\nu^*(s, \hat{x})$. Define*

$$\mathcal{T}_{s+1} = \{\nu: \nu \in \mathfrak{M}_{s+1} \text{ and } \nu'(s+1) \leqq^{\tau} \nu\} .$$

Note that $\nu'(s+1) \in \mathcal{P}_{s+1}$, and thus $\mathcal{T}_{s+1} \neq \emptyset$, by the definitions of $\mathcal{P}_{s+1}$ and d. Furthermore, for each $\nu \in \mathfrak{M}_{s+1}$, $\nu \in \mathcal{S}_{\leqq(s+1)}(D)$. Define $\nu''(s+1) = \langle e, \sigma'', \tau' \rangle$ to be the last *ν on the sequence $\mathcal{S}_{\leqq(s+1)}(D)$ such that $\nu \in \mathcal{T}_{s+1}$. Enumerate $\hat{x}$ in $\hat{U}^-_{\bar{n},s+1}$ for each $n \in \sigma'' - \sigma'$. (Hence, $\nu^*(s+1, \hat{x}) = \nu''(s+1) \in \mathfrak{M}_{s+1}$.)*

RULE $\hat{\mathrm{R}}_5$. *Suppose $\hat{x} \in \hat{Q}_s \cap \hat{Q}_{s+1}$. Let $\nu'(s+1) = \langle e, \sigma', \tau' \rangle$ denote $\nu^*(s, \hat{x})$. (Note that $d(s+1, \hat{x}) = e$ by Rule $\hat{\mathrm{R}}_8$.) Suppose that some $\nu \in \mathcal{S}_{s+1}(D)$ where $\nu \in \mathfrak{M}_{s+1}$ and $\nu'(s+1) <^{\tau} \nu$. Denote this ν (which must be unique) by $\nu''(s+1) = \langle e, \sigma'', \tau \rangle$. Enumerate x in $\hat{U}^-_{\bar{n},s+1}$ for all $n \in \sigma'' - \sigma'$. (Hence, $\nu^*(s+1, \hat{x}) = \nu''(s+1) \in \mathfrak{M}_{s+1}$.)*

Rules $\hat{\mathrm{R}}_4$ and $\hat{\mathrm{R}}_6$ immediately yield the following lemma which plays a crucial role in §6.

LEMMA 5.8. $(\forall s)(\forall \hat{x})[\hat{x} \in \hat{Q}_s \Rightarrow \nu^*(s, \hat{x}) \in \mathfrak{M}_s]$.

Proof. Fix $\hat{x} \in \hat{Q}_s$, and suppose that $\hat{x}$ last entered pocket Q at stage $t \leqq s$. By Rule $\hat{\mathrm{R}}_4$, $\nu^*(t, \hat{x}) \in \mathfrak{M}_t$. But if $\hat{x} \in \hat{Q}_u \cap \hat{Q}_{u+1}$ for some u, $t \leqq u < s$, and $\nu' = \nu^*(u, \hat{x}) \in \mathfrak{M}_u$, then $d(u+1, \hat{x}) = d(u, \hat{x})$ (by Rule $\hat{\mathrm{R}}_8$) and either $\nu' \in \mathfrak{M}_{u+1}$ or else some $\nu'' \in \mathfrak{M}_{u+1} \cap \mathcal{S}_{u+1}(D_1)$, where $\nu' <^{\tau} \nu''$ in which case $\nu^*(u+1, \hat{x}) = \nu''$ by Rule $\hat{\mathrm{R}}_5$. ⊠

LEMMA 5.9. $(\forall \hat{y})[\hat{y} \in \hat{Q}_\omega \Rightarrow \nu^*(\hat{y}) \in \mathfrak{M}_\omega]$.

Proof. Fix $\hat{y} \in \hat{Q}_\omega$, and choose s_1 such that $\hat{y} \in \hat{Q}_s$ and $\nu^*(s, \hat{y}) = \nu^*(\hat{y})$ for all $s \geqq s_1$. By Lemma 5.8, $\nu^*(s, \hat{y}) \in \mathfrak{M}_s$ for all $s \geqq s_1$. ⊠

On the other hand, Rules $\hat{\mathrm{R}}_4$ and $\hat{\mathrm{R}}_5$ have not allowed too much enumeration, for we shall now prove by a series of lemmas that C covers any stream $\hat{X}$ of $\hat{M}$. The major difficulties arise from Rule $\hat{\mathrm{R}}_4$.

LEMMA 5.10. *Fix ν_1 and ν_2. Suppose that D τ-exactly covers ν_1, and that Rule $\hat{\mathrm{R}}_4$ applies at infinitely many stages s such that $\nu'(s) = \nu_1$ and $\nu''(s) = \nu_2$. Then D τ-exactly covers ν_2 also.*

Proof. Fix $\nu_1 = \langle e, \sigma_1, \tau_1\rangle$, and $\nu_2 = \langle e, \sigma_2, \tau_1\rangle$ satisfying the hypotheses, and assume to the contrary that D fails to τ-exactly cover ν_2. Then there exists s_1 such that,

$$(\forall s)_{\geqq s_1}(\forall \nu)[\nu_2 \leqq^{\tau} [\nu]_s \Longrightarrow \nu \notin \mathcal{S}_s(D)] \ .$$

But $\nu''(s) = \nu_2$ for infinitely many s implies that $\nu_2 \in \mathfrak{M}_s$ for infinitely many s. Thus, for some $s_2 \geqq s_1$, $\nu_2 \in \mathfrak{M}_s$ for all $s \geqq s_2$.

On the other hand, some $\nu_3 \in \mathcal{S}(D)$ i.o. where $\nu_1 \leqq^{\tau} \nu_3$ and $\nu_3 \in \mathcal{K}_\omega$ because D τ-exactly covers ν_1. Furthermore, ν_3 cannot be excluded from $\mathfrak{M}_s$ at any $s \geqq s_2$ by Clause 2 else ν_2 is excluded from $\mathfrak{M}_s$ also. But $\nu_3 \in \mathcal{K}_\omega$ implies that ν_3 is excluded from $\mathfrak{M}_s$ at most finitely often under Clause 1. Thus $\nu_3 \in \mathfrak{M}_\omega$ because $\nu_3 \in \mathcal{S}(D)$ i.o. Furthermore, $\nu_2 \notin \mathcal{S}(D)$ i.o. implies that for almost all s, some occurrence of ν_3 follows the last occurrence of ν_2 on the sequence $\mathcal{S}_{\leqq s}(D)$. Thus, for almost all s if $\nu'(s) = \nu_1$ under $\hat{\text{R}}_4$, we prefer ν_3 to ν_2 in the definition of $\nu''(s)$, contrary to hypothesis. ⊠

Along with Lemma 5.8, the following lemma is crucial because it almost immediately yields that C covers (and thus D τ-exactly covers) any stream $\hat{X}$ of $\hat{M}$. The main idea is that when some $\nu_1 \in \mathcal{S}_{s+1}(\hat{X}) - \mathcal{P}_s$, we do two things. First we remove from $\mathfrak{M}_{s+1}$ all ν of length $> e_1 = |\nu_1|$, which causes $d(s + 1, \hat{y}) \leqq e_1$. Secondly, we add to $\mathcal{K}_{s+1}$ all ν_2 such that $\nu_1 \leqq^{\tau} \nu_2$ which tends to force D to τ-exactly cover ν_1 if C covers ν_1. These two features enable us to apply Lemma 5.10.

LEMMA 5.11. *For each track X of M ($\hat{X}$ of $\hat{M}$) and each ν,*

i) $\nu \in \mathcal{S}(\hat{X})$ i.o. $\Rightarrow \nu \in \mathcal{P}_\omega$, *and*

ii) $\nu \in \mathcal{S}(X)$ i.o. $\Rightarrow \nu \in \mathcal{P}'_\omega$.

Proof. The proof is by induction on the length of ν. Fix e and assume i) and ii) for all ν such that $|\nu| < e$. It suffices to prove i) for all ν of length e since ii) is dual. By inductive hypotheses ii) for ν of length $< e$, each ν of length $< e$ is added to $\mathcal{K}'$ only finitely often. Thus, by Lemma 5.5,

$$C \text{ covers } \mathcal{S}^e(\hat{C}_3) \ . \tag{5.8}$$

Now by inductive hypothesis i), each ν of length e is excluded from $\mathfrak{M}_{s+1}$ under Clause 2 for at most finitely many s. Thus, by the proof of Lemma 5.3 and by Lemma 5.7 we have for every ν of length e,

$$\nu \in \mathcal{S}(D) \text{ i.o. and } \nu \in \mathcal{K}_\omega \Longrightarrow \nu \in \mathfrak{M}_\omega \ , \tag{5.9}$$

$$\nu \in \mathcal{P}_s \text{ for infinitely many } s \Longrightarrow \nu \in \mathcal{P}_\omega, \text{ and} \tag{5.10}$$

$$\nu \in \mathcal{S}(\hat{A}) \text{ i.o.} \Longrightarrow \nu \in \mathcal{P}_\omega \ . \tag{5.11}$$

Now assume for a contradiction that for some $\hat{X}$ and ν_1, $\nu_1 \in \mathcal{S}(\hat{X})$ i.o., but $\nu_1 \notin \mathcal{P}_\omega$. Let $\nu_1 = \langle e, \sigma_1, \tau_1 \rangle$ with σ_1 minimal for e, and τ_1 minimal for e and σ_1. By (5.10), $\nu_1 \in \mathcal{P}_s$ for finitely many s, and thus

(5.12) $$\nu_1 \in \mathcal{S}_{s+1}(\hat{X}) - \mathcal{P}_s \text{ for infinitely many } s.$$

But for each such s of (5.12) all ν' of length $> e$ are excluded from $\mathcal{M}_{s+1}$ under Clause 2 and hence

(5.13) $$\nu \in \mathcal{M}_\omega \Longrightarrow |\nu| \leqq e\ .$$

Now choose elements $\hat{y}_j$ and corresponding stages $s_j + 1$ such that for all $j \in N$,

(5.14) $$\nu_1 = \nu(s_j + 1, e, \hat{y}_j) \neq \nu(s_j, e, \hat{y}_j)\ .$$

For each $j \in N$, define the finite sequence of full e-states,

$$\mathcal{T}_j = \{\nu \colon (\exists s)[v_j \leqq s \leqq s_j + 1 \text{ and } \nu = \nu(s, e, \hat{y}_j)]\}\ ,$$

where v_j is the stage when $\hat{y}_j$ entered $\hat{A}$. Let $\mathcal{T}$ be the concatenation of $\{\mathcal{T}_j \colon j \in N\}$. From (5.12) and the definition of $\mathcal{K}$, note that if $\nu \in \mathcal{T}$ i.o., then $\nu' \in \mathcal{K}$ i.o. for all ν' such that $\nu \leqq^r \nu'$. From this and Rule R_3 we have,

(5.15) $$\nu \in \mathcal{T} \text{ i.o. and } C \text{ covers } \nu \Longrightarrow D\ \tau\text{-exactly covers } \nu\ .$$

But $\nu_1 \notin \mathcal{P}_\omega$, and thus by (5.10) and (5.15), we have,

(5.16) $$C \text{ does not cover } \nu_1\ .$$

On the other hand we shall get a contradiction from (5.16) by proving that

(5.17) $$C \text{ covers } \mathcal{T}\ .$$

First note that by the minimality of σ_1 and τ_1 above we have,

(5.18) $$\nu \in \mathcal{T} \text{ i.o. and } \nu \neq \nu_1 \Longrightarrow \nu \in \mathcal{P}_\omega\ ,$$

and thus,

(5.19) $$(\text{a.a. } j)(\forall s)_{\leqq s_j}[d(s, \hat{y}_j) \geqq e]\ ,$$

by the definition of $d(s, \hat{y})$ from $\mathcal{P}_s$, by (5.18) and by induction on t for $v_j \leqq t \leqq s_j$.

Assume for a contradiction that (5.17) fails, and choose $\nu_0 = \langle e, \sigma_0, \tau_0 \rangle$ such that

(5.20) $$\nu_0 \in \mathcal{T} \text{ i.o. but } C \text{ does not cover } \nu_0\ ,$$

where σ_0 and τ_0 are minimal. Choose an infinite set J and stages $t_j + 1 \leqq s_j + 1$ such that

(5.21) $$(\forall j \in J)[\nu_0 = \nu(t_j + 1, e, \hat{y}_j) \neq \nu(t_j, e, \hat{y}_j)]\ .$$

(Of course, $t_j + 1 > v_j$ by Lemma 5.7.) By the minimality of σ_0, τ_0 and by (5.21),

(5.22) C covers the sequence $\{\nu(t_j, e, \hat{y}_j): j \in J\}$.

Now by (5.20), (5.21), (5.22), and Rule $\hat{R}_6$, for almost all $j \in J$, either Rule $\hat{R}_3$, $\hat{R}_4$, or $\hat{R}_5$ applies to $\hat{y}_j$ at stage $t_j + 1$. By (5.20) and (5.8), $\hat{R}_3$ applies for at most finitely many $j \in J$. Likewise, by (5.19) Rule $\hat{R}_5$ applies for at most finitely many $j \in J$, else D (and therefore C) covers ν_0.

Thus, for almost all $j \in J$, Rule $\hat{R}_4$ applies to $\hat{y}_j$ at stage $t_j + 1$, with $\nu_0 < \nu''(t_j + 1)$ by (5.19). However, $\mathfrak{M}_s[\nu_0] \subseteqq \{\nu_0\}$ for almost all s by (5.13) and the fact that D cannot τ-exactly cover ν_0. Hence, $\nu''(t_j + 1) = \nu_0$ for almost all $j \in J$. Fix any ν_0' such that $\nu_0' = \nu'(t_j + 1)$ in Rule $\hat{R}_4$ for infinitely many $j \in J$. By (5.22) C covers ν_0', and thus by (5.15) D τ-exactly covers ν_0'. Thus, by Lemma 5.10, D τ-exactly covers ν_0, and therefore C covers ν_0 contrary to (5.20). ⊠

LEMMA 5.12. *C covers $\hat{C}_3$ and $\hat{C}$ dual covers C_3.*

Proof. By Lemma 5.4, Lemma 5.5, Lemma 5.11, and the definitions of $\mathcal{K}$ and $\mathcal{K}'$. ⊠

LEMMA 5.13. $(\forall e)(\text{a.a. } s)(\text{a.a. } \hat{y})[\hat{y} \in \hat{M}_s \Rightarrow d(s, \hat{y}) \geqq e]$.

Proof. By Lemma 5.11 and the definition of $d(s, \hat{y})$ from $\mathcal{P}_s$. ⊠

LEMMA 5.14. *Given ν_1 and infinitely many elements $\hat{y}_j$, $j \in N$, such that for all $j \in N$ either Rule $\hat{R}_4$ or $\hat{R}_5$ applies to $\hat{y}_j$ at say stage $s_j + 1$ with $\nu_1 < \nu(s_j + 1, y_j, \hat{y}_j)$, then C covers ν_1.*

Proof. By Lemma 5.13, $d(s_j, \hat{y}_j) \geqq |\nu_1|$ for almost all j. Assume for a contradiction that C (and hence D) fails to cover ν_1. Then $\hat{R}_5$ applies at $s_j + 1$ for only finitely many j because $\nu_1 < \nu''(s_j + 1)$ and $\nu''(s_j + 1) \in \mathcal{S}_{s_j+1}(D)$ for such j. Furthermore, only finitely many ν_1' such that $\nu_1 < \nu_1'$ are ever added to $\mathfrak{M}$ because D fails to cover ν_1. But by Lemma 5.13, each $\nu = \nu''(s_j + 1)$ of $\hat{R}_4$ for at most finitely many j. Therefore, Rule $\hat{R}_4$ applies at stage $s_j + 1$ for only finitely many j contrary to hypothesis. ⊠

LEMMA 5.15. *C covers every stream $\hat{X}$ of $\hat{M}$.*

Proof. By Lemma 5.12 and Lemma 5.14.

LEMMA 5.16. *D τ-exactly covers every stream $\hat{X}$ of $\hat{M}$.*

Proof. By Lemma 5.15 and Lemma 5.6. ⊠

It easily follows from Lemmas 5.16 and 5.8 that D exactly covers Q (and hence $\hat{Q}_\omega$) although in §6 we shall need a stronger property which will be guaranteed by Lemma 5.8.

6. Proof of the Extension Theorem

Part III: Mappings

In this section we complete the proof of the Extension Theorem by giving Rules R_2, R_{10}, R_{11}, by defining mapping p_1, and by proving that the mapping $p_1: P_\omega \to \hat{Q}_\omega$ satisfies (4.2). (Rules $\hat{R}_2$, $\hat{R}_{10}$, $\hat{R}_{11}$ and the roles of pockets $\hat{P}$ and Q are similar and will be omitted.)

Intuitive, we think of the machinery in §5 as a device for "cutting down" $\nu(s, y, \hat{y})$ to $\nu^*(s, \hat{y}) \in \mathcal{P}_s$ which we hope C will cover (and hence D will τ-exactly cover). If we knew exactly which ν were covered by C, we could always cut down $\nu(s, y, \hat{y})$ to the longest initial segment which C covers, safely ignoring the rest of $\nu(s, y, \hat{y})$. Unfortunately, our construction in §5 (specifically $\mathcal{P}_s$) is only a recursive *approximation* to this ideal algorithm (which would require a $\emptyset''$ oracle), and we may have $\nu \in \mathcal{P}_\omega$ even if C fails to cover ν. Thus, we may have cut down $\nu(s, y, \hat{y})$ only as far as $\nu^*(s, \hat{y})$ which C fails to cover. In this case for some $x \in P_\omega$ we may have to match with x several (but at most finitely many) $\hat{y} \in \hat{Q}_\omega$.

Given $\hat{Q}_s$, define (in increasing order of $\prec$) $\hat{a}(s, \nu) = \mu\hat{y} \in \hat{Q}_s$ such that

$$\nu \prec \nu^*(s, \hat{y}) \text{ , and} \tag{6.1}$$

$$\hat{a}(s, \nu') \neq \hat{y} \text{ for any } \nu' \prec \nu \text{ .} \tag{6.2}$$

If $\hat{y}$ fails to exist, then $\hat{a}(s, \nu)$ is undefined. The definition clearly implies that for all s and ν,

$$\hat{a}(s, \nu) \text{ defined} \Longrightarrow (\forall \nu')[\nu' \prec \nu \Longrightarrow \hat{a}(s, \nu') \text{ defined}] \text{ .} \tag{6.3}$$

We define P_s as the disjoint union of certain P_s^ν, where each P_s^ν contains at most one element denoted by $a(s, \nu)$. If $a(s, \nu)$ is undefined, then $P_s^\nu = \emptyset$. We shall define $a(s, \nu)$ so that for all s, x and ν,

$$[a(s, \nu) \text{ defined and } = x] \Longrightarrow \nu \prec \nu(s, x, x) \text{ ;} \tag{6.4}$$

$$a(s, \nu) \text{ defined} \Longrightarrow \hat{a}(s, \nu) \text{ defined ;} \tag{6.5}$$

$$a(s, \nu) \text{ defined} \Longrightarrow (\forall \nu')[\nu' \prec \nu \Longrightarrow a(s, \nu') \text{ defined}]\text{; and} \tag{6.6}$$

$$\begin{aligned} &[a(s+1, \nu) \neq a(s, \nu) \text{ or } \hat{a}(s+1, \nu) \neq \hat{a}(s, \nu)] \\ &\Longrightarrow (\forall \nu')[\nu \prec \nu' \Longrightarrow a(s, \nu') \notin P_{s+1}] \text{ .} \end{aligned} \tag{6.7}$$

RULE R_{10}. *If $x = a(s, \nu) \in P_s$ for some $\nu = \langle e, \sigma, \tau \rangle$, and if $x \in U_{n,s+1} - U_{n,s}$ for some $n \leq e$, then at stage $s + 1$ remove from pocket P all $a(s, \nu') \in P_s$ such that $\nu \prec \nu'$.*

RULE R_{11}. *If $\hat{a}(s, \nu) \in \hat{Q}_s - \hat{Q}_{s+1}$ for some ν, then at stage $s+1$ remove from pocket P all $a(s, \nu') \in P_s$ such that $\nu < \nu'$.*

Given the sequence $\mathcal{K}_s$ defined in §5, and any ν, define the subsequence (with ordering induced by $\leqq_s^*$),

$$\mathcal{K}_s[\nu] = \{\nu' : \nu' \in \mathcal{K}_s \text{ and } \nu < \nu'\} .$$

Similarly, define $\mathfrak{M}_s[\nu]$ and $\mathfrak{M}_\omega[\nu]$ using $\mathfrak{M}_s$ and $\mathfrak{M}_\omega$ in place of $\mathcal{K}_s$. Given $x \in M_s$ and ν, the *ν-rank of x at stage s*, denoted $\rho(s, \nu, x)$, is the least ν' in the ordering $\leqq_s^*$ such that

$$\nu < \nu' < \nu(s, x, x) .$$

(Recall that the enumeration has been arranged so that if $x \in M_s$, then $x < s$, and hence $\rho(s, \nu, x)$ is defined.)

RULE R_2. *Suppose that element x enters track D at stage s. Let ν' be the first ν (in the ordering $\prec$) such that:*

i) $\nu < \nu(s+1, x, x)$;

ii) *$\hat{a}(s+1, \nu)$ is defined; and*

iii) *either $a(s, \nu)$ is undefined, or*

$$\rho(s+1, s, x) <_{s+1}^* \rho(s+1, \nu, a(s, \nu)) .$$

If ν' exists, then at stage $s+1$ define $a(s+1, \nu') = x$, place x in pocket P, and remove from P all elements $a(s, \nu'')$ such that $\nu' < \nu''$. If ν' fails to exist, place x on track D_1 at stage $s+1$.

Recall that under Rule R_{12} if $x \in P_s$, then $x \in P_{s+1}$ unless removed under either R_2, R_{10}, or R_{11} above. Since x enters P only under Rule R_2, these rules clearly establish (6.4), (6.5), (6.6), and (6.7).

Let $a(\nu)$ denote $\lim_s a(s, \nu)$ if the latter exists, and similarly for $\hat{a}(\nu)$. (Note that $\lim_s a(s, \nu)$ and $\lim_s \hat{a}(s, \nu)$ may fail to exist.) The above rules also establish that for all x,

$$x \in P_\omega \Longleftrightarrow (\exists \nu)[a(\nu) = x] . \qquad (6.8)$$

(By (6.8), (6.4), and (6.6), one may think of P_ω as corresponding to a certain tree $\subsetneqq 2^\omega$, where $a(\nu)$ corresponds to the characteristic function (restricted to $n \leqq 2e$) of the set $\{2n : n \in \sigma\} \cup \{2n+1 : n \in \tau\}$, for $\nu = \langle e, \sigma, \tau \rangle$.)

The particular difinitions in §5 of $\mathcal{K}_s$ and $\mathfrak{M}_s$ involving C and D were designed for the proof of the following lemma, which is the crux of the entire argument.

LEMMA 6.1. *Fix ν_0 and suppose that:*

i) $(\forall\nu)[\nu \prec \nu_0 \Rightarrow \lim_s a(s, \nu)$ *exists*];

ii) $(\forall\nu)[\nu \prec \nu_0 \Rightarrow \lim_s \hat{a}(s, \nu)$ *exists*]; *and*

iii) $\lim_s a(s, \nu_0)$ *fails to exist.*

Then $\mathfrak{M}_\omega[\nu_0]$ *is finite.*

Proof. Fix $\nu_0 = \langle e_0, \sigma_0, \tau_0\rangle$ satisfying i), ii), and iii) above. Choose s_0 such that

$$(\forall s)_{\geqq s_0}(\forall\nu)[\nu \prec \nu_0 \Longrightarrow a(s, \nu) = a(\nu)]\text{ ; and} \tag{6.9}$$

$$(\forall s)_{\geqq s_0}(\forall\nu)[\nu \prec \nu_0 \Longrightarrow \hat{a}(s, \nu) = \hat{a}(\nu)]\ . \tag{6.10}$$

We claim that $\mathfrak{M}_{s_0}[\nu_0] \supseteqq \mathfrak{M}_\omega[\nu_0]$. Consider any $\nu_1 \in \mathfrak{M}_{s_1}[\nu_0] - \mathfrak{M}_{s_1-1}[\nu_0]$ for some $s_1 > s_0$. Assume for a contradiction that $\nu_1 \in \mathfrak{M}_s$ for all $s \geqq s_1$. Then by Case 1 in the definition of ν_1 being excluded from $\mathfrak{M}_s$,

$$(\forall\nu)(\forall s)_{\geqq s_1}[\nu \leqq^*_s \nu_1 \Longleftrightarrow \nu \leqq^*_{s_1} \nu_1]\ . \tag{6.11}$$

Furthermore, ν_1 added to $\mathfrak{M}_{s_1}$ at stage s_1 implies that some element x_1 entered track D at stage s_1 such that

$$\nu_0 \prec \nu_1 \prec \nu(s_1, x_1, x_1)\ . \tag{6.12}$$

Therefore, Rule R_2 applies to x_1 at stage $s_1 + 1$, and x_1 becomes $a(s_1 + 1, \nu_0)$ unless $a(s_1, \nu_0)$ is defined and

$$\rho(s_1 + 1, \nu_0, a(s_1, \nu_0)) \leqq^*_{s_1} \rho(s_1 + 1, \nu_0, x_1)\ . \tag{6.13}$$

Furthermore, by (6.12), (6.13), and the definition of $\rho(s_1 + 1, \nu_0, x_1)$,

$$\rho(s_1 + 1, \nu_0, a(s_1, \nu_0)) \leqq^*_{s_1} \nu_0,\text{ and} \tag{6.14}$$

$$\rho(s_1 + 1, \nu_0, a(s_1, \nu_0)) \leqq^*_{s_1} \nu_1\ . \tag{6.15}$$

By Rule R_{12}, $a(s + 1, \nu_0) \neq a(s, \nu_0)$ only if Rule R_2, R_{10}, or R_{11} applies to $a(s, \nu_0)$ at stage $s + 1$. By (6.10), R_{11} does not apply to $a(s, \nu_0)$ at any stage $s + 1 \geqq s_1$. However, by (6.9), if R_2 applies to $a(s, \nu_0)$ at stage $s + 1 \geqq s_1$ then

$$\rho(s + 1, \nu_0, a(s + 1, \nu_0)) <^*_{s+1} \rho(s + 1, \nu_0, a(s, \nu_0))\ . \tag{6.16}$$

Hence, by (6.11), (6.15), and (6.16), Rule R_2 applies to $a(s, \nu_0)$ at most finitely often after stage s_1 (thus contradicting hypothesis iii) of the lemma) unless Rule R_{10} applies to $a(s, \nu_0)$ at some stage $s + 1 \geqq s_1$. Let s_2 be the least such s. Let x_2 denote $a(s_2, \nu_0)$ and $\nu_2 = \langle e_2, \sigma_2, \tau_2\rangle$ denote $\rho(s_2, \nu_0, x_2)$. By (6.11), (6.16), and (6.15), we have

$$\nu_2 = \rho(s_2, \nu_0, a(s_2, \nu_0)) \leqq^*_{s_2} \rho(s_1, \nu_0, a(s_1, \nu_0)) \leqq^*_{s_3} \nu_1\ . \tag{6.17}$$

But by (6.9), R_{10} applies to x_2 at stage $s_2 + 1$ only if $x_2 \in U_{n,s+1} - U_{n,s}$ for some $n \leqq e_0 \leqq e_2$. Hence, $\nu_2 <^r \nu(s_2 + 1, e_2, x_2)$, and by Rule R_{12} at stage $s_2 + 1$,

x_2 is placed above hole H_2, from which it enters track C at some stage $t > s+1$. Let ν_3 denote $\nu(s_2+1, e_2, x_2)$. Then $\nu(t, e_2, x_2) = \nu_3$ also, and thus $\nu_2 \in \mathcal{K}_t^2$ because $\nu_2 <^r \nu_3$. But $\nu_2 \in \mathcal{K}_t^2$ contradicts (6.11) and (6.17) which together assert that ν_2 is never reset after stage s_1. ⊠

LEMMA 6.2. *Fix ν_1 and suppose that there are infinitely many elements $\hat{y}_j \in \hat{Q}_\omega$, $j \in N$, such that $\nu_1 < \nu(y_j, \hat{y}_j)$, for all $j \in N$, Then*
i) *$\nu_1 < \nu^*(\hat{y}_j)$ for almost all $j \in N$; and*
ii) *$\lim_s a(s, \nu_1)$ exists.*

Proof. Fix $\nu_1 = \langle e_1, \sigma_1, \tau_1\rangle$, and $\{\hat{y}_j: j \in N\}$ satisfying the hypotheses with e_1 minimal. Then $\nu_1 \in \mathcal{S}(\hat{Q})$ i.o., and hence i) follows by Lemma 5.3 and Lemma 5.13. Hence, $\lim_s \hat{a}(s, \nu)$ exists for all $\nu < \nu_1$. By the minimality of e_1, $\lim_s a(s, \nu)$ exists for all $\nu \prec \nu_1$.

Assume for a contradiction that $\lim_s a(s, \nu_1)$ does not exist. Then by Lemma 6.1, $\mathfrak{M}_\omega[\nu_1]$ is finite. But by Lemma 5.9, $\nu^*(\hat{y}_j) \in \mathfrak{M}_\omega$ for all $j \in N$. But for each $\nu_2 \in \mathfrak{M}_\omega[\nu_1]$, by Lemma 5.13, $\nu^*(\hat{y}_j) = \nu_2$ for at most finitely many $j \in N$, thus contradicting the hypothesis. ⊠

Define $P_\omega^\nu = \{a(\nu)\}$ if $a(\nu)$ is defined and $= \varnothing$ otherwise. For ever $\hat{y} \in \hat{Q}_\omega$ define $\nu^\sharp(\hat{y})$ as follows. If $\hat{y} = \hat{a}(\nu)$ for some ν and $a(\nu)$ is defined, let $\nu^\sharp(\hat{y}) = \nu$. Otherwise, let $\nu^\sharp(\hat{y})$ be the maximum ν (in the ordering $\prec$) such that $\nu \prec \nu^*(\hat{y})$ and $a(\nu)$ is defined. Define

$$\hat{Q}_\omega^\nu = \{\hat{y}: \hat{y} \in \hat{Q}_\omega \text{ and } \nu^\sharp(\hat{y}) = \nu\} .$$

From Lemma 6.2, it immediately follows that for all ν,

$$\hat{Q}_\omega^\nu \text{ is finite .} \tag{6.18}$$

From (6.5), (6.7), and the definition of $\hat{Q}_\omega^\nu$, it follows that for all ν,

$$P_\omega^\nu \neq \varnothing \Longleftrightarrow \hat{Q}_\omega^\nu \neq \varnothing \tag{6.19}$$

and hence that the map $p_1: P_\omega \longrightarrow \hat{Q}_\omega$ can be derived from the correspondence $P_\omega^\nu \longleftrightarrow \hat{Q}_\omega^\nu$.

More precisely, define the (finite to one) map $q_1: \hat{Q}_\omega \longrightarrow \hat{P}_\omega$ by $q_1(\hat{y}) = a(\nu)$ where $\hat{y} \in \hat{Q}_\omega^\nu$. Note that for all n,

$$q_1^{-1}(U_n \cap P_\omega) =^* (\hat{U}_n \cap \hat{Q}_\omega) \text{ and } q_1(V_n \cap \hat{Q}_\omega) =^* (\hat{V}_n \cap P_\omega) , \tag{6.20}$$

because by (6.4), and the definition of $\nu^\sharp(\hat{y})$, $x \in P_\omega^\nu$ ($\hat{x} \in \hat{Q}_\omega^\nu$) only if $\nu \prec \nu(x, x)$ (respectively $\nu \prec \nu(x, \hat{x})$). Using the method of Theorem 1.3 and (6.20) one can produce a *one*: *one* map $p_1: P_\omega \leftrightarrow \hat{Q}_\omega$ satisfying (4.2), although by Corollary 1.7 (appropriately modified), (6.20) already suffices to show that Φ is an automorphism of $\mathcal{E}^*$.

7. Conclusion

Our main result is one step in the general program of deciding exactly when $A \equiv_{\mathfrak{E}} B$ for given $A, B \in \mathfrak{E}$. Following Lachlan, for each $A \in \mathfrak{E}$ define the principal filter $\mathfrak{L}(A) = \{X: X \in \mathfrak{E} \text{ and } A \subseteq X\}$. Clearly $A \equiv_{\mathfrak{E}} B$ implies $\mathfrak{L}(A) \cong \mathfrak{L}(B)$ but unfortunately the converse is false [17]. By Corollary 2.6, however, the converse holds if $\mathfrak{L}^*(A)$ is finite. We do not know in which other cases the converse holds. In particular, does it hold if $\mathfrak{L}^*(A)$ is a boolean algebra, or at least an atomless boolean algebra? What are other complete sets of invariants [10, p. 51], besides those mentioning $\mathfrak{L}^*(A)$, for characterizing the orbit of $A \in \mathfrak{E}$?

In a slightly different direction, automorphisms may be used to study the important question of the relationships between the *structure of a* r.e. set A and its degree, which will be denoted by deg A. Post's program [8] was to find a simple property on the complement of a r.e. set such that a set satisfying that property has degree strictly between $\mathbf{0}$ and $\mathbf{0}'$. Although Friedberg and Muchnik constructed r.e. sets of such degree by a different method, the existence of such a property remains an open question, as has recently been pointed out by Sacks [11, (Q3) on p. 172]. We give a partial answer to this question by proving [16] that no such property can be lattice-invariant as are the properties of simplicity, hyperhypersimplicity, and maximality.

THEOREM 7.1. *For any nonrecursive $A \in \mathfrak{E}$ there exists B of degree $\mathbf{0}'$ such that $A \equiv_{\mathfrak{E}} B$.*

A corollary is Yates' result [18] that there is a complete maximal set. We define the following subclasses of the r.e. degrees $\mathbf{R}$ for each $n \geqq 0$,

$$\mathbf{H}_n = \{\boldsymbol{d}: \boldsymbol{d} \text{ r.e. and } \boldsymbol{d}^{(n)} = \mathbf{0}^{(n+1)}\}\,, \text{ and}$$
$$\mathbf{L}_n = \{\boldsymbol{d}: \boldsymbol{d} \text{ r.e. and } \boldsymbol{d}^{(n)} = \mathbf{0}^{(n)}\}\,,$$

where $\boldsymbol{d}^{(0)} = \boldsymbol{d}$. It is well-known [10, pp. 290–294] that for each n, $\mathbf{H}_n \subsetneqq \mathbf{H}_{n+1}$ and $\mathbf{L}_n \subsetneqq \mathbf{L}_{n+1}$, and that there exists a r.e. degree $\boldsymbol{d}$ such that for all n, $\boldsymbol{d} \notin \mathbf{H}_n \cup \mathbf{L}_n$. The degrees in $\mathbf{H}_1$ ($\mathbf{L}_1$) are called *high* (*low*). (This terminology is often used with the condition "$\boldsymbol{d}$ r.e." above replaced by the weaker condition "$\boldsymbol{d} \leqq O'$".)

A class $\mathbf{C} \subseteq \mathbf{R}$ is called *$\mathfrak{E}$-definable* if $\mathbf{C} = \{\deg W: W \in \mathcal{C}\}$ for some lattice-invariant class $\mathcal{C} \subseteq \mathfrak{E}$. For example, let $\mathbf{M}$ denote the class of degrees of maximal sets, and $\mathbf{A}$ the class of degrees of atomless sets, that is, coinfinite r.e. sets which have no maximal superset. The first major results relating the structure of a r.e. set to its degree were the beautiful theorems of

Martin [7] that $\mathbf{M} = \mathbf{H}_1$ and $\mathbf{A} \supsetneqq \mathbf{H}_1$. The difficult part of Martin's results, namely that $\mathbf{M} \supseteqq \mathbf{H}_1$ and $\mathbf{A} \supseteqq \mathbf{H}_1$, can be obtained from the following theorem which was suggested by Carl Jackusch.

THEOREM 7.2. *For any nonrecursive* $A \in \mathcal{E}$ *and any* $\boldsymbol{d} \in \mathbf{H}_1$ *there exists a* r.e. $B \in \boldsymbol{d}$ *such that* $\mathcal{L}(A) \cong \mathcal{L}(B)$.

Thus, no isomorphism-invariant property on $\mathcal{L}(A)$ can guarantee either completeness or incompleteness. Another corollary is Lachlan's result [1, p. 27] that for every $\boldsymbol{d} \in \mathbf{H}_1$ and any *hh*-simple A there exists a r.e. $B \in \boldsymbol{d}$ such that $\mathcal{L}(A) \cong \mathcal{L}(B)$.

In contrast to this "complexity" of sets of high degree, we might expect those of low degree to exhibit some "uniformity" of structure like that of recursive sets which fall into only three distinct $\mathcal{E}^*$-orbits. For example, R. W. Robinson [9] verified Martin's conjecture that $\mathbf{A} \cap \mathbf{L}_1 = \varnothing$ by constructing a maximal superset B for any coinfinite $A \in \mathcal{E}$ such that $\deg A \in \mathbf{L}_1$. Lachlan showed [1, p. 27] that "B maximal" above could be replaced by "B *hh*-simple with $\mathcal{L}(B) \cong \mathcal{L}(C)$", where C is an arbitrary *hh*-simple set. Lachlan then conjectured that $\mathcal{L}(A) \cong \mathcal{L}(B)$ for any A, B, both simple and of low degree.

For any set $A \subseteq N$ (not necessarily r.e.) define $\mathcal{E}_A = \{W \cap A : W \in \mathcal{E}\}$. If A is r.e. and infinite then $\mathcal{E}_A \cong \mathcal{E}$ of course. The following result [17] extends this property to sets of low degree and yields all the above results, because if B is r.e. then clearly $\mathcal{E}_{\bar{B}} \cong \mathcal{L}(B)$.

THEOREM 7.3. *If A is infinite and* $\deg A \in \mathbf{L}_1$ *then* $\mathcal{E}_A \cong \mathcal{E}$.

Carl Jockusch has noted that our proof of Theorem 7.3 really uses only the weaker hypothesis that $\{e : W_e \cap A \neq \varnothing\}$ has degree $\mathbf{0}'$. Such sets exist in every r.e. degree.

Lachlan has shown [3] that $\mathbf{H}_2 \subseteq \mathbf{A}$ and $\mathbf{A} \cap \mathbf{L}_2 = \varnothing$. It is an open question of Lachlan and Martin whether $\mathbf{A}$ can be characterized by some condition on $\mathbf{H}_n$ or $\mathbf{L}_n$ such as $\mathbf{A} = \mathbf{H}_2$ or $\mathbf{A} = \mathbf{R} - \mathbf{L}_2$. It is unknown whether the hypothesis of Theorem 7.3 can be weakened to "$\deg A \in \mathbf{L}_2$", which is possible in view of [3, Theorem 4]. It is also unknown whether $\mathbf{H}_n$ is $\mathcal{E}$-definable for any $n > 1$. If false for say $n = 2$, one might try to use the Extension Theorem to construct an automorphism Φ corresponding to a given $A \in \mathcal{E}$ satisfying $\deg A \in \mathbf{H}_2 - \mathbf{H}_1$ so that $\deg \Phi(A) \notin \mathbf{H}_2$. Lerman [5] has shown that for any non-recursive $A \in \mathcal{E}$ and any $\boldsymbol{d} \in \mathbf{H}_1$, A has a major subset B such that $\deg B \in \boldsymbol{d}$. It is unknown whether this result may be obtained using automorphisms which fix a given $A \in \mathcal{E}$.

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A model of set-theory in which every set of reals is Lebesgue measurable*

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We show that the existence of a non-Lebesgue measurable set cannot be proved in Zermelo-Frankel set theory (ZF) if use of the axiom of choice is disallowed. In fact, even adjoining an axiom **DC** to **ZF**, which allows countably many consecutive choices, does not create a theory strong enough to construct a non-measurable set.

Let **ZFC** be Zermelo-Frankel set theory together with the axiom of choice. Let **I** be the statement: There is an inaccessible cardinal[1].

THEOREM 1. *Suppose that there is a transitive ε-model of* **ZFC** + **I**. *Then there is a transitive ε-model of* **ZF** *in which the following propositions are valid.*

(1) *The principle of dependent choice* (= **DC**, cf. III. 2.7.)

(2) *Every set of reals is Lebesgue measurable* (**LM**).

(3) *Every set of reals has the property of Baire.*[2]

(4) *Every uncountable set of reals contains a perfect subset* (**P**).

(5) *Let* $\{A_x : x \in \mathbf{R}\}$ *be an indexed family of non-empty set of reals with index set the reals. Then there are Borel functions,* h_1, h_2 *mapping* **R** *into* **R** *such that*

(a) $\{x \mid h_1(x) \notin A_x\}$ *has Lebesgue measure zero.*

(b) $\{x \mid h_2(x) \notin A_x\}$ *is of first category.*

Remarks. 1. It is known that the theory **ZFC** + **I** has a transitive ε-model if **ZF** + **DC** + **P** does; cf. [10, pp. 213-214]. Thus the hypothesis of Theorem 1 (that **ZFC** + **I** has a transitive ε-model) cannot be weakened. However it does seem likely that the existence of a transitive model of **ZF** + **DC** + **LM** is a consequence, in **ZFC**, of the existence of a transitive

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[1] In the presence of the axiom of choice, we identify cardinals with initial ordinals. A cardinal $\aleph$ is *regular*, if each order unbounded subset of $\aleph$ has power $\aleph$. A cardinal $\aleph$ is inaccessible, if it is regular, uncountable, and for $\aleph' < \aleph$, $2^{\aleph'} < \aleph$.

[2] A set of reals A has the Baire property if there is an open set U such that $(A - U) \cup (U - A)$ is of the first category.

model of ZFC.

2. Our proofs use Cohen's forcing method. In the usual way (cf. [8, pp. 132-133]), they can be recast as finitistic relative consistency proofs.

3. We take this opportunity to describe some recent work on the model of Theorem 1. Mathias has shown that the following Ramsey-like theorem holds in the model. Divide the set of all infinite subsets of ω into two disjoint pieces. Then there is an infinite subset A of ω such that every infinite subset of A lies in the same piece as A.

Levy and the author have shown that in this model every set of reals is the union of $\aleph_1$ Borel sets. This should be contrasted with the following consequence of **DC** and the axiom of determinateness (**AD**), due to Moschovakis: Every union of $\aleph_1$ Borel sets is Σ^1_2. It follows that **AD** fails in the model.

This result might seem to throw cold water on a conjecture of the author that a suitable large cardinal axiom will imply that **AD** holds in $L[\mathbf{R}]$. Closer inspection shows that there is no conflict between the conjecture and this result.

4. It is fairly easy to deduce parts (2) and (3) of Theorem 1 from part (5). Nevertheless, we have included our original proofs of (2) and (3) since they are much simpler and more natural than the proof of (5). (The ideas in our proof of (2) will be used in a forthcoming paper of the author to show that the existence of measurable cardinals implies that every Σ^1_2 set of reals is Lebesgue measurable.)

5. Proposition (5) of Theorem 1 was suggested to the author by Mycielski. According to Mycielski, (5) implies that every subset of $\mathbf{R}^3$ has a newtonian capacity. The author is totally ignorant of the theory of capacities, so will simply pass on (slightly re-phrased) the relevant portion of Mycielski's letter in the hope that some knowledgeable reader may understand it.

Let C be Choquet's paper, *Theory of capacities*, Annales de l'Institute de Fourier, 5 (1955), 131-292. Mycielski's remark is that using (5), we can establish 37.1 of C for arbitrary subsets A of $\mathbf{R}^2$. This statement generalized to the case $A \subseteq X \times Y$ where X is an arbitrary separable measure space implies capacitability of all sets with respect to the classical capacities interpreted as in 49.3 and 49.4 of C. (In the application, X is the set of brownian trajectories with the Wiener measure, and $Y = [0, \infty)$.)

6. The reader will find in [10] a detailed discussion of various forms of the axiom of choice whose failure follows from (1)-(4) of Theorem 1, e.g., the axiom of choice for families of two-element sets.

We add a brief discussion on the Hahn-Banach theorem. Of course the

Hahn-Banach theorem for *separable* Banach spaces follows readily from **DC**. On the other hand, one can deduce from (3) of Theorem 1 that there is no finitely additive probability measure on the power set of ω which vanishes on singletons. It follows that the Hahn-Banach theorem fails in the model of Theorem 1.

Of course, the axiom of choice is true, and so there are non-measurable sets. It is natural to ask if one can explicitly describe a non-Lebesgue measurable set.[3] Our next theorem bears on this question.

We say that a set of reals A is *definable* from a set x_0 if there is a set-theoretical formula $\Psi(x, y)$ (having free only the variables x and y) such that

$$A = \{y \in \mathbf{R} \colon \Psi(x_0, y)\} .$$

Because of the familiar difficulties about the "undefinability of truth" it is not clear how to express the notion "definable" by a set-theoretical formula. However, Myhill and Scott [11] have shown that the notion "A is definable from some countable sequence of ordinals" is expressible by a set-theoretical formula. Thus we can formulate in set-theory the propositions referred to in the following theorem.

THEOREM 2. *Suppose that* **ZFC** + **I** *has a transitive* ε*-model. Then so does the theory* **ZFC** + **GCH** *together with analogs of* (2) *through* (5) *of Theorem* 1. (*We state the analog of* (2):

(2′) *Every set of reals definable from a countable sequence of ordinals is Lebesgue measurable.*)

Remark. Since a real can be coded into a countable sequence of zeros and ones, every set definable from a real is, *ipso facto*, definable from a countable sequence of ordinals. In particular, every projective set of reals is definable from a countable sequence of ordinals.

McAloon has simplified the author's original proof of Theorems 1 and 2. (We present McAloon's version of the proof below.) As McAloon and the author independently noticed, McAloon's version of the proof allows one to prove the following

THEOREM 3. *Assume the hypotheses of Theorem* 1. *Then there is a transitive* ε*-model of* **ZFC** *in which* $2^{\aleph_0} = \aleph_2$ *and the analogs of* (2)-(5) *of Theorem* 1 *referred to in Theorem* 2 *are valid.*

Remarks. 1. It is clear from the proof (cf. III 3.9) that Theorem 3 remains true if "$2^{\aleph_0} = \aleph_2$" is replaced by "$2^{\aleph_0} = \aleph_\Lambda$" for a wide variety of reasonable Λ; e.g., $\Lambda = 3$, $\Lambda = \aleph_2$, etc.

[3] This question was suggested to the author by Milnor.

2. A natural question suggested by Theorem 3 is whether there are models of the sort described in Theorem 3 in which Martin's axiom holds. We can construct such models using a remark of Kunen but we need to take the large cardinal in the ground model weakly compact as well as strongly inaccessible. It seems likely that this large cardinal assumption can be appreciably weakened.

Our paper is divided into three main sections. Section I begins with general remarks on the notion of forcing. I.1 gives a precise mathematical interpretation of the concept of "generic". (Cohen did not need such a definition; he simply constructed and studied one generic filter.) The advantage of having such a precise notion is shown in I.2, where we relate generic filters on various partially ordered sets; cf. especially Lemma 2.3 relating a generic filter on the product of two partially ordered sets with the filters on the factors. I.3 describes a model due to A. Levy [9], which is the model of our Theorem 2 and gives proofs of basic facts about this model (also due to Levy). In I.4, we prove an important lemma which allows us to enlarge the ground model of Levy's construction so as to absorb a specified real of the extension.

Section II.1 contains foundational material about the relations between Borel sets of a transitive ε-model $\mathfrak{M}$ and an extension $\mathfrak{N}$ of $\mathfrak{M}$. We show that there is a natural way to prolong the Borel sets of $\mathfrak{M}$ to sets of $\mathfrak{N}$ which preserves most properties of the Borel set. II.2 defines and studies one of the main technical devices of the paper, the notion of a random real. (Roughly speaking a real is random if it avoids all the sets of measure zero that one can explicitly define. An alternative heuristic definition is that random reals are those reals whose binary expansions are obtained by tossing an honest coin infinitely many times.)

Finally III puts the material of I and II together and proves Theorems 1 through 3.

We close this introduction by thanking various people who in one way or another materially helped us in this work. The original problem of showing **ZF** + **LM** consistent was suggested to the author by Paul Cohen. (And of course Cohen's idea of forcing [2] is the *sine qua non* of our proof.) We are grateful to Levy for sending us a preprint of his work on the model $\mathfrak{N}$ (of Theorem 2) and for permission to incorporate proofs of his results into our paper. Ken McAloon made a vital simplification in our proof which reduced our original cumbersome verification of **DC** to a triviality. Finally, we are grateful to Hao Wang and the Rockefeller University for hospitality during the year when this paper was finally written.

I. The Model

1. Generic filters

1.1. We are going to review briefly the formalism of forcing in a form suitable for applications in this paper. Proofs will not be given for these results. The reader familiar with [8] and [1] should be able to reconstruct the proofs, cf. § 1.10 for some discussion of the proofs.

1.2. **ZF** is Zermelo-Frankel set theory. **ZFC** is **ZF** plus the axiom of choice. Let $\mathfrak{M}$ be a transitive model of **ZFC**. We do not assume that $\mathfrak{M}$ is countable, but we shall assume, for convenience, that $\mathfrak{M}$ is a set.

Let $\mathcal{P}$ be a non-empty partially ordered set lying in $\mathfrak{M}$. We suppose in addition that the partial order $\leqq$ on $\mathcal{P}$ is reflexive, i.e., if $x \in \mathcal{P}$, $x \leqq x$. (We assume, of course, that the ordering $\leqq$ lies in $\mathfrak{M}$.)

Two elements of $\mathcal{P}$, x and y, are *compatible* if $(\exists z \in \mathcal{P})$ $(x \leqq z$ and $y \leqq z)$. Otherwise, they are *incompatible*.

A subset X of $\mathcal{P}$ is *dense* if

(1) if $x \in X$, $y \in \mathcal{P}$, and $x \leqq y$, then $y \in X$;

(2) if $x \in \mathcal{P}$, there is a $y \in X$ with $x \leqq y$.[4]

1.3. Let G be a subset of $\mathcal{P}$. We say that G is an $\mathfrak{M}$-*generic filter*[5] on $\mathcal{P}$ if:

(1) If $x, y \in G$, then there is a $z \in G$, with $x \leqq z$, and $y \leqq z$.

(2) If $x \in G$, $y \in \mathcal{P}$, and $y \leqq x$, then $y \in G$.

(3) Let $X \subseteq \mathcal{P}$, $X \in \mathfrak{M}$, and suppose that X is dense. Then $X \cap G$ is non-void.

1.4. Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}$. Then there is a transitive model $\mathfrak{M}[G]$ of **ZF** with the following properties:

(1) $\mathfrak{M} \subseteq \mathfrak{M}[G]$;

(2) $G \in \mathfrak{M}[G]$;

(3) if $\mathfrak{N}$ is a transitive model of ZF such that $\mathfrak{M} \subseteq \mathfrak{N}$ and $G \in \mathfrak{N}$ then $\mathfrak{M}[G] \subseteq \mathfrak{N}$.

It is clear that (1)-(3) characterize $\mathfrak{M}[G]$. $\mathfrak{M}[G]$ has the following additional properties:

(4) The axiom of choice holds in $\mathfrak{M}[G]$.

(5) Let α be an ordinal. Then $\alpha \in \mathfrak{M}[G]$ if and only if $\alpha \in \mathfrak{M}$.

1.5 We introduce a first order language $\mathfrak{L}$ as follows: the predicates of

[4] Our conventions are such that if $x \leqq y$, y "gives *more* information" than x.

[5] Our original definition of generic was based on "complete sequences". The present approach is due to Levy [8].

$\mathfrak{L}$ are ε and a one-place predicate S. We interpret $\mathfrak{L}$ in $\mathfrak{M}[G]$ as follows: ε is interpreted in the obvious way; Sx holds in $\mathfrak{M}[G]$ if and only if $x \in \mathfrak{M}$.

We can formulate new instances of the replacement axiom involving the predicate S. All these instances are valid in $\mathfrak{M}[G]$.

1.6. Let $A \in \mathfrak{M}[G]$, with $A \subseteq \mathfrak{M}$. Then there is a model $\mathfrak{M}[A]$ of **ZFC** with the following properties:

(1) $\mathfrak{M} \subseteq \mathfrak{M}[A]$.

(2) $A \in \mathfrak{M}[A]$.

(3) If $\mathfrak{N}$ is a transitive model of **ZF**, $\mathfrak{M} \subseteq \mathfrak{N}$, and $A \in \mathfrak{N}$, then $\mathfrak{M}[A] \subseteq \mathfrak{N}$.

It is clear that (1)-(3) uniquely characterize $\mathfrak{M}[A]$, and that $\mathfrak{M}[A] \subseteq \mathfrak{M}[G]$. If we interpret $\mathfrak{L}$ in $\mathfrak{M}[A]$, by interpreting ε in the obvious way and interpreting S as before, then all the instances of the replacement axiom expressible in $\mathfrak{L}$ hold in $\mathfrak{M}[A]$.

1.7. If $\mathfrak{A}$ is a relational system, and Φ a sentence, we use the notation

$$\mathfrak{A} \models \Phi$$

to mean: Φ is true in $\mathfrak{A}$.

There is a formula $\Phi(v_1, v_2, v_3)$ of $\mathfrak{L}$ with the following properties:

(1) $\mathfrak{M}[G] \models \Phi(x, y, z) \rightarrow x \in \mathfrak{M},\ y \subseteq \mathfrak{M}$.

(2) If $\mathfrak{M}[G] \models \Phi(x, y, z)$ and $\mathfrak{M}[G] \models \Phi(x, y, z')$, then $z = z'$.

(3) Let $A \subseteq \mathfrak{M}$, $A \in \mathfrak{M}[G]$. Then

$$\mathfrak{M}[G] \models \Phi(x, A, z) \rightarrow z \in \mathfrak{M}[A] .$$

(4) Let A be as in (3), and let $z \in \mathfrak{M}[A]$. Then

$$\mathfrak{M}[G] \models \Phi(x, A, z) \longleftrightarrow \mathfrak{M}[A] \models \Phi(x, A, z) .$$

(5) $\mathfrak{M}[A] = \{z \mid (\exists x \in \mathfrak{M})(\mathfrak{M}[G] \models \Phi(x, A, z)\}$.

Roughly speaking Φ is constructed as follows. We can describe $\mathfrak{M}[A]$ as the set of denotations of terms of a certain ramified language $\mathfrak{L}_*$: if t is a term of $\mathfrak{L}_*$, and u is the collection of sets of $\mathfrak{M}$ of rank $\leqq$ rank (t), then we can "compute" the denotation of t from t, u, and A. Then $\Phi(\langle t, u\rangle, A, z)$ holds just in case z is the denotation of t.

The existence of Φ with the properties just stated has several important consequences:

(a) Let A be as above. Let $x \in \mathfrak{M}[A]$. Then x is definable in

$$\langle \mathfrak{M}[A]; \varepsilon, S, A\rangle$$

from some element y of $\mathfrak{M}$.

(b) The predicate "$y \in \mathfrak{M}[A]$" is expressible in $\langle \mathfrak{M}[G]; \varepsilon, S\rangle$ (by $(\exists x)\Phi(x, A, y)$). Thus we can lay our hands on $\mathfrak{M}[A]$ inside $\mathfrak{M}[G]$.

1.8. We now make the following countability assumption on $\mathfrak{M}$, $\mathcal{P}$: *there are only countably many subsets of $\mathcal{P}$ lying in $\mathfrak{M}$.* This has the following important consequence. Let $p \in \mathcal{P}$. Then there is an $\mathfrak{M}$-generic filter G on $\mathcal{P}$ with $p \in G$.

1.9. *Forcing.* Let $\mathfrak{M}$, $\mathcal{P}$ be as in § 1.8. We enlarge $\mathfrak{L}$ to a language $\mathfrak{L}'$ as follows. If $x \in \mathfrak{M}$, we introduce a term $\underline{x}$; we also have a term $\underline{G}$. If G is a generic filter on $\mathcal{P}$, we interpret $\mathfrak{L}'$ in $\mathfrak{M}[G]$ by letting $\underline{x}$ denote x, and letting $\underline{G}$ denote G.

We can arrange matters so that each formula of $\mathfrak{L}'$ is (coded by) a set of $\mathfrak{M}$, and all the usual syntactical properties relevant to $\mathfrak{L}'$ are expressible in $\mathfrak{M}$.

Let Φ be a sentence of $\mathfrak{L}'$, and $p \in \mathcal{P}$. We say that p *forces* Φ if

$$\mathfrak{M}[G] \models \Phi$$

whenever G is an $\mathfrak{M}$-generic filter containing p. (Notation: $p \Vdash \Phi$.)

A fundamental fact about forcing is the connection between forcing and truth: If G is an $\mathfrak{M}$-generic filter on $\mathcal{P}$ and

$$\mathfrak{M}[G] \models \Phi ,$$

then Φ is forced by some $p \in G$.

It follows that if p does not force Φ, there is an extension p' of p such that p' forces $\neg \Phi$.

We say that p *decides* Φ if $p \Vdash \Phi$ or $p \Vdash \neg \Phi$. (Notation: $p \parallel \Phi$.) We write $\Vdash \Phi$, if for every $p \in \mathcal{P}$, $p \Vdash \Phi$.

Suppose now that $\Phi(w_0, \cdots, w_n)$ is a formula of $\mathfrak{L}$. Then the relation

$$p \Vdash \Phi(\underline{G}, \underline{x}_1, \cdots, \underline{x}_n) \tag{1}$$

is expressible in $\mathfrak{M}$; i.e., there is a formula $\Psi(x_0, \cdots, x_n)$ such that (1) holds if and only if

$$\mathfrak{M} \models \Psi(p, x_1, \cdots, x_n) .$$

1.10. We know of no proof of the results stated above which does not require preliminary indirect definitions of $\mathfrak{M}[G]$ and $\Vdash$. For example, one can extend $\mathfrak{L}'$ to a ramified language $\mathfrak{L}''$, and define $\mathfrak{M}[G]$ for any $G \subseteq \mathcal{P}$ as the set of denotations of terms of $\mathfrak{L}''$; cf. [8] for a representative special case. One defines an auxiliary forcing relation, say $\Vdash'$, by induction on some ordinal measure of the complexity of a sentence of $\mathfrak{L}''$. (For example, if $p, q \in \mathcal{P}$, we would have

$$p \Vdash' \underline{q} \in \underline{G}$$

if and only if $q \leqq p$.) The correct forcing relation, $p \Vdash \Phi$, is defined in terms of $\Vdash'$ by $p \Vdash \Phi$ if and only if $p \Vdash' \neg\neg\Phi$ (" $\neg$ " is the negation symbol).

An alternative proof of these results can be given in terms of boolean valued models (cf. [12]).

1.11. Recall that a cardinal Ω is *regular* if each subset $A \subseteq \Omega$ of cardinality less than Ω has a sup less than Ω. A cardinal Ω is strongly inaccessible, if Ω is regular, greater than $\aleph_0$, and satisfies

$$\aleph < \Omega \rightarrow 2^{\aleph} < \Omega .$$

(Here $\aleph$ ranges over infinite cardinals.)

We shall need the following known result.

LEMMA. *Let $\mathfrak{M}$ and $\mathcal{P}$ be as in § 1.8. Let $\Omega \in \mathfrak{M}$ be such that*

$\mathfrak{M} \models$ "Ω is strongly inaccessible, and the cardinality of $\mathcal{P}$ is less than Ω".

Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}$. Then Ω is strongly inaccessible in $\mathfrak{M}[G]$.

1.12. *Collapsing a cardinal.* Now let $\mathfrak{M}$ be a countable transitive model of **ZFC**. Let λ be a non-zero ordinal of $\mathfrak{M}$. Let $\mathcal{P}_\lambda$ be the set of functions f whose domain is a finite subset of ω and whose range is a subset of λ. We partially order $\mathcal{P}_\lambda$ by inclusion: $f \leqq g$ if and only if $f \subseteq g$.

Let G be a generic filter on $\mathcal{P}_\lambda$. Then $\bigcup G$ is a function $F\colon \omega \rightarrow \lambda$ and F is surjective. It follows that λ is countable in $\mathfrak{M}[G]$; cf. the discussion of § 3.2 for proof of a related result.

One can recover G from F as follows:

$$G = \{f \in \mathcal{P}_\lambda : f \subseteq F\} .$$

It follows that $\mathfrak{M}[G] = \mathfrak{M}[F]$.

We say that $F\colon \omega \rightarrow \lambda$ is a generic collapsing function (more precisely, an $\mathfrak{M}$-generic collapsing function), if F arises from a generic filter on $\mathcal{P}_\lambda$ in the manner just described.

LEMMA. *Let $F\colon \omega \rightarrow \lambda$ be an $\mathfrak{M}$-generic collapsing map. Then there is an $s \subseteq \omega$, $s \in \mathfrak{M}[F]$, with*

$$\mathfrak{M}[F] = \mathfrak{M}[s] .$$

PROOF. We put

$$s = \{2^n 3^m \mid F(n) \leqq F(m)\} .$$

Clearly $s \in \mathfrak{M}[F]$. To complete the proof, we show that $F \in \mathfrak{M}[s]$.

Put $m \sim n$ if and only if $F(m) = F(n)$. Clearly the relation $\sim$ lies in $\mathfrak{M}[s]$. Therefore so does the set $A = \omega/\sim$ of equivalence classes. We order A by $[m] < [n]$ if and only if $2^n 3^m \notin s$. Clearly F induces a map F' of $\langle A, < \rangle$

onto $\langle \alpha, < \rangle$ in an order preserving way. Thus $\langle A, < \rangle$ is well-ordered. Hence it is well-ordered in $\mathfrak{M}[s]$. Let $F''\colon A \to \alpha'$ be an isomorphism of $\langle A, < \rangle$ with $\langle \alpha', < \rangle$ in $\mathfrak{M}[s]$. (F'' exists since

(1) $\mathfrak{M}[s]$ is a model of **ZFC**, and

(2) a theorem of **ZFC** asserts every well-ordered set is order isomorphic to an ordinal.)

The map

$$F'' \circ (F')^{-1}$$

is clearly the identity map so $F' \in \mathfrak{M}[s]$. It follows that $F \in \mathfrak{M}[s]$.

Remark. This lemma is the motivation for my paper [14].

2. Some lemmas on genericity

2.1. Throughout this section $\mathfrak{M}$ is a countable transitive model of **ZFC**.

The following lemma is trivial but useful.

LEMMA. *Let $\mathcal{P}_1$, $\mathcal{P}_2$ be non-empty reflexive partially ordered sets lying in $\mathfrak{M}$, and let $\Psi\colon \mathcal{P}_1 \to \mathcal{P}_2$ be an order isomorphism lying in $\mathfrak{M}$. If $A \subseteq \mathcal{P}_1$, let*

$$\Psi_*(A) = \{\Psi(x) \colon x \in A\} .$$

Then for $G \subseteq \mathcal{P}_1$, G is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$ if and only if $\Psi_*(G)$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}_2$. Moreover, $\mathfrak{M}[G] = \mathfrak{M}[\Psi_*(G)]$.

2.2. Now let $\mathcal{P}_1$, $\mathcal{P}_2$ be reflexive partially ordered sets lying in $\mathfrak{M}$. We suppose that $\mathcal{P}_1 \subseteq \mathcal{P}_2$, and that the order on $\mathcal{P}_1$ is the restriction of the order on $\mathcal{P}_2$.

Definition. $\mathcal{P}_1$ is *cofinal* in $\mathcal{P}_2$ if for every $x \in \mathcal{P}_2$ there is a $y \in \mathcal{P}_1$ with $x \leqq y$.

LEMMA. *Let $\mathcal{P}_1$, $\mathcal{P}_2$ be non-empty reflexive partially ordered sets lying in $\mathfrak{M}$. Suppose that $\mathcal{P}_1$ is cofinal in $\mathcal{P}_2$. Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}_2$. Then $G \cap \mathcal{P}_1$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$. The map Ψ, given by*

$$\Psi(G) = G \cap \mathcal{P}_1 ,$$

gives a bijection of the set of $\mathfrak{M}$-generic filters on $\mathcal{P}_2$ with the set of $\mathfrak{M}$-generic filters on $\mathcal{P}_1$. Moreover, $\mathfrak{M}[G] = \mathfrak{M}[\Psi(G)]$.

PROOF. Let G be a $\mathfrak{M}$-generic filter on $\mathcal{P}_2$. Then it is straightforward to verify that $G \cap \mathcal{P}_1$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$. To verify clause (1) of § 1.3, let $x, y \in G \cap \mathcal{P}_1$. Then there is a $z \in G$ with $x \leqq z$, $y \leqq z$. Since $\mathcal{P}_1$ is cofinal in $\mathcal{P}_2$ and G satisfies § 1.3, (2), we can assume $z \in \mathcal{P}_1$. For clause (3), note that if X is a dense subset of $\mathcal{P}_1$,

$$\{y \in \mathcal{P}_2 \mid (\exists x \in X)(x \leqq y)\}$$

is dense in $\mathcal{P}_1$. Clause (2) is trivial to verify.

Now suppose that G_1 and G_2 are distinct $\mathfrak{M}$-generic filters on $\mathcal{P}$. We show that

$$\Psi(G_1) \neq \Psi(G_2) . \tag{1}$$

Let $p \in G_1$, $p \notin G_2$. (If necessary, we interchange G_1 and G_2 to get such a p.) Let

$$X = \{q \in \mathcal{P}_2 \mid q \geqq p \quad \text{or } p \text{ and } q \text{ are incompatible}\} .$$

Then X lies in $\mathfrak{M}$ and X is dense in $\mathcal{P}$. Pick $q \in G_2 \cap X$. If $q \geqq p$, we would have $p \in G_2$, contradicting our choice of p. Thus q is incompatible with p. Replacing q by an extension if necessary, we may also suppose $q \in \mathcal{P}_1$. Thus $q \in G_2 \cap \mathcal{P}_1$, but $q \notin G_1 \cap \mathcal{P}_1$, since any two members of G_1 have a common extension in G_1, and $p \in G_1$. This proves (1) and shows that G is one-to-one.

Let H be an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$. Put $G = \{x \in \mathcal{P}_2 \mid (\exists y \in H)(x \leqq y)\}$. We leave to the reader to verify that G is an $\mathfrak{M}$-generic filter on $\mathcal{P}_2$ and that $\Psi(G) = H$. This shows Ψ is onto, and the lemma is proved. (The last sentence of the lemma is clear from our explicit description of Ψ^{-1}.)

2.3. We now consider the following situation: $\mathcal{P}_1$ and $\mathcal{P}_2$ are reflexive non-empty partially ordered sets lying in $\mathfrak{M}$. We define a partially ordered set $\mathcal{P}$ as follows: as a set,

$$\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 ;$$

let $\langle p_1, p_2\rangle$, $\langle p_1', p_2'\rangle$ be elements of $\mathcal{P}$. Then

$$\langle p_1, p_2\rangle \leqq \langle p_1', p_2'\rangle$$

if and only if $p_1 \leqq p_1'$ and $p_2 \leqq p_2'$.

The following lemma characterizes the $\mathfrak{M}$-generic filters on $\mathcal{P}$.

Lemma. *Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}$. Then $G = G_1 \times G_2$ where G_1 is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$, and G_2 is an $\mathfrak{M}[G_1]$-generic filter on $\mathcal{P}_2$.*

Conversely, let G_1 be an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$ and G_2 an $\mathfrak{M}[G_1]$-generic filter on $\mathcal{P}_2$. Then $G_1 \times G_2$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}$.

Proof. Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}$. Put

$$G_1 = \{x \in \mathcal{P}_1 : (\exists y \in \mathcal{P}_2)(\langle x, y\rangle \in G)\} .$$
$$G_2 = \{x \in \mathcal{P}_2 : (\exists y \in \mathcal{P}_1)(\langle y, x\rangle \in G)\} .$$

Clearly, $G \subseteq G_1 \times G_2$. Conversely, let $\langle x, y\rangle \in G_1 \times G_2$. Pick $x' \in \mathcal{P}_2$ and $y' \in \mathcal{P}_1$ with $\langle x, x'\rangle$ and $\langle y', y\rangle \in G$. Let $\langle z, z'\rangle \in G$ be a common extension of $\langle x, x'\rangle$ and $\langle y', y\rangle$. Then

$$\langle x, y\rangle \leqq \langle z, z'\rangle$$

so $\langle x, y\rangle \in G$. This proves $G = G_1 \times G_2$.

We next verify that G_1 is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$. Clauses (1) and (2) of § 1.3 are trivial to verify. We turn to clause (3). Let $X \subseteq \mathcal{P}_1$ be dense in $\mathcal{P}_1$, with $X \in \mathfrak{M}$. Then $X \times \mathcal{P}_2$ is a dense subset of $\mathcal{P}$ lying in $\mathfrak{M}$. Since G is $\mathfrak{M}$-generic, $G \cap X \times \mathcal{P}_2$ is non-empty; i.e., $G_1 \cap X$ is non-empty.

We now show that G_2 is an $\mathfrak{M}[G_1]$-generic filter on $\mathcal{P}_2$. Again, § 1.3 (1)-(2) is trivial to verify. Let $X_1 \in \mathfrak{M}[G_1]$ be a dense subset of $\mathcal{P}_2$. By § 1.9, there is a formula $\Phi(x)$ of $\mathfrak{L}'$ such that

$$\mathfrak{M}[G_1] \models \Phi(x)$$

if and only if $x = X_1$. We use $\Phi(x)$ to translate any assertion about X_1 into a statement of $\mathfrak{L}'$; i.e., replace $\Psi(X)$ by

$$(u)(\Phi(u) \rightarrow \Psi(u)) .$$

Let p_1 be an element of G_1 which forces "$(\exists!\, x)\Phi(x)$ and $\forall x(\Phi(x) \rightarrow x$ is a dense subset of $\mathcal{P}_2)$".

Let

$$X_2 = \{\langle p, q\rangle \in \mathcal{P}\text{: either } p \text{ is incompatible with } p_1 \text{ or } p_1 \leqq p \text{ and } p \Vdash q \in X_1\} .$$

We claim X_2 is a dense subset of $\mathcal{P}$.

Suppose first that $\langle p, q\rangle \in X_2$ and $\langle p, q\rangle \leqq \langle p', q'\rangle$. We must show $\langle p', q'\rangle \in X_2$. This is trivial unless $p_1 \leqq p \leqq p'$. If this is so, p' forces the following:

(1) X_1 is a dense subset of $\mathcal{P}_2$.

(2) $q \in X_1$.

(3) $q \leqq q'$.

Hence, p' forces $q' \in X_1$ and so $\langle p', q'\rangle \in X_2$.

Next let $\langle p, q\rangle \in \mathcal{P}$. We show that $\langle p, q\rangle$ has an extension $\langle p^*, q^*\rangle$ lying in X_2. If p is incompatible with p_1, $\langle p, q\rangle$ itself lies in X_2. So suppose that p and p_1 have the common extension p'. Then p' forces the following statements (since $p' \geqq p_1$):

(1) X_1 is a dense subset of $\mathcal{P}_2$.

(2) $q \in \mathcal{P}_2$.

(3) $(\exists x \in \mathcal{P}_2)(q \leqq x$ and $x \in X_1)$.

Hence there is a $q^* \in \mathcal{P}_2$, and a $p^* \geqq p'$ such that

$$p^* \Vdash \text{“}q \leqq q^* \text{ and } q^* \in X_1\text{”} .$$

It follows that $\langle p^*, q^*\rangle \in X_2$. Clearly $\langle p, q\rangle \leqq \langle p^*, q^*\rangle$.

Thus X_2 is dense. Since forcing is definable in $\mathfrak{M}$ (§ 1.9), $X_2 \in \mathfrak{M}$. Since

G is $\mathfrak{M}$-generic, there is $\langle p^*, q^*\rangle \in G \cap X_2$. Now since p^* and p_1 both lie in G_1, they are compatible. Thus

$$p_1 \leqq p^* ,$$

and $p^* \Vdash q^* \in X_1$. Since $p^* \in G_1$, we have

$$\mathfrak{M}[G_1] \models q^* \in X_1 .$$

This shows $G_2 \cap X_1$ contains q^*, and so is non-empty. Our proof that G_2 is $\mathfrak{M}[G_1]$-generic is complete. This proves the first half of the lemma.

Now suppose that G_1 is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$ and G_2 is an $\mathfrak{M}[G_1]$-generic filter on $\mathcal{P}_2$. Put $G = G_1 \times G_2$. We show G is an $\mathfrak{M}$-generic filter on $\mathcal{P}$. As usual clauses (1) and (2) of § 1.3 are trivial to verify. Now let $X \in \mathfrak{M}$ be a dense subset of $\mathcal{P}$. We show that $X \cap G$ is non-empty.

Let $X' = \{q \in \mathcal{P}_2 \mid (\exists p \in G_1)(\langle p, q\rangle \in X)\}$. Clearly $X' \subseteq \mathcal{P}_2$, $X' \in \mathfrak{M}[G_1]$. We claim X' is dense in $\mathcal{P}_2$. Clearly, if $q \in X'$, and $q \leqq q'$, then $q' \in X'$. Next, let $q \in \mathcal{P}_2$. We consider

$$X'' = \{p \in \mathcal{P}_1 \mid (\exists q' \in \mathcal{P}_2)(q \leqq q' \text{ and } \langle p, q'\rangle \in X)\} .$$

One checks easily that X'' is a dense subset of $\mathcal{P}_1$, lying in $\mathfrak{M}$. Hence $\exists p \in G_1 \cap X''$. But then there is a $q' \in \mathcal{P}_2$ with $q \leqq q'$ and $\langle p, q'\rangle \in X$; i.e., $(\exists q' \geqq q)(q' \in X')$. We have now verified that X' is dense in $\mathcal{P}_2$.

Since G_2 is $\mathfrak{M}[G_1]$-generic, there is a $q \in G_2 \cap X'$; i.e., there is a $\langle p, q\rangle$ lying in $G_1 \times G_2 \cap X$. This completes our verification that $G_1 \times G_2$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}$.

COROLLARY. *Let G_1 be an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$ and G_2 an $\mathfrak{M}[G_1]$-generic filter on $\mathcal{P}_2$. Then G_1 is an $\mathfrak{M}[G_2]$-generic filter on $\mathcal{P}_1$.*

PROOF. By the lemma, $G_1 \times G_2$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1 \times \mathcal{P}_2$. By Lemma 2.1, $G_2 \times G_1$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}_2 \times \mathcal{P}_1$. By the lemma, G_1 is an $\mathfrak{M}[G_2]$-generic filter on $\mathcal{P}_1$.

2.4. Let $\mathfrak{M}, \mathcal{P}_1, \mathcal{P}_2$ be as in § 2.3. We make the following additional assumption on $\mathcal{P}_1$ and $\mathcal{P}_2$: $\mathcal{P}_1$ and $\mathcal{P}_2$ have a minimal element (which we name 0) such that $p \in \mathcal{P}_i \rightarrow 0 \leqq p$).

This assumption is quite harmless since if a reflexive partially ordered set $\mathcal{P}$ fails to satisfy it, we can simply add a new element 0 to $\mathcal{P}$ and decree that $0 \leqq p$, for all $p \in \mathcal{P} \cup \{0\}$. Since $\mathcal{P}$ is cofinal in $\mathcal{P} \cup \{0\}$, Lemma 2.2 says that $\mathcal{P}$ and $\mathcal{P} \cup \{0\}$ are equivalent for all our purposes.

LEMMA. *Let $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$. Let Φ be a sentence of $\mathcal{L}'$. Let $p = \langle p_1, p_2\rangle$ be an element of $\mathcal{P}$. We suppose that*

$$p \Vdash \text{“}\mathfrak{M}[G_1] \models \Phi\text{”} .$$

(The sentence in quotes can be constructed by the techniques of § 1.7.) *Then*

$$\langle p_1, 0\rangle \Vdash \text{“}\mathfrak{M}[G_1] \models \Phi\text{”} .$$

PROOF. Suppose not. Then there is an element $p' = \langle p'_1, p'_2\rangle$ of $\mathcal{P}$ such that $p_1 \leqq p'_1$ and

$$p' \Vdash \text{“}\mathfrak{M}[G_1] \models \neg\, \Phi\text{”} .$$

Select G' a generic filter on $\mathcal{P}$ with $p' \in G'$. By Lemma 2.3. $G' = G'_1 \times G'_2$, where G'_1 is an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$. Since $p' \in G'$, we have

$$\mathfrak{M}[G'_1] \models \neg\, \Phi .$$

Pick an $\mathfrak{M}[G'_1]$-generic filter G''_2 on $\mathcal{P}_2$ with $p_2 \in G''_2$. By Lemma 2.3, $G'_1 \times G''_2$ is $\mathfrak{M}$-generic. By construction, $p \in G'_1 \times G''_2$. Thus, by our hypothesis on p,

$$\mathfrak{M}[G'_1] \models \Phi .$$

This is absurd, since we know that $\mathfrak{M}[G'_1] \models \neg\, \Phi$. This contradiction completes the proof.

2.5. LEMMA. *Let $\mathfrak{M}$, $\mathcal{P}_1$, $\mathcal{P}_2$ be as in § 2.4. Let $G = G_1 \times G_2$ be an $\mathfrak{M}$-generic filter on $\mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2$. Let $a \subseteq \omega$ lie in $\mathfrak{M}[G_1] \cap \mathfrak{M}[G_2]$. Then $a \in \mathfrak{M}$.*

PROOF. By § 1.7, we can find a formula $\Psi(x, y, z)$ of $\mathfrak{L}$, and elements x_1, x_2 of $\mathfrak{M}$ such that

$$a = \{n \in \omega \mid \mathfrak{M}[G_i] \models \Psi(\underline{x}_i, \underline{G}, \underline{n})\} , \qquad i = 1, 2 .$$

Hence there is a condition $p = \langle p_1, p_2\rangle$ of $\mathcal{P}_1 \times \mathcal{P}_2$ which forces the following:

$$(\alpha) \qquad (\forall n \in \omega)\big(\mathfrak{M}[G_1] \models \Psi(\underline{x}_1, \underline{G}, \underline{n}) \longleftrightarrow \mathfrak{M}[G_2] \models \Psi(\underline{x}_2, \underline{G}, \underline{n})\big) .$$

We claim that for all n, p decides $\mathfrak{M}[G_1] \models \Psi(\underline{x}_1, \underline{G}, \underline{n})$. Granting this,

$$a = \{n \in \omega \mid p \Vdash \text{“}\mathfrak{M}[G_1] \models \Psi(\underline{x}_1, \underline{G}, \underline{n})\text{”}\} ,$$

so $a \in \mathfrak{M}$ (since “forcing is expressible in the ground model”).

Suppose then that p does not decide the statement,

$$(\beta) \qquad \mathfrak{M}[G_1] \models \Psi(x_1, \underline{G}, \underline{n}) ,$$

for some $n \in \omega$. Let p', p'' be extensions of p such that $p' \Vdash (\beta)$ and $p'' \Vdash \neg\, (\beta)$. We have, say, $p' = \langle p'_1, p'_2\rangle$ and $p'' = \langle p''_1, p''_2\rangle$. Also let $p = \langle p_1, p_2\rangle$.

By Lemma 2.4, we have $\langle p'_1, p_2\rangle \Vdash (\beta)$. Since p'' extends p, and p forces (α), we see that p'' forces

$$(\gamma) \qquad \mathfrak{M}[G_2] \models \neg\, \Psi(\underline{x}_2, \underline{G}, \underline{n}) .$$

Hence, by Lemma 2.4, $\langle p_1, p''_2\rangle \Vdash (\gamma)$.

Consider now $\langle p'_1, p''_2\rangle$. As a common extension of p, $\langle p'_1, p_2\rangle$ and $\langle p_1, p''_2\rangle$, it forces ($\alpha$), ($\beta$), and ($\gamma$). Since ($\alpha$) contradicts "($\beta$) and ($\gamma$)", we have a contradiction. Thus p does decide (β), and the lemma is proved.

One can in fact show that $\mathfrak{M}[G_1] \cap \mathfrak{M}[G_2] = \mathfrak{M}$. (Otherwise pick a counter-example a of minimal rank. We have $a \subseteq \mathfrak{M}$, and the proof of Lemma 2.5 adapts to show that $a \in \mathfrak{M}$.)

3. Description of the model

3.1. The model used to prove Theorem 2 is due to Azriel Levy. In this section, we describe the model and prove some of its elementary properties. The results of this section are due to Levy and are included here with his permission.

Let $\mathfrak{M}$ be a countable transitive model of **ZFC** + "There is a strongly inaccessible cardinal". Let $\Omega \in \mathfrak{M}$ be strongly inaccessible in $\mathfrak{M}$.

3.2. Let λ be an ordinal. Let $\mathcal{P}^\lambda$ be the following set: $f \in \mathcal{P}^\lambda$ if

(1) f is a function;

(2) domain(f) is a finite subset of $\lambda \times \omega$;

(3) range(f) $\subseteq \lambda$;

(4) $f(\langle \alpha, n \rangle) < \alpha$ whenever $\langle \alpha, n \rangle \in$ domain(f).

We order $\mathcal{P}^\lambda$ by $\subseteq$. Note that if $\mathfrak{M} \subseteq \mathfrak{N}$ are transitive models of **ZF**, and $\lambda \in \mathfrak{M}$, $(\mathcal{P}^\lambda)_{\mathfrak{M}} = (\mathcal{P}^\lambda)_{\mathfrak{N}} = \mathcal{P}^\lambda$.

LEMMA. *Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}^\lambda$. Let $0 < \alpha < \lambda$. Define $f_\alpha \subseteq \omega \times \alpha$ by*

$$f_\alpha = \{\langle n, \beta \rangle : \{\langle \langle \alpha, n \rangle, \beta \rangle\} \in G\} .$$

Then f_α is a surjective map of ω onto α.

PROOF. Suppose first that $\langle n, \beta \rangle \in f_\alpha$, and $\langle n, \beta' \rangle \in f_\alpha$. Then

$$\{\langle \langle \alpha, n \rangle, \beta \rangle\} \cup \{\langle \langle \alpha, n \rangle, \beta' \rangle\}$$

for some $h \in \mathcal{P}^\lambda$. Since h is a function, $\beta = \beta'$. This proves f_α is a function.

Since $\{h \in \mathcal{P}^\lambda \mid \langle \alpha, n \rangle \in \mathrm{dom}\,(h)\}$ is dense (since $\alpha > 0$), there is an $h \in G$, $h(\langle \alpha, n \rangle) = \beta$, say. But then $\{\langle \langle \alpha, n \rangle, \beta \rangle\} \in G$, by § (1.3.2), so $n \in$ domain(f_α). Thus domain(f_α) $= \omega$. Now let $\beta < \alpha$. Since the set

$$\{h \in \mathcal{P}^\lambda \mid (\exists n < \omega)(h\langle \alpha, n \rangle) = \beta\}$$

is dense, one sees similarly that $\beta \in$ range(f_α). This proves the lemma.

COROLLARY. *Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}^\lambda$. Then $\lambda \leq \aleph_1^{\mathfrak{M}[G]}$.*

PROOF. If $0 < \alpha < \lambda$, then there is a surjective map $f_\alpha : \omega \to \alpha$ in $\mathfrak{M}[G]$ (by the lemma just proved).

3.3. The model $\mathfrak{N}$ used to prove Theorem 2 is obtained as follows. Let

G be an $\mathfrak{M}$-generic filter on $\mathcal{P}^\Omega$. Then $\mathfrak{N} = \mathfrak{M}[G]$.

We are going to show that $\Omega = \aleph_1^{\mathfrak{N}}$. We first prove the following lemma.

LEMMA. *Let $\mathcal{F} \in \mathfrak{M}$, $\mathcal{F} \subseteq \mathcal{P}^\Omega$. Suppose that any two distinct elements $\mathcal{P}$ are incompatible. Then, in $\mathfrak{M}$, $\mathcal{F}$ has cardinality less than Ω. In fact, there is a $\xi < \Omega$ such that $\mathcal{F} \subseteq \mathcal{P}^\xi$.*

PROOF. We work inside $\mathfrak{M}$. By Zorn's lemma we may assume that $\mathcal{F}$ is a maximal pairwise incompatible family of elements of $\mathcal{P}^\Omega$.

We define a sequence of ordinals $\{\xi_i, i < \omega\}$. Put $\xi_0 = \omega$. Suppose then that ξ_i has been defined, and $\xi_i < \Omega$. Then since Ω is inaccessible, $\mathcal{P}^{\xi_i}$ has cardinality less than Ω.

Let $h \in \mathcal{P}^{\xi_i}$. By the maximality of $\mathcal{F}$, there is an $f_h \in \mathcal{F}$ with f_h compatible with h. Since Ω is regular and card $(\mathcal{P}^{\xi_i}) < \Omega$, we can find ξ_{i+1} with

$$\xi_i < \xi_{i+1} < \Omega$$

and $h \in \mathcal{P}^{\xi_i} \rightarrow f_h \in \mathcal{P}^{\xi_{i+1}}$.

Let $\xi_\omega = \sup \{\xi_i, i \in \omega\}$. Then $\xi_\omega < \Omega$. We claim $\mathcal{F} \subseteq \mathcal{P}^{\xi_\omega}$.

Suppose not. Let $g \in \mathcal{F}$, $g \notin \mathcal{P}^{\xi_\omega}$. Let g' be the restriction of g to $\xi_\omega \times \omega$. Then since domain (g) is finite, $g' \in \mathcal{P}^{\xi_n}$ for some n. By construction, there is a $g'' \in \mathcal{P}^{\xi_{n+1}} \cap \mathcal{F}$ compatible with g'. But g'' is not compatible with g (since both lie in $\mathcal{F}$ and $g \notin \mathcal{P}^{\xi_\omega}$.) So there is an $\langle \alpha, n \rangle \in$ domain $(g) \cap$ domain (g'') with

$$g(\langle \alpha, n \rangle) \neq g''(\langle \alpha, n \rangle) .$$

Since $\langle \alpha, n \rangle \in \text{domain}(g'')$, $\alpha < \xi_{n+1} < \xi_\omega$. By the definition of g', $g(\langle \alpha, n \rangle) = g'(\langle \alpha, n \rangle) \neq g''(\langle \alpha, n \rangle)$. But this contradicts the fact that g' and g'' are compatible.

So $\mathcal{F} \subseteq \mathcal{P}_{\xi_\omega}$. But then the cardinality estimate of the lemma is clear.

COROLLARY. $\aleph_1^n = \Omega$.

PROOF. In view of Corollary 3.2, we have to show that if $f \in \mathfrak{M}[G]$, $f \colon \omega \rightarrow \Omega$, then f is not onto. But this follows in a known way (cf. [1, p. 132]) from the lemma.

3.4. Let $G^\lambda = G \cap \mathcal{P}^\lambda$. By § 2.3, G^λ is an $\mathfrak{M}$-generic filter on $\mathcal{P}^\lambda$. We are going to prove the following lemma.

LEMMA. Let $f \in \mathfrak{N}$ be a function such that

$$f \colon \omega \rightarrow OR$$

(OR is the class of ordinals). Then for some $\xi < \Omega$,

$$f \in \mathfrak{M}[G^\xi] .$$

PROOF. Let $\Phi(x, y)$ be a formula of $\mathcal{L}'$ such that

$$f = \{\langle x, y\rangle : \mathfrak{N} \models \Phi(x, y)\} .$$

Let $p_0 \in G$ force

"$\{\langle x, y\rangle : \Phi(x, y)\}$ is a function from ω to OR".

Let $n \in \omega$. We say that $p \geqq p_0$ decides the value of $f(n)$ if for some ordinal λ of $\mathfrak{M}$

$$p \Vdash \Phi(\underline{n}, \underline{\lambda}) .$$

Since p extends p_0, the ordinal λ is uniquely determined by p and n. Since $p_0 \in G$, and $OR^{\mathfrak{M}} = OR^{\mathfrak{N}}$, for each $n \in \omega$, there is a $p \in G$ which decides $f(n)$.

We work inside $\mathfrak{M}$. Let

$$\mathcal{G}_n = \{p \in \mathcal{P}^{\Omega} : p \geqq p_0 \text{ and } p \text{ decides } f(n)\} .$$

Let $\mathcal{F}_n$ be a maximal pairwise incompatible subfamily of $\mathcal{G}_n$. By Lemma 3.3, there is a $\xi < \Omega$ such that, for all $n \in \omega$,

$$\mathcal{F}_n \subseteq \mathcal{P}^{\xi} .$$

It follows that $p_0 \in \mathcal{P}^{\xi}$.

Claim. Let $n \in \omega$. Then there is a $p \in G^{\xi}$ such that p decides $f(n)$. In fact, let $X_n \subseteq \mathcal{P}^{\Omega}$ be the set of p such that

(1) if p is compatible with p_0, $p \geqq p_0$;

(2) If $p \geqq p_0$, then p decides $f(n)$;

(3) if $p \geqq p_0$, then $p \geqq q$ for some $q \in \mathcal{F}_n$.

One can verify easily that if $p \in X_n$, and $p \leqq p'$, then $p' \in X_n$. Let $p \in \mathcal{P}^{\Omega}$. Then either p is incompatible with p_0 (and so $p \in X_n$), or there is a $p_1 \in \mathcal{P}^{\Omega}$ extending both p and p_0. Since $p_1 \geqq p_0$, there is a $p_2 \geqq p_1$ which decides $f(n)$. Thus $p_2 \in \mathcal{G}_n$. By the maximality of $\mathcal{F}_n$, there is a $q \in \mathcal{F}_n$ with p_2 compatible with q. Let p_3 be a common extension of p_2 and q. Then by construction, p_3 is an extension of p lying in X_n. Thus X_n is dense. Since X_n clearly lies in $\mathfrak{M}$, and G is an $\mathfrak{M}$-generic filter on $\mathcal{P}^{\Omega}$, there is a $p \in G$, with $p \in X_n$.

Now $p_0 \in G$, and any two elements of G are compatible. By clause (1) of the definition of X_n, $p \geqq p_0$. By clause (3) of the definition of X_n, there is a $q \in \mathcal{F}_n$, with $q \leqq p$. Hence $q \in G$, since p is. Since $q \in G^{\xi}$, and q decides $f(n)$, our claim is clear.

But now

$$f = \{\langle n, \lambda\rangle : (\exists p \in G^{\xi})(p \Vdash \Phi(\underline{n}, \underline{\lambda})\}$$

so that lemma is clear.

COROLLARY 1. *Let $s \in \mathfrak{N}$ be a subset of ω. Then $s \in \mathfrak{M}[G^{\xi}]$, for some $\xi < \Omega$.*

COROLLARY 2. *Let $s \in \mathfrak{N}$ be a subset of ω. Then Ω is inaccessible in*

$\mathfrak{M}[s]$. *The set* $A_s = \{t \subseteq \omega \mid t \in \mathfrak{M}[s]\}$ *is countable in* $\mathfrak{N}$.

PROOF. By the lemma, we can pick $\xi < \Omega$, such that $s \in \mathfrak{M}[G^\xi]$. It follows that

$$\mathfrak{M}[s] \subseteq \mathfrak{M}[G^\xi] .$$

By Lemma 1.11, Ω is inaccessible in $\mathfrak{M}[G^\xi]$. *A fortiori*, it is inaccessible in $\mathfrak{M}[s]$. Let α be the cardinal of A_s in $\mathfrak{M}[s]$. Then $\alpha < \Omega$, since Ω is inaccessible in $\mathfrak{M}[s]$. By Corollary 3.3, α is countable in $\mathfrak{N}$.

3.5. *Symmetry*. Let π be a permutation of ω lying in $\mathfrak{M}$. Define $\pi_*: \mathcal{P}^\Omega \to \mathcal{P}^\Omega$ by

$$\pi_*(h)(\langle\alpha, n\rangle) = h(\langle\alpha, \pi(n)\rangle) .$$

Then π_* is an automorphism of $\mathcal{P}^\Omega$ lying in $\mathfrak{M}$. If G is an $\mathfrak{M}$-generic filter on $\mathcal{P}^\Omega$, so is $\pi_*[G]$, by Lemma 2.1. Clearly,

$$\mathfrak{M}[G] = \mathfrak{M}[\pi_*[G]] .$$

Let Φ be a statement of $\mathcal{L}'$, not involving G. We claim $p \Vdash \Phi$ if and only if $\pi_*(p) \Vdash \Phi$. To see this, we construct the following chain of equivalent statements.

(1) $p \Vdash \Phi$.

(2) For all $\mathfrak{M}$-generic filters G on $\mathcal{P}^\Omega$ which contain p,

$$\mathfrak{M}[G] \models \Phi .$$

Since Φ does not contain G, and $\mathfrak{M}[G] = \mathfrak{M}[\pi_*(G)]$, (2) is equivalent to

(3) For all $\mathfrak{M}$-generic filters G containing p,

$$\mathfrak{M}[\pi_*(G)] \models \Phi .$$

Now $p \in G$ if and only if $\pi_*(p) \in \pi_*(G)$. Moreover, as G ranges over the set of $\mathfrak{M}$-generic filters on $\mathcal{P}^\Omega$, so does $\pi_*(G)$. It follows that (3) is equivalent to

(4) For all $\mathfrak{M}$-generic filters G containing $\pi_*(p)$

$$\mathfrak{M}[G] \models \Phi .$$

But (4) just says that $\pi^*(p) \Vdash \Phi$.

LEMMA. *Let* Φ *be a sentence of* $\mathcal{L}'$ *not containing* G. *Let* 0 *be the minimal element of* $\mathcal{P}^\Omega$. *Then* 0 *decides* Φ.

PROOF. Otherwise there are $p_1, p_2 \in \mathcal{P}^\Omega$, with $p_1 \Vdash \Phi$, $p_2 \Vdash \neg \Phi$. We can find a permutation $\pi \in \mathfrak{M}$ such that $\pi_*(p_1)$ has domain disjoint from p_2. By our previous remark, $\pi_*(p_1) \Vdash \Phi$. But then $\pi_*(p_1)$ must be incompatible with p_2 since $p_2 \Vdash \neg \Phi$. This is absurd since $\pi_*(p_1)$ and p_2 have disjoint domains.

COROLLARY. *Let* $a \subseteq \omega$, $a \in \mathfrak{N}$. *Suppose that there is a formula* $\Phi(x)$ *of* $\mathcal{L}'$ *not containing* G *such that* a *is the unique* z *such that*

$$\mathfrak{N} \models \Phi(z) .$$

Then $a \in \mathfrak{M}$.

PROOF. There is a formula $\Psi(z)$ of $\mathfrak{L}'$, not containing G, such that

$$a = \{n \in \omega \mid \mathfrak{N} \models \Psi(\underline{n})\} .$$

By the lemma,

$$a = \{n \in \omega \mid 0 \Vdash \Psi(\underline{n})\} .$$

Since "forcing is expressible in the ground model"; (cf. § 1.9), this shows that $a \in \mathfrak{M}$.

4. An important lemma

4.1. This section is devoted to the proof of the following result. It is the key technical fact we need about $\mathfrak{N}$.

THEOREM. *Let $f: \omega \to OR$, $f \in \mathfrak{N}$. Then there is an $\mathfrak{M}[f]$-generic filter G' on $\mathcal{P}^{\Omega}$, such that $\mathfrak{N} = \mathfrak{M}[f][G']$.*

The effect of this theorem (which is the "important lemma" of the section title) is to give us excellent control on the extension $\mathfrak{N}/\mathfrak{M}[f]$.

