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Geometric Curve Evolution and Image Processing

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Preface

These lectures intend to give a self-contained exposure of some techniques for computing the evolution of plane curves. The motions of interest are the so-called motions by curvature. They mean that, at any instant, each point of the curve moves with a normal velocity equal to a function of the curvature at this point. This kind of evolution is of some interest in differential geometry, for instance in the problem of minimal surfaces. The interest is not only theoretical since the motions by curvature appear in the modeling of various phenomena as crystal growth, flame propagation and interfaces between phases. More recently, these equations have also appeared in the young field of image processing where they provide an efficient way to smooth curves representing the contours of the objects. This smoothing is a necessary step for image analysis as soon as the analysis uses some local characteristics of the contours. Indeed, natural images are very noisy and differential features are unreliable if one is not careful before computing them. A solution consists in smoothing the curves to eliminate the small oscillations without changing the global shape of the contours. What kind of smoothing is suitable for such a task? The answer shall be given by an axiomatic approach whose conclusions are that the class of admissible motions is reduced to the motions by curvature. Once this is established, the well-posedness of these equations has to be examined. For certain particular motions, this turns to be true but no complete results are available for the general existence of these motions. This problem shall be turned around by introducing a weak notion of solution using the theory of viscosity solutions of partial differential equations (PDE). A complete theory of existence and uniqueness of those equations will be presented, as self-contained as possible. (Only a technical, though important, lemma will be skipped.) The numerical resolution of the motions by curvature is the next topic of interest. After a rapid review of the most commonly used algorithms, a completely different numerical scheme is presented. Its originality is that it satisfies exactly the same invariance properties as the equations of motion by curvature. It is also inconditionally stable and its convergence can be proved in the sense of viscosity solutions. Moreover, it allows to precisely compute motions by curvature, when the normal velocity is a power of the curvature more than 3, or even 10 in some cases, which seems a priori nearly impossible in a numerical point of view. Many numerical experiments are presented.

Who this volume is addressed to?

We hope that these notes shall interest people from both communities of applied mathematics and image processing. We tried to make them as self-contained as possible. Nevertheless, we skipped the most difficult results since their proof uses techniques that would have led us too far from our main way. Indeed, these lectures are addressed to researchers discovering the common field of mathematics and image processing but also to graduate and PhD students wanting to span a theory from A to Z: from the basic axioms, to mathematical results and numerical applications. The chapters are mostly independent except Chap. 6 that uses results from Chap. 4. The bibliography on every subject we tackle is huge, and we cannot pretend to give exhaustive references on differential geometry, viscosity solutions, mathematical morphology or scale space theory. At the end of most chapters, we give short bibliographical notes detailing in a few words the main steps that produced significant advances in the theory.

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Rennes, October 2002 *Frédéric Cao*

Contents

Part II Theoretical curve evolution

1

Curve evolution and image processing

In this volume, we study some theoretical results and the numerical analysis of the motions of plane curves driven by a function of the curvature. If C is a smooth (say C^2) curve, they are described by a partial differential equation (PDE) of the type

$$\frac{\partial C}{\partial t} = G(\kappa)\mathbf{N}, \tag{1.1}$$

where κ and $\mathbf{N}$ are the curvature and the normal vector to the curve. This equation means that any point of the curve moves with a velocity which is a function of the curvature of the curve at this point. (See Fig. 1.1.)

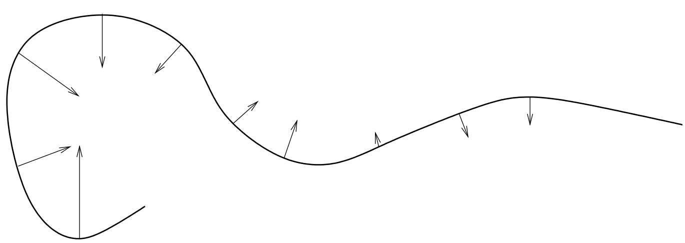

Fig. 1.1. Motion of a curve by curvature. The arrows represent the velocity at some points. Here, the velocity is a nondecreasing function of the curvature

These equations appear in differential geometry because the curvature is the variation of the area functional for hypersurfaces. (The length for curves.) In particular, the case $G(\kappa) = \kappa$ can be considered as the gradient flow of the area (length for curve), playing an important role in the theory of minimal surfaces. These equations are also related to the description of crystal growth, where the velocity may also contain an anisotropic term depending on the normal vector. Generally speaking, curvature motions often appear in the motion of interfaces driven by an inner energy or tension, as flame propagation, melting ice, or rolling stone. Surprisingly enough, the motions

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by curvature have recently appeared in the field of image processing. More precisely, the theory developed in the core of this monograph aims at solving one of the steps that belongs to what has been called low-level vision. It appeared that any automatic interpretation of an image was impossible (or at least very difficult) to perform if one does not apply some preliminary operations on the image. These operations are transformations on the image which make it easier to handle, or simplify it, in order to extract the most basic information more easily. The nature of this information itself is not so easily defined and many researches have tried to mimick the human vision for computational purpose. How vision does really work is still a controversial subject and, except in the next paragraph, we shall not enter into such considerations, but try to remain as practical as possible.

Let us examine a bit closer what an image analysis algorithm should intuitively do. The input of such an algorithm is an image taken by a camera. The output is some interpretation yielding an automatic decision. A commonly accepted method to attain this objective is to detect the objects that are present in the scene and to determine their position and possibly their movement. We could further try to determine the nature of these objects. Before the foundation of the Gestalt School in 1923 [168], it was believed that we detected objects because of the experience we had of them. On the contrary, Gestaltists proved by some psychophysical experimentations that without a priori semantic knowledge, shapes were conspicuous as the result of the collaboration or inhibition of some geometrical laws [99, 98]. Even though the Gestalt laws are rather simple (and were nearly set in mathematical terms by the Gestaltists), their formulation in a computational language is more complex, because they are nonlocal and hierarchically organized. A plan for the computational detection of perceptual information was initiated by Attneave [15], then Lowe [114] and more recently by Desolneux, Moisan and Morel [50]. In fact, most of widely used theories, as edge detection or image segmentation, without strictly following a Gestaltist program, take some part of it into account, since they assume that shapes are homogeneous regions separated from one another and the background by smooth and contrasted boundaries [26, 118, 133, 100], which is in agreement with some grouping Gestalt laws. These theories are often variational and can be formulated with elegant mathematical arguments. (We also refer to a recent book by Aubert and Kornprobst [16] exposing the mathematical substance of these theories.) Very recently, Desolneux, Moisan and Morel [49] developed a new algorithm for shape detection following the Gestalt principles. The advantage of this method is that the edges they found are level lines of images, and consequently, Jordan curves, which are the objects we shall deal with in the following.

We assume (and believe!) that this detection program is realistic but we do not cope with it. On the other hand, this does not mean that we should consider that the problem of shape extraction has been completely elucidated! Nevertheless, as the topic of these lectures follows shape detection, we are obliged to take it for granted.

1.1 Shape recognition

Determining automatically the nature of a detected object (is it a man? a vehicle? what kind of vehicle? etc...) is achieved by placing it in some pre-established classification which is the preliminary knowledge. Algorithms use some more or less large databases allowing to precise the classification and try to compare the detected shapes with known ones. Shape recognition is this classification.
Otherwise said, we have a collection of model patterns and we want to know which one the detected shape matches best. A more simple subproblem is to decide whether the observed shape matches a given model. This raises at least two questions:

1. what kind of representation do we take for a shape? (or what is our model of shape?)
2. what kind of properties a shape recognition algorithm should satisfy?

In what follows, we only consider two-dimensional images. The answer to the first question shall be simple: a shape will be a subset of the plane. If the set is regular, it shall be useful to represent it by its boundary. If the set is bounded, its boundary is a closed curve. By the Theorem of Alexandrov 2.4, it is equivalent to know the set or its boundary.
In order to answer the second question, let us follow David Marr in *Vision* [117].

> "Object recognition demands a stable shape description that depends little, if at all, on the view point. This, in turn, means that the pieces and articulation of a shape need to be described not relative to the viewer but relative to a frame of reference based on the shape itself. This has the fascinating implication that a canonical coordinate frame must be set up within the object *before* its shape is described, and there seems to be no way of avoiding this."

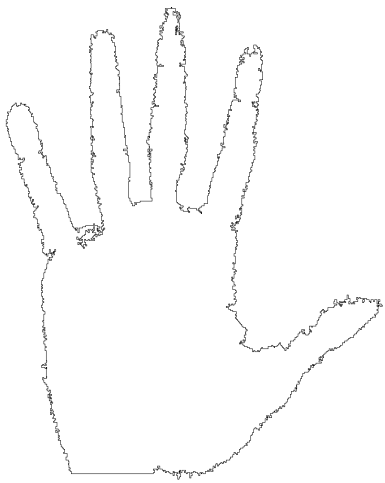
A "canonical coordinate frame" is

> "a coordinate frame uniquely determined by the shape itself."

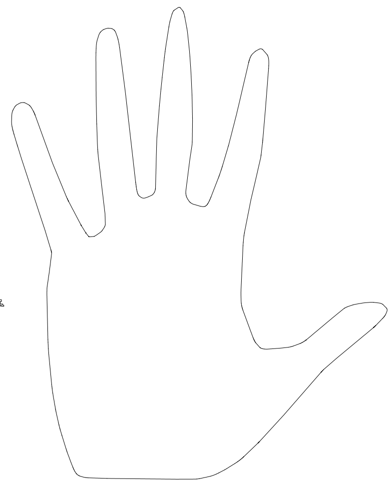
The description must be stable in the sense that it must be insensitive to noise. For instance, consider the shape given by the curve on Fig. 1.2(a). This curve has been obtained by scanning a hand and then by thresholding the grey level to a suitable value. One has no difficulty to recognize this shape immediately. However, in a computational point of view, this shape is very complicated. A quantitative measure of this complexity is that the curve has about 2000 inflexion points, most of which having no perceptual meaning! Let us now consider the shape on Fig. 1.2(b). This shape has been obtained from the original one by smoothing it with an algorithm described in the following of these lectures. The tiny oscillations have disappeared, and the curve has only 12 inflexion points. In itself, this number has no absolute significance. However, this shape is intuitively better in a computational point of view for three reasons:

1. it is very close to the first one.
2. it is smoother.

3. Fig. 1.2(b) is a good sketch of a hand in the sense that it cannot be much simplified without changing the interpretation. As a parallel, Attneave [15] showed a sketch of a cat containing only a few (carefully chosen) lines, which were sufficient to guess what the drawing was. This means that, up to some point, a shape can be considerably simplified without altering our recognition. In a sense, Fig. 1.2(b) is closer than Fig. 1.2(a) to the minimal description of a hand.



(a) Original shape



(b) Smoothed shape

Fig. 1.2. The representation of shape for recognition must be as simple and stable as possible. Both shapes represent the same object at different level of details. For a recognition task, most of details on Fig. 1.2(a) are spurious. The shape on Fig. 1.2(b) is intuitively much simpler and visually contains the same information as the noisy one

What about the "canonical coordinate frame"? Mathematicians will call such a frame *intrinsic*. It is not very complicated to imagine such a frame. For instance, the origin may be taken as the center of mass of the shape. Then, the principal directions given by the second order moments provide some intrinsic directions. Those have the nice additional properties to be invariant with respect to rotations. However, if we think of a circle, this intrinsic frame is not uniquely defined. This does not matter since each frame gives the same description, but if we now think of a noisy circle, then the orientation of the frame vector may dramatically change, contradicting the stability hypothesis.

1.1.1 Axioms for shape recognition...

Marr did not give precise any practical algorithms for shape representation and recognition but, in a sense, he initiated an axiomatic approach, that has been prolonged by many people. (A recent review with axiomatic arguments is presented by Veltkamp and Hagedoorn [164] for the shape matching problem.) Most people agree that the matching problem is equivalent to finding distances between shapes. (See for example the works of Trouvé and Younes [159, 170].)

Intrinsic distance

First, the distance between objects should be independent on the way we describe them. If an object is a set of pixels, it seems clear that a distance taking an arbitrary order of the points into account is not suitable. In the same way, for curves matching, the parameterization should not influence the matching, which should only depend on the geometry of the curves.

Invariance

Another property shall also be a cornerstone of our theory: invariance. Marr understood that the recognition should not depend on the particular position of the viewer. The mathematical formulation of invariance is a well known technique, giving extremely important results in many fields such as theoretical physics or mechanics. We consider a set of transformations (homeomorphisms), which has in general a group structure. This group models the set of modifications of the shape when the viewer moves. In a three dimensional world, images are obtained by a projection and such a group does not exist: we cannot retrieve hidden parts of objects by simply deforming the image taken by the camera. However, for “far enough” objects and “small displacements”, projective transformations are a good model to describe the modification of the silhouette of the objects since they correspond to the change of vanishing points in a perspective view. If we add some additional hypotheses on the position of the viewer, we can even consider a subgroup of the projective group. If $\mathcal{G}$ is the admissible group of deformations, and d a pseudo-distance between shapes, the invariance property can be formulated by

$$\forall g \in \mathcal{G}, \quad d(gA, B) = f(g)d(A, B), \tag{1.2}$$

where gA represents the shape A deformed by g and $f : \mathcal{G} \to \mathbb{R}$ does not depend on A and B. Notice that d is only a pseudo-distance since $d(gA, A) = f(g)d(A, A) = 0$. In a mathematical point of view, it is natural to define shape modulo a transformation, which is equivalent to define a distance between the orbits of the shape under the group action. In such a way, a true distance is retrieved instead of a pseudo-distance.

Stability

The stability may be thought as noise insensitivity. In [164], it is formulated by a set of four properties. The first one is strongly related to invariance, and we do not go further. The last three may be interpreted as follows: we modify a shape by some process and we compute the distance between the original shape and the new one. Then, it should be small in the following cases:

1. blurring: we add some parts, possibly important, but close to the shape.
2. occlusion: we hide a small part of a shape (possibly changing its topology).
3. noise addition: we add small parts possibly far from the shape.

Simplicity

This last property is not an axiom properly speaking, since it is not related to the recognition itself. However, we believe that an algorithm will be all the more efficient and fast, if it manipulates a small amount of data. Intuitively, it is certainly easier to describe the curve of Fig. 1.2(b) than the one of Fig. 1.2(a). For instance, we could think of keeping a sketch of the curve linking the points with maximal curvature. On the noisy curve, nearly all the points are maxima of curvature and the sketch is as complex as the original curve.

1.1.2 ... and their consequences

What can be deduced from the heuristic above? First, in order to get insensitivity to noise, it seems natural to smooth the shapes. Naturally, we then face the problem: what kind of oscillation can be labelled as noise, or contains real information? There is no absolute answer to this question but it shall only depend of a single parameter called *scale* representing the typical size of what will be considered as noise, or the distance at which we observe the shape. Since we cannot choose this scale a priori, smoothing will be multiscale and shape recognition will have a sense at each scale. Since the recognition must resist to occlusion, it should, at least partially, rely on local features. This is another argument for smoothing since local features are sensitive to noise. For instance, commonly used are the inflexion points and the maxima of curvature. Since they are defined from second derivatives, this is clear that a noisy curve as in Fig. 1.2(a) is totally unreliable.

1.2 Curve smoothing

We now admit the principle that shape recognition is made possible by a multiscale smoothing process removing the noise at each scale. It seems, that by adding this step, we have complicated the problem. Indeed. We do not know what kind of smoothing we have to choose, the only objective we have is to make local features reliable. It is also obvious that the smoothing has to be compatible with all the assumptions we made on the recognition task (invariance, stability, simplicity). The remaining of these lectures gives the possible ways to smooth curves and a possible implementation of the underlying equations. Once for all, we do not pretend to give *the* method for shape recognition. We do not even give any recognition algorithm since we only focus on the low-level vision preprocessing. As Guichard and Morel [81], we simply assert that *if* a smoothing has to be done, then it is sound that it satisfies some requirements, and in this case, we give the corresponding mathematical models. The conclusions are only valid because of the acceptance of a model that can always be criticized or discussed[1].

1.2.1 The linear curve scale space

Before studying in full details the axioms that curve smoothing should satisfy, we first shortly examine the most simple way of regularizing curves we can think of. This attempt will be a failure, but it will help us to better understand the final approach. It is well-known that a way to regularize a function is to compute its convolution with a smoothing kernel. If the kernel is taken to be a gaussian with a suitably chosen variance, then the function is solution of the heat equation, which is the archetype of smoothing equations[2]. In our case, consider the curve C given by its coordinates $(x(p), y(p))$. For a closed curve, p is taken on the unit circle $\mathbb{S}^1$ endowed with its usual Riemannian structure, so that we can differentiate with respect to p. We then solve the one-dimensional heat equation for each coordinate, that is

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial p^2} \quad \text{and} \quad \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial p^2}. \tag{1.3}$$

Note that the parameterization is given once for all at time $t = 0$. It is classical that, for any $t > 0$, the coordinates $x(p, t)$ and $y(p, t)$ are C^∞ functions of p. Moreover, it is quite easy to find stable numerical scheme to solve (1.3). However, the smoothness of each coordinate does not imply that the curve is smooth in the sense of Def. 2.10. Numerical evidence is shown on Fig 1.3 below, where the curve develops self-intersection and cusps.

Moreover, this flow is not intrinsic: if we change the parameterization and solve the heat equation, we obtain another family of curves. We can try to parameterize the original curve by its length parameter. This is actually what was done in the experiment. This is not satisfying though. Indeed, the curve may still develop some singularity (which is not expected from a smoothing process!) and the length parameter at initial time does not remain the length during the evolution. As a consequence,

[1] This remark is not at all pessimistic. First, this model is more than widely accepted by the community of image processing and has allowed to obtain many results. Second, we hope the analysis may be interesting also in a mathematical point of view.

[2] The result is asymptotically true, for any symmetric kernel, in the sense that, if we iterate the convolution and let the variance of the kernel go to 0, then the limit is solution of the heat equation. This makes the choice of the smoothing kernel irrelevant [81]. Let us point out that the kernel does not even need to be smooth!

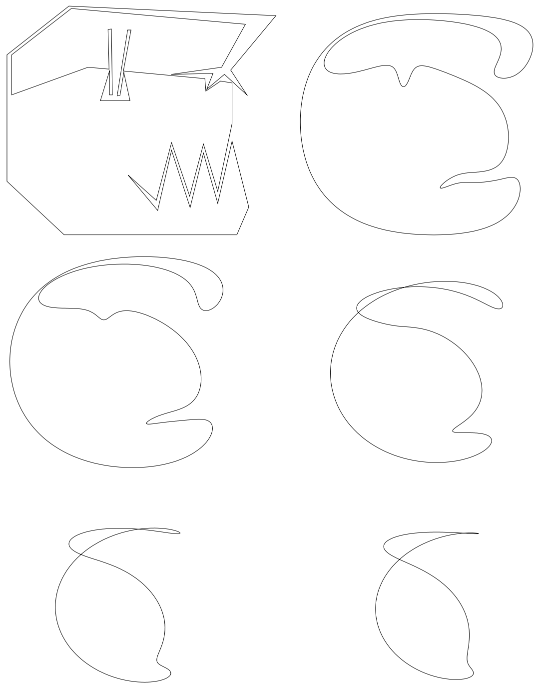


Fig. 1.3. Classical heat equation of a closed embedded curve. The initial curve is on top-left. The evolution by the classical convolution of the coordinates with a gaussian is displayed for five different scales from top to down and left to right. The first scale on top-right is still fine. The curve becomes close, then tangent to itself on middle left, and creates a self-intersection on middle right. The curve is then no longer embedded but remains locally smooth. On bottom left then right, this smoothness is lost as a cusp appears

it is not equivalent to smooth the curve up to a scale T, or to smooth it up to $t_0 < T$, renormalize the curve, and finally smooth this last one up to scale $T - t_0$. The final result depends on t_0. The conclusion is that this smoothing algorithm is not suitable because it violates the stability and the intrinsicness of the shape recognition principles.

1.2.2 Towards an intrinsic heat equation

In the previous section, we saw that the heat equation was not suitable because it is not an intrinsic evolution. Even if the initial parameterization of the curve is intrinsic, it does not remain so for any positive time. Following Mokhtarian and Mackworth [127], we can then think of renormalizing the curve at any time. On the one hand, we obtain an intrinsic evolution of the curve, but on the other hand, the time and space parameters are no longer independent. More precisely, we denote by s_0 the length parameter of the curve at time $t = 0$. We start solving the classical heat equation with this parameter s_0 up to some small time $h > 0$ and obtain a curve C_h. We stop the evolution and parameterize the new curve by computing its length parameter s_h. We then solve the classical heat equation for the scales comprised between h and $2h$ and iterate the process while it is possible. If we now let h (the time interval between two renormalizations) tend to 0, we then heuristically obtain the equation

$$\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial s^2} \quad \text{and} \quad \frac{\partial y}{\partial t} = \frac{\partial^2 y}{\partial s^2}, \tag{1.4}$$

which is called the intrinsic heat equation. In a condensed form it is written

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial s^2},$$

or by replacing the right hand term by its expression with the curvature (see Def. 2.14)

$$\frac{\partial C}{\partial t} = \kappa \mathbf{N}. \tag{1.5}$$

Contrary to what we can think at once when looking at (1.4), this equation is nonlinear, since the length parameter depends on the scale t. This equation is also known under the name *mean curvature motion* since its interpretation is that any point of the curve at time t moves in the normal direction with a velocity equal to the curvature. If we go a bit further in the examination of the equation, we see that a curve moves inward when it is locally convex and outward when it is locally concave[3]. Hollow parts are progressively filled while bumps are worn out. Thus, it seems that the general behavior is that the curve tends to become more and more convex. For a convex curve, any point move inward and the curve can only shrink, at least while it stays convex. In order to compare with the classical heat equation, we display the mean curvature motion of the same shape on Fig. 1.4. What can be rigorously said of the existence, the uniqueness and the regularity of a solution? (Remind that we look for some smoothing process.) All of the questions above were studied and eventually answered by two celebrated papers by Gage-Hamilton [69] and Grayson [77]. A part of Chap. 3 will be dedicated to their results.

Remark 1.1. At the sight of Fig. 1.4, we can wonder about the use of shortening the curve until it disappears. The asymptotic shape has nothing to do with the initial one. Gage and Hamilton even proved that the isoperimetric constant of the curve tends to the one of a circle, implying that it becomes rounder and rounder. This, in general, may seem a bit disturbing, if we understand filtering as a denoising or restoration process whose purpose is to remove spurious parts of a shape. This is not the aim of shape analysis. Indeed, the scale parameter can be interpreted as the distance from

[3] We shall say that a curve is locally convex (resp. concave) in the neighborhood of some point if its curvature is nonnegative (resp. nonpositive) around this point. Thus the definition is orientation dependent. Alternatively, if the curve is the boundary of some set K, we say that it is convex near some point x if $K \cap B$ is convex for some small ball B centered at x, and concave if the complementary of K is locally convex.

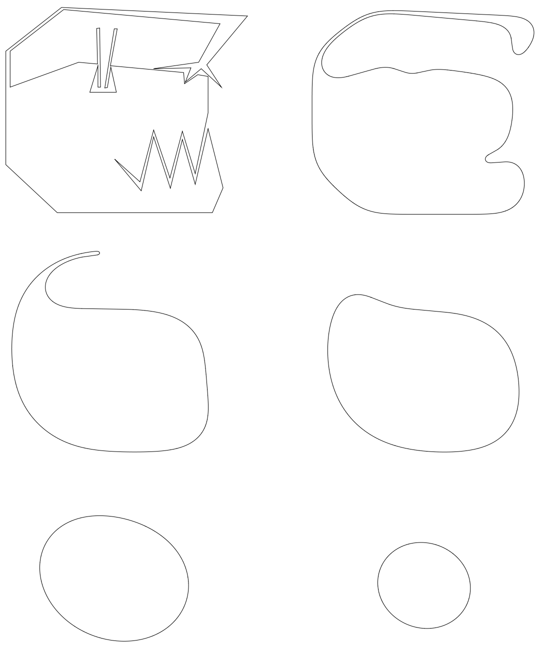


Fig. 1.4. Motion of the same curve as in Fig. 1.3 obtained by using the intrinsic heat equation. The numerical scheme is exposed in Chap. 6. The curve is smooth at any time, becomes convex and eventually disappears

which the shape is seen. Since this distance in arbitrary, all scales have to be taken in consideration, and this is the reason why multiscale analysis seems adequate. Therefore, the fact that the shape always becomes a circle is an advantage. Indeed, this means that we have gradually extracted all the geometric information; we can be sure of this because we have attained the final state which is the same for any shape. When the shape disappears, no information is left in it. Moreover, we do not create singularities in the filtering which could be thought as some important cues. In the very same paper where they introduce the renormalization scheme, Mackworth and Mokhtarian [127] gave an algorithm a multiscale shape analysis that uses all the scales, whose principle is to follow critical points of the curvature through all the scales.

1.3 An axiomatic approach of curve evolution

1.3.1 Basic requirements

The mean curvature motion seems to define a nice smoothing process for curves. It is intrinsic, and experimentally, smoothes a curve, which eventually becomes a “round point”. Nevertheless, we found this equation a little bit by chance, by renormalizing the classical heat equation at any time. A more satisfying approach is to directly examine all the criteria that the smoothing should respect. We already know some of them, but we can complete the list as follows.

1. The smoothing should be causal: the information is contained in the initial shape and the filtering can only remove some details. In particular, if we choose to look at the shape at different scales, then the shape viewed at large scale should be deduced from the same shape at any smaller scale.
2. The smoothing should be intrinsic: describing a shape or a curve by a function must not make us forget that we are interested in the geometrical object itself and not the way it is described. In particular, when dealing with curves, the smoothing should not depend on any parameterization.
3. The smoothing should be local: because of occlusions, we are often unable to match complete shapes but only parts of shapes corresponding to apparent parts. It is sound to assume that the fact that there are hidden parts does not dramatically modify the smoothing of visible parts.
4. The smoothing should be invariant with respect to some geometric transformations: we mean that the position of the observed object, or the position of the observer may have changed between the two views of a same object. This should not prevent an efficient algorithm to recognize an object. As a consequence, the smoothing itself should provide the same information whatever the relative position of the observer and the observed object is. One way to obtain this is to choose a smoothing process which is invariant with respect to the class of all admissible motions. Of course, these motions must be part of the model.
5. The smoothing should be stable: two shapes that are close to each other (in a sense willingly vague) should not diverge. A link with the locality assumption is that, if a shape locally contains another one, then this remains true for small smoothing scales.

The problem is now to translate those qualitative requirements into well posed mathematical terms; first, are they are compatible or not; what kind of mathematical models can be derived; what can be said about these models about their properties, their well-posedness; how can they be implemented?

1.3.2 First conclusions and first models

Let us first check the causality assumption. We suppose that the smoothing is achieved by an operator T_t depending on the observation scale t. The causality can be formulated as follows: if $s < t$, then T_t can be obtained from T_s by an operator $T_{s,t}$ passing from scale s to scale t. This is related to a stronger but very usual assumption which is a semi-group property, telling that $T_{s,t} = T_{t-s}$, meaning that the filtering process is stationnary. Otherwise said, for any s and t, we have $T_{s+t} = T_t \circ T_s$ and $T_0 = Id$, the identity operator. Now, if we compare the smoothing between scale t and $t + h$ for a small $h > 0$, that is we compute $T_{t+h} - T_h$, we can use the semi-group property to derive

$$T_{t+h} - T_t = (T_h - Id) \circ T_t.$$

If the evolution itself is smooth, then, up to a renormalization of the scale, there exists an operator F such that $T_h - Id = hF + o(h)$, F being called the infinitesimal generator of the semi-group. Thus, we shall look for smoothing processes such that the motion of a curve $C$is described by an equation of the type

$$\frac{\partial C}{\partial t} = F(C), \tag{1.6}$$

where F is some vector-valued function depending on C and its spatial derivatives. Each point of the curve has a velocity equal to F. Both heat equations (classical and intrinsic) presented above fit this model. But, if, in addition, we require the smoothing to be intrinsic, this implies that F should only depend on intrinsic quantities. This eliminates the classical heat equation, since whatever the initial parameter may be, it is not intrinsic for positive scale.

The locality implies that the velocity function F only depends on local features of the curve at scale t. For instance, this rejects the total length of the curve which cannot be computed without knowing the whole curve. Thus, we look for a function F depending on intrinsic differential characteristics. These characteristics cannot be chosen randomly. Indeed, they must be invariant with respect to some transformations and the way to find them is developed in Chap. 3. For instance, if we want the smoothing to be invariant with respect to isometries (translations and rotations), then we can choose the velocity F to be a function of the curvature κ. Since the length parameter is also isotropic invariant, the derivatives of κ with respect to s also are invariant and we obtain a whole family of equations which are

$$\frac{\partial C}{\partial t} = F\left(\kappa, \frac{\partial \kappa}{\partial s}, \cdots, \frac{\partial^n \kappa}{\partial s^n}\right). \tag{1.7}$$

In Chap. 3, we shall see indeed that those equations are suitable, but also that *any* suitable equation is of this type. Indeed, the theory developed by Olver, Sapiro and Tannenbaum [140, 142, 141] will allow us to classify all the intrinsic invariant equations.

We have not examined the stability property yet. In fact, the invariance approach of Olver et al. gives no such result, and each equation has to be individually studied. Mathematically, a local stability principle is equivalent to a local maximum principle. It is known that some of the equations of type (1.7) do not satisfy the maximum principle [64, 74]. On the other hand, if the velocity is an increasing function of the single curvature, then (1.7) is a parabolic equation and we can hope that a maximum principle holds.

In the following sections, we shall see that the problem of curve evolution may be considered as generic, since other points of view and purposes make them appear naturally.

1.4 Image and contour smoothing

A completely different point of view was adopted by Alvarez, Guichard, Lions and Morel [4] for who stability is a primordial axiom. The approach is to apply a smoothing *before* the shape detection. More precisely, the image smoothing is such that we can apply the smoothing and the detection in any order. As a consequence the smoothing must be compatible with an axiom of shape conservation. For instance, it is well known that a gaussian blurring makes an image smoother, but too much visual information is lost. Indeed, the contours are less and less marked and we can hardly see the edges of objects. It is then irrealistic to automatically and precisely detect the shapes. In [31], Caselles et al. argue that edges coincide with parts of level lines of images. Experimentally, this can be checked by thresholding the grey level of an image for different values. It is striking how a few grey levels allow to retrieve a large part of the objects. A consequence of this principle is that the smoothing must preserve level lines as much as possible. This can be formalized in terms of invariance with respect to contrast changes. (See [81] and Chap. 4 in this volume.) Alvarez et al. then prove that if $u : \mathbb{R}^2 \to \mathbb{R}$ is a grey level image, the suitable smoothing equations are of the type

$$\frac{\partial u}{\partial t} = |Du| G(\operatorname{curv} u), \tag{1.8}$$

where G is a nondecreasing function and $\operatorname{curv} u$ is a second-order differential operator corresponding to the curvature of the level lines of u. This equation means that the level lines move with a normal velocity equal to a function of the curvature, that is, an equation of type (1.7), a motion of the classification of Olver et al.! We shall see in Chap 4 why this is not a coincidence.

1.5 Applications

In this section we give three examples of applications of shape simplification in image processing. Of course, it is not exhaustive at all and only aims at illustrating the use of geometric motions.

1.5.1 Active contours

Active contours algorithms aim at finding the boundary of an object, by assumimg that it is smooth and that the object is contrasted with respect to the background. (Contrast can be understood in a general sense, since it can mean that the gray level, the color, the texture etc... are different in the object and on the background.) The original idea by Kass, Witkin and Terzopoulos [100] is to initialize a curve in an image and let this curve move until it adapts to the contour of a searched object. The motion of the curve is driven by the image itself. The driving force is obtained by defining a potential in the image that shall be small near objects contours. Following Marr's paradigm [117], contours coincide with irregulars parts of the image. If u is a grey level image, the external potential is defined at each point of the image and is of the type

$$g(x) = \frac{1}{1 + |DG_\sigma \star u|^2},$$

where G_σ is a gaussian with standard deviation σ. The term appearing in the denominator is the gradient of a regularized version of the image. (The star denotes a convolution.) Let C be a curve parameterized by $p \in [0, 1]$. The external potential yields an external energy on a curve C defined by

$$E_{\text{ext}}(C) = \int_0^1 g(C(p))\,dp.$$

Minimizing such a potential does not ensure to obtain a curve, since a minimizing sequence may degenerate into something very sparse. The idea is then to add some internal energy terms. This internal energy aims at controlling the rigidity of the curve. The initial model basically contained the term

$$E_{\text{int}}(C) = \alpha \int_0^1 |C'(p)|^2\,dp + \beta \int_0^1 |C''(p)|^2\,dp,$$

with $\alpha,\ \beta > 0$. Finally, the energy to minimize takes the inner an outer forces into account and is

$$E(C) = E_{\text{int}}(C) + \lambda E_{\text{ext}}(C).$$

An immediate objection to this model is that it is not intrinsic since it depends on a particular parameterization. Instead of this energy, we then consider

$$\mathcal{E}(C) = \int_C g(C(s))\,ds$$

where ds is an Euclidean parameter of the curve. We do not add any other term. The length is already penalized, and the external potential is also taken into account. It is interesting to compute the first variation of this energy. This shall give a necessary condition for a curve to be a minimizer and the gradient descent which can provide a numerical implementation. (We do not discuss its convergence at all.)

Exercise 1.2. Let $p \in \mathbb{S}^1$ an extrinsic parameterization of C and let δC_1 a small variation of C also parameterized by p.

1. By using the parameter p, prove that

$$\mathcal{E}(C + \delta C_1) - \mathcal{E}(C) = \int_C (\delta C_1)(p) \cdot (Dg(C(p))|C_p| \\ + g(C(p)) \frac{(\delta C_1)_p \cdot C_p}{|C_p|}\,dp + o(\delta)$$

2. By integrating by parts and using the length parameter, prove that a minimum curve must satisfy
$$(Dg \cdot \mathbf{N} - g\kappa)\mathbf{N} = 0,$$
and that the "gradient flow" of the energy $\mathcal{E}$ is
$$\frac{\partial C}{\partial t} = (g\kappa - (Dg \cdot \mathbf{N}))\mathbf{N}. \tag{1.9}$$

This exercise shows that a motion by curvature appears and it would be interesting to know how to solve such an equation. However, the solution may degenerate since changes of topology are likely to occur. A solution proposed by Caselles, Catté, Coll and Dibos [30] is to use a scalar formulation based on level sets methods. In Chap. 4, we shall see that their method consists in solving the evolution equation
$$\frac{\partial v}{\partial t} = g(x)|Dv| \left(\varDelta v - \frac{D^2 v(Dv, Dv)}{|Dv^2|} \right) - Dg \cdot Dv.$$

The minimization of the energy $\mathcal{E}$ above was proposed by Caselles, Kimmel and Sapiro [32], who explained the link between this approach and curvature motions of curve on non flat surfaces. Mathematical results for such motions by a direct approach were found by Angenent [12, 13].

1.5.2 Principles of a shape recognition algorithm

The recent shape recognition algorithms try to take the invariance principles into account. Most of them are limited to affine invariance. Indeed, projective invariant algorithms are in general more difficult to implement. Moreover, projective invariants are numerically less stable, because they are of higher order, and in practice, affine invariance is often sufficient. In this paragraph, we shortly describe the first affine invariant shape recognition algorithm using the affine invariant smoothing. It was achieved by Cohignac [40, 41] who defined characteristic points of shapes. They basically correspond to inflexion points and extrema of curvature while being more stable. More precisely, if C_0 is a closed Jordan curve, we denote by C_t the curve obtained by affine invariant smoothing (see Chap. 3) at scale t. We identify C_t and its interior. Now, for $a > 1$, we examine the parts of $C_{t/a}$ which are enclosed between the curve and the tangent at some point of C_t, as on Fig. 1.5. The enclosed area is algebraic, according to the local inclusion relation between $C_{t/a}$ and C_t. For instance, at x_1, the area is positive, negative at x_2, and equal to 0 at x_3. Cohignac calls characteristic points, the points of C_t for which this area is extremal or zero crossing (that is where $C_{t/a}$ and C_t cross). The characteristic regions are those regions where the area is extremal. The characteristic points provide an affine invariant collection of points which are used as a coarse representation of C for recognition. Since they are determined from area computations, they are more stable than simple curvature estimates. Other affine invariant characteristics are provided by the barycenters of the characteristic regions. If we want to match two shapes C and C', we try to find the best affine transform that maps the characteristic points of C to the points of C'.

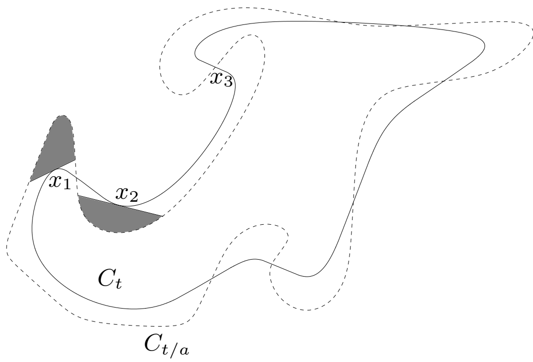


Fig. 1.5. Cohignac's shape recognition algorithm. The dashed curve is the smoothed curve at scale t/a ($a > 1$) and the solid one is the curve at scale t. Characteristics points are the points where the enclosed area is extremal or equal to 0. Matching two shapes is achieved by matching the set of characteristic points

An affine mapping is completely determined by the image of three points. For any triple of characteristic points of C, we compute the coordinates of all other characteristic points in the barycenter frame. We then compare these coordinates when the frame triple is mapped on a triple of characteristic points of C'. The procedure may be made faster by applying some learning procedure as a hashing procedure.
More recently, Lisani [111] with Moisan, Monasse and Morel [112] also used the affine invariant curve smoothing before applying some affine invariant features recognition. The principle of the algorithm is to create a dictionnary containing pieces of curves. These ones are chosen in a very stable and affine invariant way, then normalized in a reference frame. Then, being given a curve, it is splitted into pieces which are susceptible to match with codes in the dictionnary. Comparisons are made in the reference frame, which makes the algorithm affine invariant up to computational errors.

1.5.3 Optical character recognition

A particular application for which curve smoothing has turned out to be useful is optical character recognition (OCR). In this paragraph, we only show an experiment motivating efficient curve smoothing algorithms. Automatic character recognition is extremely difficult for hand-written documents, and the field is widely opened. We are here interested in the more easy case of typed characters. Even in this case, the same character may take many different appearances for instance because of the choice of different fonts. For bad quality documents, differences also arise from noise. On Fig 1.6, we display a few words taken from a fax, where the words "papers" are taken from different places. Note that this is directly the transmitted fax and not scanned after the fax was printed. As a consequence, the image is directly a binary image (which could have been different after a scan). The letters

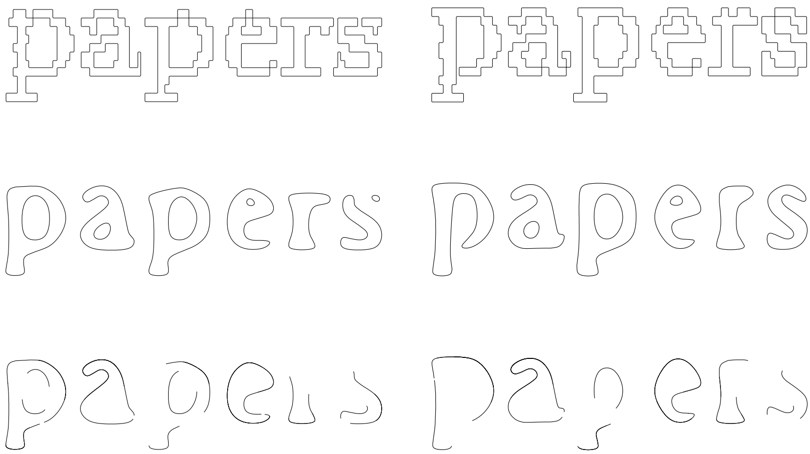


Fig. 1.6. Upper row: Some words directly taken from a fax. Remark how letters may have different topology. Middle row: smoothed curves by affine invariant smoothing. Bottom row: affine invariant matching. There are 29 matching pieces of curves, 4 of which are false matchings

are different in details, and a smoothing is necessary before applying a matching procedure [111, 112, 134].

1.6 Organization of the volume

Chapter 2 is a short introduction to the geometry of plane curves. We shall introduce the notations we shall use throughout these notes.

Chapter 3 is dedicated to the research of all the intrinsic curve evolution equations. The approach is divided in three steps. We first introduce the tools of differential geometry to formulate rigorously the problem of finding invariant evolution equations. We then derive these equations in a systematic way. Finally, existence, uniqueness and properties of these equations are exposed for the simplest cases. Concerning results about existence and uniqueness, many authors contributed to the advance of the theory, and we shall recall the main known results for the motions by curvature. All these results are given with no proof, since they are long, technical and do not fit the purpose of these notes. (Of course, we shall give references for the interested reader.) We shall concentrate on the axiomatic approach that allows to derive the invariant equations, by developing a short theory of differential invariants.

Different authors have then tried to introduce a weak notion of solution for the curve shortening problem. Chapter 4 presents the level sets approach. We first expose the connection between the curve evolution approach and the level sets one by using some results on monotone operators and see that some particular PDEs (the so-called geometric PDEs) naturally appear in the analysis. We then give a self-contained exposure of the theory of curvature motion by level sets method with existence and uniqueness results of viscosity solutions.

In Chap. 5, we briefly discuss the classical existing algorithms for curve evolution.

Finally, we present in Chap. 6 a geometric algorithm for curve evolution. This algorithm allows to solve the curvature motion when the velocity is a power of the curvature larger than $1/3$. We first propose a theoretical scheme which is unconditionally stable, consistent and convergent in the sense of level sets. We shall then give a possible implementation of this scheme. Finally, we end with many numerical experiments where we check the invariance and the stability properties of the proposed numerical scheme.

1.7 Bibliographical notes

It is certainly quite difficult to know when the first ideas of computer vision appeared. A computational program was already proposed by Attneave [15], but a commonly adopted reference is Marr's Vision [117], where he introduced the concept of *raw primal sketch*, which considerably influenced the research in computer vision. Montanari [130] had previously studied the problem of line detection in a noisy context. Shape extraction was launched with the edge detection doctrine by Marr and Hildreth [118], followed by hundreds of papers among which Canny's is one of the most famous [26]. There are basically two other classes of sketch extraction according to whether contours or regions are the objects of interest. Active contours were introduced by Kass, Witkin and Terzopoulos [100], and later improved by the use of level sets techniques (see Chap 4) by Caselles, Catté, Coll and Dibos [30] and Caselles, Kimmel and Sapiro [32]. Equation (1.9) comes from [32]. In image segmentation, edges are the boundaries of the segmented areas, in which a property is homogeneous. A general model is Mumford and Shah's [133], with mathematical developments in Morel and Solimini's book [131], and even more recently in a book by Ambrosio, Fusco and Pallara [8]. All of these methods implicitly follow some of the principles of the Gestalt school [99, 105, 168], since they assume that perceptual contours are smooth and delimit contrasted zones. More recently, an approach based on a level set decomposition was proposed by Desolneux, Moisan and Morel in [49]. A smoothing method of extracted shapes was proposed by Koenderink and Van Doorn [103], consisting in solving the heat equation for the characteristic function of the shape. The lack of locality and causality was tackled by Bence, Merriman and Osher (see [22] and Chap. 4 and 5.) For a direct curve approach, the renormalization of Sect. 1.2.2 was proposed by Mackworth and Mokhtarian [127], with a multiscale matching algorithm. An axiomatic approach for curve evolution was proposed by Lopez and Morel [113] and Sect. 1.3 is a simplified version of their work. The works by Olver, Sapiro and Tannenaum [141, 142] on invariant flows classification will be described in Chap. 3. Figure 1.2 was obtained with Moisan's affine erosion algorithm [104, 125, 126]. Shape matching is another huge subject, with an overwhelming bibliography. The axiomatic approach we gave in Sect. 1.1.1 can basically be found in Veltkamp and Hagedoorn recent review [164]. The first matching method using the affine morphological scale space (see [4] and Chap. 4) is Cohignac's [40, 41] and is described in Sect.1.5.2, while the experiments of Sect. 1.5.3 are due to an algorithm by Lisani [111, 112] recently improved by Musé, Sur and Morel [135].

2

Rudimentary bases of curve geometry

We recall in a few pages, all the elementary notions of the geometry of plane curves that we shall use in the following. The reader familiar with all those bases may skip this part and refer to it only for the main notations (which are quite usual). All the definitions shall be given in the plane. Most of them can be generalized in higher dimension but we shall keep a bidimensional point of view which shall also simplify the statements. Any proofs and generalizations in higher dimension may be found in [29, 70].

2.1 Jordan curves

Let us first start with the definition of Jordan curves which will be one of our main objects of interest for image analysis.

Definition 2.1. *Let $a, b \in \mathbb{R}$ with $a < b$. A Jordan curve is a continuous mapping from $[a, b]$ to $\mathbb{R}^2$ which is one-to-one on (a, b).*

In fact, the range $Im(C)$ of a curve C will be of more interest than the curve itself, that is, we do not mind how the curve is described. Some authors thus prefer to define a plane curve as the range of a mapping from an interval to $\mathbb{R}^2$. Nevertheless, when we want to operate on the curve, we need at least some local coordinates. As usual in geometry, we would like to describe phenomena that do not depend upon any particular system of coordinates. This leads to the following definition.

Definition 2.2. *Let I, J be closed intervals of $\mathbb{R}$ and $C : I \to \mathbb{R}^2$ be a Jordan curve. A curve C_1 will be called an admissible parameterization of C if and only if there exists an homeomorphism $\psi : I \to J$ such that $C_1 = C \circ \psi$.*

The theories we shall develop will focus on characteristics of curves that will not depend on any particular parameterization; in this case, these values will be called *intrinsic*. An intrinsic characteristic may either be a scalar or a vector, it may be local or global. In the case of a global characteristic, we say that I is intrinsic to C if for

F. Cao: LNM 1805, pp. 23–28, 2003.

any parameterization C_1 of C, we have $I(C_1) = I(C)$. In the case of a pointwise characteristic, we must have $I(C)(\psi(t)) = I(C \circ \psi)(t)$, whatever the admissible change of parameter ψ may be. In other terms, it only depends on $Im(C)$. For instance, the length of a curve and its curvature at each point are intrinsic values and only depend on the Euclidean structure of the plane. We shall come back to this in the following.
We say that a Jordan curve $C : [a, b] \to \mathbb{R}^2$ is closed if $C(a) = C(b)$. In this case, another usual and more intrinsic definition does not explicitly use any parameterization.

Definition 2.3. *A closed Jordan curve is a continuous application from the circle $\mathbb{S}^1$ to the plane which is one-to-one.*

This exactly means that the image of a closed Jordan curve is a continuous submanifold of dimension 1 without boundary. Closed Jordan curves completely inherit the properties of the unit circle. In particular, the circle embedded in the plane has a canonical Riemannian structure inherited from the Euclidean scalar product in $\mathbb{R}^2$. This allows us to define, in an intrinsic way, curves which are more than continuous. Jordan curves can be very singular objects since they are only continuous. In particular they can be fractal objects with infinite length and filling a surface (such as Peano's curve). Nevertheless, we have the following theorem.

Theorem 2.4 (Alexandrov[2]). *Let C be a closed Jordan curve. Then its range severs the plane into exactly two connected components. One is bounded and is called the interior of C denoted by $\mathrm{Int}(C)$ and the other one is unbounded and is the exterior of C (denoted by $\mathrm{Ext}(C)$).*

Of course, this result is no longer valid in higher dimension and completely relies on the geometry of the plane.

2.2 Length of a curve

We can also define the length of any part of a Jordan curve. Without any additional assumption, this length may be infinite.

Definition 2.5. *Let $C : [a, b] \to \mathbb{R}^2$ be a Jordan curve. Let $a \le s \le t \le b$. We call length of C between $C(s)$ and $C(t)$ the nonnegative value (possibly infinite)*

$$L(C, s, t) = \sup_{N \in \mathbb{N}} \sup_{s = t_0 \le t_1 \le \cdots \le t_N = t} \sum_{i=1}^{N} |C(t_i) - C(t_{i-1})|. \tag{2.1}$$

If $L(C) \equiv L(C, a, b) < +\infty$ the curve is called rectifiable.

Remark 2.6. If ψ is bijective and increasing on $[a, b]$, we can immediately check that $L(C \circ \psi, \psi^{-1}(s), \psi^{-1}(t)) = L(C, s, t)$, that is, the length is intrinsic.

If a curve admits a C^1 parameterization, then this expression reduces to what follows.

Proposition 2.7. *Let C be a closed Jordan curve such that there exists a parameterization for which C is of class C^1. We then denote by $C'(p)$ the derivative of C at the point with parameter p in $[a,b]$. Let $a \leq t_1 \leq t_2 \leq b$. Then, the length of C comprised between $C(t_1)$ and $C(t_2)$ is*

$$L(C, t_1, t_2) = \int_{t_1}^{t_2} |C'(p)|\, dp. \tag{2.2}$$

Remark 2.8. There are two remarkable facts in this result. First, the expression in (2.2) above does not depend on the parameterization. Then that for a C^1 curve, (2.1) reduces to (2.2).

2.3 Euclidean parameterization

Definition 2.9. *A locally rectifiable Jordan curve $C : [a,b] \to \mathbb{R}^2$ has a Euclidean parameterization if for all $t \in [a,b]$, the derivative $\frac{dL}{dt}(C,a,t)$ exists and is identically equal to* 1.

For closed Jordan curves, this definition stands with the usual metric on the circle and coincides with the one seen above.
Smoothness may be defined as follows.

Definition 2.10. *A Jordan curve is of class C^k if it admits a Euclidean parameterization for which it is of class C^k. Another parameterization $C_1 = C \circ \psi$ is called C^k if $\psi : I \to [a,b]$ is a C^k diffeomorphism.*

Exercise 2.11. Prove that a curve admitting an Euclidean parameterization is not necessarily smooth.

Exercise 2.12. Construct a curve and a parameterization (non Euclidean for sure!) for which the curve is C^∞, but such that the curve is not C^1 in the sense of Def. 2.10 above.

Proposition 2.13. *If $C : [a,b] \to \mathbb{R}^2$ is a C^1 Jordan curve and $t \in (a,b)$, then C admits exactly two Euclidean parameterizations near t. If C is a closed Jordan curve, then it has exactly one direct Euclidean parameterization.*

The Euclidean parameterizations can be deduced from each other by reversing the sign of the parameter. The direct parameterization is such that the interior of the curve locally lies on the "left" of the curve when we follow the parameterization. Finally, we can define the tangent, the normal and the curvature of a C^2 Jordan curve.

Definition 2.14 (and Proposition). *Let C be a C^2 Jordan curve such that C' is never equal to 0. By definition, for any t, $\mathbf{T} = C'(t)/\|C'(t)\|$ is a unit vector of $\mathbb{R}^2$ and is called the* unit tangent vector *of C. We call* unit normal vector *of C*

the vector $\mathbf{N}$ *such that* $(\mathbf{T}, \mathbf{N})$ *form a direct basis of the plane. Then, the vector* $\frac{1}{\|C'(t)\|} \frac{d}{dt} \frac{C'(t)}{\|C'(t)\|} = \frac{1}{\|C'(t)\|} \frac{d\mathbf{T}}{dt}$ *is collinear to* $\mathbf{N}$ *and may be written* $\kappa\mathbf{N}$ *with* $\kappa \in \mathbb{R}$. *The number* κ *(also denoted by* $\operatorname{curv} C$*) is called the* curvature *of* C *at the point* $C(t)$. *The tangent* $\mathbf{T}$*, the normal* $\mathbf{N}$ *and the curvature* κ *only depend on the orientation of* C.

Remark that if we reverse the parameterization, we change the sign of $\mathbf{T}$, $\mathbf{N}$ and κ. On the contrary, the mean curvature vector $\kappa\mathbf{N}$ stays unchanged and is completely invariant of the parameterization.
In what follows, it shall be very useful to express $\mathbf{T}$, and κ in a Euclidean parameterization. Let s be such a parameter. Since $\|C'(s)\| = 1$, we then have

$$\mathbf{T} = C'(s) \quad \text{and} \quad C''(s) = \kappa\mathbf{N}.$$

The curvature has a well-known geometrical interpretation: it is the inverse of the radius of the osculating circle to C at the point $C(t)$, see Fig 2.1.

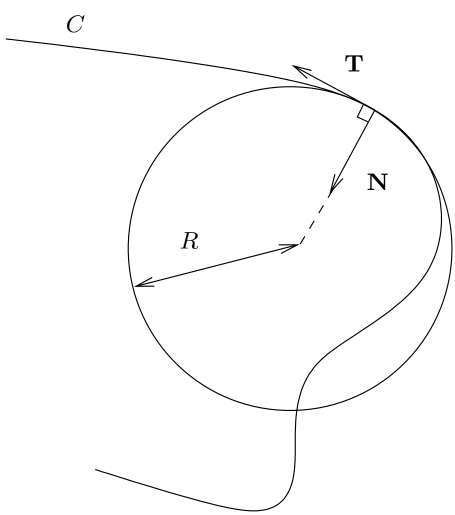


Fig. 2.1. Tangent vector, normal vector and curvature. The curvature κ is equal to $1/R$. In this case, the normal points inward the osculating circle, and the curvature is positive

Remark 2.15. The curvature will also be called the mean curvature. The reason is the following. The notion of curvature may be generalized for submanifolds of $\mathbb{R}^n$ with $n \geq 2$. For a manifold $\mathcal{M}$ of codimension k, $0 < k < n$, we define for $x \in \mathcal{M}$ the second fundamental form at x defined on any couple of tangent vectors by $B_x(\xi, \eta) = \sum_{\alpha=1}^{k} (\partial_\eta \xi, \nu_\alpha)\nu_\alpha$ where ν_α is any orthonormal basis of the normal space to $\mathcal{M}$ at x and $\partial_\eta \xi$ is the derivative of ξ in the direction η. (Hence, it is a normal vector to $\mathcal{M}$). Then, it is possible to prove that this expression does not depend on the basis ν_α and that it defines a symmetric bilinear mapping on $T_x\mathcal{M}$, the tangent plane of $\mathcal{M}$ at x. The eigenvalues of the second fundamental form are called the principal curvatures, their arithmetic mean is called the *mean curvature* and their product the *Gaussian curvature*. An essential fact is that they are intrinsic, i.e. independent of the local parameterization of the manifold. In the case of plane curves, the dimension and the codimension of the manifold are equal to 1 and the unique eigenvalue of the fundamental form is at the same time the mean and the Gaussian curvature and coincides with the curvature defined above. (See [157] for series of remarkable lectures on these geometrical topics.)

