Holm Altenbach · Johannes Altenbach
Wolfgang Kissing

Mechanics of Composite Structural Elements

Second Edition

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ISBN 978-981-10-8934-3 ISBN 978-981-10-8935-0 (eBook)
https://doi.org/10.1007/978-981-10-8935-0

Library of Congress Control Number: 2018939885

Printed on acid-free paper

This Springer imprint is published by the registered company Springer Nature Singapore Pte Ltd.
part of Springer Nature
The registered company address is: 152 Beach Road, #21-01/04 Gateway East, Singapore 189721, Singapore

Preface to the 2nd Edition

This is the second edition of the textbook *Mechanics of Composite Structural Elements* published first in 2004. Since that time the course has been delivered at several universities in Germany and abroad. Throughout the past years the authors received a lot of suggestions for improvements from students and colleagues alike. In addition, the textbook is recommended as the basic reading material in a relevant course at the Otto-von-Guericke-Universität Magdeburg.

In 2016 the first author was invited by Prof. Andreas Öchsner to present a course with the same title at the Griffith University (Gold Coast campus) for third and fourth year students of the bachelor program in the departments of Mechanical Engineering and Civil Engineering. The two weeks' course included 60 hours of lectures and tutorials. Finally, the course was concluded with a written exam and a project. Our special thanks are due to Dr. Christoph Baumann and Springer who provided personal copies of the first edition of the book for the attendants of the course. As the result of the discussions with the students the idea was born to prepare a second edition.

By and large the preliminaries of the first edition remain unchanged: the presentation of the mechanics of composite materials is based on the knowledge of the first and second year of the bachelor program in Engineering Mechanics (or in other countries the courses of General Mechanics and Strength of Materials). The focus of the students will be directed to the elementary theory as the starting point of further advanced courses.

There are some changes in the second edition in comparison with the first one:

- some problems are added or clarified (and we hope now better understandable),
- Chapter 11 is slightly shortened (some details are no more important),
- some details were adopted considering the developments of Springer's templates.

Some references for further reading, but also some original sources are added and the tables with material data are improved. Of course, we hope you will now find fewer misprints and typos.

We have to acknowledge Dr.-Ing. Heinz Köppe (Otto-von-Guericke-Universität Magdeburg) and Dipl.-Ing. Christoph Kammer (formaly at Otto-von-Guericke-

Universität Magdeburg) for finding a lot of typos. In addition, we have to thank Dr. Christoph Baumann (Executive Editor Engineering, Springer Nature Singapore) for the permanent support of the project. We appreciate for any comment and suggestion for improvements which should be sent to `holm.altenbach@ovgu.de`.

Magdeburg and Bad Kleinen,
March 2018

Holm Altenbach
Johannes Altenbach
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Preface to the 1st Edition

Laminate and sandwich structures are typical lightweight elements with rapidly expanding application in various industrial fields. In the past, these structures were used primarily in aircraft and aerospace industries. Now, they have also found application in civil and mechanical engineering, in the automotive industry, in shipbuilding, the sport goods industries, etc. The advantages that these materials have over traditional materials like metals and their alloys are the relatively high specific strength properties (the ratio strength to density, etc). In addition, the laminate and sandwich structures provide good vibration and noise protection, thermal insulation, etc. There are also disadvantages - for example, composite laminates are brittle, and the joining of such elements is not as easy as with classical materials. The recycling of these materials is also problematic, and a viable solution is yet to be developed. Since the application of laminates and sandwiches has been used mostly in new technologies, governmental and independent research organizations, as well as big companies, have spent a lot of money for research. This includes the development of new materials by material scientists, new design concepts by mechanical and civil engineers as well as new testing procedures and standards. The growing demands of the industry for specially educated research and practicing engineers and material scientists have resulted in changes in curricula of the diploma and master courses. More and more universities have included special courses on laminates and sandwiches, and training programs have been arranged for postgraduate studies.

The concept of this textbook was born 10 years ago. At that time, the first edition of ”Einführung in die Mechanik der Laminat- und Sandwichtragwerke”, prepared by H. Altenbach, J. Altenbach and R. Rikards, was written for German students only. The purpose of that book consisted the following objectives:

- to provide a basic understanding of composite materials like laminates and sandwiches,
- to perform and engineering analysis of structural elements like beams and plates made from laminates and sandwiches,
- to introduce the finite element method for the numerical treatment of composite structures and

- to discuss the limitations of analysis and modelling concepts.

These four items are also included in this textbook. It must be noted that between 1997 and 2000, there was a common education project sponsored by the European Community (coordinator T. Sadowski) with the participation of colleagues from U.K., Belgium, Poland and Germany. One of the main results was a new created course on laminates and sandwiches, and finally an English textbook ”Structural Analysis of Laminate and Sandwich Beams and Plates” written by H. Altenbach, J. Altenbach and W. Kissing.

The present textbook follows the main ideas of its previous versions, but has been significantly expanded. It can be characterized by the following items:

- The textbook is written in the style of classical courses of strength of materials (or mechanics of materials) and theory of beams, plates and shells. In this sense the course (textbook) can be recommended for master students with bachelor degree and diploma students which have finished the second year in the university. In addition, postgraduates of various levels can find a simple introduction to the analysis and modelling of laminate and sandwich structures.
- In contrast to the traditional courses referred to above, two extensions have been included. Firstly, consideration is given to the linear elastic material behavior of both isotropic and anisotropic structural elements. Secondly, the case of inhomogeneous material properties in the thickness direction was also included.
- Composite structures are mostly thin, in which case a dimension reduction of the governing equations is allowed in many applications. Due to this fact, the one-dimensional equations for beams and the two-dimensional equations for plates and shells are introduced. The presented analytical solutions can be related to the in-plane, out-of-plane and coupled behavior.
- Sandwiches are introduced as a special case of general laminates. This results in significant simplifications because sandwiches with thin or thicker faces can be modelled and analyzed in the frame of laminate theories of different order and so a special sandwich theory is not necessary.
- All analysis concepts are introduced for the global structural behavior. Local effects and their analysis must be based on three-dimensional field equations which can usually be solved with the help of numerical methods. It must be noted that the thermomechanical properties of composites on polymer matrix at high temperatures can be essentially different from those at normal temperatures. In engineering applications generally three levels of temperature are considered

  – normal or room temperature (10°–30° C)
  – elevated temperatures (30°–200° C)
  – high temperatures ($>$ 200° C)

  High temperatures yield an irreversible variation of the mechanical properties, and thus are not included in modelling and analysis. All thermal and moisture effects are considered in such a way that the mechanical properties can be assumed unchanged.

- Finite element analysis is only briefly presented. A basic course in finite elements is necessary for the understanding of this part of the book. It should be noted that the finite element method is general accepted for the numerical analysis of laminate and sandwich structures. This was the reason to include this item in the contents of this book.

The textbook is divided into 11 chapters and several appendices summarizing the material properties (for matrix and fiber constituents, etc) and some mathematical formulas:

- In the first part (Chaps. 1–3) an introduction into laminates and sandwiches as structural materials, the anisotropic elasticity, variational methods and the basic micromechanical models are presented.
- The second part (Chaps. 4–6) can be related to the modelling from single laminae to laminates including sandwiches, the improved theories and simplest failure concepts.
- The third part (Chaps. 7–9) discusses structural elements (beams, plates and shells) and their analysis if they are made from laminates and sandwiches. The modelling of laminated and sandwich plates and shells is restricted to rectangular plates and circular cylindrical shells. The individual fiber reinforced laminae of laminated structured elements are considered to be homogeneous and orthotropic, but the laminate is heterogeneous through the thickness and generally anisotropic. An equivalent single layer theory using the classical lamination theory, and the first order shear deformation theory are considered. Multilayered theories or laminate theories of higher order are not discussed in detail.
- The fourth part (Chap. 10) includes the modelling and analysis of thin-walled folded plate structures or generalized beams. This topic is not normally considered in standard textbooks on structural analysis of laminates and sandwiches, but it was included here because it demonstrates the possible application of Vlasov's theory of thin-walled beams and semi-membrane shells on laminated structural elements.
- Finally, the fifth part (Chap. 11) presents a short introduction into the finite element procedures and developed finite classical and generalized beam elements and finite plate elements in the frame of classical and first order shear deformation theory. Selected examples demonstrate the possibilities of finite element analysis.

This textbook is written for use not only in engineering curricula of aerospace, civil and mechanical engineering, but also in material science and applied mechanics. In addition, the book may be useful for practicing engineers, lectors and researchers in the mechanics of structures composed of composite materials.

The strongest feature of the book is its use as a textbook. No prior knowledge of composite materials and structures is required for the understanding of its content. It intends to give an in-depth view of the problems considered and therefore the number of topics considered is limited. A large number of solved problems are included to assess the knowledge of the presented topics. The list of references at the end of the book focuses on three groups of suggested reading:

- Firstly, a selection of textbooks and monographs of composite materials and structures are listed, which constitute the necessary items for further reading. They are selected to reinforce the presented topics and to provide information on topics not discussed. We hope that our colleagues agree that the number of recommended books for a textbook must be limited and we have given priority to newer books available in university libraries.
- Some books on elasticity, continuum mechanics, plates and shells and FEM are recommended for further reading, and a deeper understanding of the mathematical, mechanical and numerical topics.
- A list of review articles shall enable the reader to become informed about the numerous books and proceedings in composite mechanics.

The technical realization of this textbook was possible only with the support of various friends and colleagues. Firstly, we would like to express our special thanks to K. Naumenko and O. Dyogtev for drawing most of the figures. Secondly, Mrs. B. Renner and T. Kumar performed many corrections of the English text. At the same time Mrs. Renner checked the problems and solutions. We received access to the necessary literature by Mrs. N. Altenbach. Finally, the processing of the text was done by Mrs. S. Runkel. We would also like to thank Springer Publishing for their service.

Any comments or remarks are welcome and we kindly ask them to be sent to `holm.altenbach@iw.uni-halle.de`.

June 2003

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Contents

Part I
Introduction, Anisotropic Elasticity, Micromechanics

In the first part (Chaps. 1–3) an introduction into laminates and sandwiches as structural materials, the anisotropic elasticity, variational methods and the basic micromechanical models are presented.

The laminates are introduced as layered structures each of the layers is a fibre-reinforced material composed of high-modulus, high-strength fibers in a polymeric, metallic, or ceramic matrix material. Examples of fibers used include graphite, glass, boron, and silicon carbide, matrix materials are epoxies, polyamide, aluminium, titanium, and aluminium. A sandwich is a special class of composite materials consist of two thin but stiff skins and a lightweight but thick core.

The anisotropic elasticity is an extension of the isotropic elasticity. The geometrical relations are assumed to be linear. The constitutive equations contain more than two material parameters. In addition, the transition from the general three-dimensional equations to the special two-dimensional equations results in more complicated constrains. At the same time the introduction of reduced stiffness and compliance parameters result in a powerful tool for the analysis of laminates.

The variational methods are the base of many numerical solution techniques (for example, the finite element method). Here only the classical principles and methods are briefly discussed.

There are many, partly sophisticated micromechanical approaches. They are the base of a better understanding of the local behavior. In the focus of this textbook is the global structural analysis. Thats way the micromechanical models are presented only in the elementary form.

Chapter 1
Classification of Composite Materials

Fibre reinforced polymer composite systems have become increasingly important in a variety of engineering fields. Naturally, the rapid growth in the use of composite materials for structural elements has motivated the extension of existing theories in structural mechanics, therein. The main topics of this textbook are

- a short introduction into the linear mechanics of deformable solids with anisotropic material behavior,
- the mechanical behavior of composite materials as unidirectional reinforced single layers or laminated composite materials, the analysis of effective moduli, some basic mechanisms and criteria of failure,
- the modelling of the mechanical behavior of laminates and sandwiches, general assumptions of various theories, classical laminate theory (CLT), effect of stacking of the layers of laminates and the coupling of stretching, bending and twisting, first order shear deformation theory (FOSDT), an overview on refined equivalent single layer plate theories and on multilayered plate modelling,
- modelling and analysis of laminate and sandwich beams, plates and shells, problems of bending, vibration and buckling and
- modelling and analysis of fibre reinforced long thin-walled folded-plate structural elements.

The textbook concentrates on a simple unified approach to the basic behavior of composite materials and the structural analysis of beams, plates and circular cylindrical shells made of composite material being a laminate or a sandwich. The introduction into the modelling and analysis of thin-walled folded structural elements is limited to laminated elements and the CLT. The problems of manufacturing and recycling of composites will be not discussed, but to use all benefits of the new young material composite, an engineer has to be more than a material user as for classical materials as steel or alloys. Structural engineering qualification must include knowledge of material design, manufacturing methods, quality control and recycling.

In Chap. 1 some basic questions are discussed, e.g. what are composites and how they can be classified, what are the main characteristics and significance, micro-

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0_1

and macro-modelling, why composites are used, what are the advantages and the limitations. The App. F contains some values of the material characteristics of the constituents of composites.

1.1 Definition and Characteristics

Material science classifies structural materials into three categories

- metals,
- ceramics and
- polymers.

It is difficult to give an exact assessment of the advantages and disadvantages of these three basic material classes, because each category covers whole groups of materials within which the range of properties is often as broad as the differences between the three material classes. But at the simplistic level some obvious characteristic properties can be identified:

- Mostly metals are of medium to high density. They have good thermal stability and can be made corrosion-resistant by alloying. Metals have useful mechanical characteristics and it is moderately easy to shape and join. For this reason metals became the preferred structural engineering material, they posed less problems to the designer than either ceramic or polymer materials.
- Ceramic materials have great thermal stability and are resistant to corrosion, abrasion and other forms of attack. They are very rigid but mostly brittle and can only be shaped with difficulty.
- Polymer materials (plastics) are of low density, have good chemical resistance but lack thermal stability. They have poor mechanical properties, but are easily fabricated and joined. Their resistance to environmental degradation, e.g. the photomechanical effects of sunlight, is moderate.

A material is called homogeneous if its properties are the same at every point and therefore independent of the location. Homogeneity is associated with the scale of modelling or the so-called characteristic volume and the definition describes the average material behavior on a macroscopic level. On a microscopic level all materials are more or less homogeneous but depending on the scale, materials can be described as homogeneous, quasi-homogeneous or inhomogeneous. A material is inhomogeneous or heterogeneous if its properties depend on location. But in the average sense of these definitions a material can be regarded as homogeneous, quasi-homogeneous or heterogeneous if the scale decreases.

A material is isotropic if its properties are independent of the orientation, they do not vary with direction. Otherwise the material is anisotropic. A general anisotropic material has no planes or axes of material symmetry, but in Sect. 2.1.3 some special kinds of material symmetries like orthotropy, transverse isotropy, etc., are discussed in detail.

Furthermore, a material can depend on several constituents or phases, single phase materials are called monolithic. The above three mentioned classes of conventional materials are on the macroscopic level more or less monolithic, homogeneous and isotropic.

The group of materials which can be defined as composite materials is extremely large. Its boundaries depend on definition. In the most general definition we can consider a composite as any material that is a combination of two or more materials, commonly referred to as constituents, and have material properties derived from the individual constituents. These properties may have the combined characteristics of the constituents or they are substantially different. Sometimes the material properties of a composite material may exceed those of the constituents. This general definition of composites includes natural materials like wood, traditional structural materials like concrete, as well as modern synthetic composites such as fibre or particle reinforced plastics which are now an important group of engineering materials where low weight in combination with high strength and stiffness are required in structural design.

In the more restrictive sense of this textbook a structural composite consists of an assembly of two materials of different nature. In general, one material is discontinuous and is called the reinforcement, the other material is mostly less stiff and weaker. It is continuous and is called the matrix. The properties of a composite material depends on

- The properties of the constituents,
- The geometry of the reinforcements, their distribution, orientation and concentration usually measured by the volume fraction or fiber volume ratio,
- The nature and quality of the matrix-reinforcement interface.

In a less restricted sense, a structural composite can consist of two or more phases on the macroscopic level. The mechanical performance and properties of composite materials are superior to those of their components or constituent materials taken separately. The concentration of the reinforcement phase is a determining parameter of the properties of the new material, their distribution determines the homogeneity or the heterogeneity on the macroscopic scale. The most important aspect of composite materials in which the reinforcement are fibers is the anisotropy caused by the fiber orientation. It is necessary to give special attention to this fundamental characteristic of fibre reinforced composites and the possibility to influence the anisotropy by material design for a desired quality.

Summarizing the aspects defining a composite as a mixture of two or more distinct constituents or phases it must be considered that all constituents have to be present in reasonable proportions that the constituent phases have quite different properties from the properties of the composite material and that man-made composites are produced by combining the constituents by various means. Figure 1.1 demonstrates typical examples of composite materials. Composites can be classified by their form and the distribution of their constituents (Fig. 1.2). The reinforcement constituent can be described as fibrous or particulate. The fibres are continuous (long fibres) or discontinuous (short fibres). Long fibres are arranged usually in uni-

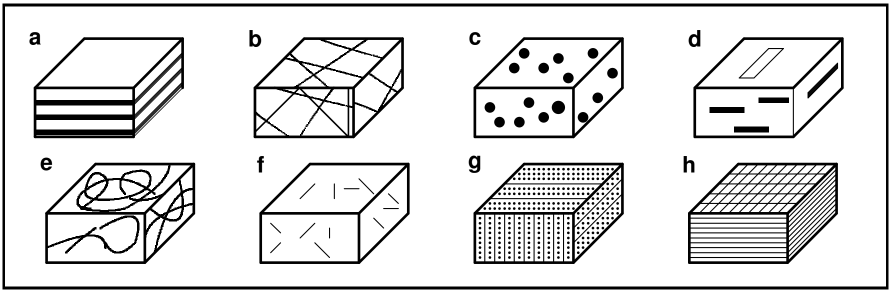

Fig. 1.1 Examples of composite materials with different forms of constituents and distributions of the reinforcements. **a** Laminate with uni- or bidirectional layers, **b** irregular reinforcement with long fibres, **c** reinforcement with particles, **d** reinforcement with plate strapped particles, **e** random arrangement of continuous fibres, **f** irregular reinforcement with short fibres, **g** spatial reinforcement, **h** reinforcement with surface tissues as mats, woven fabrics, etc.

or bidirectional, but also irregular reinforcements by long fibres are possible. The arrangement and the orientation of long or short fibres determines the mechanical properties of composites and the behavior ranges between a general anisotropy to a quasi-isotropy. Particulate reinforcements have different shapes. They may be spherical, platelet or of any regular or irregular geometry. Their arrangement may be random or regular with preferred orientations. In the majority of practical applications particulate reinforced composites are considered to be randomly oriented and

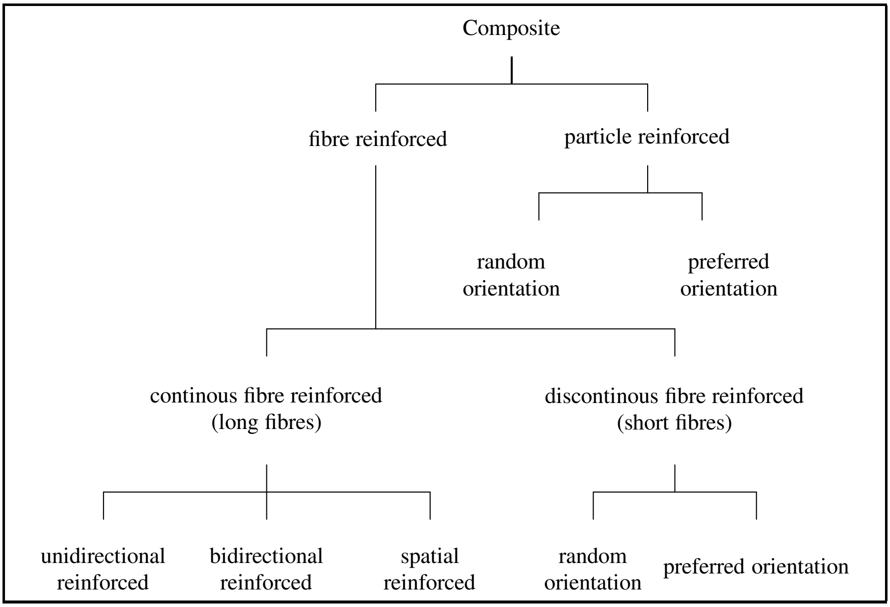

Fig. 1.2 Classification of composites

the mechanical properties are homogeneous and isotropic (for mor details Christensen, 2005; Torquato, 2002). The preferred orientation in the case of continuous fibre composites is unidirectional for each layer or lamina. Fibre reinforced composites are very important and in consequence this textbook will essentially deal with modelling and analysis of structural elements composed of this type of composite material. However, the level of modelling and analysis used in this textbook does not really differentiate between unidirectional continuous fibres, oriented short-fibres or woven fibre composite layers, as long as material characteristics that define the layer response are used. Composite materials can also be classified by the nature of their constituents. According to the nature of the matrix material we classify organic, mineral or metallic matrix composites.

- Organic matrix composites are polymer resins with fillers. The fibres can be mineral (glass, etc.), organic (Kevlar, etc.) or metallic (aluminium, etc.).
- Mineral matrix composites are ceramics with metallic fibres or with metallic or mineral particles.
- Metallic matrix composites are metals with mineral or metallic fibres.

Structural composite elements such as fibre reinforced polymer resins are of particular interest in this textbook. They can be used only in a low temperature range up to 200° to 300° C. The two basic classes of resins are thermosets and thermoplastics. Thermosetting resins are the most common type of matrix system for composite materials. Typical thermoset matrices include *Epoxy*, *Polyester*, *Polyamide* (*Thermoplastics*) and *Vinyl Ester*, among popular thermoplastics are *Polyethylene*, *Polystyrene* and *Polyether-ether-ketone (PEEK)* materials. Ceramic based composites can also be used in a high temperature range up to 1000° C and metallic matrix composites in a medium temperature range.

In the following a composite material is constituted by a matrix and a fibre reinforcement. The matrix is a polyester or epoxy resin with fillers. By the addition of fillers, the characteristics of resins will be improved and the production costs reduced. But from the mechanical modelling, a resin-filler system stays as a homogeneous material and a composite material is a two phase system made up of a matrix and a reinforcement.

The most advanced composites are polymer matrix composites. They are characterized by relatively low costs, simple manufacturing and high strength. Their main drawbacks are the low working temperature, high coefficients of thermal and moisture expansion and, in certain directions, low elastic properties. Most widely used manufacturing composites are thermosetting resins as unsaturated polyester resins or epoxy resins. The polyester resins are used as they have low production cost. The second place in composite production is held by epoxy resins. Although epoxy is costlier than polyester, approximately five time higher in price, it is very popular in various application fields. More than two thirds of polymer matrices used in aerospace industries are epoxy based. Polymer matrix composites are usually reinforced by fibres to improve such mechanical characteristics as stiffness, strength, etc. Fibres can be made of different materials (glass, carbon, aramid, etc.). Glass fibres are widely used because their advantages include high strength, low costs, high chemical resistance, etc., but their elastic modulus is very low and also their fatigue strength. Graphite or carbon fibres have a high modulus and a high strength and are very common in aircraft components. Aramid fibres are usually known by the name of Kevlar, which is a trade name. Summarizing some functional requirements of fibres and matrices in a fibre reinforced polymer matrix composite

- fibres should have a high modulus of elasticity and a high ultimate strength,
- fibres should be stable and retain their strength during handling and fabrication,
- the variation of the mechanical characteristics of the individual fibres should be low, their diameters uniform and their arrangement in the matrix regular,
- matrices have to bind together the fibres and protect their surfaces from damage,
- matrices have to transfer stress to the fibres by adhesion and/or friction and
- matrices have to be chemically compatible with fibres over the whole working period.

The fibre length, their orientation, their shape and their material properties are main factors which contribute to the mechanical performance of a composite. Their volume fraction usually lies between 0.3 and 0.7. Although matrices by themselves generally have poorer mechanical properties than compared to fibres, they influence many characteristics of the composite such as the transverse modulus and strength, shear modulus and strength, thermal resistance and expansion, etc.

An overview of the material characteristics is given in Sect. 1.4. One of the most important factors which determines the mechanical behavior of a composite material is the proportion of the matrix and the fibres expressed by their volume or their weight fraction. These fractions can be established for a two phase composite in the following way. The volume V of the composite is made from a matrix volume V_{m} and a fibre volume V_{f} ($V = V_{\mathrm{f}} + V_{\mathrm{m}}$). Then

$$v_{\mathrm{f}} = \frac{V_{\mathrm{f}}}{V}, \quad v_{\mathrm{m}} = \frac{V_{\mathrm{m}}}{V} \tag{1.1.1}$$

with

$$v_{\mathrm{f}} + v_m = 1, \quad v_{\mathrm{m}} = 1 - v_{\mathrm{f}}$$

are the fibre and the matrix volume fractions. In a similar way the weight or mass fractions of fibres and matrices can be defined. The mass M of the composite is made from M_{f} and M_{m} ($M = M_{\mathrm{f}} + M_{\mathrm{m}}$) and

$$m_{\mathrm{f}} = \frac{M_{\mathrm{f}}}{M}, \quad m_{\mathrm{m}} = \frac{M_{\mathrm{m}}}{M} \tag{1.1.2}$$

with

$$m_{\mathrm{f}} + m_{\mathrm{m}} = 1, \quad m_{\mathrm{m}} = 1 - m_{\mathrm{f}}$$

are the mass fractions of fibres and matrices. With the relation between volume, mass and density $\rho = M/V$, we can link the mass and the volume fractions

$$
\begin{aligned}
\rho = \frac{M}{V} = \frac{M_\mathrm{f} + M_\mathrm{m}}{V} &= \frac{\rho_\mathrm{f} V_\mathrm{f} + \rho_\mathrm{m} V_\mathrm{m}}{V} \\
= \rho_\mathrm{f} v_\mathrm{f} + \rho_\mathrm{m} v_\mathrm{m} &= \rho_\mathrm{f} v_\mathrm{f} + \rho_\mathrm{m}(1 - v_\mathrm{f})
\end{aligned}
\tag{1.1.3}
$$

Starting from the total volume of the composite $V = V_\mathrm{f} + V_\mathrm{m}$ we obtain

$$
\frac{M}{\rho} = \frac{M_\mathrm{f}}{\rho_\mathrm{f}} + \frac{M_\mathrm{m}}{\rho_\mathrm{m}}
$$

and

$$
\rho = \frac{1}{\dfrac{m_\mathrm{f}}{\rho_\mathrm{f}} + \dfrac{m_\mathrm{m}}{\rho_\mathrm{m}}}
\tag{1.1.4}
$$

with

$$
m_\mathrm{f} = \frac{\rho_\mathrm{f}}{\rho} v_\mathrm{f}, \quad m_\mathrm{m} = \frac{\rho_\mathrm{m}}{\rho} v_\mathrm{m}
$$

The inverse relation determines

$$
v_\mathrm{f} = \frac{\rho}{\rho_\mathrm{f}} m_\mathrm{f}, \quad v_\mathrm{m} = \frac{\rho}{\rho_\mathrm{m}} m_\mathrm{m}
\tag{1.1.5}
$$

The density ρ is determined by Eqs. (1.1.3) or (1.1.4). The equations can be easily extended to multi-phase composites.

Mass fractions are easier to measure in material manufacturing, but volume fractions appear in the theoretical equations for effective moduli (Sect. 3.1). Therefore, it is helpful to have simple expressions for shifting from one fraction to the other.

The quality of a composite material decreases with an increase in porosity. The volume of porosity should be less than 5 % for a medium quality and less than 1 % for a high quality composite. If the density ρ_exp is measured experimentally and ρ_theor is calculated with (1.1.4), the volume fraction of porosity is given by

$$
v_\mathrm{por} = \frac{\rho_\mathrm{theor} - \rho_\mathrm{exp}}{\rho_\mathrm{theor}}
\tag{1.1.6}
$$

1.2 Significance and Objectives

Development and applications of composite materials and structural elements composed of composite materials have been very rapid in the last decades. The motivations for this development are the significant progress in material science and technology of the composite constituents, the requirements for high performance materials is not only in aircraft and aerospace structures, but also in the development of very powerful experimental equipments and numerical methods and the availability of efficient computers. With the development of composite materials a new material design is possible that allows an optimal material composition in connection with the structural design. A useful and correct application of composite materials requires a close interaction of different engineering disciplines such as structural design and analysis, material science, mechanics of materials, process engineering, etc. Summarizing the main topics of composite material research and technology are

- investigation of all characteristics of the constituent and the composite materials,
- material design and optimization for the given working conditions,
- development of analytical modelling and solution methods for determining material and structural behavior,
- development of experimental methods for material characteristics, stress and deformation states, failure,
- modelling and analysis of creep, damage and life prediction,
- development of new and efficient fabrication and recycling procedures among others.

Within the scope of this book are the first three items.

The most significant driving force in the composite research and application was weight saving in comparison to structures of conventional materials such as steel, alloys, etc. However, to have only material density, stiffness and strength in mind when thinking of composites is a very narrow view of the possibilities of such materials as fibre-reinforced plastics because they often may score over conventional materials as metals not only owing to their mechanical properties. Fibre reinforced plastics are extremely corrosion-resistant and have interesting electromagnetic properties. In consequence they are used for chemical plants and for structures which require non-magnetic materials. Further carbon fibre reinforced epoxy is used in medical applications because it is transparent to X-rays.

With applications out of aerospace or aircraft, cost competitiveness with conventional materials became important. More recently requirements such as quality assurance, reproducibility, predictability of the structure behavior over its life time, recycling, etc. became significant.

Applications of polymer matrix composites range from the aerospace industry to the industry of sports goods. The military aircraft industry has mainly led the field in the use of polymer composites when compared to commercial airlines which has used composites, because of safety concerns more restrictively and frequently limited to secondary structural elements. Automotive applications, sporting goods, medical devices and many other commercial applications are examples for the application of polymer matrix composites. Also applications in civil engineering are now on the way but it will take some time to achieve wide application of composites in civil engineering as there are a lot of prescribed conditions to guarantee the reliability of structures. But it is clear that over the last decades considerable advances have been made in the use of composite materials in construction and building industries and this trend will continue.

1.3 Modelling

Composite materials consist of two or more constituents and the modelling, analysis and design of structures composed of composites are different from conventional materials such as steel. There are three levels of modelling. At the micro-mechanical level the average properties of a single reinforced layer (a lamina or a ply) have to be determined from the individual properties of the constituents, the fibres and matrix. The average characteristics include the elastic moduli, the thermal and moisture expansion coefficients, etc. The micro-mechanics of a lamina does not consider the internal structure of the constituent elements, but recognizes the heterogeneity of the ply. The micro-mechanics is based on some simplifying approximations. These concern the fibre geometry and packing arrangement, so that the constituent characteristics together with the volume fractions of the constituents yield the average characteristics of the lamina. Note that the average properties are derived by considering the lamina to be homogeneous. In the frame of this textbook only the micro-mechanics of unidirectional reinforced laminates are considered (Sect. 3).

The calculated values of the average properties of a lamina provide the basis to predict the macrostructural properties. At the macro-mechanical level, only the averaged properties of a lamina are considered and the microstructure of the lamina is ignored. The properties along and perpendicular to the fibre direction, these are the principal directions of a lamina, are recognized and the so-called on-axis stress-strain relations for a unidirectional lamina can be developed. Loads may be applied not only on-axis but also off-axis and the relationships for stiffness and flexibility, for thermal and moisture expansion coefficients and the strength of an angle ply can be determined. Failure theories of a lamina are based on strength properties. This topic is called the macro-mechanics of a single layer or a lamina (Sect. 4.1).

A laminate is a stack of laminae. Each layer of fibre reinforcement can have various orientation and in principle each layer can be made of different materials. Knowing the macro-mechanics of a lamina, one develops the macro-mechanics of the laminate. Average stiffness, flexibility, strength, etc. can be determined for the whole laminate (Sect. 4.2). The structure and orientation of the laminae in prescribed sequences to a laminate lead to significant advantages of composite materials when compared to a conventional monolithic material. In general, the mechanical response of laminates is anisotropic. One very important group of laminated composites are sandwich composites. They consist of two thin faces (the skins or sheets) sandwiching a core (Fig. 1.3). The faces are made of high strength materials having good properties under tension such as metals or fibre reinforced laminates while the core is made of lightweight materials such as foam, resins with special fillers, called syntactic foam, having good properties under compression. Sandwich composites combine lightness and flexural stiffness. The macro-mechanics of sandwich composites is considered in Sect. 4.3.

When the micro- and macro-mechanical analysis for laminae and laminates are carried out, the global behavior of laminated composite materials is known. The last step is the modelling on the structure level and to analyze the global behavior of a structure made of composite material. By adapting the classical tools of structural

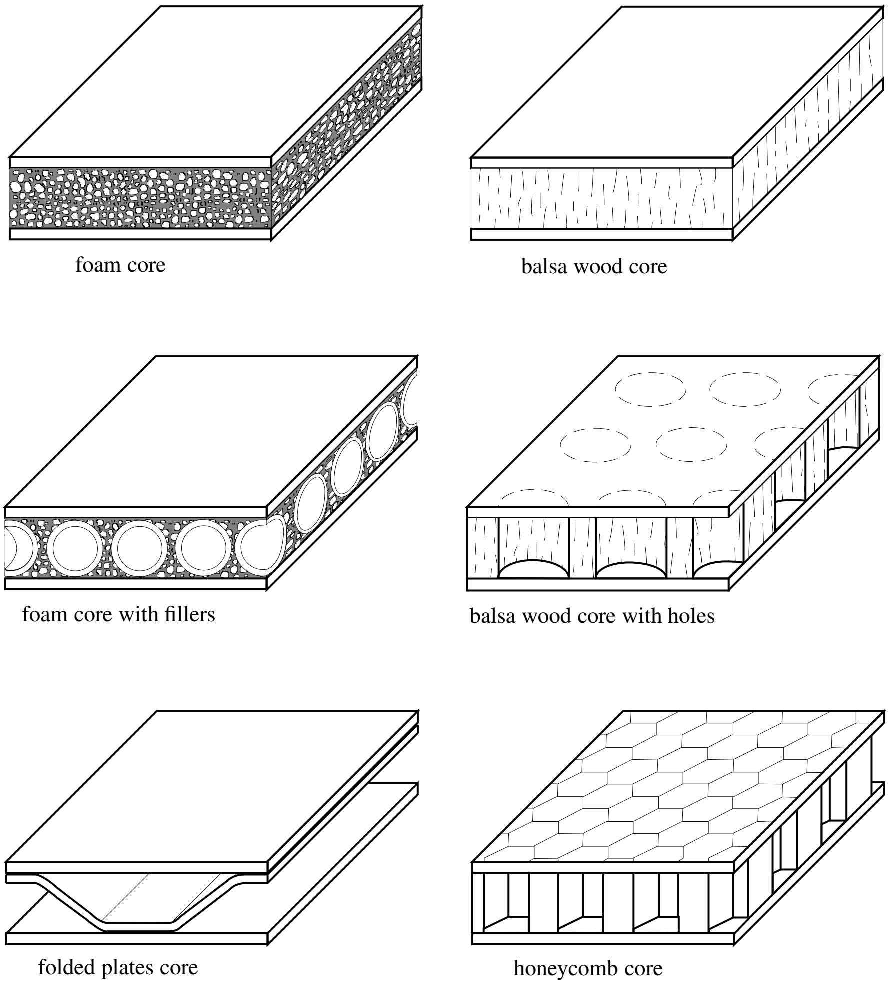

Fig. 1.3 Sandwich materials with solid and hollow cores

analysis on anisotropic elastic structure elements the analysis of simple structures as beams or plates may be achieved by analytical methods, but for more general boundary conditions and/or loading and for complex structures, numerical methods are used.

The composite structural elements in the restricted view of this textbook are laminated or sandwich composites. The motivation for sandwich composites are two-fold:

- If a beam is bent, the maximum stresses occur at the top and the bottom surface. So it makes sense using high strength materials only for the sheets and using low and lightweight materials in the middle.

- The resistance to bending of a rectangular cross-sectional beam is proportional to the cube of the thickness. Increasing the thickness by adding a core in the middle increases the resistance. The shear stresses have a maximum in the middle of a sandwich beam requiring the core to support the shear. This advantage of weight and bending stiffness makes sandwich composites more attractive for some applications than other composite or conventional materials.

The most commonly used face materials are aluminium alloys or fibre reinforced laminates and most commonly used core materials are balsa wood, foam and honeycombs (Fig. 1.3). In order to guarantee the advantages of sandwich composites, it is necessary to ensure that there is perfect bonding between the core and the sheets.

For laminated composites, assumptions are necessary to enable the mathematical modelling. These are an elastic behavior of fibres and matrices, a perfect bonding between fibres and matrices, a regular fibre arrangement in regular or repeating arrays, etc.

Summarizing the different length scales of mechanical modelling structure elements composed of fibre reinforced composites it must be noted that, independent of the different possibilities to formulate beam, plate or shell theories (Chaps. 7–9), three modelling levels must be considered:

- *The microscopic level*, where the average mechanical characteristics of a lamina have to be estimated from the known characteristics of the fibres and the matrix material taking into account the fibre volume fracture and the fibre packing arrangement. The micro-mechanical modelling leads to a correlation between constituent properties and average composite properties. In general, simple mixture rules are used in engineering applications (Chap. 3). If possible, the average material characteristics of a lamina should be verified experimentally. On the micro-mechanical level a lamina is considered as a quasi-homogeneous orthotropic material.
- The *macroscopic level*, where the effective (average) material characteristics of a laminate have to be estimated from the average characteristics of a set of laminae taking into account their stacking sequence. The macro-mechanical modelling leads to a correlation between the known average laminae properties and effective laminate properties. On the macro-mechanical level a laminate is considered generally as an equivalent single layer element with a quasi-homogeneous, anisotropic material behavior (Chap. 4).
- The *structural level*, where the mechanical response of structural members like beams, plates, shells etc. have to be analyzed taking into account possibilities to formulate structural theories of different order (Chap. 5).

In the recent years in the focus of the researchers is an additional level - the *nanoscale level*. There are two reasons for this new direction:

- composites reinforced by nanoparticles and
- nanosize structures.

Both directions are beyond this elementary textbook. Partly new concepts should be introduced since bulk effects are no more so important and the influence of surface effects is increasing. In this case the classical continuum mechanics approaches should be extended. More details are presented in Altenbach and Eremeyev (2015); Cleland (2003).

1.4 Material Characteristics of the Constituents

The optimal design and the analysis of structural elements requires a detailed knowledge of the material properties. They depend on the nature of the constituent materials but also on manufacturing.

For conventional structure materials such as metals or concrete, is available much research and construction experience over many decades, the codes for structures composed of conventional materials have been revised continuously and so design engineers pay less attention to material problems because there is complete documentation of the material characteristics.

It is quite an another situation for structures made of composites. The list of composite materials is numerous but available standards and specifications are very rare. The properties of each material used for both reinforcements and matrices of composites are very much diversified. The experiences of nearly all design engineers in civil or mechanical engineering with composite materials, are insufficient. So it should be borne in mind that structural design based on composite materials requires detailed knowledge about the material properties of the singular constituents of the composite for optimization of the material in the frame of structural applications and also detailed codes for modelling and analysis are necessary.

The following statements are concentrated on fibre reinforced composites with polymer resins. Material tests of the constituents of composites are in many cases a complicated task and so the material data in the literature are limited. In engineering applications the average data for a lamina are often tested to avoid this problem and in order to use correct material characteristics in structural analysis. But in the area of material design and selection, it is also important to know the properties of all constituents.

The main properties for the estimation of the material behavior are

- density ρ,
- Young's modulus[1] E,
- ultimate strength σ_{u} and
- thermal expansion coefficient α.

The material can be made in bulk form or in the form of fibres. To estimate properties of a material in the form of fibres, the fibre diameter d can be important.

Table F.1 gives the specific performances of selected material made in bulk form. Traditional materials, such as steel, aluminium alloys, or glass have comparable

[1] Thomas Young (*13 June 1773 Milverton, Somersetshire - †10 May 1829 London) - polymath and physician, notable scientific contributions to the fields of light, mechanics (elastic material parameter, surface tension), energy among others

specific moduli E/ρ but in contrast the specific ultimate stress σ_u/ρ of glass is significantly higher than that of steel and of aluminium alloys. Table F.2 presents the mechanical characteristics of selected materials made in the form of fibres. It should be borne in mind that the ultimate strength measured for materials made in bulk form is remarkably smaller than the theoretical strengths. This is attributed to defects or micro-cracks in the material. Making materials in the form of fibres with a very small diameter of several microns decreases the number of defects and the values of ultimate strength increases. Table F.3 gives material properties for some selected matrix materials and core materials of sandwich composites. Tables F.4 and F.5 demonstrate some properties of unidirectional fibre reinforced composite materials: E_L is the longitudinal modulus in fibre direction, E_T the transverse modulus, G_{LT} the in-plane shear modulus, ν_{LT} and ν_{TL} are the major and the minor Poisson's ratio[2], σ_{Lu}, σ_{Tu}, σ_{LTu} the ultimate stresses or strengths, α_L and α_T the longitudinal and the transverse thermal expansion coefficients.

Summarizing the reported mechanical properties, which are only a small selection, a large variety of fibres and matrices are available to design a composite material with high modulus and low density or other desired qualities. The impact of the costs of the composite material can be low for applications in the aerospace industry or high for applications such as in automotive industry. The intended performance of a composite material and the cost factors play an important role and structural design with composite materials has to be compared with the possibilities of conventional materials.

1.5 Advantages and Limitations

The main advantage of polymer matrix composites in comparison with conventional materials, such as metals, is their low density. Therefore two parameters are commonly used to demonstrate the mechanical advantages of composites:

1. The specific modulus E/ρ is the Young's modulus per unit mass or the ratio between Young's modulus and density.
2. The specific strength σ_u/ρ is the tensile strength per unit mass or the ratio between strength and density

The benefit of the low density becomes apparent when the specific modulus and the specific strength are considered. The two ratios are high and the higher the specific parameters the more weight reduction of structural elements is possible in relation to special loading conditions. Therefore, even if the stiffness and/or the strength performance of a composite material is comparable to that of a conventional alloy, the advantages of high specific stiffness and/or specific strength make composites

[2] Siméon Denis Poisson (*21 June 1781 Pithiviers - †25 April 1840 Paris) - mathematician, geometer, and physicist, with contributions to mechanics, after Poisson an elastic material parameter is named

more attractive. Composite materials are also known to perform better under cyclic loads than metallic materials because of their fatigue resistance.

The reduction of mass yields reduced space requirements and lower material and energy costs. The mass reduction is especially important in moving structures. Beware that in some textbooks the specific values are defined as $E/\rho g$ and $\sigma_u/\rho g$, where g is the acceleration due to the gravity. Furthermore it should be noted that a single performance indicator is insufficient for the material estimation and that comparison of the specific modulus and the specific strength of unidirectional composites to metals gives a false impression. Though the use of fibres leads to large gains in the properties in fibre direction, the properties in the two perpendicular directions are greatly reduced. Additionally, the strength and stiffness properties of fibre-reinforced materials are poor in another important aspect. Their strength depends critically upon the strength of the fibre, matrix interface and the strength of the matrix material, if shear stresses are being applied. This leads to poor shear properties and this lack of good shear properties is as serious as the lack of good transverse properties. For complex structure loadings, unidirectional composite structural elements are not acceptable and so-called angle-ply composite elements are necessary, i.e. the structural components made of fibre-reinforced composites are usually laminated by using a number of layers. This number of fibre-reinforced layers can vary from just a few to several hundred. While generally the majority of the layers in the laminate have their fibres in direction of the main loadings, the other layers have their fibres oriented specifically to counter the poor transverse and shear properties.

Additional advantages in the material performances of composites are low thermal expansion, high material damping, generally high corrosion resistance and electrical insulation. Composite materials can be reinforced in any direction and the structural elements can be optimized by material design or material tailoring.

There are also limitations and drawbacks in the use of composite materials:

1. The mechanical characterization of composite materials is much more complex than that of monolithic conventional material such as metal. Usually composite material response is anisotropic. Therefore, the material testing is more complicated, cost and time consuming. The evaluation and testing of some composite material properties, such as compression or shearing strengths, are still in discussion.
2. The complexity of material and structural response makes structural modelling and analysis experimentally and computationally more expensive and complicated in comparison to metals or other conventional structural materials. There is also limited experience in the design, calculating and joining composite structural elements.

Additional disadvantages are the high cost of fabrication, but improvements in production technology will lower the cost more and more, further the complicated repair technology of composite structures, a lot of recycling problems, etc.

Summarizing, it can be said that the application of composite materials in structure design beyond the military and commercial aircraft and aerospace industry and some special fields of automotive, sporting goods and medical devices is still in the early stages. But the advancing of technology and experience yields an increasing use of composite structure elements in civil and mechanical engineering and provides the stimulus to include composite processing, modelling, design and analysis in engineering education.

1.6 Problems

1. What is a composite and how are composites classified?
2. What are the constituents of composites?
3. What are the fibre and the matrix factors which contribute to the mechanical performance of composites?
4. What are polymer matrix, metal matrix and ceramic matrix composites, what are their main applications?
5. Define isotropic, anisotropic, homogeneous, nonhomogeneous.
6. Define lamina, laminate, sandwich. What is micro-mechanical and macro-mechanical modelling and analysis?
7. Compare the specific modulus, specific strength and coefficient of thermal expansion of glas fibre, epoxy resin and steel.

Exercise 1.1. A typical CFK plate (uni-directional reinforced laminae composed of carbon fibres and epoxy matrix) has the size 300 mm × 200 mm × 0.5 mm (length l × bright b × thickness d). Please show that the fibre volume fraction and the fibre mass fraction are not the same. The estimate should be based on the following data:

- density of the carbon fibres $\rho_{\mathrm{f}} = 1,8\ \mathrm{g/cm^3}$,
- diameter of the fibres is $d_{\mathrm{f}} = 6\mu\mathrm{m}$
- fibre volume fraction $v_{\mathrm{f}} = 0.8$ and
- density of the epoxy $\rho_{\mathrm{m}} = 1,1\ \mathrm{g/cm^3}$.

Solution 1.1. Let us assume that the fibres are parallel to the longer plate side. In this case the volume of one fibre V_{1f} is

$$V_{1f} = \pi\frac{d_{\mathrm{f}}^2}{4}l = \pi\frac{(6\ \mu\mathrm{m})^2}{4}\cdot 300\ \mathrm{mm} = 8,48\cdot 10^{-3}\mathrm{mm}^3$$

The volume of the plate is

$$V_{\mathrm{plate}} = 300\cdot 200\cdot 0,5\ \mathrm{mm}^3 = 3\cdot 10^4\mathrm{mm}^3$$

With the fibre volume fraction $v_{\mathrm{f}} = 0.8$ we can calculate the matrix volume fraction v_{m}

$$v_{\mathrm{m}} = 1 - v_{\mathrm{f}} = 1 - 0.8 = 0.2$$

The volume of all fibres is

$$V_{\mathrm{f}} = v_{\mathrm{f}}V_{\mathrm{plate}} = 0,8\cdot 3\cdot 10^4\mathrm{mm}^3 = 2,4\cdot 10^4\mathrm{mm}^3$$

which is equivalent to the following number of fibres n_{f}

$$n_{\mathrm{f}} = \frac{V_{\mathrm{f}}}{V_{1\mathrm{f}}} = \frac{2,4 \cdot 10^4 \mathrm{mm}^3}{8,48 \cdot 10^{-3} \mathrm{mm}^3} = 0,28 \cdot 10^7$$

The fibre mass fraction can be computed as it follows

$$m_{\mathrm{f}} = \frac{M_{\mathrm{f}}}{M} = \frac{M_{\mathrm{f}}}{M_{\mathrm{f}} + M_{\mathrm{m}}} = \frac{\rho_{\mathrm{f}} V_{\mathrm{f}}}{\rho_{\mathrm{f}} V_{\mathrm{f}} + \rho_{\mathrm{m}} V_{\mathrm{m}}}$$

With

$$V_{\mathrm{m}} = V_{\mathrm{plate}} - V_{\mathrm{f}} = 3 \cdot 10^4 \mathrm{mm}^3 - 2,4 \cdot 10^4 \mathrm{mm}^3 = 0,6 \cdot 10^4 \mathrm{mm}^3$$

fibre mass fraction is

$$m_{\mathrm{f}} = \frac{1,8 \mathrm{g/cm}^3 \cdot 2,4 \cdot 10^4 \mathrm{mm}^3}{1,8 \mathrm{g/cm}^3 \cdot 2,4 \cdot 10^4 \mathrm{mm}^3 + 1,1 \mathrm{g/cm}^3 \cdot 0,6 \cdot 10^4 \mathrm{mm}^3} = 0,867$$

This value is slightly grater than the assumed fibre volume fraction.

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Chapter 2
Linear Anisotropic Materials

The classical theory of linear elastic deformable solids is based on the following restrictions to simplify the modelling and analysis:

- The body is an ideal linear elastic body.
- All strains are small.
- The material of the constituent phases is homogeneous and isotropic.

These assumptions of classical theory of elasticity guarantee a satisfying quality of modelling and analysis of structure elements made of conventional monolithic materials. Structural analysis of elements composed of composite materials is based on the theory of anisotropic elasticity, the elastic properties of composites depend usually on the direction and the deformable solid is anisotropic. In addition, now the composite material is not homogeneous at all. It must be assumed that the material is piecewise homogeneous or quasi-homogeneous.

The governing equations of elastic bodies are nearly the same for isotropic and anisotropic material response. There are equilibrium equations, which describe the static or dynamic equilibrium of forces acting on an elastic body. The kinematic equations describe the strain-displacement relations and the compatibility equations guarantee a unique solution to the equations relating strains and displacements. All these equations are independent of the elastic properties of the material. Only the material relations (so-called constitutive equations), which describe the relations between stresses and strains are very different for an isotropic and an anisotropic body. This difference in formulating constitutive equations has a great influence on the model equations in the frame of the isotropic and the anisotropic theory of elasticity. Note that in many cases the material behavior of the constituents can assumed to be homogeneous and isotropic.

The governing equations, as defined above, including so-called initial-boundary conditions for forces/stresses and/or displacements, yield the basic model equations for linear elastic solids such as differential equations or variational and energy formulations, respectively. All equations for structural elements which are given in this textbook, are founded on these general equations for the theory of elasticity of linear elastic anisotropic solids.

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0_2

The objective of this chapter is to review the generalized Hooke's law[1], the constitutive equations for anisotropic elastic bodies, and to introduce general relations for stiffness and strains including transformation rules and symmetry relations. The constitution of a unidirectional composite material and simplified approaches for so-called effective moduli result in an engineering formulation of constitutive equations for fibre reinforced composites and will be considered in Chap. 3.

The theory of anisotropic elasticity presented in Sect. 2.1 begins with the most general form of the linear constitutive equations, and passes from all specific cases of elastic symmetries to the classical Hooke's law for an isotropic body. The only assumptions are

- all elastic properties are the same in tension and compression,
- the stress tensor is symmetric,
- an elastic potential exists and is an invariant with respect to linear orthogonal coordinate transformation.

In addition to the general three-dimensional stress-strain relationships, the plane stress and plane strain cases are derived and considered for an anisotropic body and for all the derived specific cases of elastic symmetries. The type of anisotropy considered in Sects. 2.1.1–2.1.5 can be called as rectilinear anisotropy, i.e. the homogeneous anisotropic body is characterized by the equivalence of parallel directions passing through different points of the body. Another kind of anisotropy, which can be interesting to some applications, e.g. to modelling circular plates or cylindrical tubes, is considered in a comprehensive formulation in Sect. 2.1.6. If one chooses a system of curvilinear coordinates in such a manner that the coordinate directions coincide with equivalent directions of the elastic properties at different points of the body, the elastic behavior is called curvilinear.

The chapter ends with the derivation of the fundamental equations of anisotropic elasticity and the formulation of variational solution methods. In Sect. 2.2 the differential equations for boundary and initial boundary problems are considered. The classical and generalized variational principles are formulated and approximate analytical solution methods based on variational principles are discussed.

2.1 Generalized Hooke's Law

The phenomenological modelling neglects the real on the microscopic scale discontinuous structure of the material. On the macroscopic or phenomenological scale the material models are assumed to be continuous and in general homogeneous. In the case of fibre reinforced composites, the heterogeneity of the bulk material is a consequence of the two constituents, the fibres and the matrix, but generally there exists a representative volume element of the material on a characteristic scale at which the

[1] Robert Hooke (*28$^{\text{greg.}}$/18$^{\text{jul.}}$ July 1635 Freshwater, Isle of Wight - †3 March 1703 London) - natural philosopher, architect and polymath, first constitutive law for elasticity published as an anagram *ceiiinosssttuu* which is in Latain *ut tensio sic vis* and means *as the extension, so the force*

properties of the material can be averaged to a good approximation. The composite material can be considered as macroscopically homogeneous and the problem of designing structural elements made from of composite materials can be solved in an analogous manner as for conventional materials with the help of the average material properties or the so-called effective moduli. Chapter 3 explains the calculation of effective moduli in detail.

Unlike metals or polymeric materials without reinforcements or reinforced by stochastically distributed and orientated particles or short fibres, the material behavior of an off-axis forced unidirectional lamina is anisotropic. In comparison to conventional isotropic materials, the experimental identification of the material parameters is much more complicated in the case of anisotropic materials. But anisotropic material behavior also has the advantage of material tailoring to suit the main loading cases.

2.1.1 Stresses, Strains, Stiffness, and Compliances

In preparation for the formulation of the generalized Hooke's law, a one-dimensional problem will be considered. The deformations of an elastic body can be characterized by displacements or by strains:

- Dilatational or extensional strains ε: The body changes only its volume but not its shape.
- Shear strains γ: The body changes only its shape but not its volume.

Figure 2.1 demonstrates extensional and shear strains for a simple prismatic body loaded by forces F and T, normal and tangential to the cross-section, respectively. Assuming a uniform distribution of the forces F and T on the cross-section, the elementary one-dimensional definitions for stresses and strains are given by (2.1.1)

$$\begin{array}{llll}
\sigma = \dfrac{F}{A_0} & & \text{normal stress} & \sigma, \\
\varepsilon = \dfrac{l-l_0}{l_0} & = \dfrac{\triangle l}{l_0} & \text{extensional strain} & \varepsilon, \\
\tau = \dfrac{T}{A_0} & & \text{shear stress} & \tau, \\
\gamma \approx \tan\gamma & = \dfrac{\triangle u}{l_0} & \text{shear strain} & \gamma
\end{array} \tag{2.1.1}$$

The last one definition is restricted by small strain assumption. This assumption can be accepted for many composite material applications and will be used for both types of strains.

The material or constitutive equations couple stresses and strains. In linear elasticity the one-to-one transformation of stresses and strains yield Hooke's law (2.1.2)

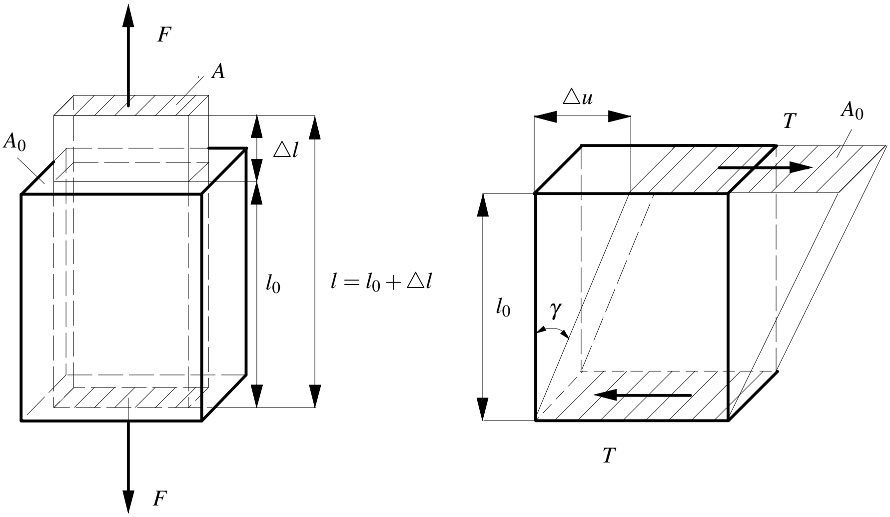


Fig. 2.1 Extensional strain ε and shear strain γ of a body with the length l_0 and the cross-section area A_0

$$\begin{aligned} \sigma &= E\varepsilon, \ E = \frac{\sigma}{\varepsilon}, \qquad E \quad \text{is the elasticity or Young's modulus}, \\ \tau &= G\gamma, \ G = \frac{\tau}{\gamma}, \qquad G \quad \text{is the shear modulus} \end{aligned} \tag{2.1.2}$$

For a homogeneous material E and G are parameters with respect to the coordinates.

For the extensionally strained prismatic body (Fig. 2.1) the phenomenon of contraction in a direction normal to the direction of the tensile loading has to be considered. The ratio of the contraction, expressed by the transverse strain ε_t, to the elongation in the loaded direction, expressed by ε, is called Poisson's ratio ν

$$\varepsilon_t = -\nu\varepsilon, \quad \nu = -\frac{\varepsilon_t}{\varepsilon} \tag{2.1.3}$$

For an isotropic bar with an extensional strain $\varepsilon > 0$ it follows that $\varepsilon_t < 0$[2].

Hooke's law can be written in the inverse form

$$\varepsilon = E^{-1}\sigma = S\sigma \tag{2.1.4}$$

$S = E^{-1}$ is the inverse modulus of elasticity or the flexibility/compliance modulus. For homogeneous material, S is an elastic parameter.

Consider a tensile loaded prismatic bar composed of different materials (Fig. 2.2). Since $\sigma = F/A$ and $\sigma = E\varepsilon$ then $\sigma A = F = EA\varepsilon$ and $\varepsilon = (EA)^{-1}F$. EA is the tensile stiffness and $(EA)^{-1}$ the tensile flexibility or compliance. The dif-

[2] The class of auxetic materials that have a negative Poisson's ratio that means when stretched, they become thicker perpendicular to the applied force, is not in the focus of the present book.

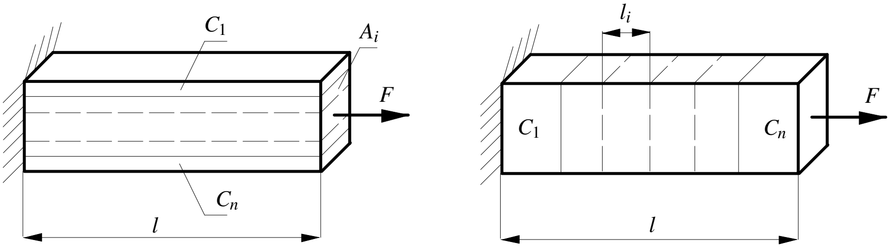


Fig. 2.2 Tensile bar with stiffness $C_i = E_i A_i$ arranged in parallel and in series

ferent materials of the prismatic bar in Fig. 2.2 can be arranged in parallel or in series. In the first case we have

$$F = \sum_{i=1}^{n} F_i, \quad A = \sum_{i=1}^{n} A_i, \quad \varepsilon = \varepsilon_i \tag{2.1.5}$$

F_i are the loading forces on A_i and the strains ε_i are equal for the total cross-section. With

$$F = EA\varepsilon, \quad F_i = E_i A_i \varepsilon, \qquad \sum_{i=1}^{n} F_i = F = \sum_{i=1}^{n} E_i A_i \varepsilon \tag{2.1.6}$$

follow the coupling equations for the stiffness $E_i A_i$ for a parallel arrangement

$$EA = \sum_{i=1}^{n} E_i A_i, \qquad (EA)^{-1} = \frac{1}{\sum_{i=1}^{n} E_i A_i} \tag{2.1.7}$$

This equal strain treatment is often described as a Voigt[3] model which is the upper-bound stiffness.

In the other case, we have

$$\triangle l = \sum_{i=1}^{n} \triangle l_i$$

and $F = F_i$, the elongation $\triangle l$ of the bar is obtained by addition of the $\triangle l_i$ of the different parts of the bar with the lengths l_i and the tensile force is equal for all cross-sectional areas. With

$$\triangle l = l\varepsilon = l(EA)^{-1}F, \quad \triangle l_i = l_i \varepsilon_i = l_i (E_i A_i)^{-1} F$$

and

$$\sum_{i=1}^{n} \triangle l_i = \left[\sum_{i=1}^{n} l_i (E_i A_i)^{-1} \right] F \tag{2.1.8}$$

[3] Woldemar Voigt (*2 September 1850 Leipzig - †13 December 1919 Göttingen) - physicist, worked on crystal physics, thermodynamics and electro-optics, the word tensor in its current meaning was introduced by him in 1898, Voigt notation

follow the coupling equations for the stiffness E_iA_i arranged in series

$$EA = \frac{l}{\sum_{i=1}^{n} l_i(E_iA_i)^{-1}}, \quad (EA)^{-1} = \frac{\sum_{i=1}^{n} l_i(E_iA_i)^{-1}}{l} \tag{2.1.9}$$

This equal stress treatment is described generally as a Reuss[4] model which is the lower bound stiffness. The coupling equations illustrate a first clear insight into a simple calculation of effective stiffness and compliance parameters for two composite structures.

The three-dimensional state of stress or strain in a continuous solid is completely determined by knowing the stress or strain tensor. It is common practice to represent the tensor components acting on the faces of an infinitesimal cube with sides parallel to the reference axes (Fig. 2.3). The sign convention is defined in Fig. 2.3. Positive stresses or strains act on the faces of the cube with an outward vector in the positive direction of the axis of the reference system and vice versa. Using the tensorial notation for the stress tensor σ_{ij} and the strain tensor ε_{ij} for the stresses and the strains we have normal stresses or extensional strains respectively for $i = j$ and shear stresses or shear strains for $i \neq j$. ε_{ij} with $i \neq j$ are the tensor shear coordinates and $2\varepsilon_{ij} = \gamma_{ij}, i \neq j$ the engineering shear strains. The first subscript of σ_{ij} and ε_{ij} indicates the plane $x_i =$ const on which the load is acting and the second subscript denotes the direction of the loading. Care must be taken in distinguishing in literature the strain tensor ε_{ij} from the tensor e_{ij} which is the tensor of the relative displacements, $e_{ij} = \partial u_i/\partial x_j$. An application of shear stresses σ_{ij} and σ_{ji} produces in the ij-plane of the infinitesimal cube (Fig. 2.3) angular rotations of the i- and j-directions by e_{ij} and e_{ji}. These relative displacements represent a combination of strain (distorsion) and rigid body rotation with the limiting cases $e_{ij} = e_{ji}$,

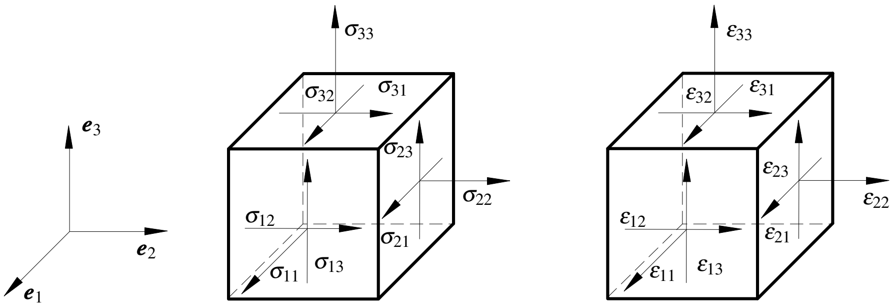


Fig. 2.3 Stress and strain components on the positive faces of an infinitesimal cube in a set of axis $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$

[4] András (Endre) Reuss (*1 July 1900 Budapest - †10 May 1968 Budapest) - mechanical engineer, contributions to the theory of plasticity

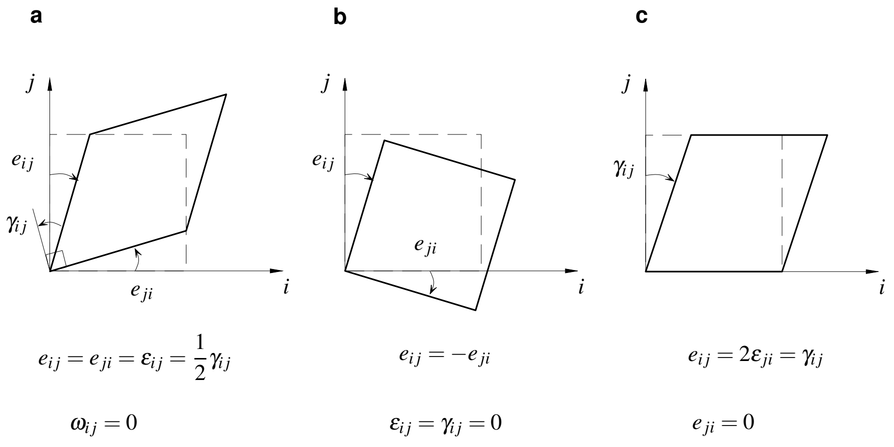


Fig. 2.4 Examples of distorsions and rigid body rotation. **a** Pure shear, **b** pure rotation, **c** simple shear

.

i.e. no rotation, and $e_{ij} = -e_{ji}$, i.e. no distorsion (Fig. 2.4). From the figure follows that simple shear is the sum of pure shear and rigid rotation. e_{ij} is positive when it involves rotating the positive j-direction towards the positive i-direction and vice versa. Writing the tensor e_{ij} as the sum of symmetric and antisymmetric tensors

$$e_{ij} = \frac{1}{2}(e_{ij} + e_{ji}) + \frac{1}{2}(e_{ij} - e_{ji}) = \varepsilon_{ij} + \omega_{ij} \tag{2.1.10}$$

where ε_{ij} is the symmetric strain tensor and ω_{ij} is the antisymmetric rotation tensor. For normal strains, i.e. $i = j$, there is $e_{ij} = \varepsilon_{ij}$, however for $i \neq j$ we have $\gamma_{ij} = 2\varepsilon_{ij} = e_{ij} + e_{ji}$ with the engineering shear strains γ_{ij} and the tensorial shear strains ε_{ij}. Careful note should be taken of the factor of two related engineering and tensorial shear strains, γ_{ij} is often more convenient for practical use but tensor operations such as rotations of the axis, Sect. 2.1.2, must be carried out using the tensor notation ε_{ij}.

The stress and the strain tensors are symmetric tensors of rank two. They can be represented by the matrices

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{12} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{13} & \varepsilon_{23} & \varepsilon_{33} \end{bmatrix} \tag{2.1.11}$$

The symmetry of the tensors (2.1.11) reduces the number of unknown components for defining these tensors to six components. For this reason, an engineering matrix notation can be used by replacing the matrix table with nine values by a column matrix or a vector with six components. The column matrices (stress and strain vector) are written in Eqs. (2.1.12) in a transposed form

$$
\begin{aligned}
&[\sigma_{11}\ \sigma_{22}\ \sigma_{33}\ \sigma_{23} \equiv \tau_{23}\ \ \sigma_{13} \equiv \tau_{13}\ \ \sigma_{12} \equiv \tau_{12}]^{\mathrm{T}},\\
&[\varepsilon_{11}\ \ \varepsilon_{22}\ \ \varepsilon_{33}\ \ 2\varepsilon_{23} \equiv \gamma_{23}\ 2\varepsilon_{13} \equiv \gamma_{13}\ 2\varepsilon_{12} \equiv \gamma_{12}]^{\mathrm{T}}
\end{aligned}
\tag{2.1.12}
$$

The stress and strain states are related by a material law which is deduced from experimental observations. For a linear elastic anisotropic material, the generalized Hooke's law relates the stress and the strain tensor

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \tag{2.1.13}$$

C_{ijkl} are the material coefficients and define the fourth rank elasticity tensor which in general case contains 81 coordinates. Due to the assumed symmetry of $\sigma_{ij} = \sigma_{ji}$ and $\varepsilon_{ij} = \varepsilon_{ji}$ the symmetry relations follow for the material tensor

$$C_{ijkl} = C_{jikl}, \quad C_{ijkl} = C_{ijlk} \tag{2.1.14}$$

and reduce the number of coordinates to 36. Introducing a contracted single-subscript notation for the stress and strain components and a double-subscript notation for the elastic parameters, the generalized relation for stresses and strains can be written in vector-matrix form

$$[\sigma_i] = [C_{ij}][\varepsilon_j]; \quad i,j = 1,2,\ldots,6; \quad C_{ij} \neq C_{ji}; \quad i \neq j \tag{2.1.15}$$

At this stage we have 36 independent material coefficients, but a further reduction of the number of independent values is possible because we have assumed the existence of an elastic potential function.

The elastic strain energy is defined as the energy expended by the action of external forces in deforming an elastic body: essentially all the work done during elastic deformations is stored as elastic energy. The strain energy per unit volume, i.e. the strain energy density function, is defined as follows

$$W = \frac{1}{2}\sigma_{ij}\varepsilon_{ij} \tag{2.1.16}$$

or in a contracted notation

$$W(\varepsilon_i) = \frac{1}{2}\sigma_i\varepsilon_i = \frac{1}{2}C_{ij}\varepsilon_j\varepsilon_i \tag{2.1.17}$$

With

$$\frac{\partial W}{\partial \varepsilon_i} = \sigma_i, \quad \frac{\partial^2 W}{\partial \varepsilon_i \partial \varepsilon_j} = C_{ij}, \quad \frac{\partial^2 W}{\partial \varepsilon_j \partial \varepsilon_i} = C_{ji}$$

and

$$\frac{\partial^2 W}{\partial \varepsilon_i \partial \varepsilon_j} = \frac{\partial^2 W}{\partial \varepsilon_j \partial \varepsilon_i}$$

follow the symmetry relations

$$C_{ij} = C_{ji}; \quad i,j = 1,2,\ldots,6 \tag{2.1.18}$$

and the number of the independent material coefficients is reduced to 21. The generalized relations for stresses and strains of an anisotropic elastic body written again in a contracted vector-matrix form have a symmetric matrix for C_{ij}

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \tag{2.1.19}$$

The transformation rules for the contraction of the subscripts of $\sigma_{ij}, \varepsilon_{ij}$ and C_{ijkl} of (2.1.13) are given in Tables 2.1 and 2.2.