COROLLARY. *Let $s \subseteq \omega$, $s \in \mathfrak{N}$. Then there is an $\mathfrak{M}[s]$-generic filter G' on $\mathcal{P}^{\Omega}$, such that $\mathfrak{N} = \mathfrak{M}[s][G']$.*

4.2. We begin with some easy lemmas.

LEMMA 1. *Let α be an ordinal of $\mathfrak{M}$. Let β be the cardinal of α in $\mathfrak{M}$. Let $F: \omega \to \alpha$ be an $\mathfrak{M}$-generic collapsing map. Then there is an $\mathfrak{M}$-generic collapsing map $G: \omega \to \beta$ such that*

$$\mathfrak{M}[F] = \mathfrak{M}[G] . \tag{1}$$

Conversely, let $G: \omega \to \beta$ be an $\mathfrak{M}$-generic collapsing map. Then there is an $\mathfrak{M}$-generic collapsing map $F: \omega \to \alpha$ such that (1) *holds.*

PROOF. Let $\psi: \alpha \to \beta$ be a bijection lying in $\mathfrak{M}$. Let $\mathcal{P}_\alpha$, $\mathcal{P}_\beta$ be as in § 1.12. Then ψ induces an order isomorphism of $\mathcal{P}_\alpha$ with $\mathcal{P}_\beta$ lying in $\mathfrak{M}$. The lemma now follows from Lemma 2.1.

LEMMA 2. *Let α be an ordinal. Let $F: \omega \to \alpha$ be a generic collapsing map. Define $F_1, F_2: \omega \to \alpha$ by*

$$F_1(n) = F(2n); \quad F_2(n) = F(2n + 1) .$$

Then F_1 is an $\mathfrak{M}$-generic collapsing map, and F_2 is an $\mathfrak{M}[F_1]$-generic collapsing map.

Conversely let $F_1, F_2: \omega \to \alpha$ be respectively an $\mathfrak{M}$-generic and a $\mathfrak{M}[F_1]$-generic collapsing map. Then if we define $F: \omega \to \alpha$ by

$$F(2n) = F_1(n);\quad F(2n+1) = F_2(n)\ ,$$

then F is an $\mathfrak{M}$-generic collapsing map. In either case, we have

$$\mathfrak{M}[F] = \mathfrak{M}[F_1, F_2]\ .$$

PROOF. Define an isomorphism

$$\psi: \mathcal{P}_\alpha \to \mathcal{P}_\alpha \times \mathcal{P}_\alpha$$

by $\psi(h) = \langle h_1, h_2\rangle$, with $h_1(n) = h(2n)$, $h_2(n) = h(2n+1)$. Let G be the $\mathfrak{M}$-generic filter on $\mathcal{P}_\alpha$ associated to F. By Lemma 2.1, $\psi_*(G)$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}_\alpha \times \mathcal{P}_\alpha$. By Lemma 2.3,

$$\psi_*(G) = G_1 \times G_2$$

with G_1 an $\mathfrak{M}$-generic filter on $\mathcal{P}_\alpha$, and G_2 an $\mathfrak{M}[G_1]$-generic filter on $\mathcal{P}_\alpha$. It is clear from the definition of ψ that

$$\bigcup G_1 = F_1;\quad \bigcup G_2 = F_2\ .$$

Thus the first half of the lemma is clear. The second half is proved similarly.

4.3. LEMMA. *Let α be an ordinal $\geqq \omega$ of $\mathfrak{M}$. Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}^{\alpha+1}$. Then there is an $\mathfrak{M}$-generic collapsing function $F: \omega \to \alpha$ such that* $\mathfrak{M}[G] = \mathfrak{M}[F]$. Conversely, if $F: \omega \to \alpha$ is an $\mathfrak{M}$-generic collapsing map, there is an $\mathfrak{M}$-generic filter G on $\mathcal{P}^{\alpha+1}$ with $\mathfrak{M}[G] = \mathfrak{M}[F]$.

PROOF. We first prove the lemma under the additional assumption that α is countable in $\mathfrak{M}$. We then show how to remove this assumption.

Since α is countable in $\mathfrak{M}$, there is a bijection $\psi: \omega \to (\alpha + 1 - \{0\}) \times \omega$ lying in $\mathfrak{M}$, such that $\psi(2n) = \langle \alpha, n\rangle$. Let $\varphi(n)$ be the first component of $\psi(n)$. Thus $\varphi: \omega \to \alpha + 1 - \{0\}$, $\varphi(2n) = \alpha$, all $n \in \omega$.

Let $\mathcal{P}'$ be the following collection of functions: $h \in \mathcal{P}'$ if and only if domain(h) is a finite subset of ω, range $(h) \subseteq \alpha$, and $h(n) < \varphi(n)$ for all $n \in$ domain(h). We order $\mathcal{P}'$ by $\subseteq$.

The map $\{h \to h \circ \psi\}$ is clearly an order isomorphism of $\mathcal{P}^{\alpha+1}$ with $\mathcal{P}'$.

Let $\mathcal{P}''$ be the following subset of $\mathcal{P}'$: $h \in \mathcal{P}''$ if and only if $h \in \mathcal{P}'$ and $(\forall n \in \omega)\,(2n \in \text{domain } (n) \to 2n + 1 \in \text{domain } (h))$. $\mathcal{P}''$ is clearly a cofinal subset of $\mathcal{P}'$. We are going to set up an isomorphism of $\mathcal{P}''$ with $\mathcal{P}_\alpha$, lying in $\mathfrak{M}$.

To describe this isomorphism, let

$$S = \{\langle \gamma_1, \gamma_2, \gamma_3\rangle: \gamma_2 < \gamma_1 \leqq \alpha \text{ and } \gamma_3 < \alpha\}\ .$$

Let $\psi': S \to \alpha$ be a map lying in $\mathfrak{M}$ such that

$$\langle \gamma_2, \gamma_3\rangle \to \psi'(\gamma_1, \gamma_2, \gamma_3)$$

is a bijection of $\gamma_1 \times \alpha$ with α whenever $0 < \gamma_1 \leqq \alpha$.

Define $\psi'': \mathcal{P}'' \to \mathcal{P}_\alpha$ as follows. $\psi''(h)$ is defined at $m \in \omega$ if and only if $h(2m)$ and $h(2m+1)$ are defined. In that case,

$$\psi''(h)(m) = \psi'(\varphi(2m+1), h(2m+1), h(2m)) .$$

A moments reflection shows that ψ'' gives an isomorphism of $\mathcal{P}''$ with $\mathcal{P}_\alpha$.

Under the assumption that α is countable in $\mathfrak{M}$ the lemma is now clear. Suppose, for example, that G is an $\mathfrak{M}$-generic filter on $\mathcal{P}^{\alpha+1}$. By applying Lemma 2.1 and 2.2 we get a filter G' on $\mathcal{P}_\alpha$ such that $\mathfrak{M}[G] = \mathfrak{M}[G']$. It suffices to take $F = \bigcup G'$.

Now drop the assumption that α is countable in $\mathfrak{M}$. Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}^{\alpha+1}$. Writing $\mathcal{P}^{\alpha+1} \cong \mathcal{P}_\alpha \times \mathcal{P}^\alpha$, we see that there is an $\mathfrak{M}$-generic collapsing map $F_1: \omega \to \alpha$ and an $\mathfrak{M}[F_1]$-generic filter, G_1, on $\mathcal{P}^\alpha$ such that

$$\mathfrak{M}[G] = \mathfrak{M}[F_1, G_1] .$$

We apply Lemma 4.2.2, to get collapsing maps F_2, F_3: $\omega \to \alpha$, generic over $\mathfrak{M}$ and $\mathfrak{M}[F_2]$ respectively, with

$$\mathfrak{M}[F_1] = \mathfrak{M}[F_2][F_3] .$$

Again using the isomorphism $\mathcal{P}^{\alpha+1} \cong \mathcal{P}_\alpha \times \mathcal{P}^\alpha$, we can coalesce F_3 and G_1 into an $\mathfrak{M}[F_1]$-generic filter on $\mathcal{P}^{\alpha+1}$, G_2. So

$$\mathfrak{M}[G] = \mathfrak{M}[F_1][G_2] .$$

But α is countable in $\mathfrak{M}[F_1]$. By the special case of the lemma previously proved there is an $\mathfrak{M}[F_1]$-generic collapsing map, F_4: $\omega \to \alpha$ with

$$\mathfrak{M}[G] = \mathfrak{M}[F_1][G_2] = \mathfrak{M}[F_1][F_4] .$$

But Lemma 4.2.2 allows us to coalesce F_1 and F_4 into a single generic collapsing map F, with

$$\mathfrak{M}[G] = \mathfrak{M}[F_1][F_4] = \mathfrak{M}[F] .$$

To prove the converse, run the argument backward.

4.4. The following lemma is the crucial step in the proof of Theorem 4.1.

LEMMA. *Let $\alpha \in \mathfrak{M}$, $\alpha \geqq \omega$. Let F_1, F_2: $\omega \to \alpha$ be collapsing maps generic over $\mathfrak{M}$ and $\mathfrak{M}[F_1]$ respectively. Let $s \subseteq OR$ be a set of $\mathfrak{M}[F_1]$. Then there is a collapsing map F: $\omega \to \alpha$, generic over $\mathfrak{M}[s]$ with*

$$\mathfrak{M}[s][F] = \mathfrak{M}[F_1, F_2] .$$

PROOF. We begin by describing a certain cofinal subset $\mathcal{P}_1$ of $\mathcal{P}_\alpha \times \mathcal{P}_\alpha$. A pair $\langle h_1, h_2 \rangle$ lies in $\mathcal{P}_1$ if and only if domain(h_1) = domain(h_2) and domain(h_1) is a finite initial segment of the integers. If $\langle h_1, h_2 \rangle \in \mathcal{P}_1$, then we put

$l(\langle h_1, h_2\rangle) = \text{domain}(h_1)$. Thus if $x \in \mathcal{P}_1$, $l(x) \in \omega$. Let G' be a generic filter on $\mathcal{P}_1$. Then there is a pair of functions, $F_1', F_2': \omega \to \alpha$, say, and

$$G' = \{\langle F_1' \mid n, F_2' \mid n\rangle : n \in \omega\} .$$

We fix a definition of $s \in \mathfrak{M}[F_1]$. Thus $\lambda \in s$ if and only if $\mathfrak{M}[F_1] \models \Phi(\underline{\lambda}, \underline{F}_1)$, and Φ is a formula involving only ε, S, and terms denoting elements of $\mathfrak{M}$.

By Lemma 2.3, F_1 and F_2 determine a generic filter, G_1, on $\mathcal{P}_\alpha \times \mathcal{P}_\alpha$. Let G be the associated filter on $\mathcal{P}_1$ given by Lemma 2.2: $G = G_1 \cap \mathcal{P}_1$.

Our next step is to define a certain subset Σ of $\mathcal{P}_1$. Roughly speaking, Σ has the following motivation: The fact that

$$s = \{\lambda \mid \mathfrak{M}[F_1] \models \Phi(\underline{\lambda}, \underline{F}_1)\}$$

gives a certain amount of information about G. This information is summed up in the fact that $G \subseteq \Sigma$.

Let $\Psi(x)$ be a formula involving only ε, S, and terms denoting elements of $\mathfrak{M}$ and G such that $\mathfrak{M}[G] \models \Psi(\underline{\lambda})$ if and only if $\mathfrak{M}[F_1] \models \Phi(\underline{\lambda}, F_1)$. Then, if Ψ is constructed in a reasonable manner, the following will be true (by Lemma 2.4):

If $\langle h_1, h_2\rangle$ and $\langle h_1, h_3\rangle \in \mathcal{P}_1$, then $\langle h_1, h_2\rangle \Vdash \Psi(\underline{\lambda})$ if and only if $\langle h_1, h_3\rangle \Vdash \Psi(\underline{\lambda})$.

We work in $\mathfrak{M}[s]$. Define a sequence of subsets of $\mathcal{P}_1$, $\{A_\alpha\}$, by transfinite induction.

(1) $p \in A_0$ if either $p \Vdash \Psi(\underline{\lambda})$ and $\lambda \notin s$ (for some $\lambda \in OR^{\mathfrak{M}}$) or $p \Vdash \neg \Psi(\underline{\lambda})$ and $\lambda \in s$).

(2) Let $\alpha = \beta + 1$. $p \in A_\alpha$ if for some dense subset X of $\mathcal{P}_1$, lying in $\mathfrak{M}$, every extension of p in X is in A_β.

(3) Let α be a limit ordinal. Then $A_\alpha = \bigcup_{\beta<\alpha} A_\beta$.

We note the following facts about $\{A_\alpha\}$.

(A1) If $p \in A_\alpha$, and $p \leqq q$, then $q \in A_\alpha$.

(This is easily checked by induction on α.)

(A2) If $\alpha < \beta$, then $A_\alpha \subseteq A_\beta$.

(The crucial case is when $\beta = \alpha + 1$. Take the dense set X to be $\mathcal{P}_1$ itself and use A1.)

(A3) Let $p = \langle h_1, h_2\rangle$, $q = \langle h_1, h_3\rangle$, and suppose $p, q \in \mathcal{P}_1$. Then $p \in A_\alpha$ if and only if $q \in A_\alpha$.

Since $\mathfrak{M}[s]$ is a model for ZF, there is an ordinal δ such that $A_\delta = A_{\delta+1}$. We put

$$\Sigma = \mathcal{P}_1 - A_\delta .$$

We next list some properties of Σ.

(Σ1) $G \subseteq \Sigma$.

Otherwise, there is an $x \in G$ such that $x \in A_\beta$. Pick x, β so that β is minimal. Clearly β is not zero, since in $\mathfrak{M}[F_1, F_2]$,

$$s = \{\lambda \mid \Psi(\underline{\lambda})\}\ .$$

Also, β is not a limit ordinal since for λ a limit ordinal $A_\lambda = \bigcup_{\gamma<\lambda} A_\gamma$. Thus $\beta = \gamma + 1$. Since $x \in A_\beta$, there is a dense set X with each extension of x in X lying in A_γ, and $X \in \mathfrak{M}$. Since G is $\mathfrak{M}$-generic, there is a $y \in G \cap X$. Let $z \in G$ be a common extension of x and y. Then $z \in X$, since y is; thus, $z \in A_\gamma$ (since z extends x). But this contradicts the minimality of β.

(Σ2) Let $p \in \Sigma$. Let X be a dense subset of $\mathcal{P}_1$ lying in $\mathfrak{M}$. Then there is a $p' \in \Sigma \cap X$ with p' extending p.

Proof. Since $p \in A_{\delta+1}$, there is a $p' \geqq p$, with $p' \in X$, $p' \in A_\delta$; i.e., $p' \in \Sigma \cap X$.

(Σ3) Let $p \in \Sigma$. Let $q \leqq p$. Then $q \in \Sigma$. (This follows from (A2) since Σ is the complement of A_δ.)

(Σ4) Let $p \in \Sigma$. Then there is an $\mathfrak{M}$-generic filter, G', on $\mathcal{P}_1$ such that $p \in G'$ and

$$s = \{\lambda \mid \mathfrak{M}[G'] \models \Psi(\underline{\lambda})\}\ .$$

Proof. Since $\mathfrak{M}$ is countable, we can enumerate the dense subsets of $\mathcal{P}_1$ in a sequence: $\{X_i,\ i \in \omega\}$. Using (Σ2), we can construct an increasing sequence of elements of Σ, $\{p_n\}$, with $p_0 = p$, and $p_{n+1} \in X_n$. Put $G' = \{x \in \mathcal{P}_1 \mid x \leqq p_n$ for some $n\}$. Then G' has the desired properties.

(Σ5) G is an $\mathfrak{M}[s]$-generic filter on Σ.

Proof. Clauses (1) and (2) of § 1.3 are clear. We turn to clause (3). Let X be a dense subset of Σ lying in $\mathfrak{M}[s]$. We must show that $X \cap G \neq \varnothing$. We assume the contrary and get a contradiction.

We fix a formula $\Phi_1(x, y)$ of $\mathcal{L}'$, not containing G, such that Φ_1 defines X from s in $\mathfrak{M}[s]$ (i.e., $\Phi_1(y, s)$ holds in $\mathfrak{M}[s]$ if and only if $y = X$). We now form a sentence Ψ_1 of $\mathcal{L}'$ such that for any generic filter G' on $\mathcal{P}_1$, we have $\mathfrak{M}[G'] \models \Psi_1$ if and only if

(1) if $s' = \{\lambda \in OR \mid \mathfrak{M}[G'] \models \Psi(\underline{\lambda})\}$ then s' is a set and there is a unique $X' \in \mathfrak{M}[s']$ such that

$$\Phi_1(X', s')\ .$$

(2) X' is a dense subset of Σ', where Σ' is the set obtained by applying our definition of Σ inside $\mathfrak{M}[s']$.

(3) $X' \cap G' = \varnothing$.

By our assumptions Ψ_1 holds in $\mathfrak{M}[G]$. Let $p \in G$ force Ψ_1. By (Σ1), $p \in \Sigma$. Since X is dense in Σ, there is a $q \in X$, with $q \geqq p$. Let G' be an $\mathfrak{M}$-generic filter on $\mathcal{P}_1$ such that $q \in G'$ and $s' = \{\lambda \in OR \mid \mathfrak{M}[G'] \models \Psi(\underline{\lambda})\} = s$. ($G'$ exists by (Σ4).) Then with notations as in our description of Ψ_1, we have

$\Sigma' = \Sigma$ and $X' = X$ (since Σ' and X' are defined in $\mathfrak{M}[s']$ by the definitions that yield Σ and X in $\mathfrak{M}[s]$, and $s' = s$). But $q \in G' \cap X'$. Thus Ψ_1 is false in $\mathfrak{M}[G']$. But this is absurd since $q \geqq p$, and $p \Vdash \Psi_1$.

Let $\mathcal{P}'_\alpha$ be the following cofinal subset of $\mathcal{P}_\alpha$: $h \in \mathcal{P}'_\alpha$ if and only if $h \in \mathcal{P}_\alpha$ and domain (h) is a finite initial segment of ω.

(Σ6) In $\mathfrak{M}[s]$, Σ is isomorphic to $\mathcal{P}'_\alpha$.

Proof. We work inside $\mathfrak{M}[s]$. Recall that if $\langle h_1, h_2\rangle \in \mathcal{P}_1$, $l(\langle h_1, h_2\rangle) =$ domain (h_1). It follows from (A3) and (Σ2) that if $p \in \Sigma$ and $l(p) = k$, then $\{q \in \Sigma \mid q \geqq p \text{ and } l(q) = k + 1\} = S_p$, has the same cardinality as α. Let ψ_p be a bijection of S_p onto α. Let now $p \in \Sigma$, with $l(p) = n$. We can find p_i, $0 \leqq i \leqq n$, with $p_i \leqq p$, and $l(p_i) = i$. Let $\chi(p)\colon n \to \alpha$ be defined by $\chi(p)(j) = \psi_{p_j}(p_{j+1})$. Then χ is easily seen to be an isomorphism of Σ with $\mathcal{P}'_\alpha$.

We can now easily prove the lemma. Let $\chi[G]$ be the image of G in $\mathcal{P}'_\alpha$. By (Σ5), (Σ6), and Lemma 2.1, $\mathfrak{M}[G] = \mathfrak{M}[s][G] = \mathfrak{M}[s][\chi[G]]$. Moreover, $\chi[G]$ is an $\mathfrak{M}[s]$-generic filter on $\mathcal{P}'_\alpha$. Hence if we put

$$F = \bigcup \chi[G] ,$$

then F is an $\mathfrak{M}[s]$-generic collapsing map of ω onto α. Since clearly, $\mathfrak{M}[s][F] = \mathfrak{M}[s][\chi[G]] = \mathfrak{M}[G] = \mathfrak{M}[F_1, F_2]$, the lemma is proved.

4.5. We can now easily prove Theorem 4.1. Let $\mathfrak{N}$ be as in Theorem 4.1, and G a generic filter on $\mathcal{P}^\Omega$ such that $\mathfrak{N} = \mathfrak{M}[G]$. Let $f\colon \omega \to OR$, $f \in \mathfrak{N}$. By Lemma 3.4, we have $f \in \mathfrak{M}[G^\beta]$, where $G^\beta = G \cap \mathcal{P}^\beta$ and $\beta < \Omega$. We may as well suppose that $\omega \leqq \beta$; put $\alpha = \beta + 2$.

We have an obvious isomorphism

$$\mathcal{P}^\Omega = \mathcal{P}^\alpha \times \mathcal{R}^\alpha .$$

Here $\mathcal{R}^\alpha = \{f \in \mathcal{P}^\Omega \mid \text{domain } (f) \cap \alpha \times \omega = \varnothing\}$. Hence, by Lemma 2.3, $\mathfrak{N} = \mathfrak{M}[G^\alpha][G_1]$, where $G^\alpha = G \cap \mathcal{P}^\alpha$ is $\mathfrak{M}$-generic and G_1 is an $\mathfrak{M}[G_1]$-generic filter on $\mathcal{R}^\alpha$.

We have $\mathcal{P}^\alpha = \mathcal{P}^{\beta+1} \times \mathcal{P}_{\beta+1}$, up to canonical isomorphism. Hence by Lemma 4.2.1 and Lemma 4.3, there are generic collapsing maps $F_1\colon \omega \to \beta$, $F_2\colon \omega \to \beta$ such that

(1) F_1 is $\mathfrak{M}$-generic and $\mathfrak{M}[F_1] = \mathfrak{M}[G^{\beta+1}]$. (Here $G^\xi = G \cap \mathcal{P}^\xi$.)

(2) F_2 is $\mathfrak{M}[F_1]$-generic and $\mathfrak{M}[F_1, F_2] = \mathfrak{M}[G^\alpha]$.

We now apply Lemma 4.4 with f in the role of s. We get an $\mathfrak{M}[f]$-generic collapsing map $F_3\colon \omega \to \beta$ such that $\mathfrak{M}[G^\alpha] = \mathfrak{M}[f][F_3]$.

We are now almost home. By Lemma 4.3 (and Lemma 4.2.1) there is an $\mathfrak{M}[f]$-generic filter, G_2, on $\mathcal{P}^\alpha$ such that

$$\mathfrak{M}[f][G_2] = \mathfrak{M}[f][F_3] = \mathfrak{M}[G^\alpha] .$$

Apply Lemma 2.3 to G_2, G_1 and the isomorphism

$$\mathcal{P}^{\Omega} = \mathcal{P}^{\alpha} \times \mathcal{R}^{\alpha} .$$

We get an $\mathfrak{M}[f]$-generic filter G' on $\mathcal{P}_{\Omega}$ such that $\mathfrak{M}[f][G'] = \mathfrak{M}[f, G_2, G_1] = \mathfrak{M}[G^{\alpha}, G_1] = \mathfrak{N}$. This proves Theorem 4.1.

4.6. Lemma. *Let $\mathfrak{M}$, $\mathfrak{N}$, Ω be as above. Let $\omega \leqq \alpha < \Omega$. Let G_1 be an $\mathfrak{M}$-generic filter on $\mathcal{P}^{\alpha+1}$ with $G_1 \in \mathfrak{N}$. Then there is an $\mathfrak{M}$-generic filter G_2 on $\mathcal{P}^{\Omega}$ with*

$$\mathfrak{N} = \mathfrak{M}[G_2]$$

and $G_1 = G_2 \cap \mathcal{P}^{\alpha+1}$.

Proof. By Lemma 4.3 and Theorem 4.1, there is an $\mathfrak{M}[G_1]$-generic filter G_3 on $\mathcal{P}^{\Omega}$ with $\mathfrak{N} = \mathfrak{M}[G_1, G_3]$.

We now write

(a) $\mathcal{P}^{\Omega} = \mathcal{P}^{\alpha+1} \times \mathcal{P}_{\alpha+1} \times \mathcal{R}_{\alpha+2}$.

Applying § 2.3, we see that G_3 determines filters G_4, G_5, G_6 on $\mathcal{P}^{\alpha+1}$, $\mathcal{P}_{\alpha+1}$, $\mathcal{R}_{\alpha+2}$ such that G_4, G_5, G_6 are generic over $\mathfrak{M}[G_1]$, $\mathfrak{M}[G_1, G_4]$, $\mathfrak{M}[G_1, G_4, G_5]$ respectively, and

$$\mathfrak{N} = \mathfrak{M}[G_1, G_4, G_5, G_6] .$$

By § 4.2 and § 4.3, there is an $\mathfrak{M}[G_1]$-generic filter G_7 on $\mathcal{P}_{\alpha+1}$ such that

$$\mathfrak{M}[G_1, G_7] = \mathfrak{M}[G_1, G_4, G_5] .$$

We now again apply § 2.3 to the isomorphism (a) and get an $\mathfrak{M}$-generic filter G_2 on $\mathcal{P}^{\Omega}$ with

$$\mathfrak{M}[G_2] = \mathfrak{M}[G_1, G_7, G_6] = \mathfrak{N}$$

and $G_2 \cap \mathcal{P}^{\alpha+1} = G_1$. This proves the lemma.

II. The Concept of a Random Real

We first discuss, in II.1, the relation between Borel sets of a countable transitive model $\mathfrak{M}$ and Borel sets of the real world. This is a preliminary to a study of the key concept of this paper, the concept of a random real. This is our main tool in adapting Cohen's method to measure theoretic problems.

1. Extending Borel sets

We let **DC** be the principle of dependent choices. A precise statement of **DC** will be given in III. For our present purposes it suffices to know the following:

(1) All the positive results of measure theory and point set topology on the real line (such as the existence of Lebesgue measure and the Baire category theorem) can be proved in **ZF** + **DC**.

(2) **DC** justifies a countable sequence of consecutive choices. In particular, it has the following corollary (known as $\mathbf{AC}_\omega$ or the countable axiom of choice):

Let $\{A_i: i \in \omega\}$ be a sequence of non-empty sets. Then there is a function f with domain ω such that $f(i) \in A_i$.

Throughout this section II.1 all theorems of mathematical nature (i.e., theorems not relating to models of set theory) will be theorems of **ZF** + DC. Therefore they will hold in any model $\mathfrak{M}$ of **ZF** + **DC**.

1.1. We use functions from ω to ω to code (or Gödel number) Borel subsets of **R**.[6]

Let $\{r_i\}$ be an arithmetical enumeration of **Q**; let J be the pairing function

$$J(a, b) = 2^a(2b + 1) .$$

The coding is defined recursively as follows:

Definition. (1) α codes $[r_i, r_j]$ if $\alpha(0) \equiv 0 \pmod 3$, $\alpha(1) = i$, and $\alpha(2) = j$.

(2) Suppose α_i codes B_i, $i = 0, 1, 2, \cdots$; then α codes $\bigcup_i B_i$ if $\alpha(0) \equiv 1 \pmod 3$ and

$$\alpha(J(a, b)) = \alpha_a(b) .$$

(3) Suppose β codes B, $\alpha(0) \equiv 2 \pmod 3$ and $\alpha(n + 1) = \beta(n)$. Then α codes the complement of B.

(4) α codes B only as required by (1)-(3).

LEMMA 1. *Suppose α codes B_1, and α codes B_2. Then $B_1 = B_2$.*

PROOF. Let $I = \{\langle\alpha, B\rangle: \alpha \text{ codes } B\}$. Let $I_1 = \{\langle\alpha, B\rangle: \alpha \text{ codes only } B\}$. Then I_1 is closed under (1)-(3) of Definition 1. By (4), $I = I_1$, q.e.d.

We write B_α for the Borel set coded by α. If $\alpha \in \mathfrak{M}$ and α codes a Borel set in $\mathfrak{M}$, we denote this set with $B_\alpha^{\mathfrak{M}}$.

LEMMA 2. *Every Borel set is coded by some α.*

PROOF. The family of sets coded by some α is closed under complements and countable unions (DC!) and contains all sets $[r, s]$ with rational endpoints. Thus it contains all Borel sets.

LEMMA 3. *Every set coded by an α is a Borel set.*

PROOF. (Similar to proof of Lemma 1 and left to the reader.)

[6] **R** is the field of real numbers; **Q** is the field of rational numbers.

1.2. THEOREM. *There are Π_1^1 predicates $A_1(\alpha)$, $A_2(\alpha, x)$, $A_3(\alpha, x)$ which are provably equivalent, (in* **ZF** + **DC**) *to the following concepts:*

(1) *α codes a Borel set;*

(2) *α codes a Borel set and $x \in B_\alpha$;*

(3) *α codes a Borel set and $x \notin B_\alpha$.*

PROOF. We let $\{s_n\}$ be a recursive enumeration without repetitions of the finite sequences of integers (including the void sequence), arranged so that if the sequence s_n is an initial segment of the sequence s_m, then $n \leqq m$. (Thus s_0 is the void sequence.) We define a function $\Phi(\alpha, n)$, taking values in ω^ω as follows:

(1) $n = 0$. Then $\Phi(\alpha, n) = \alpha$.

(2) $n > 0$. Then s_n is a non-empty sequence, of length k, say. Let s_m be the initial segment of s_n of length $k - 1$, and let r be the last element of s_n. (So $r \in \omega$.) Note that $m < n$.

Case 2.1. $\Phi(\alpha, m)(0) \equiv 0 \pmod 3$. Then put $\Phi(\alpha, n)$ equal to the identically zero function.

Case 2.2. $\Phi(\alpha, m)(0) \equiv 1 \pmod 3$. Then put

$$\Phi(\alpha, n)(x) = \Phi(\alpha, m)(J(r, x)) , \qquad (x \in \omega) .$$

(Here r, m are as in the preceding paragraph, and J is defined in § 1.1.)

Case 2.3. $\Phi(\alpha, m)(0) \equiv 2 \pmod 3$. Then put

$$\Phi(\alpha, n)(x) = \Phi(\alpha, m)(x + 1) .$$

($\Phi(\alpha, \cdot)$ allows us to recover those Borel sets from which B_α is constructed.)

The following lemma is easily checked by induction on m.

LEMMA 1. *Let α code a Borel set. Then for all m, $\Phi(\alpha, m)$ codes a Borel set.*

Let $\beta: \omega \to \omega$. Define a function $\bar{\beta}: \omega \to \omega$ by

$$s_{\bar{\beta}(n)} = \langle \beta(0), \cdots, \beta(n-1) \rangle .$$

(Here the right hand side denotes the finite sequence consisting of the first n members of β.)

We can now define the Π_1^1 predicate, A_1

$$A_1(\alpha) \equiv (\beta)(\exists n)\Phi(\alpha, \bar{\beta}(n)) = 0 .$$

An argument similar to the proof of Lemma 1.1.1 shows that if α codes a Borel set, then $A_1(\alpha)$ holds. Conversely, suppose that α fails to code a Borel set. Then one can construct a function $\beta: \omega \to \omega$ by induction on n, so that for all n, $\Phi(\alpha, \bar{\beta}(n))$ fails to code a Borel set. But then, for all n, $\Phi(\alpha, \bar{\beta}(n))(0) \neq 0$ (since otherwise, $\Phi(\alpha, \bar{\beta}(n))$ codes by Case 1 of the definition (§ 1.1)).

Now suppose that α codes a Borel set, and that x is a real. We define a function $\gamma: \omega \to \omega$, as follows: if x lies in the Borel set coded by $\Phi(\alpha, n)$, then $\gamma(n) = 1$. Otherwise, $\gamma(n) = 0$. (Lemma 1.2.1 states that $\Phi(\alpha, n)$ always codes a Borel set.)

LEMMA 2. *There is an arithmetic predicate, $A_4(\alpha, \beta, x)$ such that $A_4(\alpha, \beta, x)$ holds if and only if β is the function γ of the preceding paragraph.*

PROOF. We can describe A_4 as follows:

(1) Suppose that $\Phi(\alpha, n)(0) \equiv 0 \pmod 3$. Let $\Phi(\alpha, n)(1) = i$, $\Phi(\alpha, n)(2) = j$. Then $\beta(n) = 1$ if and only if $x \in [r_i, r_j]$.

(2) Suppose that $\Phi(\alpha, n)(0) \equiv 1 \pmod 3$. Let $\langle j \rangle$ be the length one sequence whose one element is j; let $s_{\varphi(n,j)}$ be the concatenation

$$s_n \frown \langle j \rangle .$$

Then $\beta(n) = 1$ if and only if for some j, $\beta(\varphi(n, j)) = 1$.

(3) Suppose that $\Phi(\alpha, n)(0) \equiv 2 \pmod 3$. Then $\beta(n) = 1$ if and only if $\beta(\varphi(n, 0)) = 0$.

(4) $\beta(n) = 0$ or 1 for all n.

It is clear that the predicate $A_4(\alpha, \beta, x)$ is arithmetic, and that if γ is as in the paragraph prior to the statement of Lemma 2, then $A_4(\alpha, \gamma, x)$ holds.

Now suppose that α codes a Borel set, that x is a real, that γ is as above, and that $\gamma': \omega \to \omega$ is such that $A_4(\alpha, \gamma', x)$. We want to show $\gamma' = \gamma$. Suppose not. Then for some n, $\gamma'(n) \neq \gamma(n)$. Say s_n has length r. Then we can define a function $\delta: \omega \to \omega$ with the following property.

(1) $\bar{\delta}(r) = n$

(2) If $m \geqq r$,

$$\gamma(\bar{\delta}(m)) \neq \gamma'(\bar{\delta}(m)) .$$

(One defines $\delta(m)$ for $m \geqq r$, by induction on m so that (2) holds. Indeed, if $\gamma(\bar{\delta}(m)) \neq \gamma'(\bar{\delta}(m))$, we see first that

$$\Phi(\alpha, \bar{\delta}(m))(0) \not\equiv 0 \pmod 3 .$$

(Otherwise $\gamma(\bar{\delta}(m)) = \gamma'(\bar{\delta}(m))$ by clause (1) of the definition of A_4.) Moreover, by clauses (2) and (3), of the definition of A_4, we see that for at least one extension s_n of $\bar{\delta}(m)$, of length $m + 1$, we have $\gamma(n) \neq \gamma'(n)$. We now select $\delta(m)$ so that $\bar{\delta}(m + 1) = n$.)

We have already remarked that since

$$\gamma(\bar{\delta}(n)) \neq \gamma'(\bar{\delta}(n))$$

for all $n \geqq r$, we have

$$\Phi(\alpha, \bar{\delta}(n))(0) \neq 0 , \qquad \text{for } n \geqq r.$$

If $\Phi(\alpha, \bar{\delta}(n))(0) = 0$ for some n less than r, then it would also be zero for all larger n (cf. Case 2.1 of the definition of Φ). Thus

$$(\forall n)\Phi(\alpha, \bar{\delta}(n)) \neq 0 ,$$

i.e., $A_1(\alpha)$ is false. We have already shown that this implies that α fails to code a Borel set. This contradicts our assumption on α, and shows that $\gamma = \gamma'$. The proof of Lemma 2 is complete.

We now define $A_2(\alpha, x)$ as follows:

$$A_2(\alpha, x) \equiv (\beta)(A_4(\alpha, \beta, x) \rightarrow \beta(0) = 1) \wedge A_1(\alpha) .$$

Clearly A_2 is Π_1^1. Let α code a Borel set. Let x be a real, and let γ be as in the statement of Lemma 2. Then $A_2(\alpha, x)$ if and only if $\gamma(0) = 1$, by Lemma 2. Moreover, $\gamma(0) = 1$ if and only if x lies in the Borel set coded by $\Phi(\alpha, 0)$. But $\Phi(\alpha, 0) = \alpha$. Thus $A_2(\alpha, x)$ if and only if α codes a Borel set and x lies in the Borel set coded by α.

The treatment of A_3 is similar. We put

$$A_3(\alpha, x) \equiv (\beta)(\Phi(\alpha, \beta, x) \rightarrow \beta(0) = 0) \wedge A_1(\alpha) .$$

COROLLARY. *There are* Π_1^1 *predicates* $A_4(\alpha, \beta)$, $A_5(\alpha, \beta)$ *which are provably equivalent in* **ZF** + **DC** *to the following concepts*

(4) $B_\alpha \subseteq B_\beta$.

(5) $B_\alpha = B_\beta$.

PROOF. Put

$$A_4(\alpha, \beta) \equiv A_1(\alpha) \wedge A_1(\beta) \wedge (x)(A_3(\alpha, x) \vee A_2(\beta, x))$$

and

$$A_5(\alpha, \beta) \equiv A_4(\alpha, \beta) \wedge A_4(\beta, \alpha) .$$

This suffices.

1.3. Kleene has shown that there is an extremely close relation between Π_1^1 relations and the concept of well-orderings (cf. [7]). Moreover, if $\mathfrak{M}$ is a transitive model of **ZF**, then the ordinals of $\mathfrak{M}$ are an initial segment of the ordinary ordinals (cf. [1, p. 94]). Putting these facts together, one has the following lemma (cf. [13, pp. 137-138]).

LEMMA. *Let* $\Phi(\alpha)$ *be a* Π_1^1 *predicate. Let* $\mathfrak{M}$ *be a transitive model of* **ZF**. *Let* $\alpha\colon \omega \rightarrow \omega$, *be an element of* $\mathfrak{M}$. *Then*

$$\mathfrak{M} \models \Phi(\alpha)$$

if and only if $\Phi(\alpha)$ *holds in the real world.*

1.4. We have two situations to consider simultaneously.

(a) $\mathfrak{M}$ and $\mathfrak{N}$ are transitive models of **ZF** + **DC**, and $\mathfrak{M} \subseteq \mathfrak{N}$;

(b) $\mathfrak{M}$ is a transitive model of **ZF** + **DC**, and $\mathfrak{N}$ is the universe of all sets (so the axiom of choice holds in $\mathfrak{N}$).

THEOREM. *Let* $\mathfrak{M}$, $\mathfrak{N}$ *be as in* (a) *or* (b) *above. Let* $\alpha, \beta \in (\omega^\omega)_{\mathfrak{M}}$, *and* $x \in \mathbf{R}_{\mathfrak{M}}$. *Then the following statements hold in* $\mathfrak{M}$ *if they hold in* $\mathfrak{N}$.

(1) α *codes a Borel set.*

(2) α *codes a Borel set,* B_α, *and* $x \in B_\alpha$.

(3) α, β *code Borel sets and* $B_\alpha = B_\beta$.

PROOF. Case (a) of the lemma follows easily from case (b). Case (b) follows from Lemma 1.3 and the results of § 1.2.

1.5. Theorem 1.4 implies that the assignment

$$\{B_\alpha^{\mathfrak{M}} \to B_\alpha^{\mathfrak{N}}\} \tag{1}$$

gives a 1-1 correspondence between the Borel sets of reals in $\mathfrak{M}$ and a certain subcollection of the Borel sets of reals in $\mathfrak{N}$. The map (1) is, in general, not surjective. For example, if $\mathfrak{M}$ is countable, and $\mathfrak{N}$ is the real world, (1) is certainly not surjective.

Let C be a Borel set in $\mathfrak{N}$. We say that C is *rational* over $\mathfrak{M}$, if $C = B_\alpha^{\mathfrak{N}}$ for some code α lying in $\mathfrak{M}$. By part (2) of Theorem 1.4 the Borel set in $\mathfrak{M}$ corresponding to C is then

$$C \cap \mathbf{R}_{\mathfrak{M}} .$$

Similarly, we say that a sequence of Borel sets in $\mathfrak{N}$, $\{C_i\}$, is rational over $\mathfrak{M}$ if there is a sequence of codes, $\{\alpha_i\}$, lying in $\mathfrak{M}$ such that

$$C_i = B_{\alpha_i}^{\mathfrak{N}} .$$

(Note carefully that we require not only that each α_i lie in $\mathfrak{M}$ but that the function $\{i \to \alpha_i\}$ also lie in $\mathfrak{M}$.) Since **DC** holds in $\mathfrak{M}$ it is equivalent to require that there is in $\mathfrak{M}$ a sequence of $\mathfrak{M}$-Borel sets, $\{B_i\}$, such that for each i, B_i corresponds to C_i under (1).

The correspondence just described clearly possesses the following naturality property. Let $\mathfrak{M} \subseteq \mathfrak{N}$ be transitive models of **ZF**+**DC**. Let V be the real world. Let B be a Borel set in $\mathfrak{M}$ and let $B_{\mathfrak{N}}$, B_V be the corresponding Borel sets in $\mathfrak{N}$ and V respectively. Then B_V is the Borel set in V corresponding to the Borel set $B_{\mathfrak{N}}$ of $\mathfrak{N}$.

1.6. We are going, eventually, to use the map (1) to identify the Borel sets of $\mathfrak{M}$ with certain of the Borel sets of $\mathfrak{N}$. As a temporary piece of notation, if B is a Borel set of $\mathfrak{M}$, we write $B^\sharp$ for the corresponding Borel set of

$\mathfrak{N}$. We proceed to verify that certain properties and operations are "absolute" with respect to the map $\{B \to B^{\sharp}\}$.

LEMMA 1. (1) *Boolean operations are absolute.*

(2) *Let* $\{A_n\}$ *be a sequence of Borel sets of* $\mathfrak{M}$, *with* $\{A_n\} \in \mathfrak{M}$. *Then*

$$(\bigcup_n A_n)^{\sharp} = \bigcup_n A_n^{\sharp}$$
$$(\bigcap_n A_n)^{\sharp} = \bigcap_n A_n^{\sharp} .$$

(3) *The relation* $A \subseteq B$ *is absolute.*

(4) *The relation* $A = \varnothing$ *is absolute.*

PROOF. (1) Consider for example, the intersection operation. Given A, B Borel in $\mathfrak{M}$ with codes α and β respectively. One constructs easily from α and β a code γ which codes $A \cap B$ in $\mathfrak{M}$ and $A^{\sharp} \cap B^{\sharp}$ in $\mathfrak{N}$. Thus

$$(A \cap B)^{\sharp} = A^{\sharp} \cap B^{\sharp} .$$

(2) Similar to the proof of (1).

(3) $A \subseteq B \Leftrightarrow A \cup B = B$. By Theorem 1.4 and (1) of this lemma, we have (3).

(4) Clear from Theorem 1.4.

LEMMA 2. *The following operations and notions are absolute.*

(1) *Interior* (int);

(2) *"Open"*;

(3) *Closure*; (Cl)

(4) *"Closed"*;

(5) *"Closed nowhere dense"*;

(6) *"Compact"*.

PROOF. (1) $A = \operatorname{int} B$ if and only if

$$A = \bigcup \{(r, s) : r, s \in \mathbf{Q} \quad \text{and} \quad (r, s) \subseteq B\}$$

(2) A is open $\Leftrightarrow A = \operatorname{int} A$.

(3) $\operatorname{Cl}(A) = \mathbf{R} - \operatorname{int}(\mathbf{R} - A)$.

(4) A is closed $\Leftrightarrow A = \operatorname{Cl}(A)$.

(5) A is closed nowhere dense $\Leftrightarrow A$ is closed and $\operatorname{int}(A) = \varnothing$.

(6) A is compact if and only if A is closed and for some $N \in \omega, A \subseteq [-N, N]$.

LEMMA 3. *Let* r, s *be reals of* $\mathfrak{M}$. *Then* $(r, s)^{\sharp} = (r, s)$; $[r, s]^{\sharp} = [r, s]$; $\{r\}^{\sharp} = \{r\}$.

PROOF.

$$(r, s) = \bigcup \{[t, u] : r < t \leqq u < s;\ t, u \in \mathbf{Q}\}$$
$$[r, s] = \bigcap \{(r - 1/n, s + 1/n) : n \in \omega\}$$
$$\{r\} = [r, r] .$$

LEMMA 4. *Let μ be Lebesgue measure. Let B be a Borel set in $\mathfrak{M}$. Then* $\mu_{\mathfrak{M}}(B) = \mu_{\mathfrak{N}}(B^{\sharp})$.

Case 1. B is the union of a finite number of disjoint open intervals with rational endpoints.

Say $r_1 < s_1 \leqq r_2 < s_2 \leqq \cdots \leqq r_n < s_n$, and $B = \bigcup_{i=1}^{n} (r_i, s_i)$ in both $\mathfrak{M}$ and $\mathfrak{N}$. Then $\mu(B) = \sum_{i=1}^{n} (s_i - r_i)$, which is absolute.

There are clearly only countably many sets of the sort considered in case 1; let $\{W_n\}$ be an enumeration of these sets in $\mathfrak{M}$.

Case 2. B compact.

We have $\mu(B) = \inf \{\mu(W_n): B \subseteq W_n\}$ which proves μ is absolute in this case.

Case 3. B open.

Clear since $\mu(B) = \sup \{\mu(W_n): W_n \subseteq B\}$.

Case 4. B arbitrary.

$\mu_{\mathfrak{M}}(B) = \sup\{\mu(K): K$ compact, $K \subseteq B$, and K rational over $\mathfrak{M}\} \leqq \sup\{\mu(K)$: K compact, $K \subseteq B^{\sharp}$, and K rational over $\mathfrak{N}\} = \mu_{\mathfrak{N}}(B^{\sharp})$.

Similarly $\mu_{\mathfrak{M}}(B) = \inf \{\mu(U): U$ open, $B \subseteq U$ and U rational over $\mathfrak{M}\} \geqq \inf \{\mu(U)$: open, $B^{\sharp} \subseteq U$ and U rational over $\mathfrak{N}\} = \mu_{\mathfrak{N}}(B^{\sharp})$. These two inequalities show $\mu_{\mathfrak{M}}(B) = \mu_{\mathfrak{N}}(B^{\sharp})$.

COROLLARY. *"Set of measure zero" is an absolute notion.*

Recall that the symmetric difference of two sets, A and B, (notation: $A \triangle B$) is

$$(A - B) \cup (B - A) .$$

We recall some elementary constructions from the theory of boolean algebras. This material is all contained in Halmos [4].

Let $\mathcal{I}$ be an ideal of subsets of **R**. This means that if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$, and if $A \in \mathcal{I}$, and $B \subseteq A$, then $B \in \mathcal{I}$. We say that two sets A and B are equal mod $\mathcal{I}$ if

$$A \triangle B \in \mathcal{I} .$$

Equality mod $\mathcal{I}$ is an equivalence relation. The set of equivalence classes of Borel sets is, in a natural way, a boolean algebra, since the boolean operations "pass to quotients".

If $\mathcal{I}$ is a σ-ideal (i.e., is closed under countable unions), then the boolean algebra of equivalence classes mod $\mathcal{I}$ is a σ-algebra.

The two basic examples of σ-ideals in the power set of **R** are:

(1) the σ-ideal $\mathcal{I}_1$ of sets of Lebesgue measure zero;

(2) the σ-ideal $\mathcal{I}_2$ of sets of the first category.

Two Borel sets equal mod $\mathcal{I}_1$ are said to be equal "almost everywhere"; two Borel sets equal mod $\mathcal{I}_2$ are "equal modulo a set of the first category".

The following lemma will be useful in a moment. For the proof see Halmos [4, p. 58].

LEMMA 5. *Let B be Borel. Then B is equal to an open set U modulo a set of the first category.*

LEMMA 6. *"First category" is an absolute notion.*

PROOF. Using Lemma 1 parts (2) and (3), and Lemma 2 part (5), we see that if B is first-category in $\mathfrak{M}$, B is first-category in $\mathfrak{N}$.

Suppose now that B is not first category in $\mathfrak{M}$. By Lemma 5 in $\mathfrak{M}$, there exists U open, rational over $\mathfrak{M}$ such that $B \triangle U$ is first-category in $\mathfrak{M}$. By the preceding paragraph, $B \triangle U$ is first-category in $\mathfrak{N}$. If $U = \emptyset$, $B = B \triangle U$ and so B is first category in $\mathfrak{M}$. Thus $U \neq \emptyset$, and by the Baire category theorem U is not first category in $\mathfrak{N}$. Since

$$U \subseteq B \cup (B \triangle U) ,$$

B is not first category in $\mathfrak{N}$. The proof is complete.

The following lemma is not needed in the present paper but will be used in another paper of the author [15].

LEMMA 7. *The following notions are absolute.*

(1) *X has at least two points.*

(2) *X has exactly one point.*

(3) *X is perfect.*

(4) *X is countable.*

PROOF. (1) X has at least two points if and only if there exist rationals: $r < s < t < u$ such that $X \cap (r, s) \neq \emptyset$; $X \cap (t, u) \neq \emptyset$.

(2) Immediate from (1).

(3) X is perfect if and only if X is closed, $X \neq \emptyset$, and for all $r, s \in \mathbf{Q}$, $X \cap (r, s) \neq \emptyset \Rightarrow X \cap (r, s)$ has at least two points.

(4) By (2) and (2) of Lemma 1, X countable in $\mathfrak{M}$ implies X countable in $\mathfrak{N}$. If X is not countable in $\mathfrak{M}$, X contains a perfect subset K. In $\mathfrak{N}$, K is perfect (by (3)) and $K \subseteq X$. Thus X is uncountable in $\mathfrak{N}$.

1.7. The following concept will be needed in § 2 below.

Let $C = \{\alpha \mid \alpha \text{ codes a real}\}$. Let

$$\lambda: C \rightarrow OR$$

be defined as follows.

(1) If α codes by case 1 of Definition 1.1, then $\lambda(\alpha) = 0$.

(2) If α codes by case 2, then

$$\lambda(\alpha) = \sup \{\lambda(\alpha_i) + 1\}$$

(notation as in case 2 of Definition 1.1).

(3) If α codes by case 3,

$$\lambda(\alpha) = \lambda(\beta) + 1 .$$

It is easy to see that domain $\lambda = C$ by an argument similar to the proof of Lemma 1.1.1.

We write $\lambda^{\mathfrak{M}}$ for the relativization of λ to $\mathfrak{M}$.

1.8. Our results would apply to any of the standard spaces, such as 2^ω, *mutatis mutandis.*

1.9. Let $j = 1$ or 2. Let $\mathcal{I}_j$ be the ideal described in § 1.6. Let $\mathcal{B}_j$ be the quotient algebra of the σ-algebra of Borel sets associated to the ideal $\mathcal{I}_j$. Then the following facts are proved in [4]: $\mathcal{B}_j$ is a complete boolean algebra satisfying the countable chain condition.

2.1. Let $\mathfrak{M}$ be a fixed transitive model of ZFC. A real x is *random* over $\mathfrak{M}$ if it lies in no Borel set of measure zero rational over $\mathfrak{M}$. Similarly a subset of ω is *random* over $\mathfrak{M}$ if it lies in no Borel set of measure zero of 2^ω rational over $\mathfrak{M}$.

Notice that if x is random over $\mathfrak{M}$, then $x \notin \mathfrak{M}$. (*Proof.* $x \in \mathfrak{M} \Rightarrow \{x\}$ is a Borel set of measure zero, rational over $\mathfrak{M}$, and containing x.) This definition is in accord with the usual intuitive requirements for a random real. For example if we let $\xi(x, N)$ be the number of 1's in the first N digits of the decimal expansion of x, then for x random the limit as $N \to \infty$ of $\xi(x, N)/N$ exists and equals 1/10. (A proof would show that the set of reals x for which this is false form a Borel set of measure zero rational over $\mathfrak{M}$.)

The following lemma provides for the existence of many random reals (if, for example, $\mathfrak{M}$ is countable):

LEMMA. *If* $(2^{\aleph_0})_{\mathfrak{M}}$ *is countable, then almost all reals are random over* $\mathfrak{M}$. (*In fact, the non-random reals form a Borel set of measure zero.*)

PROOF. If $(2^{\aleph_0})_{\mathfrak{M}}$ is countable, we can enumerate the Borel sets of measure zero rational over $\mathfrak{M}$ in a sequence $N_0, N_1, \cdots$. Then x is random over $\mathfrak{M}$ if and only if $x \notin \bigcup_i N_i$. But $\bigcup_i N_i$ is a Borel set of measure zero.

2.2. There is an analogous notion of a real (or set of integers) being generic over $\mathfrak{M}$. A real x is *generic* over $\mathfrak{M}$ if it lies in no Borel set of the first category rational over $\mathfrak{M}$. We shall see below that this is essentially the same as the notion introduced by Cohen. (Cohen worked with sets of integers; for reals, the conditions analogous to Cohen's have the form $r < x < s$, where r, s are rationals.)

It is true that no real x is both generic and random over $\mathfrak{M}$. We shall not stop to prove this here. (For example, the set of reals x in which 1 has the frequency 1/10 in the decimal expansion of x form a set of first category, rational over $\mathfrak{M}$, and containing all random reals.)

All the results we prove for random reals in this section have analogues for generic reals with "the same proofs". The translation consists in replacing "random" by "generic" and "measure zero" by "first category". We leave this translation to the reader.

2.3. We are going eventually to show that the random reals are in natural one-one correspondence with the generic filters on a certain partially ordered set $\mathcal{P} \in \mathfrak{M}$. The following discussion is a heuristic motivation for the correct choice of $\mathcal{P}$.

Let x be a real random over $\mathfrak{M}$. An observer stationed in $\mathfrak{M}$ cannot have total knowledge about x (since x is not in $\mathfrak{M}$). However, he can have partial knowledge about x. For example, if B is a Borel set rational over $\mathfrak{M}$, then a natural question the observer can ask about x is "Is $x \in B$?" If $\mu(B) = 0$, then the answer is certainly no. On the other hand, if $\mu(B) > 0$, it is possible for x to be in B (cf. Lemma 2.1). A similar discussion shows that if B_1 and B_2 are Borel sets rational over $\mathfrak{M}$, and $\mu(B_1 \triangle B_2) = 0$ (i.e., B_1 and B_2 are equal almost everywhere, then for x random over $\mathfrak{M}$, the questions "Is $x \in B_1$?" and "Is $x \in B_2$?" are equivalent.

We therefore make the following definition.

Definition 1. $\mathcal{P}$ is the set of equivalence classes of non-null Borel sets of reals (in $\mathfrak{M}$). Two sets B_1 and B_2 are equivalent if and only if $B_1 \triangle B_2$ has measure zero. (Let $[B]$ be the equivalence class of B. We think of the condition $[B]$ as telling us $x \in B$.)

We order $\mathcal{P}$ by an order $\leqq$ as follows. $[B] \leqq [B']$ if and only if $B' \subseteq B$ almost everywhere (i.e., $B' - B$ is a set of measure zero).

2.4. Let $\mathcal{B} \in \mathfrak{M}$. Then clearly $\mathcal{B}$ is a boolean algebra in $\mathfrak{M}$ if and only if $\mathcal{B}$ is a boolean algebra in the real world. However, $\mathcal{B}$ can be complete in $\mathfrak{M}$ (cf. [4, p. 25]) without being complete in the real world, since there may be subsets $S \subseteq \mathcal{B}$ such that $\sup S$ is not defined, but $S \in \mathfrak{M}$. We say that $\mathcal{B}$ is $\mathfrak{M}$-*complete* if and only if $\mathfrak{M} \models \mathcal{B}$ is complete.

Let now $\mathcal{B} \in \mathfrak{M}$ be a boolean algebra, and

$$h: \mathcal{B} \rightarrow 2$$

be a homomorphism. (We *do not* require that h lie in $\mathfrak{M}$.) We say that h is *$\mathfrak{M}$-completely additive* if whenever $S \subseteq \mathcal{B}$, $S \in \mathfrak{M}$, and $\sup S$ exists in $\mathcal{B}$, then

$$h(\sup S) = \sup \{h(s): s \in S)\} .$$

Let $\mathcal{B}$ be a boolean algebra lying in $\mathfrak{M}$. Let $\leqq$ be the usual order on $\mathcal{B}$: $b_1 \leqq b_2$ if and only if $b_1 \vee b_2 = b_2$. Let $\mathcal{P}$ be the set of non-zero elements of $\mathcal{B}$. We provide $\mathcal{P}$ with the order $<$ inverse to $\leqq$: $b_1 < b_2$ if and only if $b_2 \leqq b_1$.

LEMMA. *Let $h: \mathcal{B} \rightarrow 2$ be an $\mathfrak{M}$-completely additive homomorphism. Then*

$$(1) \qquad G = \{b \mid h(b) = 1\}$$

is an $\mathfrak{M}$-generic filter on $\mathcal{P}$. Conversely, if G is an $\mathfrak{M}$-generic filter on $\mathcal{P}$, there is a unique homomorphism

$$h: \mathcal{B} \rightarrow 2$$

such that (1) *holds.*

PROOF. Let $h: \mathcal{B} \rightarrow 2$ be a homomorphism. Then if G is defined by (1), then G satisfies clauses (1) and (2) of Definition I.1.3. Conversely, if $G \subseteq \mathcal{P}$ satisfies clauses (1) and (2) of Definition I.1.3, then G is, in the usual terminology of the theory of boolean algebras, a filter on $\mathcal{B}$. If G also satisfies (3) of Definition I.1.3 then G is an ultrafilter on $\mathcal{B}$. (For any $b_0 \in \mathcal{B}$, the set $\{b \in \mathcal{P}: b \leqq b_0$ or $b \cdot b_0 = 0\}$ is dense in $\mathcal{P}$.) Hence there is a unique homomorphism $h: \mathcal{B} \rightarrow 2$ such that (1) holds.

Now let X be dense in $\mathcal{P}$, $X \in \mathfrak{M}$. We say $\sup_{\mathcal{B}}(X) = 1_{\mathcal{B}}$. Otherwise there is a $b > 0$ with $b \cdot x = 0$, for all $x \in X$. But X is dense, so for some $x_0 \in X$, $x_0 \leqq b$. But then $b \cdot x_0 = x_0 \neq 0$. This contradiction proves our claim.

It follows that if $h: \mathcal{B} \rightarrow 2$ is $\mathfrak{M}$-completely additive and X is as above, we have $h(x) = 1$ for some $x \in X$. Thus $x \in G \cap X$, and G satisfies clause 3 of Definition I.1.3.

Conversely, suppose that G is $\mathfrak{M}$-generic. Let $S \subseteq \mathcal{B}$, $S \in \mathfrak{M}$, $\sup S = s_0$. We want to show

$$(2) \qquad h(s_0) = \sup \{h(s): s \in S\} .$$

We may as well suppose that $s_0 = 1_{\mathcal{B}}$. (Otherwise, replace S by $S \cup \{1_{\mathcal{B}} - s_0\}$.) Let

$$X = \{a \in \mathcal{P} \mid a \leqq b \text{ for some } b \in S\} .$$

Then X is dense in $\mathcal{P}$ (cf. Definition I.1.2). Indeed, clause (1) of the definition of "dense" is obvious; we verify clause (2). Let $x \in \mathcal{P}$. Then $x \neq 0$. Since

$\sup S = 1$, $x \wedge s \neq 0$ for some $s \in S$. Hence $x \wedge s \in X$, and $x < x \wedge s$. This verifies clause (2).

Since X is dense in $\mathcal{P}$, and X lies in $\mathfrak{M}$, we have $G \cap X \neq \emptyset$. Hence there is an $a \in G$ and a $b \in S$ with $a \leqq b$. Since $h(a) = 1$, we have $h(b) = 1$. Since $b \in S$, and $s_0 = 1$, (2) is proved.

2.5. We now suppose in addition that $\mathcal{B}$ is $\mathfrak{M}$-complete, and that only countably many subsets of $\mathcal{B}$ lie in $\mathfrak{M}$.

LEMMA. *Let Φ be a sentence of $\mathcal{L}'$. Then there is a $b_0 \in \mathcal{B}$ such that if G is an $\mathfrak{M}$-generic filter on $\mathcal{P}$, and $h: \mathcal{B} \to 2$, then $\mathfrak{M}[G] \models \Phi$ if and only if $h(b_0) = 1$. Moreover, b_0 is uniquely determined by Φ.*

PROOF. Let

$$S = \{b \in \mathcal{P} \mid b \Vdash \Phi\} .$$

Since forcing is expressible in $\mathfrak{M}$, $S \in \mathfrak{M}$. Let $b_0 = \sup S$. (If $S = \emptyset$, we take $b_0 = 0$, and the lemma is clear. So we may assume $S \neq \emptyset$.) We maintain $b_0 \Vdash \Phi$. Otherwise, there is a $c \in \mathcal{P}$ with

$$0 < c \leqq b_0 ,$$

i.e., $b_0 < c$, and $c \Vdash \neg \Phi$. Since $b_0 = \sup S$, there is a $b_1 \in S$ with $b_1 \wedge c \neq 0$. But then

$$c < b_1 \wedge c$$

so $b_1 \wedge c \Vdash \neg \Phi$. On the other hand, $b_1 \Vdash \Phi$, since $b_1 \in S$; since $b_1 < b_1 \wedge c$, $b_1 \wedge c \Vdash \Phi$. This contradiction shows $b_0 \Vdash \Phi$.

If $h(b_0) = 1$, then $b_0 \in G$, so $\mathfrak{M}[G] \models \Phi$ (since $b_0 \Vdash \Phi$.) Conversely, if $\mathfrak{M}[G] \models \Phi$, there is a $b_1 \in S \cap G$, so $h(b_1) = 1$, so $h(b_0) = 1$. Thus b_0 has the desired properties.

Suppose now that b_1 has the same relation to Φ as b_0. We show $b_1 = b_0$. Otherwise, let $b_2 = b_1 \triangle b_0$. Then if h is $\mathfrak{M}$-completely additive, $h(b_0) = h(b_1) =$ truth value of Φ, so $h(b_2) = 0$. If $b_1 \neq b_0$, then $b_2 \neq 0$. By I.1.8, there is an $\mathfrak{M}$-generic filter G with $b_2 \in G$. But then, $h(b_2) = 1$, contradicting our remark that $h(b_2) = 0$. This shows $b_1 = b_0$.

2.6. The following theorem provides the link between the abstract material of § 2.4–5 and the concept of a random real introduced in § 2.1.

Let $\mathcal{B}_1$ (resp. $\mathcal{B}_2$) be the algebra of $\mathfrak{M}$-Borel sets modulo the ideal of sets of measure zero (resp. of first category). By § 1.9, these algebras are $\mathfrak{M}$-complete.

THEOREM. *There is a canonical 1-1 correspondence between the reals random over $\mathfrak{M}$ and the $\mathfrak{M}$-completely additive homomorphisms of $\mathcal{B}_1$.*

PROOF. Let h be an $\mathfrak{M}$-completely additive homomorphism of $\mathcal{B}_1 \to \{0, 1\}$. Set

$$x_h = x = \{r \in \mathbf{Q}: h((r, \infty)) = 1\} .$$

(To abbreviate, we sometimes use the same symbol to denote an $\mathfrak{M}$-Borel set and its image in $\mathcal{B}_1$.)

LEMMA 1. *x is an irrational left Dedekind cut in $\mathbf{Q}$.*

PROOF. (1) $x \neq \emptyset$: Since $h((-\infty, \infty)) = 1$, for some $n \in \omega$, we have $h((-n, \infty)) = 1$, so $-n \in x$.

(2) $x \neq \mathbf{Q}$: Since $h(\emptyset) = 0$ and $\emptyset = \bigcap_{n=1}^{\infty}(n, \infty)$, we have $h((n, \infty)) = 0$, for some $n \in \omega$. But then $n \notin x$.

(3) $r_2 < r_1 \in x \Rightarrow r_2 \in x$: Since $(r_1, \infty) \subseteq (r_2, \infty)$, $h((r_1, \infty)) = 1 \Rightarrow h((r_2, \infty)) = 1$.

(4) x is irrational: Let $r \in \mathbf{Q}$. Since $\{r\} = \bigcap_n (r - 1/n, r + 1/n)$, $h((r - 1/n, r + 1/n)) = 0$ for some n. But then $r - 1/n \in x \Leftrightarrow r + 1/n \in x$, so x is not the Dedekind cut centered at r. (Similarly, we see that $x \notin \mathfrak{M}$.)

LEMMA 2. *Let A be a Borel set rational over $\mathfrak{M}$. Then $x \in A$ if and only if $h(A) = 1$.*

PROOF. Let α be a code for A, lying in $\mathfrak{M}$. The proof proceeds by induction on $\lambda^{\mathfrak{M}}(\alpha)$ and is straightforward (cf. § 1.7). (Note that $\lambda^{\mathfrak{M}}(\alpha) \in OR^{\mathfrak{M}} \subseteq OR$, so the induction is legitimate, even though $h \notin \mathfrak{M}$.)

It is now easy to show that x is random over $\mathfrak{M}$. Let N be a set of measure zero rational over $\mathfrak{M}$. Then $x \in N$ if and only if $h([N]) = 1$. But $[N] = 0$ in $\mathcal{B}_1$, so $h([N]) = 0$.

Now suppose that x is random over $\mathfrak{M}$. Define $h_x: \mathcal{B}_1 \to \{0, 1\}$, by $h_x([A]) = 1$ if and only if $x \in A$. (A rational over M.) (To see that h_x is well defined, let A_1 and A_2 be Borel sets rational over $\mathfrak{M}$ such that $[A_1] = [A_2]$. Then $\mu(A_1 \triangle A_2) = 0$ so $x \notin A_1 \triangle A_2$ (since x is random over $\mathfrak{M}$). It follows that $x \in A_1 \equiv x \in A_2$.)

It is not hard to check that h_x is $\mathfrak{M}$-countably additive. The proof that h_x is $\mathfrak{M}$-completely additive is based on the following lemma (Halmos [4, p. 61]).

LEMMA 3. *Let $\mathcal{B}$ be a complete boolean algebra satisfying the countable chain condition. Let $S \subseteq \mathcal{B}$. Then S has a countable subset S_0 such that*

$$\bigvee S = \bigvee S_0 .$$

Let $S \subseteq \mathcal{B}_1$, $S \in \mathfrak{M}$. Since h_x is finitely-additive, $h_x(\bigvee S) \geqq \bigvee\{h_x(s): s \in S\}$.

To get the reverse inequality, we apply Lemma 3, within $\mathfrak{M}$, to $\mathcal{B}_1$. Let $S_0 \subseteq S$, S_0 countable in $\mathfrak{M}$, such that $\bigvee S_0 = \bigvee S$. Since h_x is $\mathfrak{M}$-countably additive.

$$h_x(\bigvee S) = h_x(\bigvee S_0) = \bigvee\{h_x(s): s \in S_0\} \leqq \bigvee\{h_x(s): s \in S\} .$$

The reverse inequality has already been proved. Thus h_x is $\mathfrak{M}$-completely additive.

We have shown that if h is $\mathfrak{M}$-completely additive, x_h is random over $\mathfrak{M}$. Lemma 2 shows that h can be recovered from x_h and the discussion just completed shows that every x random over $\mathfrak{M}$ arises in this way. The theorem is proved. The theorem has an analogue for random elements of 2^ω. (In that case, Lemma 2 is unnecessary.) There is a corresponding theorem, of course, identifying reals generic over $\mathfrak{M}$ with the $\mathfrak{M}$-completely additive homomorphisms of $\mathcal{B}_2$.

2.7. Let $\mathfrak{M}$, $\mathcal{B}_1$, be as above. Let x be a real random over $\mathfrak{M}$, and G the associated filter on $\mathcal{B}_1$. It is clear from the discussion in § 2.6 that $x \in \mathfrak{M}[G]$. Moreover, in view of Lemma 2.6.2 and Lemma 2.4, it is clear that if $\mathfrak{N}$ is any transitive model of **ZFC** with $x \in \mathfrak{M}$ and $\mathfrak{M} \subseteq \mathfrak{N}$, then $G \in \mathfrak{N}$. By I.1.4, we then have $\mathfrak{M}[G] \subseteq \mathfrak{N}$.

Notations being as in the preceding paragraph, we write $\mathfrak{M}[x]$ for $\mathfrak{M}[G]$. Thus the discussion of the preceding paragraph shows that $\mathfrak{M}[x]$ is the minimal transitive model, $\mathfrak{N}$, of **ZFC** such that $\mathfrak{M} \subseteq \mathfrak{N}$ and $x \in \mathfrak{N}$.

Because we know that $\mathfrak{M}[x] = \mathfrak{M}[G]$, it is clear that $\mathfrak{M}[x]$ has the same ordinals as $\mathfrak{M}$. Moreover, using the fact that $\mathcal{B}_1$ satisfies the countable chain condition in $\mathfrak{M}$, it would be easy to show that $\mathfrak{M}$ and $\mathfrak{M}[x]$ have the same cardinals. (This is true for all reals random over $\mathfrak{M}$, and hence for almost all reals.)

2.8. We can now prove our fundamental result about random reals. We alter the language $\mathfrak{L}'$ of Chapter I slightly, by replacing the constant G by a constant $\boldsymbol{x}$. Call the resulting language $\mathfrak{L}''$. If x is a real random over $\mathfrak{M}$, we interpret $\mathfrak{L}''$ in $\mathfrak{M}[x]$ in the obvious way. In particular, we let $\boldsymbol{x}$ denote x.

THEOREM. *Let Φ be a sentence of $\mathfrak{L}''$. Then there is a Borel set A rational over $\mathfrak{M}$ such that for all x random over $\mathfrak{M}$, we have*

$$(1) \qquad \mathfrak{M}[x] \models \Phi \longleftrightarrow x \in A\ .$$

PROOF. Let $h_x\colon \mathcal{B}_1 \to 2$, G_x be the homomorphism and filter determined by x. Since $\mathfrak{M}[x] = \mathfrak{M}[G_x]$ and x is definable in $\mathfrak{M}[x]$ from $\mathcal{B}_1$, G_x, we can find a sentence Φ' of $\mathfrak{L}'$ such that for all x random over $\mathfrak{M}$,

$$\mathfrak{M}[x] \models \Phi \longleftrightarrow \mathfrak{M}[G_x] \models \Phi'\ .$$

By Lemma 2.5, there is an element b_0 of $\mathcal{B}_1$, not depending on x, such that for all x random over $\mathfrak{M}$,

$$\mathfrak{M}[G_x] \models \Phi' \longleftrightarrow h_x(b_0) = 1.$$

Let $\alpha \in \omega^\omega$ code a Borel set in $\mathfrak{M}$ whose equivalence class lies in b_0. Let A be the Borel set coded by α in the real world. Then by Lemma 2.6.2,

$$h_x(b_0) = 1 \longleftrightarrow x \in A .$$

The theorem is proved.

2.9. We show that the notion of a generic subset of ω introduced by Cohen is the same as the notion of "generic" introduced in § 4.2. The result will not be used in this paper, but is included for its historical interest.

We recall the precise definition of "generic" given by Levy in [8]. Let $\mathcal{P}_0$ be the set of Cohen conditions: an element $p \in \mathcal{P}_0$ is a function with domain a finite subset of ω and range a subset of $\{0, 1\}$.

Let $\mathfrak{M}$ be a transitive model of **ZF** + **DC**. A subset D of $\mathcal{P}_0$ is dense if $(\forall p \in \mathcal{P}_0)(\exists p' \in D)(p \subseteq p')$. An element $f \in 2^\omega$ is I-generic over $\mathfrak{M}$ if for each dense $D \in \mathfrak{M}$, there exists $p \in D$ such that $p \subseteq f$. (This definition is essentially that of Levy [8]. It comes, *via* Easton [3], from an idea of the author.) An element $f \in 2^\omega$ is II-generic over $\mathfrak{M}$ if it lies in no first category Borel set rational over $\mathfrak{M}$. We shall prove that the following properties of f are equivalent:

(1) f is I-generic over $\mathfrak{M}$;

(2) f lies in each dense open set rational over $\mathfrak{M}$;

(3) f is II-generic over $\mathfrak{M}$.

Proof. The essential point is that Lemma I.2.2 allows us to relate generic filters on the two different partially ordered sets implicit in the notions "I-generic" and "II-generic". The details are as follows.

Let $\mathcal{B}_2$ be the boolean algebra of $\mathfrak{M}$ defined (in $\mathfrak{M}$) as the quotient of the σ-algebra of Borel subsets of 2^ω modulo the σ-ideal of first category Borel sets.

We shall need an alternative description of $\mathcal{B}_2$. We recall that an open set U is *regular* if and only if U is the interior of the closure of U. Then each element of $\mathcal{B}_2$ is the representative of a unique regular open set. In this way, we get a bijective correspondence between $\mathcal{B}_2$ and the regular open sets rational over $\mathfrak{M}$. This correspondence is order preserving if we order the regular open sets by inclusion.

Let $\mathcal{P}_2$ be the set of non-zero elements of $\mathcal{B}_2$ equipped with the order, $<$, inverse to that of $\mathcal{B}_2$. Thus $\mathcal{P}_2$ is canonically isomorphic to the set of non-void regular open sets rational over $\mathfrak{M}$. Call this latter set $\mathcal{P}_2'$.

Now let G be a generic filter on $\mathcal{P}_2$, x the II-generic element of 2^ω determined by G (cf. § 2.6). Let $G' \subseteq \mathcal{P}_2'$ correspond to G. Using the analogue of Lemma 2.6.2, one sees that

$$\{x\} = \bigcap G' .$$

We map $\mathcal{P}_0$ into $\mathcal{P}_2'$ as follows: if $p \in \mathcal{P}_0$, put $W_p = \{f \in 2^\omega : p \subseteq f\}$.

Then W_p is open-closed and hence regular. It is clearly rational over $\mathfrak{M}$. The map

$$\{p \to W_p\}$$

is order-preserving. ($\mathcal{P}_2'$ carries the order induced by the given order on $\mathcal{P}_2$.) Moreover, the image, $\mathcal{P}_0'$, of $\mathcal{P}_0$ in $\mathcal{P}_2'$ is cofinal since sets of the form W_p form a basis for the open sets in 2^ω.

By definition, an element $x \in 2^\omega$ is I-generic if and only if $\{p \in \mathcal{P}_0 : p \subseteq x\}$ is a generic filter on $\mathcal{P}_0$. It follows that the I-generic elements of 2^ω are obtained as follows: take an $\mathfrak{M}$-generic filter G on $\mathcal{P}_0$, copy it onto a filter G' on $\mathcal{P}_0'$, and take $\bigcap G'$.

Now if G_1 is an $\mathfrak{M}$-generic filter on $\mathcal{P}_0$, there is an $\mathfrak{M}$-generic filter G_2 on $\mathcal{P}_2$ such that $G_1 = \{p : W_p \in G_2'\}$ (cf. Lemma I.2.2). Moreover, all generic filters on $\mathcal{P}_0$ arise in this way. Thus x is I-generic if and only if $x = \bigcap G_1'$ for a generic filter G_1 on $\mathcal{P}_0$ if and only if $x = \bigcap G_2'$ for some generic filter G_2 on $\mathcal{P}_2$ if and only if x is II-generic.

We can now drop the prefixes I, and II. Let x be generic over $\mathfrak{M}$. If U is a dense open set rational over $\mathfrak{M}$, then the complement of U, U', is first category. So $x \notin U'$. So $x \in U$.

Conversely, suppose that x lies in each dense open subset of 2^ω rational over $\mathfrak{M}$. We prove that x is generic. Let N be a first category set rational over $\mathfrak{M}$, and $N^{\mathfrak{M}}$ the corresponding Borel set of $\mathfrak{M}$. By Lemma 1.6.6, $N^{\mathfrak{M}}$ is first category in $\mathfrak{M}$. Hence, in $\mathfrak{M}$, there is a countable sequence $\{F_i^{\mathfrak{M}}\}$ of closed nowhere dense sets, with

$$N^{\mathfrak{M}} \subseteq \bigcup_{i \in \omega} F_i^{\mathfrak{M}} .$$

Let F_i be the Borel set of the real world corresponding to $F_i^{\mathfrak{M}}$ (by § 1). Then results in § 1 imply that F_i is closed nowhere dense and

$$N \subseteq \bigcup_i F_i .$$

Our assumption on x implies $x \notin F_i$, for any i. Hence $x \notin N$. So x is generic.

III. Proof of Theorems 1–3

1. Proof of Theorem 2

1.1. Let $\mathfrak{M}$ be a countable transitive model of ZFC + "There exists an inaccessible cardinal". Let Ω be inaccessible in $\mathfrak{M}$. Let $\mathcal{P}^\Omega$ be as in I § 3.2. Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}^\Omega$. We put $\mathfrak{N} = \mathfrak{M}[G]$.

1.2. Let t be a real of $\mathfrak{N}$.

LEMMA. *Almost all reals of $\mathfrak{N}$ are random over $\mathfrak{M}[t]$. Precisely, there* *is a* $B \in \mathfrak{N}$ *such that* $\mathfrak{N} \models B$ *is a Borel set of reals of measure zero and* $x \in \mathfrak{N} \cap \mathbf{R}$ *is random over* $\mathfrak{M}$ *if and only if* $x \notin B$.

PROOF. By Corollary I.3.4.2 $(2^{\aleph_0})_{\mathfrak{M}[t]}$ is countable in $\mathfrak{N}$. The lemma follows from Lemma II.2.1 applied inside $\mathfrak{N}$.

1.3. A set $x \in \mathfrak{N}$ is $\mathfrak{M}$-$\mathbf{R}$-definable if and only if there is a real $t \in \mathfrak{N}$ and an element $y \in \mathfrak{M}$ and a formula $\Phi(v_1, v_2, v_3)$ of $\mathfrak{L}$ such that x is the unique $z \in \mathfrak{N}$ for which

$$\mathfrak{N} \models \Phi(t, y, z) .$$

An argument of Scott and Myhill [11] shows that there is a predicate $\Psi(v_1)$ of $\mathfrak{L}$ such that

$$\mathfrak{N} \models \Psi(y)$$

if and only if y is $\mathfrak{M}$-$\mathbf{R}$-definable.

(There is a similar notion of $\mathfrak{M}$-definable, which is obtained by omitting all mention of the real t in the definition of "$\mathfrak{M}$-$\mathbf{R}$-definable".)

1.4. LEMMA. *Let U be a set of reals in $\mathfrak{N}$ which is $\mathfrak{M}$-$\mathbf{R}$-definable. Then $\mathfrak{N} \models U$ is Lebesgue measurable.*

PROOF. We fix a set-theoretical formula $\Phi_1(v_1, v_2, v_3)$, an element $x \in \mathfrak{M}$, and a real $t \in \mathfrak{N}$ such that for $y \in \mathfrak{N}$ we have $\mathfrak{N} \models \Phi_1(x, t, y)$ if and only if $y = U$.

Using Φ_1, we construct a set-theoretical formula $\Phi_2(v_1, v_2, v_3)$ such that for $y \in \mathfrak{N}$ we have $\mathfrak{N} \models \Phi_2(x, t, y)$ if and only if $y \in U$.

Let $\mathfrak{M}_1 = \mathfrak{M}[t]$. Then by Corollary I.3.4., Ω is inaccessible in $\mathfrak{M}_1$. By Theorem I.4.1, there is an $\mathfrak{M}_1$-generic filter G_1 on $\mathcal{P}^{\Omega}$ such that $\mathfrak{N} = \mathfrak{M}_1[G_1]$. We put $x_1 = \langle x, t \rangle$. Then $x_1 \in \mathfrak{M}_1$ and there is a set-theoretical formula $\Phi_3(v_1, v_2)$ such that for all $y \in \mathfrak{N}$,

(1) $\mathfrak{N} \models \Phi_3(x_1, y)$ if and only if $y \in U$.

Thus all our assumptions about the pair $\langle \mathfrak{N}; \mathfrak{M} \rangle$ hold for $\langle \mathfrak{N}; \mathfrak{M}_1 \rangle$ as well. All the results of I § 3–4 can be applied to $\langle \mathfrak{N}; \mathfrak{M}_1 \rangle$.

1.5. Let $t \in \mathfrak{N}$ be random over $\mathfrak{M}_1$. By Theorem I.4.1, there is an $\mathfrak{M}_1[t]$-generic filter, G_t on $\mathcal{P}^{\Omega}$ such that

$$\mathfrak{N} = \mathfrak{M}_1[t][G_t] .$$

Also, by Corollary I.3.4., Ω is inaccessible in $\mathfrak{M}_1[t]$.

We now apply Lemma I.3.5., considering $\mathfrak{N}$ as a Cohen extension of $\mathfrak{M}_1[t]$. We thus have

(2) $\mathfrak{N} \models \Phi_3(x_1, t)$ if and only if $0 \Vdash \Phi_3(\boldsymbol{x}_1, \boldsymbol{t})$.

Since "forcing is expressible in the ground model", there is a set-theoretical formula $\Phi_4(v_1, v_2)$ and an element x_2 (which we can take to be $\langle x_1, \Omega \rangle$) such that

(3) $\quad 0 \Vdash \Phi_3(x_1, t) \quad$ if and only if $\mathfrak{M}[t] \models \Phi_4(x_2, t)$.

We now invoke Theorem II.2.8. There is a Borel set B, rational over $\mathfrak{M}_1$, such that for any y random over $\mathfrak{M}_1$, $\mathfrak{M}_1[y] \models \Phi_4(x_2, y)$ if and only if $y \in B$.

Let B_1 be the Borel set of $\mathfrak{N}$ corresponding to B. Then $B_1 = B \cap \mathfrak{N}$. Hence if $t \in \mathfrak{N}$ is random over $\mathfrak{M}_1$, we have

(4) $\quad \mathfrak{M}_1[t] \models \Phi_4(x_2, t) \quad$ if and only if $t \in B_1$.

If we string (1)–(4) together, we get, for all reals t random over $\mathfrak{M}_1$,

(5) $\quad t \in U \quad$ if and only if $t \in B_1$.

Let $U \triangle B_1$ be the symmetric difference of U and B_1. Then (5) says

(6) $\quad U \triangle B_1 \subseteq \{t \in \mathfrak{N} \mid t \text{ is not random over } \mathfrak{M}_1\}$.

By Lemma 1.2, the right hand side of (6) is a Borel set of measure zero of $\mathfrak{N}$. So U differs from the Borel set B_1 by a subset of a Borel set of measure zero, i.e., U is Lebesgue measurable in $\mathfrak{N}$.

1.6. Let U be a set of reals of $\mathfrak{N}$ which is $\mathfrak{M}$-**R**-definable. By an argument similar to the proof of Lemma 1.4 (obtained by replacing "random" by "generic" and "measure zero" by "first category" we can show that every $\mathfrak{M}$-definable set of reals is equal to a Borel set modulo a set of the first category. We can in fact do slightly better. It is known (cf. [4, p. 58]) that every Borel set is equal to an open set modulo a set of the first category. Thus every $\mathfrak{M}$-**R**-definable set is equal to an open set off a set of the first category.

1.7. Now let U be a set of reals, in $\mathfrak{N}$, which is $\mathfrak{M}$-**R**-definable and which is uncountable in $\mathfrak{N}$. We are going to show that U contains a perfect subset. Before giving details, we outline the proof.

(1) By extending $\mathfrak{M}$ if necessary, we may assume that U is $\mathfrak{M}$-definable.

(2) We pick $s_1 \in U - \mathfrak{M}$. (We can do this since U is uncountable and $\mathfrak{M} \cap \mathbf{R}$ is countable.)

(3) s_1 lies in $\mathfrak{M}[G_\xi]$, for some $\xi < \Omega$. Exploiting the connection between forcing and truth, we can find $f \in \mathcal{P}_\xi$ such that for any $F\colon \omega \to \xi$ extending f which is $\mathfrak{M}$-generic, $\mathfrak{M}[F] \cap (U - \mathfrak{M}) \neq \emptyset$. In fact, we will construct an explicit $s(F) \in U - \mathfrak{M}$, with $s(F) \in \mathfrak{M}[F]$.

(4) We show that $s(F_1) \neq s(F_2)$ if F_1 is $\mathfrak{M}[F_2]$-generic.

(5) We construct a perfect set K of generic collapsing maps of ξ, and show that

$$F \longmapsto s(F)$$

maps K homeomorphically into U.

We turn to the details. Defining $\mathfrak{M}_1$ as in § 1.4, and replacing $\mathfrak{M}$ by $\mathfrak{M}_1$ if necessary, we may assume that U is $\mathfrak{M}$-definable.

By Corollary I.3.4.2, the reals of $\mathfrak{M}$ are countable in $\mathfrak{N}$. Since U is uncountable in $\mathfrak{N}$, we can select a real s_1 of U, not lying in $\mathfrak{N}$.

Let G_1 be an $\mathfrak{M}$-generic filter on $\mathcal{P}^{\Omega}$ such that $\mathfrak{N} = \mathfrak{M}[G_1]$. By Lemma I.3.4, there is a $\xi < \Omega$ such that $s_1 \in \mathfrak{M}[G_1^{\xi+1}]$. Here $G_1^{\xi+1} = G_1 \cap \mathcal{P}^{\xi+1}$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}^{\xi+1}$. We may assume that $\omega \leq \xi$.

By Lemma I.4.3, there is an $\mathfrak{M}$-generic collapsing map $F_1: \omega \to \xi$ such that

$$\mathfrak{M}[F_1] = \mathfrak{M}[G_1^{\xi+1}] .$$

By I.1.7, there is a set-theoretical formula $\psi(v_1, v_2, v_3)$ and an element x_1 of $\mathfrak{M}$ such that

$$s_1 = \{q \in Q: \mathfrak{M}[F_1] \models \psi(x_1, F_1, q)\} .$$

LEMMA. *There is an $f_1 \in \mathcal{P}_\xi$ with the following property. Let $F: \omega \to \xi$ be an $\mathfrak{M}$-generic collapsing map. Suppose $F \in \mathfrak{N}$, $f_1 \subseteq F$, and put*

$$s = s(F) = \{q \in Q: \mathfrak{M}[F] \models \psi(x_1, F, q)\} .$$

Then s is a real, $s \in U$, and $s \notin \mathfrak{M}$.

PROOF. We can construct a formula $\psi_1(v_1, v_2)$ of $\mathfrak{L}$ and an element x_2 of $\mathfrak{M}$ such that

$$\mathfrak{N} \models \psi_1(x_2, F)$$

if and only if s has the stated properties. Moreover, ψ_1 and x_2 do not depend on F.

By Theorem I.4.1 and Lemma I.3.5, there is a formula $\psi_2(v_1, v_2)$ and an element x_3 of $\mathfrak{M}$ (not depending on F) such that $\mathfrak{N} \models \psi_1(x_2, F)$ if and only if $\mathfrak{M}[F] \models \psi_2(x_3, F)$ (cf. § 1.5).

Suppose now that $F = F_1$. Then $s = s_1$, so s has the stated properties and

$$\mathfrak{M}[F_1] \models \psi_2(x_3, F_1) .$$

Hence, by the connection between forcing and truth, there is an $f_1 \subseteq F_1$, $f_1 \in \mathcal{P}_\xi$, such that

$$f_1 \Vdash \psi_2(x_3, F_1) .$$

The lemma is now clear.

1.8. Let F_1, F_2 be $\mathfrak{M}$-generic collapsing maps of ω onto ξ, lying in $\mathfrak{N}$. Suppose $f_1 \subseteq F_i$, $i = 1, 2$. Suppose further that the pair $\langle F_1, F_2\rangle$ is generic over $\mathfrak{M}$ (i.e., if G_i is the $\mathfrak{M}$-generic filter on $\mathcal{P}_\xi$ associated to F_i, then $G_1 \times G_2$ is an $\mathfrak{M}$-generic filter on $\mathcal{P}_\xi \times \mathcal{P}_\xi$). Then

$$s(F_1) \neq s(F_2) .$$

In fact, if $s(F_1) = s(F_2)$, then $s(F_1) \in \mathfrak{M}$, by Lemma I.2.5. But this contradicts Lemma 1.7.

1.9. Lemma 1.7 and § 1.8 indicate how to manufacture many elements of U. We are going to construct a "perfect set" K of $\mathfrak{M}$-generic collapsing maps of ξ. We shall also arrange that if F_1, F_2 lie in K and $F_1 \neq F_2$, then $\langle F_1, F_2 \rangle$ is $\mathfrak{M}$-generic. Finally, we shall arrange that if $F \in K$, then $f_1 \subseteq F$. It will then be shown that the map

$$\{F \rightarrow s(F)\}$$

maps K onto a perfect subset of U.

1.10. Since Ω is strongly inaccessible in $\mathfrak{M}$ and equals $\aleph_1^{\mathfrak{N}}$, we can enumerate the dense subsets of $\mathcal{P}_\xi$ lying in $\mathfrak{M}$ in a sequence $\{X_n\}$ within $\mathfrak{N}$. Similarly, let $\{W_n\}$ be an enumeration, in $\mathfrak{N}$, of the dense subsets of $\mathcal{P}_\xi \times \mathcal{P}_\xi$ lying in $\mathfrak{M}$.

Let Σ be the set of finite sequences of zeros and ones. Thus $f \in \Sigma$ if and only if f is a function, domain $f \in \omega$, and range $(f) \subseteq \{0, 1\}$. We partially order Σ by inclusion.

LEMMA. *There is in $\mathfrak{N}$ a function*

$$\psi: \Sigma \rightarrow \mathcal{P}_\xi$$

with the following properties:

(1) $\psi(\varnothing) = f_1$.

(2) *If f, f' are elements of Σ and $f \subseteq f'$, then $\psi(f) \subseteq \psi(f')$.*

(3) *If f, f' in Σ are incompatible, then $\psi(f)$ and $\psi(f')$ are incompatible.*

(4) *If* domain $(f) = n$, *and $f \in \Sigma$, then domain$(\psi(f)) \supseteq n$, and $\psi(f) \in X_m$, for $m < n$.*

(5) *If f, $f' \in \Sigma$, and* domain (f) = domain$(f') = n$, *and $f \neq f'$, then $\langle \psi(f), \psi(f') \rangle \in W_m$, for $m < n$.*

PROOF. We define $\psi(f)$ by induction on the domain of f. So put $\psi(\varnothing) = f_1$. Suppose $\psi(h)$ is defined for domain$(f) \leqq n$. We provisionally pick $\psi(f)$ for domain$(f) = n + 1$ so that

(a) $\psi(f)$ extends $\psi(f \mid n)$. ($f \mid n$ is the restriction of f to n.)

Replacing $\psi(f)$ (for f of length $n + 1$) by an extension, and relabeling, we may assume

(b) $n \in$ domain$(\psi(f))$,

(c) $\psi(f) \in X_n$ (since X_n is dense).

Continuing to extend $\psi(f)$ (for f of length $n+1$) and relabeling, we may assume

(d) if f, f' are sequences of length $n + 1$, and $f \neq f'$, then $\psi(f) \neq \psi(f')$ and

$$\langle\psi(f), \psi(f')\rangle \in W_m \qquad (m \leqq n) .$$

(We use here that W_m is dense.)

We can now "freeze" our definition of $\psi(f)$ for length $(f) = n + 1$, and turn to f of length $n + 2 \cdots$.

It is clear that ψ has the properties stated in the lemma, and that ψ can be constructed inside $\mathfrak{N}$.

1.11. We now define, inside $\mathfrak{N}$, a map

$$\psi_*: 2^\omega \rightarrow U .$$

Let $h: \omega \rightarrow 2$. Then from Lemma 1.10,

$$\bigcup_{n \in \omega} \psi(h \mid n)$$

is a function mapping ω into ξ. We denote it by $\chi(h)$. It follows from clause (4) of Lemma 1.10 that $\chi(h)$ is an $\mathfrak{M}$-generic collapsing map of ω onto ξ. Moreover, it is clear that $f_1 \subseteq \psi(h)$ from (1) of Lemma 1.10. We put $\psi_*(h) = s(\chi(h))$. (It is important to realise that ψ_* can be defined inside $\mathfrak{N}$, but this is clear) (cf. Lemma 1.7 for $s(\cdot)$). By Lemma 1.7, $\psi_*(h)$ is an element of U.

We show next that ψ_* is one-one. Indeed, if $h_1, h_2 \in 2^\omega \cap \mathfrak{N}$ and $h_1 \neq h_2$, then $\langle\chi(h_1), \chi(h_2)\rangle$ is an $\mathfrak{M}$-generic pair of collapsing functions. Hence by § 1.8, $\psi_*(h_1) \neq \psi_*(h_2)$.

We show next that ψ_* is continuous. Let $N \in \omega$, $N \geqq 1$. Consider the set, X, of $p \in \mathcal{P}_\xi$ such that

(a) if p is compatible with f_1, then $p \supseteq f_1$.

(b) if $p \supseteq f_1$, then for some $q \in Q$, p forces $\mid s(F) - q \mid < 1/(2N)$.

Using Lemma 1.7, we see that X is dense, Hence, since $X \in \mathfrak{M}$, $X = X_m$ for some m. It follows that if h_1, h_2 are functions in $2^\omega \cap \mathfrak{N}$, and $h_1 \mid m + 1 = h_2 \mid m + 1$, then

$$\mid \psi_*(h_1) - \psi_*(h_2) \mid < 1/N .$$

(In fact, let $g = h_i \mid m + 1$. Then, by Lemma 1.10 (4), $\psi(g) \in X$. It follows that for some $q \in Q$,

$$\psi(g) \Vdash \mid s(F) - q \mid < 1/(2N) .$$

Hence $\mid \psi^*(h_i) - q \mid < 1/(2N)$, $i = 1, 2$.) It is now clear that ψ_* is continuous.

So ψ_* is, in $\mathfrak{N}$, a continuous one-one map of 2^ω into U. Since 2^ω is compact, ψ_* is a homeomorphism. Hence U contains the perfect set

$$\psi_*[2^\omega] .$$

(Our proof that every $\mathfrak{M}$-**R**-definable subset of $\mathfrak{N}$ is countable or contains a perfect subset is, essentially, a slight refinement of the following result of Levy [9]: Every uncountable $\mathfrak{M}$-**R**-definable subset of $\mathfrak{N}$ has power $2^{\aleph_0}$.)

1.12. We now wish to consider the following situation. Let $A \subseteq \mathbf{R}^2$, in $\mathfrak{N}$. Suppose that

$$\forall x \exists y \langle x, y \rangle \in A$$

holds in $\mathfrak{N}$; here x, y range over **R**. Suppose finally that A is $\mathfrak{M}$-**R**-definable.

We shall show that there is a Borel function, h: $\mathbf{R} \to \mathbf{R}$ in $\mathfrak{N}$ such that

$$\langle x, h(x) \rangle \in A$$

for almost all x. Thus h is a choice function, which selects a y in

$$A_x = \{y \mid \langle x, y \rangle \in A\} ,$$

for almost all x.

Since the axiom of choice holds in $\mathfrak{N}$, there is a choice function h in $\mathfrak{N}$ defined for all x. Later, we will give an example of an A for which there is no $\mathfrak{M}$-**R**-definable h such that for *all* x, $h(x) \in A_x$.

We now give an outline of the proof.

(1) We construct a provisional h which is $\mathfrak{M}$-**R**-definable and is defined almost everywhere. Using the fact that $\mathfrak{M}$-**R**-definable subsets of **R** are Lebesgue measurable, it will then be easy to alter h on a set of measure zero to make h Borel.

(2) We may reduce ourselves to the case that A is $\mathfrak{M}$-definable.

(3) Since almost all reals in $\mathfrak{N}$ are random over $\mathfrak{M}$, we need only define $h(x)$ for x random over $\mathfrak{M}$.

(4) Using an argument similar to that of Lemma 1.7, we show that there is an $\mathfrak{M}$-definable function $\varphi(x, y)$ and an ordinal $\xi < \Omega$ such that whenever x is random over $\mathfrak{M}$, and F: $\omega \to \xi$ is an $\mathfrak{M}[x]$-generic collapsing map, then $\varphi(x, F) \in A_x$.

(5) To complete the proof, we show that there is an $\mathfrak{M}$-**R**-definable function $\psi(x)$ such that whenever x is random over $\mathfrak{M}$, $\psi(x)$ is an $\mathfrak{M}[x]$-generic collapsing map mapping ω onto ξ. (We then put $h(x) = \varphi(x, \psi(x))$.)

We turn to the details.

LEMMA 1. *Let* $h \in \mathfrak{N}$ *map* **R** *into* **R***; suppose that* h *is* $\mathfrak{M}$-**R**-*definable. Then there is a Borel function* h_1 *such that*

$$\{x \mid h(x) = h_1(x)\}$$

has measure zero.

PROOF. For $r \in \mathbf{Q}$, let

$$U_r = \{x \mid h(x) < r\} .$$

Since U_r is $\mathfrak{M}$-$\mathbf{R}$-definable, there is a Borel set B_r and a Borel set of measure zero N_r such that

$$U_r \triangle B_r \subseteq N_r \, .$$

Let $N = \bigcup_r N_r$. Then N is a Borel set of measure zero. Put $h_1(x) = h(x)$ for $x \notin N$; $h_1(x) = 0$ if $x \in N$.

If $r \leqq 0$, $r \in \mathbf{Q}$,

$$\{x \mid h_1(x) < r\} = U_r - N \, .$$

If $r > 0$, $r \in \mathbf{Q}$,

$$\{x \mid h_1(x) < r\} = U_r \cup N \, .$$

Thus h_1 is Borel. The lemma is now clear.

As usual, we may assume that A is $\mathfrak{M}$-definable (by extending $\mathfrak{M}$ if necessary).