2.4 Motion of graphs

It is sometimes useful to consider that a smooth curve is locally the graph of a smooth function $x \mapsto u(x) \in \mathbb{R}$. In this case, a parameterization is given by the variable x. In this section, we give the expression of the basic geometric characteristics of the previous section with this parameter. First, a tangent vector to the graph of u has $(1, u_x)$ for coordinates, where u_x is the derivative of u with respect to x. Thus, we can choose the tangent vector equal to

$$\mathbf{T} = \frac{1}{\sqrt{1+u_x^2}}(1, u_x).$$

The Euclidean arc-length is then given by $ds = \sqrt{1+u_x^2}dx$. The normal vector $\mathbf{N}$ is chosen orthogonal to $\mathbf{N}$ and $(\mathbf{T}, \mathbf{N})$ forms a direct basis. With the previous choice of $\mathbf{T}$, we get

$$\mathbf{N} = \frac{1}{\sqrt{1+u_x^2}}(-u_x, 1).$$

The curvature κ is obtained by the relation

$$\frac{d\mathbf{T}}{ds} = \frac{1}{\sqrt{1+u_x^2}}\frac{d\mathbf{T}}{dx} = \kappa\mathbf{N}.$$

A simple calculation leads to

$$\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}.$$

In the following chapters, we shall encounter curve evolutions defined by a normal velocity at each point. They are equations of the type

$$\frac{\partial C}{\partial t} = V\mathbf{N},$$

where V is the normal velocity that may depend on various parameters. If C is the graph of a function u, let $(x(t), u(x(t), t)$ a point of C. By differentiating with respect to t, we obtain

$$(x'(t), x'(t)u_x(x(t),t) + u_t(x(t),t)) = V\frac{(-u_x, 1)}{\sqrt{1+u_x^2}}.$$

By identifying each coordinate, we get

$$u_t(x(t),t) = V\sqrt{1 + (u_x(x(t),t))^2}.$$

Since, $x(t)$ is any point of the real line, u satisfies

$$u_t = V\sqrt{1+u_x^2}.$$

Example 2.16.

1. If the normal velocity is identically equal to 1, the corresponding equation is $u_t = \sqrt{1+u_x^2}$ which is called the dilation of the graph of u.
2. If the velocity is equal to the curvature, then u is solution of the quasilinear second order equation $u_t = u_{xx}/(1+u_x^2)$ which is called the mean curvature motion and that will be studied in the next chapters.

3

Geometric curve shortening flow

This chapter is an overview of the geometric curve evolution problem. In the first chapter, we concluded that reasonable evolution equations for a plane curve C should be of the type

$$\frac{\partial C}{\partial t} = F(C),$$

where the speed function F depends on geometrical *differential invariants* of the curve. This invariance must be understood with the datum of a transformations set $\mathcal{G}$. In what follows, we shall assume that this set is a subgroup of the real projective group. This is reasonable since it corresponds to a modification of the perspective views of objects when the observer moves. In this case, a complete classification was achieved by Olver, Sapiro and Tannenbaum [140, 141, 142] who proved that all the invariant equations were of the type

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial r^2} I\left(\chi, \frac{\partial \chi}{\partial r}, \cdots, \frac{\partial^n \chi}{\partial r^n}\right), \tag{3.1}$$

where r is the *group arc-length* and χ the *group curvature*. If $\mathcal{G}$ is the Euclidean group (generated by rotations and translations), then these notions correspond to the classical length and curvature. (See Sect. 3.1.) In particular, the mean curvature motion, is the simple case where $I = 1$ in (3.1) above. Another very interesting model will arise when the group $\mathcal{G}$ is the group of special affine functions $SL(\mathbb{R}^2)$, that is the group of affine functions preserving the surface, i.e. with determinant equal to 1. In this case, r will be the so-called affine length and χ the affine curvature. Then, there is a unique evolution equation of least order. We shall see that this evolution equation can be reformulated in terms of the Euclidean length and curvature. It is a third order equation, but it is equivalent to a second order and simple equation which is

$$\frac{\partial C}{\partial t} = \kappa^{1/3}\mathbf{N}. \tag{3.2}$$

This equation, discovered by Sapiro and Tannenbaum [152, 153], is called the affine intrinsic heat equation or the affine curve shortening. The mean curvature

F. Cao: LNM 1805, pp. 31–53, 2003.

motion (1.5) and the affine heat equation are certainly the most important geometric plane curve evolution equations. First because they are among the simplest in their respective group of invariance. (Even the simplest for the affine case.) Next, they are still the only cases for which a complete theory of existence, uniqueness, regularity and asymptotic state is known. The mean curvature motion problem was solved by Grayson [77] who used results by Gage and Hamilton [69]. (See Sect. 3.2.2.) The affine invariant case was more recently solved by Angenent, Sapiro and Tannenbaum [14]. (See Sect. 3.2.3.) In some other cases, only partial results are known as the existence for convex curves [10] or the short-time existence [12, 13, 122]. The chapter is organized as follows: by using the theory of differential invariants, we first derive the general form of invariant equations in Sect. 3.1 and 3.2. In Sect. 3.3, we focus on a particular class of Euclidean invariant equation and give an overview of the known results of existence of solutions.

3.1 What kind of equations for curve smoothing?

3.1.1 Invariant flows

In 1993, Sapiro and Tannenbaum [152, 153] discovered the affine curve evolution (3.2) as an evolution with Euclidean curvature dependent velocity which is invariant with respect to affine volume preserving mappings. Together with Olver, they made their approach systematic [141, 142] and gave a way to classify all the locally smooth flows. They completely focus on the invariance and locality requirements. The analysis relies on a deep study on differential invariants and symmetry of equations, upon which Olver consecrated several monographs [138, 139]. In this chapter, we simply intend to introduce the notions and the ideas of differential geometry and Lie groups theory leading to the classification of invariant curve evolutions. We assume a basic knowledge of the main concepts of differential geometry as manifolds and vector fields. Most of complete proofs and references can be found several reference books [29, 70, 138, 139].

3.1.2 Symmetry group of flow

An evolution equation is invariant with respect to a set of transformations $\mathcal{G}$, if any solution is transformed into another one. This shall be detailed below. Without any additional assumption on $\mathcal{G}$, this problem is very difficult to solve, because it is formulated by some equations, which are in general nonlinear. A particular case, easier to solve, occurs when $\mathcal{G}$ is a connected Lie group. Let us first give (or recall) a few basic definitions.

Definition 3.1. *We say that a group $\mathcal{G}$ is a Lie group if it is also a differential manifold and if the multiplication and the inverse operations are smooth maps.*

Definition 3.2. *Let $\mathcal{M}$ be a smooth manifold and $\mathcal{G}$ be a Lie group. A group action of $\mathcal{G}$ on $\mathcal{M}$ is a smooth map $\phi : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$, which is compatible with the group law, that is to say*

1. $\forall x \in \mathcal{M}, \quad \phi(e, x) = x$, *where* e *is the identity element of* $\mathcal{G}$.
2. $\forall x \in \mathcal{M}$, *and* $g, h \in \mathcal{G}$, $\phi(h^{-1}, \phi(g, x)) = \phi(h^{-1}g, x)$.

If there is no ambiguity, a group action will be denoted by $\phi(g, x) = g \cdot x$.

Remark that $g \cdot x$ may be defined only for g close enough to the identity, in which case we talk about *local group action*. One of the main examples we shall be interested in, is the case when $\mathcal{G}$ is the group of rotations acting on the set of derivatives of functions. In this case, the group acts locally since if the rotation angle is too large, a graph is not transformed into the graph of a function, and taking the derivatives does not make sense.

Definition 3.3. *Let* $\mathcal{G}$ *be a Lie group acting on a manifold* $\mathcal{M}$. *We consider the equation* $F(x) = 0$ *where* $x \in \mathcal{M}$. *We say that* $\mathcal{G}$ *is a symmetry group of the equation* $F(x) = 0$ *if the set of solutions is invariant under the group action, that is if any solution is transformed into another solution. Still in other words*

$$F(x) = 0 \Leftrightarrow \forall g \in \mathcal{G} \text{ for which } g \cdot x \text{ is defined },\ F(g \cdot x) = 0.$$

If $x \in \mathcal{M}$, the set of all points of the form $g \cdot x$ where $g \in \mathcal{M}$ is called the *orbit* of x. Thus a group is a symmetry group of the equation $F(x) = 0$ if the orbit of any solution is a set of solutions.
One may think that the problem is more difficult since it involves notions that are no longer elementary. In fact, the structure brought by the Lie group assumption is very rich and makes the problem solvable, if not easily. Indeed, classical results on Lie groups allow to describe the orbits under the group action. To this aim, we first need to introduce the exponential maps of vector fields.

Definition 3.4. *Let* $\mathbf{v}$ *be a vector field on a manifold* $\mathcal{M}$. *The exponential map of* $\mathbf{v}$ *associated to* $x \in \mathcal{M}$ *is the maximal integral curve of*

$$\frac{d\Phi}{dt}(t, x) = \mathbf{v}(\Phi(t, x)), \tag{3.3}$$

with initial condition $\Phi(0, x) = x$. *We denote it by* $\exp(t\mathbf{v}) \cdot x$.

Remark that the derivative in (3.3) does not have the usual meaning since $\mathcal{G}$ is not a vector space, hence $\Phi(t + h, x) - \Phi(t, x)$ does not have any sense. We first write it in a local coordinates system (local chart), solve the equation, then prove that the obtained curve is independent from the choice of the chart. The exponential map defines a one-parameter group, since we can deduce from classical results on ordinary differential equations that

$$\exp((s + t)\mathbf{v}) \cdot x = \exp(s\mathbf{v}) \cdot (\exp(t\mathbf{v}) \cdot x).$$

For this reason, $\mathbf{v}$ is called the infinitesimal generator of the flow Φ.

Remark 3.5. On a Lie group $\mathcal{G}$, a tangent vector in $T_e\mathcal{G}$ (the tangent plane to $\mathcal{G}$ at e) determines a vector field on $\mathcal{G}$. Indeed if $g \in \mathcal{G}$, $dg_{|e}(\mathbf{v})$ is a tangent vector in

$T_g\mathcal{G}$ where dg denotes the differential of the right multiplication by g. In this case, if $g \in \mathcal{G}$,

$$\Phi(t, g) = \Phi(t, e) \cdot g,$$

that we can also write

$$\exp(t\mathbf{v}) \cdot g = (\exp(t\mathbf{v}) \cdot e) \cdot g.$$

This equality means that following the vector field when starting at g is equivalent to start at the identity, follow the vector field, then multiply by g.
The following proposition explains why exponential maps are so important.

Proposition 3.6. *Let $\mathcal{G}$ be a connected Lie group. Then any $g \in \mathcal{G}$ can be written in a (non unique) form*

$$g = \exp(t_n\mathbf{v_n}) \exp(t_{n-1}\mathbf{v_{n-1}}) \cdots \exp(t_1\mathbf{v_1}), \tag{3.4}$$

where $\mathbf{v_1}, \ldots, \mathbf{v_n}$ are tangent vectors in $T_e\mathcal{G}$, defining vector fields as in Rem. 3.5.

By Remark 3.5, we can understand (3.4) in two different ways. First, we multiply the n group elements by integrating the flows given by the $\mathbf{v_i}$ starting at e, the product being equal to g. Second, we start from e, integrate the first flow with vector field $\mathbf{v_1}$, stop at parameter t_1; from this point, we integrate the second vector field up to t_2, etc... Thus it suffices to know exponential maps of tangent vectors in $T_e\mathcal{G}$ to describe all the elements of the group.

Let us now come back to our invariant curve smoothing equation; any solution must be transformed into another one by the action of any group element. By Prop. 3.6, it suffices to prove it for the exponential maps of the infinitesimal generators. Indeed, by using the above decomposition, a solution in transformed into another when we follow the first exponential map, then the second, etc... Now, the logical objection is that it is in general impossible to explicitly calculate the exponential maps. Thus, we still need to reduce the problem in a more decisive way.
Imagine that we want to prove that some function is constant in an arc-connected set. A possible solution is to prove that it is constant on any curve and we are brought back to a problem on the real line. If the function is sufficiently regular, then it is enough to prove that its derivative of the function is equal to zero at any point. This is, in simplified terms, exactly what we propose to do in the following. Otherwise said, infinitesimal invariance is sufficient to prove more global invariance, and this because the transformation group is a connected Lie group.
In what follows, we shall always make the assumption that $\mathcal{G}$ is a subgroup of the projective group. The Euclidean group and the affine group will be of particular interest.
As we are looking for local flows (understand here local smoothing), we make the assumption that the curve is the graph of a function $u : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ satisfying some partial differential evolution equation of the form

$$u_t = F(x, u, u_x, u_{xx}, \cdots, u_n), \tag{3.5}$$

u_n being the nth derivative of u with respect to x. In the following, we shall often identify u with its graph $\{x, u(x)\} \subset \mathbb{R}^2$.
Let $\mathcal{G}$ be a Lie group acting on $\mathbb{R} \times \mathbb{R}$. If $g \in \mathcal{G}$, we transform the graph of u by considering the set of points $\{\tilde{x}, \tilde{u}\} = g \cdot \{x, u(x)\}$. For instance, if $\mathcal{G}$ is the group of the rotations, this means that we rotate the graph of u. Of course, if the angle of the rotation is too large, then $\{\tilde{x}, \tilde{u}\}$ is not the graph of any function. But if u is smooth (in particular, its derivatives are bounded), and if $\mathcal{G}$ acts smoothly (this is the case since $\mathcal{G}$ is a subgroup of the projective group), then this set is the graph of a uniquely determined $\tilde{u}$, provided the transform g is close enough to the identity. As stated above, it is in general impossible to give $\tilde{u}$ otherwise than by some implicit equations. Nevertheless, we do not need that much, since only the derivatives of $\tilde{u}$, up to order n must be known, in order to check that $\tilde{u}$ is solution of (3.5). Moreover, by the arguments of Lie group theory evoked above, it is enough to prove that $\tilde{u}$ is a solution for g of the form $g = \exp(t\mathbf{v})$ for infinitesimal $t > 0$. We shall then obtain a "Taylor expansion" of the derivatives of $\tilde{u}$ as functions of t and the infinitesimal generator $\mathbf{v}$. Since $\tilde{u}$ has to be a solution of (3.5), we shall be able to derive relations between F and the coordinates of $\mathbf{v}$.
The graphs we consider are subsets of $\mathcal{M} = \mathbb{R} \times \mathbb{R}$, and the transformation group $\mathcal{G}$ acts on some sets of points (x, u). In order to signify that these sets are graphs, x is generally called the independent variable and u the dependent variable. A general vector field on $\mathcal{M}$ is of the form

$$\mathbf{v} = \xi(x, u)\partial_x + \phi(x, u)\partial_u,$$

where ξ and ϕ are smooth functions. Let us also remark that we have an additional variable in (3.5): the scale parameter t. However, scale does not play the same role as the spatial variables since the infinitesimal variations through scales do not explicitly depend on scale. (That is F in (3.5) does not depend on t.) Moreover, the group of transformation we consider only acts on the spatial variables. At any scale, we apply the *same* transform g and this should yield another solution of the equation. As a consequence, the vector fields we consider only live in the (x, u) plane. This will simplify the statements below.

Example 3.7. Let us assume that $\mathcal{G}$ is the group of rotations. A rotation of angle ε centered at the origin transforms the vector (x, u) into $(x \cos\varepsilon - u \sin\varepsilon, x \sin\varepsilon + u \cos\varepsilon)$. By taking the derivative at $\varepsilon = 0$, we obtain the infinitesimal generator of the group given by $\mathbf{v} = -u\partial_x + x\partial_u$. Conversely, $\mathbf{v}$ being given, we can obtain the expression of a rotation by integrating the system of equations

$$\begin{cases} \dfrac{dx}{d\varepsilon} = -u \\[2ex] \dfrac{du}{d\varepsilon} = x. \end{cases}$$

As we are interested in the small variations of the derivatives of $\tilde{u}$ when g is close to the identity, it is convenient to introduce the jet space J^n, which is obtained by considering the derivatives of any function up to the order n. Therefore, an element of J^n will be of the type $(x, u^{(n)}) = (x, u, u_x, \cdots, u_n)$. Generally speaking, we introduce the notion of prolongation.

Definition 3.8. *Let u be a smooth function defined on $\mathbb{R}$. We call nth prolongation of u the application $\mathbb{R} \to \mathbb{R}^n$ defined by*

$$\mathrm{pr}^{(n)}\, u(x) = (u, u_x, \cdots, u_n)(x),$$

which is simply obtained by taking the derivatives of u up to order n.

Thus, elements of the jet space J^n are of the form $(x, \mathrm{pr}^{(n)}(u))$.
The way any element of $\mathcal{G}$ transforms the derivatives of functions is called the prolongation of the group action.

Definition 3.9. *Let $(x, u^{(n)}) \in J^n$ be the derivatives of a function u at x. If g is close to the identity, it transforms the graph of u into the graph of $\tilde{u} = g \cdot u$. The prolongation of the action g is the action defined on J^n by*

$$\mathrm{pr}^{(n)}\, g \cdot (x, u^{(n)}) = (\tilde{x}, \tilde{u}^{(n)}). \tag{3.6}$$

The prolongation of the group action gives the partial derivatives up to a given order of the function obtained by transforming the graph of the original function. It is clear that it only depends of the pointwise behavior of u at x as any function with the same derivatives as u will lead to the same vector of J^n. Our purpose is to compute the derivatives of $\tilde{u}$, that is to say, find an explicit formula of the prolonged group action. In this calculation, vector fields on the jet space naturally arise. In full generality, they are objects of the type

$$\mathbf{v}^n(x, u^{(n)}) = \xi(x, u^{(n)})\partial_x + \phi(x, u^{(n)})\partial_u + \sum_{j=1}^{n} \phi^j(x, u^{(n)})\partial_{u_j},$$

but we are particularly interested in some particular vector fields. Indeed, since we examine the prolonged action of the group, we must understand how a generator of the group acts on the derivatives of functions. This means that we have to define the prolongation of a vector field defined on $\mathcal{M}$. To this purpose, we will need the concept of total derivative.

Definition 3.10. *Let $f : J^n \to \mathbb{R}$ be a smooth function defined on J^n. If $(x, u^{(n)}) \in J^n$, we choose u such that $(x, u^{(n)}) = (x, u, u_x, \cdots, u_n)$. We call total derivative of f, the function $Df : J^{n+1} \to \mathbb{R}$ defined by*

$$Df(x, u^{(n+1)}) = \frac{\partial}{\partial x}\left(f(x, u^{(n)})\right). \tag{3.7}$$

Otherwise said, Df is obtained by differentiating f while considering u and its derivatives as functions of x. Of course, this definition depends on u only through its derivatives at x.

The following definition explains how we canonically prolongate a vector field defined on $\mathcal{M} = \mathbb{R} \times \mathbb{R}$ to a vector field of the jet space J^n.

Definition 3.11 (and Theorem). *Let* $\mathbf{v} = \xi(x,u)\partial_x + \phi(x,u)\partial_u$ *be a vector field on* $\mathcal{M} = \mathbb{R} \times \mathbb{R}$. *Let also* $n \geq 1$. *We call the* n*th* prolongation *of* $\mathbf{v}$ *the vector field* $\mathrm{pr}^{(n)}\,\mathbf{v}$ *defined on the jet space* J^n *by the formula*

$$\mathrm{pr}^{(n)}\,\mathbf{v} = \mathbf{v} + \sum_{j=1}^{n} \phi^j(x, u^{(n)})\partial_{u_j}, \tag{3.8}$$

the coefficients $\phi^j(x, u^{(n)})$ *being given by*

$$\phi^j(x, u^{(n)}) = D^n(\phi - \xi u_x) + \xi u_{j+1}, \tag{3.9}$$

where $D^n = D \circ \cdots \circ D$ *is the* n*th total derivative. We then have*

$$\mathrm{pr}^{(n)}\left(\exp(\varepsilon\mathbf{v}) \cdot (x,u)\right) = \exp(\varepsilon\, \mathrm{pr}^{(n)}\,\mathbf{v}) \cdot (x, u^{(n)}). \tag{3.10}$$

The theorem part of the definition needs explanation. Let us examine the left-hand term of (3.10): $\exp(\varepsilon\mathbf{v}) \cdot (x,u)$ is the graph of the function we obtain by progressively deforming the graph u, when we follow the exponential generated by the tangent vector $\mathbf{v}$. Taking its prolongation simply consists in computing the derivatives of the resulting function. On the other hand, $(x, u^{(n)})$ describes the derivatives of the original graph. This is an element of the jet space J^n. On this metric space, we also have tangent vectors and therefore exponential maps. Starting from $(x, u^{(n)})$ and following the exponential map with initial tangent vector $\mathrm{pr}^{(n)}\,\mathbf{v}$ leads to the right-hand term of (3.10). Thus, in order to compute the derivatives of the deformed function, we first compute the prolongation of $\mathbf{v}$ and then the exponential map of this prolonged vector.
Actually, none of these expressions can be explicitly computed for any ε. Nevertheless, this is an identity which is true, in particular, for infinitely small ε. Letting ε go to zero, we obtain the following theorem that gives sufficient conditions for an equation to admit a given symmetry group.

Theorem 3.12. *Let* $\mathcal{G}$ *be a Lie group acting on* $\mathbb{R} \times \mathbb{R}$ *(the space variables). If for any infinitesimal generator* $\mathbf{v} = \xi(x,u)\partial_x + \phi(x,u)\partial_u$ *of* $\mathcal{G}$*, the relation*

$$u_t = F(x, u^{(n)}). \tag{3.11}$$

implies

$$u_t(\phi_u - \xi_u u_x) = \mathrm{pr}^{(n)}\,\mathbf{v}(F(x, u^{(n)})), \tag{3.12}$$

then $\mathcal{G}$ *is a symmetry group of (3.11).*

We do not prove this theorem completely, and in particular, the fact that it suffices that the equation is invariant by infinitely small deformations (see [141]), but we explain the idea leading to this formula. Let ε be small and $(\tilde{x}, \tilde{u})$ be the graph after deformation by $\exp(\varepsilon \mathbf{v})$. Equation (3.10) gives the spatial derivatives of u up to order n. By using Taylor expansion near $\varepsilon = 0$, we have

$$\begin{aligned}(\tilde{x}, \tilde{u}^{(n)}) &= (x, u^{(n)}) + \varepsilon(\xi, \phi, \cdots, \phi^n) + o(\varepsilon) \\ &= (x, u^{(n)}) + \varepsilon \operatorname{pr}^{(n)} \mathbf{v} \cdot (x, u^{(n)}) + o(\varepsilon).\end{aligned}$$

Thus, the small variation of $F(\tilde{x}, \tilde{u})$ when ε varies, is nothing but

$$\varepsilon \operatorname{pr}^{(n)} \mathbf{v} \cdot F(x, u^{(n)}) + o(\varepsilon).$$

Let us now compute the small variation of the left hand term, that is $\tilde{u}$. We said that the Lie group does not act through scale, so the time derivatives are left unchanged except for one thing: the point at which the derivative is computed. Indeed the time derivative of $\tilde{u}$ is computed at $\tilde{x}$ and the one of u at x. We know that

$$(\tilde{x}, \tilde{u}) = (x, u) + \varepsilon(\xi, \phi) + o(\varepsilon).$$

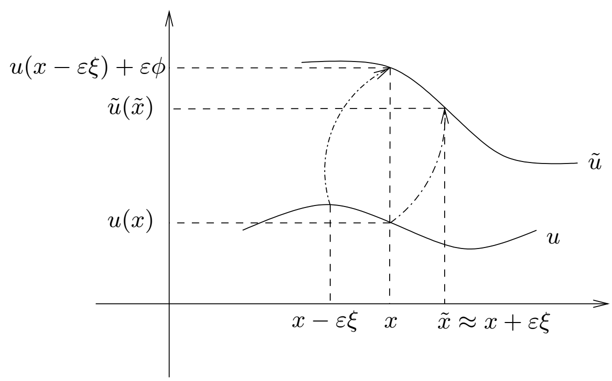


Fig. 3.1. Action of the group on a graph and value of the transformed function

From this equality, we would like to calculate $\tilde{u}(x, t)$. Let us examine Fig. 3.1. The point whose image is x after transform is $x - \varepsilon\xi + o(\varepsilon)$. As a consequence, the tranformed function at x assumes the value

$$\tilde{u}(x, t) = u(x - \varepsilon\xi, t) + \varepsilon\phi + o(\varepsilon).$$

We can differentiate with respect to t since the $o(\varepsilon)$ term does not depend on t, and we get

$$\tilde{u}_t(x, t) = u_t(x, t) + \varepsilon u_t(\phi_u - \xi_u u_x) + o(\varepsilon).$$

Identifying the ε terms gives (3.12).

Remark 3.13. The theory developed by Olver is valid in any dimension of space, but needs to define the prolongation of a vector field depending on several independent variables $x_1, \cdots, x_p$. If we are interested in the equation $F(x_1, \cdots, x_p, u^{(n)}) = 0$, (3.12) becomes

$$\mathrm{pr}^{(n)}\, \mathbf{v}(F(x_1, \cdots, x_p, u^{(n)})) = 0 \tag{3.13}$$

whenever u is solution of $F(x_1, \cdots, x_p, u^{(n)}) = 0$. The term $u^{(n)}$ contains all the partial derivatives of u with respect to the p variables $x_1, \cdots, x_p$ up to total order n. This analysis would also naturally apply in our case, where we have indeed two independent variables: the space variable x and the scale parameter t. Nevertheless, we look for equations with a very particular dependence on t since only the term u_t appears and linearly. Moreover, the Lie group does not act in the t direction. The condition (3.12) is nothing but (3.13) when these two conditions are fulfilled.

Example 3.14. Consider the group of isometries, generated by the vector fields ∂_x, ∂_u and $-u\partial_x + x\partial_u$. The two first vectors correspond to translations and the last one to rotations. Let us look for equations of second order satisfying (3.12) for these vector fields. It is trivial to check that $\mathrm{pr}^{(2)}(\partial_x) = \partial_x$. We deduce that if F does not depend on x, the equation is invariant with respect to translations in the x direction. In the same way, $\mathrm{pr}^{(2)}(\partial_u) = \partial_u$ and yields that it suffices that F does not depend on u for the equation to be invariant with respect to "vertical" translations. Note that this was not a wild guess even without using the machinery above. Things are slightly more complicated for the case of rotations. The vector field is $-u\partial_x + x\partial_u = \xi\partial_x + \phi\partial_u$, with $\xi(x, u) = -u$ and $\phi(x, u) = x$. We have

$$D(\phi - \xi u_x) = 1 + u_x^2 + u u_{xx},$$

and so $\phi^1(x, u^{(2)}) = 1 + u_x^2$. We also compute

$$D^2(\phi - \xi u_x) = 3u_x u_{xx} + u u_{xxx}$$

so that $\phi^2(x, u^{(2)}) = 3u_x u_{xx}$. We finally have

$$\mathrm{pr}^{(2)}(-u\partial_x + x\partial_u) = -u\partial_x + x\partial_u + (1 + u_x^2)\partial_{u_x} + 3u_x u_{xx}\partial_{u_{xx}}.$$

Since we assumed that F does not depend on x and u, (3.12) becomes

$$u_x u_t = (1 + u_x)^2 \partial_{u_x} F + 3u_x u_{xx} \partial_{u_{xx}} F.$$

We now replace u_t by $F(u_x, u_{xx})$ and get the equation

$$u_x F = (1 + u_x)^2 \partial_{u_x} F + 3u_x u_{xx} \partial_{u_{xx}} F,$$

where u_x and u_{xx} must be considered as two independent variables. We can check that

$$F_1(u_x, u_{xx}) = \sqrt{1 + u_x^2} \quad \text{and} \quad F_2(u_x, u_{xx}) = \frac{u_{xx}}{1 + u_x^2}$$

are solutions. The equation $u_t = F_1(u_x, u_{xx})$ is the motion of the graph with a constant normal velocity, whereas $u_t = F_2(u_x, u_{xx})$ corresponds to the mean curvature flow. Both admit the Euclidean group as a symmetry group.

3.2 Differential invariants

3.2.1 General form of invariant flows

Theorem 3.12 allows to check whether a given group is a symmetry group for a given equation of the type $u_t = F(x, u^{(n)})$. It suffices to know explicitly a complete system of infinitesimal generator of the group. As an application, we saw that the constant speed normal motion (also called dilation) and the mean curvature motion were invariant with respect to isometries. Now, this is not exactly what we are looking for, since we want to know if there is a general form for equations having a given group of symmetry. This is equivalent to search all the differential quantities that are not modified by the group action, and which are called differential invariants of the group.

Definition 3.15. *A* differential invariant I *of order* n *is a function defined on the jet space* J^n *which is constant under the prolonged group action. This means that for any* $(x, u^{(n)}) \in J^n$ *and* $g \in \mathcal{G}$*, we have*

$$I(\mathrm{pr}^{(n)}\, g \cdot (x, u^{(n)})) = I(x, u^{(n)}). \tag{3.14}$$

As usual, the Lie group structure allows to formulate this global condition in terms of local conditions (and vice-versa).

Proposition 3.16. *A function* $I : J^n \to \mathbb{R}$ *is an* n*th order differential invariant for* $\mathcal{G}$ *if and only if*

$$\mathrm{pr}^{(n)}\, \mathbf{v}(I) = 0, \tag{3.15}$$

for any infinitesimal generator of $\mathcal{G}$*.*

The proof of this proposition is essentially that I should be constant when we follow the exponential map of a prolonged vector field, so that we can use a Taylor expansion for infinitesimal displacement along the exponential flow.
Of course, if I is a differential invariant, then any function of I is also a differential invariant. Thus, there is no hope in finding a finite number of invariants, but we can wonder if all the invariants are functions of a finite number of invariants. (Such invariants form a complete system of differential invariants.) If, in a set of invariants, any invariant cannot be expressed as a function of the others, the invariants are called functionally independent. The goal is then to find a finite and complete set of functionally independent invariants, i.e. such that any invariant is a function of the invariants of the set. The result makes appear a privileged invariant form, the group arc-length. Let us first define what an invariant form is.

Definition 3.17. *An invariant form* ω *is a differential form such that for any* $g \in \mathcal{G}$ *and* $(x, u) \in \mathcal{M}$*, we have*

$$\omega_{|g\cdot(x,u)} \circ dg_{|(x,u)} = \omega_{|(x,u)}, \tag{3.16}$$

where dg *denotes the differential of the application* $(x, u) \mapsto g \cdot (x, u)$*.*

The equality (3.16) means that the infinitesimal volume given by the one-form is unchanged under the group action. For instance, if $\omega = dr$ is the Euclidean length, a rotation of a small segment changes its orientation but not its length. The invariance of a one-form can be expressed in terms of the infinitesimal generator of the group. In the case where the form is of the type $\omega = P(x, u^{(n)})\, dx$, then the conditions are as follows.

Proposition 3.18. *Let $\omega = P(x, u^{(n)})\, dx$. Then ω is $\mathcal{G}$-invariant if and only if*

$$\mathrm{pr}^{(n)}\,\mathbf{v}(P(x, u^{(n)})) + P(x, u^{(n)}) D\xi(x, u) = 0 \tag{3.17}$$

for any infinitesimal generator $\mathbf{v} = \xi(x,u)\partial_x + \phi(x,u)\partial_u$ of $\mathcal{G}$.

We now have all the elements to enounce the characterization of all differential invariants. The proof of the following result uses dimensionality arguments to construct a complete set of differentiable inavariants. (See [139].)

Theorem 3.19. *There exists a $\mathcal{G}$-invariant one form $dr = g\, dx$ and a differential invariant χ of lowest order such that any differential invariant is a function of $\chi, \cdots, \frac{d^n\chi}{dr^n} \ldots$ The parameter r is called the* group arc-length *and gives a group invariant parameterization of the curve. The simplest invariant χ is called the* group curvature.

Finally, here are the classification results of invariant flows. Any evolution equation with $\mathcal{G}$ as a symmetry group can be expressed in function of r, g and χ.

Theorem 3.20 ([141]). *Let $\mathcal{G}$ a Lie subgroup of the plane projective group. Then, any $\mathcal{G}$-invariant flow can be written*

$$\frac{\partial u}{\partial t} = \frac{1}{g^2}\frac{\partial^2 u}{\partial x^2} I\left(\chi, \frac{d\chi}{dr}, \cdots, \frac{d^n\chi}{dr^n}\right). \tag{3.18}$$

When I is identically equal to 1, the equation is called the group heat equation.

The hypothesis that $\mathcal{G}$ is a subgroup of the projective group is essential. Indeed, the proof of Thm. 3.20 makes appear the symmetry group of the equation $u_{xx} = 0$ which is precisely the projective group.
If we go back to a curve formulation (that is the complete curve is not a graph) then the equation takes the following form.

Corollary 3.21. *A curve C satisfies*

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial r^2} I\left(\chi, \frac{d\chi}{dr}, \cdots, \frac{d^n\chi}{dr^n}\right) \tag{3.19}$$

if and only if it is locally a graph satisfying

$$\frac{\partial u}{\partial t} = \frac{1}{g^2}\frac{\partial^2 u}{\partial x^2} I\left(\chi, \frac{d\chi}{dr}, \cdots, \frac{d^n\chi}{dr^n}\right).$$

Before turning to examples, we have to give a way to compute differential invariants.

Proposition 3.22. *Let* $\mathbf{v} = \xi(x,u)\partial_x + \phi(x,u)\partial_u$ *with prolongation*

$$\operatorname{pr}^{(n)} \mathbf{v} = \mathbf{v} + \sum_{i=1}^{n} \phi^j(x, u^{(n)})\partial_{u_j}.$$

Then, differential invariants under the action of the exponential map of $\operatorname{pr}^{(n)} \mathbf{v}$ *are found by solving the system of equations*

$$\frac{dx}{\xi(x,u)} = \frac{du}{\phi(x,u)} = \frac{du_j}{\phi^j(x,u^{(n)})} = \cdots = \frac{du_n}{\phi^n(x,u^{(n)})}.$$

3.2.2 The mean curvature flow is the Euclidean intrinsic heat flow

Let us go back to the Euclidean invariance. Recall that we have three infinitesimal generators, $\mathbf{v_1} = \partial_x$, $\mathbf{v_2} = \partial_u$, and $\mathbf{v_3} = -u\partial_x + x\partial_u$, with respective first order prolongations

$$\operatorname{pr}^{(1)} \mathbf{v_1} = \mathbf{v_1}, \quad \operatorname{pr}^{(1)} \mathbf{v_2} = \mathbf{v_2},$$
$$\text{and } \operatorname{pr}^{(1)} \mathbf{v_3} = \mathbf{v_3} + (1+u_x^2)\partial_{u_x}.$$

Let us look for a form $dr = g\,dx$ of the first order, that is with g depending on x, u and u_x. By applying Prop. 3.18, we find that g does not depend on x and u. The last condition gives $(1+u_x^2)g_{u_x} - gu_x = 0$ admitting $g = c\sqrt{1+u_x^2}$ as a solution, c being a constant that we can take equal to 1. As the group arc-length is also the usual arc-length, we deduce that the intrinsic Euclidean heat flow is the mean curvature motion

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial s^2} = \kappa \mathbf{N}, \tag{3.20}$$

where s is the usual arc-length. If we examine again Ex. 3.14, we see that the simplest differential invariant of the Euclidean group is $u_{xx}/(1+u_x^2)^{3/2}$, that is the usual curvature. It is worth noticing that this flow is not the simplest invariant flow, since by multiplying the right-hand term by the differential invariant $1/\kappa$, we obtain the constant velocity flow $C_t = \mathbf{N}$. Nevertheless, this flow depends on the orientation of the curve and convex and concave parts do not behave the same way. Moreover, it also develops singularities.

3.2.3 The affine invariant flow: the simplest affine invariant curve flow

We are now interested in the flows whose symmetry group is the special affine group, that is to say, the set of affine mappings with determinant equal to 1. This group is generated by the special linear group $SL(\mathbb{R}^2)$ and the translations. As in the case of the Euclidean group, we look for affine differential invariants by determining the infinitesimal generators of the special affine group. As three first generators, we choose the same vectors $\mathbf{v_1}$, $\mathbf{v_2}$ and $\mathbf{v_3}$ as for the Euclidean invariance. We then have two additional vector fields. First $\mathbf{v_4} = x\partial_x - u\partial_u$ corresponds to the surface preserving stretching $(x,u) \mapsto (\lambda x, \lambda^{-1}u)$. Finally $\mathbf{v_5} = u\partial_x$ corresponds to the shearing $(x,u) \mapsto (x+\lambda u, u)$, which is also surface preserving. Let us compute the second order prolongation of these vector fields.

$$\mathrm{pr}^{(2)}\,\mathbf{v_1} = \partial_x, \tag{3.21}$$
$$\mathrm{pr}^{(2)}\,\mathbf{v_2} = \partial_u, \tag{3.22}$$
$$\mathrm{pr}^{(2)}\,\mathbf{v_3} = -u\partial_x + x\partial_u + (1+u_x)^2\partial_{u_x} + 3u_x u_{xx}\partial_{u_{xx}}, \tag{3.23}$$
$$\mathrm{pr}^{(2)}\,\mathbf{v_4} = x\partial_x - u\partial_u - 2u_x\partial_{u_x} - 3u_{xx}\partial_{u_{xx}}, \tag{3.24}$$
$$\text{and } \mathrm{pr}^{(2)}\,\mathbf{v_5} = u\partial_x - u_x^2\partial_{u_x} - 3u_x u_{xx}\partial_{u_{xx}}. \tag{3.25}$$

We look for the group arc-length dr of the form

$$dr = P(x,u,u_x,u_{xx})dx.$$

By using the same arguments as for the Euclidean group, we conclude that P does not depend neither on x nor on u. If we try $P(u_x,u_{xx}) = P(u_x)$, then dr is necessarily the Euclidean arc-length, and we can then check that the condition (3.17) of Prop. 3.18 are not satisfied for $\mathbf{v_4}$ and $\mathbf{v_5}$. Hence, the group arc-length is necessarily at least of second order. Fortunately, this order is sufficient. Indeed, we write the Condition (3.17) on $P(u_x,u_{xx})$ for $\mathbf{v_3}, \mathbf{v_4}$ and $\mathbf{v_5}$ and we obtain the three following equations,

$$(1+u_x^2)P_{u_x} + 3u_x u_{xx}P_{u_{xx}} - u_x P = 0, \tag{3.26}$$
$$-2u_x P_{u_x} - 3u_{xx}P_{u_{xx}} + P = 0, \tag{3.27}$$
$$-u_x^2 P_{u_x} - 3u_x u_{xx}P_{u_{xx}} + u_x P = 0. \tag{3.28}$$

Multiplying (3.27) by u_x and adding it from (3.26), we obtain $P_{u_x} = 0$, which is an unexpected but very good surprise. Thus, the three equations reduce to the linear first order ordinary equation

$$3u_{xx}P_{u_{xx}} = P$$

whose general solution is $P(u_{xx}) = cu_{xx}^{1/3}$. This implies that the affine arc-length is

$$dr = cu_{xx}^{1/3}\,dx. \tag{3.29}$$

By applying Thm. 3.20, we then obtain that the affine heat equation is the remarkably simple equation given by the following theorem.

Theorem 3.23. *The affine intrinsic heat equation for a graph is*

$$u_t = u_{xx}^{1/3}. \tag{3.30}$$

We now write the affine heat equation for a curve in terms of the curvature. Here again, the equation is surprisingly simple.

Corollary 3.24. *The affine intrinsic heat equation is*

$$\frac{\partial C}{\partial t} = \kappa^{1/3}\mathbf{N}. \tag{3.31}$$

Proof. The curvature of the graph is $\kappa = \frac{u_{xx}}{(1+u_x^2)^{3/2}}$. Hence,

$$u_{xx}^{1/3} = \kappa^{1/3}\sqrt{1+u_x^2}$$

and the normal velocity of the graph is $\kappa^{1/3}$. □

The method above also permits to compute all the invariants of the group. We could check that, in the special affine case, the simplest one – the group curvature – is of fourth order.
Another classical approach leads to the affine arc-length and the affine curvature without using the analytical calculations above. Assume that a curve is parameterized by p. We define the parameter ζ such that

$$[C_\zeta, C_{\zeta\zeta}] = 1, \tag{3.32}$$

where the bracket is the two-dimensional determinant of the vectors. By computing the same determinant with ζ as a function of p, it is clear that we must have

$$d\zeta = [C_p, C_{pp}]^{1/3}\, dp.$$

We see that ζ is only defined for strictly convex curves, but this will not be a problem for curve evolution thanks to an argument given by the very useful Lemma 3.25 below. Differentiating (3.32) with respect to ζ, we find that $C_{\zeta\zeta\zeta} + \mu C_\zeta = 0$ for some real value μ. To get μ, we form the determinant with $C_{\zeta\zeta}$ and finally obtain

$$\mu = [C_{\zeta\zeta\zeta}, C_{\zeta\zeta}]. \tag{3.33}$$

By applying an affine mapping with determinant 1 to the calculation of $d\zeta$, it is easy to see that ζ is a differential invariant, and that it is *affine arc-length*. It is also possible to check that μ is the *affine curvature*, that is the simplest non-trivial invariant. We also note that $d\zeta$ is of second order and μ is of 4th order. The affine heat equation is

$$\frac{\partial C}{\partial t} = \frac{\partial^2 C}{\partial \zeta^2}, \tag{3.34}$$

which is a third order equation. It is not very easy to have a clear idea of the affine curvature, so it is useful to express the evolution in terms of the Euclidean parameters. Let us denote by s the Euclidean arc-length and κ the Euclidean curvature. The definition of the affine arc-length leads to $d\zeta = \kappa^{1/3}\, ds$ in agreement with (3.29). We also have

$$\begin{aligned}\frac{\partial^2 C}{\partial \zeta^2} &= \kappa^{-1/3}\frac{\partial}{\partial s}\left(\kappa^{-1/3}\frac{\partial C}{\partial s}\right). \\ &= \kappa^{1/3}\mathbf{N} + \kappa^{-1/3}(\kappa^{-1/3})_s\mathbf{T}.\end{aligned} \tag{3.35}$$

When the curvature is equal to 0, the tangential term is not defined, accordingly to the fact that the affine arc-length is not defined. The following lemma proved in [54] solves this problem.

Lemma 3.25. *Let β be an intrinsic quantity. If C_t evolves according to*

$$C_t = \beta\mathbf{N} + \alpha\mathbf{T},$$

for any continuous $\bar{\alpha}$, there exists another parameterization $\bar{C}$ of C such that $\bar{C}$ is solution of

$$\bar{C}_t = \bar{\beta}\mathbf{N} + \bar{\alpha}\mathbf{T},$$

where $\bar{\beta} = \beta$ at the same geometric point.

This means that the tangential velocity only changes the parameterization but not the image of the curve. In particular, we can choose $\bar{\alpha} = 0$. This lemma applied to affine heat equation leads to the equation

$$C_t = \kappa^{1/3}\mathbf{N}. \tag{3.36}$$

If C is solution of this equation, then its range is invariant with respect to special affine mappings. Thus, by some kind of miracle, we can conclude that there is only one second order evolution equation commuting with affine special transformation.

Faugeras, in [64], was interested in projective invariant equations. By using the theory of invariants, it is possible to prove that the projective heat equation is of 5th order. Moreover, singularities may develop. Thus, we shall not study this model any further, and refer the reader to [64, 101]. Olver, Sapiro and Tannenbaum [141] also considered flows preserving the surface or the length and invariant surfaces evolution in [142].

The axiomatic approach above gives all the possible equations having a given group of symmetry (which has to be a subgroup of the projective group). However, the story is not unfolded yet. Indeed, some of these equations develop some singularities, as the projective invariant ones [64], or some evolution depending on the laplacian of the curvature [74]. Thus, in what follows, we concentrate our effort on certain equations that are Euclidean invariants. Following Olver, Sapiro and Tannenbaum, we know that they are of the type

$$\frac{\partial C}{\partial t} = G(\kappa, \cdots, \frac{\partial^n \kappa}{\partial s^n})\mathbf{N}, \tag{3.37}$$

where κ is the Euclidean curvature and s the usual Euclidean arc-length. In particular, when $G(\kappa, \cdots, \kappa_n) = \kappa^{1/3}$, we know that (3.37) is geometrically equivalent to the affine heat equation. We now restrict the study to functions G that only depends on κ (and not on its derivatives) and that are moreover nondecreasing. Indeed, if we rewrite (3.37) as the local evolution of the graph of some function u, then the equation is second-order parabolic, since the dependence with respect to the second derivative is nondecreasing. For these kinds of equations, we can expect to obtain some maximum principle (implying uniqueness) and regularity [57].

3.3 General properties of second order parabolic flows: a digest

Now and after, we only consider equations of the form

$$\frac{\partial C}{\partial t} = G(\kappa)\mathbf{N}. \tag{3.38}$$

Since we also want to smooth the curve, it is also logical to assume that convex parts move inwards whereas concave parts move outwards. As we do not want to privilege convex or concave parts of curve, we shall assume that G is an odd function. For a close curve, assuming that the normal vector points inwards, we suppose that $G(\kappa) > 0$ if $\kappa > 0$. We also want long and thin parts to be removed faster than larger and rounder parts. We thus assume that G is strictly increasing. This kind of flows has led to many studies in the last fifteens years. Partial results for existence, uniqueness and regularity are now proved. Nevertheless, a general and unifying result is still unknown. In particular, if we consider the case $G(\kappa) = \kappa^\gamma$ for $\gamma > 0$, the long term existence for $\gamma > 1$ is still unknown, whereas the case $0 < \gamma < 1$ is better understood, though incompletely. These results are beautiful since their formulation is very simple. However, their proof is very technical, long, and falls out the scope of this monograph. Very recently, the original proof by Grayson [77] for the mean curvature motion has completely been rewritten by Chou and Zhu in their recent book [39], "The Curve Shortening Problem" to which we refer the reader for the most recent state of the art of shortening curve. Many results are consequences of classical results for parabolic partial differential equations.
In this section, we sum up, without proof, all the known properties on the curve shortening equation (3.38). We refer the reader to [10, 12, 13, 14, 39, 69, 77, 122] for more details. We first give some complete results for the mean curvature motion and the affine heat equation, and the partial known results on the general case. In Chap. 6, we shall see many numerical illustrations of the results below. Moreover, numerical evidences are that the general results, existing only in the Euclidean and the affine heat equations, are valid at least for all the flows where $G(\kappa) = \kappa^\gamma$ with $\gamma \geq 1/3$. In Chap. 4, we also present a weak formulation of the problem of curve evolution.

3.3.1 Existence and uniqueness for mean curvature and affine flows

There are two cases where complete results are known. The first one is the mean curvature motion, i.e. $G(\kappa) = \kappa$. The results are partially consequences of the classical theory for parabolic PDEs. Indeed, we can locally represent a curve by a graph, and the evolution equation is quasilinear and parabolic. Thanks to the maximum principle, we can prove that a curve stays smooth and two evolving curves cannot become too close from each other. The result is the following.

Theorem 3.26. *[69, 77] Let C_0 be a closed plane Jordan curve. Then, there exists a unique family of embedded curves $C(t)$ satisfying*

$$\frac{\partial C}{\partial t} = \kappa \mathbf{N}, \tag{3.39}$$

with $\lim_{t\to 0} d_H(C(t), C_0) = 0$. ($d_H$ is the Hausdorff distance[1].)
The solution is defined on $(0, T)$ where $T = A_0/2\pi$, A_0 being the area enclosed by the curve at $t = 0$ and is analytic for $t \in (0, T)$. Moreover, there exists $t_1 < T$ such that $C(t)$ is strictly convex for $t_1 < t < T$. When $t \uparrow T$, $C(t)$ shrinks to a point. The renormalized curve (by keeping the area constant) exponentially tends to a circle for the Hausdorff distance.

Remark 3.27. Gage and Hamilton [69] first proved that convex curves evolving by mean curvature remained convex and shrinked to a circular point. Grayson [77] proved the remaining part of the result by showing that a nonconvex curve stayed embedded and became convex in finite time. The fact that the curve becomes circular is based on a monotonicity formula for some isoperimetric ratio. Recently, Huisken [84] gave another proof of Grayson's theorem, but it uses a preliminary result which is the classification of all types of singularities of the mean curvature flow [83].

Later, the affine heat equation was solved by Angenent, Sapiro and Tannenbaum. The evolution of convex curves (the equivalent of Gage and Hamilton result) was proved by Sapiro and Tannenbaum [152], whereas the nonconvex curves evolution was eventually elucidated by the same authors and Angenent [14].

Theorem 3.28. *[14] Let C_0 be a closed Jordan curve. Then, there exists a unique family of embedded curves $C(t)$ satisfying*

$$C_t = \kappa^{1/3}\mathbf{N} \quad \text{and} \quad C(0) = C_0,$$

defined on some interval $(0, T)$. For $t \in (0, T)$, $C(t)$ is analytic. There exists $t_1 < T$ such that $C(t)$ is convex on (t_1, T). The renormalized curve with constant area converges to an ellipse.

[1] If A and B are subsets of $\mathbb{R}^2$, their Hausdorff distance is

$$d_H(A, B) = \max(\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)).$$

3.3.2 Short-time existence in the general case

When the normal velocity G is an increasing and smooth function of the curvature, results of short-time existence come from the theory of parabolic equations. If $G(\kappa) = \kappa^\gamma$ with $0 < \gamma < 1$, the theory does not apply directly since G is singular for $\kappa = 0$. Nevertheless, we can approximate the power function by a smooth function at the origin and obtain a solution for the approximate problem. We can pass to the limit, provided some uniform bounds are obtained on the approximation. This is true and the result is the following.

Theorem 3.29. *Assume that C_0 is a C^2 curve and that G is*

- *either increasing and smooth,*
- *or $G(\kappa) = \kappa^\gamma$ with $0 < \kappa < 1$.*

Then there exists a unique maximal solution $C(t)$, $0 < t < T$, to the evolution problem

$$C_t = G(\kappa)\mathbf{N} \quad \textit{and} \quad C(0) = C_0.$$

For any $t < T$, $C(t)$ is embedded and $\sup_{C(t)} |\kappa| \uparrow \infty$ as $t \uparrow T$.

Very recently, Chou and Zhu [39] proved a result like Grayson's when $G(\kappa) = \kappa^\gamma$ with $0 < \gamma < 1$. The difference is that it is not proved that the curve becomes convex and round before vanishing.

Theorem 3.30 ([39]). *Let $0 < \gamma < 1$ and $C(t)$ evolving by $C_t = \kappa^\gamma \mathbf{N}$. Then the flow develops no singularity until the area enclosed by the curve drops to 0. The total curvature[2] of the curve tends to 2π and the final state is either a point or a line segment.*

3.3.3 Evolution of convex curves

One major (and unsolved) difficulty in the general case is to prove that the convex components (that is the parts of the curve between inflexion points) stay away from each other and that the curve cannot cross itself. In the case of a convex curve, this is not as difficult. It is also convenient to parameterize the curve by the angle θ between the tangent and a fixed direction, that is $\mathbf{T} = (\cos\theta, \sin\theta)$. This parameter is intrinsic, but, contrary to the length parameter, is independent of time. We can derive equations satisfied by the curvature κ as a function of θ and t. These equations are parabolic and the maximum principle then implies that the curve stays convex and that θ remains an admissible parameter.

Proposition 3.31. *Let $(C(t))_{0<t<T}$ be a smooth curve evolving by $C_t = G(\kappa)\mathbf{N}$. Then, if $C(0)$ is strictly convex, $C(t)$ is also strictly convex for $0 < t < T$.*

[2] The total curvature of C is $\int_C |\kappa|\, ds$. If C is convex, then we can drop the absolute value, and we then remark that the total curvature is equal to 2π. We also see that this is a lower bound and that the total curvature is equal to 2π if and only if the curve is convex.

As stated above, Gage and Hamilton [69] first established that a convex curve evolving by the mean curvature motion remained embedded until it disappeared in a round point. Andrews [10] generalized this result as follows.

Theorem 3.32 ([10]). *Let $C(t)$ be a family of convex curves evolving by $C_t = \kappa^\gamma \mathbf{N}$ with $\gamma > 0$. Then $C(t)$ exists and disappears in a point at a time T. Moreover*

1. *if $\gamma \leq 1$, the curve is strictly convex and C^∞ for any $0 < t < T$ (in particular, it instantaneously becomes smooth).*
2. *if $\gamma > 1$, the curve is globally of class $C^{k+2,\alpha}$ where $k + \alpha = 1/(1-\gamma)$ and $\alpha \in [0, 1)$ and C^∞ at points where the curvature is positive.*

The existence until vanishing relies on some monotonicity properties of some integral of the curvature. These values are generalizations of the entropy of the curvature. It can be proved that a control on these global quantities implies a local control on the curvature. It is also not very difficult to prove that the existence up to extinction is valid when G is any smooth convex function on $\mathbb{R}_+$. The reader may check that it suffices to change Lem. 4.3.4 in [69] as follows. We define

$$f(\kappa) = \int_1^\kappa \frac{G(t)}{t^2}\, dt,$$

and we prove that $\int_0^{2\pi} f(\kappa(\theta, t))\, d\theta$ remains bounded as long as the median curvature is bounded. We can also prove that the curvature is uniformly bounded if the integral quantity above is bounded. As Gage and Hamilton, we can deduce that the curve exists as long as the inner area is bounded from below.