The elasticity equation (2.1.19) can be written in the inverse form as follows

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{2.1.20}$$

with

$$[C_{ij}][S_{jk}] = [\delta_{ik}] = \begin{cases} 1 & i = k, \\ 0 & i \neq k, \end{cases} \qquad i,j,k = 1,\ldots,6$$

In a condensed symbolic or subscript form, Eqs. (2.1.19) and (2.1.20) are (summation on double subscripts)

$$\begin{aligned} &\sigma_i = C_{ij}\varepsilon_j, \qquad \varepsilon_i = S_{ij}\sigma_j; \qquad i,j = 1,\ldots,6 \\ &\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}, \qquad \boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma} \end{aligned} \tag{2.1.21}$$

$\boldsymbol{C} \equiv [C_{ij}]$ is the stiffness matrix and $\boldsymbol{S} \equiv [S_{ij}]$ the compliance or flexibility matrix. C_{ij} and S_{ij} are only for homogeneous anisotropic materials constant material parameters

Table 2.1 Transformation of the tensor coordinates σ_{ij} and ε_{ij} to the vector coordinates σ_p and ε_p

σ_{ij}	σ_p	ε_{ij}	ε_p
σ_{11}	σ_1	ε_{11}	ε_1
σ_{22}	σ_2	ε_{22}	ε_2
σ_{33}	σ_3	ε_{33}	ε_3
$\sigma_{23} = \tau_{23}$	σ_4	$2\varepsilon_{23} = \gamma_{23}$	ε_4
$\sigma_{31} = \tau_{31}$	σ_5	$2\varepsilon_{31} = \gamma_{31}$	ε_5
$\sigma_{12} = \tau_{12}$	σ_6	$2\varepsilon_{12} = \gamma_{12}$	ε_6

Table 2.2 Transformation of the tensor coordinates C_{ijkl} to the matrix coordinates C_{pq}

C_{ijkl}	C_{pq}
ij : 11, 22, 33	p : 1, 2, 3
23, 31, 12	4, 5, 6
kl : 11, 22, 33	q : 1, 2, 3
23, 31, 12	4, 5, 6

with respect to the coordinates. Their values depend on the reference coordinate system. A change of the reference system yields a change of the constant values.

Summarizing the stiffness and the compliance relations, it can be seen that for a linear elastic anisotropic material 21 material parameters have to be measured experimentally in the general case. But in nearly all engineering applications there are material symmetries and the number of material parameters can be reduced. Section 2.1.2 describes some transformation rules for $\boldsymbol{C}$ and $\boldsymbol{S}$ following from the change of reference system and the symmetric properties of anisotropic materials discussed in Sect. 2.1.3. Furthermore the way that the material parameters C_{ij} and S_{ij} are related to the known engineering elastic parameters E_i, G_{ij} and ν_{ij} is considered.

2.1.2 Transformation Rules

If we have a reference system which is characterized by the orthonormal basic unit vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ and another reference system with the vector basis $\boldsymbol{e}'_1, \boldsymbol{e}'_2, \boldsymbol{e}'_3$. Both systems are related by a rotation of the coordinate axis (Fig. 2.5), the transformation rules are

$$\begin{array}{ll} \boldsymbol{e}'_i = R_{ij}\boldsymbol{e}_j, \quad \boldsymbol{e}_i = R_{ji}\boldsymbol{e}'_j, & \\ R_{ij} \equiv \cos(\boldsymbol{e}'_i, \boldsymbol{e}_j), \quad R_{ji} \equiv \cos(\boldsymbol{e}_i, \boldsymbol{e}'_j), & i, j = 1,2,3 \end{array} \tag{2.1.22}$$

These relationships describe a general linear orthogonal coordinate transformation and can be expressed in vector-matrix notation

$$\boldsymbol{e}' = \boldsymbol{R}\boldsymbol{e}, \quad \boldsymbol{e} = \boldsymbol{R}^{-1}\boldsymbol{e}' = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{e}' \tag{2.1.23}$$

$\boldsymbol{R}$ is the transformation or rotation matrix. In the case of an orthogonal set of axes such as given in Fig. 2.5 the matrix $\boldsymbol{R}$ is symmetric and unitary. This means the determinant of this matrix is unity (Det $\boldsymbol{R} = |R_{ij}| = 1$ and the inverse matrix $\boldsymbol{R}^{-1}$ is identical to the transposed matrix $(\boldsymbol{R}^{-1} = \boldsymbol{R}^{\mathrm{T}})$. In the special case of a rotation ϕ

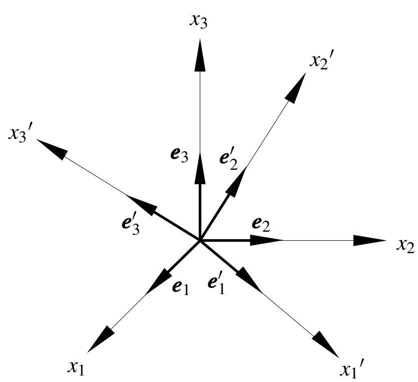


Fig. 2.5 Rotation of a reference system with the basic vectors $\boldsymbol{e}_i$ into a system with the basic vectors $\boldsymbol{e}'_i$

about the direction $\boldsymbol{e}_3$, the rotation matrix $\boldsymbol{R}$ and the inverse matrix $\boldsymbol{R}^{-1}$ are

$$\left[\overset{3}{R}_{ij}\right]=\begin{bmatrix} c & s & 0\\ -s & c & 0\\ 0 & 0 & 1\end{bmatrix},\quad \left[\overset{3}{R}_{ij}\right]^{-1}=\left[\overset{3}{R}_{ij}\right]^{\mathrm{T}}=\begin{bmatrix} c & -s & 0\\ s & c & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{2.1.24}$$

and the transformation rules are

$$\begin{bmatrix}\boldsymbol{e}'_1\\ \boldsymbol{e}'_2\\ \boldsymbol{e}'_3\end{bmatrix}=\begin{bmatrix} c & s & 0\\ -s & c & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{bmatrix},\begin{bmatrix}\boldsymbol{e}_1\\ \boldsymbol{e}_2\\ \boldsymbol{e}_3\end{bmatrix}=\begin{bmatrix} c & -s & 0\\ s & c & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\boldsymbol{e}'_1\\ \boldsymbol{e}'_2\\ \boldsymbol{e}'_3\end{bmatrix}$$

with $c=\cos\phi, s=\sin\phi$. For rotations ψ or θ about the directions $\boldsymbol{e}_2$ or $\boldsymbol{e}_1$ the rotations matrices $[\overset{2}{R}_{ij}]$ and $[\overset{1}{R}_{ij}]$ are

$$\left[\overset{2}{R}_{ij}\right]=\begin{bmatrix} c & 0 & -s\\ 0 & 1 & 0\\ s & 0 & c\end{bmatrix},\quad \left[\overset{2}{R}_{ij}\right]^{-1}=\left[\overset{2}{R}_{ij}\right]^{\mathrm{T}}=\begin{bmatrix} c & 0 & s\\ 0 & 1 & 0\\ -s & 0 & c\end{bmatrix},$$

$$\left[\overset{1}{R}_{ij}\right]=\begin{bmatrix} 1 & 0 & 0\\ 0 & c & s\\ 0 & -s & c\end{bmatrix},\quad \left[\overset{1}{R}_{ij}\right]^{-1}=\left[\overset{1}{R}_{ij}\right]^{\mathrm{T}}=\begin{bmatrix} 1 & 0 & 0\\ 0 & c & -s\\ 0 & s & c\end{bmatrix}$$

with $c=\cos\psi$ or $\cos\theta$ and $s=\sin\psi$ or $\sin\theta$ for rotations about $\boldsymbol{e}_2$ or $\boldsymbol{e}_1$, respectively.

The transformation rule (2.1.22) can be interpreted as a rule for vectors or first-rank tensors. The generalization to second-rank tensors yields e.g. for the stress tensor

$$\sigma'_{ij}=R_{ik}R_{jl}\sigma_{kl},\quad \sigma_{ij}=R_{ki}R_{lj}\sigma'_{kl} \tag{2.1.25}$$

For the following reflections the transformation rules for the contracted notation are necessary. The nine tensor coordinates σ_{ij} are shifted to six vector coordinates σ_p. The transformations

$$\sigma'_p=T^{\sigma}_{pq}\sigma_{\mathrm{q}},\quad \sigma_p=\left(T^{\sigma}_{pq}\right)^{-1}\sigma'_{\mathrm{q}},\quad p,q=1,\ldots,6 \tag{2.1.26}$$

are not tensor transformation rules. The transformation matrices T^{σ}_{pq} and $(T^{\sigma}_{pq})^{-1}$ follow by comparison of Eqs. (2.1.25) and (2.1.26). In the same manner we can find the transformation rules for the strains

$$\varepsilon'_p=T^{\varepsilon}_{pq}\varepsilon_{\mathrm{q}},\quad \varepsilon_p=\left(T^{\varepsilon}_{pq}\right)^{-1}\varepsilon'_{\mathrm{q}},\quad p,q=1,\ldots,6 \tag{2.1.27}$$

The elements of the transformation matrices $[T^{\sigma}_{pq}]$ and $[T^{\varepsilon}_{pq}]$ are defined in App. B. Summarizing, the transformation rules for stresses and strains in a condensed vector-matrix notation as follows

$$\boldsymbol{\sigma}'=\boldsymbol{T}^{\sigma}\boldsymbol{\sigma},\quad \boldsymbol{\varepsilon}'=\boldsymbol{T}^{\varepsilon}\boldsymbol{\varepsilon},\quad \boldsymbol{\sigma}=(\boldsymbol{T}^{\sigma})^{-1}\boldsymbol{\sigma}',\quad \boldsymbol{\varepsilon}=(\boldsymbol{T}^{\varepsilon})^{-1}\boldsymbol{\varepsilon}' \tag{2.1.28}$$

The comparison of

$$\sigma_{ij} = R_{ki}R_{lj}\sigma'_{kl} \quad \text{with} \quad \sigma_p = (T^{\sigma}_{pq})^{-1}\sigma'_{\mathrm{q}}$$

and

$$\varepsilon_{ij} = R_{ki}R_{lj}\varepsilon'_{kl} \quad \text{with} \quad \varepsilon_p = (T^{\varepsilon}_{pq})^{-1}\varepsilon'_{\mathrm{q}}$$

yields an important result on the linkage of inverse and transposed stress and strain transformation matrices

$$(\boldsymbol{T}^{\sigma})^{-1} = (\boldsymbol{T}^{\varepsilon})^{\mathrm{T}}, \quad (\boldsymbol{T}^{\varepsilon})^{-1} = (\boldsymbol{T}^{\sigma})^{\mathrm{T}} \tag{2.1.29}$$

The transformation relations for the stiffness and the compliance matrices $\boldsymbol{C}$ and $\boldsymbol{S}$ can be obtained from the known rules for stresses and strains. With $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma}' = \boldsymbol{C}'\boldsymbol{\varepsilon}'$, it follows that

$$\begin{aligned}
(\boldsymbol{T}^{\sigma})^{-1}\boldsymbol{\sigma}' &= \boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon} = \boldsymbol{C}(\boldsymbol{T}^{\varepsilon})^{-1}\boldsymbol{\varepsilon}',\\
\boldsymbol{\sigma}' &= \boldsymbol{T}^{\sigma}\boldsymbol{C}(\boldsymbol{T}^{\varepsilon})^{-1}\boldsymbol{\varepsilon}' = \boldsymbol{C}'\boldsymbol{\varepsilon}',\\
\boldsymbol{T}^{\sigma}\boldsymbol{\sigma} &= \boldsymbol{\sigma}' = \boldsymbol{C}'\boldsymbol{\varepsilon}' = \boldsymbol{C}'\boldsymbol{T}^{\varepsilon}\boldsymbol{\varepsilon},\\
\boldsymbol{\sigma} &= (\boldsymbol{T}^{\sigma})^{-1}\boldsymbol{C}'\boldsymbol{T}^{\varepsilon}\boldsymbol{\varepsilon} = \boldsymbol{C}\boldsymbol{\varepsilon},
\end{aligned} \tag{2.1.30}$$

respectively.

Considering (2.1.29) the transformation relations for the stiffness matrix are

$$\boldsymbol{C}' = \boldsymbol{T}^{\sigma}\boldsymbol{C}(\boldsymbol{T}^{\sigma})^{\mathrm{T}}, \quad \boldsymbol{C} = (\boldsymbol{T}^{\varepsilon})^{\mathrm{T}}\boldsymbol{C}'\boldsymbol{T}^{\varepsilon} \tag{2.1.31}$$

or in index notation

$$C'_{ij} = T^{\sigma}_{ik}T^{\sigma}_{jl}C_{kl}, \quad C_{ij} = T^{\varepsilon}_{ik}T^{\varepsilon}_{jl}C'_{kl} \tag{2.1.32}$$

The same procedure yields the relations for the compliance matrix. With

$$\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}, \quad \boldsymbol{\varepsilon}' = \boldsymbol{S}'\boldsymbol{\sigma}' \tag{2.1.33}$$

it follows

$$\begin{aligned}
(\boldsymbol{T}^{\varepsilon})^{-1}\boldsymbol{\varepsilon}' &= \boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma} = \boldsymbol{S}(\boldsymbol{T}^{\sigma})^{-1}\boldsymbol{\sigma}',\\
\boldsymbol{\varepsilon}' &= \boldsymbol{T}^{\varepsilon}\boldsymbol{S}(\boldsymbol{T}^{\sigma})^{-1}\boldsymbol{\sigma}' = \boldsymbol{S}'\boldsymbol{\sigma}',\\
\boldsymbol{T}^{\varepsilon}\boldsymbol{\varepsilon} &= \boldsymbol{\varepsilon}' = \boldsymbol{S}'\boldsymbol{\sigma}' = \boldsymbol{S}'\boldsymbol{T}^{\sigma}\boldsymbol{\sigma},\\
\boldsymbol{\varepsilon} &= (\boldsymbol{T}^{\varepsilon})^{-1}\boldsymbol{S}'\boldsymbol{T}^{\sigma}\boldsymbol{\sigma} = \boldsymbol{S}\boldsymbol{\sigma},
\end{aligned} \tag{2.1.34}$$

i.e.

$$\boldsymbol{\varepsilon}' = \boldsymbol{T}^{\varepsilon}\boldsymbol{S}(\boldsymbol{T}^{\sigma})^{-1}\boldsymbol{\sigma}', \quad \boldsymbol{\varepsilon} = (\boldsymbol{T}^{\varepsilon})^{-1}\boldsymbol{S}'\boldsymbol{T}^{\sigma}\boldsymbol{\sigma} \tag{2.1.35}$$

The comparison leads to the transformation equations for $\boldsymbol{S}$ and $\boldsymbol{S}'$

$$\boldsymbol{S}' = \boldsymbol{T}^{\varepsilon}\boldsymbol{S}(\boldsymbol{T}^{\sigma})^{-1}, \quad \boldsymbol{S} = (\boldsymbol{T}^{\varepsilon})^{-1}\boldsymbol{S}'\boldsymbol{T}^{\sigma} \tag{2.1.36}$$

or taking into account (2.1.29)

$$\boldsymbol{S}' = \boldsymbol{T}^{\varepsilon}\boldsymbol{S}(\boldsymbol{T}^{\varepsilon})^{\mathrm{T}}, \quad \boldsymbol{S} = (\boldsymbol{T}^{\sigma})^{\mathrm{T}}\boldsymbol{S}'\boldsymbol{T}^{\sigma}, \tag{2.1.37}$$

respectively, in subscript notation

$$S'_{ij} = T^{\varepsilon}_{ik} T^{\varepsilon}_{jl} S_{kl}, \quad S_{ij} = T^{\sigma}_{ik} T^{\sigma}_{jl} S'_{kl} \tag{2.1.38}$$

In the special case of a rotation ϕ about the $\boldsymbol{e}_3$-direction (Fig. 2.6) the coordinates of the transformation matrices $\boldsymbol{T}^{\sigma}$ and $\boldsymbol{T}^{\varepsilon}$ are given by the (2.1.39) and (2.1.40)

$$\left[\overset{3}{T}{}^{\sigma}_{pq}\right] = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & 2cs \\ s^2 & c^2 & 0 & 0 & 0 & -2cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -cs & cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}, \quad \left[\overset{3}{T}{}^{\sigma}_{pq}\right]^{-1} = \left[\overset{3}{T}{}^{\varepsilon}_{pq}\right]^{\mathrm{T}} \tag{2.1.39}$$

$$\left[\overset{3}{T}{}^{\varepsilon}_{pq}\right] = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & cs \\ s^2 & c^2 & 0 & 0 & 0 & -cs \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & -s & 0 \\ 0 & 0 & 0 & s & c & 0 \\ -2cs & 2cs & 0 & 0 & 0 & c^2 - s^2 \end{bmatrix}, \quad \left[\overset{3}{T}{}^{\varepsilon}_{pq}\right]^{-1} = \left[\overset{3}{T}{}^{\sigma}_{pq}\right]^{\mathrm{T}} \tag{2.1.40}$$

By all rules following from a rotation of the reference system the stresses, strains, stiffness and compliance parameters in the rotated system are known. They are summarized in symbolic notation (2.1.41)

$$\begin{aligned} &\boldsymbol{\sigma}' = \boldsymbol{T}^{\sigma}\boldsymbol{\sigma}, \quad \boldsymbol{\varepsilon}' = \boldsymbol{T}^{\varepsilon}\boldsymbol{\varepsilon}, \\ &\boldsymbol{\sigma} = (\boldsymbol{T}^{\varepsilon})^{\mathrm{T}}\boldsymbol{\sigma}', \quad \boldsymbol{\varepsilon} = (\boldsymbol{T}^{\sigma})^{\mathrm{T}}\boldsymbol{\varepsilon}', \\ &\boldsymbol{C}' = \boldsymbol{T}^{\sigma}\boldsymbol{C}(\boldsymbol{T}^{\sigma})^{\mathrm{T}}, \quad \boldsymbol{S}' = \boldsymbol{T}^{\varepsilon}\boldsymbol{S}(\boldsymbol{T}^{\varepsilon})^{\mathrm{T}}, \\ &\boldsymbol{C} = (\boldsymbol{T}^{\varepsilon})^{\mathrm{T}}\boldsymbol{C}'\boldsymbol{T}^{\varepsilon}, \quad \boldsymbol{S} = (\boldsymbol{T}^{\sigma})^{\mathrm{T}}\boldsymbol{S}'\boldsymbol{T}^{\sigma} \end{aligned} \tag{2.1.41}$$

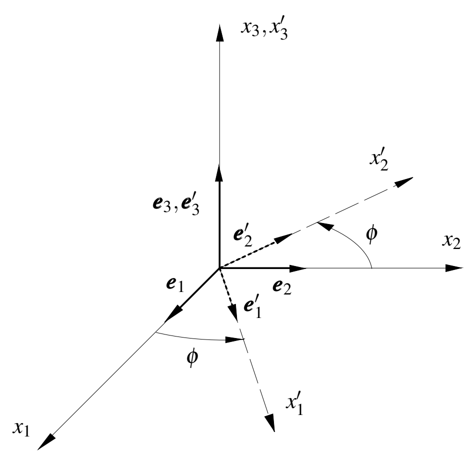


Fig. 2.6 Rotation about the $\boldsymbol{e}_3$-direction

For special cases of a rotation about a direction $\boldsymbol{e}_i$ the general transformation matrices $\boldsymbol{T}^{\sigma}$ and $\boldsymbol{T}^{\varepsilon}$ are substituted by $\overset{i}{\boldsymbol{T}}{}^{\sigma}$ or $\overset{i}{\boldsymbol{T}}{}^{\varepsilon}$. The case of a rotation about the $\boldsymbol{e}_1$-direction yields the coordinates of the transformation matrices $\boldsymbol{T}^{\sigma}$ and $\boldsymbol{T}^{\varepsilon}$ which are given in App. B.

2.1.3 Symmetry Relations of Stiffness and Compliance Matrices

In the most general case of the three-dimensional generalized Hooke's law the stiffness and the compliance matrices have 36 non-zero material parameters C_{ij} or S_{ij} but they are each determined by 21 independent parameters. Such an anisotropic material is called a triclinic material, it has no geometric symmetry properties. The experimental tests to determine 21 independent material parameters would be difficult to realize in engineering applications. So it is very important that the majority of anisotropic materials has a structure that exhibits one or more geometric symmetries and the number of independent material parameters needed to describe the material behavior can be reduced.

In the general case of 21 independent parameters, there is a coupling of each loading component with all strain states and the model equations for structure elements would be very complicated. The reduction of the number of independent material parameters results therefore in a simplifying of the modelling and analysis of structure elements composed of composite materials and impact the engineering applications. The most important material symmetries are:

2.1.3.1 Monoclinic or Monotropic Material Behavior

A monoclinic material has one symmetry plane (Fig. 2.7). It is assumed that the

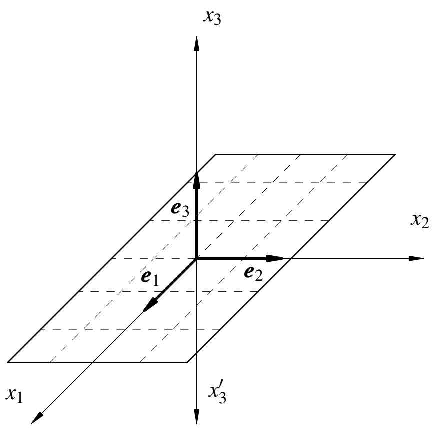


Fig. 2.7 Symmetry plane $(x_1 - x_2)$ of a monoclinic material. All points of a body which are symmetric to this plane have identical values of C_{ij} and S_{ij}. Mirror transformation ($x_1 = x_1'$, $x_2 = x_2', x_3 = -x_3'$)

symmetry plane is the $(x_1 - x_2)$ plane. The structure of the stiffness or compliance matrix must be in that way that a change of a reference system carried out by a symmetry about this plane does not modify the matrices, i.e. that the material properties are identical along any two rays symmetric with respect to the $(x_1 - x_2)$ plane. The exploitation of the transformation rules leads to a stiffness matrix with the following structure in the case of monoclinic material behavior

$$[C_{ij}]^{\mathrm{MC}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \tag{2.1.42}$$

The compliance matrix has the same structure

$$[S_{ij}]^{\mathrm{MC}} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ S_{12} & S_{22} & S_{23} & 0 & 0 & S_{26} \\ S_{13} & S_{23} & S_{33} & 0 & 0 & S_{36} \\ 0 & 0 & 0 & S_{44} & S_{45} & 0 \\ 0 & 0 & 0 & S_{45} & S_{55} & 0 \\ S_{16} & S_{26} & S_{36} & 0 & 0 & S_{66} \end{bmatrix} \tag{2.1.43}$$

The number of non-zero elements C_{ij} or S_{ij} reduces to twenty, the number of independent elements to thirteen. The loading-deformation couplings are reduced. Consider for example the stress component $\sigma_6 \equiv \tau_{12}$. There is a coupling with the extensional strains $\varepsilon_1, \varepsilon_2, \varepsilon_3$ and the shear strain $\varepsilon_6 \equiv \gamma_{12}$ but the shear stress σ_4 or σ_5 produces only shear strains.

If an anisotropic material has the plane of elastic symmetry $x_1 - x_3$ then it can be shown that

$$[C_{ij}]^{\mathrm{MC}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ C_{12} & C_{22} & C_{23} & 0 & C_{25} & 0 \\ C_{13} & C_{23} & C_{33} & 0 & C_{35} & 0 \\ 0 & 0 & 0 & C_{44} & 0 & C_{46} \\ C_{15} & C_{25} & C_{35} & 0 & C_{55} & 0 \\ 0 & 0 & 0 & C_{46} & 0 & C_{66} \end{bmatrix} \tag{2.1.44}$$

and for the plane of elastic symmetry $x_2 - x_3$

$$[C_{ij}]^{\mathrm{MC}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & 0 & 0 \\ C_{12} & C_{22} & C_{23} & C_{24} & 0 & 0 \\ C_{13} & C_{23} & C_{33} & C_{34} & 0 & 0 \\ C_{14} & C_{24} & C_{34} & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & C_{56} \\ 0 & 0 & 0 & 0 & C_{56} & C_{66} \end{bmatrix} \tag{2.1.45}$$

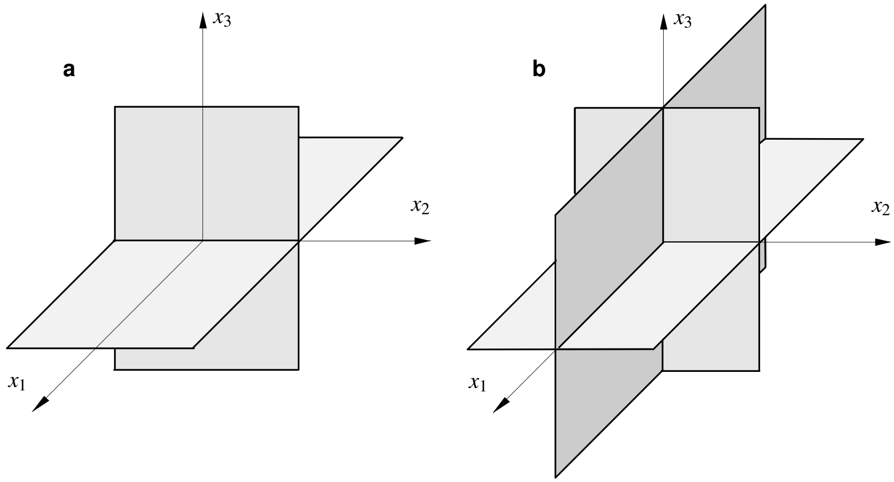


Fig. 2.8 Orthotropic material behavior. **a** Symmetry planes $(x_1 - x_2)$ and $(x_2 - x_3)$, **b** additional symmetry plane $(x_1 - x_3)$

The monoclinic compliance matrices $[S_{ij}]^{\mathrm{MC}}$ have for both cases the same structure again as the stiffness matrices $[C_{ij}]^{\mathrm{MC}}$.

2.1.3.2 Orthotropic Material Behavior

An orthotropic material behavior is characterized by three symmetry planes that are mutually orthogonal (Fig. 2.8). It should be noted that the existence of two orthogonal symmetry planes results in the existence of a third. The stiffness matrix of an orthotropic material has the following structure

$$[C_{ij}]^{\mathrm{O}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{bmatrix} \tag{2.1.46}$$

The compliance matrix has the same structure. An orthotropic material has 12 nonzero and 9 independent material parameters. The stress-strain coupling is the same as for isotropic material behavior. Normal stresses give rise to only extensional strains and shear stresses only shear strains. Orthotropic material behavior is typical for unidirectional laminae with on-axis loading.

2.1.3.3 Transversely Isotropic Material Behavior

A material behavior is said to be transversely isotropic if it is invariant with respect to an arbitrary rotation about a given axis. This material behavior is of special importance in the modelling of fibre-reinforced composite materials with coordinate axis in the fibre direction and an assumed isotropic behavior in cross-sections orthogonal to the fibre direction. This type of material behavior lies between isotropic and orthotropic.

If x_1 is the fibre direction, x_2 and x_3 are both rectangular to the fibre direction and assuming identical material properties in these directions is understandable. The structure of the stiffness matrix of a transversely isotropic material is given in (2.1.47)

$$[C_{ij}]^{\mathrm{TI}} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{12} & C_{23} & C_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(C_{22}-C_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{55} \end{bmatrix} \tag{2.1.47}$$

If x_2 is the fibre direction, x_1 and x_3 are both rectangular to the fibre direction and assuming identical material properties in these directions then

$$[C_{ij}]^{\mathrm{TI}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{12} & 0 & 0 & 0 \\ C_{13} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{13}) & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix} \tag{2.1.48}$$

If x_3 is the fibre direction, x_1 and x_2 are both rectangular to the fibre direction and assuming identical material properties in these directions then

$$[C_{ij}]^{\mathrm{TI}} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{13} & 0 & 0 & 0 \\ C_{13} & C_{13} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(C_{11}-C_{12}) \end{bmatrix} \tag{2.1.49}$$

The compliance matrix has the same structure. For example for the case (2.1.47)

$$[S_{ij}]^{\mathrm{TI}} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{22} & S_{23} & 0 & 0 & 0 \\ S_{12} & S_{23} & S_{22} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{22}-S_{23}) & 0 & 0 \\ 0 & 0 & 0 & 0 & S_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & S_{55} \end{bmatrix} \tag{2.1.50}$$

The number of non-zero elements reduces again to twelve but the independent parameters are only five.

2.1.3.4 Isotropic Material Behavior

A material behavior is said to be isotropic if its properties are independent of the choice of the reference system. There exist no preferred directions, i.e. the material has an infinite number of planes and axes of material symmetry. Most conventional materials satisfy this behavior approximately on a macroscopic scale.

The number of independent elasticity parameters is reduced to two and this leads to the following stiffness matrix in the case of isotropic material behavior

$$[C_{ij}]^{\mathrm{I}} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_* & 0 & 0 \\ 0 & 0 & 0 & 0 & C_* & 0 \\ 0 & 0 & 0 & 0 & 0 & C_* \end{bmatrix} \qquad (2.1.51)$$

with $C_* = \frac{1}{2}(C_{11} - C_{12})$. The compliance matrix has the same structure but with diagonal terms $2(S_{11} - S_{12})$ instead of $\frac{1}{2}(C_{11} - C_{12})$. Tables 2.3 and 2.4 summarize the stiffness and compliance matrices for all material models described above.

2.1.4 Engineering Parameters

2.1.4.1 Orthotropic Material Behavior

The coordinates C_{ij} and S_{ij} of the stiffness and compliance matrix are mathematical symbols relating stresses and strains. For practising engineers, a clear understanding of each material parameter is necessary and requires a more mechanical meaning by expressing the mathematical symbols in terms of engineering parameters such as moduli E_i, G_{ij} and Poisson's ratios ν_{ij}. The relationships between mathematical and engineering parameters are obtained by basic mechanical tests and imaginary mathematical experiments. The basic mechanical tests are the tension, compression and torsion test to measure the elongation, the contraction and the torsion of a specimen. In general, these tests are carried out by imposing a known stress and measuring the strains or vice versa.

It follows that the compliance parameters are directly related to the engineering parameters, simpler than those of the stiffness parameters. In the case of orthotropic materials the 12 engineering parameters are Young's moduli E_1, E_2, E_3, the shear moduli G_{23}, G_{13}, G_{12} and Poisson's ratios $\nu_{ij}, i, j = 1, 2, 3$ $(i \neq j)$. For orthotropic materials one can introduce the contracted engineering notation

Table 2.3 Three-dimensional stiffness matrices for different material symmetries

Material model	Elasticity matrix $[C_{ij}]$
Anisotropy: 21 independent material parameters	$\begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$
Monoclinic: 13 independent material parameters	Symmetry plane $x_3 = 0$ $C_{14} = C_{15} = C_{24} = C_{25} = C_{34} = C_{35} = C_{46} = C_{56} = 0$ Symmetry plane $x_2 = 0$ $C_{14} = C_{16} = C_{24} = C_{26} = C_{34} = C_{36} = C_{45} = C_{56} = 0$ Symmetry plane $x_1 = 0$ $C_{15} = C_{16} = C_{25} = C_{26} = C_{35} = C_{36} = C_{45} = C_{56} = 0$
Orthotropic: 9 independent material parameters	3 planes of symmetry $x_1 = 0, x_2 = 0, x_3 = 0$ $C_{14} = C_{15} = C_{16} = C_{24} = C_{25} = C_{26} = C_{34}$ $= C_{35} = C_{36} = C_{45} = C_{46} = C_{56} = 0$
Transversely isotropic: 5 independent material parameters	Plane of isotropy $x_3 = 0$ $C_{11} = C_{22}, C_{23} = C_{13}, C_{44} = C_{55}, C_{66} = \frac{1}{2}(C_{11} - C_{12})$ Plane of isotropy $x_2 = 0$ $C_{11} = C_{33}, C_{12} = C_{23}, C_{44} = C_{66}, C_{55} = \frac{1}{2}(C_{33} - C_{13})$ Plane of isotropy $x_1 = 0$ $C_{22} = C_{33}, C_{12} = C_{13}, C_{55} = C_{66}, C_{44} = \frac{1}{2}(C_{22} - C_{23})$ all other C_{ij} like orthotropic
Isotropy: 2 independent material parameters	$C_{11} = C_{22} = C_{33}, C_{12} = C_{13} = C_{23},$ $C_{44} = C_{55} = C_{66} = \frac{1}{2}(C_{11} - C_{12})$ all other $C_{ij} = 0$

$$\begin{aligned} \sigma_1 &= E_1\varepsilon_1, \quad \sigma_2 = E_2\varepsilon_2, \quad \sigma_3 = E_3\varepsilon_3, \\ \sigma_4 &= E_4\varepsilon_4, \quad \sigma_5 = E_5\varepsilon_5, \quad \sigma_6 = E_6\varepsilon_6 \end{aligned} \tag{2.1.52}$$

with $G_{23} \equiv E_4, G_{13} \equiv E_5, G_{12} \equiv E_6$.

The generalized Hooke's law in the form (2.1.19) and (2.1.20) leads, for example, to the relations

Table 2.4 Three-dimensional compliance matrices for different material symmetries

Material model	Compliance matrix $[S_{ij}]$
Anisotropy: 21 independent material parameters	$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ & & S_{33} & S_{34} & S_{35} & S_{36} \\ & & & S_{44} & S_{45} & S_{46} \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix}$
Monoclinic: 13 independent material parameters	Symmetry plane $x_3 = 0$ $S_{14} = S_{15} = S_{24} = S_{25} = S_{34} = S_{35} = S_{46} = S_{56} = 0$ Symmetry plane $x_2 = 0$ $S_{14} = S_{16} = S_{24} = S_{26} = S_{34} = S_{36} = S_{45} = S_{56} = 0$ Symmetry plane $x_1 = 0$ $S_{15} = S_{16} = S_{25} = S_{26} = S_{35} = S_{36} = S_{46} = S_{45} = 0$
Orthotropic: 9 independent material parameters	3 planes of symmetry $x_1 = 0, x_2 = 0, x_3 = 0$ $S_{14} = S_{15} = S_{16} = S_{24} = S_{25} = S_{26} = S_{34}$ $= S_{35} = S_{36} = S_{45} = S_{46} = S_{56} = 0$
Transversely isotropic: 5 independent material parameters	Plane of isotropy $x_3 = 0$ $S_{11} = S_{22}, S_{23} = S_{13}, S_{44} = S_{55}, S_{66} = 2(S_{11} - S_{12})$ Plane of isotropy $x_2 = 0$ $S_{11} = S_{33}, S_{12} = S_{23}, S_{44} = S_{66}, S_{55} = 2(S_{33} - S_{13})$ Plane of isotropy $x_1 = 0$ $S_{22} = S_{33}, S_{13} = S_{12}, S_{55} = S_{66}, S_{44} = 2(S_{22} - S_{23})$ all other S_{ij} like orthotropic
Isotropy: 2 independent material parameters	$S_{11} = S_{22} = S_{33}, S_{12} = S_{13} = S_{23},$ $S_{44} = S_{55} = S_{66} = 2(S_{11} - S_{12})$ all other $S_{ij} = 0$

$$\begin{aligned} \varepsilon_1 &= S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3, & \varepsilon_4 &= S_{44}\sigma_4, \\ \varepsilon_2 &= S_{12}\sigma_1 + S_{22}\sigma_2 + S_{23}\sigma_3, & \varepsilon_5 &= S_{55}\sigma_5, \\ \varepsilon_3 &= S_{13}\sigma_1 + S_{23}\sigma_2 + S_{33}\sigma_3, & \varepsilon_6 &= S_{66}\sigma_6 \end{aligned} \tag{2.1.53}$$

For uniaxial tension in x_i-direction, $\sigma_1 \neq 0, \sigma_i = 0, i = 2, \ldots, 6$. This reduces (2.1.53) to

$$\varepsilon_1 = S_{11}\sigma_1, \quad \varepsilon_2 = S_{12}\sigma_1, \quad \varepsilon_3 = S_{13}\sigma_1, \quad \varepsilon_4 = \varepsilon_5 = \varepsilon_6 = 0, \tag{2.1.54}$$

and the physical tensile tests provides the elastic parameters E_1, ν_{12}, ν_{13}

$$E_1 = \frac{\sigma_1}{\varepsilon_1} = \frac{1}{S_{11}}, \quad \nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} = -S_{12}E_1, \quad \nu_{13} = -\frac{\varepsilon_3}{\varepsilon_1} = -S_{13}E_1 \qquad (2.1.55)$$

Analogous relations resulting from uniaxial tension in x_2- and x_3-directions and all S_{ij} are related to the nine measured engineering parameters (3 Young's moduli and 6 Poisson's ratios) by uniaxial tension tests in three directions x_1, x_2 and x_3.

From the symmetry of the compliance matrix one can conclude

$$\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}, \quad \frac{\nu_{23}}{E_2} = \frac{\nu_{32}}{E_3}, \quad \frac{\nu_{31}}{E_3} = \frac{\nu_{13}}{E_1}$$

or

$$\frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}, \quad \frac{\nu_{ij}}{\nu_{ji}} = \frac{E_i}{E_j}, \quad i, j = 1,2,3 \quad (i \neq j) \qquad (2.1.56)$$

Remember that the first and the second subscript in Poisson's ratios denote stress and strain directions, respectively. Equations (2.1.56) demonstrate that the nine engineering parameters are not independent parameters and that in addition to the three tension tests, three independent shear tests are necessary to find the equations

$$\varepsilon_4 = S_{44}\sigma_4, \quad \varepsilon_5 = S_{55}\sigma_5, \quad \varepsilon_6 = S_{66}\sigma_6,$$

which yield the relations

$$S_{44} = \frac{1}{G_{23}} = \frac{1}{E_4}, \quad S_{55} = \frac{1}{G_{13}} = \frac{1}{E_5}, \quad S_{66} = \frac{1}{G_{12}} = \frac{1}{E_6} \qquad (2.1.57)$$

Now all S_{ij} in (2.1.20) can be substituted by the engineering parameters

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{12}}{E_1} & -\frac{\nu_{13}}{E_1} & 0 & 0 & 0 \\ & \frac{1}{E_2} & -\frac{\nu_{23}}{E_2} & 0 & 0 & 0 \\ & & \frac{1}{E_3} & 0 & 0 & 0 \\ & & & \frac{1}{E_4} & 0 & 0 \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & \frac{1}{E_5} & 0 \\ & & & & & \frac{1}{E_6} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \qquad (2.1.58)$$

As seen above, the relations between compliances S_{ij} and engineering parameters are very simple. This, however, is not the case for the relations between the stiffness and engineering parameters. First we need to invert the compliance matrix $\boldsymbol{S}$ and to express the stiffness C_{ij} as a function of the compliances as follows. The shear relations are uncoupled, and we obtain

$$C_{44} = \frac{1}{S_{44}} = G_{23}, \quad C_{55} = \frac{1}{S_{55}} = G_{13}, \quad C_{66} = \frac{1}{S_{66}} = G_{12} \qquad (2.1.59)$$

So only a symmetric [3x3]-matrix must be inverted. The general formula is

$$C_{ij} = S_{ij}^{-1} = \frac{(-1)^{i+j} U_{ij}}{\mathrm{Det}[S_{ij}]}, \quad \mathrm{Det}[S_{ij}] = \begin{vmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{vmatrix}, \tag{2.1.60}$$

where U_{ij} are the submatrices of $\boldsymbol{S}$ to the element S_{ij}, and yielding the relations

$$\begin{aligned} C_{11} &= \frac{S_{22}S_{33} - S_{23}^2}{\mathrm{Det}[S_{ij}]}, \quad C_{12} = \frac{S_{13}S_{23} - S_{12}S_{33}}{\mathrm{Det}[S_{ij}]}, \\ C_{22} &= \frac{S_{33}S_{11} - S_{13}^2}{\mathrm{Det}[S_{ij}]}, \quad C_{23} = \frac{S_{12}S_{13} - S_{23}S_{11}}{\mathrm{Det}[S_{ij}]}, \\ C_{33} &= \frac{S_{11}S_{22} - S_{12}^2}{\mathrm{Det}[S_{ij}]}, \quad C_{13} = \frac{S_{12}S_{23} - S_{13}S_{22}}{\mathrm{Det}[S_{ij}]} \end{aligned} \tag{2.1.61}$$

Substituting the relations between S_{ij} and engineering parameters given above in (2.1.58), we obtain

$$\begin{aligned} C_{11} &= \frac{(1 - \nu_{23}\nu_{32})E_1}{\Delta}, C_{12} = \frac{(\nu_{12} + \nu_{13}\nu_{32})E_2}{\Delta}, \\ C_{22} &= \frac{(1 - \nu_{31}\nu_{13})E_2}{\Delta}, C_{23} = \frac{(\nu_{23} + \nu_{21}\nu_{13})E_3}{\Delta}, \\ C_{33} &= \frac{(1 - \nu_{21}\nu_{12})E_3}{\Delta}, C_{13} = \frac{(\nu_{13} + \nu_{12}\nu_{23})E_3}{\Delta} \end{aligned} \tag{2.1.62}$$

with $\Delta = 1 - \nu_{21}\nu_{12} - \nu_{32}\nu_{23} - \nu_{13}\nu_{31} - 2\nu_{21}\nu_{13}\nu_{32}$. Considering $E_i/\Delta \equiv \overline{E}_i$, $1/S_i \equiv E_i$ we finally get two subsystems

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \end{bmatrix} = \begin{bmatrix} (1 - \nu_{23}\nu_{32})\overline{E}_1 & (\nu_{12} + \nu_{13}\nu_{32})\overline{E}_2 & (\nu_{13} + \nu_{12}\nu_{23})\overline{E}_3 \\ & (1 - \nu_{31}\nu_{13})\overline{E}_2 & (\nu_{23} + \nu_{21}\nu_{13})\overline{E}_3 \\ \mathrm{SYM} & & (1 - \nu_{21}\nu_{12})\overline{E}_3 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix},$$

$$\begin{bmatrix} \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} E_4 & 0 & 0 \\ & E_5 & 0 \\ \mathrm{SYM} & & E_6 \end{bmatrix} \begin{bmatrix} \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \tag{2.1.63}$$

2.1.4.2 Transversally-Isotropic Material Behavior

It should be noted that in the case of transversely isotropic material with the $(x_2 - x_3)$-plane of isotropy

$$E_2 = E_3,\ \nu_{12} = \nu_{13},\ G_{12} = G_{13},\ G_{23} = \frac{E_2}{2(1 + \nu_{23})}, \tag{2.1.64}$$

and we get

$$
\begin{aligned}
&S_{11} = \frac{1}{E_1}, \quad S_{12} = S_{13} = -\frac{\nu_{12}}{E_1}, \quad S_{44} = \frac{1}{G_{23}} = \frac{2(1+\nu_{23})}{E_2},\\
&S_{22} = S_{33} = \frac{1}{E_2}, \quad S_{23} = -\frac{\nu_{23}}{E_2}, \quad S_{55} = S_{66} = \frac{1}{G_{12}},\\
&C_{11} = \frac{(1-\nu_{23}^2)E_1}{\Delta^*}, \quad C_{12} = C_{13} = \frac{\nu_{21}(1+\nu_{23})E_1}{\Delta^*},\\
&C_{22} = C_{33} = \frac{(1-\nu_{12}\nu_{21})E_2}{\Delta^*}, \quad C_{23} = \frac{(\nu_{23}+\nu_{21}\nu_{12})E_2}{\Delta^*},\\
&C_{44} = G_{23} = \frac{E_2}{2(1+\nu_{23})}, \quad C_{55} = C_{66} = G_{12}
\end{aligned}
$$

with $\Delta^* = (1+\nu_{23})(1-\nu_{23}-2\nu_{21}\nu_{12})$.

Similar expressions one gets with the $(x_1 - x_2)$-plane of isotropy and considering

$$
E_1 = E_2,\ \nu_{13} = \nu_{23},\ G_{13} = G_{23},\ G_{12} = \frac{E_1}{2(1+\nu_{12})} \tag{2.1.65}
$$

or with the $(x_1 - x_3)$-plane of isotropy and

$$
E_1 = E_3,\ \nu_{12} = \nu_{23},\ G_{12} = G_{23},\ G_{13} = \frac{E_1}{2(1+\nu_{13})} \tag{2.1.66}
$$

The Young's modulus and the Poisson's ratio in the plane of isotropy often will be designated as E_{T} and ν_{TT}. E_{T} characterizes elongations or contractions of a transversely isotropic body in the direction of the applied load in any direction of the plane of isotropy and ν_{TT} characterizes contractions or elongations of the body in the direction perpendicular to the applied load, but parallel to the plane of isotropy. The shear modulus G_{TT} characterizes the material response under shear loading in the plane of isotropy and takes the form

$$
G_{\mathrm{TT}} = \frac{E_{\mathrm{T}}}{2(1+\nu_{\mathrm{TT}})},
$$

i.e. any two of the engineering parameters E_{T}, ν_{TT} and G_{TT} can be used to fully describe the elastic properties in the plane of isotropy. A third independent primary parameter should be E_{L}. This Young's modulus characterizes the tension respectively compression response for the direction perpendicular to the plane of isotropy. The fourth primary parameter should be the shear modulus G_{LT} in the planes perpendicular to the plane of isotropy. As a fifth primary parameter can be chosen ν_{LT} or ν_{TL}, which characterize the response in the plane of isotropy under a load in L-direction or the response in the L-direction under a load in the plane of isotropy. The stress-strain relations for an transversely isotropic body, if $(x_2 - x_3)$ is the plane of isotropy and with the reciprocity relations

$$
\frac{\nu_{\mathrm{LT}}}{E_{\mathrm{L}}} = \frac{\nu_{\mathrm{TL}}}{E_{\mathrm{T}}}
$$

can be used in the following matrix form

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_\mathrm{L}} & -\frac{\nu_\mathrm{TL}}{E_\mathrm{T}} & -\frac{\nu_\mathrm{TL}}{E_\mathrm{T}} & 0 & 0 & 0 \\ & \frac{1}{E_\mathrm{T}} & -\frac{\nu_\mathrm{TT}}{E_\mathrm{T}} & 0 & 0 & 0 \\ & & \frac{1}{E_\mathrm{T}} & 0 & 0 & 0 \\ & & & \frac{1}{G_\mathrm{TT}} & 0 & 0 \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & \frac{1}{G_\mathrm{LT}} & 0 \\ & & & & & \frac{1}{G_\mathrm{LT}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{2.1.67}$$

with the engineering parameters

$$E_1 = E_\mathrm{L}, E_2 = E_3 = E_\mathrm{T}, E_4 = G_{23} = G_\mathrm{TT} = \frac{E_\mathrm{T}}{2(1+\nu_\mathrm{TT})},$$
$$E_5 = G_{13} = E_6 = G_{12} = G_\mathrm{LT}, \nu_{12} = \nu_{13} = \nu_\mathrm{LT}, \nu_{23} = \nu_\mathrm{TT}, \frac{\nu_\mathrm{LT}}{E_\mathrm{L}} = \frac{\nu_\mathrm{TL}}{E_\mathrm{T}}$$

With the $(x_1 - x_2)$-plane of isotropy the engineering parameters are

$$E_1 = E_2 = E_\mathrm{T}, E_3 = E_\mathrm{L}, E_4 = G_{23} = E_5 = G_{13} = G_\mathrm{TL},$$
$$E_6 = G_{12} = G_\mathrm{TT} = \frac{E_\mathrm{T}}{2(1+\nu_\mathrm{TT})}, \nu_{13} = \nu_{23} = \nu_\mathrm{TL}, \nu_{12} = \nu_\mathrm{TT}$$

Notice that in literature the notations of Poisson's ratios ν_LT and ν_TL can be correspond to the opposite meaning.

2.1.4.3 Isotropic Material Behavior

For an isotropic material behavior in all directions, the number of independent engineering parameters reduces to two

$$\begin{aligned} &E_1 = E_2 = E_3 = E, \quad \nu_{12} = \nu_{23} = \nu_{13} = \nu, \\ &G_{12} = G_{13} = G_{23} = G = \frac{E}{2(1+\nu)} \end{aligned} \tag{2.1.68}$$

$$S_{11} = S_{22} = S_{33} = \frac{1}{E}, \quad S_{12} = S_{13} = S_{23} = -\frac{\nu}{E},$$
$$S_{44} = S_{55} = S_{66} = \frac{1}{G} = \frac{2(1+\nu)}{E}, \quad C_{11} = C_{22} = C_{33} = \frac{(1-\nu)E}{\Delta^{**}},$$
$$C_{12} = C_{13} = C_{23} = \frac{\nu E}{\Delta^{**}}, \quad C_{44} = C_{55} = C_{66} = G = \frac{E}{2(1+\nu)}$$

assuming $\Delta^{**} = (1+\nu)(1-2\nu)$.

With this, all three-dimensional material laws for various material symmetries interesting in engineering applications of composites are known. The relations between S_{ij}, C_{ij} and engineering parameters are summarized in Table 2.5.

Consider that the elastic properties of an isotropic material are determined by two independent parameters. The elastic parameters Young's modulus E and Poisson's ratio ν are generally used because they are determined easily in physical tests. But also the so-called Lamé coefficients[5] λ and μ, the shear modulus G or the bulk modulus K can be used if it is suitable. There are simple relations between the parameters, e.g. as a function of E, ν

$$\begin{aligned}
\lambda &= \frac{E\nu}{(1+\nu)(1-2\nu)}, & \mu &= \frac{E}{2(1+\nu)} = G, & K &= \frac{E}{3(1-2\nu)}, \\
\nu &= \frac{\lambda}{2(\lambda+\mu)}, & E &= \frac{\mu(3\lambda+2\mu)}{\lambda+\mu}, & K &= \lambda + \frac{2}{3}\mu
\end{aligned} \tag{2.1.69}$$

Summarizing the constitutive equations for isotropic, transversely isotropic and orthotropic materials, which are most important in the engineering applications of composite structural mechanics one can find that the common features of the relationships between stresses and strains for these material symmetries are that the normal stresses are not couplet with shear strains and shear stresses are not coupled with the normal strains. Each shear stress is only related to the corresponding shear strain. These features are not retained in the more general case of an monoclinic or a general anisotropic material.

2.1.4.4 Monoclinic Material Behavior

In the case of monoclinic materials we have 13 mutually independent stiffness or compliances. Therefore we have in comparison with orthotropic materials to introduce four additional engineering parameters and keeping in mind, that the monoclinic case must comprise the orthotropic case, we should not change the engineering parameters of orthotropic material behavior. Assuming that $(x_1 - x_2)$ is the plane of elastic symmetry, the additional parameters are related to the compliance matrix components S_{16}, S_{26}, S_{36} and S_{46}.

The first three pair normal strains $\varepsilon_1, \varepsilon_2, \varepsilon_3$ to the shear stress σ_6 and vice versa the shear strain ε_6 to the normal stresses $\sigma_1, \sigma_2, \sigma_3$. The fourth one couple the shear strain ε_4 to the shear stress σ_5 and vice versa the shear strain ε_5 to the shear stress σ_4. In a compact notation the strain-stress relations for an anisotropic material having one plane of elastic symmetry $(x_1 - x_2)$ are

[5] Gabriel Léon Jean Baptiste Lamé (*22 July 1795 Tours - †1 May 1870 Paris) - mathematician, who contributed to the mathematical theory of elasticity (Lamé's parameters in elasticity and fluid mechanics)

Table 2.5 Relationships between S_{ij}, C_{ij} and the engineering parameters for orthotropic, transversely-isotropic and isotropic material

Orthotropic material

$$S_{11} = E_1^{-1}, \quad S_{12} = S_{21} = -\nu_{12}E_1^{-1}, \quad S_{44} = G_{23}^{-1} = E_4^{-1}$$

$$S_{22} = E_2^{-1}, \quad S_{13} = S_{31} = -\nu_{13}E_1^{-1}, \quad S_{55} = G_{13}^{-1} = E_5^{-1}$$

$$S_{33} = E_3^{-1}, \quad S_{23} = S_{32} = -\nu_{23}E_2^{-1}, \quad S_{66} = G_{12}^{-1} = E_6^{-1}$$

$$\nu_{ij} = \nu_{ji}(E_i/E_j), \quad E_i = (\nu_{ij}/\nu_{ji})E_j \qquad i,j = 1,2,3$$

$$C_{11} = \frac{S_{22}S_{33} - S_{23}^2}{\det[S_{ij}]} = \frac{(1-\nu_{23}\nu_{32})E_1}{\Delta}, \quad C_{44} = \frac{1}{S_{44}} = G_{23} = E_4$$

$$C_{22} = \frac{S_{33}S_{11} - S_{13}^2}{\det[S_{ij}]} = \frac{(1-\nu_{31}\nu_{13})E_2}{\Delta}, \quad C_{55} = \frac{1}{S_{55}} = G_{31} = E_5$$

$$C_{33} = \frac{S_{11}S_{22} - S_{12}^2}{\det[S_{ij}]} = \frac{(1-\nu_{21}\nu_{12})E_3}{\Delta}, \quad C_{66} = \frac{1}{S_{66}} = G_{12} = E_6$$

$$C_{12} = \frac{S_{13}S_{23} - S_{12}S_{33}}{\det[S_{ij}]} = \frac{(\nu_{12}+\nu_{32}\nu_{13})E_2}{\Delta} = \frac{(\nu_{21}+\nu_{31}\nu_{23})E_1}{\Delta} = C_{21}$$

$$C_{13} = \frac{S_{12}S_{23} - S_{13}S_{22}}{\det[S_{ij}]} = \frac{(\nu_{13}+\nu_{12}\nu_{23})E_3}{\Delta} = \frac{(\nu_{31}+\nu_{21}\nu_{32})E_1}{\Delta} = C_{31}$$

$$C_{23} = \frac{S_{12}S_{13} - S_{23}S_{11}}{\det[S_{ij}]} = \frac{(\nu_{23}+\nu_{21}\nu_{13})E_3}{\Delta} = \frac{(\nu_{32}+\nu_{12}\nu_{31})E_2}{\Delta} = C_{32}$$

$$\Delta = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{13}\nu_{32}$$

Transversely-isotropic material

$(x_2 - x_3)$-plane of isotropy

$$E_1, \quad E_2 = E_3, \quad E_5 = E_6, \quad \nu_{12} = \nu_{13}, \quad E_4 = \frac{E_2}{2(1+\nu_{23})}$$

$(x_1 - x_2)$-plane of isotropy

$$E_1 = E_2, \quad E_3, \quad E_4 = E_5, \quad \nu_{13} = \nu_{23}, \quad E_6 = \frac{E_1}{2(1+\nu_{12})}$$

$(x_1 - x_3)$-plane of isotropy

$$E_1 = E_3, \quad E_2, \quad E_4 = E_6, \quad \nu_{12} = \nu_{23}, \quad E_5 = \frac{E_3}{2(1+\nu_{13})}$$

Isotropic material

$$E_1 = E_2 = E_3 = E, \quad \nu_{12} = \nu_{21} = \nu_{13} = \nu_{31} = \nu_{23} = \nu_{32} = \nu,$$

$$E_4 = E_5 = E_6 = G = \frac{E}{2(1+\nu)}$$

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & 0 & 0 & \frac{\eta_{61}}{E_6} \\ -\frac{\nu_{12}}{E_1} & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & 0 & 0 & \frac{\eta_{62}}{E_6} \\ -\frac{\nu_{13}}{E_1} & -\frac{\nu_{23}}{E_2} & \frac{1}{E_3} & 0 & 0 & \frac{\eta_{63}}{E_6} \\ 0 & 0 & 0 & \frac{1}{E_4} & \frac{\mu_{54}}{E_5} & 0 \\ 0 & 0 & 0 & \frac{\mu_{45}}{E_4} & \frac{1}{E_5} & 0 \\ \frac{\eta_{16}}{E_1} & \frac{\eta_{26}}{E_2} & \frac{\eta_{36}}{E_3} & 0 & 0 & \frac{1}{E_6} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \qquad (2.1.70)$$

with the following reciprocal relations

$$\frac{\eta_{61}}{E_6} = \frac{\eta_{16}}{E_1}, \quad \frac{\eta_{62}}{E_6} = \frac{\eta_{26}}{E_2}, \quad \frac{\eta_{63}}{E_6} = \frac{\eta_{36}}{E_3}, \quad \frac{\mu_{54}}{E_5} = \frac{\mu_{45}}{E_4} \qquad (2.1.71)$$

and the compliance components

$$S_{16} = \frac{\eta_{61}}{E_6}, \quad S_{26} = \frac{\eta_{62}}{E_6}, \quad S_{36} = \frac{\eta_{63}}{E_6}, \quad S_{45} = \frac{\mu_{54}}{E_5} \qquad (2.1.72)$$

η_{61}, η_{62} and η_{63} are extension-shear coupling coefficients indicating normal strains induced by shear stress σ_6 and η_{16}, η_{26} and η_{36} the shear-extension coupling coefficients characterizing shear strain ε_6 caused by normal stresses. μ_{45} and μ_{54} are shear-shear coupling coefficients.

The stiffness matrix for the monoclinic material can be found as the inverse of the compliance matrix, but the expressions are unreasonable to present in an explicit form. However, the inverse of a matrix can be easily calculated using standard numerical procedures. Also for a generally anisotropic material the compliance can be formulated with help of eight additional coupling parameters but the stiffness matrix should be calculated numerically.

2.1.5 Two-Dimensional Material Equations

In most structural applications the structural elements are simplified models by reducing the three-dimensional state of stress and strain approximately to a two-dimensional plane stress or plane strain state. A thin lamina for instance can be considered to be under a condition of plane stress with all stress components in the out-of-plane direction being approximately zero. The different conditions for a plane stress state in the planes $(x_1 - x_2), (x_2 - x_3)$ and $(x_1 - x_3)$ are demonstrated in Fig. 2.9.

In the following, the plane stress state with respect to the $(x_1 - x_2)$ plane (Fig. 2.9a) is considered. In addition, since in the case of unidirectional long-fibre reinforced laminae the most general type of symmetry of the material properties is the

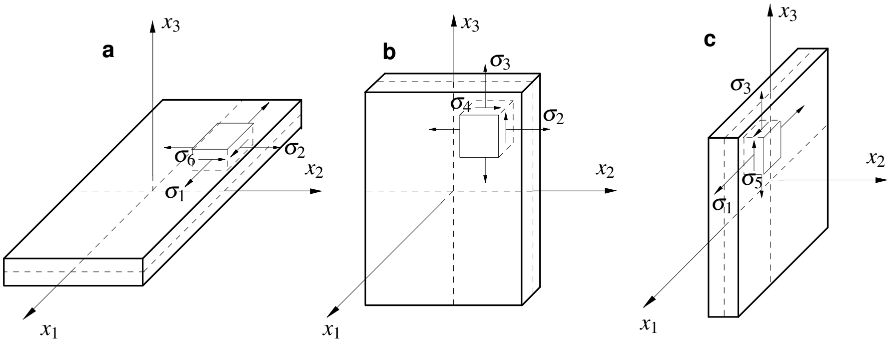


Fig. 2.9 Plane stress state. **a** $(x_1 - x_2)$-plane, $\sigma_3 = \sigma_4 = \sigma_5 = 0$, **b** $(x_2 - x_3)$-plane, $\sigma_1 = \sigma_5 = \sigma_6 = 0$, **c** $(x_1 - x_3)$-plane, $\sigma_2 = \sigma_4 = \sigma_6 = 0$

monoclinic one the generalized Hooke's law (2.1.20) is reduced taking into account (2.1.43) to

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ & S_{22} & S_{23} & 0 & 0 & S_{26} \\ & & S_{33} & 0 & 0 & S_{36} \\ \mathrm{S} & & & S_{44} & S_{45} & 0 \\ & \mathrm{Y} & & & S_{55} & 0 \\ & & \mathrm{M} & & & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \sigma_6 \end{bmatrix} \tag{2.1.73}$$

That means $\sigma_3 = \sigma_4 = \sigma_5 = 0$, and we have three in-plane constitutive equations

$$\begin{aligned} \varepsilon_1 &= S_{11}\sigma_1 + S_{12}\sigma_2 + S_{16}\sigma_6 \\ \varepsilon_2 &= S_{12}\sigma_1 + S_{22}\sigma_2 + S_{26}\sigma_6, \qquad S_{ij} = S_{ji} \\ \varepsilon_6 &= S_{16}\sigma_1 + S_{26}\sigma_2 + S_{66}\sigma_6 \end{aligned} \tag{2.1.74}$$

and an additional equation for strain ε_3 in x_3-direction

$$\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2 + S_{36}\sigma_6 \tag{2.1.75}$$

The other strains $\varepsilon_4, \varepsilon_5$ are equal to 0 considering the monoclinic material behavior.

If the plane stress assumptions are used to simplify the generalized stiffness equations (2.1.19) taking into account (2.1.42), the result is

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ 0 \\ 0 \\ 0 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ \mathrm{S} & & & C_{44} & C_{45} & 0 \\ & \mathrm{Y} & & & C_{55} & 0 \\ & & \mathrm{M} & & & C_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} \tag{2.1.76}$$

or again three in-plane equations

$$
\begin{aligned}
\sigma_1 &= C_{11}\varepsilon_1 + C_{12}\varepsilon_2 + C_{13}\varepsilon_3 + C_{16}\varepsilon_6, \\
\sigma_2 &= C_{12}\varepsilon_1 + C_{22}\varepsilon_2 + C_{23}\varepsilon_3 + C_{26}\varepsilon_6, \qquad C_{ij} = C_{ji} \\
\sigma_6 &= C_{16}\varepsilon_1 + C_{26}\varepsilon_2 + C_{36}\varepsilon_3 + C_{66}\varepsilon_6
\end{aligned}
\tag{2.1.77}
$$

There are another three equations. At first we obtain

$$\sigma_4 = S_{44}\varepsilon_4 + S_{45}\varepsilon_5 = 0, \qquad \sigma_5 = S_{45}\varepsilon_4 + S_{55}\varepsilon_5 = 0$$

Since S_{44}, S_{45}, S_{55} are arbitrary but the stiffness matrix should be positive definite it is obvious that

$$S_{44}S_{55} - S_{45}^2 > 0$$

and from the $\sigma_4 = \sigma_5 = 0$ condition it follows that $\varepsilon_4, \varepsilon_5$ must be equal to 0. Taking into account the last condition

$$\sigma_3 = C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{33}\varepsilon_3 + C_{36}\varepsilon_6 = 0$$

the strain ε_3 can be obtained

$$\varepsilon_3 = -\frac{1}{C_{33}}(C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{36}\varepsilon_6) \tag{2.1.78}$$

and eliminated and substituted in Eqs. (2.1.77). After substituting ε_3 using Eq. (2.1.78) Eqs. (2.1.77) leads to

$$\sigma_i = \left(C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}}\right)\varepsilon_j, \quad i, j = 1, 2, 6, \tag{2.1.79}$$

respectively

$$\sigma_i = Q_{ij}\varepsilon_j, \quad i, j = 1, 2, 6 \tag{2.1.80}$$

The Q_{ij} are the reduced stiffness. For the three cases in Fig. 2.9 we obtain

$$
\begin{aligned}
\sigma_i &= Q_{ij}\varepsilon_j,\ Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}},\ i, j = 1, 2, 6, \quad (x_1 - x_2) - \text{plane of symmetry}, \\
\sigma_i &= Q_{ij}\varepsilon_j,\ Q_{ij} = C_{ij} - \frac{C_{i1}C_{j1}}{C_{11}},\ i, j = 2, 3, 4, \quad (x_2 - x_3) - \text{plane of symmetry}, \\
\sigma_i &= Q_{ij}\varepsilon_j,\ Q_{ij} = C_{ij} - \frac{C_{i2}C_{j2}}{C_{22}},\ i, j = 1, 3, 5, \quad (x_1 - x_3) - \text{plane of symmetry}
\end{aligned}
$$

The number of unknown independent parameters of each of the matrices S_{ij}, C_{ij} or Q_{ij} is six. It is very important to note that the elements in the plane stress compliance matrix are simply a subset of the elements from the three-dimensional compliance matrix and their numerical values are identical. On the other hand, the elements of the reduced stiffness matrix involve a combination of elements from the three-dimensional stiffness matrix and the numerical values of the Q_{ij} differ from their counterparts C_{ij}, i.e. they are actually less than the numerical values for C_{ij}. In order to keep consistency with the generalized Hooke's law, Eq. (2.1.78) should be used when calculating the transverse normal strain ε_3 and the general case leads to consistent relations for the transverse shear strains ε_4 and ε_5.

For an orthotropic material behavior under plane stress and on-axis orientation of the reference system there are four independent parameters and for isotropic behavior there are only two. The mathematical notations S_{ij}, C_{ij} or Q_{ij} can be shifted to the engineering notation. Tables 2.6 and 2.7 summarize the compliance and stiffness matrices for the plane stress state.

Considering a plane strain state in the $(x_1 - x_2)$ plane we have the three non-zero strains $\varepsilon_1, \varepsilon_2$ and ε_6 but the four nonzero stress components $\sigma_1, \sigma_2, \sigma_3, \sigma_6$. Analogous to the plane stress state, here the stress σ_3 normal to the $(x_1 - x_2)$ plane is not an independent value and can be eliminated

$$\varepsilon_3 = S_{13}\sigma_1 + S_{23}\sigma_2 + S_{33}\sigma_3 + S_{36}\sigma_6 = 0,$$

$$\sigma_3 = -\frac{1}{S_{33}}(S_{13}\sigma_1 + S_{23}\sigma_2 + S_{36}\sigma_6)$$

Therefore in the case of plane strain, the $C_{ij}, i, j = 1, 2, 6$ can be taken directly from the three-dimensional elasticity law and instead of S_{ij} reduced compliances V_{ij} have to be used

Table 2.6 Compliance matrices for various material models, plane stress state

Material model	$\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}$
Anisotropy: 6 independentmaterial parameters	Compliances S_{ij} $\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{bmatrix} = \begin{bmatrix}S_{11} & S_{12} & S_{16}\\ & S_{22} & S_{26}\\ & & S_{66}\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{bmatrix}$
Orthotropy: 4 independent material parameters Reference system: on-axis	$S_{16} = S_{26} = 0$ $S_{11} = \frac{1}{E_1}, S_{22} = \frac{1}{E_2}$ $S_{12} = \frac{-\nu_{12}}{E_1} = \frac{-\nu_{21}}{E_2}$ $S_{66} = \frac{1}{G_{12}}$
Isotropy: 2 independent material parameters Reference system: as you like	$S_{16} = S_{26} = 0$ $S_{11} = S_{22} = \frac{1}{E}, \quad S_{12} = -\frac{\nu}{E},$ $S_{66} = 2(S_{11} - S_{12}) = \frac{2(1+\nu)}{E} = \frac{1}{G}$

Table 2.7 Stiffness matrices for various material models, plane stress state

Material model	$\boldsymbol{\sigma} = \boldsymbol{Q\varepsilon}$
Anisotropy: 6 independent material parameters	Reduced stiffness Q_{ij} $\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{bmatrix} = \begin{bmatrix}Q_{11} & Q_{12} & Q_{16}\\ & Q_{22} & Q_{26}\\ & & Q_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix}$
Orthotropy: 4 independent material parameters Reference system: on-axis	$Q_{16} = Q_{26} = 0,\ Q_{66} = \dfrac{1}{S_{66}} = G_{12}$ $Q_{11} = \dfrac{S_{22}}{\Delta} = \dfrac{E_1}{1-\nu_{12}\nu_{21}}$ $Q_{22} = \dfrac{S_{11}}{\Delta} = \dfrac{E_2}{1-\nu_{12}\nu_{21}}$ $Q_{12} = -\dfrac{S_{12}}{\Delta} = \dfrac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}}$ $\Delta = S_{11}S_{22} - S_{12}^2$
Isotropy: 2 independent material parameters Reference system: as you like	$Q_{16} = Q_{26} = 0$ $Q_{11} = Q_{22} = \dfrac{E}{1-\nu^2}$ $Q_{12} = \dfrac{\nu E}{1-\nu^2},\quad Q_{66} = \dfrac{E}{2(1+\nu)} = G$

$$\sigma_i = C_{ij}\varepsilon_j, \quad i,j = 1,2,6,$$

$$\varepsilon_i = V_{ij}\sigma_j, \quad V_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}}, \quad i,j = 1,2,6$$

Table 2.8 summarizes for the three-dimensional states and the plane stress and strain states the number of non-zero and of independent material parameters.