We use a_1, a_2, etc. to denote parameters from $\mathfrak{M}$. Since A is $\mathfrak{M}$-definable, there is a set-theoretical formula, $\psi_1(a_1, x, y)$ such that

$$\mathfrak{N} \models \psi_1(a_1, x, y) \longleftrightarrow \langle x, y \rangle \in A \, .$$

Now let x_1 be random over $\mathfrak{M}$. Then we can find an $\mathfrak{M}[x_1]$-generic filter G_1 on $\mathcal{P}^\Omega$ such that $\mathfrak{N} = \mathfrak{M}[x_1][G_1]$. Select a $y_1 \in A_{x_1}$. By the results of I. 3.4, $y_1 \in \mathfrak{M}[x_1][G_1^{\xi_1}]$ for some $\xi_1 < \Omega$. Apparently ξ_1 depends on our choice of x_1 and G_1. However, we have the following lemma.

LEMMA 2. *There is a $\xi < \Omega$ such that for all reals x random over $\mathfrak{M}$ and all filters G on $\mathcal{P}^\Omega$ generic over $\mathfrak{M}[x]$, the set*

$$A_x \cap \mathfrak{M}[x][G^\xi]$$

is non-empty.

PROOF. We extend $\mathcal{L}$ to a language $\mathcal{L}''$ as follows: for each $a \in \mathfrak{M}$, we adjoin a constant $\boldsymbol{a}$; there are two additional constants $\boldsymbol{x}$ and $\mathbf{G}$.

Let x be a real random over $\mathfrak{M}$ and G an $\mathfrak{M}[x]$-generic filter on $\mathcal{P}^\Omega$. We interpret $\mathcal{L}''$ in $\mathfrak{M}[x, G]$ in the obvious way. (Thus, variables range over $\mathfrak{M}[x, G]$; $\boldsymbol{x}$ denotes x; $\mathbf{G}$ denotes G, etc.)

Let $\Psi_2(\xi)$ be a formula of $\mathcal{L}''$ which expresses the following:

ξ is an ordinal less than Ω, and

$$A_x \cap \mathfrak{M}[x][G^\xi] \neq \varnothing \, .$$

We now fix a real x random over $\mathfrak{M}$ and an $\mathfrak{M}[x]$-generic filter G on $\mathcal{P}^\Omega$. By Lemma I.3.4, there is a $\xi < \Omega$ such that

$$\mathfrak{M}[x][G] \models \Psi_2(\xi) \, .$$

Let $p \in G$ force $\Psi_2(\xi)$. (We are viewing $\mathfrak{M}[x][G]$ as a Cohen extension of $\mathfrak{M}[x]$.)

We say that in fact

$$0 \Vdash \Psi_2(\xi) .$$

Otherwise, there is a $p' \in \mathcal{P}^\Omega$ such that $p' \Vdash \neg \Psi_2(\xi)$. We select a bijection π of ω such that $\pi_*(p')$ is compatible with p. (Notation as in I § 3.5.) A glance at the definition of π_* shows that

$$\mathfrak{M}[\pi_*(G)^\xi] = \mathfrak{M}[G^\xi] .$$

Hence an argument similar to the proof of Lemma I.3.5 will show that

$$\pi_*(p') \Vdash \neg \Psi_2(\xi) .$$

This is absurd since $p \Vdash \Psi_2(\xi)$ and p and $\pi_*(p')$ are compatible.

We can find a formula $\Psi_3(x, \xi)$ such that $\mathfrak{M}[x] \models \Psi_3(x, \xi)$ if and only if $\mathfrak{M}[x][G] \models$ "$0 \Vdash \Psi_2(\xi)$".

We know (viewing $\mathfrak{M}[x]$ as a Cohen extension of $\mathfrak{M}$) that the following are forced:

(1) $\exists \xi < \Omega \ \Psi_3(x, \xi)$,

(2) $\xi < \xi' < \Omega$ and $\Psi_3(x, \xi) \rightarrow \Psi_3(x, \xi')$.

By Zorn we pick inside $\mathfrak{M}$ a maximal family $\{\langle b_i, \xi_i \rangle : i \in I\}$ such that

(3) $\{b_i : i \in I\}$ is a pairwise disjoint family of non-zero elements of $\mathcal{B}_1$; $\xi_i < \Omega$.

(4) $b_i \Vdash \Psi_3(x, \xi_i)$.

Using (1) and (3), we see that sup $\{b_i : i \in I\}$ is the unit of $\mathcal{B}_1$. Using the fact that $\mathcal{B}_1$ satisfies C.C.C., we see that I is countable. Hence if $\xi = \sup \{\xi_i : i \in I\}$, $\xi < \Omega$. By (2),

$$b_i \Vdash \Psi_3(x, \xi) , \qquad i \in I .$$

It follows that $\Vdash \Psi_3(x, \xi)$. (For example, from Theorem 2.8 of II.) Using the relation between Ψ_3 and Ψ_2 we see that ξ satisfies the requirements of the lemma.

We let ξ_0 have the property ascribed to ξ in Lemma 2. We may assume $\xi_0 = \xi_0' + 1$.

Lemma 3. *Let x be a real of $\mathfrak{N}$ random over $\mathfrak{M}$ and let G^{ξ_0} be an $\mathfrak{M}[x]$-generic filter $\mathcal{P}^{\xi_0}$. Then*

$$\mathfrak{M}[x][G^{\xi_0}] \cap A_x \neq \varnothing .$$

Proof. Suppose not. Fix x, $G_1^{\xi_0}$ witnessing the fact that the lemma is false. By Lemma I.4.6, we can find an $\mathfrak{M}[x]$-generic filter G on $\mathcal{P}^\Omega$ with $G \cap \mathcal{P}^{\xi_0} = G_1^{\xi_0}$, and such that

$$\mathfrak{N} = \mathfrak{M}[x][G] .$$

But now our assumption on x and $G_1^{\xi_0}$ contradict Lemma 2.

LEMMA 4. *Let* $\lambda = 2^{\mathrm{card}(\xi_0)}$, *as computed in* $\mathfrak{M}$. *Then* λ *is countable in* $\mathfrak{N}$. *Let* $F\colon \omega \to \lambda$ *be surjective. Then for any real* x *random over* $\mathfrak{M}$,

$$\mathfrak{M}[x, F] \cap A_x \neq \emptyset .$$

PROOF. Since Ω is inaccessible in $\mathfrak{M}$, $\lambda < \Omega$. Hence λ is countable in $\mathfrak{N}$. It is easy to see that

$$\mathrm{card}\,(\mathcal{P}_{\xi_0}) = \mathrm{card}\,(\xi_0)$$

in $\mathfrak{M}$. Moreover, standard arguments show that if x is random over $\mathfrak{M}$, then $\mathfrak{M}$ and $\mathfrak{M}[x]$ have the same cardinals, and that

$$(2^{\mathrm{card}(\xi_0)})_{\mathfrak{M}} = (2^{\mathrm{card}(\xi_0)})_{\mathfrak{M}[x]} .$$

(The essential point is that $\mathcal{B}_1$ satisfies C.C.C. For details, cf. e.g. [12].)

Thus, in $\mathfrak{M}[F, x]$, we can enumerate the dense subsets of $\mathcal{P}^{\xi_0}$ lying in $\mathfrak{M}[x]$. It follows that there is an $\mathfrak{M}[x]$ generic filter on $\mathcal{P}^{\xi}$, G, lying in $\mathfrak{M}[F, x]$. The lemma now follows from Lemma 3.

The following lemma is standard, and we omit the proof. Let F be as in Lemma 4. Note that F is definable from a real, by I. 1.12.

LEMMA 5. *There is an* $\mathfrak{M}$-**R**-*definable function* $\psi(x)$ *such that for any real* x, $\psi(x)$ *is a well-ordering of the reals of* $\mathfrak{M}[F, x]$.

We now put it all together. Define $h\colon \mathbf{R} \to \mathbf{R}$ as follows: $h(x)$ is the $\psi(x)$-least member of $A_x \cap \mathfrak{M}[F, x]$ if this set is non-void. Otherwise, $h(x) = 0$. By Lemma 5, $h(x)$ is $\mathfrak{M}$-**R**-definable. By Lemma 4, $h(x) \in A_x$ for all x random over $\mathfrak{M}$. By Lemma II.2.1, it follows that $h(x) \in A_x$ for almost all x. By Lemma 1, we can alter h on a set of measure zero, so that it is Borel.

1.13. In a totally analogous way, we can prove that if A is as in 1.12, there is a Borel function h such that $h(x) \in A_x$ for all but a first category set of x's.

The following lemma is known.

LEMMA. *Let* $h\colon \mathbf{R} \to \mathbf{R}$ *be Borel. Then there is a set* N *of the first category such that* $h \mid \mathbf{R} - N$ *is continuous.*

The proof is similar to the proof of Lemma 1.12.1. We omit the details.

The lemma implies a similar property for $\mathfrak{M}$-**R**-definable functions.

1.14. It is now easy to complete the proof of Theorem 2. It follows from the results recalled in I.14 that $\mathfrak{N}$ is a model of **ZFC**.

From work of Gödel (cf. [1, Ch. 3]) it is known that if **ZFC** + **I** has a transitive model, then so does **ZFC** + **I** + **GCH**. We now sketch a proof that if **GCH** holds in $\mathfrak{M}$, it also holds in $\mathfrak{N}$. (Our proof would be slightly more natural in terms of the concepts of [12].)

Since **GCH** holds in $\mathfrak{M}$, Ω is strongly inaccessible in $\mathfrak{M}$.

We let $\mathfrak{S}$ be the collection of subsets of $\mathcal{P}^{\Omega}$ in $\mathfrak{M}$ of cardinality less than Ω. $\mathfrak{S}$ has cardinality Ω. Let θ be a cardinal of $\mathfrak{N}$. Let $\mathfrak{S}^{\theta}_{\mathfrak{M}}$ be the collection of maps of θ into $\mathfrak{S}$ *lying in* $\mathfrak{M}$. We define a map $h: \mathfrak{S}^{\theta}_{\mathfrak{M}} \to P(\theta)_{\mathfrak{N}}$, in $\mathfrak{N}$, as follows: if $g \in \mathfrak{S}^{\theta}_{\mathfrak{M}}$, $h(g) = \{\alpha < \theta: g(\alpha) \cap G \neq \emptyset\}$. Using the **GCH** in $\mathfrak{M}$, the cardinality of $\mathfrak{S}^{\theta}_{\mathfrak{M}}$ is easily computed. We leave this to the reader. This computation shows that **GCH** holds in $\mathfrak{N}$ provided h is surjective.

To see that h is surjective, let $A \in P(\theta)_{\mathfrak{N}}$. We fix a definition Φ of A. Thus $A = \{\alpha: \mathfrak{N} \models \Phi(\underline{\alpha}, \underline{G})\}$. For each $\alpha < \theta$, let $\mathcal{F}_{\alpha}$ be a maximal pairwise incompatible family of conditions that decide $\Phi(\underline{\alpha}, \underline{G})$ and let $\mathcal{G}_{\alpha}$ be the subset of $\mathcal{F}_{\alpha}$ consisting of conditions that force $\Phi(\underline{\alpha}, \underline{G})$. Then Lemma I.3.3 shows that $\mathcal{G}_{\alpha} \in \mathfrak{S}$. Define $g \in \mathfrak{S}^{\theta}_{\mathfrak{M}}$ by $g(\alpha) = \mathcal{G}_{\alpha}$. We leave it to the reader to verify that $h(g) = A$ (cf. the proof of Lemma I.3.4). This completes our discussion of the **GCH** in $\mathfrak{N}$.

To complete the proof of Theorem 2, we must verify that the analogues of (2) to (5) of Theorem 1 hold in $\mathfrak{N}$. In view of the results of §§ 1.1–1.13, it suffices to cite the result proved in § 2.8 below, that every set of reals definable from a countable sequence of ordinals is $\mathfrak{M}$-**R**-definable.

1.15. We now give an example of an A which has no $\mathfrak{M}$-**R**-definable cross-section. We put

$$A = \{\langle x, y\rangle: y \text{ is not } \mathfrak{M}\text{-definable from } x\} .$$

It follows from the techniques of [11] that A is $\mathfrak{M}$-definable.

LEMMA. *Let $x \in \mathbf{R}$. Then there is a y not $\mathfrak{M}$-definable from x.*

PROOF. Using the techniques of [11], one shows that

$$A'_x = \{y \mid y \text{ is } \mathfrak{M}\text{-definable from } x\}$$

has an $\mathfrak{M}$-**R**-definable well-ordering. If $A'_x = \mathbf{R}$, then we would have an $\mathfrak{M}$-**R**-definable well-ordering of **R**. Using this, one could construct an $\mathfrak{M}$-**R**-definable non-Lebesgue measurable set. This contradicts our result of 1.5. Thus $A_x \neq \emptyset$. q.e.d.

Suppose now that h is an $\mathfrak{M}$-**R**-definable function mapping **R** into **R**. Say h is $\mathfrak{M}$-definable from $x \in \mathbf{R}$. Then $h(x)$ is $\mathfrak{M}$-definable from x, i.e., $h(x) \notin A_x$.

2. Proof of Theorem 1

2.1. The present method of presenting Theorems 1 and 2, in which Theorem 1 is essentially a corollary of Theorem 2, is due to Ken McAloon.

Our original approach was to prove Theorem 1 directly. (Theorem 2 is then an easy corollary.) Our original approach had the disadvantage that the verification of DC in the model for Theorem 1 was extremely delicate. With the present approach, it is a triviality.

2.2. Let $\mathfrak{N}$ be as in § 1. We say that x is *definable from a sequence of ordinals* (in $\mathfrak{N}$), there is an $f\colon \omega \to OR$, $f \in \mathfrak{N}$, and a set-theoretical formula $\Phi(v_1, v_2)$ such that, for any $y \in \mathfrak{N}$, $\mathfrak{N} \models \Phi(f, y)$ if and only if $y = x$.

2.3. Let x be a set. It is known that there is a minimal transitive set y such that $x \in y$. (The set y consists of x, the members of x, the members of the members of x, etc.) We call y the *transitive hull* of x. We say that x *hereditarily* possesses some property P if each member of the transitive hull of x has the property P.

2.4. Let $\mathfrak{N}_1$ be the set of elements hereditarily definable from a sequence of ordinals in $\mathfrak{N}$. (Thus $\mathfrak{N}_1 \subseteq \mathfrak{N}$.)

The methods of Myhill and Scott [11] allow one to prove the following lemma.

LEMMA. *$\mathfrak{N}_1$ is a transitive model of* **ZF**. *There is a single formula, $\Phi_0(v_1, v_2)$, of set-theory such that for any $x \in \mathfrak{N}_1$, there is an $f \in \mathfrak{N}$, $f\colon \omega \to OR$, and x is the unique $y \in \mathfrak{N}$ such that*

$$\mathfrak{N} \models \Phi_0(f, y) .$$

Thus the formula "$x \in \mathfrak{N}_1$" is expressible in $\mathfrak{N}$, by a set-theoretical formula, viz.,

$$(\exists f)(f\colon \omega \to OR \wedge (y)(y = x \longleftrightarrow \Phi_0(f, y))) .$$

2.5. The following lemma is clear.

LEMMA. *Every real of $\mathfrak{N}$, and every sequence of ordinals of $\mathfrak{N}$ lies in $\mathfrak{N}_1$.*

2.6. LEMMA. *Let $h\colon \omega \to \mathfrak{N}_1$, $h \in \mathfrak{N}$. Then $h \in \mathfrak{N}_1$.*

PROOF. We work in $\mathfrak{N}$. Let $x \in \mathfrak{N}_1$. Define an ordinal, $\gamma(x)$, as follows: $\gamma(x)$ is the least ordinal λ such that for some $f\colon \omega \to \lambda$, x is the unique y such that $\Phi_0(f, y)$.

Let $\gamma = \sup\{\gamma(h(n))\colon n \in \omega\}$. Well-order the set $\{f\colon f \text{ maps } \omega \text{ into } \gamma\}$. Let $f_n\colon \omega \to \gamma$ be the least f (with respect to this well-ordering) such that $h(n)$ is the unique y such that $\Phi_0(f, y)$.

Define $g\colon \omega \to OR$ by:

$$g(2^m 3^n) = f_m(n) ;$$

otherwise $g(r) = 0$. Clearly h is definable from $\{f_m\colon m \in \omega\}$ and $\{f_m\}$ is definable from g. Thus h is definable from a sequence of ordinals. Since, by assumption, $h \subseteq \mathfrak{N}_1$, it follows that $h \in \mathfrak{N}_1$.

2.7. We now state the principle of dependent choices, **DC**.

Let X be a set, R a binary relation on X. Suppose further that $X \neq \emptyset$. Finally, we assume that

$$(\forall x \in X)(\exists y \in X)(xRy) .$$

Then there is a map $h: \omega \to X$ such that, for all $n \in \omega$, $h(n)Rh(n+1)$.

Note that **DC** follows easily from **AC** (the axiom of choice); one simply defines $h(n)$ by induction on n.

LEMMA. **DC** *holds in* $\mathfrak{N}_1$.

PROOF. Let $X, R \in \mathfrak{N}_1$ satisfy the hypotheses of **DC**. Since **AC** holds in $\mathfrak{N}$, there is an $h: \omega \to X$, $h \in \mathfrak{N}$, such that for all $n \in \omega$, $\langle h(n), h(n+1)\rangle \in R$. By Lemma 2.5, $h \in \mathfrak{N}_1$. Thus **DC** holds in $\mathfrak{N}_1$.

2.8. LEMMA. *Let* $A \in \mathfrak{N}_1$. *Then, in* $\mathfrak{N}$, A *is* $\mathfrak{M}$-**R**-*definable.*

PROOF. We may as well assume that A is a map f of ω into OR.

By Lemma I.3.4, $f \in \mathfrak{M}[G^\xi]$ where $\omega \leqq \xi < \Omega$, and $\xi = \xi' + 1$. By Lemmas I.4.3 and I.1.12, there is a real s such that $f \in \mathfrak{M}[s]$. So the lemma is clear.

2.9. LEMMA. *In* $\mathfrak{N}_1$, *every set of reals is Lebesgue measurable.*

PROOF. Let A be a set of reals in $\mathfrak{N}_1$. By Lemma 2.8, A is $\mathfrak{M}$-**R**-definable in $\mathfrak{N}$. Thus by Lemma 1.4, A is Lebesgue measurable in $\mathfrak{N}$. Thus there is a Borel set B and a Borel set N of measure zero, in $\mathfrak{N}$, such that

$$B \triangle A \subseteq N . \tag{1}$$

Let α_1, α_2 be codes for B and N in $\mathfrak{N}$. Trivially, α_1 and α_2 lie in $\mathfrak{N}_1$. (Lemma 2.5.) By Lemma 2.5, $\mathfrak{N}$ and $\mathfrak{N}_1$ have the same reals. Thus, by Theorem II 1.4, α_1 and α_2 code B and N also in $\mathfrak{N}$. Clearly (1) holds in $\mathfrak{N}$. By Lemma II.1.6.4, N has measure zero in $\mathfrak{N}_1$. Thus A is Lebesgue measurable in N_1.

2.10. The proof of the following lemma is totally analogous to that of Lemma 2.9.

LEMMA. *In* $\mathfrak{N}_1$, *every set of reals has the property of Baire.*

2.11. LEMMA. *In* $\mathfrak{N}_1$, *every uncountable set of reals contains a perfect subset.*

PROOF. Let A be a set of reals in $\mathfrak{N}_1$. By Lemma 2.8, A is $\mathfrak{M}$-**R**-definable in $\mathfrak{N}$.

Suppose first that A is countable in $\mathfrak{N}$. Then Lemma 2.6 shows that A is countable in $\mathfrak{N}_1$. On the other hand, suppose A is uncountable in $\mathfrak{N}$. Then, since A is $\mathfrak{M}$-**R**-definable in $\mathfrak{N}$, there is, in $\mathfrak{N}$, a perfect set K with $K \subseteq A$.

Let β be a code for K. Then $\beta \in \mathfrak{N}_1$ and β codes K in $\mathfrak{N}_1$ (cf. the proof of Lemma 2.9). By Lemma II.1.6.7, K is perfect in $\mathfrak{N}_1$, and the lemma is clear.

2.12. The following lemma is the key to verifying (5) in $\mathfrak{N}_1$.

LEMMA. *Let f be, in $\mathfrak{N}$, a Borel function mapping* **R** *into* **R**. *Then $f \in \mathfrak{N}_1$.*

PROOF. Using Lemma 2.5 and Theorem II.1.2 we see that every Borel set of reals of $\mathfrak{N}$ lies in $\mathfrak{N}_1$. Hence, by Lemma 2.6, the indexed family

$$\{f^{-1}((-\infty, q)) : q \in \mathbf{Q}\}$$

lies in $\mathfrak{N}_1$. It follows easily that $f \in \mathfrak{N}_1$.

It is now easy to verify that (5) holds in $\mathfrak{N}_1$. Let $\{A_x : x \in \mathbf{R}\}$ be as in the statement of (5). Applying (5a) in $\mathfrak{N}$, we get a Borel function h and a Borel set of measure zero N such that

$$x \notin N \rightarrow h(x) \in A_x .$$

Since h, N lie in $\mathfrak{N}_1$, this instance of (5a) holds in $\mathfrak{N}_1$. The verification of (5b) is similar.

2.13. The material in §§ 2.7–2.12 establishes Theorem 1.

3. Proof of Theorem 3

3.1. McAloon's idea of directly proving Theorem 2 allows one to prove Theorem 3 as well. (This fact was first noticed by McAloon.) We are going now to sketch the proof of Theorem 3. Using our sketch and the detailed proof of Theorem 2 given above, the reader should be able to fill in the details without trouble.

3.2. $\mathfrak{M}$ is a countable transitive model of ZFC + GCH + "There is an inaccessible cardinal". Ω is an inaccessible cardinal in $\mathfrak{M}$. Θ is a cardinal of $\mathfrak{M}$ with cofinality $\geqq \Omega$.

Let $\mathcal{P}'_\Theta$ be the partially ordered set appropriate to adding Θ generic sets of integers. Thus $\mathcal{P}'_\Theta$ is the set of all functions f such that

(1) domain(f) is a finite subset of $\Theta \times \omega$;

(2) range(f) $\subseteq \{0, 1\}$.

Let $\mathcal{P} = \mathcal{P}^\Omega \times \mathcal{P}'_\Theta$. Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}$. Let $\mathfrak{N}_2$ be $\mathfrak{M}[G]$.

3.3. LEMMA. *Let $X \in \mathfrak{M}$ be a pairwise incompatible family of elements of $\mathcal{P}$. Then, in $\mathfrak{M}$,*

$$\text{card}(X) < \Omega .$$

(The proof is similar to that of Lemma I.3.3.)

This lemma has the following consequences.

(1) $\Omega = \aleph_1^{\mathfrak{N}_2}$ (cf. Corollary I.3.3).

(2) If $\lambda \geqq \Omega$, λ is a cardinal in $\mathfrak{M}$ if and only if λ is a cardinal in $\mathfrak{N}_2$.

By standard methods, one can compute $2^{\aleph_0}$ in $\mathfrak{N}_2$. One gets

(3) $2^{\aleph_0} = \Theta$ in $\mathfrak{N}_2$.

(*Example.* Suppose Θ is the least cardinal of $\mathfrak{M} > \Omega$. Then in $\mathfrak{N}_2$, $2^{\aleph_0} = \aleph_2$.)

3.4. Let $A \in \mathfrak{M}$, $A \subseteq \Theta$. Let $\mathcal{P}'_A = \{f \in \mathcal{P}'_\Theta\colon \text{domain}(f) \subseteq A \times \omega\}$.

The following lemma is the analog of Lemma I.3.4 and has a similar proof.

LEMMA. *Let $f\colon \omega \to OR$, $f \in \mathfrak{N}_2$. Then there is a $\xi < \Omega$, and a subset A of Θ such that:*

(1) $A \in \mathfrak{M}$, *and in* $\mathfrak{M}$, card $(A) < \Omega$.

(2) $f \in \mathfrak{M}[G \cap (\mathcal{P}^\xi \times \mathcal{P}'_A)]$.

3.5. The following lemma is the analog of Lemma I.4.3 and has a similar proof.

LEMMA. *Let $A \in \mathfrak{M}$, $A \subseteq \Theta$, and suppose*

$$\text{card}(A) \leqq \text{card}(\xi) < \Omega$$

in $\mathfrak{M}$. Let G be an $\mathfrak{M}$-generic filter on $\mathcal{P}^{\xi+1} \times \mathcal{P}'_A$. Then there is an $\mathfrak{M}$-generic collapsing map $F\colon \omega \to \xi$ with $\mathfrak{M}[G] = \mathfrak{M}[F]$.

3.6. Using Lemmas 3.4 and 3.5, one can adapt the proof of Theorem I.4.1 to prove

LEMMA. *Let $f \in \mathfrak{N}_2$, $f\colon \omega \to OR$. Then there is an $\mathfrak{M}[f]$-generic filter, G_1, on $\mathcal{P}$ such that*

$$\mathfrak{M}[f][G_1] = \mathfrak{N}_2 .$$

3.7. The following is the analog of Lemma I.3.5 and has a similar proof.

LEMMA. *Let Φ be a sentence of $\mathcal{L}'$ not containing* G. *Let O be the minimal element of $\mathcal{P}$. Then O decides Φ.*

3.8. Using Lemmas 3.6 and 3.7, one can imitate the discussion of §1 and prove

LEMMA. *Let $A \in \mathfrak{N}_2$ be a set of reals which is $\mathfrak{M}$-R-definable in $\mathfrak{N}_2$. Then A is Lebesgue measurable and has the Baire property. If A is uncountable, A contains a perfect subset.*

The proof of Theorem 3 is now clear.

3.9. Using the product lemma (Lemma I.2.3), we see that the model of Theorem 3 is obtained from the model of Theorem 2 by the (extremely well-understood) process of adding generic reals. Hence the possible cardinalities of $2^{\aleph_0}$ in the models provided by the proof of Theorem 3 are equally well understood.

4. An extension of Theorems 1 through 3

4.1. Theorems 1 through 3 state that certain subsets of the reals are well-behaved. In this section we replace **R** by an arbitrary complete separable metric space X, and Lebesgue measure by a totally σ-finite measure space μ. We shall discuss, very sketchily, a proof of the following theorem.

THEOREM. *The following is valid in* $\mathfrak{N}_1$: *Let* X, μ *be as above, and let* $A \subseteq X$. *Then* A *is* μ *measurable,* A *has the property of Baire, and* A *is either countable or contains a perfect subset.*

PROOF. One first re-does the material of II for the space X. (There are a few technical tricks needed to re-do II in this generality, which we shall not discuss.) One then works with, e.g., the μ-random elements of X in proving that A is Lebesgue measurable (first in $\mathfrak{N}$, and then in $\mathfrak{N}_1$). Similarly, one proves A has the Baire property.

To prove that if A is uncountable, it contains a perfect subset, we invoke the following theorem of **ZF** + **DC**: Any separable metric space imbeds homeomorphically into the Hilbert cube (cf. [6, p. 125]). This reduces the problem to the special case when X is the Hilbert cube. The argument given in § 1 in the case $X = \mathbf{R}$ adapts easily to this case.

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ON DEGREES OF RECURSIVE UNSOLVABILITY*

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In Kleene-Post [4][1] a number of questions concerning the structure of the upper semi-lattice of degrees were left unanswered. The present paper contains the answers to those questions under the scope of [4] Footnote 3. With the exception of the density problem ([4], 2.2), the methods used are variations of those developed in [4]. The construction employed in showing that the degrees are not dense involves a generalization of the methods of [4], and constitutes the main result of this paper (Theorem 4) that there are minimal degrees of recursive *un*solvability.[2] Familiarity with [4] is assumed.[3]

1. Relations involving both join and jump operations

1.1. The main results of [4] are each obtained by constructing a function, say γ, satisfying certain conditions C. The method employed is to write C as a conjunction of an enumerably infinite set of conditions $C_0, C_1, C_2, \cdots$. Then, at each stage $g = 0, 1, 2, \cdots$ of the construction, a class of functions $\mathfrak{F}_g$ is defined such that every function in the class satisfies condition C_g; also $\mathfrak{F}_{g+1} \subset \mathfrak{F}_g$ for every g. Thus the functions in $\mathfrak{F}_g$ satisfy the first $g + 1$ conditions $C_0, C_1, \cdots, C_g$. It is then shown that there is a function γ which belongs to every $\mathfrak{F}_g$, and therefore satisfies C.

It is desirable to let C_0 be as simple as possible to avoid a complicated basis step. Throughout this paper we shall let C_0 specify the kind of function under consideration; e.g. $\mathfrak{F}_0$ could be the class of all functions of one argument with values 0 or 1. The classes $\mathfrak{F}_g$, for $g > 0$, employed in [4] are of the form $\mathfrak{F}(S, \sigma)$, where S is a recursive subset of the natural numbers with infinite complement, $\sigma(x)$ is a function of one argument defined for all x in S, and $\mathfrak{F}(S, \sigma)$ is the class of all functions γ in $\mathfrak{F}_0$ such that $\gamma(x) = \sigma(x)$ for every x in S. Thus each construction amounts to specifying $\mathfrak{F}_0$, the sets S_g, and the functions σ_g such that $\mathfrak{F}_g = \mathfrak{F}(S_g, \sigma_g)$. This is done by induction, i.e. S_{g+1} and σ_{g+1} are defined in terms of S_g and σ_g. Since we always have $S_g \subset S_{g+1}$ and $x \in S_g \rightarrow \sigma_{g+1}(x) = \sigma_g(x)$, it is sufficient to specify $S_{g+1} - S_g$, and to define $\sigma_{g+1}(x)$ for x in $S_{g+1} - S_g$.

In the simplest applications, $S_g = \hat{x}[x < \nu(g)]$ where $\nu(0) = 0$ and $\nu(g + 1) >$

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[1] Author's corrections: p. 387, 3 lines below (18), delete first comma; p. 404, last line of (56), change first "=" to "$\not\simeq$".

[2] For "minimal", "maximal", "least", and "greatest", as applied to partially ordered sets, see [1], p. 7.

[3] For notations not explained here see [4] and [2], bottom p. 538.

$\nu(g)$. Then we speak of γ being defined on the set S_g, and of extending γ to the set S_{g+1} by defining $\nu(g+1)$ and $\bar{\gamma}(\nu(g+1))$. We illustrate this in the following proof.

1.2. THEOREM 1. *Given a degree* **d**, *there are degrees* **a** *and* **b** *such that* $\mathbf{a}' \cup \mathbf{b}' = \mathbf{a} \cup \mathbf{b} = \mathbf{a}' = \mathbf{b}' = \mathbf{d}'$ & $\mathbf{a} \mid \mathbf{b}$ & $\mathbf{d} < \mathbf{a} < \mathbf{d}'$ & $\mathbf{d} < \mathbf{b} < \mathbf{d}'$.

PROOF (for $\mathbf{d} = 0$). We construct functions α and β of degrees **a** and **b** respectively such that

$$(1) \qquad \mathbf{a}' \cup \mathbf{b}' \leqq \mathbf{a} \cup \mathbf{b} \leqq \mathbf{0}',$$

and then show that the other properties follow. In order to fit the construction into the scheme described above, we set $\gamma(a) = 2^{\alpha(a)} 3^{\beta(a)}$, and write $\alpha(a)$ and $\beta(a)$ for $(\gamma(a))_0$ and $(\gamma(a))_1$ respectively. We shall restrict our attention to functions α and β with values <2, and therefore we let $\mathfrak{F}_0$ be the class of functions γ with values 1, 2, 3, or 6. For $g > 0$ and all e, we shall have

$$(2) \qquad \alpha(a) = \beta(a) \equiv [a \text{ is not of the form } \nu(g) - 1],$$

$$(3) \qquad (Ey)T_1^1(\bar{\alpha}(y), e, e, y) \equiv \alpha(\nu(2e+1) - 1) = 0,$$

$$(4) \qquad (Ey)T_1^1(\bar{\beta}(y), e, e, y) \equiv \beta(\nu(2e+2) - 1) = 0.$$

Then using (3), (4), and

$$(5) \qquad \nu(0) = 0, \qquad \nu(g+1) = 1 + \mu x[\alpha(x) \neq \beta(x) \;\&\; x \geqq \nu(g)],$$

it will follow that $\mathbf{a}' \cup \mathbf{b}' \leqq \mathbf{a} \cup \mathbf{b}$. Let C_{2e+1} be the condition that γ satisfies (3), C_{2e+2} that γ satisfies (4), and let each C_{g+1} include also the condition that γ satisfies (2) for $\nu(g) \leqq a < \nu(g+1)$.

Suppose $\nu(2e)$ and $\bar{\gamma}(\nu(2e))$ have been defined ($\nu(0) = 0$).

CASE 1: $\nu(2e+1)$ and $\bar{\alpha}(\nu(2e+1))$ can be defined so that $T_1^1(\bar{\alpha}(y), e, e, y)$ holds for some $y < \nu(2e+1)$. This is equivalent to the existence of an x such that

$$(6) \qquad \begin{aligned} &T_1^1((x)_0, e, e, (x)_1) \;\&\; (z)_{z<(x)_1}(x)_{0,z} < 2 \\ &\qquad \&(z)_{z<\min(\nu(2e),(x)_1)}[(x)_{0,z} = (\bar{\alpha}(\nu(2e)))_z]. \end{aligned}$$

Let X be the least such x. The functions $\alpha(a)$ and $\beta(a)$ having been defined for $a < \nu(2e)$, we set

$$(7) \qquad \alpha(a) = \beta(a) = (X)_{0,a} \qquad \text{for } \nu(2e) \leqq a < (X)_1$$

(if $\nu(2e) \geqq (X)_1$ nothing new is defined). For $m = \max(\nu(2e), (X)_1)$ we set

$$(8) \qquad \alpha(m) = 0, \qquad \beta(m) = 1, \qquad \nu(2e+1) = m + 1.$$

CASE 2: $\nu(2e+1)$ and $\bar{\alpha}(\nu(2e+1))$ cannot be so defined. Then we set

$$(9) \qquad \alpha(\nu(2e)) = 1, \qquad \beta(\nu(2e)) = 0, \qquad \nu(2e+1) = \nu(2e) + 1.$$

At the next stage, $\nu(2e+2)$ and $\tilde{\gamma}(\nu(2e+2))$ are defined by reversing the roles of α and β.

The functions $\nu(g)$ and $\tilde{\gamma}(\nu(g))$ are defined recursively in the predicate

$$\lambda euv(Ex)[T_1^1((x)_0, e, e, (x)_1) \;\&\; (z)_{z<(x)_1}(x)_{0,z} < 2 \;\&\; (z)_{z<\min(u,(x)_1)}(x)_{0,z} = (v)_z],$$

and are therefore of degree $\leqq \mathbf{0}'$. Using (2) – (5), $\mathbf{a}' \cup \mathbf{b}' \leqq \mathbf{a} \cup \mathbf{b}$, which together with $\mathbf{a} \cup \mathbf{b} \leqq \mathbf{0}'$ establishes (1).

Now by ([4]: (4), (10), and (5)),

$$(10) \qquad \mathbf{0}' \leqq \mathbf{a}' \leqq \mathbf{a}' \cup \mathbf{b}' \;\&\; \mathbf{0}' \leqq \mathbf{b}' \leqq \mathbf{a}' \cup \mathbf{b}',$$

which with (1) yields

$$(11) \qquad \mathbf{a}' \cup \mathbf{b}' = \mathbf{a} \cup \mathbf{b} = \mathbf{a}' = \mathbf{b}' = \mathbf{0}'.$$

Using $\mathbf{0} \leqq \mathbf{a} \leqq \mathbf{0}'$, $\mathbf{0} \leqq \mathbf{b} \leqq \mathbf{0}'$, (11), and ([4], (7)), if any one of $\mathbf{a} \mid \mathbf{b}$, $\mathbf{0} < \mathbf{a} < \mathbf{0}'$, $\mathbf{0} < \mathbf{b} < \mathbf{0}'$ were false, we would have $\mathbf{a} = \mathbf{a}'$ or $\mathbf{b} = \mathbf{b}'$, contradicting ([4], (11)).

Proof (for arbitrary $\mathbf{d}$). Let $\lambda a\alpha(2a)$ and $\lambda a\beta(2a)$ each be the representing function of a fixed predicate of degree $\mathbf{d}$. The values of α and β for odd arguments are determined as in the proof for $\mathbf{d} = \mathbf{0}$, except for the obvious changes in (6)–(9) to take into account that $\alpha(2a)$ and $\beta(2a)$ are determined prior to the construction. Letting $\mathbf{a}$ and $\mathbf{b}$ be the degrees of α and β respectively, we have $\mathbf{d} \leqq \mathbf{a} \leqq \mathbf{d}'$ and $\mathbf{d} \leqq \mathbf{b} \leqq \mathbf{d}'$. The proof is completed as above with $\mathbf{d}$ in place of $\mathbf{0}$.

Discussion. The theorem shows that each of the four possibilities listed in ([4], top line page 385) can occur when $\mathbf{a}' = \mathbf{b}'$. Thus for the $\mathbf{a}$ and $\mathbf{b}$ of the theorem, $\mathbf{a}' = \mathbf{b}'$ with $\mathbf{a} \mid \mathbf{b}$, which shows in addition that the converse of ([4], (10)) is false. But also $\mathbf{a}' = \mathbf{d}'$ with $\mathbf{a} > \mathbf{d}$.

To deal with the conjecture in the next paragraph of [4], in the present theorem $\mathbf{a} \mid \mathbf{b}$ and $\mathbf{a}' \cup \mathbf{b}' < (\mathbf{a} \cup \mathbf{b})'$ (since $\mathbf{d}' < \mathbf{d}''$), but in Theorem 2 below $\mathbf{b}_1 \mid \mathbf{b}_2$ and $\mathbf{b}_1' \cup \mathbf{b}_2' = (\mathbf{b}_1 \cup \mathbf{b}_2)'$.

Theorem 1 shows also that each complete degree $\mathbf{d}'$ is the l.u.b. of the set of degrees less than it, since $\mathbf{d}' = \mathbf{a} \cup \mathbf{b}$ with both $\mathbf{a}$ and $\mathbf{b} < \mathbf{d}'$. However not every degree has this property. As an example let $\mathbf{c}$ be a degree $>\mathbf{0}$ with no degree between $\mathbf{0}$ and $\mathbf{c}$ (see Theorem 4 below); the set of degrees $<\mathbf{c}$ consists of $\mathbf{0}$ alone, and thus $\mathbf{c}$ is not the l.u.b. of this set.

Given $\mathbf{a} = \mathbf{b}$ or $\mathbf{a} < \mathbf{b}$, we know now exactly which relations can hold between $\mathbf{a}'$ and $\mathbf{b}'$ (for $\mathbf{a} < \mathbf{b}$, the possibilities are $\mathbf{a}' = \mathbf{b}'$ and $\mathbf{a}' < \mathbf{b}'$). However when $\mathbf{a} \mid \mathbf{b}$ all we know is that $\mathbf{a}' = \mathbf{b}'$ is one possibility. Thus we do not know if there are degrees $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a}' \mid \mathbf{b}'$, or such that $\mathbf{a} \mid \mathbf{b} \;\&\; \mathbf{a}' < \mathbf{b}'$.

1.3. Theorem 2. *Given a degree* $\mathbf{d}$, *there are degrees* $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *such that* $\mathbf{b}_1' \cup \mathbf{b}_2' = (\mathbf{b}_1 \cup \mathbf{b}_2)' = \mathbf{b}_1' = \mathbf{b}_2' = \mathbf{d}' \;\&\; \mathbf{b}_1 \mid \mathbf{b}_2 \;\&\; \mathbf{d} < \mathbf{b}_1 < \mathbf{d}' \;\&\; \mathbf{d} < \mathbf{b}_2 < \mathbf{d}'$.

Proof (for $\mathbf{d} = \mathbf{0}$). Let $\mathbf{b}_1$ be the degree of the function $\beta_1(a) = \beta(2a)$, and $\mathbf{b}_2$ of $\beta_2(a) = \beta(2a+1)$. Then repeating the construction in the proof of Theorem 1, inserting additional steps to guarantee $\mathbf{b}_1 \mid \mathbf{b}_2$, we obtain degrees $\mathbf{a}$ and $\mathbf{b}$

satisfying Theorem 1 such that $\mathbf{b} = \mathbf{b}_1 \cup \mathbf{b}_2$ and $\mathbf{b}_1 \mid \mathbf{b}_2$. The theorem follows using $\mathbf{b}' = \mathbf{0}'$.

We need only show that these additional steps can be inserted without invalidating the arguments of the previous proof. The new conditions are that, for $(i, j) = (1, 2), (2, 1)$ and $e = 0, 1, 2, \cdots$,

$$(12) \qquad \beta_i(a) = U(\mu y T_1^1(\bar{\beta}_j(y), e, a, y))$$

does not hold for every a. Each of these conditions can be satisfied by the least value of a for which $\beta_i(a)$ has not been defined. The details of this part of the construction are similar to the treatment of [4] Theorem 1, Case 0, omitting α_2, and defining the new values of α to coincide with the new values of β. Then inserting the new steps between those of the previous proof, we observe that (3) and (4) remain valid for the function $\nu(g)$ defined by (5), and that the function γ so defined is of degree $\leqq \mathbf{0}'$. These observations suffice to prove the theorem for $\mathbf{d} = \mathbf{0}$.

PROOF (for arbitrary $\mathbf{d}$). Let $\lambda a\beta_1(2a)(=\lambda a\beta(4a))$, $\lambda a\beta_2(2a)(=\lambda a\beta(4a + 1))$, $\lambda a\alpha(4a)$, and $\lambda a\alpha(4a + 1)$ each be the representing function of a fixed predicate of degree $\mathbf{d}$. With suitable changes in the proof for $\mathbf{d} = \mathbf{0}$ to take into account that α and β are initially determined for arguments of the forms $4a$ and $4a + 1$, we obtain degrees $\mathbf{b}_1$ and $\mathbf{b}_2$ satisfying the theorem.

2. Sets of degrees without g.l.b. or without l.u.b.

In ([4], 4.2) it was shown that the upper semi-lattice of degrees is not a lattice, i.e. there are degrees $\mathbf{a}$ and $\mathbf{b}$ which have no g.l.b. However, the degrees constructed were not arithmetical. By modifying that construction we shall obtain two arithmetical degrees (both $<\mathbf{0}'$, see Footnote 4) which do not possess a g.l.b., thereby showing that the arithmetical degrees do not form a lattice. By an argument on cardinality, it was shown ([4], p. 399) that the semi-lattice is not ω-complete with respect to l.u.b. Moreover, by a slight additional argument in the proof of [4] Theorem 3 (e.g. our Corollary 2 below), it can be shown that the sequence $\mathbf{a}, \mathbf{a}', \mathbf{a}'', \cdots$ has no l.u.b. We shall show that no infinite increasing sequence of degrees has a l.u.b.

The sets S_g and $S_{g+1} - S_g$ in the proof of [4] Theorem 3 (and of our Theorem 3) are no longer all finite. However they are recursive. Indeed, S_0 is the empty set, and

$$(13) \qquad \begin{aligned} S_{g+1} = \hat{x}[x \in S_g \vee x < \nu(g + 1) \vee \{x = 2^{(x)_0}3^{(x)_1}5^{(x)_2} \\ \& \ g = 2^{(x)_0}3^{(x)_1} \ \& \ (x)_2 \geqq \nu(g + 1)\}]. \end{aligned}$$

THEOREM 3. *Given an infinite sequence of degrees* $\mathbf{a}_0 < \mathbf{a}_1 < \mathbf{a}_2 < \cdots$ *and a two-place predicate* $A(a, j)$ *of degree* $\mathbf{a}$ *such that* $\lambda a A(a, j)$ *is of degree* $\mathbf{a}_j (j = 0, 1, 2, \cdots)$, *there are degrees* $\mathbf{b}_1$ *and* $\mathbf{b}_2$ *such that* (A) $\mathbf{b}_1 < \mathbf{a}'$ *and* $\mathbf{b}_2 < \mathbf{a}'$, (B) $\mathbf{a}_j \leqq \mathbf{b}_1$ *and* $\mathbf{a}_j \leqq \mathbf{b}_2 (j = 0, 1, 2, \cdots)$, (C) *for every degree* $\mathbf{d}$ *with* $\mathbf{d} \leqq \mathbf{b}_1$ *and* $\mathbf{d} \leqq \mathbf{b}_2$, *there is a degree* $\mathbf{a}_j$ *with* $\mathbf{d} \leqq \mathbf{a}_j$.

PROOF. Construct β_1 (of degree $\mathbf{b}_1$) and β_2 (of degree $\mathbf{b}_2$) as in the proof of ([4], Theorem 3), letting our $A(a, j)$ play the role of $L^A(a, j)$. To establish (A) let

$$\psi(a, g) = 2^{\nu(g)}3^{\kappa_1(a,g)}5^{\kappa_2(a,g)}. \tag{14}$$

We define a function $\rho(g)$ of degree $\leqq \mathbf{a}'$ such that, for each g, $\lambda a\psi(a, g)$ is recursive in A with Gödel number $\rho(g)$. Let $\rho(0)$ be a Gödel number of $\lambda a\psi(a, 0)$ from A where $\psi(a, 0) = 2^0 3^2 5^2$. Suppose, for purpose of induction, that $\lambda a\psi(a, g)$ is recursive in A with Gödel number $\rho(g)$. Each subcase hypothesis used in defining $\lambda a\psi(a, g + 1)$ can be written in the form $(Ey)R^A(\rho(g), g, y)$ with R^A recursive uniformly in A, or in the dual form. Hence there is a function ξ of degree $\leqq \mathbf{a}'$ such that $\xi(\rho(g), g) = 0, 1,$ or 2 according to which one of Subcase 1.1, 1.2, or Case 2 applies. Moreover there is a uniform Gödel number f from A such that

$$\psi(a, g + 1) = \{f\}^A(i, \rho(g), g, a) \tag{15}$$

where $i = \xi(\rho(g), g)$. Thus, setting

$$\rho(g + 1) = S_1^{3,2}(f, \xi(\rho(g), g), \rho(g), g), \tag{16}$$

$\rho(g + 1)$ is a Gödel number of $\lambda a\psi(a, g + 1)$ from A. Also ρ is (primitive) recursive in ξ, and therefore the function $\psi(a, g) = \{\rho(g)\}^A(a)$ is of degree $\leqq \mathbf{a}'$. Hence $\mathbf{b}_1 \leqq \mathbf{a}'$ and $\mathbf{b}_2 \leqq \mathbf{a}'$. We obtain the strong inequality of (A) by observing that, as a consequence of (B) and (C), $\mathbf{b}_1$ and $\mathbf{b}_2$ are incomparable.

Our (B) is established by the same argument that establishes ([4], (B)). To prove (C) we can use the argument for ([4], (C)) to conclude that δ is recursive in $\lambda a\psi(a, g)$, and therefore in $\lambda a\psi(a, g + 1)$, for $g = 2^{e_0}3^{e_1}$. But, by the construction and the monotonicity of the sequence $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \cdots$, our $\lambda a\psi(a, g + 1)$ is of degree $\mathbf{a}_g$. Hence δ is of degree $\leqq \mathbf{a}_g$.

COROLLARY 1. *The upper semi-lattice of arithmetical degrees is not a lattice.*

PROOF. By ([4], Theorem 2, Corollary 3) there is a predicate $A(a, j)$ satisfying the hypothesis of our Theorem 3 with $\mathbf{a} \leqq \mathbf{0}'$.[4] Hence there are degrees $\mathbf{b}_1$ and $\mathbf{b}_2$ such that

$$\mathbf{a}_j < \mathbf{b}_k < \mathbf{a}' \leqq \mathbf{0}'' \qquad \text{(for all } j \text{ and } k = 1, 2), \tag{17}$$

and, for each $\mathbf{d}$ with $\mathbf{d} \leqq \mathbf{b}_1$ and $\mathbf{d} \leqq \mathbf{b}_2$, there is a number j such that $\mathbf{d} \leqq \mathbf{a}_j < \mathbf{a}_{j+1}$. Hence $\mathbf{b}_1$ and $\mathbf{b}_2$ have no g.l.b. Indeed we have proven the stronger result that $\mathbf{b}_1$ and $\mathbf{b}_2$ have no maximal lower bound.

COROLLARY 2. *If* $\mathbf{a}_0 < \mathbf{a}_1 < \mathbf{a}_2 < \cdots$ *is any infinite ascending sequence of degrees, the set* $\{\mathbf{a}_n\}$ *has no* l.u.b.

PROOF. Let $\mathbf{b}_1$ and $\mathbf{b}_2$ be the degrees defined in the theorem relative to the sequence $\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \cdots$ and any suitable $A(a, j)$. If $\mathbf{d}$ were a l.u.b. of this sequence, then by (B), $\mathbf{d} \leqq \mathbf{b}_k (k = 1, 2)$. Hence by (C), $\mathbf{d} < \mathbf{a}_{j+1}$ for some j, and

[4] Combining the methods of ([4], 3.2) and our §1, it is possible to obtain $A(a, j)$ such that $\mathbf{a}' = \mathbf{0}'$. Hence there are degrees $\mathbf{b}_1$ and $\mathbf{b}_2$, both $<\mathbf{0}'$, which have no g.l.b.

thus **d** could not have been a l.u.b. We have in fact shown that there is no upper bound $\leqq$ both of the upper bounds $\mathbf{b}_1$ and $\mathbf{b}_2$.

COROLLARY 3. *If* $\{\mathbf{a}_n\}$ *is any denumerable set of degrees, then either* $\mathbf{a}_0 \cup \mathbf{a}_1 \cup \cdots \cup \mathbf{a}_j$ *is a* l.u.b. *of the set for some* j, *or the set has no* l.u.b.

PROOF. Let $\mathbf{b}_n = \mathbf{a}_0 \cup \mathbf{a}_1 \cup \cdots \cup \mathbf{a}_n$. Then clearly $\{\mathbf{b}_n\}$ and $\{\mathbf{a}_n\}$ have the same upper bounds. If the set $\{\mathbf{b}_n\}$ is finite, then $\mathbf{a}_0 \cup \mathbf{a}_1 \cup \cdots \cup \mathbf{a}_j$ is a l.u.b. of $\{\mathbf{a}_n\}$ where j is the least n such that $m > n \rightarrow \mathbf{b}_m = \mathbf{b}_n$. If $\{\mathbf{b}_n\}$ is infinite, then by Corollary 2 it has no l.u.b., and therefore $\{\mathbf{a}_n\}$ has no l.u.b.

DISCUSSION. Thus no lower **a**-generable degree (see [4], 3.5) is a l.u.b. of the rational degrees constituting its cut, which settles the question left open in ([4], Footnote 27). With respect to the sequence $\mathbf{a}, \mathbf{a}', \mathbf{a}'', \cdots$, (A) of ([4], Theorem 3) is stronger than our (A), since in [4], $\mathbf{b}_k < \mathbf{a}^{(\omega)}$, whereas we obtain only $\mathbf{b}_k < (\mathbf{a}^{(\omega)})'$. The stronger result is obtained through ([4], Lemma 3). Thus $\mathbf{a}^{(\omega)}$ is not even a minimal upper bound of $\mathbf{a}, \mathbf{a}', \mathbf{a}'', \cdots$. Whether a minimal upper bound exists for this particular sequence, or for any other increasing sequence, is an open question.

3. Non-density

3.1. The remainder of this paper is concerned with the density problem ([4], p. 391). We shall show that given degrees **a** and **c** with $\mathbf{a} < \mathbf{c}$, there does not always exist a degree **b** with $\mathbf{a} < \mathbf{b} < \mathbf{c}$. In fact, for any degree **a**, there is a degree $\mathbf{c} > \mathbf{a}$ with no degree between. Thus among all degrees $> \mathbf{a}$ there is a minimal one. In particular there is a minimal degree of recursive *un*solvability.

For a fixed degree **a**, consider the set of degrees $\mathbf{c} > \mathbf{a}$ with no degree between. The construction which follows shows that this set is not empty; indeed it is infinite. For instead of constructing a single function γ (of degree **c**), we could construct two or more functions $\gamma_0, \gamma_1, \gamma_2, \cdots$, treating each γ_i as the γ of the following proof, and inserting steps to guarantee, for every e and $i \neq j$, that γ_i is not recursive in γ_j with Gödel number e.

3.2. Consider the problem of defining a function γ of minimal degree of *un*solvability. Let C_{2e+2} be the condition that γ is not recursive with Gödel number e, and C_{2e+1} that any function β recursive in γ with Gödel number e either is recursive or is of the same degree as γ. Then any function satisfying $C_1, C_2, \cdots$ will have minimal degree of *un*solvability.

As explained in 1.1, each construction in [4] (and in the first two sections of this paper) can be given in the form of an induction on classes of functions of the form $\mathfrak{F}(S, \sigma)$. With such an induction it is easy to handle the conditions C_{2e+2}; however we do not know if the induction can be applied to the conditions C_{2e+1}. This unsolved problem can be stated in the following simplified form. Given a number e, does there always exist a set S with infinite complement and a function σ on S such that

$$(18) \quad \gamma \in \mathfrak{F}(S, \sigma) \ \& \ [\beta \text{ is recursive in } \gamma \text{ with Gödel number } e] \rightarrow [\beta \text{ is recursive}] \vee [\gamma \text{ is recursive in } \beta],$$

where γ ranges over all functions of one argument with values 0 or 1, and β over all completely defined functions of one argument? A Gödel number e produced by a negative solution to this problem would define an interesting recursion. A positive solution would greatly simplify the following proof, and would be of interest in studying relative recursiveness.

The construction which follows involves classes of functions $\mathfrak{F}_g$ which cannot be written in the form $\dot{\mathfrak{F}}(S_g, \sigma_g)$. Instead, at the g^{th} stage, the natural numbers are partitioned into an infinite number of consecutive disjoint intervals $S_{g,0}$, $S_{g,1}$, $S_{g,2}$, $\cdots$, and two distinct functions are defined on each interval $S_{g,j}$. The class $\mathfrak{F}_g$ consists of all functions γ such that, on each interval $S_{g,j}(j = 0, 1, 2, \cdots)$, γ coincides with one of the two functions defined on that interval. For example, let

$$(19) \qquad S_{1,j} = \hat{x}[j^2 \leqq x < (j+1)^2].$$

On each of these intervals we define two functions, e.g. the constant function 0 and the constant function 1. Then in this case

$$(20) \qquad \gamma \in \mathfrak{F}_1 \equiv (j)(x)[j^2 \leqq x < (j+1)^2 \rightarrow \gamma(x) = \gamma(j^2) < 2].$$

Each class $\mathfrak{F}_g$ will be named by a triple of functions η, θ_0, θ_1 for which (i) $\eta(0) = 0$ & $(j)[\eta(j) < \eta(j+1)]$, and (ii) θ_0 and θ_1 are functions of one argument with values 0 or 1 which are not identical on any interval

$$\hat{x}[\eta(j) \leqq x < \eta(j+1)](=S_{g,j}).^5$$

For the example above, $\eta(j) = j^2$, $\theta_0(x) = 0$, and $\theta_1(x) = 1$. For functions η, θ_0, θ_1 satisfying (i) and (ii), we define

$$(21) \quad \mathfrak{F}(\eta, \theta_0, \theta_1) = \hat{\gamma}(j)(Ei)_{i<2}(x)[\eta(j) \leqq x < \eta(j+1) \rightarrow \gamma(x) = \theta_i(x)].$$

3.3. THEOREM 4. *For each degree* **a** *there is a degree* $\mathbf{c} < \mathbf{a}''$ *with* $\mathbf{a} < \mathbf{c}$ *such that* $\mathbf{a} < \mathbf{b} < \mathbf{c}$ *for no degree* **b**.

PROOF. In order to simplify notation, we prove the theorem with $\mathbf{a} = \mathbf{0}$, and then indicate what changes are needed to obtain the general result. As described above, classes $\mathfrak{F}_0 \supset \mathfrak{F}_1 \supset \mathfrak{F}_2 \supset \cdots$ will be defined so that

$$(22) \qquad \gamma \in \mathfrak{F}_{2e+2} \rightarrow [\gamma \text{ is not recursive with Gödel number } e],$$

$$(23) \qquad \gamma \in \mathfrak{F}_{2e+1} \rightarrow [\text{any function recursive in } \gamma \text{ with Gödel number } e \text{ either is recursive or is of the same degree as } \gamma].$$

(The class $\mathfrak{F}_0$ consists of all functions of one argument with values 0 or 1.) Furthermore, each $\mathfrak{F}_g$ will be written in the form $\mathfrak{F}(\eta, \theta_0, \theta_1)$ with η, θ_0, θ_1 recursive and satisfying (i) and (ii) of 3.2.

[5] Although the functions η, θ_0, θ_1 depend on g, we do not exhibit g as an argument for notational convenience.

CASE 0: $g = 0$. Set $\mathfrak{F}_0 = \mathfrak{F}(\eta, \theta_0, \theta_1)$ where

$$\eta(j) = j, \qquad \theta_0(x) = 0, \qquad \theta_1(x) = 1. \tag{24}$$

CASE 1: rm $(g, 2) = 0$ & $g \neq 0$. Write $g = 2e + 2$ and $\mathfrak{F}_{2e+1} = \mathfrak{F}(\eta, \theta_0, \theta_1)$. We seek recursive functions η', θ_0', θ_1' satisfying (i) and (ii) such that the class $\mathfrak{F}_{2e+2} = \mathfrak{F}(\eta', \theta_0', \theta_1')$ satisfies (22) and is a subclass of $\mathfrak{F}(\eta, \theta_0, \theta_1)$. We assume as part of the induction hypothesis that η, θ_0, θ_1 are recursive and satisfy (i) and (ii). Let

$$m = \mu x[\theta_0(x) \neq \theta_1(x)] \qquad (m < \eta(1)). \tag{25}$$

SUBCASE 1.1: $e(m)(\simeq U(\mu y T_1(e, m, y)))$ is defined, i.e. $(Ey)T_1(e, m, y)$. Let i be the smaller of 0 and 1 such that $\theta_i(m) \neq e(m)$, and set

$$\begin{aligned} &\eta'(0) = 0, \qquad \eta'(j) = \eta(j+1) && (j > 0), \\ &\theta_0'(x) = \theta_1'(x) = \theta_i(x) && (x < \eta(1)), \\ &\theta_0'(x) = \theta_0(x), \qquad \theta_1'(x) = \theta_1(x) && (x \geqq \eta(1)). \end{aligned} \tag{26}$$

SUBCASE 1.2: $e(m)$ is undefined. Proceed as above with $i = 0$.

CASE 2: rm $(g, 2) = 1$. Write $g = 2e + 1$ and $\mathfrak{F}_{2e} = \mathfrak{F}(\eta, \theta_0, \theta_1)$ where η, θ_0, θ_1 are recursive and satisfy (i) and (ii). The class $\mathfrak{F}_{2e+1} = \mathfrak{F}(\eta', \theta_0', \theta_1')$ will be defined so that

$$\begin{aligned} \gamma \in \mathfrak{F}_{2e+1} \rightarrow &[\lambda a\{e\}^{\gamma}(a) \text{ is not completely defined}] \\ &\vee\ [\lambda a\{e\}^{\gamma}(a) \text{ is recursive}]\ \vee\ [\gamma \text{ is recursive in } \lambda a\{e\}^{\gamma}(a)]. \end{aligned} \tag{27}$$

We consider three subcases corresponding to the three disjunctive members of the right side of (27). Only the third subcase requires classes of the form $\mathfrak{F}(\eta, \theta_0, \theta_1)$ as opposed to $\mathfrak{F}(S, \sigma)$.

Following Kleene [3] §24, we represent the sequence of x numbers $\gamma(0)$, $\gamma(1)$, $\cdots$, $\gamma(x-1)$ by the *sequence number*

$$\bar{\gamma}(x) = \prod_{i<x} p_i^{\gamma(i)+1} \qquad (x \geqq 0) \tag{28}$$

noting that lh $(\bar{\gamma}(x)) = x$; write Seq (a) when a is a sequence number (i.e. when a is of the form $\bar{\gamma}(x)$ for some γ and some $x \geqq 0$); and employ the normal form

$$\{e\}^{\gamma}(a) \simeq U(\mu x T_1^1(\bar{\gamma}(x), e, a)). \tag{29}$$

We say two numbers a and b are *compatible* (Cpt (a, b)) if $(a)_i = (b)_i$ whenever both $(a)_i \neq 0$ and $(b)_i \neq 0$. We say *b extends a* (Ext (b, a)) if a and b are sequence numbers, and the sequence represented by b is an extension of the one represented by a, i.e.

$$\text{Ext}\,(b, a) \equiv \text{Seq}\,(a)\ \&\ \text{Seq}\,(b)\ \&\ \text{Cpt}\,(a, b)\ \&\ b \geqq a. \tag{30}$$

Let P be the set of numbers of the form $\bar{\gamma}(x)$, and P' of the form $\bar{\gamma}(\eta(x))$, for γ in $\mathfrak{F}(\eta, \theta_0, \theta_1)$. Both of these sets are recursive.

Subcase 2.1: for some p in P' and some a, $\{e\}^{\gamma}(a)$ is undefined for all γ in $\mathfrak{F}(\eta, \theta_0, \theta_1)$ such that $\bar{\gamma}(\mathrm{lh}\,(p)) = p$. In symbols,

$$(Ep)(Ea)(s)[p \in P' \;\&\; \{s \in P \;\&\; \mathrm{Cpt}\,(s, p) \rightarrow \overline{T}_1^1(s, e, a)\}]. \tag{31}$$

Let $Z = \mu z(s)[(z)_0 \in P' \;\&\; \{s \in P \;\&\; \mathrm{Cpt}\,(s, (z)_0) \rightarrow \overline{T}_1^1(s, e, (z)_1)\}]$, $p = (Z)_0$, $a = (Z)_1$, $q = \mu x[\,\mathrm{lh}\,(p) = \eta(x)]$. It suffices to set

$$\begin{aligned} &\eta'(0) = 0, \qquad \eta'(j) = \eta(j + q) && (j > 0), \\ &\bar{\theta}_0'(\mathrm{lh}\,(p)) = \bar{\theta}_1'(\mathrm{lh}\,(p)) = p, \\ &\theta_i'(x) = \theta_i(x) && (x \geqq \mathrm{lh}\,(p);\, i = 0, 1). \end{aligned} \tag{32}$$

Then

$$\gamma \in \mathfrak{F}(\eta', \theta_0', \theta_1') \equiv \gamma \in \mathfrak{F}_{2e} \;\&\; \bar{\gamma}(\mathrm{lh}\,(p)) = p, \tag{33}$$

$$\gamma \in \mathfrak{F}(\eta', \theta_0', \theta_1') \rightarrow [\{e\}^{\gamma}(a) \text{ is undefined}]. \tag{34}$$

Subcase 2.2: there is a sequence number p in P' such that, for each a, the value of $\{e\}^{\gamma}(a)$, whenever defined, is independent of the choice of γ in $\mathfrak{F}_{2e}$ provided $\bar{\gamma}(\mathrm{lh}\,(p)) = p$. In symbols,

$$\begin{aligned} &(Ep)(s)(t)(a)[p \in P' \;\&\; \{s \in P \;\&\; t \in P \;\&\; \mathrm{Cpt}\,(s, p) \;\&\; \mathrm{Cpt}\,(t, p) \\ &\qquad \& \; T_1^1(s, e, a) \;\&\; T_1^1(t, e, a) \rightarrow U(\mathrm{lh}\,(s)) = U(\mathrm{lh}\,(t))\}]. \end{aligned} \tag{35}$$

For sake of definiteness we assume also that Subcase 2.1 does not apply. Let p be the least number asserted to exist in (35), and define q, η', θ_0', θ_1' as in Subcase 2.1. Then (33) holds.

Let γ be in $\mathfrak{F}(\eta', \theta_0', \theta_1')$. Then either $\lambda a\{e\}^{\gamma}(a)$ is only partially defined, or is general recursive. For using the definition of p and (29),

$$\{e\}^{\gamma}(a) = U(\mathrm{lh}\,(\mu s[s \in P \;\&\; \mathrm{Cpt}\,(s, p) \;\&\; T_1^1(s, e, a)])) \tag{36}$$

whenever the left side is defined.

Subcase 2.3: otherwise, i.e. (31) and (35) are both false. Indeed they remain so when P is substituted for P'. So for any p in P,

$$(a)(Es)[s \in P \;\&\; \mathrm{Cpt}\,(s, p) \;\&\; T_1^1(s, e, a)], \tag{37}$$

$$\begin{aligned} &(Es)(Et)(Ea)[s \in P \;\&\; t \in P \;\&\; \mathrm{Cpt}\,(s, p) \;\&\; \mathrm{Cpt}\,(t, p) \\ &\qquad \& \; T_1^1(s, e, a) \;\&\; T_1^1(t, e, a) \;\&\; U(\mathrm{lh}\,(s)) \neq U(\mathrm{lh}\,(t))]. \end{aligned} \tag{38}$$

We shall define η', θ_0', θ_1' so that, whenever γ is in $\mathfrak{F}(\eta', \theta_0', \theta_1')$ and the function $\beta(a) \simeq \{e\}^{\gamma}(a)$ is completely defined, γ will be the unique function δ in $\mathfrak{F}(\eta', \theta_0', \theta_1')$ such that $\beta(a) \simeq \{e\}^{\delta}(a)$ for all a. This uniqueness will be the key to showing that γ is recursive in β.

For sequence numbers p we set

$$\beta_p(a) \simeq U(\mathrm{lh}\,(\mu s[\mathrm{Ext}\,(p, s) \;\&\; T_1^1(s, e, a)])). \tag{39}$$

Then

(40) $[\beta_p(a) \text{ is defined}] \ \& \ \bar{\gamma}(\text{lh}\ (p)) = p \rightarrow \beta_p(a) = \{e\}^{\gamma}(a).$

Let p and q be arbitrary elements of P. Using (38) there are sequence numbers s and t in P, both compatible with p, and a number a such that $\beta_s(a) \neq B_t(a)$, both sides being defined. Using (37) there is a sequence number v in P, compatible with q, such that $\beta_v(a)$ is defined. Let u be the smaller of s and t such that

(41) $$\beta_u(a) \neq \beta_v(a).$$

Thus, given p and q in P, there are sequence numbers u and v in P, compatible with p and q respectively, such that (41) holds for some a. Since the value of $\beta_s(a)$ (if defined) is not changed when s is replaced by any sequence number extending s, we can extend u and v if necessary so that

(42) $$\text{lh}\ (u) = \text{lh}\ (v) \ \& \ u \in P' \ \& \ v \in P' \ \& \ \text{Ext}\ (u, p) \ \& \ \text{Ext}\ (v, q) \ \& \ \beta_u(a) \neq \beta_v(a).$$

Moreover, there is a uniform recursive procedure for obtaining u, v, and a from p, q, e, η, θ_0, θ_1. Suppressing e, η, θ_0, θ_1 as arguments in order to save space, let χ_0, χ_1, χ_2 be recursive functions (actually functionals) such that, for all p and q in P, $\chi_0(p, q)(=u)$, $\chi_1(p, q)(=v)$, $\chi_2(p, q)\ (=a)$ satisfy (42).

Next we shall define recursive functions $\psi_0(m)$, $\psi_1(m)$, $\psi_2(m, i)$ so that, if

(43) $$s_0 < s_1 < \cdots < s_{2^m-1}$$

are the 2^m sequence numbers in P' of length $\eta(m)$, then

(44) $$s_i * \psi_0(m) \in P' \ \& \ s_i * \psi_1(m) \in P' \ \& \ \text{lh}\ (\psi_0(m)) = \text{lh}\ (\psi_1(m)) > 0 \\ \& \ \beta_{s_i * \psi_0(m)}(\psi_2(m, i)) \neq \beta_{s_i * \psi_1(m)}(\psi_2(m, i))$$

for $i < 2^m$. When $m = 0$ (43) reduces to the single sequence number $s_0 = 1$ representing the empty sequence. Thus we set

(45) $$\psi_j(0) = \chi_j(1, 1), \qquad \psi_2(0, 0) = \chi_2(1, 1) \qquad (j = 0, 1).$$

When $m > 0$ we shall define a_i, u_i, v_i for $i < 2^m$ so that

(46) $$\text{Ext}\ (u_i, u_{i-1}) \ \& \ \text{Ext}\ (v_i, v_{i-1}) \ \& \ s_i * u_i \in P' \ \& \ s_i * v_i \in P' \\ \& \ \text{lh}\ (u_i) = \text{lh}\ (v_i) \ \& \ \beta_{s_i * u_i}(a_i) \neq \beta_{s_i * v_i}(a_i).$$

Let

(47) $$u_0 = \chi_0(1, 1), \qquad v_0 = \chi_1(1, 1), \qquad \psi_2(m, 0) = a_0 = \chi_2(1, 1).$$

For $i < 2^m - 1$, let u_{i+1} and v_{i+1} be the unique sequence numbers satisfying

(48) $$s_{i+1} * u_{i+1} = \chi_0(s_{i+1} * u_i, s_{i+1} * v_i) \ \& \ s_{i+1} * v_{i+1} = \chi_1(s_{i+1} * u_i, s_{i+1} * v_i),$$

and let

(49) $$\psi_2(m, i + 1) = a_{i+1} = \chi_2(s_{i+1} * u_i, s_{i+1} * v_i).$$

Finally set

(50) $\psi_0(m) = u_{2^m-1}$, $\psi_1(m) = v_{2^m-1}$, $\psi_2(m, i) = 0$ $(i \geqq 2^m)$.

Then by (46) and the fact that $\psi_0(m)$ extends each u_i, and $\psi_1(m)$ each v_i, ψ_0, ψ_1, ψ_2 satisfy (44).

Now we define η', θ_0', θ_1' from the ψ_i. For each j let

(51) $\eta'(0) = 0$, $\eta'(j + 1) = \eta'(j) + \text{lh}(\psi_0(m_j))$, $m_j = \mu k[\eta'(j) = \eta(k)]$,

and for $x < \eta'(j + 1) - \eta'(j)$, and $i = 0, 1$ let

(52) $\theta_i'(\eta'(j) + x) = (\psi_i(m_j))_x - 1$.

Each of the functions η', θ_0', θ_1' is recursive. In fact each can be written in the form $\lambda x\mathsf{F}(x, e, \eta, \theta_0, \theta_1)$ where F is a recursive functional. The functions θ_0' and θ_1' cannot be identical on any interval $\hat{x}[\eta'(j) \leqq x < \eta'(j + 1)]$, since $\psi_0(m_j) \neq \psi_1(m_j)$ by (44); and $\eta'(j + 1) > \eta'(j)$ since $\text{lh}(\psi_0(m_j)) > 0$. Hence η', θ_0', θ_1' satisfy (i) and (ii) of 3.2. This completes Subcase 2.3 and the induction step from $\mathfrak{F}_{2e} = \mathfrak{F}(\eta, \theta_0, \theta_1)$ to $\mathfrak{F}_{2e+1} = \mathfrak{F}(\eta', \theta_0', \theta_1')$ with the justification to follow.

Suppose that γ is a function in $\mathfrak{F}_{2e+1}$, and that the function $\beta(a) \simeq \{e\}^\gamma(a)$ is completely defined. Then clearly Subcase 2.1 does not apply. If Subcase 2.2 applies, β is recursive. If Subcase 2.3 applies, then γ is recursive in β; for suppose, for induction on j, that $\bar{\gamma}(\eta'(j))$ has been computed from β. Let

$$m = \mu k[\eta'(j) = \eta(k)],$$

and choose $i < 2^m$ such that $s_i = \bar{\gamma}(\eta'(j))$, where s_i is defined as in (43). Then by (52), $\bar{\gamma}(\eta'(j + 1))$ is one of the two sequence numbers $s_i * \psi_0(m)$ and $s_i * \psi_1(m)$, and by (40) and (44), each of these two possibilities for $\bar{\gamma}(\eta'(j + 1))$ would yield a different value for $\beta(\psi_2(m, i))$. Thus, noting the actual value of $\beta(\psi_2(m, i))$, we can determine the value of $\bar{\gamma}(\eta'(j + 1))$. This completes Case 2.

Let $\lambda x\theta_i(g, x)$ be the function $\lambda x\theta_i(x)$ defined at the g^{th} stage ($i = 0, 1$; $g = 0, 1, 2, \cdots$),

and let

(53) $\nu(g) = \mu x[\theta_0(g, x) \neq \theta_1(g, x)]$.

By the construction in Case 1, $\nu(g + 2) > \nu(g)$. Hence there is a unique function γ belonging to all classes $\mathfrak{F}_g$. This function has minimal degree of *un*solvability. By the methods used in treating Theorem 3 Part (A), it can be shown that the degree $\mathbf{c}$ of γ is $\leqq \mathbf{0}''$.

To prove the theorem in its general form let A be a predicate of degree $\mathbf{a}$. In the construction above, replace "recursiveness" by "recursiveness in A", and let $\mathfrak{F}_0$ be the class of functions γ with values 0 or 1 such that $\lambda x\gamma(2x)$ is the representing function of A. In this way we obtain a function γ recursive in A'' such that A is recursive in γ but γ is not recursive in A; and for every function

β recursive in γ, either β is recursive in A, or γ is recursive in β, A. Let $\mathbf{c}$ be the degree of γ. Then $\mathbf{a} < \mathbf{c} \leqq \mathbf{a}''$, and

$$\mathbf{b} \leqq \mathbf{c} \rightarrow \mathbf{b} \leqq \mathbf{a} \vee \mathbf{c} \leqq \mathbf{b} \cup \mathbf{a}. \tag{54}$$

Suppose $\mathbf{a} < \mathbf{b} < \mathbf{c}$. By (54), $\mathbf{b} \leqq \mathbf{a}$ or $\mathbf{c} \leqq \mathbf{b} \cup \mathbf{a}$. The former contradicts $\mathbf{a} < \mathbf{b}$, and using $\mathbf{b} \cup \mathbf{a} = \mathbf{b}$, the latter contradicts $\mathbf{b} < \mathbf{c}$. This concludes the proof of Theorem 4.

REMARK. For degrees $\mathbf{a}$ and $\mathbf{c}$ satisfying the theorem, every degree $\mathbf{b} < \mathbf{c}$ is either $\leqq$ or incomparable to $\mathbf{a}$. We have not touched upon the question whether we can choose $\mathbf{c}$ to exclude the second possibility.

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A DECISION METHOD FOR ELEMENTARY ALGEBRA AND GEOMETRY

INTRODUCTION

By a *decision method* for a class K of sentences (or other expressions) is meant a method by means of which, given any sentence θ, one can always decide in a finite number of steps whether θ is in K; by a *decision problem* for a class K we mean the problem of finding a decision method for K. A decision method must be like a recipe, which tells one what to do at each step so that no intelligence is required to follow it; and the method can be applied by anyone so long as he is able to read and follow directions.

The importance of the decision problem for the whole of mathematics (and for various special mathematical theories) was stressed by Hilbert, who considered this as the main task of a new field of mathematical research for which he suggested the term "metamathematics". The most important kind of decision problems is that in which K is defined to be the class of true sentences of a certain theory. When we say that there is a decision method for a certain theory, we mean that there is a decision method for the class of true sentences of the theory(1). (All superscripts in round brackets refer to Notes, pp. 47ff.)

Some decision methods have been known for a very long time. For example, Euclid's algorithm provides (among other things) a decision method for the class of all true sentences of the form "p and q are relatively prime," where p and q are integers (or polynomials with constant coefficients). And Sturm's theorem enables one to decide how many roots a given polynomial has and thus to decide on the truth of sentences of the form, "the polynomial p has exactly k roots."

Other decision methods are of more recent date. Löwenheim (1915) gave a decision method for the class of correct formulas of the lower predicate calculus involving only one variable. Post (1921) gave an exact proof of the validity of the familiar decision method (the so-called "truth-table method") for ordinary sentential calculus. Langford (1927) gave a decision method for an elementary theory of linear order. Presburger (1930) gave a decision method for the part of the arithmetic of integers which involves only the operation of addition. Tarski (1940) found a decision method for the elementary theory of Boolean algebra. McKinsey (1943) gave a decision method for the class of true universal sentences of elementary lattice theory. Mrs. Szmielew has recently found a decision method for the elementary theory of Abelian groups(2).

There are also some important negative results in this connection. From the fundamental results of Gödel (1930) and subsequent improvements of them obtained by Church (1936) and Rosser (1936), it follows that there does not exist a decision method for any theory to which belong all the sentences of elementary number theory (i.e., the arithmetic of integers with addition and multiplication) — and hence no decision method for the whole of mathematics is possible. A similar result has been obtained recently by Mrs. Robinson for theories to which belong all the sentences of the arithmetic of rationals. It is also known that there do not exist decision methods for various parts of modern algebra — in fact, for the elementary theory of rings (Mostowski and Tarski), the elementary theories of groups and lattices (Tarski), and the elementary theory of fields (Mrs. Robinson)[(3)].

In this monograph we present a method (found in 1930 but previously unpublished)[(4)] for deciding on the truth of sentences of the elementary algebra of real numbers — and hence also of elementary geometry.

By elementary algebra we understand that part of the general theory of real numbers in which one uses exclusively variables representing real numbers, constants denoting individual numbers, like "0" and "1", symbols denoting elementary operations on and elementary relations between real numbers, like "+", ".", "−", "<", ">", and "=", and expressions of elementary logic such as "and", "or", "not", "for some x", and "for all x". Among formulas of elementary algebra we find algebraic equations and inequalities; and by combining equations and inequalities by means of the logical expressions listed above, we obtain arbitrary sentences of elementary algebra. Thus, for example, the following are sentences of elementary algebra:

$0 > (1 + 1) + (1 + 1)$;

For every a, b, c, and d, where $a \neq 0$, there exists an x such that

$ax^3 + bx^2 + cx + d = 0$.

The first sentence is false, and the second is true.

On the other hand, in elementary algebra we do not use variables standing for arbitrary sets or sequences of real numbers, for arbitrary functions of real numbers, and the like. (When in this monograph we attach the qualifier "elementary" to the name of a theory, we refer to this abstention from the use of set-theoretical notions.) Hence those algebraic concepts whose definitions in terms of the fundamental notions listed above would require some set-theoretical devices cannot be represented in our system of elementary algebra. This applies, for instance, to the general notion of a polynomial, to the notion of solvability of an equation by means of radicals, and the like. For this reason it is not possible, for example, to consider as a sentence of elementary algebra the sentence:

Every polynomial has at least one root.

On the other hand, one can formulate in elementary algebra the sentences:

Every polynomial of degree 1 has a root;
Every polynomial of degree 2 has a root;
Every polynomial of degree 3 has a root;

and so on. Since we are dealing with real — not complex — algebra, the above sentences are true for odd degree but false for even degree.

It should be emphasized that the general notion of an integer (as well as that of a rational, or of an algebraic number) also belongs to those notions which cannot be represented in our system of elementary algebra — and this in spite of the fact that each individual integer can easily be represented (e.g., 2 as 1 + 1, 3 as 1 + 1 + 1, etc.)[5]. The variables in elementary algebra always stand for arbitrary real numbers and cannot be supposed to assume only integers as values. For such a supposition would imply that the class of all sentences of elementary algebra contains all sentences of elementary number theory; and, by results mentioned above, there could be no universal method for deciding on the truth of sentences of such a class. Thus, the following is not a sentence of elementary algebra:

The equation

$$x^3 + y^3 = z^3$$

has no solution in positive integral x, y, z.

This gives, we hope, an adequate idea of what is understood here by a sentence of elementary algebra. Turning now to geometry, we can say roughly that by a sentence of elementary geometry we understand one which can be translated into a sentence of elementary algebra by fixing a coordinate system. It is well known that most sentences of elementary geometry in the traditional meaning are of this kind. There are, however, exceptions. These are, for instance, statements which involve explicitly or implicitly the general notion of a natural number: for instance, statements regarding polygons with an arbitrary number of sides — such as, that in every polygon each side is shorter than the sum of the remaining sides. It goes without saying that statements which involve the general notion of a point set — of an arbitrary geometrical figure — are also not elementary in our sense, but they would hardly be regarded as elementary in the everyday understanding of the term.

On the other hand, there are sentences which are elementary according to our definition but which are not ordinarily so considered. Most sentences of analytic geometry concerning algebraic curves of any definite degree belong here: for example, the theorem that any two ellipses intersect in at most four points.

It is important to realize that only the nature of the concepts involved, and not the character of the means of proof, determines whether a geometrical theorem is a sentence of elementary geometry. For instance, the statement that every angle can be divided into three congruent angles is an elementary sentence in our sense, and of course a true elementary sentence — despite the fact that the usual proofs of this statement make essential use of the axiom of continuity. On the other hand, the general notion of constructibility by rule and compass cannot be defined in elementary geometry, and therefore the statement that an angle in general cannot be trisected by by rule and compass is not an elementary sentence — although we can express in elementary geometry the facts that, say, an angle of 30° cannot be trisected by 1, 2,..., or in general any fixed number n of applications of rule and compass.

If we now compare the theories treated in this monograph (i.e., elementary algebra and geometry) with the other theories mentioned above for which decision methods have been found, we see at once that although the logical structure in both cases is indeed equally elementary, the theories investigated here have a considerably richer mathematical content. It would be possible to mention numerous problems which can be formulated in these theories, and which played in the past an important role in the development of mathematics. In the solution of these problems, and in general in the development of the theories considered, a great variety of modes of inference have been applied — some of them of a rather intricate nature (to mention only one example: the proof of the theorem that a triangle is isosceles if the bisectors of two of its angles are congruent). Thus the fact that there exists a universal decision method for elementary algebra and geometry could hardly have been regarded as a foregone conclusion.

In the light of these remarks one should not expect that the mathematical basis for the decision method to be discussed will be of a quite obvious and trivial nature. In fact by analyzing this decision method the reader will easily see that in its mathematical content it is very closely related to a classical algebraic result — namely, the theorem of Sturm previously mentioned — and it even provides an extension of this theorem to arbitrary systems of equations and inequalities in many unknowns.

Since a decision method, by its very nature, requires no intelligence for its application, it is clear that, whenever one can give a decision method for a class K of sentences, one can also devise a machine to decide whether an arbitrary sentence belongs to K. It often happens in mathematical research, both pure and applied, that problems arise as to the truth of complicated sentences of elementary algebra or geometry. The decision method presented in this work gives the mathematician the assurance that he will be able to solve every such problem by working at it long enough. Once the machine is devised, his task will reduce to explaining the problem to the machine — or to its operator. It may be instructive to illustrate, by means of an example, the more specific ways in which a decision machine could prove helpful in the study of unsolved problems.

As is well known, any two polygons of equal area, P and Q, can be decomposed into the same finite number n of non-overlapping triangles in such a way that each triangle in P is congruent to the corresponding triangle in Q. We are interested in determining the smallest number for which such a decomposition is possible. We assume in the following that P is the unit square and Q is a rectangle of unit area whose base has x units. *Now the smallest number* n *depends exclusively on* x and is denoted by $d(x)$; our problem reduces to describing the behavior of the function d for all positive values of x.

In particular, given any x_0, we can ask what the value of $d(x_0)$ is. In most cases, even the answer to this simple question presents difficulty; e.g., it is not easily seen whether or not $d(7/2) = 8$. However, we can easily establish, by means of a direct geometrical argument, an upper bound for $d(x_0)$; in fact, if $1 \leq x_0 \leq n$, where n is an integer, we have $d(x_0) \leq 2n$. Consequently, just one of the sentences "$d(x_0) = 1$", "$d(x_0) = 2$", ..., "$d(x_0) = 2n$" is true. If, moreover, x_0 is an algebraic number, all these sentences prove to be expressible in elementary geometry. Hence, by setting the machine in motion at most $2n$ times, we could check which of the sentences is true and thus find the value of $d(x_0)$.

In turn we may consider hypotheses regarding the behavior of the function d in some intervals. For instance, offhand, it seems plausible that $5 \leq d(x) \leq 6$ whenever

$2 < x \leq 3$. This hypothesis is still expressible in elementary geometry, and hence could be confirmed or rejected by means of a machine. The situation changes when we consider hypotheses of a more general character concerning the behavior of the function in its whole domain, e.g., the following one: for any real x and integral n, if $x > n$, then $d(x) > 2n$. This hypothesis has not yet been confirmed even for small values of n. In its general form, the hypothesis cannot be formulated in elementary geometry, and hence cannot be tested by means of the machine suggested here. However, the machine would permit us to test the hypothesis for any special value of n. We could carry out such tests for a sequence of consecutive values, $n = 2, 3, \ldots,$ up to, say, $n = 100$. If the result of at least one test were negative, the hypothesis would prove to be false; otherwise, our confidence in the hypothesis would increase, and we should feel encouraged to attempt establishing the hypothesis (by means of a normal mathematical proof), instead of trying to construct a counterexample.

As is seen from the last remarks, the machine envisaged may prove useful in connection with certain problems which cannot be formulated in elementary algebra (or geometry). The most typical in this class of problems are those of the form "Is it the case that, for every integer n, the condition C_n holds?" where C_n is expressible in elementary algebra for each fixed value of n. The machine could be used to solve mechanically this sort of problem for a series of consecutive values of n; in consequence, either we would learn that the solution of the problem in its general form is negative or else the plausibility of a positive solution would increase. Many important and difficult problems belong to this class, and the applicability of the machine to such problems may greatly enhance its value for mathematical research. (The results of this work have further implications, independent of the use of the machine, for the class of problems discussed; see Supplementary Note 7.)

It will be seen later, from the detailed description of the decision method, that the machine could serve some further purposes. We are often concerned, not with a sentence of elementary algebra, but with a condition involving parameters $a, b, c, \ldots,$ and formulated in terms of elementary algebra; the condition may be very involved, and we are interested in simplifying it — and, in fact, in reducing it to a standard form, in which it appears as a combination of algebraic equations and inequalities in $a, b, c, \ldots$. To give an example, consider the condition satisfied by the numbers a, b, and c if and only if there are exactly two (real) solutions of the equation:

$$ax^2 + bx + c = 0 .$$

In this case, the reduction is very simple and is well known from high-school algebra; the condition can be given the standard form:

$$a \neq 0 \text{ and } b^2 - 4ac > 0 .$$

The decision method developed below will give the assurance that such a reduction is always possible; and the decision machine would perform the reduction mechanically.

This monograph is divided into three sections. The first section contains a description of the system of algebra to which the decision method applies. In Section 2, the decision method itself is developed in a detailed way. In Section 3, some extensions of the results obtained as well as some related open problems are discussed. The notes at the end of the monograph contain, in addition to historical and bibliographical references, the discussion of various points of theoretical interest which are not directly related to the question of constructing a decision machine. A short bibliography following the notes lists the works which are referred to in the monograph[6].

SECTION 1.

THE SYSTEM OF ELEMENTARY ALGEBRA

In this section we want to describe a formal system of elementary algebra — and in particular to define in a precise way the class of sentences of this system(7).

By a *variable* we shall mean any one of the following symbols:

$$x,\ x_1,\ x_2,\ldots;\ y,\ y_1,\ y_2,\ldots;\ z,\ z_1,\ z_2,\ldots\ .$$

We suppose that there are infinitely many variables and that they are arranged in a sequence, so that we can speak of the variable occupying the 1st, 2nd,..., nth place in the sequence. These variables are to be thought of intuitively as ranging over the set of real numbers.

By an *algebraic constant* we shall mean one of the following three symbols:

$$1,\ 0,\ -1\ .$$

By an *algebraic operation-sign* we shall mean one of the following two symbols:

$$+,\ \cdot\ .$$

The first is called the *addition sign*, and the second the *multiplication sign*.

By an *algebraic term* we understand any meaningful expression built up from variables and algebraic constants by means of the elementary operation-signs. Thus, for example,

$$x, \quad x_1 + y,\ -1\cdot x, \quad \left[\left(x_1 \cdot -1\right)\cdot x_1\right] + x_2$$

are algebraic terms. But

$$x + , \quad \sqrt{2} + x$$

are not algebraic terms: the first, because it is meaningless; the second, because it involves the sign " $\sqrt{2}$ ", which is neither a variable nor an algebraic constant (in the restricted meaning we have given to the latter term).

If one wants a precise definition of algebraic terms, they can be defined recursively as follows: An algebraic term of first order is simply a variable or one of the three algebraic constants. If α and β are algebraic terms of order at most k, and if the maximum of the orders of α and β is k, then $(\alpha\cdot\beta)$ and $(\alpha + \beta)$ are algebraic terms of order $k + 1$. An expression is called an algebraic term if, for some k, it is an algebraic term of order k.

According to the above definition, one should inclose in parentheses the results of performing operations on terms. Thus one should write, for example, always

$$(x + y) \text{ and } (x\cdot y)$$

instead of simply

$$x + y \text{ and } x \cdot y .$$

We shall often omit these parentheses, however, when no ambiguity will result from doing so; we shall use, in general, the ordinary conventions as to omitting parentheses in writing algebraic terms. Thus, we write

$$x + y \cdot z$$

instead of

$$[x + (y \cdot z)] .$$

It is convenient to introduce the operation of *subtraction* as follows[8]: if α and β are any terms, then we set

$$(\alpha - \beta) \equiv [\alpha + (-1 \cdot \beta)] .$$

We have used here the symbol "≡" to indicate that two formulas are identically the same — in the present case by definition. We shall use this symbol throughout the rest of this report. When we write

$$\alpha \equiv \beta$$

we mean that α and β are composed of exactly the same symbols, written in exactly the same order. Thus, for example, it is true that

$$0 \equiv 0 ,$$

and that

$$(0 = 1) \equiv (0 = 1) ,$$

but not that

$$(0 + 0) \equiv 0 ,$$

nor that

$$(0 = 1) \equiv (1 = 0) .$$

It is also convenient to introduce notation for sums and products of arbitrary finite length. Let $\alpha_1, \alpha_2, \ldots$ be a sequence of terms. Then we set

$$\sum_{i=1}^{1} \alpha_i \equiv \alpha_1$$

$$\sum_{i=1}^{k+1} \alpha_i \equiv \left(\sum_{i=1}^{k} \alpha_i + \alpha_{k+1} \right) ,$$

and similarly,

$$\prod_{i=1}^{1} \alpha_i \equiv \alpha_1$$

$$\prod_{i=1}^{k+1} \alpha_i \equiv \left(\prod_{i=1}^{k} \alpha_i \cdot \alpha_{k+1} \right) .$$

Instead of

$$\sum_{i=1}^{n} \alpha_i$$

we shall also sometimes use the notation

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n ,$$

or simply,

$$\alpha_1 + \ldots + \alpha_n ;$$

and instead of

$$\prod_{i=1}^{n} \alpha_i ,$$

we shall sometimes write

$$\alpha_1 \cdot \alpha_2 \cdot \ldots \cdot \alpha_n$$

or

$$\alpha_1 \cdot \ldots \cdot \alpha_n .$$

If $\alpha_1, \ldots, \alpha_n$ are all the same, and equal, say, to α, then instead of

$$\prod_{i=1}^{n} \alpha_i$$

we sometimes write simply

$$\alpha^n .$$

Thus, for example,

$$\xi^3$$

has the same meaning as

$$[(\xi \cdot \xi) \cdot \xi] .$$

Moreover, we shall sometimes write α^0 for 1.

By an *algebraic relation-symbol* we shall mean one of the two symbols:

$$= , > ,$$

called, respectively, the *equality sign* and the *greater-than sign*[8].

By an *atomic formula* we shall mean an expression of one of the forms

$$(\alpha = \beta) , (\alpha > \beta)$$

where α and β stand for arbitrary algebraic terms; according to our previous remarks, parentheses will sometimes be omitted. The first kind of expression is called an *equality*, and the second an *inequality*. Thus, for example, the following are atomic formulas:

$$1 = 1 + 1$$

$$0 + x = x$$

$$x \cdot (y + z) = 0$$

$$[x \cdot (1 + 1)] + (y \cdot y) > 0$$

$$x > (y \cdot y) + x \ .$$

By a *sentential connective* we shall mean one of the following three symbols:

$$\sim , \wedge , \vee \ .$$

The first is called the *negation sign* (and is to be read "not"), the second is called the *conjunction sign* (and is to be read "and"), and the third is called the *disjunction sign* (and is to be read "or" — in the nonexclusive sense).

By the *(existential) quantifier* we understand the symbol "*E*". If ξ is any variable, then $(E\xi)$ is called a *quantifier expression*. The expression $(E\xi)$ is to be read "there exists a ξ such that ."

By a *formula* we shall mean an expression built up from atomic formulas by use of sentential connectives and quantifiers. Thus, for example, the following are formulas:

$$0 = 0$$

$$(Ex)(x = 0) \ ,$$

$$(x = 0) \vee (Ey)(x > y) \ ,$$

$$(Ex) \sim (Ey) \sim [(x = y) \vee (x > 1 + y)] \ ,$$

$$\sim (x > 1) \wedge (Ey)(x = y \cdot y) \ .$$

If one wants a precise definition of formulas, they can be defined recursively as follows: A formula of first order is simply an atomic formula. If θ is a formula of order k, then $\sim \theta$ is a formula of order $k + 1$. If θ is a formula of order k and ξ is any variable, then $(E\xi)\ \theta$ is a formula of order $k + 1$. If θ and Φ are formulas of order at most k, and one of them is of order k, then $(\theta \wedge \Phi)$ and $(\theta \vee \Phi)$ are formulas of order $k + 1$. An expression is a formula, if, for some n, it is a formula of order n.

Among the variables occuring in a formula, it is for some purposes convenient to distinguish the so-called "free" variables. We define this notion recursively in the following way: If Φ is an atomic formula, then ξ is *free* in Φ if and only if ξ occurs in Φ; ξ is *free* in $(E\eta)\ \theta$ if and only if η is not the same variable as

ξ, and ξ is free in θ; ξ is free in $\sim\theta$ if and only if ξ is free in θ; ξ is free in $(\theta \wedge \Phi)$, and in $(\theta \vee \Phi)$, if and only if ξ is free in at least one of the two formulas θ and Φ. Thus, for example, x is free in the formulas

$$x = 1$$

$$x = x$$

$$(Ey)(y = x)$$

$$(x = 1) \vee (Ex)(x = 2)$$

but not in the formulas

$$y = 1$$

$$(Ex)(x = x)$$

$$(Ex)(Ey)(y = x) \ .$$

(For certain purposes a more subtle notion is needed: that of a variable's being free, or not free, at a certain occurrence in a formula. Thus, for instance, in the formula

$$(Ey)(y = x) \wedge (Ex)[x + y = x \cdot w] \ ,$$

the variable x is free at its first occurrence — reading from left to right — but not in its other occurrences. This notion is not necessary for our discussion, however, so we shall not give a more exact explanation of it.)

It is convenient to introduce some abbreviated notation[8]. If θ and Φ are any formulas, then we regard

$$(\theta \longrightarrow \Phi)$$

as an abbreviation for

$$(\sim\theta \vee \Phi) \ ,$$

and

$$(\theta \longleftrightarrow \Phi)$$

as an abbreviation for

$$(\theta \longrightarrow \Phi) \wedge (\Phi \longrightarrow \theta)$$

The sign $\longrightarrow$ is called the *implication sign*, and the sign $\longleftrightarrow$ is called the *equivalence sign*. If θ and Φ are any formulas, then the formula $\theta \longrightarrow \Phi$ is called an *implication*; we call θ the *antecedent* or *hypothesis* and Φ the *consequent* or *conclusion* of this implication.