3.3.4 Evolution of the length

We now give two interesting qualitative properties of Euclidean flows. Their proof is easy and is given is some form or the other in [69, 122]. In this section, we always assume that $C(t)$ is a smooth curve evolving by (3.38). The following lemma asserts that the length of C is nonincreasing. This justifies the denomination *curve shortening flow*.

Lemma 3.33. *Let C smooth satisfying $C_t = G(\kappa)\mathbf{N}$. Let $L(t)$ be the length of $C(t)$. Then*

$$\frac{dL}{dt} = -\int_{C(t)} \kappa G(\kappa)\, ds. \tag{3.40}$$

Since we assumed that G is odd and positive for positive curvature, we deduce that the length must decrease.

3.3.5 Evolution of the area

It is also useful to compute the evolution of the area surrounded by a closed curve.

Lemma 3.34. *Let $A(t)$ the area enclosed by $C(t)$. Then,*

$$\frac{\partial A}{\partial t} = -\int_{C(t)} G(\kappa)\, ds. \tag{3.41}$$

A first consequence is that the area is decreasing for convex curves. Next, by Gauss-Bonnet Theorem, the decreasing rate for the mean curvature motion is constant and equal to -2π. Therefore, the curve cannot exist on an interval longer than $A_0/2\pi$.

3.4 Smoothing staircases

We end this chapter with the study of a qualitative property of geometric flows that shall be interesting in image processing. Remember that one of the main purpose that guides our study of curve smoothing is object recognition. The objects are observed in digital images and are obtained through a sampling process, the result being a discrete set of values. Even though this should be possible to smoothly interpolate these values by Shannon theory (if the sampling is correctly performed), the model which is mostly accepted (and easiest to handle) is the one of piecewise constant images. The pixels are square-shaped in general, thus contours of shapes are staircased, that is to say, are the concatenation of segments turning by angle of $\pm\frac{\pi}{2}$. Thus, except in very particular situations, "corners" are due to the sampling and not to real phenomena. True corners, which are strong geometrical cues in man-made objects, are not purely local information since they result from the crossing of two smooth and long enough curves. Thus, in general, it may be interesting to cancel the staircasing effect, and true corners should not be too much affected. For example, if we take a broken line composed of segments of length 1, successively oriented with angle of 0 and $\frac{\pi}{2}$, this is usually nothing but a straight line oriented by an angle of $\frac{\pi}{4}$ that has been staircased by the sampling process. In [104], curve evolution was used to smooth the level lines of images (see next chapter) by using the affine invariant equation. This is natural to ask what the qualitative difference is between the different equations of curve smoothing. In order to see this, we consider the evolution of the graph of the function with initial value u_0 defined on $[-1, 1]$ by $u_0(x) = 1 - |x|$, with Dirichlet boundary conditions $u(-1) = u(1) = 0$. This corresponds to the evolution of a broken line periodically repeated. If we assume that the graph evolves by the shortening equation with a normal velocity equal to

$$V = \kappa^\gamma,$$

then u satisfies the equation

$$\frac{\partial u}{\partial t} = \sqrt{1 + u_x^2}\left(\frac{u_{xx}}{(1 + u_x^2)^{3/2}}\right)^\gamma. \tag{3.42}$$

The asymptotic state of the solution is the function identically equal to 0. What is the speed of convergence of the solution to this state? Is it reached in finite or infinite time? This is the object of the following result.

Proposition 3.35. *Let u be a smooth solution of*

$$\begin{cases} u_t = (u_{xx})^\gamma (1+u_x^2)^{\frac{1-3\gamma}{2}} \quad \textit{in } (-1,1). \\ u(x,0) = u_0(x) = 1 - |x|, \\ u(-1,t) = u(1,t) = 0 \textit{ for } t \geq 0. \end{cases}$$

Then,

1. *if $\gamma < 1$, u becomes identically 0 in finite time.*
2. *If $\gamma > 1$, there exist two constants a and b such that*

$$at^{-\frac{1}{\gamma-1}} \leq u(x,t) \leq bt^{-\frac{1}{\gamma-1}}.$$

In particular, u is positive in $(-1,1)$ at any time $t > 0$.

3. *If $\gamma = 1$, then $u(x,t)$ goes exponentially to 0.*

Proof. We first derive an equation satisfied by $v = u_x$. A simple calculation shows that

$$v_t = \gamma v_{xx}|v_x|^{\gamma-1}(1+v^2)^{\frac{1-3\gamma}{2}} + v_x^\gamma(1-3\gamma)v_x v(1+v^2)^{-\frac{1+3\gamma}{2}}.$$

Since $\gamma > 0$, this equation is still parabolic, and the maximum principle allows us to assert that $\sup v$ (*resp.* $\inf v$) is nonincreasing (*resp.* nondecreasing). Thus, we deduce that u is subsolution of the equation

$$u_t = C u_{xx}^\gamma, \tag{3.43}$$

where C can be chosen equal to 1 if $\gamma \geq 1/3$ and depends on $\sup |v|$ at time 0 but not on γ. In the same way, u is supersolution of (3.43) with a different constant C. We then look for particular solutions of these equations and use them as upper and lower barriers to bound $u(x,t)$ by applying the maximum principle.
First, assume that $\gamma < 1$. We now look for a separable solution of (3.43) of the form $w(x,t) = f(t)g(x)$. Then f and g must satisfy

$$\frac{f'}{f^\gamma} = -a \quad \text{and} \quad C\frac{(g'')^\gamma}{g} = -a,$$

for some positive constant a. Then f is explicitly solved in $f(t)^{1-\gamma} = f(0)^{1-\gamma} - a(1-\gamma)t$, which goes to 0 in finite time. Examine now the equation satisfied by g. We have

$$g'' = -\beta g^{1/\gamma},$$

with $\beta = (a/C)^{1/\gamma}$. Multiplying by g' and integrating between 0 and x gives

$$g'(x)^2 = g'(0)^2 - \beta\frac{\gamma}{\gamma+1}(g^{\frac{\gamma}{\gamma+1}}(x) - g^{\frac{\gamma}{\gamma+1}}(0)).$$

We impose $g'(0) = 0$ and $g(0) > 0$. On $(0,1)$, we choose $g' < 0$. In this case, we see that it is possible to impose $g(1) = 0$ and that this yields $g'(1) < 0$. Setting

$w(x,t) = f(t)g(x)$ gives a solution of (3.43). This solution is chosen even and satisfies $w_x(0,t) = 0$, $w(1,t) = 0$ for small t and becomes null in finite time. Moreover, since $w_x(1,0) < 0$, we can also choose w such that $w(x,0) \geq 1 - |x|$ in $(0,1)$. Indeed, if w is solution of (3.43), the homogeneity property of the equation implies that $\lambda^{1/(1-\gamma)} w(x, t/\lambda)$ is also solution. It then suffices to choose λ large enough. We then invoke the maximum principle for (3.43) and we obtain $u(x,t) \leq w(x,t)$, proving that u becomes identically equal to 0 in finite time.
We now assume that $\gamma > 1$. Since $\sup |u_x|$ is nonincreasing in virtue of the maximum principle, u is now supersolution of (3.43) with

$$C = (1 + \sup |u_x|^2)^{(1-3\gamma)/2}.$$

By still looking a separable solution $w(x,t) = f(t)g(x)$, we now remark that the "time part" f above can be explicited in

$$f(t) = (f(0)^{1-\gamma} + a(\gamma - 1)t)^{-\frac{1}{\gamma-1}}$$

which remains positive for any t though tending to 0 at infinity. The spatial part may be solved as above, and by the homogeneity of the equation, we may choose $w(x,0) \leq 1 - |x|$. In the special case $\gamma = 1$, $f(t)$ goes to zero exponentially but stays positive. We again apply the maximum principle and deduce that $u(x,t) > 0$ for $t > 0$. $\square$

The behavior of the solution is experimentally checked on Fig. 3.2, where we display the evolution of such a curve for different powers of the curvature[3]. We display the curves at scale that are multiple of a fixed value. In the case where $\gamma < 1$, there are only a finite number of curves appearing. This also holds in the case $\gamma = 1$ where the exponential decay becomes steady numerically. In the case $\gamma > 1$, the curves accumulate to a segment. What can be the conclusions of this study? If the purpose is

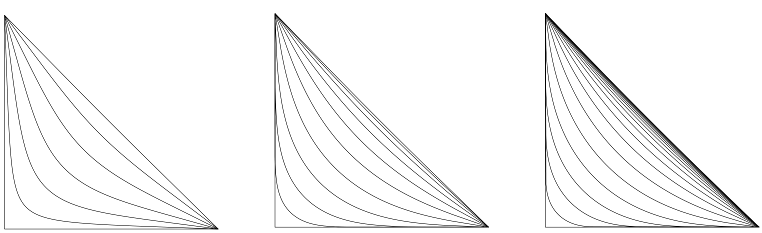


Fig. 3.2. Evolution of a corner with the normal velocity $V = -\kappa^\gamma$. Left: $\gamma = 1/3$, middle $\gamma = 1$, right $\gamma = 2$. The displayed curves correspond to scales that are multiples of a fixed value

shape analysis, we certainly want to cancel the pixel effect as fast as possible. In this

[3] The numerical scheme that we used is the object of Chap. 6

case, small powers seem to be more efficient. Nevertheless, they also remove details faster and this also may be undesirable (see Chap. 6 for numerical observation of this). In particular, if the purpose is image denoising, we may want to remove small details while keeping main features unchanged, for which large power may be more suitable.

3.5 Bibliographical notes

The mean curvature motion has been studied for quite a long time, due to its connection with geometry and physics of interfaces. (See Mullins [132].) Since the equation is quasilinear (it is the only isotropic geometric one), some standard arguments of the classical theory hold for strictly convex surfaces. (See the book by Ladyzhenskaya, Solonnikov and Ural'ceva [106] on quasilinear parabolic equations.) Huisken proved that a convex surface remained convex [82], and this holds in dimension larger than 3. The two-dimensional case was proved by Gage and Hamilton [69], and Grayson then proved that an immersed curve became convex in finite time [77] and that no singularities occur. In higher dimension, this no longer holds as was proved by Grayson [78] and also by Caselles and Sbert [34].
The affine curve shortening was discovered by Sapiro and Tannenbaum [152] at about the same time Alvarez, Guichard, Lions and Morel found the affine morphological scale space [4]. They proved the existence for strictly convex curves and generalized this with Angenent [14]. The independence of the curve with respect to its normalization (Lem. 3.25) is due to Epstein and Gage [54].
The evolution of convex curves for any power of the curvature was solved by Andrews [10], and the most Grayson's like results are exposed by Chou and Zhu [39]. The most recent results on the asymptotic shape are due to Andrews [11].
The classification of all invariant flows with respect to curve invariance was made by Olver, Sapiro and Tannenbaum [140, 141, 142, 143]. These results are also presented in Sapiro's recent book [151].

4

Curve evolution and level sets

In the previous chapter, we saw that there were no general results of existence and uniqueness for the curve evolution problem when the normal velocity is a function of the curvature. The nonexistence of topology changes is not completely solved even though such changes are unlikely to occur when the velocity is a power of the curvature. But this expected regularity does not hold any more for evolution of surfaces in three dimensions. (See for example the work of Caselles and Sbert for the evolution of a thin dumbbell [34].) In this case, it is quite natural to introduce a weak notion of solutions for which it is more easy to prove existence (as distributional solutions for some partial differential equations). There were several fruitful attempts (of which we give a few words at the end of this chapter) but we focus on the approach using the so-called level sets method, for which the notion of solution is the one of *viscosity solutions*.

The idea, which is by now classical, is to represent a curve as the level line of a function. Assume that $u : \mathbb{R}^2 \to \mathbb{R}$ has no critical points. Then, the set $u^{-1}(\{0\})$ is a one dimensional submanifold, that is to say, a curve. If u is solution of an evolution equation, its zero level sets defines a moving curve (if 0 is not a critical value). This idea of using such a *supporting function* seems to have been introduced[1] by Ohta, Jasnow and Kawasaki [137], in order to describe the motion of an interface between two phases. They derived a linear approximation from the original curvature equation and analysed the variation of the surface of the interface for small time evolution. The level set approach was then popularized by Osher and Sethian in [144] also mostly in the context of front propagation. A rigorous analysis of the motion by mean curvature was first performed by Evans and Spruck [60], and independently by Chen, Giga and Goto [38] for geometric motion with a sublinear function of the curvature tensor. Both used the theory of viscosity solutions [42] and we shall recall their main results in the following. We now look for some family of functions $u(\cdot, t)$ such that the evolving curve at time t is the zero level-line of $u(\cdot, t)$. In this case, what kind of equation has to satisfy $u(\cdot, t)$? What is the dependence of the zero level-line with respect to the chosen function u? On the other hand, we guess that

[1] We thank one of the referees for pointing out this reference.

F. Cao: LNM 1805, pp. 55–103, 2003.

this method may also have more flexibility since the set $(u(\cdot,t))^{-1}(\{0\})$ may not be a submanifold. For example, there may be some bifurcation, with the appearance of critical points; $u(\cdot,t)$ may remain smooth, while its level lines change of topology. This advantage has been used in many geometric problems with various applications: image processing (which will be detailed further), but also crystal growth or front propagation. The reader may find many others in [156] with a numerical analysis, mostly based on finite volume methods.

By using some results of mathematical morphology, it was also found that level sets (still in a sense to be precised) provided an efficient representation of images [81, 128]. The problem of curve evolution then reappeared as a problem of image filtering. We shall give the main results that are useful in the field of image processing and particularly curve evolution, so as to make the exposure as self-contained as possible. The chapter will be organized as follows: first, we rigorously define level sets and will see the equivalence between operators defined on sets and operators defined on functions. In particular, we shall eliminate the dependence of the evolving level set on the supporting function. This will make the level set representation completely rigorous and accurate in the context of image processing. After this, we shall find the evolution equation satisfied by the supporting function. Two methods will finally lead to the same results and we shall discuss them both. The first one is to examine an operator on level sets and study its behavior if we introduce a vanishing scale factor. We will see that this approach naturally leads to a particular family of partial differential equations, called the *generalized motions by curvature*. This approach makes the link between the theory of contrast invariant multiscale image filtering and mathematical morphology [102, 169, 119, 154, 27, 35, 81, 124, 126, 146]. The second approach is opposite and consists in starting from a family of PDEs, to classify them with respect to their properties [4] and to examine those which correspond to motions of level sets. Once we obtained those equations in a heuristic or axiomatic way, we then have to check that they are well posed, that is, there exists a notion of solution for which the solution exists and is unique. The suitable notion of solution to these equations is the one of viscosity solution [42, 43, 52, 96, 110]. Very general results for existence and uniqueness are proved in many papers [38, 73, 76, 94, 95, 97] and those which are useful in our context are recalled in the third section of this chapter. We end the chapter by applying this theory to some classical image filtering operators, and we give the related PDEs. Both point of views presented in this chapter are also complementary to the approach of Olver, Sapiro and Tannenbaum. (See the previous chapter.) But whereas this last puts the stress on the simplicity of the equations, both approaches in this chapter are based on the stability property which shall be expressed in terms of monotonicity and maximum principle.

4.1 From curve operators to function operators and vice versa

4.1.1 Signed distance function and supporting function

The first restriction we impose on the curves to be studied in the following is that they are closed Jordan curves. By the theorem of Alexandrov (Thm. 2.4), they sever the plane into two connected components. For a curve C, we denote by $\mathrm{Int}(C)$ the interior of C, that is the bounded connected component of $\mathbb{R}^2 \backslash C$, and we denote by $\mathrm{Ext}(C)$ its exterior. We shall sometimes identify a curve and its interior. We then define the concept of supporting function.

Definition 4.1. *Let C be a closed Jordan curve. We say that a continuous u is a* supporting function *for C if and only if $u(x) = 0 \Leftrightarrow x \in C$, $u < 0$ in $Int(C)$ and $u > 0$ in $Ext(C)$.*

First, for Jordan curves, supporting functions exist as asserts the following definition.

Definition 4.2. *Let C be a closed plane Jordan curve. The signed distance function to C is the function $\bar{d}(\cdot, C)$ which is equal to $-d(x, C) = -\inf_{y \in C} |x - y|$ if $y \in Int(C)$ and to $d(x, C)$ elsewhere.*
The signed distance function to a curve is a supporting function for the curve.

Remark 4.3. Obviously, the signed distance function is not the unique function such that C is its zero level-line. Indeed, if g is continuous, increasing and with $g(0) = 0$, then $g \circ \bar{d}$ is also a supporting function of C. In what follows, such a function g will correspond to a contrast change.

Let us now formulate the problem of level set evolution. Let C_t a family of curves following a certain evolution law. We assume that the curve at time t can be obtained from the one at time 0 by applying an operator $\mathbb{T}_t$, that is to say, we assume that $C_t = \mathbb{T}_t C_0$. We now choose a supporting function to C_0, that we denote by u_0. The problem is to find a family of functions operators T_t, such that if we call $u(\cdot, t) = T_t u_0$, then $u(\cdot, t)$ is a supporting function for C_t that is to say

$$(T_t u_0)(x) = 0 \Leftrightarrow x \in \mathbb{T}_t C_0. \tag{4.1}$$

Moreover, T should not depend on the particular choice of the supporting function u_0. If this is true, it is clear that T_t completely determines $\mathbb{T}_t$. The problem is to uniquely define T_t from $\mathbb{T}_t$. This is not true in general, that a set operator may be uniquely "extended" to a function operator. In what follows, we show that this result requires some structural hypotheses which reduce the class of sets and functions operators we put into correspondence.

4.1.2 Monotone and translation invariant operators

Definition 4.4. *Let $\mathbb{T}$ be an operator mapping some subsets of $\mathbb{R}^2$ into subsets of $\mathbb{R}^2$.*

1. *We say that* $\mathbb{T}$ *is* monotone *if and only if* $X \subset Y \Leftrightarrow \mathbb{T}(X) \subset \mathbb{T}(Y)$.
2. $\mathbb{T}$ *is* translation invariant *if and only if* $\mathbb{T}(x+X) = x + \mathbb{T}(X)$ *for any* $x \in \mathbb{R}^2$.
3. $\mathbb{T}$ *is* continuous *if and only if for any nonincreasing sequence* $(X_\lambda)_{\lambda\in\mathbb{R}}$, *we have*

$$\mathbb{T}(\bigcap_{\mu<\lambda} X_\mu) = \bigcap_{\mu<\lambda} \mathbb{T}(X_\mu).$$

A monotone, continuous, translation invariant operator is called a morphological operator.

These definitions were used by Matheron and Serra as bases of Mathematical Morphology [119, 154]. The monotone translation invariant operators are well suited for shape simplification (or multiscale analysis). Indeed, the shape recognition phase must largely be independent of the position of the object. As a consequence, any simplification should satisfy the same invariance property. The inclusion of a shape into another one should also be preserved through the filtering process, naturally yielding the monotonicity assumption.

4.1.3 Level sets and their properties

In order to extend an operator acting on sets to an operator acting on functions, we first associate to a function a family of sets (its level sets). In this section, we recall the main properties of the level sets of a function.

Definition 4.5. *Let* $u : \mathbb{R}^2 \to \mathbb{R}$. *The* upper level set *of* u *at level* $\lambda \in \mathbb{R}$ *is the set*

$$\chi_\lambda(u) = \{x \in \mathbb{R}^2 \text{ s.t. } u(x) \geq \lambda\}. \tag{4.2}$$

We call level line *at the value* λ *the topological boundary of a connected component of* $\chi_\lambda(u)$.

Remark 4.6. The lower level sets $\chi^\lambda(u)$ are defined by reversing the inequality in (4.2). This will not lead to much different results and when not specified in the sequel, level sets will always mean upper level sets.

Remark 4.7. Note also that level lines are always defined, even if the function u is constant in some domain. In particular a level line may not coincide with the set of points where the function assumes a constant value. If u is of class C^1 then by Sard's Theorem, almost every level λ is not critical, and the level line associated to λ is a Jordan curve. In addition, level sets are also defined for functions that may be discontinuous, and this is useful to prove interesting properties for functions with bounded variation [58].

We now describe the main properties of level sets. We do not detail the proof that are elementary.

Proposition 4.8. *Let* u *be a function and* $\chi_\lambda(u)$ *its family of level sets. Then* u *can be retrieved by the reconstruction formula*

$$u(x) = \sup\{\lambda \text{ s.t. } x \in \chi_\lambda(u)\}.$$

This result is interesting since it means that the datum of the level sets of u is equivalent to the knowledge of u: a function u can be represented without redundancy by the collection of its level sets.

Proposition 4.9. *The level sets of u satisfy the following properties:*

- *Monotonicity:*

$$\lambda \geq \mu \Rightarrow \chi_\lambda(u) \subset \chi_\mu(u), \quad \chi_{-\infty}(u) = \mathbb{R}^2, \quad \chi_\infty(u) = \emptyset. \tag{4.3}$$

- *Continuity:*

$$\forall \lambda \in \mathbb{R}, \quad \chi_\lambda(u) = \bigcap_{\mu<\lambda} \chi_\mu(u). \tag{4.4}$$

We have a sort of reverse proposition to this, in the sense that if a family of sets satisfies (4.3) and (4.4), then there exists a function of which they are the level sets.

Proposition 4.10. *Let $(X_\lambda)_{\lambda\in\mathbb{R}}$ a family of subsets of $\mathbb{R}^2$ satisfying (4.3) and (4.4) and let $u(x) = \sup\{\lambda \text{ s.t. } x \in X_\lambda\}$. Then, for all $\lambda \in \mathbb{R}$, the level set of u is $\chi_\lambda(u) = X_\lambda$.*

Another useful property is that level sets do not depend on contrast.

Proposition 4.11. *Let $u : \mathbb{R}^2 \to \mathbb{R}$.*

1. *Let g be an increasing function and $v = g \circ u$. Then u and v have the same level sets in the sense that*

$$\chi_\lambda(u) = \chi_{g(\lambda)}(v). \tag{4.5}$$

2. *Conversely, assume that u and v have the same level sets, that is for all $\lambda \in \mathbb{R}$, there exists $\mu \in \mathbb{R}$ such that $\chi_\lambda(u) = \chi_\mu(v)$. Let us define g by $g(\lambda) = \sup\{\mu \text{ s.t. } \chi_\lambda(u) = \chi_\mu(v)\}$. Then g is nondecreasing and $v = g \circ u$.*

Proof. The proof of 1) is trivial. We detail the proof of 2).
First g is nondecreasing comes from the monotonicity of level sets. Indeed if $\lambda \leq \lambda'$ then $\chi_{\lambda'}(u) \subset \chi_\lambda(u)$. Thus, if μ is such that $\chi_\lambda(u) \subset \chi_\mu(v)$ we also have $\chi_{\lambda'}(u) \subset \chi_\mu(v)$ and g is nondecreasing.

Now, $v \geq g \circ u$. Indeed, we have $x \in \chi_{u(x)}(u)$. Thus, if $\mu < g(u(x))$, then $x \in \chi_{u(x)}(u) \subset \chi_\mu(v)$ by definition of g. This implies $v(x) \geq \mu$. Since this is true for any $\mu < g(u(x))$ we also have $v(x) \geq g(u(x))$ which is the required inequality.

To prove that $g \circ u \geq v$, let us choose λ such that $\chi_\lambda(u) = \chi_{v(x)}(v)$ (this is possible by assumption). Since $x \in \chi_{v(x)}v$, we also have $u(x) \geq \lambda$, yielding (since g is nondecreasing) $g(u(x)) \geq g(\lambda)$. But, by definition of g, we also have $g(\lambda) \geq v(x)$. Hence, $g(u(x)) \geq v(x)$, ending the proof. □

4.1.4 Extension of sets operators to functions operators

It is in general not true that an operator defined on sets vcan be canonically extended into an operator defined on functions. In this section we shall see that this holds for morphological operators and it is natural to use level sets of functions.

Proposition 4.12 (Matheron). *Let $\mathbb{T}$ be a monotone and continuous set operator (in the sense of Def. 4.4). Then, the relation*

$$\chi_\lambda(Tu) = \mathbb{T}(\chi_\lambda(u)) \tag{4.6}$$

defines an operator defined on functions. Moreover, T satisfies

1. *monotonicity: $u \geq v \Rightarrow Tu \leq Tv$.*
2. *contrast invariance: if g is continuous and nondecreasing, then $T(g(u)) = g(T(u))$.*

Proof. In order to prove that Tu is well defined, we have to show that the sets $\mathbb{T}(\chi_\lambda(u))$ are the level sets of a function. We use the characterization of Prop. 4.10. The monotonicity of the family $\mathbb{T}(\chi_\lambda(u))$ comes from the monotonicity of the level sets of u and of $\mathbb{T}$. The continuity is obtained in the same way. From this, we deduce that there is a function that we denote by Tu and whose level sets are $\mathbb{T}(\chi_\lambda(u))$.
We now prove that T is monotone and contrast invariant. If $u \leq v$ then $\chi_\lambda(u) \supset \chi_\lambda(v)$, yielding $\mathbb{T}(\chi_\lambda(u)) \supset \mathbb{T}(\chi_\lambda(v))$. This means that $\chi_\lambda(Tu) \supset \chi_\lambda(Tv))$. Since this is true for any λ, we get $Tu \leq Tv$, meaning that T is monotone.
In order to prove contrast invariance, let us first prove the equivalence

$$g(x) \geq \lambda \Leftrightarrow \forall \mu \text{ such that } g(\mu) < \lambda \text{ implies } x \geq \mu. \tag{4.7}$$

If we can find μ such that $g(\mu) < \lambda$ and $x < \mu$, then by monotonicity of g, we have $g(x) \leq g(\mu) < \lambda$, proving the direct implication. Conversely, if $g(x) < \lambda$, the continuity of g implies that there exists a $\mu > x$ with $g(x) < g(\mu) < \lambda$. This contradicts the right-hand term. Thus, we have the set equalities

$$\begin{aligned}
\chi_\lambda(g(Tu)) &= \bigcap_{g(\mu)<\lambda} \chi_\mu(Tu), \\
&= \bigcap_{g(\mu)<\lambda} \mathbb{T}(\chi_\mu(u)) \quad \text{by definition of } T, \\
&= \mathbb{T}\left(\bigcap_{g(\mu)<\lambda} \chi_\mu(u)\right) \quad \text{by continuity of } \mathbb{T}, \\
&= \mathbb{T}(\chi_\lambda(g \circ u)), \\
&= \chi_\lambda(T(g \circ u)).
\end{aligned} \tag{4.8}$$

The first and last but one inequalities have been obtained thanks to the characterization of the level sets of g above (4.7). This proves that $g \circ Tu$ and $T(g \circ u)$ have the same level sets and are thus equal. $\square$

Definition 4.13. *We say that an operator T acting on a set of continuous functions is* morphological *if it is monotone, translation and contrast invariant.*

Remark 4.14. In [81], Guichard and Morel call these operators *weakly* morphological, because they consider larger classes of contrast changes (upper semi-continuous). The main reason to accept discontinuous contrast changes is that edges in images are generally modeled by discontinuities. In this setting, natural images are not continuous and more general contrast changes are allowed. For our purpose, images are artificially defined as supporting functions to curves which are assumed continuous, and our weaker results will be sufficient.

We have just seen that a monotone set operator uniquely defines a monotone contrast invariant function operator. Since we want to prove an equivalence between sets operators and functions operators, we have to show the reverse property. To this purpose, we prove that if T is a monotone and contrast invariant defined on continuous functions, C is a closed Jordan curve and u a supporting function for C, then the zero level set of Tu does not depend on the particular choice of u.

We first use the following lemma proved in [38, 60].

Lemma 4.15. *Let u and v be continuous. Assume that their level sets are compact. Assume also that $\chi_0(u) = \chi_0(v)$. Then, there exists a continuous, nondecreasing function g such that $g(0) = 0$ and $v \geq g \circ u$.*

Proof. For $\lambda \in \mathbb{R}$, define $\tilde{g}(\lambda) = \inf_{\chi_\lambda(u)} v$. Since the level sets of u are nonincreasing, $\tilde{g}$ is nondecreasing. We also have $\tilde{g}(0) = \inf_{\chi_0(u)} v = \inf_{\chi_0(v)} v$ since by assumption, $\chi_0(u) = \chi_0(v)$. In addition, v is continuous implies that $\tilde{g}(0) = 0$. We also prove that $v \geq \tilde{g} \circ u$. Assume $v(x) \leq \lambda$. Then $\tilde{g}(u(x)) = \inf_{\chi_{u(x)}(u)} v \leq v(x) \leq \lambda$, since $x \in \chi_{u(x)}(u)$. As λ is arbitrary, we conclude that $v \geq \tilde{g} \circ u$. The unique problem is that $\tilde{g}$ may not be continuous (though we can prove that it is continuous at least at 0). To solve this problem, we replace $\tilde{g}$ by

$$g(\lambda) = \frac{2}{\lambda} \int_{\lambda/2}^{\lambda} \tilde{g}(t)\, dt$$

if $\lambda \geq 0$, and

$$g(\lambda) = \frac{1}{|\lambda|} \int_{2\lambda}^{\lambda} \tilde{g}(t)\, dt$$

if $\lambda \leq 0$. Then g is continuous, nondecreasing, with $g(0) = 0$, and this proves the lemma since $g \leq \tilde{g}$. $\square$

Proposition 4.16. *Let C be a closed curve. Assume that u and v are supporting functions for C and that their level sets are compact. Let T be a morphological operator. Then $Tu(x) \geq 0 \Leftrightarrow Tv(x) \geq 0$.*

Proof. By applying Lemma 4.15, we find g continuous such that $g(0) = 0$ and $v \leq g \circ u$. Since T is morphological, we have $Tv \leq T(g \circ u) = g \circ Tu$. Thus $Tu(x) \geq 0$ implies $Tv(x) \geq 0$. The reverse implication is obtained by reversing the roles of u and v. $\square$

Remark 4.17. The compactness assumption on the level sets is actually not crucial. Indeed, we are not really interested in the behavior of functions at infinity. Therefore, we can assume, for example, that our functions tend to $-\infty$ at infinity, making the level sets compact. Another possibility is to deal with periodic functions, or with the torus topology where closed sets are automatically compact. This corresponds to mirror reflections which is a standard trick to avoid border effects in image processing.

Corollary 4.18. *Let T be a morphological operator defined on a set of continuous functions whose level sets are compact. Then, it is possible to extend T to a morphological operator $\mathbb{T}$ defined on compact sets.*

Proof. Let X be a compact set and u be a continuous function, positive in X and negative out of X. By applying a contrast change, we can assume that the level sets of u are compact. We then define

$$\mathbb{T}(X) = \partial\chi_0(Tu).$$

This operator is obviously monotone and translation invariant. Besides, Proposition 4.16 above makes $\mathbb{T}(X)$ independent of the choice of u. $\square$.

Let us sum up the results above and the gain of this work. We started from an operator $\mathbb{T}$ acting on curves. By restricting $\mathbb{T}$ to closed Jordan curves, we can consider that $\mathbb{T}$ acts on compact sets. If we assume that $\mathbb{T}$ is continuous and monotone, we can then define an operator T acting on functions and whose action is to apply $\mathbb{T}$ on the level sets of functions. Moreover T is contrast invariant, that is, it commutes with continuous nondecreasing functions.
Conversely, if T is monotone and contrast invariant, then it also defines an operator defined on closed curves. Indeed, if C is a Jordan curve, it is the zero level line of a function u. (Take for u a supporting function of C.) Then the zero level set of Tu does not depend on u.
We conclude that there is a direct correspondence between sets and functions morphological operators (and this also justifies a posteriori the common denomination). Remark that translation invariance has not been used so far, but we kept the definition as given in Mathematical Morphology.

4.1.5 Characterization of monotone, contrast invariant operator

We now want to examine closely the behavior of monotone contrast invariant operators. From now on, the additional assumption that the sets and functions operators we consider are translation invariant is necessary. This hypothesis is natural since it means that the analysis does not depend on the position of the objects. The main result of this section is a characterization of morphological operators. These operators are nonlinear (since linear operators are not morphological), but, as a parallel, let us examine the linear case, for which such a result is well-known: any linear, continuous, translation invariant operator is a convolution. The convolution kernel may be evaluated by testing the operator on Dirac masses or on approximation of Dirac masses, and is called the impulse response of the operator. Moreover, the operator is rotation invariant if the kernel is radial. A simple result which has important consequences is that if we scale a radial kernel, the convolution at small scales is equivalent to add a quantity which is proportional to the laplacian. (See [81] for instance.) As a consequence, an iterated convolution tends to the solution of the heat equation when the scale tends to 0.

Does this characterization in the linear case have an equivalent for morphological operators? We shall see that the answer is yes and that geometric invariance (rotation, affine) is very easy to impose as well. It will be also easy to introduce a scale parameter so that the scaled operator tends to the identity. An important question is to determine the first order term following the identity. What can then be said of the iterated operator? Does it converge to an equation, and what kind of equation?

Theorem 4.19 (Matheron). *Let T be a morphological operator acting on a set of continuous functions whose level sets are compact. Then, there exists a family $\mathbb{B}$ of subsets of $\mathbb{R}^2$ called* structuring elements *such that*

$$Tu(x) = \sup_{B\in\mathbb{B}} \inf_{x\in B} u(x+y). \tag{4.9}$$

Proof. Let us define

$$\mathbb{B} = \{X \subset \mathbb{R}^2, \text{compact and such that } T(-\bar{d}_{\partial X})(0) \geq 0\},$$

where $\bar{d}_X$ is the signed distance function to ∂X. Remark that

$$X \in \mathbb{B} \text{ and } X \subset Y \Rightarrow Y \in \mathbb{B}, \tag{4.10}$$

because $X \subset Y \Rightarrow \bar{d}X \geq \bar{d}Y$ and T is monotone. We then prove that $\mathbb{B}$ satisfies (4.9). We have indeed

$$\begin{aligned} Tu(x) \geq \lambda &\Leftrightarrow \forall \mu < \lambda \qquad Tu(x) \geq \mu, \\ &\Leftrightarrow \forall \mu < \lambda \qquad T(u(x+\cdot) - \mu)(0) \geq 0. \end{aligned} \tag{4.11}$$

Let us consider the function $-\bar{d}_{\partial\chi_\mu(u)-x}$. This function and $u(\cdot + x) - \mu$ have obviously the same (upper) zero level set. By Prop. 4.16, $T(u(x+\cdot)-\mu)(0) \geq 0$ holds if and only if $T(-\bar{d}_{\partial\chi_\mu(u)-x})(0) \geq 0$ holds too. Thus

$$\begin{aligned} Tu(x) \geq \lambda &\Leftrightarrow \forall \mu < \lambda \qquad T(-\bar{d}_{\partial\chi_\mu(u)-x})(0) \geq 0, \\ &\Leftrightarrow \forall \mu < \lambda \qquad \chi_\mu(u) - x \in \mathbb{B}, && (4.12) \\ &\Leftrightarrow \forall \mu < \lambda \quad \exists B \in \mathbb{B} \qquad B \subset \chi_\mu(u) - x && (4.13) \\ &\Leftrightarrow \forall \mu < \lambda \quad \exists B \in \mathbb{B} \qquad \inf_{y\in B} u(x+y) \geq \mu && (4.14) \\ &\Leftrightarrow \sup_{B\in\mathbb{B}} \inf_{y\in B} u(x+y) \geq \lambda. && (4.15) \end{aligned}$$

Equation (4.13) follows from (4.10).This proves

$$Tu(x) \geq \lambda \Leftrightarrow \sup_{B\in\mathbb{B}} \inf_{y\in B} u(x+y) \geq \lambda,$$

and since λ is arbitrary, Tu is equal to the sup-inf operator. □

Remark 4.20. The family $\mathbb{B}$ has been called the kernel of the operator by Serra in [154]. Remark that it is not unique, since we do not change the sup-inf operator by adding elements of the form $B' = B \cup A$ where $B \in \mathbb{B}$ and A is any subset of $\mathbb{R}^2$.

Corollary 4.21. *A morphological operator also admits the* inf-sup *form*

$$Tu(x) = \inf_{B\in\mathbb{B}} \sup_{y\in B} u(x+y).$$

The family $\mathbb{B}$ is not the same as for the sup-inf form.

Proof. Introduce $T'(u) = -T(-u)$. Let g be nondecreasing and continuous. Then $T'(g(u)) = -T(-g(u)) = -T(g'(-u))$, where $g'(s) = -g(-s)$. Since g' is also a contrast change, we have $T'(g(u)) = -g'(T(-u)) = g(-T(-u)) = g(T'(u))$. Thus T' is morphological. By Thm. 4.19, we can find a family of structuring elements $\mathbb{B}$ such that

$$T'u(x) = \sup_{B\in\mathbb{B}} \inf_{y\in B} u(x+y),$$

and

$$T(u) = -T'(-u) = -\sup_{B\in\mathbb{B}} \inf_{y\in B} (-u(x+y)) = \inf_{B\in\mathbb{B}} \sup_{y\in B} u(x+y). \qquad \square$$

The characterization of Matheron also provides a convenient way to impose additional invariance properties. Indeed, is is clear that if the family $\mathbb{B}$ is isotropic invariant (that is to say, if R is any rotation, $B \in \mathbb{B}$ is equivalent to $RB \in \mathbb{B}$) then the operator T commutes with rotations. Notice that the reciprocal is almost true. Indeed, we have seen that we could choose $\mathbb{B} = \{X,\, T(-\bar{d}_{\partial X})(0) \geq 0\}$. Therefore, we can choose $\mathbb{B}$ with the same invariance properties as T. We will also use the affine invariance property, i.e. we choose $\mathbb{B}$ invariant with respect to the special linear group (the set of linear transforms with determinant equal to 1).

4.1.6 Asymptotic behavior of morphological operators

From Matheron's theorem, it is very easy to introduce a scale factor in morphological operators. Indeed, it suffices to shrink the family of structuring elements by a small number. In the following, we shall see that the curvature of the level lines appears in the analysis in a very natural way. It is first necessary to explain how it can be calculated.

Theorem 4.22 (Implicit functions theorem). *Let $u : \mathbb{R}^2 \to \mathbb{R}$ be a C^1 function and $x = (x_1, x_2) \in \mathbb{R}^2$. Assume that $Du(x) \neq 0$. Up to a rotation, we may assume that*

$u_{x_1}(x) = 0$. *Then, there exists a neighborhood* V *of* x *and a function* $\varphi : V \to \mathbb{R}$ *such that for all* $y = (y_1, y_2) \in V$*, we have the equivalence*

$$u(y) = u(x) \Leftrightarrow y_2 = \varphi(y_1).$$

Moreover, if u *is* C^k *with* $k \geq 1$*, then* φ *is also* C^k.

This theorem states that in the neighborhood of noncritical points, isolevel sets are smooth curves since they are locally the graph of smooth functions. (See Fig. 4.1.)

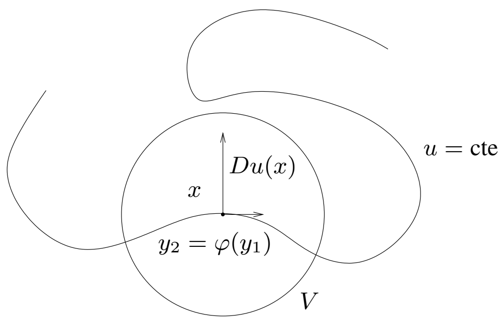


Fig. 4.1. Implicit function theorem

Proposition 4.23. *Let* u *be a* C^2 *function and* x *such that* $Du(x) \neq 0$*. Then in a small enough neighborhood of* x*, the set of points* y *such that* $u(y) = u(x)$ *defines a* C^2 *curve. The curvature of this curve at* x *is denoted* $\kappa(u(x)) = \operatorname{curv} u(x)$ *and is equal to*

$$\operatorname{curv} u(x) = \frac{1}{|Du|}\left(\Delta u - D^2 u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right)\right). \tag{4.16}$$

It is also equal to

$$\operatorname{curv} u(x) = \frac{u_{xx}u_y^2 - 2u_{xy}u_x u_y + u_{yy}u_x^2}{(u_x^2 + u_y^2)^{3/2}}. \tag{4.17}$$

Proof. The first part is a consequence of the implicit functions theorem. To prove the second part, we write the Taylor expansion of u at x. We can choose the axes of coordinates such the first axis is $Du/|Du|$. Then u can be expressed as

$$u(x+y) = u(x) + py_1 + \frac{1}{2}ay_1^2 + \frac{1}{2}by_2^2 + cy_1y_2 + o(|y|^2)$$

where $y = (y_1, y_2)$,

$$a = D^2u\left(\frac{Du}{|Du|}, \frac{Du}{|Du|}\right),\ b = D^2u\left(\frac{Du^\perp}{|Du|}, \frac{Du^\perp}{|Du|}\right),\ c = D^2u\left(\frac{Du}{|Du|}, \frac{Du^\perp}{|Du|}\right),$$

and $Du^{\perp} = (-u_y, u_x)$ is normal to Du with the same norm. Thus the equation $u(x+y) = u(x)$ becomes

$$py_1 = -\frac{1}{2}by_2^2 + o(y_2^2).$$

The curvature of the graph at $y_2 = 0$ is the same as the one of the limiting parabola i.e. $\frac{b}{p}$ which is the announced result. $\square$

Exercise 4.24. Prove that if u is C^2, we also have

$$\operatorname{curv} u = \operatorname{div} \frac{Du}{|Du|}.$$

Check that if $g \in C^2$ and increasing then $\operatorname{curv}(g \circ u) = \operatorname{curv} u$.

This expression is interesting since, in higher dimension, it is also equal to the mean curvature of the level surface of u (up to a multiplicative constant). Equation (4.16) still also holds while (4.17) is specific to the bidimensional case.

We then have the following result proved in [81].

Proposition 4.25. *Let T be a morphological operator which is also isotropic invariant. Let $\mathbb{B}$ a family of structuring elements related to T. We assume that $\mathbb{B}$ is bounded, that is, we can find $M > 0$ such that $B \subset D(0, M)$ (the disk with center 0 and radius M) for all $B \in \mathbb{B}$. For $h > 0$, let T_h the morphological operator associated to $h\mathbb{B}$. Set $H(b) = T(x_1 + bx_2^2)(0)$. Then if u is a C^3 function on $\mathbb{R}^2$, and $Du(x) \neq 0$*

1. *if $H(0) \neq 0$*
$$T_h u(x) = u(x) + hH(0)|Du(x)| + O(h^2).$$
2. *if $H(0) = 0$ then if $Du(x) \neq 0$ we have*

$$T_h u(x) = u(x) + h|Du|H(\frac{h}{2}\operatorname{curv}(u)(x)) + O(h^3).$$

Proof. By translation invariance, we assume that $x = 0$ and we shall instead use x as a variable until the end of the proof. As morphological invariance implies invariance with respect to addition of constants, we may also assume that $u(0) = 0$. Since $Du(x) \neq 0$, by rotation invariance, we may take the first axis of coordinates equal to $\frac{Du(0)}{|Du(0)|}$. First, let $H(0) \neq 0$. We write the Taylor expansion of u at the origin,

$$u(x) = px_1 + O(|x|^2).$$

Since all the structuring elements are included in the same ball, we have

$$T_h u(x) = T_h(px_1) + O(h^2) = hpT_1(x_1) + O(h^2) = h|Du|H(0) + O(h^2),$$

ending the proof in the case $H(0) \neq 0$.
From now on, we shall suppose that $H(0) = 0$. The estimate above is too crude

since we cannot say which one of $hH(O(h))$ and $O(h^2)$ is the larger. We write the Taylor expansion of u at the origin and detail the second order terms:

$$u(x) = px_1 + ax_1^2 + bx_2^2 + cx_1x_2 + O(|x|^3),$$

where $p = |Du(0)|$, $a = \frac{1}{2}u_{xx}(0)$, $b = \frac{1}{2}u_{yy}(0)$ and $c = u_{xy}(0)$. We then have

$$T_h u(0) = hpT_1\left(x_1 + \frac{h}{p}(ax_1^2 + bx_2^2 + cx_1x_2)\right) + O(h^3). \tag{4.18}$$

Before giving precise arguments, let us try to give some simple geometrical ideas to see how the infsup is attained. To this purpose, let us see the meaning of $T(x_1) = H(0) = 0$. The level lines of $(x_1, x_2) \mapsto x_1$ are vertical lines. First, no element of $\mathbb{B}$ is strictly included in a half-plane of the type $\{x_1 < a\}$ with $a < 0$, else $H(0)$ would be negative. On the other hand, for any $a > 0$, it is possible to find a structuring element which is included in $\{x_1 < a\}$, else $H(0)$ would be positive. In the computation of $T_h u(0)$, we see that the first order is controlled by x_1. Hence the infsup is attained for an element which is as close as possible to $\{x_1 < 0\}$, when x_1 and hx_2 will be of the same order. This shall eliminate the x_2^2 and x_1x_2 terms and the result shall follow. Let us now prove it.
Let us denote by $v(x)$ the polynomial $v(x) = x_1 + \frac{h}{p}(ax_1^2 + bx_2^2 + cx_1x_2)$. Since $|v - x_1| \leq Ch$ on every $B \in \mathbb{B}$ with $C = \frac{1}{p}(|a|+|b|+|c|)M^2$ which is *independent* of B, we also have by monotonicity and contrast invariance

$$-Ch \leq T_1 v(0) \leq Ch.$$

It is then clear that $x_1 < -2Ch$ implies $v(x) < -Ch$ and the supremum cannot be attained. For each structuring element, we can restrict the research of the supremum of v to the set of points such that $x_1 \geq -2Ch$. Otherwise said,

$$T_1 v(0) = \inf_{B \in \mathbb{B}} \sup_{\substack{x \in B \\ x_1 \geq -2Ch}} v(x).$$

Remark that for any B, the set $B \cap \{x_1 \geq -2Ch\}$ cannot be empty, else $H(0)$ would not be equal to 0. The supremum in the previous formula is not taken on an empty set.
On the other hand, if a structuring element meets $\{x_1 > 2Ch\}$, then the supremum on this element is larger than Ch. Thus, we can limit the research of the infsup to the structuring elements included in the half-plane $\{x_1 \leq 2Ch\}$. This means that

$$T_1 v(0) = \inf_{\substack{B \in \mathbb{B} \\ B \subset \{x_1 \leq 2Ch\}}} \sup_{\substack{x \in B \\ x_1 \geq -2Ch}} v(x).$$

On this parts of elements, we have also $v(x) = x_1 + h\frac{b}{p}x_2^2 + O(h^2)$, yielding $T_1 v(x) = hH(h\frac{b}{p}) + O(h^2)$. By replacing this into (4.18), we obtain

$$T_h u(0) = phH(h\frac{b}{p}) + O(h^3).$$

This concludes the proof since $b = \frac{1}{2}|Du| \operatorname{curv} u(0)$. $\square$

Remark 4.26. The function H has sometimes been called the impulse response of the morphological filter, but this terminology is not really used.

Remark 4.27. In the previous result, the estimate does not teach us anything at all if $H(s) = o(s^2)$ around 0. In this case, the estimates have to be thinner [145].

We can impose some stronger invariance properties. The following fundamental result gives the asymptotic behavior of affine invariant morphological operators.

Proposition 4.28 ([81]). *Let $\mathbb{B}$ a family of structuring elements. Assume that $\mathbb{B}$ is globally invariant under the action of the special linear group. We also suppose that every $B \in \mathbb{B}$ is closed, convex, symmetric with respect to the origin and with area equal to 1. For $h > 0$, we call $\mathbb{B}_h = h^{1/2}\mathbb{B}$ and IS_h and SI_h the morphological operators*

$$IS_h u(x) = \inf_{B \in \mathbb{B}_h} \sup_{y \in B} u(x+y),$$

$$SI_h u(x) = \sup_{B \in \mathbb{B}_h} \inf_{y \in B} u(x+y),$$

and $c_{\mathbb{B}} = IS_h(x_1 + x_2^2)$. Then $c_{\mathbb{B}} > 0$ and for any function $u \in C^3(\mathbb{R}^2)$, we have if $Du(x) \neq 0$

$$IS_h u(x) = u(x) + c_{\mathbb{B}} |Du| ((\operatorname{curv} u)^+)^{1/3} + o(h^{2/3}),$$

$$SI_h u(x) = u(x) - c_{\mathbb{B}} |Du| ((\operatorname{curv} u)^-)^{1/3} + o(h^{2/3}),$$

$$SI_h \circ IS_h u(x) = u(x) + c_{\mathbb{B}} |Du| (\operatorname{curv} u)^{1/3} + o(h^{2/3}).$$

As for Prop. 4.25, the proof of this result is based on the local expansion of u and we do not give it here. The tricky point is that T_h is not local a priori since the family of structuring elements is not bounded as it was in Prop. 4.25. Thus, it is not trivial to prove that T_h only depends on the gradient and the curvature of u at x. The argument used by Guichard and Morel in [81] is the localizability of the structuring elements. This is a geometric property asserting that the family of structuring elements may be restricted to a uniformly bounded one without any major effect.
More important is the meaning of this proposition. It means that nearly *all* affine invariant morphological operators are equivalent to a unique operator up to a multiplicative constant $c_{\mathbb{B}}$. As a consequence, it is not necessary to apply different types of structuring elements since they only change the constant $c_{\mathbb{B}}$.

Remark 4.29. The operators IS_h and SI_h are not self-dual in the mathematical morphology terminology. This means that they do not commute with contrast inversion $u \mapsto -u$. Precisely, $IS_h(-u) = -SI_h(u)$. The introduction of the alternate operator $SI_h \circ IS_h$ permits to partially retrieve the symmetry between u and $-u$ when the alternate operator is iterated many times.

Many other results of the kind above have been found for different families of structuring elements [27, 35, 126, 146]. We voluntarily restricted the study to $\mathbb{R}^2$ since we only consider plane curve evolution. Nevertheless, the analysis applies for the motion of hypersurfaces in higher dimensions.

4.1.7 Morphological operators yield PDEs

In this section, we give some heuristic arguments showing that a special class a PDEs emerges from the results of mathematical morphology above. The existence and uniqueness of the solutions of those PDEs and the rigorous link with mathematical morphology will be stated further.
First examine the case where the function H defined in Prop. 4.25 satisfies $H(0) \neq 0$. We define the function u_h by

$$u_h(x,t) = (T_h)^n u_0 \text{ if } t \in [nh, (n+1)h).$$

Then by Prop. 4.25, without taking any care of the smoothness of u_h, we find that u_h satisfies

$$\frac{u_h(x,t+h) - u_h(x,t)}{h} = H(0)|Du(x)| + O(h).$$

If we take h small, we see formally that u_h tends to satisfy the PDE

$$\frac{\partial u}{\partial t} = H(0)|Du|.$$

Now, if $H(0) = 0$, assume that H is homogeneous with degree α, $0 < \alpha < 2$. We also adopt the convention that the power function preserves the sign, that is $x^\alpha = x|x|^{\alpha-1}$. By the same heuristic argument as above, we see that u_h formally tends to satisfy the PDE

$$\frac{\partial u}{\partial t} = |Du|(\operatorname{curv} u)^\alpha. \tag{4.19}$$

The interpretation of this equation is the following: the level lines of the solution move with a normal velocity equal to $(\operatorname{curv} u)^\alpha$. To see this, let $u \in C^1(\mathbb{R}, C^2(\mathbb{R}^2))$ be a solution of (4.19). Let $x(t)$ be a point on a level line. Since $u(x(t),t)$ is constant, we derive

$$\frac{d}{dt}u(x(t),t) = \frac{\partial u}{\partial t} + x'(t) \cdot Du(x) = 0.$$

Since u is solution of (4.19), we obtain

$$x'(t) \cdot Du = -|Du|(\operatorname{curv} u)^\alpha(x(t),t).$$

But the normal $\mathbf{N}$ of the level line at x is $Du/|Du|$. Thus

$$x'(t) \cdot \mathbf{N} = -(\operatorname{curv} u)^\alpha(x(t),t),$$

meaning that the normal velocity of x is the desired function of the curvature. This analysis is also valid when $H(0) \neq 0$ and any point moves with a normal velocity equal to $H(0)$.

In the special case of affine invariant morphological operators, we see that the related PDE is

$$\frac{\partial u}{\partial t} = |Du|(\mathrm{curv}\, u)^{1/3}.$$

It implies that the level lines moves following Sapiro and Tannenbaum affine curve shortening. (See [152] and Chap. 3 of these notes.)

For the moment, we do not go any further in the analysis of these equations. In Sect. 4.3, we shall give the main elements of a theory giving existence and uniqueness. We shall then be able to establish some convergence results justifying the calculations above.

4.2 Curve evolution and Scale Space theory

4.2.1 Multiscale analysis are given by PDEs

Let us now adopt a different point of view which is to consider that an image filtering is described by a partial differential equation. In fact, PDEs have been used in image filtering for about twenty years [102, 117, 169]. David Marr, looking for tools to recover the *raw primal sketch* of images, understood that natural images were very noisy. Defining the edges as large gradient areas was therefore impossible without clearing out tiny oscillations like noise and texture. The first model was linear, and consequently based on convolutions. It was then remarked that all isotropic convolutions were asymptotically equivalent and all led to a limiting equation which is the heat equation. Basic numerical experiments show that the heat equation had too strong a blurring effect and that edges were completely smoothed out. From that observation, many nonlinear models were introduced to filter images while preserving edges as well as possible. The pioneering work was achieved by Perona and Malik [147], but the model they proposed does not have nice mathematical properties. (It is however numerically quite efficient!) A well posed mathematical setting was later proposed by Alvarez, Lions, Morel, Catté and Coll in [5, 36]. The approach was still to find the best possible equation that does not smooth through edges by using an anisotropic (and thus nonlinear) diffusion. Confronted to the appearance of nearly infinitely many models, Alvarez, Guichard, Lions and Morel developped in [4, 79] an axiomatic approach: being given some desired properties of invariance, what are the corresponding partial differential equations, and is it possible to find them all? In this paragraph, we simply recall these conditions and the derived PDEs. Basically, we required the same invariance properties as for morphological operators which are sound in the shape simplification point of view. Thus, logically enough, the PDEs will be the same that appear in the asymptotic behavior of morphological operators. (Prop. 4.25)

The axiomatization starts from the concept of *multiscale analysis* or *scale space*. It consists in a family $(T_t)_{t>0}$ of operators, where we assume that T_t satisfies the following properties:

1. causality: $T_0 = Id$ and if $s < t$ then there exists a transition operator $T_{s,t}$ such that $T_t = T_{s,t} \circ T_s$.