In the two-dimensional equations of anisotropic elasticity, either reduced stiffness or reduced compliances are introduced into the material laws. These equations are most important in the theory of composite single or multilayered elements, e.g. of laminae or laminates. The additional transformations rules which are necessary in laminae and laminate theories are discussed in more detail in Chap. 3. Tables 2.6 and 2.7 above shows the relationship between stress and strain through the compliance $[S_{ij}]$ or the reduced stiffness $[Q_{ij}]$ matrix for the plane stress state and how the S_{ij} and Q_{ij} are related to the engineering parameters. For a unidirectional lamina the engineering parameters are:

Table 2.8 Stiffness and compliance parameters for stress and strain equations $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}$, $\boldsymbol{C} = \boldsymbol{S}^{-1}$, plane stress state $\boldsymbol{\sigma} = \boldsymbol{Q}\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}, \boldsymbol{Q} = \boldsymbol{S}^{-1}$, plane strain state - $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}$, $\boldsymbol{\varepsilon} = \boldsymbol{V}\boldsymbol{\sigma}$, $\boldsymbol{C} = \boldsymbol{V}^{-1}$, Q_{ij} and V_{ij} are the reduced stiffness and compliances

Material model	Number of non-zero parameters	Number of independent parameters
Three-dimensional stress- or strain state	$C_{ij}; S_{ij}$ $i,j = 1,\ldots,6$	$C_{ij}; S_{ij}$ $i,j = 1,\ldots,6$
Anisotropic	36	21
Monotropic	20	13
Orthotropic	12	9
Transversely isotropic	12	5
Isotropic	12	2
Plane stress state $(x_1 - x_2)$-plane	$Q_{ij}; S_{ij}$ $i,j = 1,2,6$	$Q_{ij}; S_{ij}$ $i,j = 1,2,6$
Anisotropic	9	6
Orthotropic	5	4
Isotropic	5	2
Plane strain state $(x_1 - x_2)$-plane	$C_{ij}; V_{ij}$ $i,j = 1,2,6$	$C_{ij}; V_{ij}$ $i,j = 1,2,6$
Anisotropic	9	6
Orthotropic	5	4
Isotropic	5	2

E_1 longitudinal Young's modulus in the principal direction 1 (fibre direction)
E_2 transverse Young's modulus in direction 2 (orthogonal to the fibre direction)
ν_{12} major Poisson's ratio as the ratio of the negative normal strain in direction 2 to the normal strain in direction 1 only if normal load is applied in direction 1
G_{12} in-plane shear modulus for $(x_1 - x_2)$ plane

The four independent engineering elastic parameters are experimentally measured as follows:

- Pure tensile load in direction 1: $\sigma_1 \neq 0, \sigma_2 = 0, \sigma_6 = 0$
  With $\varepsilon_1 = S_{11}\sigma_1, \varepsilon_2 = S_{12}\sigma_1, \varepsilon_6 = 0$ are
$$E_1 = \frac{\sigma_1}{\varepsilon_1} = \frac{1}{S_{11}}, \quad \nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{S_{12}}{S_{11}}$$
- Pure tensile load in direction 2: $\sigma_1 = 0, \sigma_2 \neq 0, \sigma_6 = 0$
  With $\varepsilon_1 = S_{12}\sigma_2, \varepsilon_2 = S_{22}\sigma_2, \varepsilon_6 = 0$ are
$$E_2 = \frac{\sigma_2}{\varepsilon_2} = \frac{1}{S_{22}}, \quad \nu_{21} = -\frac{\varepsilon_1}{\varepsilon_2} = -\frac{S_{12}}{S_{22}}$$
  ν_{21} is usually called the minor Poisson's ratio and we have the reciprocal relationship $\nu_{12}/E_1 = \nu_{21}/E_2$.
- Pure shear stress in the $(x_1 - x_2)$ plane: $\sigma_1 = \sigma_2 = 0, \sigma_6 \neq 0$
  With $\varepsilon_1 = \varepsilon_2 = 0, \varepsilon_6 = S_{66}\sigma_6$ is

$$G_{12} = \frac{\sigma_6}{\varepsilon_6} = \frac{1}{S_{66}}$$

With the help of Tables 2.6 and 2.7, the relating equations of stresses and strains are given through any of the following combinations of four parameters: $(Q_{11}, Q_{12}, Q_{22}, Q_{66}), (S_{11}, S_{12}, S_{22}, S_{66}), (E_1, E_2, \nu_{12}, G_{12})$.

In Chap. 3 the evaluation of the four engineering elastic parameters is given approximately by averaging the fibre-matrix material behavior. There are different approaches for determining effective elastic moduli, e.g. in a simple way with elementary mixture rules, with semi-empirical models or an approach based on the elementary theory.

2.1.6 Curvilinear Anisotropy

The type of anisotropy considered above was characterized by the equivalence of parallel directions passing through different points of the homogeneous anisotropic body and we can speak of a rectilinear anisotropy. Another kind of anisotropy is the case, if one chooses a system of curvilinear coordinates in such a manner that the coordinate directions coincide with equivalent directions of elastic properties at different points of an anisotropic body. The elements of the body, which are generated by three pairs of coordinate surfaces possess identical elastic properties and we can speak of a curvilinear anisotropy.

In the frame of this textbook we limit the considerations to cylindrical anisotropy, which is also the most common case of this type of anisotropy. The generalized Hooke's law equations (2.1.21) are now considered in cylindrical coordinates $x_1 = r, x_2 = \theta, x_3 = z$ and we have the stress and strain vectors in the contracted single subscript notation

$$\begin{aligned} \left[\sigma_1\ \sigma_2\ \sigma_3\ \sigma_4\ \sigma_5\ \sigma_6\right]^{\mathrm{T}} &= \left[\sigma_r\ \sigma_\theta\ \sigma_z\ \sigma_{\theta z}\ \sigma_{rz}\ \sigma_{r\theta}\right]^{\mathrm{T}}, \\ \left[\varepsilon_1\ \varepsilon_2\ \varepsilon_3\ \varepsilon_4\ \varepsilon_5\ \varepsilon_6\right]^{\mathrm{T}} &= \left[\varepsilon_r\ \varepsilon_\theta\ \varepsilon_z\ \varepsilon_{\theta z}\ \varepsilon_{rz}\ \varepsilon_{r\theta}\right]^{\mathrm{T}} \end{aligned} \qquad (2.1.81)$$

In the specific cases of material symmetries the general constitutive equation in cylindrical coordinates can be simplified analogous to the case of rectilinear anisotropy.

In the specific case of an orthotropic cylindrical response there are three orthogonal planes of elastic symmetries. One plane is perpendicular to the axis z, another one is tangential to the surface $(\theta - z)$ and the third one is a radial plane (Fig. 2.10). Another case of material symmetry in possible practical situation is a transversely isotropic cylinder or cylindrical tube with the plane of isotropy $(r - \theta)$. In this case we obtain, as analog to (2.1.67)

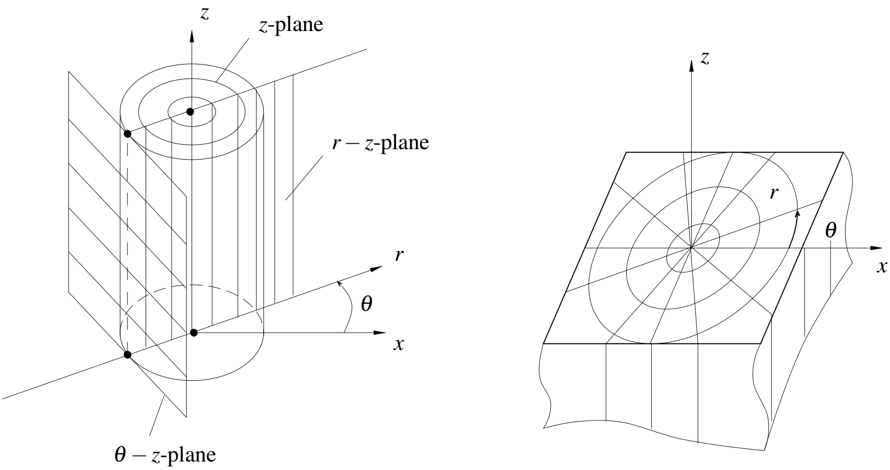


Fig. 2.10 Cylindrical orthotropic material symmetry

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_\mathrm{T}} & -\frac{\nu_\mathrm{TT}}{E_\mathrm{T}} & -\frac{\nu_\mathrm{LT}}{E_\mathrm{L}} & 0 & 0 & 0 \\ & \frac{1}{E_\mathrm{T}} & -\frac{\nu_\mathrm{LT}}{E_\mathrm{L}} & 0 & 0 & 0 \\ & & \frac{1}{E_\mathrm{L}} & 0 & 0 & 0 \\ & & & \frac{1}{G_\mathrm{LT}} & 0 & 0 \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & \frac{1}{G_\mathrm{LT}} & 0 \\ & & & & & \frac{1}{G_\mathrm{TT}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix}, \qquad (2.1.82)$$

where the index T is associated with the coordinate directions r and θ and the index L with the coordinate direction z and the reciprocal relations are

$$\frac{\nu_\mathrm{LT}}{E_\mathrm{L}} = \frac{\nu_\mathrm{TL}}{E_\mathrm{T}}$$

The stress-strain equations for the orthotropic case of cylindrical anisotropy are obtained using engineering parameters

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_r} & -\frac{\nu_{\theta r}}{E_\theta} & -\frac{\nu_{zr}}{E_r} & 0 & 0 & 0 \\ & \frac{1}{E_\theta} & -\frac{\nu_{z\theta}}{E_r} & 0 & 0 & 0 \\ & & \frac{1}{E_r} & 0 & 0 & 0 \\ & & & \frac{1}{G_{\theta z}} & 0 & 0 \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & \frac{1}{G_{rz}} & 0 \\ & & & & & \frac{1}{G_{r\theta}} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{2.1.83}$$

The indices r, θ and z of the engineering parameters are associated with the indices 1, 2 and 3 and the strain-stress equations may also be written in a different way by using the numerical subscripts. Further notice the reciprocal relations

$$E_i \nu_{ji} = E_j \nu_{ij}, \quad i, j = r, \theta, z$$

and the $G_{\theta z}, G_{rz}$ and $G_{r\theta}$ may be written in the more general form E_4, E_5, E_6.

There are two practical situations for a monoclinic material behavior. The first case can be one plane of elastic symmetry $(r - \theta)$ which is rectilinear to the z-axis. The case is interesting when considering composite discs or circular plates. The stress-strain equations follow from (2.1.70) after substituting the subscripts 1,2 and 3 by the engineering parameters to r, θ and z and the shear moduli E_4, E_5 and E_6 by $G_{\theta z}, G_{rz}$ and $G_{r\theta}$. The second case can be one plane of elastic symmetry $(\theta - z)$ as a cylindrical surface with the axis r perpendicular to this surface. This situation is of practical interest when considering e.g. filament wound cylindrical shells and we get the strain-stress relations which couple all three normal strains to the shear strain ε_4 and both shear strains $\varepsilon_5, \varepsilon_6$ to both shear stresses σ_5, σ_6

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} \frac{1}{E_1} & -\frac{\nu_{21}}{E_2} & -\frac{\nu_{31}}{E_3} & \frac{\eta_{41}}{E_4} & 0 & 0 \\ & \frac{1}{E_2} & -\frac{\nu_{32}}{E_3} & \frac{\eta_{42}}{E_4} & 0 & 0 \\ & & \frac{1}{E_3} & \frac{\eta_{43}}{E_4} & 0 & 0 \\ & & & \frac{1}{E_4} & 0 & 0 \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & \frac{1}{E_5} & \frac{\mu_{65}}{E_6} \\ & & & & & \frac{1}{E_6} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{bmatrix} \tag{2.1.84}$$

The subscripts 1, 2, 3 of the Young's moduli and the Poisson's ratios will be shifted to r, θ, z and the moduli E_4, E_5, E_6 to $G_{\theta z}, G_{rz}, G_{r\theta}$. There are as above reciprocal relations for ν_{ij}, η_{ij} and μ_{ij}

$$\frac{\eta_{41}}{E_4} = \frac{\eta_{14}}{E_1}, \qquad \frac{\eta_{42}}{E_4} = \frac{\eta_{24}}{E_2}, \qquad \frac{\eta_{43}}{E_4} = \frac{\eta_{34}}{E_3}, \qquad \frac{\mu_{65}}{E_6} = \frac{\mu_{56}}{E_5}, \\ \frac{\nu_{ij}}{E_i} = \frac{\nu_{ji}}{E_j}, \quad i,j = 1,2,3 \tag{2.1.85}$$

2.1.7 Problems

Exercise 2.1. Calculate for the tensile bar consisting of three parts (Fig. 2.2) the elongation $\triangle l$, the strain ε and the stress σ as functions of A, l and F:

1. The stiffness are arranged in parallel:
   $E_1 = E_3 = 70$ GPa, $E_2 = 3$ GPa, $A_1 = A_3 = 0,1A$, $A_2 = 0,8A$.
2. The stiffness are arranged in series:
   $E_1 = E_3 = 70$ GPa, $E_2 = 3$ GPa, $l_1 = l_3 = 0,1l$, $l_2 = 0,8l$.

Solution 2.1. The solution can be obtained considering the basic assumptions of the iso-strain and the iso-stress models.

1. Assumptions

$$\varepsilon_i = \varepsilon, \quad \triangle l_i = \triangle l, \quad i = 1,2,3, \quad A = \sum_{i=1}^{3} A_i, \quad F = \sum_{i=1}^{3} F_i$$

From $\sigma = E\varepsilon$ follows

$$\sigma_1 = E_1\varepsilon = 70\text{GPa}\ \varepsilon, \sigma_2 = E_2\varepsilon = 3\text{GPa}\ \varepsilon, \sigma_3 = E_3\varepsilon = 70\text{GPa}\ \varepsilon$$

With $F = \sigma A = EA\varepsilon$ and $F_i = E_iA_i\varepsilon$ follows

$$F = \sum_{i=1}^{3}(E_iA_i)\varepsilon = 16,4\ \text{GPa}\ \varepsilon A, \quad E = 16,4\ \text{GPa},$$

$$\varepsilon = \frac{F}{EA} = \frac{\triangle l}{l}$$

yields

$$\triangle l = \frac{Fl}{EA} = \frac{1}{16,4}\frac{Fl}{A}(\text{GPa})^{-1}$$

and the solutions are

$$\triangle l = \frac{1}{16,4}\frac{Fl}{A}(\text{GPa})^{-1} = \triangle l(F,l,A),$$

$$\varepsilon = \frac{\triangle l}{l} = \frac{1}{16,4}\frac{F}{A}(\text{GPa})^{-1} = \varepsilon(F,A),$$

$$\sigma_i = E_i\varepsilon = \frac{1}{16,4}\frac{E_iF}{A}(\text{GPa})^{-1} = \sigma_i(F,A), \quad i = 1,2,3$$

2. Assumptions

$$\triangle l = \sum_{i=1}^{3} \triangle l_i, \quad F_i = F, \quad \varepsilon_i = \frac{\triangle l_i}{l_i}, \quad i = 1,2,3$$

From $\triangle l = \sum_{i=1}^{3} \triangle l_i = \sum_{i=1}^{3} l_i \varepsilon_i$ and $F = EA\varepsilon$ follows

$$\triangle l = \sum_{i=1}^{3} \left(\frac{l_i}{E_i} \right) \frac{F}{A} = \left(\frac{0,1}{70} + \frac{0,8}{3} + \frac{0,1}{70} \right) \frac{Fl}{A} (\text{GPa})^{-1},$$

$$\triangle l = \varepsilon l = \frac{1}{E} \frac{Fl}{A} = 0,2695 \frac{Fl}{A} (\text{GPa})^{-1}, \quad E = 3,71 \text{ GPa}$$

The functions $\triangle l, \varepsilon$ and σ are

$$\triangle l = 0,2695 \frac{Fl}{A} (\text{GPa})^{-1} = \triangle l(F,l,A),$$

$$\varepsilon = 0,2695 \frac{F}{A} (\text{GPa})^{-1} = \varepsilon(F,A),$$

$$\sigma = E\varepsilon = \frac{F}{A} = \sigma(F,A)$$

Exercise 2.2. The relationship between the load F and the elongation $\triangle l$ of a tensile bar (Fig. 2.1) is

$$F = \frac{EA_0}{l_0} \triangle l = K \triangle l$$

E is the Young's modulus of the material, A_0 the cross-sectional area of the bar and l_0 is the length. The factor $K = EA_0/l_0$ is the stiffness per length and characterizes the mechanical performance of the tensile bar. In the case of two different bars with Young's moduli E_1, E_2, densities ρ_1, ρ_2, the cross-sectional areas A_1, A_2 and the lengths l_1, l_2 the ratios of the stiffness K_1 and K_2 per length and the mass of the bars are

$$\frac{K_1}{K_2} = \frac{E_1 A_1}{E_2 A_2} \frac{l_2}{l_1}, \quad \frac{m_1}{m_2} = \frac{l_1 A_1 \rho_1}{l_2 A_2 \rho_2}$$

Verify that for $l_1 = l_2$ and $m_1 = m_2$ the ratio K_1/K_2 only depends on the ratio of the specific Young's moduli E_1/ρ_1 and E_2/ρ_2.

Solution 2.2. Introducing the densities ρ_1/ρ_2 into the stiffness ratio K_1/K_2 yields

$$\frac{K_1}{K_2} = \frac{E_1/\rho_1}{E_2/\rho_2} \frac{m_1}{m_2} \left(\frac{l_2}{l_1} \right)^2$$

and with $m_1 = m_2, l_2 = l_1$

$$\frac{K_1}{K_2} = \frac{E_1/\rho_1}{E_2/\rho_2}$$

Note 2.1. A material with the highest value of E/ρ has the highest tension stiffness.

Exercise 2.3. For a simply supported beam with a single transverse load in the middle of the beam we have the following equation

$$F = 48\frac{EI}{l^3}f = Kf$$

F is the load and f is the deflection in the middle of the beam, l is the length of the beam between the supports and I the moment of inertia of the cross-section. The coefficient $K = 48EI/l^3$ characterizes the stiffness of the beam. Calculate K for beams with a circle or a square cross-sectional area (radius r or square length a) and two different materials E_1, ρ_1 and E_2, ρ_2 but of equal length l, moments of inertia and masses. Verify that for $m_1 = m_2$ the ratio of the stiffness coefficients K_1/K_2 only depends on the ratios E_1/ρ_1^2 and E_2/ρ_2^2.

Solution 2.3. Moments of inertia and masses of the two beams are

1. circle cross-sectional: $I = \pi r^4/4, m = r^2\pi l\rho$,
2. square cross-sectional: $I = a^4/12, m = a^2 l\rho$

In case 1. we have

$$K = \frac{48EI}{l^3} = \frac{48E}{l^3}\frac{\pi r^4}{4} = \frac{48E/\rho^2}{4l^3}\frac{m^2}{\pi l^2},$$

$$\frac{K_1}{K_2} = \frac{E_1/\rho_1^2}{E_2/\rho_2^2}\left(\frac{m_1}{m_2}\right)^2\left(\frac{l_2}{l_1}\right)^5$$

With $l_1 = l_2, m_1 = m_2$ we obtain

$$\frac{K_1}{K_2} = \frac{E_1/\rho_1^2}{E_2/\rho_2^2}$$

In case 2. we have

$$K = \frac{48EI}{l^3} = \frac{48E}{l^3}\frac{a^4}{12} = \frac{4E}{l^3}\frac{m^2}{\rho^2 l^2},$$

$$\frac{K_1}{K_2} = \frac{E_1/\rho_1^2}{E_2/\rho_2^2}\left(\frac{m_1}{m_2}\right)^2\left(\frac{l_2}{l_1}\right)^5$$

With $l_1 = l_2, m_1 = m_2$ we obtain

$$\frac{K_1}{K_2} = \frac{E_1/\rho_1^2}{E_2/\rho_2^2}$$

Note 2.2. The best material for an optimal bending stiffness of the beam is that with the highest value of E/ρ^2.

Exercise 2.4. Formulate explicitly the transformation matrices $(\overset{3}{\boldsymbol{T}}{}^{\boldsymbol{\sigma}})^{-1}$ and $(\overset{3}{\boldsymbol{T}}{}^{\boldsymbol{\varepsilon}})^{-1}$ for a rotation about the $\boldsymbol{e}_3$-direction (Fig. 2.6).

Solution 2.4. With Eqs. (2.1.29), (2.1.39) and (2.1.40) follows $(\overset{3}{\boldsymbol{T}}{}^{\boldsymbol{\sigma}})^{-1} = (\overset{3}{\boldsymbol{T}}{}^{\boldsymbol{\varepsilon}})^{\mathrm{T}}$ and $(\overset{3}{\boldsymbol{T}}{}^{\boldsymbol{\varepsilon}})^{-1} = (\overset{3}{\boldsymbol{T}}{}^{\boldsymbol{\sigma}})^{\mathrm{T}}$

$$(\overset{3}{\boldsymbol{T}}{}^{\boldsymbol{\sigma}})^{-1} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -2cs \\ s^2 & c^2 & 0 & 0 & 0 & 2sc \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ cs & -cs & 0 & 0 & 0 & c^2-s^2 \end{bmatrix},$$

$$(\overset{3}{\boldsymbol{T}}{}^{\boldsymbol{\varepsilon}})^{-1} = \begin{bmatrix} c^2 & s^2 & 0 & 0 & 0 & -cs \\ s^2 & c^2 & 0 & 0 & 0 & sc \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & c & s & 0 \\ 0 & 0 & 0 & -s & c & 0 \\ 2cs & -2cs & 0 & 0 & 0 & c^2-s^2 \end{bmatrix}$$

Exercise 2.5. Consider the coordinate transformation that corresponds with reflection in the plane $x_1 - x_2$: $x'_1 = x_1$, $x'_2 = x_2$, $x'_3 = -x_3$. Define for this case

1. the coordinate transformation matrix $[R_{ij}]$ and
2. the stress and strain transformation matrices $[\overset{\sigma}{T}_{pq}]$ and $[\overset{\varepsilon}{T}_{pq}]$.

Solution 2.5. The solution contains two parts: the coordinate transformation and the stress/strain transformation.

1. With $R_{ij} = \cos(e'_i, e_j)$ (2.1.22) follows
   $R_{11} = 1, R_{12} = 0, R_{13} = 0, R_{21} = 0, R_{22} = 1, R_{23} = 0,$
   $R_{31} = 0, R_{32} = 0, R_{33} = -1$
   and the transformation matrix takes the form
   $$[R_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$
2. With the help of the transformation matrix App. B we can see that both matrices are diagonal with the following nonzero elements for $\overset{\sigma}{T}_{pq}$
   $R_{11}^2 = 1, R_{22}^2 = 1, R_{33}^2 = 1,$
   $R_{22}R_{33} + R_{23}R_{32} = -1, R_{11}R_{33} + R_{13}R_{31} = -1, R_{11}R_{22} + R_{12}R_{21} = 1$
   and for $\overset{\varepsilon}{T}_{pq}$
   $R_{11}^2 = 1, R_{22}^2 = 1, R_{33}^2 = 1,$
   $R_{22}R_{33} + R_{23}R_{22} = -1, R_{11}R_{33} + R_{13}R_{31} = -1, R_{11}R_{22} + R_{12}R_{21} = 1$
   The transformation matrices take the form

$$[\overset{\sigma}{T}_{pq}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = [\overset{\varepsilon}{T}_{pq}]$$

Exercise 2.6. The engineering material parameters for an orthotropic material are given by

$$\begin{aligned} E_1 &= 173\text{GPa}, & E_2 &= 33{,}1\text{GPa}, & E_3 &= 5{,}17\text{GPa}, \\ E_4 &= 3{,}24\text{GPa}, & E_5 &= 8{,}27\text{GPa}, & E_6 &= 9{,}38\text{GPa}, \\ \nu_{12} &= 0{,}036, & \nu_{13} &= 0{,}25, & \nu_{23} &= 0{,}171 \end{aligned}$$

Calculate the stiffness matrix $\boldsymbol{C}$ and the compliance matrix $\boldsymbol{S}$.

Solution 2.6. With Table 2.5 we find the S_{ij} and the C_{ij}

$$\begin{aligned} S_{11} &= E_1^{-1} = 5{,}780\ 10^{-3}\text{GPa}^{-1}, \\ S_{12} &= S_{21} = -\nu_{12}E_1^{-1} = -0{,}208\ 10^{-3}\text{GPa}^{-1}, \\ S_{22} &= E_2^{-1} = 30{,}211\ 10^{-3}\text{GPa}^{-1}, \\ S_{13} &= S_{31} = -\nu_{13}E_1^{-1} = -1{,}445\ 10^{-3}\text{GPa}^{-1}, \\ S_{33} &= E_3^{-1} = 193{,}424\ 10^{-3}\text{GPa}^{-1}, \\ S_{23} &= S_{32} = -\nu_{23}E_2^{-1} = -5{,}166\ 10^{-3}\text{GPa}^{-1}, \\ S_{44} &= E_4^{-1} = 308{,}642\ 10^{-3}\text{GPa}^{-1}, \\ S_{55} &= E_5^{-1} = 120{,}919\ 10^{-3}\text{GPa}^{-1}, \\ S_{66} &= E_6^{-1} = 106{,}610\ 10^{-3}\text{GPa}^{-1}, \end{aligned}$$

$$\triangle = 1 - \nu_{12}\nu_{21} - \nu_{23}\nu_{32} - \nu_{31}\nu_{13} - 2\nu_{21}\nu_{13}\nu_{32},$$

$$\begin{aligned} \nu_{21} &= \nu_{12}(E_2/E_1) = 0{,}0069, & \nu_{31} &= \nu_{13}(E_3/E_1) = 0{,}0075, \\ \nu_{32} &= \nu_{23}(E_3/E_2) = 0{,}027, & \triangle &= 0{,}993, \quad \bar{E}_i = E_i/\triangle, i = 1,2,3 \end{aligned}$$

$$\begin{aligned} C_{11} &= (1 - \nu_{23}\nu_{32})\bar{E}_1 = 173{,}415\text{GPa}, \\ C_{22} &= (1 - \nu_{31}\nu_{13})\bar{E}_2 = 33{,}271\text{GPa}, \\ C_{33} &= (1 - \nu_{12}\nu_{21})\bar{E}_3 = 5{,}205\text{GPa}, \\ C_{12} &= (\nu_{12} + \nu_{13}\nu_{32})\bar{E}_2 = 1{,}425\text{GPa}, \\ C_{13} &= (\nu_{13} + \nu_{12}\nu_{23})\bar{E}_3 = 1{,}334\text{GPa}, \\ C_{23} &= (\nu_{23} + \nu_{21}\nu_{13})\bar{E}_3 = 0{,}899\text{GPa}, \end{aligned}$$

$$C_{44} = E_4, \quad C_{55} = E_5, \quad C_{66} = E_6$$

With the values for C_{ij} and S_{ij} the stiffness matrix $\boldsymbol{C}$ and the compliance matrix $\boldsymbol{S}$ can be written

$$
\boldsymbol{C} = \begin{bmatrix} 173,415 & 1,425 & 1,334 & 0 & 0 & 0 \\ 1,425 & 33,271 & 0,899 & 0 & 0 & 0 \\ 1,334 & 0,899 & 5,205 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3,24 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8,27 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9,38 \end{bmatrix} \mathrm{GPa},
$$

$$
\boldsymbol{S} = \begin{bmatrix} 5,780 & -0,208 & -1,445 & 0 & 0 & 0 \\ -0,208 & 30,211 & -5,166 & 0 & 0 & 0 \\ -1,445 & -5,166 & 193,424 & 0 & 0 & 0 \\ 0 & 0 & 0 & 308,642 & 0 & 0 \\ 0 & 0 & 0 & 0 & 120,919 & 0 \\ 0 & 0 & 0 & 0 & 0 & 106,610 \end{bmatrix} 10^{-3}\mathrm{GPa}^{-1}
$$

2.2 Fundamental Equations and Variational Solution Procedures

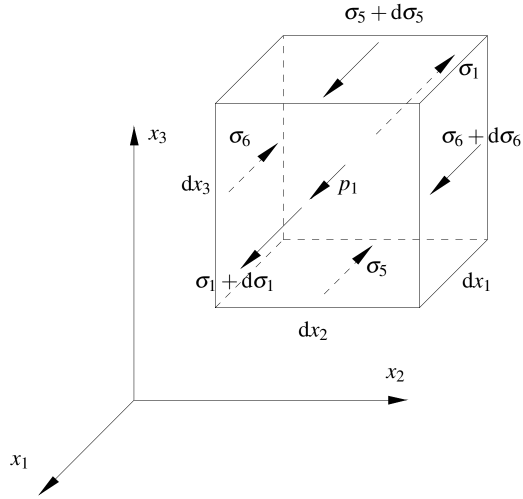
Below we discuss at first the fundamental equations of the anisotropic elasticity for rectilinear coordinates. The system of equations can be divided into two subsystems: the first one is material independent that means we have the same equations as in the isotropic case. To this subsystem belong the equilibrium equations (static or dynamic) and the kinematic equations (the strain-displacement equations and the compatibility conditions). To this subsystem one has to add the constitutive equations. In addition, we must introduce the boundary and, may be, the initial conditions to close the initial-boundary problem. At second, considering that closed solutions are impossible in most of the practical cases approximative solution techniques are briefly discussed. The main attention will be focussed on variational formulations.

2.2.1 Boundary and Initial-Boundary Value Equations

The fundamental equations of anisotropic elasticity can be formulated and solved by a displacement, a stress or a mixedapproach. In all cases the starting point are the following equations:

- The static or dynamic equilibrium equations formulated for an infinitesimal cube of the anisotropic solid which is subjected to body forces and surface forces characterized by force density per unit surface. In Fig. 2.11 the stress and the volume force components are shown in the x_1-direction. Assuming the symmetry of the stress tensor, three static equations link six unknown stress components. In the case of dynamic problems the inertia forces are expressed through displacements, therefore the equations of motion contain both, six unknown stress and three unknown displacement components

Fig. 2.11 Infinitesimal cube with lengths $\mathrm{d}x_1, \mathrm{d}x_2, \mathrm{d}x_3$: stress and volume force components in x_1-direction



$$\begin{aligned}
&\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \sigma_6}{\partial x_2} + \frac{\partial \sigma_5}{\partial x_3} + p_1 = 0,\\
&\frac{\partial \sigma_6}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \sigma_4}{\partial x_3} + p_2 = 0, \qquad \text{static equations} \qquad (2.2.1)\\
&\frac{\partial \sigma_5}{\partial x_1} + \frac{\partial \sigma_4}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + p_3 = 0,
\end{aligned}$$

$$\begin{aligned}
&\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \sigma_6}{\partial x_2} + \frac{\partial \sigma_5}{\partial x_3} + p_1 = \rho \frac{\partial^2 u_1}{\partial t^2},\\
&\frac{\partial \sigma_6}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} + \frac{\partial \sigma_4}{\partial x_3} + p_2 = \rho \frac{\partial^2 u_2}{\partial t^2}, \qquad \text{dynamic equations} \qquad (2.2.2)\\
&\frac{\partial \sigma_5}{\partial x_1} + \frac{\partial \sigma_4}{\partial x_2} + \frac{\partial \sigma_3}{\partial x_3} + p_3 = \rho \frac{\partial^2 u_3}{\partial t^2}
\end{aligned}$$

The inertial terms in (2.2.2) are dynamic body forces per unit volume. The density ρ for unidirectional laminae can be calculated e.g. using the rule of mixtures (Sect. 3.1.1).

- The kinematic equations that are six strain-displacement relations and six compatibility conditions for strains. For linear small deformation theory, the six stress-displacement equations couple six unknown strains and three unknown displacements. Figure 2.12 shows the strains of an infinitesimal cube in the $(x_1 - x_2)$-plane and we find the relations

$$\varepsilon_1 \equiv \frac{\partial u_1}{\partial x_1}, \varepsilon_2 \equiv \frac{\partial u_2}{\partial x_2}, \alpha = \frac{\partial u_2}{\partial x_1}, \beta = \frac{\partial u_1}{\partial x_2} \Rightarrow 2\varepsilon_{12} \equiv \gamma_{12} \equiv \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \equiv \varepsilon_6$$

and analogous relations for the $(x_2 - x_3)$- and $(x_1 - x_3)$-planes yield

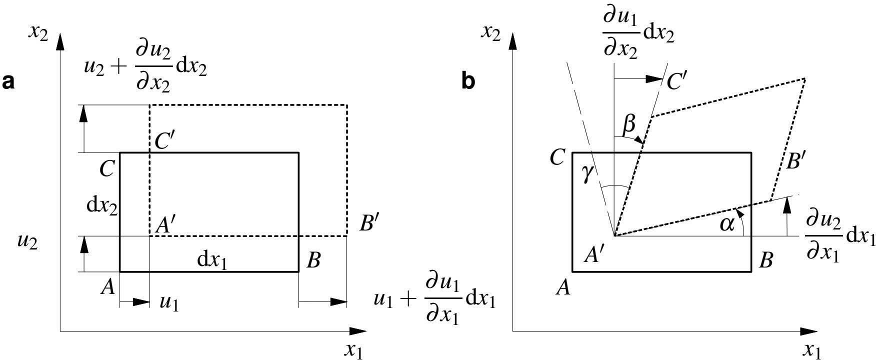


Fig. 2.12 Strains of the infinitesimal cube shown for the $(x_1 - x_2)$-plane. **a** extensional strains, **b** shear strains

$$
\begin{aligned}
\varepsilon_1 &= \frac{\partial u_1}{\partial x_1}, & \varepsilon_2 &= \frac{\partial u_2}{\partial x_2}, & \varepsilon_3 &= \frac{\partial u_3}{\partial x_3}, \\
\varepsilon_4 &= \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}, & \varepsilon_5 &= \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}, & \varepsilon_6 &= \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}
\end{aligned}
\tag{2.2.3}
$$

The displacement field of the body corresponding to a given deformation state is unique, the components of the strain tensor must satisfy the following six compatibility conditions

$$
\begin{aligned}
&\frac{\partial^2 \varepsilon_1}{\partial x_2^2} + \frac{\partial^2 \varepsilon_2}{\partial x_1^2} = \frac{\partial^2 \varepsilon_6}{\partial x_1 \partial x_2}, \quad \frac{\partial}{\partial x_3}\left(\frac{\partial \varepsilon_4}{\partial x_1} + \frac{\partial \varepsilon_5}{\partial x_2} - \frac{\partial \varepsilon_6}{\partial x_3}\right) = 2\frac{\partial^2 \varepsilon_3}{\partial x_1 \partial x_2}, \\
&\frac{\partial^2 \varepsilon_2}{\partial x_3^2} + \frac{\partial^2 \varepsilon_3}{\partial x_2^2} = \frac{\partial^2 \varepsilon_4}{\partial x_2 \partial x_3}, \quad \frac{\partial}{\partial x_1}\left(\frac{\partial \varepsilon_5}{\partial x_2} + \frac{\partial \varepsilon_6}{\partial x_3} - \frac{\partial \varepsilon_4}{\partial x_1}\right) = 2\frac{\partial^2 \varepsilon_1}{\partial x_2 \partial x_3}, \\
&\frac{\partial^2 \varepsilon_3}{\partial x_1^2} + \frac{\partial^2 \varepsilon_1}{\partial x_3^2} = \frac{\partial^2 \varepsilon_5}{\partial x_1 \partial x_3}, \quad \frac{\partial}{\partial x_2}\left(\frac{\partial \varepsilon_6}{\partial x_3} + \frac{\partial \varepsilon_4}{\partial x_1} - \frac{\partial \varepsilon_5}{\partial x_2}\right) = 2\frac{\partial^2 \varepsilon_2}{\partial x_3 \partial x_1}
\end{aligned}
$$

In the two-dimensional case the compatibility conditions reduce to a single equation

$$
\frac{\partial^2 \varepsilon_1}{\partial x_2^2} + \frac{\partial^2 \varepsilon_2}{\partial x_1^2} - 2\frac{\partial^2 \varepsilon_6}{\partial x_1 \partial x_2} = 0
$$

- The material or constitutive equations which are described in Sect. 2.1 are

$$
\varepsilon_i = S_{ij}\sigma_j, \qquad \sigma_i = C_{ij}\varepsilon_j \tag{2.2.4}
$$

The generalized Hooke's law yields six equations relating in each case six unknown stress and strain components. The elements of the stiffness matrix $\boldsymbol{C}$ and the compliance matrix $\boldsymbol{S}$ are substituted with respect to the symmetry conditions of the material.

Summarizing all equations, we have 15 independent equations for 15 unknown components of stresses, strains and displacements. In the displacement approach, the stresses and strains are eliminated and a system of three coupled partial differential equations for the displacement components are left.

In the static case we have a boundary-value problem, and we have to include boundary conditions. In the dynamic case the system of partial differential equations defines an initial-boundary-value problem and we have additional initial conditions.

A clear symbolic formulation of the fundamental equations in displacements is given in vector-matrix notation. With the transposed vectors $\boldsymbol{\sigma}^{\mathrm{T}}, \boldsymbol{\varepsilon}^{\mathrm{T}}$ and $\boldsymbol{u}^{\mathrm{T}}$

$$\boldsymbol{\sigma}^{\mathrm{T}} = [\sigma_1\ \sigma_2\ \sigma_3\ \sigma_4\ \sigma_5\ \sigma_6], \quad \boldsymbol{\varepsilon}^{\mathrm{T}} = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3\ \varepsilon_4\ \varepsilon_5\ \varepsilon_6], \quad \boldsymbol{u}^{\mathrm{T}} = [u_1\ u_2\ u_3] \qquad (2.2.5)$$

the transformation and the differential matrices $\boldsymbol{T}$ and $\boldsymbol{D}$ ($\boldsymbol{n}$ is the surface normal unit vector)

$$\boldsymbol{T} = \begin{bmatrix} n_1 & 0 & 0 \\ 0 & n_2 & 0 \\ 0 & 0 & n_3 \\ 0 & n_3 & n_2 \\ n_3 & 0 & n_1 \\ n_2 & n_1 & 0 \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} \partial_1 & 0 & 0 \\ 0 & \partial_2 & 0 \\ 0 & 0 & \partial_3 \\ 0 & \partial_3 & \partial_2 \\ \partial_3 & 0 & \partial_1 \\ \partial_2 & \partial_1 & 0 \end{bmatrix}, \quad \begin{array}{l} n_i = \cos(\boldsymbol{n}, x_i) \\ \partial_i = \dfrac{\partial}{\partial x_i} \\ i = 1,2,3 \end{array} \qquad (2.2.6)$$

and the stiffness matrix $\boldsymbol{C}$ we get

$$\begin{array}{llll} \boldsymbol{D}^{\mathrm{T}}\boldsymbol{\sigma} + \boldsymbol{p} & = \boldsymbol{0} & \in V & \text{static equilibrium equations,} \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{\sigma} + \boldsymbol{p} & = \rho \dfrac{\partial^2 \boldsymbol{u}}{\partial t^2} & \in V & \text{dynamic equilibrium equations,} \\ \boldsymbol{T}^{\mathrm{T}}\boldsymbol{\sigma} & = \boldsymbol{q} & \in A_{\mathrm{q}} & \text{surface equilibrium equations,} \\ \boldsymbol{\varepsilon} & = \boldsymbol{D}\boldsymbol{u} & \in V & \text{kinematic equations,} \\ \boldsymbol{\sigma} & = \boldsymbol{C}\boldsymbol{\varepsilon} & & \text{constitutive equations} \end{array} \qquad (2.2.7)$$

V is the volume and A_{q} the surface of the body with surface forces $\boldsymbol{q}$.

Eliminating the stresses and the strains leads to the differential equations for the displacements

$$\begin{array}{llll} \multicolumn{4}{l}{\text{Boundary-value problem - elastostatics}} \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{D}\boldsymbol{u} & = -\boldsymbol{p} & \in V & \text{equilibrium for the volume element } V, \\ \boldsymbol{u} & = \bar{\boldsymbol{u}} & \in A_u & \text{prescribed displacements } \bar{\boldsymbol{u}} \text{ on } A_u, \\ \boldsymbol{T}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{D}\boldsymbol{u} & = \bar{\boldsymbol{q}} & \in A_{\mathrm{q}} & \text{prescribed surface forces } \bar{\boldsymbol{q}} \text{ on } A_{\mathrm{q}} \end{array} \qquad (2.2.8)$$

$$\begin{array}{ll} \multicolumn{2}{l}{\text{Initial boundary-value problem - elastodynamics}} \\ \boldsymbol{D}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{D}\boldsymbol{u} - \rho\ddot{\boldsymbol{u}} = -\boldsymbol{p} \in V & \text{equilibrium equation,} \\ \boldsymbol{u} = \bar{\boldsymbol{u}} \in A_u, \quad \boldsymbol{T}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{D}\boldsymbol{u} = \bar{\boldsymbol{q}} \in A_{\mathrm{q}} & \text{boundary conditions,} \\ \boldsymbol{u}(\boldsymbol{x},0) = \bar{\boldsymbol{u}}(\boldsymbol{x},0), \quad \dot{\boldsymbol{u}}(\boldsymbol{x},0) = \dot{\bar{\boldsymbol{u}}}(\boldsymbol{x},0) & \text{initial conditions} \end{array} \qquad (2.2.9)$$

In the general case of material anisotropic behavior the three-dimensional equations are very complicated and analytical solutions are only possible for some special problems. This is independent of the approach to displacements or stresses. Some elementary examples are formulated in Sect. 2.2.4. The equations for beams and plates are simplified with additional kinematic and/or static hypotheses and the equations are deduced separately in Chaps. 7 and 8. The simplified structural equations for circular cylindrical shells and thin-walled folded structures are given in Chaps. 9 and 10.

Summarizing the fundamental equations of elasticity we have introduced stresses and displacements as static and kinematic field variables. A field is said to be statically admissible if the stresses satisfy equilibrium equations (2.2.1) and are in equilibrium with the surface traction $\bar{\boldsymbol{q}}$ on the body surface A_{q}, where the traction are given. A field is referred to as kinematically admissible if displacements and strains are restricted by the strain-displacement equations (2.2.3) and the displacement satisfies kinematic boundary conditions on the body surface A_u, where the displacements are prescribed. Admissible field variables are considered in principles of virtual work and energy formulations, Sect. 2.2.2. The mutual correspondence between static and kinematic field variables is established through the constitutive equations (2.2.4).

2.2.2 Principle of Virtual Work and Energy Formulations

The analytical description of the model equations of anisotropic elasticity may realized as above by a system of partial differential equations but also by integral statements which are equivalent to the governing equations of Sect. 2.2.1 and based on energy or variational formulations. The utility of variational formulations is in general twofold. They yield convenient methods for the derivation of the governing equations of problems in applied elasticity and provide the mathematical basis for consistent approximate theories and solution procedures. There are three variational principles which are used mostly in structural mechanics. There are the principles of virtual work, the principle of complementary virtual work, Reissner's[6] variational theorem and the related energy principles.

Restricting ourselves to static problems, extremal principles formulated for the total elastic potential energy of the problem or the complementary potential energy are very useful in the theory of elasticity and in modelling and analysis of structural elements. The fundamental equations and boundary conditions given beforehand can be derived with the extremal principles and approximate solutions are obtained by direct variational methods. Both extremal principles follow from the principle of virtual work.

If an elastic body is in equilibrium, the virtual work δW of all acting forces in moving through a virtual displacement $\delta\boldsymbol{u}$ is zero

$$\delta W \equiv \delta W_{\mathrm{a}} + \delta W_{\mathrm{i}} = 0 \qquad (2.2.10)$$

[6] Eric (Max Erich) Reissner (* January 5th, 1913 in Aachen; † November 1st, 1996 La Jolla, USA) - engineer, contributions to the theory of beams, plates and shells

δW is the total virtual work, δW_{a} the external virtual work of body or volume and surface forces and δW_{i} the internal virtual work of internal stresses, for the forces associated with the stress field of a body move the body points through virtual displacements $\delta\boldsymbol{u}$ corresponding to the virtual strain field $\delta\boldsymbol{\varepsilon}$.

A displacement is called virtual, if it is infinitesimal, and satisfies the geometric constraints (compatibility with the displacement-strain equations and the boundary conditions) and all forces are fixed at their values. These displacements are called virtual because they are assumed to be infinitesimal while time is held constant. The symbol δ is called a variational operator and in the mathematical view a virtual displacement is a variation of the displacement function. To use variational operations in structural mechanics only the following operations of the δ-operator are needed

$$\delta\left(\frac{\mathrm{d}^n f}{\mathrm{d}x^n}\right) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(\delta f), \quad \delta(f^n) = nf^{n-1}\delta f^{n-1}, \quad \delta\int f\mathrm{d}x = \int \delta f\mathrm{d}x$$

For a deformable body, the external and the internal work are given in Eqs. (2.2.11) and (2.2.12), respectively,

$$\delta W_{\mathrm{a}} = \int\limits_V p_k\delta u_k\mathrm{d}V + \int\limits_{A_{\mathrm{q}}} q_k\delta u_k\mathrm{d}A, \tag{2.2.11}$$

$$\delta W_{\mathrm{i}} = -\int\limits_V \sigma_k\delta\varepsilon_k\mathrm{d}V \tag{2.2.12}$$

p_k are the components of the actual body force vector $\boldsymbol{p}$ per unit volume and q_k the components of the actual surface force vector $\boldsymbol{q}$ (surface traction per unit area). A_{q} denotes the portion of the boundary on which surface forces are specified. σ_k and ε_k are the components of the stress and the strain vector. The negative sign in (2.2.12) indicates that the inner forces oppose the inner virtual displacements, e.g. if the virtual displacement $\delta u_1 = \delta\varepsilon_1\mathrm{d}x_1$ is subjected an inner force $(\sigma_1\mathrm{d}x_2\mathrm{d}x_3)$ the inner work is $(-\sigma_1\mathrm{d}x_2\mathrm{d}x_3)\delta\varepsilon_1\mathrm{d}x_1$. The vectors $\boldsymbol{p},\boldsymbol{q}$ and $\boldsymbol{u}$ have three components but the vectors $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ have six components. The double subscript k in $p_k\delta u_k$ and $q_k\delta u_k$ means the summarizing on 1 to 3 but in $\sigma_k\delta\varepsilon_k$ on 1 to 6.

The general formulation of the principle of virtual work for a deformable body

$$\delta W_{\mathrm{a}} + \delta W_{\mathrm{i}} \equiv \delta W = 0$$

or

$$\int\limits_V p_k\delta u_k\mathrm{d}V + \int\limits_{A_{\mathrm{q}}} q_k\delta u_k\mathrm{d}A - \int\limits_V \sigma_k\delta\varepsilon_k\mathrm{d}V = 0 \tag{2.2.13}$$

is independent of the constitutive equations. For the three-dimensional boundary-value problem of a deformable body the principle can be formulated as follow:

Theorem 2.1 (Principle of virtual work). *The sum of virtual work done by the internal and external forces in arbitrary virtual displacements satisfying the pre-*

scribed geometrical constraints and the strain-displacement relations is zero, i.e. the arbitrary field variables δu_k are kinematically admissible.

An important case is restricted to linear elastic anisotropic bodies and is known as the principle of minimum total potential energy. The external virtual work δW_{a} is stored as virtual strain energy $\delta W_{\mathrm{f}} = -\delta W_{\mathrm{i}}$, i.e. there exists a strain energy density function

$$W_{\mathrm{f}}(\boldsymbol{\varepsilon}) = \frac{1}{2}\sigma_k \varepsilon_k = \frac{1}{2} C_{kl} \varepsilon_k \varepsilon_l$$

Assuming conservative elasto-static problems, the principle of virtual work takes the form

$$\delta \Pi(\boldsymbol{u}) \equiv \delta \Pi_{\mathrm{a}}(\boldsymbol{u}) + \delta \Pi_{\mathrm{i}}(\boldsymbol{\varepsilon}) \equiv 0, \qquad \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u}) \tag{2.2.14}$$

with the total potential energy function $\Pi(\boldsymbol{u})$ of the elastic body. $\Pi_{\mathrm{a}}(\boldsymbol{u})$ and $\Pi_{\mathrm{i}}(\boldsymbol{\varepsilon})$ are the potential functions of the external and the internal forces, respectively,

$$\begin{aligned} \Pi_{\mathrm{i}} &= \Pi(\varepsilon_k) = \frac{1}{2} \int\limits_V C_{kl} \varepsilon_k \varepsilon_l \mathrm{d}V, \\ \Pi_{\mathrm{a}} &= \Pi_{\mathrm{a}}(u_k) = - \int\limits_V p_k u_k \mathrm{d}V - \int\limits_{A_{\mathrm{q}}} q_k u_k \mathrm{d}A \end{aligned} \tag{2.2.15}$$

The principle of minimum total potential energy may be stated for linear elastic bodies with the constraints $\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{u})$ as follows:

Theorem 2.2 (Principle of minimum of the total potential energy). *Of all the admissible displacement functions satisfying strain-stress relations and the prescribed boundary conditions, those that satisfy the equilibrium equations make the total potential energy an absolute minimum.*

The Euler[7]-Lagrange[8] equations of the variational problem yield the equilibrium and mechanical boundary conditions of the problem. The minimum total potential energy is widely used in solutions to problems of structural mechanics.

The principle of virtual work can be formulated in a complementary statement. Then virtual forces are introduced and the displacements are fixed. In analogy to (2.2.13) we have the principle of complementary virtual work as

$$\delta W_{\mathrm{a}}^* + \delta W_{\mathrm{i}}^* \equiv \delta W^* = 0$$

with the complimentary external and internal virtual works

$$\delta W_{\mathrm{a}}^* = \int\limits_{A_u} u_k \delta q_k \mathrm{d}A, \delta W_{\mathrm{i}}^* = - \int\limits_V \varepsilon_k \delta \sigma_k \mathrm{d}V \tag{2.2.16}$$

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[7] Leonhard Euler (*15 April 1707 Basel - †7$^{\mathrm{jul.}}$/18$^{\mathrm{greg.}}$ September 1783 St. Petersburg) - mathematician, physicist, astronomer, logician and engineer, introducing beam theory elements and the equations of motion

[8] Joseph-Louis Lagrange, born Giuseppe Lodovico Lagrangia or Giuseppe Ludovico De la Grange Tournier, also reported as Giuseppe Luigi Lagrange or Lagrangia (*25 January 1736 Turin - †10 April 1813 Paris) - mathematician and astronomer, variational calculations, general mechanics

A_u denotes the portion of the boundary surface on which displacements are specified.

With the complementary stress energy density function

$$W_{\mathrm{f}}^{*}(\boldsymbol{\sigma}) = \frac{1}{2}S_{kl}\sigma_k\sigma_l, \quad \delta W_{\mathrm{f}}^{*}(\boldsymbol{\sigma}) = S_{kl}\sigma_l\delta\sigma_k = \varepsilon_k\delta\sigma_k$$

and assuming conservative elasto-static problems, the principle of complementary work can be formulated as principle of minimum total complementary energy

$$\delta\Pi^{*} = \delta\Pi_{\mathrm{i}}^{*} + \delta\Pi_{\mathrm{a}}^{*}$$

or

$$\delta\left(\int\limits_V W^{*}(\sigma_k)\mathrm{d}V - \int\limits_{A_u} u_k q_k \mathrm{d}A\right) = 0 \qquad (2.2.17)$$

The principle of minimum total complementary energy may be stated for linear elastic bodies with constraints $\boldsymbol{\varepsilon} = \boldsymbol{S\sigma}$ as follows:

Theorem 2.3 (Principle of minimum total complementary energy). *Of all admissible stress states satisfying equilibrium equations and stress boundary conditions, those which are kinematically admissible make the total complementary energy an absolute minimum.*

The Euler-Lagrange equations of the variational statement yield now the compatibility equations and the geometrical boundary conditions.

The both well-known principles of structure mechanics, the principle of virtual displacements (displacement method) and the theorem of Castigliano (principle of virtual forces, force method) correspond to the principle of minimum potential energy and complementary energy. The principle of minimum potential energy is much more used in solution procedures, because it is usually far easier to formulate assumptions about functions to represent admissible displacements as to formulate admissible stress functions that ensure stresses satisfying mechanical boundary conditions and equilibrium equations. It should be kept in mind that from the two principles considered above no approximate theory can be obtained in its entirety. One must either satisfy the strain-displacement relations and the displacement boundary conditions exactly and get approximate equilibrium conditions or vice versa. Both principles yield the risk to formulate approximate theories or solution procedures which may be mathematically inconsistent. Reissner's variational statement yields as Euler-Lagrange equations both, the equilibrium equations and the strain-displacement relations, and has the advantage that its use would yield approximate theories and solution procedures which satisfy both requirements to the same degree and would be consistent. Reissner's variational theorem (Reissner, 1950) can be formulated as follows:

Theorem 2.4 (Reissner's variational theorem, 1950). *Of all sets of stress and displacement functions of an elastic body* $\boldsymbol{\varepsilon} = \boldsymbol{C\sigma}$ *which satisfy the boundary* *conditions, those which also satisfy the equilibrium equations and the stress-displacements relations correspond to a minimum of the functional* Ψ_{R} *defined as*

$$\Psi_{\mathrm{R}}(\boldsymbol{u},\boldsymbol{\sigma}) = \int\limits_V [\sigma_k\varepsilon_k - W_{\mathrm{f}}(\sigma_k)]\mathrm{d}V - \int\limits_V p_k u_k \mathrm{d}V - \int\limits_{A_{\mathrm{q}}} q_k u_k \mathrm{d}A \tag{2.2.18}$$

$W_{\mathrm{f}}(\sigma_k)$ *is the strain energy density function in terms of stresses only,* Ψ_{R} *is Reissner's functional.*

It should be noted that all stress and strain components must be varied while p_k and q_k are prescribed functions and therefore fixed. The variation of the functional $\Psi_{\mathrm{R}}(\boldsymbol{u},\boldsymbol{\sigma})$ yields

$$\begin{aligned}\delta\Psi_{\mathrm{R}} = &\int\limits_V \left[\sigma_k\delta\varepsilon_k + \varepsilon_k\delta\sigma_k - \frac{\partial W_{\mathrm{f}}}{\partial\sigma_k}\delta\sigma_k\right]\mathrm{d}V - \int\limits_V p_k\delta u_k\mathrm{d}V\\ &- \int\limits_{A_{\mathrm{q}}} q_k\delta u_k\mathrm{d}A,\end{aligned} \tag{2.2.19}$$

where ε_k is determined by (2.2.3). $\delta\Psi_{\mathrm{R}}(\boldsymbol{u},\boldsymbol{\sigma})$ can be rearranged and we obtain finally

$$\delta\Psi_{\mathrm{R}} = \int\limits_V \left\{\left[\boldsymbol{\varepsilon} - \frac{\partial W_{\mathrm{f}}}{\partial\boldsymbol{\sigma}}\right]\delta\boldsymbol{\sigma}^{\mathrm{T}} - \left[\boldsymbol{D}^{\mathrm{T}}\boldsymbol{\sigma} + \boldsymbol{p}\right]\delta\boldsymbol{u}^{\mathrm{T}}\right\}\mathrm{d}V - \int\limits_{A_{\mathrm{q}}} \boldsymbol{q}\delta\boldsymbol{u}^{\mathrm{T}}\mathrm{d}A \tag{2.2.20}$$

Since $\delta\boldsymbol{\sigma}$ and $\delta\boldsymbol{u}$ are arbitrary variations $\delta\Psi_{\mathrm{R}} = 0$ is satisfied only if

$$\varepsilon_{ij} = \frac{\partial W_{\mathrm{f}}(\sigma_{ij})}{\partial\sigma_{ij}}, \quad \frac{\partial\sigma_{ij}}{\partial x_j} + p_i = 0 \tag{2.2.21}$$

Summarizing we have considered two dual energy principles with u_k or σ_k as admissible functions which have to be varied and one generalized variational principle, where both, u_k and σ_k, have to be varied. The considerations are limited to linear problems of elasto-statics, i.e. the generalized Hooke's law describes the stress-strain relations.

Expanding the considerations on dynamic problems without dissipative forces following from external or inner damping effects the total virtual work has in the sense of the d'Alambert principle an additional term which represents the inertial forces

$$\delta W = -\int\limits_V \rho\ddot{u}_k\delta u_k\mathrm{d}V - \int\limits_V \sigma_k\delta\varepsilon_k\mathrm{d}V + \int\limits_V p_k\delta u_k dV + \int\limits_{A_{\mathrm{q}}} q_k\delta u_k\mathrm{d}A \tag{2.2.22}$$

Equation (2.2.22) represents an extension of the principle of virtual work from statics to dynamics. ρ is the density of the elastic body.

For conservative systems of elasto-dynamics, the Hamilton[9] principle replaces the principle of the minimum of the total potential energy

$$\delta \int_{t_1}^{t_2} (T - \Pi)\mathrm{d}t \equiv \delta \int_{t_1}^{t_2} L(u_k)\mathrm{d}t = 0, \quad T = \frac{1}{2}\int_V \rho \dot{u}_k \dot{u}_k \mathrm{d}V \tag{2.2.23}$$

$\Pi(\boldsymbol{u})$ is the potential energy given beforehand and $T(\boldsymbol{u})$ is the so-called kinetic energy. $L = T - \Pi$ is the Lagrangian function.

Theorem 2.5 (Hamilton's principle for conservative systems). *Of all possible paths between two points at time interval t_1 and t_2 along which a dynamical system may move, the actual path followed by the system is the one which minimizes the integral of the Lagrangian function.*

In the contracted vector-matrix notation we can summarize:
Conservative elasto-static problems

$$\begin{aligned} \Pi(\boldsymbol{u}) &= \frac{1}{2}\int_V \boldsymbol{\sigma}\boldsymbol{\varepsilon}^{\mathrm{T}}\mathrm{d}V - \int_V \boldsymbol{p}\boldsymbol{u}^{\mathrm{T}}\mathrm{d}V - \int_{A_{\mathrm{q}}} \boldsymbol{q}\boldsymbol{u}^{\mathrm{T}}\mathrm{d}A, \\ \delta\Pi &= \int_V \boldsymbol{\sigma}\delta\boldsymbol{\varepsilon}^{\mathrm{T}}\mathrm{d}V - \int_V \boldsymbol{p}\delta\boldsymbol{u}^{\mathrm{T}}\mathrm{d}V - \int_{A_{\mathrm{q}}} \boldsymbol{q}\delta u^{\mathrm{T}}\mathrm{d}A = 0 \end{aligned} \tag{2.2.24}$$

Conservative elasto-dynamic problems

$$\begin{aligned} &L(\boldsymbol{u}) = T(\boldsymbol{u}) - \Pi(\boldsymbol{u}), \quad T(\boldsymbol{u}) = \frac{1}{2}\int_V \rho\dot{\boldsymbol{u}}^{\mathrm{T}}\dot{\boldsymbol{u}}\mathrm{d}V, \\ &\delta\int_{t_1}^{t_2} L(\boldsymbol{u})\mathrm{d}t = 0 \end{aligned} \tag{2.2.25}$$

All variations are related to the displacement vector $\boldsymbol{u}$. For the stress and the strain vector we have to take into consideration that $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{\varepsilon}) = \boldsymbol{\sigma}[\boldsymbol{\varepsilon}(\boldsymbol{u})]$ and for the time integrations

$$\delta\boldsymbol{u}(\boldsymbol{x}, t_1) = \delta\boldsymbol{u}(\boldsymbol{x}, t_2) = 0 \tag{2.2.26}$$

For non-conservative systems of elastodynamics, the virtual work δW includes an approximate damping term

$$-\int_V \mu\dot{\boldsymbol{u}}^{\mathrm{T}}\delta\boldsymbol{u}\mathrm{d}V$$

with μ as a damping parameter and Eq. (2.2.22) is substituted by

[9] William Rowan Hamilton (*4 August 1805 Dublin - †2 September 1865 Dublin) - physicist, astronomer, and mathematician, who made important contributions to classical mechanics, optics, and algebra

$$\delta W = -\int\limits_V (\rho \ddot{\boldsymbol{u}}^{\mathrm{T}} + \mu \dot{\boldsymbol{u}}^{\mathrm{T}}) \delta \boldsymbol{u} \mathrm{d}V - \int\limits_V \delta(\boldsymbol{\sigma}^{\mathrm{T}} \boldsymbol{\varepsilon}) \mathrm{d}V - \int\limits_V \boldsymbol{p} \delta \boldsymbol{u} \mathrm{d}V \\ - \int\limits_{A_{\mathrm{q}}} \boldsymbol{q} \delta \boldsymbol{u} \mathrm{d}A \tag{2.2.27}$$

A generalized Hamilton's principle in conjunction with the Reissner's variational statement can be presented as

$$\delta \chi(\boldsymbol{u}, \boldsymbol{\sigma}) = \delta \int\limits_{t_1}^{t_2} [T(\boldsymbol{u}) - \psi_{\mathrm{R}}(\boldsymbol{u}, \boldsymbol{\sigma})] \mathrm{d}t = 0,$$

where $T(\boldsymbol{u})$ is the kinetic energy as above, $\psi_{\mathrm{R}}(\boldsymbol{u}, \boldsymbol{\sigma})$ the Reissner's functional (2.2.18).

2.2.3 Variational Methods

The variational principles can be used to obtain, in a mathematical way, the governing differential equations and associated boundary conditions as the Euler-Lagrange equations of the variational statement. Now we consider the use of the variational principles in the solution of the model equations. We seek in the sense of the classical variational methods, approximate solutions by direct methods, i.e. the approximate solution is obtained directly by applying the same variational statement that are used to derive the fundamental equations.

2.2.3.1 Rayleigh-Ritz Method

Approximate methods are used when exact solutions to a problem cannot be derived. Among the approximation methods, Ritz[10] method is a very convenient method based on a variational approach. The variational methods of approximation described in this textbook are limited to Rayleigh[11]-Ritz method for elasto-statics and elasto-dynamics problems of anisotropic elasticity theory and to some extent on weighted-residual methods.

The Rayleigh-Ritz method is based on variational statements, e.g. the principle of minimum total potential energy, which is equivalent to the fundamental differential equations as well as to the so-called natural or static boundary conditions including

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[10] Walter Ritz (* February 22nd, 1878, Sion, Switzerland; † July 7th, 1909, Göttingen) - theoretical physicist, variational methods

[11] John William Strutt, 3rd Baron Rayleigh (* November 12th, 1842, Langford Grove, Maldon, Essex, UK; † June 30th, 1919, Terling Place, Witham, Essex, UK) - physicist, Nobel Prize in Physics winner (1904), variational method

force boundary conditions . This variational formulation is known as the weak form of the model equations. The method was proposed as the direct method by Rayleigh and a generalization was given by Ritz.

The starting point for elasto-static problems is the total elastic potential energy functional

$$\begin{aligned} \Pi &= \frac{1}{2}\int\limits_V \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{\varepsilon}\mathrm{d}V - \int\limits_V \overline{\boldsymbol{p}}^{\mathrm{T}}\boldsymbol{u}\mathrm{d}V - \int\limits_{A_{\mathrm{q}}} \overline{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{u}\mathrm{d}A \\ &= \frac{1}{2}\int\limits_V (\boldsymbol{D}\boldsymbol{u})^{\mathrm{T}}\boldsymbol{C}\boldsymbol{D}\boldsymbol{u}\mathrm{d}V - \int\limits_V \overline{\boldsymbol{p}}^{\mathrm{T}}\boldsymbol{u}\mathrm{d}V - \int\limits_{A_{\mathrm{q}}} \overline{\boldsymbol{q}}^{\mathrm{T}}\boldsymbol{u}\mathrm{d}A \end{aligned} \tag{2.2.28}$$

The variations are related to the displacements $\boldsymbol{u}$ and the strains $\boldsymbol{\varepsilon}$ which have to be substituted with help of the differential matrix $\boldsymbol{D}$, (2.2.6), by the displacements. The approximate solution is sought in the form of a finite linear combination. Looking first at a scalar displacement approach, the approximation of the scalar displacement function $u(x_1,x_2,x_2)$ is given by the Ritz approximation

$$\tilde{u}(x_1,x_2,x_3) = \sum_{i=1}^{N} a_i\varphi_i(x_1,x_2,x_3)$$

or

$$\tilde{u}(x_1,x_2,x_3) = \sum_{i=1}^{N} a_i\varphi_i(x_1,x_2,x_3) + \varphi_0(x_1,x_2,x_3) \tag{2.2.29}$$

The φ_i are known functions chosen a priori, named approximation functions or coordinate functions. The a_i denote undetermined constants named generalized coordinates. The approximation $\tilde{u}$ has to make (2.2.28) extremal

$$\tilde{\Pi}(\tilde{u}) = \tilde{\Pi}(a_i), \qquad \delta\tilde{\Pi}(a_i) = 0 \tag{2.2.30}$$

This approximation is characterized by a relative extremum. From (2.2.30) comes $\tilde{\Pi}$ in form of a function of the constants a_i and $\delta\tilde{\Pi}(a_i) = 0$ yields N stationary conditions

$$\frac{\partial\tilde{\Pi}(a_i)}{\partial a_i} = 0, \quad i = 1,2,\ldots,N \tag{2.2.31}$$

$\tilde{\Pi}$ may be written as a quadratic form in a_i and from Eqs. (2.2.31) follows a system of N linear equations allowing the N unknown constants a_i to be determined. In order to ensure a solution of the system of linear equations and a convergence of the approximate solution to the true solution as the number N of the a_i is increased, the φ_i values have to fulfill the following requirements:

- φ_0 satisfies specified inhomogeneous geometric boundary conditions, the so-called essential conditions of the variational statement and $\varphi_i, i = 1,2,\ldots,N$ satisfy the homogeneous form of the geometric boundary conditions.
- φ_i are continuous as required in the variational formulation, e.g. they should have a non-zero contribution to $\tilde{\Pi}$.
- φ_i are linearly independent and complete.

The completeness property is essential for the convergence of the Ritz approximation. Polynomial and trigonometric functions are selected examples of complete systems of functions.

Generalizing the considerations to three-dimensional problems and using vector-matrix notation it follows

$$\tilde{\boldsymbol{u}}(x_1,x_2,x_3) \equiv \begin{bmatrix} \tilde{u}_1 \\ \tilde{u}_2 \\ \tilde{u}_3 \end{bmatrix} = \begin{bmatrix} \boldsymbol{a}_1^{\mathrm{T}}\boldsymbol{\varphi}_1 \\ \boldsymbol{a}_2^{\mathrm{T}}\boldsymbol{\varphi}_2 \\ \boldsymbol{a}_3^{\mathrm{T}}\boldsymbol{\varphi}_3 \end{bmatrix} \equiv \begin{bmatrix} \boldsymbol{\varphi}_1 & \boldsymbol{o} & \boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{\varphi}_2 & \boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{o} & \boldsymbol{\varphi}_3 \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{bmatrix} \tag{2.2.32}$$

or

$$\tilde{\boldsymbol{u}}(x_1,x_2,x_3) = \boldsymbol{G}^{\mathrm{T}}\boldsymbol{a} \tag{2.2.33}$$

with

$$\boldsymbol{G}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{\varphi}_1 & \boldsymbol{o} & \boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{\varphi}_2 & \boldsymbol{o} \\ \boldsymbol{o} & \boldsymbol{o} & \boldsymbol{\varphi}_3 \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{\varphi}_1^{\mathrm{T}} & \boldsymbol{o}^{\mathrm{T}} & \boldsymbol{o}^{\mathrm{T}} \\ \boldsymbol{o}^{\mathrm{T}} & \boldsymbol{\varphi}_2^{\mathrm{T}} & \boldsymbol{o}^{\mathrm{T}} \\ \boldsymbol{o}^{\mathrm{T}} & \boldsymbol{o}^{\mathrm{T}} & \boldsymbol{\varphi}_3^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \boldsymbol{a}_3 \end{bmatrix}$$

$\boldsymbol{G}$ is the matrix of the approximation functions, $\boldsymbol{\varphi}_i$ and $\boldsymbol{o}$ are N-dimensional vectors and $\boldsymbol{a}_i$ are N-dimensional subvectors of the vector $\boldsymbol{a}$ of the unknown coordinates. The application of the Ritz method using the minimum principle of elastic potential energy $\varPi$ has the following steps:

1. Choose the approximation function $\tilde{\boldsymbol{u}} = \boldsymbol{G}^{\mathrm{T}}\boldsymbol{a}$.
2. Substitute $\tilde{\boldsymbol{u}}$ into $\varPi$

$$\begin{aligned} \tilde{\varPi}(\tilde{\boldsymbol{u}}) &= \frac{1}{2}\int\limits_V (\boldsymbol{D}\tilde{\boldsymbol{u}})^{\mathrm{T}}\boldsymbol{C}\boldsymbol{D}\tilde{\boldsymbol{u}}\,\mathrm{d}V - \int\limits_V \overline{\boldsymbol{p}}^{\mathrm{T}}\tilde{\boldsymbol{u}}\,\mathrm{d}V - \int\limits_{A_{\mathrm{q}}} \overline{\boldsymbol{q}}^{\mathrm{T}}\tilde{\boldsymbol{u}}\,\mathrm{d}A \\ &= \frac{1}{2}\boldsymbol{a}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{a} - \boldsymbol{a}^{\mathrm{T}}\boldsymbol{f} \end{aligned} \tag{2.2.34}$$

with

$$\boldsymbol{K} = \int\limits_V (\boldsymbol{D}\boldsymbol{G})^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{D}\boldsymbol{G})\,\mathrm{d}V = \int\limits_V \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{B}\,\mathrm{d}V,$$

$$\boldsymbol{f} = \int\limits_V \boldsymbol{G}^{\mathrm{T}}\overline{\boldsymbol{p}}\,\mathrm{d}V + \int\limits_{A_{\mathrm{q}}} \boldsymbol{G}^{\mathrm{T}}\overline{\boldsymbol{q}}\,\mathrm{d}A$$

3. Formulate the stationary conditions of $\tilde{\varPi}(\boldsymbol{a})$

$$\frac{\partial\tilde{\varPi}(\boldsymbol{a})}{\partial\boldsymbol{a}} = \boldsymbol{o}$$

i.e. with

$$\frac{\partial}{\partial\boldsymbol{a}}(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{a}) = 2\boldsymbol{K}\boldsymbol{a}, \quad \frac{\partial}{\partial\boldsymbol{a}}(\boldsymbol{a}^{\mathrm{T}}\boldsymbol{f}) = \boldsymbol{a}$$

follows

$$\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f} \tag{2.2.35}$$

$\boldsymbol{K}$ is called the stiffness matrix, $\boldsymbol{a}$ the vector of unknowns and $\boldsymbol{f}$ the force vector. These notations are used in a generalized sense.