If θ is any formula, and ξ is any variable, then

$$(A\xi)\theta$$

is an abbreviation for

$$\sim(E\xi)\sim\theta \ .$$

We also introduce notation to represent disjunctions and conjunctions of arbitrarily many formulas. In the simplest case the formulas in question are arranged in a finite sequence $\theta_1, \theta_2, \ldots, \theta_n$; we then denote the disjunction of the formulas by

$$\bigvee_{1 \leq i \leq n} \theta_i$$

or

$$\theta_1 \vee \theta_2 \vee \ldots \vee \theta_n$$

and their conjunction by

$$\bigwedge_{1 \leq i \leq n} \theta_i$$

or

$$\theta_1 \wedge \theta_2 \wedge \ldots \wedge \theta_n .$$

A recursive definition of these symbolic expressions hardly needs to be formulated explicitly here. Sometimes we are confronted with more involved cases: for instance, we may have a finite set S of ordered couples (n,p), a formula $\theta_{n,p}$ being correlated with each member (n,p) of S. To denote the disjunction and conjunction of all such formulas $\theta_{n,p}$ we use the symbolic expressions

$$\bigvee_{(n,p) \text{ in } S} \theta_{n,p}$$

and

$$\bigwedge_{(n,p) \text{ in } S} \theta_{n,p}$$

(where "(n,p) *in* S" may be replaced by formulas defining the set S). To ascribe to these symbolic expressions an unambiguous meaning we have of course to specify the order in which the formulas $\theta_{n,p}$ are taken in forming the disjunction or conjunction. The way in which this order is specified is immaterial for our purposes; we can, if we wish, specify once and for all that the formulas are taken in lexicographical order of their indices; thus, for instance, the symbolic expression

$$\bigvee_{\substack{n+p \leq 4 \\ 1 \leq n,p}} \theta_{n,p}$$

will denote the disjunction

$$\left(\left[\left\{\left[\left(\theta_{1,1} \vee \theta_{1,2}\right) \vee \theta_{1,3}\right] \vee \theta_{2,1}\right\} \vee \theta_{2,2}\right] \vee \theta_{3,1}\right).$$

Analogous notations are used for disjunctions and conjunctions of finite systems of formulas which are correlated, not with couples, but with triples, quadruples,..., or even arbitrary finite sequences, of integers.

We also need a symbolism to denote arbitrarily long sequences of quantifier expressions. For this purpose we shall use exclusively the "three-dot" notation:

$$(E\xi_1)\ldots(E\xi_n)\,\theta$$

and

$$(A\xi_1)\ldots(A\xi_n)\,\theta\,,$$

where $\xi_1,\ldots,\xi_n$ are arbitrary variables, and θ is an arbitrary formula.

A formula is called a *sentence*, if it contains no free variables. Thus, for example, the following are sentences:

$$0 = 0$$
$$0 = 1$$
$$(0 = 0) \wedge (1 = 1)$$
$$(Ex)(0 = 0)$$
$$(Ex)(x = 0) \wedge (Ey) \sim (y = 0)$$
$$(Ex)(Ey)(y > x)\ .$$

On the other hand, the following are not sentences since they contain free variables:

$$x > 0$$
$$(Ey)(x > 0)$$
$$(Ex)\big(x > [(x + 1) + (y \cdot y)]\big).$$

It should be noticed that while a sentence is either true or false, this is not the case for a formula with free variables, which in general will be satisfied by some values of the free variables and not satisfied by others.

The notion of the truth of a sentence will play a fundamental role in our further discussion. It will occur either explicitly or, more often, implicitly; in fact, in terms of the notion of truth we shall define that of the equivalence of two formulas, and the latter notion will be involved in numerous theorems of Section 2, which are essential for establishing the decision method. We shall use the notion of truth intuitively, without defining it in a formal way. We hope, however, that the correctness of the theorems involving the notion of truth will be apparent to anyone who grasps the mathematical content of our arguments. No one will doubt, for instance, that a sentence of elementary algebra like

$$(Ax)(Ay)\left[(x + y) = (y + x)\right]$$

is true, and that the sentence

$$(Ax)(Ay)\left[(x - y) = (y - x)\right]$$

is false[9].

As examples of general laws involving the notion of truth, we give the following:

If θ is a sentence, then $\sim\theta$ is true if and only if θ is not true. If θ and Φ are sentences, then $(\theta \wedge \Phi)$ is true if and only if θ and Φ are both true. If θ and Φ are sentences, then $(\theta \vee \Phi)$ is true if and only if at least one of the sentences θ and Φ is true; $\theta \longrightarrow \Phi$ is true if and only if either θ is not true, or Φ is true; and $\theta \longleftrightarrow \Phi$ is true if and only if θ and Φ are either both true or both false.

Let θ and Φ be any formulas, and let $\xi_1, \xi_2, \ldots, \xi_n$ be the totality of free variables that occur in θ or Φ or both.

Then if the sentence

$$(A\xi_1)\ldots(A\xi_n)(\theta \longleftrightarrow \Phi)$$

is true, we say that θ and Φ are *equivalent.*

Thus, for example, the following two formulas are equivalent:

$$(x > 0) \vee (x = 0)\ , \quad (Ey)(x = y\cdot y)\ .$$

(Notice that neither of these formulas is equivalent to

$$(z > 0) \vee (z = 0)$$

since the latter contains "z" instead of "x".)

We now have some simple but very useful theorems regarding this notion of equivalence; they will be used in the subsequent discussion without explicit references.

A. *The relation of equivalence is symmetric, reflexive, and transitive.*

B. *Let θ_1 and θ_2 be equivalent formulas, and suppose that the formula Ψ_2 arises from the formula Ψ_1 by replacing θ_1 by θ_2 at one or more places. Then Ψ_1 is equivalent to Ψ_2.*

The proof of A and B presents no difficulty; in establishing B we apply induction with respect to the order of Ψ_1.

It is also convenient to define an equivalence relation for terms. Let α and β be any terms, and let ξ_1, $\xi_2, \ldots,$ ξ_n be the variables which occur in α or β or both. Then if the sentence

$$(A\xi_1)\ldots(A\xi_n)(\alpha = \beta)$$

is true, we say that α and β are *equivalent.*

The fundamental theorems regarding the equivalence of terms are analogous to those concerning the equivalence of formulas. In fact we have:

C. *The relation of equivalence of terms is symmetric, reflexive, and transitive.*

D. *If* α_1 *and* α_2 *are equivalent terms, and if the term* β_2 *arises from the term* β_1 *by replacing* α_1 *by* α_2 *at one or more places, then* β_1 *is equivalent to* β_2.

E. *If* α_1 *and* α_2 *are equivalent terms, and if the formula* Ψ_2 *arises from the formula* Ψ_1 *by replacing* α_1 *by* α_2 *at one or more places, then* Ψ_1 *is equivalent to* Ψ_2.

SECTION 2.

DECISION METHOD FOR ELEMENTARY ALGEBRA

The decision method for elementary algebra which will be explained in this section can be properly characterized as the "method of eliminating quantifiers"[(10),(11)]. It falls naturally into two parts. The first, essential, part consists in a procedure by means of which, given any formula θ, one can always find in a mechanical way an equivalent formula which involves no quantifiers, and no free variables besides those already occurring in θ; in particular, this procedure enables us, given any sentence, to find an equivalent sentence without quantifiers. Mathematically, this part of the decision method coincides with the extension of Sturm's theorem mentioned in the Introduction. The second part consists in a procedure by means of which, given any sentence θ without quantifiers, one can always decide in a mechanical way whether θ is true. It is obvious that these two procedures together provide the desired decision method.

In order to establish the first half of the decision method, we proceed by induction on the order of a formula. As is easily seen (using the elementary properties of equivalence of formulas mentioned in Section 1) it suffices to describe a procedure by means of which, given a formula $(E\xi)\ \theta$, where θ contains no quantifiers, one can always find an equivalent formula Φ, without quantifiers, and such that every variable in Φ is free in $(E\xi)\ \theta$; i.e., to give a method of eliminating the quantifier from $(E\xi)\ \theta$. Actually, it turns out to be convenient to do slightly more: i.e., to give a method of eliminating the quantifier from $(\underset{k}{E}\xi)\ \theta$, where the prefix "$(\underset{k}{E}x)$" is to be read "there exist exactly k values of x such that."

Definition 1. *Let $\alpha_0, \alpha_1, \ldots, \alpha_n$ be terms which do not involve ξ. Then the term*

$$\alpha_0 + \alpha_1 \cdot \xi + \ldots + \alpha_n \cdot \xi^n$$

is called a polynomial *in ξ. We say that the* degree *of this polynomial is n, and that $\alpha_0, \ldots, \alpha_n$ are its coefficients: α_n is called the* leading coefficient.

Remark. Our definition of the degree of polynomials differs slightly from the one usually given in algebra, in that we do not require that the leading coefficient be different from zero. Thus, we call

$$1 + (1 + 1)x + (1 - 1)x^2$$

a polynomial of the second degree, not of the first degree.

Definition 2. *Let α and β be polynomials in ξ of degrees m and n respectively: i.e., let*

$$\alpha \equiv \alpha_0 + \alpha_1 \cdot \xi + \ldots + \alpha_m \cdot \xi^m$$

$$\beta \equiv \beta_0 + \beta_1 \cdot \xi + \ldots + \beta_n \cdot \xi^n ,$$

where $\alpha_0, \ldots, \alpha_m$ *and* $\beta_0, \ldots, \beta_n$ *are terms which do not involve* ξ. *Let* r *be the minimum of the integers* m *and* n, *and let* s *be their maximum. Let*

$$\gamma_i \equiv \alpha_i + \beta_i \text{ for } i \leq r \ .$$

If $m < n$, *let*

$$\gamma_i \equiv \beta_i \text{ for } r < i \leq s \ .$$

If $m > n$, *let*

$$\gamma_i \equiv \alpha_i \text{ for } r < i \leq s \ .$$

Then we set

$$\alpha +_\xi \beta \equiv \gamma_0 + \gamma_1 \cdot \xi + \ldots + \gamma_s \cdot \xi^s \ .$$

DEFINITION 3. *Let*

$$\alpha \equiv \alpha_0 + \alpha_1 \cdot \xi + \ldots + \alpha_m \cdot \xi^m$$

be a polynomial in ξ. *Then by the* **first reductum** *(or, simply, the* **reductum***) of* α *we mean the polynomial obtained by leaving off the term* $\alpha_m \cdot \xi^m$*: i.e., we set*

$$Rd_\xi(\alpha) \equiv \alpha_0 + \alpha_1 \cdot \xi + \ldots + \alpha_{m-1} \cdot \xi^{m-1} \ ;$$

if $m = 0$ *(so that* α *does not involve* ξ *at all) we set*

$$Rd_\xi(\alpha) \equiv 0 \ .$$

We define reducta of all orders recursively, by setting

$$Rd_\xi^0(\alpha) \equiv \alpha$$

$$Rd_\xi^{k+1}(\alpha) \equiv Rd_\xi\left[Rd_\xi^k(\alpha)\right] \ .$$

The following theorem is easily established by an induction on the degree of α.

THEOREM 4. *If* α *is a polynomial in* ξ, *then* $Rd_\xi(\alpha)$ *is also a polynomial in* ξ *(whose coefficients, of course, are the same as certain of the coefficients of* α *— and hence contain no variables except those occurring in the coefficients of* α*). If* α *is of a degree* $m > 0$, *then* $Rd_\xi(\alpha)$ *is of degree* $m - 1$.

We make use of Theorem 4 in defining recursively the product of two polynomials:

DEFINITION 5. *Let*

$$\alpha \equiv \alpha_0 + \alpha_1 \cdot \xi + \ldots + \alpha_m \cdot \xi^m$$

$$\beta \equiv \beta_0 + \beta_1 \cdot \xi + \ldots + \beta_n \cdot \xi^n$$

be polynomials in ξ of degrees m and n respectively. If $m = 0$, then we set

$$\alpha \cdot_\xi \beta \equiv (\alpha\cdot\beta_0) + (\alpha\cdot\beta_1)\xi + \ldots + (\alpha\cdot\beta_n)\xi^n .$$

If $m > 0$, let

$$\begin{aligned} \gamma_i &\equiv 0 \text{ for } i < m \\ \gamma_m &\equiv \alpha_m \cdot \beta_0 \\ \gamma_{m+1} &\equiv \alpha_m \cdot \beta_1 \\ &\vdots \\ \gamma_{m+n} &\equiv \alpha_m \cdot \beta_n , \end{aligned}$$

and we set

$$\alpha \cdot_\xi \beta \equiv \left[Rd_\xi(\alpha) \cdot_\xi \beta\right] +_\xi \left(\gamma_0 + \gamma_1\cdot\xi + \ldots + \gamma_{m+n}\cdot\xi^{m+n}\right).$$

Definition 6. *If α and β are polynomials in ξ, then we set*

$$\alpha -_\xi \beta \equiv \alpha +_\xi \left[(-1) \cdot_\xi \beta\right] .$$

Theorem 7. *If α and β are polynomials in ξ, then $\alpha +_\xi \beta$, $\alpha \cdot_\xi \beta$, and $\alpha -_\xi \beta$ are polynomials in ξ.*

Proof. Obvious from the definitions.

Definition 8. *If $\alpha \equiv \xi$, then we set*

$$P_\xi(\alpha) \equiv 0 + 1\cdot\xi .$$

If α is a constant (0, 1, or $^-1$), or a variable different from ξ, then we set

$$P_\xi(\alpha) \equiv \alpha .$$

If α and β are arbitrary terms, then we set

$$\begin{aligned} P_\xi(\alpha + \beta) &\equiv P_\xi(\alpha) +_\xi P_\xi(\beta) \\ P_\xi(\alpha\cdot\beta) &\equiv P_\xi(\alpha) \cdot_\xi P_\xi(\beta) . \end{aligned}$$

Theorem 9. *If α is any term, and ξ is any variable, then $P_\xi(\alpha)$ is a polynomial in ξ, and is equivalent to α.*

Proof. By an induction on the order of α, making use of Theorem 7.

REMARK. It will be seen that if α is any term, then $P_\xi(\alpha)$ is the polynomial which results from "multiplying out" and "arranging in increasing powers of ξ." It is convenient to extend the definition of P_ξ so that it will be defined not only for all terms but for all formulas without quantifiers. This is done in our next definition. The intuitive significance of $P_\xi(\theta)$, when θ is a formula, will become clear in Theorem 11.

DEFINITION 10. *For all terms α and β, and for all formulas θ and Φ, we set*

(i) $$P_\xi(\alpha = \beta) \equiv P_\xi(\alpha - \beta) = 0$$

(ii) $$P_\xi(\alpha > \beta) \equiv P_\xi(\alpha - \beta) > 0$$

(iii) $$P_\xi[\sim(\alpha = \beta)] \equiv [P_\xi(\alpha > \beta) \vee P_\xi(\beta > \alpha)]$$

(iv) $$P_\xi[\sim(\alpha > \beta)] \equiv [P_\xi(\alpha = \beta) \vee P_\xi(\beta > \alpha)]$$

(v) $$P_\xi(\theta \vee \Phi) \equiv [P_\xi(\theta) \vee P_\xi(\Phi)]$$

(vi) $$P_\xi(\theta \wedge \Phi) \equiv [P_\xi(\theta) \wedge P_\xi(\Phi)]$$

(vii) $$P_\xi[\sim(\theta \vee \Phi)] \equiv P_\xi(\sim\theta) \wedge P_\xi(\sim\Phi)$$

(viii) $$P_\xi[\sim(\theta \wedge \Phi)] \equiv P_\xi(\sim\theta) \vee P_\xi(\sim\Phi)$$

(ix) $$P_\xi(\sim\sim\theta) \equiv P_\xi(\theta) \ .$$

THEOREM 11. *Let θ be any formula without quantifiers, and let ξ be any variable. Then $P_\xi(\theta)$ is equivalent to θ. Moreover, $P_\xi(\theta)$ is a formula built up by means of conjunction and disjunction signs (but without using negation signs) from atomic formulas of the form*

$$\alpha = 0$$

and

$$\alpha > 0 \ ,$$

where α is a polynomial in ξ.

PROOF. We prove that $P_\xi(\theta)$ is equivalent to θ by an induction on the order of θ. If θ is of first order, the theorem is obvious by 10 (i), 10 (ii), 9, and the following facts: $\alpha = \beta$ is equivalent to $\alpha - \beta = 0$; and $\alpha > \beta$ is equivalent to $\alpha - \beta > 0$. In order to carry out the recursive step, we make use of the facts: that $\sim(\alpha = \beta)$ is equivalent to $(\alpha > \beta) \vee (\beta > \alpha)$; that $\sim(\alpha > \beta)$ is equivalent to $(\alpha = \beta) \vee (\beta > \alpha)$; that $\sim(\theta \vee \Phi)$ is equivalent to $\sim\theta \wedge \sim\Phi$; that $\sim(\theta \wedge \Phi)$ is equivalent to $\sim\theta \vee \sim\Phi$; and that $\sim\sim\theta$ is equivalent to θ.

The second part of the theorem can also be proved by an induction on the order of θ, making use of Theorem 9.

Given any formula θ without quantifiers, we have thus obtained an equivalent formula $\Psi \equiv P_{\xi}(\theta)$ which contains no quantifiers and no negation signs. We are now going to define an operator Q which subjects any formula Ψ of this kind to further transformations by applying mainly the distributive law of sentential calculus, so as to bring Ψ to the so-called "disjunctive normal form."

DEFINITION 12. *If Φ is an atomic formula, then we set*

$$Q(\Phi) \equiv \Phi .$$

If

$$Q(\Phi_1) \equiv \bigvee_{i \le m} \bigwedge_{j \le m_i} \Psi_{i,j} ,$$

and

$$Q(\Phi_2) \equiv \bigvee_{m < i \le m+n} \bigwedge_{j \le m_i} \Psi_{i,j}$$

where $\psi_{i,j}$ (for $i \le m + n$ and $j \le m_i$) *is an atomic formula, then we set*

$$Q(\Phi_1 \vee \Phi_2) \equiv \bigvee_{i \le m+n} \bigwedge_{j \le m_i} \Psi_{i,j}$$

and

$$Q(\Phi_1 \wedge \Phi_2) \equiv \bigvee_{\substack{i \le m \\ m < j \le m+n}} \left(\Psi_{i,1} \wedge \ldots \wedge \Psi_{i,m_i} \wedge \Psi_{j,1} \wedge \ldots \wedge \Psi_{j,m_j}\right) .$$

THEOREM 13. *If Φ is any formula which involves no negation signs or quantifiers, then $Q(\Phi)$ is a disjunction of conjunctions of atomic formulas. Moreover, $Q(\Phi)$ is equivalent to Φ.*

PROOF. By induction on the order of Φ, making use of the following fact: for any formulas Φ, θ, and Ψ, the formulas $\Phi \wedge (\theta \vee \Psi)$ and $(\Phi \wedge \theta) \vee (\Phi \wedge \Psi)$ are equivalent, as are also the formulas $\Phi \wedge (\theta \wedge \Psi)$ and $(\Phi \wedge \theta) \wedge \Psi$, as well as the formulas $\Phi \vee (\theta \vee \Psi)$ and $(\Phi \vee \theta) \vee \Psi$.

Theorem 14. *Let Φ be any formula without quantifiers, and let ξ be any variable. Then $QP_\xi(\Phi)$ is a disjunction of conjunctions of atomic formulas — each of the atomic formulas in question having a polynomial in ξ for its left member and 0 for its right member. Moreover, $QP_\xi(\Phi)$ is equivalent to Φ.*

Proof. By Theorems 11 and 13; $QP_\xi(\Phi)$ is of course used here to mean $Q[P_\xi(\Phi)]$.

We now introduce the notion of a derivative (with respect to a given variable).

Definition 15. *If*

$$\alpha \equiv \alpha_0 + \alpha_1 \cdot \xi + \ldots + \alpha_n \cdot \xi^n$$

is a polynomial in ξ, of degree $n > 0$, then we put (writing "2" for "1 + 1", etc.)

$$D_\xi(\alpha) \equiv \alpha_1 + (2 \cdot \alpha_2) \cdot \xi + \ldots + (n \cdot \alpha_n)\xi^{n-1} .$$

If α is of degree zero in ξ, we set

$$D_\xi(\alpha) \equiv 0 .$$

Remark. The notion of a derivative can of course be extended to arbitrary terms which are not formally polynomials in ξ according to Definition 1 by putting

$$D_\xi(\alpha) \equiv D_\xi P_\xi(\alpha) .$$

Theorem 16. *If α is a polynomial in ξ, so is $D_\xi(\alpha)$.*

Proof. By Definitions 1 and 15.

We also define derivatives of arbitrary order as follows:

Definition 17. *If α is any term, and ξ is any variable, we set*

$$D_\xi^0(\alpha) \equiv \alpha$$

$$D_\xi^{k+1}(\alpha) \equiv D_\xi\left[D_\xi^k(\alpha)\right] .$$

Theorem 18. *If α is a polynomial in ξ, and k is a non-negative integer, then $D_\xi^k(\alpha)$ is a polynomial in α.*

Proof. By Theorem 16 and Definition 17.

The operator M which will be introduced next correlates, with every polynomial α, every variable ξ, and every non-negative integer n, a formula $M_\xi^n(\alpha)$, which in intuitive interpretation means that ξ is a root of α of order n. In case $n = 0$, this formula means simply that ξ is not a root of α. In this connection the formula $M_\xi^n(\alpha)$ will be read "the number ξ is of order n in the polynomial α," independent of whether n is positive or equal to zero.

Definition 19. *Let* α *be any polynomial in* ξ. *If* n *is any positive integer, we set*

$$M^n_\xi(\alpha) \equiv \left\{ \left(\bigwedge_{1 \leq i \leq n} \left[D^{i-1}_\xi(\alpha) = 0 \right] \right) \wedge \sim \left[D^n_\xi(\alpha) = 0 \right] \right\} .$$

We set, in addition,

$$M^0_\xi(\alpha) \equiv \sim(\alpha = 0) .$$

We now introduce by definition a new kind of existential quantifier, which may be called the *numerical* existential quantifier. If n is any non-negative integer, ξ any variable, and Φ any formula, then $(\underset{n}{E}\xi)\Phi$ is to be interpreted intuitively as meaning that there exist exactly n values of ξ which make Φ true.

Definition 20. *Let* ξ *be any variable, and let* Φ *be any formula. We set*

$$(\underset{0}{E}\xi)\Phi \equiv (A\xi)\sim\Phi .$$

Let n *be any positive integer, and let* $\eta_1, \ldots, \eta_n$ *be the first* n *variables (in the sequence of all variables) which do not occur in* Φ *and are different from* ξ. *Then we set*

$$(\underset{n}{E}\xi)\Phi \equiv \left\{ (E\eta_1)\ldots(E\eta_n) \left(\bigwedge_{1 \leq i < j \leq n} \sim(\eta_i = \eta_j) \wedge \right.\right.$$

$$\left.\left. (A\xi) \left[\Phi \leftrightarrow \bigvee_{1 \leq i \leq n} (\eta_i = \xi) \right] \right) \right\} .$$

We next introduce an operator F with a more complicated and technical interpretation. If n is an integer, ξ a variable, and α and β any polynomials, then $F^n_\xi(\alpha,\beta)$ is a formula to be intuitively interpreted as meaning that there are exactly n numbers ξ which satisfy the following conditions: (1) ξ is a root of higher order of α than of β, and the difference between these two orders is an odd integer; (2) there exists an open interval, whose right end-point is ξ, within which α and β have the same sign. The exact form of the symbolic expression used to define $F^n_\xi(\alpha,\beta)$ will probably seem strange at first glance even to those who are acquainted with logical symbolism; we have chosen this form so as to avoid the necessity of introducing a notation to indicate the result of replacing one variable by another in a given term. (An analogous remark applies to some other symbolic formulations given elsewhere in this work — in particular, in Note 9.) It will be noticed that the variable ξ is not free in $F_\xi^n(\alpha,\beta)$.

Definition 21. *Let α be a polynomial of degree p in ξ, and let β be a polynomial of degree q in ξ. Let η_1 and η_2 be the first two variables which are different from ξ and which do not occur in α or β. Then we set*

$$F_\xi^n(\alpha, \beta) \equiv (E\xi)_n \left\{ \bigvee_{\substack{0 \le k \le q \\ 0 \le 2m \le p-k-1}} \left[M_\xi^{k+2m+1}(\alpha) \wedge M_\xi^k(\beta)\right] \wedge \right.$$

$$\left. (E\eta_1)(E\eta_2) \left[(\eta_1 = \xi) \wedge (\xi > \eta_2) \wedge (A\xi)\left\{\left[(\xi > \eta_2) \wedge (\eta_1 > \xi)\right] \rightarrow (\alpha\cdot\beta > 0)\right\}\right]\right\}$$

The operator G which will now be defined is closely related to the operator F. In fact, α and β being polynomials in ξ, $G_\xi^n(\alpha,\beta)$ has the following meaning: if n_1 is the integer for which $F_\xi^{n_1}(\alpha,\beta)$ holds and n_2 is the integer for which $F_\xi^{n_2}(\alpha,\beta')$ holds (where β' is the negative of β — i.e., the polynomial obtained by multiplying β by -1), then $n = n_1 - n_2$; the integer n may be positive, zero, or negative. Remembering the intuitive meaning of $F_\xi^n(\alpha,\beta)$, the intuitive meaning of $G_\xi^n(\alpha,\beta)$ now becomes clear.

Definition 22. *Let n be any integer (positive, negative, or zero), and let α and β be any polynomials in ξ. Let k be the maximum of the degrees of α and β. Then we set*

$$G_\xi^n(\alpha,\beta) \equiv \bigvee_{\substack{0 \le m \le k \\ 0 \le m+n \le k}} \left[F_\xi^{n+m}(\alpha,\beta) \wedge F_\xi^m(\alpha,(-1)\cdot_\xi\beta)\right].$$

We need also the notion of the *remainder* obtained by dividing one polynomial by another. For our purposes, however, it turns out to be slightly more convenient to introduce a notation for the *negative of the remainder*.

Definition 23. *Let ξ be a variable. Let α be a polynomial of degree m in ξ, whose leading coefficient is α_m. Let β be a polynomial of degree n in ξ, whose leading coefficient is β_n. If $m < n$, we set*

$$R_\xi(\alpha,\beta) \equiv (-1)\cdot_\xi\alpha .$$

If $m = n$, *we set*

$$R_\xi(\alpha,\beta) \equiv Rd_\xi P_\xi\left(\alpha_m\cdot\beta_n\cdot\beta - \beta_n^2\cdot\alpha\right) .$$

If $m > n$, *we set*

$$R_\xi(\alpha,\beta) \equiv R_\xi\left\{ Rd_\xi P_\xi\left(\beta_n^2\cdot\alpha - \alpha_m\cdot\beta_n\cdot\xi^{m-n}\cdot\beta\right),\ \beta \right\} .$$

THEOREM 24. *If* α *and* β *are two polynomials in a variable* ξ, *then* $R_\xi(\alpha,\beta)$ *is again a polynomial in* ξ *whose coefficients contain no variables except those occurring in the coefficients of* α *and* β. *If* β *is a polynomial of degree* $n > 0$, *then* $R_\xi(\alpha,\beta)$ *is of degree less than* n. *If* β *is of degree zero, then* $R_\xi(\alpha,\beta) \equiv 0$.

It should be noticed that our definition of the negative of the remainder diverges somewhat from that which would normally be given in a textbook of algebra. According to the usual definition of the remainder, the negative of the remainder obtained by dividing a polynomial α of degree m by a polynomial β of degree n — both in the variable ξ — is a polynomial δ of a degree lower than n, such that, for some polynomial γ, the equation

$$\alpha = \beta\cdot\gamma - \delta$$

is satisfied identically. The coefficients of γ and δ can be obtained from those of α and β by means of the four rational operations, division included. We have modified this definition so as to eliminate division, which, as we know, is not available in our system. In consequence, we cannot construct for the negative remainder in our sense a polynomial γ which satisfies the above equation. We have instead:

THEOREM 25. *Let* α *and* β *be any polynomials in a variable* ξ, *of degrees* m *and* n, *respectively, and let* β_n *be the leading coefficient of* β. *We set* $q = 0$ *in case* $m < n$, *and* $q = m - n + 1$ *in case* $m \geq n$. *Then there is a polynomial* γ *in* ξ, *whose coefficients contain no variables except those occurring in the coefficients of* α *and* β, *and for which* $\alpha\cdot\beta_n^{2q}$ *and* $\beta\cdot\gamma - R_\xi(\alpha,\beta)$ *are equivalent.*

PROOF. By induction on the difference of the degrees of α and β.

One rather undesirable consequence of our modification of the notion of a negative remainder is that, in case the leading coefficient β_n of β is 0, all the coefficients of $R_\xi(\alpha,\beta)$ prove to be terms equivalent to 0. No difficulty will arise from this fact, however, since we shall never use $R_\xi(\alpha,\beta)$ except when conjoined with the hypothesis $\sim\left(\beta_n = 0\right)$.

It should be pointed out that our negative remainder $R_\xi(\alpha,\beta)$ is still a polynomial of lower degree than β — except when β is of degree zero, in which case $R_\xi(\alpha,\beta) \equiv 0$. This circumstance, together with the analogous property of the reductum of a polynomial (see Theorem 4) will be the basis for some recursive definitions given in our later discussion.

The following three definitions (26, 28, and 30) and the theorems which follow them (27, 29, and 31) are of crucial importance for the decision method under discussion. In these definitions we introduce three operators S, T, and U which correlate, with certain formulas Φ, new formulas $S(\Phi)$, $T(\Phi)$, and $U(\Phi)$ containing no quantifiers; and in the subsequent theorems we show that the correlated formulas are always equivalent to the original ones. The operator S is defined only for rather special formulas — in fact, for those of the form $G^k_\xi(\alpha,\beta)$. The operator T, which is constructed with the help of S, is defined for a rather extensive class of formulas, which contains formulas like

$$(\underset{k}{E}\xi)(\alpha = 0), \qquad (\underset{k}{E}\xi)[(\alpha = 0) \wedge (\beta > 0)]$$

(where α and β are any polynomials in ξ), and some related but more complicated types of formulas. The operator U, finally, constructed in terms of T, is defined for all possible formulas; hence Theorem 31, which establishes the equivalence of Φ and $U(\Phi)$, provides us with a universal method of eliminating quantifiers. It may be pointed out that the operator U, though constructed with the help of T, and thus indirectly of S, is not an extension of either of these operators; thus, if Φ is a formula for which S is defined, then $S(\Phi)$ and $U(\Phi)$ are in general formally different, though equivalent, formulas.

Definition 26. *Let k be an integer, and let α and β be polynomials in a variable ξ of degrees m and n respectively and having leading coefficients α_m and β_n; and let*

$$\Phi \equiv G^k_\xi(\alpha,\beta) \quad .$$

(i) *If α or β is the polynomial 0, we set*

$$S(\Phi) \equiv (0 = 0), \qquad \text{for } k = 0$$

and

$$S(\Phi) \equiv (0 = 1), \qquad \text{for } k \neq 0 \quad .$$

(ii) *If neither α nor β is the polynomial 0, and if $m + n$ is even, we set*

$$S(\Phi) \equiv \Big\{ \Big[\big(\alpha_m = 0\big) \wedge SG^k_\xi\big(Rd_\xi(\alpha),\beta\big)\Big] \vee \Big[\big(\beta_n = 0\big) \wedge SG^k_\xi\big(\alpha, Rd_\xi(\beta)\big)\Big] \vee \Big[\sim\!\big(\alpha_m\cdot\beta_n = 0\big) \wedge SG^k_\xi\big(\beta, R_\xi(\alpha,\beta)\big)\Big] \Big\} \; .$$

(iii) *If neither α nor β is the polynomial 0, and $m + n$ is odd, we set*

$$S(\Phi) \equiv \Big\{ \Big[\big(\alpha_m = 0\big) \wedge SG^k_\xi\big(Rd_\xi(\alpha),\beta\big)\Big] \vee \Big[\big(\beta_n = 0\big) \wedge SG^k_\xi\big(\alpha, Rd_\xi(\beta)\big)\Big] \vee \Big[\big(\alpha_m\cdot\beta_n > 0\big) \wedge SG^{k+1}_\xi\big(\beta, R_\xi(\alpha,\beta)\big)\Big] \vee \Big[\big(0 > \alpha_m\cdot\beta_n\big) \wedge SG^{k-1}_\xi\big(\beta, R_\xi(\alpha,\beta)\big)\Big] \Big\} .$$

Theorem 27. *Let Φ be one of the formulas for which the operator S is defined (by 26). Then $S(\Phi)$ is a formula which contains no quantifiers, and no variables except those that occur free in Φ. Moreover, Φ is equivalent to $S(\Phi)$.*

Proof. The first part follows immediately from Definition 26.

To show the second part we consider the recursive definition for S given in 26. In view of this definition, we easily see that it suffices to establish what follows (for arbitrary polynomials α and β of degrees m and n respectively in a variable ξ, with leading coefficients α_m and β_n, and for an arbitrary integer k):

(1) if α or β is the polynomial 0, then $G^k_\xi(\alpha,\beta)$ is equivalent to $(0 = 0)$ for $k = 0$, and to $(0 = 1)$ for $k \neq 0$;

(2) if $m + n$ is even, then $G^k_\xi(\alpha,\beta)$ is equivalent to

$$\left\{\left[(\alpha_m = 0) \wedge G^k_\xi(Rd_\xi(\alpha),\beta)\right] \vee \left[(\beta_n = 0) \wedge G^k_\xi(\alpha,Rd_\xi(\beta))\right] \vee \left[\sim(\alpha_m\cdot\beta_n = 0) \wedge G^k_\xi(\beta,R_\xi(\alpha,\beta))\right]\right\};$$

(3) if $m + n$ is odd, then $G^k_\xi(\alpha,\beta)$ is equivalent to

$$\left\{\left[(\alpha_m = 0) \wedge G^k_\xi(Rd_\xi(\alpha),\beta)\right] \vee \left[(\beta_n = 0) \wedge G^k_\xi(\alpha,Rd_\xi(\beta))\right] \vee \left[(\alpha_m\cdot\beta_n > 0) \wedge G^{k+1}_\xi(\beta,R_\xi(\alpha,\beta))\right] \vee \left[(0 > \alpha_m\cdot\beta_n) \wedge G^{k-1}_\xi(\beta,R_\xi(\alpha,\beta))\right]\right\}.$$

Let $\xi_1, \ldots, \xi_s$ be all the variables which occur in the coefficients of α or β or both.

It is easily seen that the proof of (1) reduces to showing that, in case $\alpha \equiv 0$ or $\beta \equiv 0$, the following sentences are true:

$$(A\xi_1)\ldots(A\xi_s)G^k_\xi(\alpha,\beta), \text{ for } k = 0;$$

$$(A\xi_1)\ldots(A\xi_s)\sim G^k_\xi(\alpha,\beta), \text{ for } k \neq 0.$$

Now, we notice by Definition 15 that, for every non-negative integer p, $D^p_\xi(0) \equiv 0$. Hence we conclude by Definition 19 that, in case $\alpha \equiv 0$ or $\beta \equiv 0$, the formula $F^k_\xi(\alpha,\beta)$ is satisfied by all values of $\xi_1, \ldots, \xi_s$ if $k = 0$, and by no such values if $k \neq 0$. From Definition 22 we then easily see that the same applies to the formula $G^k_\xi(\alpha,\beta)$; and this is just what we wanted to show.

Analogously, by means of easy transformations, we see that the proof of (2) and (3) reduces to showing that the following sentences are true:

$$(4)\quad (A\xi_1)\ldots(A\xi_s)\left\{\sim(\alpha_m\cdot\beta_n = 0) \rightarrow \left[G_\xi^k(\alpha,\beta) \leftrightarrow G_\xi^k\big(\beta, R_\xi(\alpha,\beta)\big)\right]\right\}, \text{ for } m+n \text{ even;}$$

$$(5)\quad (A\xi_1)\ldots(A\xi_s)\left\{(\alpha_m\cdot\beta_n > 0) \rightarrow \left[G_\xi^k(\alpha,\beta) \leftrightarrow G_\xi^{k+1}\big(\beta, R_\xi(\alpha,\beta)\big)\right]\right\}, \text{ for } m+n \text{ odd:}$$

$$(6)\quad (A\xi_1)\ldots(A\xi_s)\left\{(0 > \alpha_m\cdot\beta_n) \rightarrow \left[G_\xi^k(\alpha,\beta) \leftrightarrow G_\xi^{k-1}\big(\beta, R_\xi(\alpha,\beta)\big)\right]\right\}, \text{ also for } m+n \text{ odd.}$$

Actually it turns out to be more convenient to establish, instead of (4), (5) and (6), certain stronger statements. For this purpose we introduce the formula $H_\xi^p(\alpha,\beta)$ expressing the fact that there are just p numbers ξ such that the difference between the order of ξ in α and the order of ξ in β is an odd integer, not necessarily positive. A precise formal definition of $H_\xi^p(\alpha,\beta)$ hardly needs to be given here. The sentences whose truth we want to establish can now be formulated as follows (letting γ and δ be arbitrary polynomials in ξ, whose coefficients involve no variables except $\xi_1, \ldots, \xi_s$, and letting p and q be arbitrary non-negative integers):

$$(7)\quad (A\xi_1)\ldots(A\xi_s)\left\{\left[H_\xi^p(\alpha,\beta) \wedge (A\xi)\big(\alpha\cdot\beta_n^{2q} = \beta\cdot\gamma - \delta\big) \wedge \sim(\alpha_m\cdot\beta_n = 0)\right] \rightarrow \left[G_\xi^k(\alpha,\beta) \leftrightarrow G_\xi^k(\beta,\delta)\right]\right\}, \text{ for } p \text{ even;}$$

$$(8)\quad (A\xi_1)\ldots(A\xi_s)\left\{\left[H_\xi^p(\alpha,\beta) \wedge (A\xi)\big(\alpha\cdot\beta_n^{2q} = \beta\cdot\gamma - \delta\big) \wedge (\alpha_m\cdot\beta_n > 0)\right] \rightarrow \left[G_\xi^k(\alpha,\beta) \leftrightarrow G_\xi^{k+1}(\beta,\delta)\right]\right\}, \text{ for } p \text{ odd;}$$

$$(9)\quad (A\xi_1)\ldots(A\xi_s)\left\{\left[H_\xi^p(\alpha,\beta) \wedge (A\xi)\big(\alpha\cdot\beta_n^{2q} = \beta\cdot\gamma - \delta\big) \wedge (0 > \alpha_m\cdot\beta_n)\right] \rightarrow \left[G_\xi^k(\alpha,\beta) \leftrightarrow G_\xi^{k-1}(\beta,\delta)\right]\right\}, \text{ also for } p \text{ odd.}$$

It is easily seen that the truth of (7), (8), and (9) implies that of (4), (5), and (6) respectively. We shall sketch the proof of this for the cases (7) and (4). Thus, assume (7) to be true and $m + n$ to be even. Consider any fixed but arbitrary set of values of $\xi_1, \ldots, \xi_s$ and suppose the hypothesis of (4) to be satisfied. Let p be the (uniquely determined) integer for which $H_\xi^p(\alpha,\beta)$ is satisfied (by the given values of $\xi_1, \ldots, \xi_s$). An elementary algebraic argument shows that p is congruent to $m + n$ modulo two; therefore p is even, and (7) may be applied. We now set $q = 0$ if $m < n$, and $q = m - n + 1$ otherwise; and we construct a polynomial γ in ξ, with coefficients involving no variables except those occurring in the coefficients of α or β, and such that

$$\alpha\cdot\beta_n^{2q} = \beta\cdot\gamma - R_\xi(\alpha,\beta)$$

holds for every value of ξ. (Regarding the possibility of constructing such a γ, see Theorem 25.) We then see that the hypothesis of (7) is satisfied with δ replaced by $R_\xi(\alpha,\beta)$. Hence the conclusion of (7) is also satisfied. This conclusion, however, with the indicated replacement, coincides with the conclusion of (4). The proof now reduces to establishing the truth of (7), (8), and (9). It is convenient in this part of proof to avail ourselves of customary mathematical language and symbolism. Also, we shall not be too meticulous in trying to avoid possible confusions between mathematical and metamathematical formulations.

Given a polynomial α and a number λ, we shall denote by $f(\lambda,\alpha)$ the order of λ in α: i.e., the uniquely determined non-negative integer r such that $M^r_\lambda(\alpha)$ holds. The function f is thus defined for every number λ, and for every polynomial α which does not vanish identically.

Similarly, for any given polynomials α and β in ξ we denote by $g(\alpha,\beta)$ the integer k for which $G^k_\xi(\alpha,\beta)$ holds. From the definition of $G^k_\xi(\alpha,\beta)$ (see Definition 22) it follows that such an integer always exists and is uniquely determined. It can be computed in the following way. We consider all these numbers λ for which $f(\lambda,\alpha) - f(\lambda,\beta)$ is positive and odd, and we divide them into two sets, P and N; λ belongs to P (or N) if there is an open interval whose right-hand end-point is λ, within which the values of α and β have always the same sign (or always different signs). Both sets P and N are clearly finite, and the difference between the number of elements in P and the number of elements in N is just $g(\alpha,\beta)$. Thus $g(\alpha,\beta)$ can be positive, negative, or zero; in case α or β vanishes identically, $g(\alpha,\beta) = 0$.

Finally we introduce the symbol $h(\alpha,\beta)$ to denote the integer p for which $H^p_\xi(\alpha,\beta)$ holds; in other words, $h(\alpha,\beta)$ is the number of all those numbers λ for which $f(\lambda,\alpha) - f(\lambda,\beta)$ is odd — though not necessarily positive.

For later use we state here without proof (which would be quite elementary) the following property of the function f defined above:

(10) Let α, β, γ, and δ be polynomials in ξ, such that

$$\alpha\cdot\beta_n^{2q} = \gamma\cdot\beta - \delta$$

holds for every value of ξ, β_n being the (nonvanishing) leading coefficient of β and q some integer. If, for any given number λ, $f(\lambda,\beta) > f(\lambda,\alpha)$ — so that α, as well as β, does not vanish identically — then $f(\lambda,\alpha) = f(\lambda,\delta)$ (so that δ does not vanish identically either). Similarly, if $f(\lambda,\beta) > f(\lambda,\delta)$, then $f(\lambda,\alpha) = f(\lambda,\delta)$.

We now take up the proof of (7), (8), and (9), which will be done by a simultaneous induction on $h(\alpha,\beta) = p$. The reader can easily verify that (7), (8), and (9) hold in case the polynomial δ vanishes identically; therefore we shall assume henceforth that δ does not vanish identically.

Assume first that $h(\alpha,\beta) = 0$. Thus there are no numbers λ such that $f(\lambda,\alpha) - f(\lambda,\beta)$ is odd. *A fortiori* there are no numbers λ such that $f(\lambda,\alpha) - f(\lambda,\beta)$ is positive and odd; and hence $g(\alpha,\beta) = 0$. Furthermore, there are no numbers λ such that $f(\lambda,\beta) - f(\lambda,\delta)$ is positive and odd; for if such a number λ existed, we should have $f(\lambda,\alpha) =$

$f(\lambda,\delta)$ by (10), and hence $f(\lambda,\alpha) - f(\lambda,\beta)$ would be odd. Consequently, $g(\beta,\delta) = 0$ and therefore $g(\alpha,\beta) = g(\beta,\delta)$. Thus in this case (7) proves to hold, while (8) and (9) are of course vacuously satisfied.

Assume now that (7), (8), and (9) have been established for arbitrary polynomials α and β with $h(\alpha,\beta) = p$ (p any given integer). Consider any polynomials α and β in ξ with nonvanishing coefficients α_m and β_n, and with

(11) $$h(\alpha,\beta) = p + 1 \ ,$$

as well as two further polynomials γ and δ in ξ such that

(12) $\alpha \cdot \beta_n^{2q} = \gamma \cdot \beta - \delta$ holds identically for some non-negative integer q .

Two cases can be distinguished here, according as $\alpha_m \cdot \beta_n > 0$ or $0 > \alpha_m \cdot \beta_n$; since the arguments are entirely analogous in both cases, however, we restrict ourselves to the case

(13) $$\alpha_m \cdot \beta_n > 0 \ .$$

Our assumption (11) implies that there are exactly $p + 1$ numbers λ for which $f(\lambda,\alpha) - f(\lambda,\beta)$ is odd. Let

(14) λ_0 = the largest λ such that $f(\lambda,\alpha) - f(\lambda,\beta)$ is odd.

Condition (13) implies that for sufficiently large numbers $\xi > \lambda_0$ the values of α and β are of the same sign. This can be extended to every number $\xi > \lambda_0$ (not a root of α or β), since, by (14), there is no number $\xi > \lambda_0$ for which $f(\xi,\alpha) - f(\xi,\beta)$ is odd (and therefore at which one of the polynomials α and β changes sign while the other does not). Hence, and from the fact that $f(\lambda_0,\alpha) - f(\lambda_0,\beta)$ is odd, we conclude:

(15) There is an open interval whose right-hand end-point is λ_0, within which the values of α and β are everywhere of different signs.

We now introduce three new polynomials α', γ', and δ' by stipulating that the equations

(16) $$\alpha' = \alpha \cdot (\lambda_0 - \xi), \qquad \gamma' = \gamma \cdot (\lambda_0 - \xi), \qquad \delta' = \delta \cdot (\lambda_0 - \xi)$$

hold identically. By (12), (13), and (16) we obviously have:

(17) $\alpha' \cdot \beta_n^{2q} = \gamma' \cdot \beta - \delta'$ holds identically for some non-negative integer q ;

(18) $\alpha'_{m+1} \cdot \beta_n < 0$, where α'_{m+1} is the leading coefficient of α' ;

(19) $f(\lambda_0,\alpha') = f(\lambda_0,\alpha) + 1$, and also $f(\lambda_0,\delta') = f(\lambda_0,\delta) + 1$ (since δ and δ' do not vanish identically);

(20) $f(\xi,\alpha') = f(\xi,\alpha)$ for every $\xi \neq \lambda_0$, and similarly $f(\xi,\delta') = f(\xi,\delta)$ for every $\xi \neq \lambda_0$ (since δ and δ' do not vanish identically).

From (19) and (20) we conclude that the set of numbers λ such that $f(\lambda,\alpha') - f(\lambda,\beta)$ is odd differs from the analogous set for α and β only by the absence of λ_0; thus, using also (11),

$$h(\alpha',\beta) = h(\alpha,\beta) - 1 = p .$$

Consequently, our inductive premise applies to the polynomials α' and β: i.e., sentences (7), (8), and (9) are true if α is replaced by α'. Remembering the meaning of $g(\alpha,\beta)$, and taking account of (17) and (18), we conclude:

(21) $$g(\alpha',\beta) = g(\beta,\delta') \text{ in case } p \text{ is even;}$$

(22) $$g(\alpha',\beta) = g(\beta,\delta') + 1 \text{ in case } p \text{ is odd.}$$

We now want to show that

(23) $$g(\alpha,\beta) - g(\beta,\delta) = g(\alpha',\beta) - g(\beta,\delta') - 1 .$$

To do this, we first notice that by (16):

(24) The values of α and α' are of the same sign for every $\xi < \lambda_0$ (not a root of α); similarly for the values of δ and δ'.

We also observe that:

(25) There is no $\xi > \lambda_0$ such that $f(\xi,\beta) - f(\xi,\delta)$ is positive and odd; similarly for β and δ'.

For, if $f(\xi,\beta) - f(\xi,\delta)$ were positive and odd for some $\xi > \lambda_0$, then, by (10) and (12), $f(\xi,\alpha) - f(\xi,\beta)$ would be odd for the same $\xi > \lambda_0$, and this would contradict (14). The argument for β and δ' is analogous; instead of (12) we use (17), and when stating the final contradiction we refer to (14) combined with the first part of (20), instead of merely to (14).

We now distinguish two cases, dependent on the sign of $f(\lambda_0,\alpha) - f(\lambda_0,\beta)$. In view of (20), (24), and (25), the only number which can cause a difference in the values of $g(\alpha,\beta)$ and $g(\alpha',\beta)$, or in the values of $g(\beta,\delta)$ and $g(\beta,\delta')$, is the number λ_0. If now

(A) $$f(\lambda_0,\alpha) - f(\lambda_0,\beta) > 0 ,$$

then by (14) and (15) the number λ_0 effects a decrease of $g(\alpha,\beta)$ by 1; while, as a result of (19), it has no effect on the value of $g(\alpha',\beta)$. Hence

(26) $$g(\alpha,\beta) = g(\alpha',\beta) - 1 .$$

Furthermore, in the case (A) considered, $f(\lambda_0,\beta) - f(\lambda_0,\delta)$ cannot be positive by (10); and therefore $f(\lambda_0,\beta) - f(\lambda_0,\delta')$ cannot *a fortiori* be positive by (19). Thus in this case the number λ_0 proves to have no effect on the values of $g(\beta,\delta)$ and $g(\beta,\delta')$; and consequently

(27) $$g(\beta,\delta) = g(\beta,\delta') .$$

Equations (26) and (27) at once imply (23).

Turning to the case

(B) $$f(\lambda_0,\alpha) - f(\lambda_0,\beta) < 0 ,$$

we first notice that under this assumption λ_0 does not affect the value of $g(\alpha,\beta)$. Nor does it affect the value of $g(\alpha',\beta)$, since, by (14) and (19), $f(\lambda_0,\alpha') - f(\lambda_0,\beta)$ is even. Therefore,

(28) $$g(\alpha,\beta) = g(\alpha',\beta) .$$

Moreover, in the case (B) under consideration, we see from (10) that $f(\lambda_0,\alpha) = f(\lambda_0,\delta)$, and hence, using (14), that

(29) $$f(\lambda_0,\beta) - f(\lambda_0,\delta) \text{ is positive and odd.}$$

Let
$$f(\lambda_0,\alpha) = f(\lambda_0,\delta) = r .$$

Thus λ_0 is of order r in α and δ, and of a higher order in β. Consequently there are three polynomials α'', β'', and δ'' such that the equations

(30) $$\alpha = \alpha''\cdot(\lambda_0 - \xi)^r, \qquad \beta = \beta''\cdot(\lambda_0 - \xi)^r, \qquad \delta = \delta''\cdot(\lambda_0 - \xi)^r$$

hold identically; λ_0 is a root of β'', but not of α'' or δ''. We obtain from (12) and (30):

$$\alpha''\cdot\beta_n^{2q} = \gamma\cdot\beta'' - \delta'' .$$

Consequently, the values of α'' and δ'', for $\xi = \lambda_0$, have different signs. Therefore there is an open interval, whose right-hand end-point is λ_0, within which the values of α'' and δ'' have different signs; and, by (30), this applies also to α and δ. By comparing this result with (15), we conclude that there is an open interval whose right-hand end-point is λ_0, within which the values of β and δ have the same sign. Hence, and by (29), λ_0 contributes to the increase of $g(\beta,\delta)$ by 1. On the other hand, by (19) and (29), $f(\lambda_0,\beta) - f(\lambda_0,\delta')$ is even, so that λ_0 has no effect on the value of $g(\beta,\delta')$. Thus, finally,

(31) $$g(\beta,\delta) = g(\beta,\delta') + 1 .$$

Equations (28) and (31) again imply (23). Hence (23) holds in both the cases (A) and (B).

From (21), (22), and (23) we obtain at once:

$$g(\alpha,\beta) = g(\beta,\delta) \text{ in case } p+1 \text{ is even ;}$$

$$g(\alpha\ \beta) = g(\beta,\delta) - 1 \text{ in case } p+1 \text{ is odd .}$$

Thus we have shown that (7), (8), and (9) hold for polynomials α and β with $h(\alpha,\beta) = p+1$; and hence by induction they hold for arbitrary polynomials α and β. This completes the proof.

DEFINITION 28. *Let*

$$\alpha \equiv \alpha_0 + \alpha_1\xi + \ldots + \alpha_m\xi^m$$

$$\beta \equiv \beta_0 + \beta_1\xi + \ldots + \beta_n\xi^n$$

$$\gamma_1 \equiv \gamma_{1,0} + \gamma_{1,1}\xi + \ldots + \gamma_{1,n_1}\xi^{n_1}$$

$$\vdots$$

$$\gamma_r \equiv \gamma_{r,0} + \gamma_{r,1}\xi + \ldots + \gamma_{r,n_r}\xi^{n_r}$$

be arbitrary polynomials in ξ. We define the function T as follows:

(i) *If Φ is a formula of the form*

$$(\underset{k}{E}\xi)\,[\alpha = 0] \ ,$$

then we set

$$T(\Phi) \equiv \left[\sim\left(\alpha_0 = 0\right) \vee \ldots \vee \sim\left(\alpha_m = 0\right)\right] \wedge SG_\xi^{-k}\left(\alpha, D_\xi(\alpha)\right) \ .$$

(ii) *If Φ is a formula of the form*

$$\left[\sim\left(\alpha_0 = 0\right) \vee \ldots \vee \sim\left(\alpha_m = 0\right)\right] \wedge (\underset{k}{E}\xi)\left[(\alpha = 0) \wedge (\beta > 0)\right] \ ,$$

then we set

$$T(\Phi) \equiv \left\{ \left[\sim\left(\alpha_0 = 0\right) \vee \ldots \vee \sim\left(\alpha_m = 0\right)\right] \wedge \bigvee_{\substack{2k = r_1 - r_2 + r_3 \\ 0 \le r_1,\, r_2 \le m \\ -m \le r_3 \le m}} \left(SG_\xi^{-r_1}\left[\alpha, D_\xi(\alpha)\right] \wedge SG_\xi^{-r_2}\left[P_\xi(\alpha^2 + \beta^2), D_\xi P_\xi(\alpha^2 + \beta^2)\right] \wedge SG_\xi^{-r_3}\left[\alpha, D_\xi(\alpha) \cdot_\xi \beta\right] \right) \right\} \ .$$

(iii) *If Φ is a formula of the form*

$$\left[\sim\left(\alpha_0 = 0\right) \vee \ldots \vee \sim\left(\alpha_m = 0\right)\right] \wedge (\underset{k}{E}\xi)\left[(\alpha = 0) \wedge \left(\gamma_1 > 0\right) \wedge \ldots \wedge \left(\gamma_r > 0\right)\right] ,$$

where $r \geq 2$, *then we set*

$$T(\Phi) \equiv \bigvee_{\substack{2k = r_1 + r_2 - r_3 \\ 0 \leq r_1, r_2, r_3 \leq m}} \left\{ T(\Phi_1) \wedge T(\Phi_2) \wedge T(\Phi_3) \right\},$$

where

$$\Phi_1 \equiv \left\{ \left[\sim (a_0 = 0) \vee \ldots \vee \sim (a_m = 0) \right] \wedge \right.$$
$$\left. (E\xi)_{r_1} \left[(a = 0) \wedge (\gamma_1 > 0) \wedge \ldots \wedge (\gamma_{r-2} > 0) \wedge P_\xi (\gamma_{r-1} \cdot \gamma_r^2) > 0 \right] \right\},$$

$$\Phi_2 \equiv \left\{ \left[\sim (a_0 = 0) \vee \ldots \vee \sim (a_m = 0) \right] \wedge \right.$$
$$\left. (E\xi)_{r_2} \left[(a = 0) \wedge (\gamma_1 > 0) \wedge \ldots \wedge (\gamma_{r-2} > 0) \wedge P_\xi (\gamma_{r-1}^2 \cdot \gamma_r) > 0 \right] \right\},$$

$$\Phi_3 \equiv \left\{ \left[\sim (a_0 = 0) \vee \ldots \vee \sim (a_m = 0) \right] \wedge \right.$$
$$\left. (E\xi)_{r_3} \left((a = 0) \wedge (\gamma_1 > 0) \wedge \ldots \wedge (\gamma_{r-2} > 0) \wedge P_\xi \left[(-1) \cdot \gamma_{r-1} \cdot \gamma_r \right] > 0 \right) \right\}$$

in the case $r = 2$ *we omit the expression*

$$(\gamma_1 > 0) \wedge \ldots \wedge (\gamma_{r-2} > 0) \wedge$$

from the formulas defining Φ_1, Φ_2, *and* Φ_3.

(iv) *If* Φ *is a formula of the form*

$$\sim (a_m = 0) \wedge (E\xi)_k \left[(a = 0) \wedge (\gamma_1 > 0) \wedge \ldots \wedge (\gamma_r > 0) \right],$$

then we set

$$T(\Phi) \equiv \left(\sim (a_m = 0) \wedge T \left\{ \left[\sim (a_0 = 0) \vee \ldots \vee \sim (a_m = 0) \right] \wedge (E\xi)_k \left[(a = 0) \wedge (\gamma_1 > 0) \wedge \ldots \wedge (\gamma_r > 0) \right] \right\} \right).$$

(v) *If* Φ *is a formula of the form*

$$\left[\sim (\gamma_{1,n_1} = 0) \wedge \sim (\gamma_{2,n_2} = 0) \wedge \ldots \wedge \sim (\gamma_{r,n_r} = 0) \right] \wedge (E\xi)_k \left[(\gamma_1 > 0) \wedge \ldots \wedge (\gamma_r > 0) \right],$$

then, if $k > 0$, we set

$$T(\Phi) \equiv (0 = 1) \;;$$

while if $k = 0$ and $n_1 + \cdots + n_r = 0$, we set

$$T(\Phi) \equiv \left\{ \left[\sim(\gamma_{1,0} = 0) \wedge \ldots \wedge \sim(\gamma_{r,0} = 0) \right] \wedge \left[(0 > \gamma_{1,0}) \vee \ldots \vee (0 > \gamma_{r,0}) \right] \right\} ;$$

and if $k = 0$ and $n_1 + \ldots + n_r > 0$, we set

$$\begin{aligned} T(\Phi) \equiv & \left\{ \sim(\gamma_{1,n_1} = 0) \wedge \ldots \wedge \sim(\gamma_{r,n_r} = 0) \wedge \left[(0 > \gamma_{1,n_1}) \vee \ldots \vee (0 > \gamma_{r,n_r}) \right] \right\} \wedge \\ & \left\{ \left[0 > (-1)^{n_1} \cdot \gamma_{1,n_1} \right] \vee \ldots \vee \left[0 > (-1)^{n_r} \cdot \gamma_{r,n_r} \right] \right\} \wedge \\ & T \left\{ \sim(\delta = 0) \wedge (\underset{0}{E}\xi) \left(\left[D_\xi P_\xi (\gamma_1 \cdot \ldots \cdot \gamma_r) = 0 \right] \wedge (\gamma_1 > 0) \wedge \ldots \wedge (\gamma_r > 0) \right) \right\} \end{aligned}$$

where δ is the leading coefficient of $D_\xi P_\xi(\gamma_1 \cdot \ldots \cdot \gamma_r)$.

(vi) *If Φ is a formula of the form*

$$(\underset{k}{E}\xi) \left[(\gamma_1 > 0) \wedge \ldots \wedge (\gamma_r > 0) \right] ,$$

and if $k \neq 0$, we set

$$T(\Phi) \equiv (0 = 1) \;;$$

while if $k = 0$, we set

$$\begin{aligned} T(\Phi) \equiv & \left\{ \left[(\gamma_{1,0} = 0) \wedge \ldots \wedge (\gamma_{1,n_1} = 0) \right] \vee \ldots \vee \left[(\gamma_{r,0} = 0) \wedge \ldots \wedge (\gamma_{r,n_r} = 0) \right] \right\} \vee \\ & \bigvee_{(s_1, \ldots, s_r) \text{ in } S} \left\{ \Psi_{1,s_1} \wedge \ldots \wedge \Psi_{r,s_r} \wedge T\left(\left[\sim(\gamma_{1,s_1} = 0) \wedge \ldots \wedge \sim(\gamma_{r,s_r} = 0) \right] \wedge \right. \right. \\ & \left. \left. (\underset{0}{E}\xi) \left[(Rd_\xi^{n_1 - s_1}(\gamma_1) > 0) \wedge \ldots \wedge (Rd_\xi^{n_r - s_r}(\gamma_r) > 0) \right] \right) \right\} . \end{aligned}$$

where S is the set of all ordered r-tuples $(s_1, \ldots, s_r)$ with $0 \le s_1 \le n_1, \ldots, 0 \le s_r \le n_r$,

$\Psi_{k,l} \equiv \left[(\gamma_{k,l+1} = 0) \wedge \ldots \wedge (\gamma_{k,n_k} = 0) \right]$ *for $0 \le l < n_k$, and $\Psi_{k,l} \equiv (0 = 0)$ for $l = n_k$.*

(vii) *If Φ is a formula of the form*

$$(\underset{k}{E\xi})\left[(\alpha = 0) \wedge (\gamma_1 > 0) \wedge \ldots \wedge (\gamma_r > 0)\right]$$

then we set

$$T(\Phi) \equiv T\left\{\left[\sim(\alpha_0 = 0) \vee \ldots \vee \sim(\alpha_m = 0)\right] \wedge \Phi\right\} \vee$$

$$\left(\left[(\alpha_0 = 0) \wedge \ldots \wedge (\alpha_m = 0)\right] \wedge T\left\{(\underset{k}{E\xi})\left[(\gamma_1 > 0) \wedge \ldots \wedge (\gamma_r > 0)\right]\right\}\right).$$

(viii) *If Φ is a formula of the form*

$$(\underset{k}{E\xi})\left[(\gamma_1 = 0) \wedge \ldots \wedge (\gamma_r = 0)\right],$$

then we set

$$T(\Phi) \equiv T\left\{(\underset{k}{E\xi})\left[P_\xi(\gamma_1^2 + \ldots + \gamma_r^2) = 0\right]\right\}.$$

(ix) *If Φ is a formula of the form*

$$(\underset{k}{E\xi})\left[(\gamma_1 = 0) \wedge \ldots \wedge (\gamma_s = 0) \wedge (\gamma_{s+1} > 0) \wedge \ldots \wedge (\gamma_r > 0)\right]$$

where $1 < s < r$, then we set

$$T(\Phi) \equiv T\left\{(\underset{k}{E\xi})\left[P_\xi(\gamma_1^2 + \ldots + \gamma_s^2) = 0 \wedge (\gamma_{s+1} > 0) \wedge \ldots \wedge (\gamma_r > 0)\right]\right\}.$$

(x) *If Φ is a formula, not of any of the previous forms, but such that*

$$\Phi \equiv (\underset{k}{E\xi})\left(\Phi_1 \wedge \Phi_2 \wedge \ldots \wedge \Phi_r\right),$$

where each Φ_i is of one of the forms $\gamma_i = 0$ or $\gamma_i > 0$, and if $j_1, \ldots, j_u$ are the values of i (in increasing order) for which $\Phi_i \equiv (\gamma_i = 0)$; and if $j_{u+1}, \ldots, j_r$ are the values of i (in increasing order) for which $\Phi_i \equiv (\gamma_i > 0)$, then we set

$$T(\Phi) \equiv T(\underset{k}{E\xi})\left(\Phi_{j_1} \wedge \ldots \wedge \Phi_{j_r}\right)$$

THEOREM 29. *Let ξ be any variable, and let Φ be any formula such that $T(\Phi)$ is defined (by Definition 28). Then $T(\Phi)$ is a formula which contains no quantifiers, and no variables except those that occur free in Φ. Moreover, Φ is equivalent to $T(\Phi)$.*

Proof. The first part follows immediately from Theorem 27 and Definition 28.

We shall prove the second part by considering separately the ten possible forms Φ can have according to Definition 28. As in the proof of Theorem 27, we shall use here partially ordinary mathematical modes of expression, without taking any great pains to distinguish sharply mathematical from metamathematical notions.

Suppose first, then, that Φ is of the form 28 (i): i.e., that $\Phi \equiv (\underset{k}{E}\xi)(\alpha = 0)$, where α is a polynomial in ξ. We are to show that Φ is equivalent to the formula

$$\left[\sim\left(\alpha_0 = 0\right)\vee\ldots\vee\sim\left(\alpha_m = 0\right)\right]\wedge SG_{\xi}^{-k}\left(\alpha,D_{\xi}(\alpha)\right) ,$$

where $\alpha_0,\ldots,\alpha_m$ are the coefficients of α. Since by Theorem 27 the latter formula is equivalent to

$$\left[\sim\left(\alpha_0 = 0\right)\vee\ldots\vee\sim\left(\alpha_m = 0\right)\right]\wedge G_{\xi}^{-k}\left(\alpha,D_{\xi}(\alpha)\right) ,$$

we see that our task reduces to establishing the following: if α is any polynomial in ξ which is not identically zero, then α has k distinct roots if and only if $G_{\xi}^{-k}\left(\alpha,D_{\xi}(\alpha)\right)$ holds. Let s_1 be the number of numbers ξ such that: (1) the order of ξ in α is by a positive odd integer higher than its order in $D_{\xi}(\alpha)$; (2) there exists an open interval whose right-hand end-point is ξ, within which the values of α and $D_{\xi}(\alpha)$ have the same sign. Let s_2 be the number of numbers ξ which satisfy condition (1) above and moreover the condition: (3) there exists an open interval whose right-hand end-point is ξ, within which the values of α and $D_{\xi}(\alpha)$ have different signs. By the remark preceding Definition 22, we see that $G_{\xi}^{-k}(\alpha,D_{\xi}(\alpha))$ is true if and only if $-k = s_1 - s_2$. Moreover, it is readily seen that $s_1 = 0$, and that s_2 is simply the number of distinct roots of α. Thus $G_{\xi}^{-k}(\alpha,D_{\xi}(\alpha))$ holds if and only if k is the number of distinct roots of α, as was to be shown.

In order to treat the case where Φ is of the form 28 (ii), it is convenient first to establish the following: Let α and β be polynomials in ξ, let t_1, be the number of roots of α at which β is positive, and let t_2 be the number of roots of α at which β is negative; then $SG_{\xi}^{-c}(\alpha,D_{\xi}(\alpha)\cdot_{\xi}\beta)$ is true if and only if $c = t_1 - t_2$. This can easily be done by the sort of argument applied in the preceding paragraph — making use of Theorem 27 and the remark preceding Definition 22. We notice also that, since we are dealing with the algebra of real numbers, the common roots of two polynomials α and β coincide with the roots of $\alpha^2 + \beta^2$. Now let Φ be a formula of the form given in 28 (ii). To show that Φ is equivalent to $T(\Phi)$, it suffices to show that, if k is a non-negative integer, and α and β are polynomials, where α is of degree m and not identically zero, then the following conditions are equivalent: (1) there are exactly k roots of α at which β is positive; (2) there are integers r_1, r_2, and r_3 satisfying $2k = r_1 - r_2 + r_3$, $0 \leq r_1 \leq m$, $0 \leq r_2 \leq m$, $-m \leq r_3 \leq m$, such that r_1 is the number of roots of α, r_2 is the number of roots common to α and β, and r_3 is the difference between the number of roots of α at which β is positive and the number of roots of α at which β is negative. Now α has at most m roots; hence, if r_1, r_2, and r_3 have the meanings indicated in (2) — i.e., if r_1 is the number of roots of α, etc. — we obviously have

$$0 \leq r_1 \leq m,\ 0 \leq r_2 \leq m,\ \text{and}\ -m \leq r_3 \leq m .$$

Let k be the number of roots of α at which β is positive, and let r_4 be the number of roots of α at which β is negative. We see immediately from the definitions of r_1, r_2, r_3, k, and r_4 that

$$r_1 - r_2 = k + r_4$$

$$r_3 = k - r_4$$

Eliminating r_4 between these two equations, we obtain

$$2k = r_1 - r_2 + r_3 .$$

Thus (1) implies (2); the proof in the opposite direction is almost obvious.

To prove our theorem for formulas of the form 28 (iii), it suffices to show that if α is a polynomial of degree m, and not identically zero, and if $\gamma_1, \ldots, \gamma_r$ are any polynomials, then the following conditions are equivalent: (1) there are exactly k roots of α at which $\gamma_1, \ldots, \gamma_r$ are all positive; and (2) there are three integers r_1, r_2, r_3 satisfying $2k = r_1 + r_2 - r_3$, $0 \leq r_1, r_2, r_3 \leq m$, such that r_1 is the number of roots of α at which $\gamma_1, \ldots, \gamma_{r-2}$, and $\gamma_{r-1} \cdot \gamma_r^2$ are all positive, r_2 is the number of roots of α at which $\gamma_1, \ldots, \gamma_{r-2}$, and $\gamma_{r-1}^2 \cdot \gamma_r$ are all positive, and r_3 is the number of roots of α at which $\gamma_1, \ldots, \gamma_{r-2}$, and $-1 \cdot \gamma_{r-1} \cdot \gamma_r$ are all positive. In fact, if r_1, r_2, and r_3 have the meanings just indicated, we obviously have

$$0 \leq r_1, r_2, r_3 \leq m .$$

Let r_4 be the number of roots of α at which $\gamma_1, \ldots, \gamma_{r-2}, \gamma_{r-1}$ are all positive, and γ_r is negative. Let r_5 be the number of roots of α at which $\gamma_1, \ldots, \gamma_{r-2}$ and γ_r are all positive, and γ_{r-1} is negative. Let k be the number of roots of α at which $\gamma_1, \ldots, \gamma_r$ are all positive. From the definitions of r_1, r_2, r_3, r_4, r_5 and k we see that

$$k + r_4 = r_1$$

$$k + r_5 = r_2$$

$$r_4 + r_5 = r_3$$

Eliminating r_4 and r_5 from these equations, we obtain

$$2k = r_1 + r_2 - r_3 ,$$

which completes this part of the proof.

To prove our theorem for formulas of the form 28 (iv), we need only notice that the formula

$$\sim(\alpha_m = 0) \wedge [\sim(\alpha_0 = 0) \vee \ldots \vee \sim(\alpha_m = 0)]$$

is equivalent to

$$\sim(\alpha_m = 0) .$$

Now suppose that Φ is of the form 28 (v): i.e., that Φ is

$$\left[\sim\left(\gamma_{1,n_1} = 0\right)\wedge\ldots\wedge\sim\left(\gamma_{r,n_r} = 0\right)\right] \wedge \underset{k}{(E\xi)}\left[\left(\gamma_1 > 0\right)\wedge\ldots\wedge\left(\gamma_r > 0\right)\right] .$$

We notice first that if $k > 0$, then the formula

$$\underset{k}{(E\xi)}\left[\left(\gamma_1 > 0\right)\wedge\ldots\wedge\left(\gamma_r > 0\right)\right]$$

is never satisfied (i.e., is satisfied by no values of the free variables occurring in it), so that Φ is never satisfied either — and hence is equivalent to $(0 = 1)$. If $k = 0$, and $n_1 + \ldots + n_r = 0$, then $n_1 = n_2 = \ldots = n_r = 0$, and hence Φ reduces to

$$\left[\sim\left(\gamma_{1,0} = 0\right)\wedge\ldots\wedge\sim\left(\gamma_{r,0} = 0\right)\right] \wedge \underset{0}{(E\xi)}\left[\left(\gamma_{1,0} > 0\right)\wedge\ldots\wedge\left(\gamma_{r,0} > 0\right)\right] ,$$

where $\gamma_{1,0}, \ldots, \gamma_{r,0}$ are terms which do not involve ξ; since

$$\underset{0}{(E\xi)}\left[\left(\gamma_{1,0} > 0\right)\wedge\ldots\wedge\left(\gamma_{r,0} > 0\right)\right]$$

is then equivalent to

$$\sim\left[\left(\gamma_{1,0} > 0\right)\wedge\ldots\wedge\left(\gamma_{r,0} > 0\right)\right] ,$$

we see that Φ is equivalent to $T(\Phi)$, as was to be shown.

Thus we are left with the case that Φ is of the form 28 (v) where $k = 0$ and $n_1 + \ldots + n_r > 0$. To establish in this case that Φ is equivalent to $T(\Phi)$, it suffices to prove: If $\gamma_1, \ldots, \gamma_r$ are polynomials in ξ not all of which are of degree zero, and whose leading coefficients are all different from zero, then a necessary and sufficient condition that there exist no value of ξ which makes all these polynomials positive is that the following three conditions hold: (1) at least one of the polynomials have a negative leading coefficient; (2) at least one of the polynomials satisfy $(-1)^{n_i}\gamma_{i,ni} < 0$ (where n_i is the degree of the polynomial, and $\gamma_{i,ni}$ its leading coefficient); (3) there exist no value of ξ which is a root of the derivative of the product of the polynomials, and which makes them all positive. To see that the condition is necessary, suppose that $\gamma_1, \ldots, \gamma_r$ are polynomials which are never all positive for the same value of ξ; then it is immediately apparent that (3) is satisfied; to see that (1) is satisfied, we remember that, if the leading coefficient of a polynomial is positive, then we can find a number μ such that the polynomial is positive for all values of the variable ξ greater than μ; the proof of (2) is similar, by considering large negative values of the variable. Now suppose, if possible, that the condition is not sufficient: i.e., suppose that (1), (2), and (3) are satisfied, and that there exists some ξ which makes all the polynomials positive. Let λ be a value of ξ at which $\gamma_1 > 0, \ldots, \gamma_r > 0$. Then we see that, for $\xi = \lambda$,

$$\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_r > 0 .$$

On the other hand, since (1) is true, there exists an i such that γ_i has a negative leading coefficient. Hence we can find a λ' which is larger than λ and sufficiently large that γ_i is negative at λ'. Then γ_i is positive at λ and negative at λ' and hence has a root between these numbers. Since every root of γ_i is also a root of $\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_r$, we conclude that $\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_r$ has a root to the right of λ. Similarly, making use of (2), we see that $\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_r$ has a root to the left of λ. Now let μ_1 be the largest root of $\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_r$ to the left of λ and let μ_2 be the smallest root of $\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_r$ to the right of λ. Then $\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_r$ is positive in the open interval (μ_1, μ_2) and zero at its end-points. We see that no γ_i can have a root within the open interval (μ_1, μ_2); since each γ_i is positive at λ, which lies within this interval, we conclude that each γ_i is positive throughout the whole open interval. On the other hand, since $\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_r$ is zero at μ_1 and at μ_2, we see by Rolle's theorem that there is a point ν within (μ_1, μ_2) at which the derivative of $\gamma_1 \cdot \gamma_2 \cdot \ldots \cdot \gamma_r$ vanishes. Since this contradicts (3), we conclude that the condition is also sufficient.

Now suppose that Φ is of the form 28 (vi). If $k \neq 0$, it is obvious that Φ is equivalent to $T(\Phi)$. Hence suppose that $k = 0$. Two cases are logically possible: either one of the polynomials $\gamma_1, \ldots, \gamma_r$ vanishes identically, or none of them does. In the first case, Φ obviously holds. In the second case, let s_i, for $i = 1, \ldots, r$, be the largest index j such that $\gamma_{i,j}$ (i.e., the j^{th} coefficient of γ_i) does not vanish. Then Φ is obviously equivalent to the formula

$$\Psi \equiv \left[\sim\left(\gamma_{1,s_1} = 0\right) \wedge \ldots \wedge \sim\left(\gamma_{r,s_r} = 0\right)\right] \wedge (E\xi)\left(\left[Rd_\xi^{n_1 - s_1}(\gamma_1) > 0\right] \wedge \ldots \wedge \left[Rd_\xi^{n_r - s_r}(\gamma_r) > 0\right]\right).$$

However Ψ is of the form 28 (v), and hence, as we have shown above, is equivalent to $T(\Psi)$. In view of these remarks, by looking at the formula defining $T(\Phi)$ in 28 (vi), we see at once that Φ and $T(\Phi)$ are equivalent.

If Φ is of the form 28 (vii), our theorem follows from the obvious fact that Φ is equivalent to

$$\left\{\left[\sim\left(\alpha_0 = 0\right) \vee \ldots \vee \sim\left(\alpha_m = 0\right)\right] \wedge \Phi\right\} \vee \left\{\left[\left(\alpha_0 = 0\right) \wedge \ldots \wedge \left(\alpha_m = 0\right)\right] \wedge \Phi\right\}.$$

If Φ is of the form 28 (viii), or of the form 28 (ix), our theorem follows from the fact — which was already used in discussing 28 (ii) — that the common roots of r polynomials $\gamma_1, \ldots, \gamma_r$ coincide with the roots of $\gamma_1^2 + \ldots + \gamma_r^2$.

If Φ is of the form 28 (x), our theorem follows from the associative and commutative laws for conjunction, familiar from elementary logic.

DEFINITION 30. *Let Φ, Ψ, and θ be any formulas, and ξ any variable.*

(i) *If Φ is an atomic formula, we set*

$$U(\Phi) \equiv \Phi .$$

(ii) *If $\Phi \equiv (\Psi \vee \theta)$, then we set*

$$U(\Phi) \equiv \left[U(\Psi) \vee U(\theta)\right] .$$

(iii) *If* $\Phi \equiv (\Psi \wedge \Theta)$, *then we set*

$$U(\Phi) \equiv \left[U(\Psi) \wedge U(\Theta) \right] .$$

(iv) *If* $\Phi \equiv \sim \Psi$, *then we set*

$$U(\Phi) \equiv \sim U(\Psi) .$$

(v) *If* $\Phi \equiv (E\xi)\,\Psi$, *and*

$$QP_{\xi}\,U(\Psi) \equiv \Psi_1 \vee \Psi_2 \vee \ldots \vee \Psi_n ,$$

where Ψ_i, *for* $i = 1, \ldots, n$, *is a conjunction of atomic formulas, then we set*

$$U(\Phi) \equiv \sim T\left[\underset{0}{(E\xi)}\,\Psi_1\right] \vee \sim T\left[\underset{0}{(E\xi)}\,\Psi_2\right] \vee \ldots \vee \sim T\left[\underset{0}{(E\xi)}\,\Psi_n\right] .$$

THEOREM 31. *If* Φ *is any formula, then* $U(\Phi)$ *is a formula which contains no quantifiers, and no free variables except variables which occur free in* Φ. *Moreover,* Φ *is equivalent to* $U(\Phi)$.

PROOF. By induction on the order of Φ, making use of Theorems 14 and 29.

COROLLARY 32. *If* Φ *is any sentence, then* $U(\Phi)$ *is an equivalent sentence without any variables or quantifiers.*

The first part of our task as outlined at the beginning of this section has thus been completed. We have established a general procedure which permits us to transform every formula (and in particular every sentence) into an equivalent formula (or sentence) without quantifiers[12],[13]. Before continuing the discussion, we should like to give a few relatively simple examples in which such a transformation has actually been carried out. The equivalent transformations $U'(\Phi)$ which are given below for some formulas Φ do not coincide with $U(\Phi)$ but can be obtained from the latter by means of elementary simplifications.

Let ξ be any variable, and α_0, α_1, α_2, α_3, β_0, β_1, and β_2 any terms which do not involve ξ. If

$$\Phi \equiv (E\xi)\left[\alpha_0 + \alpha_1 \cdot \xi + \alpha_2 \cdot \xi^2 + \alpha_3 \cdot \xi^3 = 0\right]$$

we obtain an equivalent formula by setting

$$U'(\Phi) \equiv \Big\{ (\alpha_0 = 0) \vee \left[\sim(\alpha_1 = 0) \wedge (\alpha_2 = 0) \right] \vee$$
$$\left[\sim(\alpha_2 = 0) \wedge \sim(4 \cdot \alpha_0 \cdot \alpha_2 > \alpha_1^2) \right] \vee \sim(\alpha_3 = 0) \Big\}$$

(where, as can easily be guessed, 4 stands for 1 + 1 + 1 + 1). If

$$\Phi \equiv (E\xi)\left[\alpha_0 + \alpha_1\cdot\xi + \alpha_2\cdot\xi^2 + \alpha_3\cdot\xi^3 > 0\right],$$

we can put

$$U'(\Phi) \equiv \left\{ \left(\alpha_0 > 0\right) \vee \left(\alpha_1^2 > 4\cdot\alpha_0\cdot\alpha_2\right) \vee \left(\alpha_2 > 0\right) \vee \sim\left(\alpha_3 = 0\right) \right\}.$$

If, finally,

$$\Phi \equiv (E\xi)\left[\left(\alpha_0 + \alpha_1\cdot\xi + \alpha_2\cdot\xi^2 = 0\right) \wedge \left(\beta_0 + \beta_1\cdot\xi + \beta_2\cdot\xi^2 > 0\right)\right],$$

we can put

$$\begin{aligned} U'(\Phi) \equiv \Big\{ & \left[\left(\alpha_0 = 0\right) \wedge \left(\alpha_1 = 0\right) \wedge \left(\alpha_2 = 0\right) \wedge \left(\left(\beta_0 > 0\right) \vee \left(\beta_1^2 > 4\cdot\beta_0\cdot\beta_2\right) \vee \left(\beta_2 > 0\right)\right)\right] \vee \\ & \left[\sim\left(\alpha_1 = 0\right) \wedge \left(\alpha_2 = 0\right) \wedge \left(\alpha_0^2\cdot\beta_2 + \alpha_1^2\cdot\beta_0 > \alpha_0\cdot\alpha_1\cdot\beta_1\right)\right] \vee \\ & \left[\sim\left(\alpha_2 = 0\right) \wedge \sim\left(4\cdot\alpha_0\cdot\alpha_2 > \alpha_1^2\right) \wedge \left(\alpha_1^2\cdot\beta_2 + 2\cdot\alpha_2^2\cdot\beta_0 > 2\cdot\alpha_0\cdot\alpha_2\cdot\beta_2 + \alpha_1\cdot\alpha_2\cdot\beta_1\right)\right] \vee \\ & \Big[\sim\left(\alpha_2 = 0\right) \wedge \left(\alpha_1^2 > 4\alpha_0\cdot\alpha_2\right) \wedge \\ & \quad \left(\alpha_0^2\cdot\beta_2^2 + \alpha_0\cdot\alpha_2\cdot\beta_1^2 + \alpha_1^2\cdot\beta_0\cdot\beta_2 + \alpha_2^2\cdot\beta_0^2 > \alpha_0\cdot\alpha_1\cdot\beta_1\cdot\beta_2 + 2\cdot\alpha_0\cdot\alpha_2\cdot\beta_0\cdot\beta_2 + \alpha_1\cdot\alpha_2\cdot\beta_0\cdot\beta_1\right)\Big]\Big\}. \end{aligned}$$

We now turn to the second part of our task. We want to correlate, with every sentence Φ which contains no variables or quantifiers, an equivalent sentence of a very special form: in fact, one of the two sentences

$$0 = 0$$

and

$$0 = 1.$$

We first consider terms which occur in such sentences. As is easily seen, every such term is obtained from the algebraic constants 0, I, and −1 by combining them by means of addition and multiplication. Hence we can correlate with every such term α an integer $n(\alpha)$ in the following way.

DEFINITION 33. *We set*

$$n(1) = 1,$$

$$n(-1) = -1,$$

$$n(0) = 0.$$

If $\alpha \equiv (\beta + \gamma)$, *then we set*

$$n(\alpha) = n(\beta) + n(\gamma).$$

If $\alpha \equiv (\beta\cdot\gamma)$, *then we set*

$$n(\alpha) = n(\beta)\cdot n(\gamma).$$

Remark. It should be emphasized that the above definition correlates integers, not expressions, with terms. It is for this reason that we have written, for example,

$$n(1) = 1,$$

instead of

$$n(1) \equiv 1;$$

$n(1)$ is the integer 1, not a name of that integer. In the equation

$$n(\alpha) = n(\beta) + n(\gamma)$$

the addition sign indicates the sum of the two integers $n(\beta)$ and $n(\gamma)$. $n(\alpha)$ is what would ordinarily be called the "value" of the expression α; thus, if

$$\alpha \equiv 1 + (1 + 1)\cdot(1 + 1), \text{ then } n(\alpha) = 5.$$

On the other hand, we could use for our purposes, instead of integers, certain expressions of our formal system of algebra — in fact one of the terms of the following sequence

$$\ldots, \quad (-1) + (-1), \quad -1, \quad 0, \quad 1, \quad 1 + 1, \ldots .$$

We can use these special terms since they can obviously be put in one-to-one correspondence with arbitrary integers. As a result of this modification, however, Definition 33 and the subsequent Definition 34 would assume a more complicated form.

Definition 34. *Let α and β be terms, and Φ, Ψ, and θ formulas, none of which contain any variables.*

(i) *If $\Phi \equiv (\alpha = \beta)$, we set*

$$W(\Phi) \equiv (0 = 0)$$

in case $n(\alpha) = n(\beta)$, and otherwise

$$W(\Phi) \equiv (0 = 1) .$$

(ii) *If $\Phi \equiv (\alpha > \beta)$, we set*

$$W(\Phi) \equiv (0 = 0)$$

in case $n(\alpha) > n(\beta)$, and otherwise

$$W(\Phi) \equiv (0 = 1) .$$

(iii) *If $\Phi \equiv (\Psi \vee \theta)$, we set*

$$W(\Phi) \equiv (0 = 0)$$

in case either $W(\Psi) \equiv (0 = 0)$ or $W(\theta) \equiv (0 = 0)$, and otherwise

$$W(\Phi) \equiv (0 = 1) .$$

(iv) *If* $\Phi \equiv (\Psi \wedge \Theta)$, *we set*

$$W(\Phi) \equiv (0 = 0)$$

in case both $W(\Psi) \equiv (0 = 0)$ *and* $W(\Theta) \equiv (0 = 0)$, *and otherwise*

$$W(\Phi) \equiv (0 = 1)\ .$$

(v) *If* $\Phi \equiv \sim\Psi$, *we set*

$$W(\Phi) \equiv (0 = 0)$$

in case $W(\Psi) \equiv (0 = 1)$, *and otherwise*

$$W(\Phi) \equiv (0 = 1)\ .$$

Theorem 35. *If* Φ *is any sentence which involves no quantifiers or variables, then* $W(\Phi)$ *is one or the other of the two sentences* $0 = 0$ *and* $0 = 1$. *Moreover,* Φ *is equivalent to* $W(\Phi)$.

Proof. By induction on the order of Φ.

Theorem 36. *If* Φ *is any sentence, then* $WU(\Phi)$ *is one or the other of the two sentences* $0 = 0$ *and* $0 = 1$. *Moreover,* Φ *is equivalent to* $WU(\Phi)$.

Proof. By 32 and 35.

Now by analyzing the definitions of W, U, and the preceding functions, we notice that for any given sentence Φ we can actually find the value of $WU(\Phi)$ in a finite number of steps[14]. By combining this with the result stated in Theorem 36, we obtain

Theorem 37. *There is a decision method for the class of all true sentences of elementary algebra*[15].

In concluding this section we should like to remark that the minimum number of steps which are necessary for the evaluation of $WU(\Phi)$ is of course a function of the form of Φ — in particular, this number depends on the length of Φ, on the number of quantifiers occurring in it, and so on. The problem of estimating the order of increase of this function is of primary importance in connection with the question of the feasibility of constructing a decision machine for elementary algebra.

SECTION 3.

EXTENSIONS TO RELATED SYSTEMS

In this section we shall discuss some applications to other systems of the results obtained in Section 2, as well as some problems that are still open.

The decision method found for the algebra of real numbers can be extended to various algebraic systems built upon real numbers — thus to the elementary algebra of complex numbers, that of quaternions, and that of n-dimensional vectors. We can think of the elementary algebra of complex numbers, for example, as a formal system very closely related to that described in Section 1: variables are now thought of as representing arbitrary complex numbers; the logical and mathematical constants remain unchanged; but now the greater-than relation is thought of as holding exclusively between real numbers — thus we can define real numbers within the system by saying that x is real if, for some y, $x > y$. If one wishes, one can enrich the system by a new predicate $Rl(x)$, agreeing that $Rl(x)$ will mean that x is real[16].

The results obtained can furthermore be extended to the elementary systems of n-dimensional Euclidean geometry. Since the methods of extending the results to the algebraic systems and the geometric systems are essentially the same, we shall consider a little more closely the case of 2-dimensional Euclidean geometry.

We first give a sketchy description of the formal system of 2-dimensional Euclidean geometry. We use infinitely many *variables*, which are to be thought of as representing arbitrary points of the Euclidean plane. We use three constants denoting relations between points: the binary relation of *identity*, symbolized by "="; the ternary relation of *betweenness*, symbolized by "B", so that "$B(x, y, z)$" is to be read "y is between x and z" (i.e., y lies between x and z on the straight line connecting them; it is not necessary that the three points all be distinct; $B(x, y, z)$ is always true if $x = y$ or if $y = z$; but we cannot have $x = z$ unless $x = y = z$); and the quaternary *equidistance* relation, symbolized by "D", so that "$D(x,y;\ x',y')$" is to be read "x is just as far from y as x' is from y'" (or, "the distance from x to y equals the distance from x' to y'")[17]. The only *terms* of this system are variables. An *atomic formula* is an expression of one of the forms

$$\xi = \eta, \qquad B(\xi_1, \xi_2, \xi_3), \qquad D(\xi, \eta; \xi', \eta'),$$

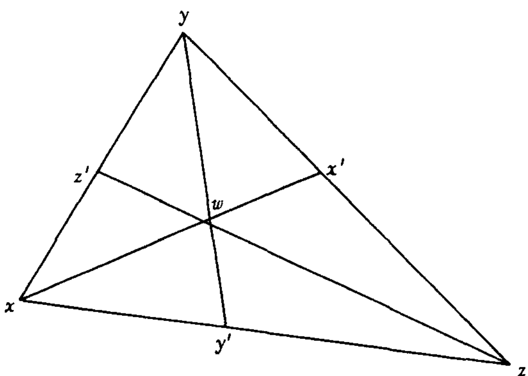
where

$$\xi, \eta, \xi_1, \xi_2, \xi_3, \xi', \text{ and } \eta'$$

are arbitrary variables. As in the formal system of elementary algebra we build up *formulas*, from atomic formulas by means of negation, conjunction, disjunction, and the application of quantifiers; we also introduce here as abbreviations the symbols $\rightarrow$ and $\leftrightarrow$.

Sentences of elementary geometry, in our formulation, express certain facts about points and relation between them. On the other hand, most theorems which one finds in high-school textbooks on this subject involve also such notions as triangle, plane, circle, line, and the like. It is easy, however, to convince oneself that a considerable part of these notions can be translated into the language of our system. Thus, for example, the theorem that the medians of a triangle are concurrent can be expressed as follows (cf. the figure immediately following the formula):

$$(Ax)(Ay)(Az)(Ax')(Ay')(Az')\Big\{\Big[\sim B(x,y,z)\wedge\sim B(y,z,x)\wedge\sim B(z,x,y)\wedge B(x,y',z)\wedge B(y,z',x)\wedge B(z,x',y)\wedge D(x,z';z',y)\wedge D(y,x';x',z)\wedge D(z,y';y',x)\Big]\longrightarrow (Ew)\Big[B(x,w,x')\wedge B(y,w,y')\wedge B(z,w,z')\Big]\Big\}.$$



On the other hand, it would not be difficult to enrich our system of geometry so as to enable us to refer to these elementary figures directly. Regarding more essential limitations of our system, see the remarks in the Introduction.

In order to obtain a decision procedure for elementary geometry, we correlate with every sentence Φ of elementary geometry a sentence Φ^* of elementary algebra in the sense of Section 1. The construction of Φ^* can be roughly described in the following way. With every (geometric) variable ξ in Φ we correlate two different (algebraic) variables $\bar{\xi}$ and $\bar{\bar{\xi}}$, in such a way that if ξ and η are two different variables in Φ, then $\bar{\xi}$, $\bar{\bar{\xi}}$, $\bar{\eta}$, and $\bar{\bar{\eta}}$ are all distinct. Next we replace in Φ every quantifier expression $(E\xi)$ by $(E\bar{\xi})(E\bar{\bar{\xi}})$; every partial formula $\xi = \eta$ by $(\bar{\xi} = \bar{\eta}) \wedge (\bar{\bar{\xi}} = \bar{\bar{\eta}})$; every formula $B(\xi,\eta,\mu)$ by

$$\Big[(\bar{\bar{\eta}}-\bar{\bar{\xi}})\cdot(\bar{\mu}-\bar{\eta}) = (\bar{\bar{\mu}}-\bar{\bar{\eta}})\cdot(\bar{\eta}-\bar{\xi})\Big]\wedge\Big[((\bar{\xi}-\bar{\eta})\cdot(\bar{\eta}-\bar{\mu}) > 0)\vee((\bar{\xi}-\bar{\eta})\cdot(\bar{\eta}-\bar{\mu}) = 0)\Big]\wedge \Big[((\bar{\bar{\xi}}-\bar{\bar{\eta}})\cdot(\bar{\bar{\eta}}-\bar{\bar{\mu}}) > 0)\vee((\bar{\bar{\xi}}-\bar{\bar{\eta}})\cdot(\bar{\bar{\eta}}-\bar{\bar{\mu}}) = 0)\Big];$$

and every partial formula $D(\xi,\eta;\mu,\nu)$ by

$$(\bar{\xi}-\bar{\eta})^2 + (\bar{\bar{\xi}}-\bar{\bar{\eta}})^2 = (\bar{\mu}-\bar{\nu})^2 + (\bar{\bar{\mu}}-\bar{\bar{\nu}})^2.$$

It is now obvious to anyone familiar with the elements of analytic geometry that whenever Φ is true then Φ^* is true, and conversely. And since we can always decide in a mechanical way about the truth of Φ^*, we can also do this for Φ.

The decision method just outlined applies with obvious changes to Euclidean geometry of any number of dimensions[18]. And, since it depends exclusively on the possibility of introducing into geometry a system of real coordinates, it will apply as well to various systems of non-Euclidean and projective geometry[19].

We can attempt to extend the results concerning elementary algebra in still another way: in fact, by introducing into the system of algebra new mathematical terms which cannot be defined by means of those occurring in the original system. The new terms may denote certain properties of numbers, certain relations between numbers, or certain operations on numbers (in particular, unary operations — i.e., functions of one real variable). In consequence of any such extension of the original system we are presented with a new decision problem. In some cases, from the results known in the literature it easily follows that the solution of the problem is negative — i.e., that no decision method for the enlarged system can ever be found, and that no decision machine can be constructed. In view of the Gödel-Church-Rosser result mentioned in the Introduction, this applies, for instance, if we introduce into the system of real algebra the predicate *In*, to denote the property of being an integer (so that $In(x)$ is read: "x in an integer"); and, by the result of Mrs. Robinson, the same applies to the predicate *Rt*, denoting the property of being rational. The situation is still the same if we introduce a symbol for some periodic function, for instance, *sine*; this is seen if only from the fact that the notion of an integer and of a rational number can easily be defined in terms of *sine* and the notions of our original system; thus we can say that x is a rational if and only if it satisfies the formula

$$(Ey)(Ez)\left[(x\cdot y = z)\wedge\sim(y = 0)\wedge(\sin y = 0)\wedge(\sin z = 0)\right].$$

In other cases, by introducing a new symbol we arrive at a system for which the decision problem is open. This applies, for instance, to the system obtained by introducing the operation of exponentiation (of course restricted to the cases where it yields a definite real result), or — what amounts essentially to the same thing — the symbol *Exp* to denote an exponential with a fixed base, for example, 2[20]. The decision problem for the system just mentioned is of a great theoretical and practical interest. But its solution seems to present considerable difficulties. These difficulties appear, however, to be of a purely mathematical (not logical) nature: they arise from the fact that our knowledge of conditions for the solvability of equations and inequalities in the enlarged system is far from adequate[21].

In this connection it may be worth while to mention that by introducing the operation of exponentiation into the system of elementary complex algebra, we arrive at a system for which the solution of the decision problem is negative. In fact it is well known that the exponential function in the complex domain is periodic, and hence, like the function *sine* in the real domain, it allows us to define the notion of an integer.

NOTES

1. When dealing with theories presented as formal axiomatized systems, one often uses the term "decision method" for a theory in a different sense, by referring it to the class, not of all true sentences, but of all theorems of the theory: i.e., of all sentences of the theory which can be derived from the axioms by means of certain prescribed rules of inference.

2. See Löwenheim [11], Post [14], Langford [10], Presburger [15], and McKinsey [12]. (The numbers in square brackets refer to items in the Bibliography following these Notes.) The results of Tarski and Mrs. Szmielew are unpublished.

3. See Gödel [4], Church [3], and Rosser [16]. The results of Mostowski, Tarski, and Mrs. Robinson are unpublished.

4. This result was mentioned, though in an implicit form and without proof, in Tarski [19], pp.233 and 234; see also Tarski [22]. Some partial results tending in the same direction — e.g., decision methods for elementary algebra with addition as the only operation, and for the geometry of the straight line — are still older, and were obtained by Tarski and presented in his university lectures in the years 1926-1928; cf. Presburger [15], p.95, footnote 4, and Tarski [21], p.324, footnote 53.