2. local monotonicity or comparison principle: if $u(y) < v(y)$ in a small neighborhood of x except maybe at x, then for any t and for h small enough, we have $T_{t,t+h}u(x) \leq T_{t,t+h}v(x)$.
3. regularity: there exists a function $F : \mathcal{S}^2 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R}$ such that if $u(x) = c + (p, x - x_0) + \frac{1}{2}A(x - x_0, x - x_0)$ is a quadratic form then

$$\lim_{h \to 0} \frac{(T_{t,t+h}u)(x_0) - u(x_0)}{h} = F(A, p, c, x_0, t). \tag{4.20}$$

   We assume that F is continuous everywhere except at the points where its second argument is equal to 0.

These conditions are quite natural: the first one asserts that no information is created. The second locally preserves the order given by brightness. The last one requires that second order characteristics locally determine the filtering process. With these conditions fulfilled, the regularity axioms holds more generally for C^2 functions.

Lemma 4.30. *Assume that $T_{t,t+h}$ satisfies the local monotonicity and regularity axioms. Let u be a C^2 function with $Du(x) \neq 0$. Then*

$$\lim_{h \to 0} \frac{(T_{t,t+h}u)(x) - u(x)}{h} = F(D^2u(x), Du(x), u(x), x, t). \tag{4.21}$$

Proof. Set $c = u(x), p = Du(x)$ and $A = D^2u(x)$. Denote by I the identity matrix and let v_ε be defined by

$$v_\varepsilon(y) = c + (p, y - x) + \frac{1}{2}(A - \varepsilon I)(y - x, y - x).$$

In the same way, let

$$w_\varepsilon(y) = c + (p, y - x) + \frac{1}{2}(A + \varepsilon I)(y - x, y - x).$$

Then $u(x) = v_\varepsilon(x) = w_\varepsilon(x)$ and for $y \neq x$ close to x, we have

$$v_\varepsilon(y) < u(y) < w_\varepsilon(y).$$

For small h, the local comparison principle implies that

$$T_{t,t+h}v_\varepsilon(x) \leq T_{t,t+h}u(x) \leq T_{t,t+h}w_\varepsilon(x).$$

Subtracting $u(x)$, dividing by h and passing to the limit yields

$$\begin{aligned} F(A - \varepsilon I, p, c, x, t) &\leq \liminf \frac{T_{t,t+h}u(x) - u(x)}{h} \\ &\leq \limsup \frac{T_{t,t+h}u(x) - u(x)}{h} \\ &\leq F(A + \varepsilon I, p, c, x, t). \end{aligned} \tag{4.22}$$

Since, this is true for any $\varepsilon > 0$, we can let ε tend to 0 and obtain the result by using the continuity of F. $\square$

The local comparison and the regularity also give the following basic fact, which is fundamental for the equations we shall study.

Proposition 4.31 (and Definition). *If $T_{t,t+h}$ satisfies the local monotonicity and the regularity axioms then F is nondecreasing with respect to its first argument, that is to say, $A \leq B$ (for the partial order on symmetric matrices) implies*

$$F(A, p, c, x, t) \leq F(B, p, c, x, t).$$

We say that F is an elliptic degenerate operator.

Proof. In fact, we nearly proved it in the previous lemma. We consider

$$u(y) = c + (p, y - x) + \frac{1}{2} A(y - x, y - x),$$

and

$$v_\varepsilon(y) = c + (p, y - x) + \frac{1}{2}(B + \varepsilon I)(y - x, y - x).$$

We can apply the local monotonicity to these quadratic forms. We then form the quotient appearing in the regularity assumption and get

$$F(A, p, c, x, t) \leq F(B + \varepsilon I, p, c, x, t).$$

We let $\varepsilon \to 0$ and use the continuity of F. $\square$

Remark 4.32. Let us shortly comment the terminology "degenerate elliptic" operator. This has to be opposed to "uniform ellipticity". We say that F is uniformly elliptic if we can find two positive constants λ and Λ such that, for any (A, p, c, x, t),

$$\lambda I \leq \frac{\partial F}{\partial A}(A, p, c, x, t) \leq \Lambda I. \tag{4.23}$$

Most of the uniqueness and regularity results for elliptic equations of the type

$$F(D^2 u, Du, u, x) = 0$$

are known for uniformly elliptic equations. (See [75, 160, 161, 162, 163].)

The three basics axioms above suffice to claim that a multiscale analysis is given by a PDE. We give the following result a bit prematurely since we have not defined yet the good notion of solution for the equation in question.

Proposition 4.33. *Let $u_0 \in BUC(\mathbb{R}^2)$ (bounded and uniformly continuous). Let $u : \mathbb{R}^2 \times (0, +\infty) \to \mathbb{R}$ be defined by $u(x, t) = (T_t u_0)(x)$. Let us assume that the scale space is invariant with respect to addition of constants, that is $T_{t,t+h}(u + c) = c + T_{t,t+h} u$. Then u is viscosity solution of the equation*

$$\frac{\partial u}{\partial t} = F(D^2 u, Du, u, x, t). \tag{4.24}$$

The concept of viscosity solution will be the topic of the next section and we do not enter into details here. What one should remind is that this equation is an evolution equation where the evolution parameter is the scale (i.e. the amount of filtering). It is also *degenerate parabolic* since the second member is elliptic. We prove the result p. 79.

4.2.2 Morphological scale space

The purpose of this section is to push the axiomatic approach forward by selecting, among all the possible parabolic degenerate equations, the ones that may give interesting image filtering. As in the case of curve evolution, the requirements are given in terms of invariance, and precisely

- translation invariance: by denoting $\tau_a u$ the function $u(\cdot - a)$, we require that $T_{t,t+h}(\tau_a u) = \tau_a(T_{t,t+h}u)$.
- isotropic invariance: if R is a plane rotation and Ru the function $u(R^t\cdot)$, then $T_{t,t+h}(Ru) = R(T_{t,t+h}u)$.
- affine invariance: if A is an invertible affine transform, and $Au(x) = u(A^{-1}x)$, then $T_{s,t}(Au) = AT_{\lambda(A,s),\lambda(A,t)}u$ for some function λ increasing with respect to t.
- contrast invariance: if g is a continuous nondecreasing function then $T_{t,t+h}(g \circ u) = g \circ (T_{t,t+h}u)$.
- contrast reversal invariance: $T_{t,t+h}(-u) = -T_{t,t+h}u$.

In the image processing viewpoint, these properties means that the filtering processing should not depend on the position and the sensibility of the camera. Affine mappings are a fine approximation of projective mappings [51, 65] for small deformations or far objects. We say that a scale space in *morphological* if it is translation and contrast invariant.
Note that the affine invariance property needs special attention concerning the filtering scale. Indeed, objects can be shrinked or expanded and this influences the filtering scale. (See Rem. 4.35 below.)
In [4], Alvarez, Guichard, Lions and Morel gave necessary conditions for the equations of the type of (4.24) to satisfy the invariance properties above.[2] The computations are a bit long but straightforward. In particular, they do not use the Lie groups techniques, but only an extensive use of the chain rule. What really matters is that the class of invariant equations is remarkably small. In order not to cut the thread of the exposure, we give this proof in annex.

Theorem 4.34 ([4]). *The isotropic morphological scale spaces are given by equations of the form*
$$\frac{\partial u}{\partial t} = |Du| G(t \operatorname{curv} u), \tag{4.25}$$
where G is continuous and nondecreasing with respect to its first argument.
In particular, there exists a unique affine morphological scale space. It is described by the equation
$$\frac{\partial u}{\partial t} = |Du|(t \operatorname{curv} u)^{1/3}. \tag{4.26}$$

[2] The approach of Chen, Giga and Goto [38] may also be considered as axiomatic by their definition of geometric equations.

Remark 4.35. In fact, the equations are unique up to a renormalization of the scale. To impose real uniqueness, we can add some additional invariance property with respect to scale changes. Guichard and Morel [81] proved that under very mild conditions, the normalization of the scale as in Thm. 4.34 was always possible. For instance [79], the affine invariant equation is uniquely determined if we impose that the renormalization of the scale in the definition of affine invariance is given by $t'(A,t) = |\det A|^{1/2} t$. In the proof of Thm. 4.34 we give in appendix, we shall skip this problem of renormalization.

Remark 4.36. Equation (4.26) has been independently discovered at about the same time as the affine scale space of Sapiro and Tannenbaum [152]. The approaches were quite different since Sapiro and Tannenbaum used some arguments of affine geometry to derive their equation (they only later used a systematic approach with Olver), while Alvarez et al. focuses on monotonicity (in which case the equations must be of second order) and started from axiomatic arguments. The fact that both completely different points of views led to the same equation is all the more beautiful.

Exercise 4.37. We are going to prove that the curvature is degenerate elliptic (not uniformly elliptic).

1. By using (4.16), prove that

$$\operatorname{curv} u = F(D^2 u, Du, u, x),$$

   with

$$F(A,p,c,x) = \frac{1}{|p|}\left(\operatorname{Tr} A - A\left(\frac{p}{|p|}, \frac{p}{|p|}\right)\right).$$

2. Prove that for all $\lambda \in \mathbb{R}$

$$F(A + \lambda p \otimes p, p, c, x) = F(A,p,c,x). \tag{4.27}$$

3. Conclude.
4. By using the usual scalar product in the set of symmetric matrices $(A,B) = \operatorname{Tr}(AB)$, prove that (4.27) is a consequence of the more general result

$$\frac{\partial F}{\partial A} = \frac{1}{|p|}\left(I - \frac{p \otimes p}{|p|^2}\right). \tag{4.28}$$

We also conclude that the morphological scale spaces are degenerate parabolic equations.

4.3 Viscosity solutions

We are now interested with mathematical results of existence and uniqueness of the solution of the equation

$$\frac{\partial u}{\partial t} = |Du| G(\operatorname{curv} u), \tag{4.29}$$

where G is continuous and nondecreasing[3]. The second order term is nondecreasing with respect to the second derivative. (The equation is degenerate parabolic of second order.) It may be singular, i.e. the right hand term may not be defined even for very smooth functions. This is the case at critical points if G has a superlinear growth. Equations of the type of (4.29) are also called "generalized motion by curvature". A general theory was made for general elliptic second order, nonlinear equations of the form

$$F(D^2 u, Du, u, x) = 0, \tag{4.30}$$

where F is nonincreasing with respect to its first argument. The first order case (that is, F does not depend on $D^2 u$) was introduced by Crandall and Lions [44, 110] for solving Hamilton-Jacobi equations. A very nice monograph by Barles [17] gives most of the first order case results. The breakthrough for the second order case was achieved by Jensen [95] who proved that, under large conditions, (4.30) satisfied a maximum principle. His results were afterwards completed by many important papers. (See e.g. [88, 92, 97].) Regularity was later studied by Trudinger [160, 161, 162, 163] by extending results of linear theory for uniformly elliptic equations. There also exist several reviews or books for the second order case. Maybe the first synthesis article is the famous User's guide [43]. The reader will also find complements in a book by Fleming and Soner [67] and an excellent review by Crandall (as well as examples of applications) in [52]. Even more recently, Giga also dedicated a book to generalized motion by curvature [72].

Curvature evolutions of level sets were studied in the beginning of the 90s'. The mean curvature motion corresponding to the particular case $G(x) = x$ in (4.29) was particularly under focus. It is the critical behavior for G between regularity and singularity. In this case, the equation is quasilinear. Results of existence and uniqueness were proved independently in [38] and [60]. (In fact the results in [38] handles any motion by curvature with a velocity with sublinear growth.) See also [61, 62, 63, 71, 73] for extended and additional results. Those results are no longer valid when G has a superlinear growth at infinity. This case was eventually solved in [76, 94] by using two different methods.
In this section, we give the proof of existence and uniqueness of viscosity solution for equations of type (4.29). The proof of uniqueness is a consequence of a maximum principle, which is a very standard property for linear and quasilinear elliptic and parabolic equations and which turns to be true for (4.29). We adopt the same point of view as in [94]. Existence will also follow from the maximum principle by the so-called Perron's Method.

[3] The equations of Thm. 4.34 do not exactly follow the model of (4.29) where the scale factor t completely disappeared from the right hand side. Nevertheless, existence and uniqueness results stay unchanged. Moreover, when the normal velocity is a power of the curvature, a change of variable in t eliminates the dependence of G upon t.

4.3.1 Definition of viscosity solution

The necessity to introduce viscosity solutions is that classical solutions (that is twice differentiable) of (4.29) do not exist in general. For the particular case of geometric motions, we do not even want any solution to be smooth. For instance, $g \circ u$ and u have the same level sets even if g is not smooth. Since the evolution of u shall be given by the motion of its level sets, we would like to say that $g \circ u$ is a (non smooth) solution, whenever u is solution. Therefore, it is natural to look for a notion of weak solution. We could think at first of distributional solutions. However, (4.29) cannot be put into a divergence form and integration by parts, when we multiply the equation by a smooth test function, does not lead to interesting results, mainly because the equation is nonlinear. Hence, we have to find another notion of weak solution. There is a similarity in the sense that the definition will consist in applying the differentiations on smooth and adequate test functions. Nevertheless, the definition of solutions (the so-called viscosity solutions) will completely rely on a comparison principle, which itself holds because the equation is parabolic. In order to motivate the definition, let us consider the academic case of the heat equation for which we know that a smooth solution exists. Let u satisfy

$$\frac{\partial u}{\partial t} = \Delta u,$$

and let φ be a C^2 function such that $u - \varphi < 0$ everywhere except at some point (x, t) at which it is equal to 0. Since, $u - \varphi$ is smooth, then, at (x, t), $u_t = \varphi_t$ and $Du = D\varphi$. Now, the Laplacian is elliptic and Taylor expansion yields $\Delta(u - \varphi) \leq 0$. Since u is solution of the heat equation, this implies

$$\frac{\partial \varphi}{\partial t}(x, t) - \Delta\varphi(x, t) \leq 0. \tag{4.31}$$

Remark that u has vanished from this expression. We say that u is subsolution if 4.31 holds whenever $u - \varphi$ has a maximum at (x, t). Similarly, we say that u is supersolution if the reverse inequality holds when φ touches u from above at (x, t). The important result of this section is that those inequalities ensure the existence and the uniqueness of a solution. (A function which is both sub- and supersolution.) The definition was introduced by Crandall and Lions [44] and many papers on uniqueness (which essentially relies on the maximum principle) were published in the ten following years. Possible discontinuities of the involved differential operators were also studied and this allowed to solve the existence and uniqueness for (4.29). In particular, concerning the motions by curvature, existence and uniqueness results were first proved by Evans and Spruck [60] for the mean curvature motion and at the same time by Chen, Giga and Goto [38] when G (the velocity function in (4.29)) has sublinear growth. These results were next extended by Ishii and Souganidis when G has arbitrary growth. In these lectures, we shall directly give the definition of viscosity solution which is adapted to the possible superlinear growth of G that makes the elliptic operator undefined at critical points. This definition was given by Ishii and Souganidis in [20].

In all what follows, we assume that G is a nondecreasing and continuous function. We first introduced the class of test functions that will be compared to solutions.

Let $\mathcal{F}(G)$ the set of functions $f \in C^2([0, +\infty))$, such that

- $f(0) = f'(0) = f''(0) = 0$,
- for $r > 0$, $f''(r) > 0$,
- $\lim_{r\to 0} f'(|r|)G(\frac{1}{r}) = 0$.

Remark 4.38. The fact that $\mathcal{F}(G)$ is nonempty is obvious since G is continuous.

Remark 4.39. If G has a sublinear growth at infinity, the last condition is useless and the functions in $\mathcal{F}$ are simply radial with zero second derivative at the origin and strictly convex. This is the case for the affine morphological scale space as well as for the mean curvature motion.

Definition 4.40. *Let $\varphi : \mathbb{R}^2 \times [0, +\infty)$ be a C^2 compactly supported function. We say that φ is admissible for G (or is a test function for G) if for any point (x, t) such that $D\varphi(x, t) = 0$ there exists a disk $D((x, t), \delta)$, a function $f \in \mathcal{F}(G)$ and $\omega \in C((0, +\infty))$ satisfying $\lim_{r\to 0} \frac{\omega(r)}{r} = 0$, such that*

$$\forall (y, s) \in D((x, t), \delta) \quad |\varphi(y, s) - \varphi(x, t) - \varphi_t(x, t)(s-t)| \le f(|y-x|) + \omega(|s-t|).$$

We denote by $\mathcal{A}(G)$ the set of admissible functions.

The set $\mathcal{A}(G)$ is defined in such a way that the flatness of test functions absorbs the singularity of G at critical points. Again, if G grows sublinearly, this precaution is useless.
For any $u : \mathbb{R}^2 \to \mathbb{R}$, we denote by u^* and u_* its upper and lower semicontinuous envelope (that is the smaller (resp. larger) upper (resp. lower) semicontinuous function that is larger (resp. smaller) than u)

$$u^*(x, t) = \limsup_{(y,s)\to(x,t)} u(y, s) \text{ and } u_*(x, t) = \liminf_{(y,s)\to(x,t)} u(y, s).$$

We are now ready to define the notion of solution for (4.29).

Definition 4.41. *We say that a bounded function $u : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ is a viscosity subsolution of (4.29) if for all admissible $\varphi \in \mathcal{A}(G)$, if $u^* - \varphi$ admits a strict maximum at (x, t), then*

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(x, t) - D\varphi(x, t)G(\mathrm{curv}(\varphi)(x, t)) \le 0 & \text{if } D\varphi(x, t) \ne 0 \\[2ex] \dfrac{\partial \varphi}{\partial t}(x, t) \le 0 & \text{if } D\varphi(x, t) = 0. \end{cases}$$

In the same way, we say that u is supersolution if for all admissible $\varphi \in \mathcal{A}(G)$, if $u_* - \varphi$ admits a strict minimum at (x, t), then

$$\begin{cases} \dfrac{\partial \varphi}{\partial t}(x,t) - D\varphi(x,t)G(\mathrm{curv}(\varphi)(x,t)) \geq 0 & \textit{if } D\varphi(x,t) \neq 0 \\ \\ \dfrac{\partial \varphi}{\partial t}(x,t) \geq 0 & \textit{if } D\varphi(x,t) = 0. \end{cases}$$

We say that u is solution if it is both super- and subsolution.

In the following, we shall say that a test function φ touches u from above (resp. below) if it satisfies the condition for testing the subsolution (resp. supersolution) statement.

It is natural to check that viscosity solutions are consistent with classical solutions.

Proposition 4.42. *Let u be a C^2 viscosity solution of (4.29). Then u is a classical solution.*

Proof. Let $(x,t) \in \mathbb{R}^2$. If $Du(x) = 0$ and the equation is degenerate, we have nothing to prove since it is not classically defined. Thus assume that either the equation is not degenerate or that $Du(x,t) \neq 0$. Introduce

$$\varphi^+(y,s) = u(y,s) + |s-t|^2 + |y-x|^4.$$

Then $\varphi^+ > u$ in a neighborhood of (x,t) except at (x,t). Since $Du(x) = D\varphi^+(x)$, $u_t(x) = \varphi^+_t(x)$ and $D^2u(x) = D^2\varphi^+(x)$ and u is subsolution, we obtain

$$\frac{\partial u}{\partial t}(x,t) \leq |Du|G(\mathrm{curv}\, u)(x,t).$$

In the same way, we introduce φ^- touching u from below and prove that u is a classical supersolution. □

Remark that we did not use the parabolicity of the equation, whereas it is necessary in the following opposite proposition.

Proposition 4.43. *Let u be a classical solution of (4.29). Then u is a viscosity solution.*

Proof. Let us prove that u is subsolution. Let φ be an admissible test function touching u from above at (x,t). Then $Du(x,t) = D\varphi(x,t)$, $u_t(x,t) = \varphi(x,t)$ and $D^2u(x,t) \leq D^2\varphi(x,t)$. The parabolicity yields

$$\frac{\partial u}{\partial t} - |Du|G(\mathrm{curv}\, u) \geq \frac{\partial \varphi}{\partial t} - |D\varphi|G(\mathrm{curv}\, \varphi).$$

But the left-hand term is equal to 0 because u is a classical solution. Hence, u is subsolution.
In the same way, we prove that u is supersolution, and this implies that it is a viscosity solution. □

Finally, we can prove why a causal, monotone and regular multiscale analysis is given by the viscosity solution of a parabolic equation.

Proof (of Prop. 4.33). Assume, to simplify, that F is continuous everywhere. Let $u(x,t) = (T_t u_0)(x)$ and let us prove that it is viscosity solution of

$$\frac{\partial u}{\partial t} = F(D^2u, Du, u, x, t).$$

Let φ touching u from above at (x,t). For (y,s) close to (x,t), we then have $u(y,s) < \varphi(y,s)$. We apply $T_{t-h,t}$ on both sides of this inequality and take the value at $(x, t-h)$. By local monotonicity, we have

$$T_{t-h,t}u(x,t-h) \leq T_{t-h,t}\varphi(x,t-h).$$

But, by definition of u and the transition operator (here, we use the causality principle), the left hand term is nothing but $u(x,t)$, which is also equal to $\varphi(x,t)$. Thus

$$\varphi(x,t) \leq T_{t-h,t}\varphi(x,t-h).$$

We subtract $\varphi(x,t-h)$ on both sides and divide by h. The left hand term tends to $\varphi_t(x,t)$. It is not possible to pass to the limit by using (4.21), because we take the value at a point that depends on h. But since φ is smooth, we can use an approximation argument. For any $\varepsilon > 0$, we write that

$$\varphi(y,t-h) \leq \varphi(x,t) + (Du, y-x) + \frac{1}{2}(D^2u + \varepsilon Id)(y-x, y-x) - hu_t + o(h),$$

in a neighborhood of (x,t), the $o(h)$ terms being independent of y. (All the derivatives of u are taken at (x,t).) We then apply $T_{t-h,t}$ on each side and subtract $u(x,t-h) = u(x,t) - hu_t + o(h)$. By using the invariance by addition of constant, the u_t terms cancel on the right hand side. By letting h then ε tend to 0, we conclude all the same that

$$\frac{\partial \varphi}{\partial t} \leq F(D^2\varphi, D\varphi, \varphi, x, t).$$

This proves that u is viscosity subsolution. The same arguments can be adapted to prove that u is supersolution. If the equation is degenerate, then we can easily check (by taking admissible test functions) that the result remains unchanged. □

Remark 4.44. Another simple thing that shall be useful in the sequel is the following: in the definition of viscosity subsolution (resp. supersolution), when the gradient of the test function is equal to 0 at (x,t), then it suffices to check the desired property for test functions of the form

$$\psi(y,s) = g(s) + f(|y-x|)(\text{ resp. } -f(|y-x|)).$$

where $f \in \mathcal{F}(G)$. Indeed, by assumption, if φ is admissible with $D\varphi(x,t) = 0$, we have

$$
\begin{aligned}
\varphi(y,s) \leq & \varphi(x,t) + \varphi_t(x,t)(s-t) + \omega(|s-t|) + f(|y-x|) \\
& (\text{ resp. } \geq \varphi(x,t) + \varphi_t(x,t)(s-t) - \omega(|s-t|) - f(|y-x|)),
\end{aligned}
\tag{4.32}
$$

for some $f \in \mathcal{F}(G)$ and some continuity modulus ω such that $\omega(r) = o(r)$. Let us call $\psi(x,t)$ the right hand term. This function is of the desired form and $\psi_t(x,t) = \varphi_t(x,t)$. Moreover if $u - \varphi$ has a local maximum (resp. minimum) at (x,t), the same holds for $u - \psi$.

4.3.2 Proof of uniqueness: the maximum principle

In order to prove this fundamental result, we follow [94]. To simplify the notations, we denote by Ω_T the set $\mathbb{R}^2 \times (0,T)$ and by $\partial_p \Omega_T$ the set $\mathbb{R}^2 \times \{0\}$.
Before going to the result, we give the following lemma, known as the Theorem on Sums in the excellent review by Crandall [42]. It has first been proved by Ishii [88] and can also be found in the classical User's Guide [43] and in Giga's recent lecture notes [72]. In this result, we denote by $\nabla u \in \mathbb{R}^3$ the gradient of u with respect to its three variables. In the same way, $\nabla^2 u$ is obtained by taking all the second derivatives (not only the spatial ones).

Lemma 4.45 (Theorem on Sums). *Let u and v be bounded and uniformly continuous. Let $\phi : \Omega_T^2 \to \mathbb{R}$ be a C^2 function and*

$$
w(x,t,y,s) = u(x,t) - v(y,s) - \phi(x,t,y,s)
$$

be such that w has a strict maximum at $(\hat{x}, \hat{t}, \hat{y}, \hat{s})$. Then, for all $\varepsilon > 0$, there are two 3×3 symmetric matrices X and Y, two sequences of C^2 functions u_n and v_n satisfying

- *u_n touches u from above at (x_n, t_n) and $(x_n, t_n) \to (\hat{x}, \hat{t})$,*
- *v_n touches v from below at (y_n, s_n) and $(y_n, s_n) \to (\hat{y}, \hat{s})$,*

and such that

$$
(\nabla u_n, \nabla^2 u_n) \to (\nabla_{x,t}\phi, X), \tag{4.33}
$$

$$
(\nabla v_n, \nabla^2 v_n) \to (-\nabla_{y,s}\phi, Y), \tag{4.34}
$$

and

$$
-\left(\frac{1}{\varepsilon} + \|A\|\right) I \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq A + \varepsilon A^2, \tag{4.35}
$$

where $A = \nabla^2_{x,t,y,s}\phi$.

Let us give a few words about this result. If u and v are smooth, then the proof is quite obvious. But u and v are only semi-continuous, and the characterization of a maximum of w is not so simple. The proof of the Theorem on Sums relies on a regularization of u and v by inf-convolution [107]. The inf-convolution of a function is only Lipschitz continuous, but it is semi-convex. Hence, by a classical result of

Alexandrov [1], it is twice differentiable almost everywhere and we can apply the trivial smooth version of the Theorem on Sums. The difficulty is to show that we can pass to the limit when the regularization parameter tends to 0.

Finally, we shall use the following notation in the proof of the maximum principle. We call $F : \mathcal{S}^3 \times \mathbb{R}^3 \to \mathbb{R}$ the unique function such that for any admissible u, we have

$$F(\nabla^2 u, \nabla u) = |Du| G(\operatorname{curv} u). \tag{4.36}$$

Theorem 4.46 (Maximum principle). *Let u and v be bounded and respectively upper and lower semicontinuous in $\mathbb{R}^2 \times [0,T]$. Assume that u is subsolution of (4.29) and that v is supersolution of (4.29) in $\mathbb{R}^2 \times (0,T)$.*
Assume that

$$\limsup_{r\downarrow 0} \{u(\xi) - v(\zeta),\ (\xi,\zeta) \in \partial_p \Omega_T \times \Omega_T \cup \Omega_T \times \partial_p \Omega_T,\ |\xi - \zeta| \le r\} \le 0. \tag{4.37}$$

Then

$$\forall (x,t) \in \mathbb{R}^2 \times [0,T] \quad u(x,t) \le v(x,t).$$

We first prove a preliminary lemma allowing to extend the conditions required for viscosity solutions up to T.

Lemma 4.47. *[37] Let u be bounded on $\mathbb{R}^2 \times [0,T]$ and subsolution of (4.29) in $\mathbb{R}^2 \times (0,T)$. Then u is subsolution up to T, that is if φ is admissible for G and $u - \varphi$ has a local maximum at $(x,t) \in \mathbb{R}^2(0,T]$, then*

$$\begin{cases} \varphi_t(x,T) + |D\varphi| G(\operatorname{curv} \varphi)(x,T) \le 0 & \textit{if } D\varphi(x,T) \neq 0 \\ \varphi_t(x,T) \le 0 & \textit{otherwise}. \end{cases}$$

Proof. The interest of the lemma is for maximum points at (x,T) for $x \in \mathbb{R}^2$. In this case, we define $u_\varepsilon(x,t) = u(x,t) - \frac{\varepsilon}{T-t}$. If ε is small enough, $u_\varepsilon - \varphi$ admits a maximum at $(x_\varepsilon, t_\varepsilon)$, with $(x_\varepsilon, t_\varepsilon) \to (x,T)$ when ε tends to 0. Since u is subsolution, we have

$$\varphi_t(x_n,t_n) + \frac{\varepsilon}{(T-t_n)^2} + |D\varphi| G(\operatorname{curv} \varphi)(x_n,t_n) \le 0$$

if $D\varphi(x_n,t_n) \neq 0$, implying

$$\varphi_t(x_n,t_n) + |D\varphi| G(\operatorname{curv} \varphi)(x_n,t_n) \le 0.$$

If $D\varphi(x_n,t_n) = 0$, we have $\varphi_t(x_n,t_n) + \frac{\varepsilon}{(T-t_n)^2} \le 0$, yielding $\varphi_t(x_n,t_n) \le 0$. We then let n go to $+\infty$ and we obtain the result. □

Remark 4.48. The same result is valid for supersolution by reversing the sign of the inequalities.

Proof (of Thm. 4.46). The proof relies on some standard arguments [38, 42, 43] which were adapted in [94] to the possible singularity of the operator at critical points. Assume that

$$\theta \equiv \sup_{(x,t)\in\mathbb{R}^2\times[0,T)} (u(x,t) - v(x,t)) > 0. \tag{4.38}$$

Let us choose a function $f \in \mathcal{F}(G)$. We then define

$$\phi_1(x,t,y,s) = u(x,t) - v(y,s) - \alpha f(|x-y|) - \alpha(s-t)^2 - \lambda t - \lambda s \tag{4.39}$$

where α and λ are positive. This dedoubling of the variables is usual for the proof of the maximum principle and the second part of the function (after the $v(y,s)$ term) plays the role of test function both for u and v. Thus, it must be adapted to the possible singularity of the curvature term. Choosing $f \in \mathcal{F}(G)$ is thus completely natural.
Since $s,t \leq T$, by using (4.38) and (4.39), we may assume that

$$\theta_1 \equiv \sup\{\phi_1(x,t,y,s), (x,t),\ (y,s) \in \mathbb{R}^2 \times (0,T)\} > 0 \tag{4.40}$$

if λ is chosen small enough.
Let M be a bound of $|u|$ and $|v|$ (which are bounded by assumption).
If $\phi(x,t,y,s) \geq 0$, we then have

$$f(|x-y|) \leq \frac{2M}{\alpha} \text{ and } |t-s|^2 \leq \frac{2M}{\alpha}. \tag{4.41}$$

By using the hypothesis on the values of u and v near the parabolic boundary together with the fact that u and v are upper and lower semi-continuous, we can choose α large enough and t_0 small enough such that

$$\sup\{\phi_1(x,t,y,s),\ t,s \leq t_0\} < \frac{\theta_1}{2}.$$

Thus, we may assume that s and t are larger than t_0. From now on, α and λ are fixed. We then have to distinguish two cases whether or not the supremum of ϕ_1 is approached for x close to y, since in this case the gradient of the test function goes to zero.

1) In the first step, we assume that the supremum θ_1 of ϕ_1 satisfies

$$\sup \phi_1 = \limsup_{r\downarrow 0} \sup\{\phi_1(x,t,y,s),\ |x-y| < r\}. \tag{4.42}$$

Then, we can find a sequence (x_n, t_n, y_n, s_n), such that

1. $\phi_1(x_n, t_n, y_n, s_n) > \theta_1 - \frac{1}{n}$.
2. $|x_n - y_n| < \frac{1}{n}$.

Up to an extraction, we may assume that $(t_n, s_n) \to (\bar{t}, \bar{s})$ in $[t_0, T]^2$.
We now consider the function

$$\psi(x,t) = u(x,t) - (\alpha+1)f(|x-y_n|) - \alpha(t-s_n)^2 - (t-\bar{t})^2 - \lambda t.$$

Let (ξ_n, τ_n) a maximum of this function (there exists at least one since f tends to $+\infty$ at infinity). By comparing the value at (ξ_n, τ_n) and (x_n, t_n) and subtracting $v(y_n, s_n)$, we obtain

$$\begin{aligned} &\phi_1(x_n, t_n, y_n, s_n) - f(|x_n - y_n|) - (t_n - \bar{t})^2 \\ &\quad \le \phi_1(\xi_n, \tau_n, y_n, s_n) - f(|\xi_n - y_n|) - (\tau_n - \bar{t})^2. \end{aligned} \tag{4.43}$$

From $\phi(\xi_n, \tau_n, y_n, s_n) \le \theta_1 < \phi(x_n, t_n, y_n, s_n) + \frac{1}{n}$, $x_n - y_n \to 0$ and $t_n \to \bar{t}$, we obtain that $\xi_n - y_n \to 0$ and $\tau_n \to \bar{t}$. In particular, τ_n is positive for n large enough. Finally, since u is subsolution, we also have

$$\begin{cases} 2(\tau_n - \bar{t}) + 2\alpha(\tau_n - s_n) + \lambda + (\alpha+1) & f'(|\xi_n - x_n|)G(\frac{1}{|\xi_n - x_n|}) \le 0 \\ & \text{if } \xi_n \neq x_n \\ 2(\tau_n - \bar{t}) + 2\alpha(\tau_n - s_n) + \lambda & \text{otherwise.} \end{cases}$$

We let n tend to $+\infty$ and get

$$2\alpha(\bar{t} - \bar{s}) + \lambda \le 0. \tag{4.44}$$

In the same way, we consider the function

$$\psi'(y,s) = -v(y,s) - (\alpha+1)f(|y-x_n|) - \alpha(t_n - s)^2 - (s-\bar{s})^2 - \lambda s.$$

We denote by (η_n, σ_n) a maximum of ψ'. By the same argument as above, we also have $x_n - \eta_n \to 0$ and $\sigma_n \to \bar{t}$. Since v is supersolution, we obtain by letting n tend to $+\infty$ the inequality

$$0 \le 2\alpha(\bar{t} - \bar{s}) - \lambda.$$

By subtracting this to (4.44), we get $2\lambda \le 0$, which contradicts $\lambda > 0$.

2) Assume now that

$$\limsup_{r \downarrow 0} \sup\{\phi_1(x,t,y,s), |x-y| < r\} < \sup \phi_1.$$

This yields that there is a $\gamma > 0$ such that

$$\sup\{\phi_1(x,t,y,s), |x-y| < \gamma\} < \theta_1 = \sup \phi_1. \tag{4.45}$$

Introduce also the function

$$\phi_2(x,t,y,s) = \phi_1(x,t,y,s) - \delta|x|^2 - \delta|y|^2.$$

Since u and v are bounded, there exists at least a maximum point of ϕ_2 that we denote $(x_\delta, t_\delta, y_\delta, s_\delta)$. If δ is chosen small enough, then we may assume that

$$\sup \phi_2 > \sup\{\phi_1(x,t,y,s), |x-y| < \gamma\}.$$

Thus, by (4.45), we may assume that $|x_\delta - y_\delta| > \gamma$, for any δ. From (4.41), we also obtain an upper bound for $x_\delta - y_\delta$. Moreover, since $\delta(|x_\delta|^2 + |y_\delta|^2)$ is bounded, we get $\delta(|x_\delta| + |y_\delta|) \to$ when δ goes to 0.
We apply the Theorem on Sums 4.45 to ϕ_2 that we shall denote for convenience

$$\phi_2(x,t,y,s) = u(x,t) - v(y,s) - \psi(x,t,y,s).$$

We call u_n and v_n the two C^2 functions resulting from this lemma, touching u (resp. v) from above (resp. from below) at (x_n, t_n) (resp. (y_n, s_n)) tending to (x_δ, t_δ) (resp. (y_δ, s_δ)). We also call $X_n \to X_\delta$ and $Y_n \to Y_\delta$ the matrices given by the Theorem on Sums. Let us call $p_n = x_n - y_n$. Then $p_n \to p_\delta = x_\delta - y_\delta$. We have

$$D_x\psi(x_\delta, t_\delta, y_\delta, s_\delta) = \alpha f'(|p_\delta|)\frac{p_\delta}{|p_\delta|} + 2\delta x_\delta, \tag{4.46}$$

$$\psi_t(x_\delta, t_\delta, y_\delta, s_\delta) = 2\alpha(t_\delta - s_\delta) + \lambda, \tag{4.47}$$

$$D_y\psi(x_\delta, t_\delta, y_\delta, s_\delta) = -\alpha f'(|p_\delta|)\frac{p_\delta}{|p_\delta|} + 2\delta y_\delta, \tag{4.48}$$

$$\psi_s(x_\delta, t_\delta, y_\delta, s_\delta) = -2\alpha(t_\delta - s_\delta) + \lambda. \tag{4.49}$$

Since $|p_\delta| > \gamma$, we also have $|p_n| > \frac{\gamma}{2}$ for n large enough. Moreover, since $\delta(|x_\delta| + |y_\delta|)$ goes to 0, for δ small enough $D_x\psi$ and $D_y\psi$ are different from 0 at $(x_\delta, t_\delta, y_\delta, s_\delta)$, and this non zero gradient condition also holds for u_n and v_n at their contact points with u and v. We then apply the definition of viscosity sub- and supersolution for u and v with test functions u_n and v_n. Since gradients stay away from 0, the involved differential operators are not singular; we can directly let n tend to $+\infty$, and u is subsolution yields

$$2\alpha(t_\delta - s_\delta) + \lambda + F\left(X_\delta, \alpha f'(|p_\delta|)\frac{p_\delta}{|p_\delta|} + 2\delta x_\delta, 2\alpha(t_\delta - s_\delta)\right) \le 0, \tag{4.50}$$

while v is supersolution yields

$$2\alpha(t_\delta - s_\delta) - \lambda + F\left(Y_\delta, \alpha f'(|p_\delta|)\frac{p_\delta}{|p_\delta|} - 2\delta y_\delta, 2\alpha(t_\delta - s_\delta)\right) \ge 0. \tag{4.51}$$

(The function F has been defined in (4.36).) Now, p_δ, t_δ, s_δ are bounded because the penalization introduced in ϕ_1 tends to infinity at infinity. By the Theorem on Sums, X_δ and Y_δ are also bounded. Thus, up to a subsequence we may assume that

$\delta \to 0$ implies $p_\delta \to \bar{p}$, $t_\delta \to \bar{t}$, $s_\delta \to \bar{s}$, $X_\delta \to \bar{X}$ and $Y_\delta \to \bar{Y}$ with $\bar{X} \leq \bar{Y}$. Since $\delta(|x_\delta| + |y_\delta|)$ tends to 0, (4.50) and (4.51) imply

$$2\alpha(\bar{t} - \bar{s}) + \lambda + F\left(\bar{X}, \alpha f'(|\bar{p}|)\frac{\bar{p}}{|\bar{p}|}, 2\alpha(\bar{t} - \bar{s})\right) \leq 0$$

and

$$2\alpha(\bar{t} - \bar{s}) - \lambda + F\left(\bar{Y}, \alpha f'(|\bar{p}|)\frac{\bar{p}}{|\bar{p}|}, 2\alpha(\bar{t} - \bar{s})\right) \geq 0.$$

By subtracting these inequalities, and using $\bar{X} \leq \bar{Y}$, we obtain $2\lambda \leq 0$ which is a contradiction. □

The maximum principle directly implies the uniqueness of the solution to (4.29).

Corollary 4.49. *Let u and v in $BUC(\mathbb{R}^2 \times \mathbb{R})$ be two solutions of (4.29) with $u(x, 0) = v(x, 0) = g(x)$. Then $u = v$.*

Proof. Since $u \leq v$ on $\mathbb{R}^2 \times \{0\}$ and u is subsolution and v supersolution, we have $u \leq v$ by the maximum principle on $\mathbb{R}^2 \times [0, T]$ for any $T > 0$. Conversely, we also have $v \leq u$ by symmetry. Hence $u = v$. □

4.3.3 Existence of solution by Perron's Method

The existence of a solution of (4.29) is simpler than the proof of uniqueness. It relies on the two following propositions. The proof of these results in the viscosity framework were due to Ishii [89] and can also be found in [38].

Proposition 4.50. *Let $\mathcal{S}$ be a set of subsolutions of (4.29). Assume that they are uniformly bounded. Define*

$$v(x, t) = \sup\{u(x, t),\ u \in \mathcal{S}\}.$$

Then v is also a subsolution of (4.29).

Proof. Let φ an admissible function for G and assume that $v^* - \varphi$ has a maximum at (x, t). We can always assume that $v^*(x, t) - \varphi(x, t) = 0$ and that the maximum is strict. By definition of v, there exists a sequence $u_k \in \mathcal{S}$ such that $u_k(x, t) \geq v(x, t) - \frac{1}{k}$. Moreover, since the u_k are uniformly bounded, we can assume that $u_k^* - \varphi$ admits a maximum in (x_k, t_k) where the sequence (x_k, t_k) is bounded. Since

$$\begin{aligned}
-\frac{1}{k} &\leq u_k^*(x, t) - \varphi(x, t) \\
&\leq u_k^*(x_k, t_k) - \varphi(x_k, t_k) \\
&\leq v^*(x_k, t_k) - \varphi(x_k, t_k) \\
&\leq v^*(x, t) - \varphi(x, t) \\
&\leq 0,
\end{aligned}$$

we can also assume that $(x_k, t_k) \to (x, t)$ when k tends to $+\infty$ (since the maximum is strict). If $D\varphi(x, t) \neq 0$, then $D\varphi(x_k, t_k) \neq 0$ for k large enough. We obtain

$$\frac{\partial \varphi}{\partial t}(x_k, t_k) - |D\varphi(x_k, t_k)| G(\operatorname{curv} \varphi(x_k, t_k)) \leq 0.$$

Passing to the limit, we get $\varphi_t(x, t) - |Du| G(\operatorname{curv} u)(x, t) \leq 0$. If $D\varphi(x, t) = 0$, by Remark 4.44, we may assume that $\varphi(y, s) = g(s) + f(|y - x|)$ with $f \in \mathcal{F}(G)$. If $D\varphi(x_k, t_k) = 0$, we have $\varphi_t(x_k, t_k) \leq 0$ and $g'(t_k) + f'(|x_k - x|) G(\frac{1}{|y - x_k|})$ otherwise. Since $g'(t) = \varphi_t(x, t)$, we get the result is both cases by letting $k \to +\infty$. □

Proposition 4.51. *Let v and w be bounded and respectively sub- and super-solution of (4.29). Assume that $v \leq w$ in $\mathbb{R}^2 \times [0, +\infty)$. Define*

$$u(x) = \sup\{h(x), \text{ with } h \text{ subsolution s.t. } v \leq h \leq w\}.$$

Then u is a solution of (4.29).

Proof. We know by Prop. 4.50 that u is subsolution. We prove that it is also supersolution. We prove the result by contradiction. Assume that u is not supersolution. By definition, we can find φ admissible for G such that $u_* - \varphi$ has a strict minimum at some point (x, t) and

$$\varphi_t(x, t) - |D\varphi| G(\operatorname{curv} \varphi)(x, t) < 0 \tag{4.52}$$

if $D\varphi(x, t) \neq 0$ and $\varphi_t(x, t) < 0$ otherwise. We can assume without restriction that $u_*(x, t) - \varphi(x, t) = 0$. We then construct a subsolution h such that $v \leq h \leq w$ and $h(x, t) > u(x, t)$ contradicting the definition of u. We have $u_* \leq w_*$ by construction. But $u_*(x, t) < w_*(x, t)$, otherwise $w_* - \varphi$ would attain a minimum at (x, t). Since w is supersolution, this would imply

$$\varphi_t - |D\varphi| G(\operatorname{curv} \varphi) \geq 0,$$

if $D\varphi(x, t) \neq 0$ and $\varphi_t(x, t) \geq 0$ otherwise, contradicting (4.52) above. Therefore, we can choose $\delta > 0$ such that $\varphi(y, s) + \delta < w_*(y, s)$ in a small neighborhood A_δ of (x, t). By continuity of φ and by (4.52), we also have for $(y, s) \in A_\delta$

$$\varphi_t(y, s) - |D\varphi| G(\operatorname{curv} \varphi)(y, s) \leq 0$$

if $D\varphi(y, s) \neq 0$ or $\varphi_t(y, s) \leq 0$ if $D\varphi(y, s) = 0$. By choosing δ small enough, we can also assume that $u_* \geq \varphi + \delta$ out of a ball $B_\delta \subset A_\delta$.
Since φ is a classical subsolution in A_δ, it is also a viscosity subsolution in A_δ. (This is a simple consequence of Prop. 4.43.) We then define

$$h(y, s) = \begin{cases} \max(u_*(y, s), \varphi(y, s) + \delta) & \text{in } B_\delta \\ u(y) & \text{otherwise.} \end{cases}$$

By Prop. 4.50, h is also a subsolution. Since $h(x, t) = u(x, t) + \delta$, we get a contradiction with the definition of u. □

Thus we see that the proof of existence for the initial value problem bring us to the construction of “barriers” that is a sub- and a supersolution with the desired initial value. By Prop. 4.51 above, the solution will exist. We shall finally obtain the main result of this section.

Theorem 4.52. *Let $u_0 \in BUC(\mathbb{R}^2)$. Then there exists a unique solution to the initial value problem*

$$\begin{cases} \dfrac{\partial u}{\partial t} = |Du| G(\operatorname{curv} u) & \text{in } \mathbb{R}^2 \times \mathbb{R}. \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}^2. \end{cases} \tag{4.53}$$

Proof ([94]). The uniqueness is a consequence of the maximum principle (see Cor. 4.49). We now construct an upper and lower barrier. Let $f \in \mathcal{F}(G)$ such that f' is bounded. Let $\varepsilon > 0$. For $y \in \mathbb{R}^2$, we look for a subsolution of the form

$$v_{\varepsilon,y}(x,t) = u_0(y) - \varepsilon - A(\varepsilon) f(|x-y|) - B(\varepsilon)t,$$

where $A(\varepsilon)$ and $B(\varepsilon)$ are positive and chosen in the following. First, we require that $v_{\varepsilon,y}(x,0) \le u_0(x)$. It suffices that

$$\forall x, y \in \mathbb{R}^2,\ |u_0(x) - u_0(y)| \le \varepsilon + A(\varepsilon) f(|x-y|),$$

which is possible for $A(\varepsilon) \to +\infty$ when ε tends to 0, since u_0 is uniformly continuous and bounded. For $v_{\varepsilon,y}$ to be subsolution, it is enough that

$$\forall r > 0,\ -B(\varepsilon) + A(\varepsilon) f'(r) G(\frac{1}{r}) \le 0.$$

This is possible if we choose $B(\varepsilon)$ large enough because f' is bounded and $f \in \mathcal{F}(G)$. We now call

$$v(x,t) = \sup_{0<\varepsilon<1} \sup_{y \in \mathbb{R}^2} v_{\varepsilon,y}(x,t).$$

By Prop. 4.50, v is also subsolution. Moreover $v(x,0) = u_0(x)$. Indeed, since $v_{\varepsilon,y}(x,0) \le u_0(x)$, we also have $v(x,0) \le u_0(x)$, but by taking $y = x$ we obtain

$$v(x,0) \ge v_{\varepsilon,x}(x,0) = u_0(x) - \varepsilon.$$

It simply remains to add that v is bounded on $\mathbb{R}^2 \times [0,T]$ for all $T > 0$, since it is bounded from above by $\|u_0\|_\infty$ and from below by $-\|u_0\| - 1 - \inf_\varepsilon B(\varepsilon)T$.
Conversely, we define $w_{\varepsilon,y}(x,t) = u_0(y) + \varepsilon + A(\varepsilon) f(|x-y|) + B(\varepsilon)t$ with the same $A(\varepsilon)$ and $B(\varepsilon)$. We have $w_{\varepsilon,y}(x,0) \ge u_0(x)$ and $w_{\varepsilon,y}$ is a classical supersolution. We then define w by $w(x,t) = \inf_{0<\varepsilon<1} \inf_{y \in \mathbb{R}^2} w_{\varepsilon,y}(x,t)$ such that $w(x,0) = u_0(x)$. Since for all $(x,t) \in \mathbb{R}^2 \times [0,T]$ and $0 < \varepsilon < 1$,

$$v_{\varepsilon,y}(x,t) \le v_{\varepsilon,y}(x,0) \le u_0(x) \le w_{\varepsilon,y}(x,0) \le w_{\varepsilon,y}(x,t),$$

we also have $v(x,t) \le u_0(x) \le w(x,t)$ and we can take v and w as lower and upper barriers. We conclude by applying Perron's Method (Prop. 4.51). □

4.3.4 Contrast invariance of level sets flow

Equation (4.29) was introduced since it was the best candidate for the contrast invariance property. Remind that this condition is essential since, as we represent our moving curve by a supporting function, this last is non unique and the flow may depend on this particular choice. For instance, if we choose the signed distance function to the curve, we can always compose the result with a nondecreasing function to obtain another supporting function. The invariance of (4.29) with respect to contrast change is a sufficient condition for the zero level set to be independent from the contrast change apply to the supporting function. There are two intuitive reasons why the level set flow should commute with a nondecreasing contrast change. The first one is that its generator [24, 25] appeared as the asymptotic limit of a contrast invariant operator. The second one is given by the axiomatic approach of Alvarez et al. [4]. Nevertheless, nothing is proved and we have to check the invariance in full rigor, as first derived in [38, 60].

Proposition 4.53. *Let u be the unique solution of (4.29) with initial condition $u_0 \in BUC(\mathbb{R}^2)$. Let $g \in BUC(\mathbb{R})$ be nondecreasing. Then $g \circ u$ is the unique solution of (4.29) with initial condition $g \circ u_0$.*

Proof. Let call $v = g \circ u$. We prove that if u is subsolution, then v is also subsolution. The same will be valid for supersolution.
We first prove the result when g is C^2 with $g' > 0$. Assume that $v - \varphi$ has a maximum at (x,t), $\varphi \in \mathcal{F}(G)$. Let $\psi = g^{-1} \circ \varphi$. First, it is obvious that ψ is an admissible test function. Since g is strictly increasing and $g \circ u - g \circ \psi$ has a maximum at (x,t), $u - \psi$ is also maximal at (x,t). Moreover $D\varphi(x,t) = 0 \Leftrightarrow D\psi(x,t) = 0$.

1. if $D\varphi(x,t) = 0$, we have $\psi_t(x,t) \leq 0$ (since u is subsolution), yielding $\varphi_t(x,t) = g'(\psi(x,t))\psi(x,t) \leq 0$.
2. If $D\varphi \neq 0$, we have of course $\operatorname{curv} \varphi = \operatorname{curv} \psi$. (See Ex. 4.24.) Then, from

$$\varphi_t - |D\varphi| G(\operatorname{curv} \varphi) = g'(\psi)(\psi_t - |D\psi| G(\operatorname{curv} \psi)), \tag{4.54}$$

we deduce that v is also subsolution.

Assume now simply that $g \in BUC(\mathbb{R})$ and nondecreasing. By using nonnegative mollifiers, we can approximate g by some $g_\varepsilon \in C^2$. By adding to g_ε a term $\varepsilon \arctan$, we can also assume that $g'_\varepsilon > 0$. By construction $g_\varepsilon \to g$ uniformly. Let $v_\varepsilon = g_\varepsilon \circ u$. By the result above, the v_ε are subsolutions. Now, $v_\varepsilon \to v$ uniformly. This simply implies that v is subsolution.
Indeed, if $v - \varphi$ has a maximum at (x,t), $v_\varepsilon - \varphi$ has a strict maximum at $(x_\varepsilon, t_\varepsilon) \to (x,t)$.

1. If $D\varphi(x,t) \neq 0$, then $D\varphi(x_\varepsilon, t_\varepsilon) \neq 0$ for ε small enough. Then

$$\varphi_t(x_\varepsilon, t_\varepsilon) - |D\varphi| G(\operatorname{curv} \varphi)(x_\varepsilon, t_\varepsilon) \leq 0.$$

We let ε go to 0 and obtain the same inequality at (x,t).

2. If $D\varphi(x,t) = 0$, then $\varphi_t(x_\varepsilon, t_\varepsilon) \le 0$ if $D\varphi(x_\varepsilon, t_\varepsilon) = 0$ and $\varphi_t(x_\varepsilon, t_\varepsilon) - |D\varphi| G(\operatorname{curv}\varphi)(x_\varepsilon, t_\varepsilon) \le 0$ otherwise. Since by Rem. 4.44, we can choose $\varphi(y,s) = h(t) + f(|y-x|)$ with $f \in \mathcal{F}(G)$, we also obtain $\varphi_t(x,t) \le 0$ by letting ε tend to 0.

Thus $v = g \circ u$ is subsolution. In the same way, it is supersolution, thus it is solution. □

By combining Prop. 4.53 and 4.16, we obtain the following proposition that justifies the viscosity solution method.