4. Solve the system of linear equations $\boldsymbol{K}\boldsymbol{a} = \boldsymbol{f}$. The vector $\boldsymbol{a}$ of unknown coefficients is known.
5. Calculate the approximation solution $\tilde{\boldsymbol{u}} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{\varphi}$ and the $\tilde{\boldsymbol{\varepsilon}} = \boldsymbol{D}\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{\sigma}} = \boldsymbol{C}\tilde{\boldsymbol{\varepsilon}}, \ldots$

For an increasing number N, the previously computed coefficients of $\boldsymbol{a}$ remain unchanged provided the previously chosen coordinate functions are not changed. Since the strains are calculated from approximate displacements, strains and stresses are less accurate than displacements.

The Ritz approximation of elasto-dynamic problems is carried out in an analogous manner and can be summarized as follows. For conservative problems we start with the variational statement (2.2.23). The displacement vector $\boldsymbol{u}$ is now a function of $\boldsymbol{x}$ and t and the $\boldsymbol{a}$-vector a function of t. The stationary condition yields

$$\frac{\partial}{\partial \boldsymbol{a}} \left\{ \frac{1}{2}\boldsymbol{a}^{\mathrm{T}}(t)\boldsymbol{K}\boldsymbol{a}(t) - \boldsymbol{a}^{\mathrm{T}}(t)\boldsymbol{f}(t) + \ddot{\boldsymbol{a}}^{\mathrm{T}}(t)\boldsymbol{M}\boldsymbol{a}(t) \right\} = \boldsymbol{0} \tag{2.2.36}$$

$$\boldsymbol{M}\ddot{\boldsymbol{a}}(t) + \boldsymbol{K}\boldsymbol{a}(t) = \boldsymbol{f}(t), \tag{2.2.37}$$

$$\begin{aligned}
\boldsymbol{M} &= \int_V \rho \boldsymbol{G}^{\mathrm{T}}\boldsymbol{G}\,\mathrm{d}V, \\
\boldsymbol{K} &= \int_V (\boldsymbol{D}\boldsymbol{G})^{\mathrm{T}}\boldsymbol{C}(\boldsymbol{D}\boldsymbol{G})\,\mathrm{d}V, \\
\boldsymbol{f} &= \int_V \boldsymbol{G}^{\mathrm{T}}\overline{\boldsymbol{p}}\,\mathrm{d}V + \int_{A_{\mathrm{q}}} \boldsymbol{G}^{\mathrm{T}}\overline{\boldsymbol{q}}\,\mathrm{d}A
\end{aligned}$$

The matrix $\boldsymbol{G}$ depends on $\boldsymbol{x}, \boldsymbol{p}$ and $\boldsymbol{q}$ on $\boldsymbol{x}$ and t. $\boldsymbol{M}$ is called the mass matrix.

An direct derivation of a damping matrix from the Ritz approximation analogous to the $\boldsymbol{K}$- and the $\boldsymbol{M}$-matrix of (2.2.37) is not possible. In most engineering applications (2.2.37) has an additional damping term and the damping matrix is formulated approximately as a linear combination of mass- and stiffness-matrix (modal damping)

$$\boldsymbol{M}\ddot{\boldsymbol{a}}(t) + \boldsymbol{C}_{\mathrm{D}}\dot{\boldsymbol{a}}(t) + \boldsymbol{K}\boldsymbol{a}(t) = \boldsymbol{f}(t), \quad \boldsymbol{C}_{\mathrm{D}} \approx \alpha\boldsymbol{M} + \beta\boldsymbol{K} \tag{2.2.38}$$

In the case of the study of free vibrations, we write the time dependence of $\boldsymbol{a}(t)$ in the form

$$\boldsymbol{a}(t) = \hat{\boldsymbol{a}}\cos(\omega t + \varphi) \tag{2.2.39}$$

and from (2.2.38) with $\boldsymbol{C}_{\mathrm{D}} = \boldsymbol{0}, \boldsymbol{f}(t) = \boldsymbol{0}$ comes the matrix eigenvalue problem (App. A.3)

$$(\boldsymbol{K} - \omega^2\boldsymbol{M})\hat{\boldsymbol{a}} = \boldsymbol{o}, \quad \det[\boldsymbol{K} - \omega^2\boldsymbol{M}] = 0 \tag{2.2.40}$$

For N coordinate functions the algebraic equation (2.2.40) yields N eigenfrequencies of the deformable body.

The Rayleigh-Ritz method approximates the continuous deformable body by a finite number of degree of freedoms, i.e. the approximated system is less flexible than the actual body. Consequently for the approximated energy potential $\tilde{\Pi} \leq \Pi$. The energy potential converges from below. The approximate displacements satisfy the equilibrium equations only in the energy sense and not pointwise, unless the solution converges to the exact solution. The Rayleigh-Ritz method can be applied to all mechanical problems since a virtual statement exists, i.e. a weak form of the model equations including the natural boundary conditions. If the displacements are approximate, the approximate eigenfrequencies are higher than the exact, i.e. $\tilde{\omega} \geq \omega$.

2.2.3.2 Weighted Residual Methods

Finally some brief remarks on weighted residual methods are given. The fundamental equation in the displacement approach may be formulated in the form

$$A(u) = f \tag{2.2.41}$$

A is a differential operator. We seek again an approximate solution (2.2.29), where now the constants a_i are determined by requiring the residual

$$R_N = A\left(\sum_{i=1}^{N} a_i \varphi_i + \varphi_0\right) - f \neq 0 \tag{2.2.42}$$

be orthogonal to N linear independent weight function ψ_i

$$\int\limits_V R_N \psi_i \mathrm{d}\boldsymbol{x} = 0, \quad i = 1,2,\ldots,N \tag{2.2.43}$$

φ_0, φ_i should be linear independent and complete and fulfill all boundary conditions. Various known special methods follow from (2.2.43). They differ from each other due to the choice of the weight functions ψ_i:

- Galerkin's method[12] $\psi_i \equiv \varphi_i$,
- Least-squares method $\psi_i \equiv A(\varphi_i)$,
- Collocation method $\psi_i \equiv \delta(\boldsymbol{x} - \boldsymbol{x}_i)$ ($\delta(\boldsymbol{x} - \boldsymbol{x}_i)$ = 1 if $\boldsymbol{x} = \boldsymbol{x}_i$ otherwise 0)

The Galerkin method is a generalization of the Ritz method, if it is not possible to construct a weak form statement. Otherwise the Galerkin and the the Ritz method for weak formulations of problems yield the same solution equations, if the coordinate functions φ_i in both are the same.

[12] Boris Grigorjewitsch Galerkin, surname more accurately romanized as Galyorkin (*4 March$^{\text{greg.}}$/20 February$^{\text{jul.}}$ 1871 Polozk - 12 July 1945 Leningrad) - mathematician and engineer, contributions to the theory of approximate solutions of partial differential equations

The classical variational methods of Ritz and Galerkin are widely used to solve problems of applied elasticity or structural mechanics. When applying the Ritz or Galerkin method to special problems involving, e.g. a two-dimensional functional $\Pi[u(x_1,x_2)]$ or a two-dimensional differential equation $A[u(x_1,x_2)] = f(x_1,x_2)$, an approximative solution is usually assumed in the form

$$\tilde{u}(x_1,x_2) = \sum_{i=1}^{N} a_i \varphi_i(x_1,x_2) \quad \text{or} \quad \tilde{u}(x_1,x_2) = \sum_{i=1}^{N}\sum_{j=1}^{M} a_{ij}\varphi_{1i}(x_1)\varphi_{2j}(x_2), \qquad (2.2.44)$$

where $\varphi_i(x_1,x_2)$ or $\varphi_{1i}(x_1), \varphi_{2j}(x_2)$ are a priori chosen trial functions and the a_i or a_{ij} are unknown constants. The approximate solution depends very strongly on the assumed trial functions.

To overcome the shortcoming of these solution methods Vlasov[13] and Kantorovich[14] suggested an approximate solution in the form

$$\tilde{u}(x_1,x_2) = \sum_{i=1}^{N} a_i(x_1)\varphi_i(x_1,x_2) \qquad (2.2.45)$$

The φ_i are again a priori chosen trial functions but the $a_i(x_1)$ are unknown coefficient functions of one of the independent variables. The condition $\delta\Pi[\tilde{u}(x_1,x_2)] = 0$ or with $\mathrm{d}A = \mathrm{d}x_1\mathrm{d}x_2$

$$\int_A R_N(\tilde{u})\varphi_i \mathrm{d}A = 0, i = 1,2,\ldots,N$$

lead to a system of N ordinary differential equations for the unknown functions $a_i(x_1)$. Generally it is advisable to choose if possible the trial functions φ_i as functions of one independent variable, i.e. $\varphi_i = \varphi_i(x_2)$, since otherwise the system of ordinary differential equations will have variable coefficients. The approximate solution $\tilde{u}(x_1,x_2)$ tends in regard of the arbitrariness of the assumed trial function $\varphi_i(x_2)$ to a better solution in the x_1-direction. The obtained approximative solution can be further improved in the x_2-direction in the following manner. In a first step the assumed approximation

$$\tilde{u}(x_1,x_2) = \sum_{i=1}^{N} a_i(x_1)\varphi_i(x_2) \qquad (2.2.46)$$

yields the functions $a_i(x_1)$ by solving the resulting set of ordinary differential equations with constant coefficients. In the next step, with $a_i(x_1) \equiv a_{i1}(x_1)$ and $\varphi_i(x_2) \equiv a_{i2}(x_2)$, i.e.

$$\tilde{u}^{[I]}(x_1,x_2) = \sum_{i=1}^{N} a_{i2}(x_2)a_{i1}(x_1),$$

13 Vasily Zakharovich Vlasov ($^*11^{greg.}/24^{jul.}$ February 1906 Kareevo - †7 August 1958 Moscow) - civil engineer

14 Leonid Vitaliyevich Kantorovich (*19 January 1912 St. Petersburg - †7 April 1986 Moscow) - mathematician and economist, Nobel prize winner in economics in 1975

the $a_{i1}(x_1)$ are the given trial functions and the unknown functions $a_{i2}(x_2)$ can be determined as before the $a_{i1}(x_1)$ by solving a set of ordinary differential equations. After completing the first cycle, which yields $\tilde{u}^{[I]}(x_1,x_2)$, the procedure can be continued iteratively. This iterative solution procedure is denoted in literature as variational iteration or extended Vlasov-Kantorowich method. The final form of the generated solution is independent of the initial choice of the trial function $\varphi_i(x_2)$ and the iterative procedure converges very rapidly. It can be demonstrated, that the iterative generated solutions $\tilde{u}(x_1,x_2)$ agree very closely with the exact analytical solutions $u(x_1,x_2)$ even with a single term approximation

$$\tilde{u}(x_1,x_2) = a_1(x_1)\varphi_1(x_2) \tag{2.2.47}$$

In engineering applications, e.g. for rectangular plates, the single term approximations yield in general sufficient accuracy.

Summarizing it should be said that the most difficult problem in the application of the classical variational methods or weighted residual methods is the selection of coordinate functions, especially for structures with irregular domains. The limitations of the classical variational methods can be overcome by numerical methods, e.g. the finite element method which is discussed in more detail in Chap. 11.

2.2.4 Problems

Exercise 2.7. An anisotropic body is subjected to a hydrostatic pressure $\sigma_1 = \sigma_2 = \sigma_3 = -p$, $\sigma_4 = \sigma_5 = \sigma_6 = 0$.

1. Calculate the strain state $\boldsymbol{\varepsilon}$.
2. Calculate the stress state $\boldsymbol{\sigma}$ for a change of the coordinate system obtained by a rotation $\boldsymbol{T}^{\sigma}$.

Solution 2.7. The solution can be presented in two parts:

1. The generalized Hooke's law yields (Eq. 2.1.20)

$$\begin{aligned}
\varepsilon_1 &= -(S_{11}+S_{12}+S_{13})p, & \varepsilon_4 &= -(S_{14}+S_{24}+S_{34})p,\\
\varepsilon_2 &= -(S_{12}+S_{22}+S_{23})p, & \varepsilon_5 &= -(S_{15}+S_{25}+S_{35})p,\\
\varepsilon_3 &= -(S_{13}+S_{23}+S_{33})p, & \varepsilon_6 &= -(S_{16}+S_{26}+S_{36})p
\end{aligned}$$

Note 2.3. A hydrostatic pressure in an anisotropic solid yields extensional and shear strains.

2. From (2.1.39) follows

$$\begin{aligned}
\sigma_1' &= -p(c^2+s^2) = -p, & \sigma_4' &= 0,\\
\sigma_2' &= -p(s^2+c^2) = -p, & \sigma_5' &= 0,\\
\sigma_3' &= -p, & \sigma_6' &= -p(-cs+cs) = 0
\end{aligned}$$

Exercise 2.8. An anisotropic body has a pure shear stress state
$\sigma_1 = \sigma_2 = \sigma_3 = \sigma_5 = \sigma_6 = 0$, $\sigma_4 = t$.

1. Calculate the strain state $\boldsymbol{\varepsilon}$.
2. Compare the strain state for the anisotropic case with the isotropic case.

Solution 2.8. The solution can be presented again in two parts:

1. Equations (2.1.20) yield

$$\begin{array}{lll} \varepsilon_1 = S_{14}t, & \varepsilon_2 = S_{24}t, & \varepsilon_3 = S_{34}t, \\ \varepsilon_4 = S_{44}t, & \varepsilon_5 = S_{45}t, & \varepsilon_6 = S_{46}t \end{array}$$

   The anisotropic body has extensional and shear strains in all coordinate planes.
2. In an isotropic body a pure shear stress state yields only shearing strains: $S_{14} = S_{24} = S_{34} = 0$

Exercise 2.9. Consider a prismatic homogeneous anisotropic bar which is fixed at one end. The origin of the coordinates x_1, x_2, x_3 is placed in the centroid of the fixed section and the x_3-axis is directed along the bar axis, l and A are the length and the cross-section of the undeformed bar. Assume that the bar at the point $x_1 = x_2 = x_3 = 0$ has no displacement and torsion:

$$u_1 = u_2 = u_3 = 0, u_{1,3} = u_{2,3} = u_{2,1} - u_{1,2} = 0$$

1. A force F acts on the bar on the cross-section $x_3 = l$ and the stress state is determined by $\sigma_1 = \sigma_2 = \sigma_4 = \sigma_5 = \sigma_6 = 0$, $\sigma_3 = F/A$. Determine the strains, the displacements and the extension of the axis.
2. The fixed bar is deformed only under its own weight: $p_1 = p_2 = 0$, $p_3 = g\rho$. Determine the strain state and the displacements and calculate the displacements in the point $(0,0,l)$.

Solution 2.9. Now one has

1. The generalized Hooke's law (2.1.20) with

$$\sigma_1 = \sigma_2 = \sigma_4 = \sigma_5 = \sigma_6 = 0, \quad \sigma_3 = F/A = \sigma \neq 0,$$

   gives

$$\begin{array}{lll} \varepsilon_1 = S_{13}\sigma, & \varepsilon_2 = S_{23}\sigma, & \varepsilon_3 = S_{33}\sigma, \\ \varepsilon_4 = S_{34}\sigma, & \varepsilon_5 = S_{35}\sigma, & \varepsilon_6 = S_{36}\sigma \end{array}$$

   The displacements can be determined by the introduction of the following equations

$$\begin{array}{lll} S_{13}\sigma = u_{1,1}, & S_{23}\sigma = u_{2,2}, & S_{33}\sigma = u_{3,3}, \\ S_{34}\sigma = u_{3,2} + u_{2,3}, & S_{35}\sigma = u_{3,1} + u_{1,3}, & S_{36}\sigma = u_{2,1} + u_{1,2} \end{array}$$

   The equations

$$\begin{aligned}u_1 &= \sigma(S_{13}x_1 + 0.5S_{36}x_2),\\ u_2 &= \sigma(0.5S_{36}x_1 + S_{23}x_2),\\ u_3 &= \sigma(S_{35}x_1 + S_{34}x_2 + S_{33}x_3)\end{aligned}$$

satisfy the displacement differential equations and the boundary conditions which are prescribed at the point $x_1 = x_2 = x_3 = 0$.

Note 2.4. The stress-strain formulae show that an anisotropic tension bar does not only lengthens in the force direction x_3 and contracts in the transverse directions, but also undergoes shears in all planes parallel to the coordinate planes. The cross-sections of the bar remain plane. The stress states of an isotropic or an anisotropic bar are identical, the anisotropy effects the strain state only.

2. The stress in a bar under its own weight is

$$\sigma_1 = \sigma_2 = \sigma_4 = \sigma_5 = \sigma_6 = 0, \sigma_3 = \rho g(l - x_3)$$

The stress-strain displacement equations are

$$\begin{aligned}\varepsilon_1 &= u_{1,1} = S_{13}\rho g(l - x_3), & \varepsilon_4 &= u_{3,2} + u_{2,3} = S_{34}\rho g(l - x_3),\\ \varepsilon_2 &= u_{2,2} = S_{23}\rho g(l - x_3), & \varepsilon_5 &= u_{3,1} + u_{1,3} = S_{35}\rho g(l - x_3),\\ \varepsilon_3 &= u_{3,3} = S_{33}\rho g(l - x_3), & \varepsilon_6 &= u_{2,1} + u_{1,2} = S_{36}\rho g(l - x_3)\end{aligned}$$

The boundary conditions are identical to case a) and the following displacement state which satisfies all displacement differential equations and the conditions at the point $x_1 = x_2 = x_3 = 0$ can be calculated by integration

$$\begin{aligned}u_1 &= \rho g[-0.5S_{35}x_3^2 + S_{13}x_1(l - x_3) + 0.5S_{36}x_2(l - x_3)],\\ u_2 &= \rho g[-0.5S_{34}x_3^2 + S_{23}x_2(l - x_3) + 0.5S_{36}x_1(l - x_3)],\\ u_3 &= \rho g[-0.5S_{13}x_1^2 + 0.5S_{23}x_2^2\\ &\quad + 0.5S_{36}x_1x_2 + (S_{34}x_2 + S_{35}x_1)l + 0.5S_{33}x_3(2l - x_3)]\end{aligned}$$

Note 2.5. The cross-section does not remain plane, it is deformed to the shape of a second-order surface and the bar axis becomes curved. The centroid of the cross-section $x_3 = l$ is displaced in all three directions

$$\begin{aligned}u_1(0,0,l) &= -0.5\rho g S_{35} l^2,\\ u_2(0,0,l) &= -0.5\rho g S_{34} l^2,\\ u_3(0,0,l) &= 0.5\rho g S_{33} l^2\end{aligned}$$

Exercise 2.10. Show that for a composite beam subjected to a distributed continuous load $q(x_1)$ the differential equation and the boundary conditions can be derived using the extremal principle of potential energy.

Solution 2.10. The beam is of the length l, width b and height h. $q(x_1)$ is the lateral load per unit length. The average elasticity modulus of the beam is E_1. With the stress-strain-displacement relations from Bernoulli[15] beam theory

[15] Jakob I. Bernoulli (*27 December 1654$^{\text{jul.}}$/6 January 1655$^{\text{greg.}}$ Basel - †16 August 1705 Basel) - mathematician, beam theory

$$\sigma_1 = E_1\varepsilon_1, \quad \varepsilon_1 = -x_3\frac{\mathrm{d}^2u_3}{\mathrm{d}x_1^2}$$

the strain energy function W_f is seen to be

$$W_\mathrm{f} = \frac{1}{2}\sigma_1\varepsilon_1 = \frac{1}{2}E_1\varepsilon_1^2 = \frac{1}{2}E_1x_3^2\left(\frac{\mathrm{d}^2u_3}{\mathrm{d}x_1^2}\right)^2$$

and

$$\Pi_\mathrm{i} = \int\limits_0^l\int\limits_{-b/2}^{b/2}\int\limits_{-h/2}^{h/2}\frac{E_1}{2}\left(\frac{\mathrm{d}^2u_3}{\mathrm{d}x_1^2}\right)^2x_3^2\mathrm{d}x_3\mathrm{d}x_2\mathrm{d}x_1 = \frac{E_1I}{2}\int\limits_0^l\left(\frac{\mathrm{d}^2u_3}{\mathrm{d}x_1^2}\right)^2\mathrm{d}x_1$$

with inertial moment $I = bh^3/12$ for a rectangular cross-section.

In the absence of body forces, the potential function Π_a of the external load $q(x_1)$ is

$$\Pi_\mathrm{a} = -\int\limits_0^l q(x_1)u_3(x_1)\mathrm{d}x_1$$

and the total elastic potential energy is

$$\Pi(u_3) = \frac{E_1I}{2}\int\limits_0^l\left(\frac{\mathrm{d}^2u_3}{\mathrm{d}x_1^2}\right)^2\mathrm{d}x_1 - \int\limits_0^l q(x_1)u_3(x_1)\mathrm{d}x_1$$

From the stationary condition $\delta\Pi(u_3) = 0$ it follows that

$$\delta\Pi(u_3) = \frac{E_1I}{2}\int\limits_0^l\delta\left(\frac{\mathrm{d}^2u_3}{\mathrm{d}x_1^2}\right)^2\mathrm{d}x_1 - \int\limits_0^l q(x_1)\delta u_3(x_1)\mathrm{d}x_1 = 0$$

There is no variation of E_1I or $q(x_1)$, because they are specified. From the last equation one gets

$$\delta\Pi(u_3) = \frac{E_1I}{2}\int\limits_0^l 2\frac{\mathrm{d}^2u_3}{\mathrm{d}x_1^2}\delta\left(\frac{\mathrm{d}^2u_3}{\mathrm{d}x_1^2}\right)^2\mathrm{d}x_1 - \int\limits_0^l q(x_1)\delta u_3(x_1)\mathrm{d}x_1 = 0$$

The first term can be integrated by parts. The first integration can be performed with the following substitution

$$u = \frac{\mathrm{d}^2u_3}{\mathrm{d}x_1^2}, \quad u' = \frac{\mathrm{d}^3u_3}{\mathrm{d}x_1^3}, \quad v' = \delta\left(\frac{\mathrm{d}^2u_3}{\mathrm{d}x_1^2}\right), \quad v = \delta\left(\frac{\mathrm{d}u_3}{\mathrm{d}x_1}\right)$$

and yields

$$\delta\Pi(u_3) = E_1 I \left\{ \left[\frac{\mathrm{d}^2 u_3}{\mathrm{d}x_1^2} \delta\left(\frac{\mathrm{d}u_3}{\mathrm{d}x_1}\right) \right]_0^l - \int\limits_0^l \frac{\mathrm{d}^3 u_3}{\mathrm{d}x_1^3} \delta\left(\frac{\mathrm{d}u_3}{\mathrm{d}x_1}\right) \mathrm{d}x_1 \right\} - \int\limits_0^l q\delta u_3 \mathrm{d}x_1 = 0$$

The second integration can be performed with the following substitution

$$u = \frac{\mathrm{d}^3 u_3}{\mathrm{d}x_1^3}, \quad u' = \frac{\mathrm{d}^4 u_3}{\mathrm{d}x_1^4}, \quad v' = \delta\left(\frac{\mathrm{d}u_3}{\mathrm{d}x_1}\right), \quad v = \delta u_3$$

and yields

$$\begin{aligned} \delta\Pi(u_3) &= \left[E_1 I \frac{\mathrm{d}^2 u_3}{\mathrm{d}x_1^2} \delta\left(\frac{\mathrm{d}u_3}{\mathrm{d}x_1}\right) \right]_0^l - \left[E_1 I \frac{\mathrm{d}^3 u_3}{\mathrm{d}x_1^3} \delta u_3 \right]_0^l \\ &\quad + \int\limits_0^l E_1 I \frac{\mathrm{d}^4 u_3}{\mathrm{d}x_1^4} \mathrm{d}x_1 - \int\limits_0^l q(x_1)\delta u_3 \mathrm{d}x_1 = 0 \end{aligned}$$

Finally we obtain

$$\begin{aligned} \delta\Pi(u_3) &= \left[E_1 I \frac{\mathrm{d}^2 u_3}{\mathrm{d}x_1^2} \delta\left(\frac{\mathrm{d}u_3}{\mathrm{d}x_1}\right) \right]_0^l - \left[E_1 I \frac{\mathrm{d}^3 u_3}{\mathrm{d}x_1^3} \delta u_3 \right]_0^l \\ &\quad + \int\limits_0^l \left[E_1 I \frac{\mathrm{d}^4 u_3}{\mathrm{d}x_1^4} - q(x_1) \right] \delta u_3 \mathrm{d}x_1 = 0 \end{aligned}$$

Since the variations are arbitrary the equation is satisfied if

$$E_1 I (u_3)'''' = q$$

and either

$$E_1 I (u_3)'' = 0$$

or u_3' is specified and

$$E_1 I (u_3)''' = 0$$

or u_3 is specified at $x_1 = 0$ and $x_1 = l$.

The beam differential equation is the Euler-Lagrange equation of the variational statement $\delta\Pi = 0$, u_3, u_3' represent essential boundary conditions, and $E_1 I u_3'', E_1 I u_3'''$ are natural boundary conditions of the problem. Note that the boundary conditions include the classical conditions of simply supports, clamped and free edges.

Exercise 2.11. The beam of Exercise 2.10 may be moderately thick and the effects of transverse shear deformation ε_5 and transverse normal stress are taken into account. Show that the differential equations and boundary condition can be derived using the Reissner's variational principle. The beam material behavior may be isotropic.

Solution 2.11. To apply Reissner's variational statement one must assume admissible functions for the displacements $u_1(x_1), u_3(x_1)$ and for the stresses $\sigma_1(x_1,x_3)$, $\sigma_3(x_1,x_3), \sigma_5(x_1,x_3)$. For the beam with rectangular cross-section and lateral loading $q(x_1)$ follow in the frame of beam theory

$$u_2 = 0, \quad \sigma_2 = \sigma_4 = \sigma_6 = 0$$

As in the classical beam theory (Bernoulli theory) we assume that the beam cross-sections undergo a translation and a rotation, the cross-sections are assumed to remain plane but not normal to the deformed middle surface (Timoshenko[16] theory). Therefore we can assume in the simplest case

$$u_1 = x_3\psi(x_1), \quad u_3 = w(x_1)$$

and the strain-displacement relations may be written

$$\varepsilon_1 = \frac{\partial u_1}{\partial x_1} = x_3\psi'(x_1), \varepsilon_3 = \frac{\partial u_3}{\partial x_3} = 0, \varepsilon_5 = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} = w'(x_1) + \psi(x_1)$$

For the stresses σ_1, σ_3 and σ_5 the following functions are assumed

$$\begin{aligned}
\sigma_1 &= \frac{M}{I}x_3, \quad I = \frac{bh^3}{12}, \\
\sigma_3 &= \frac{3q}{4b}\left[\frac{x_3}{h/2} + \frac{2}{3} - \frac{1}{3}\left(\frac{x_3}{h/2}\right)^2\right], \\
\sigma_5 &= \frac{3Q}{2A}\left[1 - \left(\frac{x_3}{h/2}\right)^2\right], \quad A = bh
\end{aligned}$$

The assumed functions for σ_1 and σ_5 are identical to those of the Bernoulli beam theory and σ_3 may be derived from the stress equation of equilibrium in the thickness direction, Eq. (2.2.1), with

$$\sigma_3(+h/2) = q, \sigma_1(-h/2) = 0$$

The bending moment M and the shear force resultant Q will be defined in the usual manner

$$M(x_1) = \int\limits_{-h/2}^{+h/2} b\sigma_1 \mathrm{d}x_3, \quad Q(x_1) = \int\limits_{-h/2}^{+h/2} b\sigma_5 \mathrm{d}x_3$$

Now Reissner's functional $\Psi_{\mathrm{R}}(\boldsymbol{u}, \boldsymbol{\sigma})$, e.g. (2.2.18), takes with the assumption above the form

[16] Stepan Prokopovich Timoshenko (*22 December, 1878 Schpotiwka - † 29 May, 1972 Wuppertal-Elberfeld) - engineer, founder of the modern applied mechanics

$$
\begin{aligned}
\Psi_{\mathrm{R}}(\boldsymbol{u},\boldsymbol{\sigma}) &= \int\limits_0^l \int\limits_{-h/2}^{+h/2} \left\{ \sigma_1\varepsilon_1 + \sigma_5\varepsilon_5 - \frac{1}{2E}\left[\sigma_1^2 + \sigma_3^2 + 2(1+\nu)\sigma_5^2\right] \right\} b\mathrm{d}x_3\mathrm{d}x_1 \\
&\quad - \int\limits_0^l qw\mathrm{d}x_1 \\
&= \int\limits_0^l \int\limits_{-h/2}^{+h/2} \left\{ \sigma_1 x_3 \psi'(x_1) + \sigma_5\left[w'(x_1) + \psi(x_1)\right] \right. \\
&\quad \left. - \frac{1}{2E}\left[\sigma_1^2 + \sigma_3^2 + 2(1+\nu)\sigma_5^2\right] \right\} b\mathrm{d}x_1 - \int\limits_0^l qw\mathrm{d}x_1
\end{aligned}
$$

Substituting M and Q and neglecting the term σ_3 which only depends on q and not on the basic unknown functions ψ, w respectively, M and Q, and yields no contribution to the variation $\delta\Psi$ we obtain

$$
\Psi(w,\psi,M,Q) = \int\limits_0^l \left[M\psi' + Q(w'+\psi) - \frac{M^2}{2EI} + \frac{6\nu qM}{5EA} - \frac{3Q^2}{5GA} - qw \right] \mathrm{d}x_1
$$

$$
\begin{aligned}
\delta\Psi_{\mathrm{R}} &= \int\limits_0^l \left[M\delta\psi' + \psi'\delta M + Q(\delta w' + \delta\psi) + (w'+\psi)\delta Q \right. \\
&\quad \left. - \frac{M}{EI}\delta M + \frac{6\nu q}{5EA}\delta M - \frac{6Q}{5GA}\delta Q - q\delta w \right] \mathrm{d}x_1 = 0
\end{aligned}
$$

Integration the terms $M\delta\psi'$ and $Q\delta w'$ by parts and rearranging the equation

$$
\begin{aligned}
\delta\Psi_{\mathrm{R}} &= \left[M\delta\psi + Q\delta w\right]_0^l + \int\limits_0^l \left\{ [Q + M']\delta\psi - [Q' - q]\delta w \right. \\
&\quad \left. + \left[\psi' - \frac{M}{EI} + \frac{6\nu q}{5EA}\right]\delta M + \left[\psi + w' - \frac{6Q}{5GA}\right]\delta Q \right\} \mathrm{d}x_1 = 0
\end{aligned}
$$

The first term yields the natural boundary conditions of the variational statement:

1. Either $M = 0$ or ψ must be prescribed at $x_1 = 0, l$.
2. Either $Q = 0$ or w must be prescribed at $x_1 = 0, l$.

The variations $\delta\psi, \delta w, \delta M$ and δQ are all arbitrary independent functions of x_1 and therefore $\delta\Psi_{\mathrm{R}} = 0$ only if

$$
\begin{aligned}
&-\frac{\mathrm{d}M(x_1)}{\mathrm{d}x_1} + Q(x_1) = 0, \qquad \frac{\mathrm{d}Q(x_1)}{\mathrm{d}x_1} + q(x_1) = 0, \\
&\frac{\mathrm{d}\psi(x_1)}{\mathrm{d}x_1} - \frac{M(x_1)}{EI} + \frac{6\nu q(x_1)}{5EA} = 0, \qquad \frac{\mathrm{d}w(x_1)}{\mathrm{d}x_1} + \psi(x_1) - \frac{6Q(x_1)}{5GA} = 0
\end{aligned}
$$

The both equations for the stress resultants are identical with the equations of the classical Bernoulli's theory. The term $(w' + \psi)$ in the fourth equation describes the change in the angle between the beam cross-section and the middle surface during the deformation. The term $(w' + \psi)$ is proportional to the average shear stress Q/A and so a measure of shear deformation. With $GA \to \infty$ the shear deformation tends to zero and ψ to $-w'$ as assumed in the Bernoulli's theory. The third term in the third equation depends on the lateral load q and Poisson's ratio ν and tends to zero for $\nu \to 0$. This term described the effect of the transverse normal stress σ_3. which will be vanish if $\nu = 0$ as in the classical beam theory.

Substituting the differential equations for ψ and w into the differential relations for M and Q leads to

$$EI\left[\psi''(x_1) + \nu\frac{6}{5}\frac{Q'(x_1)}{EA}\right] - Q(x_1) = 0, \quad \frac{5}{6}GA[\psi'(x_1) + w''(x_1)] + q(x_1) = 0$$

or

$$EI\left[\psi''(x_1) - \nu\frac{6}{5}\frac{q(x_1)}{EA}\right] - \frac{5}{6}GA[\psi + w'] = 0, \quad \frac{5}{6}GA[\psi'(x_1) + w''(x_1)] + q(x_1) = 0$$

Derivation and rearrangement yield a differential equation of 3rd order for $\psi(x_1)$

$$EI\psi'''(x_1) = -q(x_1) + \nu\frac{6}{5}\frac{q'(x_1)}{EA}$$

With

$$Q(x_1) = \frac{\mathrm{d}M(x_1)}{\mathrm{d}x_1} = EI\left[\psi''(x_1) + \nu\frac{6}{5}\frac{Q'(x_1)}{EA}\right]$$

and

$$Q(x_1) = \frac{5}{6}\frac{EI}{GA}[\psi(x_1) + w'(x_1)]$$

follows an equation for $w'(x_1)$

$$w'(x_1) = -\psi(x_1) + \frac{6}{5}\frac{EI}{GA}\left[\psi''(x_1) + \nu\frac{6}{5}\frac{q(x_1)}{EA}\right]$$

Neglecting with $\nu = 0$ the effect of the transverse normal stress σ_3 we get the Timoshenko's beam equation

$$\begin{aligned}&EI\psi'''(x_1) = -q(x_1), \quad M(x_1) = EI\psi'(x_1), \quad Q(x_1) = EI\psi''(x_1)\\ &w'(x_1) = -\psi(x_1) + \frac{EI\psi''(x_1)}{k^sGA}, \quad k^s = \frac{5}{6}\end{aligned}$$

One can note that as $G \to \infty$ the shear deformation tends to vanish as assumed in classical beam theory, i.e. $\psi(x_1) = -w'(x_1)$ and the classical beam equations follow to

$$EIw''''(x_1) = q(x_1), \quad M(x_1) = -EIw''(x_1), \quad Q(x_1) = -EIw'''(x_1)$$

The derivation of the Timoshenko's beam equation for laminated beams one can find in more detail in Sect. 7.3.

Exercise 2.12. Derive the free vibration equations for the moderately thick beam, Exercise 2.11, using Hamilton's principle in conjunction with the Reissner variational theorem.

Solution 2.12. In order to derive the free vibration equations we apply the generalized Hamilton's principle, i.e. the Hamilton's principle in conjunction with the Reissner variational statement

$$\delta\chi(\boldsymbol{u},\boldsymbol{\sigma}) = \delta\int\limits_{t_1}^{t_2}[T(\boldsymbol{u}) - \Psi_{\mathrm{R}}(\boldsymbol{u},\boldsymbol{\sigma})]\mathrm{d}t = 0$$

$T(\boldsymbol{u})$ is the kinetic energy and $\Psi_{\mathrm{R}}(\boldsymbol{u},\boldsymbol{\sigma})$ the Reissner functional of the moderately thick beam. $\Psi_{\mathrm{R}}(\boldsymbol{u},\boldsymbol{\sigma})$ is known from Exercise 2.11 and the kinetic energy for the beam may be written as

$$\begin{aligned}
T(\boldsymbol{u}) &= \int\limits_0^l\int\limits_{-h/2}^{h/2}\frac{1}{2}\rho\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]b\mathrm{d}x_3\mathrm{d}x_1\\
&= \int\limits_0^l\frac{1}{2}\rho\left[I\left(\frac{\partial\psi}{\partial t}\right)^2 + A\left(\frac{\partial w}{\partial t}\right)^2\right]\mathrm{d}x_1
\end{aligned}$$

with

$$I = \int\limits_{-h/2}^{h/2}bx_3^2\mathrm{d}x_3,\quad A = \int\limits_{-h/2}^{h/2}b\mathrm{d}x_3$$

and the mass density ρ of the beam material. The substitution of $T[\boldsymbol{u}]$ above and $\Psi(w,\psi,M,Q)$ of Exercise 2.11 in the functional χ yields

$$\begin{aligned}
\delta\chi(\boldsymbol{u},\boldsymbol{\sigma}) &= \delta\int\limits_{t_1}^{t_2}\int\limits_0^l\left\{\frac{1}{2}\rho\left[I\left(\frac{\partial\psi}{\partial t}\right)^2 + A\left(\frac{\partial w}{\partial t}\right)^2\right]\right.\\
&\quad\left. - \left[M\psi' + Q(w'+\psi) - \frac{M^2}{2EI} + \frac{6\nu qM}{5EA} - \frac{3Q^2}{5GA} - qw\right]\right\}\mathrm{d}t\mathrm{d}x_1\\
&= \int\limits_{t_1}^{t_2}\int\limits_0^l\left[\left(\rho I\frac{\partial^2\psi}{\partial t^2}\delta\psi + A\frac{\partial^2 w}{\partial t^2}\right)\delta w\right.\\
&\quad - M\delta\psi' - \psi'\delta M - Q(\delta w' + \delta\psi) - (w'+\psi)\delta Q\\
&\quad\left. + \frac{M}{EI}\delta M - \frac{6\nu q}{5EA}\delta M - \frac{6Q}{5GA}\delta Q - q\delta w\right]\mathrm{d}t\mathrm{d}x_1
\end{aligned}$$

Integration the terms $M\delta\psi' + Q\delta w'$ by parts

$$\int_0^l M\delta\psi' dx_1 = M\delta\psi\Big|_0^l - \int_0^l M'\delta\psi \mathrm{d}x_1, \quad \int_0^l Q\delta w' dx_1 = Q\delta w\Big|_0^l - \int_0^l Q'\delta w \mathrm{d}x_1,$$

rearranging the equation $\delta\chi$ and setting $\delta\chi = 0$ yield

$$\begin{aligned}
&\int_{t_1}^{t_2} (M\delta\psi + Q\delta w)_0^l \mathrm{d}t + \int_{t_1}^{t_2}\int_0^l \left[\left(\rho I \frac{\partial^2\psi}{\partial t^2} + Q - M'\right)\delta\psi\right.\\
&-\left[Q' + qA\left(\frac{\partial^2 w}{\partial t^2}\right)\right]\delta w + \left(\psi' - \frac{M}{EI} + \frac{6\nu q}{5EA}\right)\delta M\\
&+\left.\left(\psi + w' - \frac{6Q}{5GA}\right)\delta A\right]\mathrm{d}t\mathrm{d}x_1 = 0
\end{aligned}$$

and the equations for free vibrations follow with $q \equiv 0$

$$Q - \frac{\partial M}{\partial x_1} + \underline{\rho I \frac{\partial^2\psi}{\partial t^2}} = 0, \qquad \frac{\partial Q}{\partial x_1} - \rho A \frac{\partial^2 w}{\partial t^2} = 0,$$

$$\frac{\partial\psi}{\partial x_1} - \frac{M}{EI} = 0, \qquad \psi + \frac{\partial w}{\partial x_1} - \underline{\frac{6Q}{5GA}} = 0$$

The underlined terms represent the contribution of rotatory inertia and the effect of transverse shear deformation.

The system of four equations can be reduced to a system of two equations for the unknowns w and ψ. Substitution of

$$Q = EI\frac{\partial^2\psi}{\partial x_1^2} - \rho I \frac{\partial^2\psi}{\partial t^2}, \quad I = \frac{bh^3}{12}, \quad A = bh$$

in the second and fourth equation leads

$$\frac{\partial^3\psi}{\partial x_1^3} - \frac{\rho}{E}\frac{\partial^3\psi}{\partial t^2\partial x_1} - \rho\frac{A}{E}\frac{\partial^2 w}{\partial t^2} = 0, \quad \psi + \frac{\partial w}{\partial x_1} - \frac{h^2}{10}\left[\frac{E}{G}\frac{\partial^2\psi}{\partial x_1^2} - \frac{\rho}{G}\frac{\partial^2\psi}{\partial t^2}\right] = 0$$

The equations for forced or free vibrations are given in Sect. 7.3 in a more general form.

References

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Chapter 3
Effective Material Moduli for Composites

Composite materials have at least two different material components which are bonded. The material response of a composite is determined by the material moduli of all constituents, the volume or mass fractions of the single constituents in the composite material, by the quality of their bonding, i.e. of the behavior of the interfaces, and by the arrangement and distribution of the fibre reinforcement, i.e. the fibre architecture.

The basic assumptions made in material science approach models of fibre reinforced composites are:

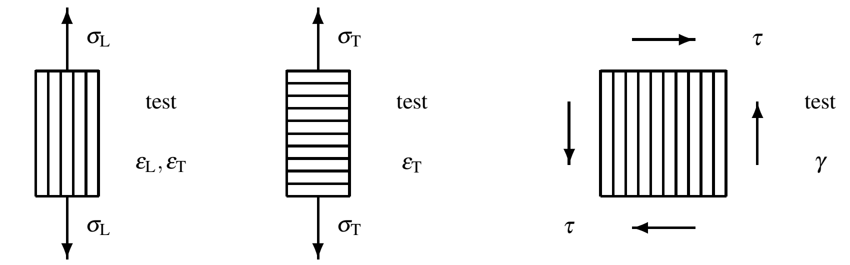
- The bond between fibres and matrix is perfect.
- The fibres are continuous and parallel aligned in each ply, they are packed regularly, i.e. the space between fibres is uniform.
- Fibre and matrix materials are linear elastic, they follow approximately Hooke's law and each elastic modulus is constant.
- The composite is free of voids.

Composite materials are heterogeneous, but in simplifying the analysis of composite structural elements in engineering applications, the heterogeneity of the material is neglected and approximately overlayed to a homogeneous material. The most important composites in structural engineering applications are laminates and sandwiches. Each single layer of laminates or sandwich faces is in general a fibre reinforced lamina. For laminates we have therefore two different scales of modelling:

- The modelling of the mechanical behavior of a lamina, is called the micromechanical or microscopic approach of a composite. The micro-mechanical modelling leads to a correlation between constituent properties and average effective composite properties. Most simple mixture rules are used in engineering applications. Whenever possible, the average properties of a lamina should be verified experimentally by the tests described in Sect. 3.1 or Fig. 3.1.
- The modelling of the global behavior of a laminate constituted of several quasi-homogeneous laminae is called the macroscopic approach of a composite.

Fibre reinforced material is in practice neither monolithic nor homogeneous, but it is impossible to incorporate the real material structure into design and analysis of

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0_3



$$E_L = \frac{\sigma_L}{\varepsilon_L}, \quad \nu_{LT} = \frac{\varepsilon_T}{\varepsilon_L} \qquad E_T = \frac{\sigma_T}{\varepsilon_T} \qquad G_{LT} = \frac{\tau}{\gamma}$$

Fig. 3.1 Experimental testing of the mechanical properties of an UD-layer: $E_L = E_1$, $E_T = E_2$, $G_{LT} = G_{12}, \nu_{LT} = \nu_{12}$

composite or any other structural component. Therefore the concept of replacing the heterogeneous material behavior with an effective material which is both homogeneous and monolithic, thus characterized by the generalized Hooke's law, will be used in engineering applications. We assume that the local variations in stress and strain state are very small in comparison to macroscopical measurements of material behavior.

In the following section some simple approaches to the lamina properties are given with help of the mixture rules and simple semi-empirical consideration. The more theoretical modelling in Sect. 3.2 has been developed to establish bounds on effective properties. The modelling of the average mechanical characteristics of laminates will be considered in Chap. 4.

3.1 Elementary Mixture Rules for Fibre-Reinforced Laminae

In Sect. 1.1 the formulas for volume fraction, mass fraction and density for fibre reinforced composites are given by (1.1.1) - (1.1.5). The rule of mixtures and the inverse rule of mixtures is based on the statement that the composite property is the weighted mean of the properties or the inverse properties of each constituent multiplied by its volume fraction. In the first case we have the upper-bound effective property, in the second - the lower-bound. In composite mechanics these bounds are related to W. Voigt and A. Reuss. In crystal plasticity the bounds were indroduced by G. Taylor[1] and O. Sachs[2]. The notation used is as follows:

[1] Geoffrey Ingram Taylor (*7 March 1886 St. John's Wood, England - †27 June 1975 Cambridge) - physicist and mathematician, contributions to the theory of plasticity, fluid mechanics and wave theory

[2] Oscar Sachs (*5 April 1896 Moscow - †30 October 1960 Syracus, N.Y.) - metallurgist, contributions to the theory of plasticity

E Young's modulus
ν Poisson's ratio
G Shear modulus
σ Stress
ε Strain
V, M Volume, mass
v, m Volume fraction, mass fraction
A Cross-section area
ρ Density
The subscripts f and m refer to fibre and matrix, the subscripts $\mathrm{L} \equiv 1, \mathrm{T} \equiv 2$ refer to the principal direction (fibre direction) and transverse to the fibre direction.

3.1.1 Effective Density

The derivation of the effective density of fibre reinforced composites in terms of volume fractions is given in Sect. 1.1

$$\begin{aligned} \rho = \frac{M}{V} = \frac{M_\mathrm{f} + M_\mathrm{m}}{V} &= \frac{\rho_\mathrm{f} V_\mathrm{f} + \rho_\mathrm{m} V_\mathrm{m}}{V} \\ &= \rho_\mathrm{f} v_\mathrm{f} + \rho_\mathrm{m} v_\mathrm{m} = \rho_\mathrm{f} v_\mathrm{f} + \rho_\mathrm{m}(1 - v_\mathrm{f}) \end{aligned} \tag{3.1.1}$$

In literature we also find $v_\mathrm{f} \equiv \phi$ for the fibre volume fraction and we have

$$\rho = \rho_\mathrm{f}\phi + \rho_\mathrm{m}(1 - \phi) \tag{3.1.2}$$

In an actual lamina the fibres are randomly distributed over the lamina cross-section and the lamina thickness is about 1 mm and much higher than the fibre diameter (about 0,01 mm). Because the actual fibre cross-sections and the fibre packing generally are not known and can hardly be predicted exactly typical idealized regular fibre arrangements are assumed for modelling and analysis, e.g. a layer-wise, square or a hexagonal packing, and the fibre cross-sections are assumed to have circular form. There exists ultimate fibre volume fractions $v_{\mathrm{f}_\mathrm{max}}$, which are less than 1 and depend on the fibre arrangements:

- square or layer-wise fibre packing - $v_{\mathrm{f}_\mathrm{max}} = 0.785$,
- hexagonal fibre packing - $v_{\mathrm{f}_\mathrm{max}} = 0.907$

For real UD-laminae we have $v_{\mathrm{f}_\mathrm{max}}$ about 0.50 - 0.65. Keep in mind that a lower fibre volume fraction results in lower laminae strength and stiffness under tension in fibre-direction, but a very high fibre volume fraction close to the ultimate values of v_f may lead to a reduction of the lamina strength under compression in fibre direction and under in-plane shear due to the poor bending of the fibres.

3.1.2 Effective Longitudinal Modulus of Elasticity

When an unidirectional lamina is acted upon by either a tensile or compression load parallel to the fibres, it can be assumed that the strains of the fibres, matrix and composite in the loading direction are the same (Fig. 3.2)

$$\varepsilon_{\mathrm{Lf}} = \varepsilon_{\mathrm{Lm}} = \varepsilon_{\mathrm{L}} = \frac{\Delta l}{l} \tag{3.1.3}$$

The mechanical model has a parallel arrangement of fibres and matrix (Voigt model, Sect. 2.1.1) and the resultant axial force F_{L} of the composite is shared by both fibre and matrix so that

$$F_{\mathrm{L}} = F_{\mathrm{Lf}} + F_{\mathrm{Lm}} \quad \text{or} \quad F_{\mathrm{L}} = \sigma_{\mathrm{L}} A = \sigma_{\mathrm{Lf}} A_{\mathrm{f}} + \sigma_{\mathrm{Lm}} A_{\mathrm{m}} \tag{3.1.4}$$

With Hooke's law it follows that

$$\sigma_{\mathrm{L}} = E_{\mathrm{L}} \varepsilon_{\mathrm{L}}, \quad \sigma_{\mathrm{Lf}} = E_{\mathrm{Lf}} \varepsilon_{\mathrm{Lf}}, \quad \sigma_{\mathrm{Lm}} = E_{\mathrm{Lm}} \varepsilon_{Lm}$$

or

$$E_{\mathrm{L}} \varepsilon_{\mathrm{L}} A = E_{\mathrm{f}} \varepsilon_{\mathrm{Lf}} A_{\mathrm{f}} + E_{\mathrm{m}} \varepsilon_{\mathrm{Lm}} A_{\mathrm{m}} \tag{3.1.5}$$

Since the strains of all phases are assumed to be identical (iso-strain condition), (3.1.5) reduces to

$$E_{\mathrm{L}} = E_{\mathrm{f}} \frac{A_{\mathrm{f}}}{A} + E_{\mathrm{m}} \frac{A_{\mathrm{m}}}{A} \tag{3.1.6}$$

with

$$\frac{A_{\mathrm{f}}}{A} = \frac{A_{\mathrm{f}} l}{A l} = \frac{V_{\mathrm{f}}}{V} = v_{\mathrm{f}}, \quad \frac{A_{\mathrm{m}}}{A} = \frac{A_{\mathrm{m}} l}{A l} = \frac{V_{\mathrm{m}}}{V} = v_{\mathrm{m}} \tag{3.1.7}$$

and the effective modulus E_{L} can be written as follows

$$E_{\mathrm{L}} = E_{\mathrm{f}} v_{\mathrm{f}} + E_{\mathrm{m}} v_{\mathrm{m}} = E_{\mathrm{f}} v_{\mathrm{f}} + E_{\mathrm{m}} (1 - v_{\mathrm{f}}) = E_{\mathrm{f}} \phi + E_{\mathrm{m}} (1 - \phi) \tag{3.1.8}$$

Equation (3.1.8) is referred to the Voigt estimate or is more familiarly known as the rule of mixture. The predicted values of E_{L} are in good agreement with experimental results. The stiffness in fibre direction is dominated by the fibre modulus. The ratio of the load taken by the fibre to the load taken by the composite is a measure of the load shared by the fibre

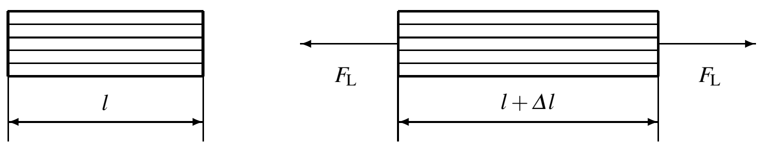


Fig. 3.2 Mechanical model to calculate the effective Young's modulus E_{L}

$$\frac{F_{\mathrm{Lf}}}{F_{\mathrm{L}}} = \frac{E_{\mathrm{Lf}}}{E_{\mathrm{L}}} v_{\mathrm{f}} \tag{3.1.9}$$

Since the fibre stiffness is several times greater than the matrix stiffness, the second term in (3.1.8) may be neglected

$$E_{\mathrm{L}} \approx E_{\mathrm{f}} v_{\mathrm{f}} \tag{3.1.10}$$

3.1.3 Effective Transverse Modulus of Elasticity

The mechanical model in Fig. 3.3 has an arrangement in a series of fibre and matrix (Reuss model, Sect. 2.1.1). The stress resultant F_{T} respectively the stress σ_{T} is equal for all phases (iso-stress condition)

$$F_{\mathrm{T}} = F_{\mathrm{Tf}} = F_{\mathrm{Tm}}, \sigma_{\mathrm{T}} = \sigma_{\mathrm{Tf}} = \sigma_{\mathrm{Tm}} \tag{3.1.11}$$

From Fig. 3.3 it follows that

$$\Delta b = \Delta b_{\mathrm{f}} + \Delta b_{\mathrm{m}}, \quad \varepsilon_{\mathrm{T}} = \frac{\Delta b}{b} = \frac{\Delta b_{\mathrm{f}} + \Delta b_{\mathrm{m}}}{b} \tag{3.1.12}$$

and with

$$b = v_{\mathrm{f}} b + (1 - v_{\mathrm{f}}) b = b_{\mathrm{f}} + b_{\mathrm{m}} \tag{3.1.13}$$

and

$$\varepsilon_{\mathrm{T}} = \frac{\Delta b_{\mathrm{f}}}{v_{\mathrm{f}} b} \frac{v_{\mathrm{f}} b}{b} + \frac{\Delta b_{\mathrm{m}}}{(1 - v_{\mathrm{f}}) b} \frac{(1 - v_{\mathrm{f}}) b}{b} = v_{\mathrm{f}} \varepsilon_{\mathrm{Tf}} + (1 - v_{\mathrm{f}}) \varepsilon_{\mathrm{Tm}} \tag{3.1.14}$$

with

$$\varepsilon_{\mathrm{Tf}} = \frac{\Delta b_{\mathrm{f}}}{v_{\mathrm{f}} b}, \quad \varepsilon_{\mathrm{Tm}} = \frac{\Delta b_{\mathrm{m}}}{(1 - v_{\mathrm{f}}) b}$$

Using Hooke's law for the fibre, the matrix and the composite

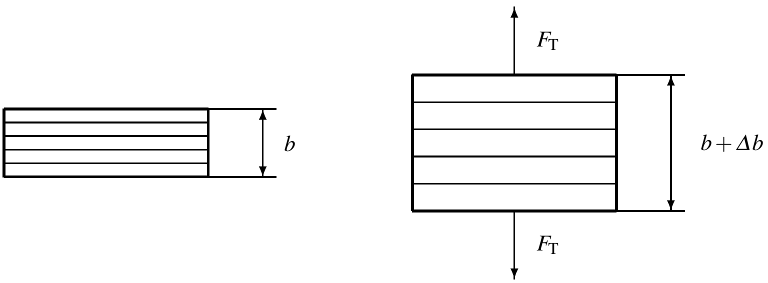


Fig. 3.3 Mechanical model to calculate the effective transverse modulus E_{T}

$$\sigma_{\mathrm{T}} = E_{\mathrm{T}}\varepsilon_{\mathrm{T}}, \quad \sigma_{\mathrm{Tf}} = E_{\mathrm{Tf}}\varepsilon_{\mathrm{Tf}}, \quad \sigma_{\mathrm{Tm}} = E_{\mathrm{Tm}}\varepsilon_{\mathrm{Tm}} \tag{3.1.15}$$

substituting Eqs. (3.1.15) in (3.1.14) and considering (3.1.11) gives the formula of E_{T}

$$\frac{1}{E_{\mathrm{T}}} = \frac{v_{\mathrm{f}}}{E_{\mathrm{f}}} + \frac{1-v_{\mathrm{f}}}{E_{\mathrm{m}}} = \frac{v_{\mathrm{f}}}{E_{\mathrm{f}}} + \frac{v_{\mathrm{m}}}{E_{\mathrm{m}}} \quad \text{or} \quad E_{\mathrm{T}} = \frac{E_{\mathrm{f}}E_{\mathrm{m}}}{(1-v_{\mathrm{f}})E_{\mathrm{f}} + v_{\mathrm{f}}E_{\mathrm{m}}} \tag{3.1.16}$$

Equation (3.1.16) is referred to Reuss estimate or sometimes called the inverse rule of mixtures. The predicted values of E_{T} are seldom in good agreement with experimental results. With $E_{\mathrm{m}} \ll E_{\mathrm{f}}$ follows from (3.1.16) $E_{\mathrm{T}} \approx E_{\mathrm{m}}(1-v_{\mathrm{f}})^{-1}$, i.e. E_{T} is dominated by the matrix modulus E_{m}.

3.1.4 Effective Poisson's Ratio

Assume a composite is loaded in the on-axis direction (parallel to the fibres) as shown in Fig. 3.4. The major Poison's ratio is defined as the negative of the ratio of the normal strain in the transverse direction to the normal strain in the longitudinal direction

$$\nu_{\mathrm{LT}} = -\frac{\varepsilon_{\mathrm{T}}}{\varepsilon_{\mathrm{L}}} \tag{3.1.17}$$

With

$$-\varepsilon_{\mathrm{T}} = \nu_{\mathrm{LT}}\varepsilon_{\mathrm{L}} = -\frac{\Delta b}{b} = -\frac{\Delta b_{\mathrm{f}} + \Delta b_{\mathrm{m}}}{b} = -[v_{\mathrm{f}}\varepsilon_{\mathrm{Tf}} + (1-v_{\mathrm{f}})\varepsilon_{Tm}],$$

$$\nu_{\mathrm{f}} = -\frac{\varepsilon_{\mathrm{Tf}}}{\varepsilon_{\mathrm{Lf}}}, \quad \nu_{\mathrm{m}} = -\frac{\varepsilon_{\mathrm{Tm}}}{\varepsilon_{\mathrm{Lm}}}$$

it follows that

$$\varepsilon_{\mathrm{T}} = -\nu_{\mathrm{LT}}\varepsilon_{\mathrm{L}} = -v_{\mathrm{f}}\nu_{\mathrm{f}}\varepsilon_{\mathrm{Lf}} - (1-v_{\mathrm{f}})\nu_{\mathrm{m}}\varepsilon_{\mathrm{Lm}} \tag{3.1.18}$$

The longitudinal strains in the composite, the fibres and the matrix are assumed to be equal (Voigt model of parallel connection) and the equation for the major Poisson's ratio reduces to

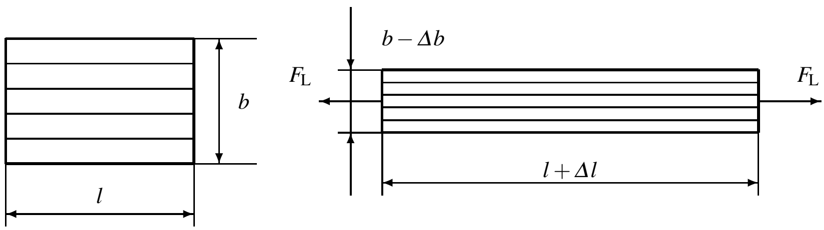


Fig. 3.4 Mechanical model to calculate the major Poisson's ration ν_{LT}

$$\nu_{\mathrm{LT}} = v_{\mathrm{f}}\nu_{\mathrm{f}} + (1 - v_{\mathrm{f}})\nu_{\mathrm{m}} = v_{\mathrm{f}}\nu_{\mathrm{f}} + v_{\mathrm{m}}\nu_{\mathrm{m}} = \phi\nu_{\mathrm{f}} + (1 - \phi)\nu_{\mathrm{m}} \tag{3.1.19}$$

The major Poisson's ratio ν_{LT} obeys the rule of mixture. The minor Poisson's ratio $\nu_{\mathrm{TL}} = -\varepsilon_{\mathrm{L}}/\varepsilon_{\mathrm{T}}$ can be derived with the symmetry condition or reciprocal relationship

$$\frac{\nu_{\mathrm{TL}}}{E_{\mathrm{T}}} = \frac{\nu_{\mathrm{LT}}}{E_{\mathrm{L}}},$$

$$\nu_{\mathrm{TL}} = \nu_{\mathrm{LT}}\frac{E_{\mathrm{T}}}{E_{\mathrm{L}}} = (v_{\mathrm{f}}\nu_{\mathrm{f}} + v_{\mathrm{m}}\nu_{\mathrm{m}})\frac{E_{\mathrm{f}}E_{\mathrm{m}}}{(v_{\mathrm{f}}E_{\mathrm{m}} + v_{\mathrm{m}}E_{\mathrm{f}})(v_{\mathrm{f}}E_{\mathrm{f}} + v_{\mathrm{m}}E_{\mathrm{m}})} \tag{3.1.20}$$

The values of Poisson's ratios for fibres or matrix material rarely differ significantly, so that neither matrix nor fibre characteristics dominate the major or the minor elastic parameters ν_{LT} and ν_{TL}.

3.1.5 Effective In-Plane Shear Modulus

Apply a pure shear stress τ to a lamina as shown in Fig. 3.5. Assuming that the shear stresses on the fibre and the matrix are the same, but the shear strains are different

$$\gamma_{\mathrm{m}} = \frac{\tau}{G_{\mathrm{m}}}, \quad \gamma_{\mathrm{f}} = \frac{\tau}{G_{\mathrm{f}}}, \quad \gamma = \frac{\tau}{G_{\mathrm{LT}}} \tag{3.1.21}$$

The model is a connection in series (Reuss model) and therefore

$$\tau = \tau_{\mathrm{f}} = \tau_{\mathrm{m}}, \quad \Delta = \Delta_{\mathrm{f}} + \Delta_{\mathrm{m}}, \quad \Delta = \hat{b}\tan\gamma = \gamma_{\mathrm{f}}\hat{b}_{\mathrm{f}} + \gamma_{\mathrm{m}}\hat{b}_{\mathrm{m}} \tag{3.1.22}$$

and with

$$\hat{b} = \hat{b}_{\mathrm{f}} + \hat{b}_{\mathrm{m}} = (v_{\mathrm{f}} + v_{\mathrm{m}})\hat{b} = v_{\mathrm{f}}\hat{b} + (1 - v_{\mathrm{f}})\hat{b} \tag{3.1.23}$$

it follows that

$$\Delta_{\mathrm{f}} = \gamma_{\mathrm{f}}v_{\mathrm{f}}\hat{b}, \quad \Delta_{\mathrm{m}} = \gamma_{\mathrm{m}}(1 - v_{\mathrm{f}})\hat{b} \tag{3.1.24}$$

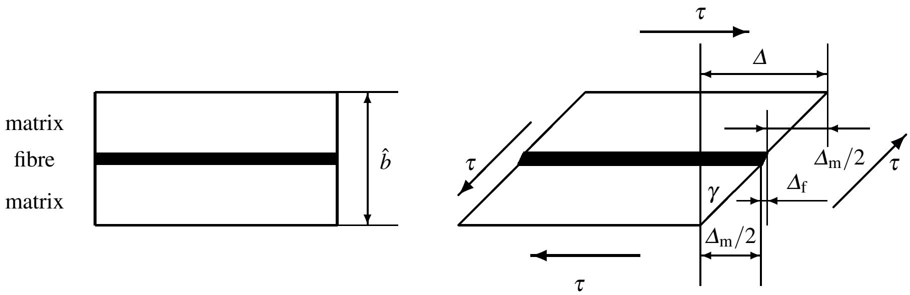


Fig. 3.5 Mechanical model to calculate the effective in-plane shear modulus G_{LT}

Using Hooke's law we have $\tau/G_{\mathrm{LT}} = (\tau/G_{\mathrm{m}})v_{\mathrm{m}} + (\tau/G_{\mathrm{f}})v_{\mathrm{f}}$ which yields

$$G_{\mathrm{LT}} = \frac{G_{\mathrm{m}}G_{\mathrm{f}}}{(1-v_{\mathrm{f}})G_{\mathrm{f}} + v_{\mathrm{f}}G_{\mathrm{m}}} = \frac{G_{\mathrm{m}}G_{\mathrm{f}}}{(1-\phi)G_{\mathrm{f}} + \phi G_{\mathrm{m}}} \qquad (3.1.25)$$

or by analogy to (3.1.16)

$$\frac{1}{G_{\mathrm{LT}}} = \frac{v_{\mathrm{f}}}{G_{\mathrm{f}}} + \frac{1-v_{\mathrm{f}}}{G_{\mathrm{m}}} = \frac{v_{\mathrm{f}}}{G_{\mathrm{f}}} + \frac{v_{\mathrm{m}}}{G_{\mathrm{m}}}$$

which is again a Reuss estimate. Note that assuming isotropic fibres and matrix material one gets

$$G_{\mathrm{f}} = \frac{E_{\mathrm{f}}}{2(1+\nu_{\mathrm{f}})}, \quad G_{\mathrm{m}} = \frac{E_{\mathrm{m}}}{2(1+\nu_{\mathrm{m}})} \qquad (3.1.26)$$

3.1.6 Discussion on the Elementary Mixture Rules

Summarizing the rule of mixtures as a simple model to predict effective engineering moduli it must be kept in mind that there is no interaction between fibres and matrix. There are only two different types of material response: the Voigt or iso-strain model in which applied strain is the same in both material phases (parallel response) and the Reuss or iso-stress model in which the applied stress is the same in both material phases (series response).

For an aligned fibre composite the effective material behavior may be assumed as transversally isotropic and five independent effective engineering moduli have to be estimated. With $x_2 - x_3$ as the plane of isotropy (Table 2.5) we have

$$\begin{aligned}
&E_1 = E_{\mathrm{L}}, \quad E_2 = E_3 = E_{\mathrm{T}}, \quad E_4 = G_{23} = G_{\mathrm{TT}} = E_{\mathrm{T}}/[2(1+\nu_{\mathrm{TT}})],\\
&E_5 = G_{13} = E_6 = G_{12} = G_{\mathrm{LT}}, \quad \nu_{12} = \nu_{13} = \nu_{\mathrm{LT}}, \quad \nu_{23} = \nu_{\mathrm{TT}},\\
&\nu_{\mathrm{LT}}E_{\mathrm{T}} = \nu_{\mathrm{TL}}E_{\mathrm{L}}
\end{aligned}$$

If we make choice of $E_{\mathrm{L}}, E_{\mathrm{T}}, G_{\mathrm{LT}}, G_{\mathrm{TT}}, \nu_{\mathrm{LT}}$ as the five independent moduli the rules of mixture yield

$$\begin{aligned}
E_{\mathrm{L}} &= v_{\mathrm{f}}E_{\mathrm{f}} + (1-v_{\mathrm{f}})E_{\mathrm{m}},\\
E_{\mathrm{T}} &= \frac{E_{\mathrm{f}}E_{\mathrm{m}}}{v_{\mathrm{f}}E_{\mathrm{m}} + (1-v_{\mathrm{f}})E_{\mathrm{f}}},\\
G_{\mathrm{LT}} &= \frac{G_{\mathrm{f}}G_{\mathrm{m}}}{v_{\mathrm{f}}G_{\mathrm{m}} + (1-v_{\mathrm{f}})G_{\mathrm{f}}},\\
\nu_{\mathrm{LT}} &= v_{\mathrm{f}}\nu_{\mathrm{f}} + (1-v_{\mathrm{f}})\nu_{\mathrm{m}}
\end{aligned} \qquad (3.1.27)$$

The shear modulus G_{TT} corresponds to an iso-shear strain model and is analogous to the axial tensile modulus case

$$G_{\mathrm{TT}} = v_{\mathrm{f}}G_{\mathrm{f}} + (1-v_{\mathrm{f}})G_{\mathrm{m}} \qquad (3.1.28)$$

It may be noted that neither the iso-shear stress nor the iso-shear strain condition for G_{LT} and G_{TT} estimation are close to the real situation of shearing loaded fibre reinforced composites. Therefore the equations for G_{LT} and G_{TT} cannot be expected as very reliable.

If one considers the approximative predictions for the effective moduli E_{L} and E_{T} as a function of the fibre volume fraction v_{f}, i.e., $E_{\mathrm{L}} = E_{\mathrm{L}}(v_{\mathrm{f}})$, $E_{\mathrm{T}} = E_{\mathrm{T}}(v_{\mathrm{f}})$, and the ratio $E_{\mathrm{f}}/E_{\mathrm{m}}$ is fixed, it is clear that reinforcing a matrix by fibres mainly influences the stiffness in fibre direction (E_{L} is a linear function of v_{f}) and rather high fibre volume fractions are necessary to obtain a significant stiffness increase in the transverse direction (E_{T} is a non-linear function of v_{f} and rather constant in the interval $0 < v_{\mathrm{f}} < 0{,}5$).

Very often fibres material behavior is transversally isotropic but the matrix material is isotropic. For such cases simple alternative relations for the effective engineering moduli of the UD-lamina can be given

$$\begin{aligned} E_{\mathrm{L}} &= v_{\mathrm{f}}E_{\mathrm{f}} + (1-v_{\mathrm{f}})E_{\mathrm{m}}, & \nu_{\mathrm{LT}} &= v_{\mathrm{f}}\nu_{\mathrm{LTf}} + (1-v_{\mathrm{f}})\nu_{\mathrm{m}}, \\ E_{\mathrm{T}} &= \frac{E_{\mathrm{Tf}}E_{\mathrm{m}}}{v_{\mathrm{f}}E_{\mathrm{m}} + (1-v_{\mathrm{f}})E_{\mathrm{Tf}}}, & \nu_{TT} &= \frac{\nu_{\mathrm{TTf}}\nu_{\mathrm{m}}}{v_{\mathrm{f}}\nu_{\mathrm{m}} + (1-v_{\mathrm{f}})\nu_{\mathrm{TTf}}}, \\ G_{\mathrm{LT}} &= \frac{G_{\mathrm{LTf}}G_{\mathrm{m}}}{v_{\mathrm{f}}G_{\mathrm{m}} + (1-v_{\mathrm{f}})G_{\mathrm{LTf}}}, & G_{TT} &= v_{\mathrm{f}}G_{\mathrm{f}} + (1-v_{\mathrm{f}})G_{\mathrm{m}} \end{aligned} \tag{3.1.29}$$

In Eq. (3.1.29) $E_{\mathrm{m}}, \nu_{\mathrm{m}}, G_{\mathrm{m}} = E_{\mathrm{m}}/2(1+\nu_{\mathrm{m}})$ are the isotropic matrix moduli and $E_{\mathrm{f}}, E_{\mathrm{Tf}}, G_{\mathrm{LTf}}, G_{\mathrm{f}} = E_{\mathrm{Tf}}/2(1+\nu_{\mathrm{TTf}}), \nu_{\mathrm{TTf}}, \nu_{\mathrm{LTf}}$ are transversally isotropic fibre moduli. $E_{\mathrm{m}}, \nu_{\mathrm{m}}$ of the matrix material and $E_{\mathrm{f}}, E_{\mathrm{Tf}}, G_{\mathrm{LTf}}, \nu_{\mathrm{LTf}}, \nu_{\mathrm{TTf}}$ or G_{TTf} of the fibre material can be chosen as the independent moduli.