5. In this connection A. Mostowski has pointed out the following. Although the general concept of an integer is lacking in our system of elementary algebra, yet it can easily be shown that a "general arithmetic" in the sense of Carnap [2], p.206, is "contained" in this system. Since the language in question is consistent and decidable (again in the sense of Carnap [2], pp.207 and 209), it provides an example against Carnap's assertion that "every consistent language which contains a general arithmetic is irresoluble" (*ibid.*, p.210). Carnap's definition of the phrase "contains a general arithmetic" is therefore certainly too wide.

6. Among the works listed in the Bibliography, Hilbert-Bernays [7] may be consulted for various logical and metamathematical notions and results involved in our discussion, and van der Waerden [23] will provide necessary information in the domain of algebra.

7. In this monograph we establish certain results concerning various mathematical theories, such as elementary algebra and elementary geometry. Hence our discussion belongs to the general theory of mathematical theories: i.e., to what is called "metamathematics". To give our discussion a precise form we have to use various metamathematical symbols and notions. Since, however, we do not want to create any special difficulties for the reader, we apply the following method: when referring to individual symbols of the mathematical theory being discussed, or to expressions involving these symbols, we use the symbols and expressions themselves. We could thus say that the symbols and expressions play in our discussion the role of metamathematical constants. On the other

hand, when referring to arbitrary symbols and expressions, or to arbitrary expressions of a certain form, we use special metamathematical variables. In fact, small Greek letters, as for instance "α", "β", "γ", are used to represent arbitrary terms, and in particular the letters "ξ", "η", "λ", "μ", "ν", will be used to represent arbitrary variables; on the other hand, Greek capitals "Φ", "Θ", "Ψ" will be used to represent arbitrary formulas and sentences. With these exceptions we do not introduce any special metamathematical symbolism. Various metamathematical notions whose intuitive meaning is clear will be used without any explanation; this applies, for instance, to such a phrase as "the variable ξ occurs in the formula Φ." Also, we do not consider it necessary to set up an axiomatic foundation for our metamathematical discussion, and we avoid a strictly formal exposition of metamathematical arguments. We assume that we can avail ourselves in metamathematics of elementary number theory; we use variables "m", "n", "p", and so on to represent arbitrary integers; and we employ the ordinary notation for individual integers, arithmetical relations between integers, and operations on them.

The reader who is interested in the deductive foundations, and a precise development, of metamathematical discussion, may be referred to Carnap [2] (part II, pp.55 ff.), Gödel [4], Tarski [21] (Section 2, pp.279 ff., in particular p.289), and Tarski [20] (especially p.100).

8. In choosing symbols for the formalized system of algebra, we have been interested *in presenting the metamathematical results in the simplest possible form.* For this reason we have not introduced into the system various mathematical and logical symbols which are ordinarily used in expressing mathematical theorems: such as the subtraction symbol "$-$", the symbol "$<$", the implication sign "$\rightarrow$", the equivalence sign "$\leftrightarrow$", and the universal quantifier "A". Nevertheless, some of these symbols are made available for our use, since they are introduced as metamathematical abbreviations. If we wished, we could reduce the number of symbols still further; we could, for instance, dispense with the "$>$" sign, by treating

$$x > y$$

merely as an abbreviation for

$$(Ez)\left[\sim(z = 0) \wedge (x = y + z^2)\right] .$$

In an analogous way we could dispense with the symbols 0, 1, and -1, and with one of the two logical connectives $\vee$ and $\wedge$. But such a reduction in the number of symbols would hardly be advantageous from our point of view.

It should be pointed out that, in order to increase the efficiency of the decision machine which may be constructed on the basis of this monograph, it might very well turn out to be useful to enrich the symbolism of our system, even if this carried with it certain complications in the description of the decision method.

9. A formal definition of truth can be found in Tarski [21]. It should be pointed out that we can eliminate the notion of truth from our whole discussion by subjecting the system of elementary algebra to the process of axiomatization. For this purpose, we single out certain sentences of our system which

we call "axioms". They are divided into logical and algebraic axioms. The logical axioms (or rather, axiom schemata) are those of the sentential calculus and the lower predicate calculus with identity; they can be found, for instance, in Hilbert-Bernays [7] (see sections 3, 4 and 5 in vol.1, and supplement 1 in vol.2). Among algebraic axioms we find, in the first place, those which characterize the set of real numbers as a commutative ordered field with the operations + and · and the relation >, and which single out in a familiar way the three special elements 0, 1, and -1. These axioms are supplemented by one additional axiom schema comprehending all sentences of the form

$$\text{(i)}\quad (A\xi_1)\dots(A\xi_n)(A\eta)(A\zeta)\Big\{\Big[(\eta>\zeta)\wedge(E\xi)\big((\xi=\eta)\wedge(\alpha>0)\big)\wedge (E\xi)\big((\xi=\zeta)\wedge(0>\alpha)\big)\Big]\rightarrow(E\xi)\big((\eta>\xi)\wedge(\xi>\zeta)\wedge(\alpha=0)\big)\Big\}.$$

where $\xi_1,\dots,\xi_n$, η, ζ are arbitrary variables, ξ is any variable different from η and ζ, and α is any term — which, in the non-trivial cases, of course involves the variable ξ. Intuitively speaking, this axiom schema expresses the fact that every function which is represented by a term of our symbolism (i.e., every rational integral function) and which is positive at one point and negative at another, vanishes at some point in between.

From what can be found in the literature (see van der Waerden [23], in particular pp. 235 f.), it is seen that this axiom schema can be equivalently replaced by the combination of an axiom expressing the fact that every positive number has a square root, with an axiom schema comprehending all sentences to the effect that every polynomial of odd degree has a zero: i.e., all sentences of the form

$$\text{(ii)}\quad (A\eta_0)(A\eta_1)\dots(A\eta_{2n+1})\Big[\sim(\eta_{2n+1}=0)\rightarrow(E\xi)\big(\eta_0+\eta_1\xi+\dots+\eta_{2n+1}\xi^{2n+1}=0\big)\Big],$$

where $\eta_0, \eta_1,\dots,\eta_{2n+1}$ are arbitrary variables, and ξ is any variable different from all of them. It is also possible to use, instead of (ii), a schema comprehending all sentences to the effect that every polynomial of degree at least three has a quadratic factor. Finally, it turns out to be possible to replace equivalently schema (i) by the seemingly much stronger axiom schema comprehending all those particular cases of the continuity axiom which can be expressed in our symbolism. (By the continuity axiom we may understand the statement that every set of real numbers which is bounded above has a least upper bound; when expressing particular cases of this axiom in our symbolism, we speak, not of elements of a set, but of numbers satisfying a given formula.) The possibility of this last replacement, however, is a rather deep result, which is a by-product of other results presented in this work: in fact, of those discussed below in Note 15.

After having selected the axioms, we describe the operations by means of which new sentences can be derived from given ones. These operations are expressed in the so-called "rules of inference" familiar from mathematical logic. A sentence which can be derived from axioms by repeated applications of the rules of inference is called a provable sentence. In our further discussion — in particular, in defining the notions of equivalence of terms and equivalence of formulas — we replace everywhere the notion of a true sentence by that of a provable one. Hence, when establishing certain of the results given later — in particular, the theorems about equivalent formulas — we have to show that the sentences involved are formally derivable from the selected axioms (and not that they are true in any intuitive sense); otherwise the discussion does not differ from that in the text.

10. We use the term "decision method" here in an intuitive sense, without giving a formal definition. Such a procedure is possible because our result is of a positive character: we are actually going to establish a decision method, and no one who understands our discussion will be likely to have any doubt that this method enables us to decide in a finite number of steps whether any given sentence of elementary algebra is true. The situation changes radically, however, if one intends to obtain a result of a negative character — i.e., to show for a given theory that no decision method can be found; a precise definition of a decision method then becomes indispensable. The way in which such a definition is to be given is of course known from the contemporary literature. Using one of the familiar methods — for instance the method due to Gödel — one establishes a one-to-one correspondence between expressions of the system and positive integers, and one agrees to treat the phrase "there exists a decision method for the class A of expressions" as equivalent with the phrase "the set of numbers correlated with the expressions of A is general recursive." (When the set of numbers correlated with a class A of sentences is general recursive, we sometimes say simply that A is general recursive.) For a discussion of the notion of general recursiveness, see Hilbert-Bernays [7] and Kleene [8].

11. The *method of eliminating quantifiers* occurs in a more or less explicit form in the papers Löwenheim [11] (section 3), Skolem [18] (section 4), Langford [10], and Presburger [15]. In Tarski's university lectures for the years 1926-1928 this method was developed in a general and systematic way; cf. Presburger [15], p.95, footnote 4, and p.97, footnote 1.

12. The results obtained in Theorems 27 and 29, and culminating in Theorem 31, seem to deserve interest even from the purely mathematical point of view. They are closely related to the well-known theorem of Sturm, and in proving them we have partly used Sturm's methods.

The theorem most closely related to Sturm's ideas is Theorem 27. In fact, by analyzing, and slightly generalizing, the proof of this theorem we arrive at the following formulation. Let α and β be any two polynomials in a variable ξ, and κ and μ any two real numbers with $\kappa < \mu$. We construct a sequence of polynomials $\gamma_1, \gamma_2, \ldots, \gamma_n$ — which may be called the Sturm chain for α and β — by taking α for γ_1, β for γ_2, and assuming that γ_i, with $i > 2$, is the negative remainder of γ_{i-2} and γ_{i-1}; we discontinue the construction when we reach a polynomial γ_n which is a divisor of γ_{n-1}. Let $\kappa_1, \ldots, \kappa_n$ and $\mu_1, \ldots, \mu_n$ be the sequences of values of $\gamma_1, \ldots, \gamma_n$ at $\xi = \kappa$ and $\xi = \mu$, respectively; let k be the number of changes in sign of the sequence $\kappa_1, \ldots, \kappa_n$, and let m be the number of changes in sign of the sequence $\mu_1, \ldots, \mu_n$. Then it turns out that k-m is just the number $g(\alpha,\beta)$ defined as in the proof of Theorem 27, but with the roots assumed to lie between κ and μ. (In Theorem 27 we were dealing, not with the arbitrary interval (κ,μ), but with the interval $(-\infty, +\infty)$.)

Sturm himself considered two particular cases of this general theorem: the case where β is the derivative of α — when the number k-m proves to be simply the number of distinct roots of α in the interval (κ,μ); and the case where β is arbitrary but α is a polynomial without multiple roots — when k-m proves to be the difference between the number of roots of α at which β agrees in sign with the derivative of α, and the number of roots of α at which β disagrees in sign with the derivative of α — the roots being taken from the interval (κ,μ). These two special cases easily follow from the theorem, and we have made an essential use of this fact in the proof of Theorem 29. The general formulation was found recently by J.C.C. McKinsey; it contributed to a simplification, not of the original decision method itself, but of its mathematical description.

Apart, however, from technicalities connected with the notion and construction of Sturm chains, the mathematical content of Sturm's theorem essentially consists in the following: given any algebraic equation in one variable x, and with the coefficients a_0, $a_1,\ldots,$ a_n, there is an elementary criterion for this equation to have exactly k real solutions (which may be in addition subjected to the condition that they lie in a given interval): such a criterion is obtained by constructing a certain finite sequence of systems, each consisting of finitely many equations and inequalities which involve the coefficients a_0, $a_1,\ldots,$ a_n of the given equation (and possibly the end-points b and c of the interval); it is shown that the equation has exactly k roots if and only if its coefficients satisfy all the equations and inequalities of at least one of these systems. (When applied to an equation with constant coefficients, the criterion enables us actually to determine the number of roots of the equation, but this is only a by-product of Sturm's theorem.) By applying Sturm's theorem we obtain in particular an elementary condition for an algebraic equation in one unknown to have at least one real solution. Theorem 31 gives directly an extension of this special result to an arbitrary system of algebraic equations and inequalities with arbitrarily many unknowns. It is easily seen, however, that from our theorem one can obtain stronger consequences: in fact, criteria for such systems to have exactly k real solutions. To clear up this point, let us consider the simple case of a system consisting of one equation in two unknowns

(i) $$F(x,y) = 0\ .$$

We form the following system of equations and inequalities

(ii) $$\begin{cases} F(x,y) = 0 \\ F(x',y') = 0 \\ (x - x')^2 + (y - y')^2 > 0\ . \end{cases}$$

By Theorem 31 we have an elementary criterion for the system (ii) to have at least one solution. But it is obvious that this criterion is at the same time a criterion for (i) to have at least two solutions. In the same way, we can obtain criteria for (i) to have at least $3,4,\ldots,$ k real solutions. Hence we also obtain a criterion for (i) to have exactly k solutions (since an equation has exactly k solutions if it has at least k, but not at least $k + 1$, solutions).

The situation does not change if the solutions are required to satisfy additional conditions — namely, to lie within given bounds. We can thus say that *Theorem 31 constitutes an extension of Sturm's theorem* (or, at least, of the essential part of this theorem) *to arbitrary systems of equations and inequalities with arbitrarily many unknowns.*

It may be noticed that by Sturm's theorem a criterion for solvability (in the real domain) involves systems which contain inequalities as well as equations. Hence, to obtain an extension of this theorem to systems of equations in many unknowns, it seemed advisable to consider inequalities from the beginning, and in the first step to extend the theorem to arbitrary systems of equations and inequalities in one unknown. As a result of this preliminary extension, the subsequent induction with respect to the number of unknowns becomes almost trivial.

In its most general form the mathematical result obtained above seems to be new, although, in view of the extent of the literature involved, we have not been able to establish this fact with absolute certainty. At any rate some precedents are known in the literature. From what can be found in Sturm's original paper, the extension of his result to the case of one equation and one inequality with one unknown can easily be obtained; Kronecker, in his theory of characteristics, concerned himself with the case of n (independent) equations with n unknowns. It seems, on the other hand, that such a simple problem as that of finding an elementary criterion for the solvability in the real domain of one equation in two unknowns has not been previously treated; the same applies to the case of a system of inequalities (without equations) in one unknown — although this case is essential for the subsequent induction. (Cf. in this connection, Weber [24], pp.271 ff., and Runge [17], pp.416 ff., where further references to the literature are also given.)

13. The result established in Theorem 31 and discussed in the preceding note has various interesting consequences. To formulate them, we can use, for instance, a geometric language and refer the result to n-dimensional analytic space with real coordinates — or, what is slightly more convenient, to the infinite-dimensional space S_ω, in which, however, every point has only finitely many coordinates different from zero. By an elementary algebraic domain in S_ω we understand the set of all points $\langle x_0, x_1, \ldots, x_n, \ldots \rangle$ in which the coordinates $x_{k_1}, x_{k_2}, \ldots, x_{k_m}$ satisfy a given algebraic equation or inequality

$$F(x_{k_1}, \ldots, x_{k_m}) = 0$$

or

$$F(x_{k_1}, \ldots, x_{k_m}) > 0 \quad ,$$

and the remaining coordinates are zeros. Let $\mathfrak{J}$ be the smallest family of point sets in S_ω which contains among its elements all elementary algebraic domains and is closed under the operations of finite set-addition, finite set-multiplication, set-complementation, and projection parallel to any axis. (The projection of a set A parallel to the n^{th} axis is the set obtained by replacing by zero the n^{th} coordinate of every point of A.) Now Theorem 31 in geometric formulation implies that the family $\mathfrak{J}$ consists of those and only those sets in S_ω which are finite sums of finite products of elementary algebraic domains. The possibility of passing from the original formulation to the new one is a consequence of the known relations between projection and existential quantifiers.

Theorem 31 has also some implications concerning the notion of arithmetical (or elementary) definability. A set A of real numbers is called arithmetically definable if there is a formula Φ in our system containing one free variable and such that A consists of just those numbers which satisfy Φ. In a similar way we define an arithmetically definable binary, ternary, and in general an n-ary, relation between real numbers. Now Theorem 31 gives us a simple characterization of those sets of real numbers, and relations between real numbers, which are arithmetically definable. We see, for instance, that a set of real numbers is arithmetically definable if and only if it is a set-theoretical sum of a finite number of intervals (bounded, or unbounded; closed, open, or half-closed, half-open) with algebraic end-points; in particular, a real number (i.e., the set consisting of this number alone) is arithmetically definable if and only if it is algebraic. Hence it follows that an arithmetically definable set of real numbers which is bounded above has an arithmetically definable least upper bound — a consequence which is relevant in connection with a result mentioned near the end of Note 9. As further consequences we conclude that the sets of all integers, of all rationals, etc., are not arithmetically definable, which justifies some remarks made in the Introduction.

As a simple corollary of Theorem 31 we obtain: For every formula Φ there is an equivalent formula Ψ with the same free variables of the following form:

$$\Psi \equiv (E\xi_1)\ldots(E\xi_n)\left[\alpha = 0\right] .$$

This corollary can also be interpreted geometrically.

For the notions used in this note, cf. Tarski [19] and Kuratowski-Tarski [9].

14. In other words, using terminology introduced in Note 10, we state that the number-theoretic function correlated with WU is general recursive. Actually this function is easily seen to be a general recursive function of a very simple type — what is called a "primitive" recursive function.

15. If we take the axiomatic point of view outlined in Note 9 and replace in our whole discussion the notion of truth by that of provability, then the meaning and extent of the fundamental results obtained in Section 2 undergo some essential changes. In the new interpretation, Theorem 36 implies that every sentence of elementary algebra is provably equivalent to one of the sentences $0 = 0$ or $0 = 1$. In addition, we can easily show that $WU(\Phi) \equiv (0 = 0)$ if and only if $WU(\sim\Phi) \equiv (0 = 1)$, and that for any provable sentence Φ we have $WU(\Phi) \equiv (0 = 0)$. By combining these results, we arrive at the conclusion that the axiomatic system of elementary algebra is consistent and complete, in the sense that one and only one of any pair Φ and $\sim\Phi$ of contradictory sentences is provable. The proof of this fact has what is called a constructive character. The completeness of the system implies by itself the existence of a decision method for the class of all provable sentences (even without the knowledge that the number-theoretical function correlated with WU is general recursive); cf. Kleene [8].

We further notice that all the axioms listed in Note 9 are satisfied, not only by real numbers, but by the elements of any real closed field in the sense of Artin and Schreier (cf. van der Waerden [23], chapter IX). Thus all the results just mentioned can be extended to the elementary theory of real closed fields. From the fact that this theory is complete it follows that there is no sentence expressible in our formal system of elementary algebra which would hold in one real closed field and fail in another. In other words, any arithmetically definable property (in the sense of Note 13) which applies to one real closed field also applies to all other such fields: i.e., any two real closed fields are arithmetically indistinguishable.

In general, when applied to axiomatized theories, the notions of truth and provability do not have the same extension. Usually it can be shown only that every provable sentence is true. Since, however, in the case of elementary algebra the class of provable sentences turns out to be complete, we conclude that in this particular case the converse holds, and hence that the two classes coincide. Thus in the case of elementary algebra three equivalent definitions of a true sentence are available: (i) the definition of a true sentence as a sentence Φ such that $WU(\Phi) \equiv (0 = 0)$; (ii) the definition of a true sentence as a provable sentence; (iii) the definition based on the general method of defining truth developed in Tarski [21]. Correspondingly, when starting to develop elementary algebra, we have three methods of stipulating which sentences will be accepted in this algebra — i.e., recognized as true. Apart from any educational and psychological considerations, the first method has in principle a great advantage: it implies directly that the class of sentences recognized as true is general recursive. Hence it provides us from the beginning with a mechanical device to decide in each particular case whether a sentence should be accepted, and serves as a basis for the construction of a decision machine. The second method — which is the usual axiomatic method — is less advantageous: it has as a direct consequence only the fact that the class of sentences recognized as true is what is called recursively enumerable (not necessarily general recursive). It leads to the construction of a machine which would be much less useful — to a machine which would construct, so to speak, blindly, the infinite sequence of all sentences accepted as true, without being able to tell in advance whether a given sentence would ever appear in this sequence. The third method, though very important for certain theoretical considerations, is even less advantageous than the second. It does not show that the class of accepted sentences is recursively enumerable; it can hardly be applied to a practical construction of a theory unless it is combined on a metamathematical level with the first or the second method. It goes without saying that in the particular case with which we are concerned — that is, in the case of elementary algebra — by establishing the equivalence of these possible definitions of truth we have *eo ipso* shown that in this case the three methods determine eventually the same class of sentences.

16. We can also consider a more restricted elementary system of complex algebra, from which the symbols > and *Rl* have been eliminated. The decision method applies to such a system as well, and even becomes much simpler. By taking the axiomatic point of view and basing the discussion on the notion of provability, we can carry over to this restricted system of complex algebra all the results pointed out in Note 15. Since the axioms of this system prove to

be satisfied by elements of an arbitrary algebraic closed field with characteristic zero (thus, in particular, by the complex algebraic numbers), the results apply to the general elementary theory of such fields; in particular, any two algebraic closed fields with characteristic zero turn out to be arithmetically indistinguishable. A slight change in the argument permits us further to extend the results just mentioned to algebraic closed fields with any given characteristic p. (For these notions, cf. van der Waerden [23], chapter 5.)

On the other hand, as was mentioned in the Introduction, no decision method can be given for the arithmetic of rationals, nor for the elementary theory of arbitrary fields. For most special fields the decision problem still remains open. This applies, for instance, to finite algebraic extensions of the field of rational numbers and to the field of all numbers expressible by means of radicals. It would be interesting to solve the decision problem for some of these special fields, or even to obtain a simple mathematical characterization of all those fields for which the solution of the decision problem is positive.

17. As in the case of elementary algebra (see Note 8), some of the symbols listed could be eliminated from the system of elementary geometry and treated merely as abbreviations. It is known, for example, that in n-dimensional geometry with $n \geq 2$ the symbol "B" of the betweenness relation can be defined in terms of the symbol "D" of the equidistance relation.

18. Exactly as in the case of elementary algebra, we can treat the system of elementary geometry in an axiomatic way, and base our discussion of the decision problem on the notion of provability. If we restrict ourselves to the case of two dimensions, we can take, for instance (in addition to the general logical axioms mentioned in Note 9), the following geometrical axioms:

(i) $(Ax)(Ay)B(x,y,y)$:

(ii) $(Ax)(Ay)\left[B(x,y,x) \rightarrow (x = y)\right]$;

(iii) $(Ax)(Ay)(Az)\left[B(x,y,z) \rightarrow B(z,y,x)\right]$;

(iv) $(Ax)(Ay)(Az)(Au)\left\{\left[B(x,y,u) \wedge B(y,z,u)\right] \rightarrow B(x,y,z)\right\}$;

(v) $(Ax)(Ay)(Az)(Au)\left\{\left[B(x,y,z) \wedge B(y,z,u) \wedge \sim(y = z)\right] \rightarrow B(x,y,u)\right\}$;

(vi) $(Ax)(Ay)(Az)(Au)\left\{\left[B(x,y,u) \wedge B(x,z,u)\right] \rightarrow \left[B(x,y,z) \vee B(x,z,y)\right]\right\}$;

(vii) $(Ax)(Ay)(Az)(Au)\left\{\left[B(x,y,z) \wedge B(x,y,u) \wedge \sim(x = y)\right] \rightarrow \left[B(x,z,u) \vee B(x,u,z)\right]\right\}$;

(viii) $(Ex)(Ey)(Ez)\left[\sim B(x,y,z) \wedge \sim B(y,z,x) \wedge \sim B(z,x,y)\right]$;

(ix) $(Ax)(Ay)(Az)(Az')(Au)\left\{\left[B(x,z',z) \wedge B(y,z,u)\right] \rightarrow (Ey')\left[B(x,y',y) \wedge B(y',z',u)\right]\right\}$;

(x) $(Ax)(Ay)(Az)(Az')(Au)\Big\{\big[B(x,z,z') \wedge B(y,z,u) \wedge \sim (x = z)\big] \longrightarrow$
$(Ey')(Eu')\big[B(x,y,y') \wedge B(x,u,u') \wedge B(y',z',u')\big]\Big\}$;

(xi) $(Ax)(Ay)(Az)(Au)(Ev)\Big\{\big[(B(x,u,v) \vee B(u,v,x) \vee B(v,x,u)) \wedge B(y,v,z)\big] \vee$
$\big[(B(y,u,v) \vee B(u,v,y) \vee B(v,y,u)) \wedge B(z,v,x)\big] \vee$
$\big[(B(z,u,v) \vee B(u,v,z) \vee B(v,z,u)) \wedge B(x,v,y)\big]\Big\}$;

(xii) $(Ax)(Ay)\ D(x,y;y,x)$

(xiii) $(Ax)(Ay)(Az)\big[D(x,y;z,z) \longrightarrow (x = y)\big]$

(xiv) $(Ax)(Ay)(Az)(Au)(Av)(Aw)\Big\{\big[D(x,y;z,u) \wedge D(x,y;v,w)\big] \longrightarrow D(z,u;v,w)\Big\}$;

(xv) $(Ax)(Ay)(Az)(Az')(Au)\Big\{\big[\sim (x = y) \wedge D(x,z;x,z') \wedge D(y,z;y,z') \wedge$
$B(y,u,z') \wedge (B(x,u,z) \vee B(x,z,u))\big] \longrightarrow (z = z')\Big\}$;

(xvi) $(Ax)(Ax')(Ay)(Ay')(Az)(Az')(Au)(Au')\Big\{\big[D(x,y;x',y') \wedge D(y,z;y',z') \wedge$
$D(x,u;x',u') \wedge D(y,u;y',u') \wedge$
$B(x,y,z) \wedge B(x',y',z') \wedge$
$\sim (x = y) \wedge \sim (y = z)\big] \longrightarrow D(z,u;z',u')\Big\}$;

(xvii) $(Ax)(Ay)(Ay')(Az')(Ez)\Big\{B(x,y,z) \wedge D(y,z;y',z')\Big\}$;

(xviii) $(Ax)(Ax')(Ay)(Ay')(Az')(Av)\Big\{D(x,y;x',y') \longrightarrow$
$(Ez)(Eu)\big[D(x,z;x',z') \wedge D(y,z;y',z') \wedge B(z,u,v) \wedge$
$(B(x,y,u) \vee B(y,u,x) \vee B(u,x,y))\big]\Big\}$.

To these is added the axiom schema which comprehends all particular cases of the axiom of continuity (e.g., in the Dedekind form) that are expressible in our system: i.e., all sentences of the following form:

(xix) $(A\xi_1)\ldots(A\xi_n)\Big\{(E\mu)(A\eta_1)(A\eta_2)\big[(\Phi \wedge \Psi) \longrightarrow B(\mu,\eta_1,\eta_2)\big] \longrightarrow$,
$(E\mu)(A\eta_1)(A\eta_2)\big[(\Phi \wedge \Psi) \longrightarrow B(\eta_1,\mu,\eta_2)\big]\Big\}$,

where neither μ nor η_2 is free in the formula Φ, and neither μ nor η_1 is free in the formula Ψ.

The reader will notice the formal simplicity of most of the axioms just given—which we have tried to put into evidence by avoiding (contrary to the prevailing custom) the use of any defined terms in formulating the axioms. On the other hand, however, the reader will easily recognize a close similarity between our axiom system and various systems which can be found in the comprehensive literature of the foundations of geometry; see, e.g., Hilbert [6].

By means of some obvious changes in (viii) and (xi) one can obtain from this axiom system a system of axioms for elementary geometry of any number of dimensions.

Again as in the case of algebra, one of the achievements attained by the axiomatic treatment of the subject is a constructive consistency proof for the whole of elementary geometry. This improves a result to be found in Hilbert-Bernays [7] (vol.2, pp.38 ff.). It may also be mentioned that in Hilbert [6] (section 35, pp. 96-98) a result is given which is closely connected with the decision method for elementary geometry, but which has a rather restricted character.

19. As is known, ordinary projective geometry can be treated as a specialized branch of lattice theory — more specifically, of the theory of modular lattices: see Birkhoff [1], where references to earlier papers of Menger can also be found. The decision method applies to this branch of the theory of modular lattices as well.

20. In the axiomatic presentation, the introduction of the new symbol *Exp* would require the addition of new axioms. The following three axioms can be used, for instance, for this purpose:

$$(Ax)(Ay)\Big[(x > y) \longrightarrow \big(Exp(x) > Exp(y)\big)\Big]$$

$$(Ax)(Ay)\Big[\big(Exp(x)\cdot Exp(y)\big) = Exp(x + y)\Big]$$

$$Exp(1) = 1 + 1.$$

21. Similar decision problems arise if we introduce into our system of elementary algebra the symbol *Al* to denote the property of being an algebraic number, or the symbol *Cn* to denote the property of being a constructible number (i.e., a number which can be obtained from the number 1 by means of the rational operations, together with the operation of extracting square roots). If the solution of the decision problem for elementary algebra with the addition of the symbol *Cn* were positive, this result would have an interesting application for geometry: in fact, we should obtain a decision method which would enable us, not only to decide on the truth of every sentence of elementary geometry, but also — in the case of existential sentences (like the sentence stating the possibility of trisecting an arbitrary angle) — to decide whether the truth of such a sentence can be established using only the so-called elementary constructions: i.e., constructions by means of rule and compass. It seems unlikely, however, that the solution of the problem in question is indeed positive; probably we shall be able to show that such a sharper decision method for elementary geometry cannot be found.

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SUPPLEMENTARY NOTES

1. The references to decision methods previously established given on p. 1 and in Note 2, p. 47, were not intended to be complete. For some further results and additional references compare the series of abstracts by Mostowski, Mrs. Szmielew, and Tarski in the *Bulletin of the American Mathematical Society,* vol. 55, pp. 63-66 and 1192, 1949, as well as the following papers:

Gentzen, G., "Untersuchungen über das logische Schliessen". *Mathematische Zeitschrift,* vol. 39, pp. 176-210, 1934.

McKinsey, J. C. C., "A solution of the decision problem for the Lewis systems S2 and S4, with an application to topology". *Journal of symbolic logic,* vol. 6, pp. 117-134, 1941.

McKinsey, J. C. C., and Tarski, A., "The algebra of topology". *Annals of mathematics,* vol. 45, pp. 141-191, 1944.

Skolem, T., *Über einige Satzfunktionen in der Arithmetik.* [Skrifter utgitt av det Norske Videnskaps-Akademi i Oslo, I. klasse 1930, no. 7.] Oslo, 1931.

Szmielew, W., "Decision problem in group theory". *Proceedings of the Tenth International Congress of Philosophy,* fasc. 2, pp. 763-766, Amsterdam, 1949.

2. The results of Mrs. Robinson, Mostowski, and the author mentioned in the first paragraph of p. 2 appeared in print (some only in outline form) after the first edition of this monograph. See the series of abstracts in the *Journal of symbolic logic,* vol. 14, pp. 75-78, 1949, as well as the article:

Robinson, J., "Definability and decision problems in arithmetic". *Journal of symbolic logic,* vol. 14, pp. 98-114, 1949.

Some related results can be found in the article:

Robinson, R. M., "Undecidable rings". *Transactions of the American Mathematical Society,* vol. 70, pp. 137-159, 1951.

3. The following remarks refer to the discussion on pp. 4 and 5. Many examples of open problems in elementary algebra and geometry are known; one comes across discussions of such problems by looking through any issue of the *American mathematical monthly.* However, the problem of describing the behavior of the function d does not seem to have been previously treated in the literature. For a discussion of a related problem — involving the decomposition of P and Q, not in triangles, but in arbitrary polygons — see the following article (where references to earlier papers of Moese and the author can also be found):

Tarski, A., "Uwagi o stopniu równoważności wielokątów". (Remarks on the degree of equivalence of polygons, in Polish.) *Parametr,* vol. 2, 1932.

It may be interesting to mention that some conclusions concerning the function d can be derived from the general results stated in Note 13, p. 53. In fact, it can be shown that every bounded interval (a, b) can be divided into finitely many subintervals such that the function d is constant within each of these subintervals; all the endpoints of these subintervals are algebraic, with the possible exception of a and b.

4. The statement in Note 12, p. 52, to the effect that the case of a system of inequalities in one unknown was not previously treated in the literature, seems to be correct when applied to the situation which existed at the time when the results of this work were found and first mentioned in print (1931), as well as for many years thereafter. However, the author's attention has been called to the fact that this case has recently been treated in the paper:

Meserve, B. E., "Inequalities of higher degree in one unknown". *American journal of mathematics,* vol. 49, pp. 357-370, 1947.

5. The discussion in Note 13, pp. 52-53, may convey the impression that the notions considered in the first paragraph have but little in common with those considered in the second paragraph. Actually, these notions are very closely related to each other. In fact, if the notion of arithmetical definability is applied to arbitrary sets of sequences of real numbers, i.e., to point sets in S_ω, then the family of all arithmetically definable point sets simply coincides with the family $\mathfrak{J}$.

6. It was stated in Note 13, p. 53, that every real number which is arithmetically definable is algebraic. An interesting application of this result to the theory of games has recently been found by O. Gross and is discussed in his paper "On certain games with transcendental values" (to appear in the *American mathematical monthly).*

7. As was pointed out in Note 15, p. 54, the completeness theorem for elementary algebra leads to the following result: every arithmetically definable property which applies to one real closed field also applies to all other such fields. It is important to realize that the result just mentioned extends to a comprehensive class of properties which are not arithmetically definable (i.e., which are not expressible in our formal system of elementary algebra). This class includes in particular all the properties expressed by sentences of the form $(Am)\Phi_m$, $(Am)(En)\Psi_{m,n}$, . . . where $m, n,$ . . . are variables assumed to range over all positive integers and where Φ_m, $\Psi_{m,n}$, . . . are formulas which involve $m, n,$ . . . (as free variables) and which, for any particular values of $m, n,$ . . ., are equivalent in any real closed fields to sentences of elementary algebra. In fact, consider a sentence of this kind, say, $(Am)\Phi_m$. If this sentence holds in a given real closed field, the same obviously applies to all the particular sentences of the form Φ_m, i.e., to Φ_1, Φ_2, Φ_3, ... Each of these particular sentences is equivalent to a sentence of elementary algebra and hence, by the result discussed, it holds in every real closed field. Consequently, the universal sentence $(Am)\Phi_m$ also holds in every real closed field. Various theorems of these types are known which were originally established for the field of real numbers using essentially the continuity of this field (sometimes with the help of difficult topological methods) and whose extension to arbitrary real closed fields presented a new and difficult problem; in view of our general result such an extension now becomes automatic. As examples the following three theorems may be mentioned.

I. *Let R be an m-dimensional region defined as the set of all points $\langle x_0, x_1,$ . . ., $x_{m-1}\rangle$ satisfying a finite system of inequalities $P_i(x_0, x_1, \ldots, x_{m-1}) \geq 0$ where the P_i's for $i = 0, 1, \ldots, n-1$ are polynomials of degree at most p; let F be a rational function whose denominator does not vanish on R. Then there is a positive integer q (dependent exclusively on m, n, and p) such that the set S of all function* *values of F on R is a sum of at most q closed intervals; if R is bounded, then all these intervals are also bounded, and hence F reaches a maximum and minimum on R.*

II. *For every system of m polynomials* $P_0, P_1, \ldots, P_{m-1}$ *in m variables there are real numbers* $c \geq 0, x_0, x_1, \ldots, x_{m-1}$ *such that* $P_i(x_0, x_1, \ldots, x_{m-1}) = c \cdot x_i$ *for* $i = 0, 1, \ldots, m-1$.

III. *Every commutative division algebra — whether associative or not — over the field of real numbers is of order 1 or 2; if it has a unit, it coincides either with the field of real numbers or with the algebraic closure of this field (i.e., with the field of complex numbers).*

While I is simply a particular case of a familiar theorem concerning continuous functions, and the same applies to the "eigenvalue theorem" II, Theorem III has a specifically algebraic character; it was proved, with the help of topology, in the article:

Hopf, H., "Systeme symmetrischer Bilinearformen und euklidische Modelle der projectiven Räume". *Vierteljahrsschrift der Naturforschenden Gesellschaft in Zürich*, vol. 85, supplement No. 32, pp. 165-177, 1940.

The research to extend these and similar results, obtained by means of topological methods, to arbitrary real closed fields was initiated by H. Hopf. Compare the following papers where partial results in this direction (in particular, extensions of some special cases of Theorem I) have been achieved directly, without the help of our general method:

Behrend, F., "Über Systeme algebraischer Gleichungen". *Compositio mathematica*, vol. 7, pp. 1-19, 1939.

Habicht, W., "Ein Existenzsatz über reelle definite Polynome". *Commentarii mathematici helvetici*, vol. 18, pp. 331-348, 1946.

Habicht, W., "Über die Lösbarkeit gewisser algebraischer Gleichungssysteme". *Commentarii mathematici helvetici*, vol. 18, pp. 154-175, 1946.

Kaplansky, I., "Polynomials in topological fields". *Bulletin of the American Mathematical Society*, vol. 54, pp. 909-916, 1948.

8. In view of the results stated in Note 16, pp. 54-55, the remarks made in the preceding note will still hold if, instead of real closed fields, we consider the class of algebraically closed fields with a given characteristic.

Denumerable models of complete theories *

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Introduction. The following theorem, which characterizes a certain type of complete theories, was established by Ryll-Nardzewski.

0.1. *A necessary and sufficient condition for a complete theory T, having infinite models, to be $\aleph_0$-categorical* (¹) *is that, for each n, there are only finitely many formulas, with free variables* $v_0, \dots, v_{n-1}$, *which are inequivalent in T.*

A simplification of the proof (of necessity) was found by Ehrenfeucht (²).

In this paper we shall apply methods closely related to those used in proving 0.1 to the study of the denumerable models of some other types of complete theories.

Before the work can be described more fully, some notions must be defined. Let $\mathfrak{A}$ be an infinite model of a theory T. We say that $\mathfrak{A}$ is *homogenenous* if, whenever $a_0, \dots, a_n$ and $a'_0, \dots, a'_n$ satisfy in $\mathfrak{A}$ exactly the same formulas of T, there is an automorphism of $\mathfrak{A}$ carrying a_i into a'_i $(i = 0, \dots, n)$. $\mathfrak{A}$ is $\aleph_0$-*universal* if $\mathfrak{A}$ is denumerable and is an elementary extension (cf. § 1) of an isomorph of each denumerable model of T. $\mathfrak{A}$ is *prime* if every model of T is an elementary extension of an isomorph of $\mathfrak{A}$ (³).

* Many of the results in this paper were announced in [22].

(¹) A theory is said to be categorical in the power $\aleph_\alpha$ or, simply, $\aleph_\alpha$-categorical if all its models of that power are isomorphic (cf. [9]). The exact meaning we ascribe to various familiar terms such as "theory", will be specified in § 1; but let it be said now that, herein, "complete" implies "consistent".

(²) Cf. [13] and, also, [10], p. 24. Later, independently, 0.1 was established by L. Svenonius, and by E. Engeler [3].

(³) Notions more or less closely related to "$\aleph_0$-universal" and "homogeneous" have been employed by various authors. Cf. e.g., [1], [4], and [8]; also, see footnote 15. A. Robinson [12] defined "prime" as above, but with "elementary" omitted; however, for the "model-complete" theories he was studying, this omission does not change the extension of the notion.

The two types of complete theories we discuss are those having prime models and those having $\aleph_0$-universal models—or, what, turns out to be the same—those having $\aleph_0$-universal, homogeneous models. It is shown that a model of T of the last sort is unique up to isomorphism, and that the same applies to a prime model. A number of necessary and sufficient conditions for a model to be such a model, or for a theory to have such a model are given in 3.4, 3.5, 4.6, and 4.7; these are the principal results of the paper.

According to a theorem of Ehrenfeucht [2], certain of these conditions are satisfied by theories categorical in a non-denumerable power. Consequently, our results may be applied to show that such theories possess prime models and $\aleph_0$-universal, homogeneous models. Some additional conclusions regarding these theories are also derived in § 5.

In § 6, it is shown that a complete theory cannot have exactly two non-isomorphic denumerable models, answering a question of Raphael Robinson.

§ 1. Preliminaries. The *theories* we consider are formalized in the first order logic with identity, and are assumed to have at most $\aleph_0$ non-logical constants (4), each of which is either a relation symbol or an individual constant. (When more than $\aleph_0$ non-logical constants occur, we speak of a *generalized theory*.) A theory specifies a non-repeating list $X_0, \ldots, X_\xi, \ldots (\xi < \eta)$ of its non-logical constants. The distinct individual variables of every theory T are $v_0, v_1, \ldots, v_n, \ldots$ *The set of all formulas of T whose free variables are among* $v_0, \ldots, v_{n-1}$ *is called* $F_n(T)$ (5). Let $\varphi, \varphi' \in F_m(T)$. We write $\vdash_T \varphi$ to mean that φ is valid in T (i.e., the sentence $\bigwedge v_0 \ldots \bigwedge v_{m-1}\varphi$ is valid in T). φ and φ' are *equivalent in T* if $\vdash_T \varphi \leftrightarrow \varphi'$; and φ is *consistent with T* if $\sim\varphi$ is not valid in T. It $\tau_0, \ldots, \tau_{p-1}$ are terms, $\Phi(\tau_0, \ldots, \tau_{p-1})$ is the formula obtained by the proper simultaneous substitution of τ_i for the free occurrences of v_i in φ $(i = 0, \ldots, p-1)$. ("Proper" means that bound variables should be changed to avoid collisions.)

If a system $\mathfrak{A}$ is a realization of T and $a_0, \ldots, a_{m-1} \in |\mathfrak{A}|$ (6), we write $\models_{\mathfrak{A}} \Phi[a_0, \ldots, a_{m-1}]$ to mean that φ is satisfied in $\mathfrak{A}$ by

(4) Terminology which is explained only partly or not at all is that of [17] and [18].

(5) Letters "i", ..., "r" denote natural numbers 0, 1, ..., i.e., members of ω. "0" also denotes the empty set. "ξ", "η", "ζ" denote ordinals.

(6) $|\mathfrak{A}|$ is the *universe* of $\mathfrak{A} = \langle A, \ldots\rangle$, i. e., the set A.

the assignment of a_0 to $v_0, \ldots, a_{m-1}$ to v_{m-1}. Thus, when $m = 0$, $\models_{\mathfrak{A}} \varphi$ means that φ is true in $\mathfrak{A}$, or $\mathfrak{A}$ is a model of φ. Realizations of the same theory are called *similar*. The similar systems $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent* (in symbols, $\mathfrak{A} \equiv \mathfrak{B}$) if they have the same true sentences, or in other words, if they are models of the same complete theory. $\mathfrak{A}$ is said to be an *elementary extension* of $\mathfrak{B}$ and $\mathfrak{B}$ an *elementary subsystem* of $\mathfrak{A}$ (in symbols, $\mathfrak{A} \succ \mathfrak{B}$) if (in addition) $\mathfrak{B}$ is a subsystem of $\mathfrak{A}$ and, in general, $\models_{\mathfrak{B}} \Phi[b_0, \ldots, b_{n-1}]$ implies $\models_{\mathfrak{A}} \varphi[b_0, \ldots, b_{n-1}]$.

Let $\mathfrak{A}_0, \ldots, \mathfrak{A}_\xi, \ldots$ $(\xi < \eta)$ be similar systems such that $\mathfrak{A}_{\xi'} \succ \mathfrak{A}_\xi$ whenever $\eta > \xi' > \xi$ (such a sequence of systems is said to be *elementarily increasing*). Then the union $\bigcup \{\mathfrak{A}_\xi | \ \xi < \eta\}$ is the system whose universe is the ordinary union of $\{|\mathfrak{A}_\xi| / \xi < \eta\}$ and whose ζth relation or distinguished element is, respectively, the union of the ζth relations of all the $\mathfrak{A}_\xi$, or the (common) ζth distinguished element of all the $\mathfrak{A}_\xi$. Given a system $\mathfrak{A} = \langle A, X_0, \ldots, X_\xi, \ldots \rangle_{\xi<\eta}$ and a sequence $Y_0, \ldots, Y_\xi, \ldots$ $(\xi < \zeta)$ of further relations or distinguished elements over A, the system $\langle A, X_0, \ldots, X_\xi, \ldots, Y_0, \ldots, Y_{\xi'}, \ldots \rangle_{\xi<\eta, \xi'<\zeta}$ will be indicated by the notation (of S. Feferman) $(\mathfrak{A}, Y_0, \ldots, Y_\xi, \ldots)_{\xi<\zeta}$.

To simplify the description of the next notion, we deal with the case where T has only one non-logical constant, a ternary relation symbol R; from this illustration the general situation will be clear. By a *possible relative interpretation* of T in another theory T_1 we understand a system $I = \langle \theta, \gamma \rangle$, where $\theta \in F_1(T_1)$ and $\gamma \in F_3(T_1)$. For any formula φ of T, Φ^I is the formula (of T_1) obtained from φ by replacing each atomic formula $\mathrm{R} v_{k_0} v_{k_1} v_{k_2}$ by $\gamma(v_{k_0}, v_{k_1}, v_{k_2})$, and then replacing subformulas of the form $\bigvee v_j \psi$ or $\bigwedge v_j \psi$ by $\bigvee v_j (\theta(v_j) \wedge \psi)$ or $\bigwedge v_j (\theta(v_j) \rightarrow \psi)$, respectively. I is a *relative interpretation* of T in T_1 if σ^I is valid in T_1 whenever σ is a sentence valid in T. If $\mathfrak{B}$ is a realization of T_1, then the *denotation of I in $\mathfrak{B}$* is the system $\langle A, R \rangle$, where $A = \{x / \models_{\mathfrak{B}} \theta[x]\}$ and $R = \{\langle x, y, z \rangle / x, y, z \in A$ and $\models_{\mathfrak{B}} \gamma[x, y, z]\}$.

For later reference we state here the following, easily proved facts:

LEMMA 1.1. (.1) *If $\mathfrak{A}_\xi \succ \mathfrak{A}_{\xi'}$ whenever $\eta > \xi > \xi'$, then, for each $\xi < \eta$, $\bigcup \{\mathfrak{A}_\xi / \xi < \eta\} \succ \mathfrak{A}_\xi$.*

(.2) *If $\mathfrak{B}$ is a model of T_1, $\mathfrak{B} \succ \mathfrak{B}'$, and $\mathfrak{A}$ and $\mathfrak{A}'$ are the respective denotations in $\mathfrak{B}$ and $\mathfrak{B}'$ of a relative interpretation I of T in T_1, then $\mathfrak{A} \succ \mathfrak{A}'$.*

(.3) *Suppose that* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are similar systems,* $|\mathfrak{A}| = \{a_n / n \in \omega\}$, *and, for each* n, $(\mathfrak{A}, a_0, \ldots, a_{n-1}) \equiv (\mathfrak{B}, b_0, \ldots, b_{n-1})$. *Then* $\{\langle a_n, b_n\rangle / n \in \omega\}$ *is a function mapping* $\mathfrak{A}$ *isomorphically onto an elementary subsystem of* $\mathfrak{B}$ (7).

We turn now to some less familiar notions. *Henceforth it is assumed that* T *is a complete theory having infinite models* (8). (One or both of these assumptions is often dispensable, but usually with little serious gain in generality.) For each n, the set $F_n(T)$, together with the operations $\wedge$ (which, when applied to φ and ψ, yields $\varphi\wedge\psi$), $\vee$, and $\sim$, and the relation of equivalence in T, constitutes a Boolean algebra (9)—also denoted by $F_n(T)$. Consequently, the ordinary terminology for Boolean algebras may be employed:

1.2.1. A member a of $F_n(T)$ is an *atom* of $F_n(T)$ provided that a is consistent with T and, for any $\varphi \in F_n(T)$, if $a \wedge \varphi$ is consistent with T then $\vdash_T a \to \varphi$.

1.2.2. φ is an *atomless* element of $F_n(T)$ if φ is consistent with T and $\vdash_T a \to \varphi$ holds for no atom a of $F_n(T)$.

1.2.3. $F_n(T)$ is *atomistic* if it has no atomless element.

1.2.4. A *prime ideal* of $F_n(T)$ is a non-empty, proper subset P of $F_n(T)$ such that, for any $\varphi, \psi \in F_n(T)$: $\varphi\wedge\psi \in P$ if $\varphi, \psi \in P$; $\psi \in P$, if $\varphi \in P$ and $\vdash_T \varphi \to \psi$; and either $\varphi \in P$ or $\sim\varphi \in P$. (This is what is usually called a "dual prime ideal".)

1.2.5. A prime ideal P is *principal* if, for some $\theta \in F_n(T)$, $P = \{\varphi/\varphi \in F_n(T)$ and $\vdash_T \theta \to \varphi\}$ —or, equivalently, if P contains an atom of $F_n(T)$.

The set of all prime ideals of $F_n(T)$ will be denoted by $\mathcal{P}_n(T)$. If $P \in \mathcal{P}_n(T)$ and $\mathfrak{A}$ is a model of T, then we denote by $P(\mathfrak{A})$ the set of all n-tuples $\langle a_0, \ldots, a_{n-1}\rangle$ such that, for every $\varphi \in P$, $\vdash_{\mathfrak{A}} \varphi[a_0, \ldots, a_{n-1}]$.

1.3. Clearly there is a natural one-to-one correspondence between $F_n(T)$ and the set $\overline{F}_n(T)$ of all sentences which involve the non-logical constants of T plus the distinct, new, individual

(7) For (.1), cf. [18], Theorem 1.9. For (.3), cf. the proof of 1.12 of [18].

(8) Any reference to the power of A is to be understood as referring to the power of $|\mathfrak{A}|$.

(9) Thus for us a Boolean algebra is a system of the form $\langle A, +, \cdot, -, \approx\rangle$, whose quotient modulo $\approx$ is a Boolean algebra in the more usual sense. Note also that, T being complete, $F_0(T)$ has always only two inequivalent elements; nonetheless, it is included in the discussion, for technical convenience.

constants $c_0, \ldots, c_{n-1}$. A formula $\varphi \in F_n(T)$ goes into the sentence $\bar{\varphi} = \varphi(c_0, \ldots, c_{n-1})$. The induced correspondence (also denoted by $^-$) maps the prime ideals of $F_n(T)$ onto the complete theories involving the non-logical constants of T plus $c_0, \ldots, c_{n-1}$. Clearly, $\langle a_0, \ldots, a_{n-1} \rangle \in P(\mathfrak{A})$ if and only if $(\mathfrak{A}, a_0, \ldots, a_{n-1})$ is a model of $\bar{P}$. It is sometimes convenient to think of the theory $\bar{P}$ in place of the prime ideal P of $F_n(T)$.

Lemma 1.4, below, gives an obvious, alternative characterization of the notion of "atom".

LEMMA 1.4. *A necessary (and sufficient) condition for a member a of $F_n(T)$ to be an atom of $F_n(T)$ is that, for any models $\mathfrak{A}$ and $\mathfrak{B}$ of T, if $\models_{\mathfrak{A}} a[a_0, \ldots, a_{n-1}]$ and $\models_{\mathfrak{B}} a[b_0, \ldots, b_{n-1}]$, then $(\mathfrak{A}, a_0, \ldots, a_{n-1}) \equiv (\mathfrak{B}, b_0, \ldots, b_{n-1})$.*

§ 2. Existence of models. In this section we shall prove the following

THEOREM 2.1. (.1) *There is a denumerable model $\mathfrak{A}$ of T such that*

($*$) *every finite sequence of elements of $|\mathfrak{A}|$, of any length $m+1$, satisfies in $\mathfrak{A}$ either an atom or an atomless member of $F_{m+1}(T)$.*

(.2) *If, for each $j \in \omega$, P_j is a non-principal prime ideal of $F_{p_j+1}(T)$, then there is a denumerable model $\mathfrak{A}$ of T such that*

($**$) *$P_0(\mathfrak{A}), P_1(\mathfrak{A}), \ldots$ are all empty.*

(.3) *Indeed, under the hypothesis of (.2), a denumerable model $\mathfrak{A}$ of T can be found for which both ($*$) and ($**$) hold.*

2.1.2 was proved by Ehrenfeucht. Its special case in which there is only one P_j was used by him to give a simple proof of the necessity in Ryll-Nardzewski's theorem, 0.1 [2], and will be used in § 3 and § 6, below. 2.1.1 was established by the author. In § 3 it will be applied in the special case in which each $F_n(T)$ is atomistic. Thus, 2.1 is, in a number of ways, stronger than what is needed in the rest of the paper. The strong form has been stated because it is no more difficult to prove and may, perhaps, be of some intrinsic interest. Note that (.2) allows us to assume that countably many, *arbitrary*, non-principal prime ideals will be empty for $\mathfrak{A}$. On the other hand, (.1) says that *certain* non-principal prime ideals, possibly $2^{\aleph_0}$ in number, can be made empty for $\mathfrak{A}$ [10].

[10] As one easily verifies, these are, in fact, those non-principal prime ideals of each $F_n(T)$ which—as points of the topological space corresponding to $F_n(T)$ by Stone's representation theorem ([15])—are the limit of a sequence of isolated points.

Proof. We proceed by a modification of Henkin's proof of the completeness theorem (Cf. [6] and pp. 42-43 of [5]). Let $c_0, c_1, \ldots$ be distinct, new individual constants. Clearly all entities of the form $\Pi = \langle P, c_{k_0}, \ldots, c_{k_q}\rangle$ —where q is arbitrary, $k_0, \ldots, k_q$ are distinct, and either $P = 0$ or else P is a P_i with $p_i = q$ —may be enumerated in a list $\Pi_0, \Pi_1, \ldots$ Let T_1 be the theory whose constants are those of T plus $c_0, c_1, \ldots$, and whose axioms are the valid sentences of T. The members of $F_1(T_1)$ may be enumerated in a list $\varphi_0, \varphi_1, \ldots$ For later reference, we note the well-known principle, which holds for any $\psi \in F_r(T)$:

(1) ψ, or $\bigvee v_0 \ldots \bigvee v_{r-1}\psi$, is consistent with T if and only if $\psi(c_0, \ldots, c_{r-1})$ is consistent with T_1.

We are going to define recursively sentences $\sigma_0, \sigma_1, \ldots$ of T_1 in such a way that, for each n, $\sigma_0 \wedge \ldots \wedge \sigma_{n-1}$ is consistent with T_1. Suppose that $\sigma_0, \ldots, \sigma_{n-1}$ have been defined and $\sigma_0 \wedge \ldots \wedge \sigma_{n-1}$ is consistent with T_1. Let ν be the smallest number such that c_ν occurs in none of $\varphi_n, \sigma_0, \ldots, \sigma_{n-1}$. Then as is well known, the sentence

$$\gamma:\ [\bigvee v_0\varphi_n \rightarrow \varphi_n(c_\nu)] \wedge \sigma_0 \wedge \ldots \wedge \sigma_{n-1}$$

is (yb (1)) consistent with T_1. Let Π_n be $\langle P, d_0, \ldots, d_q\rangle$, and let $e_0, \ldots, e_{r-1}$ be the distinct c_i's occurring in γ and not equal to any of $d_0, \ldots, d_q$. Clearly, there is a formula $\theta \in F_{q+1+r}(T)$ such that $\theta(d_0, \ldots, d_q, e_0, \ldots, e_{r-1})$ is logically equivalent to γ. Then the formula

$$\beta:\ \bigvee v_{q+1} \ldots \bigvee v_{q+r}\theta$$

of $F_{q+1}(T)$ is consistent with T. We now distinguish two cases:

Case (i). For some atom α of $F_{q+1}(T)$, $\vdash_T \alpha \rightarrow \beta$. Choosing a definite α, we take for σ_n the sentence $\alpha(d_0, \ldots, d_q) \wedge \gamma$. Since α (as an atom) is consistent with T, one easily sees (applying principle (1)) that $\sigma_0 \wedge \ldots \wedge \sigma_n$ is consistent with T_1.

Case (ii). β is an atomless element of $F_{q+1}(T)$. If P is the empty set, we take for σ_n simply the sentence γ; then, certainly, $\sigma_0 \wedge \ldots \wedge \sigma_n$ is consistent with T_1. Otherwise, P is a non-principal prime ideal of $F_{q+1}(T)$. Then, clearly, there is a formula $\delta \in F_{q+1}(T)$ such that $\sim\delta \in P$, while $\beta \wedge \delta$ is consistent with T. For σ_n we take $\delta(d_0, \ldots, d_q) \wedge [\bigvee v_0\varphi_n \rightarrow \varphi_n(c_\nu)]$. Again, applying (1), we see that $\sigma_0 \wedge \ldots \wedge \sigma_n$ is consistent with T_1.

Thus, $\sigma_0, \sigma_1, \ldots$ are defined, and the theory T_2, obtained from T_1 by taking $\sigma_0, \sigma_1, \ldots$ as additional axioms, is consistent. Moreover, the construction assured that

(2) for each n, there is a c_i such that $\vdash_{T_2} \bigvee v_0\varphi_n \rightarrow \varphi_n(c_i)$.

By a well-known result of Henkin ([6]), it follows from (2) that T_2 has a model $(\mathfrak{A}, c_0, \ldots, c_n, \ldots)$ such that $\mathfrak{A}$ is a model of T and $|\mathfrak{A}| = \{c_0, \ldots, c_n, \ldots\}$. We shall see that our construction above has also ensured that $\mathfrak{A}$ fulfills $(*)$ and $(**)$.

Indeed, suppose that $a_0, \ldots, a_q \in |\mathfrak{A}|$. Then there are *distinct* $c_{k_0}, \ldots, c_{k_q}$ such that $a_0 = c_{k_0}, \ldots$, and $a_m = c_{k_q}$. (This is because any given c_j is of the form c_i, for infinitely many i. The latter fact is easily seen by noting that there are infinitely many φ_i's of the form $\mathrm{v}_0 = \mathrm{c}_j \wedge \sigma$, where σ is tautologous [11]). Now, at some point in defining $\sigma_0, \sigma_1, \ldots$, we had $\Pi_n = \langle 0, \mathrm{c}_{k_0}, \ldots, \mathrm{c}_{k_q}\rangle$. If case (i) occurred, then clearly $c_{k_0}, \ldots, c_{k_q}$ satisfy in $\mathfrak{A}$ an atom of $F_q(T)$—namely, the α of case (i). Otherwise, β was atomless, and, since the construction insured that $\vdash_{T_2} \beta(\mathrm{c}_{k_0}, \ldots, \mathrm{c}_{k_q})$, we see that $c_{k_0}, \ldots, c_{k_q}$ satisfy in $\mathfrak{A}$ an atomless member of $F_n(T)$. Thus, $(*)$ holds. Now suppose further that $p_j = q$, so that for some n', $\Pi_{n'} = \langle P_j, \mathrm{c}_{k_0}, \ldots, \mathrm{c}_{k_q}\rangle$. If at the n'th step case (i) occurred, then $c_{k_0}, \ldots, c_{k_q}$ satisfy in $\mathfrak{A}$ an atom, say α'. Since P_j is non-principal, $\sim\alpha' \in P$, and hence $\langle c_{k_0}, \ldots, c_{k_q}\rangle \notin P_j(\mathfrak{A})$. If, instead, case (ii) occurred in the n'th step, then we have explicitly ensured in that step that $\langle c_{k_0}, \ldots, c_{k_q}\rangle \notin P_j(\mathfrak{A})$. Thus $(**)$ holds, and the theorem is proved.

§ 3. **Prime models.** A model $\mathfrak{A}$ of T will be called *atomic* if each finite sequence of elements of $|\mathfrak{A}|$, of any length n, satisfies in $\mathfrak{A}$ an atom of $F_n(T)$ [12].

LEMMA 3.1. *Suppose that $\mathfrak{A}$ is a denumerable, atomic model of T and $\mathfrak{B}$ is an arbitrary model of T. Then $\mathfrak{A}$ can be mapped isomorphically onto an elementary subsystem of $\mathfrak{B}$. Moreover, if $a_0, \ldots, a_{m-1} \in |\mathfrak{A}|$, $b_0, \ldots, b_{m-1} \in |\mathfrak{B}|$, and $(\mathfrak{A}, a_0, \ldots, a_{m-1}) \equiv (\mathfrak{B}, b_0, \ldots, b_{m-1})$, then the mapping may be so chosen that it carries a_i into b_i for each $i < m$.*

Proof. Let $a_0, \ldots, a_n, \ldots$ be a list, possibly with repetitions, of the elements of $|\mathfrak{A}|$ (commencing with the given $a_0, \ldots, a_{m-1}$). Suppose that $b_m, b_{m+1}, \ldots, b_{n-1}$ $(n > m)$ have been defined in such a way that

$$(1) \qquad (\mathfrak{A}, a_0, \ldots, a_{n-1}) \equiv (\mathfrak{B}, b_0, \ldots, b_{n-1}) .$$

Since $\mathfrak{A}$ is atomic, there exists an atom α of $F_n(T)$ and an atom α' of $F_{n+1}(T)$ such that $\models_{\mathfrak{A}} \alpha[a_0, \ldots, a_{n-1}]$ and

[11] This detail seems less bothersome than those required in the applications of (1) above, had $\mathrm{d}_0, \ldots, \mathrm{d}_q$ not there been assumed distinct.

[12] This terminology is due to L. Svenonius.

$\models_{\mathfrak{A}} \alpha'[a_0, \ldots, a_n]$. It follows that $\models_{\mathfrak{B}} \alpha[b_0, \ldots, b_{n-1}]$ and that formula $\alpha \wedge \bigvee v_n \alpha'$ is consistent. By 1.2.1, the latter implies that $\vdash_T \alpha \rightarrow \bigvee v_n \alpha'$. $\mathfrak{B}$ being a model of T, we infer that $\models_{\mathfrak{B}} (\bigvee v_n \alpha')\ [b_0, \ldots, b_{n-1}]$. Thus, we may choose for b_n an element of $|\mathfrak{B}|$ such that $\models_{\mathfrak{B}} \alpha'[b_0, \ldots, b_n]$. Then, by 1.4, $(\mathfrak{A}, a_0, \ldots, a_n) \equiv (\mathfrak{B}, b_0, \ldots, b_n)$.

Thus, $b_m, b_{m+1}, \ldots$ can be defined recursively in such a way that (1) holds for every n. Lemma 3.1 now follows immediately from 1.1.3.

THEOREM 3.2. *If $\mathfrak{A}$ and $\mathfrak{B}$ are denumerable, atomic models of T, then $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$* ([13]).

THEOREM 3.3. *If $\mathfrak{A}$ is a denumerable, atomic model of T, then $\mathfrak{A}$ is homogenous.*

Proof of 3.2 and 3.3. The proof of 3.1, above, resembles Cantor's argument showing that any denumerable, simply ordered system is a subsystem of a denumerable, densly ordered system without extreme points. The proof of 3.2 is analogously related to Cantor's proof that any two systems of the latter sort are isomorphic. Roughly, to prove 3.2, we let $|\mathfrak{A}| = \{a_m / m \in \omega\}$ and $|\mathfrak{B}| = \{b_n / n \in \omega\}$ and define recursively couples $\langle a_{j_n}, b_{k_n} \rangle$, $n = 0, 1, \ldots$ The passage from n to $n+1$ is like that in the proof of 3.1 (j_n being defined to be the first $i \neq j_0, \ldots, j_{n-1}$) when n is even; when n is odd, the rôles of $\mathfrak{A}$ and $\mathfrak{B}$ are reversed. The proof of 3.3 is analogously related to the argument proving the second conclusion of 3.1.

It may be noted that the second part of 3.1 could have been derived from the first part, and 3.3 from 3.2, by noting the following easily proved fact: If $\mathfrak{A}$ is atomic and $a_0, \ldots, a_n \in |\mathfrak{A}|$, then $(\mathfrak{A}, a_0, \ldots, a_n)$ is atomic.

THEOREM 3.4. *$\mathfrak{A}$ is prime if and only if $\mathfrak{A}$ is denumerable and atomic.*

Proof. If $\mathfrak{A}$ is atomic and denumerable, then, by 3.1, $\mathfrak{A}$ is prime. On the other hand, if $\mathfrak{A}$ is prime, then clearly (by the Löwenheim-Skolem theorem) $\mathfrak{A}$ is denumerable. Suppose now, that $\mathfrak{A}$ is not atomic. Then some $a_0, \ldots, a_m \in |\mathfrak{A}|$ satisfies no atom (of $F_m(T)$). Hence, clearly, $P = \{\varphi / \models_{\mathfrak{A}} \varphi[a_0, \ldots, a_m]\}$ is a non-principal prime ideal of $F_m(T)$. By 2.1.2, T has a model $\mathfrak{B}$ with $P(\mathfrak{B})$ empty. It obviously follows that $\mathfrak{B}$ cannot be an

([13]) This result was also (indeed, earlier) established by Svenonius; of course, a closely related result was proved by Ryll-Nardzewski [13].

elementary extension of an isomorph of $\mathfrak{A}$, a contradiction. This proves 3.4.

THEOREM 3.5. *The following are equivalent:*
(.1) *T has a prime model;*
(.2) *T has an atomic model;*
(.3) *Each $F_n(T)$ is atomistic.*

Proof. It follows immediately from 2.1.1 that, if each $F_n(T)$ is atomistic, then T has a denumerable, atomic model; by 3.4, such a model is prime. Thus, (.3) implies (.1). By 3.4, (.1) implies (.2). The remaining implication is nearly obvious. Indeed, suppose that T has an atomic model $\mathfrak{A}$. Let φ be any member of $F_n(T)$, consistent with T. Since T is complete, $\vdash_T \bigvee v_0 \dots \bigvee v_{n-1} \varphi$; hence, there are $a_0, \dots, a_{n-1} \in |\mathfrak{A}|$ such that $\models_{\mathfrak{A}} \varphi[a_0, \dots, a_{n-1}]$. $\mathfrak{A}$ being atomic, there is an atom α of $F_n(T)$ such that $\models_{\mathfrak{A}} \alpha[a_0, \dots, a_{n-1}]$. Then $\alpha \wedge \varphi$ is consistent with T_1 so that, by 1.2.1, $\vdash_T \alpha \rightarrow \varphi$. Thus, φ is not atomless. This argument shows that $F_n(T)$ is atomistic, completing the proof.

As was to be expected the results in § 2 and § 3 have as a consequence 0.1, the theorem of Ryll-Nardzewski. Indeed, the proposed condition clearly implies that any model of T is atomic; the $\aleph_0$-categoricity of T then follows, by 3.2. As already remarked (after 2.1) the reverse implication is easily derived by using 2.1.2 and the completeness theorem—as was noted by Ehrenfeucht. (It may also rather easily be derived from 2.1.1.) The argument depends on the well-known fact:

3.6. *A Boolean algebra contains infinitely many inequivalent elements if and only if it has a non-principal prime ideal.*

In Ryll-Nardzewski's theorem a semantical condition is shown to be equivalent to a purely syntactical statement. The equivalence, proved above, between 3.5.1 and 3.5.3 has the same character.

§ 4. Saturated models. A model $\mathfrak{A}$ of T will be called *weakly saturated* if, for any $P \in \mathcal{P}_n(T)$ (n arbitrary), $P(\mathfrak{A})$ is not empty. $\mathfrak{A}$ is said to be *saturated* if (in addition) the conditions $P \in \mathcal{P}_n(T)$, $P \subseteq Q \in \mathcal{P}_{n+1}(T)$, and $\langle a_0, \dots, a_{n-1}\rangle \in P(\mathfrak{A})$ imply that there exists an element x such that $\langle a_0, \dots, a_{n-1}, x\rangle \in Q(\mathfrak{A})$.

LEMMA 4.1. *Suppose that I is a relative interpretation of T in a theory T_1, $\mathfrak{B}$ is a denumerable model of T_1, and $\mathfrak{A}$ is the denotation of I in $\mathfrak{B}$. Let $n \in \omega$, $P \in \mathcal{P}_n(T)$, $P \subseteq Q \in \mathcal{P}_{n+1}(T)$, and*

$\langle b_0, \ldots, b_{n-1}\rangle \in P(\mathfrak{A})$. *Then there exists a denumerable, elementary extension* $\mathfrak{B}^*$ *of* $\mathfrak{B}$, *in which the denotation of* I *is a system* $\mathfrak{A}^*$ *having an element* d *such that* $\langle b_0, \ldots, b_{n-1}, d\rangle \in Q(\mathfrak{A}^*)$.

Proof. We employ a type of argument which has been used by A. Robinson ([14]). Let $b_0, \ldots, b_{n-1}, b_n, \ldots$ be all the elements of $|\mathfrak{B}|$, and let $\mathrm{d}, \mathrm{b}_0, \ldots, \mathrm{b}_n, \ldots$ be distinct, new individual constants. Let the axioms of the theory T_1' be all sentences of the form $\psi(\mathrm{b}_{j_0}, \ldots, \mathrm{b}_{j_{k-1}})$, such that $\psi \in F_k(T_1)$ and $\models_{\mathfrak{B}}\psi[b_{j_0}, \ldots, b_{j_{k-1}}]$, plus all sentences $\Phi^I(\mathrm{b}_0, \ldots, \mathrm{b}_{n-1}, \mathrm{d})$ for which $\Phi \in Q$. If $\Phi_0, \ldots, \Phi_p \in Q$ then, clearly, the formula $\bigvee v_n(\Phi_0 \wedge \ldots \wedge \Phi_p) \in P$; therefore, there is a y such that, for each $i \leqslant p$, $\models_{\mathfrak{A}}\Phi_i[b_0, \ldots, b_{n-1}, y]$, i.e. $\models_{\mathfrak{B}}\Phi_i^I[b_0, \ldots, b_{n-1}, y]$. It easily follows that any finite set of axioms of T_1' has a model, so that T_1' is consistent. By the completeness theorem, T_1' has a model B_1'. B_1' is of the form $(\mathfrak{B}_1, u_0, \ldots, u_n, \ldots)$, where $\mathfrak{B}_1$ is a model of T_1. Clearly, $\mathfrak{B}_1$ is isomorphic to a system $\mathfrak{B}^*$ having the desired properties.

THEOREM 4.2. *Suppose that each* $\mathcal{P}_n(T)$ *is countable. Let* T_1 *be a consistent theory, and let* $I_0, \ldots, I_n, \ldots$ *be relative interpretations of* T *in* T_1. *Then* T_1 *has a denumerable model in which the denotation of each* I_n *is a saturated model of* T.

Proof. We note first that given any denumerable model $\mathfrak{B}$ of T_1, a system $\mathfrak{B}^*$ can be found, with the following properties:

(1) $\mathfrak{B}^*$ is a denumerable elementary extension of $\mathfrak{B}$; (2) if $k, m \in \omega$, if $\mathfrak{A}$ and $\mathfrak{A}^*$ are the (respective) denotations of I_k in $\mathfrak{B}$ and $\mathfrak{B}^*$, and if $P \in \mathcal{P}_m(T)$, $\langle a_0, \ldots, a_{m-1}\rangle \in P(\mathfrak{A})$, and $P \subseteq Q \in \mathcal{P}_{m+1}(T)$, then there exists an x such that $\langle a_0, \ldots, a_{m-1}, x\rangle \in Q(\mathfrak{A}^*)$. Indeed, since each $P_n(T)$ is countable, we may enumerate all tuples $\langle I_k, P, Q, a_0, \ldots, a_{m-1}\rangle$ for which the hypothesis of (2) holds. Let $\mathfrak{C}_0 = \mathfrak{B}$ and (recursively) let $\mathfrak{C}_{n+1}$ be a system of the type whose existence is asserted in Lemma 4.1—as applied to $\mathfrak{C}_n$ and to the entities $I_k, P, Q, a_0, \ldots, a_{n-1}$ constituting the nth-tuple in our enumeration. One easily sees, using 1.1.1 and 1.1.2, that the system $\mathfrak{B}^* = \bigcup\{\mathfrak{C}_k/k \in \omega\}$ has the properties (1) and (2).

Now let $\mathfrak{B}_0$ be a denumerable model of T_1, and (recursively) let $\mathfrak{B}_{m+1}$ be to $\mathfrak{B}_m$ as $\mathfrak{B}^*$ is to $\mathfrak{B}$ in (1) and (2). Then, clearly, the system $\bigcup\{\mathfrak{B}_k/k \in \omega\}$ is as demanded in 4.2.

([14]) Cf. theorems 2.1, 2.2, and 2.5 of [11]. One could, indeed, derive 4.1 from these.

A special case of 4.2 is the following: a sufficient condition for T to have a denumerable, saturated model is that each $\mathcal{P}_n(T)$ be countable. Later, in 4.7 and 4.4, we shall see that this condition is also necessary, and that there is at most one denumerable, saturated model of T up to isomorphism. Consequently, the full Theorem 4.2 states, roughly, that denumerable, saturated models have further "saturation properties"—in addition to those in the definition of "saturated"; in particular, a kind of second order saturation.

LEMMA 4.3. *Suppose that $\mathfrak{B}$ is a denumerable, saturated model of T and $\mathfrak{A}$ is an arbitrary, denumerable model of T. Then $\mathfrak{A}$ can be mapped isomorphically onto an elementary subsystem of $\mathfrak{B}$. Moreover, if $a_0, \ldots, a_{m-1} \in |\mathfrak{A}|$, $b_0, \ldots, b_{m-1} \in |\mathfrak{B}|$, and $(\mathfrak{A}, a_0, \ldots, a_{m-1}) \equiv (\mathfrak{B}, b_0, \ldots, b_{m-1})$, then the mapping can be chosen so that it carries a_i into b_i for each $i < m$.*

Proof. Let $a_0, \ldots, a_{m-1}, a_m, \ldots$ be all the elements of $|\mathfrak{A}|$. We can define by recursion a sequence $b_m, b_{m+1}, \ldots$ of elements of $|\mathfrak{B}|$ such that, for any n,

$$(3) \qquad (\mathfrak{A}, a_0, \ldots, a_{n-1}) \equiv (\mathfrak{B}, b_0, \ldots, b_{n-1}) .$$

Indeed, (3) holds for $n < m$, by hypothesis. Suppose that (3) holds for a given $n \geqslant m$. Then, clearly (cf. 1.3), for some $P \in \mathcal{P}_n(T)$, $\langle a_0, \ldots, a_{n-1}\rangle \in P(\mathfrak{A})$ and $\langle b_0, \ldots, b_{n-1}\rangle \in P(\mathfrak{B})$. Now, for some $Q \in \mathcal{P}_{n+1}(T)$, $\langle a_0, \ldots, a_n\rangle \in Q(\mathfrak{A})$. But then $P \subseteq Q$, so that, since $\mathfrak{B}$ is saturated, we may choose $b_n \in |\mathfrak{B}|$ such that $\langle b_0, \ldots, b_n\rangle \in Q(\mathfrak{B})$. Then (3) holds with "$n+1$" for "$n$", by 1.3.

4.3 now follows from 1.1.3.

THEOREM 4.4. *Any two denumerable, saturated models of T are isomorphic.*

LEMMA 4.5. *Any denumerable, saturated model of T is homogeneous.*

Proof. The proofs of 4.4 and 4.5 are obtained by modifying that of 4.3 in a manner completely analogous to the one in which the proofs of 3.2 and 3.3 were obtained from that of 3.1.

THEOREM 4.6. *For a denumerable model $\mathfrak{A}$ of T, the following conditions are equivalent:*

(.1) *$\mathfrak{A}$ is saturated;*

(.2) *$\mathfrak{A}$ is $\aleph_0$-universal and homogeneous;*

(.3) *$\mathfrak{A}$ is weakly saturated and homogeneous.*

Proof. By 4.3 and 4.5, (.1) implies (.2). From the Gödel-Löwenheim-Skolem theorem, one sees at once that an $\aleph_0$-universal model is weakly saturated, so that (.2) implies (.3). Suppose (.3) holds, $a_0, \dots, a_{n-1} \in P(\mathfrak{A})$, $P \in \mathcal{P}_n(T)$, and $P \subseteq Q \in \mathcal{P}_{n+1}(T)$. Since $\mathfrak{A}$ is weakly saturated, some $\langle a'_0, \dots, a'_n \rangle \in Q(\mathfrak{A})$. Then $\langle a'_j, \dots, a'_{n-1} \rangle \in P(\mathfrak{A})$, and hence, $\mathfrak{A}$ being homogeneous, there is an automorphism f of $\mathfrak{A}$ taking a'_i into a_i for $i < n$. Consequently, $\langle a_0, \dots, a_{n-1}, f(a'_n) \rangle \in Q(\mathfrak{A})$. Thus, (.3) implies (.1).

THEOREM 4.7. *The following conditions are equivalent:*

(.1) *Each* $\mathcal{P}_n(T)$ *is countable;*

(.2) *T has a denumerable, saturated model;*

(.3) *T has an $\aleph_0$-universal model;*

(.4) *T has a weakly saturated, denumerable model* [15].

Proof. We have already remarked that "(.1) implies (.2)" is a special case of 4.2. By 4.3 or 4.6, (.2) implies (.3). As noted above, (.3) implies (.4). Finally, that (.4) implies (.1) is obvious.

COROLLARY 4.8. *If T has an $\aleph_0$-universal model, then T has a prime model.*

Proof. As is well-known, Boolean algebras with countably many prime ideals are atomistic. Hence 4.8 follows from 4.7, and 3.5. (While this proof of 4.8 depends on 2.1.1, it may be noted that another proof could be constructed depending, rather, on 2.1.2—since, under the hypothesis of 4.8, $\bigcup \{\mathcal{P}_n(T)/n \in \omega\}$ is countable.)

One is tempted to say, by analogy with the discussion in the last paragraph of § 3, that condition 4.7.1 is purely syntactical. Indeed, in 4.7.1, no reference to any semantical concept, such as "model", is made. However, a little thought convinces one that a notion of "purely syntactical condition" wide enough to include (.1) would be so broad as to be pointless.

In § 5 and § 6, we will see that the results of § 2-§ 4 can be applied to establish some general properties of models of certain kinds of theories. On the other hand, the chances that these results can be usefully applied in the study of a particular

(15) In view of 4.6, it follows that 4.7.1 or 4.7.3 is, also, a necessary and sufficient condition for T to have an $\aleph_0$-universal, homogeneous model. In [20] and [21], the author announced some results concerning the existence in powers $\aleph_\alpha > \aleph_0$ of "$\aleph_\alpha$-universal models" for arbitrary theories and of $\aleph_\alpha$-universal, "homogeneous" models for complete theories. (For the meanings of "$\aleph_\alpha$-universal" and of "homogeneous" intended here, cf. [20] and [21].) The author takes this opportunity to state that he has learned that results very closely related to those in [20] and [21] were obtained several years earlier by Mr. Michael Morley. Morley's work is not yet published.

relational system or complete theory seem not too good. This is due at least in part to the fact that the notion "elementary subsystem" rather than "elementarily equivalent subsystem" is involved in such notions as "prime" or "$\aleph_0$-universal". Thus, for example, to establish that a theory T fulfills any one of the conditions in 3.5 or 4.8 one would need to have already a good deal of metamathematical (and not just algebraic) information concerning T.

It may, however, be worthwhile, for the sake of illustration, to give some examples of theories which fulfill the condition of 3.5 or 4.7. (But it should be noted that the results of § 3 and § 4 yield no new information about these examples.) The theory T_1 of infinite, discretely ordered systems with, say, a first but no last element, is one in which each $\mathcal{P}_n(T_1)$ is countable. That this is so is easily verified, because the known decision procedure for T_1 provides a description of all possible definable sets and relations in models of T_1 ([16]). The $\aleph_0$-universal, homogeneous model of T_1 is the system of order type $\omega+(\omega^*+\omega)\cdot\eta$. It may be remarked that T_1 has $2^{\aleph_0}$ non-isomorphic denumerable models.

For the theory T_2 of real closed fields, the set $\mathcal{P}_1(T_2)$ obviously has $2^{\aleph_0}$ members. However, as is known (cf. [16], [12]) the field of real, algebraic numbers is a prime model of T_2 and each $F_n(T_2)$ is atomistic.

§ 5. $\aleph_{1+a}$-categorical theories ([17]). Ehrenfeucht has proved that

5.1. *If, for some a, T has less than $2^{\aleph_0}$ non-isomorphic models of power $\aleph_a$, then each $\mathcal{P}_n(T)$ is countable* ([18]).

Consequently, the results of § 3 and § 4 may be applied to such theories T.

An immediate consequence of 3.5, 4.8, and 5.1 is

([16]) For a brief discussion of T_1 and references, cf. [18], pp. 90-91.

([17]) Examples of $\aleph_a$-categorical theories are given in [9] and [19]. It may be noted that, as pointed out in [9] and [19], any such theory, which has no finite models, is necessarily complete.

([18]) For 5.1, cf. [2] (where only the case $n=1$ is stated). Earlier, in [1], Ehrenfeucht had shown that a $2^{\aleph_a}$-categorical theory T has an "$\aleph_0$-universal" model.

5.1 generalizes its own case where $a=0$, which is much more easily proved. This case was established in [10], p. 25-26, for some special theories T; the method, however, is adequate for any T. (One should note the well-known fact that a denumerable Boolean algebra has either countably many or $2^{\aleph_0}$ prime ideals.)

THEOREM 5.2. *If the hypothesis of 5.1 holds, then T has an $\aleph_0$-universal, homogeneous model and a prime model* (18).

The hypothesis of 5.1 is satisfied, in particular, by $\aleph_\alpha$-categorical theories T. Of course, 5.2 is of no interest when T is $\aleph_0$-categorical. A typical example of a theory which is categorical in non-denumerable powers, but not in $\aleph_0$, is the theory of algebraically closed fields. Here, as is known (cf. [12] and, also, [18], p. 101), the field of complex, algebraic numbers is a prime model, while the algebraically closed fields of transcendence degree $\aleph_0$ are $\aleph_0$-universal, homogeneous models.

THEOREM 5.3. *Under the assumption that T is $\aleph_1$-categorical, but not $\aleph_0$-categorical, we may say further that a prime model $\mathfrak{A}$ of T is minimal—i.e., that $\mathfrak{A}$ has no proper elementary subsystems.*

Proof. Suppose, on the contrary, that a prime model $\mathfrak{A}$ of T has a proper elementary subsystem $\mathfrak{A}'$. It is clear that $\mathfrak{A}'$ is also prime. Consequently, we can define recursively a transfinite sequence $\mathfrak{A} = \mathfrak{A}_0, \mathfrak{A}_1, \ldots, \mathfrak{A}_\xi, \ldots$ $(\xi < \omega_1)$ of prime models of T such that, for any $\xi < \omega_1$, $\mathfrak{A}_\xi$ is a proper elementary subsystem of $\mathfrak{A}_{\xi+1}$ and $\mathfrak{A}_\xi = \bigcup\{\mathfrak{A}_\eta/\eta < \xi\}$ if ξ is a limit number. Indeed, since all prime models of T are isomorphic, our assumption guarantees that such an $\mathfrak{A}_{\xi+1}$ can be found, given $\mathfrak{A}_\xi$; and, when ξ is a limit number less than ω_1, then $\mathfrak{A}_\xi$ is prime, by 3.4 and 1.1.1. Again by 3.4 and 1.1.1, the model $\mathfrak{B} = \bigcup\{\mathfrak{A}_\xi/\xi < \omega_1\}$, of the power $\aleph_1$, is atomic. On the other hand, since T is not $\aleph_0$-categorical, some $F_n(T)$ has a non-principal prime ideal P, by 1.1 and 3.6. By the completeness theorem and the generalized Löwenheim-Skolem theorem (19) (and 1.3), T has a model $\mathfrak{C}$, of power $\aleph_1$, in which $P(\mathfrak{C})$ is not empty. Since $P(\mathfrak{B})$ is empty, $\mathfrak{B}$ and $\mathfrak{C}$ are not isomorphic, contrary to our hypothesis. Thus, 5.3 is established.

A conjecture of Łoś [9] is that a theory T which is categorical in some non-denumerable power is categorical in all such powers. From this it would follow that, in 5.2, "$\aleph_1$", could be replaced by "$\aleph_{1+\alpha}$". We have been unable to prove this stronger version of 5.3.

In the following theorem we establish a result which would easily follow from Łoś' conjecture (and which, it would seem, might possibly be useful in establishing it).

(19) Cf. [18], p. 92, line 5 and, for references, footnote 4 on the same page.

THEOREM 5.4. *Suppose that T is $\aleph_1$-categorical and $\theta \in F_1(T)$. Then, in any model $\mathfrak{A}$ of T, the set $\{x / \models_{\mathfrak{A}} \theta[x]\}$ is either finite or of the same power as $\mathfrak{A}$.*

Proof. To simplify the notation we assume that T has only one non-logical constant, the ternary relation symbol R; the extension to an arbitrary T is obvious. Suppose that the conclusion is false, so that T has a model $\mathfrak{A}^1$ in which the set $U = \{x / \models_{\mathfrak{A}^1} \theta[x]\}$ has an infinite power, smaller than that of $\mathfrak{A}^1$. By the generalized Löwenheim-Skolem Theorem ([20]), $\mathfrak{A}^1$ has an elementary subsystem $\mathfrak{A}^2$, having the power of U, such that $U \subseteq |\mathfrak{A}^2|$. Clearly, the system $(\mathfrak{A}^1, |\mathfrak{A}^2|)$ is a model of the theory T', whose symbols are those of T plus a new singulary predicate V, and whose axioms are the valid sentences of T, the sentences

(1) $\quad \bigvee \mathrm{v}_0 \mathrm{V}\mathrm{v}_0, \quad \bigvee \mathrm{v}_1 \sim \mathrm{V}\mathrm{v}_1, \quad$ and $\quad \bigwedge \mathrm{v}_0[\theta(\mathrm{v}_0) \rightarrow \mathrm{V}\mathrm{v}_0]$,

and all sentences of the form

(2) $\quad \bigwedge \mathrm{v}_0 \ldots \bigwedge \mathrm{v}_{n-1}[\mathrm{V}\mathrm{v}_0 \wedge \ldots \wedge \mathrm{V}\mathrm{v}_{n-1} \rightarrow (\varphi \leftrightarrow \varphi^{\mathrm{V}})]$

—where $\varphi \in F_n(T)$ and φ^{V} is obtained from φ by "relativizing the quantifiers to V" ([21]).

In the theory T' there are two relative interpretations of T—namely, $\langle \mathrm{v}_0 = \mathrm{v}_0, \mathrm{R}\mathrm{v}_0\mathrm{v}_1\mathrm{v}_2 \rangle$ and $\langle \mathrm{V}\mathrm{v}_0, \mathrm{R}\mathrm{v}_0\mathrm{v}_1\mathrm{v}_2 \rangle$. Since T' is consistent, it follows from 4.2 and Ehrenfeucht's theorem, 5.1, that T' has a denumerable model $\mathfrak{B} = \langle B, R, V \rangle$, such that the system $\mathfrak{A}^* = \langle B, R \rangle$ and its subsystem $\mathfrak{A}$ with universe V are both saturated. Since all sentences of (1) and (2) are true in $\mathfrak{B}$, $\mathfrak{A}^*$ is obviously a proper, elementary extension of $\mathfrak{A}$, and

(3) $$\{x / \models_{\mathfrak{A}^*} \theta[x]\} = \{x / \models_{\mathfrak{A}} \theta[x]\}.$$

All denumerable, saturated models being isomorphic (by 4.4), we conclude that an arbitrary such system $\mathfrak{A}$ is a proper elementary subsystem of some other such system $\mathfrak{A}^*$ in such a way that (3) holds.

Now take for $\mathfrak{A}_0$ an arbitrary denumerable, saturated model of T (by 5.1), for $\mathfrak{A}_{\xi+1}$ ($\xi < \omega_1$) a system related to $\mathfrak{A}_\xi$ as $\mathfrak{A}^*$ is to $\mathfrak{A}$ above, and for $\mathfrak{A}_\eta$ —when η is a limit number $\leqslant \omega_1$ —the system $\bigcup \{\mathfrak{A}_\xi / \xi < \eta\}$. It is clear from 1.1.1 that

([20]) —in the form given in [18], p. 92, Theorem 2.1.

([21]) —in the sense of [17], p. 24-25.

the union of an elementarily increasing sequence of saturated systems is saturated, so that our construction is justified. By 1.1.1 and induction on ξ, we see that for any $\xi \leqslant \omega_1$ and for any x, $\models_{\mathfrak{A}_0} \theta[x]$ if and only if $\models_{\mathfrak{A}_\xi} \theta[x]$. Thus $\mathfrak{A}_{\omega_1}$ is a model of T, of power $\aleph_1$, in which $\{x / \models_{\mathfrak{A}_{\omega_1}} \theta[x]\}$ has the power $\aleph_0$. On the other hand, by applying the generalized completeness theorem (to the generalized theory T_1 obtained by adding to T individual constants $c_0, \ldots, c_\xi, \ldots$ $(\xi < \omega_1)$ and axioms $c_\xi \neq c_\eta$ $(\xi \neq \eta)$ and $\theta(c_\xi)$—for $\xi, \eta < \omega_1$), we see that T has a model $\mathfrak{C}$, of power $\aleph_1$, in which $\{x / \models_{\mathfrak{C}} \theta[x]\}$ has the power $\aleph_1$. Thus $\mathfrak{A}_{\omega_1}$ and $\mathfrak{C}$ are not isomorphic, contrary to the hypothesis that T is $\aleph_1$-categorical. This completes the proof (22).

§ 6. The number of non-isomorphic denumerable models. Consider the complete theory T^0 whose models are all systems $\langle A, R \rangle$ such that R is an equivalence relation over A, R has exactly two equivalence classes, and each of these is infinite. Some time ago, Raphael Robinson remarked that T^0 has the following property: There are exactly two non-isomorphic models of T^0 having the power $\aleph_1$. He raised the question whether there exists a complete theory T having exactly two non-isomorphic models of power $\aleph_0$ (23).

As is well-known, the theory T_1 of densely ordered systems without extreme points is complete and, by Cantor's theorem, has exactly one denumerable model, up to isomorphism. Ehrenfeucht constructed an example, which he showed to the author, of a complete theory, T_3, having exactly three non-isomorphic denumerable models. (He has kindly allowed me to reproduce it here.) T_3 has a binary relation symbol $<$ and individual constants $c_0, \ldots, c_n, \ldots$ The axioms of T_3 assure that, if $\langle A, <, c_0, \ldots, c_n, \ldots \rangle$ is a model of T_3, then $\langle A, < \rangle$ is a model of T_1, and $c_i < c_{i+1}$, for $i = 0, 1, \ldots$ That T_3 has the stated properties is easily shown; the three isomorphism types of denumerable models $\langle A, c_0, \ldots, c_n, \ldots \rangle$ of T_3 are those in which (i) $c_0, c_1, \ldots$ are confinal, (ii) $c_0, c_1, \ldots$ are not confinal and have no limit in A, or (iii) $c_0, c_1, \ldots$ have a limit in A.

(22) Again we are unable to establish that "$\aleph_1$" can be replaced by "$\aleph_{1+\alpha}$", in 5.4. However, an argument similar to, but simpler than, the proof just given does show that, in 5.4, "$\aleph_1$-categorical" can be replaced by "$\aleph_\alpha$-categorical and $\aleph_{\alpha+1}$-categorical".

(23) Robinson formulated this question during a conversation in 1957 with several people including the author.

By a simple modification of T_3 we may obtain, for $n = 4, 5, \ldots$, a complete theory T_n having exactly n non-isomorphic models. The non-logical constants of T_n are $<$, $U_0, \ldots, U_{n-3}, c_0, c_1, \ldots$, the U_i being singulary relation symbols. The axioms are those of T_3 plus axioms assuring that, in any model

$$(1) \qquad \langle A, <, U_0, \ldots, U_{n-3}, c_0, c_1, \ldots \rangle ,$$

the sets $U_0, \ldots, U_{n-3}$ form a partition of A, each U_i is dense in A, and, for each n, $c_n \in U_0$. It is a theorem of Skolem [14] that any two denumerable models of T_n are isomorphic if we ignore their lists of distinguished elements. Using this fact it is easily seen that T_n is complete and that the possible (isomorphism) types of denumerable models (1) of T_n are the two in which (i) or (ii) holds, plus the $n-2$ in which $c_0, c_1, \ldots$ have a limit belonging to U_i, $i = 0, \ldots, n-3$.

The theories $T_3, T_4, \ldots$ have infinitely many non-logical constants, but they can easily be converted into complete theories $T'_3, T'_4, \ldots$ having only finitely many—and still having exactly 3, 4, etc., non-isomorphic denumerable models. How to do this, in general, will be clear from the description of T'_3. The non-logical constants of T'_3 are the binary relation symbols $<$ and R. Its axioms assure that, in any model $\langle A, <, R \rangle$: R is an equivalence relation over A having the substitution property with respect to $<$; the quotient of $\langle A, < \rangle$ modulo R is a densely ordered system without extreme points; for $n = 1, 2, \ldots$, there exists exactly one R-equivalence class having exactly n elements; and, for $n = 1, 2, \ldots$, and any $x, y \in A$, *if* the R-equivalence classes of x and y have exactly n and $n+1$ elements, respectively, then $x < y$.

Of various simple proofs omitted in the above discussion, perhaps one showing that T'_3 is complete should be briefly indicated. Granted that T'_3 has only three non-isomorphic models, we may proceed as follows: Let $\mathfrak{A} = \langle A, <, R \rangle$ be a denumerable model of T'_3 of the type in which the R-equivalence classes having $1, 2, \ldots$ elements are confinal (in $\mathfrak{A}$ modulo R). To the theory T whose valid sentences are all true sentences in $\mathfrak{A}$, add an individual constant c and axioms assuring that in any model $\langle A', <', R', c \rangle$, if xRy holds for exactly n elements y, then $x <' c$ $(n = 1, 2, \ldots)$. Clearly the resultant theory T' is consistent, so that by the completeness theorem, it has a denumerable model $\langle A'', <'', R'', c'' \rangle$;

$\langle A'', <'', R'' \rangle$ is then a model of T_3' of one of the other two types. By further applications of the completeness theorem, together with a construction which involves taking the union of an elementarily increasing family of systems, one can show that a denumerable model of T_3' in which the finite equivalence classes do have a limit is elementarily equivalent to one in which they do not have a limit and are not confinal. It then follows easily from the Löwenheim-Skolem theorem that T_3' is complete.

We shall now show that the situation is quite different for $n = 2$, completing the proof of

THEOREM 6.1. *There exists a complete theory having exactly n non-isomorphic denumerable models if and only if $n \neq 2$.*

Remainder of proof. Assume, on the contrary, that T has exactly 2 non-isomorphic denumerable models. Then, by 5.1,

(2) each $\mathcal{P}_n(T)$ is countable.

On the other hand, by 1.1 and 3.6, some $F_m(T)$ has a non-principal prime ideal P. By 2.1.2, T has a model $\mathfrak{A}_1$ in which $P(\mathfrak{A}_1) = 0$. If $\mathfrak{A}$ is any system such that

(3) for some $c_0, \ldots, c_{m-1}$, $(\mathfrak{A}, c_0, \ldots, c_{m-1})$ is a denumerable model of the theory $\bar{P}$ (cf. 1.3),

then $P(\mathfrak{A}) \neq 0$, and, hence, $\mathfrak{A}$ is not isomorphic to $\mathfrak{A}_1$. Consequently, all systems $\mathfrak{A}$ for which (3) holds must be isomorphic. Moreover, any such system $\mathfrak{A}$ must be saturated since $\mathfrak{A}_1$ is not saturated and, by 4.7 and (2), some denumerable model of T is. By applying 4.5, we easily conclude that all denumerable models $(\mathfrak{A}, c_0, \ldots, c_m)$ of $\bar{P}$ are isomorphic. Hence, by 1.1, $F_m(\bar{P})$ contains only finitely many formulas inequivalent in $\bar{P}$. This contradicts the fact that the subset $F_m(T)$ of $F_m(\bar{P})$ already contains infinitely many formulas inequivalent in T and, hence, T being complete, in $\bar{P}$.

We may mention here the following unresolved conjecture (which appears to have been made by a number of people):

A theory which is $\aleph_{1+\alpha}$-categorical but not $\aleph_0$-categorical has exactly $\aleph_0$ non-isomorphic denumerable models.

A second problem is this:

Can it be proved, without the use of the continuum hypothesis, that there exists a complete theory having exactly $\aleph_1$ non-isomorphic denumerable models?