Corollary 4.54. *Let C be an embedded curve. Let also u_0 and v_0 be two supporting functions of C and u and v the solutions of the evolution with initial condition u_0 and v_0. Then, for any $t > 0$,*

$$\{x \textit{ such that } u(x,t) \le 0\} = \{x \textit{ such that } v(x,t) \le 0\}.$$

4.3.5 Viscosity solutions shorten level lines

The viscosity solutions allow us to define curve evolution even if the classical existence is not known. Numerous authors tried to establish a rigorous equivalence between the different available notions of solutions: classical solutions, viscosity solutions but also varifold solutions [23, 85], reaction-diffusion approximation [85, 123] or minimal barriers [21]. (See the end of this chapter for a rapid overview of these notions.) It is interesting at least to check that qualitative properties for classical solutions are satisfied. In this section, we follow the analysis of Evans in [56] to formally show that viscosity solutions of the generalized curve shortening flow shortens the perimeter of level sets. In order to see this, assume that u is a smooth solution of

$$\frac{\partial u}{\partial t} = |Du| G(\operatorname{curv} u).$$

By the coarea formula [58], the sum of the perimeter of the level sets of u may be computed by

$$\int_{\lambda \in \mathbb{R}} \mathcal{H}^1(\partial \chi_\lambda(u)) d\lambda = \int_{\mathbb{R}^2} |Du|.$$

Differentiating the right-hand term gives

$$\frac{d}{dt} \int_{\mathbb{R}^2} |Du| = \int_{\mathbb{R}^2} \frac{Du \cdot Du_t}{|Du|}. \tag{4.55}$$

Integrations by part yields

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^2} |Du| &= -\int_{\mathbb{R}^2} \operatorname{div}\left(\frac{Du}{|Du|}\right) u_t & (4.56)\\ &= -\int_{\mathbb{R}^2} (\operatorname{curv} u)\, G(\operatorname{curv} u) |Du| & (4.57)\\ &\le 0 & (4.58) \end{aligned}$$

because $\kappa G(\kappa) \geq 0$. We then integrate these inequalities between t_1 and $t_2 > t_1$ and find that

$$\int_{\lambda \in \mathbb{R}} \mathcal{H}^1(\partial \chi_\lambda(u(\cdot, t_2)))d\lambda \leq \int_{\lambda \in \mathbb{R}} \mathcal{H}^1(\partial \chi_\lambda(u(\cdot, t_2)))d\lambda.$$

This is not much information though, since it only asserts that the total sum of the perimeters decreases. In order to retrieve information for each level line, we use the contrast invariance property. Let $\lambda \in \mathbb{R}$ and $h > 0$. Let g be defined by $g(t) = 0$ if $t \leq \lambda$, $g(t) = \frac{1}{h}(t - \lambda)$ for $t \in (\lambda, \lambda + h)$ and $g(t) = 1$ elsewhere. By applying the same analysis as above for $g \circ u$, we find that

$$\frac{1}{h}\int_{\lambda}^{\lambda+h} \mathcal{H}^1(\partial \chi_\lambda(u(\cdot, t_2)))d\lambda \leq \frac{1}{h}\int_{\lambda}^{\lambda+h} \mathcal{H}^1(\partial \chi_\lambda(u(\cdot, t_2)))d\lambda.$$

Letting h goes to 0 implies that

$$\mathcal{H}^1(\partial \chi_\lambda(u(\cdot, t_2)))d\lambda \leq \mathcal{H}^1(\partial \chi_\lambda(u(\cdot, t_2))) \tag{4.59}$$

which is the shortening property we are looking for. In general, solutions are not smooth and the calculation above does not make sense. However, the result still stands but requires more elaborate arguments of geometric measure theory. In [62, 63], Evans and Spruck indeed prove that the level set solution is a varifold (or geometric theoretical) solution as introduced by Brakke [23] and that (4.59) holds for almost every level λ.

4.4 Morphological operators and viscosity solution

In order to illustrate the very strong link between mathematical morphology and scale space theory, we give two results of convergence of iterated morphological operators to solution of generalized curvature motion. We do not give the proofs which can be found in [81]. Moreover, in the last chapter of this monograph, a complete proof of convergence of an algorithm for curve evolution will be given. As the structure of the proof is the same, we do not go into details for the moment. Related proofs of convergence of morphological schemes, all based on consistency arguments, may also be found in [18, 27, 35, 55, 90, 125, 146].

4.4.1 Median filter and mean curvature motion

Let $h > 0$ and $D(0, h)$ the disk centered at 0 with radius h. Let $u : \mathbb{R}^2 \to \mathbb{R}$.

Definition 4.55. *We call median value of u in $D(x, h)$ the value*

$$\operatorname{med}_h u(x) = \sup\{\lambda \textit{ s.t. } |\chi_\lambda(u) \cap D(x, h)| \geq \frac{\pi h^2}{2}\}. \tag{4.60}$$

(Here, $|A|$ is the Lebesgue measure of A.) Otherwise said, in $D(0, h)$ half of the points have a value larger that $\operatorname{med}_h u(x)$ *and half have a smaller value.*

The median filter has been widely used as an alternative of a mere convolution. Roughly speaking, it is a nonlinear mean value, but it is more robust to outlier values (typically impulse noise in image processing). The fact that it is morphological is completely obvious since it only depends on the level sets of u. The median filter can be written in and sup-inf form with a certain family of structuring elements. The choice is not unique but the following one is suitable.

Proposition 4.56. *Let* $\mathbb{B} = \{B \in \mathbb{R}^2, \quad B \subset D(0,1), |B| \geq \frac{\pi^2}{2}\}$ *and* $\mathbb{B}_h = h\mathbb{B}$. *Then*

$$\operatorname{med}_h u(x) = \sup_{B \in \mathbb{B}_h} \inf_{y \in B} u(x+y).$$

Proof. By using the translation invariance, we may assume that $x = 0$. We have the following equivalences:

$$\begin{aligned}
\operatorname{med}_h u(0) \geq \lambda &\Leftrightarrow \forall \mu < \lambda \quad |\chi_\mu(u) \cap D(x,h)| \geq \frac{\pi h^2}{2} \\
&\Leftrightarrow \forall \mu < \lambda \quad \chi_\mu(u) \in \mathbb{B}_h \\
&\Leftrightarrow \forall \mu < \lambda \quad \exists B \in \mathbb{B}_h \quad B \subset \chi_\mu(u) \\
&\Leftrightarrow \forall \mu < \lambda \quad \exists B \in \mathbb{B}_h \quad \inf_{y \in B} u(y) \geq \mu \\
&\Leftrightarrow \sup_{B \in \mathbb{B}_h} \inf_{y \in B} u(y) \geq \lambda.
\end{aligned}$$

The third equivalence follows from $X \subset Y$ and $X \in \mathbb{B} \Rightarrow Y \in \mathbb{B}$. □

Whereas the behavior of convolution with a gaussian was understood (it is asymptotically equivalent to the heat equation), the same kind of result for the median filter is much more recent.

Proposition 4.57 ([81]). *Let u be a C^3 function and $x \in \mathbb{R}^2$ such that $Du(x) \neq 0$. Then*

$$T_h u(x) = u(x) + \frac{h^2}{6}|Du| \operatorname{curv} u + o(h^2). \tag{4.61}$$

This proof of this result is not very difficult and we only sketch it. We first remark that by Prop. 4.25, it suffices to approximate $T_1(u)(0)$ when $u(x) = x_1 + hx_2^2$. The median value is then geometrically observed: for $m > 0$ small enough, the unit circle is cut into two parts by the parabola $x_1 = m - hx_2^2$. The median value $m(h)$ is obtained when both parts have the same area, and geometric approximations leads to (4.61). (See Fig. 4.2.)

From Prop. 4.57, it is possible to prove the following convergence result.

Theorem 4.58. *Let $u_0 \in BUC(\mathbb{R}^2)$. Let also u_h be defined by*

$$u_h(x,t) = (\operatorname{med}_h)^n u_0(x) \text{ if } t \in [\frac{nh^2}{6}, \frac{(n+1)^2 h}{6}).$$

Then, when h tends to 0, u_h converges locally uniformly towards the unique viscosity solution of the mean curvature motion

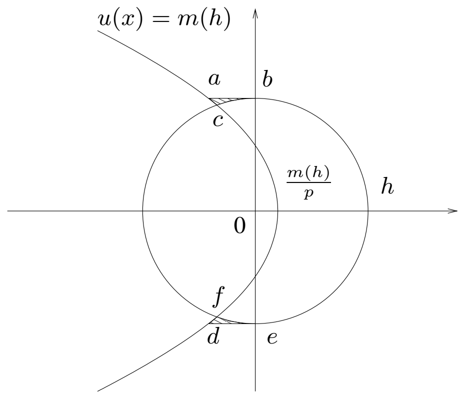


Fig. 4.2. Consistency of the median filter. When $u(x) = px_1 + hx_2^2$, the median value at the origin is the value of $m(h)$ such that the parabola severs the disk in parts with equal area. Equivalently, the algebraic area between the parabola and the axis $x_1 = 0$ and the lines $x_2 = \pm h$ must be equal to the areas of the triangles abc and def

$$\frac{\partial u}{\partial t} = |Du| \operatorname{curv} u,$$

with initial value u_0.

This theorem results from (4.61) and a theorem by Barles and Souganidis [20] proving that a consistent monotone scheme is convergent. Those arguments shall be detailed in Chap. 6, where we construct and prove the convergence of such a scheme.

In fact, there is not a unique median filter since we can imagine to define a median filter with a different kernel from the characteristic function of a disk. For instance, Bence, Merriman and Osher [22] used a Gaussian kernel. A general result by Ishii [90], shows that the limiting equation is still the mean curvature motion. We shall come back on this in Chap. 5.

4.4.2 Affine invariant schemes

We also have a convergence result for the affine invariant scheme of Prop. 4.28.

Theorem 4.59. *Let* $u_0 \in BUC(\mathbb{R}^2)$. *Let* u_h *be defined by*

$$u_h(x,t) = (IS_h \circ SI_h)^n u_0(x) \text{ if } t \in [nc_{\mathbb{B}}h^{2/3}, (n+1)c_{\mathbb{B}}h^{2/3}),$$

where IS_h, SI_h *and* $c_{\mathbb{B}}$ *are defined in Prop. 4.28. Then, when* h *goes to 0,* u_h *locally uniformly converges to the solution of the affine invariant equation*

$$\frac{\partial u}{\partial t} = |Du|(\operatorname{curv} u)^{1/3},$$

with initial value u_0.

4.5 Conclusions

Putting all things together, we saw that operating on sets or on functions was essentially the same for morphological operators. Any morphological scheme could be written in an inf-sup form (Matheron's Theorem 4.19) by introducing a family of structuring elements. When we scale this family by a small factor, then general results of consistency with motion by curvature can be obtained. We gave the two particular cases of the median filter and affine invariant morphological schemes. Then, it is not very difficult to prove that the consistency and monotonicity of the schemes imply their convergence to a morphological scale space. (See Chap. 6.)
In the particular case of affine invariance, with very mild hypotheses, any scaled morphological scheme is consistent with a motion by the power $1/3$ of the curvature. On the other hand, there is a unique affine invariant scale space. As a consequence, if one wants to perform a local, causal, affine invariant and contrast invariant smoothing, then we have to solve this equation as accurately as possible. This was the approach of Moisan for his geometrical scheme called affine erosion [126] and that we shall see in Chap. 6.

4.6 Curvature thresholding

One operator of interests for mathematical morphologists is an operator allowing to threshold curvatures. Curvature thresholding was introduced as a "simple" way to simplify curves. Nevertheless, it appears that such an operation is not so easy to define as soon as we add some additional requirements. First, it should preserve inclusion as any operator acting on shape. Now, it must also preserve the topology and precisely the connectedness of a shape. In Sect. 4.6.1, we shall see how it is possible to define an operator which should satisfy these requirements. In Sect. 4.6.2, we prove that this operator reduces to the well-known *opening* operator which was among the best candidates of Mathematical Morphology.

4.6.1 Viscosity approach

The operator we shall define obviously relies on the PDE we have studied in all the beginning of this chapter. Let us indeed consider the curvature evolution equation

$$\frac{\partial u}{\partial t} = |Du|(t\,\mathrm{curv}\,u)^{\gamma}. \tag{4.62}$$

Equation (4.62) is not exactly of the type presented in Sect. 4.3 above, since the scale explicitly appears on the right-hand side. Nevertheless, because of the special form of the speed (a power function), it is easily seen that the usual equation (without "t" on the right) can be retrieved by applying a change of variable. Now, why do we introduce this rescaling? The answer is twofold.
The first point of view is that (4.62) has a scale covariance property (because of the homogeneity of the power function) and which is independent of γ. More precisely,

let us denote by S_t^γ the operator mapping a function u_0 to the solution of (4.62) at time t. Remark that this is no longer a semi-group because of the dependence on t. For $\lambda > 0$, also denote by H_λ the homothety by a factor λ, that is $(H_\lambda u)(x) = u(x/\lambda)$ for all x. Then, a direct calculation shows that

$$H_\lambda \circ S_t^\gamma = S_{\lambda t}^\gamma \circ H_\lambda, \tag{4.63}$$

which means that zooming on a smoothed image is equivalent to smooth a zoomed image but with a scale multiplied by the zoom factor (and thus independent of γ). The second argument is even more explicit. Let us consider the curve shortening associated to (4.62), that is

$$C_t = (t\kappa)^\gamma \mathbf{N}. \tag{4.64}$$

We do not discuss the existence of such an evolution but only consider the evolution of a circle with initial radius R_0. We know that the evolution exists in this case and that the solution is still a circle whose radius is solution of

$$R'(t)R^\gamma(t) = -t^\gamma,$$

and thus equal to $R(t) = (R_0^{\gamma+1} - t^{\gamma+1})^{1/(\gamma+1)}$. What is particularly interesting in this expression is the vanishing time of the circle: it is equal to R_0 for all γ. Moreover, if we let γ tend to infinity, then $R(t)$ tends to R_0 for any $t < R_0$ and instantaneously collapses to 0 precisely at time R_0. Since the evolution (4.62) is local, we may hope that this behavior will also stand for more complex shapes, i.e. that shapes with small curvature are not modified at low scale then brutally shrink at some critical scale. We do not know strong result of existence and uniqueness for the curve shortening equation (4.64), but the viscosity solution for (4.62) has these properties. This leads us to the following definition.

Definition 4.60 (and Proposition). *Let $u_0 \in BUC(\mathbb{R}^2)$ and let u^γ be the unique solution of (4.62) with initial value u_0. Define*

$$u^\infty(x,t) = \limsup_{\substack{\gamma\to+\infty \\ (y,s)\to(x,t)}} u^\gamma(y,s) \quad \textit{and} \quad u_\infty(x,t) = \liminf_{\substack{\gamma\to+\infty \\ (y,s)\to(x,t)}} u^\gamma(y,s). \tag{4.65}$$

Then u^∞ and u_∞ are both subsolution of

$$t \operatorname{curv} u(x,t) = -1, \tag{4.66}$$

and supersolution of

$$t \operatorname{curv} u(x,t) = 1. \tag{4.67}$$

In classical terms, they both satisfy

$$-1 \le t \operatorname{curv} u(x,t) \le 1. \tag{4.68}$$

Proof. We prove the result for u^∞, and it will also hold for u_∞. We show that u^∞ satisfies $t\,\mathrm{curv}(u^\infty) \geq -1$. To this purpose, let $(x,t) \in \mathbb{R}^2 \times (0,+\infty)$ such that $u^\infty - \varphi$ admits a strict maximum at (x,t). If $D\varphi(x,t) = 0$, there is nothing to be proved, and we thereafter assume that $D\varphi(x,t) \neq 0$. We know that we can find a sequence (γ_n, x_n, t_n) tending to $(+\infty, x, t)$ such that $u(x,t) = \lim u^{\gamma_n}(x_n, t_n)$. By possibly modify φ and taking a subsequence, we may assume that $u^{\gamma_n} - \varphi$ has a strict maximum at a (ξ_n, τ_n) tending to a point (ξ, τ). Then, the fact that (x,t) is a strict maximum of $u^\infty - \varphi$ and the inequalities

$$\begin{aligned} u^\infty(x,t) - \varphi(x,t) &= \lim(u^{\gamma_n}(x_n,t_n) - \varphi(x_n,t_n)) \\ &\leq \limsup(u^{\gamma_n}(\xi_n,\tau_n) - \varphi(\xi_n,\tau_n)) \\ &\leq u^\infty(\xi,\tau) - \varphi(\xi,\tau) \end{aligned}$$

prove that $(\xi,\tau) = (x,t)$. Since u^{γ_n} satisfies (4.62), we have $D\varphi(\xi_n,\tau_n) \neq 0$ for large enough n and

$$\left(\frac{1}{|Du|}\frac{\partial\varphi}{\partial t}\right)^{1/\gamma}(\xi_n,\tau_n) \leq (t\,\mathrm{curv}\,\varphi)(\xi_n,\tau_n).$$

Letting n tend to $+\infty$ we see that the $\liminf$ of the left-hand term is always larger than -1 and we are done.
By using the same arguments we prove that u^∞ also satisfies $t\,\mathrm{curv}(u^\infty) \leq 1$. □

In fact, the above proof is valid for any convergent subsequence of u^γ. The point is that we cannot establish that this limit is also equal to u^∞ which is equivalent to say that the whole family converge to a single function. We are only able to consider upper and lower limits. As for curve evolution, it is likely that level sets do not change topology until they disappear but we saw that such a result is still unproved. We must also add that this operator is no longer described by a PDE and the level sets as function of time are not continuous for the Hausdorff distance.

4.6.2 Opening and closing

In Mathematical Morphology, there are classically two candidates for curvature thresholding: the classical opening and closing operators.

Definition 4.61. *Let $X \subset \mathbb{R}^2$ be a closed set. The opening of X at scale t is the set*

$$O_t X = \bigcup_{\overline{B(x,t)} \subset X} \overline{B(x,t)}, \tag{4.69}$$

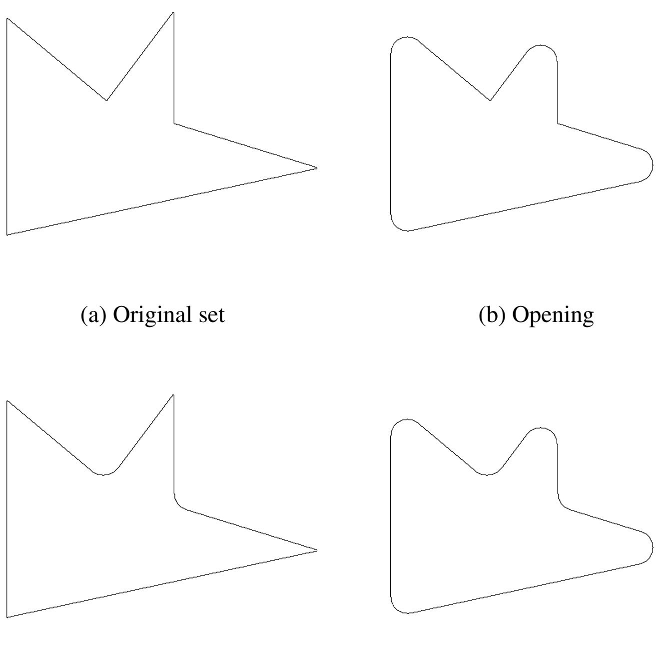
that is the union of close balls of radius t contained in X. The closing is defined by duality by

$$C_t X = (O_t(X^c))^c.$$

In fact, this definition is only an exemple of opening and closing corresponding to a structuring element equal to a ball. For the general case, see [154]. In order to be complete, we also have to define a weak notion of curvature for a set. This is formulated in terms of the inclusion principle.

Definition 4.62. *Let X be a closed set in the plane. If $\kappa_0 > 0$, we say that* $\mathrm{curv}(\partial X) \leq \kappa_0$ *at x, if we can find a neighborhood N of x and a close ball B of radius $1/\kappa_0$ such that $B \cap N \subset X \cap N$ and their intersection is reduced to $\{x\}$.*
Conversely, we say that $\mathrm{curv}(\partial X) \geq -\kappa_0$ *if* $\mathrm{curv}\, \partial X^c \leq \kappa_0$ *in the above sense.*

It is then obvious that the curvature of the boundary of $O_t X$ is smaller than $1/t$, whereas it is not bounded from below as can be seen on Fig. 4.3. On the contrary, closing bounds the curvature from below but not from above. On Fig. 4.4, we



(a) Original set (b) Opening

(c) Closing (d) Opening then closing

Fig. 4.3. Opening and closing of a set. The opening thresholds positive curvatures and the closing negative ones. If we alternate a closing and an opening, then the result has a controlled curvature

computed the evolution of the same shape with a velocity equal to κ to the power 7. In particular, doubling the curvature multiplies the velocity by 128.

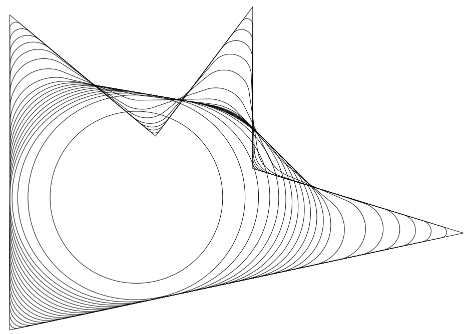


Fig. 4.4. Evolution of a set with $V = \kappa^7$. (CPU: 430s)

It is also well known that opening and closing may change the topology of a set, for instance a thin dumbbell is broken into two pieces. On Fig. 4.5 and Fig. 4.6 we display the evolution of such a dumbbell when $\gamma = 5$ and $\gamma = 10$. As foreseen, the connectedness is preserved. Moreover, when γ increases, the balls of the dumbbell hardly changes before the shape collapses. The shape is initially symmetric and should remain so. This is clearly the case except for the last curve (just before the disappearance), where the symmetry is broken. It is very difficult to say whether this is a numerical artifact due to a convexification step (see 6.6.4) that slightly breaks the symmetry or because the initial curve is only symmetric up to some numerical accuracy. In any case, when γ is large, the filtering is less and less continuous in time and small details may have dramatic consequences. If for instance one ball of the dumbbell is a bit smaller than the other one, then it should disappear nearly instantaneously while the other one has not changed yet. Both figures contains 10 curves taken at regular time steps. The last one is obtained just before the curves shrink. As can be expected, the curve are smoothed at the junction of the bar of the dumbbell and the balls, while the balls do not change. Afterwards, the shrinking becomes very fast with high values of γ. The curves have four inflexion points (as the initial data) during all the evolution then become convex and vanish nearly instantaneously.

When the set is convex, then the situation is clearer and the next result asserts that the opening is equivalent to the scale space curvature thresholding introduced in the previous section. Let X be a convex closed set in $\mathbb{R}^2$ and $X^\gamma(t)$ the interior of the evolving curve solution of (4.64) (which exists by Andrews' result [10]) with initial value ∂X. Let $X^\infty(t) = \limsup X^\gamma(t)$. We then have

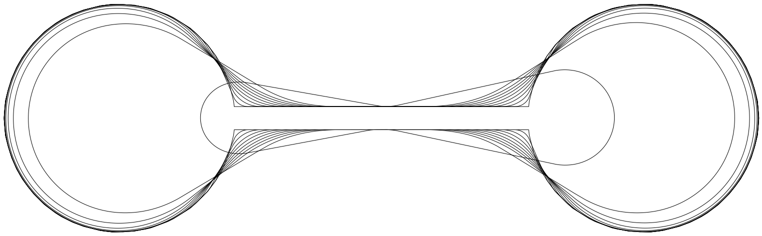


Fig. 4.5. Evolution of a dumbbell shape with $V = \kappa^5$. CPU: 84s

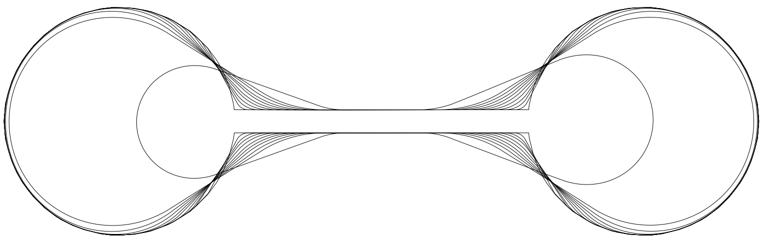


Fig. 4.6. Evolution of a dumbbell shape with $V = \kappa^{10}$. CPU: 189s

Proposition 4.63.

$$X^\infty(t) = O_t(X). \tag{4.70}$$

We first start with some simple statements.

Lemma 4.64. *For any t,* $X^\infty(t)$ *is convex and* $\sup(\mathrm{curv}\,\partial(X^\infty(t))) \leq 1/t$.

Proof. For any γ, X^γ is convex, thus X^∞ is also convex. The second part is proved by considering the X as the zero level set of its distance function and applying the result in Def. 4.60. □

Lemma 4.65. *Let* X *a convex set satisfying* $\sup \mathrm{curv}(\partial X) \leq M < +\infty$ *in the weak sense. Then* ∂X *is differentiable.*

Proof. Indeed, if X is convex, its boundary blows up to a convex cone at each of its point. If this cone were not flat at a point x, then we could find a ball with a radius smaller than $1/M$, that contains X in a neighborhood of x. This contradicts the fact that the curvature is bounded by M. □

Lemma 4.66. *Let* X *be a compact convex set such that* $\mathrm{curv}(\partial X) < 1/t$*. Then* $O_t(X) = X$*.*

Proof. Since $O_t(X) \subset X$ by definition, we only have to prove the reverse inclusion. Assume that $X \neq O_t(X)$ and first that $O_t(X) \neq \emptyset$. Let $x \in X \backslash O_t(X)$ such that $d(x, O_t(X))$ is positive and maximal. This is possible since X and $O_t(X)$ are compact. Since $O_t(X)$ is also convex, there is a unique orthogonal projection of x on $O_t(X)$ (Fig 4.7). Let denote it by $x^\perp$. By definition of $O_t(X)$, $x^\perp$ also belongs to the boundary of a closed ball $B(y,t) \subset X$. Then, this is clear that $B(y,t)$ is bitangent to ∂X at two points z_1 and z_2, else we could find another ball with radius t contained in X which is closer to x than $B(y,t)$. Moreover, we can sever $\partial B(y,t)$ into two parts with the diameter orthogonal to $x - x^\perp$. Then, z_1 and z_2 can be chosen on the same half circle as $x^\perp$. If it was not the case, we would be able to find y' between x and y such that $B(y',t) \subset X$, contradicting again that $x^\perp$ is the orthogonal projection of x onto $O_t(X)$. We now consider the part of the boundary

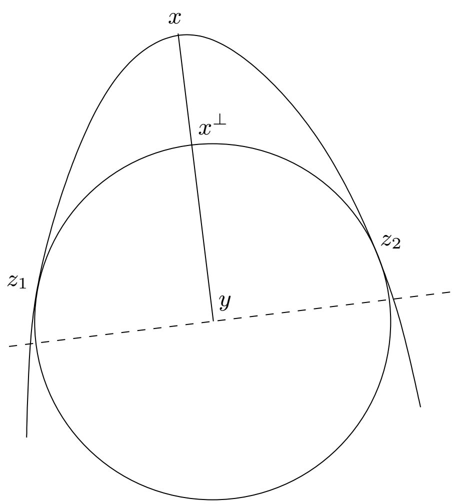


Fig. 4.7. If $\mathrm{curv}(\partial X) \leq 1/t$, then $O_t(X) = X$. See text

of ∂X between z_1 and z_2. With suitable axis of coordinates, it can be written as the graph of a convex function u which is solution of

$$\frac{u''}{(1+u'^2)^{3/2}} \geq -1/t$$

since the left hand term is the curvature of the graph of u. Otherwise said, u is a subsolution of the equation obtained by replacing "$\geq$" by "$=$". But u also lies above the function whose graph is a circle with radius $1/t$ and endpoints z_1 and z_2. This contradicts the maximum principle.
If $O_t(X) = \emptyset$, then the inner circle of X has a radius strictly smaller than t. We then conclude as above by invoking the maximum principle. □

Proof (of Prop 4.63). First, $O_t(X) \subset X_\infty(t)$ follows from the inclusion principle applied to balls with radius t. Conversely, since the curvature of $X_\infty(t)$ is less than $1/t$, we then have $O_t(X_\infty(t)) = X_\infty(t)$. But, since $X_\infty(t) \subset X$, we also have $O_t(X_\infty(t)) \subset O_t(X)$. □

4.7 Alternative weak solutions of curve evolution

The viscosity solution theory was first introduced in the first order case by Lions and Crandall to solve optimal control problems involving Hamilton-Jacobi equations. It was only later with the works of Chen, Giga and Goto [38] and Evans and Spruck [60] that second-order equations found their usefulness in front propagation. These authors were the first to introduce the notion of geometric equations. The concept was axiomatic and the authors proved that the underlying equations were contrast invariant (though they were not concerned with image processing). The analysis of geometric equations was also made possible by the uniqueness results by Jensen [95]. Nevertheless, it is very interesting that other notions of weak solutions had been previously introduced to solve geometric evolutions. In fact, the mean curvature motion was, from far, the focus of interest of many researchers because of its connection with geometry and the physics of phase transition. Indeed, the mean curvature flow is the “gradient” flow of the area functional and thus naturally appear in the theory of minimal surfaces.

4.7.1 Brakke’s varifold solution

Brakke [23] first introduced a weak solution using geometric measure theory. He replaces the classical notion of surfaces by the generalized notion of varifold. Varifolds are defined by duality and are to surfaces what measures are to continuous functions. More precisely, a varifold in the plane is a measure on the set $\mathbb{R}^2 \times \mathbb{S}^1$; each point of this set represents a point of the plane attached to a tangent line. Still by duality, one can define the curvature of a varifold as its second variation, exactly as the curvature is the second variation of a curve, then a weak notion of motion by mean curvature. Brakke was then capable to prove that the mean curvature motion of a varifold always exists and also gave partial regularity results. Nevertheless, several solutions may exist in some cases. Brakke’s approach also has the advantage that it can be generalized in any dimension and for surfaces of any codimension, whereas the level sets approach is natural for surfaces of codimension 1. (Ambrosio and Soner [9] removed this assumption for the mean curvature flow by considering the codimension-one surface obtained from a general submanifold by taking the ε-level set of the distance function to the manifold and letting ε go to 0.) Evans and Spruck proved in [62, 63] that level sets of viscosity solutions were a Brakke’s varifold solution with unit density.

4.7.2 Reaction diffusion approximation

Another notion of solution comes from a model of Allen and Cahn for front propagation [3]. Let us consider the functional

$$F_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^2} |Du|^2\,dx + \frac{1}{\varepsilon^2}\int_{\mathbb{R}^2} W(u)\,dx, \tag{4.71}$$

where W is the two-wells potential $W(u) = \frac{1}{2}(u^2 - 1)^2$. We call u_ε a minimizer of this energy functional. When letting ε go to 0, the W term forces u_ε to be equal to -1 or 1. Since by classical theory, u_ε is smooth, there is a transition layer passing from -1 to 1. It is then possible to prove that this layer has a width proportional to ε and that $F_\varepsilon(u_\varepsilon)$ is proportional to the length of Γ_ε, where Γ_ε is the zero-level line of u_ε. It is rigorously possible to prove this convergence by using the notion of Γ-convergence introduced by De Giorgi and Franzoni [47]. In particular, the gradient flow of (4.71),

$$\frac{\partial u_\varepsilon}{\partial t} = \Delta u_\varepsilon - \frac{1}{\varepsilon} u_\varepsilon (u_\varepsilon^2 - 1) \tag{4.72}$$

provides an approximation of the mean curvature flow when we look the evolution of the set $\{u_\varepsilon = 0\}$. We refer the reader to [7, 19, 59, 158] for more details. This approach is also valid for hypersurfaces evolution and Ilmanen [85, 86] studied the equivalence between this and the Brakkes's varifold solutions.

4.7.3 Minimal barriers

A completely geometric approach was developed following another idea of De Giorgi [46], based on the maximum principle. Contrary to viscosity solutions where the requirements are local, barriers are based on a global comparison principle with classical "test functions". More precisely, let $\mathcal{F}$ be the sets of $\mathbb{R}^2$ whose boundary are classical solutions of the mean curvature flow. An element of $\mathcal{F}$ is a function $t \mapsto C(t)$ where $C(t) \subset \mathbb{R}^2$ moves by mean curvature. We say that $E(t)$ is a barrier for $\mathcal{F}$ if $C(0) \subset E(0)$ implies $C(t) \subset E(t)$ for any $t \geq 0$ and $C(t) \in \mathcal{F}$. We denote by $\mathcal{B}(\mathcal{F})$ the set of barriers for $\mathcal{F}$; $\mathcal{B}(\mathcal{F})$ corresponds to supersolutions. If $X \subset \mathbb{R}^2$,we define

$$\mathcal{M}(\mathcal{F}, X)(t) = \inf \left\{ E(t) \text{ such that } E \in \mathcal{B}(\mathcal{F}),\ X \subset E(0) \right\}, \tag{4.73}$$

which is, in some sense, the smallest supersolution. Consequently, this is a good candidate for defining a generalized mean curvature motion, but it can be proved that it is not appropriate for passages to the limit. Hence, De Giorgi considered the set $\mathcal{M}^*(\mathcal{F}, X)(t)$ defined by

$$\mathcal{M}^*(\mathcal{F}, X)(t) = \bigcap_{\rho > 0} \mathcal{M}(\mathcal{F}, N_\rho(X))(t), \tag{4.74}$$

where $N_\rho(X) = \{x \in \mathbb{R}^2 \text{ such that } d(x, X) < \rho\}$ is the ρ-tubular neighborhood of X. Some equivalence between the minimal barriers and the viscosity solutions approaches has been established by Bellettini and Novaga in [21].

4.8 Bibliographical notes

The most ancient reference we have for a level set representation of an interface is an article by Ohta, Jasnow and Kawasaki [137], for modeling an interface. It was then

used in a systematic way by Osher and Sethian [144, 155], but not exclusively for geometric equations. The level set decomposition is a classical tool of Mathematical Morphology founded by Matheron. All the results about the level sets decomposition and the characterization of morphological operators exist in one form or the other in Matheron's or Serra's books [119, 154]. Some results are also due to Maragos [116]. A more complete and accessible theory for image analysis is presented by Guichard and Morel [81], who also proved the first consistency results for isotropic operators (Prop. 4.25), and affine invariant operators (Prop 4.28). Generalizations of consistency proofs for morphological operators were then proved by different authors in [27, 28, 35, 125, 146].
Gaussian image filtering was explicitely used by Marr [117] for edge detection and Scale Space Theory was introduced by Witkin [169]. Many papers are available on the only topic of scale space (see Lindeberg's and Weickert's books [109, 167] and references therein). Malik and Perona [147] first had the idea of edge preserving anisotropic diffusion, mathematically studied by Catté, Lions, Morel and Coll [36]. The axiomatic approach of Sect. 4.2 was developed by Alvarez, Guichard, Lions and Morel [4], who concluded the uniqueness of the affine morphological scale space (Thm. 4.34) by a completely different way from Sapiro and Tannenbaum's [152]. They also proved that scale space gave a viscosity solution (Prop. 4.33). Such an axiomatic approach was later developped by Guichard for movie analysis [4, 79, 80], Moisan for depth recovery [124, 125], and Dibos for projective analysis [51]. (In Dibos' case, the model is not an equation but a family of equations.)
At that time, the path for viscosity solutions was ready. The notion was introduced by Lions and Crandall [44, 110] for first order equations, with proof of uniqueness and existence. First order theory is exposed in a book by Barles [17]. The uniqueness for the second order case is a work by Jensen [95] for proper operators. Jensen's proof of the maximum principle was simplified by Ishii [88] who proved the Theorem on Sums (Lemma 4.45). Curvature motions are degenerate equations and the extension of the maximum principle was independently proved by Chen, Giga and Goto [38] and Evans and Spruck [60]. In [38], the notion of geometric equation is introduced with axiomatic arguments and uniqueness is proved to hold for velocities which are sublinear functions of the curvature. In [60], the authors prove the result for the mean curvature motion by using an approximation argument. The superlinear case was solved in two different ways by Goto [76] and Ishii and Souganidis [94]. (Theorem. 4.46 follows from [94].) Existence of viscosity solutions by Perron's method was proved by Ishii [87]. Contrast invariance of level set flows is basically taken from [38]. Classical overviews of viscosity solutions of second order are the User's guide by Crandall, Ishii and Lions [43]. Crandall also published a remarkable review in a series of lectures by different authors [52]. The book by Fleming and Soner [67] also gives a clear and detailed proof of the maximum principle with the key argument of Jensen. The results we exposed are also fully detailed in Giga's forthcoming book [72].
Convergence results of morphological operators to motions by curvature were proved by Guichard and Morel [81], Catté, Dibos and Koepfler [35], Cao and Moisan [28, 125], Pasquignon [146]. In the mean curvature case, the convergence

of weighted median filters was independently proved by Barles and Georgelin [18], Evans [55], Cao [27], the most general results being Ishii's [90]. More recent and general results of convergence are proved by Leoni [108] and Vivier [166].
The geometric measure theory solution of the mean curvature motion was defined by Brakke [23]. Existence holds in any codimension but there is no result of uniqueness. The nonuniqueness is related to the *fattening* phenomenon, meaning that a curve (or a surface) may develop a nonempty interior. It was first observed by Evans and Spruck [60] and is also well described in [71]. Evans and Spruck proved in [63] that almost every level set of a viscosity solution is a Brakke's solution with unit density, hence Brakke's result for almost everywhere regularity holds. The reaction-diffusion model was used by Allen and Cahn [3]. Its convergence to Brakke's motion was proved by Ilmanen [85]. Its relation with level set motion was established by Evans, Soner and Souganidis [59]. The notion of minimal barrier was introduced by De Giorgi [46]. Connection with level sets were made by Bellettini and Novaga [21] and more recently by Giga [72].

5

Classical numerical methods for curve evolution

In this chapter, we give a short review of the main existing numerical algorithms for curvature motions. Of course, it is not exhaustive, but we shall try to summarize the advantages and the drawbacks of these algorithms. In any case, they will be completely different in form and spirit to the scheme we shall develop in the next chapter. There are basically two classes of algorithms, as can be expected from the analysis of the first part of this monograph: parametric ones, that is to say direct curve discretization, and nonparametric models such as level sets methods (but not only). An immediate advantage of the direct approach is that the amount of data is in general smaller since it is proportional to the number of points of the curve. On the contrary, nonparametric methods are in general more costly since they require a discretization of the whole plane. But, it is often easier to prove that they are stable and they also can be generalized in higher dimension very easily. Throughout the chapter, we consider the equation

$$\frac{\partial C}{\partial t} = G(\kappa)\mathbf{N},$$

for an odd, continuous, nondecreasing function G. Due to its strong connections with physics, the case of the mean curvature motion was particularly under focus. Some specifics algorithms were discovered, and it seems hard to generalize them in a simple way. We shall precise whenever there are some additional restrictions on G.

5.1 Parametric methods

5.1.1 Finite difference methods

This approach is certainly the most natural, since it consists in approaching the curve by a polygon with possibly many vertices and to use finite differences to approximate the normal and the curvature. Each vertex then moves in the approximate normal direction by a function of the approximate curvature. In a series of papers [120,

F. Cao: LNM 1805, pp. 107–110, 2003.

121, 122], Mikula and Ševčovič describe very interesting finite differences schemes that are certainly the state of the art for this approach. In [121], they first define an evolution scheme, continuous in space and discrete in time and they show that the unique solution is indeed a curve shortening. (That is, the length decreases at each time step.) Finally, the problem is also discretized in time by using an implicit finite difference scheme. The conditions of stability cannot be determined easily and the time step must be chosen ad hoc. When $G(\kappa) = \kappa^\gamma$ with $\gamma > 1$, the scheme also develop some singularities because the points are attracted in areas where the curvature is high. As a consequence, the parameterization becomes highly nonuniform and numerical errors appear. In [122], Mikula and Ševčovič propose to automatically reparametrize the curve by adding a tangential velocity term. As we saw in Chap. 3, Lem. 3.25, this term does not change the geometry of the curve. It is chosen such that the local length parameter stays uniform, and this experimentally gives fine results. Nevertheless, the choice of the time step still is very tricky. Moreover, it seems very difficult to prove that the space continuous and time discrete scheme satisfies an inclusion principle as it should do, at least when $G(\kappa) = \kappa^\gamma$ with $\gamma \leq 1$.

5.1.2 Finite element schemes

Finite elements schemes are used in general for equations coming from variational problems. Curve evolution does not enter this class of problems, but if $G(\kappa) = \kappa$, the curve evolution problem is quasilinear and a finite elements approximation has been proposed by Dziuk [53]. The main interest here is that error bounds can be computed and the convergence established. Nevertheless, no inclusion principle seems to be available.

5.2 Non parametric methods

5.2.1 Sethian's level sets methods

The idea to use nonparametric methods appeared in [137] but was undoubtedly popularized by Osher and Sethian [144] who represented a curve by the level line of a function. At that time, their subject of interest was flame propagation and crystal growth. Sethian developed many refined algorithms for level sets evolution, mostly by using finite volume methods. The reader may find a complete description in Sethian's recent book [156]. The problems studied therein are different from ours, since there is not a constant will to make evolve all the level lines of a function at the same time by a geometric motion. In particular, motions violating the inclusion principle are allowed. The implementation is based on classical approximations of conservation laws. It is rather heavy in a computational point of view if the plane is discretized uniformly. Sethian then proposes to refine the grid near the propagation front and to allows a coarse grid far from it. These schemes can be generalized in higher dimensions. They are consistent but there is no proof of convergence.

Moreover, for motions by a power of the curvature larger than 1, numerical problems appear and remain unsolved to our knowledge.

5.2.2 Alvarez and Guichard's finite differences scheme

Alvarez and Guichard were explicitly interested in the resolution of level sets motion for image analysis. They proposed a finite differences explicit scheme described in [6] and which is very efficient in the case of the mean curvature motion and the affine morphological scale space. As any finite differences scheme, it cannot be contrast invariant, but it is consistent. It cannot be strictly rotation invariant neither, but Alvarez and Guichard smartly choose the finite differences such that rotation invariance is respected as much as possible. (And this is numerically effective.) It is not known whether it is monotone or not, though it is experimentally stable for small time steps. Convergence is also unknown. Approximations of the curvature also become inaccurate when the gradient is small.

5.2.3 A monotone and convergent finite difference schemes

Crandall and Lions [45] proposed another finite differences scheme for the mean curvature motion. To our knowledge, this is the only scheme of that kind for which a convergence result is available. The convergence is a consequence of the monotonicity and the consistency. The main difficulty is to obtain monotonicity. The authors use a discrete elliptic regularization, that is to say they add to the approximation of the curvature a small Laplacian-like term that makes the scheme monotone. Deckelnick [48] even gave some error bounds for the convergence when the parameters of Crandall and Lions' scheme are suitably chosen[1]. The scheme is valid in any dimension but only applies to the mean curvature motion. As for any finite difference schemes, the rotation invariance is only asymptotically satisfied.

5.2.4 Bence, Merriman and Osher scheme for mean curvature motion

An original scheme was discovered by Bence, Merriman and Osher who were interested in approximations of motion of interfaces by mean curvature. They were also interested in the motion of junctions of these interfaces between different phases. The scheme, applied to the motion of the boundary of a set X, consists in the following. Assume that we know $X(t)$, the evolving set at instant t. For $h > 0$, we then define $X(t+h)$ by the following procedure.

1. Solve the heat equation $u_t = \Delta u$ on $\mathbb{R}^2 \times [0,h]$ with initial value $u_0 = \chi_{X(t)}$.
2. define $X(t+h) = \{x \in \mathbb{R}^2 \text{ such that } u(x,h) \geq \frac{1}{2}\}$.

[1] At the time being, we do not know any other convergent scheme with a known rate of convergence. The scheme we give in the next chapter follows this rule since convergence is established without any rate.

Of course, we take $X(0) = X$. This operator is nothing but an iterated median filter where the probability measure is a standard gaussian distribution. (See Chap 4.) This scheme defines a time-discrete operator and we retrieve a time continuous operator by letting h go to 0. The advantage of this scheme is that it is very easy to implement, since we only have to numerically solve the heat equation. The monotonicity allows us to extend it to real valued functions by using the Mathematical Morphology results of Chap. 4 (Prop. 4.12). We then obtain a scheme that is completely monotone and contrast invariant. Its convergence has been proved by Barles and Georgelin [18] and Evans [55]. A generalization has been proved by Ishii [90] who proved that the heat kernel can be replaced by any kernel respecting some mild decay conditions. See also [27] for a proof of a slightly weaker result using the Mathematical Morphology formalism. Contrast invariance is theoretically satisfying but, on the other hand, it may stop the numerical evolution before the steady state is really attained. Indeed, since no new value is created, a point must move of at least a step of the grid. Ruuth [148, 149, 150] studied ways to locally refine the grid to avoid this artificial stopping of the evolution. Bence et. al's scheme is unfortunately only valid for the mean curvature motion. Very recently though, two different papers by Leoni [108] and Vivier [166] generalized this approach for more general motions. It consists in iterating the resolution of a parabolic equation on a small time interval and a thresholding.

5.2.5 Elliptic regularization

The reaction diffusion approximation presented in Sect. 4.7.2 has also been implemented. Classical numerical schemes are appropriate since the functional to minimize is quadratic and the equation is semilinear. A finite element implementation for this variational approximation has been proposed in [136]. Such schemes can be generalized in higher dimensions though becoming computationally heavy.

6

A geometrical scheme for curve evolution

Motion by curvature have a large group of invariance since they commute with isometries and contrast changes. Moreover, they define causal families of monotone operators. Finding numerical schemes for curvature motion with the same properties is a real challenge. All the scheme we presented in the previous chapter, violate one or several of these principles. Scalar schemes on a grid do not usually respect rotation invariance (let alone affine invariance) and the nonlinearity of the equation is hardly compatible with the maximum principle (except the scheme by Crandall and Lions [45]). Causality is also limited by the size of the time step. To our knowledge, the first completely invariant and monotone scheme for the affine scale space was designed by Moisan [125, 126] and was called affine erosion. This scheme was later simplified in a very fast algorithm [104]. We end this monograph with a generalization of Moisan's scheme, found by Cao and Moisan[28], and following an idea of Ishii [91]. It shall use the main elements of the theories we developed in the previous chapters. We first define a translation invariant operator on sets and prove that it is continuous and monotone in the sense of Def. 4.4. From Prop. 4.12, it corresponds to a unique morphological function operator. We then prove some consistency results of the type of Prop.4.28, the normal velocity being this time a power a the curvature. From this, we shall then prove that the iterated scheme converges to the solution of the curvature motion in the sense of viscosity solution. This scheme will completely be free of curvature estimates by finite differences. Instead, the curvature will be approximated by some function of the area comprised between the curve and a chord of the curve. In this way, the estimate shall be more stable since area computations are less sensitive to little oscillations of the curve than finite differences. The monotonicity also implies that the scheme is inconditionally stable and the time step and the space discretization are completely uncorrelated. This scheme is quite general since it allows any normal velocity equal to a power γ of the curvature for any $\gamma \geq 1/3$, and numerically, this power may be equal to 5 and nearly 10 in certain cases. Just think that if this power is 7, then doubling the curvature is equivalent to multiply the velocity by a factor 128, which is nearly infinite in a numerical point of view!
The plan of the chapter will follow the problematic above: we first give some no-

F. Cao: LNM 1805, pp. 111–166, 2003.

tations in order to define an operator, that will be called erosion[1] since it consists in removing some material to the set. It will depend on a single scale parameter, related to the thinness of the removed part. By duality, we define a dilation operator adding material to a set. Both operators satisfy the nice properties exposed in Chap. 4: there are morphological, that is monotone, continuous and translation invariant. This allows us to use the Mathematical Morphology machinery and to extend them to real valued functions. We shall see that, when the scale tends to 0, the erosion is consistent with a generalized motion by curvature. We then jump to the viscosity solution theory and show that the consistency implies the convergence of the iterated erosion-dilation to the semi-group of a motion of level sets by a function of the curvature. This chapter will contain a proof of convergence from A to Z. In fact, a result proved by Barles and Souganidis [20], brings back the proof of convergence to consistency results. Indeed, they proved that convergence holds for any monotone, stable and (of course) consistent operators. The final proof of convergence (Thm. 6.27) is nothing that their result in the special case of curvature motions, and the reader will be easily convinced that the convergence theorems given without proof in Chap. 4 hold because of monotonicity and consistency. Most of the time, this last only uses elementary analysis and geometry. There again, the process is quite general but needs to be adapted for any morphological operator, as can be seen in different cases treated in [27, 35, 81, 146].
The implementation of the erosion is not completely straightforward and we propose a possible one which has been tested on the numerous numerical experiments we end with. Throughout the chapter, we will use bold faced letters for points of $\mathbb{R}^2$ and regular fonts for their coordinates, that is $\mathbf{x} = (x, y) \in \mathbb{R}^2$.

6.1 Preliminary definitions

The algorithm we detail will belong to the class of level sets method. As a consequence, we are more interested in the evolution of sets rather than curve evolution, and a curve shall be viewed as the boundary of a set which is licit by the theorem of Alexandrov 2.4. In the following, we assume that the plane is counterclockwise oriented (this orientation is obviously arbitrary, and the results shall not depend upon it). We say that an oriented simple curve C in $\mathbb{R}^2$ is a *semi-closed* curve, if it divides the plane in exactly two connected components that we call its interior and its exterior. At this state, let us simply remark that, fortunately, any closed curve is also a semi-closed curve. Introducing semi-closed curves will allow us to consider the evolution of unbounded sets that will be defined as the interior of their boundary (which might be a closed curve). We assume that curves are oriented and piecewise smooth. (Lipschitz is enough in practice.) As in Chap. 2, we assume that the tangent vector and the inner normal vector form a direct orthonormal basis. This uniquely defines the inner normal and the interior is the part of $\mathbb{R}^2 \backslash C$ this normal points to.

[1] We keep the same name as Moisan [126] since our operator is a generalization of his affine erosion.

Otherwise said, if K is the interior of C, we suppose that C is oriented such that K lies on “the left” when C is positively described. Beware that this definition implies that the interior of a closed curve shall not always be the bounded component of $\mathbb{R}^2 \backslash C$ since it depends on the curve orientation.

Definition 6.1. *A* chord *is a segment of the form* $]C(s), C(t)[$ *that does not intersect* C *for any point with parameter between* s *and* t*, that is to say*

$$]C(s), C(t)[\cap C(]s, t[) = \emptyset.$$

With this definition, remark that the segment on Fig. 6.1 is actually a chord even though it does not cut the curve only at its endpoints. We sometimes denote a chord $]C(s), C(t)[$ by the parameters of its endpoints that is (s, t).

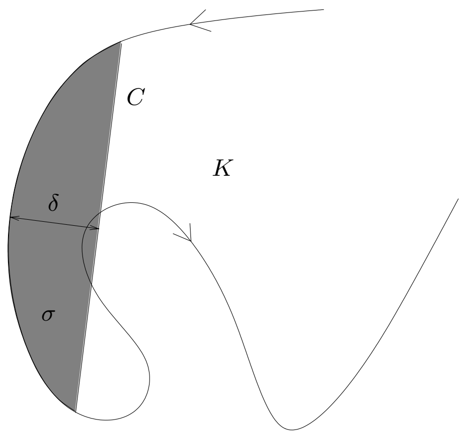


Fig. 6.1. A σ-chord set. δ is the chord-arc distance

Definition 6.2. *A* chord set $C_{s,t}$ *is the connected set enclosed by a chord* $\mathcal{C} =]C(s), C(t)[$ *and the curve* $C(]s, t[)$. *We say that* $C_{s,t}$ *is a* σ-chord set *and* (s, t) *is a* σ-chord *if its area* $|C_{s,t}|$ *is equal to* $|\sigma|$ *and if the area of any chord set strictly included in* $C_{s,t}$ *is strictly less than* $|\sigma|$. *We denote by* $\mathcal{K}_\sigma(C)$ *the set of* σ*-chord sets.*

Let us also remark the absolute value around σ in the definition. Since the curve is oriented, the area included in a closed curve is algebraic.

Let $\mathcal{C} =]C(s), C(t)[$ a σ-chord of K and $C_{s,t}$ the associated σ-chord set.

Definition 6.3. *We call* chord arc distance *of* $\mathcal{C}$ *(or of* $C_{s,t}$*) the number*

$$\delta(C([s, t]), [C(s), C(t)]),$$

where δ *is the Hausdorff semi-distance (in particular it is not commutative) defined by*

$$\delta(A,B) = \sup_{\mathbf{x}\in A}\inf_{\mathbf{y}\in B} |\mathbf{x}-\mathbf{y}|.$$

(see Figure 6.1 for an illustration of this definition.)

For $\mathbf{x}$ in $\mathbb{R}^2$, we also denote by $\delta_{s,t}(\mathbf{x})$ the distance from $\mathbf{x}$ to the oriented line $]C(s), C(t)[$, that is

$$\delta_{s,t}(\mathbf{x}) = \frac{[\mathbf{x}-C(s), C(t)-C(s)]}{|C(t)-C(s)|}.$$

(If $\mathbf{x}, \mathbf{y}$ are in $\mathbb{R}^2$, we denote by $[\mathbf{x}, \mathbf{y}]$ the determinant of the 2×2 matrix with columns $\mathbf{x}$ and $\mathbf{y}$). We also denote by $\mathcal{K}_\sigma^+(C)$ the sets of positive σ-chords i.e the chord-sets $C_{s,t}$ satisfying

$$\forall \mathbf{x}\in C(]s,t[), \quad \delta_{s,t}(\mathbf{x}) \geq 0.$$

In the same way, we can define $\mathcal{K}_\sigma^-$ the set of negative chord sets. Remark that for positive chord sets, the chord-arc distance is nothing but $\sup \delta_{s,t}(\mathbf{x})$ for $\mathbf{x}\in C([s,t])$ and for negative chord sets, the chord-arc distance is $-\inf \delta_{s,t}(\mathbf{x})$ for $\mathbf{x}\in C([s,t])$. Finally, we set $\omega = \frac{1}{2}\left(\frac{3}{2}\right)^{2/3}$.