3.2 Improved Formulas for Effective Moduli of Composites

Effective elastic moduli related to loading in the fibre direction, such as E_{L} and ν_{LT}, are dominated by the fibres. All estimations in this case and experimental results are very close to the rule of mixtures estimation. But the values obtained for transverse Young's modulus and in-plane shear modulus with the rule of mixtures which can be reduced to the two model connections of Voigt and Reuss, do not agree well with experimental results. This establishes a need for better modelling techniques based on elasticity solutions and variational principle models and includes analytical and numerical solution methods.

Unfortunately, the theoretical models are only available in the form of complicated mathematical equations and the solution is very limited and needs huge effort. Semi-empirical relationships have been developed to overcome the difficulties with purely theoretical approaches.

The most useful of those semi-empirical models are those of Halpin and Tsai[3] which can be applied over a wide range of elastic properties and fibre volume fractions. The Halpin-Tsai relationships have a consistent form for all properties. They are developed as simple equations by curve fitting to results that are based on the theory of elasticity.

Starting from results obtained in theoretical analysis, Halpin and Tsai proposed equations that are general and simple in formulation. The moduli of a unidirectional composite are given by the following equations

- E_{L} and ν_{LT} by the law of mixtures Eqs. (3.1.8), (3.1.19)
- For the other moduli by

$$\frac{M}{M_{\mathrm{m}}} = \frac{1+\xi\eta v_{\mathrm{f}}}{1-\eta v_{\mathrm{f}}} \tag{3.2.1}$$

  M is the modulus under consideration, e.g. E_{T}, G_{LT}, ..., η is a coefficient given by

$$\eta = \frac{(M_{\mathrm{f}}/M_{\mathrm{m}})-1}{(M_{\mathrm{f}}/M_{\mathrm{m}})+\xi} \tag{3.2.2}$$

  ξ is called the reinforcement factor and depends on

  - the geometry of the fibres
  - the packing arrangement of the fibres
  - the loading conditions.

The main difficulty in using (3.2.1) is the determination of the factor ξ by comparing the semi-empirical values with analytical solutions or with experimental results.

In addition to the rule of mixtures and the semi-empirical solution of Halpin and Tsai there are some solutions available which are based on elasticity models, e.g. for the model of a cylindrical elementary cell subjected to tension. The more complicated formulas for E_{L} and ν_{LT} as the formulas given above by the rule of mixtures yields practically identical values to the simpler formulas and are not useful. But the elasticity solution for the modulus G_{LT} yields much better results and should be applied

$$G_{\mathrm{LT}} = G_{\mathrm{m}}\frac{G_{\mathrm{f}}(1+v_{\mathrm{f}})+G_{\mathrm{m}}(1-v_{\mathrm{f}})}{G_{\mathrm{f}}(1-v_{\mathrm{f}})+G_{\mathrm{m}}(1+v_{\mathrm{f}})} = G_{\mathrm{m}}\frac{G_{\mathrm{f}}(1+\phi)+G_{\mathrm{m}}(1-\phi)}{G_{\mathrm{f}}(1-\phi)+G_{\mathrm{m}}(1+\phi)} \tag{3.2.3}$$

Summarizing the results of Sects. 3.1 and 3.2 the following recommendations may be possible for an estimation of effective elastic moduli of unidirectional laminae

- E_{L} and ν_{LT} should be estimated by the rule of mixtures
- ν_{TL} follows from the reciprocal condition
- G_{LT} should be estimated from (3.2.3) or the Halpin/Tsai formulas (3.2.1) and (3.2.2)

[3] Stephen Wei-Lun Tsai (*6 July 1929, Beijing) - US-American engineer, strength criteria for composites, founder of the Journal of Composite Materials

- $E_{\rm T}$ may be estimated with help of the Halpin/Tsai formulas. But only when reliable experimental values of $E_{\rm T}$ and $G_{\rm LT}$ are available for a composite the ξ-factor can be derived for this case and can be used to predict effective moduli for a range of fibre volume ratios of the same composite. It is also possible to look for numerical or analytical solutions for ξ based on elasticity theory. In general, ξ may vary from zero to infinity, and the Reuss and Voigt models are special cases, e.g.

$$E_{\rm T} = \frac{1+\xi\eta v_{\rm f}}{1-\eta v_{\rm f}} E_{\rm m}$$

for $\xi = 0$ and $\xi = \infty$, respectively. In the case of circular cross-sections of the fibres $\xi = 2$ or $\xi = 1$ can be recommended for Halpin-Tsai equation for $E_{\rm T}$ or $G_{\rm LT}$. But it is dangerous to use uncritically these values for any given composite.

3.3 Problems

Exercise 3.1. Determine for a glass/epoxy lamina with a 70 % fibre volume fraction

1. the density and the mass fractions of the fibre and matrix,
2. the Young's moduli $E'_1 \equiv E_{\rm L}$ and $E'_2 \equiv E_{\rm T}$ and determine the ratio of a tensile load in L-direction taken by the fibres to that of the composite,
3. the major and the minor Poisson's ratio $\nu_{\rm LT}, \nu_{\rm TL}$,
4. the in-plane shear modulus

The properties of glass and epoxy are taken approximately from Tables F.1 and F.2 as $\rho_{\rm glass} \equiv \rho_{\rm f} = 2,5\ {\rm gcm}^{-3}$, $v_{\rm f} = 0,7, v_{\rm m} = 0,3$, $\rho_{\rm epoxy} \equiv \rho_{\rm m} = 1,35\ {\rm gcm}^{-3}$, $E_{\rm f} = 70$ GPa, $E_{\rm m} = 3,6$ GPa.

Solution 3.1. Taking into account the material parameters and the volume fraction of the fibres one obtains:

1. Using Eq. (1.1.3) the density of the composite is

$$\rho = \rho_{\rm f} v_{\rm f} + \rho_{\rm m} v_{\rm m} = 2,5 \cdot 0,7\ {\rm gcm}^{-3} + 1,35 \cdot 0,3\ {\rm gcm}^{-3} = 2,155\ {\rm gcm}^{-3}$$

Using Eq. (1.1.4) the mass fractions are

$$m_{\rm f} = \frac{\rho_{\rm f}}{\rho} v_{\rm f} = \frac{25}{2,155} 0,7 = 0,8121, \qquad m_{\rm m} = \frac{\rho_{\rm m}}{\rho} v_{\rm m} = \frac{1,35}{2,155} 0,3 = 0,1879$$

Note that the sum of the mass fractions must be 1

$$m_{\rm f} + m_{\rm m} = 0,8121 + 0,1879 = 1$$

2. Using Eqs. (3.1.8), (3.1.16) and (3.1.9) we have

$$E_{\rm L} = 70 \cdot 0,7\ {\rm GPa} + 3,6 \cdot 0,3\ {\rm GPa} = 50,08\ {\rm GPa},$$

$$\frac{1}{E_{\mathrm{T}}} = \frac{0,7}{70\mathrm{GPa}} + \frac{0,3}{3,6\mathrm{GPa}} = 0,09333\ \mathrm{GPa}^{-1}, \qquad E_{\mathrm{T}} = 10,71\ \mathrm{GPa},$$
$$\frac{F_{\mathrm{Lf}}}{F_{\mathrm{L}}} = \frac{E_{\mathrm{Lf}}}{E_{\mathrm{L}}} v_{\mathrm{f}} = \frac{70}{50,08} 0,7 = 0,9784$$

The ratio of the tensile load F_{L} taken by the fibre is $0,9784$.

3. Using Eqs. (3.1.19) and (3.1.20) follows

$$v_{\mathrm{LT}} = 0.2 \cdot 0.7 + 0.3 \cdot 0.3 = 0.230, \qquad v_{\mathrm{TL}} = 0.230 \frac{10.71}{50.08} = 0,049$$

4. Using (3.1.26) and (3.1.25)

$$G_{\mathrm{f}} = \frac{70\ \mathrm{GPa}}{2(1+0.2)} = 29.17\ \mathrm{GPa}, \qquad G_{\mathrm{m}} = \frac{3.6\ \mathrm{GPa}}{2(1+0,3)} = 1.38\ \mathrm{GPa},$$
$$G_{\mathrm{LT}} = \frac{1.38\ \mathrm{GPa} \cdot 29.17\ \mathrm{GPa}}{29.17\ \mathrm{GPa} \cdot 0.3 + 1,38\ \mathrm{GPa} \cdot 0.7)} = 4.14\ \mathrm{GPa}$$

Using (3.2.3)

$$G_{\mathrm{LT}} = 1.38\ \mathrm{GPa} \frac{29.17\ \mathrm{GPa} \cdot 1.7 + 1.38\ \mathrm{GPa} \cdot 0.3}{29.17\ \mathrm{GPa} \cdot 0.3 + 1.38\ \mathrm{GPa} \cdot 1.7} = 6.22\ \mathrm{GPa}$$

Conclusion 3.1. The difference between the both formulae for G_{LT} is significant. The improved formulae should be used.

Exercise 3.2. Two composites have the same matrix materials but different fibre material. In the first case $E_{\mathrm{f}}/E_{\mathrm{m}} = 60$, in the second case $E_{\mathrm{f}}/E_{\mathrm{m}} = 30$. The fibre volume fraction for both cases is $v_{\mathrm{f}} = 0,6$. Compare the stiffness values E_{L} and E_{T} by $E_{\mathrm{L}}/E_{\mathrm{T}}, E_{\mathrm{L}}/E_{\mathrm{m}}, E_{\mathrm{T}}/E_{\mathrm{m}}$.

Solution 3.2. First case

$$E_{\mathrm{L1}} = E_{\mathrm{f1}} v_{\mathrm{f}} + E_{\mathrm{m}}(1 - v_{\mathrm{f}}), \qquad \frac{E_{\mathrm{f1}}}{E_{\mathrm{m}}} = 60, \quad E_{\mathrm{f1}} = 60 E_{\mathrm{m}},$$
$$E_{\mathrm{L1}} = 36 E_{\mathrm{m}} + 0,4 E_{\mathrm{m}} = 36,4 E_{\mathrm{m}}, \qquad E_{\mathrm{T1}} = \left(\frac{v_{\mathrm{f}}}{E_{\mathrm{f1}}} + \frac{1 - v_{\mathrm{f}}}{E_{\mathrm{m}}} \right)^{-1} = 2,439 E_{\mathrm{m}},$$
$$\frac{E_{\mathrm{L1}}}{E_{\mathrm{T1}}} = 14,925, \qquad \frac{E_{\mathrm{L1}}}{E_{\mathrm{m}}} = 36,4, \qquad \frac{E_{\mathrm{T1}}}{E_{\mathrm{m}}} = 2,439$$

Second case

$$E_{\mathrm{L2}} = E_{\mathrm{f2}} v_{\mathrm{f}} + E_{m}(1 - v_{\mathrm{f}}), \qquad E_{\mathrm{f2}} = 30 E_{\mathrm{m}},$$
$$E_{\mathrm{L2}} = 18 E_{\mathrm{m}} + 0,4 E_{\mathrm{m}} = 18,4 E_{\mathrm{m}}, \qquad E_{\mathrm{T2}} = \left(\frac{v_{\mathrm{f}}}{E_{\mathrm{f2}}} + \frac{1 - v_{\mathrm{f}}}{E_{\mathrm{m}}} \right)^{-1} = 2,3810 E_{\mathrm{m}},$$
$$\frac{E_{\mathrm{L2}}}{E_{\mathrm{T2}}} = 7,728, \qquad \frac{E_{\mathrm{L2}}}{E_{\mathrm{m}}} = 18,4, \qquad \frac{E_{\mathrm{T2}}}{E_{\mathrm{m}}} = 2,381$$

Conclusion 3.2. The different fibre material has a significant influence on the Young's moduli in the fibre direction. The transverse moduli are nearly the same.

Exercise 3.3. For a composite material the properties of the constituents are $E_\mathrm{f} = 90$ GPa, $\nu_\mathrm{f} = 0,2, G_\mathrm{f} = 35$ GPa, $E_\mathrm{m} = 3,5$ GPa, $\nu_\mathrm{m} = 0,3$, $G_\mathrm{m} = 1,3$ GPa. The volume fraction $v_\mathrm{f} = \phi = 60\%$. Calculate $E_\mathrm{L}, E_\mathrm{T}, G_\mathrm{LT}, \nu_\mathrm{LT}, \nu_{TL}$ with the help of the rule of mixtures and also G_LT with the improved formula.

Solution 3.3.

$$
\begin{aligned}
&E_\mathrm{L} = E_\mathrm{f} v_\mathrm{f} + E_\mathrm{m}(1 - v_\mathrm{f}) = 55,4\ \mathrm{GPa}, \qquad E_\mathrm{T} = \left(\frac{v_\mathrm{f}}{E_\mathrm{f}} + \frac{v_\mathrm{m}}{E_\mathrm{m}}\right)^{-1} = 8,27\ \mathrm{GPa},\\
&\nu_\mathrm{LT} = \nu_\mathrm{f} v_\mathrm{f} + \nu_\mathrm{m}(1 - v_\mathrm{f}) = 0,24, \qquad \nu_\mathrm{TL} = \frac{\nu_\mathrm{LT} E_\mathrm{T}}{E_\mathrm{L}} = 0,0358,\\
&G_\mathrm{LT} = \frac{G_\mathrm{m} G_\mathrm{f}}{(1 - v_\mathrm{f}) G_\mathrm{f} + v_\mathrm{f} G_\mathrm{m}} = 3,0785\ \mathrm{GPa}
\end{aligned}
$$

The improved formula (3.2.3) yields

$$
G_\mathrm{LT} = G_\mathrm{m} \frac{(1 + v_\mathrm{f}) G_\mathrm{f} + (1 - v_\mathrm{f}) G_\mathrm{m}}{(1 - v_\mathrm{f}) G_\mathrm{f} + (1 + v_\mathrm{f}) G_\mathrm{m}} = 4,569\ \mathrm{GPa}
$$

Conclusion 3.3. The difference in the G_LT values calculated using the rule of mixtures and the improved formula is again significant.

Exercise 3.4. A unidirectional glass/epoxy lamina is composed of 70% by volume of glass fibres in the epoxy resin matrix. The material properties are $E_\mathrm{f} = 85$ GPa, $E_\mathrm{m} = 3,4$ GPa.

1. Calculate E_L using the rule of mixtures.
2. What fraction of a constant tensile force F_L is taken by the fibres and by the matrix?

Solution 3.4. Withe the assumed material parameters and fibre volume fraction one obtains

1. $E_\mathrm{L} = E_\mathrm{f} v_\mathrm{f} + E_\mathrm{m}(1 - v_\mathrm{f}) = 60,52$ GPa,
2. $F_\mathrm{L} = \sigma_\mathrm{L} A = E_\mathrm{L} \varepsilon A, F_\mathrm{f} = \sigma_\mathrm{f} A = E_\mathrm{f} \varepsilon A_\mathrm{f}, F_\mathrm{m} = \sigma_\mathrm{m} A = E_\mathrm{m} \varepsilon A_\mathrm{m}$
   With $F_\mathrm{L} = F_\mathrm{f} + F_\mathrm{m}$ it follows that

$$
\begin{aligned}
&E_\mathrm{L} \varepsilon A = E_\mathrm{f} \varepsilon A_\mathrm{f} + E_\mathrm{m} \varepsilon A_\mathrm{m} \Rightarrow E_\mathrm{L} = 0,7 E_\mathrm{f} + 0,3 E_\mathrm{m},\\
&60,52 = 0,7 \cdot 85 + 3 \cdot 3,4 = 59,5 + 1,02
\end{aligned}
$$

   and therefore $F_\mathrm{L} = 60,52$ N, $F_\mathrm{f} = 59,5$ N, $F_\mathrm{m} = 1,02$ N.

   Conclusion 3.4. The fractions of a constant tensile load in the fibres and the matrix are: Fibres: 98,31 %, Matrix: 1,69 %

Exercise 3.5. The fibre and the matrix characteristics of a lamina are $E_\mathrm{f} = 220$ GPa, $E_\mathrm{m} = 3,3$ GPa, $G_\mathrm{f} = 25$ GPa, $G_\mathrm{m} = 1,2$ GPa, $\nu_\mathrm{f} = 0,15$, $\nu_\mathrm{m} = 0,37$. The fibre is transversally isootropic and transverse Young's modulus is $E_\mathrm{Tf} = 22$ GPa. The fibre volume fraction is $v_\mathrm{f} = 0,56$. The experimentally measured effective moduli are $E_\mathrm{L} = 125$ GPa, $E_\mathrm{T} = 9,1$ GPa, $G_\mathrm{LT} = 5$ GPa, $\nu_\mathrm{LT} = 0,34$.

1. compare the experimental values with predicted values based on rule of mixtures.
2. Using the Halpin-Tsai approximate model for calculating E_T and G_LT, what value of ξ must be used in order to obtain moduli that agree with experimental values?

Solution 3.5. The comparison can be be performed by results from the mixture rules and improved equations.

1. The rule of mixtures (3.1.27) yields

$$
\begin{aligned}
E_\mathrm{L} &= 0,56 \cdot 220\,\mathrm{GPa} + (1-0,56)\cdot 3,3\,\mathrm{GPa} = 124,65\,\mathrm{GPa},\\
E_\mathrm{T} &= \frac{220\,\mathrm{GPa}\cdot 3,3\,\mathrm{GPa}}{0,56\cdot 3,3\,\mathrm{GPa} + (1-0,56)\cdot 220\,\mathrm{GPa}} = 7,36\,\mathrm{GPa},\\
G_\mathrm{LT} &= \frac{25\,\mathrm{GPa}\cdot 1,2\,\mathrm{GPa}}{0,56\cdot 1,2\,\mathrm{GPa} + (1-0,56)\cdot 25\,\mathrm{GPa}} = 2,57\,\mathrm{GPa},\\
\nu_\mathrm{LT} &= 0,56\cdot 0,15 + (1-0,56)\cdot 0,37 = 0,25
\end{aligned}
$$

   It is seen that the fibre dominated modulus E_L is well predicted by the rule of mixtures, while $E_\mathrm{T}, G_\mathrm{LT}$ and ν_LT are not exactly predicted.
2. The Halpin-Tsai approximation yields with $E_\mathrm{Tf} = 22$ GPa, $\xi = 2$ for E_T

$$
E_\mathrm{T} = \frac{1+\xi\eta v_\mathrm{f}}{1-\eta v_\mathrm{f}} E_\mathrm{m}, \quad \eta = \frac{E_\mathrm{Tf}/E_\mathrm{m} - 1}{E_\mathrm{Tf}/E_\mathrm{m} + \xi} = 0,6538, \quad E_\mathrm{T} = 9,02\,\mathrm{GPa}
$$

   With a value $\xi = 2.5$ follows for G_LT

$$
G_\mathrm{LT} = \frac{1+\xi\eta v_\mathrm{f}}{1-\eta v_\mathrm{f}} G_\mathrm{m}, \quad \eta = \frac{G_\mathrm{f}/G_\mathrm{m} - 1}{G_\mathrm{f}/G_\mathrm{m} + \xi} = 0,8339, \quad G_\mathrm{LT} = 4,88\,\mathrm{GPa}
$$

   The predicted value for E_T is nearly accurate for $\xi = 2$ which is the recommended value in literature but the recommended value of $\xi = 1$ for G_LT would underestimate the predicted value significantly

$$
G_\mathrm{LT} = \frac{1+\eta v_\mathrm{f}}{1-\eta v_\mathrm{f}} G_\mathrm{m}, \quad \eta = \frac{G_\mathrm{f}/G_\mathrm{m} - 1}{G_\mathrm{f}/G_\mathrm{m} + 1} = 0,9084, \quad G_\mathrm{LT} = 3,68\,\mathrm{GPa}
$$

   It can be seen that it is dangerous to accept these values uncritically without experimental measurements.

Exercise 3.6. Let us assume the following material parameters for the shear modulus of the fibre and the matrix: $G_\mathrm{f} = 2,5\cdot 10^4$ Nmm^{-2}, $G_\mathrm{m} = 1,2\cdot 10^3$ Nmm^{-2}. The experimental value of the effective shear modulus is $G_\mathrm{LT}^\mathrm{exp} = 5\cdot 10^3$ Nmm^{-2}, the Poisson's ratios are $\nu_\mathrm{f} = 0,15, \nu_\mathrm{m} = 0,37$ and the fibre volume fraction is $v_\mathrm{f} = 0,56$. For the improvement in the sense of Halpin-Tsai we assume $\xi = 1\ldots 2$ for a circular cross-section of the fibres. Compute:

1. the effective shear modulus,
2. the improved effective shear modulus,

3. the improved effective shear modulus in the sense of Halpin-Tsai

Solution 3.6. The three problems have the following solutions:

1. the effective shear modulus after Eq. (3.1.25)

$$
\begin{aligned}
G_{\mathrm{LT}} &= \frac{1,2\cdot 10^3\,\mathrm{Nmm}^{-2}\cdot 2,5\cdot 10^4\,\mathrm{Nmm}^{-2}}{(1-0,56)\cdot 2,5\cdot 10^4\,\mathrm{Nmm}^{-2}+0,56\cdot 1,2\cdot 10^3\,\mathrm{Nmm}^{-2}}\\
&= 2,57\cdot 10^3\,\mathrm{Nmm}^{-2}
\end{aligned}
$$

2. the improved effective shear modulus after Eq. (3.2.3)

$$
\begin{aligned}
G_{\mathrm{LT}} &= 1,2\cdot 10^3\frac{\mathrm{N}}{\mathrm{mm}^2}\,\frac{2,5\cdot 10^4\,\mathrm{Nmm}^{-2}(1+0,56)+1,2\cdot 10^3\,\mathrm{Nmm}^{-2}(1-0,56)}{2,5\cdot 10^4\,\mathrm{Nmm}^{-2}(1-0,56)+1,2\cdot 10^3\,\mathrm{Nmm}^{-2}(1+0,56)}\\
&= 3,68\cdot 10^3\,\mathrm{Nmm}^{-2}
\end{aligned}
$$

3. For the improved effective shear modulus in the sense of Halpin-Tsai at first should be estimated η with Eq. (3.2.2)

$$
\eta = \frac{(2,5\cdot 10^4\,\mathrm{Nmm}^{-2}/1,2\cdot 10^3\,\mathrm{Nmm}^{-2})-1}{(2,5\cdot 10^4\,\mathrm{Nmm}^{-2}/1,2\cdot 10^3\,\mathrm{Nmm}^{-2})+\xi}
$$

In the case of $\xi=1$ we get $\eta=0,9084$, if $\xi=2$ we get $\eta=0,8686$. The shear modulus itself follows from Eq. (3.2.1)

$$
G_{\mathrm{LT}} = \frac{1+\xi\eta\cdot 0,15}{1-\eta\cdot 0,15}1,2\cdot 10^3\,\mathrm{Nmm}^{-2}
$$

Finally we get

$$
G_{\mathrm{LT}}(\xi=1) = 3,68\cdot 10^3\,\mathrm{Nmm}^{-2},\quad G_{\mathrm{LT}}(\xi=2) = 4,61\cdot 10^3\,\mathrm{Nmm}^{-2}
$$

The last result is the closest one to the experimental value. Let us compute the deviation

$$
\delta = \frac{G_{\mathrm{LT}}^{\mathrm{exp}}-G_{\mathrm{LT}}^{\mathrm{comp}}}{G_{\mathrm{LT}}^{\mathrm{exp}}}\cdot 100\%
$$

The results are

$$
\begin{aligned}
\delta(\text{case a}) &= \frac{5\cdot 10^3\,\mathrm{Nmm}^{-2}-2,57\cdot 10^3\,\mathrm{Nmm}^{-2}}{5\cdot 10^3\,\mathrm{Nmm}^{-2}}\cdot 100\% = 46,6\%\\
\delta(\text{case b}) &= \frac{5\cdot 10^3\,\mathrm{Nmm}^{-2}-2,66\cdot 10^3\,\mathrm{Nmm}^{-2}}{5\cdot 10^3\,\mathrm{Nmm}^{-2}}\cdot 100\% = 26,8\%\\
\delta(\text{case c}, \xi=1) &= \frac{5\cdot 10^3\,\mathrm{Nmm}^{-2}-3,68\cdot 10^3\,\mathrm{Nmm}^{-2}}{5\cdot 10^3\,\mathrm{Nmm}^{-2}}\cdot 100\% = 26,3\%\\
\delta(\text{case c}, \xi=2) &= \frac{5\cdot 10^3\,\mathrm{Nmm}^{-2}-4,61\cdot 10^3\,\mathrm{Nmm}^{-2}}{5\cdot 10^3\,\mathrm{Nmm}^{-2}}\cdot 100\% = 7,8\%
\end{aligned}
$$

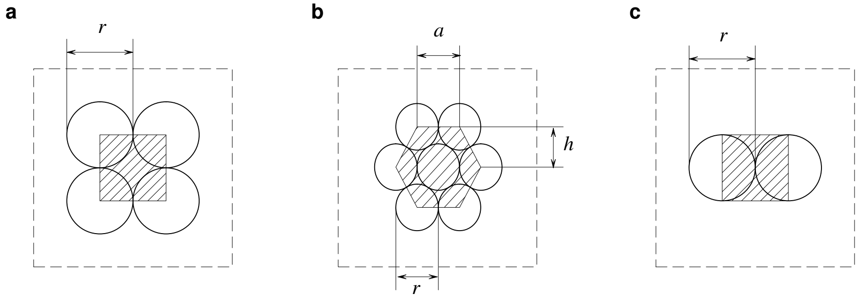


Fig. 3.6 Fibre arrangements. **a** Square array, **b** hexagonal array, **c** layer-wise array

The result can be improved if the ξ-value will be increased.

Exercise 3.7. Calculate the ultimate fibre volume fractions v_f^u for the following fibre arrangements:

1. square array,
2. hexagonal array,
3. layer-wise array.

Assume circular fibre cross-sections.

Solution 3.7. For the three fibre arrangements one gets:

1. Square array (Fig. 3.6 a)

$$v_f^u = \frac{A_f}{A_c} = \frac{4(r^2\pi)/4}{4r^2} = \frac{\pi}{4} = 0.785$$

2. Hexagonal array (Fig. 3.6 b) with $a \to 2r$, $h \to \sqrt{3}r$ follow

$$A_c = hr, \quad A_f = 6(1/3)\pi r^2 + \pi r^2$$

and

$$v_f^u = \frac{A_f}{A_c} = \frac{6(1/3)\pi r^2 + \pi r^2)/4}{6\sqrt{3}r^2} = \frac{\pi}{2\sqrt{3}} = 0.9069$$

3. Layer-wise array (Fig. 3.6 c)

$$v_f^u = \frac{A_f}{A_c} = \frac{r^2\pi}{(2r)^2} = \frac{\pi}{4} = 0.785$$

Part II
Modelling of a Single Laminae, Laminates and Sandwiches

The second part (Chaps. 4–6) can be related to the modelling from single laminae to laminates including sandwiches, the improved theories and simplest failure concepts.

The single layer (lamina) is modelled with the help of the following assumptions

- linear-elastic isotropic behaviour of the matrix and the fibre materials,
- the fibres are unidirectional oriented and uniformly distributed

These assumptions result in good stiffness approximations in the longitudinal and transverse directions. The stiffness under shear is not well-approximated.

After the transfer from the local to the global coordinates the stiffness parameters for the laminate can be estimated. For the classical cases the effective stiffness parameters of the laminate is a simple sum up over the laminate thickness of the weighted laminae reduced stiffness parameters transferred into the global coordinate system. Some improved theories are briefly introduced.

The failure concepts are at the moment a research topic characterised by a large amount of suggestions for new criteria. With respect to this only some classical concepts are discussed here.

Chapter 4
Elastic Behavior of Laminate and Sandwich Composites

A lamina has been defined as a thin single layer of composite material. A lamina or ply is a typical sheet of composite materials, which is generally of a thickness of the order 1 mm. A laminate is constructed by stacking a number of laminae in the direction of the lamina thickness. The layers are usually bonded together with the same matrix material as in the single lamina. A laminate bonded of n ($n \geq 2$) laminae of nearly the same thickness. A sandwich can be defined as a special case of a laminate with $n = 3$. Generally, the sandwich is made of a material of low density for the inner layer, the core or the supporting pith respectively, and of high strength material for the outer layers, the cover or face sheets. The thickness of the core is generally much greater than the thickness of the sheets and core and sheets are bonded to each other at the surfaces.

The design and analysis of structures composed of composite materials demands knowledge of the stresses and strains in laminates or sandwiches. However, the laminate elements are single laminae and so understanding the mechanical behavior of a lamina precedes that of a laminate. Section 4.1 introduces elastic behavior of laminae. For in-plane and out-of-plane loading, the stress resultants are formulated and basic formulae for stress analysis are derived. These considerations are expanded to laminates and sandwiches in Sects. 4.2 and 4.3. The governing equations of the classical laminate theory, the shear deformation theory and of a layer-wise theory are discussed in Chap. 5. A successful design of composite structures requires knowledge of the strength and the reliability of composite materials. Strength failure theories have to be developed in order to compare the actual stress states in a material to a failure criteria. Chapter 6 gives an overview on fracture modes of laminae. For laminates, the strength is related to the strength of each individual lamina. Various failure theories are discussed for laminates or sandwiches based on the normal and shear strengths of unidirectional laminae.

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0_4

4.1 Elastic Behavior of Laminae

The macro-mechanical modelling and analysis of a lamina is based on average material properties and by considering the lamina to be homogeneous. The methods to find these average properties based on the individual mechanical values of the constituents are discussed in Sect. 3.1. Otherwise the mechanical characterization of laminae can be determined experimentally but it demands special experimental equipment and is costly and time-consuming. Generally the modelling goal is to find the minimum of parameters required for the mechanical characterization of a lamina.

For the considerations on the elastic behavior of laminae in the following Sects. 4.1.1 - 4.1.3 one has to keep in mind that two assumptions are most important to model the mechanics of fibre reinforced laminae:

- The properties of the fibres and the matrix can be smeared into an equivalent homogeneous material with orthotropic behavior. This assumption allows to develop the stress-strain relations and to formulate the response of a fibre-reinforced lamina sufficient simply to deal with the structural level response in a tractable manner.
- Three of the six components of stress state are generally much smaller than the other three, i.e. the plane stress assumption, which is based on the manner in which fibre-reinforced materials are used in such structural elements as beams, plates or shells, will be sufficient accurately. With the assumption that the $(x_1 - x_2)$-plane of the principal co-ordinate system is in-plane stress state, the in-plane stress components $\sigma_1, \sigma_2, \sigma_6$ are considered to be much larger in value than the out-of-plane stress components $\sigma_3, \sigma_4, \sigma_5$ and the last ones are set approximately to zero.

Using the plane stress assumption it has to be in mind that some serious inaccuracies in the mechanical response of laminates can be occurred, Sect. 4.2.

Therefore, together with the plane stress assumption two major misconceptions should be avoided:

- The stress components $\sigma_3, \sigma_4, \sigma_5$ equated to zero have to be estimated to their magnitude and effect. Fibre reinforced material is often very poor in resisting stresses transversely to the plane $(x_1 - x_2)$ and therefore out-of-plane stresses may be small but large enough to cause failure of the composite material.
- With assuming σ_3 is zero does not follow that the associated strain ε_3 is also zero and ignorable, for the stresses in the $(x_1 - x_2)$-plane can cause a significant strain response in the x_3-direction.

4.1.1 On-Axis Stiffness and Compliances of UD-Laminae

A thin lamina is assumed to be in a plane stress state (Sect. 2.1.5). Three cases of material behavior of laminae are of special interest for engineering applications:

1. Short fibres or particle reinforced components with random orientation in the matrix
   The elastic behavior has no preferred direction and is macroscopically quasi-homogenous and isotropic. The effective elastic moduli are E and ν and the relations of the in-plane stress components with the in-plane strain components are described (Tables 2.6 and 2.7) by

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_6\end{bmatrix}=\begin{bmatrix}Q_{11}&Q_{12}&0\\Q_{12}&Q_{11}&0\\0&0&Q_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_6\end{bmatrix},$$
$$\begin{bmatrix}\varepsilon_1\\\varepsilon_2\\\varepsilon_6\end{bmatrix}=\begin{bmatrix}S_{11}&S_{12}&0\\S_{12}&S_{11}&0\\0&0&S_{66}\end{bmatrix}\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_6\end{bmatrix}\tag{4.1.1}$$

   with

$$\begin{aligned}Q_{11}&=\frac{E}{1-\nu^2}, & S_{11}&=\frac{1}{E},\\Q_{12}&=\frac{E\nu}{1-\nu^2}, & S_{12}&=-\frac{\nu}{E},\\Q_{66}&=G=\frac{E}{2(1+\nu)}, & S_{66}&=\frac{1}{G}=\frac{2(1+\nu)}{E}\end{aligned}$$

2. Long fibres with one unidirectional fibre orientation, so-called unidirectional laminae or UD-laminae, with loading along the material axis (on-axis case)
   This type of material forms the basic configuration of fibre composites and is the main topic of this textbook. The elastic behavior of UD-laminae depends on the loading reference coordinate systems. In the on-axis case the reference axes $(1',2')$ are identical to the material or principal axes of the lamina parallel and transverse to the fibre direction (Fig. 4.1). The $1'$-axis is also denoted as L-axis and the $2'$-axis as T-axis (on-axis case). The elastic behavior is macroscopically quasi-homogeneous and orthotropic with four independent material moduli (Table 2.6)

$$E'_1\equiv E_{\mathrm{L}},\quad E'_2\equiv E_{\mathrm{T}},\quad E'_6\equiv G'_{12}\equiv G_{\mathrm{LT}},\quad \nu'_{12}\equiv\nu_{\mathrm{LT}}\tag{4.1.2}$$

   and the in-plane stress-strain relations are

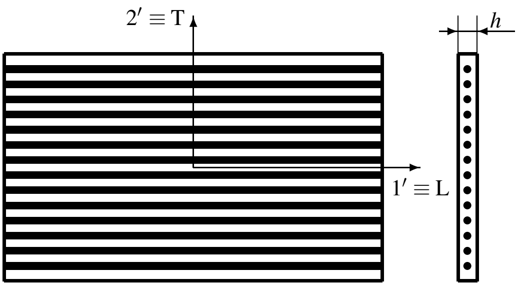


Fig. 4.1 Unidirectional lamina with principal material axis L and T (on-axis)

$$\begin{bmatrix}\sigma'_1\\ \sigma'_2\\ \sigma'_6\end{bmatrix} = \begin{bmatrix}Q'_{11} & Q'_{12} & 0\\ Q'_{12} & Q'_{22} & 0\\ 0 & 0 & Q'_{66}\end{bmatrix}\begin{bmatrix}\varepsilon'_1\\ \varepsilon'_2\\ \varepsilon'_6\end{bmatrix},$$
$$\begin{bmatrix}\varepsilon'_1\\ \varepsilon'_2\\ \varepsilon'_6\end{bmatrix} = \begin{bmatrix}S'_{11} & S'_{12} & 0\\ S'_{12} & S'_{22} & 0\\ 0 & 0 & S'_{66}\end{bmatrix}\begin{bmatrix}\sigma'_1\\ \sigma'_2\\ \sigma'_6\end{bmatrix} \qquad (4.1.3)$$

with

$$\begin{aligned} Q'_{11} &= E'_1/(1-\nu'_{12}\nu'_{21}), & S'_{11} &= 1/E'_1,\\ Q'_{22} &= E'_2/(1-\nu'_{12}\nu'_{21}), & S'_{22} &= 1/E'_2,\\ Q'_{66} &= G'_{12} = E'_6, & S'_{66} &= 1/G'_{12} = 1/E'_6,\\ Q'_{12} &= E'_2\nu'_{12}/(1-\nu'_{12}\nu'_{21}), & S'_{12} &= -\nu'_{12}/E'_1 = -\nu'_{21}/E'_2 \end{aligned}$$

3. UD-laminae with loading along arbitrary axis (x_1, x_2) different from the material axis (off-axis case). The elastic behavior is macroscopically quasi-homogeneous and anisotropic. The in-plane stress-strain relations are formulated by fully populated matrices with all Q_{ij} and S_{ij} different from zero but the number of independent material parameters is still four as in case 2. The transformation rules are given in detail in Sect. 4.1.2.

A UD-lamina has different stiffness in the direction of the material axes. With $E_\mathrm{f} \gg E_\mathrm{m}$ the stiffness in the L-direction is fibre dominated and for the effective moduli (Sect. 3.1) $E_\mathrm{L} \gg E_\mathrm{T}$. Figure 4.2 illustrates qualitatively the on-axis elastic behavior of the UD-lamina. In thickness direction $x_3 \equiv \mathrm{T}'$ orthogonal to the (L-T)-plane a UD-lamina is macro-mechanically quasi-isotropic. The elastic behavior in the thickness direction is determined by the matrix material and a three-dimensional model of a single UD-layer yields a transversely-isotropic response with five independent material engineering parameters:

$$\begin{aligned} &E'_1 \equiv E_\mathrm{L}, \quad E'_2 \equiv E_\mathrm{T} = E'_3 \equiv E_{\mathrm{T}'}, \quad E'_5 = E'_6 \equiv G_\mathrm{LT},\\ &\nu'_{12} \equiv \nu_\mathrm{LT} = \nu'_{13} \equiv \nu_{\mathrm{LT}'},\\ &E'_4 \equiv G_{\mathrm{TT}'} = E'_2/[2(1+\nu'_{23})] \equiv E_\mathrm{T}/[2(1+\nu_{\mathrm{TT}'})] \end{aligned} \qquad (4.1.4)$$

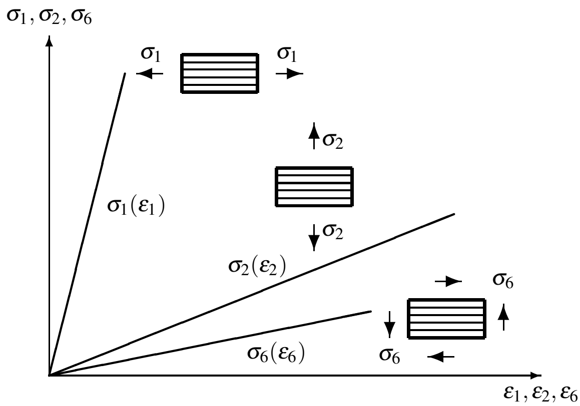


Fig. 4.2 On-axis stress-strain equations for UD-lamina (qualitative)

The material behavior in the $2' \equiv \mathrm{T}$ and $3 = \mathrm{T}'$ directions is equivalent. Therefore, the notation of the engineering parameters is given by $E'_2 = E'_3 \equiv E_\mathrm{T}$, $E'_5 = E'_6 \equiv G_\mathrm{LT}$, $\nu'_{12} = \nu'_{13} \equiv \nu_\mathrm{LT}$, $E'_4 = E'_2/[2(1+\nu'_{23})] \equiv G_\mathrm{TT} = E_\mathrm{T}/[2(1+\nu_\mathrm{TT})]$. Summarizing, the stress-strain relations for on-axis loading of UD-laminae in a contracted vector-matrix notation leads the equations

$$\begin{array}{ll} \boldsymbol{\sigma}' = \boldsymbol{Q}'\boldsymbol{\varepsilon}' \text{ or } \sigma'_i = Q'_{ij}\varepsilon'_j,\ Q'_{ij} = Q'_{ji} & \\ \boldsymbol{\varepsilon}' = \boldsymbol{S}'\boldsymbol{\sigma}' \text{ or } \varepsilon'_i = S'_{ij}\sigma'_j,\ S'_{ij} = S'_{ji} & \end{array} \qquad i,j = 1,2,6 \tag{4.1.5}$$

The values Q'_{ij} of the reduced stiffness matrix $\boldsymbol{Q}'$ and the S'_{ij} of the compliance matrix $\boldsymbol{S}'$ depend on the effective moduli of the UD-lamina. The term *reduced stiffness* is used in relations given by Eqs. (2.1.76) and (4.1.3). These relations simplify the problem from a three-dimensional to a two-dimensional or plane stress state. Also the numerical values of the stiffness Q'_{ij} are actually less than the numerical values of their respective counterparts C'_{ij}, see Eq. (2.1.79), of the three-dimensional problem and therefore the stiffness are reduced in that sense also. For on-axis loading the elastic behavior is orthotropic and with $Q'_{16} = Q'_{26} = 0$ and $S'_{16} = S'_{26} = 0$, there is as in isotropic materials no coupling of normal stresses and shear strains and also shear stresses applied in the (L-T)-plane do not result in any normal strains in the L and T direction. The UD-lamina is therefore also called a specially orthotropic lamina.

Composite materials are generally processed at high temperature and then cooled down to room temperature. For polymeric fibre reinforced composites the temperature difference is in the range of $200^0 - 300^0$ C and due to the different thermal expansion of the fibres and the matrix, residual stresses result in a UD-lamina and expansion strains are induced. In addition, polymeric matrix composites can generally absorb moisture and the moisture change leads to swelling strains and stresses similar to these due to thermal expansion. Therefore we speak of hygrothermal stresses and strains in a lamina. The hygrothermal strains in the longitudinal direction and transverse the fibre direction of a lamina are not equal since the effective elastic moduli E_L and E_T and also the thermal and moisture expansion coefficients $\alpha_\mathrm{L}^\mathrm{th}, \alpha_\mathrm{T}^\mathrm{th}$ and $\alpha_\mathrm{L}^\mathrm{mo}, \alpha_\mathrm{T}^\mathrm{mo}$ respectively, are different.

The stress-strain relations of a UD-lamina, including temperature and moisture differences are given by

$$\begin{bmatrix} \varepsilon'_1 \\ \varepsilon'_2 \\ \varepsilon'_6 \end{bmatrix} = \begin{bmatrix} S'_{11} & S'_{12} & 0 \\ S'_{12} & S'_{22} & 0 \\ 0 & 0 & S'_{66} \end{bmatrix} \begin{bmatrix} \sigma'_1 \\ \sigma'_2 \\ \sigma'_6 \end{bmatrix} + \begin{bmatrix} \varepsilon_1^{'\mathrm{th}} \\ \varepsilon_2^{'\mathrm{th}} \\ 0 \end{bmatrix} + \begin{bmatrix} \varepsilon_1^{'\mathrm{mo}} \\ \varepsilon_2^{'\mathrm{mo}} \\ 0 \end{bmatrix} \tag{4.1.6}$$

with

$$\begin{bmatrix} \varepsilon_1^{'\mathrm{th}} \\ \varepsilon_2^{'\mathrm{th}} \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1^{'\mathrm{th}} \\ \alpha_2^{'\mathrm{th}} \\ 0 \end{bmatrix} T, \qquad \begin{bmatrix} \varepsilon_1^{'\mathrm{mo}} \\ \varepsilon_2^{'\mathrm{mo}} \\ 0 \end{bmatrix} = \begin{bmatrix} \alpha_1^{'\mathrm{mo}} \\ \alpha_2^{'\mathrm{mo}} \\ 0 \end{bmatrix} M^* \tag{4.1.7}$$

T is the temperature change and M^* is weight of moisture absorption per unit weight of the lamina. $\alpha_\mathrm{L}^\mathrm{mo}, \alpha_\mathrm{T}^\mathrm{mo}$ are also called longitudinal and transverse swelling coefficients. Equation (4.1.6) can be inverted to give

$$\begin{bmatrix}\sigma_1'\\ \sigma_2'\\ \sigma_6'\end{bmatrix}=\begin{bmatrix}Q_{11}' & Q_{12}' & 0\\ Q_{12}' & Q_{22}' & 0\\ 0 & 0 & Q_{66}'\end{bmatrix}\begin{bmatrix}\varepsilon_1'-\varepsilon_1'^{\mathrm{th}}-\varepsilon_1'^{\mathrm{mo}}\\ \varepsilon_2'-\varepsilon_2'^{\mathrm{th}}-\varepsilon_2'^{\mathrm{mo}}\\ \varepsilon_6'\end{bmatrix}\tag{4.1.8}$$

Note that the temperature and moisture changes do not have any shear strain terms since no shearing is induced in the material axes. One can see that the hygrothermal behavior of an unidirectional lamina is characterized by two principal coefficients of thermal expansion, $\alpha_1'^{\mathrm{th}}, \alpha_2'^{\mathrm{th}}$, and two of moisture expansion, $\alpha_1'^{\mathrm{mo}}, \alpha_2'^{\mathrm{mo}}$. These coefficients are related to the material properties of fibres and matrix and of the fibre volume fraction.

Approximate micro-mechanical modelling of the effective hygrothermal coefficients were given by Schapery[1] and analogous to the micro-mechanical modelling of elastic parameters, Chap. 3, for a fibre reinforced lamina and isotropic constituents the effective thermal expansion coefficients are

$$\begin{aligned}\alpha_{\mathrm{L}}^{\mathrm{th}} &= \frac{\alpha_{\mathrm{f}}^{\mathrm{th}} v_{\mathrm{f}} E_{\mathrm{f}} + \alpha_{\mathrm{m}}^{\mathrm{th}}(1-v_{\mathrm{f}})E_{\mathrm{m}}}{v_{\mathrm{f}} E_{\mathrm{f}} + (1-v_{\mathrm{f}})E_{\mathrm{m}}},\\ \alpha_{\mathrm{T}}^{\mathrm{th}} &= \alpha_{\mathrm{f}}^{\mathrm{th}} v_{\mathrm{f}}(1+\nu_{\mathrm{f}}) + \alpha_{\mathrm{m}}^{\mathrm{th}}(1-v_{\mathrm{f}})(1+\nu_{\mathrm{m}}) - [v_{\mathrm{f}}\nu_{\mathrm{f}} + (1-v_{\mathrm{f}})\nu_{\mathrm{m}}]\alpha_{\mathrm{L}}^{\mathrm{th}}\end{aligned}\tag{4.1.9}$$

If the fibres are not isotropic but have different material response in axial and transverse directions, e.g. in the case of carbon or aramid fibres, the relations for $\alpha_{\mathrm{L}}^{\mathrm{th}}$ and $\alpha_{\mathrm{T}}^{\mathrm{th}}$ have to be changed to

$$\begin{aligned}\alpha_{\mathrm{L}}^{\mathrm{th}} &= \frac{\alpha_{\mathrm{Lf}}^{\mathrm{th}} v_{\mathrm{f}} E_{\mathrm{Lf}} + \alpha_{\mathrm{m}}^{\mathrm{th}}(1-v_{\mathrm{f}})E_{\mathrm{m}}}{v_{\mathrm{f}} E_{\mathrm{Lf}} + (1-v_{\mathrm{f}})E_{\mathrm{m}}},\\ \alpha_{\mathrm{T}}^{\mathrm{th}} &= (\alpha_{\mathrm{Tf}}^{\mathrm{th}} + \nu_{\mathrm{Tf}}\alpha_{\mathrm{Lf}}^{\mathrm{th}})v_{\mathrm{f}} + (1+\nu_{\mathrm{m}})\alpha_{\mathrm{m}}^{\mathrm{th}}(1-v_{\mathrm{f}})\\ &\quad - [v_{\mathrm{f}}\nu_{\mathrm{Tf}} + (1-v_{\mathrm{f}})\nu_{\mathrm{m}}]\alpha_{\mathrm{L}}^{\mathrm{th}}\end{aligned}\tag{4.1.10}$$

In most cases the matrix material can be considered isotropic and therefore the orientation designation L, T of the matrix material parameters can be dropped.

Discussion and conclusions concerning effective moduli presented in Chap. 3 are valid for effective thermal expansion coefficients too. The simple micro-mechanical approximations of effective moduli yield proper results for $\alpha_{\mathrm{L}}^{\mathrm{th}}$ but fails to predict $\alpha_{\mathrm{T}}^{\mathrm{th}}$ with the required accuracy. For practical applications $\alpha_{\mathrm{L}}^{\mathrm{th}}$ and $\alpha_{\mathrm{T}}^{\mathrm{th}}$ should be normally determined by experimental methods.

Micro-mechanical relations for effective coefficients of moisture expansion can be modelled analogously. However, some simplification can be taken into consideration. Usually the fibres, e.g. glass, carbon, boron, etc., do not absorb moisture that means $\alpha_{\mathrm{f}}^{\mathrm{mo}} = 0$. For isotropic constituents the formulae for $\alpha_{\mathrm{L}}^{\mathrm{mo}}$ and $\alpha_{\mathrm{T}}^{\mathrm{mo}}$ are

[1] Richard Allan Schapery (*3 March 1935 Duluth, Minnesota, United States) - engineering educator, contributions in the field of mechanics of composite materials

$$
\begin{aligned}
\alpha_{\mathrm{L}}^{\mathrm{mo}} &= \frac{\alpha_{\mathrm{m}}^{\mathrm{mo}} E_{\mathrm{m}}(1-v_{\mathrm{f}})}{v_{\mathrm{f}} E_{\mathrm{f}} + (1-v_{\mathrm{f}})E_{\mathrm{m}}} = \frac{\alpha_{\mathrm{m}}^{\mathrm{mo}} E_{\mathrm{m}}(1-v_{\mathrm{f}})}{E_1}, \\
\alpha_{\mathrm{T}}^{\mathrm{mo}} &= \frac{\alpha_{\mathrm{m}}^{\mathrm{mo}}(1-v_{\mathrm{f}})[(1+\nu_{\mathrm{m}})E_{\mathrm{f}} v_{\mathrm{f}} + (1-v_{\mathrm{f}})E_{\mathrm{m}} - \nu_{\mathrm{f}} v_{\mathrm{f}} E_{\mathrm{m}}]}{v_{\mathrm{f}} E_{\mathrm{f}} + (1-v_{\mathrm{f}})E_{\mathrm{m}}} \\
&= \frac{\alpha_{\mathrm{m}}^{\mathrm{mo}}(1-v_{\mathrm{f}})}{E_1}\{(1+\nu_{\mathrm{m}})E_{\mathrm{f}} v_{\mathrm{f}} + [(1-v_{\mathrm{f}}) - \nu_{\mathrm{f}} v_{\mathrm{f}}]E_{\mathrm{m}}\}
\end{aligned}
\tag{4.1.11}
$$

and for a composite with isotropic matrix but orthotropic fibres the effective moduli are given by

$$
\begin{aligned}
\alpha_{\mathrm{L}}^{\mathrm{mo}} &= \alpha_{\mathrm{m}}^{\mathrm{mo}} \frac{E_{\mathrm{m}}(1-v_{\mathrm{f}})}{E_{\mathrm{L}}}, \\
\alpha_{\mathrm{T}}^{\mathrm{mo}} &= \alpha_{\mathrm{m}}^{\mathrm{mo}} \frac{1-v_{\mathrm{f}}}{E_{\mathrm{L}}}\{(1+\nu_{\mathrm{m}})E_{\mathrm{Lf}} v_{\mathrm{f}} + [(1-v_{\mathrm{f}}) - \nu_{\mathrm{LTf}} v_{\mathrm{f}} E_{\mathrm{m}}\}
\end{aligned}
\tag{4.1.12}
$$

The formulae (4.1.9) - (4.1.12) completes the discussion about micro-mechanics in Chap. 3. Note that for a great fibre volume fraction negative $\alpha_{\mathrm{L}}^{\mathrm{th}}$ values can be predicted reflecting the dominance of negative values of fibre expansion α_L^{f}, e.g. for graphite-reinforced material.

Summarizing one has to keep in mind that with the plane stress assumption referred to the principal material axis L, T the mechanical shear strains and the total shear strain are identical, i.e. $\varepsilon_6^{\prime\mathrm{th}} = \varepsilon_6^{\prime\mathrm{mo}} \equiv 0$, Eqs. (4.1.6) - (4.1.8). Also the through-the-thickness total strains ε_4', ε_5' are zero and there are no mechanical, thermal or moisture strains. The conclusion regarding the normal strains ε_3 is not the same. Using the condition $\sigma_3 = 0$ follows

$$
\varepsilon_3 = S_{13}'\sigma_1' + S_{23}'\sigma_2' + \alpha_3^{\prime\mathrm{th}} T + \alpha_3^{\prime\mathrm{mo}} M^*
$$

This equation is the basis for determining the out-of-plane or through-the-thickness thermal and moisture effects of a laminate.

4.1.2 Off-Axis Stiffness and Compliances of UD-Laminae

A unidirectional lamina has very low stiffness and strength properties in the transverse direction compared with these properties in longitudinal direction. Laminates are constituted generally of different layers at different orientations. To study the elastic behavior of laminates, it is necessary to take a global coordinate system for the whole laminate and to refer the elastic behavior of each layer to this reference system. This is necessary to develop the stress-strain relationship for an angle lamina, i.e. an off-axis loaded UD-lamina.

The global and the local material reference systems are given in Fig. 4.3. We consider the ply material axes to be rotated away from the global axes by an angle θ, positive in the counterclockwise direction. This means that the (x_1, x_2)-axes are at an angle θ clockwise from the material axes. Thus transformation relations are

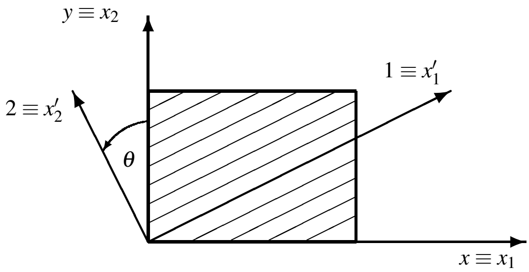


Fig. 4.3 UD-lamina with the local material principal axis $(1,2) \equiv (\mathrm{L},\mathrm{T})$ and the global reference system (x_1,x_2)

needed for the stresses, the strains, the stress-strain equations, the stiffness and the compliance matrices.

The transformation equations for a rotating the reference system (x'_1,x'_2) or (x_1,x_2) counterclockwise or clockwise by an angle θ follow from Sect. 2.1.2 and are given in Table 4.1. Note the relations for the transformation matrices derived in Sect. 2.1.2

Table 4.1 Transformation rules of the coordinates, displacements, strains and stresses of a lamina

a) Rotation of the reference systems

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix}$$

$$\boldsymbol{x}' = \boldsymbol{R}\boldsymbol{x}, \quad \boldsymbol{x} = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{x}'$$

b) Transformation of displacements

$$\begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \quad \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix}$$

$$\boldsymbol{u}' = \boldsymbol{R}\boldsymbol{u}, \quad \boldsymbol{u} = \boldsymbol{R}^{\mathrm{T}}\boldsymbol{u}'$$

c) Transformation of strains

$$\begin{bmatrix} \varepsilon'_1 \\ \varepsilon'_2 \\ \varepsilon'_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & sc \\ s^2 & c^2 & -sc \\ -2sc & 2sc & c^2-s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix}, \quad \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -sc \\ s^2 & c^2 & sc \\ 2sc & -2sc & c^2-s^2 \end{bmatrix} \begin{bmatrix} \varepsilon'_1 \\ \varepsilon'_2 \\ \varepsilon'_6 \end{bmatrix}$$

$$\boldsymbol{\varepsilon}' = \boldsymbol{T}^{\varepsilon}\boldsymbol{\varepsilon} = \left(\boldsymbol{T}^{\sigma'}\right)^{\mathrm{T}}\boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \boldsymbol{T}^{\varepsilon'}\boldsymbol{\varepsilon}' = (\boldsymbol{T}^{\sigma})^{\mathrm{T}}\boldsymbol{\varepsilon}'$$

d) Transformation of stresses

$$\begin{bmatrix} \sigma'_1 \\ \sigma'_2 \\ \sigma'_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2-s^2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix}, \quad \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2-s^2 \end{bmatrix} \begin{bmatrix} \sigma'_1 \\ \sigma'_2 \\ \sigma'_6 \end{bmatrix}$$

$$\boldsymbol{\sigma}' = \boldsymbol{T}^{\sigma}\boldsymbol{\sigma} = \left(\boldsymbol{T}^{\varepsilon'}\right)^{\mathrm{T}}\boldsymbol{\sigma}, \quad \boldsymbol{\sigma} = \boldsymbol{T}^{\sigma'}\boldsymbol{\sigma}' = (\boldsymbol{T}^{\varepsilon})^{\mathrm{T}}\boldsymbol{\sigma}'$$

with $s \equiv \sin\theta, c \equiv \cos\theta$

$$
\begin{aligned}
\boldsymbol{T}^{\varepsilon} &= \left(\boldsymbol{T}^{\varepsilon'}\right)^{-1} = \left(\boldsymbol{T}^{\sigma'}\right)^{\mathrm{T}}, \ \boldsymbol{T}^{\varepsilon'} = \left(\boldsymbol{T}^{\varepsilon}\right)^{-1} = \left(\boldsymbol{T}^{\sigma}\right)^{\mathrm{T}}, \\
\boldsymbol{T}^{\sigma} &= \left(\boldsymbol{T}^{\sigma'}\right)^{-1} = \left(\boldsymbol{T}^{\varepsilon'}\right)^{\mathrm{T}}, \ \boldsymbol{T}^{\sigma'} = \left(\boldsymbol{T}^{\sigma}\right)^{-1} = \left(\boldsymbol{T}^{\varepsilon}\right)^{\mathrm{T}}
\end{aligned} \tag{4.1.13}
$$

Table 4.2 summarizes the transformation rules for the stress-strain relations and for the values of the stiffness and the compliance matrices. The transformation matrices of Table 4.2 follow from Sect. 2.1.2. Using the relations (4.1.13) the transformation rules can be formulated in matrix notation

Table 4.2 Transformation of the reduced stiffness matrix Q'_{ij} and compliance matrix S'_{ij} in the reference system (x'_1, x'_2) to the reduced stiffness matrix Q_{ij} and compliance matrix S_{ij} in the (x_1, x_2)-system

a) Constitutive equations in the (x_1, x_2)-reference system

$$
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} \\ Q_{12} & Q_{22} & Q_{26} \\ Q_{16} & Q_{26} & Q_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix}, \quad \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix}
$$

$$
\boldsymbol{\sigma} = \boldsymbol{Q}\boldsymbol{\varepsilon}, \quad \boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma},
$$

$$
\boldsymbol{Q} = (\boldsymbol{T}^{\varepsilon})^{\mathrm{T}} \boldsymbol{Q}' \boldsymbol{T}^{\varepsilon}, \quad \boldsymbol{S} = (\boldsymbol{T}^{\sigma})^{\mathrm{T}} \boldsymbol{S}' \boldsymbol{T}^{\sigma},
$$

$$
Q_{ij} = Q_{ji}, \quad Q'_{ij} = Q'_{ji}, \quad S_{ij} = S_{ji}, \quad S'_{ij} = S'_{ji}
$$

b) Transformation of the reduced stiffnesses

$$
\begin{bmatrix} Q_{11} \\ Q_{12} \\ Q_{16} \\ Q_{22} \\ Q_{26} \\ Q_{66} \end{bmatrix} = \begin{bmatrix} c^4 & 2c^2s^2 & s^4 & 4c^2s^2 \\ c^2s^2 & c^4+s^4 & c^2s^2 & -4c^2s^2 \\ c^3s & -cs(c^2-s^2) & -cs^3 & -2cs(c^2-s^2) \\ s^4 & 2c^2s^2 & c^4 & 4c^2s^2 \\ cs^3 & cs(c^2-s^2) & -c^3s & 2cs(c^2-s^2) \\ c^2s^2 & -2c^2s^2 & c^2s^2 & (c^2-s^2)^2 \end{bmatrix} \begin{bmatrix} Q'_{11} \\ Q'_{12} \\ Q'_{22} \\ Q'_{66} \end{bmatrix}
$$

c) Transformation of the compliances

$$
\begin{bmatrix} S_{11} \\ S_{12} \\ S_{16} \\ S_{22} \\ S_{26} \\ S_{66} \end{bmatrix} = \begin{bmatrix} c^4 & 2c^2s^2 & s^4 & c^2s^2 \\ c^2s^2 & c^4+s^4 & c^2s^2 & -c^2s^2 \\ 2c^3s & -2cs(c^2-s^2) & -2cs^3 & -cs(c^2-s^2) \\ s^4 & 2c^2s^2 & c^4 & c^2s^2 \\ 2cs^3 & 2cs(c^2-s^2) & -2c^3s & cs(c^2-s^2) \\ 4c^2s^2 & -8c^2s^2 & 4c^2s^2 & (c^2-s^2)^2 \end{bmatrix} \begin{bmatrix} S'_{11} \\ S'_{12} \\ S'_{22} \\ S'_{66} \end{bmatrix}
$$

with $s \equiv \sin\theta, c \equiv \cos\theta$

$$
\begin{aligned}
\boldsymbol{Q}' &= \left(\boldsymbol{T}^{\varepsilon'}\right)^{\mathrm{T}} \boldsymbol{Q}\boldsymbol{T}^{\varepsilon'} = \boldsymbol{T}^{\sigma}\boldsymbol{Q}\,(\boldsymbol{T}^{\sigma})^{\mathrm{T}},\\
\boldsymbol{S}' &= \left(\boldsymbol{T}^{\sigma'}\right)^{\mathrm{T}} \boldsymbol{S}\boldsymbol{T}^{\sigma'} = \boldsymbol{T}^{\varepsilon}\boldsymbol{S}\,(\boldsymbol{T}^{\varepsilon})^{\mathrm{T}},\\
\boldsymbol{Q} &= (\boldsymbol{T}^{\varepsilon})^{\mathrm{T}}\boldsymbol{Q}'\boldsymbol{T}^{\varepsilon},\\
\boldsymbol{S} &= (\boldsymbol{T}^{\sigma})^{\mathrm{T}}\boldsymbol{S}'\boldsymbol{T}^{\sigma}
\end{aligned}
\tag{4.1.14}
$$

Starting with the stiffness equation $\boldsymbol{\sigma} = \boldsymbol{Q}\boldsymbol{\varepsilon}$ and introducing $\boldsymbol{\sigma}' = \boldsymbol{Q}'\boldsymbol{\varepsilon}'$ in the transformation $\boldsymbol{\sigma} = \boldsymbol{T}^{\sigma'}\boldsymbol{\sigma}' = \left(\boldsymbol{T}^{\varepsilon'}\right)^{\mathrm{T}}\boldsymbol{\sigma}'$ it follows that $\boldsymbol{\sigma} = \left(\boldsymbol{T}^{\varepsilon'}\right)^{\mathrm{T}}\boldsymbol{Q}'\boldsymbol{\varepsilon}'$ and with $\boldsymbol{\varepsilon}' = \boldsymbol{T}^{\varepsilon}\boldsymbol{\varepsilon}$ this gives $\boldsymbol{\sigma} = (\boldsymbol{T}^{\varepsilon})^{\mathrm{T}}\boldsymbol{Q}'\boldsymbol{T}^{\varepsilon}\boldsymbol{\varepsilon}$. Comparison of equations $\boldsymbol{\sigma} = \boldsymbol{Q}\boldsymbol{\varepsilon}$ and $\boldsymbol{\sigma} = (\boldsymbol{T}^{\varepsilon})^{\mathrm{T}}\boldsymbol{Q}'\boldsymbol{T}^{\varepsilon}\boldsymbol{\varepsilon}$ yields

$$
\begin{bmatrix} Q_{11} & Q_{12} & Q_{16}\\ Q_{12} & Q_{22} & Q_{26}\\ Q_{16} & Q_{26} & Q_{66}\end{bmatrix} = (\boldsymbol{T}^{\varepsilon})^{\mathrm{T}} \begin{bmatrix} Q'_{11} & Q'_{12} & 0\\ Q'_{12} & Q'_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix} \boldsymbol{T}^{\varepsilon}
\tag{4.1.15}
$$

In an analogous way

$$
\begin{bmatrix} S_{11} & S_{12} & S_{16}\\ S_{12} & S_{22} & S_{26}\\ S_{16} & S_{26} & S_{66}\end{bmatrix} = (\boldsymbol{T}^{\sigma})^{\mathrm{T}} \begin{bmatrix} S'_{11} & S'_{12} & 0\\ S'_{12} & S'_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix} \boldsymbol{T}^{\sigma}
\tag{4.1.16}
$$

can be derived. Note that in (4.1.15) and (4.1.16) the matrices $[Q_{ij}]$ and $[S_{ij}]$ have six different elements but the matrices $[Q'_{ij}]$ and $[S'_{ij}]$ have only four independent elements. The elements in Q_{ij} or S_{ij} are functions of the four independent material characteristics Q'_{ij} or S'_{ij} and the angle θ. The experimental testing is therefore more simple than for a real anisotropic material with 6 independent material values, if the material axes of the lamina are known.

From the transformation c) in Table 4.2 follows the transformation of the engineering parameters $E_{\mathrm{L}}, E_{\mathrm{T}}, G_{\mathrm{LT}}, \nu_{\mathrm{LT}}$ of the UD-lamina in the on-axis-system to the engineering parameters in the global system (x_1, x_2). From equation a) in Table 4.2 for an angle lamina it can be seen that there is a coupling of all normal and shear terms of stresses and strains. In Fig. 4.4 these coupling effects in an off-axes loaded UD-lamina are described.