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MODEL COMPLETENESS RESULTS FOR EXPANSIONS OF THE ORDERED FIELD OF REAL NUMBERS BY RESTRICTED PFAFFIAN FUNCTIONS AND THE EXPONENTIAL FUNCTION

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1. Introduction

Recall that a subset of $\mathbb{R}^n$ is called *semi-algebraic* if it can be represented as a (finite) boolean combination of sets of the form $\{\vec{\alpha} \in \mathbb{R}^n : p(\vec{\alpha}) = 0\}$, $\{\vec{\alpha} \in \mathbb{R}^n : q(\vec{\alpha}) > 0\}$ where $p(\vec{x})$, $q(\vec{x})$ are n-variable polynomials with real coefficients. A map from $\mathbb{R}^n$ to $\mathbb{R}^m$ is called semi-algebraic if its graph, considered as a subset of $\mathbb{R}^{n+m}$, is so. The geometry of such sets and maps ("semi-algebraic geometry") is now a widely studied and flourishing subject that owes much to the foundational work in the 1930s of the logician Alfred Tarski. He proved ([11]) that the image of a semi-algebraic set under a semi-algebraic map is semi-algebraic. (A familiar simple instance: the image of $\{\langle a,b,c,x\rangle \in \mathbb{R}^4 : a \neq 0 \text{ and } ax^2+bx+c = 0\}$ under the projection map $\mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ is $\{\langle a,b,c\rangle \in \mathbb{R}^3 : a \neq 0 \text{ and } b^2 - 4ac \geq 0\}$.) Tarski's result implies that the class of semi-algebraic sets is closed under first-order logical definability (where, as well as boolean operations, the quantifiers "$\exists x \in \mathbb{R} \ldots$" and "$\forall x \in \mathbb{R} \ldots$" are allowed) and for this reason it is known to logicians as "quantifier elimination for the ordered ring structure on $\mathbb{R}$". Immediate consequences are the facts that the closure, interior and boundary of a semi-algebraic set are semi-algebraic. It is also the basis for many inductive arguments in semi-algebraic geometry where a desired property of a given semi-algebraic set is inferred from the same property of projections of the set into lower dimensions. For example, the fact (due to Hironaka) that any bounded semi-algebraic set can be triangulated is proved this way.

In the 1960s the analytic geometer Lojasiewicz extended the above theory to the analytic context ([8]). The definition of a *semi-analytic* subset of $\mathbb{R}^n$ is the same as above except that for the basic sets the $p(\vec{x})$'s and $q(\vec{x})$'s are allowed to be *analytic* functions and we only insist that the boolean representations work locally around each point of $\mathbb{R}^n$ (allowing different representations around different points). It is also necessary to restrict the maps to be *proper* (with semi-analytic graph). With this restriction it is true that the image of a semi-analytic set, known as a *sub-analytic* set, is semi-analytic *provided that* the target space is either $\mathbb{R}$ or $\mathbb{R}^2$. Counterexamples have been known since the beginning of this century for maps to $\mathbb{R}^m$ for $m \geq 3$. (They are due to Osgood, see [8].) However, the situation was clarified in 1968 by Gabrielov ([5]) who showed that the class of *sub*-analytic sets

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is closed under taking complements. Gabrielov's theorem can be reformulated in terms of logical definability as follows.

For each m and each analytic function $f : U \to \mathbb{R}$, where U is some open neighbourhood of the closed box $[0,1]^m$ in $\mathbb{R}^m$, let $\tilde{f} : \mathbb{R}^m \to \mathbb{R}$ be defined by

$$\tilde{f}(\vec{x}) = \begin{cases} f(\vec{x}) & \text{if } \vec{x} \in [0,1]^m, \\ 0 & \text{if } \vec{x} \in \mathbb{R}^m \backslash [0,1]^m. \end{cases}$$

Let L_{an} denote the language extending that of ordered rings obtained by adding a function symbol for each such function $\tilde{f}$. Then Gabrielov's theorem is readily seen to be equivalent to the assertion that for each n, every subset of $\mathbb{R}^n$ which can be defined by some logical formula of the language L_{an} can, in fact, be defined by an *existential* formula of L_{an}, that is, one of the form $\exists y_1, \ldots, y_r\, \phi(x_1, \ldots, x_n, y_1, \ldots, y_r)$ where the L_{an}-formula ϕ contains no occurrences of quantifiers. (Further, the class of all such subsets which are *bounded* is exactly the class of bounded sub-analytic subsets of $\mathbb{R}^n$. At first sight it might seem that the former class is richer because the projection map implicit in the existential quantification is not restricted to a compact set. The fact that we do not obtain non-sub-analytic sets is due, of course, to our original truncation of the analytic functions.) In this form Gabrielov's theorem was given a fairly straightforward treatment, based on the Weierstrass preparation theorem and Tarski's elimination theory, by Denef and van den Dries ([3]).

Thus, although we do not have full quantifier elimination for this local analytic structure (together with the ordered ring structure) on $\mathbb{R}$, we do have elimination down to existential formulas. Such structures are called *model complete*, a term introduced by Abraham Robinson. Actually, whether or not a structure is model complete only depends on the *theory* of the structure, that is on the set of all sentences of its language (a *sentence* is a formula without free variables) that are true in the structure. More generally, if T is a consistent set of sentences of some language L, then T is called *model complete* if for every formula $\psi(\vec{x})$ of L there is an existential formula $\theta(\vec{x})$ of L such that the sentence $\forall \vec{x}(\psi(x) \leftrightarrow \theta(\vec{x}))$ is a formal consequence of T. Further, if $\theta(\vec{x})$ can always be chosen to contain no occurrences of quantifiers at all, then T is said to admit *elimination of quantifiers*.

To summarize the above discussion, then, let $\overline{\mathbb{R}} = \langle \mathbb{R}; +, \cdot, -, 0, 1, < \rangle$ and $\mathbb{R}_{an} = \langle \overline{\mathbb{R}}; \mathcal{F} \rangle$ where $\mathcal{F}$ consists of all functions of the form $\tilde{f}$ as described above. Let $\overline{T}$ and T_{an} denote the theories of these structures respectively. Then $\overline{T}$ admits elimination of quantifiers (Tarski) and T_{an} is model complete (Gabrielov) but does not admit elimination of quantifiers (Osgood).

My aim in this paper is to give two variations of Gabrielov's theme. The first is in response to the following natural question: when can the analytic functions needed to describe the complement of a given sub-analytic set be chosen from the ring generated by functions used to describe the given set? Or, in model theoretic terms, for which subsets $\mathcal{G}$ of $\mathcal{F}$ is (the theory of) the structure $\langle \overline{\mathbb{R}}; \mathcal{G} \rangle$ model complete? I shall show that this is the case when $\mathcal{G}$ is a *Pfaffian chain* of functions. Let me make this more precise.

Firstly, fix $m, l \in \mathbb{N}, m, l \geq 1$, and an open set $U \subseteq \mathbb{R}^m$ such that the closed box $[0,1]^m$ is contained in U. Let $G_1, \ldots, G_l : U \to \mathbb{R}$ be analytic functions and suppose that there exist polynomials $p_{i,j} \in \mathbb{R}[z_1, \ldots, z_{m+i}]$ (for $i = 1, \ldots, l, j = 1, \ldots, m$)

such that

$$\frac{\partial G_i}{\partial x_j}(\vec{x}) = p_{i,j}(\vec{x}, G_1(\vec{x}), \ldots, G_i(\vec{x})) \quad (\text{for all } \vec{x} \in U). \tag{1}$$

The sequence $G_1, \ldots, G_l$ is called a Pfaffian chain on U. Let $F_1, \ldots, F_l$ be the corresponding truncations. That is,

$$F_i(\vec{x}) = \begin{cases} G_i(\vec{x}) & \text{if } \vec{x} \in [0,1]^m, \\ 0 & \text{if } \vec{x} \in \mathbb{R}^m \backslash [0,1]^m. \end{cases} \tag{2}$$

Now let C be any subset of $\mathbb{R}$ such that each coefficient of each $p_{i,j}$ is the value of some term (without free variables) in the structure $\langle \overline{\mathbb{R}}; F_1, \ldots, F_l; r\rangle_{r\in C}$. (For example, we could take C to be the set of all such coefficients.) I denote this structure by $\widetilde{\mathbb{R}}$ and its language and theory by $\widetilde{L}, \widetilde{T}$, respectively.

Obviously $\widetilde{L}$ is a sublanguage of L_{an} and every subset of $\mathbb{R}^n$ that can be defined by a formula of $\widetilde{L}$ can be defined by a formula, and hence an existential formula, of L_{an}. I shall prove the following.

First Main Theorem. *Every subset of $\mathbb{R}^n$ (for any n) that can be defined by some formula of $\widetilde{L}$ can be defined by an existential formula of $\widetilde{L}$. That is, $\widetilde{T}$ is model complete.*

Examples. (A) Take $m = l = 1$, $U = \mathbb{R}$, $G_1(x_1) = \exp(x_1)$, $p_{1,1}(z_1, z_2) = z_2$ and $C = \emptyset$. Then the theorem tells us that the theory of the structure $\langle \overline{\mathbb{R}}; \exp \restriction [0,1]\rangle$ is model complete. Of course the convention (2) dictates that $\exp \restriction [0,1]$ is defined to be 0 outside $[0,1]$. If one prefers to have only functions that are analytic throughout $\mathbb{R}$ (or $\mathbb{R}^m$) in the basic language, then one can always invoke the following cosmetic trick. Define $e : \mathbb{R} \to \mathbb{R}, x \to \exp((1+x^2)^{-1})$. Then the structures $\langle \overline{\mathbb{R}}; \exp \restriction [0,1]\rangle$ and $\langle \overline{\mathbb{R}}; e\rangle$ are essentially the same, i.e. they have the same definable sets and, more to the point, the same existentially definable sets. It follows that the theory of the structure $\langle \overline{\mathbb{R}}; e\rangle$ is model complete.

(B) Sometimes the cosmetic trick comes for free. Take $m = 1$, $l = 2$, $U = \mathbb{R}$, $G_1(x_1) = (1 + x_1^2)^{-1}$, $G_2(x_1) = \tan^{-1}(x_1)$, $p_{1,1}(z_1, z_2) = 2z_1z_2^2$, $p_{2,1}(z_1, z_2, z_3) = z_2$ and $C = \emptyset$. Since the graph of the function G_1 (or rather F_1) is already definable in $\overline{\mathbb{R}}$ by a quantifier-free formula, the theorem implies that the theory of the structure $\langle \overline{\mathbb{R}}; \tan^{-1} \restriction [0,1]\rangle$ is model complete. But in this case we have functional equations at $\pm\infty$, namely

$$\tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \tan^{-1}(x) \quad \text{for } x > 0,$$

and

$$\tan^{-1}\left(\frac{1}{x}\right) = -\frac{\pi}{2} - \tan^{-1}(x) \quad \text{for } x < 0,$$

which, together with the equations

$$\tan^{-1}(-x) = -\tan^{-1}(x) \quad \text{and} \quad \frac{\pi}{2} = 2\tan^{-1}(1),$$

clearly imply that the theory of the structure $\langle \overline{\mathbb{R}}; \tan^{-1}\rangle$ (with $\tan^{-1}$ unrestricted) is model complete.

(C) As far as I know the first result along these lines was obtained by van den Dries ([15]) who showed that the theory of the structure

$$\langle \overline{\mathbb{R}}; \sin \restriction [0,1], \exp \restriction [0,1]; r\rangle_{r\in\mathbb{R}}$$

is model complete. This also follows from the first main theorem by combining the chains used in (A) and (B) and then invoking elementary trigonometric identities. I leave the details to the reader. (The reason van den Dries actually *needs* the sine function is that his proof uses *complex* power series methods and the required model completeness is then deduced, by "taking real parts", from a corresponding result for *complex* exponentiation restricted to the unit disc. The key point in this approach is that the complex analytic functions cropping up as coefficients in the Weierstrass Preparation Theorem can be existentially defined from the initial data (and, possibly, extra parameters—hence the choice $C = \mathbb{R}$ here)—a fact that, interestingly, seems to be unknown in the case of the Preparation Theorem for real analytic functions.)

Whether one uses analytic or model-theoretic terminology and methods, the proofs of all the above results work because one only ever has to deal with analytic functions restricted to compact subsets of their natural domains or, equivalently (via the cosmetic trick), with total analytic functions that are also analytic at infinity. The second result of this paper removes this restriction in one particular case.

Second Main Theorem. *The theory of the structure* $\langle \overline{\mathbb{R}}; \exp\rangle$, *where* exp *is the usual exponential function* $x \mapsto e^x$ *with domain* $\mathbb{R}$, *is model complete.*

Thus, if we define a subset of $\mathbb{R}^n$ to be *semi-EA* ("semi-exponential-algebraic") if it can be represented as a boolean combination of sets of the form $\{\vec{\alpha} \in \mathbb{R}^n : p(\vec{\alpha}) = 0\}$, $\{\vec{\alpha} \in \mathbb{R}^n : q(\vec{\alpha}) > 0\}$, where the $p(\vec{x})$'s and $q(\vec{x})$'s are exponential polynomials (i.e. polynomials in $x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}$ with real (or, more generally (!), integer) coefficients), and a map from $\mathbb{R}^n$ to $\mathbb{R}^m$ to be semi-EA if its graph is so (and we do not demand that the map be proper) and, finally, a set to be *sub-EA* if it is the image of a semi-EA set under a semi-EA map, *then* the theorem is equivalent to the assertion that the complement of a sub-EA set is a sub-EA set. This, as for the semi-algebraic case, implies that the class of sub-EA sets is also closed under taking closures, interiors and boundaries.

It is difficult to see how conventional analytic or differential geometric methods could be used to establish this result because of the essential singularity of the exponential function at infinity. The proof given here uses model-theoretic methods to analyse large zeros of systems of exponential-algebraic equations.

Before giving a plan of the paper I should make a few remarks concerning effectivity. For it was Tarski's main purpose in his paper to show not only that every formula of $\overline{L}$ is equivalent (modulo $\overline{T}$) to a quantifier-free formula, but also that the latter could be found *effectively* from the former. From this he deduced that $\overline{T}$ is a decidable theory, i.e. there exists an (explicitly given) algorithm to decide whether or not an arbitrary sentence of $\overline{L}$ is true in $\overline{\mathbb{R}}$ (hence the title of the paper). Tarski asked whether this holds for $\langle \overline{\mathbb{R}}; \exp\rangle$ and while this question was the motivation for the work in this paper, I feel it would have obscured the arguments here had I paid constant attention to effectivity considerations. Such problems will be discussed in a forthcoming paper of A. J. Macintyre and the author, where they will be shown to be intimately linked with the conjecture of Schanuel in transcendental number theory.

For an introduction to the general notion of model completeness I refer the reader to [1]. Several equivalent formulations are mentioned there (and, in fact, the definition I have given is not Robinson's original one but one of these equivalents) including the following: if T is a consistent set of sentences in a language L, then T is model complete if and only if whenever $\mathfrak{A}, \mathfrak{B}$ are models of T with $\mathfrak{A} \subseteq \mathfrak{B}$ (i.e. $\mathfrak{A}$ is an L-substructure of $\mathfrak{B}$), then $\mathfrak{G}$ is *existentially closed* in $\mathfrak{B}$. For the theories involved in the two main theorems above (or, indeed, for the theory of any structure expanding $\overline{\mathbb{R}}$ by functions and constants) establishing the latter is equivalent to showing that any finite set of equations (involving the basic functions of the given language) with parameters from $\mathfrak{A}$ is solvable in $\mathfrak{A}$ provided that it is solvable in $\mathfrak{B}$. This is how I shall go about proving the theorems.

The next section clarifies this approach and organizes the equations that we need to solve into manageable form. After summarizing known finiteness theorems for the solutions of such equations in section 3 I develop, in sections 4 and 5, a theory of Noetherian rings of differentiable germs that works for *arbitrary* (possibly nonstandard) models of suitable theories. (As an application we give a proof (see 5.3) of the theorem of Khovanskii stating that Pfaffian varieties have only finitely many connected components.) Sections 6 and 7 are rather tedious. This is because I need to develop some very elementary, but *global*, existence theorems from the differential calculus that apply to, as above, *arbitrary* models of the theories under consideration and this can only be accomplished, as far as I can see, by exhibiting explicit definitions. Many algebraic manipulations (especially of Jacobian matrices) are involved here, the details of which may be safely skipped without loss of understanding of the main arguments.

For all the results of sections 2 to 7 (apart from 3.4 and 3.5) it is irrelevant whether or not the basic functions are restricted to the closed unit box and so they apply to the situations of both main theorems. I have, however, concentrated on those structures to which the first theorem applies because the truncation actually introduces extra difficulties (of a rather superficial nature). Hereafter the proofs diverge because we need to confront the problem, briefly referred to above, of large solutions of the equations under consideration. This is done in section 8 for the first theorem, thus completing its proof. Sections 9 to 11 are devoted to the completion of the proof of the second theorem. These may be read independently as I restate the necessary results from earlier sections.

Tarski's problem on the real exponential function has been the focus of papers by many authors. Apart from those mentioned above I refer the reader to the pioneering work of Dahn ([2]) and Wolter ([17]). For the crucial inequalities needed in the proof of the model completeness of the structure $\langle\overline{\mathbb{R}}; \exp\rangle$ can be viewed as a generalization to many variables of the Dahn bounding theorem ([2]).

2. Towards the proof of the first main theorem

The symbols $\widetilde{K}, \tilde{k}$ will denote $\widetilde{L}$-structures with domains K, k respectively, although I shall sometimes use K, k to denote the underlying fields or ordered fields. If $\tilde{k} \subseteq \widetilde{K}$, then $\widetilde{L}_k$ (respectively $\overline{L}_k$) denotes the expansion of $\widetilde{L}$ (respectively $\overline{L}$) obtained by adding a new constant symbol for each element of k. The corresponding $\widetilde{L}_k$-expansion of $\widetilde{K}$ will be denoted simply $\widetilde{K}^+$ when it is clear which k is intended. I shall also adopt the usual practice of not distinguishing notationally between non-logical symbols of a language and their interpretation in a structure under consideration. In particular, for $i = 1, \ldots, l$, the symbol F_i, which was introduced in section 1 as a particular function from $\mathbb{R}^m$ to $\mathbb{R}$, will also denote the corresponding function symbol of $\widetilde{L}$ as well as the function from K^m to K that the function symbol is interpreted as in a given $\widetilde{L}$-structure $\widetilde{K}$.

Now for $1 \leq i \leq l$, $1 \leq j \leq m$ and $n \geq 1$, (1) and (2) imply

$$(3) \qquad F_i \text{ is } n\text{-times differentiable on the open box } (0,1)^m,$$

and

$$(4) \qquad \forall \vec{x} \in (0,1)^m, \quad \frac{\partial F_i}{\partial x_j} = p_{i,j}(\vec{x}, F_1(\vec{x}), \ldots, F_i(\vec{x})).$$

Clearly (3) and (4) can be expressed by sentences of $\widetilde{L}$ (note the property of the set C) and these sentences are therefore in $\widetilde{T}$. But there is no obvious way to express the fact that F_i (restricted to $(0,1)^m$) has an analytic continuation (namely G_i) to an open set containing the closed box $[0,1]^m$ (namely U) such that (1) holds. However, as the remarks following example (C) above were intended to suggest, we *must* use this fact and, indeed, there are several consequences of it that *are* first-order expressible and I need to mention one such here.

Let $S \subseteq \{1, \ldots, m\}$ and suppose $a_j \in \{0, 1\}$ for $j \in S$. Define the functions $F_i^* : \mathbb{R}^m \to \mathbb{R}$ by $F_i^*(x_1, \ldots, x_m) = F_i(x_1', \ldots, x_m')$ where

$$x_j' = \begin{cases} x_j & \text{if } j \notin S, \\ a_j & \text{if } j \in S. \end{cases}$$

Let

$$J_j = \begin{cases} (0,1) & \text{if } j \notin S, \\ \mathbb{R} & \text{if } j \in S. \end{cases}$$

Then (1) and (2) imply (with i, j, n as above)

$$(5) \qquad F_i^* \text{ is } n\text{-times differentiable on the open set } \prod_{j=1}^{m} J_j,$$

and

$$(6) \quad \forall \vec{x} \in \prod_{j=1}^{m} J_j, \quad \frac{\partial F_i^*}{\partial x_j}(\vec{x}) = \begin{cases} P_{i,j}(x_1', \ldots, x_m', F_1^*(\vec{x}), \ldots, F_i^*(\vec{x})), & \text{if } j \notin S, \\ 0 \quad \text{if } j \in S, \end{cases}$$

and these facts are expressible by sentences of $\widetilde{L}$ (which are therefore in $\widetilde{T}$).

Now to prove the first main theorem it suffices, by remarks in section 1 (see [1]), to show that if $\tilde{k}, \widetilde{K} \models \widetilde{T}$, $\tilde{k} \subseteq \widetilde{K}$ and χ is an existential sentence of $\widetilde{L}_k$ such that $\widetilde{K}^+ \models \chi$, then $\tilde{k}^+ \models \chi$. We may also suppose here that χ has the form

$$\exists x_1, \ldots, x_r \bigwedge_{s=1}^{n} \tau_s = 0,$$

where each τ_s is either a term of $\overline{L}_k$ or else has the form $F_i(x_{i_1}, \ldots, x_{i_m}) - x_{i_{m+1}}$. This is because of standard logical equivalences and the facts that the formulas $x \neq y$ and $x < y$ are equivalent in $\overline{T}$ to the formulas $\exists z\, (y - x) \cdot z - 1 = 0$ and $\exists z\, (y - x) \cdot z^2 - 1 = 0$ respectively and that composite terms may be unravelled by introducing new variables (e.g. replace $\tau(\sigma) = 0$ by $\exists x\,(\sigma - x = 0 \wedge \tau(x) = 0)$). Now notice also that the formula

$$F_i(y_1, \ldots, y_m) - y_{m+1} = 0 \wedge y_j \geq 1$$

(the y_i's being variables or constants) where $1 \leq j \leq m$ and y_j is a variable, is equivalent in $\widetilde{T}$ to the formula

$$(y_j > 1 \wedge y_{m+1} = 0) \vee (F_i(y_1, \ldots, y_m)(y_j/1) - y_{m+1} = 0 \wedge y_j = 1),$$

and that a similar equivalence holds with "$y_j \leq 0$" in place of "$y_j \geq 1$". Thus by repeated use of all these equivalences we may suppose that χ actually has the form

$$\exists x_1, \ldots, x_r \bigwedge_{s=1}^{n} \chi_s(x_1, \ldots, x_r),$$

where each $\chi_s(x_1, \ldots, x_s)$ is either of the form $\tau(x_1, \ldots, x_r) = 0$ for some term $\tau(x_1, \ldots, x_r)$ of $\overline{L}_k$ (i.e. a polynomial in $x_1, \ldots, x_r$ over k) or of the form

$$\bigwedge_{j \notin S} 0 < x_{i_j} < 1 \wedge F_i(x'_{i_1}, \ldots, x'_{i_m}) - x_{i_{m+1}} = 0$$

for some $S \subseteq \{1, \ldots, m\}$, where $1 \leq i_1, \ldots, i_{m+1} \leq r$ and where

$$x'_{i_j} = \begin{cases} x_{i+j} & \text{for } j \notin S, \\ 0 \text{ or } 1 & \text{for } j \in S. \end{cases}$$

The proof of the first main theorem will be essentially by induction on the number of χ_s's of this second form that occur in χ although it is convenient first to pad out the set of such χ_s's. This the purpose of the following

2.1. Definition. Let $n, r \in \mathbb{N}$.

(i) A sequence $\langle \sigma_1, \ldots, \sigma_n \rangle$ of terms of $\widetilde{L}$ in the variables $x_1, \ldots, x_r$ is called an (n, r)-*sequence* if
  (a) for $s = 1, \ldots, n$, σ_s has the form $F_i(y_1, \ldots, y_m)$ for some $i = 1, \ldots, l$ and some $y_1, \ldots, y_m \in \{0, 1, x_1, \ldots, x_r\}$, and
  (b) if $1 \leq s \leq n$, $1 < i \leq l$ and σ_s is $F_i(y_1, \ldots, y_m)$ (as in (a)), then $s > 1$ and for some $t = 1, \ldots, s-1$, σ_t is $F_{i-1}(y_1, \ldots, y_m)$.

(ii) Those variables actually occurring in some term of an (n, r)-sequence $\vec{\sigma}$ are called $\vec{\sigma}$-*bounded.*

Clearly any (n, r)-sequence $\vec{\sigma}$ is also an (n, r')-sequence for any $r' \geq r$ (and the set of $\vec{\sigma}$-bounded variables is the same), and any initial segment of an (n, r)-sequence is an (n', r)-sequence for the appropriate $n' \leq n$. Further, any sequence satisfying (i) (a) may be clearly rearranged and padded out to an (n', r)-sequence for some n'. Now let $\widetilde{K} \models \widetilde{T}$.

2.2. Definition. Suppose $\vec{\sigma} = \langle \sigma_1, \ldots, \sigma_n \rangle$ is an (n, r)-sequence. The *natural domain of $\vec{\sigma}$ on $\widetilde{K}$*, denoted $D^r(\vec{\sigma}, \widetilde{K})$, is defined to be $\prod_{i=1}^{r} I_i$ where

$$I_i = \begin{cases} \{x \in K : \widetilde{K} \models 0 < x < 1\} & \text{if } x_i \text{ is } \vec{\sigma}\text{-bounded}, \\ K \quad \text{otherwise}. \end{cases}$$

Clearly $D^r(\vec{\sigma}, \widetilde{K})$ is a definable open (in the sense of $\widetilde{K}$) subset of K^r.
Suppose now that $\tilde{k} \models \widetilde{T}$ and $\tilde{k} \subseteq \widetilde{K}$.

2.3. Definition. I denote by $M^r(\tilde{k}, \tilde{K}, \vec{\sigma})$, where $\vec{\sigma}$ is an (n, r)-sequence, the ring of all those functions $f : D^r(\vec{\sigma}, \tilde{K}) \to K$ for which there exists a polynomial $p(X_1, \ldots, X_r, Y_1, \ldots, Y_n) \in k[X_1, \ldots, X_r, Y_1, \ldots, Y_n]$ such that

$$f(\vec{\alpha}) = p(\vec{\alpha}, \sigma_1(\vec{\alpha}), \ldots, \sigma_n(\vec{\alpha})) \text{ for all } \vec{\alpha} \in D^r(\vec{\sigma}, \tilde{K}), \tag{7}$$

where $\vec{\sigma} = \langle \sigma_1, \ldots, \sigma_n \rangle$.

The reductions preceding Definition 2.1 clearly imply

2.4. Lemma. *In order to prove the main theorem it is sufficient to show that for all* $\tilde{k}, \tilde{K} \models \tilde{T}$ *with* $\tilde{k} \subseteq \tilde{K}$, *all* $n, r \in \mathbb{N}$, *all* (n, r)*-sequences* $\vec{\sigma}$, *and all* $g_1, \ldots, g_q \in M^r(\tilde{k}, \tilde{K}, \vec{\sigma})$, *if* $g_1, \ldots, g_q$ *have a common zero in* $D^r(\vec{\sigma}, \tilde{K})$, *then they have one in* $D^r(\vec{\sigma}, \tilde{k})$. *(Note that we clearly have* $D^r(\vec{\sigma}, \tilde{k}) \subseteq D^r(\vec{\sigma}, \tilde{K})$.*)* □

Of course our reductions show that the polynomials p of (7) representing the g_i's of 2.4 may be taken to be either independent of the Y_i's or of the form $Y_i - X_j$ (for some $i = 1, \ldots, n$, $j = 1, \ldots, r$). However, while this observation will play a role later (in somewhat disguised form) it is much more convenient to work with *rings* of functions, and I now want to establish some elementary properties of these rings.

Fix, for the rest of this section, models $\tilde{k}, \tilde{K}$ of $\tilde{T}$ such that $\tilde{k} \subseteq \tilde{K}$.

Suppose that $n, r \in \mathbb{N}$ and that $\vec{\sigma} = \langle \sigma_1, \ldots, \sigma_n \rangle$ is an (n, r)-sequence. Let $g \in M^r(\tilde{k}, \tilde{K}, \vec{\sigma})$. Then by (5) and the comments immediately following (6), g is a C^∞ function on $D^r(\vec{\sigma}, \tilde{K})$ in the sense of $\tilde{K}$. That is, for each $q \in \mathbb{N}$, $\tilde{K}$ satisfies the usual ε-δ definition for the existence of continuous qth partial derivatives of g at all points of $D^r(\vec{\sigma}, \tilde{K})$. Further, it clearly follows from (6) and (i)(b) of Definition 2.1 that these partial derivatives of g all lie in $M^r(\tilde{k}, \tilde{K}, \vec{\sigma})$. Thus $M^r(\tilde{k}, \tilde{K}, \vec{\sigma})$ is a differential ring. It is also an integral domain. This is because $M^r(\tilde{\mathbb{R}}, \tilde{\mathbb{R}}, \vec{\sigma})$ is certainly an integral domain (since it is a ring of functions analytic on an open connected set) and this fact clearly transfers to $M^r(\tilde{K}, \tilde{K}, \vec{\sigma})$ (just represent elements of $M^r(\tilde{K}, \tilde{K}, \vec{\sigma})$ in the form (7) and quantify out the coefficients of p), which contains $M^r(\tilde{k}, \tilde{K}, \vec{\sigma})$ as a subring.

Suppose now that $p, q \leq r$ and $1 \leq i_1 < \cdots < i_q \leq r$. For $g_1, \ldots, g_p \in M^r(\tilde{k}, \tilde{K}, \vec{\sigma})$ consider the (Jacobian) matrix

$$\begin{pmatrix} \frac{\partial g_1}{\partial x_{i_1}} & \cdots & \frac{\partial g_1}{\partial x_{i_q}} \\ \vdots & & \vdots \\ \frac{\partial g_p}{\partial x_{i_1}} & \cdots & \frac{\partial g_p}{\partial x_{i_q}} \end{pmatrix}.$$

It is a matrix over $M^r(\tilde{k}, \tilde{K}, \vec{\sigma})$ and I denote it by $\frac{\partial(g_1, \ldots, g_p)}{\partial(x_{i_1}, \ldots, x_{i_q})}$. Note that if $p = q$, then

$$\det\left(\frac{\partial(g_1, \ldots, g_p)}{\partial(x_{i_1}, \ldots, x_{i_p})}\right) \in M^r(\tilde{k}, \tilde{K}, \vec{\sigma}).$$

If $p = q = r$, I write $J(g_1, \ldots, g_r)$ for $\det(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)})$.

2.5. Definition. Suppose $n, r \in \mathbb{N}$ and let $\vec{\sigma}$ be an (n, r)-sequence. Then a point $P \in K^r$ is called $(\tilde{k}, \vec{\sigma})$*-definable* if there exist $g_1, \ldots, g_r \in M^r(\tilde{k}, \tilde{K}, \vec{\sigma})$ such that

(i) $P \in D^r(\vec{\sigma}, \tilde{K})$,

(ii) $g_1(P) = \cdots = g_r(P) = 0$, and
(iii) $J(g_1, \ldots, g_r)(P) \neq 0$.

Examples. (D) Let $r \in \mathbb{N}$. Note that the empty sequence, $\varnothing$, is a $\langle 0, r\rangle$ sequence, that $D^r(\varnothing, \tilde{K}) = K^r$ and that $M^r(\tilde{k}, \tilde{K}, \varnothing)$ may be identified with the polynomial ring $k[x_1, \ldots, x_r]$. Now suppose $P \in k^r$, say $P = \langle p_1, \ldots, p_r\rangle$. For $i = 1, \ldots, r$ define $g_i(x_1, \ldots, x_r) = x_i - p_i$. Then $g_1, \ldots, g_r \in M^r(\tilde{k}, \tilde{K}, \varnothing)$, $g_1(P) = \cdots = g_r(P) = 0$ and $J(g_1, \ldots, g_r)(P) = 1 \neq 0$. Hence P is a $(\tilde{k}, \varnothing)$-definable point of K^r. Conversely, suppose Q is a $(\tilde{k}, \varnothing)$-definable point of K^r. Then elementary algebra tells us that each coordinate of Q is algebraic over (the field) k. Since k is algebraically closed in K (both being models of $\overline{T}$) it follows that $Q \in k^r$.

(E) More generally, suppose $n, r \in \mathbb{N}$ and that $\vec{\sigma} = \langle \sigma_1, \ldots, \sigma_n\rangle$ is an (n, r)-sequence. Let $s \geq 1$ and regard $\vec{\sigma}$ as an $(n, r+s)$-sequence. Then clearly $D^{r+s}(\vec{\sigma}, \tilde{K}) = D^r(\vec{\sigma}, \tilde{K}) \times K^s$ (cf. 2.1(ii) and 2.2) and $M^{r+s}(\tilde{k}, \tilde{K}, \vec{\sigma})$ may be identified with the polynomial ring $M^r(\tilde{k}, \tilde{K}, \vec{\sigma})[x_{r+1}, \ldots, x_{r+s}]$ over the domain $M^r(\tilde{k}, \tilde{K}, \vec{\sigma})$. Suppose $P \in D^r(\vec{\sigma}, \tilde{K})$ and $Q \in K^s$ and that $\langle P, Q\rangle$ is $(\tilde{k}, \vec{\sigma})$-definable. Then elementary algebra again tells us that each coordinate of Q is algebraic over the subfield $k(p_1, \ldots, p_r, \sigma_1(P), \ldots, \sigma_n(P))$ of K (where $P = \langle p_1, \ldots, p_r\rangle$).

Example (D) shows that a point of K^r is $(\tilde{k}, \varnothing)$-definable if and only if it lies in k^r. In fact:

2.6. Main Lemma. *For any $n, r \in \mathbb{N}$ and any (n, r)-sequence $\vec{\sigma}$, every $(\tilde{k}, \vec{\sigma})$-definable point of K^r lies in k^r.*

We shall also prove the following

2.7. Lemma. *Let $n, r \in \mathbb{N}$ and let $\vec{\sigma}$ be an (n, r)-sequence. Suppose $g \in M^r(\tilde{k}, \tilde{K}, \vec{\sigma})$ and $g(P) = 0$ for some $P \in D^r(\vec{\sigma}, \tilde{K})$. Then for some $s \in \mathbb{N}$ there exists $Q_0 \in D^r(\vec{\sigma}, \tilde{K})$ and $Q_1 \in K^s$ such that $g(Q_0) = 0$ and $\langle Q_0, Q_1\rangle$ is $(\tilde{k}, \vec{\sigma})$-definable (cf. example (E) above).*

Clearly the first main theorem follows from 2.4, 2.6 and 2.7 by taking the g of 2.7 to be $\sum_{i=1}^q g_i^2$ with $g_1, \ldots, g_q$ as in 2.4.

It is convenient to split the main lemma into two statements, the proofs of which are entirely different. They are:

2.8. Lemma. *Suppose $n, r \in \mathbb{N}$ and that $\vec{\sigma}$ is an (n, r)-sequence. Suppose further that for each $s \geq r$ and each $(\tilde{k}, \vec{\sigma})$-definable point $\langle p_1, \ldots, p_s\rangle$ of K^s there is some $B \in k$ such that $\tilde{K} \models \bigwedge_{i=1}^s -B < p_i < B$. Then every $(\tilde{k}, \vec{\sigma})$-definable point of K^r lies in k^r.*

2.9. Lemma. *Suppose that $n, r \in \mathbb{N}$ and that $\vec{\sigma}' = \langle \sigma_1, \ldots, \sigma_n, \sigma_{n+1}\rangle$ is an $(n+1, r)$-sequence. Let $\vec{\sigma}$ denote the (n, r)-sequence $\langle \sigma_1, \ldots, \sigma_n\rangle$. Suppose that for each $s \geq r$ every $(\tilde{k}, \vec{\sigma})$-definable point of K^s lies in k^s. Then for each $s \geq r$ and each $(\tilde{k}, \vec{\sigma}')$-definable point $\langle p_1, \ldots, p_s\rangle$ of K^s, there is some $B \in k$ such that $\tilde{K} \models \bigwedge_{i=1}^s -B < p_i < B$.*

Clearly the main lemma follows by induction on n (for all values of r) from 2.8 and 2.9, the base step of the induction being provided by example (D).

We have now reduced the task of proving the first main theorem to that of proving Lemmas 2.7, 2.8 and 2.9. In fact, 2.7 and 2.8 require only minor modifications of the techniques developed in [16] but I prefer to deduce them from a general theory of

Noetherian differential rings of definable functions which I shall develop in section 4. I shall prove 2.7 and 2.8 in sections 5 and 7 respectively. These proofs do not depend on the fact that the F_i's have continuations to an open set containing $[0,1]^m$ and so we can deduce a modified model completeness result for *unrestricted* Pfaffian functions in situations where 2.9 holds trivially (e.g. when $\tilde{K}$ is a cofinal extension of $\tilde{k}$) and I conclude section 7 with such a result. Section 8 is devoted to a proof of 2.9 which needs van den Dries's work on the model theory of finitely sub-analytic sets ([13]). I shall also rely heavily throughout most of this paper on Khovanskii's work on Pfaffian functions ([6]). The exact results needed from these two papers as well as some immediate corollaries are described in the next section.

3. Results of Khovanskii and van den Dries

3.1. Proposition (Khovanskii [6]). *Suppose that $h_1, \ldots, h_l$ is any Pfaffian chain of functions on $\mathbb{R}^{m+n}$. Suppose further that $g_1, \ldots, g_m \in \mathbb{R}[x_1, \ldots, x_{m+n}, h_1, \ldots, h_l]$ (where $x_i : \mathbb{R}^{m+n} \to \mathbb{R}$ denotes the ith projection function). Then there is a natural number N such that for any $Q \in \mathbb{R}^n$ the set*

$$\left\{ P \in \mathbb{R}^m : g_1(P,Q) = \cdots = g_m(P,Q) = 0 \text{ and } \det\left(\frac{\partial(g_1, \ldots, g_m)}{\partial(x_1, \ldots, x_m)}\right)(P,Q) \neq 0 \right\}$$

contains at most N elements.

The reader may have already observed that some such result has to be true if we are to have any chance of proving 2.6. In fact we need a version of 3.1 where $\mathbb{R}^{m+n}$ is replaced with sets of the form $\prod_{i=1}^{m+n} J_i$ where each J_i is either $\mathbb{R}$ or $(0,1)$. That such a modification holds can be seen by inspecting Khovanskii's proof. Alternatively we may argue as follows.

Suppose $h_1, \ldots, h_l$ is a Pfaffian chain on $\prod_{i=1}^{m+n} J_i$. Define the functions $\alpha_i, \beta_i : \mathbb{R}^{m+n} \to \mathbb{R}$ (for $i = 1, \ldots, m+n$) by

$$\alpha_i(\vec{x}) = \begin{cases} 1 & \text{if } J_i = \mathbb{R}, \\ \frac{1}{\pi(1+x_i^2)} & \text{if } J_i = (0,1), \end{cases}$$

$$\beta_i(\vec{x}) = \begin{cases} x_i & \text{if } J_i = \mathbb{R}, \\ \frac{1}{2} + \frac{1}{\pi} \cdot \tan^{-1}(x_i) & \text{if } J_i = (0,1). \end{cases}$$

Then clearly the map $\vec{\beta} : \vec{x} \mapsto \langle \beta_1(\vec{x}), \ldots, \beta_{m+n}(\vec{x}) \rangle$ is an analytic bijection from $\mathbb{R}^{m+n}$ to $\prod_{i=1}^{m+n} J_i$ so the functions $h_i \circ \vec{\beta} : \mathbb{R}^{m+n} \to \mathbb{R}$ (for $i = 1, \ldots, l$) are defined and analytic throughout $\mathbb{R}^{m+n}$. Further, by the chain rule (and see also example B), the sequence $\alpha_1, \beta_1, \ldots, \alpha_{m+n}, \beta_{m+n}, h_1 \circ \vec{\beta}, \ldots, h_l \circ \vec{\beta}$ is a Pfaffian chain on $\mathbb{R}^{m+n}$.

Let M denote the ring of functions (defined on $\prod_{i=1}^{m+n} J_i$) $\mathbb{R}[x_1, \ldots, x_{m+n}, h_1, \ldots, h_l]$ and M^* the ring of functions (defined on $\mathbb{R}^{m+n}$) $\mathbb{R}[x_1, \ldots, x_{m+n}, \alpha_1, \ldots, \alpha_{m+n}, \beta_1, \ldots, \beta_{m+n}, h_1 \circ \vec{\beta}, \ldots, h_l \circ \vec{\beta}]$. Suppose $g_1, \ldots, g_m \in M$, $P \in \prod_{i=1}^{m} J_i$, $Q \in \prod_{i=m+1}^{m+n} J_i$, $g_1(P,Q) = \cdots = g_m(P,Q) = 0$ and $\det(\frac{\partial(g_1, \ldots, g_m)}{\partial(x_1, \ldots, x_m)})(P,Q) \neq 0$. Then clearly $g_1 \circ \vec{\beta}, \ldots, g_m \circ \vec{\beta} \in M^*$ and $g_1 \circ \vec{\beta}(P',Q') = \cdots = g_m \circ \vec{\beta}(P',Q') = 0$

where $\langle P', Q'\rangle = \vec{\beta}^{-1}(P,Q)$. Further, as an easy calculation using the chain rule shows, we have

$$(*) \qquad \frac{\partial(g_1 \circ \vec{\beta}, \ldots, g_m \circ \vec{\beta})}{\partial(x_1, \ldots, x_m)}(P', Q') = \frac{\partial(g_1, \ldots, g_m)}{\partial(x_1, \ldots, x_m)}(P, Q) \times \frac{\partial(\beta_1, \ldots, \beta_m)}{\partial(x_1, \ldots, x_m)}(P', Q').$$

Now

$$\det\left(\frac{\partial(\beta_1, \ldots, \beta_m)}{\partial(x_1, \ldots, x_m)}\right)(P', Q') = \prod_{i=1}^{m} \alpha_i(P', Q') \neq 0,$$

so the left hand side of (*) has non-zero determinant. Since P' depends only on P and Q' only on Q we can now use 3.1 to conclude:

3.2. Corollary. *Proposition 3.1 holds with* $\prod_{i=1}^{m+n} J_i$ *in place of* $\mathbb{R}^{m+n}$ *where each* J_i *is either* $\mathbb{R}$ *or* $(0,1)$. □

The fact that the upper bound N is independent of Q here can now be used to transfer this result to the situation we are interested in. The easy formal details required for the proof of the following result are left to the reader.

3.3. Corollary. *Suppose* $n, r_1, r_2 \in \mathbb{N}$ *and that* $\vec{\sigma}$ *is an* $(n, r_1 + r_2)$*-sequence. Suppose further that* $\tilde{k}, \tilde{K} \models \tilde{T}$, $\tilde{k} \subseteq \tilde{K}$, *and that* $g_1, \ldots, g_{r_1} \in M^{r_1+r_2}(\tilde{k}, \tilde{K}, \vec{\sigma})$. *Then there is* $N \in \mathbb{N}$ *such that for each* $Q \in K^{r_2}$ *the set*

$$\left\{P \in K^{r_1} : \langle P, Q\rangle \in D^{r_1+r_2}(\vec{\sigma}, K),\ g_1(P,Q) = \cdots = g_{r_1}(P,Q) = 0 \right.$$
$$\left. \text{and } \det\left(\frac{\partial(g_1, \ldots, g_{r_1})}{\partial(x_1, \ldots, x_{r_1})}\right)(P,Q) \neq 0\right\}$$

contains at most N *elements.*

I now turn to a result of van den Dries concerning sets and functions definable in the structure $\mathbb{R}_{an}$ (cf. section 1). The result we need can be found in [13] where it is formulated in terms of so-called "finitely sub-analytic sets". Since these are exactly the sets definable in $\mathbb{R}_{an}$ (see [3]) we may reformulate the result as follows

3.4. Proposition (van den Dries [13]). (i) $\mathbb{R}_{an}$ *is 0-minimal (i.e. every subset of* $\mathbb{R}$ *definable (with parameters) in* $\mathbb{R}_{an}$ *is a finite union of open intervals and points).*

(ii) *If* $e \in \mathbb{R}$ *and* $f : (e, \infty) \to \mathbb{R}$ *is any function definable (with parameters) in* $\mathbb{R}_{an}$, *then there exists* $d \geq e$ *such that on* (d, ∞) *the function* f *may be represented by a convergent Puiseux expansion:*

$$(*) \qquad f(x) = \sum_{i=p}^{\infty} a_i \cdot x^{-i/q}$$

where $q \in \mathbb{N}$, $q \geq 1$, $p \in \mathbb{Z}$, $a_i \in \mathbb{R}$ *(for* $i \in \mathbb{Z}$, $i \geq p$*) and* $a_p \neq 0$ *provided* f *is not (eventually) identically zero.*

Now, as pointed out in section 1, every subset of $\mathbb{R}^n$ (for any n) definable in the structure $\tilde{\mathbb{R}}$ is definable in $\mathbb{R}_{an}$. Hence 3.4 holds with $\tilde{\mathbb{R}}$ in place of $\mathbb{R}_{an}$. I need the following consequence of this fact.

3.5. Corollary. *Suppose $\widetilde{K} \models \widetilde{T}$, $e \in K$ and $g : (e, \infty) \to K$ is a $\widetilde{K}$ definable function which is not eventually identically zero. Then there is a rational number s and a non-zero element $a \in K$ such that $g(x)x^s \to a$ as $x \to \infty$ (in the sense of $\widetilde{K}$).*

Proof. Suppose $\phi(\vec{b}, x, y)$ defines the graph of g in $\widetilde{K}$ where $\phi(\vec{z}, x, y)$ is an $\widetilde{L}$-formula. Let $\psi(\vec{z})$ be the $\widetilde{L}$-formula

$$\exists u(\forall x > u\ \exists! y\ \phi(\vec{z}, x, y) \wedge \forall x > u\ \exists w > x\ \neg\phi(\vec{z}, w, 0)),$$

and note that $\widetilde{K} \models \psi(\vec{b})$.

Now suppose that $\vec{\alpha}$ is a tuple of reals such that $\widetilde{\mathbb{R}} \models \psi(\vec{\alpha})$ and let $f_{\vec{\alpha}} : (\beta, \infty) \to \mathbb{R}$ be the function defined by $\phi(\vec{\alpha}, x, y)$ in $\widetilde{\mathbb{R}}$ (for suitable $\beta \in \mathbb{R}$). By (3.4)(ii) $f_{\vec{\alpha}}$ may be represented in the form (*) for sufficiently large x, and we clearly have $a_p \neq 0$ and $f_{\vec{\alpha}}(x)x^{p/q} \to a_p$ as $x \to \infty$.

Now by elementary real analysis the series (*) may be differentiated term by term to obtain the convergent representation

$$f'_{\vec{\alpha}}(x) = \sum_{i=p}^{\infty} -\frac{ia_i}{q} x^{(-i/q)-1}$$

(for sufficiently large $x \in \mathbb{R}$), and we have that $f'_{\vec{\alpha}}(x)x^{(p/q)+1} \to -\frac{pa_p}{q}$ as $x \to \infty$. It follows that $\lim_{x\to\infty} -(f'_{\vec{\alpha}}(x)x)/(f_{\vec{\alpha}}(x))$ exists and equals p/q. By using the usual ε-δ definition of derivatives and limits we may clearly write down an $\widetilde{L}$-formula $\chi(\vec{z}, y)$ expressing (in $\widetilde{\mathbb{R}}$) : "$\psi(\vec{z})$ and $\lim_{x\to\infty} -(f'_{\vec{z}}(x)x)/f_{\vec{z}}(x) = y$." We have shown that the $\widetilde{L}$-formula $\exists\vec{z}\chi(\vec{z}, y)$ defines in $\widetilde{\mathbb{R}}$ a set of rationals, and since, by the comments above, $\widetilde{\mathbb{R}}$ is a 0-minimal structure, it follows that this set is finite, say $\{s_1, \ldots, s_n\}$. We have also shown that the $\widetilde{L}$-sentence expressing: "$\forall\vec{z}(\psi(\vec{z}) \to \bigvee_{i=1}^{n}(\lim_{x\to\infty} f_{\vec{z}}(x) \cdot x^{s_i}$ exists and is non-zero))" is true in $\widetilde{\mathbb{R}}$, and hence in $\widetilde{K}$. Since $\widetilde{K} \models \psi(\vec{b})$ and $f_{\vec{b}} = g$ (eventually in $\widetilde{K}$) the result follows. □

4. Differentiable germs in arbitrary expansions of $\overline{\overline{\mathbb{R}}}$

Throughout this section $\overline{\overline{\mathbb{R}}}$ denotes *any* expansion of the ordered field $\overline{\mathbb{R}}$, $\overline{\overline{L}}$ its language and $\overline{\overline{T}}$ its theory. We employ conventions analogous to those set out at the beginning of section 2 concerning models of $\overline{\overline{T}}$.

Let $\overline{\overline{K}} \models \overline{\overline{T}}$. As we have already seen many local notions from topology and calculus can be immediately transferred from $\overline{\overline{\mathbb{R}}}$ to $\overline{\overline{K}}$ and I will assume the reader is familiar with this process. It should always be clear how (and where) these notions are to be interpreted. The implicit function theorem, however, requires some comment.

Suppose $r, m \in \mathbb{N}$, $r, m \geq 1$ and $\langle P, Q\rangle = \langle p_1, \ldots, p_r, q_1, \ldots, q_m\rangle \in K^{r+m}$. Let U be a definable (i.e. K-definable with parameters) open neighbourhood of $\langle P, Q\rangle$ and suppose $f_1, \ldots, f_m : U \to K$ are definable functions which are infinitely differentiable throughout U. Suppose further that $\langle P, Q\rangle$ is a *non-singular zero* of $f_1, \ldots, f_m$ *with respect to* $x_{r+1}, \ldots, x_{r+m}$. This means, by definition, that

$f_i(P,Q) = 0$ for $i = 1, \ldots, m$ and that the determinant of the Jacobian matrix

$$\Delta = \Delta(x_1, \ldots, x_{r+m}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_{r+1}} & \cdots & \frac{\partial f_1}{\partial x_{r+m}} \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_{r+1}} & \cdots & \frac{\partial f_m}{\partial x_{r+m}} \end{pmatrix}$$

does not vanish at $\langle P, Q \rangle$.

Now if $\overline{\overline{K}} = \overline{\overline{\mathbb{R}}}$ we can apply the implicit function theorem (see e.g. [4]) to obtain open neighbourhoods V_1 of P (in K^r) and V_2 of Q (in K^m) such that

4.1. $V_1 \times V_2 \subseteq U$.

4.2. For each $\vec{x} \in V_1$ there is a unique point $\langle y_1, \ldots, y_m \rangle = \langle y_1(\vec{x}), \ldots, y_m(\vec{x}) \rangle \in V_2$ such that $f_i(\vec{x}, \vec{y}) = 0$ for $i = 1, \ldots, m$, and this point satisfies $J(\vec{x}, \vec{y}) \neq 0$.

4.3. The functions $y_i : V_1 \to K$ (for $i = 1, \ldots, m$) are infinitely differentiable and for each $l = 1, \ldots, r$ and $\vec{x} \in V_1$

$$\begin{pmatrix} \frac{\partial y_1}{\partial x_l} \\ \vdots \\ \frac{\partial y_m}{\partial x_l} \end{pmatrix} = -\Delta^{-1} \begin{pmatrix} \frac{\partial f_1}{\partial x_l} \\ \vdots \\ \frac{\partial f_m}{\partial x_l} \end{pmatrix}$$

where the right hand side is evaluated at the point $\langle \vec{x}, y_1(\vec{x}), \ldots, y_m(\vec{x}) \rangle$.

We require 4.1–4.3 to hold for arbitrary $\overline{\overline{K}}$ and that this is the case can be argued as follows. Firstly, the existence of V_1 and V_2 satisfying 4.1 and 4.2 can be guaranteed since we may suppose they are *box* neighbourhoods (i.e. of the form $\{\langle z_1, \ldots, z_t \rangle \in K^t : |\alpha_i - z_i| < \varepsilon$ for $i = 1, \ldots, t\}$ for some $\alpha_1, \ldots, \alpha_t$, $\varepsilon \in K$ with $\varepsilon > 0$). Having fixed such V_1 and V_2 the uniqueness in 4.2 guarantees that the y_i's are definable functions which, by transfer, are continuously differentiable throughout V_1 and satisfy the formula in 4.3. But this formula implies (simply by arguing in $\overline{\overline{K}}$) that the y_i's are infinitely differentiable throughout V_1.

I now turn to germs of differentiable definable functions in (an arbitrary given model of $\overline{\overline{T}}$) $\overline{\overline{K}}$.

4.4. Definition. Let $n \in \mathbb{N}$, $n \geq 1$.

(i) A *neighbourhood system* (n.s.) *in* K^n is a non-empty collection of non-empty, definable open subsets of K^n which is closed under (finite) intersection.

(ii) For $\mathfrak{G}$ a n.s. in K^n, $\mathfrak{D}^{(n)}(\mathfrak{G})^-$ denotes the set of all pairs $\langle f, U \rangle$ where $U \in \mathfrak{G}$ and $f : U \to K$ is an infinitely differentiable definable function.

(iii) For $\langle f_1, U_1 \rangle$, $\langle f_2, U_2 \rangle \in \mathfrak{D}^{(n)}(\mathfrak{G})^-$, $\langle f_1, U_1 \rangle \sim \langle f_2, U_2 \rangle$ means that there is some $U \in \mathfrak{G}$ with $U \subseteq U_1 \cap U_2$ such that $f_1(\vec{x}) = f_2(\vec{x})$ for all $\vec{x} \in U$. This is clearly an equivalence relation and the equivalence class of $\langle f, U \rangle$ ($\in \mathfrak{D}^{(n)}(\mathfrak{G})^-$) is denoted $[f, U]$.

(iv) The set of equivalence classes, or *germs*, is denoted $\mathfrak{D}^{(n)}(\mathfrak{G})$.

Clearly $\mathfrak{D}^{(n)}(\mathfrak{G})$ is naturally a differential (unital) ring and I continue to write $\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$ for the obvious induced derivatives on $\mathfrak{D}^{(n)}(\mathfrak{G})$.

4.5. Lemma. *Let $n \in \mathbb{N}$, $n \geq 1$ and suppose $\mathfrak{G}$ is a n.s. in K^n. Suppose further that M is a subring of $\mathfrak{D}^{(n)}(\mathfrak{G})$ closed under differentiation and that I is a finitely generated ideal of M also closed under differentiation. Let* $\{[g_1, U_1], \ldots, [g_s, U_s]\}$

be any finite set of generators for I and set $Z = \{P \in \bigcap_{i=1}^{s} U_i : g_i(P) = 0$ for $i = 1, \ldots, s\}$. Then for some $U \in \mathfrak{G}$, $U \cap Z$ is an open (definable) subset of K^n.

Proof. Since I is closed under differentiation and $\mathfrak{G}$ under finite intersection there exist $U \in \mathfrak{G}$ and definable functions $a_{i,j}^{(r)}$ $(1 \le i, j \le s, 1 \le r \le n)$ such that $g_1, \ldots, g_s$ and the $a_{i,j}^{(r)}$'s all have domains containing U, are infinitely differentiable throughout U, and satisfy the equations

$$(*) \qquad \frac{\partial g_i}{\partial x_r} = \sum_{j=1}^{s} a_{i,j}^{(r)} \cdot g_j \qquad (1 \le i \le s, 1 \le r \le n)$$

on U.

I claim that $U \cap Z$ is open in K^n. For suppose $P = \langle p_1, \ldots, p_n \rangle \in U \cap Z$ and let U_0 be an open box neighbourhood of P contained in U. It suffices to show that each g_i vanishes on U_0 so suppose that this is not the case. Since each g_i certainly vanishes at P we can clearly find $Q, S \in U_0$ such that $g_i(Q) = 0$ for all $i = 1, \ldots, s$ and $g_i(S) \neq 0$ for some $i = 1, \ldots, s$ and such that Q and S differ in exactly one coordinate, which we suppose for convenience is the first. Say $Q = \langle q_1, q_2, \ldots, q_n \rangle$ and $S = \langle q_1', q_2, \ldots, q_n \rangle$ where $q_1 \neq q_1'$. Let (a, b) be an open interval in K such that $q_1, q_2' \in (a, b)$ and $(a, b) \times \{\langle q_2, \ldots, q_n \rangle\} \subseteq U_0$. For any definable function $f : U_0 \to K$ let $\overline{f}$ be the result of substituting q_i for x_i in f for $i = 2, \ldots, n$. Then by $(*)$ (for $r = 1$) we have

$$\begin{pmatrix} \overline{g}_1' \\ \vdots \\ \overline{g}_s' \end{pmatrix} = A \begin{pmatrix} \overline{g}_1 \\ \vdots \\ \overline{g}_s \end{pmatrix}$$

for all $x_1 \in (a, b)$, where A is the matrix $(\overline{a_{i,j}^{(1)}}(x_1))_{1 \le i,j \le s}$ and where $'$ denotes $\frac{d}{dx_1}$.

We now transfer this situation to $\overline{\overline{\mathbb{R}}}$ (by quantifying out parameters) and obtain a real interval (c, d), continuously differentiable functions $h_i, b_{i,j} : (c, d) \to \mathbb{R}$ (for $1 \le i, j \le s$) and points $\alpha, \beta \in (c, d)$ such that (setting $B = (b_{i,j}(x))_{1 \le i,j \le s}$)

$$\begin{pmatrix} h_1' \\ \vdots \\ h_s' \end{pmatrix} = B \begin{pmatrix} h_1 \\ \vdots \\ h_s \end{pmatrix} \quad \text{for all } x \in (c, d),$$

$$h_i(\alpha) = 0 \quad \text{for all } i = 1, \ldots, s,$$

and

$$h_i(\beta) \neq 0 \quad \text{for some } i = 1, \ldots, s.$$

The theory of linear differential equations (see e.g. [9], Theorem 11.4.1 and its proof) now tells us that for all $x \in (c, d)$

$$\begin{pmatrix} h_1(x) \\ \vdots \\ h_s(x) \end{pmatrix} = E(x)^{-1} E(\alpha) \begin{pmatrix} h_1(\alpha) \\ \vdots \\ h_s(\alpha) \end{pmatrix}$$

for some $s \times s$ matrix E of functions on (c, d) which is invertible for all $x \in (c, d)$. Setting $x = \beta$ here gives the required contradiction. □

4.6. *Notation*. For $n \in \mathbb{N}$, $n \geq 1$ and $P \in K^n$, $\mathfrak{G}_P$ denotes the set of all definable open neighbourhoods of P. It is clearly a n.s. in K^n. Write $\mathfrak{D}^{(n)}(P)^-$ and $\mathfrak{D}^{(n)}(P)$ for $\mathfrak{D}^{(n)}(\mathfrak{G}_P)^-$ and $\mathfrak{D}^{(n)}(\mathfrak{G}_P)$ respectively. If $g \in \mathfrak{D}^{(n)}(P)$, say $g = [f, U]$, $g(P)$ denotes the element $f(P)$ of K. It is clearly well defined. Finally, depending on convenience, $d_P g$ or $d_P f$ denotes $\langle \frac{\partial f}{\partial x_1}(P), \ldots, \frac{\partial f}{\partial x_n}(P) \rangle$ considered as an element of the K-vector space K^n.

I now wish to return to the situation of 4.1–4.3 to discuss some classical results applied to the present context. So let $r, m, P, Q, f_1, \ldots, f_m, U$ be as in the discussion of the implicit function theorem at the beginning of this section. Let $n = r+m$. Define $\phi_1, \ldots, \phi_n$ by $\phi_i(\vec{x}) = x_i$ for $i = 1, \ldots, r$, and $\phi_i(\vec{x}) = y_{i-r}(\vec{x})$ for $i = r+1, \ldots, n$ (cf. 4.2, 4.3) where $\vec{x} = \langle x_1, \ldots, x_r \rangle$. These functions are defined and are infinitely differentiable on a set in $\mathfrak{G}_P$ (namely V_1) and hence determine germs in $\mathfrak{D}^{(r)}(P)$. Notice also that $\langle \phi_1(P), \ldots, \phi_n(P) \rangle = \langle P, Q \rangle$ so we have an induced mapping $\hat{}: \mathfrak{D}^{(n)}(P,Q) \to \mathfrak{D}^{(r)}(P)$ defined (on functions) by $\hat{f}(\vec{x}) = f(\phi_1(\vec{x}), \ldots, \phi_n(\vec{x}))$, $\vec{x} \in W'$, where $\langle f, W \rangle \in \mathfrak{D}^{(n)}(P,Q)^-$ and $W' = \{\vec{x} \in V_1 : \langle \phi_1(\vec{x}), \ldots, \phi_n(\vec{x}) \rangle \in W\}$. (Clearly $W' \in \mathfrak{G}_P$.) This mapping is clearly a (unital) ring homomorphism and its kernel consists exactly of those germs $[f, W]$ such that f vanishes on $V \cap Z$ for some $V \in \mathfrak{G}_{P,Q}$ (with $V \subseteq W$), where $Z = \{\langle x_1, \ldots, x_n \rangle \in U : f_i(x_1, \ldots, x_n) = 0$ for $i = 1, \ldots, m\}$. In particular $\widehat{[f_i, U]} = 0$ (in $\mathfrak{D}^{(r)}(P)$) for $i = 1, \ldots, m$, so $\frac{\partial \hat{f}_i}{\partial x_j} = 0$ (in $\mathfrak{D}^{(r)}(P)$) for $i = 1, \ldots, m$ and $j = 1, \ldots, r$.

4.7. Lemma. *With the above notation we have that for all $g \in \mathfrak{D}^{(n)}(P,Q)$, the sequence of vectors $d_{P,Q}f_1, \ldots, d_{P,Q}f_m$, $d_{P,Q}g$ is linearly independent over K if and only if $d_P \hat{g} \neq 0$ (in K^r).*

Proof. Notice first that the sequence $d_{P,Q}f_1, \ldots, d_{P,Q}f_m$ is certainly linearly independent since $J(P,Q) \neq 0$ (cf. the discussion at the beginning of this section). Write $g = [f_{m+1}, W]$.

Suppose that $\sum_{i=1}^{m+1} a_i \cdot d_{P,Q}f_i = 0$ with not all the a_i's zero. Then $a_{m+1} \neq 0$. By the chain rule

$$(*) \qquad \frac{\partial \hat{f}_i}{\partial x_j}(P) = \sum_{l=1}^{n} \frac{\partial f_i}{\partial x_l}(P,Q) \cdot \frac{\partial \phi_l}{\partial x_j}(P)$$

for $j = 1, \ldots, r$, $i = 1, \ldots, m+1$. Now by the remark before the lemma, our assumption and $(*)$ we have

$$\frac{\partial \hat{f}_{m+1}}{\partial x_j}(P) = a_{m+1}^{-1} \sum_{i=1}^{m+1} a_i \frac{\partial \hat{f}_i}{\partial x_j}(P)$$
$$= a_{m+1}^{-1} \cdot \sum_{l=1}^{n} \left(\frac{\partial \phi_l}{\partial x_j}(P) \sum_{i=1}^{m+1} a_i \frac{\partial f_i}{\partial x_l}(P,Q) \right) = 0$$

for $j = 1, \ldots, r$, as required.

Suppose now that the sequence $d_{P,Q}f_1, \ldots, d_{P,Q}f_{m+1}$ is linearly independent. Let A denote the $n \times (m+1)$ matrix (over K) with columns $d_{P,Q}f_i$ for $1 \leq i \leq m+1$. Then A determines a K-linear map from K^n onto K^{m+1} with kernel of dimension $n - (m+1) = r - 1$. Moreover, by $(*)$ and the remark before the lemma

$$\left\langle \frac{\partial \phi_1}{\partial x_j}(P), \ldots, \frac{\partial \phi_n}{\phi x_j}(P) \right\rangle A = \left\langle 0, \ldots, 0, \frac{\partial \hat{f}_{m+1}}{\partial x_j}(P) \right\rangle \quad \text{for } j = 1, \ldots, r.$$

But the sequence of vectors $\langle\langle \frac{\partial \phi_1}{\partial x_j}(P), \ldots, \frac{\partial \phi_n}{\partial x_j}(P)\rangle : 1 \leq j \leq r\rangle$ is linearly independent (since $\frac{\partial \phi_i}{\partial x_j} = \delta_{i,j}$ for $1 \leq i, j \leq r$), so not all of them are in Ker(A). Thus $\frac{\partial f_{m+1}}{\partial x_j}(P) \neq 0$ for some $j = 1, \ldots, r$, as required. □

4.8. Definition. Let $n, s \in \mathbb{N}$, $n \geq 1$. Suppose $g_1, \ldots, g_s$ are infinitely differentiable definable functions with domains open in K^n. Then

$$V(g_1, \ldots, g_s) \stackrel{\text{def}}{=} \left\{ Q \in \bigcap_{i=1}^{s} \operatorname{dom}(g_i) : g_i(Q) = 0 \text{ for } i = 1, \ldots, s \right\},$$

and

$$V^{ns}(g_1, \ldots, g_s) \stackrel{\text{def}}{=} \{Q \in V(g_1, \ldots, g_s) : \langle d_Q g_i : 1 \leq i \leq s\rangle \text{ is linearly independent}\}.$$

(For $s = 0$, $V = V^{ns} = K^n$.)

The following theorem will be used repeatedly throughout this paper.

4.9. Theorem. *Let $n \in \mathbb{N}$, $n \geq 1$, $P_0 \in K^n$, and suppose M is a Noetherian (unital) subring of $\mathfrak{D}^{(n)}(P_0)$ closed under differentiation. Let $m \in \mathbb{N}$ and suppose $[f_i, U_i] \in M$ for $i = 1, \ldots, m$. Suppose further that $P_0 \in V^{ns}(f_1, \ldots, f_m)$. Then one of the following is true:*

(i) *$n = m$, or*
(ii) *$m < n$ and for any $[h, W] \in M$ with $h(P_0) = 0$, h vanishes on $U \cap V^{ns}(f_1, \ldots, f_m)$ for some $U \in \mathfrak{G}_{P_0}$ (with $U \subseteq W$), or*
(iii) *$m < n$ and for some $[h, W] \in M$, $P_0 \in V^{ns}(f_1, \ldots, f_m, h)$.*

Proof. If $m \neq n$, then $m < n$ since $P_0 \in V^{ns}(f_1, \ldots, f_m)$. Say $r + m = n$ where $1 \leq r \leq n$.

Now since $\langle d_{P_0} f_i : 1 \leq i \leq m\rangle$ is linearly independent there exists an m-element subset of $\{1, \ldots, n\}$, S say, such that the matrix $(\frac{\partial f_i}{\partial x_j}(P_0))_{1 \leq i \leq m, j \in S}$ is non-singular. There is no harm here in supposing that $S = \{r + 1, \ldots, n\}$, so if we denote by λ the function

$$\langle x_1, \ldots, x_n\rangle \mapsto \det\left(\frac{\partial f_i}{\partial x_j}(x_1, \ldots, x_n)\right)_{1 \leq i \leq m, r+1 \leq j \leq n},$$

then clearly $[\lambda, U_0] \in M$ (for some $U_0 \in \mathfrak{G}_{P_0}$) and $[\lambda, U_0]$ ($= \Lambda$, say) is invertible in $\mathfrak{D}^{(n)}(P_0)$. Let $M^* = M[\Lambda^{-1}]$. Now write P_0 as $\langle P, Q\rangle$, where $P \in K^r$ and $Q \in K^m$, and consider the map $\hat{}: \mathfrak{D}^{(n)}(P, Q) \to \mathfrak{D}^{(r)}(P)$ described above. Clearly $\widehat{M}^*$, the image of M^* under $\hat{}$, is a Noetherian (unital) subring of $\mathfrak{D}^{(r)}(P)$ and is closed under differentiation. This latter fact follows easily from the chain rule and 4.3 (this is why we consider M^*—the entries of Δ^{-1} (in 4.3) determine germs in M^*, but not necessarily in M). Now let I denote the ideal $\{g \in \widehat{M}^* : g(P) = 0\}$ of $\widehat{M}^*$.

Case 1. $I = \{0\}$. Suppose $[h, W] \in M$ and $h(P_0) = 0$. Let $g = [h, W]$. Then $g(P_0) = 0$ so $\hat{g}(P) = 0$, i.e. $\hat{g} \in I$. Hence $\hat{g} = 0$ in $\mathfrak{D}^{(r)}(P)$. The conclusion of (ii) in the statement of the theorem now follows from the comments before 4.7.

Case 2. $I \neq \{0\}$. Since I is finitely generated it clearly follows from 4.5 (with $\mathfrak{G} = \mathfrak{G}_P, M = \widehat{M}^*$) that I is not closed under differentiation. Hence there is some

$g \in M^*$ such that $\hat{g} \in I$, i.e. $\hat{g}(P) = 0$ so $g(P_0) = 0$, and some i with $1 \leq i \leq r$ such that

$$\frac{\partial \hat{g}}{\partial x_i} \notin I, \quad \text{i.e. } \frac{\partial \hat{g}}{\partial x_i}(P) \neq 0.$$

Now for some $s \in \mathbb{N}$, $\Lambda^s \cdot g \in M$. Let $f = \Lambda^s \cdot g$. Then $f(P_0) = 0$ and, further,

$$\begin{aligned}\frac{\partial \hat{f}}{\partial x_i}(P) &= (s\hat{\Lambda}^{s-1} \cdot \frac{\partial \hat{\Lambda}}{\partial x_i} \cdot \hat{g})(P) + \left(\hat{\Lambda}^s \cdot \frac{\partial \hat{g}}{\partial x_i}\right)(P) \\ &= \hat{\Lambda}^s(P) \cdot \frac{\partial \hat{g}}{\partial x_i}(P) \neq 0.\end{aligned}$$

Thus $d_P\hat{f} \neq 0$ and hence, by 4.7, the conclusion of (iii) in the statement of the theorem holds (with $[h, W] = f$). □

Before leaving this section I need to mention one more result that follows (either directly, or by using 4.7) from the corresponding classical theorem in elementary calculus. (It will be used in the next section to find "definable points" on the zero sets of Pfaffian functions (cf. 2.7).) The easy details of the transfer (from $\overline{\overline{\mathbb{R}}}$) required for the proof are left to the reader.

4.10. Proposition. *Suppose $n, s, g_1, \ldots, g_s$ are as in 4.8, $s < n$ and $P \in V^{ns}(g_1, \ldots, g_s)$. Let $[g, W] \in \mathfrak{D}^{(n)}(P)$ and suppose that for some $U \in \mathfrak{G}_P$ (with $U \subseteq W \cap \bigcap_{i=1}^{s} \mathrm{dom}(g_i)$) we have $g(\vec{x}) \geq g(P)$ for all $\vec{x} \in U \cap V^{ns}(g_1, \ldots, g_s)$ (i.e. P is a local minimum of g on $V^{ns}(g_1, \ldots, g_s)$). Then the sequence of vectors $\langle d_P g_1, \ldots, d_P g_s, d_P g\rangle$ is linearly dependent.*

5. Definable points on components and the proof of Lemma 2.7

I continue to use the notation of section 4. In particular, $\overline{\overline{K}}$ denotes an arbitrary model of $\overline{\overline{T}}$.

Fix $n \in \mathbb{N}$, $n \geq 1$, and let U be a definable open subset of K^n. Clearly $\{U\}$ is a n.s. in K^n and we may safely identify both $\mathfrak{D}^{(n)}(\{U\})^-$ and $\mathfrak{D}^{(n)}(\{U\})$ with the differential unital ring of all definable, infinitely differentiable functions from U to K, which we denote by $\mathfrak{D}^{(n)}(U)$. If $P \in U$, then clearly the map $R_P : \mathfrak{D}^{(n)}(U) \to \mathfrak{D}^{(n)}(P) : f \mapsto [f, U]$ is a differential ring homomorphism which need be neither injective nor surjective in general. It is, however, clearly injective on the unital subring generated by the n projection functions (restricted to U) and I use the usual notation, $\mathbb{Z}[x_1, \ldots, x_n]$, for this subring and for its R_P-image in $\mathfrak{D}^{(n)}(P)$.

5.1. Theorem. *With the above notation let M be a Noetherian subring of $\mathfrak{D}^{(n)}(U)$ which contains $\mathbb{Z}[x_1, \ldots, x_n]$ and which is closed under differentiation. Let $f \in M$ and suppose that S is a non-empty definable subset of $V(f)$ which is both open in $V(f)$ (in the subspace topology) and closed in K^n. Then there exist $f_1, \ldots, f_n \in M$ such that $S \cap V^{ns}(f_1, \ldots, f_n) \neq \varnothing$.*

Proof. For each $Q \in S$ let I_Q be the ideal $\{g \in M : g(Q) = 0\}$ of M. Since M is Noetherian we may choose $P \in S$ such that I_P is maximal in $\{I_Q : Q \in S\}$. Let $\{g_1, \ldots, g_N\}$ be a finite generating set for I_P and set $g = \sum_{i=1}^{N} g_i^2$. Then $P \in V(g) \cap S$ and, further, we have

$$(*) \qquad I_Q = I_P \quad \text{for any } Q \in V(g) \cap S.$$

Now choose m maximal so that for some $f_1, \dots, f_m \in M$, $P \in V^{ns}(f_1, \dots, f_m)$. If $m = n$ we are done, so suppose, for a contradiction, that $m < n$ and fix such $f_1, \dots, f_m$.

Claim 1. $V(g) \cap S \subseteq V^{ns}(f_1, \dots, f_m)$.

Proof. Since $P \in V^{ns}(f_1, \dots, f_m)$ we have $f_1, \dots, f_m \in I_P$ and $\det(E) \notin I_P$ where E is some $m \times m$ submatrix of the $m \times n$ matrix with rows $\langle \frac{\partial f_i}{\partial x_1}, \dots, \frac{\partial f_i}{\partial x_n} \rangle$ for $1 \le i \le m$ (note that $\det(E) \in M$ since M is closed under differentiation). Hence by $(*)$ we have that for any $Q \in V(g) \cap S$, $f_1, \dots, f_m \in I_Q$ and $\det(E) \notin I_Q$ which immediately implies that $Q \in V^{ns}(f_1, \dots, f_m)$, as required.

Claim 2. *Let $Q \in V(g) \cap S$ and $h \in M$. Then $Q \notin V^{ns}(f_1, \dots, f_m, h)$.*

Proof. Suppose $Q \in V^{ns}(f_1, \dots, f_m, h)$. Then arguing as in the proof of Claim 1 we would have $P \in V^{ns}(f_1, \dots, f_m, h)$ which contradicts the maximality of m.

Claim 3. *Let $Q \in V(g) \cap S$. Then there exists $W \in \mathfrak{G}_Q$ (with $W \subseteq U$) such that $W \cap V(g) \cap S = W \cap V^{ns}(f_1, \dots, f_m)$.*

Proof. Since $g \in I_P$ we have by $(*)$ that $g(Q) = 0$. Hence, by Claim 2 and 4.9 (applied to the image of M under the map R_Q), there exists $W' \in \mathfrak{G}_Q$ (with $W' \subseteq U$) such that g vanishes on $W' \cap V^{ns}(f_1, \dots, f_m)$. It follows that every element of I_P, and in particular f, vanishes on $W' \cap V^{ns}(f_1, \dots, f_m)$. Thus $W' \cap V^{ns}(f_1, \dots, f_m) \subseteq W' \cap V(g) \cap V(f)$. But S is open in $V(f)$ (by hypothesis) so for some $W'' \in \mathfrak{G}_Q$, $W'' \cap S = W'' \cap V(f)$. Thus $W \cap V^{ns}(f_1, \dots, f_m) \subseteq W \cap V(g) \cap S$ where $W = W' \cap W''$. Claim 3 now follows from Claim 1.

Claim 4. *$S \cap V(g)$ is closed in K^n.*

Proof. This is immediate from the facts that S is closed in K^n (by hypothesis), $S \subseteq U$ and g is (defined and) continuous on the open set U.

Now let $\vec{\eta} = \langle \eta_1, \dots, \eta_n \rangle \in \mathbb{Z}^n$. By Claim 4 there is a point $Q \in S \cap V(g)$ whose distance from $\vec{\eta}$ is minimal (note also that $S \cap V(g) \neq \emptyset$ since $P \in S \cap V(g)$), i.e. $h(Q) \le h(\vec{x})$ for all $\vec{x} \in S \cap V(g)$, where $h(x_1, \dots, x_n) = \sum_{i=1}^{n} (x_i - \eta_i)^2$. Note that h (restricted to U) is an element of M since $\mathbb{Z}[x_1, \dots, x_n] \subseteq M$. Further, by Claim 3, Q is actually a local minimum of h on $V^{ns}(f_1, \dots, f_m)$, so by 4.10 the sequence of vectors $\langle d_Q f_1, \dots, d_Q f_m, d_Q h \rangle$ is linearly dependent. Arguing as in the proof of Claim 1 it follows that $\langle d_P f_1, \dots, d_P f_m, d_P h \rangle$ is linearly dependent. Since the sequence $\langle d_P f_1, \dots, d_P f_m \rangle$ is linearly independent it follows that $d_P h$ lies in the subspace, call it X, of K^n spanned (over K) by $d_P f_1, \dots, d_P f_m$, for *any* $\vec{\eta} \in \mathbb{Z}$. Write $h = h_{\vec{\eta}}$. By an easy calculation, $\vec{\eta} = \frac{1}{2}(d_P h_{\vec{0}} - d_P h_{\vec{\eta}})$. Hence $\mathbb{Z}^n \subseteq X$, which is impossible since $m < n$. □

5.2. *Proof of 2.7.* We shall apply 5.1 with $\overline{\overline{\mathbb{R}}} = \widetilde{\mathbb{R}}$, $\overline{\overline{T}} = \widetilde{T}$ and $\widetilde{K}$ an arbitrary model of $\widetilde{T}$ (cf. the beginning of section 2). Let $n, r \in \mathbb{N}$ and suppose that $\vec{\sigma}$ is an (n, r)-sequence. Let $\tilde{k} \models \widetilde{T}$, $\tilde{k} \subseteq \widetilde{K}$, and set $U = D^r(\vec{\sigma}, \widetilde{K})$ so that U is an open definable subset of K^r (cf. 2.2). Further, by 2.3 and the comments between 2.4 and 2.5, $M^r(\tilde{k}, \widetilde{K}, \vec{\sigma})$ is a subring of $\mathfrak{D}^{(r)}(U)$ which is closed under differentiation. It is also Noetherian, because it is finitely generated over the field k, and it clearly contains $\mathbb{Z}[x_1, \dots, x_r]$ (in fact, $k[x_1, \dots, x_r]$) as a subring.

Now to prove 2.7, suppose $g \in M^r(\tilde{k}, \widetilde{K}, \vec{\sigma})$ and $g(P) = 0$ for some $P \in U$. If we knew that $V(g)$ were closed in K^r, then we could apply 5.1 directly (with $n = r$, $M = M^r(\tilde{k}, \widetilde{K}, \vec{\sigma})$, $U = D^r(\vec{\sigma}, \widetilde{K})$, $f = g$ and $S = V(g)$) to obtain a $(\tilde{k}, \vec{\sigma})$-definable point $Q \in D^r(\vec{\sigma}, \widetilde{K})$ such that $Q \in V(g)$, thus completing the proof of 2.7 (with $s = 0$). Unfortunately, there is no reason to suppose that $V(g)$ does not have limit points on the boundary of U. However, this problem can be easily overcome by the standard geometric technique of pushing such points out to infinity. To do this we regard $\vec{\sigma}$ as an $(n, r+s)$-sequence, where $s = 2r$, in the sense of example E of section 2. Now for $1 \leq i \leq r$ define

$$g_i(x_1, \ldots, x_{r+s}) = \begin{cases} x_i \cdot x_{r+i} - 1 & \text{if } x_i \text{ is } \vec{\sigma}\text{-bounded}, \\ x_{r+i} - x_i & \text{otherwise}, \end{cases}$$

$$g_{r+i}(x_1, \ldots, x_{r+s}) = \begin{cases} (x_i - 1)x_{2r+i} - 1 & \text{if } x_i \text{ is } \vec{\sigma}\text{-bounded}, \\ x_{2r+i} - x_i & \text{otherwise}. \end{cases}$$

Now set $f = g^2 + \sum_{i=1}^{2r} g_i^2$ and notice that if $\langle p_1, \ldots, p_r \rangle \in V(g)$, then $\langle p_1, \ldots, p_{r+s} \rangle \in V(f)$ where $p_{r+i} = p_{2r+i} = p_i$ if x_i is not $\vec{\sigma}$-bounded, and $p_{r+i} = p_i^{-1}, p_{2r+i} = (p_i - 1)^{-1}$ if x_i is $\vec{\sigma}$-bounded (in which case we necessarily have that $0 < p_i < 1$). Thus $V(f) \neq \emptyset$ and it is easy to see that $V(f)$ is closed in K^{r+s}. We may therefore argue as above (this time using 5.1 with $n = r+s$, $M = M^{r+s}(\tilde{k}, \tilde{K}, \vec{\sigma})$, $U = D^{r+s}(\vec{\sigma}, \tilde{K}) = D^r(\vec{\sigma}, \tilde{K}) \times K^s$, and $S = V(f)$) to obtain the conclusion of 2.7. □

Recall now Proposition 3.1. This states that 0-dimensional Pfaffian varieties are uniformly finite. Khovanskii has proved a natural generalization of this fact for arbitrary zero-sets of Pfaffian functions which turns out to follow from 3.1 and 5.1 using a simple model theoretic argument. Thus rather than simply quoting the result it seems worthwhile to include the proof here.

5.3. Theorem (Khovanskii). *Suppose that $h_1, \ldots, h_l$ is any Pfaffian chain of functions on $\mathbb{R}^{m+n}$. Let $g \in \mathbb{R}[x_1, \ldots, x_{m+n}, h_1, \ldots, h_l]$. Then there is $N \in \mathbb{N}$ such that for any $Q \in \mathbb{R}^n$ the set $\{P \in \mathbb{R}^m : g(P,Q) = 0\}$ has at most N components.*

(A *component* of a set $S \subseteq \mathbb{R}^m$ is a set $X \subseteq S$ such that X is clopen in (the subspace) S. Clearly the collection of all components of S forms a Boolean algebra.)

Proof. Suppose the theorem is false. Then for each $i \in \mathbb{N}$, there exist $Q^{(i)} \in \mathbb{R}^n$ and pairwise disjoint non-empty components, $C_0^{(i)}, \ldots, C_i^{(i)}$, of the set $\{P \in \mathbb{R}^m : g(P, Q^{(i)}) = 0\}$.

Let $\overline{\overline{L}}$ be any expansion of $\overline{L}$ that includes symbols for the functions $h_1, \ldots, h_l$, the set $\mathbb{N}$, the map $i \to Q^{(i)} (i \in \mathbb{N})$ and the $(m+2)$-ary relation "$P \in C_j^{(i)}$". Let $\overline{\overline{\mathbb{R}}}$ be the corresponding expansion of $\overline{\mathbb{R}}$ and suppose $\overline{\overline{K}}$ is a $(2^{\aleph_0})^+$-saturated elementary extension of $\overline{\overline{\mathbb{R}}}$. Let a be a nonstandard natural number in K. Then (the $\overline{\overline{K}}$ interpretations of) each $C_i^{(a)}$ (for $i \leq a$, $\overline{\overline{K}} \models$ "$i \in \mathbb{N}$") is a non-empty subset of $Z \stackrel{\text{def}}{=} \{P \in K^m : g(P, Q^{(a)}) = 0\}$ which is both open and closed in Z, and hence also closed in K^m. Suppose $Q^{(a)} = \langle q_1, \ldots, q_n \rangle$ and let

$$M = \mathbb{R}[x_1, \ldots, x_m, q_1, \ldots, q_n, h_1(x_1, \ldots, x_m, Q^{(a)}), \ldots, h_l(x_1, \ldots, x_m, Q^{(a)})].$$

Then M is a Noetherian ring of $\overline{\overline{K}}$-definable, infinitely differentiable functions on K^m which contains $\mathbb{Z}[x_1, \ldots, x_m]$ and is closed under differentiation. Hence, by 5.1, for each $i \leq a$ (with $\overline{\overline{K}} \models$ "$i \in \mathbb{N}$") there exist $f_1^{(i)}, \ldots, f_m^{(i)} \in M$ such that $C_i^{(a)} \cap V^{ns}(f_1^{(i)}, \ldots, f_m^{(i)}) \neq \emptyset$. But there are at most $2^{\aleph_0}$ possibilities for $f_1^{(i)}, \ldots, f_m^{(i)}$ and, by 3.1, each $V^{ns}(f_1^{(i)}, \ldots, f_m^{(i)})$ is finite. However, the collection $\{C_i^{(a)} : i \leq a,$

$\overline{\overline{K}} \models$ "$i \in \mathbb{N}$"$\}$ consists of at least $(2^{\aleph_0})^+$ pairwise disjoint sets. This contradiction proves the theorem. □

5.4. Corollary. *Let $H_1, \ldots, H_l$ be a Pfaffian chain of functions on $\mathbb{R}^m$ ($m \in \mathbb{R}, m \geq 1$) and let $\widetilde{\mathbb{R}}'$ be the structure $\langle \overline{\mathbb{R}}; H_1, \ldots, H_l; r \rangle_{r \in C}$ (where C is any subset of $\mathbb{R}$) and $\widetilde{L}'$ its language. Suppose that $\phi(x_1, x_2, \ldots, x_p)$ is an existential formula of $\widetilde{L}'$. Then there exists $N \in \mathbb{N}$ such that for all $r_2, \ldots, r_p \in \mathbb{R}$ the set $\{r_1 \in \mathbb{R} : \widetilde{\mathbb{R}}' \models \phi(r_1, r_2, \ldots, r_p)\}$ is a union of at most N open intervals and N points.*

Proof. By the usual tricks (cf. section 2 before Definition 2.1) we may suppose that $\phi(x_1, \ldots, x_p)$ has the form $\exists y_1, \ldots, y_n \; f(x_1, \ldots, x_p, y_1, \ldots, y_n) = 0$, where f is a term of $\widetilde{L}'$. Now it is easy to construct a Pfaffian chain of functions on $\mathbb{R}^{p+n}$, $h_1, \ldots, h_{l'}$ say, such that $f \in \mathbb{R}[\vec{x}, \vec{y}, h_1, \ldots, h_{l'}]$. Thus by 5.3 there exists $N_0 \in \mathbb{N}$ such that for all $r_2, \ldots, r_p \in \mathbb{R}$ the set $Z(r_2, \ldots, r_p) \stackrel{\text{def}}{=} \{\langle p, q_1, \ldots, q_n \rangle \in \mathbb{R}^{1+n} : f(p, r_2, \ldots, r_p, q_1, \ldots, q_n) = 0\}$ has at most N_0 components. But then clearly this is also true for $\pi[Z(r_2, \ldots, r_p)]$ where $\pi : \mathbb{R}^{1+n} \to \mathbb{R}$ is the projection map onto the first coordinate. □

6. One dimensional varieties

In this section $\overline{\overline{\mathbb{R}}}$ denotes an expansion of $\overline{\mathbb{R}}$ which is either of the form $\widetilde{\mathbb{R}}$ (as described in section 1), or of the form $\widetilde{\mathbb{R}}'$ as described in the hypothesis of Corollary 5.4. In the latter case the set C of distinguished elements should be chosen to satisfy an analogous condition to the former case (cf. section 1, just after equation (2)). Clearly all the definitions from section 2 can be applied to the $\widetilde{\mathbb{R}}'$-$\widetilde{L}'$-$\widetilde{T}'$-$\widetilde{K}'$-$\widetilde{k}'$ case and are, in fact, somewhat less complicated. For example, there is no need to allow $y_1, \ldots, y_m$ to be 0 or 1 in Definition 2.1 (i) (a), and $D^r(\vec{\sigma}, \widetilde{K}')$ is simply K^r for any (n, r)-sequence $\vec{\sigma}$ and $\widetilde{K}' \models \widetilde{T}'$ (cf. Definition 2.2).

My aim in this section is to show that non-singular (space-) curves implicitly defined by terms in models of $\overline{\overline{T}}$ can be explicitly parameterized by finitely many infinitely differentiable definable functions having open intervals for domains. I first require the following combinatorial result.

6.1. Lemma. *Let $n, N \in \mathbb{N}$ with $n, N \geq 1$. Then there exist $Q_1, \ldots, Q_s \in \mathbb{Z}^n$, where $s = n \cdot N^2 + 1$, with the property that for any field K of characteristic 0 and any distinct elements $P_1, \ldots, P_m \in K^n$ (where $m \leq N$), there exists an i, $1 \leq i \leq s$, such that $Q_i \cdot P_1, \ldots, Q_i \cdot P_m$ are distinct elements of K. (Here "$\cdot$" denotes the usual scalar product.)*

Proof. Choose $Q_1, \ldots, Q_s \in \mathbb{Z}^n$ in general position, i.e. any n of them are linearly independent over $\mathbb{Q}$ (and hence over any field of characteristic 0). Suppose, for a contradiction, that there exist K, m ($m \leq N$) and distinct $P_1, \ldots, P_m \in K^n$ such that for each $i = 1, \ldots, s$, $Q_i \cdot P_{\alpha_i} = Q_i \cdot P_{\beta_i}$ for some α_i, β_i with $1 \leq \alpha_i < \beta_i \leq m$. Since the map $i \to \langle \alpha_i, \beta_i \rangle$ has domain of size $> n \cdot N^2$ and range of size $\leq N^2$, there exist α, β with $1 \leq \alpha < \beta \leq m$ such that $Q_i \cdot (P_\alpha - P_\beta) = 0$ for n distinct values of i. This contradicts the choice of the Q_i's since $P_\alpha - P_\beta \neq 0$. □

6.2. Theorem. *Suppose $n, r \in \mathbb{N}$, $r \geq 2$, and that $\vec{\sigma}$ is an (n,r)-sequence. Let $\overline{\overline{k}}, \overline{\overline{K}} \models \overline{\overline{T}}$ with $\overline{\overline{k}} \subseteq \overline{\overline{K}}$ and suppose that $g_1, \ldots, g_{r-1} \in M^r(\overline{\overline{k}}, \overline{\overline{K}}, \vec{\sigma})$. Let $V = \{P \in D^r(\vec{\sigma}, \overline{\overline{K}}) : g_1(P) = \cdots = g_{r-1}(P) = 0\}$ and suppose that*

(a) *V is a closed subset of K^r, and*

(b) *for all $P \in V$, $\det(\frac{\partial(g_1,\ldots,g_{r-1})}{\partial(x_2,\ldots,x_r)})(P) \neq 0$ (cf. the notation described before 2.5).*

Then there exists a finite set $\mathcal{S}$ of pairs $\langle I, \phi\rangle$ such that

(i) *for each $\langle I, \phi\rangle \in \mathcal{S}$, I is an open interval in K and $\phi : I \to K^{r-1}$ is an infinitely differentiable definable function;*

(ii) *for each $\langle I, \phi\rangle \in \mathcal{S}$, if $\sup I \in K$ (i.e. $\sup I \neq \infty$), then $\|\phi(x)\| \to \infty$ as $x \to \sup I$ (from below), where $\|\cdot\|$ denotes the usual norm on K^{r-1}, and similarly for $\inf I$;*

(iii) *$V = \bigcup\{\text{graph}(\phi) : \langle I, \phi\rangle \in \mathcal{S}\}$ and the union is disjoint.*

Proof. By 3.3 (or the analogous result in the case $\overline{\overline{\mathbb{R}}} = \widetilde{\mathbb{R}}'$, which follows directly from 3.1) and (b) it follows that there is some $N \in \mathbb{N}$ such that for each $p_1 \in K$ the set V_{p_1} contains at most N elements, where $V_{p_1} = \{\langle p_2, \ldots, p_r\rangle \in K^{r-1} : \langle p_1, \ldots, p_r\rangle \in V\}$. Let $s = (r-1)\cdot N^2 + 1$ and let $Q_1, \ldots, Q_s \in \mathbb{Z}^{r-1}$ be as in Lemma 6.1 (with $n = r-1$). For each $m = 1, \ldots, N$ and $i = 1, \ldots, s$ set $A_{m,i} = \{p_1 \in K : \text{card}(V_{p_1}) = \text{card}(Q_i \cdot V_{p_1}) = m\}$.

Now it is easy to see that each $A_{m,i}$ can be defined in $\overline{\overline{K}}$ by a boolean combination of existential formulas (with parameters) and hence, either by the comments following 3.5 (in the case $\overline{\overline{\mathbb{R}}} = \widetilde{\mathbb{R}}$) or by 5.4 (in the case $\overline{\overline{\mathbb{R}}} = \widetilde{\mathbb{R}}'$), it is a finite union of open intervals and points. It clearly follows from this that there exist $t \in \mathbb{N}$ and $a_1, \ldots, a_t \in K$ such that (setting $a_0 = -\infty$, $a_{t+1} = +\infty$) :

$$(*)\qquad \begin{array}{l} a_0 < a_1 < \cdots < a_t < a_{t+1} \text{ and for each } j = 0, \ldots, t, \\ i = 1, \ldots, s,\ m = 1, \ldots, N \text{ and } p, q \in (a_j, a_{j+1}), \\ p \in A_{m,i} \text{ if and only if } q \in A_{m,i}. \end{array}$$

Now for $p \in K$ let $m(p) = \text{card}(V_p)$ and $i(p) =$ the least i such that $\text{card}(Q_i \cdot V_p) = m(p)$. Then $m(p) \leq N$ and $i(p)$ exists by the conclusion of 6.1. Further, it clearly follows from $(*)$ that for each $j = 0, \ldots, t$, if $a_j < p, q < a_{j+1}$, then $m(p) = m(q)$ and $i(p) = i(q)$ so we may denote these numbers by m_j and i_j respectively. Hence we may define functions $\phi_{j,l} : (a_j, a_{j+1}) \to K^{r-1}$ (for those $j = 0, \ldots, t$ with $m_j \geq 1$ and $l = 1, \ldots, m_j$) by

$$\phi_{j,l}(x) = \vec{y} \Leftrightarrow \exists \vec{y}^{(1)}, \ldots, \exists \vec{y}^{(m_j)}(\langle x, y^{(1)}\rangle \in V \wedge \cdots \wedge \langle x, \vec{y}^{(m_j)}\rangle \in V$$
$$\wedge\, Q_{i_j} \cdot \vec{y}^{(1)} < \cdots < Q_{i_j} \cdot \vec{y}^{(m_j)} \wedge \vec{y} = \vec{y}^{(l)}).$$

Now since the map $K^{r-1} \to K : \vec{y} \to Q_{i_j} \cdot \vec{y}$ is continuous it follows that each $\phi_{j,l}$ coincides locally with a function given by the implicit function theorem for V (cf. the discussion at the beginning of section 4) and hence is infinitely differentiable on (a_j, a_{j+1}). We also clearly have that $\{\langle p_1, \ldots, p_r\rangle \in V : a_j < p_1 < a_{j+1}\} = \bigcup\{\text{graph}(\phi_{j,l}) : 1 \leq l \leq m_j\}$ where the union is disjoint.