6.2 Erosion

Let $G : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function. We also assume that G is 1-Lipschitz and that $G(0) = 0$.
Let K be a smooth set in $\mathbb{R}^2$ (for example piecewise C^1) with oriented boundary $C = \partial K$. Let $\sigma > 0$. Assume that $\mathcal{L}^2(K) > \sigma$. Let $C_{s,t}$ be a chord set of K with chord-arc distance h. We define

$$\tau(C_{s,t}) = \left\{ \mathbf{x}\in K \,/\, \delta_{s,t}(\mathbf{x}) > h - \omega\sigma^{2/3} G(\frac{h}{\omega\sigma^{2/3}}) \right\}. \tag{6.1}$$

In the sequel, we shall briefly say that $\tau(C_{s,t})$ is a modified σ-chord set. We remark that the right-hand term of the inequality above is nonnegative because of the Lipschitz assumption on G. Hence, a modified σ-chord set is always included in its associated σ-chord set. On Figure 6.2, we represent a σ-chord and its modified chord. The modified σ-chord set $\tau(C_{s,t})$ is filled.

Definition 6.4. *Let $\sigma > 0$. Let K be the interior of a piecewise C^1 semi-closed curve. We define*

$$E_\sigma(K) = K \setminus \left(\bigcup_{\substack{A\in\mathcal{K}_{\sigma'}^+(K)\\ \sigma'\leq\sigma}} \tau(A) \right), \tag{6.2}$$

if $|K| > \sigma$ and $E_\sigma(K) = \emptyset$ else. We will refer to E_σ as an erosion operator *and to $E_\sigma(K)$ as the eroded of K at scale σ.*

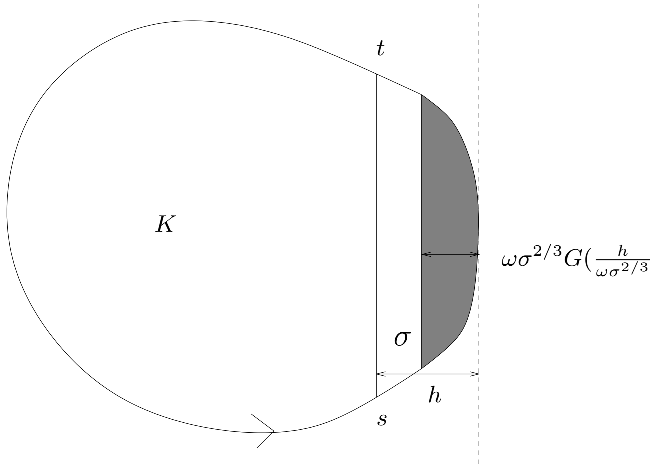


Fig. 6.2. A σ-chord set before and after transform

Remark 6.5. The algorithm in [126] corresponds to $G(x) = x$ and has been named *affine erosion*. It consists in "peeling" the set K such that any peeling has an area equal to σ. As a direct and obvious consequence, this operator admits the special affine group as symmetry group.

Remark 6.6. We can generalize Def. 6.4 to sets with several connected components by applying the erosion to each component.

Remark 6.7. Note also that $G(x) \leq x$ because G is 1-Lipschitz. Therefore, the modified σ-chord set is always included in the original chord-set. The situation of Fig. 6.2 is not a coincidence.

6.3 Properties of the erosion

We give some properties of the erosion operator defined in the previous section. In what follows, we always assume that the sets have an area larger that σ, else the results obviously hold.

Lemma 6.8. *Let K be a convex set of $\mathbb{R}^2$. Then $E_\sigma(K)$ is also convex.*

Proof. Indeed, $E_\sigma(K)$ can also be written as $\cap_A(K\backslash\tau(A))$ where the intersection is taken over all the positive σ-chord sets. Then, we remark that $E_\sigma(K)$ is an intersection of convex sets and we conclude. □

Lemma 6.9. *If K is a smooth compact set, then $E_\sigma(K)$ is a compact set.*

Proof. This is obvious since $E_\sigma(K)$ is an intersection of compact sets. □

In this lemma, we do not precise what smooth exactly means since the compactness is really the relevant information. The reader may replace smooth by Lipschitz.
We now give a far more important property which is a comparison principle. As will be noticed in the proof, as soon as G is 1-Lipschitz, this stability result shall be inconditionally true, no matter what the value of σ and the regularity of the sets are. We also make full use of the fact that G is nondecreasing and 1-Lipschitz.

Proposition 6.10 (Monotonicity). *Let $K_1 \subset K_2$. Assume that G is nondecreasing and* 1*-Lipschitz. Then*

$$E_\sigma(K_1) \subset E_\sigma(K_2).$$

Proof. Assume that $\mathbf{x} \in K_2$ and $\mathbf{x} \notin E_\sigma(K_2)$. We prove that $\mathbf{x} \notin E_\sigma(K_1)$. If $\mathbf{x} \notin K_1$ then $\mathbf{x} \notin E_\sigma(K_1)$ since $E_\sigma(K_1) \subset K_1$, and this is over. Assume now that $\mathbf{x} \in K_1$. By assumption, there exists a σ'-chord of K_2 (that we denote by $\mathcal{C}$) with $\sigma' \leq \sigma$ such that $\mathbf{x}$ belongs to the modified chord-set. The Lipschitz condition on G implies that $\mathbf{x}$ also belongs to the σ'-chord set. The same σ'-chord delimits in K_1 a chord-set with area $\sigma'' \leq \sigma'$. It then suffices to prove that this chord excludes $\mathbf{x}$ from $E_\sigma(K_1)$. Consider the situation illustrated in Fig. 6.3 where

- h_1 and h_2 are the chord-arc distances of $\mathcal{C}$ in K_1 and K_2.
- l_1 and l_2 are the chord-arc distances of the associated modified chords.
- l is the difference of length between K_1 and K_2 in the direction that is orthogonal to the chords, i.e. $l = h_2 - h_1$.

It is enough to prove that $l + l_1 \geq l_2$. But, we know that $l_1 = \sigma^{2/3}\omega G(\frac{h_1}{\sigma^{2/3}\omega})$ and $l_2 = \sigma^{2/3}\omega G(\frac{h_2}{\sigma^{2/3}\omega})$. Hence,

$$\begin{aligned}
l_2 &= \sigma^{2/3}\omega G\left(\frac{h_1 + l}{\sigma^{2/3}\omega}\right) \\
&\leq \sigma^{2/3}\omega G\left(\frac{h_1}{\sigma^{2/3}\omega}\right) + l \\
&= l_1 + l,
\end{aligned}$$

the middle inequality being true since G is 1-Lipschitz. $\square$

Let now turn to the continuity property.

Proposition 6.11 (Continuity). *Let K_n a sequence of compact sets whose boundary is a Jordan curve. Then*

$$E_\sigma(\bigcap_{n\in\mathbb{N}} K_n) = \bigcap_{n\in\mathbb{N}} E_\sigma(K_n). \tag{6.3}$$

Proof. Let $K = \cap_n K_n$. Since $K \subset K_n$ for any n, by monotonicity we have also $E_\sigma(K) \subset E_\sigma(K_n)$, implying the first part of the equality.
In order to prove the reverse inclusion, we can assume that the family K_n is non-increasing. Without loss of generality, we also suppose that (K_n) converges to K

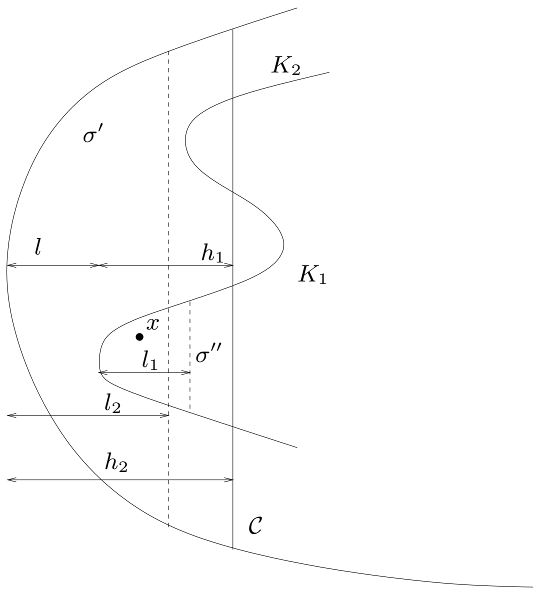


Fig. 6.3. The inclusion principle

for the Hausdorff distance between compact sets. Assume that $\mathbf{x} \notin E_\sigma(K)$. By definition, there is a chord (s,t) with area not more than σ such that the modified chord excludes $\mathbf{x}$. Since $E_\sigma(K)$ is closed, its complementary is an open set; thus we can assume that the area of $C_{s,t}$ is strictly less than σ. (Indeed, it suffices to take a slightly smaller chord-set still excluding $\mathbf{x}$. By definition of σ chord-sets, the area is strictly less than the previous one.) In K_n it also defines a chord and for n large enough, the area of this chord is also less than σ (by using convergence of measures). Moreover, as K_n tends to K for the Hausdorff distance, the chord-arc distance also converges. This implies that the chord excludes $\mathbf{x}$ in K_n for n large enough. Hence $\mathbf{x} \notin \cap E_\sigma(K_n)$ and this ends the proof. $\square$

Remark 6.12. The compactness assumption in the proposition above is far from necessary. It suffices for example that the boundary of the sets is locally convex or concave. This ensures that the erosion is local when σ is small. We can then conclude by the same arguments where we restrict the sets in a ball with fixed area. Indeed, it also suffices that K_n locally converge to K for the Hausdorff distance.

The continuity and the monotonicity permit to define the erosion on a more general class of sets.

Definition 6.13. *Let X a subset of $\mathbb{R}^2$. Let $\mathcal{K}(X)$ be the set of subsets of $\mathbb{R}^2$ whose boundary is piecewise C^1 and that contains X. We define*

$$E_\sigma(X) = \bigcap_{K \in \mathcal{K}(X)} E_\sigma(K). \tag{6.4}$$

Then E_σ is a morphological operator.

The behavior of the erosion will be interesting when the scale tends to zero. Indeed, in the limit, the erosion does not see the global shape of the eroded set and should have a generic behavior depending on the local features of the set. This is the object of the following proposition.

Proposition 6.14. *Assume that $C = \partial K$ is of class C^2. Let $\mathbf{x} \in C$ such that the curvature of C at $\mathbf{x}$ is not equal to 0. Then*

$$\lim_{\sigma \to 0} \frac{d(\mathbf{x}, E_\sigma(K))}{\omega \sigma^{2/3}} = G((\kappa(\mathbf{x})^+)^{1/3}).$$

Proof. Assume that C is concave at $\mathbf{x}$ (that is the curvature at $\mathbf{x}$ is strictly negative in a neighborhood of $\mathbf{x}$). Then for σ small enough, any σ-chord (s,t) such that $\mathbf{x} \in C([s,t])$ is a (strictly) negative σ-chord. Hence $\mathbf{x} \in E_\sigma(K)$ and the proposition follows from $G(0) = 0$. Assume now that C is strictly convex at $\mathbf{x}$ (thus in a neighborhood of $\mathbf{x}$). Then for any σ-chord with chord-arc distance equal to h, we have

$$h = \omega \cdot \kappa^{1/3} \cdot \sigma^{2/3} + o(\sigma^{2/3}), \tag{6.5}$$

which can be easily established for a parabola, then for any regular curve by approximation. Consider the σ-chords (s,t) such that $\mathbf{x}$ belongs to the associated chord set $C_{s,t}$. Since the curve is strictly convex, as σ tends to 0, the endpoints of the σ-chords above tend to $\mathbf{x}$. We use approximation (6.5) above and the fact that the curvature is continuous and conclude. □

If $G(x) = x$, we recognize the most simple affine invariant flow of Sapiro and Tannenbaum. This is also coherent with the results of Alvarez, Guichard, Lions and Morel. Indeed, the erosion defines a morphological affine invariant operator. However, the erosion is not self dual (that is $E_\sigma(K^c) \neq (E_\sigma(K))^c$, where the superscript denotes the complementary set) and the erosion has not the same behavior on convex and concave sets. To reestablish the symmetry, we define the dual operator D_σ (called dilation operator) by

$$D_\sigma(K) = (E_\sigma(K^c))^c. \tag{6.6}$$

Exercise 6.15. By using the properties of the erosion, prove that

1. D_σ is monotone,
2. D_σ is continuous,
3. if $\mathbf{x} \in \partial K$, where K has a C^2 boundary and curvature $\kappa(\mathbf{x}) \neq 0$, then

$$d(\mathbf{x}, \partial(D_\sigma(K))) = \omega \sigma^{2/3} \left(G((\kappa(\mathbf{x})^-)^{1/3}) + o(1) \right).$$

This operator satisfies the same properties as E_σ except the consistency result where the positive part of the curvature has to be replaced by the negative part.

6.4 Erosion and level sets

As we have seen in Chap. 3, general results on existence and uniqueness of curve evolution are not known. As a consequence, instead of directly studying the convergence of the erosion, we exploit the monotonicity and continuity property and extend the erosion and dilation to continuous functions.

Proposition 6.16 (Prop. 4.12 applied to erosion). *Let $\mathcal{F}$ a set of real valued continuous functions in $\mathbb{R}^2$. Then, we can extend E_σ to elements of $\mathcal{F}$ by setting*

$$\chi_\lambda(E_\sigma(u)) = E_\sigma(\chi_\lambda(u)). \tag{6.7}$$

This uniquely defines a function morphological operator.

We now face the same problematic we encountered in Chap. 4, that is studying the asymptotic behavior of the erosion acting on functions, when the scale parameter σ tends to 0.

6.4.1 Consistency

Consistency is generally proved in two steps. (See [27, 81, 145] for the same kind of approximations.) First, the analyzed function is a simple quadratic form. Then, we prove that the general case follows if the morphological operator can be localized, in which case a function can be approximated by a quadratic form.

Lemma 6.17. *Let be $p > 0$, $b < 0$ and $u(\mathbf{x}) = px + by^2$. Then*

$$E_\sigma(u)(0) = p\omega G\left(\frac{b}{2p}\right)\sigma^{2/3} + o(\sigma^{2/3}).$$

Proof. Let us consider the level lines of u. They all are parabolas with parameter $b/2$. For any σ, the chord-arc set with the largest chord-arc distance is the one whose chord is perpendicular to the axis of the parabola. Thus, the value of $E_\sigma(0)$ can be determined by examining the modified chord-arc distance for all the level lines for the unique σ-chord normal to the axis of the parabola. We can then use the consistency result (Prop. 6.14). Since the curvature at the corresponding point is $b/2p$, we obtain the result. □

Note that $p = |Du|$ and $b/2p = \operatorname{curv} u(0)$. Thus, the consistency can be written

$$E_\sigma(u) = |Du|G(\operatorname{curv} u^-)\omega\sigma^{2/3} + o(\sigma^{2/3}).$$

Notice that we take the negative part of $\operatorname{curv} u$ contrary to what is proved in Prop. 6.14. This simply comes from the fact that we consider upper level sets, and because of this convention, $\operatorname{curv} u$ is negative if the boundary is convex.
Let us now prove the general case.

Proposition 6.18. *Let u be a C^3 function and x a point such that $Du(x) \neq 0$ and* $\operatorname{curv} u \neq 0$. *Then*

$$E_\sigma(u)(\mathbf{x}) = u(\mathbf{x}) - \omega\sigma^{2/3}|Du|G((\kappa(u)^-)^{1/3}) + o(\sigma^{2/3}). \tag{6.8}$$

$$D_\sigma(u)(\mathbf{x}) = u(\mathbf{x}) + \omega\sigma^{2/3}|Du|G((\kappa(u)^+)^{1/3}) + o(\sigma^{2/3}). \tag{6.9}$$

In both equations, the term $o(\sigma^{2/3})$ is locally uniform in $\mathbf{x}$.

Proof. The proof may seem a bit long and tedious but it is not very difficult and uses elementary geometrical arguments. The aim is to prove that $E_\sigma(u)$ only depends on local features of u. In general, difficulties may arise when the differential operator we want to approximate is singular. We see that the curvature is not defined at critical points, but the growth of G at infinity is absorbed by the term $|Du|$. Consequently, the differential operator is defined and continuous everywhere. By using translation invariance and contrast invariance, we may assume that $\mathbf{x} = 0$ and $u(0) = 0$. We choose $r = \sigma^{1/4}$, yielding $\sigma^{1/3} = o(r)$ and $r^3 = o(\sigma^{2/3})$ when σ tends to 0. The key of the proof is the following. If r is small enough, the curvature of the level lines of u has a strict sign in $D(0, r)$. As a consequence, for any σ-chord, we can estimate the chord-arc distance by (6.5). Moreover the same kind of approximation (made on a circle or a parabola) shows that the length of the chord is equivalent to $(\frac{\sigma}{\kappa})^{1/3}$. Hence, any σ-chord intersecting $D(0, \frac{r}{2})$ is asymptotically included in $D(0, \frac{r}{2}(1+r^\varepsilon))$ for some $\varepsilon > 0$ (because of the choice of r) and the erosion operator is local.
Assume first that $\kappa(u)(0) > 0$. Define u_+ and u_- by

$$\begin{cases} \forall \mathbf{x} \in D(0, r) \quad u_+(\mathbf{x}) = u_-(\mathbf{x}) = u(\mathbf{x}), \\ u_+(\mathbf{x}) = +\infty, \quad u_-(\mathbf{x}) = -\infty \text{ elsewhere.} \end{cases}$$

(We can replace the infinite value by very large numbers). The global inequalities $u_- \leq u \leq u_+$ yield $E_\sigma(u_-) \leq E_\sigma(u) \leq E_\sigma(u_+)$. If σ is small enough, the level lines of u are uniformly strictly concave in $D(0, r)$. Thus, for u_- and u_+, there is no positive σ-chord intersecting $D(0, \frac{r}{2})$. (See Fig 6.4.) Hence

$$\forall \mathbf{x} \in D(0, \frac{r}{2}) \quad u(\mathbf{x}) = E_\sigma(u_+)(\mathbf{x}) = E_\sigma(u)(\mathbf{x}) = E_\sigma(u_-)(\mathbf{x}).$$

Assume now that $\kappa < 0$. Define

$$\begin{cases} \forall \mathbf{x} \in D(0, r) \quad v(\mathbf{x}) = Du(0) \cdot \mathbf{x} + \frac{1}{2}D^2u(0)(\mathbf{x}, \mathbf{x}) - kr^3, \\ v(\mathbf{x}) = -\infty \text{ elsewhere.} \end{cases} \tag{6.10}$$

Define also

$$\begin{cases} \forall \mathbf{x} \in D(0, r) \quad w(\mathbf{x}) = Du(0) \cdot \mathbf{x} + \frac{1}{2}D^2u(0)(\mathbf{x}, \mathbf{x}) + kr^3, \\ w(\mathbf{x}) = +\infty \text{ elsewhere.} \end{cases} \tag{6.11}$$

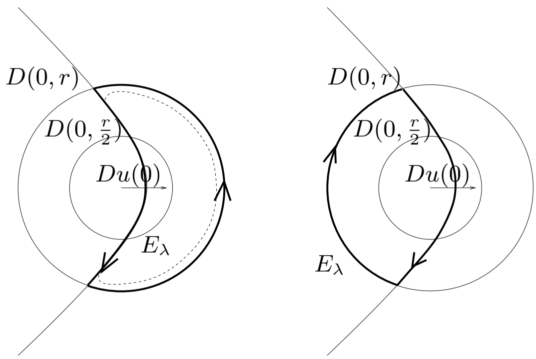


Fig. 6.4. Case $\kappa > 0$. On the left, a level set of u_-. The oriented boundary is the bold line. The level set is the bounded connected component delimited by the curve. The dashed line is the boundary of the eroded set. On the right, a level set of u_+: the boundary is the bold oriented line and the level set is the unbounded component. In both cases, there is no σ-chord set intersecting $D(0,\frac{r}{2})$. Hence, the erosion has no effect

In both case, k is a constant chosen such that for σ small enough, we have

$$\forall \mathbf{x} \in \mathbb{R}^2 \quad v(\mathbf{x}) \le u(\mathbf{x}) \le w(\mathbf{x})$$

yielding

$$\forall \mathbf{x} \in \mathbb{R}^2 \quad E_\sigma(v)(\mathbf{x}) \le E_\sigma(u)(\mathbf{x}) \le E_\sigma(w)(\mathbf{x}).$$

This is possible since we assumed that u is C^3. As v and w have trivial level sets out of $D(0,r)$, their image by the erosion is determined by its effect in $D(0,r)$. We now use the consistency result (Prop. 6.14). The only trick is that the level lines of u and v are not parabolas in the canonical form like in Lem. 6.17 above. By choosing the coordinates axes, we can write v as

$$v(\mathbf{x}) = px + ax^2 + by^2 + cxy + kr^3, \tag{6.12}$$

with $p > 0$ and $b < 0$ (since the curvature at the origin is $\frac{b}{2p}$). We first eliminate the xy term by using

$$xy \ge -\frac{1}{\varepsilon}x^2 - \varepsilon y^2,$$

where ε is a parameter depending on σ that we will choose later. We then cancel the cross term in (6.12) by replacing v by v_ε for which the coefficients a and b becomes $a(\varepsilon) = O(\varepsilon^{-1})$ and $b(\varepsilon) = b + O(\varepsilon)$. (To simplify the notations, we shall not note the dependency of a and b upon ε in the following.) There is still the bothering x^2 term. We remark that $E_\sigma v_\varepsilon(0)$ tends to 0 when σ tend to 0. We then introduce the function $g(s) = s - \frac{a}{p^2}s^2$ which is nondecreasing (and thus a contrast change) for s smaller than $O(\varepsilon)$. We then choose $\varepsilon = \sigma^{-\theta}$ for $0 < \theta < 1/4$ and small such that in $D(0,r)$, the considered quadratic form v_ε is $O(r) + O(r^2\varepsilon^{-1}) = o(\varepsilon)$. Thus, it assumes values for which g is invertible. We then compute

$$E_\sigma v_\varepsilon(0) = g^{-1}(E_\sigma(g \circ v_\varepsilon)(0)).$$

But,

$$g \circ v_\varepsilon(\mathbf{x}) = px + by^2 - kr^3 + O(\varepsilon^{-2}r^3).$$

We choose θ such that $3/4 - 2\theta > 2/3$, that is $\theta < 1/24$. We then get $g \circ v_\varepsilon(\mathbf{x}) = px + by^2 - o(\sigma^{2/3})$ the $o(\sigma^{2/3})$ being uniform in $D(0,r)$. This time, we can apply Lem. 6.17 and get

$$E_\sigma v_\varepsilon(0) = g^{-1}\left(G\left(\frac{b}{2p}\right)\sigma^{2/3} + o(\sigma^{2/3})\right).$$

Finally, we remark that $g^{-1}(s) = s + O(\varepsilon^{-1}s^2)$. Since $\sigma^{4/3}\varepsilon^{-1} = o(\sigma^{2/3})$, we obtain

$$E_\sigma(v)(0) \geq -\omega|Du(0)|G(\kappa^-) + o(\sigma^{2/3}).$$

By applying exactly the same method to w (we eliminate the xy term by using $xy < \varepsilon^{-1}x^2 + \varepsilon y^2$ and apply the same contrast change), we finally obtain an upper bound for $E_\sigma(u)$ which is equal to the lower bound, up to a $o(\sigma^{2/3})$ term. This ends the proof of the consistency of the erosion.
The case of the dilation operator D_σ follows from $D_\sigma(u) = -E_\sigma(-u)$. □

The previous proposition means that the erosion makes upper level sets move inward when they are locally convex and does not modify them if they are concave. The dilation has the dual behavior (replace concave by convex). In order to retrieve the symmetry between erosion and dilation, a natural idea is to alternate them. We first extend G so as to make it odd by setting

$$G(x) = -G(-x)$$

for any $x \leq 0$.

Proposition 6.19. *Let u be a C^3 function. Suppose that $Du(x) \neq 0$ and $\kappa(u)(x) \neq 0$. Then*

$$T_\sigma u(\mathbf{x}) = u(\mathbf{x}) + \omega\sigma^{2/3}|Du|G((\kappa(u))^{1/3})(\mathbf{x}) + o(\sigma^{2/3}), \tag{6.13}$$

where T_σ is either $E_\sigma \circ D_\sigma$ or $D_\sigma \circ E_\sigma$.

Proof. This follows from the fact that near a point with gradient and curvature different from zero, the arguments developed in the previous proposition are uniform. □

The reader would have remarked that nothing precise has been described at critical points. In fact, it suffices to prove that we can control the erosion-dilation well enough. This is the object of the following lemma.

Lemma 6.20. *Let u be a C^3 function with $u(0) = 0$, $Du(0) = 0$ and $D^2u(0) = 0$. Then*

$$\lim_{\substack{\mathbf{x}\to 0\\ \sigma\to 0}} \frac{E_\sigma(u)(\mathbf{x}) - u(\mathbf{x})}{\sigma^{2/3}} = 0. \tag{6.14}$$

The limit is taken for $\mathbf{x}$ and σ tending to 0 independently.

Proof. We use a strong property of the affine erosion operator defined in [125]. Let A_σ be this operator. We recall that it coincides with E_σ when $G(\kappa) = \kappa$. As G is 1-Lipschitz, for any set K, we have

$$A_\sigma(K) \subset E_\sigma(K) \subset K.$$

To see this, refer to Equation (6.1). For the affine erosion, the modified σ chord-sets coincide with the σ chord-sets. In any other case, the σ-chord sets contain their modified version.
When transposing this to functions, we obtain

$$A_\sigma(u) \leq E_\sigma u \leq u.$$

It is proven in [125], that the affine erosion satisfies a strong locality property[2]. As a consequence, consistency is proved to be uniform without any additional conditions. Thus

$$\begin{aligned} 0 &\geq E_\sigma u(\mathbf{x}) - u(\mathbf{x}) \\ &\geq A_\sigma u(\mathbf{x}) - u(\mathbf{x}) \\ &= |Du|(\mathbf{x})(\kappa(u)(\mathbf{x}))^{1/3} + o(\sigma^{2/3}) \end{aligned} \tag{6.15}$$

where the $o(\sigma^{2/3})$ term in the last equality is uniform with respect to $\mathbf{x}$. By letting σ and $\mathbf{x}$ tend to 0 independently, we obtain the result. □

Remark 6.21. The same result is obviously valid for the dilation D_σ but also for the alternate operator $D_\sigma \circ E_\sigma$. Indeed, $E_\sigma(u) \leq u$ also yields

$$E_\sigma(u) \leq D_\sigma \circ E_\sigma(u) \leq D_\sigma(u).$$

[2] In fact, this is the true difficulty of the lemma. In [81], Guichard and Morel introduce the notion of localizability of structuring elements. It is possible to approximate in a precise geometric sense unbounded families of structuring elements by bounded ones. Uniform consistency then easily follows since everything can be proved with bounded families. In particular, Taylor expansions hold with uniform bound. What is far from obvious is to prove that a given family of structuring elements is localizale, and this is a result shown in [125] for the elements associated with the affine erosion. In the next section, we shall use two results (Prop 6.24 and Lem. 6.26) allowing not to use the localizability property and also give the announced result.

6.4.2 Convergence

Consistency and monotonicity imply the convergence of the iterated erosion/dilation to the solution of a curvature motion.

Theorem 6.22. *Let $u_0 \in BUC(\mathbb{R}^2)$. For any $\sigma > 0$, let T_σ be either $D_\sigma \circ E_\sigma$ or $E_\sigma \circ D_\sigma$. We define u_σ by*

$$\forall t \in [n\omega\sigma^{2/3}, (n+1)\omega\sigma^{2/3}) \quad u_\sigma(\mathbf{x}, t) = (T_\sigma)^n(u_0)(\mathbf{x}). \tag{6.16}$$

Then, when σ tends to 0, u_σ converges locally uniformly towards the unique solution of the equation

$$\frac{\partial u}{\partial t} = |Du|G((\mathrm{curv}\, u)^{1/3}) \tag{6.17}$$

with initial value u_0.

Thanks to the theory developed in Chap. 4, we know that the solution of (6.17) exists and is unique. The proof of convergence follows exactly the same sketch as the one of Thm. 6.27 below, to which we refer the reader. A bit in advance, let us qualitatively explain how all the elements assemble. First, it is easily seen that the approximate solution u_σ are equicontinuous in space since they have the same modulus of continuity as u_0. They are not continuous in time since they are piecewise constant in time on intervals of length $\omega\sigma^{2/3}$. Thus, we cannot directly apply Ascoli-Arzela's Theorem though this is essentially the idea. Indeed, when σ tends to 0, we prove that we can control the time variation of u_σ and that we have limit values up to subsequence extraction. We then prove that the limit of any subsequence is viscosity solution by comparing it to smooth test functions. When the gradient of the test function is not equal to 0, we can apply the consistency. When it is null, we use Lem. 6.20 or an equivalent result.

6.5 Evolution by a power of the curvature

Here and thereafter, we shall be interested in the case where the normal velocity is a power of the curvature ($V = \kappa^\gamma$). The results above do not apply directly except for the only case $G(x) = x$, since no power function except $G(x) = x$ is globally 1-Lipschitz continous. Thus the power $1/3$ plays a central role. If $\gamma < 1/3$, the corresponding erosion would not respect inclusion principle for small curvatures. Thus, we have no hope to shorten nonconvex curves. If $\gamma > 1$, then the points with high curvature are problematic. Nevertheless, we can approximate the function $x^{3\gamma}$ (which naturally arises) by replacing it by a 1-Lipschitz function. Namely, we set

$$G(x) = \begin{cases} x^{3\gamma} \text{ if } x \leq \alpha_\gamma \\ \alpha_\gamma^{3\gamma} + (x - \alpha_\gamma) \text{ if } x > \alpha_\gamma. \end{cases} \tag{6.18}$$

where

$$\alpha_\gamma = \left(\frac{1}{3\gamma}\right)^{\frac{1}{3\gamma-1}}$$

is the largest positive number such that the power function $x^{3\gamma}$ has a derivative smaller than 1.

6.5.1 Consistency

Of course, we made everything such that G satisfies the hypotheses of the previous section. The associated evolution is a motion by curvature to the right power for small enough curvatures. For high curvatures, this is no longer true.
The trick is to use the homogeneity properties of the power functions and to adequately adapt the definition of the erosion. Thus, we slightly change the definition of E_σ. For a σ-set $C_{s,t}$, we now set

$$\tau(C_{s,t}) = \{x \in K \,/\, \delta_{s,t}(x) > h - G(h)\} \tag{6.19}$$

and we still define the erosion operator by

$$E_\sigma(K) = K \backslash \left(\bigcup_{\substack{A \in \mathcal{K}^+_{\sigma'}(K) \\ \sigma' \leq \sigma}} \tau(A) \right). \tag{6.20}$$

We do not repeat the whole proofs with this new definition of the erosion since they are the same as above. We just recall that

- the modified erosion operator E_σ still satisfies the inclusion principle.
- In the same way, we can prove continuity with the same hypotheses as above.

Nevertheless, consistency changes a bit and uses the covariance of power functions with respect to dilations. To be precise, we write the new consistency result.

Proposition 6.23. *Assume that $C = \partial K$ is of class C^2. Let $\mathbf{x} \in C$ such that the curvature of C at $\mathbf{x}$ negative. Then*

$$\lim_{\sigma \to 0} \frac{d(\mathbf{x}, E_\sigma(K))}{\left(\omega\sigma^{2/3}\right)^{3\gamma}} = (\kappa(\mathbf{x})^+)^\gamma.$$

Proof. The body of the proof is the same as in the proof of Proposition 6.14. The chord-arc distance is still equivalent to $\omega\sigma^{2/3}\kappa^{1/3}$. Thus, the modified chord-arc distance is $G(\omega\sigma^{2/3}\kappa^{1/3}(1 + o(1)))$. If σ is small enough (it might depend on the point since the $o(1)$ term depends on the point), this value is $(\omega\sigma^{2/3}\kappa^{1/3}(1 + o(1)))^{3\gamma}$, yielding the result. □

The proof of monotonicity and continuity are analogous to the previous case. We conclude that the erosion is morphological and can be extended to continuous functions. We define the dual dilation operator as above and we now prove consistency. As can be foreseen, the arguments of the regular case are much too rough for these singular operators (as soon as $\gamma > 1$). Thus, we shall need more precise calculations still relying on the inclusion principle.

Proposition 6.24. *Let u be a C^3 function. Let $\mathbf{x}$ be such that $Du(\mathbf{x}) \neq 0$ and $\kappa(u)(\mathbf{x}) \neq 0$. Then*

$$E_\sigma(u)(\mathbf{x}) = u(\mathbf{x}) - \omega^{3\gamma}\sigma^{2\gamma}((\kappa(u)(\mathbf{x}))^-)^\gamma + o(\sigma^{2\gamma}),$$

$$D_\sigma(u)(\mathbf{x}) = u(\mathbf{x}) + \omega^{3\gamma}\sigma^{2\gamma}((\kappa(u)(\mathbf{x}))^+)^\gamma + o(\sigma^{2\gamma}).$$

Moreover, consistency is locally uniform in $\mathbf{x}$.

Proof. As usual, we assume that $\mathbf{x} = 0$ and $u(0) = 0$. In a first step, assume that $\kappa(u)(0) > 0$. We use the same locality argument as in the first proof of consistency in this chapter. Since the curvature of u in a small ball with radius $r > 0$ is bounded from below by a positive constant, say $\kappa - \eta$ with $\eta > 0$, the level sets of u have no positive σ-chord in $D(0, r)$. We set $u_-(\mathbf{x}) = u(\mathbf{x})$ for $\mathbf{x} \in D(0, r)$ and $u_-(\mathbf{x}) = -\infty$ elsewhere. We use the same locality argument as in the first proof of consistency as above: For σ small enough, the level sets of u_- have no positive σ-chord in $D(0, \frac{r}{2})$. Thus the erosion has no effect upon u_- in $D(0, \frac{r}{2})$. By using monotonicity, we have

$$0 = u_-(0) = E_\sigma(u_-)(0) \leq E_\sigma(u)(0) \leq u(0) = 0$$

and the result is proved in the case $\kappa > 0$.
Let us now come to the most difficult case: Assume that $\kappa(u)(0) < 0$. Let also $Q(\mathbf{x}) = px + ax^2 + by^2 + cxy$ be the Taylor expansion of u at the origin with $b < 0$. Let $\varepsilon > 0$ be a small parameter and let

$$v(\mathbf{x}) = px + (a - \frac{p\varepsilon}{2})x^2 + (b - \frac{p\varepsilon}{2})y^2 + 2cxy = Q(\mathbf{x}) - \frac{p\varepsilon}{2}(x^2 + y^2).$$

If r is chosen small enough, we have $v(\mathbf{x}) \leq u(\mathbf{x})$ in $D(0, r)$. By extending v by $-\infty$ out of $D(0, r)$ this remains true everywhere. We now estimate $E_\sigma(v)(0)$ and obtain a lower bound for $E_\sigma(u)$ by invoking the monotonicity. By choosing ε small enough, we can assume that the curvature of the level lines of v is still strictly negative. Indeed, its value is $\kappa - \varepsilon + O(r)$ where κ is the value at the origin. Let now $\eta > 0$ be also small, such that the curvatures of the level lines of v is larger than $\kappa - \varepsilon - \eta$ (recall that in this part of the proof the curvatures are all negative, hence a circle with a small curvature will also have a small radius). We can again invoke the same locality property of the erosion in the case of a curve with a strictly negative curvature: We know that the chord-arc distance is equal to $O(\sigma^{2/3}|\kappa - \varepsilon - \eta|^{1/3})$ and the length of the chord is a $O(|\kappa - \varepsilon - \eta|^{-1/3}\sigma^{1/3})$. We deduce from this that the σ-chord sets containing 0 must be included in a ball with radius $O(\sigma^{1/3})$. A short calculation shows that the modified corresponding σ-chord sets are included in a ball with radius $O(\sigma^\gamma)$. The constant in these terms are clearly uniform because the curvature is bounded from above by a negative constant. In particular, they do *not* depend on ε and η. Let now $\mathbf{x}$ be in a $D(0, r)$. We call $C_{\kappa-\varepsilon-\eta}(\mathbf{x})$ the disk of curvature $\kappa - \varepsilon - \eta$ which is tangent to the level line of v at $\mathbf{x}$ and which is on the same side as $\chi_{v(\mathbf{x})}(\mathbf{x})$. Because of the comparison of the curvatures, we can still assume that r is small enough such that we have the inclusion (see Fig. 6.5)

$$D(0, r) \cap C_{\kappa-\varepsilon-\eta}(\mathbf{x}) \subset D(0, r) \cap \chi_{v(\mathbf{x})}(v).$$

Now, since the erosion operator is local, we also have

$$D(0, \frac{r}{2}) \cap E_\sigma(C_{\kappa-\varepsilon-\eta}(\mathbf{x})) \subset D(0, \frac{r}{2}) \cap E_\sigma(\chi_{v(\mathbf{x})}(v)).$$

Assume that $E_\sigma(v)(0) < \lambda$. By definition, this means that $0 \notin E_\sigma(\chi_\lambda(v))$. By using the inclusion principle, we deduce that $0 \notin E_\sigma(C_{\kappa-\varepsilon-\eta}(\mathbf{x}))$ for any point such that $v(\mathbf{x}) = \lambda$. Let $\mathbf{x}_\lambda$ be the orthogonal projection of the origin on the level line of v with level λ. It is uniquely defined since the level lines are strictly convex. This point is also characterized by the fact that the distance between the origin and the tangent to the level line is minimal. Hence, the modified σ-chord set of $C_{\kappa-\varepsilon-\eta}(\mathbf{x}_\lambda)$ at $\mathbf{x}_\lambda$ also contains the origin. Let (x_λ, y_λ) be the coordinates of $\mathbf{x}_\lambda$. Since $\mathbf{x}_\lambda$ is also characterized by the fact that $\mathbf{x}_\lambda$ and $Dv(\mathbf{x}_\lambda)$ are collinear, a simple calculation gives

$$Dv(\mathbf{x}_\lambda) = (p + 2(a - \frac{p\varepsilon}{2})x_\lambda + 2cy_\lambda, 2(b - \frac{p\varepsilon}{2})y_\lambda + 2cx_\lambda).$$

Since $\mathbf{x}_\lambda$ and $Du(\mathbf{x}_\lambda)$ are collinear, we have

$$|\mathbf{x}_\lambda| = -\mathbf{x}_\lambda \cdot \frac{Dv}{|Dv|}(\mathbf{x}_\lambda). \tag{6.21}$$

By using consistency on disks, we have $|\mathbf{x}_\lambda| \leq \omega^{3\gamma}\sigma^{2\gamma}(-\kappa + \varepsilon + \eta)^\gamma(1 + o(1))$. On the other hand,

$$\begin{aligned}
|\mathbf{x}_\lambda| &= -\mathbf{x}_\lambda \cdot \frac{Dv}{|Dv|}(\mathbf{x}_\lambda) \\
&= -\frac{1}{|Dv(\mathbf{x}_\lambda)|}(px_\lambda + 2(a - \frac{p\varepsilon}{2})x_\lambda^2 + 2(b - \frac{p\varepsilon}{2})y_\lambda^2 + 4cx_\lambda y_\lambda) \\
&= -\frac{1}{|Dv(\mathbf{x}_\lambda)|}(v(x_\lambda) + O(\sigma^{4\gamma})) \\
&= -\frac{1}{|Dv(\mathbf{x}_\lambda)|}(\lambda + O(\sigma^{4\gamma})).
\end{aligned}$$

From this, we deduce that

$$\lambda > -p\omega^{3\gamma}(|\kappa - \varepsilon - \eta|)^\gamma\sigma^{2\gamma}(1 + o(1)),$$

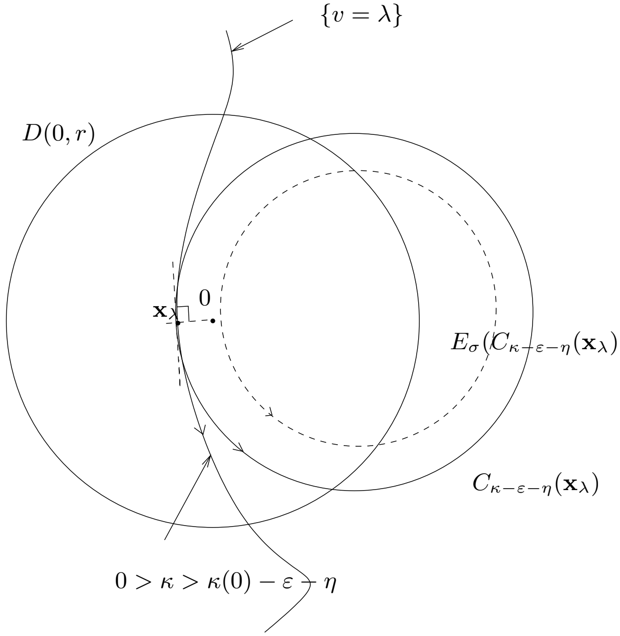


Fig. 6.5. If $\kappa < 0$, in a small ball, one can small disk tangent to the level sets with a fixed radius. By using comparison principle, if $E_\sigma(\chi_\lambda(v))$ does not contain the origin, the eroded disk do not neither. In particular, this is true for the closer disk to the origin. By consistency on a disk, one can deduce a lower bound for $E_\sigma(v)$

where we have approximated $|Dv(\mathbf{x}_\lambda)|$ by p up to a $O(\sigma^{2\gamma})$ term. This analysis can be performed for any $\varepsilon > 0$ and $\eta > 0$, since even though the constant were not explicited, we already stressed that they do not depend upon ε and η. We then deduce that

$$E_\sigma(u)(0) \geq -p\omega^{3\gamma}|\kappa|^\gamma\sigma^{2\gamma}(1 + o(1)).$$

Let now search an upper bound to $E_\sigma(u)(0)$. We do not repeat all the arguments since there will be some similarity with the research of a lower bound. We approximate u by its Taylor expansion and define $w(\mathbf{x}) = px + (a + \frac{p\varepsilon}{2})x^2 + (b + \frac{p\varepsilon}{2})y^2 + 2cxy$ such that $u \leq w$ is a small ball of radius r with $0 < \frac{p\varepsilon}{2} < -b$. For $\eta > 0$ small enough, the curvature of the level lines of w is smaller than $\kappa - \varepsilon - \eta$ which can also be chosen negative if r is small enough. The locality of the σ-chords still holds. We now define $C_{\kappa+\varepsilon+\eta}(\mathbf{x})$ as above; its radius is equal to $|\kappa + \varepsilon + \eta|^{-1}$. If r is small enough, for any $\mathbf{x}$ the level set $\chi_{w(\mathbf{x})}(w)$ is included in $C_{\kappa+\varepsilon+\eta}(\mathbf{x})$ inside $D(0, r)$. The rest of the proof is still an application of comparison principle and the asymptotic behavior of the erosion on disks. Assume that $E_\sigma(w)(0) \geq \lambda$. This means that $0 \in E_\sigma(\chi_\lambda(w))$. In particular $0 \in E_\sigma(C_{\kappa+\varepsilon+\eta}(\mathbf{x}_\lambda))$ where $\mathbf{x}_\lambda$ is a above. This implies that $\mathbf{x}_\lambda \geq \omega^{3\gamma}|\kappa + \varepsilon + \eta|^\gamma + \sigma^{2\gamma}(1 + o(1))$. We use the characterization (6.21) of $\mathbf{x}_\lambda$ and deduce that we must have

$$\lambda \le -p\omega^{3\gamma}|\kappa+\varepsilon+\eta|^{\gamma}+\sigma^{2\gamma}(1+o(1)).$$

Since, this is true for any $\varepsilon > 0$ and $\eta > 0$, we also obtain

$$E_\sigma(u)(0) \le -p\omega^{3\gamma}|\kappa|^{\gamma}\sigma^{2\gamma}(1+o(1)).$$

Again, the $o(1)$ term is uniform in ε and η, since all the curvatures may be taken bounded by above by a strictly negative constant. Thus, if $\kappa < 0$, we have

$$E_\sigma(u)(0)-u(0) = -|Du|\omega^{3\gamma}(\kappa^-)^{\gamma}\sigma^{2\gamma}(1+o(1)).$$

Together with the simpler case $\kappa > 0$, we obtain the result.
The case of the dilation can be deduced by the relation $D_\sigma(u) = -E_\sigma(-u)$. Again, the uniform consistency follows from the fact that the σ-chords are uniformly bounded in some ball with radius $O(\sigma^{2/3})$ and the constants of these terms are bounded as soon as the curvature have an absolute value strictly more that a positive constant. □

Uniform consistency yields consistency for the alternate operators.

Corollary 6.25. *Let* $u \in C^3$ *with* $Du(\mathbf{x}) \neq 0$ *and* $\kappa(u)(\mathbf{x}) \neq 0$. *Then*

$$T_\sigma u(\mathbf{x}) = u(\mathbf{x}) + \omega^{3\gamma}(\kappa(u)(\mathbf{x}))^{\gamma}\sigma^{2\gamma} + o(\sigma^{2\gamma}),$$

for either $T_\sigma = D_\sigma \circ E_\sigma(u)$ *or* $T_\sigma = E_\sigma \circ D_\sigma(u)$.

Proof. Let $\mathbf{x}_0 \in \mathbb{R}^2$. For some $\varepsilon > 0$ and any $\mathbf{x} \in B(\mathbf{x}_0, \varepsilon)$, we have

$$E_\sigma(u)(\mathbf{x}) = u(\mathbf{x}) + \omega^{3\gamma}(\kappa^-)^{\gamma}(\mathbf{x})\sigma^{2\gamma} + o_{\mathbf{x}_0}(\sigma^{2\gamma}).$$

The last term $o_{\mathbf{x}_0}(\sigma^{2\gamma})$ is uniform in $B(\mathbf{x}_0,\varepsilon)$. Now, u is C^3 and $\kappa(\mathbf{x}) = \kappa(\mathbf{x}_0) + O(\varepsilon)$ in $B(\mathbf{x}_0,\varepsilon)$, the error term being uniform. By applying the dilation, we obtain

$$D_\sigma \circ E_\sigma(u)(\mathbf{x}_0) = u(\mathbf{x}_0) + \omega^{3\gamma}\kappa^{\gamma}(\mathbf{x}_0)\sigma^{2\gamma} + \varepsilon O(\sigma^{2\gamma}) + o(\sigma^{2\gamma}).$$

We end the proof by taking ε as a function of σ and tending to 0. □

6.5.2 Convergence

We can now prove the results of convergence in the case of power functions. As foretold in Chap. 4, the notion of viscosity solution has to be taken in the sense of Ishii and Souganidis [94] as soon as the power of the curvature is larger than 1. We know that the solution is exists and is unique. We construct approximate solutions by iterating the erosion/dilation and prove that any converging subsequence tends to a solution when the erosion scale tends to 0. By uniqueness, this yields convergence. To this purpose we compare these limits to smooth test functions and use the consistency. At points which are not critical, the uniform consistency of the scheme gives the result. One has to check that critical points do not raise any trouble. This is the object of the following lemma.

Lemma 6.26. *Let $u(\mathbf{x}) = -f(|\mathbf{x}|)$ where $f \in \mathcal{F}(x^\gamma)$ (defined p. 77). Then, for any sequence of points $\mathbf{x}_n$ tending to 0, we have*

$$\lim_{\substack{n\to+\infty\\ \sigma\to 0}} \frac{T_\sigma u(\mathbf{x}_n) - u(\mathbf{x}_n)}{\sigma^{2\gamma}} = 0,$$

where the limit is taken as n and σ independently tend to their respective limit and T_h designs either E_σ or D_σ or $D_\sigma \circ E_\sigma$.

Proof. Let consider a point $\mathbf{x}$ with coordinates $(x, 0)$. Let $r > x$ be such that $E_\sigma(u)(\mathbf{x}) = -f(r)$. For any $x < t < r$, we can find a modified σ-chord set of the disk with radius t containing $\mathbf{x}$. We can choose this chord parallel to the y axis. We call θ the semi-angle joining the origin and the two endpoints of the chord (see Figure 6.6). We can calculate σ in function of t and θ.

$$\sigma = t^2(\theta - \frac{1}{2}\sin 2\theta).$$

The chord arc distance is equal to $h = t(1 - \cos\theta)$. We obviously have $h \leq t\frac{\theta^2}{2}$

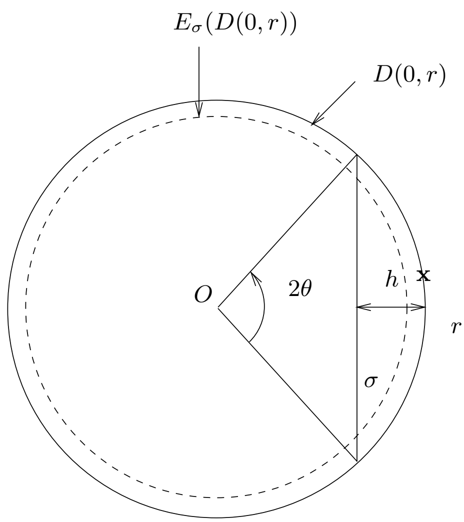


Fig. 6.6. Calculation of an exact bound of the erosion for a circle

and on $(0, \pi)$ we can find a constant c such that $\sigma \geq t^2\theta^3 c$. Thus $t - x = h^{3\gamma} \leq At^{-\gamma}\sigma^{2\gamma}$ where A is a constant depending on c. By letting t tend to r, we also obtain $r - x = h^{3\gamma} \leq Ar^{-\gamma}\sigma^{2\gamma}$. By convexity of f, we have

$$\frac{-f(r) + f(x)}{\sigma^{2\gamma}} \geq -\frac{f'(r)(r - x)}{\sigma^{2\gamma}},$$

yielding

$$\frac{-f(r)+f(x)}{\sigma^{2\gamma}} \geq -Ar^{-\gamma} f'(r).$$

Thus

$$\begin{aligned} 0 &\geq \frac{E_\sigma(u)(\mathbf{x}) - u(\mathbf{x})}{\sigma^{2\gamma}} \\ &= \frac{-f(r)+f(x)}{\sigma^{2\gamma}} \\ &\geq -Ar^{-\gamma} f'(r). \end{aligned}$$

We can let x and σ tend to 0. Then r also tends to 0 and the right hand term also tends to 0 because $f \in \mathcal{F}(x^\gamma)$. The composition with the dilation operator is trivial because, since the curvature of the level lines is negative, the dilation has no effect on u or on $E_\sigma(u)$. $\square$

We can conclude this section with the expected convergence theorem.

Theorem 6.27. *Let $\gamma \geq \frac{1}{3}$. Let $T_\sigma = D_\sigma \circ E_\sigma$ or $E_\sigma \circ D_\sigma$ be the alternate dilation-erosion for the curvature power function x^γ. Let u_0 in $BUC(\mathbb{R}^2)$ and define u_σ by*

$$\forall t \in [n\omega^{3\gamma}\sigma^{2\gamma}, (n+1)\omega^{3\gamma}\sigma^{2\gamma}) \qquad u_\sigma(\mathbf{x}, t) = (T_\sigma)^n u_0(\mathbf{x}).$$

Then, when σ tends to 0, u_σ tends locally uniformly to the unique viscosity solution (in the sense of Ishii-Souganidis) of the equation

$$\frac{\partial u}{\partial t} = |Du|(\operatorname{curv} u)^\gamma \tag{6.22}$$

with initial value u_0.