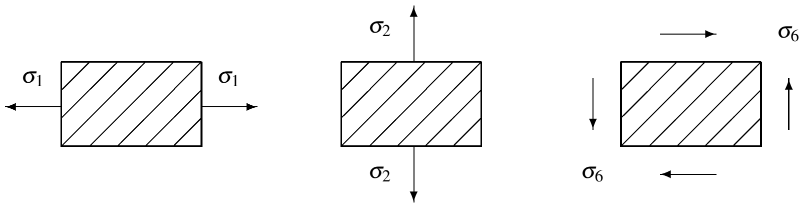
The coupling coefficients

$$
\nu_{12} = -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{S_{21}}{S_{11}}, \quad \nu_{21} = -\frac{\varepsilon_1}{\varepsilon_2} = -\frac{S_{12}}{S_{22}}
\tag{4.1.17}
$$

are the known Poisson's ratios and the ratios

$$
\nu_{16} = \frac{\varepsilon_6}{\varepsilon_1} = \frac{S_{61}}{S_{11}}, \quad \nu_{26} = \frac{\varepsilon_6}{\varepsilon_2} = \frac{S_{62}}{S_{22}}
\tag{4.1.18}
$$

are so called shear coupling values. They are non-dimensional parameters like Poisson's ratio and relate normal stresses to shear strains or shear stresses to normal strains.



$$\frac{\varepsilon_1}{\sigma_1} = S_{11} = \frac{1}{E_1} \qquad \frac{\varepsilon_2}{\sigma_2} = S_{22} = \frac{1}{E_2} \qquad \frac{\varepsilon_6}{\sigma_6} = S_{66} = \frac{1}{E_6}$$

$$\frac{\varepsilon_2}{\sigma_1} = S_{21} = -\frac{\nu_{12}}{E_1} \qquad \frac{\varepsilon_1}{\sigma_2} = S_{12} = -\frac{\nu_{21}}{E_2} \qquad \frac{\varepsilon_1}{\sigma_6} = S_{16} = \frac{\nu_{61}}{E_6}$$

$$\frac{\varepsilon_6}{\sigma_1} = S_{61} = \frac{\nu_{16}}{E_1} \qquad \frac{\varepsilon_6}{\sigma_2} = S_{62} = \frac{\nu_{26}}{E_2} \qquad \frac{\varepsilon_2}{\sigma_6} = S_{26} = \frac{\nu_{62}}{E_6}$$

Fig. 4.4 Off-axis loaded UD-lamina with one stress component in each case

Hence the strain-stress equation of an angle lamina can be written in terms of engineering parameters of the off-axis case as

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & \nu_{61}/E_6 \\ -\nu_{12}/E_1 & 1/E_2 & \nu_{62}/E_6 \\ \nu_{16}/E_1 & \nu_{26}/E_2 & 1/E_6 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} \tag{4.1.19}$$

With the compliance engineering parameters

$$\begin{aligned} S_{11} &= \frac{1}{E_1}, & S_{12} &= -\frac{\nu_{21}}{E_2}, & S_{21} &= -\frac{\nu_{12}}{E_1}, \\ S_{66} &= \frac{1}{E_6}, & S_{16} &= \frac{\nu_{61}}{E_6}, & S_{61} &= \frac{\nu_{16}}{E_1}, \\ S_{26} &= \frac{\nu_{62}}{E_6}, & S_{62} &= \frac{\nu_{26}}{E_2}, & S_{22} &= \frac{1}{E_2} \end{aligned} \tag{4.1.20}$$

it follows from the symmetry considerations of the compliance matrix that

$$S_{ij} = S_{ji}, i, j = 1,2,6; \quad \text{i.e.} \quad \frac{\nu_{12}}{E_2} = \frac{\nu_{21}}{E_1}, \ \frac{\nu_{16}}{E_1} = \frac{\nu_{61}}{E_6}, \ \frac{\nu_{26}}{E_2} = \frac{\nu_{62}}{E_6} \tag{4.1.21}$$

but the anisotropic coupling coefficients are

$$\nu_{ij} \neq \nu_{ji}, i, j = 1,2,6 \tag{4.1.22}$$

Equation (4.1.19) can be inverted to yield the stress-strain equations in terms of engineering parameters but these relations would be more complex than (4.1.19). Using the relationships between engineering parameters and compliances (4.1.20) in the compliance transformation rule (Table 4.2) we obtain the following transformations for the engineering parameters of the angle lamina including shear coupling ratios ν_{16} and ν_{26}

$$\begin{aligned}
\frac{1}{E_1} &= \frac{1}{E_1'}c^4 + \left(\frac{1}{G_{12}'} - \frac{2\nu_{12}'}{E_1'}\right)s^2c^2 + \frac{1}{E_2'}s^4, \\
\frac{1}{E_2} &= \frac{1}{E_1'}s^4 + \left(\frac{1}{G_{12}'} - \frac{2\nu_{12}'}{E_1'}\right)s^2c^2 + \frac{1}{E_2'}c^4, \\
\frac{1}{G_{12}} &= 2\left(\frac{2}{E_1'} + \frac{2}{E_2'} + \frac{4\nu_{12}'}{E_1'} - \frac{1}{G_{12}'}\right)s^2c^2 + \frac{1}{G_{12}'}(c^4+s^4), \\
\frac{\nu_{12}}{E_1} &= \frac{\nu_{21}}{E_2} = \left[\frac{\nu_{12}'}{E_1'}(c^4+s^4) - \left(\frac{1}{E_1'} + \frac{1}{E_2'} - \frac{1}{G_{12}'}\right)c^2s^2\right], \\
\nu_{16} &= E_1'\left[\left(\frac{2}{E_1'} + \frac{2\nu_{12}'}{E_1'} - \frac{1}{G_{12}'}\right)sc^3 - \left(\frac{2}{E_2'} + \frac{2\nu_{12}'}{E_1'} - \frac{1}{G_{12}'}\right)s^3c\right], \\
\nu_{26} &= E_2'\left[\left(\frac{2}{E_1'} + \frac{2\nu_{12}'}{E_1'} - \frac{1}{G_{12}'}\right)s^3c - \left(\frac{2}{E_2'} + \frac{2\nu_{12}'}{E_1'} - \frac{1}{G_{12}'}\right)sc^3\right]
\end{aligned} \tag{4.1.23}$$

Equation (4.1.23) can be also written in the following form

$$\begin{aligned}
E_1 &= \frac{E_{\mathrm{L}}}{c^4 + \left(\frac{E_{\mathrm{L}}}{G_{\mathrm{LT}}} - 2\nu_{\mathrm{LT}}\right)s^2c^2 + \frac{E_{\mathrm{L}}}{E_{\mathrm{T}}}s^4}, \\
E_2 &= \frac{E_{\mathrm{T}}}{c^4 + \left(\frac{E_{\mathrm{T}}}{G_{\mathrm{LT}}} - 2\nu_{\mathrm{TL}}\right)s^2c^2 + \frac{E_{\mathrm{T}}}{E_{\mathrm{L}}}s^4}, \\
G_{12} &= \frac{G_{\mathrm{LT}}}{c^4 + s^4 + 2\left[2\frac{G_{\mathrm{LT}}}{E_{\mathrm{L}}}(1+2\nu_{\mathrm{LT}}) + 2\frac{G_{\mathrm{LT}}}{E_{\mathrm{T}}} - 1\right]s^2c^2}, \\
\nu_{12} &= \frac{\nu_{\mathrm{LT}}(c^4+s^4) - \left(1 + \frac{E_{\mathrm{L}}}{E_{\mathrm{T}}} - \frac{E_{\mathrm{L}}}{G_{\mathrm{LT}}}\right)c^2s^2}{c^4 + \left(\frac{E_{\mathrm{L}}}{G_{\mathrm{LT}}} - 2\nu_{\mathrm{LT}}\right)c^2s^2 + \frac{E_{\mathrm{L}}}{E_{\mathrm{T}}}s^4}, \\
\nu_{21} &= \frac{\nu_{\mathrm{TL}}(c^4+s^4) - \left(1 + \frac{E_{\mathrm{T}}}{E_{\mathrm{L}}} - \frac{E_{\mathrm{T}}}{G_{\mathrm{LT}}}\right)c^2s^2}{c^4 + \left(\frac{E_{\mathrm{T}}}{G_{\mathrm{LT}}} - 2\nu_{\mathrm{TL}}\right)c^2s^2 + \frac{E_{\mathrm{T}}}{E_{\mathrm{L}}}s^4}, \\
\nu_{16} &= E_{\mathrm{L}}\left[\left(\frac{2}{E_{\mathrm{L}}} + \frac{2\nu_{\mathrm{LT}}}{E_{\mathrm{L}}} - \frac{1}{G_{\mathrm{LT}}}\right)sc^3 - \left(\frac{2}{E_{\mathrm{T}}} + \frac{2\nu_{\mathrm{LT}}}{E_{\mathrm{L}}} - \frac{1}{G_{\mathrm{LT}}}\right)s^3c\right], \\
\nu_{26} &= E_{\mathrm{T}}\left[\left(\frac{2}{E_{\mathrm{L}}} + \frac{2\nu_{\mathrm{LT}}}{E_{\mathrm{L}}} - \frac{1}{G_{\mathrm{LT}}}\right)s^3c - \left(\frac{2}{E_{\mathrm{T}}} + \frac{2\nu_{\mathrm{LT}}}{E_{\mathrm{L}}} - \frac{1}{G_{\mathrm{LT}}}\right)sc^3\right]
\end{aligned} \tag{4.1.24}$$

The engineering parameters can change rapidly with angel θ. This can be interpreted as if the fibres are not oriented exactly as intended the values of engineering parameters are very less or more than expected.

A computational procedure for calculating the elastic parameters of a UD-lamina in off-axis loading can be illustrated by the following steps:

1. Input the basic engineering parameters $E_{\mathrm{L}}, E_{\mathrm{T}}, G_{\mathrm{LT}}, \nu_{\mathrm{LT}}$ referred to the material axes of the lamina and obtained by material tests or mathematical modelling.
2. Calculate the compliances S'_{ij} and the reduced stiffness Q'_{ij}.
3. Application of transformations to obtain the lamina stiffness Q_{ij} and compliances S_{ij}.
4. Finally calculate the engineering parameters $E_1, E_2, G_{12}, \nu_{12}, \nu_{21}, \nu_{16}, \nu_{26}$ referred to the (x_1, x_2)-system.

Otherwise the engineering parameters referred to the (x_1, x_2)-coordinate can be calculated directly by Eqs. (4.1.24).

Analogous to the on-axis loaded UD-lamina also the off-axis lamina is in thickness direction $x_3 \equiv T'$ orthogonal to the (x_1, x_2)-plane macro-mechanically quasi-isotropic and the three-dimensional material behavior is transversely-isotropic. The mechanical properties transverse to the fibre direction are provided by weaker matrix material and the effects of transverse shear deformation may be significant. For such cases, the stress and the strain vector should include all six components

$$\boldsymbol{\sigma}^{\mathrm{T}} = [\sigma_1\ \sigma_2\ \sigma_3\ \sigma_4\ \sigma_5\ \sigma_6], \quad \boldsymbol{\varepsilon}^{\mathrm{T}} = [\varepsilon_1\ \varepsilon_2\ \varepsilon_3\ \varepsilon_4\ \varepsilon_5\ \varepsilon_6]$$

For a rotation about the direction $\boldsymbol{e}_3$ (Fig. 2.6) the transformation matrices (2.1.39) and (2.1.40) are valid and relations for stress and strain vectors in the on-axis and the off-axis reference system are given by

$$\begin{aligned}
&\boldsymbol{\sigma}' = \overset{3}{\boldsymbol{T}}{}^{\sigma}\,\boldsymbol{\sigma}, \qquad \boldsymbol{\varepsilon}' = \overset{3}{\boldsymbol{T}}{}^{\varepsilon}\,\boldsymbol{\varepsilon}, \qquad \overset{3}{\boldsymbol{T}}{}^{\varepsilon'} = \left(\overset{3}{\boldsymbol{T}}{}^{\sigma}\right)^{\mathrm{T}} = \left(\overset{3}{\boldsymbol{T}}{}^{\varepsilon}\right)^{-1},\\
&\boldsymbol{\sigma} = \left(\overset{3}{\boldsymbol{T}}{}^{\sigma}\right)^{-1}\boldsymbol{\sigma}', \ \boldsymbol{\varepsilon} = \left(\overset{3}{\boldsymbol{T}}{}^{\varepsilon}\right)^{-1}\boldsymbol{\varepsilon}, \ \overset{3}{\boldsymbol{T}}{}^{\sigma'} = \left(\overset{3}{\boldsymbol{T}}{}^{\varepsilon}\right)^{\mathrm{T}} = \left(\overset{3}{\boldsymbol{T}}{}^{\sigma}\right)^{-1}
\end{aligned} \tag{4.1.25}$$

When the stiffness matrix $\boldsymbol{C}'$ corresponding to an orthotropic material behavior, see Eq. (2.1.46), the transformed stiffness matrix $\boldsymbol{C}$ may be written in detail as for monoclinic material behavior, see Eq. (2.1.42)

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ & C_{22} & C_{23} & 0 & 0 & C_{26}\\ & & C_{33} & 0 & 0 & C_{36}\\ & & & C_{44} & C_{45} & 0\\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & C_{55} & 0\\ & & & & & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix} \tag{4.1.26}$$

The C_{ij} are the transformed stiffness, i.e. in vector-matrix notation

$$\begin{aligned}\boldsymbol{\sigma}' &= \boldsymbol{C}'\boldsymbol{\varepsilon}'\\ \boldsymbol{\sigma} &= \boldsymbol{C}\boldsymbol{\varepsilon}\end{aligned}, \quad \boldsymbol{\sigma} = \overset{3}{\boldsymbol{T}}{}^{\sigma'}\boldsymbol{\sigma}' = \left(\overset{3}{\boldsymbol{T}}{}^{\varepsilon}\right)^{\mathrm{T}}\boldsymbol{C}'\boldsymbol{\varepsilon}' = \left(\overset{3}{\boldsymbol{T}}{}^{\varepsilon}\right)^{\mathrm{T}}\boldsymbol{C}'\,\overset{3}{\boldsymbol{T}}{}^{\varepsilon}\,\boldsymbol{\varepsilon} = \boldsymbol{C}\boldsymbol{\varepsilon}$$

we finally obtain

$$\boldsymbol{C} = \left(\overset{3}{\boldsymbol{T}}{}^{\varepsilon}\right)^{\mathrm{T}} \boldsymbol{C}' \, \overset{3}{\boldsymbol{T}}{}^{\varepsilon}, \tag{4.1.27}$$

in which the C_{ij}

$$\begin{aligned}
C_{11} &= C'_{11}c^4 + 2C'_{12}c^2s^2 + C'_{22}s^4 + 4C'_{66}c^2s^2,\\
C_{12} &= C'_{11}c^2s^2 + C'_{12}(c^4+s^4) + C'_{22}c^2s^2 - 4C'_{66}c^2s^2,\\
C_{13} &= C'_{13}c^2 + C'_{23}s^2, C_{14} = 0, \quad C_{15} = 0,\\
C_{16} &= C'_{11}c^3s - C'_{12}cs(c^2-s^2) - C'_{22}cs^3 - 2C'_{66}cs(c^2-s^2),\\
C_{22} &= C'_{11}s^4 + 2C'_{12}c^2s^2 + C'_{22}c^4 + 4C'_{66}c^2s^2,\\
C_{23} &= C'_{13}s^2 + C'_{23}c^2, C_{24} = 0, \quad C_{25} = 0,\\
C_{26} &= C'_{11}cs^3 + C'_{12}cs(c^2-s^2) - C'_{22}c^3s + 2C'_{66}cs(c^2-s^2),\\
C_{33} &= C'_{33}, C_{34} = 0, C_{35} = 0,\\
C_{36} &= C'_{13}cs - C'_{23}cs,\\
C_{44} &= C'_{44}c^2 + C'_{55}s^2,\\
C_{45} &= -C'_{44}cs + C'_{55}cs, \quad C_{46} = 0,\\
C_{55} &= C'_{44}s^2 + C'_{55}c^2, \quad C_{56} = 0,\\
C_{66} &= C'_{11}c^2s^2 - 2C'_{12}c^2s^2 + C'_{22}c^2s^2 + C'_{66}(c^2-s^2)^2
\end{aligned} \tag{4.1.28}$$

The 13 non-zero stiffness of C_{ij} are not independent material values. They are functions of 9 C'_{ij} for a three-dimensional orthotropic material, i.e. of

$$C'_{11}, C'_{12}, C'_{13}, C'_{22}, C'_{23}, C'_{33}, C'_{44}, C'_{55}, C_{66}$$

and of 5 C'_{ij} for a transverse-isotropic behavior, i.e. of

$$C'_{11}, C'_{12}, C'_{22}, C'_{23}, C'_{55}$$

because

$$C'_{13} = C'_{12}, C'_{33} = C'_{22}, C'_{44} = \frac{1}{2}(C'_{22} - C'_{23}), C'_{66} = C'_{55}$$

With $\boldsymbol{\varepsilon}' = \boldsymbol{S}'\boldsymbol{\sigma}', \boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma}$ follow analogously the transformed compliances

$$\boldsymbol{S} = \left(\overset{3}{\boldsymbol{T}}{}^{\sigma}\right)^{\mathrm{T}} \boldsymbol{S}' \, \overset{3}{\boldsymbol{T}}{}^{\sigma}, \tag{4.1.29}$$

in which the S_{ij}

$$\begin{aligned}
S_{11} &= S'_{11}c^4 + 2S'_{12}c^2s^2 + S'_{22}s^4 + S'_{66}c^2s^2,\\
S_{12} &= S'_{11}c^2s^2 + S'_{12}(c^4+s^4) + S'_{22}c^2s^2 - S'_{66}c^2s^2,\\
S_{13} &= S'_{13}c^2 + S'_{23}s^2, S_{14} = 0, S_{15} = 0,\\
S_{16} &= 2S'_{11}c^3s - 2S'_{12}cs(c^2-s^2) - 2S'_{22}cs^3 - S'_{66}cs(c^2-s^2),\\
S_{22} &= S'_{11}s^4 + 2S'_{12}c^2s^2 + S'_{22}c^4 + S'_{66}c^2s^2,\\
S_{23} &= S'_{13}s^2 + S'_{23}c^2, S_{24} = 0, S_{25} = 0,\\
S_{26} &= 2S'_{11}cs^3 - 2S'_{12}cs(c^2-s^2) - 2S'_{22}c^3s - S'_{66}cs(c^2-s^2),\\
S_{33} &= S'_{33}, S_{34} = 0, S_{35} = 0,\\
S_{36} &= 2S'_{13}cs - 2S'_{23}cs,\\
S_{44} &= S'_{44}c^2 + S'_{55}s^2,\\
S_{45} &= -S'_{44}cs + S'_{55}cs, S_{46} = 0,\\
S_{55} &= S'_{44}s^2 + S'_{55}c^2, S_{56} = 0,\\
S_{66} &= 4S'_{11}c^2s^2 - 4S'_{12}c^2s^2 + 2S'_{22}c^2s^2 + S'_{66}(s^2-c^2)^2
\end{aligned} \tag{4.1.30}$$

There are again 13 non-zero compliances, but only 9 independent material values for the orthotropic and 5 independent material values for the transversal-isotropic case.

The stress-strain relationship for an angle lamina, i.e. an off-axis loaded UD-lamina, including hygrothermal effects takes the following form

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{12} & S_{22} & S_{26} \\ S_{16} & S_{26} & S_{66} \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} + \begin{bmatrix} \varepsilon_1^{\mathrm{th}} \\ \varepsilon_2^{\mathrm{th}} \\ \varepsilon_6^{\mathrm{th}} \end{bmatrix} + \begin{bmatrix} \varepsilon_1^{\mathrm{mo}} \\ \varepsilon_2^{\mathrm{mo}} \\ \varepsilon_6^{\mathrm{mo}} \end{bmatrix}, \tag{4.1.31}$$

where

$$\begin{bmatrix} \varepsilon_1^{\mathrm{th}} \\ \varepsilon_2^{\mathrm{th}} \\ \varepsilon_6^{\mathrm{th}} \end{bmatrix} = \begin{bmatrix} \alpha_1^{\mathrm{th}} \\ \alpha_2^{\mathrm{th}} \\ \alpha_6^{\mathrm{th}} \end{bmatrix} T, \begin{bmatrix} \varepsilon_1^{\mathrm{mo}} \\ \varepsilon_2^{\mathrm{mo}} \\ \varepsilon_6^{\mathrm{mo}} \end{bmatrix} = \begin{bmatrix} \alpha_1^{\mathrm{mo}} \\ \alpha_2^{\mathrm{mo}} \\ \alpha_6^{\mathrm{mo}} \end{bmatrix} M^* \tag{4.1.32}$$

with the thermal and moisture expansion coefficients $\alpha_i^{\mathrm{th}}, \alpha_i^{\mathrm{mo}}, i = 1,2,6$ and the temperature change T or the weight of moisture absorption per unit weight M^*, respectively. It should be remembered that although the coefficients of both thermal and moisture expansions are pure dilatational in the material coordinate system (L,T), rotation into the global (x_1,x_2) system results in coefficients $\alpha_6^{\mathrm{th}}, \alpha_6^{\mathrm{mo}}$. Furthermore if there are no constraints placed on a UD-lamina, no mechanical strains will be included in it and therefore no mechanical stresses are induced. But in laminates, even if there are no constraints on the laminate, the difference in thermal and moisture expansion coefficients of the various laminae of a laminate induces different expansions in each layer and results in residual stresses. This will be explained fully in Sects. 4.2.4 and 4.2.5. With

$$\begin{aligned}
\varepsilon'^{\mathrm{th}}_1 &= \alpha'^{\mathrm{th}}_1 T,\\
\varepsilon'^{\mathrm{th}}_2 &= \alpha'^{\mathrm{th}}_2 T,\\
\varepsilon'^{\mathrm{th}}_6 &= 0,
\end{aligned} \tag{4.1.33}$$

and

$$\begin{aligned}
\varepsilon_1^{\text{th}} &= \varepsilon_1^{\prime\text{th}} c^2 + \varepsilon_2^{\prime\text{th}} s^2, \\
\varepsilon_2^{\text{th}} &= \varepsilon_1^{\prime\text{th}} s^2 + \varepsilon_2^{\prime\text{th}} c^2, \\
\varepsilon_6^{\text{th}} &= 2(\varepsilon_1^{\prime\text{th}} - \varepsilon_2^{\prime\text{th}}) cs,
\end{aligned} \tag{4.1.34}$$

follow

$$\begin{aligned}
\alpha_1^{\text{th}} &= \alpha_1^{\prime\text{th}} c^2 + \alpha_2^{\prime\text{th}} s^2, \\
\alpha_2^{\text{th}} &= \alpha_1^{\prime\text{th}} s^2 + \alpha_2^{\prime\text{th}} c^2, \\
\alpha_6^{\text{th}} &= 2(\alpha_1^{\prime\text{th}} - \alpha_2^{\prime\text{th}}) cs
\end{aligned} \tag{4.1.35}$$

In an anisotropic layer, uniform heating induces not only normal strains, but also shear thermal strains. For a transversal isotropic material behavior there is additional $\varepsilon_3^{\text{th}} = \alpha_3^{\text{th}} T$, $\alpha_3^{\text{th}} = \alpha_3^{\prime\text{th}}$, $\alpha_4^{\text{th}} = \alpha_5^{\text{th}} = 0$. Because $\varepsilon_3 = \varepsilon_3'$ the strain ε_3 can be obtained directly from

$$\varepsilon_3 = \alpha_3^{\text{th}} T + \alpha_3^{\text{mo}} M^* + S_{13}' \sigma_1' + S_{23}' \sigma_2'$$

However, the stresses σ_1', σ_2' can be written in terms $\sigma_1, \sigma_2, \sigma_6$ referred to the off-axis coordinate system to obtain an expression for ε_3 that represents the normal strain in the x_3-direction in terms of the global coordinate system

$$\begin{aligned}
\varepsilon_3 = {} & \varepsilon_3^{\text{th}} + \varepsilon_3^{\text{mo}} + (S_{13} c^2 + S_{23} s^2)\sigma_1 + (S_{13} s^2 + S_{23} c^2)\sigma_2 \\
& + 2(S_{13} - S_{23} s^2) sc\sigma_1
\end{aligned} \tag{4.1.36}$$

4.1.3 Stress Resultants and Stress Analysis

Sections 4.1.1 and 4.1.2 describe the constitutive equations for UD-laminae in an on-axis and an off-axis reference system as a relation between stresses and strains. For each lamina, the stress components can be integrated across their thickness h and yield stress resultants. Stress resultants can be in-plane forces, transverse forces and resultant moments. The constitutive equations may then be formulated as relations between mid-plane strain and in-plane forces, transverse shear strains and transverse forces and mid-plane curvatures and resultant moments, respectively.

The in-plane stress resultant force vector, denoted by

$$\boldsymbol{N} = [N_1 \; N_2 \; N_6]^{\text{T}}, \tag{4.1.37}$$

is defined by

$$\boldsymbol{N} = \int_{-h/2}^{h/2} \boldsymbol{\sigma} \mathrm{d}x_3 \tag{4.1.38}$$

The N_i are forces per unit length, N_1, N_2 are normal in-plane resultants and N_6 is a shear in-plane resultant, respectively. They are illustrated in Fig. 4.5 for constant in-plane stresses $\sigma_1, \sigma_2, \sigma_6$ across the thickness. In this case we have

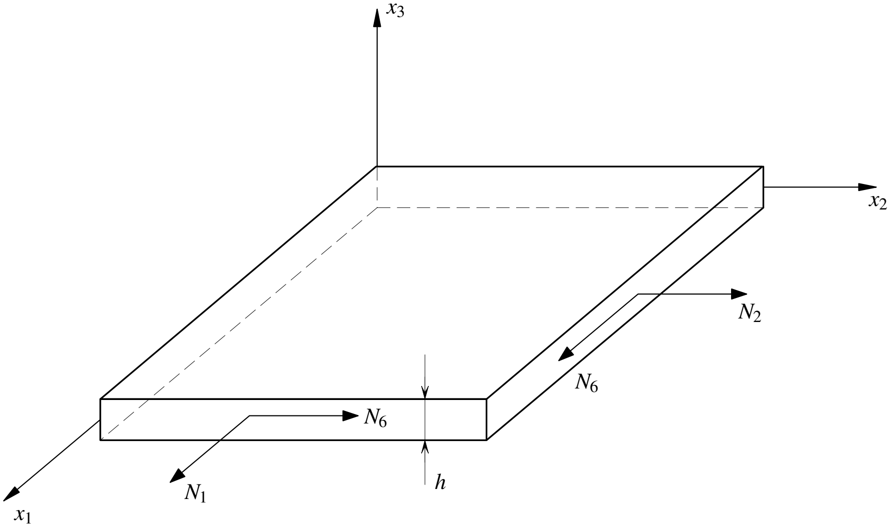


Fig. 4.5 In-plane force resultants per unit length $\boldsymbol{N}^{\mathrm{T}} = [N_1\ N_2\ N_6]$

$$\boldsymbol{N} = \boldsymbol{\sigma} h \tag{4.1.39}$$

The reduced stiffness matrix $\boldsymbol{Q}$ of the lamina has also constant components across h. The strains of the midplane $x_3 = 0$ of the lamina are given by

$$\boldsymbol{\varepsilon}(x_1, x_2, x_3 = 0) = \boldsymbol{\varepsilon}(x_1, x_2)$$

and Eq. (4.1.39) yields

$$\boldsymbol{N} = \boldsymbol{Q}\boldsymbol{\varepsilon} h = \boldsymbol{A}\boldsymbol{\varepsilon}, \quad \boldsymbol{A} = \boldsymbol{Q} h, \quad \boldsymbol{\varepsilon}^{\mathrm{T}} = [\varepsilon_1\ \varepsilon_2\ \varepsilon_6] \tag{4.1.40}$$

$\boldsymbol{Q}$ is the reduced stiffness matrix (Table 4.2 a) and $\boldsymbol{A}$ is the off-axis stretching or extensional stiffness matrix of the lamina. From (4.1.40) it follows that

$$\boldsymbol{\varepsilon} = \boldsymbol{A}^{-1}\boldsymbol{N} = \boldsymbol{a}\boldsymbol{N}, \quad \boldsymbol{a} = \boldsymbol{A}^{-1} = \boldsymbol{S} h^{-1} \tag{4.1.41}$$

$\boldsymbol{a}$ is the off-axis in-plane compliance matrix. $\boldsymbol{A}$ and $\boldsymbol{a}$ are, like $\boldsymbol{Q}$ and $\boldsymbol{S}$ symmetric matrices, which have in the general case only non-zero elements. In the special cases of on-axis reference systems or isotropic stiffness and compliances, respectively, the structure of the matrices is simplified. $\boldsymbol{A}$ is the extensional stiffness and $\boldsymbol{a}$ the extensional compliance matrix expressing the relationship between the in-plane stress resultant $\boldsymbol{N}$ and the mid-plane strain $\boldsymbol{\varepsilon}$:

Off-axis extensional stiffness and compliance matrices

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \tag{4.1.42}$$

On-axis extensional stiffness and compliance matrices

$$\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad \boldsymbol{a} = \begin{bmatrix} a_{11} & a_{12} & 0 \\ a_{12} & a_{22} & 0 \\ 0 & 0 & a_{66} \end{bmatrix} \tag{4.1.43}$$

If the stresses are not constant across h, resultant moments can be defined

$$\boldsymbol{M} = \int_{-h/2}^{h/2} \boldsymbol{\sigma} x_3 \mathrm{d}x_3 \tag{4.1.44}$$

The resultant moment vector is denoted by

$$\boldsymbol{M} = [M_1 \; M_2 \; M_6]^{\mathrm{T}} \tag{4.1.45}$$

The M_i are moments per unit length, M_1, M_2 are bending moments and M_6 is a torsional or twisting moment. Figure 4.6 illustrates these moments and a linear stress distribution across h. The resultant moments yield flexural strains, e.g. bending and

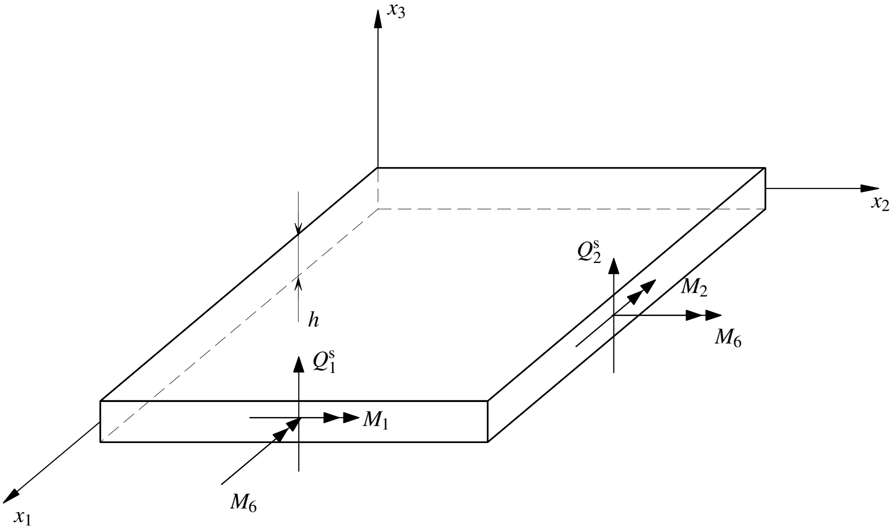


Fig. 4.6 Resultant moment vector $\boldsymbol{M}^{\mathrm{T}} = [M_1 \; M_2 \; M_6]$ and transverse shear resultants $\boldsymbol{Q}^{\mathrm{s}\,\mathrm{T}} = [Q_1^{\mathrm{s}} \; Q_2^{\mathrm{s}}]$

twisting strains, which are usually expressed by the relationship

$$\boldsymbol{\varepsilon}(x_1,x_2,x_3) = x_3\boldsymbol{\kappa}, \quad \boldsymbol{\kappa}^{\mathrm{T}} = [\kappa_1\ \kappa_2\ \kappa_6] \tag{4.1.46}$$

$\boldsymbol{\kappa}$ is the vector of curvature, κ_1, κ_2 correspond to the bending moments M_1, M_2 and κ_6 to the torsion moment M_6, respectively. The flexural strains are assumed linear across h. With $Q_{ij} =$ const across h, $i,j = 1,2,6$ follow

$$\boldsymbol{M} = \boldsymbol{Q}\boldsymbol{\kappa} \int\limits_{-h/2}^{h/2} x_3^2 \mathrm{d}x_3 = \boldsymbol{Q}\boldsymbol{\kappa}\frac{h^3}{12} = \boldsymbol{D}\boldsymbol{\kappa}, \quad \boldsymbol{\kappa} = \boldsymbol{D}^{-1}\boldsymbol{M} = \boldsymbol{d}\boldsymbol{M} \tag{4.1.47}$$

$\boldsymbol{D} = \boldsymbol{Q}h^3/12$ is the flexural stiffness matrix and $\boldsymbol{d}$ the flexural compliance matrix expressing the relations between stress couples $\boldsymbol{M}$ and the curvatures. For off-axis and on-axis reference systems the matrices are given by:
Off-axis flexural stiffness and compliance matrices

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} d_{11} & d_{12} & d_{16} \\ d_{12} & d_{22} & d_{26} \\ d_{16} & d_{26} & d_{66} \end{bmatrix} \tag{4.1.48}$$

On-axis flexural stiffness and compliance matrices

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}, \quad \boldsymbol{d} = \begin{bmatrix} d_{11} & d_{12} & 0 \\ d_{12} & d_{22} & 0 \\ 0 & 0 & d_{66} \end{bmatrix} \tag{4.1.49}$$

The transverse shear resultants can be defined in the same way by

$$\boldsymbol{Q}^{\mathrm{s}} = \begin{bmatrix} Q_1^{\mathrm{s}} \\ Q_2^{\mathrm{s}} \end{bmatrix} = \int\limits_{-h/2}^{h/2} \begin{bmatrix} \sigma_5^{\mathrm{s}} \\ \sigma_4^{\mathrm{s}} \end{bmatrix} \mathrm{d}x_3 \tag{4.1.50}$$

$\boldsymbol{Q}^{\mathrm{s}}$ is (like the in-plane resultants) a load vector per unit length in the cross section of the lamina $x_1 =$ const or $x_2 =$ const, respectively. The transverse shear resultant vector $\boldsymbol{Q}^{\mathrm{s}}$ is written with a superscript s to distinguish it from the reduced stiffness matrix $\boldsymbol{Q}$. When modelling a plane stress state, there are no constitutive equations for σ_4, σ_5 and the shearing stresses are calculated with the help of the equilibrium equations, Eq. (2.2.1), or with help of equilibrium conditions of stress resultant, e.g. Chap. 8. In three-dimensional modelling, including transverse shear strains, however, constitutive equations for transverse shear resultant can be formulated.

For a lamina with resultant forces $\boldsymbol{N}$ and moments $\boldsymbol{M}$ the in-plane strain $\boldsymbol{\varepsilon}$ and the curvature term $\boldsymbol{\kappa}$ have to be combined

$$\boldsymbol{\varepsilon}(x_1,x_2,x_3) = \boldsymbol{\varepsilon}(x_1,x_2) + x_3\boldsymbol{\kappa}(x_1,x_2) \tag{4.1.51}$$

For the stress vector is

$$\begin{aligned}\boldsymbol{\sigma}(x_1,x_2,x_3) &= \boldsymbol{Q}[\boldsymbol{\varepsilon}(x_1,x_2)+x_3\boldsymbol{\kappa}(x_1,x_2)]\\ &= \boldsymbol{Q}\boldsymbol{\varepsilon}(x_1,x_2)+\boldsymbol{Q}x_3\boldsymbol{\kappa}(x_1,x_2)\end{aligned} \tag{4.1.52}$$

and by integration through the lamina thickness h follow

$$\begin{aligned}\boldsymbol{N} &= \int\limits_{-h/2}^{h/2}\boldsymbol{\sigma}\mathrm{d}x_3 = \boldsymbol{Q}\boldsymbol{\varepsilon}h+\boldsymbol{Q}\boldsymbol{\kappa}\int\limits_{-h/2}^{h/2}x_3\mathrm{d}x_3 = \boldsymbol{A}\boldsymbol{\varepsilon}+\boldsymbol{B}\boldsymbol{\kappa},\\ \boldsymbol{M} &= \int\limits_{-h/2}^{h/2}\boldsymbol{\sigma}x_3\mathrm{d}x_3 = \boldsymbol{Q}\boldsymbol{\varepsilon}\int\limits_{-h/2}^{h/2}x_3\mathrm{d}x_3+\boldsymbol{Q}\boldsymbol{\kappa}\frac{h^3}{12} = \boldsymbol{B}\boldsymbol{\varepsilon}+\boldsymbol{D}\boldsymbol{\kappa}\end{aligned} \tag{4.1.53}$$

The coupling stiffness matrix $\boldsymbol{B}$ is zero for a lamina, which is symmetric to the midplane $x_3 = 0$, i.e. there are no coupling effects between the $\boldsymbol{N}$ and $\boldsymbol{\kappa}$ or $\boldsymbol{M}$ and $\boldsymbol{\varepsilon}$, respectively. In Table 4.3 the constitutive equations of the lamina resultants are summarized for a symmetric general angle lamina, for a UD-orthotropic lamina and for an isotropic layer. In a contracted vector-matrix notation, we can formulate the constitutive equation of a lamina by

$$\begin{bmatrix}\boldsymbol{N}\\ \boldsymbol{M}\end{bmatrix} = \begin{bmatrix}\boldsymbol{A} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{D}\end{bmatrix}\begin{bmatrix}\boldsymbol{\varepsilon}\\ \boldsymbol{\kappa}\end{bmatrix}, \tag{4.1.54}$$

where the in-plane stiffness submatrix $\boldsymbol{A} = \boldsymbol{Q}h$ and the plate stiffness submatrix $\boldsymbol{D} = \boldsymbol{Q}h^3/12$. $\boldsymbol{0}$ are zero submatrices. The inverted form of (4.1.54) is important for stress analysis

$$\begin{bmatrix}\boldsymbol{\varepsilon}\\ \boldsymbol{\kappa}\end{bmatrix} = \begin{bmatrix}\boldsymbol{a} & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{d}\end{bmatrix}\begin{bmatrix}\boldsymbol{N}\\ \boldsymbol{M}\end{bmatrix}, \quad \begin{aligned}\boldsymbol{a} &= \boldsymbol{A}^{-1}, & \boldsymbol{A} &= \boldsymbol{Q}h,\\ \boldsymbol{d} &= \boldsymbol{D}^{-1}, & \boldsymbol{D} &= \boldsymbol{Q}(h^3/12)\end{aligned} \tag{4.1.55}$$

Equation (4.1.52) yields the stress components $\sigma_i, i = 1,2,6$

$$\begin{aligned}\sigma_1 &= Q_{11}(\varepsilon_1+x_3\kappa_1)+Q_{12}(\varepsilon_2+x_3\kappa_2)+Q_{16}(\varepsilon_6+x_3\kappa_6) = \sigma_{1\mathrm{M}}+\sigma_{1\mathrm{B}},\\ \sigma_2 &= Q_{21}(\varepsilon_1+x_3\kappa_1)+Q_{22}(\varepsilon_2+x_3\kappa_2)+Q_{26}(\varepsilon_6+x_3\kappa_6) = \sigma_{2\mathrm{M}}+\sigma_{2\mathrm{B}},\\ \sigma_6 &= Q_{16}(\varepsilon_1+x_3\kappa_1)+Q_{62}(\varepsilon_2+x_3\kappa_2)+Q_{66}(\varepsilon_6+x_3\kappa_6) = \sigma_{6\mathrm{M}}+\sigma_{6\mathrm{B}}\end{aligned}$$

$\sigma_{i\mathrm{M}}$ are the membrane or in-plane stresses coupled with N_i and $\sigma_{i\mathrm{B}}$ the curvature or plate stresses coupled with M_i. The membrane stresses are constant and the bending stresses linear through the lamina thickness (Fig. 4.7).

The transverse shear stresses σ_4, σ_5 for plane stress state condition are obtained by integration of the equilibrium equations (2.2.1). If the volume forces $p_1 = p_2 = 0$, Eqs. (2.2.1) yield

$$\sigma_5(x_3) = -\int\limits_{-h/2}^{x_3}\left(\frac{\partial\sigma_1}{\partial x_1}+\frac{\partial\sigma_6}{\partial x_2}\right)\mathrm{d}x_3, \quad \sigma_4(x_3) = -\int\limits_{-h/2}^{x_3}\left(\frac{\partial\sigma_6}{\partial x_1}+\frac{\partial\sigma_2}{\partial x_2}\right)\mathrm{d}x_3 \tag{4.1.56}$$

Table 4.3 Stiffness matrices of laminae

Anisotropic single layer or UD-lamina, off-axis
$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & 0 & 0 & 0 \\ A_{12} & A_{22} & A_{26} & 0 & 0 & 0 \\ A_{16} & A_{26} & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix}, \quad \begin{aligned} A_{ij} &= Q_{ij}h, \\ D_{ij} &= Q_{ij}\frac{h^3}{12} \end{aligned}$
Orthotropic single layer or UD-laminae, on-axis
$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix}, \quad \begin{aligned} Q_{11} &= \frac{E_1}{1-\nu_{12}\nu_{21}}, \\ Q_{22} &= \frac{E_2}{1-\nu_{12}\nu_{21}}, \\ Q_{12} &= \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} \\ &= \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}, \\ Q_{66} &= G_{12} \end{aligned}$
Isotropic single layer
$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix}, \quad \begin{aligned} Q_{11} &= \frac{E}{1-\nu^2}, \\ Q_{12} &= \frac{\nu E}{1-\nu^2}, \\ Q_{66} &= \frac{E}{2(1+\nu)} \end{aligned}$

and with Eq. (4.1.52) follows

$$\begin{aligned} \sigma_5(x_3) = -\int\limits_{-h/2}^{x_3} \Big\{ & \frac{\partial}{\partial x_1}\left[Q_{11}(\varepsilon_1 + x_3\kappa_1) + Q_{12}(\varepsilon_2 + x_3\kappa_2) + Q_{16}(\varepsilon_6 + x_3\kappa_6)\right] \\ & + \frac{\partial}{\partial x_2}\left[Q_{61}(\varepsilon_1 + x_3\kappa_1) + Q_{62}(\varepsilon_2 + x_3\kappa_2) + Q_{66}(\varepsilon_6 + x_3\kappa_6)\right]\Big\} \mathrm{d}x_3, \end{aligned} \tag{4.1.57}$$

$$\frac{N_1}{h} = \sigma_{1\mathrm{M}} \qquad \frac{M_1}{h^3/12}x_3 = \sigma_{1\mathrm{B}} \qquad \sigma_1 = \sigma_{1\mathrm{M}} + \sigma_{1\mathrm{B}}$$

$$\frac{N_2}{h} = \sigma_{2\mathrm{M}} \qquad \frac{M_2}{h^3/12}x_3 = \sigma_{2\mathrm{B}} \qquad \sigma_2 = \sigma_{2\mathrm{M}} + \sigma_{2\mathrm{B}}$$

$$\frac{N_6}{h} = \sigma_{6\mathrm{M}} \qquad \frac{M_6}{h^3/12}x_3 = \sigma_{6\mathrm{B}} \qquad \sigma_6 = \sigma_{6\mathrm{M}} + \sigma_{6\mathrm{B}}$$

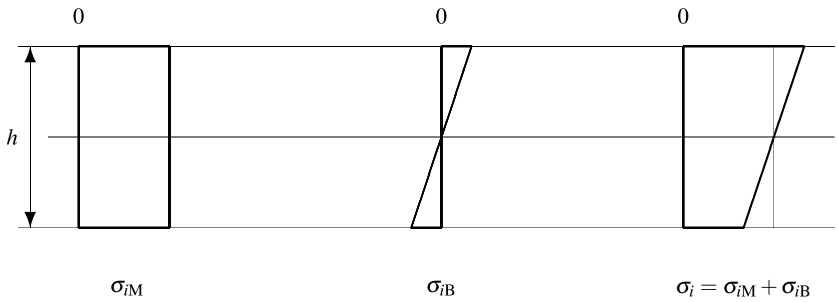


Fig. 4.7 In-plane membrane stresses $\sigma_{i\mathrm{M}}$, bending stresses $\sigma_{i\mathrm{B}}$ and total stresses σ_i across h (qualitative)

$$\begin{aligned}\sigma_4(x_3) = -\int\limits_{-h/2}^{x_3} & \left\{ \frac{\partial}{\partial x_1}\left[Q_{61}(\varepsilon_1 + x_3\kappa_1) + Q_{62}(\varepsilon_2 + x_3\kappa_2) + Q_{66}(\varepsilon_6 + x_3\kappa_6)\right] \right. \\ & \left. + \frac{\partial}{\partial x_2}\left[Q_{21}(\varepsilon_1 + x_3\kappa_1) + Q_{22}(\varepsilon_2 + x_3\kappa_2) + Q_{26}(\varepsilon_6 + x_3\kappa_6)\right] \right\} \mathrm{d}x_3\end{aligned}$$

Substituting the midplane strain $\boldsymbol{\varepsilon}$ and the curvature $\boldsymbol{\kappa}$ by the resultants $\boldsymbol{N}$ and $\boldsymbol{M}$ Eqs. (4.1.57) takes the form

$$\begin{aligned}\sigma_5(x_3) = -\int\limits_{-h/2}^{x_3} & \left\{ Q_{11}\frac{\partial}{\partial x_1}[a_{11}N_1 + a_{12}N_2 + a_{16}N_6 + x_3(d_{11}M_1 + d_{12}M_2 + d_{16}M_6)] \right. \\ & + Q_{12}\frac{\partial}{\partial x_1}[a_{21}N_1 + a_{22}N_2 + a_{26}N_6 + x_3(d_{21}M_1 + d_{22}M_2 + d_{26}M_6)] \\ & + Q_{16}\frac{\partial}{\partial x_1}[a_{61}N_1 + a_{62}N_2 + a_{66}N_6 + x_3(d_{61}M_1 + d_{62}M_2 + d_{66}M_6)] \\ & + Q_{61}\frac{\partial}{\partial x_2}[a_{11}N_1 + a_{12}N_2 + a_{16}N_6 + x_3(d_{11}M_1 + d_{12}M_2 + d_{16}M_6)] \\ & + Q_{62}\frac{\partial}{\partial x_2}[a_{21}N_1 + a_{22}N_2 + a_{26}N_6 + x_3(d_{21}M_1 + d_{22}M_2 + d_{26}M_6)]\end{aligned}$$

$$+Q_{66}\frac{\partial}{\partial x_2}[a_{16}N_1+a_{26}N_2+a_{66}N_6+x_3(d_{16}M_1+d_{26}M_2+d_{66}M_6)]\Big\}\mathrm{d}x_3, \tag{4.1.58}$$

$$\begin{aligned}\sigma_4(x_3)=-\int\limits_{-h/2}^{x_3}\Big\{&Q_{61}\frac{\partial}{\partial x_1}[a_{11}N_1+a_{12}N_2+a_{16}N_6+x_3(d_{11}M_1+d_{12}M_2+d_{16}M_6)]\\&+Q_{62}\frac{\partial}{\partial x_1}[a_{21}N_1+a_{22}N_2+a_{26}N_6+x_3(d_{21}M_1+d_{22}M_2+d_{26}M_6)]\\&+Q_{66}\frac{\partial}{\partial x_1}[a_{61}N_1+a_{62}N_2+a_{66}N_6+x_3(d_{61}M_1+d_{62}M_2+d_{66}M_6)]\\&+Q_{21}\frac{\partial}{\partial x_2}[a_{11}N_1+a_{12}N_2+a_{16}N_6+x_3(d_{11}M_1+d_{12}M_2+d_{16}M_6)]\\&+Q_{22}\frac{\partial}{\partial x_2}[a_{21}N_1+a_{22}N_2+a_{26}N_6+x_3(d_{21}M_1+d_{22}M_2+d_{26}M_6)]\\&+Q_{26}\frac{\partial}{\partial x_2}[a_{61}N_1+a_{62}N_2+a_{66}N_6+x_3(d_{61}M_1+d_{62}M_2+d_{66}M_6)]\Big\}\mathrm{d}x_3\end{aligned}$$

The distribution of the transverse shear stresses through the thickness h is obtained by integration from the bottom surface $x_3=-h/2$ of the lamina to x_3

$$\begin{aligned}\int\limits_{-h/2}^{x_3}\boldsymbol{Q}\mathrm{d}x_3 &= \tilde{\boldsymbol{A}}(x_3)=\boldsymbol{Q}(x_3+h/2),\\ \int\limits_{-h/2}^{x_3}\boldsymbol{Q}x_3\mathrm{d}x_3 &= \tilde{\boldsymbol{B}}(x_3)=\boldsymbol{Q}\frac{1}{2}\left(x_3^2-\frac{h^2}{4}\right)\end{aligned} \tag{4.1.59}$$

and we have

$$\tilde{\boldsymbol{A}}(-h/2)=\tilde{\boldsymbol{B}}(-h/2)=\boldsymbol{0},\quad \tilde{\boldsymbol{A}}(h/2)\equiv\boldsymbol{A},\quad \tilde{\boldsymbol{B}}(h/2)\equiv\boldsymbol{B}$$

Finally, the transverse shear stress equations (4.1.57) take the form

$$\begin{aligned}\sigma_5(x_3)=-&\Big[\tilde{A}_{11}(x_3)\frac{\partial\varepsilon_1}{\partial x_1}+\tilde{A}_{12}(x_3)\frac{\partial\varepsilon_2}{\partial x_1}+\tilde{A}_{16}(x_3)\frac{\partial\varepsilon_6}{\partial x_1}\\&+\tilde{B}_{11}(x_3)\frac{\partial\kappa_1}{\partial x_1}+\tilde{B}_{12}(x_3)\frac{\partial\kappa_2}{\partial x_1}+\tilde{B}_{16}(x_3)\frac{\partial\kappa_6}{\partial x_1}\\&+\tilde{A}_{61}(x_3)\frac{\partial\varepsilon_1}{\partial x_2}+\tilde{A}_{62}(x_3)\frac{\partial\varepsilon_2}{\partial x_2}+\tilde{A}_{66}(x_3)\frac{\partial\varepsilon_6}{\partial x_2}\\&+\tilde{B}_{61}(x_3)\frac{\partial\kappa_1}{\partial x_2}+\tilde{B}_{62}(x_3)\frac{\partial\kappa_2}{\partial x_2}+\tilde{B}_{66}(x_3)\frac{\partial\kappa_6}{\partial x_2}\Big],\end{aligned}$$

$$\begin{aligned}\sigma_4(x_3) = - \quad & \left[\tilde{A}_{61}(x_3)\frac{\partial \varepsilon_1}{\partial x_1} + \tilde{A}_{62}(x_3)\frac{\partial \varepsilon_2}{\partial x_1} + \tilde{A}_{66}(x_3)\frac{\partial \varepsilon_6}{\partial x_1}\right.\\ & + \tilde{B}_{61}(x_3)\frac{\partial \kappa_1}{\partial x_1} + \tilde{B}_{62}(x_3)\frac{\partial \kappa_2}{\partial x_1} + \tilde{B}_{66}(x_3)\frac{\partial \kappa_6}{\partial x_1}\\ & + \tilde{A}_{21}(x_3)\frac{\partial \varepsilon_1}{\partial x_2} + \tilde{A}_{22}(x_3)\frac{\partial \varepsilon_2}{\partial x_2} + \tilde{A}_{26}(x_3)\frac{\partial \varepsilon_6}{\partial x_2}\\ & \left. + \tilde{B}_{21}(x_3)\frac{\partial \kappa_1}{\partial x_2} + \tilde{B}_{22}(x_3)\frac{\partial \kappa_2}{\partial x_2} + \tilde{B}_{26}(x_3)\frac{\partial \kappa_6}{\partial x_2}\right]\end{aligned}$$

or an analogous equation to (4.1.59) by substituting the in-plane strains $\boldsymbol{\varepsilon}$ and the mid-plane curvatures by the stress resultants $\boldsymbol{N}$ and $\boldsymbol{M}$.

The transverse shear stresses σ_4, σ_5 are parabolic functions across the lamina thickness. If there are no surface edge shear stresses, the conditions $\sigma_5(h/2) = \sigma_4(h/2) = 0$ are controlling the performance of the equilibrium equations and the accuracy of the stress analysis.

4.1.4 Problems

Exercise 4.1. For a single layer unidirectional composite, the on-axis elastic behavior is given by $E_{\mathrm{L}} = 140\,\mathrm{GPa}$, $E_{\mathrm{T}} = 9\,\mathrm{GPa}$, $\nu_{\mathrm{LT}} = 0.3$. Calculate the reduced stiffness matrix $\boldsymbol{Q}'$ and the reduced compliance matrix $\boldsymbol{S}'$.

Solution 4.1. $E_{\mathrm{L}} \equiv E'_1, E_{\mathrm{T}} \equiv E'_2, \nu_{\mathrm{LT}} = \nu'_{12}$. Equation (4.1.3) yields

$$\begin{bmatrix}\sigma'_1\\ \sigma'_2\\ \sigma'_6\end{bmatrix} = \begin{bmatrix}Q'_{11} & Q'_{12} & 0\\ Q'_{12} & Q'_{22} & 0\\ 0 & 0 & Q'_{66}\end{bmatrix}\begin{bmatrix}\varepsilon'_1\\ \varepsilon'_2\\ \varepsilon'_6\end{bmatrix}, \quad \begin{bmatrix}\varepsilon'_1\\ \varepsilon'_2\\ \varepsilon'_6\end{bmatrix} = \begin{bmatrix}S'_{11} & S'_{12} & 0\\ S'_{12} & S'_{22} & 0\\ 0 & 0 & S'_{66}\end{bmatrix}\begin{bmatrix}\sigma'_1\\ \sigma'_2\\ \sigma'_6\end{bmatrix},$$

$$\begin{aligned}Q'_{11} &= E'_1/(1-\nu'_{12}\nu'_{21}), & S'_{11} &= 1/E'_1,\\ Q'_{22} &= E'_2/(1-\nu'_{12}\nu'_{21}), & S'_{22} &= 1/E'_2,\\ Q'_{12} &= E'_2\nu'_{12}/(1-\nu'_{12}\nu'_{21}), & S'_{12} &= -\nu'_{12}/E'_2,\\ Q'_{66} &= G'_{12} = E'_2/2(1+\nu'_{12}), & S'_{66} &= 1/G'_{12},\end{aligned}$$

$$\nu'_{21} = \nu'_{12}E'_2/E'_1 = 0,0192, \quad G'_{12} = E'_1/2(1+\nu'_{12}) = 53,846\,\mathrm{GPa},$$

$$\begin{aligned}&Q'_{11} = 140,811\,\mathrm{GPa}, && Q'_{22} = 9,052\,\mathrm{GPa},\\ &Q'_{12} = 2,716\,\mathrm{GPa}, && Q'_{66} = 53,846\,\mathrm{GPa},\\ &S'_{11} = 7,143\;10^{-3}\,\mathrm{GPa}^{-1}, && S'_{22} = 111,111\;10^{-3}\,\mathrm{GPa}^{-1},\\ &S'_{12} = -2,143\;10^{-3}\,\mathrm{GPa}^{-1}, && S'_{66} = 18,571\;10^{-3}\,\mathrm{GPa}^{-1},\end{aligned}$$

$$\boldsymbol{Q}' = \begin{bmatrix} 140,811 & 2,716 & 0 \\ 2,716 & 9,052 & 0 \\ 0 & 0 & 53,846 \end{bmatrix} \text{GPa},$$

$$\boldsymbol{S}' = \begin{bmatrix} 7,143 & -2,143 & 0 \\ -2,143 & 111,111 & 0 \\ 0 & 0 & 18,576 \end{bmatrix} 10^{-3}\text{GPa}^{-1}$$

Exercise 4.2. A composite panel is designed as a single layer lamina with the following properties $E_{\mathrm{L}} = 140$ GPa, $E_{\mathrm{T}} = 10$ GPa, $G_{\mathrm{LT}} = 6,9$ GPa, $\nu_{\mathrm{LT}} = 0,3$ and $\theta = 45^0$. Calculate the strains $\varepsilon_1, \varepsilon_2$ and ε_6 when the panel is loaded by a shear stress $\sigma_6 = \pm\tau = \pm$ 10 MPa.

Solution 4.2. From Fig. 4.4 follows

$$\varepsilon_1 = \pm S_{16}\sigma_6, \quad \varepsilon_2 = \pm S_{26}\sigma_6, \quad \varepsilon_6 = \pm S_{66}\sigma_6$$

With $E_{\mathrm{L}} \equiv E'_1, E_{\mathrm{T}} \equiv E'_2, G_{\mathrm{LT}} \equiv G'_{12}$ and $\nu_{\mathrm{LT}} \equiv \nu'_{12}$ and Eq. (4.1.17) is
$S'_{11} = 1/E'_1 = 7,143\ 10^{-3}$ GPa^{-1},
$S'_{22} = 1/E'_2 = 100,000\ 10^{-3}$ GPa^{-1},
$S'_{12} = -\nu'_{12}/E'_1 = -2,143\ 10^{-3}$ GPa^{-1},
$S'_{66} = 1/G'_{12} = 144,928\ 10^{-3}$ GPa^{-1}
The transformation rule, Table 4.2, yields
$S_{16} = S_{26} = -0,46\ 10^{-1}$ GPa^{-1}, $S_{66} = 1,11\ 10^{-1}$ GPa^{-1}
The strains are $\varepsilon_1 = \mp 0,46 \cdot 10^{-3}$, $\varepsilon_2 = \varepsilon_1$, $\varepsilon_6 = \pm 1,11 \cdot 10^{-3}$.

Conclusion 4.1. A positive shear load $\sigma_6 = +\tau$ shortens the composite panel in both directions, a negative shear load $\sigma_6 = -\tau$ would enlarge the panel in both directions.

Exercise 4.3. In a unidirectional single layer is a strain state $\varepsilon_{11} = 1\% = 10^{-2}$, $\varepsilon_{22} = -0.5\% = -0.5\ 10^{-2}, \gamma_{12} = 2\% = 2\ 10^{-2}$. In the principal directions, the following engineering parameters of the composite material are $E'_1 = E_{\mathrm{L}} = 40$ GPa, $E'_2 = E_{\mathrm{T}} = 10$ GPa, $G_{12} = G_{\mathrm{LT}} = 5$ GPa, $\nu_{\mathrm{LT}} = 0.3$. Determine the plane stress state for the axis (x_1, x_2) and $(x'_1, x'_2) \equiv (\mathrm{L}, \mathrm{T})$ and $\theta = 45^0$.

Solution 4.3. The stress states $\boldsymbol{\sigma}'$ and $\boldsymbol{\sigma}$ are to calculate for a given strain state $\varepsilon_1 = 10^{-2}$, $\varepsilon_2 = -0,5\ 10^{-2}$, $\varepsilon_6 = 2\ 10^{-2}$ for a UD lamina with $E'_1 = 40$ GPa, $E'_2 = 10$ GPa, $G'_{12} = 5$ GPa, $\nu'_{12} = 0,3$ and a fibre angle $\theta = 45^0$. With Table 4.1 the strains for the off-axis system x_1, x_2 are transferred to the strains for the on-axis system x'_1, x'_2. Taking into account $\cos 45^0 = \sin 45^0 = \sqrt{2}/2 = 0,707107$ we obtain

$$\begin{bmatrix} \varepsilon'_1 \\ \varepsilon'_2 \\ \varepsilon'_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & sc \\ s^2 & c^2 & -sc \\ -2sc & 2sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix}$$

$$= \begin{bmatrix} 0,5 & 0,5 & 0,5 \\ 0,5 & 0,5 & -0,5 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 10 \\ -5 \\ 20 \end{bmatrix} 10^{-3} = \begin{bmatrix} 12,5 \\ -7,5 \\ -15 \end{bmatrix} 10^{-3},$$

$$\varepsilon_1' = 12{,}5\ 10^{-3}, \varepsilon_2' = -7{,}5\ 10^{-3}, \varepsilon_6' = -15\ 10^{-3}$$

and

$$\nu_{21}' = \frac{\nu_{12}' E_2'}{E_1'} = 0{,}075$$

The reduced stiffness Q_{ij}' and the stresses σ_i' in the on-axis system follow from (4.1.3)
$Q_{11}' = E_1'/(1-\nu_{12}'\nu_{21}') = 40{,}9207$ GPa, $Q_{22}' = E_2'/(1-\nu_{12}'\nu_{21}') = 10{,}2302$ GPa,
$Q_{12}' = E_2'\nu_{12}'/(1-\nu_{12}'\nu_{21}') = 3{,}0691$ GPa, $Q_{66}' = G_{12}' = 5{,}0$ GPa,

$$\begin{bmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_6' \end{bmatrix} = \begin{bmatrix} Q_{11}' & Q_{12}' & 0 \\ Q_{12}' & Q_{22}' & 0 \\ 0 & 0 & Q_{66}' \end{bmatrix} \begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \\ \varepsilon_6' \end{bmatrix} = \begin{bmatrix} 488{,}5 \\ -38{,}36 \\ -75{,}0 \end{bmatrix} 10^{-3}\ \text{GPa}$$

The stresses σ_i in the off-axis system are calculated with the help of the transformation rules Table 4.1

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & -2sc \\ s^2 & c^2 & 2sc \\ sc & -sc & c^2-s^2 \end{bmatrix} \begin{bmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_6' \end{bmatrix}$$

$$= \begin{bmatrix} 0{,}5 & 0{,}5 & -1 \\ 0{,}5 & 0{,}5 & 1 \\ 0{,}5 & -0{,}5 & 0 \end{bmatrix} \begin{bmatrix} 488{,}5 \\ -38{,}36 \\ -75{,}0 \end{bmatrix} 10^{-3}\text{GPa} = \begin{bmatrix} 0{,}300 \\ 0{,}150 \\ 0{,}263 \end{bmatrix} \text{GPa}$$

Exercise 4.4. Sketch the variation curves E_1/E_2' and G_{12}/E_2' against the fibre orientation θ for a carbon-epoxy and glass-epoxy lamina using the following material data:
carbon-epoxy $E_1' = 140$ GPa, $E_2' = 10$ GPa, $G_{12}' = 7$ GPa, $\nu_{12}' = 0{,}3$
glass-epoxy $E_1' = 43$ GPa, $E_2' = 9$ GPa, $G_{12}' = 4{,}5$ GPa, $\nu_{12}' = 0{,}27$
Discuss the curves.

Solution 4.4. From (4.1.23) follows

$$\begin{aligned} (E_1)^{-1} &= \frac{c^4}{E_1'} + \left(\frac{1}{G_{12}'} - \frac{2\nu_{12}'}{E_2'}\right) s^2 c^2 + \frac{s^4}{E_2'}, \\ (G_{12})^{-1} &= 2\left(\frac{2}{E_1'} + \frac{2}{E_2'} + \frac{4\nu_{12}'}{E_1'} - \frac{1}{G_{12}'}\right) s^2 c^2 + \frac{1}{G_{12}'}(c^4 + s^4) \end{aligned}$$

Now the functions $f_1(\theta) = E_1(\theta)/E_2'$ and $f_2(\theta) = G_{12}(\theta)/E_2'$ can be sketched. The results are shown on Figs. 4.8 and 4.9. Discussion of the functions $f_1(\theta)$ and $f_2(\theta)$:

1. The anisotropic ratio E_1/E_2' is higher for carbon- than for glass-epoxy.
2. The longitudinal effective modulus E_1 of the lamina drops sharply as the loading direction deviates from the fibre direction, especially for-carbon-epoxy.
3. The effective shear modulus of the lamina attains a maximum value at $\theta = 45^0$.

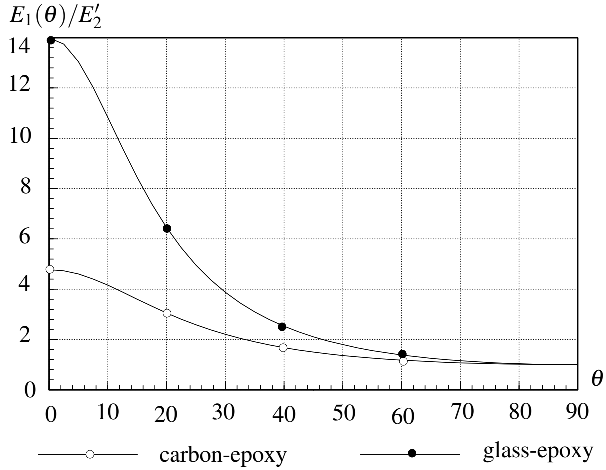


Fig. 4.8 Variation of $E_1(\theta)/E_2'$ against the fibre orientation for two composites

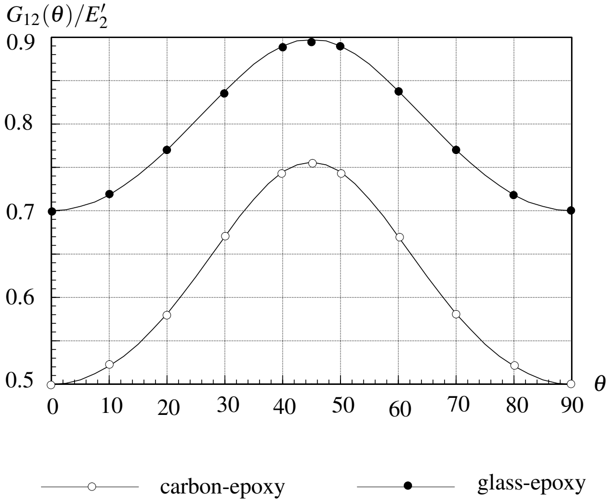


Fig. 4.9 Variation of $G_{12}(\theta)/E_2'$ against the fibre orientation for two composites

Exercise 4.5. For a UD-lamina with the elastic properties $E_1' = 180$ GPa, $E_2' = 10$ GPa, $G_{12}' = 7$ GPa, $\nu_{12}' = 0,3$ calculate

1. the on-axis compliances S_{ij}' and the on-axis strains, if the applied on-axis stresses are $\sigma_1' = 2$ MPa, $\sigma_2' = -3$ MPa, $\sigma_6' = 4$ MPa,
2. the off-axis compliances S_{ij} and the off-axis and on-axis strains $\boldsymbol{\varepsilon}, \boldsymbol{\varepsilon}'$ if $\theta = 45^0$ (Fig. 4.3) and the applied off-axis stresses are
   $\sigma_1 = 2$ MPa, $\sigma_2 = -3$ MPa, $\sigma_6 = 4$ MPa,
3. the coefficients of thermal expansion in the off-axis system if
   $\alpha_1^{\mathrm{th}'} = 9\ 10^{-6}/^0\mathrm{K}$, $\alpha_2^{\mathrm{th}'} = 22\ 10^{-6}/^0\mathrm{K}$.

Solution 4.5. 1. Using (4.1.3) follows

$$\begin{aligned}
S'_{11} &= (E'_1)^{-1} = 5,556\ 10^{-12}\,\mathrm{Pa}^{-1},\\
S'_{12} &= -\nu'_{12}/E'_1 = -1,667\ 10^{-12}\,\mathrm{Pa}^{-1},\\
S'_{22} &= (E'_2)^{-1} = 100\ 10^{-12}\,\mathrm{Pa}^{-1},\\
S'_{66} &= (G'_{12})^{-1} = 142,86\ 10^{-12}\,\mathrm{Pa}^{-1},
\end{aligned}$$

$$\begin{aligned}
\begin{bmatrix}\varepsilon'_1\\ \varepsilon'_2\\ \varepsilon'_6\end{bmatrix} &= \begin{bmatrix}5,556 & -1,667 & 0\\ -1,667 & 100 & 0\\ 0 & 0 & 142,86\end{bmatrix} 10^{-12}\mathrm{Pa}^{-1} \begin{bmatrix}2\\ -3\\ 4\end{bmatrix} 10^{6}\mathrm{Pa}\\
&= \begin{bmatrix}16,113\\ -303,334\\ 571,440\end{bmatrix} 10^{-6}
\end{aligned}$$

2. With Table 4.2 the transformed compliances S_{ij} can be calculated (note that $c = \cos 45^0 = 0.7071, s = \sin 45^0 = 0.7071$)
$S_{11} = 61,270\ 10^{-12}\ \mathrm{Pa}^{-1}$, $S_{22} = 61,271\ 10^{-12}\ \mathrm{Pa}^{-1}$,
$S_{12} = -10,160\ 10^{-12}\ \mathrm{Pa}^{-1}$, $S_{66} = 108,89\ 10^{-12}\ \mathrm{Pa}^{-1}$,
$S_{16} = S_{26} = -47,222\ 10^{-12}\ \mathrm{Pa}^{-1}$
Equations (4.1.19) and (4.1.20) yield the strains ε_i

$$\begin{aligned}
\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{bmatrix} &= \begin{bmatrix}S_{11} & S_{12} & S_{16}\\ S_{12} & S_{22} & S_{26}\\ S_{16} & S_{26} & S_{66}\end{bmatrix} \begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{bmatrix}\\
&= \begin{bmatrix}61,270 & -10,160 & -47,222\\ -10,160 & 61,271 & -47,222\\ -47,222 & -47,222 & 108,89\end{bmatrix} 10^{-12}\mathrm{Pa}^{-1} \begin{bmatrix}2\\ -3\\ 4\end{bmatrix} 10^{6}\mathrm{Pa}\\
&= \begin{bmatrix}-35,868\\ -393,01\\ 482,782\end{bmatrix} 10^{-6}
\end{aligned}$$

Using Table 4.1 the strains $\boldsymbol{\varepsilon}$ can be transformed to the strains $\boldsymbol{\varepsilon}'$

$$\begin{aligned}
\begin{bmatrix}\varepsilon'_1\\ \varepsilon'_2\\ \varepsilon'_6\end{bmatrix} &= \begin{bmatrix}0.5 & 0.5 & 0.5\\ 0.5 & 0.5 & -0.5\\ -1 & 1 & 0\end{bmatrix} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{bmatrix}\\
&= \begin{bmatrix}0.5 & 0.5 & 0.5\\ 0.5 & 0.5 & -0.5\\ -1 & 1 & 0\end{bmatrix} \begin{bmatrix}-35,868\\ -393,01\\ 482,782\end{bmatrix} 10^{-6}\\
&= \begin{bmatrix}26,95\\ -455,83\\ -357,14\end{bmatrix} 10^{-6}
\end{aligned}$$

3. The transformed thermal expansion coefficients α_i^{th} follow like the strains with Table 4.1 to

$$\begin{bmatrix} \alpha_1^{\rm th} \\ \alpha_2^{\rm th} \\ \alpha_6^{\rm th} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & 0.5 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} \alpha_1'^{\rm th} \\ \alpha_2'^{\rm th} \\ \alpha_6'^{\rm th} \end{bmatrix} = \begin{bmatrix} 15,5 \\ 15,5 \\ -13,0 \end{bmatrix} 10^{-6}/{\rm K}$$

Note that in the off-axis system $\alpha_6^{\rm th} \neq 0$.

Exercise 4.6. The micro-mechanical material parameters of a carbon-epoxy composite are
$E_{\rm Lf} = 411$ GPa, $E_{\rm Tf} = 6.6$ GPa, $\nu_{\rm TLf} = 0.06$, $\nu_{\rm LTf} = 0.35$,
$\alpha_{\rm Lf}^{\rm th} = -1.2\ 10^{-6}$ 1/K, $\alpha_{\rm Tf}^{\rm th} = 27.3\ 10^{-6}$ 1/K,
$E_{\rm m} = 5.7$ GPa, $\nu_{\rm m} = 0.316$, $\alpha_{\rm m} = 45\ 10^{-6}$ 1/K, $v_{\rm f} = 0.5$
The experimental tested values of the lamina are
$E_{\rm L} = 208.6$ GPa, $E_{\rm T} = 6.3$ GPa, $\nu_{\rm LT} = 0.33$,
$\alpha_{\rm L}^{\rm th} = -0.5\ 10^{-6}$ 1/^{0}C, $\alpha_{\rm T}^{\rm th} = 29.3\ 10^{-6}$ 1/^{0}C,
Predict the lamina values using the micro-mechanical modelling and compare the calculated and the experimental measured values.

Solution 4.6. Using Eqs. (3.1.27) and (4.1.9)

$$\begin{aligned}
E_{\rm L} &= v_{\rm f} E_{\rm Lf} + (1 - v_{\rm f}) E_{\rm m} = 208.35 \text{ GPa}, \\
\nu_{\rm LT} &= v_{\rm f} \nu_{\rm LTf} + (1 - v_{\rm f}) \nu_{\rm m} = 0.33, \\
E_{\rm T} &= E_{\rm Tf} E_{\rm m} / [v_{\rm f} E_{\rm m} + (1 - v_{\rm f}) E_{\rm Tf}] = 6.12 \text{ GPa}, \\
\alpha_{\rm L}^{\rm th} &= [\alpha_{\rm Lf}^{\rm th} v_{\rm f} E_{\rm Lf} + \alpha_{\rm m}^{\rm th} (1 - v_{\rm f}) E_{\rm m}] / E_{\rm L} = -0.57\ 10^{-6}\ 1/\text{K}, \\
\alpha_{\rm T}^{\rm th} &= (\alpha_{\rm Tf}^{\rm th} + \nu_{\rm Tf} \alpha_{\rm Lf}^{\rm th}) v_{\rm f} + (1 + \nu_{\rm m}) \alpha_{\rm m}^{\rm th} (1 - v_{\rm f}) - \nu_{\rm LT} \alpha_{\rm L}^{\rm th} = 4.43\ 10^{-6}\ 1/\text{K}
\end{aligned}$$

It can be concluded that the simple rules of mixture providing proper results for longitudinal material characteristics $E_{\rm L}$, $\nu_{\rm LT}$ and $\alpha_{\rm L}^{\rm th}$. In this case also $E_{\rm T}$ is predicted quite well but the formula for α_T fails to predict the transverse thermal expansion coefficient with required accuracy. For engineering applications the thermal expansion coefficients should be normally determined by experimental methods.