Now suppose that $j < t$ (so $a_{j+1} \neq \infty$) and $1 \leq l \leq m_j$. Then either $\|\phi_{j,l}(x)\| \to \infty$ as $x \to a_{j+1}$ (from below) or else there is some $\langle p_2, \ldots, p_r\rangle \in K^{r-1}$ such that $\langle a_{j+1}, p_2, \ldots, p_r\rangle$ is a limit point of $\text{graph}(\phi_{j,l})$. For this is clear if $\overline{\overline{K}} = \overline{\overline{\mathbb{R}}}$, and since only the continuity of $\phi_{j,l}$ is required the result may be transferred to a general $\overline{\overline{K}}$. It follows from hypothesis (a) of the theorem that $\langle a_{j+1}, p_2, \ldots, p_r \rangle \in V$ (notice that the fact that V is a closed subset of $D^r(\vec{\sigma}, \overline{\overline{K}})$ is not sufficient here) and hence, by (b) and the implicit function theorem, there is an open box neighbourhood, U say, of $\langle p_2, \ldots, p_r \rangle$ in K^{r-1}, an $\varepsilon \in K$ with $a_j < a_{j+1} - \varepsilon < a_{j+1} < a_{j+1} + \varepsilon < a_{j+2}$, and a K-definable infinitely differentiable function $\phi : (a_{j+1} - \varepsilon, a_{j+1} + \varepsilon) \to U$ such that $\phi(a_{j+1}) = \langle p_2, \ldots, p_r \rangle$ and $V \cap ((a_{j+1} - \varepsilon, a_{j+1} + \varepsilon) \times U) = \text{graph}(\phi)$. It must be the case that ϕ coincides with $\phi_{j,l}$ on $(a_{j+1} - \varepsilon, a_{j+1})$ (because the set $\{p \in (a_{j+1} - \varepsilon, a_{j+1}) : \phi(p) = \phi_{j,l}(p)\}$ is both open and closed in $(a_{j+1} - \varepsilon, a_{j+1})$ and is non-empty since $(a_{j+1} - \varepsilon, a_{j+1}) \times U$ contains a point of graph$(\phi_{j,l})$ which is necessarily a point of V and hence of graph(ϕ)) and, indeed, that there exists l' with $1 \leq l' \leq m_{j+1}$ such that ϕ coincides with $\phi_{j+1,l'}$ on $(a_{j+1}, a_{j+1} + \varepsilon)$. Thus $\phi_{j,l}, \phi_{j+1,l'}$, and $\{\langle a_{j+1}, p_2, \ldots, p_r \rangle\}$ may be glued together to form a definable, infinitely differentiable function from (a_j, a_{j+2}) to K^{r-1} whose graph is contained in V. The theorem now follows by repeating this process until no further glueing across the a_j's is possible. □

I shall refer to the set $\mathcal{S}$ given by 6.2 as a *parameterization of* V *in* $\overline{\overline{K}}$. Of course, if $V \cap k^r$ is also closed in k^r, we may apply 6.2 with $\overline{\overline{K}} = \overline{\overline{k}}$ and obtain a parameterization, $\mathcal{S}'$ say, of $V \cap k^r$ in $\overline{\overline{k}}$ but at the moment we cannot infer any relationship between $\mathcal{S}$ and $\mathcal{S}'$. The following lemma clarifies the situation somewhat.

6.3. Lemma. *Suppose that, in addition to the hypotheses of 6.2, every* $(\overline{\overline{k}}, \vec{\sigma})$-*definable point of* $K^r \cap V$ *(cf. 2.5) lies in* k^r. *Let* $K^- = \{\alpha \in K : -\beta < \alpha < \beta$ *for some* $\beta \in k\}$ *and suppose that* $\alpha \in K^-$, $P \in K^{r-1}$, $\|P\| \in K^-$ *and* $\langle \alpha, P \rangle \in V$. *Then there exist* $\gamma_1, \gamma_2, \beta_1, \beta_2, B_1, B_2 \in k$ *with* $\gamma_2 < \gamma_1 < \alpha < \beta_1 < \beta_2$ *and* $\|P\| < B_1 < B_2$, $m \in \mathbb{N}$ $(m \geq 1)$, *and* $\overline{\overline{K}}$-*definable infinitely differentiable functions* $\phi_i : (\gamma_2, \beta_2) \to K^{r-1}$ *(for* $i = 1, \ldots, m$*) such that*

(i) $\|\phi_i(p)\| < B_1$ *for* $i = 1, \ldots, m$ *and* $p \in (\gamma_2, \beta_2)$;
(ii) $V \cap ((\gamma_2, \beta_2) \times \{Q \in K^{r-1} : \|Q\| < B_2\}) = \bigcup_{i=1}^m \text{graph}(\phi_i)$, *and the union is disjoint.*

Further, if $V \cap k^r$ *is closed in* k^r, *there exist* $\overline{\overline{k}}$-*definable infinitely differentiable functions* $\psi_i : (\gamma_2, \beta_2) \to k^{r-1}$ *(for* $i = 1, \ldots, m$*) such that* (i) *and* (ii) *hold with* ψ_i *in place of* ϕ_i *where all notions are interpreted in* $\overline{\overline{k}}$.

Remark. As I shall show below, it follows from the additional assumption on $\overline{\overline{k}}$ and $\overline{\overline{K}}$, and (b) of 6.2, that if $1 \leq i \leq m$, $p \in k$ and $\gamma_2 < p < \beta_2$, then $\phi_i(p) \in k^{r-1}$. However, there is still no guarantee that the function $\phi_i \restriction k$ is equal to some $\psi_{i'}$, or even that it is $\overline{\overline{k}}$-definable.

Proof. With the notation of the proof of 6.2 choose $m \in \mathbb{N}$ such that there are exactly m points $Q \in V_\alpha$ such that $\|Q\| \in K^-$. Let $P_1, \ldots, P_m$ be these points and note that $m \geq 1$ since P is one of them. Choose $B \in k$ such that $\|P_i\| < B$ for $i = 1, \ldots, m$ and let $B' \in k$, $B' > B$. Then certainly $\|Q\| > B'$ for all $Q \in V_\alpha \backslash \{P_1, \ldots, P_m\}$. For each $i = 1, \ldots, m$ let $\langle I_i, \phi_i \rangle$ be the (unique) element of $\mathcal{S}$ such that $\alpha \in I_i$ and $\phi_i(\alpha) = P_i$. This is possible by (iii) of 6.2. Now consider the ($\overline{\overline{K}}$-definable) set $A^+(= A^+(B, B'))$ given by:

$$A^+ = \{p \in \bigcap_{i=1}^{m} I_i : p \geq \alpha \text{ and for all } q \in [\alpha, p] \text{ and } i = 1, \ldots, m,$$
$$\|\phi_i(q)\| < B, \text{ and } \phi_1(q), \ldots, \phi_m(q) \text{ are the only}$$
$$\text{points } Q \in V_q \text{ satisfying } \|Q\| \leq B'\}.$$

By (i), (ii) and (iii) of 6.2, A^+ has the form $[\alpha, \beta)$ where $\beta \in K \cup \{\infty\}$ and $\beta > \alpha$. If $\beta = \infty$, let β_1, β_2 be any elements of k satisfying $\alpha < \beta_1 < \beta_2$. This is possible since $\alpha \in K^-$. If $\beta \in K$, I claim that $\beta \in k$. For certainly $\beta \in \bigcap_{i=1}^m I_i$ since otherwise we clearly contradict (i) and (ii) of 6.2. It follows that there is some $Q \in V_\beta$ such that either $\|Q\| = B$ or $\|Q\| = B'$. Define $g : D^r(\vec{\sigma}, \overline{\overline{K}}) \to K$ by $g(x_1, \ldots, x_r) = \sum_{i=2}^r x_i^2 - B^2$ (in the former case) or $\sum_{i=2}^r x_i^2 - (B')^2$ (in the latter case). Then $g \in M^r(\overline{\overline{k}}, \overline{\overline{K}}, \vec{\sigma})$ and g vanishes at the point $\langle \beta, Q\rangle$ but does not vanish on $V \cap W$ for any open neighbourhood W of $\langle \beta, Q\rangle$. It now follows from 4.9 (with $n = r$, $P_0 = \langle \beta, Q\rangle$, $M = \{[f, D^r(\vec{\sigma}, \overline{\overline{K}})] : f \in M^r(\tilde{k}, \tilde{K}, \vec{\sigma})\}$ and $\{f_1, \ldots, f_m\} = \{g_1, \ldots, g_{r-1}\}$) that $\langle \beta, Q\rangle$ is a $(\overline{\overline{k}}, \vec{\sigma})$-definable point of $K^r \cap V$ and hence lies in k^r. This proves the claim. Now let $\beta_1 = \beta$ and choose $B_1, B_2 \in k$ such that $B < B_1 < B_2 < B'$. Then $A^+(B_1, B_2) = [\alpha, \beta')$ for some $\beta' \in k \cup \{\infty\}$ and clearly $\beta' > \beta_1$. If $\beta' \in k$ set $\beta_2 = \beta'$. If $\beta' = \infty$ set $\beta_2 = \beta_1 + 1$. By the definition of A^+ we now have that (i) and (ii) hold with α in place of γ_2. However, the elements γ_1, γ_2 of k can be obtained by a similar argument by considering A^- (where "$p \geq \alpha$", "$[\alpha, p]$" are replaced by "$p \leq \alpha$", "$[p, \alpha]$" in the definition of A^+) with the same B, B', B_1, B_2.

To establish the last part of the lemma first observe that the result mentioned in the remark follows from 4.9 since if $\langle p, Q\rangle \in V$ and $p \in k$, then the function $D^r(\vec{\sigma}, \overline{\overline{K}}) \to K : \langle x_1, \ldots, x_r\rangle \to x_1 - p$ is in $M^r(\overline{\overline{k}}, \overline{\overline{K}}, \vec{\sigma})$, vanishes at $\langle p, Q\rangle$, but certainly does not vanish locally on V at $\langle p, Q\rangle$. Hence, since V has a quantifier-free definition (with parameters in k), it follows from (i) and (ii) that for each $p \in k$ with $\gamma_2 < p < \beta_2$ there are exactly m points $Q \in k^{r-1}$ such that $\overline{\overline{k}} \models (\langle p, Q\rangle \in V \wedge \|Q\| < B_2)$, and each such point satisfies $\|Q\| < B_1$. Let $Q_1, \ldots, Q_m$ be these points for the choice $p = \frac{\gamma_2 + \beta_2}{2}$. Now let $\mathcal{S}'$ be a parameterization of V in $\overline{\overline{k}}$ and for $i = 1, \ldots, m$ choose (the unique) $\langle I_i', \psi_i\rangle \in \mathcal{S}'$ such that $\psi_i(\frac{\gamma_2+\beta_2}{2}) = Q_i$. Then since each map $x \mapsto \|\psi_i(x)\|$ is continuous on $(\gamma_2, \beta_2) \cap I_i'$, it follows from the intermediate value theorem (interpreted in $\overline{\overline{k}}$) that it takes no value $\geq B_1$. In particular (by (ii) of 6.2) $(\gamma_2, \beta_2) \subseteq I_i'$. This proves the lemma. □

7. The proof of Lemma 2.8

I shall in fact prove 2.8 for both the $\tilde{T}$ and $\tilde{T}'$ situation, so let $\overline{\overline{\mathbb{R}}}, \overline{\overline{T}}$ etc. be as described at the beginning of section 6. The proof is by induction on n. The base step is provided by example (D) of section 2. For the induction step suppose $n, r \in \mathbb{N}$, $\overline{\overline{K}} \models \overline{\overline{T}}$, $\overline{\overline{k}} \models \overline{\overline{T}}$, $\overline{\overline{k}} \subseteq \overline{\overline{K}}$ and that $\langle \vec{\sigma}, \sigma_{n+1}\rangle$ is an $(n+1, r)$-sequence (where $\vec{\sigma}$ is an (n, r)-sequence) such that

(8) for all $s \geq r$, every $(\overline{\overline{k}}, \langle \vec{\sigma}, \sigma_{n+1}\rangle)$-definable point of K^s lies in $(K^-)^s$.

(Here, K^- is as defined in the statement of Lemma 6.3.)

Now suppose $s \geq r$. Since every $\vec{\sigma}$-bounded variable (cf. 2.1 (ii)) is also $\langle \vec{\sigma}, \sigma_{n+1} \rangle$-bounded we have that $D^s(\langle \vec{\sigma}, \sigma_{n+1} \rangle, \overline{\overline{K}}) \subseteq D^s(\vec{\sigma}, \overline{\overline{K}})$. Further, if $g \in M^s(\overline{\overline{k}}, \overline{\overline{K}}, \vec{\sigma})$, then $g \restriction D^s(\langle \vec{\sigma}, \sigma_{n+1} \rangle, \overline{\overline{K}}) \in M^s(\overline{\overline{k}}, \overline{\overline{K}}, \langle \vec{\sigma}, \sigma_{n+1} \rangle)$ and there will be no harm in identifying $M^s(\overline{\overline{k}}, \overline{\overline{K}}, \vec{\sigma})$ with its image in $M^s(\overline{\overline{k}}, \overline{\overline{K}}, \langle \vec{\sigma}, \sigma_{n+1} \rangle)$ under this restriction mapping. (Similar remarks apply with $\overline{\overline{K}}$ replaced everywhere by $\overline{\overline{k}}$.) Clearly our inductive hypothesis and (8) imply

(9) for all $s \geq r$ and $P \in K^s$, if P is $(\overline{\overline{k}}, \vec{\sigma})$-definable and $P \in D^s(\langle \vec{\sigma}, \sigma_{n+1} \rangle, \overline{\overline{K}})$, then $P \in k^s$.

Now let Q be any $(\overline{\overline{k}}, \langle \vec{\sigma}, \sigma_{n+1} \rangle)$-definable point of K^r. We must show that $Q \in k^r$.

Now, by definition, there exist $g_1, \ldots, g_r \in M^r(\overline{\overline{k}}, \overline{\overline{K}}, \langle \vec{\sigma}, \sigma_{n+1} \rangle)$ such that

$$g_1(Q) = \cdots = g_r(Q) = 0, \tag{10}$$

and

$$\det \left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)} \right)(Q) \neq 0. \tag{11}$$

Further,

$$Q \in D^r(\langle \vec{\sigma}, \sigma_{n+1} \rangle, \overline{\overline{K}}). \tag{12}$$

I shall now deduce that $Q \in k^r$ under several extra assumptions on the sequence of functions $g_1, \ldots, g_r$. These will be justified later. For the moment, set $V = \{P \in K^r : g_1(P) = \cdots = g_{r-1}(P) = 0\}$ (we may clearly suppose that $r \geq 2$), and *assume that*

$$g_1, \ldots, g_{r-1} \in M^r(\overline{\overline{k}}, \overline{\overline{K}}, \vec{\sigma}); \tag{13}$$

(14) V is a closed subset of K^r and $V \cap k^r$ is a closed subset of k^r;

$$V \subseteq D^r(\langle \vec{\sigma}, \sigma_{n+1} \rangle, \overline{\overline{K}}); \tag{15}$$

$$\det \left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)} \right)(P) \neq 0 \quad \text{for all } P \in V; \tag{16}$$

(17) for all $P \in V$, if $g_r(P) = 0$ then $\det \left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)} \right)(P) < 0$.

Now notice that all the hypotheses of 6.3 are satisfied. (The fact that every $(k, \vec{\sigma})$-definable point of $K^r \cap V$ lies in k^r follows from (15) and (9).) Further, since Q is $(\overline{\overline{k}}, \langle \vec{\sigma}, \sigma_{n+1} \rangle)$-definable, we have $Q \in (K^-)^r$ by (8). Hence we may apply 6.3 with $\alpha = q_1$ and $P = \langle q_2, \ldots, q_r \rangle$ (where $Q = \langle q_1, \ldots, q_r \rangle$) to obtain $\gamma_1, \gamma_2, \beta_1, \beta_2, B_1, B_2 \in k$, $\phi_i : (\gamma_2, \beta_2) \to K^{r-1}$ and $\psi_i : (\gamma_2, \beta_2) \cap k \to k^{r-1}$ (for $i = 1, \ldots, m$) satisfying the conclusions of that lemma.

Now let ϕ be any one of the ϕ_i's. Notice that for $t \in (\gamma_2, \beta_2)$ we have $\langle t, \phi(t) \rangle \in V$ and hence (by (15)) $\langle t, \phi(t) \rangle \in D^r(\langle \vec{\sigma}, \sigma_{n+1} \rangle, \overline{\overline{K}})$. Therefore we may define, for any

$g \in M^r(\overline{\overline{k}}, \overline{\overline{K}}, \langle \vec{\sigma}, \sigma_{n+1} \rangle)$, a function $\overline{g} : (\gamma_2, \beta_2) \to K$ by $\overline{g}(t) = g(t, \phi(t))$. Clearly $\overline{g}$ is $\overline{\overline{K}}$-definable and infinitely differentiable. Its first derivative is given by

$$\frac{d\overline{g}}{dt}(t) = \overline{\frac{\partial g}{\partial x_1}}(t) + \sum_{i=2}^{r} \overline{\frac{\partial g}{\partial x_i}}(t) \cdot \frac{d\phi^{(i)}}{dt}(t),$$

where $\phi(t) = \langle \phi^{(2)}(t), \ldots, \phi^{(r)}(t) \rangle$. Of course this formula holds for $\overline{g} = \overline{g}_1, \ldots, \overline{g}_{r-1}$, which are identically zero, and I leave the reader to perform the linear algebra required to eliminate the $\frac{d\phi^{(i)}}{dt}(t)$ terms and arrive at

$$\frac{d\overline{g}}{dt}(t) = (-1)^{r+1} \overline{J}(t) \cdot \overline{J}_1(t)^{-1} \quad \text{for all } t \in (\gamma_2, \beta_2), \tag{18}$$

where

$$J(x_1, \ldots, x_r) = \det \left(\frac{\partial(g_1, \ldots, g_{r-1}, g)}{\partial(x_1, \ldots, x_r)} \right)$$

and

$$J_1(x_1, \ldots, x_r) = \det \left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)} \right).$$

(Notice that (18) makes sense since $J, J_1 \in M^r(\overline{\overline{k}}, \overline{\overline{K}}, \langle \vec{\sigma}, \sigma_{n+1} \rangle)$ and, by (16), $\overline{J}_1(t) \neq 0$ for all $t \in (\gamma_2, \beta_2)$.)

I shall now assume that r is even and leave the reader to make the obvious modifications to the argument for the case of r odd.

7.1. Claim. (i) *If $p \in (\gamma_2, \beta_2)$ and $\overline{g}_r(p) = 0$, then $\frac{d\overline{g}_r}{dt}(p)$ has the same sign as $\overline{J}_1(p)$.*

(ii) *$\overline{g}_r$ has at most one zero.*

Proof. (i) By (17) $\overline{J}(p) < 0$, where J is defined as in (18) with $g = g_r$. Thus (i) now follows immediately from (18) and the fact that r is even.

(ii) Note that, by (16), $\overline{J}_1$ is non-zero throughout (γ_2, β_2) and hence has constant sign on (γ_2, β_2). It follows from (i) that $\frac{d\overline{g}_r}{dt}(p_1)$ and $\frac{d\overline{g}_r}{dt}(p_2)$ have the same (non-zero) sign whenever $\overline{g}_r(p_1) = \overline{g}_r(p_2) = 0$. This is impossible (by transfer from $\mathbb{R}$) unless $\overline{g}_r$ has at most one zero. □

Now notice that (13)–(17) all hold with $\overline{\overline{k}}$ in place of $\overline{\overline{K}}$ and $V \cap k^r$ in place of V. This is because each of these statements actually implies the corresponding statement for $\overline{\overline{k}}$ and $V \cap k^r$. Hence the discussion above holds good in $\overline{\overline{k}}$ if we take ϕ to be one of the ψ_i's.

Now for any $g \in M^r(\overline{\overline{k}}, \overline{\overline{K}}, \langle \sigma, \sigma_{n+1} \rangle)$, let $\overline{g}(\phi_i; \cdot)$ be the ($\overline{\overline{K}}$-definable) function from $\{t \in K : \gamma_2 < t < \beta_2\}$ to K obtained as above with $\phi = \phi_i$ and let $\overline{g}(\psi_i; \cdot)$ be the ($\overline{\overline{k}}$-definable) function from $\{t \in k : \gamma_2 < t < \beta_2\}$ to k obtained with $\phi = \psi_i$ (note that $\gamma_2, \beta_2 \in k$). We complete the proof of 2.8 (under the extra assumptions) as follows.

Let i_0 be the (unique) number such that $1 \leq i_0 \leq m$ and $\phi_{i_0}(q_1) = \langle q_2, \ldots, q_r \rangle$. Suppose that $\overline{J}_1(\phi_{i_0}; q_1) > 0$. (The proof is similar if $\overline{J}_1(\phi_{i_0}; q_1) < 0$.) Let $S = \{i : 1 \leq i \leq m \text{ and } \overline{J}_1(\phi_i; q_1) > 0\}$. Then, just as in the proof of 7.1, it follows from (16) that $\overline{J}_1(\phi_i; t) > 0$ for all $i \in S$ and all $t \in (\gamma_2, \beta_2)$ and that $\overline{J}_1(\phi_i; t) < 0$ for all $i \in \{1, \ldots, m\} \backslash S$ and all $t \in (\gamma_2, \beta_2)$. In particular $\overline{J}_1(\phi_i; \gamma_1) > 0$ for $i \in S$ and $\overline{J}_1(\phi_i; \gamma_1) < 0$ for $i \in \{1, \ldots, m\} \backslash S$. It now follows from 6.3 (and the remark there) that there is a subset, S' say, of $\{1,\dots,m\}$ such that $\{\psi_i(\gamma_1) : i \in S'\} = \{\phi_i(\gamma_1) : i \in S\}$, and hence that $\overline{J}_1(\psi_i;t) > 0$ (respectively < 0) for all $i \in S'$ (respectively $i \in \{1,\dots,m\}\backslash S'$) and $t \in (\gamma_2,\beta_2)\cap k$. Now $\overline{\overline{k}}$ is a substructure of $\overline{\overline{K}}$ so we clearly have (again using 6.3) that for all $t \in (\gamma_2,\beta_2)\cap k$, $\{\psi_i(t) : i \in S'\} = \{\phi_i(t) : i \in S\}$. Now choose $\gamma_3, \beta_3 \in k$ such that $\gamma_2 < \gamma_3 < \gamma_1$ and $\beta_1 < \beta_3 < \beta_2$ and such that for no $i = 1,\dots,m$ does either $\overline{g}_r(\phi_i;\cdot)$ or $\overline{g}_r(\psi_i;\cdot)$ have a zero at γ_3 or β_3. This is possible since there are at most a finite number of points to be avoided. By 7.1 (and its version for $\overline{\overline{k}}$) it clearly follows that if $i \in S$ (respectively $i \in S'$), then $\overline{g}_r(\phi_i;\cdot)$ has a zero in (γ_3,β_3) (respectively, $\overline{g}_r(\psi_i;\cdot)$ has one in $(\gamma_3,\beta_3)\cap k$) if and only if $\overline{g}_r(\phi_i;\gamma_3) < 0$ and $\overline{g}_r(\phi_i;\beta_3) > 0$ (respectively $\overline{g}_r(\psi_i;\gamma_3) < 0$ and $\overline{g}_r(\psi_i;\beta_3) > 0$). Hence

$$\begin{aligned}&\operatorname{card}\{i \in S : \exists t \in (\gamma_3,\beta_3)\ \overline{g}_r(\phi_i;t) = 0\}\\ &\quad = \operatorname{card}\{i \in S : \overline{g}_r(\phi_i;\gamma_3) < 0\} - \operatorname{card}\{i \in S : \overline{g}_r(\phi_i;\beta_3) < 0\}\end{aligned}$$

and

$$\begin{aligned}&\operatorname{card}\{i \in S' : \exists t \in (\gamma_3,\beta_3)\cap k\ \overline{g}_r(\psi_i;t) = 0\}\\ &\quad = \operatorname{card}\{i \in S' : \overline{g}_r(\psi_i;\gamma_3) < 0\} - \operatorname{card}\{i \in S' : \overline{g}_r(\psi_i;\beta_3) < 0\}.\end{aligned}$$

However, by 6.3 (and the fact that $\overline{\overline{k}} \subseteq \overline{\overline{K}}$) the two right hand sides here are equal. It now follows (again using 6.3) that every point $P = \langle p_1,\dots,p_r\rangle \in K^r$ satisfying $P \in V$, $g_r(P) = 0$, $J_1(P) > 0$, $\gamma_3 < p_1 < \beta_3$ and $\|\langle p_2,\dots,p_r\rangle\| < B_1$ actually lies in k^r. But Q is such a point!

I must now show why (13)–(17) may be assumed. So suppose that $g_1,\dots,g_r$ and Q satisfy (10)–(12). I shall modify $\langle\vec{\sigma},\sigma_{n+1}\rangle$ (to $\langle\vec{\sigma}',\sigma'_{n+1}\rangle$) so that (8) and (9) are still satisfied, and produce $h_1,\dots,h_s \in M^s(\overline{\overline{k}},\overline{\overline{K}},\langle\vec{\sigma}',\sigma'_{n+1}\rangle)$ for some $s \geq r$) and a point $Q' \in K^s$ such that (10)–(17) are satisfied with $h_1,\dots,h_s$, Q' in place of $g_1,\dots,g_r,Q$. Further, $q_1,\dots,q_r$ will occur amongst the coordinates of Q'. This is clearly sufficient. The new functions and point will be produced in several stages but to avoid a proliferation of notation I shall revert to the original notation (i.e. $g_1,\dots,g_r,Q$) at the end of the justification of each stage. The conditions (10)–(12) will be satisfied at each stage.

Stage 1. We may assume that for each $\langle\vec{\sigma},\sigma_{n+1}\rangle$-bounded variable x, there are variables y,z such that both $x\cdot y^2 - 1$ and $(1-x)\cdot z^2 - 1$ occur amongst $g_1,\dots,g_r$.

Justification. Suppose that x_i is $\langle\vec{\sigma},\sigma_{n+1}\rangle$-bounded (where $1 \leq i \leq r$). Define $g_{r+1}, g_{r+2} \in M^{r+2}(\overline{\overline{k}},\overline{\overline{K}},\langle\vec{\sigma},\sigma_{n+1}\rangle)$ by $g_{r+1}(x_1,\dots,x_{r+1}) = x_i\cdot x_{r+1}^2 - 1$, $g_{r+2}(x_1,\dots,x_{r+2}) = (1-x_i)\cdot x_{r+2}^2 - 1$. Then, since $0 < q_i < 1$ (because of (12)), we may set $q_{r+1} = +q_i^{-\frac{1}{2}}$ and $q_{r+2} = +(1-q_i)^{-\frac{1}{2}}$ so that (10) and (12) are clearly satisfied for $g_1,\dots,g_{r+2},\langle Q,q_{r+1},q_{r+2}\rangle$. Further, as a simple calculation shows,

$$\det\left(\frac{\partial(g_1,\dots,g_{r+2})}{\partial(x_1,\dots,x_{r+2})}\right)(Q,q_{r+1},q_{r+2}) = \det\left(\frac{\partial(g_1,\dots,g_r)}{\partial(x_1,\dots,x_r)}\right)(Q)\cdot 4\cdot q_i^{\frac{1}{2}}\cdot(1-q_i)^{\frac{1}{2}}$$

and the right hand side is non-zero by (11) (for $g_1,\dots,g_r,Q$). Hence (11) holds for the new system.

Stage 2. We may assume that $g_1,\dots,g_{r-1} \in M^r(\overline{\overline{k}},\overline{\overline{K}},\vec{\sigma})$ and that g_r has the form $\sigma_{n+1}(x_1,\dots,x_r) - x_e$, where x_e is not $\langle\vec{\sigma},\sigma_{n+1}\rangle$-bounded (and hence does not actually occur in the term $\sigma_{n+1}(x_1,\dots,x_r)$).

Justification. By definition of $M^r(\overline{\overline{k}}, \overline{\overline{K}}, \langle\vec{\sigma}, \sigma_{n+1}\rangle)$ there exist $h_1, \dots, h_r \in M^r(\overline{\overline{k}}, \overline{\overline{K}}, \vec{\sigma})[x_{r+1}]$ $(= M^{r+1}(\overline{\overline{k}}, \overline{\overline{K}}, \vec{\sigma}))$ such that

$$g_i(x_1, \dots, x_r) = h_i(x_1, \dots, x_r, \sigma_{n+1}(x_1, \dots, x_r))$$

for $i = 1, \dots, r$. Let $q_{r+1} = \sigma_{n+1}(q_1, \dots, q_r)$, $Q' = \langle Q, q_{r+1}\rangle$ and $h_{r+1}(x_1, \dots, x_{r+1}) = \sigma_{n+1}(x_1, \dots, x_r) - x_{r+1}$. Clearly (10) and (12) are satisfied for $h_1, \dots, h_{r+1}, Q'$ as well as stage 1 and stage 2. For (11), consider the matrix $\frac{\partial(h_1, \dots, h_{r+1})}{\partial(x_1, \dots, x_{r+1})}(Q')$. For each $i = 1, \dots, r$, multiply row $r+1$ by $\frac{\partial h_i}{\partial x_{r+1}}(Q')$ and add the result to row i. By the chain rule, the resulting matrix has determinant $-\det(\frac{\partial(g_1, \dots, g_r)}{\partial(x_1, \dots, x_r)})(Q)$ which is non-zero by (the old) (11).

Stage 3. We may assume that for all $P \in D^r(\langle\vec{\sigma}, \sigma_{n+1}\rangle, \overline{\overline{K}})$, if $g_i(P) = 0$ for $i = 1, \dots, r-1$, then $\det(\frac{\partial(g_1, \dots, g_{r-1})}{\partial(x_2, \dots, x_r)})(P) \neq 0$.

Justification. By (11) there is some i $(1 \le i \le r)$ such that

$$\det\left(\frac{\partial(g_1, \dots, g_{r-1})}{\partial(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_r)}\right)(Q) \neq 0.$$

By relabelling the variables we may suppose that $i = 1$. (Note that for every $n, r \in \mathbb{N}$, the notion of an (n, r)-sequence is invariant under permutation of variables. Further, the definable points for the permuted sequence are just coordinate permutations of definable points for the original sequence. Thus (8) and (9) are still true for the permuted sequence. Clearly so are (10)–(12) and stages 1 and 2 for the corresponding transformation of $g_1, \dots, g_r$ and Q.)

Now let

$$h(x_1, \dots, x_{r+1}) = x_{r+1} \cdot \det\left(\frac{\partial(g_1, \dots, g_{r-1})}{\partial(x_2, \dots, x_r)}\right)(x_1, \dots, x_r) - 1,$$

and set

$$q_{r+1} = \det\left(\frac{\partial(g_1, \dots, g_{r-1})}{\partial(x_2, \dots, x_r)}\right)(Q)^{-1}, \qquad Q' = \langle Q, q_{r+1}\rangle.$$

Then $g_1, \dots, g_{r-1}, h, g_r$ and Q' still satisfy stages 1 and 2 and also, clearly, (10) and (12). For (11), a simple calculation shows that

$$\det\left(\frac{\partial(g_1, \dots, g_{r-1}, h, g_r)}{\partial(x_1, \dots, x_{r+1})}\right)(Q') = -\det\left(\frac{\partial(g_1, \dots, g_r)}{\partial(x_1, \dots, x_r)}\right)(Q) \cdot q_{r+1}^{-1}$$

which is non-zero by (the old) (11).

Finally, to see that stage 3 is satisfied suppose that $P \in D^{r+1}(\langle\vec{\sigma}, \sigma_{n+1}\rangle, \overline{\overline{K}})$ and that $g_1(P) = \dots = g_{r-1}(P) = h(P) = 0$. Say $P = \langle p_1, \dots, p_{r+1}\rangle$. Since $h(P) = 0$ we have $p_{r+1} \neq 0$ and routine calculation gives $\det(\frac{\partial(g_1, \dots, g_{r-1}, h)}{\partial(x_2, \dots, x_{r+1})})(P) = p_{r+1}^{-2} \neq 0$, as required.

Stage 4. We may assume that for all $P \in D^r(\langle\vec{\sigma}, \sigma_{n+1}\rangle, \overline{\overline{K}})$, if $g_i(P) = 0$ for $i = 1, \dots, r$, then $\det(\frac{\partial(g_1, \dots, g_r)}{\partial(x_1, \dots, x_r)})(P) < 0$.

Justification. As in the proof of stage 2, there is some $h \in M^r(\overline{\overline{k}}, \overline{\overline{K}}, \vec{\sigma})[z]$ such that

$$(*) \qquad \det\left(\frac{\partial(g_1, \dots, g_r)}{\partial(x_1, \dots, x_r)}\right)(x_1, \dots, x_r) = h(x_1, \dots, x_r, \sigma_{n+1}(x_1, \dots, x_r)).$$

Define $H \in M^{r+1}(\overline{\overline{k}}, \overline{\overline{K}}, \vec{\sigma})$ by $H(x_1, \ldots, x_{r+1}) = x_{r+1} \cdot h(x_1, \ldots, x_r, x_e) - 1$ where e is as given by stage 2 (so $1 \leq e \leq r$). Now since $g_r(q_1, \ldots, q_r) = 0$, i.e. $\sigma_{n+1}(q_1, \ldots, q_r) = q_e$, it follows from $(*)$ that $h(q_1, \ldots, q_r, q_e) = \det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_2, \ldots, x_r)}\right)(Q)$ which is non-zero by (11). Hence we may set $q_{r+1} = h(q_1, \ldots, q_r, q_e)^{-1}$ and $Q' = \langle Q, q_{r+1}\rangle$ so that (10), (12), stages 1 and 2 are clearly satisfied for the system $g_1, \ldots, g_{r-1}, H, g_r, Q'$. To see that stage 4 (and hence (11)) are also satisfied, suppose that $\langle p_1, \ldots, p_{r+1}\rangle = P \in D^{r+1}(\langle\vec{\sigma}, \sigma_{n+1}\rangle, \overline{\overline{K}})$ and that $g_1(P) = \cdots = g_{r-1}(P) = H(P) = g_r(P) = 0$. Then by routine calculation we obtain

$$\begin{aligned}\det\left(\frac{\partial(g_1, \ldots, g_{r-1}, H, g_r)}{\partial(x_1, \ldots, x_{r+1})}\right)(P) &= -\det\left(\frac{\partial(g_1, \ldots, g_{r-1}, g_r, H)}{\partial(x_1, \ldots, x_{r+1})}\right)(P)\\ &= -\det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)}\right)(p_1, \ldots, p_r) \cdot h(p_1, \ldots, p_r, p_e)\\ &= -h(p_1, \ldots, p_r, p_e)^2\end{aligned}$$

(by $(*)$ and the fact that $g_r(p_1, \ldots, p_r) = 0$). Since $H(P) = 0$, $h(p_1, \ldots, p_r, p_e) \neq 0$ so the conclusion of stage 4 follows. Finally, stage 3 is still satisfied because if $P = \langle p_1, \ldots, p_{r+1}\rangle$ is any point in $D^{r+1}(\langle\vec{\sigma}, \sigma_{n+1}\rangle, \overline{\overline{K}})$ such that $g_1(P) = \cdots = g_{r-1}(P) = H(P) = 0$, then

$$\begin{aligned}&\det\left(\frac{\partial(g_1, \ldots, g_{r-1}, H)}{\partial(x_2, \ldots, x_{r+1})}\right)(P)\\ &= \det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right)(p_1, \ldots, p_r) \cdot h(p_1, \ldots, p_r, p_e)\end{aligned}$$

which is non-zero by (the old) stage 3 and the fact that $H(P) = 0$.

The proof of Lemma 2.8 is now complete, for (13) follows from stage 2, (14) and (15) from stage 1, (16) from stage 3 and (17) from stage 4. Further, (8)–(12) were preserved throughout. □

I shall prove Lemma 2.9 for the $\widetilde{T}$ situation (and hence complete the proof of the first main theorem) in the next section. I conclude this section, however, with the best result I know for the unrestricted ($\widetilde{T}'$) case. The proof follows immediately from 2.8 (for the $\widetilde{T}'$ situation) and 2.7 (the proof of which—given in 5.2—clearly also works (in fact, more smoothly) for the $\widetilde{T}'$-situation).

7.2. Theorem. *Let $H_1, \ldots, H_l$ be a Pfaffian chain of functions on $\mathbb{R}^m$ ($m \in \mathbb{N}$, $m \geq 1$) and let $\widetilde{\mathbb{R}}'$ be the structure $\langle\mathbb{R}; H_1, \ldots, H_l; r\rangle_{r\in C}$ where the set C is chosen as at the beginning of section 6. Let $\tilde{k}', \widetilde{K}' \models \widetilde{T}'$, $\tilde{k}' \subseteq \widetilde{K}'$, and suppose that for all $n, r \in \mathbb{N}$ and all (n,r)-sequences $\vec{\sigma}$, every $(\tilde{k}', \vec{\sigma})$-definable point $\langle p_1, \ldots, p_r\rangle$ of $(K')^r$ satisfies $-B < p_i < B$ ($i = 1, \ldots, r$) for some $B \in k'$. (In particular, this is satisfied if $\widetilde{K}'$ is a cofinal extension of $\tilde{k}'$.) Then for any existential formula $\phi(x_1, \ldots, x_e)$ of $\widetilde{L}'$, and any $a_1, \ldots, a_e \in k'$ we have $\tilde{k}' \models \phi(a_1, \ldots, a_e)$ if and only if $\widetilde{K}' \models \phi(a_1, \ldots, a_e)$.* □

8. The proof of Lemma 2.9

In this section I revert to the $\widetilde{T}$-situation of section 1. The proof of 2.9 that I shall give here does not work for the $\widetilde{T}'$-situation because it relies heavily on 3.5 and

I know of no analogue of this result for this situation. (I say "analogue" because 3.5 as it stands obviously fails for, e.g., $\widetilde{T}' = \text{Theory}(\langle\overline{\mathbb{R}}, \exp\rangle)$ with exp unrestricted.)

So suppose $\widetilde{K}, \tilde{k} \models \widetilde{T}$, $\tilde{k} \subseteq \widetilde{K}$ and $n, r \in \mathbb{N}$. Let $\vec{\sigma} = \langle \sigma_1, \ldots, \sigma_n \rangle$ be an (n, r)-sequence and suppose that for each $s \geq r$ every $(\tilde{k}, \vec{\sigma})$-definable point of K^s lies in k^s. Suppose σ_{n+1} is such that $\langle \vec{\sigma}, \sigma_{n+1} \rangle$ is an $(n+1, r)$-sequence. It is clearly sufficient to show that every $(\tilde{k}, \langle \vec{\sigma}, \sigma_{n+1} \rangle)$-definable point of K^r lies in $(K^-)^r$ where, as before, $K^- = \{\alpha \in K : -\beta < \alpha < \beta \text{ for some } \beta \in k\}$.

Let $Q = \langle q_1, \ldots, q_r \rangle$ be a $(\tilde{k}, \langle \vec{\sigma}, \sigma_{n+1} \rangle)$-definable point of K^r. Then, by applying the stages described in the previous section, we may assume that ($r \geq 2$ and) there are functions $g_1, \ldots, g_r \in M^r(\tilde{k}, \widetilde{K}, \langle \vec{\sigma}, \sigma_{n+1} \rangle)$ such that:

(19) $$g_1, \ldots, g_{r-1} \in M^r(\tilde{k}, \widetilde{K}, \vec{\sigma});$$

(20) $$g_r \text{ has the form } \sigma_{n+1}(x_1, \ldots, x_r) - x_e, \\ \text{where } x_e \text{ is not } \langle \vec{\sigma}, \sigma_{n+1} \rangle\text{-bounded;}$$

(21) $$g_i(Q) = 0 \text{ for } i = 1, \ldots, r \text{ and } \det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)}\right)(Q) \neq 0;$$

and, setting $V = \{P \in D^r(\vec{\sigma}, \widetilde{K}) : g_i(P) = 0 \text{ for } i = 1, \ldots, r-1\}$:

(22) $$V \subseteq D^r(\langle \vec{\sigma}, \sigma_{n+1} \rangle, \widetilde{K}) \text{ and } V \text{ (respectively } V \cap k^r) \\ \text{is a closed subset of } K^r \text{ (respectively } k^r);$$

(23) $$\text{for all } P \in V, \quad \det\left(\frac{\partial(g_1, \ldots, g_{r-1})}{\partial(x_2, \ldots, x_r)}\right)(P) \neq 0;$$

(24) $$\text{for all } P \in V, \text{ if } g_r(P) = 0 \text{ then } \det\left(\frac{\partial(g_1, \ldots, g_r)}{\partial(x_1, \ldots, x_r)}\right)(P) \neq 0.$$

The hypothesis of 2.9 may now be strengthened as follows.

8.1. Claim. *Suppose $\chi(x_1, \ldots, x_r)$ is a formula of $\overline{L}$ (the language of ordered rings) possibly containing parameters from k. Suppose further that for some $\langle p_1, \ldots, p_r \rangle \in V$, $\widetilde{K} \models \chi(p_1, \ldots, p_r)$. Then for some $\langle p_1, \ldots, p_r \rangle \in V \cap k^r$, $\tilde{k} \models \chi(p_1, \ldots, p_r)$.*

Proof. By quantifier elimination and the usual tricks we may suppose that $\chi(x_1, \ldots, x_r)$ has the form $\exists x_{r+1}, \ldots, \exists x_{r+t}\, \rho(x_1, \ldots, x_{r+t}) = 0$ where ρ is a polynomial with coefficients in k. Let $g = \rho^2 + \sum_{i=1}^{r-1} g_i^2$. Then $g \in M^{r+t}(\tilde{k}, \widetilde{K}, \vec{\sigma})$ (by (19)) and $g(P) = 0$ for some $P \in D^{r+t}(\vec{\sigma}, \widetilde{K})$. Hence by 2.7 there is some $\langle P, P' \rangle \in D^{(r+t)+s}(\vec{\sigma}, \widetilde{K})$ (for some $s \in \mathbb{N}$) such that $g(P) = 0$ and $\langle P, P' \rangle$ is $(\tilde{k}, \vec{\sigma})$-definable. By the hypothesis of 2.9, $\langle P, P' \rangle \in k^{(r+t)+s}$. Clearly if $P = \langle p_1, \ldots, p_{r+t} \rangle$, then $\langle p_1, \ldots, p_r \rangle$ satisfies the conclusion of the claim. □

I now suppose, for a contradiction, that $Q \notin (I^-)^r$.

8.2. Claim. $q_1 \notin k$.

Proof. Suppose $q_1 \in k$. Let $h(x_1, \dots, x_r) = x_1 - q_1$. Then $h \in M^r(\tilde{k}, \widetilde{K}, \vec{\sigma})$, $h(Q) = g_1(Q) = \cdots = g_{r-1}(Q) = 0$ and

$$\det\left(\frac{\partial(h, g_1, \dots, g_{r-1})}{\partial(x_1, \dots, x_r)}\right)(Q) = \det\left(\frac{\partial(g_1, \dots, g_{r-1})}{\partial(x_2, \dots, x_r)}\right)(Q) \neq 0$$

(by (23)). Thus Q is a $(\tilde{k}, \vec{\sigma})$-definable point of K^r and so lies in $k^r (\subseteq (K^-)^r)$—a contradiction. □

Now by (19), (22), (23) and 6.2 (see also the comments following the proof of 6.2) there exists a parameterization, $\{\langle I_j, \psi_j\rangle : 1 \leq j \leq N\}$ say, of $V \cap k^r$ in $\tilde{k}$. (Note, by the way, that $V \cap k^r \neq \emptyset$ by 8.1.) Let $I_j = (a_j, b_j)$ where $a_j \in k \cup (-\infty\}$, $b_j \in k \cup \{+\infty\}$ for $j = 1, \dots, N$.

8.3. Claim. *Either $q_1 \notin K^-$, or else there is some $j = 1, \dots, N$ such that either $0 < q_1 - a_j < \alpha$ or $0 < b_j - q_1 < \alpha$ for all $\alpha \in k$ with $\alpha > 0$.*

Proof. Suppose $q_1 \in K^-$. Now we must have $a_j < q_1 < b_j$ for some $j = 1, \dots, N$, for otherwise we could find $a, b \in k$ with $a < q_1 < b$ such that no $\langle p_1, \dots, p_r\rangle \in V \cap k^r$ satisfies the formula $a < x_1 < b$, and this contradicts 8.1. Let $a = \max\{a_j : 1 \leq j \leq N$ and $a_j < q_1 < b_j\}$ and $b = \min\{b_j : 1 \leq j \leq N$ and $a_j < q_1 < b_j\}$. Suppose, for a contradiction, that there is some $\alpha \in k$, $\alpha > 0$ such that $q_1 - a > \alpha$ and $b - q_1 > \alpha$. Then $a < a + \alpha < q_1 < b - \alpha < b$ so clearly $[a + \alpha, b - \alpha] \subseteq I_j$ for all j such that $a_j < q_1 < b_j$. (In the case $a = -\infty$, replace $a + \alpha$ by any element of k which is less than q_1. This is possible since $q_1 \in K^-$. Proceed similarly if $b = \infty$.)

Now since each ψ_j is continuous, there is some $B \in k$ such that $\|\psi_j(t)\| < B$ for all j such that $a_j < q_1 < b_j$ and for all $t \in k$ with $a + \alpha \leq t \leq b - \alpha$. Now let $c = \max(\{a+\alpha\} \cup \{b_j : b_j < q_1\})$ and $d = \min(\{b-\alpha\} \cup \{a_j : a_j > q_1\})$. Then by 8.2 (and 6.2), there is no $\langle p_1, \dots, p_r\rangle \in V \cap k^r$ with $c < p_1 < d$ and $\|\langle p_2, \dots, p_r\rangle\| \geq B$. This contradicts 8.1 since Q is such a point in V. □

I now claim that in addition to (19)–(24) we may assume that:

(25) $$q_1 > \alpha \quad \text{for all } \alpha \in k.$$

For if this is not already the case, then by 8.3 we have (for some $a, b \in k$) either (a) $q_1 < \alpha$ for all $\alpha \in k$, or (b) $0 < q_1 - a < \alpha$ for all $\alpha \in k$, $\alpha > 0$, or (c) $0 < b - q_1 < \alpha$ for all $\alpha \in k$, $\alpha > 0$. Define $h \in M^{r+1}(\tilde{k}, \widetilde{K}, \vec{\sigma})$ by

$$h(x_1, \dots, x_{r+1}) = \begin{cases} x_1 + x_{r+1} & \text{in case (a)}, \\ x_{r+1}(x_1 - a) - 1 & \text{in case (b)}, \\ x_{r+1}(b - x_1) - 1 & \text{in case (c)}. \end{cases}$$

In all cases there is a unique $q_{r+1} \in K$ such that $\langle Q, q_{r+1}\rangle$ ($= Q'$, say) satisfies $g_1(Q') = \cdots = g_{r-1}(Q') = h(Q') = g_r(Q') = 0$, and clearly $q_{r+1} > \alpha$ for all $\alpha \in k$. Further, by immediate inspection or routine calculation, (19)–(22) and (24) all hold for the system $g_1, \dots, g_{r-1}, h, g_r, Q'$. Actually, (23) holds too, but more relevant for present purposes is the fact (again proved by direct calculation) that if $P \in K^{r+1}$ and $g_1(P) = \cdots = g_{r-1}(P) = h(P) = 0$, then $\det(\frac{\partial(g_1, \dots, g_{r-1}, h)}{\partial(x_1, \dots, x_r)})(P) \neq 0$. Now relabel variables (as in the justification of stage (3) in section 7) so that x_{r+1} becomes x_1. Then (19)–(25) are satisfied for the new system, and we revert to the original notation.

8.4. Claim. *There exists a finite subset S of k, an element B of k and a positive rational number θ such that:*

(i) $0 \le \alpha \le 1$ *for all* $\alpha \in S$;

(ii) *for any* $\langle p_1, \ldots, p_r \rangle \in K^r$ *with* $p_1 > B$ *and* $\langle p_1, \ldots, p_r \rangle \in V$, *and any* i *such that the variable* x_i *is* $\langle \vec{\sigma}, \sigma_{n+1} \rangle$*-bounded, there exists* $a \in S$ *such that* $|p_i - a| < p_1^{-\theta}$.

Proof. By 8.1 it is sufficient to prove this claim with K replaced by k in (ii), so we work in $\tilde{k}$. Let $\mathcal{S}$ be a parameterization of $V \cap k^r$ in $\tilde{k}$ and suppose $\langle I, \psi \rangle \in \mathcal{S}$ is such that I is unbounded to the right. Say $\psi = \langle \psi_2, \ldots, \psi_r \rangle$. Suppose that the variable x_i is $\langle \vec{\sigma}, \sigma_{n+1} \rangle$-bounded. Then by (22) and (25), $2 \le i \le r$ and $0 < \psi_i(t) < 1$ for all $t \in I$. It clearly follows from 3.5 (with $\tilde{K} = \tilde{k}$, $g = \psi_i$) that $\psi_i(t) \to a_i$ as $t \to \infty$ for some $a_i \in k$ with $0 \le a_i \le 1$. Further, by applying 3.5 again with $g = \psi_i - a_i$, there exists a positive rational, θ_i say, such that $|\psi_i(t) - a_i| < t^{-\theta_i}$ for all sufficiently large $t \in k$. The claim now follows since there are only finitely many possibilities for $\langle I, \psi \rangle$ and i. □

8.5. Claim. *There exists a positive integer μ and an element B' of k such that for any* $\langle p_1, \ldots, p_r \rangle \in V \cap k^r$ *with* $p_1 > B'$ *we have* $|g_r(p_1, \ldots, p_r)| > p_1^{-\mu}$.

Proof. By (24) and 3.3, g_r has only finitely many zeros on $V \cap k^r$. The claim now follows from 3.5 by an argument similar to that of 8.4. (Consider $g(t) = g_r(t, \psi_2(t), \ldots, \psi_r(t))$.) □

Of course we would be done if we could show that 8.5 remained true with V in place of $V \cap k^r$. To achieve this we shall approximate g_r by a polynomial (uniformly in both $\tilde{K}$ and $\tilde{k}$) and apply 8.1.

By (20), $g_r(x_1, \ldots, x_r)$ has the form $\sigma_{n+1}(x_1, \ldots, x_r) - x_e$, and by 2.1, $\sigma_{n+1}(x_1, \ldots, x_r)$ has the form $F_i(y_1, \ldots, y_m)$ for some $i = 1, \ldots, l$ and some $y_1, \ldots, y_m \in \{0, 1, x_1, \ldots, x_r\}$. Now (working in $\mathbb{R}$) consider the function $G_i : U \to \mathbb{R}$ (cf. section 1). Recall that U is an open set containing $[0,1]^m$, G_i is C^∞ (in fact analytic) on U and $G_i \restriction [0,1]^m = F_i \restriction [0,1]^m$. From now on I write F, G for F_i, G_i respectively.

Since $[0,1]^m$ is compact, there exists a positive rational number ε such that for each $P \in [0,1]^m$, $B_\varepsilon(P)$ ($\stackrel{\text{def}}{=}$ the open Euclidean ball in $\mathbb{R}^m$ with centre P and radius ε) is contained in U. We may further assume that G and all its derivatives are bounded (though not necessarily uniformly) on $\bigcup\{B_\varepsilon(P) : P \in [0,1]^m\}$. Now by Taylor's theorem with Lagrange's form of the remainder, we have

$$G(p_1 + t_1, \ldots, p_m + t_m) = \sum_{i=0}^{\lambda} \left[\frac{1}{i!} \left(\sum_{j=1}^{m} t_j \frac{\partial}{\partial x_j} \right)^i G \right] (P) + R_\lambda, \tag{26}$$

for all $P = \langle p_1, \ldots, p_m \rangle \in [0,1]^m$, $\langle t_1, \ldots, t_m \rangle \in B_\varepsilon(0)$ and $\lambda \in \mathbb{N}$, where

$$R_\lambda = \left[\frac{1}{(\lambda+1)!} \left(\sum_{j=1}^{m} t_j \frac{\partial}{\partial x_j} \right)^{\lambda+1} G \right] (P') \tag{27}$$

for some $P' \in B_\varepsilon(P)$.

By our boundedness assumption on G it follows that for all $\lambda \in \mathbb{N}$, there exists $C_\lambda \in \mathbb{N}$ such that for all $\langle t_1, \ldots, t_m \rangle \in B_\varepsilon(0)$,

$$|R_\lambda| < C_\lambda \cdot (\max\{|t_i| : 1 \le i \le m\})^{\lambda+1}. \tag{28}$$

Now by (1), (2) (see the beginning of section 1) and (26), (28) it follows that for all $\lambda \in \mathbb{N}$ and all monomials $\pi(x_1, \ldots, x_m)$ of degree $\le \lambda$, there exist terms $\tau_\pi^\lambda(x_1, \ldots, x_m)$ of $\widetilde{L}$ such that for all $P = \langle p_1, \ldots, p_m \rangle \in [0,1]^m$ and $\langle t_1, \ldots, t_m \rangle \in B_\varepsilon(0)$ with $\langle p_1 + t_1, \ldots, p_m + t_m \rangle \in B_\varepsilon(P) \cap [0,1]^m$, we have

$$\begin{aligned} &|\lambda! F(p_1 + t_1, \ldots, p_m + t_m) - \sum_\pi^{(\lambda)} \tau_\pi^\lambda(P) \cdot \pi(t_1, \ldots, t_m)| \\ &\qquad < \lambda! C_\lambda (\max\{|t_i| : 1 \le i < m\})^{\lambda+1} \end{aligned} \tag{29}$$

where the summation is over the monomials of degree $\le \lambda$.

I now want to apply (29) in $\widetilde{K}$ (and $\tilde{k}$). Recall that $\sigma_{n+1}(x_1, \ldots, x_r)$ has the form $F(y_1, \ldots, y_m)$ for some $y_1, \ldots, y_m \in \{0, 1, x_1, \ldots, x_r\}$. I therefore define, for each $\langle p_1, \ldots, p_r \rangle \in K^r$ and $i = 1, \ldots, m$,

$$p_i' = \begin{cases} 0 & \text{if } y_i = 0, \\ 1 & \text{if } y_i = 1, \\ p_j & \text{if } y_i = x_j. \end{cases}$$

Thus, if $\langle p_1, \ldots, p_r \rangle \in D^r(\langle \vec{\sigma}, \sigma_{n+1} \rangle, \widetilde{K})$ (in particular, if $\langle p_1, \ldots, p_r \rangle \in V$—see (22)), then $0 \le p_i' \le 1$ for $i = 1, \ldots, m$ and $\sigma_{n+1}(p_1, \ldots, p_r) = F(p_1', \ldots, p_m')$.

Now let S, θ, B be as in 8.4, μ, B' as in 8.5, and let λ_0 be an integer greater than $\frac{\mu+1}{\theta}$.

Consider the point $Q \in K^r$. Then $Q \in V$ and $q_1 > B$ (by (25)) so we may define a_i (for $i = 1, \ldots, m$) as the unique $a \in S \cup \{0,1\}$ such that $|q_i' - a| < q_1^{-\theta}$. Note that $a_i \in k$ and $0 \le a_i \le 1$ for $i = 1, \ldots, m$. Further, $\langle q_1' - a_1, \ldots, q_m' - a_m \rangle \in B_\varepsilon(0)$ (since $0 \le q_1^{-\theta} < \varepsilon$, ε, θ being positive rationals) and $\langle q_1', \ldots, q_m' \rangle \in B_\varepsilon(\langle a_1, \ldots, a_m \rangle) \cap [0,1]^m$. Also, $g_r(Q) = 0$ so $F(q_1', \ldots, q_m') = q_e$. Hence, by (29) applied in $\widetilde{K}$, we obtain:

$$\left| \lambda_0! q_e - \sum_\pi^{(\lambda_0)} \tau_\pi^{\lambda_0}(a_1, \ldots, a_m) \cdot \pi(q_1' - a_1, \ldots, q_m' - a_m) \right| < \lambda_0! C_{\lambda_0} \cdot q_1^{-\theta(\lambda_0+1)}. \tag{30}$$

We also clearly have

$$q_1 > \max(B', 2C_\lambda, \varepsilon^{-\theta^{-1}}) \tag{31}$$

and

$$|q_i' - a_i| < q_1^{-\theta}, \quad \text{for } i = 1, \ldots, m. \tag{32}$$

Now since $\tilde{k} \subseteq \widetilde{K}$, all the $\tau_\pi^{\lambda_0}(a_1, \ldots, a_m)$'s are elements of k (and the evaluation of the term is absolute between $\widetilde{K}$ and $\tilde{k}$). We may therefore express the conjunction of (30), (31) and (32) as $\chi(q_1, \ldots, q_r)$, where $\chi(x_1, \ldots, x_r)$ is a formula of $\overline{L}$ with parameters in k. It follows from 8.1 that (30), (31) and (32) hold in $\tilde{k}$ for some $\langle \overline{p}_1, \ldots, \overline{p}_r \rangle \in V \cap k^r$ in place of $\langle q_1, \ldots, q_r \rangle$. However, we may also apply (29) in $\tilde{k}$ with $p_i = a_i$ and $t_i = \overline{p}_i' - a_i$. (Note that $\langle t_1, \ldots, t_m \rangle \in B_\varepsilon(0)$ by the new (31) and (32), and $\langle a_1 + t_1, \ldots, a_m + t_m \rangle \in B_\varepsilon(\langle a_1, \ldots, a_m \rangle) \cap [0,1]^m$ since $\langle \overline{p}_1, \ldots, \overline{p}_r \rangle \in V$, where all these notions are being interpreted in $\tilde{k}$.) Combining this with the new (30) (and using the new (32)) gives:

$$\begin{aligned}|F(\overline{p}_1',\ldots,\overline{p}_m') - p_e| &< 2C_\lambda \cdot \overline{p}_1^{-\theta(\lambda_0+1)} \\ &< 2C_\lambda \cdot \overline{p}_1^{-\mu-1} \quad \text{(by choice of } \lambda_0) \\ &< \overline{p}_1^{-\mu} \quad \text{(by the new (31))},\end{aligned}$$

i.e. $|g_r(\overline{p}_1,\ldots,\overline{p}_r)| < \overline{p}_1^{-\mu}$ is true in $\tilde{k}$. Since $\overline{p}_1 > B'$ (by the new (31)), this contradicts 8.5 and establishes 2.9. □

The proof of the first main theorem is now complete.

9. Towards the proof of the second main theorem

Recall that this states that the theory of the structure $\langle \overline{\mathbb{R}}; \exp\rangle$ is model complete. Here, exp denotes the exponential function $x \mapsto e^x$ defined for *all* $x \in \mathbb{R}$ and, as before, $\overline{\mathbb{R}}$ denotes the ordered field of real numbers (in the language of ordered rings). Let us denote the theory and language of $\langle \overline{\mathbb{R}}; \exp\rangle$ by T_{exp} and L_{exp} respectively. Then, by the brief discussion of model completeness in section 1, we must show that if $k, K \models T_{\text{exp}}$ and k is a substructure of K, then any existential sentence with parameters in k which is true in K is also true in k.

Let us fix $k, K \models T_{\text{exp}}$ with k a substructure of K for the rest of this section. Henceforth, I shall not distinguish notationally between structures and their domains, nor between terms of a language and their interpretations in given structures.

Now consider Theorem 7.2 in the case $m = l = 1$, $C = \emptyset$, $H_1 = \exp$, $\tilde{K}' = K$ and $\tilde{k}' = k$. This result tells us that it is sufficient to show that whenever $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in k[x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_n)]$ then there exists $b \in k$ such that if $\vec{\alpha} = \langle \alpha_1, \ldots, \alpha_n\rangle \in K^n$ satisfies $f_1(\vec{\alpha}) = \cdots = f_n(\vec{\alpha}) = 0$ and $J(f_1, \ldots, f_n)(\vec{\alpha}) \neq 0$ (where, as before, $J(f_1, \ldots, f_n)$ denotes the determinant of the Jacobian matrix $(\frac{\partial f_i}{\partial x_j})_{1 \le i,j \le n}$), then $|\alpha_i| < b$ for $i = 1, \ldots, n$.

(This reduction of the problem of proving the model completeness of $\langle \overline{\mathbb{R}}; \exp\rangle$ was already established in [16] (Theorem 2).)

We shall prove this by induction on the number of exponentials actually occurring in $f_1, \ldots, f_n$. However, in eliminating an exponential we shall introduce new variables and their exponentials but in such a way that only values of the new variables lying between 0 and 1 will be relevant. This will cause no problems at the base step of the induction because of the model completeness of the structure $\langle \overline{\mathbb{R}}; \exp \restriction [0,1]\rangle$ (which follows from the first main theorem—see section 1, example (A)). Now it turns out to be technically more convenient to avoid the use of truncated functions, so I define the function e (in any model of T_{exp}) by $e(x) = \exp((1 + x^2)^{-1})$ (see section 1, example (A) again). We are thus led to the following

9.1. Definition. Let $n \in \mathbb{N}$, $s \subseteq \{1, \ldots, n\}$. Then M_n^s denotes the ring of functions from K^n to K generated (as a ring) over k (considered as a field of constant functions) by x_i, $(1 + x_i^2)^{-1}$, $e(x_i)$ (for $i = 1, \ldots, n$) and $\exp(x_i)$ (for $i \in s$).

Notice that, for any $n \in \mathbb{N}$ and $s \subseteq \{1, \ldots, n\}$, M_n^s is a Noetherian ring of K-definable, C^∞ (in the sense of K) functions from K^n to K. Further, M_n^s is closed under differentiation and so, in particular, for any $f_1, \ldots, f_n \in M_n^s$ we have

$J(f_1, \ldots, f_n) \in M_n^s$. The results of sections 4 and 5 are therefore applicable and we summarize them now in a form suitable for application here.

9.2. Proposition. *Let* $n \in \mathbb{N}$, $s \subseteq \{1, \ldots, n\}$.

(i) *Suppose* $f \in M_n^s$, $\vec{\alpha} \in K^n$ *and* $f(\vec{\alpha}) = 0$. *Then there exist* $f_1, \ldots, f_n \in M_n^s$ *and* $\vec{\beta} \in K^n$ *such that* $f(\vec{\beta}) = f_1(\vec{\beta}) = \cdots = f_n(\vec{\beta}) = 0$ *and* $J(f_1, \ldots, f_n)(\vec{\beta}) \neq 0$.

(ii) *If, in* (i), $\vec{\alpha}$ *is an isolated zero of* f, *then we may take* $\vec{\beta} = \vec{\alpha}$.

(iii) *Let* $f_1, \ldots, f_n \in M_n^s$. *Then there are only finitely many* $\vec{\gamma} \in K^n$ *such that* $f_1(\vec{\gamma}) = \cdots = f_n(\vec{\gamma}) = 0$ *and* $J(f_1, \ldots, f_n)(\vec{\gamma}) \neq 0$.

Proof. For (i) apply Theorem 5.1 with $\overline{\overline{T}} = T_{\exp}$, $\overline{\overline{K}} = K$, $M = M_n^s$, $U = K^n$ and $S = \{\vec{\gamma} \in K^n : f(\vec{\gamma}) = 0\} (= V(f))$.

For (ii) apply Theorem 4.9 (with $\overline{\overline{T}} = T_{\exp}$, $\overline{\overline{K}} = K$, $P_0 = \vec{\alpha}$ and $M = \{[g \restriction U, U] : U$ an open neighbourhood of $\vec{\alpha}$ in K^n and $g \in M_n^s\}$) repeatedly for $m = 0, \ldots, n-1$. We must eventually find $f_1, \ldots, f_n \in M_n^s$ such that $\vec{\alpha} \in V^{ns}(f_1, \ldots, f_n)$ because otherwise we would have (ii) of Theorem 4.9 holding for some $f_1, \ldots, f_m$ with $m < n$ and, in particular, for $[h, W] = [f, K^n]$. But this contradicts the implicit function theorem applied in K (see the beginning of section 4) and the fact that $\vec{\alpha}$ is an isolated zero of f.

Finally, note that the sequence $(1 + x_1^2)^{-1}, \ldots, (1 + x_n^2)^{-1}, e(x_1), \ldots, e(x_n)$, $\exp(x_{i_1}), \ldots, \exp(x_{i_m})$ (where $s = \{i_1, \ldots, i_m\}$) is a Pfaffian chain on $\mathbb{R}^n$. Statement (iii) follows upon transferring Theorem 3.1 to K. □

Let us now assume that the second main theorem is false. By the discussion above it follows that there exists $m \in \mathbb{N}$ such that:

$$(*)_m \quad \begin{array}{l} \text{for some } n \in \mathbb{N}, n \geq m, \text{ there exist } \vec{\alpha} = \langle \alpha_1, \ldots, \alpha_n \rangle \in K^n, \\ l \in \{1, \ldots, n\} \text{ and } s \subseteq \{1, \ldots, n\} \text{ with } \operatorname{card}(s) = m \text{ such that for some} \\ f_1, \ldots, f_n \in M_n^s, f_1(\vec{\alpha}) = \cdots = f_n(\vec{\alpha}) = 0 \neq J(f_1, \ldots, f_n)(\vec{\alpha}). \text{ Further,} \\ |\alpha_l| > b \text{ for all } b \in k, \text{ and if } m > 0, \text{ then } l \in s. \end{array}$$

(Of course, the comments above imply that we could take $n = m$ and $s = \{1, \ldots, n\}$ here, but, as already mentioned, the point is that we shall be reducing m at the expense of extra variables and e-terms.)

Choose m minimal such that $(*)_m$ holds. I first claim that $m > 0$. For consider the structure with the same domain and ordered ring structure as K, but with exp replaced by e. Call the resulting structure K' and proceed similarly to obtain k' from k. Clearly k' is a substructure of K' and they are both models of the complete theory of the structure $\langle \mathbb{R}; x \mapsto \exp((1 + x^2)^{-1}) \rangle$. But by example (A) of section 1, this theory is model complete. This contradicts $(*)_0$ and 9.2 (iii).

Now for our minimal (non-zero) m, choose $n, \vec{\alpha}, l, s$ and $f_1, \ldots, f_n$ witnessing $(*)_m$. In the final section of this paper I shall establish (independently of all assumptions being made here) a property of elements of models of $T_{\exp}$ and their exponentials which implies the following:

9.3. There exist integers n_i (for $i \in s$), not all zero, and $c \in k$ such that $0 < c + \sum_{i \in s} n_i \alpha_i < 1$.

Assuming 9.3, note that since $|\alpha_l| > b$ for all $b \in k$, we cannot have $n_i = 0$ for all $i \in s \backslash \{l\}$. Suppose, for convenience, that $1 \in s$, $n_1 \neq 0$, and $1 \neq l$. We may assume that $n_1 > 0$ for if $n_1 < 0$ simply replace n_i by $-n_i$ (for $i \in s$) and c by $1 - c$ in 9.3. Now set $\alpha_{n+1} = \exp(\alpha_1)$ and choose $\alpha_{n+2} \in K$ so that $\alpha_{n+2} > 0$ and $(1 + \alpha_{n+2}^2)^{-1} = c + \sum_{i \in s} n_i \alpha_i$. This is possible since K, as a field, is real closed.

Now let $g_i(x_1, \ldots, x_{n+1})$ be the result of replacing $\exp(x_1)$ by x_{n+1} in $f_i(x_1, \ldots, x_n)$. Then $g_i \in M_{n+1}^{s \setminus \{1\}}$ and clearly $\langle \alpha_1, \ldots, \alpha_{n+2} \rangle$ is a solution of the following system of equations:

$$\Lambda(x_1, \ldots, x_{n+2}) : \begin{cases} g_1(x_1, \ldots, x_{n+1}) = 0, \\ \vdots \\ g_n(x_1, \ldots, x_{n+1}) = 0, \\ (1 + x_{n+2}^2)^{-1} - c - \sum_{i \in s} n_i x_i = 0, \\ x_{n+1}^{n_1} \cdot \exp(c) \cdot \prod_{j \in s^+} \exp(x_j)^{n_j} \\ \qquad -e(x_{n+2}) \cdot \prod_{j \in s^-} \exp(x_j)^{-n_j} = 0, \end{cases}$$

where $s^{\pm} = \{j \in s : j > 1 \text{ and } \pm n_j > 0\}$ (respectively). (The last equation is obtained by exponentiating the previous one, substituting x_{n+1} for $\exp(x_1)$ and rearranging. An empty product is interpreted as 1.)

Now by 9.2 (iii) there exists a K-definable open neighbourhood, U say, of $\langle \alpha_1, \ldots, \alpha_n \rangle$ (in K^n) such that $\langle \alpha_1, \ldots, \alpha_n \rangle$ is the only solution of $f_1(x_1, \ldots, x_n) = \cdots = f_n(x_1, \ldots, x_n) = 0 \neq J(f_1, \ldots, f_n)(x_1, \ldots, x_n)$ in U. Since

$$J(f_1, \ldots, f_n)(\alpha_1, \ldots, \alpha_n) \neq 0$$

we may actually suppose that $\langle \alpha_1, \ldots, \alpha_n \rangle$ is the only solution of $f_1(x_1, \ldots, x_n) = \cdots = f_n(x_1, \ldots, x_n) = 0$ in U. I now claim that $\langle \alpha_1, \ldots, \alpha_{n+2} \rangle$ is the only solution of the system $\Lambda(x_1, \ldots, x_{n+2})$ lying in the open subset $U \times K^+ \times K^+$ of K^{n+2} (where $K^+ = \{a \in K : a > 0\}$). For suppose that $\langle \beta_1, \ldots, \beta_{n+2} \rangle$ is such a solution. Since $\beta_{n+1} > 0$ and $n_1 \neq 0$ the last two equations force $\beta_{n+1} = \exp(\beta_1)$. The first n equations now force $f_i(\beta_1, \ldots, \beta_n) = 0$ for $i = 1, \ldots, n$ and hence, since $\langle \beta_1, \ldots, \beta_n \rangle \in U$, $\beta_i = \alpha_i$ for $i = 1, \ldots, n$. Further, $\beta_{n+1} = \exp(\beta_1) = \exp(\alpha_1) = \alpha_{n+1}$. Finally, the penultimate equation and the condition $\beta_{n+2} > 0$ force $\beta_{n+2} = \alpha_{n+2}$.

Now let f be the sum of the squares of the $n + 2$ functions appearing in $\Lambda(x_1, \ldots, x_{n+2})$. Then $f \in M_{n+2}^{s \setminus \{1\}}$ (note that $c, \exp(c) \in k$) and we have shown that $\langle \alpha_1, \ldots, \alpha_{n+2} \rangle$ is an isolated zero of f. By 9.2 (ii) it follows that there exist $h_1, \ldots, h_{n+2} \in M_{n+2}^{s \setminus \{1\}}$ such that $h_1(\alpha_1, \ldots, \alpha_{n+2}) = \cdots = h_{n+2}(\alpha_1, \ldots, \alpha_{n+2}) = 0 \neq J(h_1, \ldots, h_{n+2})(\alpha_1, \ldots, \alpha_{n+2})$. Since $l \in s \setminus \{1\}$, this implies that $(*)_{m-1}$ holds which contradicts the minimality of m and establishes the second main theorem modulo 9.3.

10. Smooth 0-minimal theories

We touched on the notion of 0-minimality in section 2 where it was needed to establish asymptotic formulas for definable functions in structures covered by the first main theorem. We now require a deeper asymptotic analysis and I must assume that the reader is familiar with the basic general properties of 0-minimal structures. These can be found in the foundational papers [10] and [7]. (See also [15] for more recent developments.)

For this section let $\widetilde{\mathbb{R}}$ be any 0-minimal expansion of the real ordered field $\overline{\mathbb{R}}$ and let $\widetilde{T}$ denote the complete theory of $\widetilde{\mathbb{R}}$. Then $\widetilde{T}$ admits definable Skolem functions and the closure of $\{0\}$ under these functions in any model K of $\widetilde{T}$ is an Archimedean-ordered elementary substructure of K (because $\widetilde{T}$ is complete and has an Archimedean ordered model).

Suppose that $K \models \widetilde{T}$, $k \preceq K$ (i.e. k is an elementary substructure of K) and $n \in \mathbb{N}$.

I shall say that a function from K^n to K or a subset of K^n is *k-definable* if it is definable by a formula of the language of $\widetilde{T}$ possibly involving parameters from k.

Consider now the following condition on $\widetilde{T}$:

(S1) For any $K \models \widetilde{T}$ and any K-definable function $f : K \to K$, there exists $N \in \mathbb{N}$ such that $|f(x)| \leq x^N$ for all sufficiently large $x \in K$.

10.1. Theorem. *Suppose $\widetilde{T}$ satisfies* (S1). *Let $K \models \widetilde{T}$ and suppose that R is a convex subring of K. Let I be the (convex) ideal of R consisting of those elements of R which are not invertible in R (i.e. I is the unique maximal ideal of R). Then there exists $k_0 \preceq K$ such that $k_0 \subseteq R$ and such that for each $a \in R$, $k_0 \cap (a + I)$ contains exactly one element, i.e. k_0* splits *R.*

Proof. Clearly the set $\mathcal{S} = \{k : k \preceq K \text{ and } k \subseteq R\}$ satisfies the hypotheses of Zorn's lemma ($\mathcal{S}$ is nonempty since it contains the Skolem closure of $\{0\}$ in K). Let k_0 be a maximal element of $\mathcal{S}$. Then $k_0 \preceq K$ (so $k_0 \models \widetilde{T}$), $k_0 \subseteq R$ and clearly, since k_0 is a field, $k_0 \cap (a + I)$ contains at most one element for each $a \in R$.

I claim that for all $a \in R$ there exists $\alpha \in k_0$ such that $\alpha > a$. For suppose a is a counterexample. Since $\widetilde{T}$ has definable Skolem functions the set $\{f(a) : f : K \to K,\ f \text{ a } k_0\text{-definable function}\}$ is the domain of an elementary substructure of K (containing k_0) which, by the maximality of k_0, contains an element greater than every element of R. Suppose $f(a)$ is such an element (where $f : K \to K$ is k_0-definable). By (S1) there is an element $b \in k_0$ and $N \in \mathbb{N}$ such that $k_0 \models \forall x > b(|f(x)| \leq x^N)$. Since $k_0 \preceq K$ and $a > b$ we have $|f(a)| \leq a^N$ (in K), contradicting the fact that R is a subring of K.

Now suppose that $a \in R$ and that $k_0 \cap (a + I) = \emptyset$. It is again sufficient (for a contradiction) to show that $f(a) \in R$ for any k_0-definable function $f : K \to K$. So let f be such a function. By a result of [10] there are elements $a_1 < a_2 < \cdots < a_n$ of k_0 such that (setting $a_0 = -\infty$, $a_{n+1} = +\infty$) f is (weakly) monotonic (in k_0, and hence in K) on each open interval (a_i, a_{i+1}) for $i = 0, \ldots, n$. Thus, by the claim above, there exist $b, c \in k_0$ with $b < a < c$ and f (weakly) monotonic on (b, c). Since $k_0 \cap (a + I) = \emptyset$ we have $c - a, a - b > \beta$ for all $\beta \in I$, and hence $(c - a)^{-1}$, $(a - b)^{-1} \in R$. By the claim there exists $d \in k_0$ such that $d > (c - a)^{-1}, (a - b)^{-1}$. But then $d^{-1} \in k_0$ and $b < b + d^{-1} < a < c - d^{-1} < c$. It follows that $f(a)$ lies between the elements $f(b + d^{-1})$, $f(c - d^{-1})$ of k_0 and hence $f(a) \in R$, as required. □

Let $K \models \widetilde{T}$. For any subset A of K denote by $\mathcal{C}\ell(A)$ the closure of A (in K) under the Skolem functions of $\widetilde{T}$. The set A is said to *generate* K if $\mathcal{C}\ell(A) = K$, and is called *independent* if $a \notin \mathcal{C}\ell(A \backslash \{a\})$ for each $a \in A$. An independent set that generates K is called a *basis* for K. It was shown in [10] that this notion of independence has the exchange property, and hence any independent subset of K can be extended to a basis for K and all bases for K have the same cardinality. The cardinality of any basis for K is denoted $\dim(K)$.

If $k \preceq K$, then all the above remarks apply "over k", i.e. to the notion of closure under k-definable functions. I denote by $\dim_k(K)$ the cardinality of any basis for K over k. If $k_0 \preceq k_1 \preceq K$ it is easy to verify that $\dim_{k_0}(K) = \dim_{k_0}(k_1) + \dim_{k_1}(K)$ provided that $\dim_{k_0}(K)$ is finite.

I now introduce another notion of dimension for models of $\widetilde{T}$. Indeed, let K be any real-closed ordered field. Recall that an element a of K is called *finite* if $|a| < n$ for some $n \in \mathbb{N}$, and *infinitesimal* if $|a| < \frac{1}{n}$ for all $n \in \mathbb{N}\backslash\{0\}$. The set $Fin(K)$ of finite elements of K forms a convex subring of K with unique maximal ideal $\mu(K)$, the set of infinitesimals. Further, the set $Fin(K)\backslash\mu(K)$ is a multiplicative subgroup of $K\backslash\{0\}$, and I call the quotient of the latter by the former the *value group* of K and denote it by $V(K)$. It is usual to write $V(K)$ as an additive group and as such it can be ordered by setting $a/(Fin(K)\backslash\mu(K)) > 0$ if and only if $a \in \mu(K)$. Further, $V(K)$ is a divisible group (since mth roots of positive elements exist in K for all $m \in \mathbb{N}\backslash\{0\}$) and is therefore an ordered $\mathbb{Q}$-vector space. I denote its dimension over $\mathbb{Q}$ by valdim(K).

The map $\nu_K : K \to V(K) \cup \{\infty\}$, extending the natural map $K\backslash\{0\} \to V(K)$ by setting $\nu_K(0) = \infty$, is called the *valuation map* of K. It is easy to verify the following (where we set $\infty > \alpha$ for all $\alpha \in V(K)$ and $\infty + \alpha = \alpha + \infty = \infty$ for all $\alpha \in V(K) \cup \{\infty\}$) :

(v1) $\nu_K(x \cdot y) = \nu_K(x) + \nu_K(y)$ for all $x, y \in K$;

(v2) $\nu_K(x + y) \geq \min(\nu_K(x), \nu_K(y))$ for all $x, y \in K$, with equality if $\nu_K(x) \neq \nu_K(y)$;

(v3) for all $x \in K$, $\nu_K(x) \geq 0$ if and only if $x \in Fin(K)$, and $\nu_K(x) > 0$ if and only if $x \in \mu(K)$.

My present aim is to formulate a condition on $\widetilde{T}$ that guarantees (if $\widetilde{T}$ also satisfies (S1)) that valdim$(K) \leq \dim(K)$ for all models K of $\widetilde{T}$ for which $\dim(K)$ is finite. Notice that this inequality is satisfied if $\widetilde{T}$ is just the theory of real-closed ordered fields (i.e. $\widetilde{\mathbb{R}} = \overline{\mathbb{R}}$). For in this case $\dim(K)$ is the transcendence degree (over $\mathbb{Q}$) of K and it is easy to check that if $\alpha_1, \ldots, \alpha_n \in K$ and $p(\alpha_1, \ldots, \alpha_n) = 0$ for some non-trivial polynomial p with rational coefficients, then $\nu_K(\alpha_1), \ldots, \nu_K(\alpha_n)$ are linearly dependent over $\mathbb{Q}$.

Consider the following condition on $\widetilde{T}$:

(S2) For any formula $\phi(x_1, \ldots, x_n)$ of the language of $\widetilde{T}$ there are $m, p \in \mathbb{N}$ and C^∞ functions $F_i : \mathbb{R}^{n+m} \to \mathbb{R}$ (for $i = 1, \ldots, p$), which are definable without parameters in $\widetilde{\mathbb{R}}$, and are such that

$$\widetilde{\mathbb{R}} \models \forall \vec{x} \left(\phi(\vec{x}) \leftrightarrow \exists \vec{y} \left(\|\vec{y}\| \leq 1 \wedge \bigvee_{i=1}^{p} (N_i(\vec{y}) \wedge F_i(\vec{x}, \vec{y}) = 0) \right) \right),$$

where, if $\vec{y} = y_1, \ldots, y_m$, $\|\vec{y}\| = \max\{|y_i| : i = 1, \ldots, m\}$ and $N_i(\vec{y})$ is a formula of the form $\bigwedge_{j \in s_i} y_j \neq 0$ for some $s_i \subseteq \{1, \ldots, m\}$.

□

10.2. Definition. If $\widetilde{T}$ satisfies (S1) and (S2) (and $\widetilde{T}$ is the complete theory of a 0-minimal expansion $\widetilde{\mathbb{R}}$ of $\mathbb{R}$), then $\widetilde{T}$ is called *smooth*.

10.3. Theorem. *Suppose $\widetilde{T}$ is smooth and $K \models \widetilde{T}$. If* $\dim(K)$ *is finite, then* valdim$(K) \leq \dim(K)$.

Proof. The proof is by induction on $\dim(K)$.

If K is Archimedean (i.e. $\mu(K) = \{0\}$), then $\mathrm{valdim}(K) = 0$ so the result is clear. Notice that this covers the case $\dim(K) = 0$ (by the remarks at the beginning of this section), so suppose that $\dim(K) = n > 0$ and $\mu(K) \neq \{0\}$.

By an argument similar to the one used in the claim in the proof of 10.1, there is some $a_0 \in \mu(K)$ with $a_0 > 0$ such that for all $b \in K$ with $b > 0$ we have $a_0^m < b$ for some $m \in \mathbb{N}$. Let $R = \{b \in K : |b| < a_0^{-1/m} \text{ for all } m \in \mathbb{N}\backslash\{0\}\}$. Then R is a convex subring of K and its maximal ideal, I say, is *Archimedean* in the sense that for all $a, b \in I\backslash\{0\}$, there is some $m \in \mathbb{N}$ such that $|b|^m < |a|$.

By 10.1 we may choose $k \preceq K$ such that k splits R. Since $k \neq K$ we have $\dim(k) < n$. Say $\dim(i) = n - r$ (where $r \in \mathbb{N}\backslash\{0\}$) and choose $c_1, \ldots, c_r \in K$ such that $\{c_1, \ldots, c_r\}$ is a basis for K over k. We may suppose that $c_1, \ldots, c_r \in I$ since if $c_i \notin R$ then we may replace c_i by c_i^{-1}, and if $c_i \in R$ we may replace c_i by the unique $\eta \in I$ such that $c_i + \eta \in k$ (using the splitting property of k).

Now let k^* be the algebraic closure of the field $k(c_1, \ldots, c_r)$ in K. Then clearly $\nu_K[k^*]$ is a subspace of $V(K)$ and by an argument similar to the one discussed before the formulation of (S2) we have $\dim_{\mathbb{Q}} \nu_K[k^*] \leq \dim_{\mathbb{Q}} \nu_K[k] + r$ (where $\dim_{\mathbb{Q}}$ means $\mathbb{Q}$-vector space dimension here). However, clearly $\nu_K[k] \cong V(k)$ (as $\mathbb{Q}$-vector spaces) so $\dim_{\mathbb{Q}} \nu_K[k^*] \leq \mathrm{valdim}(k) + r$ which, by the inductive hypothesis, implies $\dim_{\mathbb{Q}} \nu_K[k^*] \leq \dim(k) + r = n$. Hence it is sufficient to show that ν_K maps $k^*\backslash\{0\}$ surjectively onto $V(K)$.

Let $d \in K\backslash\{0\}$. We must show that there is some $\alpha \in k^*$ such that $\nu_K(\alpha) = \nu_K(d)$. Since $\nu_K(-\beta) = \nu_K(\beta)$ and $\nu_K(\beta^{-1}) = -\nu_K(\beta)$ for any $\beta \in K\backslash\{0\}$, and $\nu_K(\beta) \in \nu_K[k]$ for any $\beta \in R\backslash I$ (as k splits R), we may suppose that $d > 0$ and $d \in I$. Let $f : K^r \to K$ be a k-definable function such that $f(c_1, \ldots, c_r) = d$.

By (S2) there exists a k-definable, C^∞ (in the sense of K) function $F : K^{r+1+m} \to K$ (for some $m \in \mathbb{N}$) and $s \subseteq \{1, \ldots, m\}$ such that:

$$\text{for all } x \in K,\ f(c_1, \ldots, c_r) = x \text{ if and only if there exist}$$
$$(33) \qquad b_1, \ldots, b_m \in K \text{ with } b_i \neq 0 \text{ for } i \in s, \text{ and } |b_i| \leq 1 \text{ for } i = 1, \ldots, m$$
$$\text{such that } F(c_1, \ldots, c_r, x, b_1, \ldots, b_m) = 0.$$

(In applying (S2) I have replaced the parameters from k occurring in the formula defining he graph of f by variables, to obtain $\phi(\vec{z}, x_1, \ldots, x_r, x)$ say, and then taken F to be that F_i for which the corresponding disjunct holds in K when these parameters are replaced for $\vec{z}$ in ϕ, and x_i is set to c_i for $i = 1, \ldots, r$ and x is set to d.)

Now fix $\beta_1, \ldots, \beta_m \in K$ such that $\beta_i \neq 0$ for $i \in s$, $|\beta_i| \leq 1$ for $i = 1, \ldots, m$, and $F(c_1, \ldots, c_r, d, \beta_1, \ldots, \beta_m) = 0$. Since $\beta_1, \ldots, \beta_m \in R$ we may choose (uniquely) $\beta_1^0, \ldots, \beta_m^0 \in k$ such that $\beta_i - \beta_i^0 \in I$ for $i = 1, \ldots, m$ (using the splitting property of k). Further, by the Archimedean property of I we may choose $N \in \mathbb{N}$ so large that $|\beta_i| > |c_1|^N$ for $i \in s$. (We cannot have $c_1 = 0$ since c_1 occurs in a basis for K over k.)

Let $A = \{\langle x_1, \ldots, x_m\rangle \in K^m : |c_1|^N \leq |x_i| \text{ for } i \in s,\ |x_i| \leq 1 \text{ for } i = 1, \ldots, m\}$. Consider the function

$$h : K^{1+m} \to K : \langle x, x_1, \ldots, x_m\rangle \mapsto |F(c_1, \ldots, c_r, x, x_1, \ldots, x_m)|.$$

Since (in the sense of K) h is continuous, it must achieve its minimum on any closed, bounded, K-definable subset of K^{1+m}. Let γ be the minimum of h on

$([0,1]\backslash(\frac{d}{2},\frac{3d}{2}))\times A$. By (33) and the preceding remark we have that $\gamma > 0$, so we may choose $N' \in \mathbb{N}$ so large that $\gamma > |c_1|^{N'}$. Then clearly:

$$\text{(34)}\qquad \begin{aligned}&\text{for all } \alpha\in[0,1] \text{ and } \langle\beta_1',\dots,\beta_m'\rangle\in A,\\ &\text{if } |F(c_1,\dots,c_r,\alpha,\beta_1',\dots,\beta_m')|\le |c_1|^{N'},\\ &\text{then } \frac{d}{2}<\alpha<\frac{3d}{2} \text{ and hence } \nu_K(\alpha)=\nu_K(d).\end{aligned}$$

Now let $\lambda\in\mathbb{N}$ and consider the Taylor expansion to degree λ of the function $F: K^{r+1+m}\to K$, with Lagrange's form of the remainder, about the point

$$\vec{\omega}=\langle\underbrace{0,\dots,0}_{r+1},\beta_1^0,\dots,\beta_m^0\rangle(\in k^{r+1+m}).$$

This clearly provides us with (by either transferring the classical result from $\widetilde{\mathbb{R}}$ to K, or else by just proving Taylor's theorem in K) a polynomial

$$\rho_\lambda(y_1,\dots,y_r,x,x_1,\dots,x_m)$$

with coefficients in k and an element $B_\lambda\in k$ $(B_\lambda>0)$ such that:

$$\text{(35)}\qquad \begin{aligned}&\text{for all } t\in K \text{ with } 1>t>0, \text{ and all}\\ &\vec{z}\in K^{r+1+m} \text{ with } \|\vec{z}-\vec{\omega}\|<t, |F(\vec{z})-\rho_\lambda(\vec{z})|<B_\lambda\cdot t^{\lambda+1}.\end{aligned}$$

Let $t_0=2\cdot\max(|c_1|,\dots,|c_r|,d,|\beta_1-\beta_1^0|,\dots,|\beta_m-\beta_m^0|)$. Then $t_0\in I$ and $t_0>0$, so we may choose $\lambda_0\in\mathbb{N}$ so large that:

$$\text{(36)}\qquad t_0^{\lambda_0+1}<(2B_{\lambda_0})^{-1}\cdot|c_1|^{N'}.$$

Now setting $\lambda=\lambda_0$, $t=t_0$ and $\vec{z}=\langle c_1,\dots,c_r,d,\beta_1,\dots,\beta_m\rangle$ in (35), and then using (36), gives:

$$\text{(37)}\qquad |\rho(c_1,\dots,c_r,d,\beta_1,\dots,\beta_m)|<\tfrac{1}{2}\cdot|c_1|^{N'},\quad \text{where } \rho=\rho_{\lambda_0}.$$

We also clearly have:

$$\text{(38)}\qquad \langle d,\beta_1,\dots,\beta_m\rangle\in[0,1]\times A$$

and

$$\text{(39)}\qquad \|\langle c_1,\dots,c_r,d,\beta_1,\dots,\beta_m\rangle-\vec{\omega}\|<((2B_{\lambda_0})^{-1}\cdot|c_1|^{N'})^{(\lambda_0+1)^{-1}}.$$

Now (37), (38) and (39) can be expressed in the language of ordered rings and can be viewed as conditions on the point $\langle d,\beta_1,\dots,\beta_m\rangle$ with parameters in k^* (= the algebraic closure of the field $k(c_1,\dots,c_r)$ in K). Since k^* is an elementary substructure of K for the language of ordered rings (both being real-closed ordered fields) it follows that there are $\alpha,\beta_1',\dots,\beta_m'\in k^*$ such that:

$$\text{(40)}\qquad |\rho(c_1,\dots,c_r,\alpha,\beta_1',\dots,\beta_m')|<\tfrac{1}{2}\cdot|c_1|^{N'},$$

$$\text{(41)}\qquad \langle\alpha,\beta_1',\dots,\beta_m'\rangle\in[0,1]\times A,$$

and

$$\text{(42)}\qquad \|\langle c_1,\dots,c_r,\alpha,\beta_1',\dots,\beta_m'\rangle-\vec{\omega}\|<((2B_{\lambda_0})^{-1}\cdot|c_1|^{N'})^{(\lambda_0+1)^{-1}}.$$

Now by (42), we may apply (35) with $\lambda = \lambda_0$, $t = ((2B_{\lambda_0})^{-1} \cdot |c_1|^{N'})^{(\lambda_0+1)^{-1}}$ and $\vec{z} = \langle c_1, \dots, c_r, \alpha, \beta'_1, \dots, \beta'_m \rangle$ which, together with (40) gives:

$$(43) \qquad |F(c_1, \dots, c_r, \alpha, \beta'_1, \dots, \beta'_m)| < |c_1|^{N'}.$$

The required conclusion now follows from (43), (41) and (34). □

Suppose $K \models \widetilde{T}$ and $k \preceq K$. Then $\nu_K[k\setminus\{0\}]$ is a $\mathbb{Q}$-vector subspace of $V(K)$ (because k is certainly a real-closed ordered subfield of K) and I denote the dimension of $V(K)$ over $\nu_K[k\setminus\{0\}]$ by $\mathrm{valdim}_k(K)$.

I require the following generalization of 3.4.

10.4. Theorem. *Suppose $\widetilde{T}$ is smooth, $K \models \widetilde{T}$, $k \preceq K$ and that* $\dim_k(K)$ *is finite. Then* $\mathrm{valdim}_k(K) \leq \dim_k(K)$.

Proof. It is clearly sufficient to consider the case $\dim_k(K) = 1$, so let a be a generator for K over k. Suppose, for a contradiction, that $\mathrm{valdim}_k(K) \geq 2$. Then there exist k-definable functions $f, g : K \to K$ such that $\nu_K(f(a))$, $\nu_K(g(a))$ are $\mathbb{Q}$-linearly independent over $\nu_K[k\setminus\{0\}]$.

Now consider the structure $\langle K, P\rangle$, where P is the unary relation on K interpreted as (the domain of) k. Let $\langle {}^*K, {}^*P\rangle$ be an $\aleph_0$-saturated elementary extension of $\langle K, P\rangle$. Let ${}^*k$ be the elementary substructure of ${}^*K$ with domain ${}^*P$. Then ${}^*k$ is certainly $\aleph_0$-saturated. I claim that $\nu_{{}^*k}(f(a)), \nu_{{}^*k}(g(a))$ are $\mathbb{Q}$-linearly independent over $\nu_{{}^*k}[{}^*k\setminus\{0\}]$. For suppose not. Then for some $p, q \in \mathbb{Q}$ not both zero, and some $b \in {}^*k\setminus\{0\}$ we have $p\nu_{{}^*k}(f(a)) + q\nu_{{}^*k}(g(a)) + \nu_{{}^*k}(b) = 0$. This implies that $i^{-1} < |f(a)|^p \cdot |g(a)|^q \cdot |b| < i$ for some $i \in \mathbb{N}\setminus\{0\}$. Since $a \in K$ and $\langle K, P\rangle \preceq \langle {}^*K, {}^*P\rangle$ it follows that there is some $b' \in k\setminus\{0\}$ such that

$$i^{-1} < |f(a)|^p \cdot |g(a)|^q \cdot |b'| < i,$$

which contradicts the fact that $\nu_K(f(a))$, $\nu_K(g(a))$ are $\mathbb{Q}$-linearly independent over $\nu_K[k\setminus\{0\}]$.

This shows that we may suppose that k is $\aleph_0$-saturated (by taking $k = {}^*k$ and K to be the elementary substructure of ${}^*K$ generated over ${}^*k$ by a).

Now let k_0 be an elementary substructure of k such that $\dim(k_0)$ is finite and such that f, g are both k_0-definable. Consider the following set of formulas over k :

$$\Theta(x): \quad \{|f(x)|^p \cdot |g(x)|^q \cdot |b| \leq i^{-1} \vee |f(x)|^p \cdot |g(x)|^q \cdot |b| \geq i : \\ i \in \mathbb{N}\setminus\{0\}, p, q \in \mathbb{Q}, \text{ not both zero, } b \in k_0\setminus\{0\}\}.$$

Clearly $\Theta(x)$ is realised in K by a and hence $\Theta(x)$ is finitely satisfiable in k. Further, since $\dim(k_0)$ is finite, $\Theta(x)$ can be rewritten so that it contains only finitely many parameters from k_0 (namely, the elements of a basis) and hence from k. Thus $\Theta(x)$ is realized in k, by a_1 say. Let k_1 be the elementary substructure of k generated over k_0 by a_1. Then clearly $\dim(k_1) = \dim(k_0) + 1$ and $\mathrm{valdim}(k_1) \geq \mathrm{valdim}(k_0) + 2$ (since $\nu_{k_1}(f(a_1)), \nu_{k_1}(g(a_1))$ are $\mathbb{Q}$-linearly independent over $\nu_{k_1}[k_0\setminus\{0\}]$ by the definition of $\Theta(x)$). But we may repeat this argument with k_1 in place of k_0 and, indeed, continue to do so to obtain, for each $l \in \mathbb{N}$, an elementary substructure, k_l say, of k such that $\dim(k_l) = \dim(k_0) + l$ and $\mathrm{valdim}(k_l) \geq \mathrm{valdim}(k_0) + 2l$. But this contradicts 10.4 when $l = \dim(k_0) + 1$. □

Before applying 10.4 to the situation of section 9, I require the following result on ordered $\mathbb{Q}$-vector spaces.

10.5. Lemma. *Let V be an ordered $\mathbb{Q}$-vector space and U a subspace of V such that V has dimension n ($\in \mathbb{N}$) over U. Then there exists a basis $0 < v_1 < v_2 < \cdots < v_n$ for V over U such that if v is any element of V with $v > u$ for all $u \in U$ and if $v = (\sum_{i=1}^{n} q_i v_i + u_0)$ (where $q_1, \ldots, q_n \in \mathbb{Q}, u_0 \in U$), then $|v| > q \cdot v_j$ for some positive $q \in \mathbb{Q}$ where $j = \max\{i : q_i \neq 0\}$.*

Proof. Let $\overline{U}$ be the convex closure of U in V. The result is trivial if $\overline{U} = V$. Otherwise, simply observe that there exists $l \in \mathbb{N}$, with $1 \leq l \leq n$, and Archmidean-ordered $\mathbb{Q}$-vector spaces $A_1, \ldots, A_l$ such that V is isomorphic (as an ordered $\mathbb{Q}$-vector space) to $\overline{U} \times A_1 \times \cdots \times A_l$ with reverse lexicographic ordering. □

11. Bounding the solutions to exponential-polynomial equations and the completion of the proof of the second main theorem

Recall from section 9 that $L_{\exp}$ and $T_{\exp}$ denote the language and theory of the structure $\langle \overline{\mathbb{R}}; \exp \rangle$, respectively. I denote by L_e and T_e the language and theory of the structure $\langle \overline{\mathbb{R}}; e; \rangle$, where $e : \mathbb{R} \to \mathbb{R} : x \mapsto \exp((1+x^2)^{-1})$.