Proof. Let

$$\overline{u}(x,t) = \limsup_{\substack{(y,s)\to(x,t)\\ \sigma\to 0}} u_\sigma(y,s)$$

and

$$\underline{u}(x,t) = \liminf_{\substack{(y,s)\to(x,t)\\ \sigma\to 0}} u_\sigma(y,s).$$

By definition, $\overline{u}$ and $\underline{u}$ are upper and lowersemicontinuous and bounded. We prove that $\overline{u}$ and $\underline{u}$ are respectively sub- and supersolution. By using the maximum principle, we shall then obtain $\overline{u} \leq \underline{u}$. Since by definition, $\underline{u} \leq \overline{u}$, this will imply $\underline{u} = \overline{u}$ and the convergence of all the family u_σ toward the solution of (6.22). Before this, we check that the initial conditions of (4.37) in Thm. 4.46 are satisfied. In order to see this, we prove that we can control the variation of $\overline{u}$ and $\underline{u}$. We use the uniform continuity of u_0. To this purpose, let us call α a modulus of continuity of u, that is to say a positive, increasing continuous function with $\alpha(0) = 0$ and such that

$$|u_0(\mathbf{x}) - u_0(\mathbf{y})| \leq \alpha(|\mathbf{x} - \mathbf{y}|).$$

Then for any $t > 0$, $u_\sigma(\cdot, t)$ is in $BUC(\mathbb{R}^2)$ with α as modulus of continuity. Indeed, for all $\mathbf{x}$ and $\mathbf{y}$

$$u_0(\mathbf{x}) - \alpha(|\mathbf{y}|) \leq u_0(\mathbf{x}+\mathbf{y}) \leq u_0(\mathbf{x}) + \alpha(|\mathbf{y}|)$$

implies

$$T_\sigma u_0(\mathbf{x}) - \alpha(|\mathbf{y}|) \leq T_\sigma u_0(\mathbf{x}+\mathbf{y}) \leq T_\sigma u_0(\mathbf{x}) + \alpha(|\mathbf{y}|),$$

where we used that T_σ is both contrast and translation invariant. By a trivial induction, this result holds with $(T_\sigma)^n$, and consequently for $\underline{u}$ and $\overline{u}$. We now give a bound on the variations of u_σ in the time variable. Of course, u_σ is not continuous in t, but this is nearly true when σ is small. Indeed, from $|u(\mathbf{x}+\mathbf{y}) - u(\mathbf{x})| \leq \alpha(|\mathbf{y}|)$, we obtain

$$|(T_\sigma)^n u(\mathbf{x}) - u(\mathbf{x})| \leq \alpha\big((T_\sigma^n(|\mathbf{y}|))(0)\big),$$

by iterating n times T_σ when $\mathbf{x}$ is fixed. (Again, we have used the contrast and translation invariance.) We want to estimate the right-hand term when $n\sigma^{2\gamma}$ tends to t. We then introduce the function f defined for $r \geq 0$ by $f(r) = T_\sigma(|\mathbf{y}|)(\mathbf{x})$ where $|\mathbf{x}| = r$. We are interested in the numerical sequence $(r_n)_{n\in\mathbb{N}}$ defined by $r_0 = r$ and $r_{n+1} = f(r_n)$. By Lem. 6.26, there exists a constant A such that

$$r_{n+1} \leq r_n + \frac{A\sigma^{2\gamma}}{r_n^\gamma} \leq r + \frac{An\sigma^{2\gamma}}{r^\gamma}. \tag{6.23}$$

The last inequality is obtained by induction and using that r_n is increasing. Let us assume that $n\sigma^{2\gamma} \leq t$. We then have

$$r_n \leq r + A\frac{t}{r^\gamma}. \tag{6.24}$$

The minimum of the right-hand term is attained for $r^{\gamma+1} = A\gamma t$ and assume the value $2(A\gamma t)^{1/(\gamma+1)}$. Since f is nondecreasing, $r^{\gamma+1} \leq A\gamma t$ implies

$$f^n(r) \leq 2(A\gamma t)^{1/(\gamma+1)}. \tag{6.25}$$

Therefore, we also have

$$f^n(0) \leq 2(A\gamma t)^{1/(\gamma+1)}. \tag{6.26}$$

Now, $|u(\mathbf{x}+\mathbf{y}) - u(\mathbf{x})| \leq \alpha(|\mathbf{y}|)$ implies

$$|T_\sigma^n(u)(\mathbf{x}) - u(\mathbf{x})| \leq \alpha(f^n(0)) \leq \alpha(2(A\gamma t)^{1/(\gamma+1)}), \tag{6.27}$$

as soon as $n\sigma^{2\gamma} \leq t$. Note that this is nearly a condition of equicontinuity, the difference lying only on the first iteration that induces a small term tending to 0 when σ tends to 0. As a matter of fact, we could get the same conclusion as for Ascoli-Arzela's Theorem. We could then prove that any limit is viscosity solution and conclude by uniqueness. Coming back to our proof, (6.27) implies that $\overline{u}$ and satisfy the initial value condition (4.37). Le us now apply the maximum principle 4.46, and first prove that $\overline{u}$ is subsolution. Let φ be admissible for G and $(\mathbf{x}, t)$ be a maximum point of $\overline{u} - \varphi$. Up to a subsequence, $u_\sigma - \varphi$ admits a maximum at $(\mathbf{x}_\sigma, t_\sigma) \to (\mathbf{x}, t)$. Then, for all $(\mathbf{y}, s)$, we have

$$u_\sigma(\mathbf{y}, s) - \varphi(\mathbf{y}, s) \leq u_\sigma(\mathbf{x}_\sigma, t_\sigma) - \varphi_(\mathbf{x}_\sigma, t_\sigma),$$

which we rewrite

$$u_\sigma(\mathbf{y}, s) - u_\sigma(\mathbf{x}_\sigma, t_\sigma) \leq \varphi(\mathbf{y}, s) - \varphi_(\mathbf{x}_\sigma, t_\sigma).$$

By applying T_σ on each side and taking the value at $(\mathbf{x}_\sigma, t_\sigma - \omega^{3\gamma}\sigma^{2\gamma})$, we obtain by definition of u_σ

$$0 \leq T_\sigma\varphi(\mathbf{x}_\sigma, t_\sigma - \omega^{3\gamma}\sigma^{2\gamma}) - \varphi(\mathbf{x}_\sigma, t_\sigma),$$

yielding

$$\frac{\varphi(\mathbf{x}_\sigma, t_\sigma) - \varphi(\mathbf{x}_\sigma, t_\sigma - \omega^{3\gamma}\sigma^{2\gamma})}{\omega^{3\gamma}\sigma^{2\gamma}} \leq \frac{(T_\sigma\varphi - \varphi)(\mathbf{x}_\sigma, t_\sigma - \omega^{3\gamma}\sigma^{2\gamma})}{\omega^{3\gamma}\sigma^{2\gamma}}.$$

The left hand term tends to $\varphi_t(\mathbf{x}, t)$. Assume that $D\varphi(\mathbf{x}, t) \neq 0$. Then for σ small enough, $D\varphi(\mathbf{x}_\sigma, t_\sigma - \omega^{3\gamma}\sigma^{2\gamma}) \neq 0$. If $\operatorname{curv}\varphi(\mathbf{x}, t) \neq 0$, we use the consistency result (Prop 6.24) and letting σ go to 0, we obtain

$$\varphi_t(\mathbf{x}, t) \leq |D\varphi|(\operatorname{curv}\varphi)^\gamma(\mathbf{x}, t). \tag{6.28}$$

If $\operatorname{curv}\varphi(\mathbf{x}, t) = 0$, we replace $\varphi(\mathbf{y})$ by $\varphi_\varepsilon(\mathbf{y}) = \varphi(\mathbf{y}) + \varepsilon|\mathbf{y} - \mathbf{x}|^2$. Then φ_ε is an admissible test function and its curvature is different from zero near $\mathbf{x}$. Equation (6.28) holds with φ_ε. We let ε tends to 0 and obtain the same result. If $D\varphi(x, t) = 0$, we use Lem. 6.26, proving that the right hand side term tends to 0. We then obtain $\varphi_t(x, t) \leq 0$. This proves that $\overline{u}$ is subsolution.
By the same way, we prove that $\underline{u}$ is supersolution. The maximum principle 4.46 then implies $\overline{u} \leq \underline{u}$. Since $\overline{u} \geq \underline{u}$ by construction, we deduce that $\overline{u} = \underline{u}$. This shows that the whole family u_σ converges to the solution of (6.22). □

6.6 A numerical implementation of the erosion

6.6.1 Scale covariance

The erosion operator that we defined above satisfies all the desired invariance properties. Precisely, it is Euclidean invariant and in the special case where $\gamma = 1/3$, it is also affine invariant, as it was already known by Moisan [126]. Nevertheless, because of the homogeneity of the power functions, the curve evolution (if it exists) is invariant with respect to scale changes. More precisely, assume that $C(\cdot, t)$ is a shortening flow, following the evolution law

$$\frac{\partial C}{\partial t} = \kappa^\gamma \mathbf{N}. \tag{6.29}$$

We denote by S_t the semi-group mapping the initial curve $C = C(\cdot, 0)$ to $C(\cdot, t)$. In order not to introduce another notation, S_t will be applied to curves as well as to sets with smooth enough boundary. We also denote by H_λ the scaling with factor λ, that is $H_\lambda(\mathbf{x}) = \lambda\mathbf{x}$. Then, we have the commutation relation

$$S_t \circ H_\lambda = H_\lambda \circ S_{\frac{t}{\lambda^{\gamma+1}}}. \tag{6.30}$$

Indeed, let $C_1(t)$ the evolving curve defined by

$$C_1(t) = H_\lambda C(\frac{t}{\lambda^{\gamma+1}}).$$

Then, (6.30) asserts that C_1 satisfies (6.29) with initial condition $H_\lambda C(0)$, and it can be checked by a simple calculation. The erosion operator E_σ does not satisfy the same covariance property. We thus define a modified operator

$$O_t = H_a \circ E_{\frac{\sigma}{a^2}} \circ H_{a^{-1}}, \tag{6.31}$$

where $a > 0$ is a free parameter depending on t, σ and (possibly) C. It is chosen such that O_t satisfies the same invariance property as S_t. If we consider a and σ as functions of C and t, then for any λ, we must have

$$a(H_\lambda C, t) = \lambda a\left(C, \frac{t}{\lambda^{\gamma+1}}\right) \quad \text{and} \quad \sigma(H_\lambda C, t) = \lambda^2\sigma\left(C, \frac{t}{\lambda^{\gamma+1}}\right). \tag{6.32}$$

Obviously, we also want O_t to be consistent with the semi-group generator, that is to say

$$O_t(K) = S_t(K) + o(t)$$

for any set K with a C^3 boundary. The term $o(t)$ is measured with respect to the Hausdorff distance. By using the consistency (Prop. 6.14), we see that a, t and σ must satisfy

$$t = \omega^{3\gamma} a^{1-3\gamma} \sigma^{2\gamma}. \tag{6.33}$$

Assume now that $\sigma > 0$ is fixed. If a is chosen large enough such that, for any σ chord of K with chord-arc distance h the inequality

$$\frac{h}{a} \le \alpha_\gamma \tag{6.34}$$

holds, then the modified chord-arc distance is then given by G near the origin (where G is still the truncation of the power function defined in (6.18)). Precisely, for $M \in \partial K$ (with smooth boundary) consistency writes down

$$\begin{aligned} d(M, O_t K) &= aG(\frac{h}{a}) \\ &= a(\frac{h}{a})^{3\gamma} \\ &= t\kappa^\gamma + o(t). \end{aligned} \tag{6.35}$$

Notice that it is interesting to take the smallest possible value of a (the saturation length given by the case of equality in (6.34)) in order to get the largest possible scale step t from (6.33). We summarize the above discussion.

Proposition 6.28. *Let $h(A)$ denote the chord-arc distance of a chord-set A. Then, the operator O_t defined by (6.31) with*

$$a = \frac{1}{\alpha_\gamma} \sup_{\substack{A \in \mathcal{K}^+_{\sigma'}(\partial K) \\ \sigma' \leq \sigma}} h(A) \quad \text{and} \quad \sigma = \left(t\omega^{-3\gamma} a^{3\gamma-1}\right)^{1/2\gamma} \tag{6.36}$$

is consistent with (6.29) for convex sets and satisfy the same scaling property as S_t in (6.30).

Remark 6.29. Consistency is an asymptotic property and holds for σ tending to 0. As a consequence, we do not need that $h/a < \alpha_\gamma$ for all (σ, a). But, if this inequality does not hold, then the error is $O(t)$, whereas it is $O(t^2)$ in the opposite case.

6.6.2 General algorithm

Each iteration of the operator defined above involves three parameters: the scale step t (that can be viewed as a *time* step), the erosion area σ and the saturation length a. These three quantities have to satisfy (6.36), which leaves only one degree of freedom. A usual numerical scheme would consider t as the free parameter (the time step, related to the required accuracy), and then define a and σ from t. In the present case, this would not be a good choice for two reasons. First, a is defined as an explicit function of σ but as an implicit function of t, which suggests that σ may be a better (or at least simpler) free parameter than t. Second, t has no geometrical interpretation in the scheme we defined, contrarily to σ which corresponds in some way to "the scale at which we look at the curve". In particular, σ is constrained by the numerical accuracy at which the curve is known: roughly speaking, if C is approximated by a polygon with accuracy ε (corresponding, for example, to the Hausdorff distance between the both of them), then we must have $\sigma \gg \varepsilon^2$, in order that the effect of the erosion at each iteration overcomes the effect of the spatial quantization. For all of these reasons, we choose to fix σ as the free parameter, and then compute t and a by using (6.36). If the scale step t obtained in this way is too large, we can simply adjust it by reducing σ while keeping the same value of a. We propose the following algorithm for the evolution of a convex set K at final scale T with area accuracy σ.

1. Let $t = 0$ and $K_0 = K$.
2. While $t < T$
   - For each σ-set of K_t, compute the chord-arc distance.
   - Set a to the maximal value of these distances.
   - Let $\delta t = \omega^{3\gamma} a^{1-3\gamma} \sigma^{2\gamma}$.
   - If $\delta t > T - t$, take $\delta t = T - t$ and decrease σ in order to keep the previous equality.
   - Apply operator O_t to $K(t)$, yielding $K(t + \delta t)$.
   - Increment t by δt.

6.6.3 Eroded set and envelope

We know that the boundary of the eroded set is included in the envelope of the modified σ-chords. It is impossible to really compute this envelope and, in any case, the curves are sampled since we can store only a finite amount of data. Thus, in a computational point of view, a curve is a polygon. We then choose to only consider the σ-chords with an end-point being a vertex of the curve. A possibility to approximate the envelope is to compute the coordinates of the intersection point of two consecutive σ-chords when we run along the curve. This solution is not numerically stable, since computing the intersection point of two lines which are nearly parallel involves a division by a number close to 0, which increases the numerical errors, because of the limited computational accuracy. In [125, 126], the exact affine erosion ($\gamma = 1/3$) was computed. Indeed, it can be seen that the affine erosion of a quadrant is an hyperbola. Thus, the affine erosion of a polygon is obtained by gathering pieces of hyperbolas. This new curve is then sampled in an affine invariant way, and the affine erosion is applied to the new polygon. The computation is very precise, but a bit heavy. Thus, a simplified (but only approximate) version was proposed in [104]. It consists in explicitly computing the coordinates of the point of a chord belonging to the searched envelope. In the affine erosion case, this is remarkably simple, as asserts the following lemma.

Lemma 6.30 (Middle point Property). *Let K be a convex set. Let A_1 be a σ-set of K with σ-chord $\mathcal{C}_1$. Let A_2 be another σ-set, and let $\mathcal{C}_2$ be its σ-chord. Then, when $d_H(A_1, A_2)$ tends to 0, the intersection point of $\mathcal{C}_1$ and $\mathcal{C}_2$ tends to the middle point of $\mathcal{C}_1$.*

Proof ([125]). Let θ be the geometrical angle between $\mathcal{C}_1$ and $\mathcal{C}_2$. If A_1 and A_2 are close enough, $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at a point that we call $I(\theta)$. We also call $r_1(\theta)$ and $r_2(\theta)$ the length of the part of the chord $\mathcal{C}_2$ on each side of $I(\theta)$. Since $\mathcal{C}_1$ and $\mathcal{C}_2$

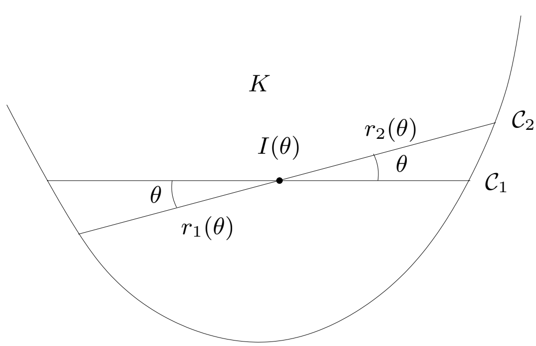


Fig. 6.7. The middle point property

are σ-chords, we have (see Fig. 6.7)

$$\frac{1}{2}r_1^2(\theta)\cdot\theta = \frac{1}{2}r_2^2(\theta)\cdot\theta + o(\theta).$$

This implies that $\lim(r_1(\theta) - r_2(\theta)) = 0$ when θ tends to 0, so that $I(\theta)$ tends to the middle point of $\mathcal{C}_1$. □

For the general case, the result shall not be so simple. Nevertheless, it is not very difficult, in a local frame, to calculate the coordinates of the coordinates of the point belonging to a modified σ-chord and to the envelope.

Proposition 6.31. *Let K be a strictly convex set, and $\mathcal{C} = C_{s,t}$ a σ-chord of K. Consider P the farthest point of $C([s,t])$ from $\mathcal{C}$. If (L,h) are the coordinates of P in the direct orthonormal referential whose origin is the middle point of $\mathcal{C}$ and whose first axis is directed by C(t)-C(s), then the contribution of the modified chord arising from $\mathcal{C}$ is either void or the unique point with coordinates*

$$L\left(1-3\gamma\left(\frac{h}{a}\right)^{3\gamma-1}\right)\ ,\ h\left(1-\left(\frac{h}{a}\right)^{3\gamma-1}\right) \tag{6.37}$$

in the same referential.

Proof. Let us examine the situation on Figure 6.8. Let $\mathcal{C}_\theta$ the σ-chord making an

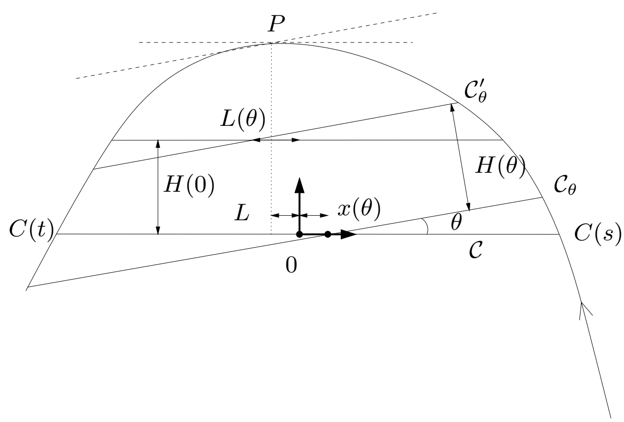


Fig. 6.8. The modified middle point property

angle θ with $\mathcal{C}$. We search the coordinates of the intersection point of the modified chord of $\mathcal{C}$ and $\mathcal{C}_\theta$ when θ tends to 0. We set $x(\theta)$ the abscissa of the point $\mathcal{C}\cap\mathcal{C}_\theta$. By the middle point property, we know that $x(\theta)\to 0$ as $\theta\to 0$. Let $\mathcal{C}'_\theta$ the modified chord of $\mathcal{C}_\theta$. The distance between these two chords is $H(\theta) = h - a(\frac{h}{a})^{3\gamma}$ where $h = h(\theta)$ is the chord-arc distance of $\mathcal{C}_\theta$. Let $(L(\theta), H(0))$ be the coordinates of the common point of $\mathcal{C}'$ and $\mathcal{C}'_\theta$. Elementary but a bit fastidious geometry proves that

$$L(0) = L\left(1 - 3\gamma\left(\frac{h}{a}\right)^{3\gamma-1}\right), \tag{6.38}$$

implying that the limit point we are looking for has coordinates given by (6.37). □

Remark 6.32. In the case of a C^1 function of the curvature and without introducing the scaling preventing saturation phenomenon, the coordinates of the point are

$$L\left(1 - G'\left(\frac{h}{\omega\sigma^{2/3}}\right)\right),\ h - \omega\sigma^{2/3}G\left(\frac{h}{\omega\sigma^{2/3}}\right).$$

6.6.4 Swallow tails

The boundary of the eroded set is in general strictly included in the envelope of the modified σ-chord. In the case of the affine erosion, Moisan [125] gave some conditions for the inclusion to be strict. In this case, some spurious parts have to be removed. This parts are known as *swallow tails* and are a usual numerical artifact for many curve evolution algorithms. The following lemma shows that two lines crossing with a right angle always creates swallow tails even if σ is very small.

Lemma 6.33. *Let* $K = \{(x, y) \in \mathbb{R}^2,\ x \geq 0,\ y \geq 0\}$. *Then for* $\gamma > 1/2$, *for any* $\sigma > 0$, *the boundary of the eroded set is strictly included in the envelope of the modified* σ*-chords.*

Proof. The proof relies on the explicit calculation of the envelope of the modified σ-chords. We shall skip the details of the calculations and only interpret their results. By scale invariance, we assume that $2\sigma = 1$. The equation of a σ-chord is $x/t + ty = 1$ where $t > 0$ is a parameter. Thus, the equation of the modified σ-chord is $x/t+ty = (t^2 + 1/t^2)^{1-3\gamma}$. From this expression, we derive the equation of the envelop of the σ-chord parameterized by t. We have

$$x(t) = y\left(\frac{1}{t}\right) = f_\gamma(t),$$

where

$$f_\gamma(t) = \frac{t}{2}\left(t^2 + \frac{1}{t^2}\right)^{-3\gamma}\left(t^2(6\gamma - 1) + \frac{3}{t^2}(1 - 2\gamma)\right).$$

The derivative of f_γ is

$$f'_\gamma(t) = \frac{1}{2}\left(t^2 + \frac{1}{t^2}\right)^{-3\gamma-1}\left(3(6\gamma - 1)(1 - 2\gamma)\left(t^4 + \frac{1}{t^4}\right) + 6(12\gamma^2 - 1)\right).$$

Thus, we deduce that if $\gamma \leq 1/2$, f is increasing. If $\gamma \geq 1/2$, we see that f'_γ has two positive zeros. Moreover $f_\gamma \uparrow 0$ when $t \downarrow 0$ and $f_\gamma \downarrow 0$ when $t \uparrow +\infty$. Thus, f_γ is first decreasing and negative, then increasing, becomes positive and is again decreasing. The parts of the envelope near 0 and ∞ correspond to the swallow-tail as on Fig. 6.9. □

On Fig. 6.9, we present the approximated envelope of a modified σ-chord and observe the swallow tails which have to be removed. If the original curve is convex, we then know that the eroded set is also convex. We use this to remove the spurious parts to retrieve a convex set.

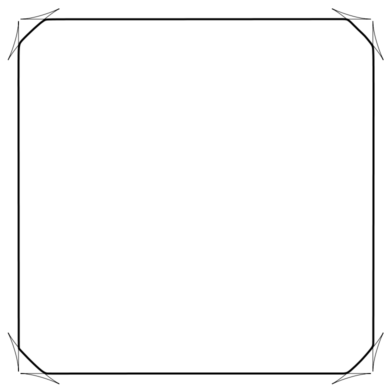


Fig. 6.9. Envelop of the modified σ-chord for a square ($\gamma = 2$). The result is not convex. The bold line is what is to be kept

Flat areas of the curves are also problematic. From the coordinates of the points of the envelope we computed in (6.37), we see that points are localized near corners when γ becomes large. This could create numerical instability.
Note that there is no condition between the discretization of the curve and the time step. This allows to take the time step as large as possible. In fact, the time step is limited de facto since we first start the iteration by dilating the curve to decrease the curvature. The time step is then proportional to $a^{1-3\gamma}$, a being of the order of the larger chord-arc distance. As a consequence, for very curvy sets, the time step will be very small at the beginning of the evolution. Moreover, the larger γ will be, the smaller the time step, with a decrease with a power -3γ which becomes problematic for very large values of γ.

6.7 Numerical experiments

We now show some experiments obtained with the scheme described above. In fact, we also made a scale change and implement the equation

$$\frac{\partial C}{\partial t} = (t\kappa)^{\gamma}\mathbf{N}. \tag{6.39}$$

With this convention, the scale covariance of the equation does no longer depend on γ. Explicitly, if we denote by T_t the operator mapping C to $C(t)$ (the loss is that it is no longer a semi-group), we have

$$T_t(H_\lambda C) = H_\lambda(T_{t/\lambda} C). \tag{6.40}$$

Let S_t be the semi-group of the evolution by the same power of the curvature. We have

$$T_t = S_{t^{\gamma+1}/(\gamma+1)}.$$

Numerically we approximate S_t, so that a renormalization is necessary to obtain T_t. Observe that when γ is large, the scale at which we compute S_t becomes huge and a very large number of iterations is necessary.
All the experiments were achieved with the MegaWave2 environment, developed at CMLA, ENS Cachan [68].

6.7.1 Evolving circles

Circles are the simplest curves to make evolve. There are convex and swallow tails do not appear in general. Moreover, explicit calculations are feasible for any $\gamma > 0$. The radius of a circle evolving by (6.39) is

$$R(t) = (R_0^{\gamma+1} - t^{\gamma+1})^{\frac{1}{\gamma+1}}. \tag{6.41}$$

Remark that the vanishing time does not depend on γ. Even though this is not true for general curve, the normalized equation thus seems better to compare the behavior of the shortening when γ varies.
In Table 6.1, we display the radius of a circle for different values of γ. On Fig.6.10,

Table 6.1. For time $t = 9$ and different values of γ, we give the theoretical value of the radius of the shortening circle (R_{theo}), the corresponding computed value R_{comp}, the number of iterations and the CPU time. The initial radius is 10 for all experiments

γ	R_{theo}	fast computation			slow computation		
		R_{comp}	# iter	CPU (s)	R_{comp}	# iter	CPU (s)
0.34	2.20	2.18	101	1.36	2.21	101	16
1.0	4.36	4.28	236	3.25	4.35	234	41
2.0	6.47	6.37	324	5.02	6.45	319	68
3.0	7.65	7.56	342	5.69	7.64	336	75
10.0	9.66	9.63	219	4.12	9.66	209	41

we display the evolution of a circle with radius 10, for $t \leq 9$. As expected, at a given scale, the radius is an increasing function of γ.

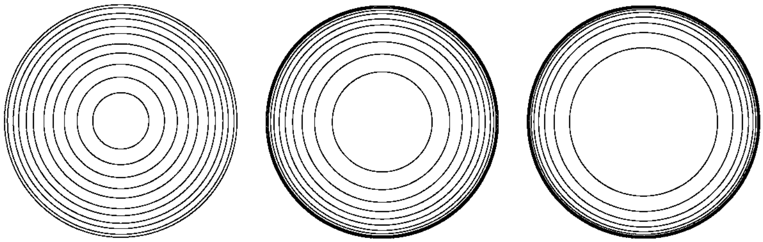


Fig. 6.10. Evolution of circles for $\gamma = 0.4,\ 1,\ 2$ displayed at time 0, 1, ..., 9. CPU times are displayed in Tab. 6.1

6.7.2 Convex polygons

For a convex polygon, we have to remove the swallow tails appearing at each step of the evolution. From this, we obtain a new convex polygon and we iterate. In Figures 6.11 and 6.12, we display the evolution of convex closed polygons. On each figure, the display times are the same for all the different values of γ.

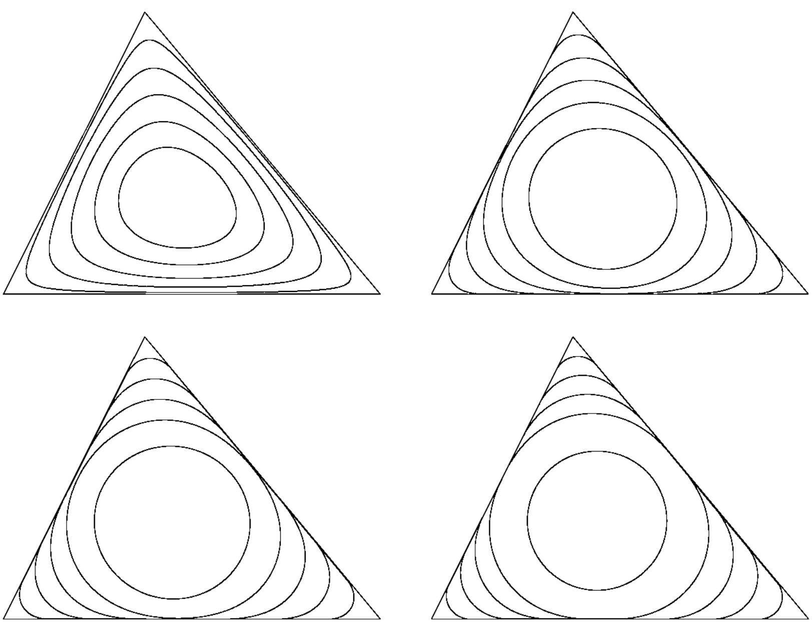


Fig. 6.11. Evolution of a triangle with power (from top to down and left to right), $\gamma = 1/3$, $\gamma = 1$, $\gamma = 2$, $\gamma = 5$. CPU times are respectively 4.1s, 13.3s, 31.9s and 96.2s

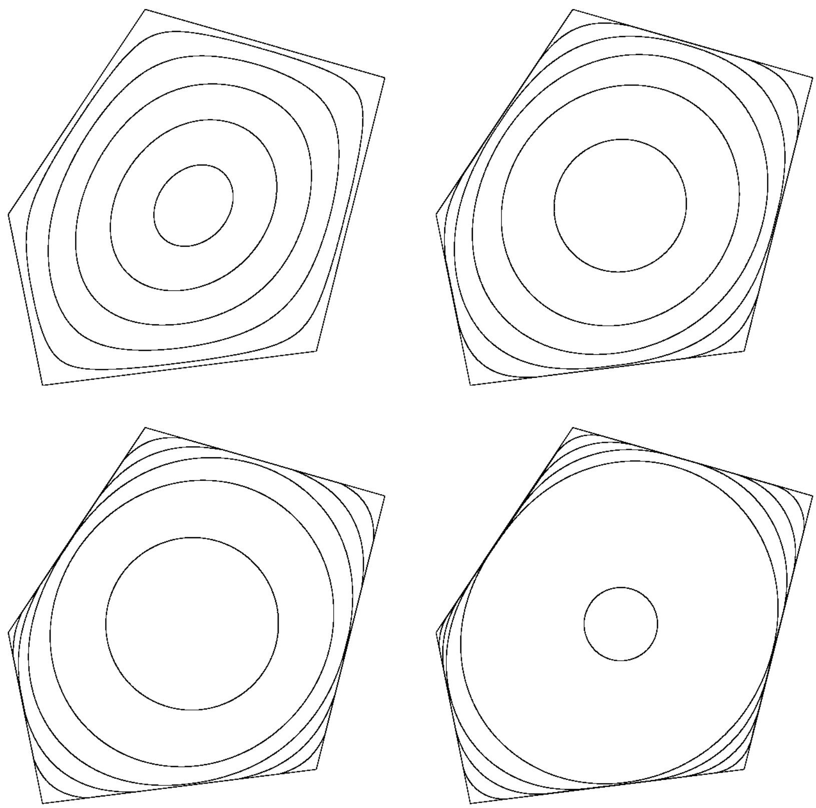


Fig. 6.12. Evolution of a pentagon with power (from top to down and left to right), $\gamma = 1/3$, $\gamma = 1$, $\gamma = 2$, $\gamma = 5$. CPU times are respectively 3.6s, 3.7s, 6.0s and 13.3s

6.7.3 Unclosed curve

Unclosed curves can be shortened by fixing the endpoints. If the curve is the graph of a function, fixing the endpoints is equivalent to apply a mirror effect and make the curve periodic. The final state is then a straight line.
As was proved in Prop. 3.35, the graph of the function $1 - |x|$ becomes flat in finite time if $\gamma < 1$ and in infinite time if $\gamma \geq 1$. We have checked this on the experiments of Fig 3.2, p. 52.

6.7.4 Evolving non convex curves

The numerical algorithm for a nonconvex curve starts by a cutting of the curve in convex components. Numerically speaking, the angle of the sides of a polygonal curve compared to a fixed direction may not be monotonous. Segments for which it goes from increasing to decreasing (or the reverse) are called *inflexion segments*. We call inflexion points the middle of inflexion segments. The convex components are the pieces of curve between two inflexion points. The algorithm we use for evolving general curves is the following.

1. Cut the curve in convex components.
2. For each convex components:
   a) For each chord having an endpoint as a vertex of the curve, compute the point belonging to the envelope of the modified σ-chord by using (6.37).
   b) Make the envelope convex by removing swallow tails.
3. Gather the new convex components and remove the old inflexion points.

We do not really know how inflexion points move. Indeed, at these points, the velocity is equal to zero. Experimentally, their velocity is smaller and smaller when γ increases, which is sound. They are not completely still (which would rapidly stop the evolution), since they are removed and computed at each iteration of the above algorithm.
A consequence of the Sturm oscillation theorem for parabolic equations, is that the number of inflexion points cannot increase. In the special cases $\gamma = 1/3$ and $\gamma = 1$, it eventually comes to 0 and the convex curve continues to shrink until it disappears. We observe that, numerically, this is also true for any power of the curvature. Even if this has still to be proved, it is experimentally check that the results of Grayson [77] and Angenent et al. [14] holds for any power of the curvature larger than $1/3$.

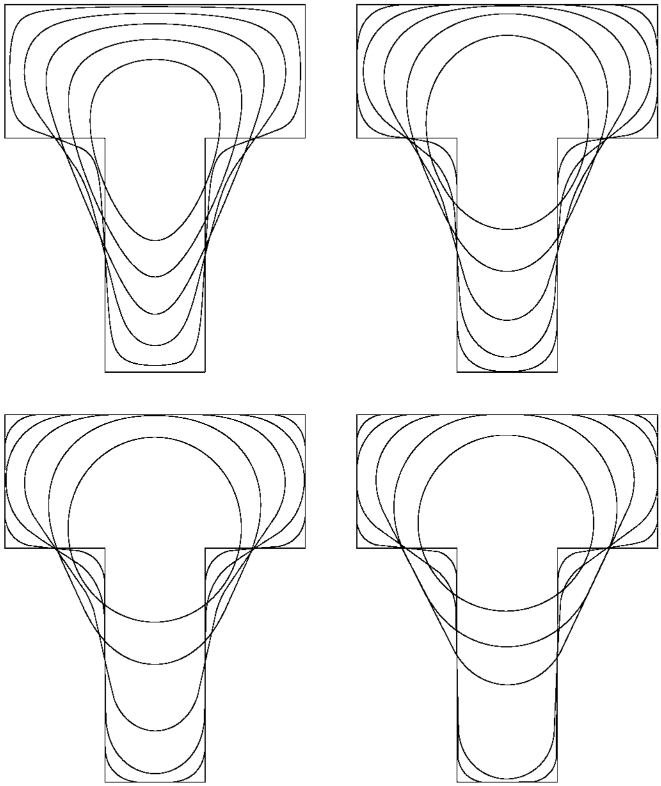


Fig. 6.13. Evolution of a "T" shape for values of $\gamma = 1/3$, 1, 2 and 5. CPU times are 5.3s, 8.6s, 22.2s and 79s

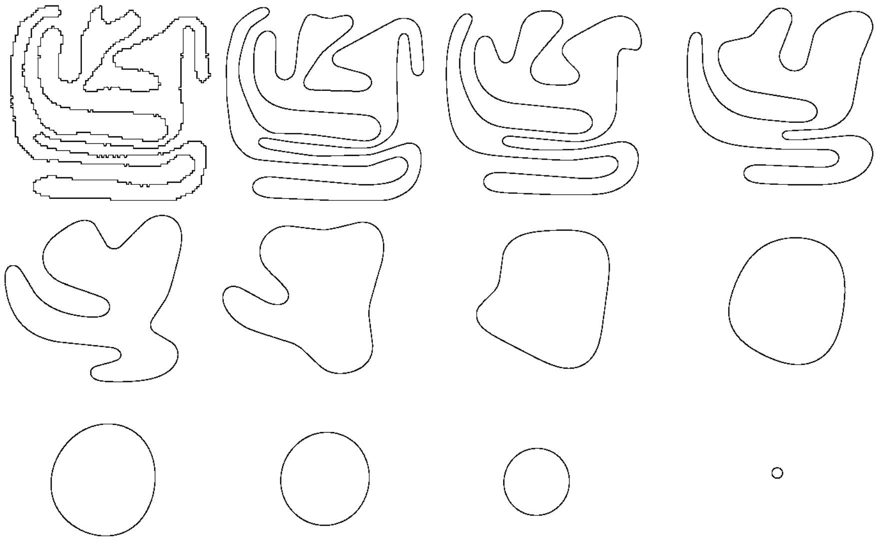


Fig. 6.14. Evolution of a nonconvex set with $\gamma = 1$. CPU: 20s

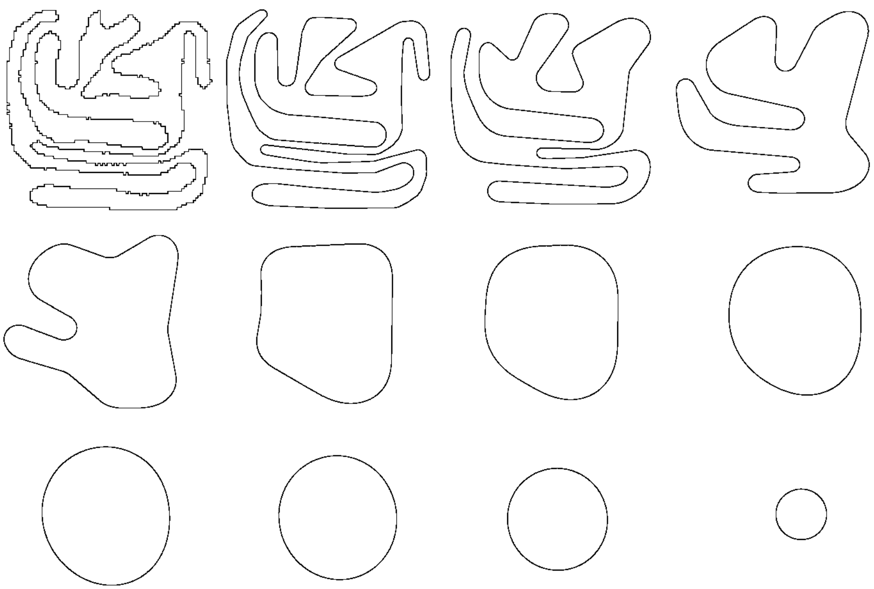


Fig. 6.15. Evolution of a nonconvex set with $\gamma = 2$. CPU: 53s

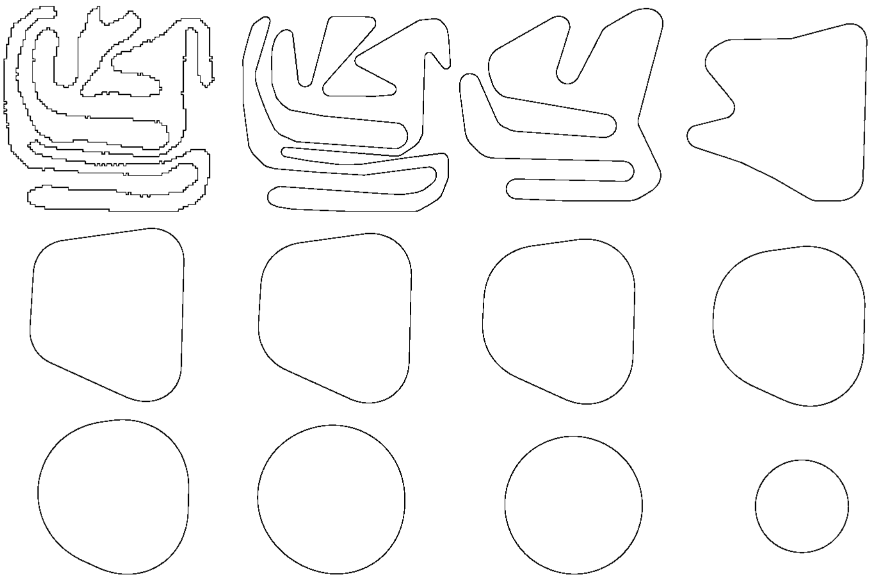


Fig. 6.16. Evolution of a nonconvex set with $\gamma = 5$ (CPU 5min)

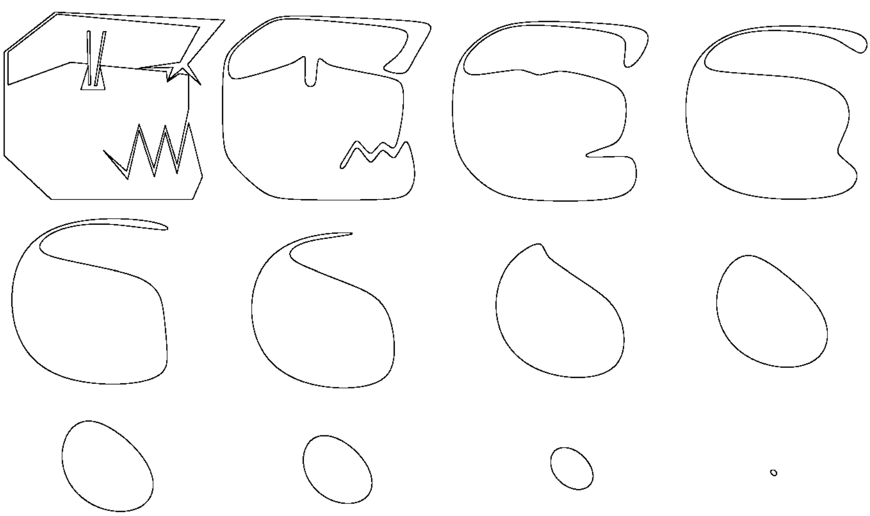


Fig. 6.17. Evolution of a weird polygon with $\gamma = 1/3$ (CPU: 2.6s)

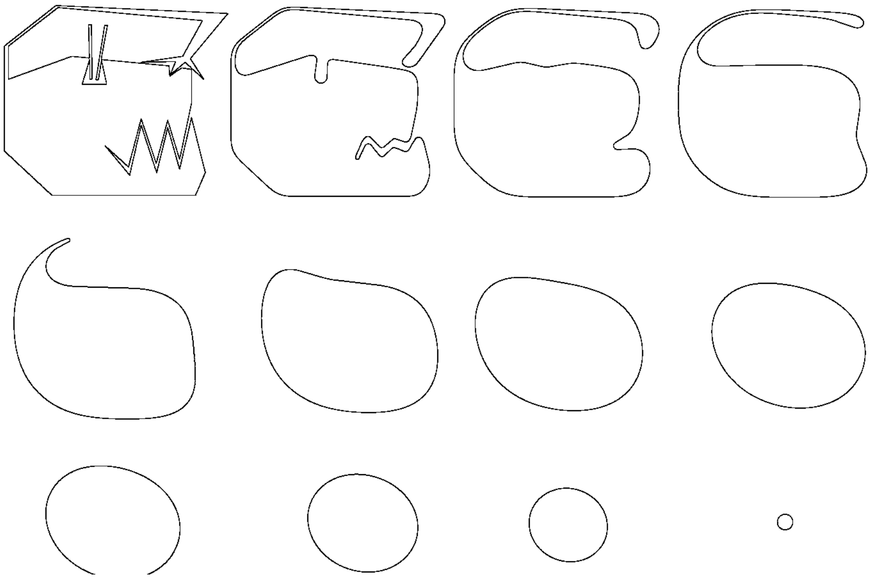


Fig. 6.18. Evolution of a weird polygon with $\gamma = 1$ (CPU: 19.7s)

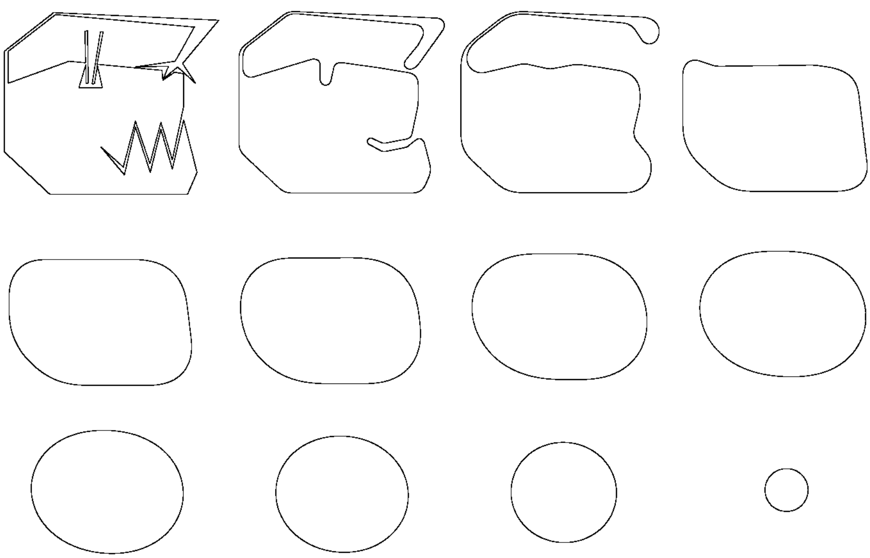


Fig. 6.19. Evolution of a weird polygon with $\gamma = 2$ (CPU: 262s)

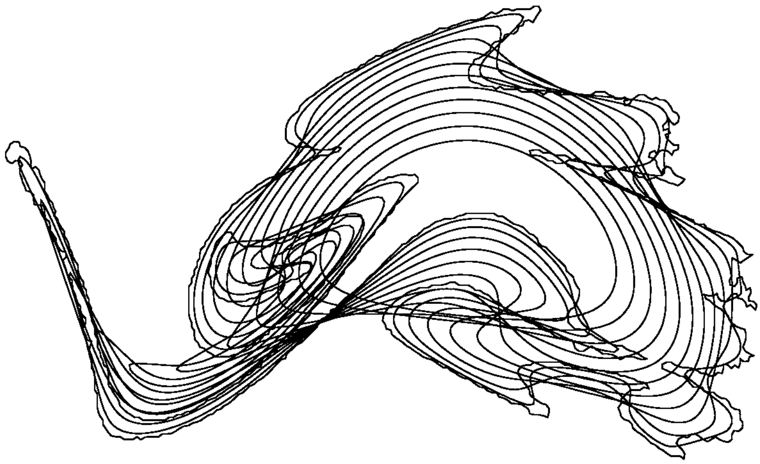


Fig. 6.20. Evolution of a scanned curve with $\gamma = 1/3$ (CPU: 5.5s)

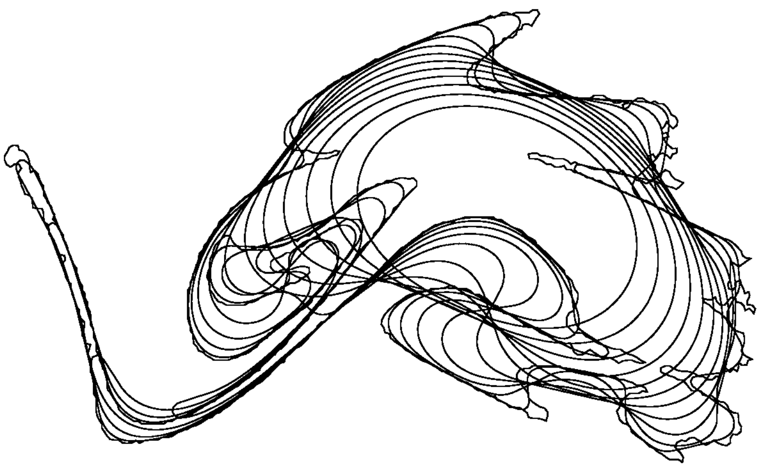


Fig. 6.21. Evolution of a scanned curve with $\gamma = 1$ (CPU: 50s)

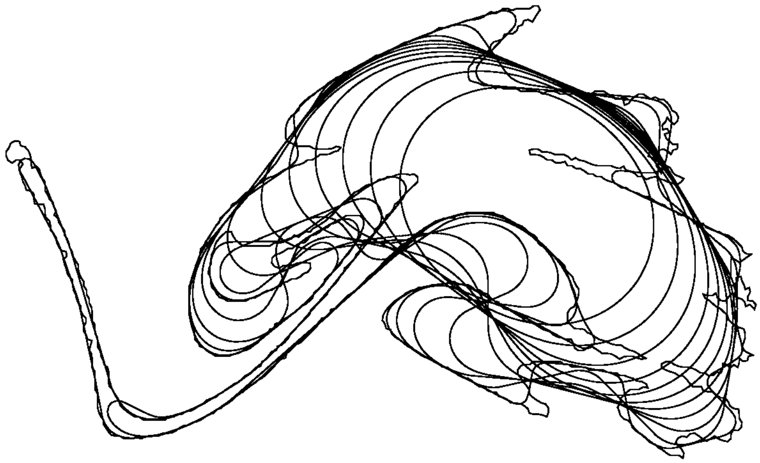


Fig. 6.22. Evolution of a scanned curve with $\gamma = 2$ (CPU: 141s)

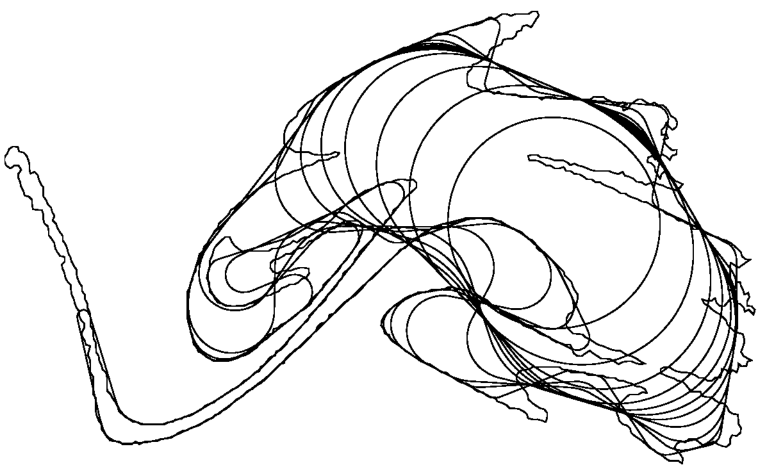


Fig. 6.23. Evolution of a scanned curve with $\gamma = 5$ (CPU: 1600s)

6.7.5 Invariance

All the equations we consider are Euclidean invariant. Moreover, in the special case $\gamma = 1/3$, the evolution is affine invariant. The schemes we used have the same properties. Some errors may come from the numerical approximation of floating numbers. In the following experiments, we check the invariance of the numerical erosion.
The first interesting test is the affine invariance of the case $\gamma = 1/3$ and was previously examined by Moisan [126]. We apply to a curve[3] the linear mapping given by the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 0.5 & 1.5 \end{pmatrix}.$$

We apply the evolution algorithm to this new curve. Note the A is area preserving since its determinant is equal to 1. As a consequence, we shorten the original and the deformed curves up to the same scale (Fig. 6.24 and 6.25). We then apply the inverse of A to the curve which has been deformed by A, then shortened and we check that the results are the same. The invariance is very well satisfied, and we must zoom of a factor at least 10 to start to see the difference between both curves.

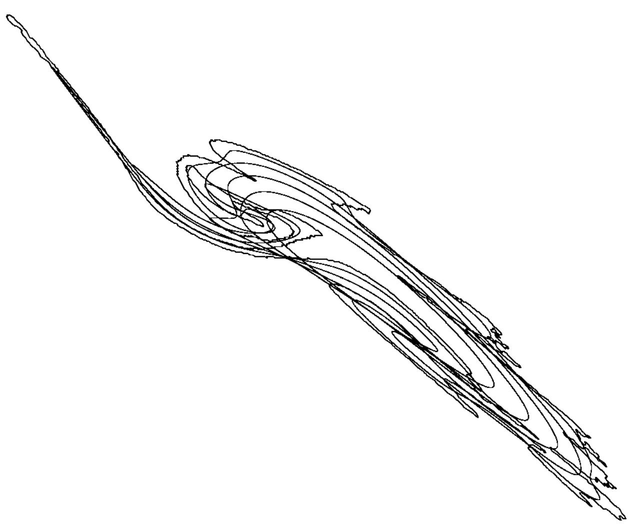


Fig. 6.24. Evolution of the curve after applying an affine area preserving mapping, and with $\gamma = 1/3$.

As a comparison, we make the same experiment for the mean curvature motion flow which is not affine invariant. On Fig. 6.26, we see clearly see the difference between both flows (with and without shear) and no zooming is even necessary.

[3] This curve is a level-line in an image of a smoke cloud.

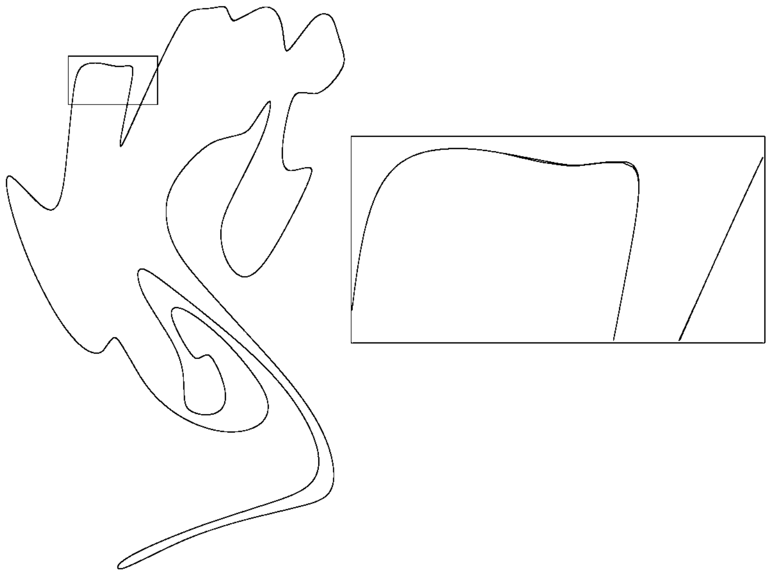


Fig. 6.25. Experimental checking of affine invariance. One curve is smoothed by the affine erosion, while the other one is first deformed by $A \in SL(\mathbb{R}^2)$, then smoothed and finally goes under A^{-1}. Only by zooming the part in the box and can we see that there is still a very slight difference between the curves

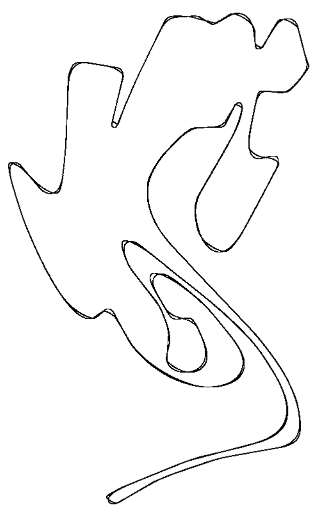


Fig. 6.26. One of the curve is obtained by mean curvature flow. We get the other one by applying an affine transform, the mean curvature flow and the inverse affine transform. We check that the mean curvature flow is not affine invariant since the curves do not coincide

Nonetheless, all the flow we considered are Euclidean invariant. We check this on the mean curvature motion. We first apply a similitude with ratio 0.8 and angle 30^o and shorten the new curve to scale 4. Next, we apply the inverse similitude. On the other hand, we smooth the initial curve up to scale 5. On Fig. 6.27, we superimpose both curve. We hardly see any difference, and this becomes clear only by zooming on curvy parts. Again, the distance between both curves is less than 0.1 pixel.