4.2 Elastic Behavior of Laminates

In Sect. 4.1, stress-strain equations were developed for a single lamina. Mostly important in engineering applications are isotropic, quasi-isotropic (stochastic distribution of short fibres or particles) and quasi-orthotropic (unidirectional fibre reinforced) laminae. Reduced stiffness Q_{ij}, compliances S_{ij}, membrane or in-plane stiffness A_{ij} and plate or out-of-plane stiffness D_{ij} were defined. Assuming symmetry about the midplane of a lamina in-plane and plate responses are uncoupled in the form of a first order theory (linear force-displacement relations).

The mechanics of laminated composite materials is generally studied at two distinct levels, commonly called micromechanics and macromechanics. In Chap. 3 the micromechanics was used to study the interaction between the fibres and matrix in a lamina such that the mechanical behavior of the lamina could be predicted from the known behavior of the constituents. Micromechanics establishes the relationship between the properties of the constituents and those of the lamina. All micromechanics approaches suffer from the problem of measuring the material properties of the constituents and generally require correction factors to correlate with measured lamina properties. For most engineering design applications an analysis that addressed to the micro-mechanical level is unrealistic.

At the macro-mechanical level the properties of the individual layers of a laminate are assumed to be known a priori. Macromechanics involves investigation of the interaction of the individual layers of a laminate with one another and their effects on the overall response quantities, e.g. elastic stiffness, influence of temperature and moisture on the response of laminated composites, etc. Such global response quantities can be predicted well on this level. Thus, the use of macromechanical formulations in designing composite laminates for desired material characteristics is well established. Macromechanics is based on continuum mechanics, which models each lamina as homogeneous and orthotropic and ignoring the fibre/matrix interface.

The lamination theory is the mathematical modelling to predict the macromechanical behavior of a laminate based on an arbitrary assembly of homogeneous orthotropic laminae. A two-dimensional modelling is most common, a three-dimensional theory is very complex and should be limited to selected problems, e.g. the analysis of laminates near free edges.

A real structure generally will not consist of a single lamina. A laminate consisting of more than one lamina bonded together through their thickness, for a single lamina is very thin and several laminae will be required to take realistic structural loads. Furthermore the mechanical characteristics of a UD-lamina are very limited in the transverse direction and by stacking a number of UD-laminae it may be an optimal laminate for unidirectional loading only. One can overcome this restriction by making laminates with layers stacked at different fibre angles corresponding to complex loading and stiffness requirements. To minimize the increasing costs and weights for such approach one have to optimize the laminae angles. It may be also useful to stack layers of different composite materials.

4.2.1 General Laminates

In the following section the macro-mechanical modelling and analysis of laminates will be considered. The behavior of a multidirectional laminate is a function of the laminae properties, i.e. their elastic moduli, thickness, angle orientations, and the stacking sequence of the individual layers. The macro-mechanical modelling may be in the framework of the following assumptions:

- There is a monolithic bonding of all laminae i.e. there is no slip between laminae at their interface and no special interface layers are arranged between the angle plies.

- Each layer is quasi-homogeneous and orthotropic, but the angle orientations may be different.
- The strains and displacements are continuous throughout the laminate. The in-plane displacements and strains vary linearly through the laminate thickness.

We will see that the stacking codes of laminates have a great influence on the global mechanical laminate response (Sect. 4.2.3), but there are some rules to guarantee an optimal global laminate behavior:

- Symmetric laminate stacking yields an uncoupled modelling and analysis of in-plane and bending/torsion stress-strain relations and avoids distorsions in the processing.
- Laminates should be made up of at least three UD-laminae with different fibre angle orientation.
- The differences of the mechanical properties and the fibre orientations between two laminae following in the stacking sequence should not be so large that the so-called interlaminar stresses are small.
- Although it is possible to determine an optimum orientation sequence of laminates for any given load condition, it is more practical from a fabrication standpoint and from effective experimental lamina testing to limit the number of fibre orientations to a few specific laminae types, e.g. fibre orientations of 0^0, $\pm 45^0$ and 90^0, etc.

Consider a laminate made of n plies shown in Fig. 4.10. Each lamina has a thickness of $h^{(k)}, k = 1, 2, \ldots, n$, and we have

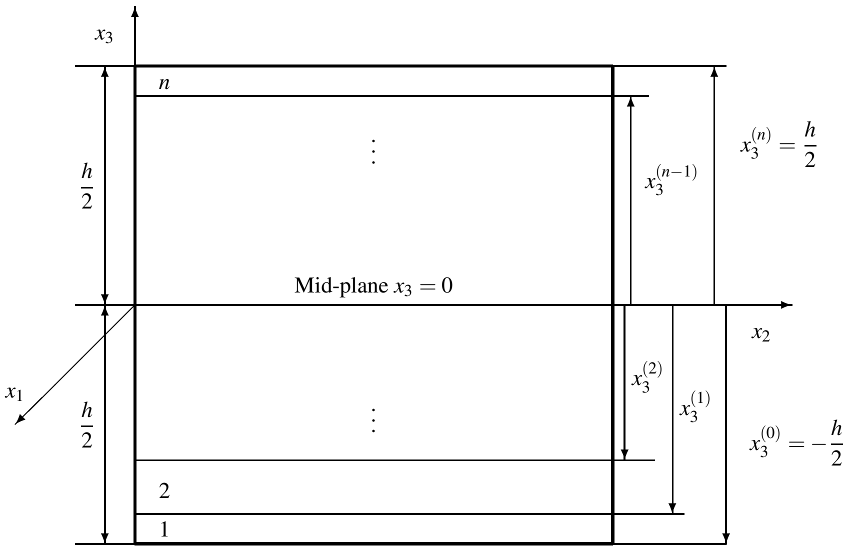


Fig. 4.10 Laminate made of n single layers, coordinate locations

$$
\begin{aligned}
h^{(k)} &= x_3^{(k)} - x_3^{(k-1)}, k = 1,2,\ldots,n && \text{thickness of a lamina,} \\
h &= \sum_{k=1}^{n} h^{(k)} && \text{thickness of the laminate,} \\
x_3^{(k)} &= -\frac{h}{2} + \sum_{i=1}^{k} h^{(i)} && \text{distance from the mid-plane,}
\end{aligned}
\tag{4.2.1}
$$

Then

$$x_3^{(n)} = \frac{h}{2} \quad \text{and} \quad x_3^{(0)} = -\frac{h}{2}$$

are the coordinates of the top and the bottom surface of the laminate,

$$x_3^{(k)} \quad \text{and} \quad x_3^{(k-1)}, \; k = 1,2,\ldots,n$$

are the location coordinates of the top and the bottom surface of lamina k. Each layer of a laminate can be identified by its location in the laminate, its material and its fibre orientation. The layers of the laminate may be symmetric, antisymmetric or asymmetric to the midplane $x_3 = 0$. $h^{(k)}$ and the reduced stiffness $\boldsymbol{Q}^{(k)}$ may be different for each lamina, but $\boldsymbol{Q}^{(k)}$ is constant for the kth lamina. The following examples illustrate the laminate code. In Fig. 4.11 the laminate codes for an unsymmetric laminate with four layers and a symmetric angle-ply laminate with eight layers are illustrated. A slash sign separates each lamina. The codes in Fig. 4.11 imply that each lamina is made of the same material and is of the same thickness. Regular symmetric are those laminates which have an odd number of UD-laminae of equal thicknesses and alternating angle orientations (Fig. 4.12). Since the number of laminae is odd and symmetry exists at the mid-surface, the 90^0 lamina is denoted with a bar on the top. The subscript S outside the code brackets, e.g., in Fig. 4.11 b), represents that the four plies are repeated in the reverse order.

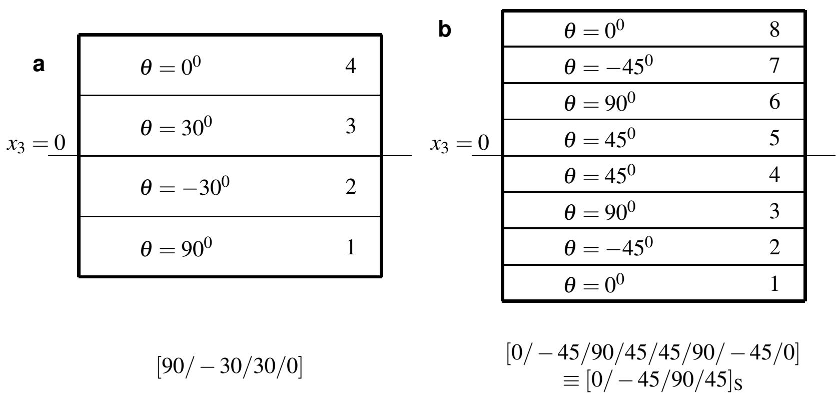
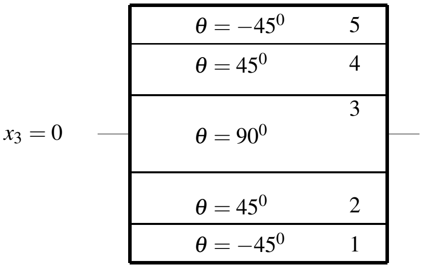


Fig. 4.11 Angle-ply laminates. **a** unsymmetric 4-layer laminate, **b** symmetric 8-layer-laminate



$[-45/45/90/45/-45] \equiv [-45/45/\bar{90}]_S$

Fig. 4.12 Regular symmetric angle-ply laminate: orientation of the midplane $\theta = 90^0$

A general laminate has layers of different orientations θ with $-90^0 \leq \theta \leq 90^0$. An angle-ply laminate has ply orientations of θ and $-\theta$ with $0^0 \leq \theta \leq 90^0$ and at least one lamina has an orientation other than 0^0 or 90^0. Cross-ply laminates are those which have only ply orientations of 0^0 and 90^0.

A laminate is balanced when it consists of pairs of layers with identical thickness and elastic properties but have $+\theta$ and $-\theta$ orientations of their principal material axes with respect to the laminate reference axes. A balanced laminate can be symmetric, antisymmetric or asymmetric

$$
\begin{array}{ll}
[+\theta_1/-\theta_1/+\theta_2/-\theta_2]_S & \text{symmetric lay-up,} \\
[\theta_1/\theta_2/-\theta_2/-\theta_1] & \text{antisymmetric lay-up,} \\
[\theta_1/\theta_2/-\theta_1/-\theta_2] & \text{asymmetric lay-up}
\end{array}
\tag{4.2.2}
$$

Antisymmetric laminates are a special case of balanced laminates, having the balanced $+\theta$ and $-\theta$ pairs of layers symmetrically situated about the middle surface. Generally each layer of a laminate can have different fibre angles, different thicknesses and different composite materials. The influence of the laminate codes, i.e the properties and the stacking sequences, on the elastic behavior of laminates will be considered in Sect. 4.2.3.

4.2.2 Stress-Strain Relations and Stress Resultants

The stiffness matrix of a single lamina referred to the reference system $x_i, i = 1,2,3$, has been formulated in Sect. 4.1.2, Eq. (4.1.26). Extending the assumption of a plane stress state to laminates with in-plane and out-of-plane loading, the stress-strain relation (4.1.26) can be rewritten by separating the transverse shear stresses and strains. The stresses in the kth layer are expressed by means of the reduced stiffness coefficients Q_{ij}

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\ Q_{21} & Q_{22} & Q_{26} & 0 & 0 \\ Q_{61} & Q_{62} & Q_{66} & 0 & 0 \\ 0 & 0 & 0 & Q_{44} & Q_{45} \\ 0 & 0 & 0 & Q_{54} & Q_{55} \end{bmatrix}^{(k)} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix}^{(k)} \tag{4.2.3}$$

or in contracted notation

$$\begin{aligned} \sigma_i^{(k)} &= Q_{ij}^{(k)} \varepsilon_j^{(k)},\ i,j = 1,2,6, \\ \sigma_i^{(k)} &= Q_{ij}^{(k)} \varepsilon_j^{(k)},\ i,j = 4,5 \end{aligned} \tag{4.2.4}$$

with (see also 2.1.78)

$$\sigma_3^{(k)} = 0, \quad \varepsilon_3^{(k)} = -\frac{1}{C_{33}}(C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{36}\varepsilon_6) \tag{4.2.5}$$

$Q_{ij}^{(k)}, i,j = 1,2,6$ are the reduced stiffness of the kth layer and functions of ${Q'_{ij}}^{(k)}$ and the fibre orientation angle, the $Q_{44}^{(k)}, Q_{45}^{(k)} = Q_{54}^{(k)}, Q_{55}^{(k)}$ are identical to the material coefficients $C_{44}^{(k)}, C_{45}^{(k)} = C_{54}^{(k)}, C_{55}^{(k)}$, which are not reduced by the assumption of a plane stress state. The discontinuity of $Q_{ij}^{(k)}$ from layer to layer implies the discontinuity of the stresses when passing from one lamina to another.

From the assumption of macro-mechanical modelling of laminates it follows that

$$\boldsymbol{\varepsilon}(x_1,x_2,x_3) = \boldsymbol{\varepsilon}(x_1,x_2) + x_3\boldsymbol{\kappa}(x_1,x_2) \tag{4.2.6}$$

i.e the strains $\varepsilon_1,\varepsilon_2,\varepsilon_6$ vary linearly through the laminate thickness. $\boldsymbol{\varepsilon}(x_1,x_2)$ is the vector of the in-plane or membrane strains and $x_3\boldsymbol{\kappa}(x_1,x_2)$ the vector of flexural strains (bending and twisting). $\boldsymbol{\kappa}(x_1,x_2)$ is the vector of curvature subjected to bending and twisting. We shall see later (Sect. 5.4) that there are different curvature components in the classical and the shear deformation theory of laminates.

The in-plane stress resultant force vector $\boldsymbol{N}$ of a laminate follows by summarizing the adequate vectors of all laminae

$$\boldsymbol{N} = \sum_{k=1}^{n} \boldsymbol{N}^{(k)}, \quad \boldsymbol{N}^{\mathrm{T}} = [N_1\ N_2\ N_6], \quad {\boldsymbol{N}^{(k)}}^{\mathrm{T}} = [N_1^{(k)}\ N_2^{(k)}\ N_6^{(k)}] \tag{4.2.7}$$

By analogy it follows that the resultant moment vector is

$$\boldsymbol{M} = \sum_{k=1}^{n} \boldsymbol{M}^{(k)}, \quad \boldsymbol{M}^{\mathrm{T}} = [M_1\ M_2\ M_6], \quad {\boldsymbol{M}^{(k)}}^{\mathrm{T}} = [M_1^{(k)}\ M_2^{(k)}\ M_6^{(k)}] \tag{4.2.8}$$

The positive directions are corresponding to Figs. 4.5 and 4.6 for a single layer. The transverse shear resultants given in (4.1.50)

$$\boldsymbol{Q}^{\mathrm{s}}(x_1,x_2)=\sum_{k=1}^{n}\boldsymbol{Q}^{\mathrm{s}(k)},\quad \boldsymbol{Q}^{\mathrm{sT}}=[Q_1^{\mathrm{s}}\ Q_2^{\mathrm{s}}],\quad {\boldsymbol{Q}^{\mathrm{s}(k)}}^{\mathrm{T}}=[Q_1^{\mathrm{s}(k)}\ Q_2^{\mathrm{s}(k)}] \tag{4.2.9}$$

Equations (4.2.4) and (4.2.6) yield

$$\boldsymbol{\sigma}^{(k)}=\boldsymbol{Q}^{(k)}\boldsymbol{\varepsilon}=\boldsymbol{Q}^{(k)}(\boldsymbol{\varepsilon}+x_3\boldsymbol{\kappa}) \tag{4.2.10}$$

and the resultants $\boldsymbol{N}$ and $\boldsymbol{M}$ for the laminate are $(k=1,2,\dots,n)$

$$\begin{aligned}\boldsymbol{N}^{(k)}&=\int\limits_{(h^{(k)})}\boldsymbol{\sigma}^{(k)}\mathrm{d}x_3=\boldsymbol{\sigma}^{(k)}h^{(k)},\quad h^{(k)}=x_3^{(k)}-x_3^{(k-1)},\\ \boldsymbol{N}&=\sum_{k=1}^{n}\boldsymbol{\sigma}^{(k)}h^{(k)}\end{aligned} \tag{4.2.11}$$

and

$$\begin{aligned}\boldsymbol{M}^{(k)}&=\int\limits_{(h^{(k)})}\boldsymbol{\sigma}^{(k)}x_3\mathrm{d}x_3=\boldsymbol{\sigma}^{(k)}\frac{1}{2}\left(x_3^{(k)^2}-x_3^{(k-1)^2}\right)=\boldsymbol{\sigma}^{(k)}h^{(k)}\overline{x}^{(k)},\\ \boldsymbol{M}&=\sum_{k=1}^{n}\boldsymbol{\sigma}^{(k)}h^{(k)}\overline{x}^{(k)}\end{aligned} \tag{4.2.12}$$

with

$$\overline{x}^{(k)}=\frac{1}{2}\left(x_3^{(k)}+x_3^{(k-1)}\right)$$

For each layer the membrane strains $\varepsilon_1,\varepsilon_2,\varepsilon_6$, the curvatures $\kappa_1,\kappa_2,\kappa_6$ and the reduced stiffness $Q_{11}^{(k)},Q_{12}^{(k)},Q_{16}^{(k)},Q_{22}^{(k)},Q_{26}^{(k)},Q_{66}^{(k)}$ are constant through each thickness $h^{(k)}$ and (4.2.11) and (4.2.12) reduces to:

$$\begin{aligned}\boldsymbol{N}&=\left(\sum_{k=1}^{n}\boldsymbol{Q}^{(k)}\int\limits_{x_3^{(k-1)}}^{x_3^{(k)}}\mathrm{d}x_3\right)\boldsymbol{\varepsilon}+\left(\sum_{k=1}^{n}\boldsymbol{Q}^{(k)}\int\limits_{x_3^{(k-1)}}^{x_3^{(k)}}x_3\mathrm{d}x_3\right)\boldsymbol{\kappa}\\ &=\boldsymbol{A}\boldsymbol{\varepsilon}+\boldsymbol{B}\boldsymbol{\kappa},\\ \boldsymbol{M}&=\left(\sum_{k=1}^{n}\boldsymbol{Q}^{(k)}\int\limits_{x_3^{(k-1)}}^{x_3^{(k)}}x_3\mathrm{d}x_3\right)\boldsymbol{\varepsilon}+\left(\sum_{k=1}^{n}\boldsymbol{Q}^{(k)}\int\limits_{x_3^{(k-1)}}^{x_3^{(k)}}x_3^2\mathrm{d}x_3\right)\boldsymbol{\kappa}\\ &=\boldsymbol{B}\boldsymbol{\varepsilon}+\boldsymbol{D}\boldsymbol{\kappa}\end{aligned} \tag{4.2.13}$$

$\boldsymbol{A},\boldsymbol{B},\boldsymbol{D}$ are the extensional, coupling and bending stiffness matrices, respectively. From (4.2.4) and (4.2.9), the relations for the transverse shear resultants are

$$\boldsymbol{Q}^{\mathrm{s}} = \sum_{k=1}^{n} \boldsymbol{C}^{\mathrm{s}(k)} \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \boldsymbol{\sigma}^{\mathrm{s}(k)} \mathrm{d}x_3 = \left(\sum_{k=1}^{n} \boldsymbol{C}^{\mathrm{s}(k)} \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \mathrm{d}x_3 \right) \boldsymbol{\gamma}^{\mathrm{s}} = \boldsymbol{A}^{\mathrm{s}} \boldsymbol{\gamma}^{\mathrm{s}} \tag{4.2.14}$$

with

$$\boldsymbol{C}^{\mathrm{s}} = \begin{bmatrix} C_{44} & C_{45} \\ C_{54} & C_{55} \end{bmatrix}, \quad \boldsymbol{\sigma}^{\mathrm{s}} = \begin{bmatrix} \sigma_4 \\ \sigma_5 \end{bmatrix}, \quad \boldsymbol{\gamma}^{\mathrm{s}} = \begin{bmatrix} \varepsilon_4 \\ \varepsilon_5 \end{bmatrix}, \quad \boldsymbol{A}^{\mathrm{s}} = \begin{bmatrix} A_{44} & A_{45} \\ A_{54} & A_{55} \end{bmatrix}$$

Equation (4.2.14) is a first approach and consists of taking the transverse shear strain independent of the coordinate x_3. $\boldsymbol{A}^{\mathrm{s}}$ is the transverse shear stiffness matrix. An improvement is possible by replacing the transverse shear stiffness A_{ij}^{s} by $(kA)_{ij}^{\mathrm{s}}$. k_{ij}^{s} are so called shear correction factors (Sect. 5.4). The elements of the matrices $\boldsymbol{A},\boldsymbol{B},\boldsymbol{D},\boldsymbol{A}^{\mathrm{s}}$ are

$$\begin{aligned}
A_{ij} &= \sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)} - x_3^{(k-1)} \right) = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)}, \quad i,j = 1,2,6, \\
B_{ij} &= \frac{1}{2} \sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)2} - x_3^{(k-1)2} \right) = \sum_{k=1}^{n} Q_{ij}^{(k)} \overline{x}_3^{(k)} h^{(k)}, \\
D_{ij} &= \frac{1}{3} \sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)3} - x_3^{(k-1)3} \right) = \sum_{k=1}^{n} Q_{ij}^{(k)} \left(\overline{x}_3^{(k)2} + \frac{h^{(k)2}}{12} \right) h^{(k)}, \\
A_{ij}^{\mathrm{s}} &= \sum_{k=1}^{n} C_{ij}^{(k)} \left(x_3^{(k)} - x_3^{(k-1)} \right) = \sum_{k=1}^{n} C_{ij}^{(k)} h^{(k)}, \quad i,j = 4,5
\end{aligned} \tag{4.2.15}$$

The constitutive equation for laminates including extensional, bending/torsion and transverse shear strains is the superposition from the so-called classical equations for $\boldsymbol{N}$ and $\boldsymbol{M}$ and the equation that involves the transverse shear resultant $\boldsymbol{Q}^{\mathrm{s}}$. The constitutive equation can be written in the following contracted hypermatrix form

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \\ \boldsymbol{Q}^{\mathrm{s}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}^{\mathrm{s}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma}^{\mathrm{s}} \end{bmatrix} \tag{4.2.16}$$

The stiffness $Q_{ij}^{(k)}$ and $C_{ij}^{(k)}$ in (4.2.15) referred to the laminate's global reference coordinate system $x_i, i = 1,2,3$, are given in Table 4.2 as functions of the ${Q'_{ij}}^{(k)}$ and in (4.2.17) as functions of the ${C'_{ij}}^{(k)}$ referred to the material principal directions of each lamina (k)

$$\begin{aligned}
C_{44} &= C'_{44}c^2 + C'_{55}s^2, \\
C_{45} &= (C'_{55} - C'_{44})sc, \\
C_{55} &= C'_{44}s^2 + C'_{55}c^2
\end{aligned} \tag{4.2.17}$$

Equation (4.2.16) illustrates the coupling between stretching and bending/twisting of a laminate, i.e. in-plane strains result in in-plane resultants but also bending and/or torsion moments and vice versa. Since there are no coupling effects with the transverse shear strains or shear resultants we consider the in-plane and flexural simultaneous equations, (4.2.18), separately

$$\begin{bmatrix} \boldsymbol{N} \\ \cdots \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \vdots & \boldsymbol{B} \\ & \cdots & \\ \boldsymbol{B} & \vdots & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \cdots \\ \boldsymbol{\kappa} \end{bmatrix},$$

or

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ \cdots \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & \vdots & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & \vdots & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & \vdots & B_{16} & B_{26} & B_{66} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{11} & B_{12} & B_{16} & \vdots & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & \vdots & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & \vdots & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \cdots \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} \tag{4.2.18}$$

The following steps are necessary for analyzing a laminated composite subjected to forces and moments:

- Calculate the values of the reduced stiffness Q'_{ij} for each lamina k using the four elastic moduli, $E_L, E_T, \nu_{LT}, G_{LT}$ (4.1.2) and (4.1.3).
- Calculate the values of the transformed reduced stiffness Q_{ij} for each lamina k (Table 4.2).
- Knowing the thickness $h^{(k)}$ of each lamina k calculate the coordinates $x_3^{(k)}, x_3^{(k-1)}$ to the top and the bottom surface of each ply.
- Calculate all A_{ij}, B_{ij} and D_{ij} from (4.2.15).
- Substitute the calculated stiffness and the applied resultant forces and moments in (4.2.18) and calculate the midplane strains ε_i and curvatures κ_i.
- Calculate the global strains $\boldsymbol{\varepsilon}^{(k)}$ in each lamina using (4.2.6) and then the global stresses $\boldsymbol{\sigma}^{(k)}$ for each lamina k using (4.2.10).
- Calculate the local strains $\boldsymbol{\varepsilon}'^{(k)}$ and the local stresses $\boldsymbol{\sigma}'^{(k)}$ for each lamina k using Table 4.1

The inverted relation (4.2.18) leads to the compliance hypermatrix for the in-plane and flexural resultants

$$\begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & \boldsymbol{D} \end{bmatrix}^{-1} \tag{4.2.19}$$

The compliance submatrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ follow from the stiffness submatrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}$. With

$$\boldsymbol{N} = \boldsymbol{A}\boldsymbol{\varepsilon} + \boldsymbol{B}\boldsymbol{\kappa} \tag{4.2.20}$$

it follows that

$$\boldsymbol{\varepsilon} = \boldsymbol{A}^{-1}(\boldsymbol{N} - \boldsymbol{B}\boldsymbol{\kappa}) \tag{4.2.21}$$

and using (4.2.13)

$$\boldsymbol{M} = \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{N} - (\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B} - \boldsymbol{D})\boldsymbol{\kappa} \tag{4.2.22}$$

The first result is a mixed-type constitutive equation

$$\begin{gathered}
\begin{bmatrix} \boldsymbol{\varepsilon} \\ \cdots \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}^* & \vdots & \boldsymbol{B}^* \\ \cdots & \cdots & \cdots \\ \boldsymbol{C}^* & \vdots & \boldsymbol{D}^* \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \cdots \\ \boldsymbol{\kappa} \end{bmatrix}, \\
\begin{aligned}
&\boldsymbol{A}^* = \boldsymbol{A}^{-1}, \ \boldsymbol{B}^* = -\boldsymbol{A}^{-1}\boldsymbol{B}, \\
&\boldsymbol{C}^* = \boldsymbol{B}\boldsymbol{A}^{-1} = -\boldsymbol{B}^{*\mathrm{T}}, \ \boldsymbol{D}^* = \boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}
\end{aligned}
\end{gathered} \tag{4.2.23}$$

With

$$\boldsymbol{\kappa} = \boldsymbol{D}^{*-1}\boldsymbol{M} - \boldsymbol{D}^{*-1}\boldsymbol{C}^*\boldsymbol{N} \tag{4.2.24}$$

it follows that

$$\boldsymbol{\varepsilon} = (\boldsymbol{A}^* - \boldsymbol{B}^*\boldsymbol{D}^{*-1}\boldsymbol{C}^*)\boldsymbol{N} + \boldsymbol{B}^*\boldsymbol{D}^{*-1}\boldsymbol{M} \tag{4.2.25}$$

and the compliance relation has in contracted notation the form

$$\begin{gathered}
\begin{bmatrix} \boldsymbol{\varepsilon} \\ \cdots \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} & \vdots & \boldsymbol{b} \\ \cdots & \cdots & \cdots \\ \boldsymbol{c} & \vdots & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \cdots \\ \boldsymbol{M} \end{bmatrix}, \\
\begin{aligned}
&\boldsymbol{a} = \boldsymbol{A}^* - \boldsymbol{B}^*\boldsymbol{D}^{*-1}\boldsymbol{C}^* = \boldsymbol{A}^* + \boldsymbol{B}^*\boldsymbol{D}^{*-1}\boldsymbol{B}^{*\mathrm{T}}, \\
&\boldsymbol{b} = \boldsymbol{B}^*\boldsymbol{D}^{*-1}, \\
&\boldsymbol{c} = -\boldsymbol{D}^{*-1}\boldsymbol{C}^* = \boldsymbol{D}^{*-1}\boldsymbol{B}^{*\mathrm{T}} = \boldsymbol{b}^{\mathrm{T}}, \\
&\boldsymbol{d} = \boldsymbol{D}^{*-1}
\end{aligned}
\end{gathered} \tag{4.2.26}$$

Equations (4.2.18) and (4.2.26) are inverse relations of the constitutive equation for the resultants and the strains of a laminate. The elements of the submatrices $\boldsymbol{A},\boldsymbol{B},\boldsymbol{D}$ and $\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}$ are functions of the geometry, the material properties and the structure of a laminate and therefore averaged effective elastic laminate moduli. The submatrices $\boldsymbol{A},\boldsymbol{B},\boldsymbol{D},\boldsymbol{a},\boldsymbol{d}$ are symmetric submatrices. That is not the case for the submatrices $\boldsymbol{b}$ and $\boldsymbol{c}$ but with $\boldsymbol{c} = \boldsymbol{b}^{\mathrm{T}}$ the compliance hypermatrix is symmetric. The coupling of different deformation states is a very important quality of the constitutive equations of laminates. In the general case, considered in this section, all coupling effects are present. Figure 4.13 illustrates for example the coupling states for the resultant force N_1 and the resultant moment M_1.

In the next Sect. 4.2.3 we shall see that the stacking sequence of a laminate influences the coupling behavior of loaded laminates. In engineering applications it is desired to specify the stacking sequence such that a number of coefficients of the stiffness matrix will be zero and undesirable couplings between stretching, bending

$$
\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ & A_{22} & A_{26} & \cdot & B_{22} & B_{26} \\ & & A_{66} & \cdot & \cdot & B_{66} \\ & & & D_{11} & D_{12} & D_{16} \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & \cdot & D_{22} & D_{26} \\ & & & \cdot & \cdot & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix}
$$

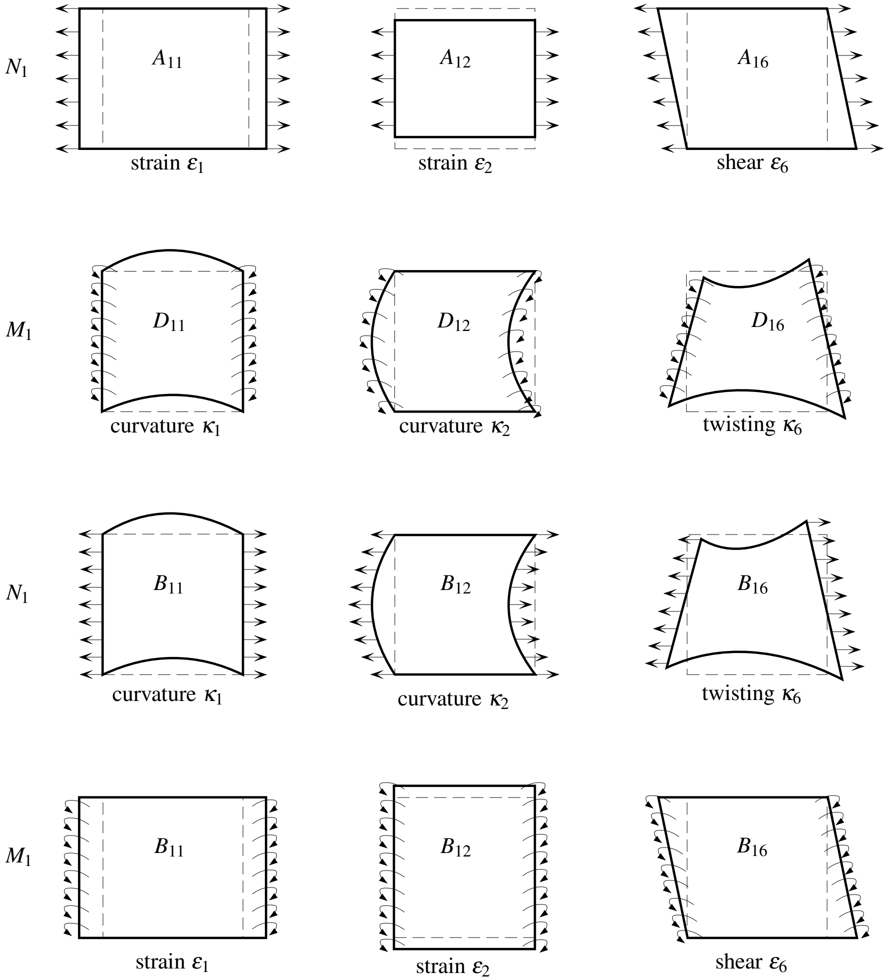


Fig. 4.13 Coupling of strain states: Influence of the stiffness A_{1j}, D_{1j} and $B_{1j} (j = 1,2,6)$ on the strains ε_j and the curvature κ_j of the middle surface of a general laminate loading with N_1 or M_1. In each case 6 deformation states of N_1 and M_1 have to be superposed.

and/or twisting will be avoided. But it is rather difficult to specify an optimum stacking sequence without detailed information about the performance requirements.

Engineering composite design has continued to evolve over many years. Most early applications of composite materials were aimed particularly at weight reduction. Metals were replaced by composites with little or no emphasis placed on tailoring the composite properties. Engineering design created quasi-isotropic laminates that largely suppressed the directional material properties of unidirectional laminae and made the laminate material response similar to that of isotropic materials, e.g. of metals. We shall see in the following discussion that one of such quasi-isotropic laminate is given if it has equal percentages of $0^0, +45^0, -45^0$ and 90^0 layers placed symmetrically with respect to the laminate mid-plane. Quasi-isotropic laminates have elastic properties that are independent of the direction in the plane of the laminate, like traditional isotropic engineering materials. Therefore, quasi-isotropic laminates were in the first applications of composites a convenient replacement for steel or alloys in weight critical applications, e.g. in aerospace industries. Weight saving could be achieved by simple replacing the isotropic metal with a similar stiffness laminate that was lighter and probably stronger. If we compare a graphite/epoxy laminate with an quasi-isotropic stacking sequence of laminae and aluminium we find nearly the same elastic moduli, e.g. $E \approx 70$ GPa, but the density values ρ and the specific stiffness E/ρ differ significantly. The specific stiffness of graphite/epoxy laminate can be twice that of aluminium. Such applications of quasi-isotropic laminates required a minimal amount of redesign effort and therefore minimal changes in structural configuration.

By the time the number of design engineers which are trained in composite materials increased and the tailoring of material properties gained more acceptance. To maximize the utility of the non-isotropic nature of laminates, the influence of the stacking sequence on the structural behavior must be investigated in detail and optimized. Particularly the coupling effects of in-plane and out-of-plane responses affect the effort of laminate structural analysis.

4.2.3 Laminates with Special Laminae Stacking Sequences

Now special cases of laminates which are important in the engineering design of laminated structures will be introduced. Quite often the design of laminates is done by using laminae that have the same constituents, the same thicknesses, etc. but have different orientations of their fibre reinforcement direction with respect to the global reference system of the laminate and a different stacking sequence of these layers. In other cases layers with different materials or thicknesses are bonded to a laminate. The stacking sequence of the layers may result in reducing the coupling of normal and shear forces, of bending and twisting moments etc. It can simplify the mechanical analysis but also gives desired mechanical performance. In the following, the mechanical behavior of special symmetric and unsymmetric laminates are considered.

4.2.3.1 Symmetric Laminates

A laminate is called symmetric if the material, angle and thickness of laminae are the same above and below the midplane, i.e. two symmetric arranged layers to the midplane have the same reduced stiffness matrix $\boldsymbol{Q}^{(k)} \equiv \boldsymbol{Q}^{(k')}$ and the same thickness $h^{(k)} \equiv h^{(k')}$ for opposite coordinates $\bar{x}^{(k)}$ and $\bar{x}^{(k')} = -\bar{x}^{(k)}$ (Fig. 4.14). It follows that the coefficients B_{ij} of the coupling submatrix $\boldsymbol{B}$ are zero and there are no coupling relations of stretching and bending

$$\begin{aligned} B_{ij} &= \frac{1}{2}\sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)^2} - x_3^{(k-1)^2}\right) \\ &= \frac{1}{2}\sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)} + x_3^{(k-1)}\right)\left(x_3^{(k)} - x_3^{(k-1)}\right) \\ &= \sum_{k=1}^{n} Q_{ij}^{(k)} \overline{x}_3^{(k)} h^{(k)} = 0, \quad i,j = 1,2,6 \end{aligned} \tag{4.2.27}$$

With $\overline{x}_3^{(k')} = -\overline{x}_3^{(k)}$ the sum above have two pairs of equal absolute values but opposite signs. Thus the $\boldsymbol{ABD}$-matrix of symmetric laminates is uncoupled, i.e. all terms of the coupling submatrix $[B_{ij}]$ are zero, see following equation

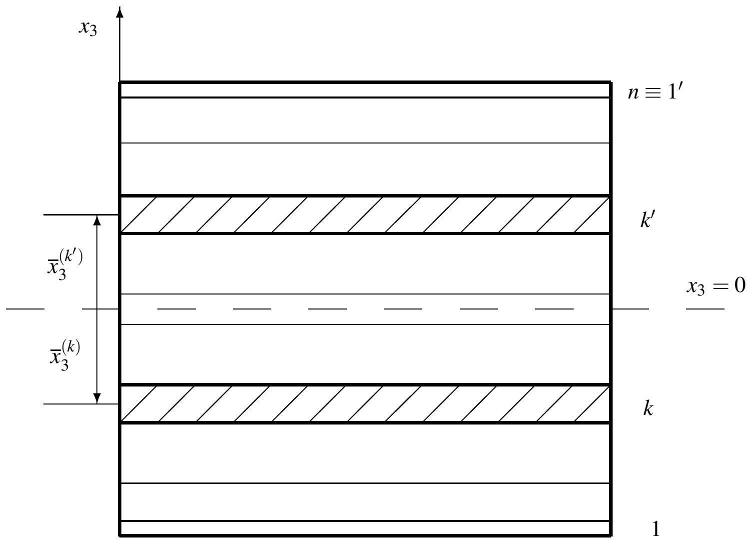


Fig. 4.14 Symmetric laminate with identical layers k and k' opposite to the middle surface ($h^{(k)} = h^{(k')}, \boldsymbol{Q}^{(k)} = \boldsymbol{Q}^{(k')}$)

$$
\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ \cdot\cdot \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & \vdots & 0 & 0 & 0 \\ A_{12} & A_{22} & A_{26} & \vdots & 0 & 0 & 0 \\ A_{16} & A_{26} & A_{66} & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & \vdots & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & \vdots & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \cdot\cdot \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} \tag{4.2.28}
$$

The extensional submatrix $\boldsymbol{A}$ and the bending submatrix $\boldsymbol{D}$ are in the case of symmetric angle ply laminates fully populated and we have in-plane normal and shear strain and out-of-plane bending and torsion couplings. Since the coupling submatrix $\boldsymbol{B}$ is zero the elastic behavior of symmetric laminates is simpler to analyze than that of general laminates and symmetric laminates have no tendency to warp as a result of thermal contractions induced during the composite processing. Some important special cases of symmetric laminates are:

- *Symmetric laminate with isotropic layers*

$$
\begin{aligned}
&Q_{11}^{(k)} = Q_{22}^{(k)} = Q_{11}^{(k')} = Q_{22}^{(k')} = \frac{E^{(k)}}{1-\nu^{(k)}}, \\
&Q_{12}^{(k)} = Q_{12}^{(k')} = \frac{\nu^{(k)}E^{(k)}}{1-\nu^{(k)}},
\end{aligned}
$$

$$
\begin{aligned}
&Q_{16}^{(k)} = Q_{26}^{(k)} = Q_{16}^{(k')} = Q_{26}^{(k')} = 0, \\
&Q_{66}^{(k)} = Q_{66}^{(k')} = \frac{E^{(k)}}{2(1+\nu^{(k)})} = G^{(k)},
\end{aligned}
$$

$$
\begin{aligned}
&A_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)} \\
&\Longrightarrow A_{11} = A_{22}, A_{16} = A_{26} = 0, \\
&D_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)} \left(\bar{x}_3^{(k)^2} + \frac{h^{(k)^2}}{12} \right) \\
&\Longrightarrow D_{11} = D_{22}, D_{16} = D_{26} = 0,
\end{aligned} \tag{4.2.29}
$$

$$
\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ .. \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & \vdots & 0 & 0 & 0 \\ A_{12} & A_{11} & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & A_{66} & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & \vdots & D_{12} & D_{11} & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ .. \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} \tag{4.2.30}
$$

This type of symmetric laminates has no stretching-shearing or bending-torsion coupling.

- *Symmetric cross-ply laminate*
  A laminate is called a cross-ply laminate or a laminate with specially orthotropic layers if only 0^0 and 90^0 plies were used. The material principal axes and the global reference axes are identical. If for example for the kth layer the fibre orientation and the x_1-direction of the global reference system coincide, we have

$$
\begin{aligned}
Q_{11}^{(k)} &\equiv Q_{11}^{(k')} = \frac{E_1^{(k)}}{1 - \nu_{12}^{(k)} \nu_{21}^{(k)}}, & Q_{22}^{(k)} &\equiv Q_{22}^{(k')} = \frac{E_2^{(k)}}{1 - \nu_{12}^{(k)} \nu_{21}^{(k)}}, \\
Q_{12}^{(k)} &\equiv Q_{12}^{(k')} = \frac{\nu_{21}^{(k)} E_1^{(k)}}{1 - \nu_{12}^{(k)} \nu_{21}^{(k)}}, & Q_{16}^{(k)} &\equiv Q_{16}^{(k')} = 0, \\
Q_{66}^{(k)} &\equiv Q_{66}^{(k')} = G_{12}^{(k)}, & Q_{26}^{(k)} &\equiv Q_{26}^{(k')} = 0
\end{aligned} \tag{4.2.31}
$$

and with (4.2.15) $A_{16} = A_{26} = 0, D_{16} = D_{26} = 0$. The stiffness matrix of the constitutive equation has an adequate structure as for isotropic layers, but now $A_{11} \neq A_{22}$ and $D_{11} \neq D_{22}$, i.e. the laminate has an orthotropic structure

$$
\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ \cdots \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & \vdots & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & \vdots & 0 & 0 & 0 \\ 0 & 0 & A_{66} & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \vdots & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & \vdots & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \cdots \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} \tag{4.2.32}
$$

Figure 4.15 illustrates examples of symmetric cross-ply laminates. With $A_{16} = A_{26} = 0, D_{16} = D_{26} = 0$ there is uncoupling between the normal and shear in-plane forces and also between the bending and the twisting moments.

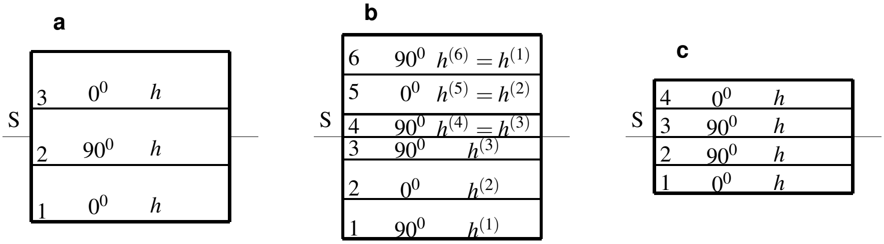


Fig. 4.15 Symmetric cross-ply laminate. **a** 3-layer laminate with equal layer thickness, **b** 6-layer laminate with equal layer thickness in pairs, **c** 4-layer laminate with equal layer thickness

- *Symmetric balanced laminate*
  A laminate is balanced when it consists of pairs of layers of the same thickness and material where the angles of plies are $+\theta$ and $-\theta$. An example is the 8 - layer-laminate $[\pm\theta_1/\pm\theta_2/]_s$. The stiffness coefficients A_{ij} and D_{ij} will be calculated from

$$
\begin{aligned}
&A_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)},\ h^{(k)} = h^{(k')},\ \theta^{(k)} = -\theta^{(k')} \\
&\Longrightarrow A_{16} = A_{26} = 0,
\end{aligned}
$$

$$
\begin{aligned}
D_{ij} &= \frac{1}{3}\sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)^3} - x_3^{(k-1)^3} \right) \\
&= \sum_{k=1}^{n} Q_{ij}^{(k)} \left(\bar{x}_3^{(k)^2} + \frac{h^{(k)^2}}{12} \right) h^{(k)}
\end{aligned}
\tag{4.2.33}
$$

with

$$
h^{(k)} = h^{(k')},\ \theta^{(k)} = \theta^{(k')},\ \bar{x}_3^{(k')} = -\bar{x}_3^{(k)}
$$

and the constitutive equation yields

$$
\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ \cdot\cdot \\ M_1 \\ M_2 \\ M_6 \end{bmatrix}
=
\begin{bmatrix}
A_{11} & A_{12} & 0 & \vdots & 0 & 0 & 0 \\
A_{12} & A_{22} & 0 & \vdots & 0 & 0 & 0 \\
0 & 0 & A_{66} & \vdots & 0 & 0 & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & 0 & 0 & \vdots & D_{11} & D_{12} & D_{16} \\
0 & 0 & 0 & \vdots & D_{12} & D_{22} & D_{26} \\
0 & 0 & 0 & \vdots & D_{16} & D_{26} & D_{66}
\end{bmatrix}
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \cdot\cdot \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix}
\tag{4.2.34}
$$

The fact that the in-plane shear coupling stiffness A_{16} and A_{26} are zero is a defining characteristic of a balanced laminate. In general the bending/twisting coupling stiffness D_{16} and D_{26} are not zero unless the laminate is antisymmetric.

Summarizing the results on symmetric laminates above, it is most important that all components of the $\boldsymbol{B}$-matrix are identical to zero and the full (6×6) $\boldsymbol{ABD}$ - matrix decouples into two (3×3) matrices, namely

$$\boldsymbol{N} = \boldsymbol{A}\boldsymbol{\varepsilon}, \quad \boldsymbol{M} = \boldsymbol{D}\boldsymbol{\kappa} \tag{4.2.35}$$

Therefore also the inverse relations degenerates from (6×6) into two (3×3) relations

$$\boldsymbol{\varepsilon} = \boldsymbol{a}\boldsymbol{N}, \quad \boldsymbol{\kappa} = \boldsymbol{d}\boldsymbol{M} \tag{4.2.36}$$

In these matrix equations $\boldsymbol{a}$ is the inverse of $\boldsymbol{A}$ and $\boldsymbol{d}$ the inverse of $\boldsymbol{D}$

$$\begin{aligned}
a_{11} &= \frac{A_{22}A_{66} - A_{26}^2}{\mathrm{Det}[\boldsymbol{A}]}, & d_{11} &= \frac{D_{22}D_{66} - D_{26}^2}{\mathrm{Det}[\boldsymbol{D}]},\\
a_{12} &= \frac{A_{26}A_{16} - A_{12}A_{66}}{\mathrm{Det}[\boldsymbol{A}]}, & d_{12} &= \frac{D_{26}D_{16} - D_{12}D_{66}}{\mathrm{Det}[\boldsymbol{D}]},\\
a_{16} &= \frac{A_{12}A_{26} - A_{22}A_{16}}{\mathrm{Det}[\boldsymbol{A}]}, & d_{16} &= \frac{D_{12}D_{26} - D_{22}D_{26}}{\mathrm{Det}[\boldsymbol{D}]},\\
a_{22} &= \frac{A_{11}A_{66} - A_{16}^2}{\mathrm{Det}[\boldsymbol{A}]}, & d_{22} &= \frac{D_{11}D_{66} - D_{16}^2}{\mathrm{Det}[\boldsymbol{D}]},\\
a_{26} &= \frac{A_{12}A_{16} - A_{11}A_{26}}{\mathrm{Det}[\boldsymbol{A}]}, & d_{26} &= \frac{D_{12}D_{16} - D_{11}D_{26}}{\mathrm{Det}[\boldsymbol{D}]},\\
a_{66} &= \frac{A_{11}A_{22} - A_{12}^2}{\mathrm{Det}[\boldsymbol{A}]}, & d_{66} &= \frac{D_{11}D_{22} - D_{12}^2}{\mathrm{Det}[\boldsymbol{D}]}
\end{aligned} \tag{4.2.37}$$

with

$$\begin{aligned}
\mathrm{Det}[\boldsymbol{X}] &= X_{11}(X_{22}X_{66} - X_{26}^2) - X_{12}(X_{12}X_{66} - X_{26}X_{16})\\
&\quad + X_{16}(X_{12}X_{26} - X_{22}X_{16}); X_{ij} = A_{ij}, D_{ij}
\end{aligned}$$

For the special symmetric stacking cases one can find

1. Isotropic layers

$$\begin{aligned}
&a_{11} = a_{22}, && a_{16} = a_{26} = 0, && A_{11} = A_{22}, && A_{16} = A_{26} = 0,\\
&d_{11} = d_{22}, && d_{16} = d_{26} = 0, && D_{11} = D_{22}, && D_{16} = D_{26} = 0,
\end{aligned}$$

$$\begin{aligned}
A_{11} &= \frac{Eh}{1-\nu}, & A_{12} &= \nu A_{11}, & A_{66} &= \frac{Eh}{2(1+\nu)},\\
D_{11} &= \frac{Eh^3}{12(1-\nu^2)}, & D_{12} &= \nu D_{11}, & D_{66} &= \frac{Eh^3}{24(1+\nu)}
\end{aligned}$$

with

$$h = \sum_{k=1}^{n} h^{(k)}$$

2. Cross-play layers

$$
\begin{aligned}
a_{11} &= \frac{A_{22}}{A_{11}A_{22}-A_{12}^2}, & d_{11} &= \frac{D_{22}}{D_{11}D_{22}-D_{12}^2},\\
a_{12} &= \frac{-A_{12}}{A_{11}A_{22}-A_{12}^2}, & d_{12} &= \frac{-D_{12}}{D_{11}D_{22}-D_{12}^2},\\
a_{22} &= \frac{A_{11}}{A_{11}A_{22}-A_{12}^2}, & d_{22} &= \frac{D_{11}}{D_{11}D_{22}-D_{12}^2},\\
a_{66} &= \frac{1}{A_{66}}, & d_{66} &= \frac{1}{D_{66}}
\end{aligned}
$$

3. Balanced layers
   The a_{ij} are identical to cross-ply layers. The d_{ij} are identical to the general symmetric case.

4.2.3.2 Antisymmetric Laminates

A laminate is called antisymmetric if the material and thickness of the laminae are the same above and below the midplane but the angle orientations at the same distance above and below of the midplane are of opposite sign, i.e two symmetric arranged layers to the midplane with the coordinates $\bar{x}^{(k)}$ and $x^{(k')} = -x^{(k)}$ having the same thickness $h^{(k)} = h^{(k')}$ and antisymmetric orientations $\theta^{(k)}$ and $\theta^{(k')} = -\theta^{(k)}$ (Fig. 4.14).

- *Antisymmetric cross-ply laminate*
  Antisymmetric cross-ply laminates consist of 0^0 and 90^0 laminae arranged in such a way that for all 0^0-laminae (k) at a distance $\bar{x}^{(k)}$ from the midplane there are 90^0-laminae (k') at a distance $\bar{x}^{(k')} = -\bar{x}^{(k)}$ and vice versa. By definition these laminates have an even number of plies. The reduced stiffness fulfills the conditions

$$
\begin{aligned}
&Q_{11}^{(k)} = Q_{22}^{(k')}, Q_{22}^{(k)} = Q_{11}^{(k')}, Q_{12}^{(k)} = Q_{12}^{(k')},\\
&Q_{16}^{(k)} = Q_{16}^{(k')} = Q_{26}^{(k)} = Q_{26}^{(k')} = 0,
\end{aligned}
$$

  which yield considering (4.2.15) and the 0^0 and 90^0 layers have the same thickness

$$
\begin{aligned}
A_{11} &= A_{22}, A_{16} = A_{26} = 0,\\
B_{11} &= -B_{22}, B_{12} = B_{16} = B_{26} = B_{66} = 0,\\
D_{11} &= D_{22}, D_{16} = D_{26} = 0
\end{aligned}
\tag{4.2.38}
$$

  and the constitutive equation has the form

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ \cdot\cdot \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & \vdots & B_{11} & 0 & 0 \\ A_{12} & A_{11} & 0 & \vdots & 0 & -B_{11} & 0 \\ 0 & 0 & A_{66} & \vdots & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{11} & 0 & 0 & \vdots & D_{11} & D_{12} & 0 \\ 0 & -B_{11} & 0 & \vdots & D_{12} & D_{11} & 0 \\ 0 & 0 & 0 & \vdots & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \cdot\cdot \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} \tag{4.2.39}$$

The constitutive equation (4.2.39) shows that antisymmetric cross-ply laminates only have a tension/bending coupling. It is important to note that the coupling coefficient B_{11} approaches zero as the number of plies increases for a constant laminate thickness since it is inversely proportional to the total number of layers.

- *Antisymmetric balanced laminate*
  Antisymmetric balanced laminates consist of pairs of laminae (k) and (k') at a distance $\bar{x}^{(k)}$ and $\bar{x}^{(k')} = -\bar{x}^{(k)}$ with the same material and thickness but orientations $\theta^{(k)}$ and $\theta^{(k')} = -\theta^{(k)}$. Examples of these laminates are $[\theta_1 \ / \ - \ \theta_1]$, $[\theta_1/\theta_2/-\theta_2/-\theta_1]$, etc. As for all balanced laminates $A_{16} = A_{26} = 0$ and with

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)^3} - x_3^{(k-1)^3} \right) \tag{4.2.40}$$

and

$$\left(x_3^{(k)^3} - x_3^{(k-1)^3} \right) = \left(x_3^{(k')^3} - x_3^{(k'-1)^3} \right),$$

$$Q_{16}^{(k)} = -Q_{16}^{(k')}, \ Q_{26}^{(k)} = -Q_{26}^{(k')}$$

it follows that

$$D_{16} = D_{26} = 0$$

Note that $\overline{x}_3^{(k)} = -\overline{x}_3^{(k')}, h^{(k)} = h^{(k')}$ and

$$\begin{aligned} &Q_{11}^{(k)} = Q_{11}^{(k')}, Q_{22}^{(k)} = Q_{22}^{(k')}, Q_{12}^{(k)} = Q_{12}^{(k')}, \\ &Q_{66}^{(k)} = Q_{66}^{(k')}, Q_{16}^{(k)} = -Q_{16}^{(k')}, Q_{26}^{(k)} = -Q_{26}^{(k')} \end{aligned}$$

Equation (4.2.15) yields $B_{11} = B_{22} = B_{12} = B_{66} = 0$. Balanced antisymmetric laminates have no in-plane shear coupling and also no bending/twisting coupling but a coupling of stretching/twisting and bending/shearing. The constitutive equation has the following structure

$$
\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ \cdots \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & 0 & \vdots & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & \vdots & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & \vdots & B_{16} & B_{26} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & B_{16} & \vdots & D_{11} & D_{12} & 0 \\ 0 & 0 & B_{26} & \vdots & D_{12} & D_{22} & 0 \\ B_{16} & B_{26} & 0 & \vdots & 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \cdots \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} \qquad (4.2.41)
$$

4.2.3.3 Stiffness Matrices for Symmetric and Unsymmetric Laminates in Engineering Applications

Table 4.4 summarizes the stiffness matrices for symmetric and unsymmetric laminates which are used in engineering applications. Symmetric laminates avoid the stretching/bending coupling. But certain applications require the use of nonsymmetric laminates. If possible symmetric balanced laminates should be used. The bending and shearing couplings are eliminated and one can show that for symmetric laminates with a constant total thickness h the values of the bending or flexural stiffness D_{16} and D_{26} decrease with an increasing number of layers and approach zero for $k \longrightarrow \infty$. If the stiffness A_{ij}, B_{ij} and D_{ij} are calculated, the compliances $a_{ij}, b_{ij}, c_{ij} = b_{ij}^{\mathrm{T}}, d_{ij}$ follow from (4.2.26) or for symmetric laminates from (4.2.37). The experimental identification of the compliances is simpler than for the stiffness parameters.

The coupling stiffness B_{ij} and A_{16}, A_{26}, D_{16} and D_{26} complicate the analysis of laminates. To minimize coupling effects symmetric balanced laminates should be created with a fine lamina distribution. Then all B_{ij} and the A_{16}, A_{26} are identical to zero and the D_{16}, D_{26} couplings are relatively low because of the fine lamina distribution. Whenever possible it is recommended to limit the number of fibre orientations to a few specific one, that are $0^0, \pm 45^0, 90^0$ to minimize the processing and experimental testing effort and to select a symmetric and balanced lay-up with a fine lamina interdispersion in order to eliminate in-plane and out-of-plane coupling and the in-plane tension/shearing coupling and to minimize torsion coupling.

There is furthermore a special class of quasi-isotropic laminates. The layers of the laminate can be arranged in such a way that the laminate will behave as an isotropic layer under in-plane loading. Actually, such laminates are not isotropic, because under transverse loading normal to the laminate plane and under interlaminar shear their behavior is different from real isotropic layer. That is why one use the notation quasi-isotropic layer. Because all quasi-isotropic laminates are symmetric and balanced the shear coupling coefficients A_{16}, A_{26} are zero. It can be checked in general any laminate with a lay-up of

Table 4.4 Stiffness matrices for symmetric and unsymmetric laminates

Symmetric laminate	Unsymmetric laminate
Isotropic layers	Balanced laminate
$\begin{matrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{11} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{matrix}$	$\begin{matrix} A_{11} & A_{12} & 0 & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & 0 & B_{12} & B_{22} & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{matrix}$
Eq. (4.2.30)	
Cross-ply laminate	Antimetric balanced laminate
$\begin{matrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{matrix}$	$\begin{matrix} A_{11} & A_{12} & 0 & 0 & 0 & B_{16} \\ A_{12} & A_{22} & 0 & 0 & 0 & B_{26} \\ 0 & 0 & A_{66} & B_{16} & B_{26} & 0 \\ 0 & 0 & B_{16} & D_{11} & D_{12} & 0 \\ 0 & 0 & B_{26} & D_{12} & D_{22} & 0 \\ B_{16} & B_{26} & 0 & 0 & 0 & D_{66} \end{matrix}$
Eq. (4.2.32)	Eq. (4.2.41)
Balanced laminate	Cross-ply
$\begin{matrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{16} & D_{26} & D_{66} \end{matrix}$	$\begin{matrix} A_{11} & A_{12} & 0 & B_{11} & 0 & 0 \\ A_{12} & A_{11} & 0 & 0 & -B_{11} & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ B_{11} & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & -B_{11} & 0 & D_{12} & D_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{matrix}$
Eq. (4.2.34)	Eq. (4.2.39)
Angle-ply laminate	Cross-ply (approximate solution $k \to \infty$)
$\begin{matrix} A_{11} & A_{12} & A_{16} & 0 & 0 & 0 \\ A_{12} & A_{22} & A_{26} & 0 & 0 & 0 \\ A_{16} & A_{26} & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & D_{16} \\ 0 & 0 & 0 & D_{12} & D_{22} & D_{26} \\ 0 & 0 & 0 & D_{16} & D_{26} & D_{66} \end{matrix}$	$\begin{matrix} A_{11} & A_{12} & 0 & 0 & 0 & 0 \\ A_{12} & A_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & A_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{11} & D_{12} & 0 \\ 0 & 0 & 0 & D_{12} & D_{11} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{matrix}$
Eq. (4.2.28)	

$$\left[0/\frac{\pi}{n}/\frac{2\pi}{n}/\dots/\frac{(n-1)\pi}{n}\right]_S$$

or

$$\left[\frac{\pi}{n}/\frac{2\pi}{n}/\dots/\pi\right]_S$$

is quasi-isotropic for any integer n greater than 2. The simplest types are laminates with the following lay-up

$$[0/60/-60]_S\,, n = 3, 120^0 \equiv -60^0$$

and

$$[0/+45/-45/90]_S\,, n = 4, 135^0 \equiv -45^0$$

Summarizing the mechanical performance of laminates with special laminae stacking sequences which are used in laminate design, we have considered the following classification:

1. *General laminates*
   The stacking sequence, the thickness, the material and the fibre orientations of all laminae is quite general. All extensional stiffness B_{ij} are not zero.
2. *Symmetric laminates*
   For every layer to one side of the laminate reference surface there is a corresponding layer to the other side of the reference surface at an equal distance and with identical thickness, material and fibre orientation. All coupling stiffness B_{ij} are zero.
3. *Antisymmetric laminates*
   For every layer to one side of the laminate reference surface there is a corresponding layer to the other side of the reference surface at an equal distance, with identical thickness and material, but opposite fibre orientation. The stiffness A_{16}, A_{26}, D_{16} and D_{26} are zero.
4. *Balanced laminates*
   For every layer with a specified thickness, specific material properties and specific fibre orientation there is another layer with identical thickness and material properties, but opposite fibre orientation anywhere in the laminate, i.e. the corresponding layer with opposite fibre orientation does not have to be on the opposite side of the reference surface, nor immediately adjacent to the other layer nor anywhere particular. A balanced laminate can be

   - General or unsymmetric: $A_{16} = A_{26} = 0$
   - Symmetric $A_{16} = A_{26} = 0$, $B_{ij} = 0$
   - Antisymmetric $A_{16} = A_{26} = 0, D_{16} = D_{26} = 0$

   An antisymmetric laminate is a special case of a balanced laminate, having its balanced $\pm$ pairs of layers symmetrically situated to the middle surface.
5. *Cross-ply laminates*
   Every layer of the laminate has its fibers oriented at either 0^0 or 90^0. Cross-ply laminates can be

   - General or unsymmetric: $A_{16} = A_{26} = 0$, $B_{16} = B_{26} = 0$, $D_{16} = D_{26} = 0$
   - Symmetric $A_{16} = A_{26} = 0$, $D_{16} = D_{26} = 0$, $B_{ij} = 0$
   - Antisymmetric $A_{16} = A_{26} = 0, D_{16} = D_{26} = 0$, $B_{12} = B_{16} = B_{26} = B_{66} = 0$, $A_{11} = A_{22}$, $B_{11} = B_{22}$, $D_{11} = D_{22}$

Symmetric cross-ply laminates are orthotropic with respect to both in-plane and bending behavior and all coupling stiffness are zero.

6. *Quasi-isotropic laminates*
For every laminate with a symmetric lay-up of

$$[0/\frac{\pi}{n}/\ldots/\frac{(n-1)\pi}{n}]_S \quad \text{or} \quad [0/\frac{\pi}{n}/\frac{2\pi}{n}/\ldots/\pi]_S$$

the in-plane stiffness are identical in all directions. Because all these quasi-isotropic laminates are also balanced we have $A_{11} = A_{22} = \text{const}, A_{12} = \text{const}$, $A_{16} = A_{26} = 0, B_{ij} \equiv 0, D_{ij} \neq 0$

7. *Laminates with isotropic layers*
If isotropic layers of possible different materials properties and thicknesses are arranged symmetrically to the middle surface the laminate is symmetric isotropic and we have $A_{11} = A_{22}, D_{11} = D_{22}$, $A_{16} = A_{26} = 0, D_{16} = D_{26} = 0$, $B_{ij} = 0$, i.e. the mechanical performance is isotropic.

For symmetric laminates the in-plane and flexural moduli can be defined with help of effective engineering parameters. We start with (4.2.26). $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} = \boldsymbol{b}^{\mathrm{T}}, \boldsymbol{d}$ are the extensional compliance matrix, coupling compliance matrix and bending compliance matrix, respectively. For a symmetric laminate $\boldsymbol{B} = \boldsymbol{0}$ and it can be shown that $\boldsymbol{a} = \boldsymbol{A}^{-1}$ and $\boldsymbol{d} = \boldsymbol{D}^{-1}$. The in-plane and the flexural compliance matrices $\boldsymbol{a}$ and $\boldsymbol{d}$ are uncoupled but generally fully populated

$$\boldsymbol{\varepsilon} = \boldsymbol{aN}, \quad \boldsymbol{\kappa} = \boldsymbol{dM} \tag{4.2.42}$$

Equations (4.2.42) lead to effective engineering moduli for symmetric laminates.

1. Effective in-plane engineering moduli $E_1^N, E_2^N, G_{12}^N, \nu_{12}^N$:
Substitute $N_1 \neq 0, N_2 = N_6 = 0$ in $\boldsymbol{\varepsilon} = \boldsymbol{aN}$ as

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{16} \\ a_{12} & a_{22} & a_{26} \\ a_{16} & a_{26} & a_{66} \end{bmatrix} \begin{bmatrix} N_1 \\ 0 \\ 0 \end{bmatrix} \tag{4.2.43}$$

which gives

$$\varepsilon_1 = a_{11} N_1$$

and the effective longitudinal modulus E_1^N is

$$E_1^N \equiv \frac{\sigma_1}{\varepsilon_1} = \frac{N_1/h}{a_{11}N_1} = \frac{1}{ha_{11}} \tag{4.2.44}$$

In an analogous manner with $N_1 = 0, N_2 \neq 0, N_6 = 0$ or $N_1 = N_2 = 0, N_6 \neq 0$, the effective transverse modulus E_2^N or the effective shear modulus G_{12}^N are

$$E_2^N \equiv \frac{\sigma_2}{\varepsilon_2} = \frac{N_2/h}{a_{22}N_2} = \frac{1}{ha_{22}}, \tag{4.2.45}$$

$$G_{12}^N \equiv \frac{\sigma_6}{\varepsilon_6} = \frac{N_6/h}{a_{66}N_6} = \frac{1}{ha_{66}} \tag{4.2.46}$$

The effective in-plane Poisson's ratio ν_{12}^N can be derived in the following way. With $N_1 \neq 0, N_2 = N_6 = 0$ (4.2.43) yields $\varepsilon_2 = a_{12}N_1, \varepsilon_1 = a_{11}N_1$ and ν_{12} is defined as

$$\nu_{12}^N = -\frac{\varepsilon_2}{\varepsilon_1} = -\frac{a_{12}N_1}{a_{11}N_1} = -\frac{a_{12}}{a_{11}} \tag{4.2.47}$$

The Poisson's ratio ν_{21}^N can be derived directly by substituting $N_1 = N_6 = 0$, $N_2 \neq 0$ in (4.2.42) and define $\nu_{21}^N = -\varepsilon_1/\varepsilon_2$ or by using the reciprocal relationship $\nu_{12}^N/E_1^N = \nu_{21}^N/E_2^N$. In both cases ν_{21}^N is given as

$$\nu_{21}^N = -\frac{a_{12}}{a_{22}} \tag{4.2.48}$$

The effective in-plane engineering moduli can be also formulated in terms of the elements of the $\boldsymbol{A}$-matrix

$$\begin{aligned} &E_1^N = \frac{A_{11}A_{22} - A_{12}^2}{A_{22}h}, E_2^N = \frac{A_{11}A_{22} - A_{12}^2}{A_{11}h}, G_{12}^N = \frac{A_{66}}{h}, \\ &\nu_{12}^N = \frac{A_{12}}{A_{22}}, \nu_{21}^N = \frac{A_{12}}{A_{11}} \end{aligned} \tag{4.2.49}$$

2. Effective flexural engineering moduli $E_1^M, E_2^M, G_{12}^M, \nu_{12}^M, \nu_{21}^M$:
   To define the effective flexural moduli we start with $\boldsymbol{\kappa} = \boldsymbol{dM}$. Apply $M_1 \neq 0$, $M_2 = 0, M_6 = 0$ and substitute in the flexural compliance relation to give

$$\kappa_1 = d_{11}M_1 = \frac{M_1}{E_1^M I}, \quad I = \frac{h^3}{12}$$

   and the effective flexural longitudinal modulus E_1^M is

$$E_1^M = \frac{12M_1}{\kappa_1 h^3} = \frac{12}{h^3 d_{11}} \tag{4.2.50}$$

   Similarly, one can show that the other flexural elastic moduli are given by

$$E_2^M = \frac{12}{h^3 d_{22}}, \quad \nu_{12}^M = -\frac{d_{12}}{d_{11}}, \quad G_{12}^M = \frac{12}{h^3 d_{66}}, \quad \nu_{21}^M = -\frac{d_{12}}{d_{22}} \tag{4.2.51}$$

   Flexural Poisson's ratios also have a reciprocal relationship

$$\frac{\nu_{12}^M}{E_1^M} = \frac{\nu_{21}^M}{E_2^M} \tag{4.2.52}$$

   In terms of the elements of the $\boldsymbol{D}$ matrix we find

$$
\begin{aligned}
E_1^M &= \frac{12(D_{11}D_{22}-D_{12}^2)}{D_{22}h^3}, E_2^M = \frac{12(D_{11}D_{22}-D_{12}^2)}{D_{11}h^3}, \\
G_{12}^M &= \frac{12D_{66}}{h^3}, \nu_{12}^M = \frac{D_{12}}{D_{22}}, \nu_{21}^M = \frac{D_{12}}{D_{11}}
\end{aligned} \tag{4.2.53}
$$

Consider unsymmetric laminates, the laminate stiffness or compliance matrices are not uncoupled and therefore it is not meaningful to use effective engineering laminate moduli.