11.1. Theorem. *The theory T_e is smooth and model complete.*

Proof. That T_e is model complete follows from the first main theorem (see example (A) of section 1), and 0-minimality and condition (S1) follow from results in [13] (see also Corollary 3.5). For (S2), consider the function

$$e^* : \mathbb{R} \to \mathbb{R} : x \mapsto \exp(x^2 \cdot (1+x^2)^{-1})$$

and note that $e(x^{-1}) = e^*(x)$ for all $x \in \mathbb{R}\backslash\{0\}$. It follows that e^* is definable in $\langle \mathbb{R}; e \rangle$ without parameters. Notice also that e and e^* are both C^∞ throughout $\mathbb{R}$. Now let $\phi(x_1, \ldots, x_n)$ be any formula of L_e. It easily follows from the model completeness of T_e that there is a polynomial $\rho \in \mathbb{Z}[z_1, \ldots, z_{2m+2n}]$ (for some $m \in \mathbb{N}$) such that

$$(*) \qquad \begin{aligned} \langle \overline{\mathbb{R}}; e \rangle \models \forall x_1, \ldots, x_n\, (\phi(x_1, \ldots, x_n) \leftrightarrow \\ \exists y_1, \ldots, y_m\, \rho(y_1, \ldots, y_m, e(y_1), \ldots, e(y_m), \\ x_1, \ldots, x_n, e(x_1), \ldots, e(x_n)) = 0). \end{aligned}$$

Condition (S2) now follows by considering, for each $s \subseteq \{1, \ldots, m\}$, the result of replacing y_j by y_j^{-1} and $e(y_j)$ by $e^*(y_j)$ (for each $j \in s$) in the function on the right hand side of (*), and then multiplying by a suitably high power of $\prod_{j \in s} y_j$ to obtain a C^∞ function (on $\mathbb{R}$), F_s say. Thus, in the notation of (S2), p is 2^m, $N_i(\vec{y})$ is $\bigwedge_{j \in s_i} y_j \neq 0$ and F_i is F_{s_i}, where $\{s_i : i < 2^m\}$ is an enumeration of all the subsets of $\{1, \ldots, m\}$. □

Suppose now that k and K are models of $T_{\exp}$ with $k \subseteq K$. Clearly, k, K determine models of T_e with the same underlying ordered field and I denote these ("restricted") models by k', K' respectively. Certainly $k' \subseteq K'$ so $k' \preceq K'$ by the model completeness of T_e (Theorem 11.1).

Now suppose that k^* is any model of T_e such that $k' \subseteq k^* \subseteq K'$. Then for each $a \in k^*, \exp(a)$ is an element of (the domain of) K which may or may not lie in k^*. Let $E(k^*) = \{a \in k^* : \exp(a) \in k^*\}$. Clearly $E(k^*)$ is a $\mathbb{Q}$-vector subspace of the additive group of k^* (because k^* is a real-closed ordered field and hence closed under taking rational powers of positive elements) and contains the additive group of k as a subspace. It also contains $Fin(k^*)$ as a subspace because if $a \in Fin(k^*)$, then there exist $m \in \mathbb{Z}$ and $b \in k^*$ such that $\frac{m}{1+b^2} = a$, and then $\exp(a) = e(b)^m$, and $e(b)^m \in k^*$.

11.2. Lemma. *In the above situation, suppose that* $\dim_{k'}(k^*) = n$ *(as models of* T_e*), where* $n \in \mathbb{N}$*. Suppose further that* $E(k^*)$ *is at least* n *dimensional over its subspace* $k + Fin(k^*)$*. Then for each* $a \in E(k^*)$ *there is some* $b \in k$ *such that* $|a| < b$*.*

Proof. Suppose not. Let $U = k + Fin(k^*)$ and choose some subspace V of $E(k^*)$ such that $U \subseteq V$, V is exactly n-dimensional over U, and such that V contains an element α with $\alpha > b$ for all $b \in k$. Clearly this implies that $\alpha > b$ for all $b \in U$.

Let $0 < v_1 < \cdots < v_n$ be a basis for V over U as given by Lemma 10.5. Let j be minimal such that $v_j > b$ for all $b \in U$.

Now consider the elements $\nu_K(\exp(v_1)), \ldots, \nu_K(\exp(v_n))$ of the value group $V(K)$ of K. I claim that they are linearly independent over $\nu_K[k\backslash\{0\}]$. For if not there exist $q_1, \ldots, q_n \in \mathbb{Q}$, not all zero, and $c \in k\backslash\{0\}$ such that $c\exp(\sum_{i=1}^n q_i v_i) \in Fin(K)\backslash\mu(K)$ (using (v1) and (v3) of section 10). Clearly we may suppose that $c > 0$, so $c = \exp(d)$ for some $d \in k$ (since $k \models T_{\exp}$). We thus have $\exp(d + \sum_{i=1}^n q_i v_i) \in Fin(K)\backslash\mu(K)$ which, by standard properties of the exponential function, implies that $d + \sum_{i=1}^n q_i v_i \in Fin(K)$, and hence $d + \sum_{i=1}^n q_i v_i \in Fin(k^*)$. But this contradicts the linear independence of $v_1, \ldots, v_n$ over U.

Now by Theorems 10.4 and 11.1 and the hypothesis that $\dim_{k'}(k^*) = n$ it follows that $\mathrm{valdim}_{k'}(k^*) \leq n$ and hence that $\nu_K(\exp(v_1)), \ldots, \nu_K(\exp(v_n))$ span $\nu_K[k^*\backslash\{0\}]$ over $\nu_K[k\backslash\{0\}]$ (note that $\exp(v_1), \ldots, \exp(v_n) \in k^*$). In particular

$$\nu_K(v_j) = \nu_K(c) + \sum_{i=1}^n p_i \nu_K(\exp(v_i))$$

for some $c \in k\backslash\{0\}$ and $p_1, \ldots, p_n \in \mathbb{Q}$. Again, we may suppose that $c = \exp(d)$ for some $d \in k$ and hence, $\nu_K(v_j) = \nu_K(\exp(d + \sum_{i=1}^n p_i v_i))$. By (v3) of section 10, this implies that $\frac{v_j}{N} < \exp(d + \sum_{i=1}^n p_i v_i) < N v_j$ for some $N \in \mathbb{N}\backslash\{0\}$. Now the left hand inequality here implies that $d + \sum_{i=1}^n p_i v_i > 0$ (since certainly $\frac{v_j}{N} > 1$). Further, if $p_j = p_{j+1} = \cdots = p_n = 0$ we would have $0 < d + \sum_{i=1}^n p_i v_i < b$ for some $b \in k$ and hence $\frac{v_j}{N} < \exp(b)$, which contradicts the choice of v_j since $N \cdot \exp(b) \in k$. Thus $p_i \neq 0$ for some $i = j, \ldots, n$ and so, by the choice of $v_i, \ldots, v_n$, there exists $q \in \mathbb{Q}$ with $q > 0$ such that $d + \sum_{i=1}^n p_i v_i > q v_j$ (see Lemma 10.5). But, by the right hand inequality above, this implies that $N v_j > \exp(q v_j)$. However, this is absurd since certainly $v_j > r$ for all $r \in \mathbb{N}$. □

I now complete the proof of the main theorem. Recall from section 2 that we must consider the following situation:

We are given $n, m \in \mathbb{N}$ with $n \geq m > 0$, $\vec{\alpha} = \langle \alpha_1, \ldots, \alpha_n \rangle \in K^n$, $l \in \{1, \ldots, n\}$, $s \subseteq \{1, \ldots, n\}$ with $|s| = m, l \in s$, and $f_1, \ldots, f_n \in M_n^s$ (cf. Definition 2.1) such that $f_1(\vec{\alpha}) = \cdots = f_n(\vec{\alpha}) = 0$ and $J(f_1, \ldots, f_n)(\vec{\alpha}) \neq 0$. Further, $|\alpha_l| > b$ for all $b \in k$.

We must establish 9.3, for which it is clearly sufficient to show that $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly dependent over the subspace $k + Fin(K)$ of (the $\mathbb{Q}$-vector space) K.

To do this, consider the submodel, k^* say, of K' generated over k' by $\{\alpha_i : 1 \leq i \leq n\} \cup \{\exp(\alpha_i) : i \in s\}$ using the Skolem functions of T_e. Then $k^* \models T_e$ and $k' \subseteq k^* \subseteq K'$. Obviously we have $\dim_{k'}(k^*) \leq n+m$. I claim that, in fact, we have $\dim_{k'}(k^*) \leq m$. Granted this claim, it follows from 11.2 that $E(k^*)$ has dimension at most $m-1$ over $k + Fin(k^*)$ (because $\alpha_l \in E(k^*)$). But $\{\alpha_i : i \in s\} \subseteq E(k^*)$ so $\{\alpha_i : i \in s\}$ is a $\mathbb{Q}$-linearly dependent set over $k + Fin(k^*)$. A fortiori, $\alpha_1, \ldots, \alpha_n$ are $\mathbb{Q}$-linearly dependent over $k + Fin(K)$. To prove the claim let us suppose, for convenience, that $s = \{1, \ldots, m\}$. Set $\alpha_{n+i} = \exp(\alpha_i)$ for $i = 1, \ldots, m$.

Now for $1 \leq i \leq n$, pick $g_i \in M_n^{\emptyset}[x_{n+1}, \ldots, x_{n+m}]$ such that $g_i(x_1, \ldots, x_n, \exp(x_1), \ldots, \exp(x_m)) \equiv f_i(x_1, \ldots, x_m)$, and set $g_{n+i}(x_1, \ldots, x_{n+m}) = \exp(x_i) - x_{n+i}$ for $1 \leq i \leq m$. Then clearly $\langle \alpha_1, \ldots, \alpha_{n+m} \rangle$ is a solution to the system $g_i(x_1, \ldots, x_{n+m}) = 0$ $(1 \leq i \leq n+m)$. It is also easy to show, using the chain rule, elementary matrix algebra and the fact that $J(f_1, \ldots, f_n)(\alpha_1, \ldots, \alpha_n) \neq 0$, that $J(g_1, \ldots, g_{n+m})(\alpha_1, \ldots, \alpha_{n+m}) \neq 0$. It follows that the row vectors $\langle \frac{\partial g_1}{\partial x_i} : 1 \leq i \leq n+m \rangle, \ldots, \langle \frac{\partial g_n}{\partial x_i} : 1 \leq i \leq n+m \rangle$ evaluated at $\langle \alpha_1, \ldots, \alpha_{n+m} \rangle$ are linearly independent over K and hence that there exists a subset $u \subseteq \{1, \ldots, n+m\}$ of size n such that the matrix

$$\left(\frac{\partial g_i}{\partial x_j} \right)_{\substack{1 \leq i \leq n \\ j \in u}}$$

is non-singular when evaluated at $\langle \alpha_1, \ldots, \alpha_{n+m} \rangle$. Now notice that $g_1, \ldots, g_n$ are k'-definable functions (i.e. they are L_e-definable with parameters in k) so it clearly follows from Proposition 9.2(iii) (with $s = \emptyset$) that for each $j \in u$, α_j is k'-definable from $\{\alpha_i : 1 \leq i \leq n+m, i \notin u\}$. Thus the submodel of K' generated over k' by $\{\alpha_i : 1 \leq i \leq n+m, i \notin u\}$ contains $\alpha_1, \ldots, \alpha_{n+m}$ and is therefore equal to k^*. Thus $\dim_{k'}(k^*) \leq m$ as required.

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Mathematics

Supercompact cardinals, sets of reals, and weakly homogeneous trees

(set theory/descriptive set theory/large cardinals)

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ABSTRACT **It is shown that if there exists a supercompact cardinal then every set of reals, which is an element of $L(\mathbb{R})$, is the projection of a weakly homogeneous tree. As a consequence of this theorem and recent work of Martin and Steel [Martin, D. A. & Steel, J. R. (1988) *Proc. Natl. Acad. Sci. USA* 85, 6582–6586], it follows that (if there is a supercompact cardinal) every set of reals in $L(\mathbb{R})$ is determined.**

The subtle relationships between the existence of certain large cardinals and regularity properties of various simple sets of reals is one of the striking developments in modern set theory.

One of the first results in this direction is that of R. Solovay (cf. ref. 1), which states that if a measurable cardinal exists then every Σ^1_2 set of reals is Lebesgue measurable and has the property of Baire. An unusual aspect of Solovay's argument is the use of the method of forcing. Prior to this the uses of forcing had been limited to obtaining independence results. Martin (2) refined Solovay's result in showing that if there exists a measurable cardinal then every Π^1_1 set of reals is determined, briefly Π^1_1 determinacy holds. Martin actually proved a slightly stronger result: Suppose for every real, x, $x^\#$ exists. Then Π^1_1 determinacy holds. The assertion of Π^1_1 determinacy itself implies that every Σ^1_2 set of reals is Lebesgue measurable and has the property of Baire. Thus, Martin's theorem may be regarded as a strengthening of Solovay's. The methods Martin used eliminate the need for forcing.

Shelah and Woodin have shown that if a supercompact cardinal exists then every set of reals that belongs to $L(\mathbb{R})$, the smallest inner model of set theory containing the reals and the ordinals, is Lebesgue measurable and has the property of Baire. Since these properties are absolute to $L(\mathbb{R})$, if there exists a supercompact cardinal then the inner model $L(\mathbb{R})$ is a model of set theory without choice (i.e., of Zermelo–Fraenkel) in which every set of reals is Lebesgue measurable and has the property of Baire. The results of Shelah and Woodin were motivated by those of ref. 3.

I show here that the existence of a supercompact cardinal implies that every set of reals that belongs to $L(\mathbb{R})$ has a certain structural representation from which the regularity results, such as measurability, easily follow. This is made more precise through a sequence of definitions.

For our purposes the set of reals, $\mathbb{R}$, is the set ω^ω of all functions $f : \omega \to \omega$, where $\omega = \{0, 1, \ldots, k, \ldots\}$ is the set of nonnegative integers. We let $\omega^{<\omega}$ denote the set of all finite sequences of elements of ω and for $s \in \omega^{<\omega}$ let N_s be the set, $N_s = \{f \in \omega^\omega : f \restriction l(s) = s\}$, where $l(s) = \text{length}(s)$. The set $\{N_s : s \in \omega^{<\omega}\}$ generates a topology on ω^ω; it is the product topology derived from the discrete topology on ω. Endowed with this topology ω^ω is homeomorphic to the Euclidean space of irrationals. Suppose X is a set. We denote by X^ω the set of all functions $f : \omega \to X$ and we denote by $X^{<\omega}$ the set of all finite sequences of elements of X. We adopt the usual convention that $X^{<\omega}$ is the set of all functions f : dom $f \to X$ such that dom $f \in \omega$ and if $s \in X^{<\omega}$ then dom $s = l(s) = \text{length}(s)$. Suppose λ is an ordinal, $\lambda > 0$. A tree on $\omega \times \lambda$ is a subset $T \subset \omega^{<\omega} \times \lambda^{<\omega}$ such that for all pairs $(s, t) \in T$, $l(s) = l(t)$ and $(s \restriction i, t \restriction i) \in T$ for all $i < l(s)$, $i \in \omega$. Suppose T is a tree on $\omega \times \lambda$. For $s \in \omega^{<\omega}$ and for $x \in \omega^\omega$

$$T_s = \{t \in \lambda^{<\omega} : (s, t) \in T\} \text{ and } T_x = \cup\{T_{x \restriction k} : k \in \omega\}.$$

For each $x \in \omega^\omega$, $T_x \subset \lambda^{<\omega}$ and is naturally viewed as a tree on λ. $[T] = \{(x, f) : x \in \omega^\omega, f \in \lambda^\omega \text{ and } (x \restriction k, f \restriction k) \in T \text{ for all } k \in \omega\}$. We also define $p[T] = \{x : (x, f) \in [T] \text{ for some } f \in \lambda^\omega\}$. Thus $p[T] \subset \omega^\omega$, it is the *projection* of T, and $p[T] = \{x \in \omega^\omega : T_x \text{ is not well-founded}\}$.

Suppose X is a nonempty set. We denote by $m(X)$ the set of countably complete ultrafilters on the Boolean algebra $P(X)$. μ is a *measure* on X if $\mu \in m(X)$. For $\mu \in m(X)$ and $A \subset X$ we write $\mu(A) = 1$ to indicate $A \in \mu$. Suppose that $X = Y^{<\omega}$ and that $\mu \in m(Y^{<\omega})$. Since μ is countably additive, there is a unique $k \in \omega$ such that $\mu(Y^k) = 1$. Suppose $\mu_1, \mu_2 \in m(Y^{<\omega})$, $\mu_1(Y^{k_1}) = 1$, and $\mu_2(Y^{k_2}) = 1$. Then $\mu_1 < \mu_2$ (μ_2 *projects* to μ_1) if $k_1 < k_2$ and, for all $A \subset Y^{k_1}$, $\mu_1(A) = 1$ if and only if $\mu_2(A^*) = 1$ where $A^* = \{s \in Y^{k_2} : s \restriction k_1 \in A\}$.

For each $\mu \in m(X)$ there is a canonical elementary embedding $j_\mu : V \to M_\mu$ of the universe, V, into an inner model, M_μ, where M_μ is the transitive collapse of V^X/μ. Suppose $\mu_1, \mu_2 \in m(Y^{<\omega})$ and $\mu_1 < \mu_2$. Then there is also a canonical elementary embedding $j_{\mu_1\mu_2} : M_{\mu_1} \to M_{\mu_2}$ such that $j_{\mu_2} = j_{\mu_1\mu_2} \circ j_{\mu_1}$.

Suppose $\langle \mu_k : k \in \omega \rangle$ is a sequence of measures in $m(Y^{<\omega})$ such that for each $k \in \omega$, $\mu_k(Y^k) = 1$. The sequence $\langle \mu_k : k \in \omega \rangle$ is a *tower* if for all $k_1 < k_2$, $\mu_{k_1} < \mu_{k_2}$. The tower, $\langle \mu_k : k \in \omega \rangle$, is *countably complete* if for any sequence $\langle A_k : k \in \omega \rangle$ such that for each $k \in \omega$, $A_k \subset Y^k$ and $\mu_k(A_k) = 1$, there exists $f \in Y^\omega$ such that $f \restriction k \in A_k$ for all $k \in \omega$. A tower of measures in $m(Y^{<\omega})$, $\langle \mu_k : k \in \omega \rangle$, is countably complete if and only if the direct limit of the sequence $\langle M_{\mu_k} : k \in \omega \rangle$ under the maps, $j_{\mu_{k_1}\mu_{k_2}} : M_{\mu_{k_1}} \to M_{\mu_{k_2}}$ (where $k_1 < k_2$) is well-founded.

Definition 1: Suppose λ is an ordinal, $\lambda > 0$. A tree, T, on $\omega \times \lambda$ is *weakly homogeneous* if there is a partial function $\pi : \omega^{<\omega} \times \omega^{<\omega} \to m(\lambda^{<\omega})$ such that

(*i*) if $(s, t) \in \text{dom } \pi$ then $\pi(s, t)(T_s) = 1$ and

(*ii*) for all $x \in \omega^\omega$, $x \in p[T]$ if and only if there exists $y \in \omega^\omega$ such that for all $k \in \omega$, $(x \restriction k, y \restriction k) \in \text{dom } \pi$ and $\langle \pi(x \restriction k, y \restriction k) : k \in \omega \rangle$ is a countably complete tower. □

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Abbreviations: AD, axiom of determinacy; ZFC, Zermelo–Fraenkel with the axiom of choice.

Definition 2: Suppose λ is an ordinal, $\lambda > 0$. A tree, T, on $\omega \times \lambda$ is *homogeneous* if there is a partial function $\pi : \omega^{<\omega} \to m(\lambda^{<\omega})$ such that

(*i*) if $s \in \text{dom}\ \pi$ then $\pi(s)(T_s) = 1$ and

(*ii*) for all $x \in \omega^\omega$, $x \in p[T]$ if and only if for all $k \in \omega$, $x \restriction k \in \text{dom}\ \pi$, and $\langle \pi(x \restriction k) : k \in \omega \rangle$ is a countably complete tower. □

The notions of homogeneous and weakly homogeneous trees arose in the study of descriptive set theory in the context of the axiom of determinacy (AD). The concept of a homogeneous tree is implicit in early work of Martin and was formally isolated by Kechris (4).

Any tree on $\omega \times \omega$ is weakly homogeneous. Thus any Σ^1_1 subset of ω^ω is the projection of a weakly homogeneous tree. Similarly any tree on $\omega \times 1$ is homogeneous and so any closed subset of ω^ω is the projection of a homogeneous tree.

The primary interest in homogeneous or weakly homogeneous trees exists because of the structural representations they provide for their projections that are sets of reals. Suppose $A \subset \omega^\omega$. If A is the projection of a homogeneous tree then A is determined as is any preimage of A via a continuous function $F : \omega^\omega \to \omega^\omega$. If A is the projection of a weakly homogeneous tree then enough of the continuous preimages of A are determined so that one can show that many regularity properties hold for A—for example, that A is Lebesgue measurable and has the property of Baire.

Notice that if $A \subset \omega^\omega$ is the projection of a homogeneous tree and $F : \omega^\omega \to \omega^\omega$ is a continuous function then the image of A under F, $B = F''(A)$, is the projection of a weakly homogeneous tree. The converse is also true. If $B \subset \omega^\omega$ is the projection of a weakly homogeneous tree then there is a continuous function $F : \omega^\omega \to \omega^\omega$ and a set $A \subset \omega^\omega$ such that $B = F''(A)$ and A is the projection of a homogeneous tree.

If there are no measurable cardinals then $A \subset \omega^\omega$ is the projection of a homogeneous tree if and only if A is closed. This is essentially because if there are no measurable cardinals then for any λ the only elements of $m(\lambda^{<\omega})$ are the atomic measures (i.e., principal ultrafilters). Similarly without measurable cardinals $B \subset \omega^\omega$ is the projection of a weakly homogeneous tree if and only if B is a Σ^1_1 set.

For our purposes an easier formulation of weak homogeneity is actually more relevant. This is given in the easily verified lemma below.

LEMMA. *Suppose λ is an ordinal and that* T *is a tree on $\omega \times \lambda$. The tree,* T, *is weakly homogeneous if and only if there exists a countable set $\sigma \subset$ m$(\lambda^{<\omega})$ such that for all* x $\in$ p[T] *there is a countably complete tower, $\langle \mu_k : k \in \omega \rangle$, of measures in σ such that for all* k $\in \omega$, $\mu_k(T_{x \restriction k}) = 1$. □

There are two minor points. First, if T is a tree on $\omega \times \lambda$ and $\langle \mu_k : k \in \omega \rangle$ is a countably complete tower of measures in $m(\lambda^{<\omega})$, where for some $x \in \omega^\omega$, $\mu_k(T_{x \restriction k}) = 1$ for each $k \in \omega$, then x is necessarily an element of $p[T]$. The second point is that in the case of weak homogeneity (following the notation in the definition) it is only the range of π that is important.

If T is a weakly homogeneous tree then σ is a witness for this if σ satisfies the conditions in the statement of the lemma. Suppose μ is a measure in $m(X)$ and that κ is an ordinal. The measure μ is *κ-complete* if for any $S \subset \mu$ with $|S| < \kappa$, $\cap S \in \mu$; i.e., if $S \subset P(X)$ is a set of cardinality $< \kappa$ such that for all $Z \in S$, $\mu(Z) = 1$ then $\mu(\cap \{Z : Z \in S\}) = 1$. A tree T on $\omega \times \lambda$ is *κ-weakly homogeneous* if there exists a witness σ for the weak homogeneity of T containing only κ-complete measures. T is *$< \kappa$-weakly homogeneous* if T is α-weakly homogeneous for each $\alpha > \kappa$.

Martin's proof of Π^1_1 determinacy from the existence of a measurable cardinal is in essence a proof of the following. Assume κ is a measurable cardinal. Then every Π^1_1 set of reals is the projection of a κ-homogeneous tree. As an immediate consequence, if κ is a measurable cardinal then every Σ^1_2 set of reals is the projection of a κ-weakly homogeneous tree.

THEOREM 1. *Suppose κ is a supercompact cardinal. Then every set of reals that belongs to* **L(R)** *is the projection of a κ-weakly homogeneous tree.*

Proof of *Theorem 1*

An essential ingredient in the proof of *Theorem 1* is the use of generic elementary embeddings. We develop some of the necessary machinery. We use standard notation. For example, $X^Y = \{f \mid f : Y \to X\}$. If $f : Y \to X$ and $Z \subset Y$ then $f''(Z) = \{f(t) : t \in Z\}$. When considering an elementary embedding $j : V \to M$ it is always assumed that j is an elementary embedding from (V, ε) into (M, ε) and that M is transitive. If we write $j : V \to (M, E)$ then nothing is implicitly assumed except that j is an elementary embedding from (V, ε) into (M, E); in particular, M need not be transitive. Suppose $j : V \to M$ is an elementary embedding. Then the critical point of j, denoted cp(j), is the least ordinal α, if it exists, such that $j(\alpha) \neq \alpha$. Suppose X is a transitive set. Recall that κ is X-supercompact if there exists an elementary embedding $j : V \to M$ such that $\kappa = \text{cp}(j)$, $M^X \subset M$ and $X \in j(V_\kappa)$. κ is supercompact if κ is X-supercompact for each X.

Definition 3: (*i*) A nonempty set b is *stationary* if for any function $F : (\cup b)^{<\omega} \to \cup b$, $F''(a^{<\omega}) \subset a$ for some $a \in b$;

(*ii*) a set c is *closed* if for some function $F : (\cup c)^{<\omega} \to \cup c$, $c = \{a \subset \cup c : F''(a^{<\omega}) \subset a\}$;

(*iii*) a set a is *closed and unbounded in* b ($b \neq \emptyset$) if $a = c \cap b$ for some closed set c with $\cup c = \cup b$;

(*iv*) a set $a \subset b$ is *stationary in* b if a is stationary and $\cup a = \cup b$. □

Notice that if $\cup a \in a$ then a is stationary (these are the degenerate stationary sets). There is some conflict with standard usage: a cofinal subset of a singular limit ordinal is never stationary in the preceding sense.

Lemmas 1 and *2* are easy consequences of the definition. A function $f : b \to \cup b$ is a *choice function* if for all $A \in b$, if $A \neq \emptyset$, then $f(A) \in A$.

LEMMA 1. *Suppose* x $\subset \cup$b *and* b *is stationary. Then* $\{$A $\cap$ x : A $\in$ b$\}$ *is stationary.* □

LEMMA 2. *Suppose* b *is stationary and* f : b $\to \cup$b *is a choice function. Then for some set* a *stationary in* b, f $\restriction$ a *is constant.* □

Suppose a is a set. Let $ht(a) = \sup\{rk(x) : x \in a\}$ = least α such that $\cup a \subset V_\alpha$.

For each ordinal α define a partial order $\mathbb{P}_{<\alpha}$ as follows: $\mathbb{P}_{<\alpha} = \{a : a$ is stationary and $ht(a) < \alpha\}$ and for all $a, b \in \mathbb{P}_{<\alpha}$, $a \leq b$ if (*i*) $\cup b \subset \cup a$ and (*ii*) for each $Z \in a$, $Z \cap (\cup b) \in b$. $\mathbb{Q}_{<\alpha}$ is the suborder of $\mathbb{P}_{<\alpha}$ given by $\mathbb{Q}_{<\alpha} = \{a \in \mathbb{P}_{<\alpha} : a \subset P_{\omega_1}(\cup a)\} = \{a \in \mathbb{P}_{<\alpha}$: each $Z \in a$ is countable$\}$.

The following definition is of central importance. A set X end-extends Y if $Y \subset X$ and for all $Z \in Y$, $Z \cap Y = Z \cap X$.

Definition 4: Suppose δ is an inaccessible cardinal. $A \subset \mathbb{P}_{<\delta}$ is *semiproper in* $\mathbb{P}_{<\delta}$ if

$$\text{sp}(A) = \{X \subset V_{\delta+1} : \text{For some } Y < V_{\delta+1}, X \subset Y, Y \text{ end-extends } X \cap V_\delta \text{ and } Y \cap (\cup a) \in a \text{ for some } a \in A \cap Y\}$$

contains a set closed and unbounded in $P_\delta(V_{\delta+1})$.

Similarly, $A \subset \mathbb{Q}_{<\delta}$ is semiproper in $\mathbb{Q}_{<\delta}$ if sp(A) contains a set closed and unbounded in $P_{\omega_1}(V_{\delta+1})$.

We shall restrict our attention to $\mathbb{Q}_{<\delta}$ though much of what we prove holds in an analogous form for $\mathbb{P}_{<\delta}$. Suppose $A \subset \mathbb{Q}_{<\delta}$ and let $B = P_{\omega_1}(V_{\delta+1}) \setminus \text{sp}(A)$. Then A is not semiproper if and only if B is stationary in $P_{\omega_1}(V_{\delta+1})$. One can show that if $A \subset \mathbb{Q}_{<\delta}$ is semiproper then A is predense in $\mathbb{Q}_{<\delta}$—i.e., $\{a \in \mathbb{Q}_{<\delta} : a \leq b$ for some $b \in A\}$ is dense in $\mathbb{Q}_{<\delta}$.

LEMMA 3. *Suppose* M *is a transitive set,* $M^{V_\delta} \subset$ M *and* F $\in$

M *for all* $F : V_{\delta+1}^{<\omega} \to V_{\delta+1}$. *Suppose* $A \subset \mathbb{Q}_{<\delta}$. *The following are equivalent.*

(*i*) A *is semiproper in* $\mathbb{Q}_{<\delta}$.

(*ii*) *For any* $X < M$ *with* A, $\delta \in X$ *and* $|X| < \omega_1$, *there exists* $Y < M$ *such that* $X \subset Y$, Y *end-extends* $X \cap V_\delta$ *and* $Y \cap (\cup a) \in a$ *for some* $a \in Y \cap A$.

(*iii*) *For any* $X < M$ *with* A, $\delta \in X$ *and* $|X| < \omega_1$, *there exists* $Y \subset V_\delta$ *such that* Y *end-extends* $X \cap V_\delta$, $Y \cap (\cup a) \in a$ *for some* $a \in Y \cap A$, *and* $f''(Y^{<\omega}) \subset Y$ *for all* $f : V_\delta^{<\omega} \to V_\delta$, $f \in X$.

Proof: We shall only use this lemma in the case $M \models$ Zermelo–Fraenkel, with the axiom of choice (ZFC). Clearly statement *ii* of *Lemma 3* → statement *i*. To see that *iii* → *ii* observe that since $M^{V_\delta} \subset M$ it follows that if $X < M$ and if $Y \subset V_\delta$ then $Y^* < M$, where $Y^* = \{f(s) : s \in Y^{<\omega}$ and $f \in X\}$. To see that *i* → *ii* suppose $A \subset \mathbb{Q}_{<\delta}$ is semiproper, $X < M$, X is countable and $A \in X$. Since $X < M$ there exists $F : V_{\delta+1}^{<\omega} \to V_{\delta+1}$, $F \in X$, such that $c_F \cap P_{\omega_1}(V_{\delta+1}) \subseteq \text{sp}(A)$, where $c_F = \{Y \subset V_{\delta+1} : F''(Y^{<\omega}) \subset Y\}$. Clearly $F''((X \cap V_{\delta+1})^{<\omega}) \subset X$ and so $X \cap V_{\delta+1} \in \text{sp}(A)$. Choose $Y < V_{\delta+1}$ such that $X \cap V_{\delta+1} \subset Y$, Y end-extends $X \cap V_\delta$ and $Y \cap (\cup a) \in a$ for some $a \in Y \cap A$. Let

$$Y^* = \{f(s) : f \in X \text{ and } s \in (Y \cap V_\delta)^{<\omega}\}.$$

Then $Y^* \cap V_\delta = Y \cap V_\delta$, $X \subset Y^*$ and $Y^* < M$. This proves statement *ii* assuming *i*. Finally *ii* → *iii* is trivial. □

THEOREM 2. *Suppose* κ *is* $V_{\kappa+1}$*-supercompact and that* $A \subset \mathbb{Q}_{<\kappa}$ *is predense. Then* A *is semiproper in* $\mathbb{Q}_{<\kappa}$.

Proof: Fix an elementary embedding $j : V \to M$ such that $\text{cp}(j) = \kappa$ and $M^{V_{\kappa+1}} \subset M$. Suppose $A \subset \mathbb{Q}_{<\kappa}$ is predense. Let $B = P_{\omega_1}(V_{\kappa+1}) \setminus \text{sp}(A)$. Assume toward a contradiction that A is not semiproper in $\mathbb{Q}_{<\kappa}$. Hence B is stationary in $P_{\omega_1}(V_{\kappa+1})$. $M^{V_{\kappa+1}} \subset M$ and so $B \in M$. Thus $M \models$ "B is stationary" and so $B \in j(\mathbb{Q}_{<\kappa})$. Choose $a \in j(A)$ and $b \in j(\mathbb{Q}_{<\kappa})$ such that $b \le a$ and $b \le B$. Let $\lambda = j(\kappa + 4)$. Choose $Z < M_\lambda = V_\lambda \cap M$ such that $\{j \restriction V_{\kappa+1}, A, a, b\} \subset Z$, $Z \cap (\cup b) \in b$ and $Z \in M$. Since $M \models$ "b is stationary" and $M^{V_{\kappa+1}} \subset M$, Z exists. Let $Z_0 = Z \cap V_{\kappa+1}$. Since $b \le B$ in $j(\mathbb{Q}_{<\kappa})$, $Z_0 \in B$. Therefore $j(Z_0) \in j(B)$. But Z_0 is countable, hence $j(Z_0) = j''(Z_0)$. But then $Z \cap j(V_{\kappa+1})$ end-extends $j(Z_0) \cap j(V_\kappa)$ since $j(Z_0) \cap j(V_\kappa) = (j''(Z_0)) \cap j(V_\kappa) = Z_0 \cap V_\kappa = Z \cap V_\kappa$. Finally $Z \cap (\cup a) \in a$ and $a \in j(A) \cap Z$. Thus $M \models$ "$Z \cap j(V_{\kappa+1})$ is a witness that $j(Z_0) \in j(\text{sp}(A))$." Therefore, $j(Z_0) \in j(\text{sp}(A))$ and so $Z_0 \in \text{sp}(A)$ contradicting $Z_0 \in B$ since $B = P_{\omega_1}(V_{\kappa+1}) \setminus \text{sp}(A)$. □

COROLLARY. *Suppose* κ *is* $V_{\kappa+1}$*-supercompact. Then* $\{\delta : \delta < \kappa$, δ *is inaccessible, and each predense* $A \subset \mathbb{Q}_{<\delta}$ *is semiproper in* $\mathbb{Q}_{<\delta}\}$ *is stationary in* κ.

Proof: Let $j : V \to M$ be an elementary embedding with $\text{cp}(j) = \kappa$ and $V_{\kappa+2} \subset M$. By *Theorem 2* and since $V_{\kappa+2} \subset M$, $M \models$ "each predense $A \subset \mathbb{Q}_{<\kappa}$ is semiproper in $\mathbb{Q}_{<\kappa}$." The corollary follows. □

THEOREM 3. *Suppose* κ *is* $V_{\kappa+1}$*-supercompact and that* $G \subset \mathbb{Q}_{<\kappa}$ *is* V*-generic. Then there is a generic elementary embedding* $j : V \to M \subset V[G]$ *such that* $M^{<\kappa} \subset M$ *in* V[G] *and for each* $a \in G$, $j''(\cup a) \in j(a)$.

Proof: For $a \in \mathbb{Q}_{<\kappa}$ let $NS_a \subset P(a)$ be the ideal $NS_a = \{b \subset a : b$ is not stationary in $a\}$. We work in $V[G]$ except that for $a \in G$; $P(a)$, NS_a and V^a are as computed in V. Thus for $a \in G$, $U_a = P(a) \cap G$ defines a V-ultrafilter on $P(a)/NS_a$. This gives an elementary embedding $j_a : V \to (N_a, E_a) \sim V^a/U_a$. Suppose $a \in G$, $b \in G$ and $\cup a \subset \cup b$. Then there is a canonical elementary embedding $j_{ab} : (N_a, E_a) \to (N_b, E_b)$ such that $j_b = j_{ab} \circ j_a$. Taking the direct limit over a in G yields $j : V \to (N, E)$. For $a \in G$ and $f \in V^a$ let $[f]$ denote the element f defines in N. Hence if $f, g \in V^a$, $[f]\ E\ [g]$ if and only if for some $b \in G$, $b \le a$ and $f(Y \cap (\cup a)) \in g(Y \cap (\cup a))$ for all $Y \in b$. Clearly $N = \{[f] : f \in V^a$ for some $a \in G\}$. Fix $a \in G$ and let $e_a : a \to a$ be the identity function. Clearly $[e_a]\ E\ j(a)$. I claim that $[e_a] = j''(\cup a)$ in the sense that for all $c \in N$, $c\ E\ [e_a]$ iff $c = j(X)$ for some $X \in \cup a$. To see this, fix $c \in N$ such that $c\ E\ [e_a]$. Choose $b \in G$ and $g \in V^b$ such that $b \le a$, $c = [g]$, and $g(Y) \in Y \cap (\cup a)$ for all $Y \in b$. Hence g is a choice function and so by genericity there exists $b^* \in G$ such that $b^* \le b$ and $g \restriction \{Y \cap (\cup b) : Y \in b^*\}$ is constant. Take for X, this constant value. We now prove that (N, E) is well-founded and that its transitive collapse, M, is closed under $< \kappa$ sequences in $V[G]$. This is equivalent to showing that if $X \subset N$ and $|X| < \kappa$ then for some $Z \in N$, $X = \{t : t\ E\ Z\}$. Suppose $\langle \tau_\alpha : \alpha < \lambda \rangle$ is a λ sequence of terms for elements of N where $\lambda < \kappa$ and the sequence is in V. Fix $a_0 \in G$ and we now work in V. Assume that for $\alpha < \lambda$, $[[\tau_\alpha \in N]] = 1$. For each $\alpha < \lambda$ fix a maximal antichain $A_\alpha \subset \mathbb{Q}_{<\kappa}$ and a function $F_\alpha : A_\alpha \to V$ such that for $b \in A_\alpha$, $F_\alpha(b) \in V^b$ and $b \Vdash$ "$\tau_\alpha = [F_\alpha(b)]$." By *Corollary* to *Theorem 2* there is an inaccessible cardinal $\delta < \kappa$ such that $\delta > \lambda$, $\delta > ht(a_0)$ and such that $A_\alpha \cap V_\delta$ is semiproper in $\mathbb{Q}_{<\delta}$ for each $\alpha < \lambda$. Let $a = \{X \in P_{\omega_1}(V_\delta) : X < V_\delta$ and for each $\alpha \in X \cap \lambda$, $X \cap (\cup b) \in b$ for some $b \in X \cap A_\alpha\}$.

CLAIM. $a_0^* = \{X \in a : X \cap (\cup a_0) \in a_0\}$ *is stationary in* $P_{\omega_1}(V_\delta)$.

Proof of Claim: Suppose $H : V_\delta^{<\omega} \to V_\delta$. We find $X \in a_0^*$ such that $H''(X^{<\omega}) \subset X$. Choose $X_0 \in P_{\omega_1}(V_\kappa)$ such that $X_0 < V_\kappa$, $X_0 \cap (\cup a_0) \in a_0$ and $\{H, a_0, \langle A_\alpha : \alpha < \lambda \rangle\} \subset X_0$. Since a_0 is stationary X_0 exists. Choose an elementary chain $\langle X_\gamma : \gamma \in X_0 \cap \lambda \rangle$, starting with X_0, such that (*i*) for all $\gamma \in X_0 \cap \lambda$, $X_\gamma < V_\kappa$, $|X_\gamma| = \omega$ and $X_{\gamma+1} \cap (\cup b) \in b$ for some $b \in A_\gamma \cap X_{\gamma+1} \cap V_\delta$ and (*ii*) for all $\gamma_1 < \gamma_2$, if $\{\gamma_1, \gamma_2\} \subset X_0 \cap \lambda$ then X_{γ_2} end-extends $X_{\gamma_1} \cap V_\delta$. Using part *ii* of *Lemma 3*, the chain is easily constructed. Let $X = \cup\{X_\gamma \cap V_\delta : \gamma \in X_0 \cap \lambda\}$. Note $\lambda < \delta$ and $\lambda \in X_0$. Hence $X \cap \lambda = X_0 \cap \lambda$ and so $X \in a_0^*$ and $H''(X^{<\omega}) \subset X$. □ proof of claim

For each $\alpha < \lambda$ if $X < V_\delta$ then $\{b : b \in A_\alpha \cap X$ and $X \cap (\cup b) \in b\}$ contains at most one element. This is because A_α is an antichain. Define $f \in V^{a_0^*}$ by $f(Y) = \{F_\gamma(b)(Y \cap (\cup b)) : \gamma \in Y \cap \lambda, b \in Y \cap A_\gamma \cap V_\delta$ and $Y \cap (\cup b) \in b\}$. Thus $a_0^* \Vdash$ "$\{c \in N \mid c\ E\ [f]\} = \{\tau_\alpha : \alpha < \lambda\}$." By genericity we can choose a_0 so that $a_0^* \in G$. □

Henceforth an inner model is a transitive class, possibly a set, closed under the primitive recursive set functions. If M is an inner model then $M(X)$ is the smallest inner model containing $M \cup \{X\}$. For ordinals, α, let $\text{Coll}(\omega, < \alpha)$ be the partial order of finite conditions for the Levy collapse of ordinals $< \alpha$ to ω. Suppose M is an inner model of ZFC, $\alpha \in M$, and $G \subset \text{Coll}(\omega, < \alpha)$ is M-generic. Let $\tau = \cup\{M(G \cap \text{Coll}(\omega, < \beta)) \cap \mathbb{R} : \beta < \alpha\}$ and let $N_G = M(\tau)$. Standard arguments show that if G_1 and G_2 are each M-generic for $\text{Coll}(\omega, < \alpha)$ then $N_{G_1} \equiv_M N_{G_2}$; i.e., N_{G_1} and N_{G_2} satisfy the same formulas of the language of set theory with parameters from M. We say an inner model, N, is a symmetric extension of M for $\text{Coll}(\omega, < \alpha)$ if $M \subset N$ and $N \equiv_M N_G$ in $V[G]$, where $G \subset \text{Coll}(\omega, < \alpha)$ is V-generic. This is a first-order property of N in a predicate for M.

LEMMA 4. *Suppose* M *is an inner model of ZFC*, $\lambda \in M$ *and* $M \models$ "λ *is a strong limit cardinal.*" *Suppose* $\tau \subset \mathbb{R}$ *is such that* (i) *for all* $x \in \tau$, *there exits* $\mathbb{P} \in M$ *and* $g \subset \mathbb{P}$ *such that* $M \models$ "$|\mathbb{P}| < \lambda$," g *is* M*-generic for* $\mathbb{P}$, *and* $x \in M[g]$, (ii) *for all* x, $y \in \tau$, $\mathbb{R} \cap M(x, y) \subset \tau$, *and* (iii) $\sup\{\omega_1^{M(x)} : x \in \tau\} = \lambda$. *Then* $\mathbb{R} \cap M(\tau) = \tau$ *and* $M(\tau)$ *is a symmetric extension of* M *for* $\text{Coll}(\omega, < \lambda)$. □

THEOREM 4. *Suppose* κ *is supercompact and* $G \subset \mathbb{Q}_{<\kappa}$ *is* V*-generic. Then* $\{\delta : \delta < \kappa$ *and* $G \cap \mathbb{Q}_{<\delta}$ *is* V*-generic for* $\mathbb{Q}_{<\delta}\}$ *is unbounded in* κ.

Proof: We work in V. Fix $a_0 \in \mathbb{Q}_{<\kappa}$. By *Corollary* to *Theorem 2* there exists $\delta > ht(a_0)$ such that δ is inaccessible, $\delta < \kappa$, and each dense $A \subset \mathbb{Q}_{<\delta}$ is semiproper. Let $a_0^* = \{X \in P_{\omega_1}(V_{\delta+1}) : X \cap (\cup a_0) \in a_0$ and for each dense $A \subset \mathbb{Q}_{<\delta}$ if $A \in X$ then $X \cap (\cup b) \in b$ for some $b \in X \cap A\}$. By an elementary chain argument, using *Lemma 3*, as above (*Proof* of *Theorem 3*), a_0^* is stationary. It follows that $a_0^* \le a_0$ and $a_0^* \Vdash$ "$G \cap \mathbb{Q}_{<\delta}$ is V-generic." □

COROLLARY. *Suppose κ is supercompact and $G \subset \mathbb{Q}_{<\kappa}$ is V-generic. Let $\tau = (\mathbb{R})^{V[G]}$. Then $V(\tau)$ is a symmetric extension of V for Coll(ω, < κ).*

Proof: Suppose $\tau \in V^{\mathbb{Q}_{<\kappa}}$ is a term, $a \in \mathbb{Q}_{<\kappa}$, and $a \Vdash$ "$\tau \subset \omega$." Then there exists $\tau^* \in V^{\mathbb{Q}_{<\kappa}} \cap V_\kappa$ and $b \le a$ such that $b \Vdash$ "$\tau = \tau^*$." Given this it follows by *Theorem 4* that for each $x \subset \omega$ with $x \in V[G]$, there exists $\delta < \kappa$ such that $x \in V[G \cap \mathbb{Q}_{<\delta}]$ and $G \cap \mathbb{Q}_{<\delta}$ is V-generic for $\mathbb{Q}_{<\delta}$. The corollary now follows by *Lemma 4* provided $\kappa = \omega_1^{V[G]}$. By *Theorem 3*, $\kappa \le \omega_1^{V[G]}$. This combined with the preceding shows $\kappa = \omega_1^{V[G]}$.

To see that b and τ^* exist define for each $i < \omega$ a set $A_i \subset \mathbb{Q}_{<\kappa}$ by $A_i = \{c \le a : c \Vdash$ "$i \in \tau$" or $c \Vdash$ "$i \notin \tau$"$\}$. Thus A_i is dense below a. Choose $b \le a$, as above, such that for all $Z \in b$ and for all $i < \omega$, $Z \cap (\cup c) \in c$ for some $c \in A_i \cap b$. τ^* is easily defined from b. □

Recall $\mathbb{R}^\#$ is the theory of $L(\mathbb{R})$ in parameters from $\mathbb{R} \cup \{\gamma_k : k \in \omega\}$, where $\langle \gamma_k : k \in \omega\rangle$ is any increasing sequence of Silver indiscernibles for $L(\mathbb{R})$ see (ref. 5). We uniformly view $\mathbb{R}^\# \subset \mathbb{R}$. Suppose $\mathbb{R}^\#$ exists, M is an inner model of ZFC and that $M \models$ "$\mathbb{R}^\#$ exists." Then $(\mathbb{R})^M$, $(\mathbb{R}^\#)^M$ are as computed in M. Of course $(\mathbb{R})^M \subset \mathbb{R}$ but if $(\mathbb{R}^\#)^M \subset \mathbb{R}^\#$ then $(L(\mathbb{R}))^M \equiv L(\mathbb{R})$ in a very strong sense.

THEOREM 5. *Suppose κ is supercompact and suppose $V(\tau)$ is a symmetric extension of V for Coll(ω, < κ). Then $(\mathbb{R}^\#)^V \subset (\mathbb{R}^\#)^{V(\tau)}$.*

Proof: Suppose $G \subset \mathbb{Q}_{<\kappa}$ is V-generic and let $j : V \to M \subseteq V[G]$ be the induced embedding. Let $\tau_G = (\mathbb{R})^{V[G]}$. Thus $V(\tau_G)$ is a symmetric extension of V for Coll(ω, < κ). Clearly $(\mathbb{R}^\#) \subset (\mathbb{R}^\#)^M$. Since $M^\omega \subset M$ in $V[G]$, $(\mathbb{R}^\#)^M = (\mathbb{R}^\#)^{V[G]} = (\mathbb{R}^\#)^{V(\tau_G)}$. So $(\mathbb{R}^\#)^V \subset (\mathbb{R}^\#)^{V(\tau_G)}$ and therefore by homogeneity $(\mathbb{R}^\#)^V \subset (\mathbb{R}^\#)^{V(\tau)}$. □

THEOREM 6. *Suppose κ is supercompact. Suppose $\mathbb{P}_1$ is a partial order and $G_1 \subset \mathbb{P}_1$ is V-generic. Suppose $\mathbb{P}_2 \in V[G_1]$ is a partial order and $G_2 \subset \mathbb{P}_2$ is $V[G_1]$-generic. Then $(\mathbb{R}^\#)^{V[G_1]} \subset (\mathbb{R}^\#)^{V[G_1][G_2]}$.*

Proof: By reflection (κ is supercompact) we can reduce to the case $\mathbb{P}_1 \in V_\kappa$ and $\mathbb{P}_2 \in V_\kappa[G_1]$. Hence $V[G_1] \models$ "κ is supercompact" and $V[G_1][G_2] \models$ "κ is supercompact." Let $g \subset$ Coll(ω, < κ) be $V[G_1][G_2]$-generic and let $\tau = (\mathbb{R})^{V[G_1][G_2][g]}$. $\mathbb{P}_1 \in V_\kappa$ and $\mathbb{P}_2 \in V_\kappa[G_1]$ and so by *Lemma 4*, $V(\tau)$ is a symmetric extension of *both* $V[G_1]$ and $V[G_1][G_2]$ for Coll(ω, < κ). Hence by *Theorem 5*, $(\mathbb{R}^\#)^{V[G_1]} \subset (\mathbb{R}^\#)^{V(\tau)}$ and $(\mathbb{R}^\#)^{V[G_1][G_2]} \subset (\mathbb{R}^\#)^{V(\tau)}$. Hence $(\mathbb{R}^\#)^{V[G_1]} \subset (\mathbb{R}^\#)^{V[G_1][G_2]}$. □

THEOREM 7. *Suppose κ is supercompact. There are trees T, T* on $\omega \times \kappa$ such that for any partial order $\mathbb{P} \in V_\kappa$, if $G \subset \mathbb{P}$ is V-generic then $V[G] \models$ "$p[T] = \mathbb{R}^\#$ and $p[T^*] = \omega^\omega \setminus p[T]$."*

Proof: Suppose $\delta \le \kappa$ is inaccessible and let $S_\delta = \{X \in P_{\omega_1}(V_\delta) : X \prec V_\delta$ and $(\mathbb{R}^\#)^{N[g]} \subset \mathbb{R}^\#$ for *any* g such that for some $\mathbb{P} \in N$, $g \subset \mathbb{P}$ and g is N-generic for $\mathbb{P}$, where N = coll(X) = transitive collapse of $X\}$. If $\delta < \kappa$ I claim S_δ contains a set closed and unbounded in $P_{\omega_1}(V_\delta)$. If not then $a_\delta = P_{\omega_1}(V_\delta) \setminus S_\delta$ is stationary in $P_{\omega_1}(V_\delta)$. Let $G \subset \mathbb{Q}_{<\kappa}$ be V-generic with $a_\delta \in G$ and let $j : V \to M \subset V[G]$ be the induced embedding. Thus $j''(\cup a_\delta) \in j(a_\delta)$. However, $\cup a_\delta = V_\delta$ and so $j''(V_\delta) \in j(a_\delta)$. But V_δ = coll($j''(V_\delta)$); hence, for some $\mathbb{P} \in V_\delta$ and $g \in M$, $g \subset \mathbb{P}$, g is V_δ-generic and $(\mathbb{R}^\#)^{V_\delta[g]} \not\subset (\mathbb{R}^\#)^M$. δ is inaccessible so g is V-generic for $\mathbb{P}$ and $(\mathbb{R}^\#)^{V_\delta[g]} = (\mathbb{R}^\#)^{V[g]}$. Further $(\mathbb{R}^\#)^M = (\mathbb{R}^\#)^{V[G]}$. But by *Theorem 6*, $(\mathbb{R}^\#)^{V[g]} \subset (\mathbb{R}^\#)^{V[G]}$, a contradiction. Hence for each inaccessible $\delta < \kappa$, S_δ contains a set closed and unbounded in $P_{\omega_1}(V_\delta)$. κ is supercompact; hence, by reflection S_κ contains a set closed and unbounded in $P_{\omega_1}(V_\kappa)$. Fix $F_0 : V_\kappa^{<\omega} \to V_\kappa$ such that $\{X \in P_{\omega_1}(V_\kappa) : F_0''(X^{<\omega}) \subset X\} \subset S_\kappa$. For $h \in (\omega \times \omega)^\omega$ let E_h = range(h). Let $A = \{(x, h) : x \in \omega^\omega, h \in (\omega \times \omega)^\omega$, $(\omega, E_h) \models$ ZFC, E_h is well-founded, and if N = coll(ω, E_h) then for some $\mathbb{P} \in N$ and for some $g \subset \mathbb{P}$, g is N-generic for $\mathbb{P}$ and $x \in (\mathbb{R}^\#)^{N[g]}\}$. One can show that A is Π_1^1. Fix a Π_1 formula $\varphi(t_1\ t_2)$ such that $A = \{(x, h) \mid V_{\omega+1} \models \varphi[x, h]\}$. Choose a tree T_0 on $(\omega \times \omega) \times \kappa$ such that for any $\mathbb{P} \in V_\kappa$, if $G \subset \mathbb{P}$ is V-generic then $V[G] \models$ "$p[T_0] = \{(x, h) \mid (V[G])_{\omega+1} \models \varphi[x, h]\}$." Fix an enumeration $\langle s_k : k \in \omega\rangle$ of $\omega^{<\omega}$ such that for each $k \in \omega$, dom $s_k \cup$ rng $s_k \subset k$. Define T as a tree on $\omega \times (\omega \times \omega) \times \kappa \times V_\kappa$ such that $(x, h, g, f) \in [T]$ iff (*i*) $(x, h, g) \in [T_0]$, (*ii*) $E_h = \{(i, j) : f(i) \in f(j)\}$, (*iii*) for each $i \in \omega$, $E_h \cap i \times i \subset$ range($h \restriction i^2$), and (*iv*) for each $k \in \omega$, $f(2k + 1) = F_0(f \circ s_k)$. Similarly define T^* from $A^* = \{(x, h) : x \in \omega^\omega, \ldots$ and $x \notin (\mathbb{R}^\#)^{N[g]}\}$. Thus $p[T] = \mathbb{R}^\#$ and $p[T^*] = \omega^\omega \setminus \mathbb{R}^\# = \omega^\omega \setminus p[T]$. Finally for any $\mathbb{P} \in V_\kappa$, if $G \subset \mathbb{P}$ is V-generic then

$$V[G] \models \text{"}\mathbb{R}^\# \subset p[T]\text{"}$$

and

$$V[G] \models \text{"}(\omega^\omega \setminus \mathbb{R}^\#) \in p[T^*].\text{"}$$

By absoluteness

$$V[G] \models \text{"}p[T] \cap p[T^*] = \emptyset\text{"}$$

and so

$$V[G] \models \text{"}p[T] = \mathbb{R}^\# \text{ and } p[T^*] = \omega^\omega \setminus p[T].\text{"} \quad \square$$

THEOREM 8. *Suppose κ is supercompact. Suppose T and T* are trees on $\omega \times \kappa$ such that for any partial order $\mathbb{P} \in V_\kappa$, if $G \subset \mathbb{P}$ is V-generic then $V[G] \models$ "$p[T^*] = \omega^\omega \setminus p[T]$." Then T and T* are each κ-weakly homogeneous.*

Proof: Let $m_\kappa(\kappa^{<\omega}) \subset m(\kappa^{<\omega})$ be the set of κ-complete measures on $\kappa^{<\omega}$. Choose $\tau \prec V_{\kappa+2}$, $|\tau| < \kappa$, with $\{T, T^*\} \subset \tau$ such that (*i*) if $\nu \in m_\kappa(\kappa^{<\omega})$ then $\nu \cap \tau = \mu \cap \tau$ for some (necessarily unique) $\mu \in \tau$ and (*ii*) if $\langle \mu_k : k \in \omega\rangle$ is a tower of measures in $\tau \cap m_\kappa(\kappa^{<\omega})$ then the tower is countably complete iff for some $f \in \kappa^\omega$, $f \restriction k \in \cap \{A \in \tau : \mu_k(A) = 1\}$ for all $k \in \omega$. To see that such a set τ exists, consider in M the set $j''(V_{\kappa+2}) \prec j(V_{\kappa+2})$, where $j : V \to M$ is an elementary embedding, cp(j) = κ and $M^{V_{\kappa+2}} \subset M$. For $s \in \kappa^{<\omega}$ let $\mu(\tau, s)$ denote that measure $\mu \in \tau$ such that $\mu \cap \tau = \{A \in \tau : A \subset \kappa^{<\omega}$ and $s \in A\}$. Note that for any $f \in \kappa^\omega$, $\langle \mu(\tau, f \restriction k) : k \in \omega\rangle$ is a countably complete tower.

Fix $\kappa_0 < \kappa$, $|\tau| < \kappa_0$, such that κ_0 is V_{κ_0+1}-supercompact. Let $G_0 \subset \mathbb{Q}_{<\kappa_0}$ be V-generic and let $j_0 : V \to M_0 \subset V[G_0]$ be the induced embedding. Since $|\tau| < \kappa_0$, $j_0''(\tau) \in M_0$ and $M_0 \models$ "$j_0''(\tau)$ is countable." I shall show that $M_0 \models$ "for each $x_0 \in p[j_0(T)]$ there exists a countably complete tower $\langle \nu_k : k \in \omega\rangle$ of $j_0(\kappa)$-complete measures in $j_0''(\tau)$ such that for each $k \in \omega$, $\nu_k(j_0(T)_{x_0}) = 1$." Given this it follows that T is κ-weakly homogeneous. We work in $V[G_0]$. Fix $x_0 \in M_0$ such that $x_0 \in p[j_0(T)]$. Clearly $p[T] \subset p[j_0(T)]$ and $p[T^*] \subset p[j_0(T^*)]$. But $p[T^*] = \omega^\omega \setminus p[T]$ since $\mathbb{Q}_{<\kappa_0} \in V_\kappa$. Hence $p[T] = p[j_0(T)]$ and so $x_0 \in p[T]$. Choose $f_0 \in \kappa^\omega$ such that $(x_0, f_0) \in [T]$. For each $k \in \omega$ let $\mu_k = \mu(\tau, f_0 \restriction k)$ and so $\mu_k \in V$ and $V \models$ "μ_k is a κ-complete measure." Define $\bar{f}_0$ by $\bar{f}_0(k) = j_0(f_0(k))$. $M_0^\omega \subset M_0$ in $V[G_0]$, hence $\bar{f}_0 \in M_0$ and $\langle j_0(\mu_k) : k \in \omega\rangle \in M_0$. Clearly $M_0 \models$ "$j_0(\mu_\kappa)\ (j_0(T)_{x_0}) = 1$ for all $k \in \omega$" and $M_0 \models$ "$\langle j_0(\mu_k) : k \in \omega\rangle$ is a tower of measures." Thus it suffices to show $M_0 \models$ "the tower $\langle j_0(\mu_k) : \kappa \in \omega\rangle$ is countably complete." But this is immediate since $M_0 \models$ "for each $k \in \omega$, $j_0(\mu_k) = \mu(j_0(\tau), \bar{f}_0 \restriction k)$." □

A similar argument shows that T^* is κ-weakly homogeneous. □

Theorem 1 now follows easily. Suppose κ is supercompact. By *Theorems 7* and *8*, $\mathbb{R}^\#$ is the projection of a κ-weakly homogeneous tree. Every set of reals, A, with $A \in L(\mathbb{R})$ is continuously reducible to $\mathbb{R}^\#$. Therefore each such set is the projection of a κ-weakly homogeneous tree. □

Remarks: All of the theorems here can be proved from substantially weaker large cardinal assumptions. Further, the potential influence is well beyond the sets of reals in $L(\mathbb{R})$. See, for example, the following theorem.

THEOREM. *Assume κ is supercompact. Assume the continuum hypothesis. Then every Σ_1^2-definable set of reals is the projection of a κ-weakly homogeneous tree.* □

These related results and results for $\mathbb{P}_{<\kappa}$ will be published in a forthcoming paper on large cardinals and determinacy.

The assertion that every set of reals, in $L(\mathbb{R})$, is the projection of a weakly homogeneous tree has consequences beyond the usual regularity properties such as Lebesgue measurability. For example by results of Kechris it follows that $L(\mathbb{R}) \models$ "ω_1 is measurable." It may even be that this alone implies $L(\mathbb{R}) \models AD$.

Question: Suppose every set of reals that belongs to $L(\mathbb{R})$ is the projection of a weakly homogeneous tree. Does AD hold in $L(\mathbb{R})$?

The main question left open here is whether the existence of a supercompact cardinal implies $L(\mathbb{R}) \models AD$. If so, this would be a dramatic reduction in the large cardinal hypothesis sufficient to prove $AD^{L(\mathbb{R})}$. Martin and Steel (6) have answered this question. They show, among other things, that if κ is supercompact then the projection of every κ-weakly homogeneous tree is determined and so by *Theorem 1*, $L(\mathbb{R}) \models AD$.

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STRUCTURAL PROPERTIES OF MODELS OF $\aleph_1$-CATEGORICAL THEORIES

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1.

The structural theory of categoricity (in uncountable powers) began with the works of BALDWIN [1972] and BALDWIN & LACHLAN [1971], in which the notions of a strongly minimal set and algebraic closure were introduced and it was shown that the structure of a strongly minimal set with respect to algebraic closure (acl) affects essentially the structure of the model itself.

The structure of a strongly minimal set S with respect to the closure operator acl can be essentially characterized by the *geometry* associated with S. The geometry associated with S over a subset A is given by its points, which are the sets of the form $\mathrm{acl}(a, A)$ for $a \in S\text{-}\mathrm{acl}(A)$, and its n-dimensional subspaces, which are $\mathrm{acl}(a_0, \ldots, a_n, A)$, where $a_0, \ldots, a_n$ are algebraically independent over A. We omit "over A", if $A = \emptyset$.

If the geometry associated with S over any non-algebraic element is isomorphic to a geometry of a projective space over a division ring then the geometry associated with S is called *locally projective*.

If the division ring in the definition is finite, then the main result of DOYEN & HUBAUT [1971] describes the locally projective geometry as an affine or projective geometry over the division ring.

Call a strongly minimal structure S disintegrated if $\mathrm{acl}(X \cup Y) = \mathrm{acl}(X) \cup \mathrm{acl}(Y)$ for every $X, Y \subseteq S$. This is equivalent to the degeneracy of the geometry associated with S (i.e. all subsets of the geometry are subspaces).

Natural examples of strongly minimal structures with projective geometries are strongly minimal abelian groups and, more generally, modules. Affine spaces over division rings have locally projective geometries which are not projective. The natural numbers with the successor operation is a typical example of a strongly minimal disintegrated structure.

On the other hand such strongly minimal structures as algebraically closed fields can hardly be characterized in terms of their geometries. More adequate in this situation seems the following notion introduced by LACHLAN [1973/74].

A pseudoplane is a triple $\langle P, L, I\rangle$, where P is a set of "points", L is a set of "lines" and $I \subseteq P \times L$ is an incidence relation satisfying the following:

(1) every line is incident to an infinite set of points;

(2) every point is incident to an infinite set of lines;

(3) any two distinct points are incident in common to at most finite number of lines;

(4) any two distinct lines are incident in common to at most finite number of points.

CONJECTURE. For any uncountably categorical pseudoplane there is an algebraically closed field such that the field is definable in the pseudoplane and the pseudoplane is definable in the field.

In the paper the following theorem will be proved:

TRICHOTOMY THEOREM. *For an uncountably categorical structure M one and only one of the following holds:*

(1) *An uncountably categorical pseudoplane is definable in M.*

(2) *For every strongly minimal structure S definable in M the geometry associated with S is locally projective.*

(3) *Every strongly minimal structure definable in M is disintegrated.*

In the connection with the Trichotomy Theorem the following theorem is of special interest.

THEOREM 2. *There is no totally categorical pseudoplane (i.e. one the complete theory of which is categorical in all infinite powers).*

Theorem 2 was proved independently by CHERLIN et al. [1981] and the author [1977] (the complete proof is to appear in Sibirsk. M.Ž.). The proofs are quite different, that of CERLIN et al. [1981] relies on the classification of all finite simple groups. The proof of the author is rather long but does not use any deep results outside model theory.

As was shown in ZIL'BER [1980a] the global properties of an uncountably categorical structure M depend essentially on the structure of groups definable in M. Therefore the following theorems, which will be proved in the paper, are of much importance for the structural theory.

THEOREM 3. *Let M be an uncountably categorical structure satisfying* (2) *of the Trichotomy Theorem and G a group definable in M. Then*

(i) *G is abelian-by-finite.*

(ii) *If G is infinite and has no proper infinite definable subgroup, then G is strongly minimal.*

THEOREM 4. *If M is an uncountably categorical structure satisfying* (3) *of the Trichotomy Theorem, then no infinite group is definable in M. It follows from this that M is almost strongly minimal.*

Note that Theorem 3(i) contains the known theorem of BAUR, CHERLIN and MACINTYRE [1979] which states that totally categorical groups are abelian-by-finite.

2. Proofs

An *incidence sytem* is a triple $\langle P, L, T\rangle$, where P is a set of "points", L is a set of "lines" and $I \subseteq P \times L$ is an arbitrary relation called an incidence relation.

For a binary relation R and an element x we denote

$$xR = \{y: xRy\}, \qquad Rx = \{y: yRx\}.$$

Thus, for $p_0 \in P$, $l_0 \in L$

$$p_0 I = \{l \in L: p_0 I l\}, \qquad I l_0 = \{p \in P: p I l_0\}.$$

Let $A \subseteq M^n$ be an X-definable subset of a structure M and E be an X-definable in M equivalence relation on A. Sets of the form A/E are called *X-definable sets in M.* Definable means X-definable for some $X \subseteq M$.

An *X-definable structure in M* is an X-definable set with X-definable relations.

A natural construction considered in SHELAH [1978, III, §6], ZIL'BER [1980a], CHERLIN et al. [1981] allows us to treat definable sets in M as definable subsets of some larger structure M^* which contains M and preserves categoricity, ranks and definability.

Now we begin with the proof of the Trichotomy Theorem. From now on M is an uncountable categorical structure.

LEMMA 1. *Let $\langle P, L, I\rangle$ be an incidence system 0-definable in M,*

$$\langle p_0, l_0 \rangle \in I,$$

$$\text{rank}(l_0, \emptyset) = \text{rank}(L), \qquad \text{rank}(p_0, \emptyset) = \text{rank}(P),$$

$$\text{rank}(p_0, \{l_0\}) = \text{rank}(Il_0), \qquad \text{rank}(l_0, \{p_0\}) = \text{rank}(p_0 I).$$

Then there exist an 0*-definable incidence system* $\langle P', L', I' \rangle$ *in* M, $l_0' \in L'$ *and a mapping* $m : L' \to L$ *such that*

$$P' = P, \qquad \text{rank}(L') = \text{rank}(L),$$

$$m^{-1}(l) \textit{ is finite} \quad \textit{for all } l \in L, \qquad m(l_0') = l_0,$$

$$\text{rank}(I'l_0') = \text{rank}(Il_0), \qquad \deg(I'l_0') = 1,$$

$$\textit{for all } p \in P \quad m(pI') \subseteq pI, \quad \text{rank}(pI') = \text{rank}(pI).$$

PROOF. By the Finite Equivalence Relation Theorem in SHELAH [1978, III, T2.28] there is a two-variable formula E_{l_0} with constant l_0, which defines an equivalence relation on Il_0 with finite number of classes, each of the classes having degree 1 or rank less than $r_0 = \text{rank}(Il_0)$. Let the number of classes be k_0. Put

$$L_1 = \{l \in L : E_l \text{ is an equivalence relation on } Il \text{ with } k_0 \text{ classes}\}.$$

Evidently, L_1 is 0-definable, $l_0 \in L_1$, therefore $\text{rank}(L_1) = \text{rank}(L)$.

Define an equivalence relation E on $I \cap (P \times L_1)$:

$$\langle p, l \rangle E \langle p', l' \rangle \quad \text{iff} \quad l = l' \;\&\; p E_l p',$$

and put

$$L' = I \cap (P \times L_1)/E.$$

It is easy to see that for every $l \in L_1$ there are precisely k_0 elements $l' \in L'$ of the form $l' = \langle p, l \rangle E$ for some $p \in P$. Define $m(l') = l$ in this case. Evidently, $l' \in \text{acl}(l)$, therefore, in particular, $\text{rank}(L') = \text{rank}(L)$. Put

$$pI'l' \quad \text{iff} \quad l' = \langle p, l \rangle E \;\&\; \text{rank}(\langle p, l \rangle E) = r_0.$$

Note that the last condition is definable in M since M is uncountably categorical. Put

$$l_0' = \langle p_0, l_0 \rangle E.$$

It is clear that $I'l_0' \subseteq Il_0$, $I'l_0'$, is an E_{l_0}-equivalence class and $p_0 \in I'l_0'$, therefore $\text{rank}(I'l_0') = r_0$, $\deg(I'l_0') = 1$.

LEMMA 2. *Let* M *be a strongly minimal structure. If there are elements* a_1, a_2,

b_1, b_2, c in M, every four of which are algebraically independent, $c \in \mathrm{acl}(a_1, a_2, b_1, b_2)$ *and* $\mathrm{acl}(a_1, a_2, c) \cap \mathrm{acl}(b_1, b_2, c) = \mathrm{acl}(c)$, *then there is an incidence system* $\langle P, L, I\rangle$ *which is 0-definable in M and:*

$$\mathrm{rank}(P) = 2, \qquad \mathrm{rank}(L) \geq 2, \qquad \deg(P) = 1;$$

$$\mathrm{rank}(Il) = 1 \quad \textit{for every } l \in L;$$

$$\textit{if } l_1, l_2 \in L,\ l_1 \neq l_2, \quad \textit{then } \mathrm{rank}(Il_1 \cap Il_2) = 0.$$

PROOF. Let $P_0 = M \times M$, $L_0 = M \times M \times M$ and $I_0 \subseteq P_0 \times L_0$ be an arbitrary 0-definable relation such that

$$\langle b_1, b_2\rangle\, I_0 \langle a_1, a_2, c\rangle$$

and

$$\langle x_1, x_2\rangle\, I_0 \langle y_1, y_2, z\rangle \rightarrow z \in \mathrm{acl}(x_1, x_2, y_1, y_2).$$

It is easy to check that putting $p_0 = \langle b_1, b_2\rangle$, $l_0 = \langle a_1, a_2, c\rangle$ we have all the assumptions of Lemma 1 satisfied. Hence for some L_0', $l_0' \in L_0$, I_0' we have

$$\mathrm{rank}(L_0') = \mathrm{rank}(L_0) = 3, \qquad l_0' \in \mathrm{acl}(l_0), \quad l_0 \in \mathrm{acl}(l_0'),$$

$$\mathrm{rank}(I_0'l_0') = \mathrm{rank}(I_0 l_0) = 1, \qquad \deg(I_0'l_0') = 1.$$

Put

$$L_1 = \{l_1 \in L_0' \colon \mathrm{rank}(I_0'l_1) = 1\ \&\ (\forall l_2 \in L_0')$$
$$(\mathrm{rank}(I_0'l_1 \cap I_0'l_2) > 0 \rightarrow \mathrm{rank}(I_0'l_1 - I_0'l_2) = 0).$$

Since $I_0'l_0'$ is strongly minimal, hence $l_0' \in L_1$, therefore

$$\mathrm{rank}(L_1) = \mathrm{rank}(l_0', \emptyset) = \mathrm{rank}(L_0').$$

Define an equivalence relation E on L_1:

$$l_1 E l_2 \quad \text{iff} \quad \mathrm{rank}(I_0'l_1 \dot{-} I_0'l_2) = 0.$$

Now put $P = P_0$, $L = L_1/E$ and for $p \in P$, $l \in L_1$

$$pI(lE) \quad \text{iff} \quad \mathrm{rank}(pI_0' - lE) < \mathrm{rank}(lE).$$

It follows from Proposition 1.5 of ZIL'BER [1980a] that for every l of L_1 there is p of $I_0'l$ such that $pI(lE)$ (consider $\gamma = I_1 l$, $\varphi = lE$, $\psi = I$). Moreover it follows from the same proposition that $pI(lE)$ holds for almost all p of $I_0'l$, i.e.

$$\mathrm{rank}(I(lE) \dot{-} I_0'l) = 0, \qquad \mathrm{rank}(I(lE)) = 1.$$

In particular for our p_0 and l_0, if we put $\bar{l}_0 = l'_0E$ we get the strong minimality of $I\bar{l}_0$ and $p_0I\bar{l}_0$.

If $l_1, l_2 \in L_1$, $\bar{l}_1 = l_1E$, $\bar{l}_2 = l_2E$, $\operatorname{rank}(I\bar{l}_1 \cap I\bar{l}_2) > 0$, then

$$\operatorname{rank}(I'_0l_1 \cap I'_0l_2) > 0 \quad \text{and} \quad \operatorname{rank}(I'_0l_1 \dot{-} I'_0l_2) = 0,$$

which follows from the definition of L_1, hence $\bar{l}_1 = \bar{l}_2$.

We show now that $\operatorname{rank}(\bar{l}_0, \emptyset) \geqslant 2$ and therefore $\operatorname{rank}(L) \geqslant 2$.

Suppose $\operatorname{rank}(\bar{l}_0, \emptyset) \leqslant 1$. Then, since $\operatorname{rank}(p_0, \{\bar{l}_0\}) = 1 < \operatorname{rank}(p_0, \emptyset)$, $\operatorname{rank}(\bar{l}_0, \{p_0\}) < \operatorname{rank}(\bar{l}_0, \emptyset) \leqslant 1$. Thus $\bar{l}_0 \in \operatorname{acl}(p_0) = \operatorname{acl}(b_1, b_2)$. Evidently $c \notin \operatorname{acl}(\bar{l}_0)$, therefore there is c' of M such that

$$t(\langle c, c' \rangle, \{\bar{l}_0\}) = t(\langle b_1, b_2 \rangle, \{\bar{l}_0\}).$$

Since $\operatorname{rank}(\langle c, c' \rangle, \{\bar{l}_0\}) = 1$,

$$c' \in \operatorname{acl}(\bar{l}_0, c) \subseteq \operatorname{acl}(b_1, b_2, c) \cap \operatorname{acl}(a_1, a_2, c).$$

By the definition $c' \notin \operatorname{acl}(c)$. This contradicts the assumptions of the lemma. Hence, $\operatorname{rank}(\bar{l}_0, \emptyset) \geqslant 2$.

Lemma 3. *Let for an incidence system* $\langle P, L, I \rangle$: $p_0 \in P$, $\operatorname{rank}(p_0, \emptyset) \geqslant \operatorname{rank}(L)$, $\operatorname{rank}(p_0I) > 0$ *and if* $p_1, p_2 \in P$, $p_1 \neq p_2$, *then* $\operatorname{rank}(p_1I \cap p_2I) = 0$.

Then there is $l_0 \in L$ *such that*

$$\langle p_0, l_0 \rangle \in I, \qquad \operatorname{rank}(p_0, \{l_0\}) > 0,$$

$$\operatorname{rank}(l_0, \emptyset) > \operatorname{rank}(p_0I), \qquad \operatorname{rank}(Il_0) > 0.$$

Proof. It follows from the assumptions of the lemma that there is no 0-definable subset L' of L such that $L' \supseteq p_0I$, $\operatorname{rank}(L') = \operatorname{rank}(p_0I)$. Thus, by the Compactness Theorem there is

$$l_0 \in p_0I - \operatorname{acl}(p_0), \qquad \operatorname{rank}(l_0, \emptyset) > \operatorname{rank}(p_0I).$$

Now counting

$$\operatorname{rank}(\langle p_0, l_0 \rangle, \emptyset) = \operatorname{rank}(p_0, \{l_0\}) + \operatorname{rank}(l_0, \emptyset)$$
$$= \operatorname{rank}(l_0, \{p_0\}) + \operatorname{rank}(p_0, \emptyset),$$

we have

$$\operatorname{rank}(p_0, \{l_0\}) = \operatorname{rank}(l_0, \{p_0\}) + \operatorname{rank}(p_0, \emptyset)$$
$$- \operatorname{rank}(l_0, \emptyset) \geqslant \operatorname{rank}(l_0, \{p_0\}) > 0.$$

Hence, in particular, $\operatorname{rank}(Il_0) > 0$.

LEMMA 4. *If all the assumptions of Lemma 2 hold then an uncountably categorical pseudoplane* $\langle P, L, I\rangle$ *is definable in* M *with* $\operatorname{rank}(P) = \operatorname{rank}(L) = 2$, $\deg(P) = \deg(L) = 1$.

PROOF. Using Lemma 2 and the symmetry of the definition we get an incidence system $\langle P_0, L_0, I_0\rangle$ definable in M such that the following hold:

$$\text{(i)} \qquad \operatorname{rank}(P_0) \geq 2, \qquad \operatorname{rank}(L_0) = 2, \qquad \deg(L_0) = 1;$$

$$\text{(ii)} \qquad \operatorname{rank}(pI_0) = 1 \quad \text{for all } p \text{ of } P_0;$$

$$\text{(iii)} \qquad \text{if } p_1, p_2 \in P_0,\ p_1 \neq p_2, \quad \text{then } \operatorname{rank}(p_1 I_0 \cap p_2 I_0) = 0.$$

Considering a definable subset of rank 2 degree 1 of P_0 instead of P_0 and taking an inessential expansion of M we preserve (i), (ii), (iii) having $\operatorname{rank}(P_0) = 2$ and the incidence structure 0-definable in M.

Apply now Lemma 3 to find $\langle p_0, l_0\rangle \in I_0$ such that

$$\operatorname{rank}(p_0, \emptyset) = \operatorname{rank}(l_0, \emptyset) = 2,$$

$$\operatorname{rank}(p_0 I) = 1 = \operatorname{rank}(l_0, \{p_0\}),$$

$$\operatorname{rank}(I l_0) = 1 = \operatorname{rank}(p_0, \{l_0\}).$$

Now by Lemma 1 we get an incidence system $\langle P_0, L_0', I_0'\rangle$ and $l_0' \in L_0'$. $I_0' l_0'$ is strongly minimal in the system. In addition for different p_1, p_2, of P_0 the set $p_1 I_0' \cap p_2 I_0'$ is finite, since it lies in $m^{-1}(p_1 I_0 \cap p_2 I_0)$.

Put as in the proof of Lemma 2

$$L_1 = \{l_1 \in L_0' \colon \operatorname{rank}(I_0' l_1) = 1\ \&\ (\forall l_2 \in L_0')$$
$$(\operatorname{rank}(I_0' l_1 \cap I_0' l_2) > 0 \to \operatorname{rank}(I_0' l_1 - I_0' l_2) = 0)\};$$

$$l_1 E l_2 \quad \text{iff} \quad \operatorname{rank}(I_0' l_1 \dot{-} I_0' l_2) = 0;$$

$$P = P_0, \qquad L = L_1/E,$$

$$pI(lE) \quad \text{iff} \quad \operatorname{rank}(pI_0' - lE) < \operatorname{rank}(lE).$$

Observe that every class lE is finite, since if lEl_1, then there are $p_1, p_2 \in I_0' l \cap I_0' l_1$,

$$\operatorname{rank}(p_1, \{l\}) = \operatorname{rank}(p_1, \{l, l_1\}) = 1,$$

$$\operatorname{rank}(p_2, \{p_1, l\}) = \operatorname{rank}(p_2, \{p_1, l, l_1\}) = 1,$$

and by the reciprocity principle

$$\operatorname{rank}(l_1, \{l\}) = \operatorname{rank}(l_1, \{l, p_1\}),$$

$$\operatorname{rank}(l_1, \{l, p_1\}) = \operatorname{rank}(l_1, \{l, p_1, p_2\}),$$

thus

$$\operatorname{rank}(l_1, \{l\}) \leq \operatorname{rank}(l_1, \{p_1, p_2\}) = 0$$

(since $l_1 \in p_1 I'_0 \cap p_2 I'_0$), i.e.

$$l_1 \in \operatorname{acl}(l).$$

Granting the finiteness of lE,

$$pI(lE) \quad \text{iff} \quad lE \subseteq pI'_0.$$

Hence

$$pI \subseteq pI'_0/E,$$

$$\operatorname{rank}(p_1 I \cap p_2 I) = 0 \quad \text{for distinct } p_1, p_2 \in P.$$

As is shown in the proof of Lemma 2 for distinct l_1, l_2 of L

$$\operatorname{rank}(Il_1 \cap Il_2) = 0, \qquad \operatorname{rank}(Il_1) = 1.$$

To get $\operatorname{rank} pI = 1$ for all $p \in P$ remove all $p \in P$ with $\operatorname{rank} pI = 0$ from P. Since $\operatorname{rank}(p_0, \emptyset) = 2$, $\operatorname{rank}(p_0 I) = 1$ and $\operatorname{rank}(P) = 2$, $\deg(P) = 1$, the set of the points removed has rank not greater than 1, therefore removing these points we diminsh the rank of only a finite number of Il, $l \in L$. Remove these lines too, denote by the same letters P, L the new sets, and the construction of the pseudoplane $\langle P, L, I\rangle$ is finished. In ZIL'BER [1980b, Proposition 11] it is proved that $\langle P, L, I\rangle$ is an uncountably categorical pseudoplane.

PROPOSITION 5. *If no uncountably catagorical pseudoplane is definable in M then for every strongly minimal structure S definable in M the geometry associated with S is locally projective or degenerate.*

The proof of the proposition follows from Lemma 4 as is shown in ZIL'BER [1980b, Section 2].

PROPOSITION 6. *If a pseudoplane is definable in M, $A \subseteq M$, then for every A-definable in M strongly minimal structure S the geometry associated with S over a is neither projective ,nor degenerate.*

PROOF. Let $\langle P, L, I\rangle$ be a C-definable in M incidence system satisfying the following:

(i) $\quad \text{rank}(Il_1 \cap Il_2) = 0, \qquad \text{rank}(p_1 I \cap p_2 I) = 0$

for $l_1, l_2 \in L$, $p_1, p_2 \in P$, $l_1 \neq l_2$, $p_1 \neq p_2$.

(ii) For some pair $\langle p_0, l_0 \rangle \in I$

$$\text{rank}(l_0, C) = \text{rank}(L), \qquad \text{rank}(p_0, C) = \text{rank}(P),$$

$$\text{rank}(l_0, \{p_0\} \cup C) > 0, \qquad \text{rank}(Il_0) > 0, \qquad \text{rank}(p_0 I) > 0.$$

(iii) The four-tuple $\langle \text{rank}(P), \deg(P), \text{rank}(L), \deg(L) \rangle$ is lexicographically minimal among such four-tuples for every $C \subseteq M$ and systems $\langle P, L, I \rangle$ satisfying (i) and (ii).

Observe that it follows from Lemma 3 and the minimality condition that

$$\text{rank}(P) = \text{rank}(L), \qquad \deg(P) = 1, \qquad \deg(L) = 1.$$

Condition (ii) implies the existence of such C-definable P', L', I', $p_0 \in P' \subseteq P$, $l_0 \in L' \subseteq L$, $\langle p_0, l_0 \rangle \in I' \subseteq I$ that for every $p \in P'$, $l \in L'$

(iv) $\text{rank}(pI') > 0, \qquad \text{rank}(I'l) > 0$.

We assume that $P' = P$, $L' = L$, $I' = I$. Put $r = \text{rank}(P) = \text{rank}(L)$.

Observe also that any extension of C preserves conditions (i)–(iv), thus we may assume C contains all the parameters required in what follows.

Let S be a C-definable in M set and $\psi(x, y)$ a formula with parameters in C which is a stratification of L over S of rank less than r, i.e.:

(v) For every $l \in L$ there is an $s \in S$ such that $\psi(s, l)$.

(vi) For every $s \in S$, $\text{rank}(\psi(s, M)) < r$.

The stratification ψ exists if C is sufficiently large, ZIL'BER [1974] (see also another version of the statement in SHELAH [1978, V. 6.1]).

Let us prove

(vii) For every $s \in S$, $\text{rank}(p_0 I \cap \psi(s, M)) = 0$. Indeed, otherwise, putting

$$L_0 = \psi(s, M) \cap L, \qquad I_0 = I \cap (P \times L_0)$$

we get

$$\text{rank}(p_0, C \cup \{s\}) \geq r - 1 \geq \text{rank}(L_0), \qquad \text{rank}(p_0 I_0) > 0.$$

Evidently $\langle P, L_0, I_0 \rangle$ over $C \cup \{s\}$ satisfies (i), and (ii) follows from Lemma 3. This contradicts the minimality of $\langle P, L, I \rangle$.

Observe again that if we take

$$P' = \{p \in P \colon \forall s \in S, \ \text{rank}(pI \cap \psi(s, M)) = 0\},$$
$$I = I \cap (P' \times L),$$

we get system $\langle P', L, I' \rangle$ satisfying (i)–(iii), therefore it can be assumed $P' = P$.

Let Q be a definable subset of $P \cup L$ of the rank maximal among all such Q that

$$Q \subseteq \mathrm{acl}(S \cup C'), \quad C' \supseteq C, \quad C' \text{ is finite.}$$

Since C was assumed to be sufficiently large, hence $C' = C$. Also $Q \neq \emptyset$, since $Q \supseteq (P \cup L) \cap C$. Let

$$\mathrm{rank}(Q \cap L) \geqslant \mathrm{rank}(Q \cap P),$$

$$P^* = P \cap Q, \qquad L^* = \bigcup\{pI\colon p \in Q \cap P\}.$$

For every l from L^* there are $p \in P^*$ and $s \in S$ such that $l \in pI \cap \psi(s, M)$. Since the last set is of rank 0,

$$l \in \mathrm{acl}(p, s, C); \qquad L^* \subseteq \mathrm{acl}(S \cup C),$$

i.e. $\mathrm{rank}(L^*) \leqslant \mathrm{rank}(P^*)$. Choose $p_0^* \in P^*$ so that $\mathrm{rank}(p_0^*, C) \geqslant \mathrm{rank}(L^*)$ and by Lemma 3 we get (ii) for $\langle P^*, L^*, I \cap (P^* \times L^*)\rangle$. Since (i) for this system follows from that of $\langle P, L, I\rangle$,

$$\mathrm{rank}(P^*) = \mathrm{rank}(L^*) = r.$$

We will assume $P^* = P$, $L^* = L$, i.e. $P \cup L \subseteq \mathrm{acl}(S \cup C)$.

It can be easily proved by induction on k, that for every C-definable set Q of rank k, if $Q \subseteq \mathrm{acl}(S \cup C)$, then C can be extended so that for every $q \in Q$ there are $d_1, \ldots, d_k \in S$

$$\mathrm{acl}(q, C) = \mathrm{acl}(d_1, \ldots, d_k, C).$$

Let now Q be $P \cup L$. Assume for simplicity $C = \emptyset$. Then

$$\mathrm{acl}(p_0) = \mathrm{acl}(s_1, \ldots, s_r), \qquad \mathrm{acl}(l_0) = \mathrm{acl}(t_1, \ldots, t_r),$$

for $s_1, \ldots, s_r$, $t_1, \ldots, t_r \in S$. Since $\mathrm{rank}(p_0, \emptyset) = r$, $\mathrm{rank}(l_0, \emptyset) = r$,

$$\dim(s_1, \ldots, s_r) = r = \dim(t_1, \ldots, t_r).$$

It follows from (i) that

$$\mathrm{rank}(p_0, \{l_0\}) < r,$$

therefore

$$\dim(s_1, \ldots, s_r, t_1, \ldots, t_r) < 2r.$$

If the geometry associated with S over $\emptyset$ is degenerate or projective, then the last condition implies the existence of $u_0 \in S - \mathrm{acl}(\emptyset)$

$$u_0 \in \mathrm{acl}(s_1, \ldots, s_r) \cap \mathrm{acl}(t_1, \ldots, t_r),$$

i.e. $u_0 \in \mathrm{acl}(p_0) \cap \mathrm{acl}(l_0)$.

It is easy to get, using the last fact, a formula $\psi(x, y)$ without parameters such that

$$\psi(u_0, l_0) \quad \text{and} \quad \text{rank}(\psi(u_0, M)) < r.$$

Such a formula can be easily touched up so that (v) and (vi) be satisfied. Therefore $p_0 I \cap \psi(u_0, M)$ is finite. This set contains l_0, hence

$$l_0 \in \text{acl}(p_0, u_0) = \text{acl}(p_0).$$

This contradicts condition (ii). Thus the geometry associated with S over C is neither projective nor degenerate. Since $A \subseteq C$, the proposition is proved.

Proof of the Trichotomy Theorem. By Propositions 5 and 6 the non-definability of pseudoplanes in M is equivalent to the fact that all strongly minimal structures in M have locally projective or degenerate geometries. Adding a new constant for any locally projective strongly minimal structure we can assume that the locally projective structure is projective. Since every two strongly minimal sets in an uncountably categorical structure are nonorthogonal, their geometries are isomorphic provided they are projective or degenerate, as is proved in CHERLIN et al. [1981, 2.8].

Now we begin with the proof of Theorem 3. M is the structure which does not satisfy (i) of the Trichotomy Theorem, G is the group definable in M.

Proof of Theorem 3(ii). We will prove that if G has no proper infinite definable subgroups and Q is its strongly minimal subset, then $G - Q$ is finite. Let

$$H = \{h \in G \colon Qh \dot{-} Q \text{ is finite}\},$$

$$H' = \{h \in G \colon hQ \dot{-} Q \text{ is finite}\}.$$

It is known from ZIL'BER [1977, Lemma 10] that H and H' are definable subgroups of G and for some $g, g' \in Q$, $\text{rank}(gH - Q) < \text{rank}(H)$, $\text{rank}(H'g' - Q) < \text{rank}(H)$. By our assumptions H and H' are finite or equal to G. If $H = G$ or $H' = G$, then $G - Q$ is finite. Thus we may assume that H and H' are finite.

Now by the definition $Q \dot{-} H'QH$ is finite, we assume $Q = H'QH$. Put

$$P = \{gH' \colon g \in G\}, \qquad L = \{Hg \colon g \in G\},$$

$$I = \{\langle g'H', Hg \rangle \colon g' \in Qg\}.$$

If H and H' are finite, then all the axioms of a pseudoplane are satisfied by $\langle P, L, I\rangle$, which contradicts the assumptions of the theorem. Theorem 3(ii) is proved.

LEMMA 7. *Let $J \subseteq U \times G$ be a binary definable relation such that for every $u \in U$ the set uJ is a strongly minimal subgroup of G and for any distinct $u_1, u_2 \in U$, $u_1J \neq u_2J$. Then U is finite.*

PROOF. Suppose not. Then we may assume U is strongly minimal. Put

$$H = \{h \in G: Jh \text{ is infinite}\},$$

$$P = \{Hg: g \in G\}, \qquad L = \{g \cdot uJ: g \in G, u \in U\},$$

$$I = \{\langle gH, g \cdot uJ\rangle: g \in G, u \in U\}.$$

H is finite for otherwise, since for any $h_1, \ldots, h_k \in H$ $Jh_1 \cap \cdots \cap Jh_k$ is infinite we can find distinct $u_1, u_2 \in U$ such that $u_1J \cap u_2J$ contains at least k elements $h_1, \ldots, h_k$, therefore distinct u_1, u_2 can be found with $u_1J \cap u_2J$ infinite, which contradicts assumptions of the lemma.

H is a subgroup of G, since for $h_1, h_2 \in H$, $Jh_1 \cdot h_2^{-1} \supseteq Jh_1 \cap Jh_2$ is infinite. Now it can be directly verified that $\langle P, L, I\rangle$ is a pseudoplane, which is a contradiction.

LEMMA 8. *G possesses a definable normal nilpotent subgroup of finite index.*

PROOF. We may assume that G is connected (i.e. has no proper definable subgroup of finite index, see CHERLIN [1979] or ZIL'BER [1977]). Then $G \times G$ is also connected.

Let H be a strongly minimal subgroup of G, which exist by Theorem 3(ii), if $G \neq 1$. Let $gI\langle h, h'\rangle$ mean $h \in H$ & $h' = g^{-1}hg$.

Clearly for every $g \in G$ the set gI is a subgroup of $G \times G$ isomorphic to H, i.e. gI is strongly minimal.

Let g_1Eg_2 denote $g_1I = g_2I$. Then, by Lemma 7, G/E is finite. This means that the centralizer $C(H)$ of H in G has a finite index. Since G is connected, $C(H) = G$ and thus H lies in the center of G. It follows by induction on rank(G) that G is nilpotent.

LEMMA 9. *G possesses a definable normal abelian subgroup of finite index.*

PROOF. Now we may assume G is connected and nilpotent of class 2 (i.e. $G/C(G)$ is abelian). It is sufficient to prove that $G = C(G)$.

Denote $\bar{G} = G/C(G)$ and supposing $\bar{G} \neq 1$ we get by the connectedness of G that $\bar{G}$ is infinite and by Theorem 3(ii) $\bar{G}$ has a strongly minimal subgroup H. Put

$$gI\langle h, h'\rangle \quad \text{iff} \quad h, h' \in H \;\&\; g \in G \;\&\; h' = hgh^{-1}g^{-1},$$

$$g_1 E g_2 \quad \text{iff} \quad g_1 I = g_2 I.$$

It is evident that gI is a strongly minimal subgroup of $\bar{G} \times C(G)$. By Lemma 7, G/E is finite, in other words the subgroup

$$\{g \in G : \forall h \in H,\ hgh^{-1} = g\}$$

has a finite index in G and thus coincides with G. This means $H = 1$ in $\bar{G}$, contradiction.

This proves the lemma and concludes the proof of Theorem 3.

Proof of Theorem 4. First we suppose $G \subseteq \text{acl}(S)$ for some strongly minimal set S. We shall prove that G is finite.

Let $\text{rank}(G) = k$. It is easy to prove for any set $G \subseteq \text{acl}(S)$ by induction on k that there is a finite $A \subseteq M$ such that for every $g \in G$ there are $s_1, \ldots, s_k \in S$ with

$$\text{acl}(g, A) = \text{acl}(s_1, \ldots, s_k, A).$$

Assume for simplicity $A = \emptyset$ and choose $g, h \in G$ independent over $\emptyset$ with $\text{rank}(g, \emptyset) = \text{rank}(h, \emptyset) = k$. We have

$$\text{acl}(g) = \text{acl}(s_1, \ldots, s_k), \qquad \text{acl}(h) = \text{acl}(t_1, \ldots, t_k)$$

for some $s_1, \ldots, s_k, t_1, \ldots, t_k \in S$. It follows from the independence of g and h that

$$\text{acl}(s_1, \ldots, s_k) \cap \text{acl}(t_1, \ldots, t_k) = \text{acl}(\emptyset).$$

Let $g \cdot h = f$, $\text{acl}(f) = \text{acl}(u_1, \ldots, u_k)$, $u_1, \ldots, u_k \in S$. Since $f \in \text{acl}(g, h)$, $h \in \text{acl}(g, f)$, $g \in \text{acl}(h, f)$ we have, granting S is disintegrated,

$$\text{acl}(s_1, \ldots, s_k) \cup \text{acl}(t_1, \ldots, t_k) \supseteq \text{acl}(u_1, \ldots, u_k),$$

$$\text{acl}(s_1, \ldots, s_k) \cup \text{acl}(u_1, \ldots, u_k) \supseteq \text{acl}(t_1, \ldots, t_k),$$

$$\text{acl}(t_1, \ldots, t_k) \cup \text{acl}(u_1, \ldots, u_k) \supseteq \text{acl}(s_1, \ldots, s_k).$$

This is possible only if all the sets lie in $\text{acl}(\emptyset)$. Thus $k = 0$ and G is finite.

Now if $M \not\subseteq \text{acl}(S)$, then an infinite group G is definable in M with $G \subseteq \text{acl}(S)$, as is shown in ZIL'BER [1980a, Proposition 4.3]. This is impossible as was shown above, and it follows that $G \subseteq \text{acl}(S)$ for every group G definable in M and G is finite.

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