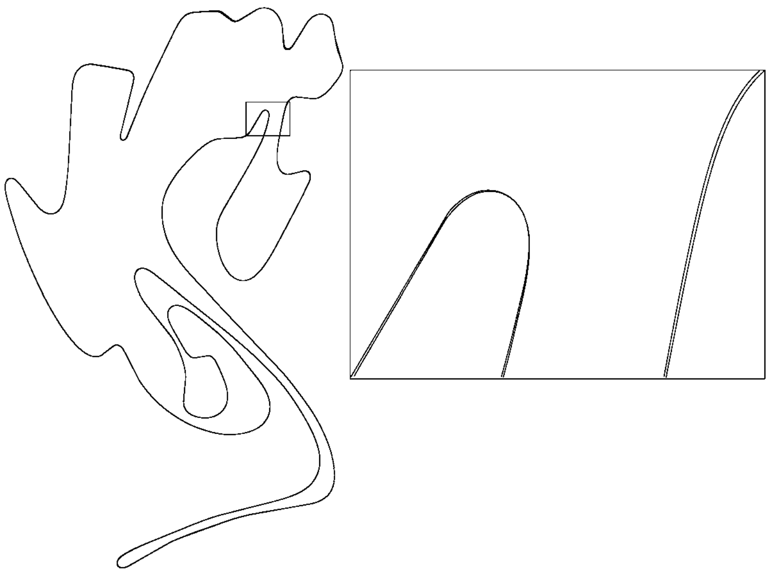


Fig. 6.27. Similitude covariance of the shortening flow driven by a power of the curvature (here $\gamma = 1$). We superimpose a curve and its correspondent after application of the same similitude as above. A slight difference appears on the middle finger, the index and the little finger. Moreover, we also have the zoom invariance property

6.7.6 Numerical maximum principle

In virtue of the maximum principle, it is known that if a curve surrounds another one at initial time, then this remains true if the curves evolve by a parabolic flow. In this section, we show that the maximum principle is numerically true. This is also the occasion to check the inclusion principle property 6.10. In the following experiments, we independently apply the evolution to two curves. As can be seen, the inclusion principle is not violated. In the first series of figures, we display two intricated evolving spirals. As time goes by, they unfold without crossing each other. Remark that one of the spirals is thinner than the other one. As a consequence, it disappears faster and faster as the power of the curvature increases. We display the evolution for different values of γ until the first curve disappears. When γ increases, the largest curve hardly moves.

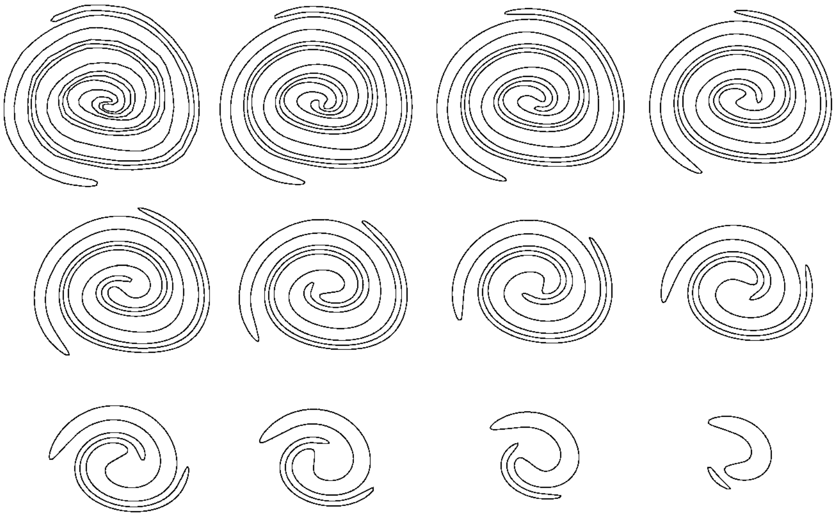


Fig. 6.28. Evolution of two spirals, $\gamma = 1/3$, CPU: 11s

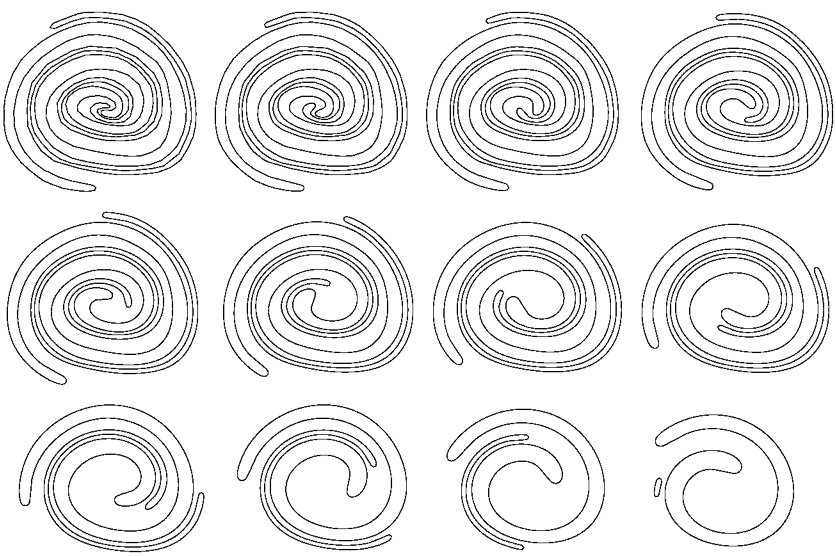


Fig. 6.29. Evolution of two spirals, $\gamma = 1$, CPU: 64s

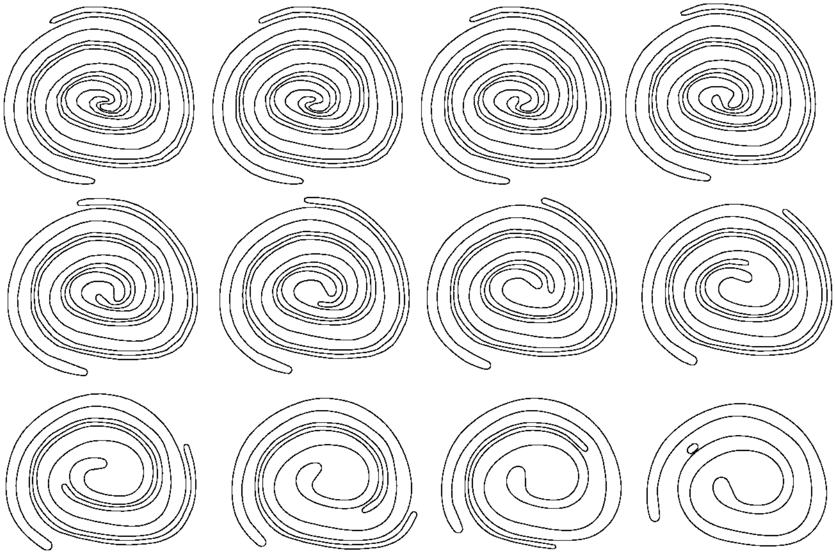


Fig. 6.30. Evolution of two spirals, $\gamma = 2$, CPU: 158s

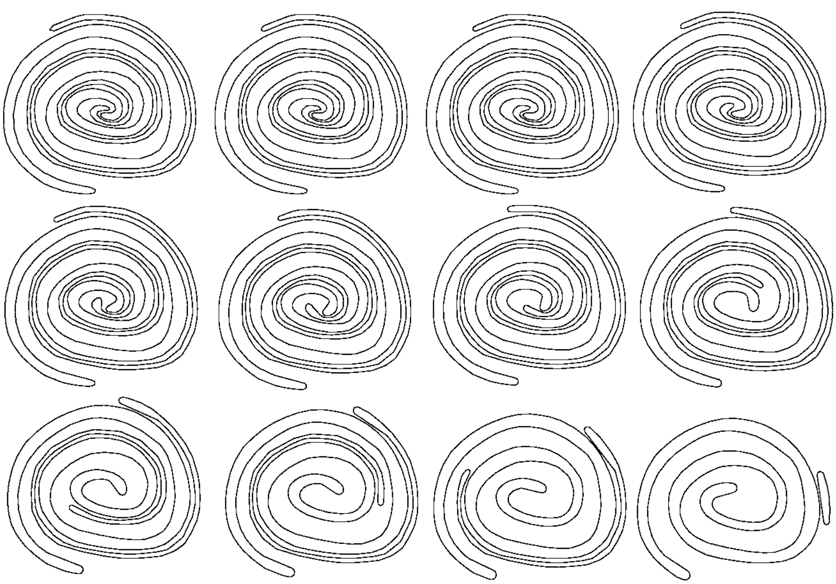


Fig. 6.31. Evolution of two spirals, $\gamma = 5$, CPU: 1700s

We show other experiments of simultaneous curves evolution. Again, two initial curves are intricated and the algorithm is applied to them independently. As foreseen, the curves never cross.

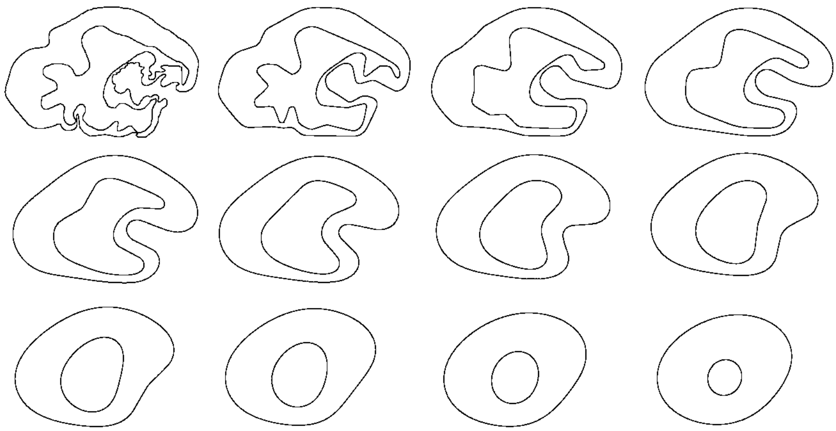


Fig. 6.32. Evolution of two shapes, $\gamma = 1$, CPU: 4s

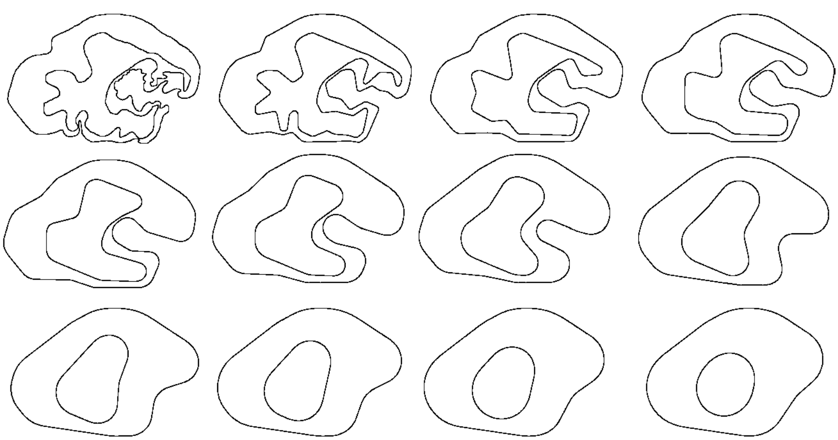


Fig. 6.33. Evolution of two shapes, $\gamma = 5$, CPU: 600s

6.7.7 Image filtering

By Matheron's result (Prop. 4.12), we deduce an algorithm of image filtering obtained by shortening all the level lines by the iterated erosion/dilation. In this case, we lose the gain of time computation. Indeed, we first have to decompose an image in level sets(See Koepfler and Moisan [104]). The decomposition can be made quite rapidly thanks to a very recent algorithm by Monasse [128, 129]. We independently make level lines evolve by the erosion algorithm. We then reconstruct the filtered image from its level sets. Remember that this makes sense only because the erosion is monotonous and continuous. In this method of filtering, the subpixel resolution is lost as soon as we reconstruct the image. If we wanted to obtain an image with subpixel resolution, it might be useful to create new gray levels by using some adequate interpolation method [33, 115].

For shape recognition application, this is certainly not optimal to filter all the level lines before trying to registrate them. It is more logical to start extracting lines which are geometrically meaningful and to filter only the necessary level lines.

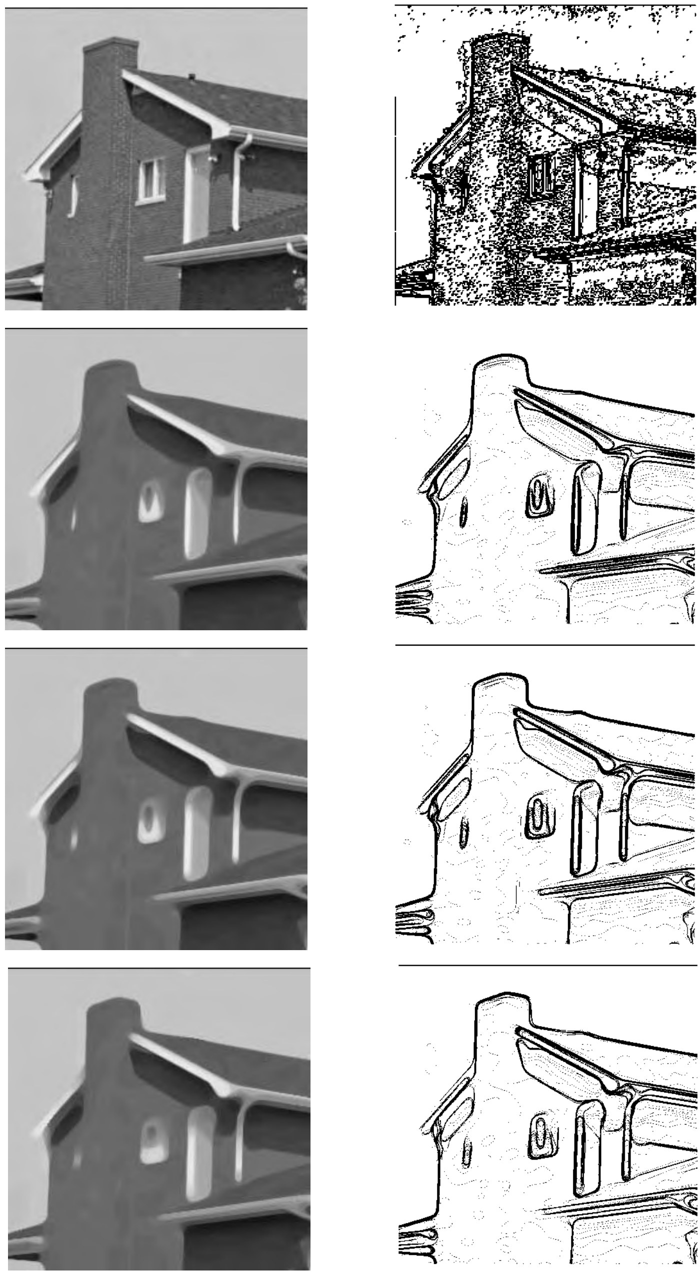


Fig. 6.34. Filtering of level lines. Top Left: Original image. Top Right: level lines (multiple of 20). New three rows. From top to bottom: level lines filtering with normal velocity equal to κ^γ with $\gamma = 1/3,\ 1,\ 2$. On the right, level lines (multiple of 5)

We apply the same filtering on Fig. 6.35. We see that the texture on the cushion is lost, as expected. Nevertheless, level lines that correspond to contours of real object are kept. On Figs. 6.38, we display the level lines of the head of "Mona Lisa". Remark how the topographic map contains most of the information of the image. On Fig. 6.36 and 6.37, we display experiments of filtering of a noisy image (25%

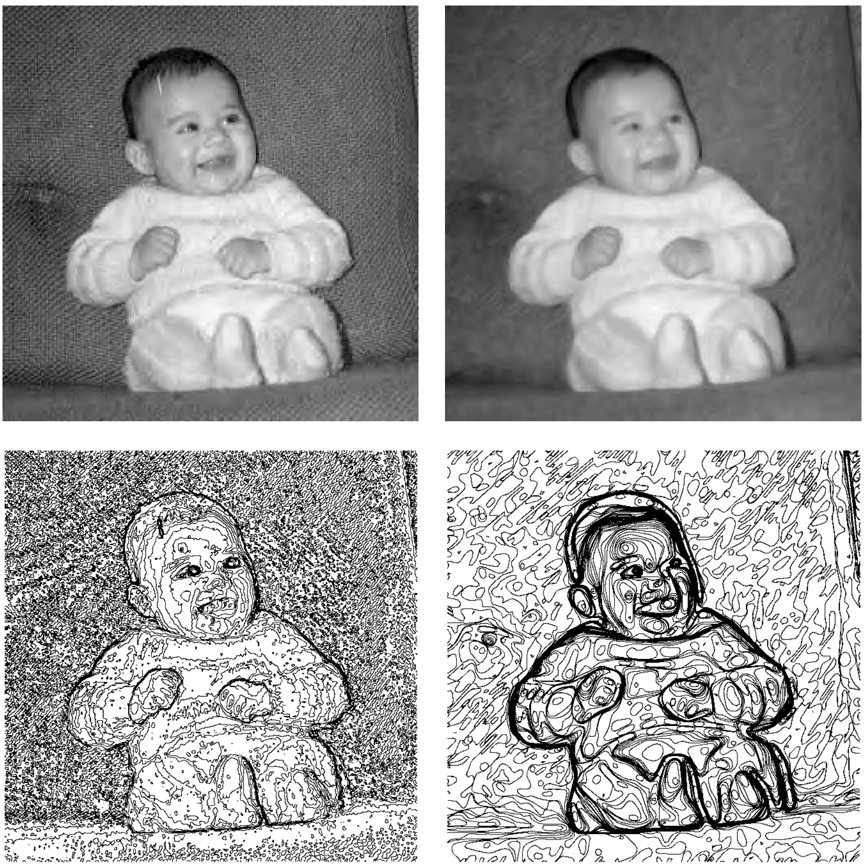


Fig. 6.35. Filtering of images obtained by smoothing all the level lines. Left: original image. Right: filtering of all level lines by the mean curvature motion. Down: level lines of the images above. The level which are multiples of 20 are displayed for the original and multiples of 5 on the filtered one. Remark how the level lines concentrate around the edges. Contrary to the classical heat equation, the level lines do not separate and stay close to each other

impulse noise) with six different methods: curve shortening, grain filters [165, 129], Alvarez and Guichard's scheme, Bence, Merriman and Osher's scheme (See [22] and Chap. 5), median filter and the heat equation.

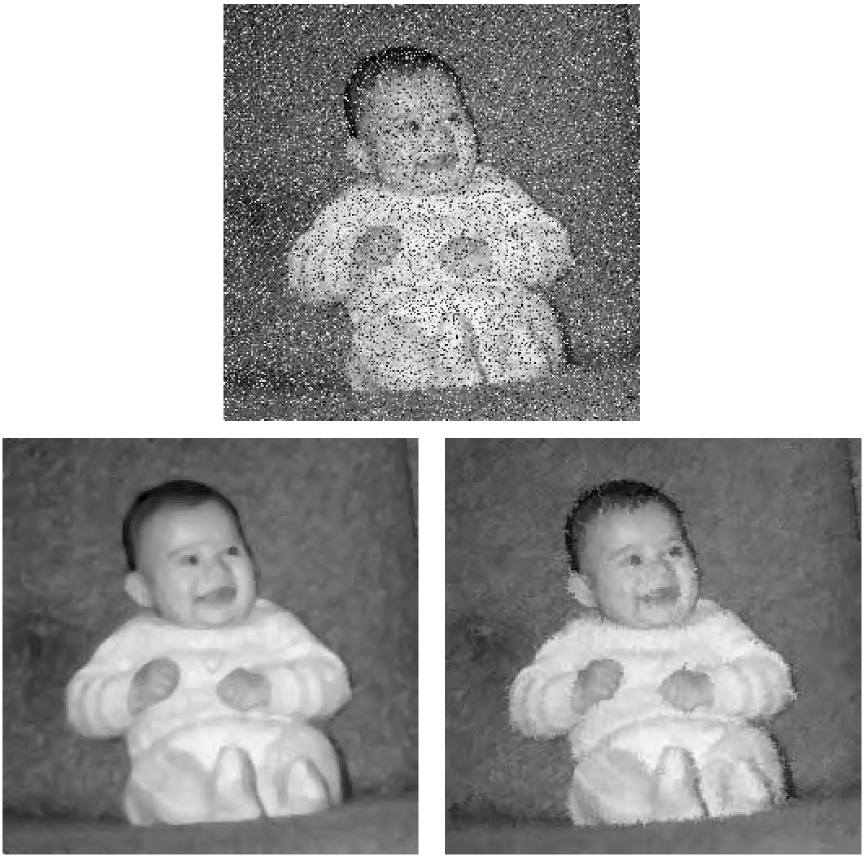


Fig. 6.36. Top: image with 25% impulse noise. Second row. Left: filtering of level lines with $V = \kappa^2$. It destroys T-junctions at occlusions and computations are slow. Right: grain filter. We removed all the connected components of level sets with an area less than 20. It preserves junctions but contours stay noisy

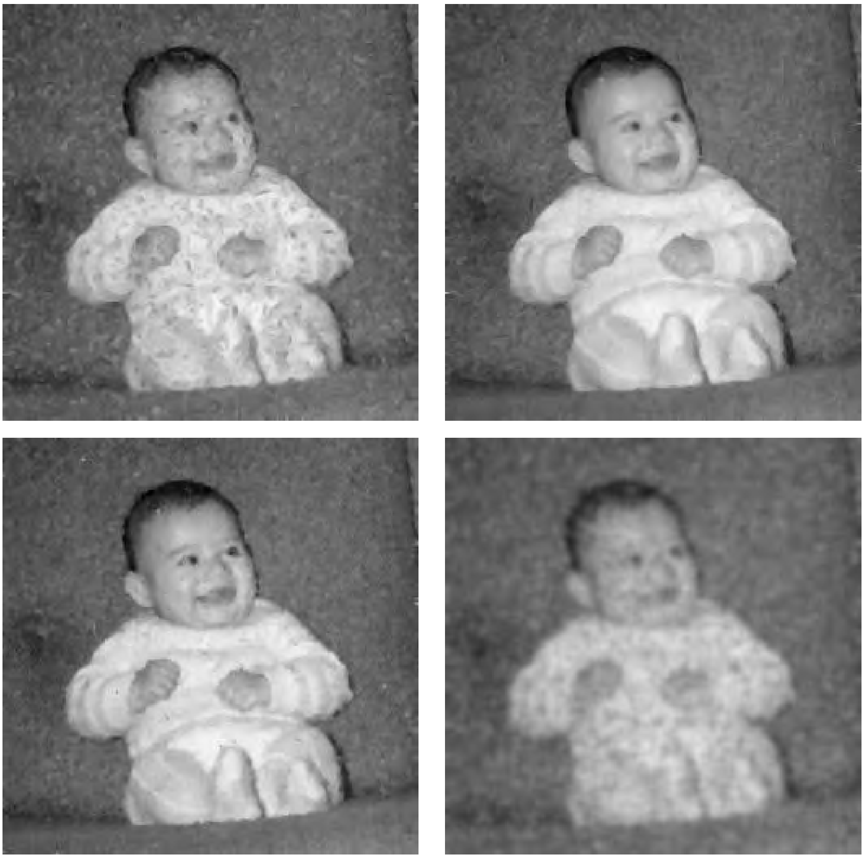


Fig. 6.37. Filtering of a 25% impulse noise image (following). Top left: Alvarez and Guichard's scheme is inefficient in this case because it badly estimates the gradient direction. Right: Bence, Merriman and Osher's scheme works better and is contrast invariant, whereas it is slower since it requires to solve the heat equation for every level sets. Down: the median filter is a quite good approximation of the mean curvature motion. Right: the classical heat equation completely blurs the image before the noise has been removed

Fig. 6.38. Detail of "La Joconda". Top row. Left: level lines multiple of 5. Right: affine scale space. Bottom row. Left: mean curvature motion. Right: $V = \kappa^2$

On the last series of experiments, we applied some impulse noise on an image. We display the level lines of both images (original and filtered) for a quantization step of 20 and for values of $\gamma = 1/3$, 1 and 2 and for scales equal to $t = 1$, 2 and 3. The level lines of the filtered images (both original and noisy) becomes really similar beyond $t = 2$.

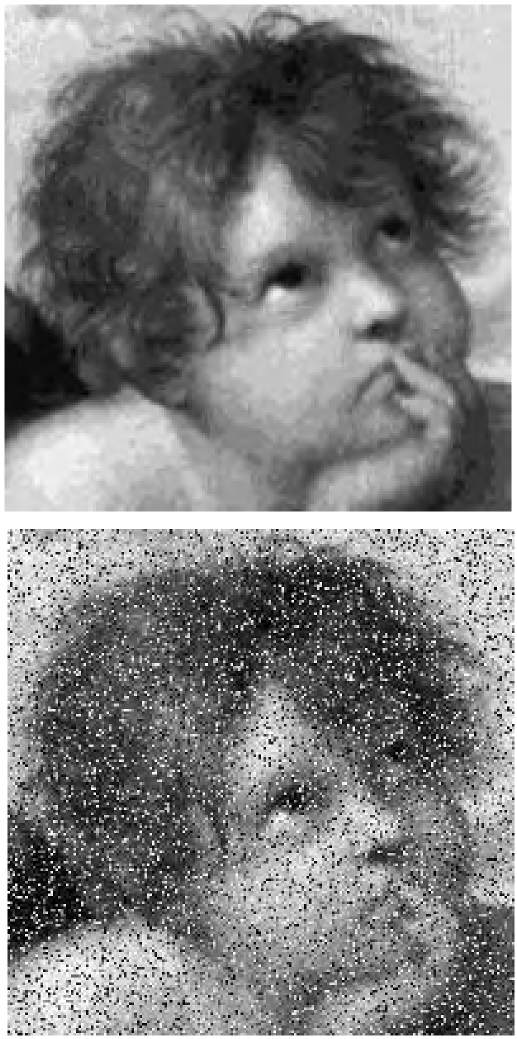


Fig. 6.39. Detail of "La madonna sistine" (Raffaello). On the bottom image 25% of impulse noise is added

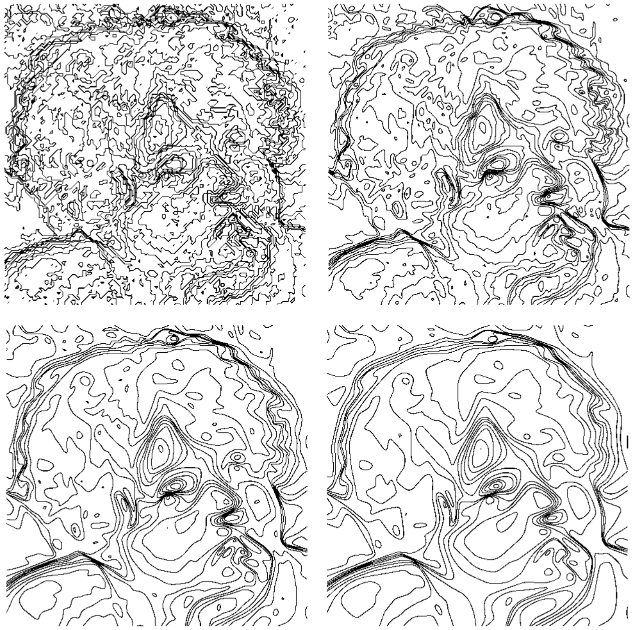


Fig. 6.40. Scale Space of original image, $\gamma = 1/3$. CPU: 22s. 413 lines over 547 disappeared

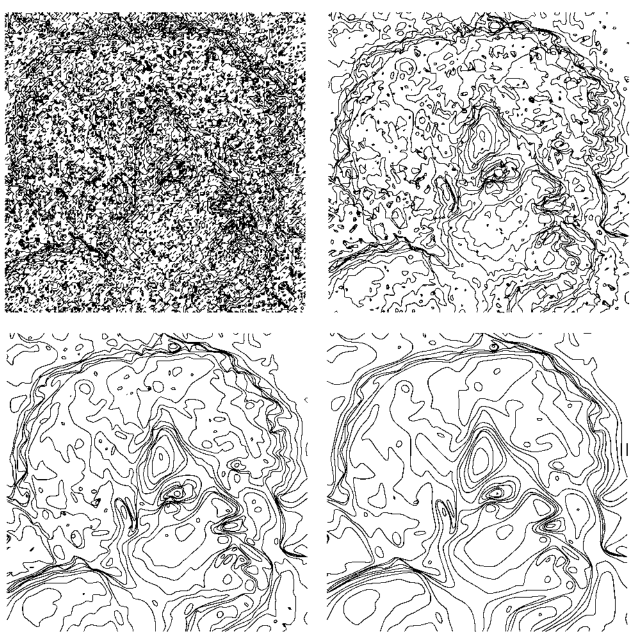


Fig. 6.41. Scale Space of noisy image, $\gamma = 1/3$. CPU: 33s. 8201 lines over 8587 disappeared

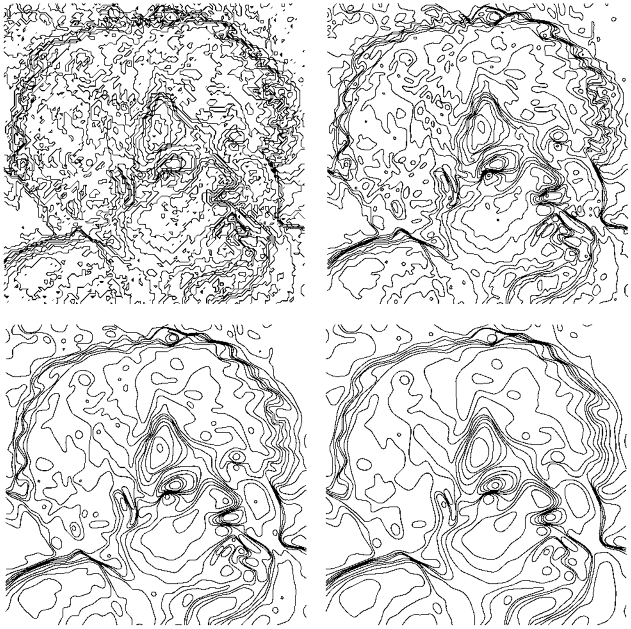


Fig. 6.42. Scale Space of original image, $\gamma = 1$. CPU: 189s. 423 lines over 547 disappeared

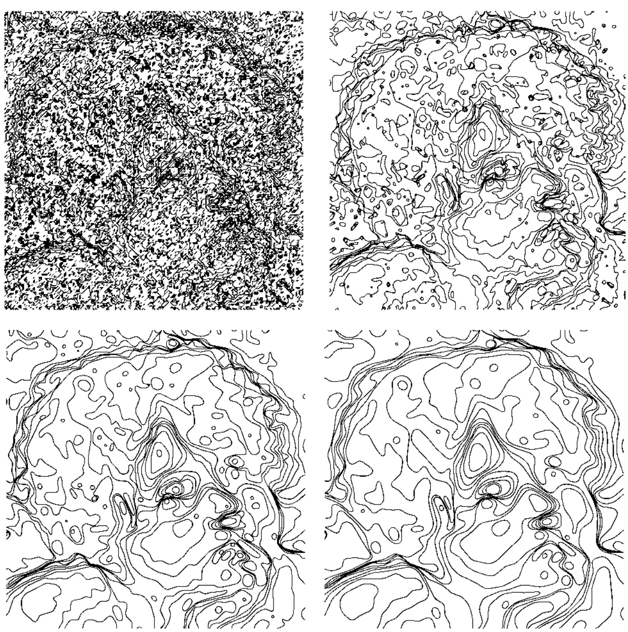


Fig. 6.43. Scale Space of noisy image, $\gamma = 1$. CPU: 314s. 8210 lines over 8587 disappeared

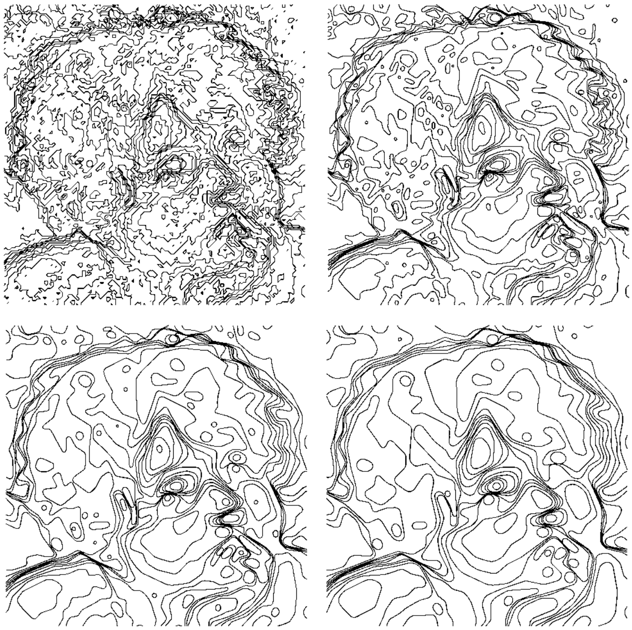


Fig. 6.44. Scale Space of original image, $\gamma = 2$. CPU: 369s. 426 lines over 547 disappeared

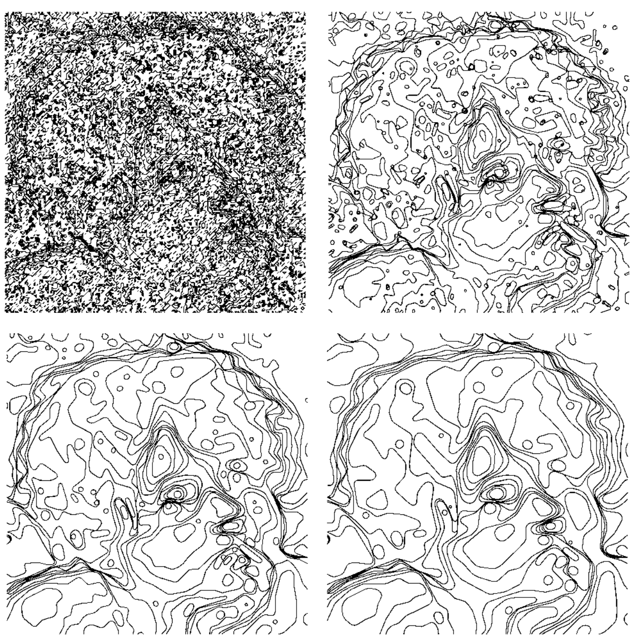


Fig. 6.45. Scale Space of noisy image, $\gamma = 2$. CPU: 579s. 8215 lines over 8587 disappeared

6.8 Bibliographical notes

The affine erosion, which is the first affine invariant and monotone scheme, was discovered by Moisan [125, 126]. He also proved the consistency and the convergence of the scheme. It was later simplified in a paper with Koepfler [104] where they used the middle-point property (Lem. 6.30) to improve the scheme speed. In this paper, they also use the level set decomposition to filter the level lines of an image, then reconstruct it thanks to monotonicity. The decomposition was later made easier by Monasse and Guichard's algorithm [128, 129], that we used for our experiments. Uniform consistency for the Lipschitz case (Sect. 6.4.1 and Lem. 6.20) is obtained thanks to the localizability of the affine erosion (Moisan [125]). Localizability was introduced by Guichard and Morel [81]. The convergence Theorem 6.27 is essentially due to Barles and Souganidis [20], adapted to the case of degenerate equations. The algorithm we describe was previously published by Cao and Moisan [28], based on an idea of Ishii [91] for a completely different topic [66, 93].

Conclusion and perspectives

Discussion

In this volume, we have seen that two different approaches of shape simplification naturally led to the same class of equations; the curvature motions. The first one is based on geometrical invariance and proves that being given a subgroup of the projective group, all the invariant equations could be expressed in terms of the group length and the group curvature which are respectively the simplest invariant 1-form and the simplest differential invariant. This allows to find *all* the invariant equations, sorted with respect to their simplicity.
The second approach focuses on a stability principle expressed in terms of monotonicity, and use the embedding of the curves into graphs of functions. (Level sets methods.) This implies that not all the operators are suitable for our analysis but only the morphological ones. It is then seen, that scaled morphological operators are consistent with a generalized motion by curvature.
Results of existence, uniqueness and regularity on geometric evolutions of curves are very difficult to prove, and unknown in general. The second order case is the simplest, due to the parabolic nature of the curvature motion but some cases are not elucidated yet, and we simply enounced what we believe to be the most up-to-date results. As a matter of fact, a reasonable conjecture is that any curve becomes convex before it shrinks to a point. This result is only proved in the cases of the mean curvature motion and the affine shortening flow. With the level sets formulation, the existence and uniqueness is more easy. The strength of the method is also a more simple implementation that can be generalized in any dimension. On the other hand, as curves are defined implicitly, the link between classical evolution and the generalized level set approach is not obvious.
Then comes the implementation topic. It is natural to look for schemes with the same invariance properties as the curvature motions. Although the level sets evolutions are more natural to implement, they are not geometrically invariant in most cases because schemes are generally defined on a fixed square grid. In particular affine and contrast invariance are very hard constraints excluding all the finite differences scheme. Moreover, as the curvature motions are nonlinear, schemes are

F. Cao: LNM 1805, pp. 167–169, 2003.

usually not monotone, and the proof of their convergence unknown. The evolution scheme we described in Chap 6 has *exactly* the same invariance properties as the underlying equations. Moreover, it is also monotone, which naturally implies the convergence. We proposed an implementation of this scheme which is numerically and experimentally stable, and allows to compute motions by curvature when the normal velocity is a power of the curvature. It seems reliable for powers up to 6 or 7 which were previously impossible to handle. In particular, invariance is experimentally respected.

Nevertheless, as could have said Billy Wilder, nothing is perfect. The first thing is that beyond those values of the power of the curvature, some numerical computations may have no sense. Indeed, small numbers to the power 3γ appear in the algorithm. (The normal velocity being $V = \kappa^\gamma$.) For large γ, this does not only raise problems of numerical accuracy but also makes the algorithm considerably slow. Another algorithmic problem we had is the convexification step. Indeed, we saw that the numerical approximation of the erosion scheme we proposed usually transform a convex curve in a nonconvex one. Since we know that the result should be convex, it is theoretically easy to remove spurious points. Numerically, we want to find a procedure in linear computational time. The successive versions we implemented were more and more stable, though clearly imperfect. In particular, the numerical parameters as the area erosion and the discretization step may be chosen with care, even if the algorithm works for a quite large range of values.

The motion of inflexion points is also unclear. Experimentally, their speed decreases with γ which seems logical. Numerically, inflexion points are badly defined since any point except the vertices of a polygon is an inflexion point! We rather define inflexion segments as segments for which the convexity of the curve changes and inflexion points as the middle-points of inflexion segments. Therefore, inflexion points are never vertices of the polygon. This is crucial since this allow them to move! A higher order study could maybe tell if this procedure is licit.

What value of γ should be chosen for image processing tasks? For shape recognition, the properties that are absolutely necessary are numerical speed, stability, and geometrical invariance. For all these reasons, the affine invariant case $\gamma = 1/3$ should be used, since it is the fastest and the most invariant. Powers tending to infinity certainly provide an operator whose effect is to threshold curvature while preserving the topology, which is an old problem for mathematical morphologists. Large powers are maybe more adapted to noise removal applications. Nevertheless, our algorithm works with a subpixel accuracy which is certainly useless if the purpose is only to suppress the noise. Moreover, the computational time becomes completely prohibitive if we smoothed all the curves in an image for very large γ. (A very simple operation is efficient before smoothing: we can first remove the level lines with small area or small length.) Simpler algorithms (like a grain filter followed by a median filter) may be as efficient and more rapid.

Another usual question is the choice of the smoothing scale. There is a usual answer consisting in considering the curve at *any* scale. This was the original purpose of multiscale analysis. Local features were supposed to be reliably detected at large scale then tracked back to small scale. Multiscale analysis makes the recognition

stabler but naturally increases the computation time. For this reason, only a few values of the scale are practically used and this is generally efficient.
One should not be surprised that a smoothed curve eventually becomes circular or elliptic before disappearing. Multiscale analysis such as we presented it is not aimed at denoising curves (which assumes at least that we know the typical scale of noise), but to extract the geometric information of curves at *any* scale. The fact that a curve disappears means that it has a finite size, and the fact that it becomes rounder and rounder means that it contains a finite amount of geometrical information which is empty beyond a certain scale. When the curve disappears, we can be sure that all the information has been extracted through the process.
We also emphasize the fact that we never pretended that a smoothing was absolutely necessary for applications. It is indeed, for many existing algorithms in image analysis, up to now, mainly because shape matching algorithms are local and noise sensitive. What we only say is that if a smoothing is to be used then it should be invariant and stable, and its approximation too, as much as possible.

Open problems

It is very unlikely that curvature motions of embedded plane curves develop singularities before the curves vanish. However, this still remains to be proved in full generality. The correspondence between the level set approach and the direct geometric evolution is established in a weak sense. In [62, 63], Evans and Spruck indeed prove that the level set solution is a Brakke's varifold solution [23] for the mean curvature motion. For more general motion, an equivalence between level set flows and a purely geometric notion of solution has to be found.
The higher dimension case is much more difficult to handle. It is known that singularities in the motion of surfaces do occur [34, 78]. Numerical approximations of surfaces are far more complex than curves approximations. Even though a theoretical erosion scheme exists for functions of the Gaussian curvature [91], it seems very complex to implement, let alone the topology changes problem. All the algorithm we know use a level set representation, but geometrical invariant schemes are very challenging.

A

Proof of Thm. 4.34

In this chapter, we give a proof of the result of Alvarez, Guichard, Lions and Morel [4] that the isotropic morphological scale spaces are given by equations of the form

$$\frac{\partial u}{\partial t} = |Du| G(\operatorname{curv} u, t),$$

where G is continuous and nondecreasing with respect to its first argument.
In particular, there exists a unique affine morphological scale space which is described by the equation

$$\frac{\partial u}{\partial t} = |Du| (t \operatorname{curv} u)^{1/3}.$$

We shall admit the scale normalization which is detailed in [81]. In this chapter, we do not cope with this problem. In order to simplify the notations, we shall not mark the dependence on t. We thus assume that translation, rotation, contrast, contrast reversal and special affine invariances hold for the operator $T_{t,t+h}$. We shall use these axioms and progressively reduce the set of possible equations. We shall also use Lem. 4.30 and write the regularity axiom for C^2 functions.

First step. First use of contrast invariance: $F(A, p, c, x)$ does not depend on c.
Let u be a C^2 function and $v = u + c$ where $c \in \mathbb{R}$. The function $g(s) = s + c$ is a contrast change, thus $T_{t,t+h} v(x) = c + T_{t_t+h} u(x)$. Hence

$$\frac{T_{t,t+h} v(x) - v(x)}{h} = \frac{T_{t,t+h} u(x) - u(x)}{h}.$$

We let $h \to 0$. The left-hand term tends to $F(D^2u, Du, u + c, x)$ and the right-hand term to $F(D^2u, Du, u, x)$. Since c is arbitrary, F does not depend on the third argument.
Thus, there exists a function F_1 such that $F(A, p, c, x) = F_1(A, p, x)$.

Second step. Use of translation invariance: $F_1(A, p, x)$ does not depend on x. Let u be a C^2 function and let $v(x) = \tau_a u(x) = u(x - a)$. The regularity assumption implies that

F. Cao: LNM 1805, pp. 171–176, 2003.

$$\begin{aligned}\frac{(T_{t,t+h}v)(x+a)-v(x+a)}{h} &\to F_1(D^2v(x+a), Dv(x+a), x+a)\\ &= F_1(D^2u(x), Du(x), x+a).\end{aligned}$$

But, by translation invariance

$$\begin{aligned}\frac{(T_{t,t+h}v)(x+a)-v(x+a)}{h} &= \frac{\tau_{-a}(T_{t,t+h}\tau_a u)(x)-u(x)}{h}\\ &= \frac{\tau_{-a}\tau_a(T_{t,t+h}u)(x)-u(x)}{h}\\ &\to F_1(D^2u(x), Du(x), x).\end{aligned}$$

Since a is arbitrary, F does not depend on x, and we can write $F(A,p,c,x) = F_1(A,p,x) = F_2(A,p)$.

Third step. Use of rotation invariance. Let R be a plane rotation and $v(x) = u(R^t x)$. We have

$$\begin{aligned}v(x+y) &= u(R^t x) + (Du(R^t x), R^t y) + \frac{1}{2}D^2u(R^t y, R^t y) + o(y^2)\\ &= u(R^t x) + (RDu(R^t x), y) + \frac{1}{2}\left(RD^2u(R^t x)R^t\right)(y,y) + o(y^2),\end{aligned}$$

which by uniqueness of Taylor expansion implies

$$Dv(x) = RDu(R^t x), \quad D^2v(x) = RD^2u(R^t x)R^t.$$

Then, by regularity

$$\frac{T_{t,t+h}v(x)-v(x)}{h} \to F_2(RD^2u(R^t x)R^t, RDu(R^t x)),$$

but also by rotation invariance

$$\begin{aligned}\frac{T_{t,t+h}v(x)-v(x)}{h} &= \frac{T_{t,t+h}(Ru)(x)-Ru(x)}{h}\\ &= \frac{R(T_{t,t+h}u)(x)-Ru(x)}{h}\\ &= \frac{T_{t,t+h}u(R^t x)-u(R^t x)}{h}\\ &= F_2(D^2u(R^t x), Du(R^t x)).\end{aligned}$$

This yields for any R, A and p

$$F_2(A,p) = F_2(RAR^t, Rp). \tag{A.1}$$

If $p \in \mathbb{R}^2\backslash\{0\}$, we denote by R_p the unique rotation mapping p onto $|p|e_1$, where e_1 is the first vector of a fixed chosen basis. By applying this rotation in (A.1), we obtain

$$F_2(A,p) = F_2(R_pAR_p^t, |p|e_1).$$

Since e_1 is fixed once for all, we have

$$F(A,p,c,x) = F_2(A,p) = F_3(R_pAR_p^t, |p|),$$

for some function F_3.

Fourth step. Use of contrast invariance. Let g be a C^2 increasing contrast change and $v = g(u)$. A straighforward application of the chain rule yields $Dv = g'(u)Du$ and $D^2v = g''(u)Du \otimes Du + g'(u)D^2u$. We have

$$\frac{T_{t,t+h}v(x) - v(x)}{h} \to F_2(g'(u)D^2u + g''(u)Du \otimes Du, g'(u)Du).$$

But, by contrast invariance,

$$\begin{aligned}\frac{T_{t,t+h}v(x) - v(x)}{h} &= \frac{g(T_{t,t+h}u(x)) - g(u(x))}{h} \\ &\to g'(u)F_2(D^2u, Du).\end{aligned}$$

Since g is arbitrary and increasing, g'' can take any value and g' can take any positive value. Thus

$$\forall\lambda > 0,\ \mu \in \mathbb{R} \quad F_2(\lambda A + \mu p \otimes p, \lambda p) = \lambda F_2(A,p). \tag{A.2}$$

Remark that by taking $\mu = 0$, we obtain

$$F_2(\lambda A, \lambda p) = \lambda F(A,p).$$

We plug this result in the form of a rotation invariant operator and obtain the identity

$$F_3(R_{\lambda p}(\lambda A + \mu p \otimes p)R_{\lambda p}^t, |\lambda p|) = \lambda F_3(R_pAR_p^t, |p|).$$

Since $\lambda > 0$, $R_{\lambda p} = R_p$ and $R_pp \otimes pR_p^t = e_1 \otimes e_1$. The equality above becomes

$$F_3(\lambda R_pAR_p^t + \mu e_1 \otimes e_1, \lambda|p|) = \lambda F_3(R_pAR_p^t, |p|). \tag{A.3}$$

We choose the particular value $\lambda = 1/|p|$ and we get

$$F_3(R_pAR_p^t, |p|) = |p|F_3\left(\frac{1}{|p|}R_pAR_p^t + \mu e_1 \otimes e_1, 1\right). \tag{A.4}$$

The first argument of F_3 is a 2×2 symmetric matrix and thus depends on three scalar parameters. Let us explicit these parameters. The rotation R_p^t sends the vector $|p|e_1$ onto p. Thus, we can write R_p^t with two column vectors equal to $p/|p|$ and $p^\perp/|p|$ where $p^\perp$ is orthogonal to p such that $(p, p^\perp)$ forms a direct basis. We can then write $R_pAR_p^t$ in the basis (e_1, e_2) and obtain

$$\frac{1}{|p|}R_pAR_p^t + \mu e_1 \otimes e_1 = \frac{1}{|p|^3}\begin{pmatrix} A(p,p) + \mu|p|^3 & A(p,p^\perp) \\ A(p,p^\perp) & A(p^\perp,p^\perp)\end{pmatrix},$$

where A is written in the basis (e_1, e_2). We know that F_3 only depends on the three independent parameters of this matrix. Moreover, μ is arbitrary, hence F_3 is independent on the upper left coefficient. This means that

$$F(A,p,c,x) = |p| F_4\left(\frac{1}{|p|^3} A(p^\perp, p^\perp), \frac{1}{|p|^3} A(p, p^\perp)\right),$$

where F_4 depends on two scalar parameters.

Fifth step. Use of monotonicity. We saw that the monotonicity implies that F is nondecreasing with respect to A. We take the special case $p = e_1$ for which

$$F(A, e_1, x, c) = F_4(a_{22}, a_{12}),$$

where the a_{ij} are the coefficient of A in the basis (e_1, e_2). Let $\varepsilon > 0$, $\lambda \in \mathbb{R}$ and consider the matrix

$$A_{\varepsilon,\lambda} = \begin{pmatrix} a_{11} + \frac{\lambda^2}{\varepsilon^2} & a_{12} + \lambda \\ a_{12} + \lambda & a_{22} + \varepsilon^2 \end{pmatrix}.$$

Let us compare $A_{\varepsilon,\lambda}$ and A. For $(x, y) \in \mathbb{R}^2$

$$A_{\varepsilon,\lambda}((x,y),(x,y)) = A((x,y),(x,y)) + \frac{\lambda^2}{\varepsilon^2} x^2 + 2\lambda xy + \varepsilon^2 y^2.$$

But $2\lambda xy = 2(\lambda x/\varepsilon)(\varepsilon y) \geq -\lambda^2 x^2/\varepsilon^2 - \varepsilon^2 y^2$, thus $A_{\varepsilon,\lambda} \geq A$ for any $\varepsilon > 0$ and $\lambda \in \mathbb{R}$. Hence $F(A_{\varepsilon,\lambda}, e_1, c, x) \geq F(A, e_1, c, x)$ and also

$$F_4(a_{22} + \varepsilon^2, a_{12} + \lambda) \geq F_4(a_{22}, a_{12}).$$

Since F is continuous ($p = e_1 \neq 0$), we obtain by letting ε go to 0

$$F_4(a_{22}, a_{12} + \lambda) \geq F_4(a_{22}, a_{12}).$$

Since λ is arbitrary, this implies that F_4 does not depend upon a_{12}. Hence

$$F(A,p,c,x) = |p| F_5\left(\frac{1}{|p|^3} A(p^\perp, p^\perp)\right),$$

where F_5 is a continuous and nondecreasing function. This proves that if u is C^2 then

$$F(D^2u, Du, u, x) = |Du| F_5(\operatorname{curv} u).$$

Conversely, we can check that this kind of operator is rotation, translation and contrast invariant. This proves the first part of the theorem.

Sixth step. Finally, we use the affine invariance and prove that $F_5(s) = s^{1/3}$. For now, the operator is invariant by translation, rotation, and contrast change. Let now A be an affine mapping such that $\det A = 1$. Let $v(x) = u(Ax)$. The chain rule implies

$$Dv(x) = A^t Du(Ax) \text{ and } D^2 v(x) = A^t D^2 u(A^t x) A.$$

As above, we combine the affine invariance and the regularity axioms and obtain

$$
\begin{aligned}
&|Du|F_5\left(\frac{1}{|Du|^3}D^2u(Du^\perp, Du^\perp)\right) = \\
&\qquad |A^tDu|F_5\left(\frac{1}{|A^tDu|^3}(A^tD^2uA)((A^tDu)^\perp, (A^tDu)^\perp)\right).
\end{aligned}
\tag{A.5}
$$

The expressions above are taken at the point Ax. For this fixed x, we choose A such that $A^tDu = \lambda Du$ and $A^tDu^\perp = Du^\perp/\lambda$. The expression above gives

$$
|Du|F_5\left(\frac{1}{|Du|^3}D^2u(Du^\perp, Du^\perp)\right) = |\lambda Du|F_5\left(\frac{1}{|\lambda Du|^3}D^2u(Du^\perp, Du^\perp)\right).
$$

This yields

$$
F_5(\lambda^3 s) = \lambda F_5(s),
$$

for any nonnegative λ. Thus, there exists two nonnegative constants a and b such that $F_5(s) = as^{1/3}$ for $s \geq 0$ and $F_5(s) = bs^{1/3}$ for $s \leq 0$, where we take the convention $s^{1/3} = s|s|^{-2/3}$.

Seventh and last step. Use of contrast reversal invariance. By taking $v(x) = u(-x)$ we obtain that F_5 is an odd function. Then

$$
F(A, p, c, x) = a|p|\left(\frac{A(p^\perp, p^\perp)}{|p|^3}\right)^{1/3}.
$$

Up to a rescaling, we conclude that the only possible affine morphological parabolic second order equation, namely

$$
\frac{\partial u}{\partial t} = |Du|(\operatorname{curv} u)^{1/3}.
$$

Conversely this equation is morphological, rotation invariant and by the same arguments as above, we can prove that it commutes with pure stretchings, that is affine mapping that are diagonal in some basis. Before concluding, it must be checked that this equation is also invariant with respect to pure shearings, that is affine mappings with matrix of the type

$$
A = \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix}.
$$

Let us write it in a coordinates system. If u is C^2, and $v(x) = u(Ax)$. The chain rules gives

$$
Dv(x) = \begin{pmatrix} u_x \\ \lambda u_x + u_y \end{pmatrix}(Ax),
$$

$$
D^2v(x) = \begin{pmatrix} u_{xx} & \lambda u_{xx} + u_{xy} \\ \lambda u_{xx} + u_{xy} & \lambda^2 u_{xx} + 2\lambda u_{xy} + u_{yy} \end{pmatrix}(Ax).
$$

A straightforward computation gives

$$(u_{xx}u_y^2 - 2u_{xy}u_xu_y + u_{yy}u_x^2)(Ax) = (v_{xx}v_y^2 - 2v_{xy}v_xv_y + v_{yy}v_x^2)(x).$$

and thus that $|Du|(\operatorname{curv} u)^{1/3}(Ax) = |Dv|(\operatorname{curv} v)^{1/3}(x)$. This proves that the equation is affine invariant. $\square$

It can proved that this equation is not projective invariant. By uniqueness of the affine invariant equation, we conclude that projective invariance is incompatible with the set of axioms we use. We have to let down either morphological invariance or the comparison principle. Any morphological projective invariant equation is of order strictly larger than 2 and no maximum principle holds.

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