4.2.4 Stress Analysis

Laminate stresses may be subdivided into in-plane stresses, which are calculated below with the classical assumption of linear strain functions of x_3, and the through-the-thickness stresses, which are calculated approximately by integration of the equilibrium conditions. Taking into account the assumptions of macro-mechanical modelling of laminates the strains $\varepsilon_1, \varepsilon_2, \varepsilon_6$ vary linearly across the thickness of the laminate

$$
\boldsymbol{\varepsilon}(x_1,x_2,x_3) = \boldsymbol{\varepsilon}(x_1,x_2) + x_3\boldsymbol{\kappa}(x_1,x_2), \quad h = \sum_{k=1}^{n} h^{(k)} \tag{4.2.54}
$$

These global strains can be transformed to the local strains in the principal material directions of the kth layer through the transformation equations (Table 4.1)

$$
\boldsymbol{\varepsilon}'^{(k)} = \boldsymbol{T}^{\varepsilon}\boldsymbol{\varepsilon}^{(k)}, \quad x_3^{(k-1)} \le x_3 \le x^{(k)} \tag{4.2.55}
$$

If the strains are known at any point along the thickness of the laminate, the stress-strain relation (Table 4.2) calculates the global stress in each lamina

$$
\boldsymbol{\sigma}^{(k)} = \boldsymbol{Q}^{(k)}\boldsymbol{\varepsilon}^{(k)} = \boldsymbol{Q}^{(k)}(\boldsymbol{\varepsilon} + x_3\boldsymbol{\kappa}), \quad x_3^{(k-1)} \le x_3 \le x^{(k)} \tag{4.2.56}
$$

By applying the transformation equation for the stress vector (Table 4.1) the stresses expressed in the principal material axes can be calculated

$$
\boldsymbol{\sigma}'^{(k)} = \boldsymbol{T}^{\sigma}\boldsymbol{\sigma}^{(k)} \tag{4.2.57}
$$

Starting from the strains $\boldsymbol{\varepsilon}'^{(k)}$, the stresses in the kth layer are expressed as follows

$$
\boldsymbol{\sigma}'^{(k)} = \boldsymbol{Q}'^{(k)}\boldsymbol{\varepsilon}'^{(k)} \tag{4.2.58}
$$

From (4.2.56), the stresses vary linearly through the thickness of each lamina and may jump from lamina to lamina since the reduced stiffness matrix $\boldsymbol{Q}^{(k)}$ changes from ply to ply since $\boldsymbol{Q}^{(k)}$ depends on the material and orientation of the lamina (k). Figure 4.16 illustrates qualitatively the stress jumps of the membrane stresses $\boldsymbol{\sigma}_{\mathrm{M}}^{(k)}$

$$\begin{array}{lll}
\sigma_{1M}^{(k)} = Q_{11}^{(k)} \varepsilon_1 & \sigma_{1B}^{(k)} = Q_{11}^{(k)} x_3 \kappa_1 & \sigma_1^{(k)} = \sigma_{1M}^{(k)} + \sigma_{1B}^{(k)} \\
\sigma_{2M}^{(k)} = Q_{22}^{(k)} \varepsilon_2 & \sigma_{2B}^{(k)} = Q_{22}^{(k)} x_3 \kappa_2 & \sigma_2^{(k)} = \sigma_{2M}^{(k)} + \sigma_{2B}^{(k)} \\
\sigma_{6M}^{(k)} = Q_{66}^{(k)} \varepsilon_6 & \sigma_{6B}^{(k)} = Q_{66}^{(k)} x_3 \kappa_6 & \sigma_6^{(k)} = \sigma_{6M}^{(k)} + \sigma_{6B}^{(k)}
\end{array}$$

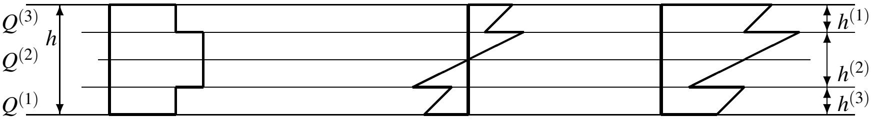


Fig. 4.16 Qualitatively variation of the in-plane membrane stresses σ_{iM}, the bending stresses σ_{iB} and the total stress σ_i through the thickness of the laminate. Assumptions $h^{(1)} = h^{(3)}$, $Q^{(1)} = Q^{(3)} < Q^{(2)}, i = 1,2,6$

which follow from the in-plane resultants $\boldsymbol{N}$ and are constant through each lamina and the bending/torsion stresses $\boldsymbol{\sigma}_{\mathrm{B}}^{(k)}$ following from the moment resultants $\boldsymbol{M}$ and vary linearly through each ply thickness. The transverse shear stresses σ_4, σ_5 follow for a plane stress state assumptions that is in the framework of the classical laminate theory, Sect. 5.1, not from a constitutive equation but as for the single layer, (4.1.56), by integration of the equilibrium equations. For any lamina m of the laminate by analogy to (4.1.57) can be established

$$\begin{aligned}
\sigma_5^{(m)}(x_3) = -\sum_{k=1}^{m-1} \Bigg\{ \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} & \Bigg[Q_{11}^{(k)} \frac{\partial}{\partial x_1}(\varepsilon_1 + x_3\kappa_1) + Q_{12}^{(k)} \frac{\partial}{\partial x_1}(\varepsilon_2 + x_3\kappa_2) \\
& + Q_{16}^{(k)} \frac{\partial}{\partial x_1}(\varepsilon_6 + x_3\kappa_6) + Q_{61}^{(k)} \frac{\partial}{\partial x_2}(\varepsilon_1 + x_3\kappa_1) \\
& + Q_{62}^{(k)} \frac{\partial}{\partial x_2}(\varepsilon_2 + x_3\kappa_2) + Q_{66}^{(k)} \frac{\partial}{\partial x_2}(\varepsilon_6 + x_3\kappa_6) \Bigg] \mathrm{d}x_3 \Bigg\} \\
- \int\limits_{x_3^{(m-1)}}^{x_3} & \Bigg[Q_{11}^{(m)} \frac{\partial}{\partial x_1}(\varepsilon_1 + x_3\kappa_1) + Q_{12}^{(m)} \frac{\partial}{\partial x_1}(\varepsilon_2 + x_3\kappa_2) \\
& + Q_{16}^{(m)} \frac{\partial}{\partial x_1}(\varepsilon_6 + x_3\kappa_6) + Q_{61}^{(m)} \frac{\partial}{\partial x_2}(\varepsilon_1 + x_3\kappa_1) \\
& + Q_{62}^{(m)} \frac{\partial}{\partial x_2}(\varepsilon_2 + x_3\kappa_2) + Q_{66}^{(m)} \frac{\partial}{\partial x_2}(\varepsilon_6 + x_3\kappa_6) \Bigg] \mathrm{d}x_3, \qquad (4.2.59)
\end{aligned}$$

$$\sigma_4^{(m)}(x_3) = -\sum_{k=1}^{m-1} \Bigg\{ \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \Bigg[Q_{61}^{(k)} \frac{\partial}{\partial x_1}(\varepsilon_1 + x_3\kappa_1) + Q_{62}^{(k)} \frac{\partial}{\partial x_1}(\varepsilon_2 + x_3\kappa_2)$$

$$
\left.\begin{aligned}
&+Q_{66}^{(k)}\frac{\partial}{\partial x_1}(\varepsilon_6+x_3\kappa_6)+Q_{21}^{(k)}\frac{\partial}{\partial x_2}(\varepsilon_1+x_3\kappa_1)\\
&+Q_{22}^{(k)}\frac{\partial}{\partial x_2}(\varepsilon_2+x_3\kappa_2)\mathrm{d}x_3+Q_{26}^{(k)}\frac{\partial}{\partial x_2}(\varepsilon_6+x_3\kappa_6)\mathrm{d}x_3
\end{aligned}\right\}
\tag{4.2.60}
$$

$$
\begin{aligned}
&-\int\limits_{x_3^{(m-1)}}^{x_3}\left[Q_{61}^{(m}\frac{\partial}{\partial x_1}(\varepsilon_1+x_3\kappa_1)+Q_{62}^{(m)}\frac{\partial}{\partial x_1}(\varepsilon_2+x_3\kappa_2)\right.\\
&+Q_{66}^{(m)}\frac{\partial}{\partial x_1}(\varepsilon_6+x_3\kappa_6)+Q_{21}^{(m)}\frac{\partial}{\partial x_2}(\varepsilon_1+x_3\kappa_1)\\
&\left.+\,Q_{22}^{(m)}\frac{\partial}{\partial x_2}(\varepsilon_2+x_3\kappa_2)+Q_{26}^{(m)}\frac{\partial}{\partial x_2}(\varepsilon_6+x_3\kappa_6)\right]\mathrm{d}x_3,
\end{aligned}
$$

$$
\sigma_i^{(m)}(x_3=x_3^{(m)})=\sigma_i^{(m+1)}(x_3=x_3^{(m)}),i=4,5;m=1,2,\ldots,n,
$$
$$
x_3^{(m+1)}\leq x_3\leq x_3^{(m)}
$$

With the relationships

$$
\begin{aligned}
\int\limits_{x_3^{(k-1)}}^{x_3^{(k)}}\boldsymbol{Q}^{(k)}\mathrm{d}x_3 &=\boldsymbol{Q}^{(k)}\left(x_3^{(k)}-x_3^{(k-1)}\right)=\boldsymbol{Q}^{(k)}h^{(k)},\\
\int\limits_{x_3^{(k-1)}}^{x_3^{(k)}}\boldsymbol{Q}^{(k)}x_3\mathrm{d}x_3 &=\boldsymbol{Q}^{(k)}\frac{1}{2}\left({x_3^{(k)}}^2-{x_3^{(k-1)}}^2\right)=\boldsymbol{Q}^{(k)}h^{(k)}\bar{x}_3^{(k)}=\boldsymbol{Q}^{(k)}s^{(k)}
\end{aligned}
\tag{4.2.61}
$$

$\bar{x}_3^{(k)}$ is the distance of lamina k from the midplane. The shear stresses $\sigma_i^{(m)}(x_3 = x_3^{(m)}), i=4,5$, at the top surface of the mth lamina can be formulated by

$$
\begin{bmatrix}\sigma_5^{(m)}\left(x_3=x_3^{(m)}\right)\\ \sigma_4^{(m)}\left(x_3=x_3^{(m)}\right)\end{bmatrix}=-\sum_{k=1}^{m}\begin{bmatrix}{\boldsymbol{F}_1^{(k)}}^{\mathrm{T}} & {\boldsymbol{F}_6^{(k)}}^{\mathrm{T}}\\ {\boldsymbol{F}_6^{(k)}}^{\mathrm{T}} & {\boldsymbol{F}_2^{(k)}}^{\mathrm{T}}\end{bmatrix}\begin{bmatrix}\boldsymbol{\eta}_1\\ \boldsymbol{\eta}_2\end{bmatrix}
\tag{4.2.62}
$$

with

$$
\begin{aligned}
{\boldsymbol{F}_1^{(k)}}^{\mathrm{T}} &=[h^{(k)}Q_{(11)}^{(k)}\ h^{(k)}Q_{(12)}^{(k)}\ h^{(k)}Q_{(16)}^{(k)}\quad s^{(k)}Q_{(11)}^{(k)}\ s^{(k)}Q_{(12)}^{(k)}\ s^{(k)}Q_{(16)}^{(k)}],\\
{\boldsymbol{F}_2^{(k)}}^{\mathrm{T}} &=[h^{(k)}Q_{(21)}^{(k)}\ h^{(k)}Q_{(22)}^{(k)}\ h^{(k)}Q_{(26)}^{(k)}\quad s^{(k)}Q_{(21)}^{(k)}\ s^{(k)}Q_{(22)}^{(k)}\ s^{(k)}Q_{(26)}^{(k)}],\\
{\boldsymbol{F}_6^{(k)}}^{\mathrm{T}} &=[h^{(k)}Q_{(61)}^{(k)}\ h^{(k)}Q_{(62)}^{(k)}\ h^{(k)}Q_{(66)}^{(k)}\quad s^{(k)}Q_{(61)}^{(k)}\ s^{(k)}Q_{(62)}^{(k)}\ s^{(k)}Q_{(66)}^{(k)}]
\end{aligned}
$$

and

$$\boldsymbol{\eta}_1^{\mathrm{T}} = \left[\frac{\partial \varepsilon_1}{\partial x_1}\ \frac{\partial \varepsilon_2}{\partial x_1}\ \frac{\partial \varepsilon_6}{\partial x_1}\ \frac{\partial \kappa_1}{\partial x_1}\ \frac{\partial \kappa_2}{\partial x_1}\ \frac{\partial \kappa_6}{\partial x_1}\right], \quad \boldsymbol{\eta}_2^{\mathrm{T}} = \left[\frac{\partial \varepsilon_1}{\partial x_2}\ \frac{\partial \varepsilon_2}{\partial x_2}\ \frac{\partial \varepsilon_6}{\partial x_2}\ \frac{\partial \kappa_1}{\partial x_2}\ \frac{\partial \kappa_2}{\partial x_2}\ \frac{\partial \kappa_6}{\partial x_2}\right]$$

The transverse shear stresses only satisfy the equilibrium conditions but violate the other fundamental equations of anisotropic elasticity. They vary in a parabolic way through the thicknesses $h^{(k)}$ of the laminate layers and there is no stress jump if one crosses the interface between two layers.

4.2.5 Thermal and Hygroscopic Effects

In Sect. 4.1.2 the hygrothermal strains were calculated for unidirectional and angle ply laminae. As mentioned above, no residual mechanical stresses would develop in the lamina at the macro-mechanical level, if the lamina is free to expand. Free thermal strain, e.g., refers to the fact that fibres and matrix of an UD-lamina are smeared into a single equivalent homogeneous material and that the smeared elements are free of any stresses if temperature is changed. When one considers an unsmeared material and deals with the individual fibres and the surrounding matrix, a temperature change can create significant stresses in the fibre and matrix. When such selfbalanced stresses are smeared over a volume element, the net result is zero. However, in a laminate with various laminae of different materials and orientations each individual lamina is not free to deform. This results in residual stresses in the laminate. As in Eqs. (4.1.31) and (4.1.32) $\alpha^{\mathrm{th}}, \alpha^{\mathrm{mo}}$ are the thermal and moisture expansion coefficients, T is the temperature change and M^* the weight of moisture absorption per unit weight. In the following equations, T and M^* are independent of the x_3-coordinate, i.e. they are constant not only through the thickness $h^{(k)}$ of a single layer but through the thickness h of the laminate. Heat transfer in thin laminates, e.g., is generally quite rapid and, hence, thermal gradients in x_3-direction are seldom taken into account and the temperature change T is then approximately independent of x_3. Analogous considerations are valid for changes in moisture.

For a single layer in off-axis coordinates, the hygrothermal strains and stresses are given by

$$\boldsymbol{\varepsilon} = \boldsymbol{S}\boldsymbol{\sigma} + \boldsymbol{\alpha}^{\mathrm{th}} T + \boldsymbol{\alpha}^{\mathrm{mo}} M^*, \quad \boldsymbol{\sigma} = \boldsymbol{Q}(\boldsymbol{\varepsilon} - \boldsymbol{\alpha}^{\mathrm{th}} T - \boldsymbol{\alpha}^{\mathrm{mo}} M^*) \tag{4.2.63}$$

or substituting $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} + x_3 \boldsymbol{\kappa}$

$$\boldsymbol{\sigma} = \boldsymbol{Q}(\boldsymbol{\varepsilon} + x_3 \boldsymbol{\kappa} - \boldsymbol{\alpha}^{\mathrm{th}} T - \boldsymbol{\alpha}^{\mathrm{mo}} M^*) \tag{4.2.64}$$

The definitions of the force and moment resultants $\boldsymbol{N}$ and $\boldsymbol{M}$

$$
\begin{aligned}
\boldsymbol{N} &= \int\limits_{(h)} \boldsymbol{\sigma} \mathrm{d}x_3 &&= \int\limits_{(h)} \boldsymbol{Q}(\boldsymbol{\varepsilon} + x_3\boldsymbol{\kappa} - \boldsymbol{\alpha}^{\mathrm{th}} T - \boldsymbol{\alpha}^{\mathrm{mo}} M^*)\mathrm{d}x_3, \\
\boldsymbol{M} &= \int\limits_{(h)} \boldsymbol{\sigma} x_3 \mathrm{d}x_3 &&= \int\limits_{(h)} \boldsymbol{Q}(\boldsymbol{\varepsilon} + x_3\boldsymbol{\kappa} - \boldsymbol{\alpha}^{\mathrm{th}} T - \boldsymbol{\alpha}^{\mathrm{mo}} M^*)x_3\mathrm{d}x_3
\end{aligned}
\qquad (4.2.65)
$$

yield the equations

$$
\begin{aligned}
\boldsymbol{N} &= \boldsymbol{A}\boldsymbol{\varepsilon} + \boldsymbol{B}\boldsymbol{\kappa} - \boldsymbol{N}^{\mathrm{th}} - \boldsymbol{N}^{\mathrm{mo}}, \\
\boldsymbol{M} &= \boldsymbol{B}\boldsymbol{\varepsilon} + \boldsymbol{D}\boldsymbol{\kappa} - \boldsymbol{M}^{\mathrm{th}} - \boldsymbol{M}^{\mathrm{mo}},
\end{aligned}
\qquad (4.2.66)
$$

$$
\boldsymbol{A} = \boldsymbol{Q}h, \quad \boldsymbol{B} = \boldsymbol{0}, \quad \boldsymbol{D} = \boldsymbol{Q}\frac{h^3}{12}
$$

$\boldsymbol{B} = \boldsymbol{0}$ follows from the symmetry of a single layer to its midplane. $\boldsymbol{N}^{\mathrm{th}}$, $\boldsymbol{N}^{\mathrm{mo}}$, $\boldsymbol{M}^{\mathrm{th}}, \boldsymbol{M}^{\mathrm{mo}}$ are fictitious hygrothermal resultants which are defined in (4.2.67). If T and M^* are independent of x_3 one can introduce unit thermal and unit moisture stress resultants $\hat{N}^{\mathrm{th}}, \hat{M}^{\mathrm{th}}, \hat{N}^{\mathrm{mo}}, \hat{M}^{\mathrm{mo}}$, i.e. resultants per unit temperature or moisture change

$$
\begin{aligned}
\boldsymbol{N}^{\mathrm{th}} &= \int\limits_{(h)} \boldsymbol{Q}\boldsymbol{\alpha}^{\mathrm{th}} T \mathrm{d}x_3 &&= \boldsymbol{Q}\boldsymbol{\alpha}^{\mathrm{th}} T h = \hat{N}^{\mathrm{th}} T, \\
\boldsymbol{M}^{\mathrm{th}} &= \int\limits_{(h)} \boldsymbol{Q}\boldsymbol{\alpha}^{\mathrm{th}} T x_3 \mathrm{d}x_3 &&= \frac{1}{2}\boldsymbol{Q}\boldsymbol{\alpha}^{\mathrm{th}} T h^2 = \hat{M}^{\mathrm{th}} T, \\
\boldsymbol{N}^{\mathrm{mo}} &= \int\limits_{(h)} \boldsymbol{Q}\boldsymbol{\alpha}^{\mathrm{mo}} M^* \mathrm{d}x_3 &&= \boldsymbol{Q}\boldsymbol{\alpha}^{\mathrm{mo}} M^* h = \hat{N}^{\mathrm{mo}} M^*, \\
\boldsymbol{M}^{\mathrm{mo}} &= \int\limits_{(h)} \boldsymbol{Q}\boldsymbol{\alpha}^{\mathrm{mo}} M^* x_3 \mathrm{d}x_3 &&= \frac{1}{2}\boldsymbol{Q}\boldsymbol{\alpha}^{\mathrm{mo}} M^* h^2 = \hat{M}^{\mathrm{mo}} M^*.
\end{aligned}
\qquad (4.2.67)
$$

N^{th} and N^{mo} have the units of the force resultant, namely N/m, and M^{th} and M^{mo} the units of the moment resultants, namely Nm/m. The integral form of the resultant definitions makes these definitions quite general, i.e. if T or M^* are known functions of x_3, the integration can be carried out. But for the temperature changes with x_3 and if the material properties change with temperature, the integration can be complicated, but in general the simple integrated form, Eqs. (4.2.67) can be used. With the total force and moment resultants $\tilde{\boldsymbol{N}}, \tilde{\boldsymbol{M}}$, equal to the respective sums of their mechanical and hygrothermal components

$$
\tilde{\boldsymbol{N}} = \boldsymbol{N} + \boldsymbol{N}^{\mathrm{th}} + \boldsymbol{N}^{\mathrm{mo}}, \ \tilde{\boldsymbol{M}} = \boldsymbol{M} + \boldsymbol{M}^{\mathrm{th}} + \boldsymbol{M}^{\mathrm{mo}}, \qquad (4.2.68)
$$

the extended hygrothermal constitutive equation for a lamina can be written

$$
\begin{bmatrix} \tilde{\boldsymbol{N}} \\ \cdots \\ \tilde{\boldsymbol{M}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \vdots & \boldsymbol{0} \\ \cdots & & \cdots \\ \boldsymbol{0} & \vdots & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \cdots \\ \boldsymbol{\kappa} \end{bmatrix} \qquad (4.2.69)
$$

This constitutive equation is identical to that derived for mechanical loading only, (4.1.54), except for the fact that here the hygrothermal forces and moments are added to the mechanically applied forces and moments. The inversion of (4.2.69) yields the compliance relation

$$\begin{bmatrix} \boldsymbol{\varepsilon} \\ \cdots \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} & \vdots & \boldsymbol{0} \\ \cdots & & \cdots \\ \boldsymbol{0} & \vdots & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{N}} \\ \cdots \\ \tilde{\boldsymbol{M}} \end{bmatrix}, \quad \boldsymbol{a} = \boldsymbol{A}^{-1}, \quad \boldsymbol{d} = \boldsymbol{D}^{-1} \tag{4.2.70}$$

The values of the stiffness A_{ij}, D_{ij} and compliances a_{ij}, d_{ij} are the same as for pure mechanical loading (Table 4.3) and unit stress resultants $\hat{N}^{\text{th}}, \hat{M}^{\text{th}}, \hat{N}^{\text{mo}}, \hat{M}^{\text{mo}}$ are

$$\begin{aligned}
\hat{N}_1^{\text{th}} &= (Q_{11}\alpha_1^{\text{th}} + Q_{12}\alpha_2^{\text{th}} + Q_{16}\alpha_6^{\text{th}})h, \\
\hat{N}_2^{\text{th}} &= (Q_{12}\alpha_1^{\text{th}} + Q_{22}\alpha_2^{\text{th}} + Q_{26}\alpha_6^{\text{th}})h, \\
\hat{N}_6^{\text{th}} &= (Q_{16}\alpha_1^{\text{th}} + Q_{26}\alpha_2^{\text{th}} + Q_{66}\alpha_6^{\text{th}})h, \\
\hat{M}_1^{\text{th}} &= (Q_{11}\alpha_1^{\text{th}} + Q_{12}\alpha_2^{\text{th}} + Q_{16}\alpha_6^{\text{th}})\frac{1}{2}h^2, \\
\hat{M}_2^{\text{th}} &= (Q_{12}\alpha_1^{\text{th}} + Q_{22}\alpha_2^{\text{th}} + Q_{26}\alpha_6^{\text{th}})\frac{1}{2}h^2, \\
\hat{M}_6^{\text{th}} &= (Q_{16}\alpha_1^{\text{th}} + Q_{26}\alpha_2^{\text{th}} + Q_{66}\alpha_6^{\text{th}})\frac{1}{2}h^2
\end{aligned} \tag{4.2.71}$$

and analogous for unit moisture stress resultant with $\alpha_i^{\text{mo}}, i = 1,2,6$. In the more general case the integral definitions have to used.

When a laminate is subjected to mechanical and hygrothermal loading, a lamina k within the laminate is under a state of stress $\boldsymbol{\sigma}^{(k)}$ and strain $\boldsymbol{\varepsilon}^{(k)}$. The hygrothermoelastic superposition principle shows that the strains $\boldsymbol{\varepsilon}^{(k)}$ in the lamina k are equal to the sum of the strains produced by the existing stresses and the free, i.e. unrestrained, hygrothermal strains and the stresses $\boldsymbol{\sigma}^{(k)}$ follow by inversion

$$\begin{aligned}
\boldsymbol{\varepsilon}^{(k)} &= \boldsymbol{S}^{(k)}\boldsymbol{\sigma}^{(k)} + \boldsymbol{\alpha}^{\text{th}^{(k)}}T + \boldsymbol{\alpha}^{\text{mo}^{(k)}}M^*, \\
\boldsymbol{\sigma}^{(k)} &= \boldsymbol{Q}^{(k)}(\boldsymbol{\varepsilon}^{(k)} - \boldsymbol{\alpha}^{\text{th}^{(k)}}T - \boldsymbol{\alpha}^{\text{mo}^{(k)}}M^*) \\
&= \boldsymbol{Q}^{(k)}(\boldsymbol{\varepsilon}^{(k)} + x_3\boldsymbol{\kappa}^{(k)} - \boldsymbol{\alpha}^{\text{th}^{(k)}}T - \boldsymbol{\alpha}^{\text{mo}^{(k)}}M^*)
\end{aligned} \tag{4.2.72}$$

When in the lamina k all strains are restrained, then $\boldsymbol{\varepsilon}^{(k)} = \boldsymbol{0}$ and the hygrothermal stresses are

$$\boldsymbol{\sigma}^{(k)} = \boldsymbol{Q}^{(k)}\left(-\boldsymbol{\alpha}^{\text{th}^{(k)}}T - \boldsymbol{\alpha}^{\text{mo}^{(k)}}M^*\right) \tag{4.2.73}$$

Integration of the stresses $\boldsymbol{\sigma}^{(k)}$ and the stresses $\boldsymbol{\sigma}^{(k)}$ multiplied by the x_3-coordinate across the thickness $h^{(k)}$ and summation for all laminae gives the force and moment resultants of the laminate

$$
\begin{aligned}
\boldsymbol{N} &= \sum_{k=1}^{n} \boldsymbol{\sigma}^{(k)} h^{(k)} \\
&= \left(\sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \mathrm{d}x_3 \right) \boldsymbol{\varepsilon} + \left(\sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} x_3 \mathrm{d}x_3 \right) \boldsymbol{\kappa} \\
&+ \left(\sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \boldsymbol{\alpha}^{\mathrm{th}^{(k)}} T \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \mathrm{d}x_3 \right) + \left(\sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \boldsymbol{\alpha}^{\mathrm{mo}^{(k)}} M^{*} \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \mathrm{d}x_3 \right), \\
\boldsymbol{M} &= \sum_{k=1}^{n} \boldsymbol{\sigma}^{(k)} h^{(k)} \overline{x}_3^{(k)} \\
&= \left(\sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} x_3 \mathrm{d}x_3 \right) \boldsymbol{\varepsilon} + \left(\sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} x_3^2 \mathrm{d}x_3 \right) \boldsymbol{\kappa} \\
&+ \left(\sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \boldsymbol{\alpha}^{\mathrm{th}^{(k)}} T \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} x_3 \mathrm{d}x_3 \right) + \left(\sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \boldsymbol{\alpha}^{\mathrm{mo}^{(k)}} M^{*} \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} x_3 \mathrm{d}x_3 \right)
\end{aligned}
\tag{4.2.74}
$$

With the stiffness matrices $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{D}$, Eqs. (4.2.74), can be rewritten in a brief matrix form

$$
\begin{aligned}
\boldsymbol{N} &= \boldsymbol{A}\boldsymbol{\varepsilon} + \boldsymbol{B}\boldsymbol{\kappa} - \boldsymbol{N}^{\mathrm{th}} - \boldsymbol{N}^{\mathrm{mo}}, \\
\boldsymbol{M} &= \boldsymbol{B}\boldsymbol{\varepsilon} + \boldsymbol{D}\boldsymbol{\kappa} - \boldsymbol{M}^{\mathrm{th}} - \boldsymbol{M}^{\mathrm{mo}}
\end{aligned}
\tag{4.2.75}
$$

The fictitious hygrothermal resultants are given by

$$
\begin{aligned}
\boldsymbol{N}^{\mathrm{th}} &= T \sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \boldsymbol{\alpha}^{\mathrm{th}^{(k)}} h^{(k)} = T \sum_{k=1}^{n} \hat{N}^{(k)\mathrm{th}}, \\
\boldsymbol{N}^{\mathrm{mo}} &= M^{*} \sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \boldsymbol{\alpha}^{\mathrm{mo}^{(k)}} h^{(k)} = M^{*} \sum_{k=1}^{n} \hat{N}^{(k)\mathrm{mo}}, \\
\boldsymbol{M}^{\mathrm{th}} &= \frac{1}{2} T \sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \boldsymbol{\alpha}^{\mathrm{th}^{(k)}} \left(x_3^{(k)^2} - x_3^{(k-1)^2} \right) \\
&= T \sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \boldsymbol{\alpha}^{\mathrm{th}^{(k)}} \overline{x}_3^{(k)} h^{(k)} = T \sum_{k=1}^{n} \hat{M}^{(k)\mathrm{th}}, \\
\boldsymbol{M}^{\mathrm{mo}} &= \frac{1}{2} M^{*} \sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \boldsymbol{\alpha}^{\mathrm{mo}^{(k)}} \left(x_3^{(k)^2} - x_3^{(k-1)^2} \right) \\
&= M^{*} \sum_{k=1}^{n} \boldsymbol{Q}^{(k)} \boldsymbol{\alpha}^{\mathrm{mo}^{(k)}} \overline{x}_3^{(k)} h^{(k)} = M^{*} \sum_{k=1}^{n} \hat{M}^{(k)\mathrm{mo}}
\end{aligned}
\tag{4.2.76}
$$

By analogy to the single layer one can introduce total force and moment resultants $\tilde{\boldsymbol{N}}$ and $\tilde{\boldsymbol{M}}$ the stiffness and compliance equations expanded to the hygrothermal components

$$\boldsymbol{N}+\boldsymbol{N}^{\mathrm{th}}+\boldsymbol{N}^{\mathrm{mo}}=\tilde{\boldsymbol{N}}, \quad \boldsymbol{M}+\boldsymbol{M}^{\mathrm{th}}+\boldsymbol{M}^{\mathrm{mo}}=\tilde{\boldsymbol{M}},$$

$$\begin{bmatrix}\tilde{\boldsymbol{N}}\\ \cdots\\ \tilde{\boldsymbol{M}}\end{bmatrix}=\begin{bmatrix}\boldsymbol{A} & \vdots & \boldsymbol{B}\\ & \cdots & \\ \boldsymbol{B} & \vdots & \boldsymbol{D}\end{bmatrix}\begin{bmatrix}\boldsymbol{\varepsilon}\\ \cdots\\ \boldsymbol{\kappa}\end{bmatrix}, \quad \begin{bmatrix}\boldsymbol{\varepsilon}\\ \cdots\\ \boldsymbol{\kappa}\end{bmatrix}=\begin{bmatrix}\boldsymbol{a} & \vdots & \boldsymbol{b}\\ & \cdots & \\ \boldsymbol{b}^{\mathrm{T}} & \vdots & \boldsymbol{d}\end{bmatrix}\begin{bmatrix}\tilde{\boldsymbol{N}}\\ \cdots\\ \tilde{\boldsymbol{M}}\end{bmatrix},$$

$$\begin{aligned} \boldsymbol{a} &= \boldsymbol{A}^{*}-\boldsymbol{B}^{*}\boldsymbol{D}^{*-1}\boldsymbol{C}^{*}, & \boldsymbol{A}^{*} &= \boldsymbol{A}^{-1},\\ \boldsymbol{b} &= \boldsymbol{B}^{*}\boldsymbol{D}^{*-1}, & \boldsymbol{B}^{*} &= -\boldsymbol{A}^{-1}\boldsymbol{B},\\ \boldsymbol{d} &= \boldsymbol{D}^{*-1}, & \boldsymbol{D}^{*} &= \boldsymbol{D}-\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B} \end{aligned} \tag{4.2.77}$$

The coupling effects discussed in Sects. 4.2.2 and 4.2.3 stay unchanged and the stiffness matrices in Table 4.4 can be transferred.

If we can classify a laminate as symmetric, balanced, cross-plied or some combinations of these three laminate stacking types, some of the thermal or moisture force or moment resultant coefficients may be zero. For temperature or moisture change that depends on x_3, no general statements can be given. However for changes independent of x_3, the following simplifications for the unit stress resultants can be considered, $J=\mathrm{th},\mathrm{mo}$

Symmetric laminates

$$\begin{aligned} \hat{N}_1^J &\neq 0, & \hat{M}_1^J &= 0,\\ \hat{N}_2^J &\neq 0, & \hat{M}_2^J &= 0,\\ \hat{N}_6^J &\neq 0, & \hat{M}_6^J &= 0 \end{aligned}$$

Balanced laminates

$$\begin{aligned} \hat{N}_1^J &\neq 0, & \hat{M}_1^J &\neq 0,\\ \hat{N}_2^J &\neq 0, & \hat{M}_2^J &\neq 0,\\ \hat{N}_6^J &= 0, & \hat{M}_6^J &\neq 0 \end{aligned}$$

Symmetric balanced laminates

$$\begin{aligned} \hat{N}_1^J &\neq 0, & \hat{M}_1^J &= 0,\\ \hat{N}_2^J &\neq 0, & \hat{M}_2^J &= 0,\\ \hat{N}_6^J &= 0, & \hat{M}_6^J &= 0 \end{aligned}$$

Cross-ply laminates

$$\begin{aligned} \hat{N}_1^J &\neq 0, & \hat{M}_1^J &\neq 0,\\ \hat{N}_2^J &\neq 0, & \hat{M}_2^J &\neq 0,\\ \hat{N}_6^J &= 0, & \hat{M}_6^J &= 0 \end{aligned}$$

Symmetric cross-ply laminates

$$\begin{aligned} \hat{N}_1^J &\neq 0, & \hat{M}_1^J &= 0, \\ \hat{N}_2^J &\neq 0, & \hat{M}_2^J &= 0, \\ \hat{N}_6^J &= 0, & \hat{M}_6^J &= 0 \end{aligned}$$

Summarizing the hygrothermal effects one can see that if both mechanical and hygrothermal loads are applied the mechanical and fictitious hygrothermal loads can be added to find ply by ply stresses and strains in the laminate, or the mechanical and hygrothermal loads can be applied separately and then the resulting stresses and strains of the two problems are added.

4.2.6 Problems

Exercise 4.7. A symmetric laminate under in-plane loading can be considered as an equivalent homogeneous anisotropic plate in plane stress state by introducing average stress $\boldsymbol{\sigma} = \boldsymbol{N}/h$ and $\boldsymbol{N} = \boldsymbol{A}\boldsymbol{\varepsilon}$. Calculate the effective moduli for general symmetric laminates and for symmetric cross-ply laminates.

Solution 4.7. Equations (4.2.37) yields $\boldsymbol{\varepsilon} = \boldsymbol{a}\boldsymbol{N}, \boldsymbol{a} = \boldsymbol{A}^{-1}$. The components of the inverse matrix $\boldsymbol{a}$ are
$a_{11} = (A_{22}A_{66} - A_{26}^2)/\Delta, a_{12} = (A_{16}A_{26} - A_{12}A_{66})/\Delta,$
$a_{22} = (A_{11}A_{66} - A_{16}^2)/\Delta, a_{16} = (A_{12}A_{26} - A_{22}A_{16})/\Delta,$
$a_{26} = (A_{12}A_{16} - A_{11}A_{26})/\Delta, a_{66} = (A_{11}A_{22} - A_{12}^2)/\Delta,$

$$\Delta = \mathrm{Det}(A_{ij}) = A_{11}\begin{vmatrix} A_{22} & A_{26} \\ A_{26} & A_{66} \end{vmatrix} - A_{12}\begin{vmatrix} A_{12} & A_{26} \\ A_{16} & A_{66} \end{vmatrix} + A_{16}\begin{vmatrix} A_{12} & A_{22} \\ A_{16} & A_{26} \end{vmatrix}$$

The comparison $\boldsymbol{\varepsilon} = \boldsymbol{a}\boldsymbol{N} = h\boldsymbol{a}\boldsymbol{\sigma}$ with (4.1.19) leads to
$E_1 = 1/ha_{11}, E_2 = 1/ha_{22}, E_6 = 1/ha_{66},$
$\nu_{12} = -a_{12}/a_{11}, \nu_{21} = -a_{12}/a_{22}, \nu_{16} = a_{16}/a_{11},$
$\nu_{61} = a_{16}/a_{66}, \nu_{26} = a_{26}/a_{22}, \nu_{62} = a_{26}/a_6$
These are the effective moduli in the general case. For cross-ply laminates is $A_{16} = A_{26} = 0$ (Eqs. 4.2.31). The effective moduli can be explicitly expressed in terms of the in-plane stiffness A_{ij}. With $\mathrm{Det}(A_{ij}) = A_{11}A_{22}A_{66} - A_{12}^2A_{66}$ follow the effective moduli
$E_1 = (A_{11}A_{22} - A_{12}^2)/hA_{22}, \nu_{12} = A_{12}/A_{22},$
$E_2 = (A_{11}A_{22} - A_{12}^2)/hA_{11}, \nu_{21} = A_{12}/A_{11},$
$G_{12} \equiv E_6 = A_{66}/h, \nu_{16} = \nu_{61} = \nu_{26} = \nu_{62} = 0$
Note that these simplified formulae are not only valid for symmetric cross-ply laminates $(0/90)_S$ but also for laminates $(\pm 45)_S$.

Exercise 4.8. Show that a symmetric laminate $[\pm 45^0/0^0/90^0]_S$ with $E'_1 = 140$ GPa, $E'_2 = 10$ GPa, $E'_6 = 7$ GPa, $\nu'_{12} = 0,3$ has a quasi-isotropic material behavior. The thicknesses of all plies are onstant $h^{(k)} = 0,1$ mm.

Solution 4.8. The solution will obtained in four steps:

1. Calculation of the on-axis reduced stiffness Q'_{ij}
   With respect to (2.1.56) we obtain
$$\nu'_{12}/E'_1 = \nu'_{21}/E'_2 \Longrightarrow \nu'_{21} = (\nu'_{12}E'_2)/E'_1 = 0,0214$$
   and finally from Eqs. (4.1.3) follow
   $Q'_{11} = E'_1(1 - \nu'_{12}\nu'_{21}) = 140,905$ GPa,
   $Q'_{22} = E'_2(1 - \nu'_{12}\nu'_{21}) = 10,065$ GPa,
   $Q'_{12} = E'_2\nu'_{12}/(1 - \nu'_{12}\nu'_{21}) = 3,019$ GPa,
   $Q'_{66} = E'_6 = 7$ GPa
2. Calculation of the reduced stiffness in the laminae (Table 4.2)
$$\begin{aligned}
Q_{ij[0^0]} &\equiv Q'_{ij},\\
Q_{11[90^0]} &= Q'_{22} = 10,065 \text{ GPa},\\
Q_{12[90^0]} &= Q'_{12} = 3,019 \text{ GPa},\\
Q_{22[90^0]} &= Q'_{11} = 140,905 \text{ GPa},\\
Q_{66[90^0]} &= Q'_{66} = 7 \text{ GPa},\\
Q_{16[90^0]} &= Q_{26[90^0]} = 0,\\
Q_{11[\pm 45^0]} &= 46,252 \text{ GPa},\\
Q_{12[\pm 45^0]} &= 32,252 \text{ GPa},\\
Q_{22[\pm 45^0]} &= 46,252 \text{ GPa},\\
Q_{66[\pm 45^0]} &= 36,233 \text{ GPa},\\
Q_{16[\pm 45^0]} &= \pm 32,71 \text{ GPa},\\
Q_{26[\pm 45^0]} &= \pm 32,71 \text{ GPa}
\end{aligned}$$
3. Calculation of the axial stiffness A_{ij} (4.2.15)
$$A_{ij} = \sum_{n=1}^{8} Q_{ij}^{(k)} h^{(k)} = 2\sum_{n=1}^{4} Q_{ij}^{(k)} h^{(k)},$$
   $A_{11} = 48,695\ 10^6\ \text{Nm}^{-1} = A_{22}, A_{12} = 14,108\ 10^6\ \text{Nm}^{-1}$,
   $A_{66} = 17,29310^6\ \text{Nm}^{-1}, A_{16} = A_{26} = 0$
4. Calculation of the effective moduli (example 1)
   $E_1 = E_2 = (A_{11}^2 - A_{12}^2)/hA_{22} = 446,1$ GPa, $E_6 = A_{66}/h = 172,9$ GPa,
   $\nu_{12} = \nu_{21} = A_{12}/A_{22} = 0,29$
   Note that $E = 2(1 + \nu)G = 446,1$ GPa, i.e. the isotropy condition is satisfied.

Exercise 4.9. Calculate the laminar stresses σ and σ' in the laminate of previous example loaded by uniaxial tension N_1.

Solution 4.9. The following reduced stiffness matrices are calculated

$$\boldsymbol{Q}_{[0^0]} = \begin{bmatrix} 140,9 & 3,02 & 0 \\ 3,02 & 10,06 & 0 \\ 0 & 0 & 7,0 \end{bmatrix} 10^9\text{Pa},$$

$$\boldsymbol{Q}_{[90^0]} = \begin{bmatrix} 10,06 & 3,02 & 0 \\ 3,02 & 140,9 & 0 \\ 0 & 0 & 7,0 \end{bmatrix} 10^9\text{Pa},$$

$$\boldsymbol{Q}_{[\pm 45^0]} = \begin{bmatrix} 46,25 & 32,25 & \pm 32,71 \\ 32,25 & 46,25 & \pm 32,71 \\ \pm 32,71 & \pm 32,71 & 36,23 \end{bmatrix} 10^9\text{Pa}$$

The axial stiffness matrix $\boldsymbol{A}$ is also calculated

$$\boldsymbol{A} = \begin{bmatrix} 48,70 & 14,11 & 0 \\ 14,11 & 48,70 & 0 \\ 0 & 0 & 17,29 \end{bmatrix} 10^6\text{N/m}$$

With

$$\begin{aligned} a_{11} &= A_{22}/(A_{11}^2 - A_{12}^2) = 0,02241\ 10^{-6}\text{m/N}, \\ a_{22} &= A_{11}/(A_{11}^2 - A_{12}^2) = a_{11}, \\ a_{66} &= 1/A_{66} = 0,05784\ 10^{-6}\text{m/N}, \\ a_{12} &= -A_{12}/(A_{11}A_{22} - A_{12}^2) = -0,00645\ 10^{-6}\text{m/N} \end{aligned}$$

follows the inverse matrix $\boldsymbol{a} = \boldsymbol{A}^{-1}$

$$\boldsymbol{a} = \begin{bmatrix} 22,41 & -6,49 & 0 \\ -6,49 & 22,41 & 0 \\ 0 & 0 & 57,84 \end{bmatrix} 10^{-9}\text{m/N}$$

The strains are with (4.2.42)

$$\boldsymbol{\varepsilon} = \boldsymbol{aN} \Longrightarrow \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \end{bmatrix} = \boldsymbol{a} \begin{bmatrix} N_1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 22,41 \\ -6,49 \\ 0 \end{bmatrix} 10^{-9}(\text{m/N})N_1$$

N_1 is given in N/m, i.e. ε_i are dimensionless.

Now the laminar stresses are (Table 4.2)

$$\begin{aligned} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix}_{[0^0]} &= \boldsymbol{Q}_{[0^0]}\boldsymbol{\varepsilon} = \begin{bmatrix} 3138 \\ 2,4 \\ 0 \end{bmatrix} N_1[\text{N/m}^2], \\ \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix}_{[90^0]} &= \boldsymbol{Q}_{[90^0]}\boldsymbol{\varepsilon} = \begin{bmatrix} 205,8 \\ -846,8 \\ 0 \end{bmatrix} N_1[\text{N/m}^2], \end{aligned}$$

$$\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_6\end{bmatrix}_{[\pm 45^0]} = \boldsymbol{Q}_{[\pm 45^0]}\boldsymbol{\varepsilon} = \begin{bmatrix}827,2\\422,6\\\pm 520,7\end{bmatrix} N_1\,[\mathrm{N/m^2}]$$

The stresses jump from lamina to lamina. Verify that the resultant force $N_2 = 0$.

The stress components in reference to the principal material axes follow with the transformation rule (Table 4.1)

$$\begin{bmatrix}\sigma_1'\\\sigma_2'\\\sigma_6'\end{bmatrix} = \begin{bmatrix}c^2 & s^2 & 2sc\\ s^2 & c^2 & -2sc\\ -cs & cs & c^2-s^2\end{bmatrix}\begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_6\end{bmatrix},$$

$$\begin{aligned}\begin{bmatrix}\sigma_1'\\\sigma_2'\\\sigma_6'\end{bmatrix}_{[0^0]} &= \begin{bmatrix}3138\\2,4\\0\end{bmatrix} N_1\,[\mathrm{N/m^2}],\\ \begin{bmatrix}\sigma_1'\\\sigma_2'\\\sigma_6'\end{bmatrix}_{[90^0]} &= \begin{bmatrix}-846,8\\205,8\\0\end{bmatrix} N_1\,[\mathrm{N/m^2}],\\ \begin{bmatrix}\sigma_1\\\sigma_2\\\sigma_6\end{bmatrix}_{[\pm 45^0]} &= \begin{bmatrix}1146\\104,2\\\mp 202,3\end{bmatrix} N_1\,[\mathrm{N/m^2}]\end{aligned}$$

Note 4.1. These stresses are used in failure analysis of a laminate.

Exercise 4.10. A laminate with an unsymmetric layer stacking $[-45^0/30^0/0^0]$ has three layers of equal thickness $h^{(1)} = h^{(2)} = h^{(3)} = 5$ mm. The mechanical properties of all UD-laminae are $E_1' = 181$ GPa, $E_2' = 10,30$ GPa, $G_{12}' = 7,17$ GPa, $\nu_{12}' = 0,28$ GPa. Determine the laminate stiffness A_{ij}, B_{ij}, D_{ij}.

Solution 4.10. Using (4.1.3) the elements S_{ij}' of the compliance matrix $\boldsymbol{S}'$ are

$$\begin{aligned}S_{11}' &= 1/E_1' = 0,0055\ \mathrm{GPa}^{-1},\\ S_{12}' &= -\nu_{12}'/E_1' = -0,0015\ \mathrm{GPa}^{-1},\\ S_{22}' &= 1/E_2' = 0,0971\ \mathrm{GPa}^{-1},\\ S_{66}' &= 1/G_{12}' = 0,1395\ \mathrm{GPa}^{-1}\end{aligned}$$

The minor Poisson's ratio follows with

$$\nu_{21}' = \nu_{12}'E_2'/E_1' = 0,01593$$

Using (4.1.3) the elements Q_{ij}' of the reduced stiffness matrix $\boldsymbol{Q}'$ are

$$\begin{aligned}Q_{11}' &= E_1'/(1-\nu_{12}'\nu_{21}') = 181,8\ 10^9\mathrm{Pa},\\ Q_{12}' &= \nu_{12}'E_2'/(1-\nu_{12}'\nu_{21}') = 2,897\ 10^9\mathrm{Pa},\\ Q_{22}' &= E_2'/(1-\nu_{12}'\nu_{21}') = 10,35\ 10^9\mathrm{Pa},\\ Q_{66}' &= G_{12}' = 7,17\ 10^9\mathrm{Pa}\end{aligned}$$

Verify that the reduced stiffness matrix could also be obtained by inverting the compliance matrix, i.e. $\boldsymbol{Q}' = \boldsymbol{S}'^{-1}$. Now the transformed reduced stiffness matrices $\boldsymbol{Q}_{[0^0]}, \boldsymbol{Q}_{[30^0]}, \boldsymbol{Q}_{[-45^0]}$ have to be calculated with the help of Table 4.2 taking into account that $c = \cos 0^0 = 1, \cos 30^0 = 0,8660, \cos(-45^0) = 0,7071$ and $s = \sin 0^0 = 0, \sin 30^0 = 0,5, \sin(-45^0) = -0,7071$

$$\boldsymbol{Q}_{[0^0]} = \begin{bmatrix} 181,8 & 2,897 & 0 \\ 2,897 & 10,35 & 0 \\ 0 & 0 & 7,17 \end{bmatrix} 10^9\text{Pa},$$

$$\boldsymbol{Q}_{[30^0]} = \begin{bmatrix} 109,4 & 32,46 & 54,19 \\ 32,46 & 23,65 & 20,05 \\ 54,19 & 20,05 & 36,74 \end{bmatrix} 10^9\text{Pa},$$

$$\boldsymbol{Q}_{[-45^0]} = \begin{bmatrix} 56,66 & 42,32 & -42,87 \\ 42,32 & 56,66 & -42,87 \\ -42,87 & -42,87 & 46,59 \end{bmatrix} 10^9\text{Pa}$$

The location of the lamina surfaces are $x_3^{(0)} = -7,5$ mm, $x_3^{(1)} = -2,5$ mm, $x_3^{(2)} = 2,5$ mm, $x_3^{(3)} = 7,5$ mm. The total thickness of the laminate is 15 mm. From (4.2.15) the extensional stiffness matrix $\boldsymbol{A}$ follows with

$$A_{ij} = \sum_{k=1}^{3} Q_{ij}^{(k)} h^{(k)} = 5 \sum_{k=1}^{3} Q_{ij}^{(k)} \text{mm},$$

the coupling matrix $\boldsymbol{B}$ follows with

$$B_{ij} = \sum_{k=1}^{3} Q_{ij}^{(k)} h^{(k)} \bar{x}_3^{(k)} = 5 \sum_{k=1}^{3} Q_{ij}^{(k)} \bar{x}_3^{(k)} \text{mm}$$

and the bending stiffness matrix D_{ij} follows with

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{3} Q_{ij}^{(k)} h^{(k)} \left((\bar{x}_3^{(k)})^2 + \frac{h^{(k)^2}}{12} \right) = \frac{5}{3} \sum_{k=1}^{3} Q_{ij}^{(k)} \left(\frac{25}{12} + (\bar{x}_3^{(k)})^2 \right) \text{mm}^3$$

with

$$\bar{x}_3^{(k)} = \frac{1}{2}(x_3^{(k)} + x_3^{(k-1)}),$$

i.e. $\bar{x}_3^{(1)} = 5$ mm, $\bar{x}_3^{(2)} = 0$ mm, $\bar{x}_3^{(3)} = -5$ mm. Summarizing the formulas for A_{ij}, B_{ij} and D_{ij} we have the equations

$$\begin{aligned} A_{ij} &= 5[Q_{ij}^{(1)} + Q_{ij}^{(2)} + Q_{ij}^{(3)}]\text{mm}, \\ B_{ij} &= 5[5Q_{ij}^{(1)} + 0Q_{ij}^{(2)} - 5Q_{ij}^{(3)}\text{mm}^2], \\ D_{ij} &= 5[(25 + 25/12)Q_{ij}^{(1)} + (25/12)Q_{ij}^{(2)} + (25 + 25/12)Q_{ij}^{(3)}]\text{mm}^3 \end{aligned}$$

and the stiffness matrices follow to

$$\boldsymbol{A} = \begin{bmatrix} 17,39 & 3,884 & 0,566 \\ 3,884 & 4,533 & -1,141 \\ 0,566 & -1,141 & 4,525 \end{bmatrix} 10^8 \text{Pa m},$$

$$\boldsymbol{B} = \begin{bmatrix} -3,129 & 0,986 & -1,072 \\ 0,986 & 1,158 & -1,072 \\ -1,072 & -1,072 & 0,986 \end{bmatrix} 10^6 \text{Pa m}^2,$$

$$\boldsymbol{D} = \begin{bmatrix} 33,43 & 6,461 & -5,240 \\ 6,461 & 9,320 & -5,596 \\ -5,240 & -5,596 & 7,663 \end{bmatrix} 10^3 \text{Pa m}^3$$

4.3 Elastic Behavior of Sandwiches

One special group of laminated composites used extensively in engineering applications is sandwich composites. Sandwich panels consist of thin facings, also called skins or sheets, sandwiching a core. The facings are made of high strength material while the core is made of thick and lightweight materials, Sect. 1.3. The motivation for sandwich structure elements is twofold. First for beam or plate bending the maximum normal stresses occur at the top and the bottom surface. So it makes sense using high-strength materials at the top and the bottom and using low and lightweight strength materials in the middle. The strong and stiff facings also support axial forces. Second, the bending resistance for a rectangular cross-sectional beam or plate is proportional to the cube of the thickness. Increasing the thickness by adding a core in the middle increases the resistance. The maximum shear stress is generally in the middle of the sandwich requiring a core to support shear. The advantages in weight and bending stiffness make sandwich composites attractive in many applications.

The most commonly used facing materials are aluminium alloys and fibre reinforced plastics. Aluminium has a high specific modulus, but it corrodes without treatment and can be prone to denting. Therefore fibre reinforced laminates, such as graphite/epoxy or glass/epoxy are becoming more popular as facing materials. They have high specific modulus and strength and corrosion resistance. The fibre reinforced facing can be unidirectional or woven laminae.

The most commonly used core materials are balsa wood, foam, resins with special fillers and honeycombs (Fig. 1.3). These materials must have high compressive and shear strength. Honeycombs can be made of plastics, paper, card-boards, etc. The strength and stiffness of honeycomb sandwiches depend on the material and the cell size and thickness. The following sections consider the modelling and analysis of sandwiches with thin and thick cover sheets.

4.3.1 General Assumptions

A sandwich can be defined as a special laminate with three layers. The thin cover sheets, i.e. the layers 1 and 3, are laminates of the thicknesses $h^{(1)}$ for the lower skin and $h^{(3)}$ for the upper skin. The thickness of the core is $h^{(2)} \equiv h^{\mathrm{c}}$. In a general case $h^{(1)}$ does not have to be equal to $h^{(3)}$, but in the most important practical case of symmetric sandwiches $h^{(1)} = h^{(3)} \equiv h^{\mathrm{f}}$.

The assumptions for macro-mechanical modelling of sandwiches are:

1. The thickness of the core is much greater than that of the skins, $h^{(2)} \gg h^{(1)}, h^{(3)}$ or $h^{\mathrm{c}} \gg h^{\mathrm{f}}$
2. The strains $\varepsilon_1, \varepsilon_2, \varepsilon_6$ vary linearly through the core thickness h^{c}
$$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} + x_3 \boldsymbol{\kappa}$$
3. The sheets only transmit stresses $\sigma_1, \sigma_2, \sigma_6$ and the in-plane strains are uniform through the thickness of the skins. The transverse shear stresses σ_4, σ_5 are neglected within the skin.
4. The core only transmits transverse shear stresses σ_4 and σ_5, the stresses σ_1, σ_2 and σ_6 are neglected.
5. The strain ε_3 is neglected in the sheets and the core.

With these additional assumptions in the frame of linear anisotropic elasticity, the stresses and strains can be formulated.
Strains in the lower and upper sheets:

$$\boldsymbol{\varepsilon}^{(l)} = \boldsymbol{\varepsilon} \mp \frac{1}{2} h^{\mathrm{c}} \boldsymbol{\kappa}, \quad \varepsilon_i^{(l)} = \varepsilon_i \mp \frac{1}{2} h^{\mathrm{c}} \kappa_i, \quad l = 1,3, \quad i = 1,2,6 \tag{4.3.1}$$

The transverse shear strains $\varepsilon_4, \varepsilon_5$ are neglected.
Strains in the sandwich core:

$$\boldsymbol{\varepsilon}^{(2)} = \boldsymbol{\varepsilon}^{\mathrm{c}} = \boldsymbol{\varepsilon} + x_3 \boldsymbol{\kappa}, \quad -\frac{h^{\mathrm{c}}}{2} \leq x_3 \leq +\frac{h^{\mathrm{c}}}{2} \tag{4.3.2}$$

The transverse shear strains are, in a first approach, independent of the coordinate x_3 (4.2.14)

$$\boldsymbol{\gamma}^{sc} = [\varepsilon_5^{\mathrm{c}} \quad \varepsilon_4^{\mathrm{c}}]^{\mathrm{T}} \tag{4.3.3}$$

We shall see in Chap. 5 that in the classical laminate theory and the laminate theory including transverse shear deformations the strain vector $\boldsymbol{\varepsilon}$ is written in an analogous form, and only the expressions for the curvatures are modified.
Stresses in the lower and upper sheets:
In the sheets a plane stress state exists and with assumption 3. the transverse shear stresses σ_4 and σ_5 are neglected. These assumptions imply that for laminated sheets in all layers of the lower and the upper skins

$$\sigma_4^{(k)} = \sigma_5^{(k)} = 0$$

The other stresses are deduced from the constant strains

$$\varepsilon_1^{(l)}, \varepsilon_2^{(l)}, \varepsilon_6^{(l)}, \quad l = 1,3$$

by the relationships

$$\sigma_i^{(k)} = Q_{ij}^{(k)} \varepsilon_j^{(l)}, \quad i,j = 1,2,6, \quad l = 1,3 \tag{4.3.4}$$

for the kth layer of the lower ($l = 1$) or the upper ($l = 3$) skin.
Stresses in the sandwich core:
From assumption 4. it follows

$$\sigma_1^{\mathrm{c}} = \sigma_2^{\mathrm{c}} = \sigma_6^{\mathrm{c}} = 0$$

and the core transmits only the transverse shear stresses

$$\begin{bmatrix} \sigma_5^{\mathrm{c}} \\ \sigma_4^{\mathrm{c}} \end{bmatrix} = \begin{bmatrix} C_{55} & C_{45} \\ C_{45} & C_{44} \end{bmatrix} \begin{bmatrix} \varepsilon_5 \\ \varepsilon_4 \end{bmatrix} \tag{4.3.5}$$

or in matrix notation (Eq. 4.2.14)

$$\boldsymbol{\sigma}^{\mathrm{s}} = \boldsymbol{C}^{\mathrm{s}} \boldsymbol{\gamma}^{\mathrm{s}} \tag{4.3.6}$$

The coefficients C_{ij}^{c} of $\boldsymbol{C}^{\mathrm{s}}$ are expressed as functions of the coefficients $C_{ij}^{c'}$ referred to the principal directions by the transformation equation (4.2.17). The coefficients $C_{ij}^{c'}$ in the principal directions are themselves written as functions of the shear moduli of the core (Sect. 2.1, Table 2.5), measured in principal directions as follows

$$C_{44}^{c'} = G_{23}^{\mathrm{c}}, \quad C_{55}^{\mathrm{c}} = G_{13}^{c'} \tag{4.3.7}$$

For an isotropic core material a transformation is not required.

4.3.2 Stress Resultants and Stress Analysis

The in-plane resultants $\boldsymbol{N}$ for sandwiches are defined by

$$\boldsymbol{N} = \int\limits_{-\left(\frac{1}{2}h^{\mathrm{c}} + h^{(1)}\right)}^{-\frac{1}{2}h^{\mathrm{c}}} \boldsymbol{\sigma} \mathrm{d}x_3 + \int\limits_{\frac{1}{2}h^{\mathrm{c}}}^{\frac{1}{2}h^{\mathrm{c}} + h^{(3)}} \boldsymbol{\sigma} \mathrm{d}x_3, \tag{4.3.8}$$

the moment resultants by

$$\boldsymbol{M} = \int\limits_{-\left(\frac{1}{2}h^{\mathrm{c}}+h^{(1)}\right)}^{-\frac{1}{2}h^{\mathrm{c}}} \boldsymbol{\sigma} x_3 \mathrm{d}x_3 + \int\limits_{\frac{1}{2}h^{\mathrm{c}}}^{\frac{1}{2}h^{\mathrm{c}}+h^{(3)}} \boldsymbol{\sigma} x_3 \mathrm{d}x_3 \tag{4.3.9}$$

and the transverse shear force by

$$\boldsymbol{Q}^{\mathrm{s}} = \int\limits_{-\frac{1}{2}h^{\mathrm{c}}}^{\frac{1}{2}h^{\mathrm{c}}} \boldsymbol{\sigma}^{\mathrm{s}} \mathrm{d}x_3 \tag{4.3.10}$$

For the resultants $\boldsymbol{N}$ and $\boldsymbol{M}$ the integration is carried out over the sheets only and for the transverse shear force over the core.

By substituting Eqs. (4.3.4) - (4.3.7) for the stresses into the preceding expressions for the force and moment resultants, we obtain analogous to (4.2.16) the constitutive equation

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \\ \boldsymbol{Q}^{\mathrm{s}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{C} & \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}^{\mathrm{s}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma}^{\mathrm{s}} \end{bmatrix} \tag{4.3.11}$$

with the stiffness coefficients

$$\begin{aligned} A_{ij} &= A_{ij}^{(1)} + A_{ij}^{(3)},\ B_{ij} = \frac{1}{2}h^{\mathrm{c}}\left(A_{ij}^{(3)} - A_{ij}^{(1)}\right), \\ C_{ij} &= C_{ij}^{(1)} + C_{ij}^{(3)},\ D_{ij} = \frac{1}{2}h^{\mathrm{c}}\left(C_{ij}^{(3)} - C_{ij}^{(1)}\right) \end{aligned} \tag{4.3.12}$$

and

$$\begin{aligned} A_{ij}^{(1)} &= \int\limits_{-\left(\frac{1}{2}h^{\mathrm{c}}+h_1\right)}^{-\frac{1}{2}h^{\mathrm{c}}} Q_{ij}^{(k)} \mathrm{d}x_3 = \sum_{k=1}^{n_1} \int\limits_{h^{(k)}} Q_{ij}^{(k)} \mathrm{d}x_3 = \sum_{k=1}^{n_1} Q_{ij}^{(k)} h^{(k)}, \\ A_{ij}^{(3)} &= \int\limits_{\frac{1}{2}h^{\mathrm{c}}}^{\frac{1}{2}h^{\mathrm{c}}+h_3} Q_{ij}^{(k)} \mathrm{d}x_3 = \sum_{k=1}^{n_2} \int\limits_{h^{(k)}} Q_{ij}^{(k)} \mathrm{d}x_3 = \sum_{k=1}^{n_2} Q_{ij}^{(k)} h^{(k)}, \\ C_{ij}^{(1)} &= \int\limits_{-\left(\frac{1}{2}h^{\mathrm{c}}+h_1\right)}^{-\frac{1}{2}h^{\mathrm{c}}} Q_{ij}^{(k)} x_3 \mathrm{d}x_3 = \sum_{k=1}^{n_1} \int\limits_{h^{(k)}} Q_{ij}^{(k)} x_3 \mathrm{d}x_3 = \sum_{k=1}^{n_1} Q_{ij}^{(k)} h^{(k)} \bar{x}_3^{(k)}, \\ C_{ij}^{(3)} &= \int\limits_{\frac{1}{2}h^{\mathrm{c}}}^{\frac{1}{2}h^{\mathrm{c}}+h_3} Q_{ij}^{(k)} x_3 \mathrm{d}x_3 = \sum_{k=1}^{n_2} \int\limits_{h^{(k)}} Q_{ij}^{(k)} x_3 \mathrm{d}x_3 = \sum_{k=1}^{n_2} Q_{ij}^{(k)} h^{(k)} \bar{x}_3^{(k)} \end{aligned} \tag{4.3.13}$$

with $\bar{x}_3^{(k)} = \frac{1}{2}(x_3^{(k)} + x_3^{(k-1)}$ and

$$A_{ij}^{\mathrm{s}} = h^{\mathrm{c}} C_{ij}^{\mathrm{s}}, \quad i,j = 4,5 \tag{4.3.14}$$

n_1 and n_2 are the number of layers in the lower and the upper sheet respectively and C_{ij}^{s} are the transverse shear moduli of the core. The constitutive equations (4.3.11) for a sandwich composite has a form similar to the constitutive equation for laminates including transverse shear. It differs only by the terms C_{ij} instead of B_{ij} which induce an unsymmetry in the stiffness matrix.

In the case of symmetric sandwiches with identical sheets $h^{(1)} = h^{(3)} = h^{\mathrm{f}}$, $A_{ij}^{(1)} = A_{ij}^{(3)} = A_{ij}^{\mathrm{f}}, C_{ij}^{(1)} = -C_{ij}^{(3)} = C_{ij}^{\mathrm{f}}$ and from this it results that the stiffness coefficients Eq. (4.3.12) are

$$A_{ij} = 2A_{ij}^{\mathrm{f}}, \quad D_{ij} = h^{\mathrm{c}} C_{ij}^{\mathrm{f}}, \quad B_{ij} = 0, \quad C_{ij} = 0 \tag{4.3.15}$$

As developed for laminates including shear deformations, the coefficients A_{ij}^{s} can be corrected by shear correction factors k_{ij}^{s} and replaced by shear constants $(k^{\mathrm{s}}A^{\mathrm{s}})_{ij}$ to improve the modelling.

In the case of symmetric sandwiches there is no coupling between stretching and bending and the form of the constitutive equation is identical to the constitutive equation for symmetric laminates including transverse shear.

4.3.3 Sandwich Materials with Thick Cover Sheets

In the case of thick cover sheets it is possible to carry out the modelling and analysis with the help of the theory of laminates including transverse shear. Considering the elastic behavior of sandwich composites we have:

- The stretching behavior is determined by the skins.
- The transverse shear is imposed by the core.

The modelling assumption 1. of Sect. 4.3.1 is not valid. Restricting the modelling to the case of symmetric sandwich composites and to the case where the core's principal direction is in coincidence with the directions of the reference coordinate system. The elastic behavior of the composite material is characterized by

- the reduced stiffness parameters Q_{ij}^{f} for the face sheets,
- the reduced stiffness parameters Q_{ij}^{c} and the transverse shear moduli C_{ij}^{c} for the core

Application of the sandwich theory, Sect. 4.3.2, leads to the following expressions for the stiffness coefficients of the constitutive equation (upper index Sa), one lamina

$$A_{ij}^{\mathrm{Sa}} = 2h^{\mathrm{f}} Q_{ij}^{\mathrm{f}}, B_{ij}^{\mathrm{Sa}} = 0, C_{ij}^{\mathrm{Sa}} = 0, D_{ij}^{\mathrm{Sa}} = \frac{1}{2} Q_{ij}^{\mathrm{f}} (h^{\mathrm{f}} + h^{\mathrm{c}}) h^{\mathrm{f}} h^{\mathrm{c}}, i,j = 1,2,6 \tag{4.3.16}$$

The shear stiffness coefficients A_{ij}^{s} are in the sandwich theory

$$A_{ij}^{sSa} = h^{\mathrm{c}} C_{ij}, \; i,j = 4,5, \quad C_{44} = G_{23}^{\mathrm{c}}, \; C_{55} = G_{13}^{\mathrm{c}}, C_{45} = 0 \tag{4.3.17}$$

Application of the laminate theory including transverse shear, Sect. 4.2.2, (4.2.15) leads (upper index La) to

$$\begin{aligned} A_{ij}^{\mathrm{La}} &= 2h^{\mathrm{f}}Q_{ij}^{\mathrm{f}} + h^{\mathrm{c}}Q_{ij}^{\mathrm{c}}, \quad B_{ij}^{\mathrm{La}} = 0, \\ D_{ij}^{\mathrm{La}} &= \frac{1}{2}Q_{ij}^{\mathrm{f}}h^{\mathrm{f}}\left[(h^{\mathrm{f}}+h^{\mathrm{c}})^2 + \frac{1}{3}(h^{\mathrm{f}})^2\right] + \frac{1}{12}Q_{ij}^{\mathrm{c}}(h^{\mathrm{c}})^3, \ i,j = 1,2,6 \end{aligned} \tag{4.3.18}$$

The shear stiffness coefficients A_{ij}^{s} are now

$$A_{ij}^{\mathrm{sLa}} = 2h^{\mathrm{f}}C_{ij}^{\mathrm{f}} + h^{\mathrm{c}}C_{ij}^{\mathrm{c}}, \ i,j = 4,5, C_{44}^{\mathrm{f/c}} = G_{23}^{\mathrm{f/c}}, \ C_{55}^{\mathrm{f/c}} = G_{13}^{\mathrm{f/c}}, C_{45}^{\mathrm{f/c}} = 0 \tag{4.3.19}$$

For symmetric faces with n laminae Eqs. (4.3.16) - (4.3.19) yield

$$\begin{aligned} A_{ij}^{\mathrm{Sa}} &= 2\sum_{k=1}^{n} Q_{ij}^{f(k)}h^{f(k)}, \\ D_{ij}^{\mathrm{Sa}} &= h^{\mathrm{c}}\sum_{k=1}^{n} Q_{ij}^{f(k)}h^{f(k)}\bar{x}_3^{(k)}, \\ A_{ij}^{\mathrm{La}} &= 2\sum_{k=1}^{n} Q_{ij}^{f(k)}h^{f(k)} + h^{\mathrm{c}}Q_{ij}^{\mathrm{c}}, \\ D_{ij}^{\mathrm{La}} &= \sum_{k=1}^{n} Q_{ij}^{f(k)}\left(h^{f(k)}(\bar{x}_3^{(k)})^2 + \frac{h^{f(k)^3}}{12}\right) + Q_{ij}^{\mathrm{c}}\frac{h^{\mathrm{c}3}}{12} \end{aligned}$$

The comparison of the analysis based on the sandwich or the laminate theory yields

$$\begin{aligned} A_{ij}^{\mathrm{La}} &= A_{ij}^{\mathrm{Sa}}\left(1 + \frac{h^{\mathrm{c}}Q_{ij}^{\mathrm{c}}}{2h^{\mathrm{f}}Q_{ij}^{\mathrm{f}}}\right), \\ D_{ij}^{\mathrm{La}} &= D_{ij}^{\mathrm{Sa}}\left(1 + \frac{h^{\mathrm{f}}}{h^{\mathrm{c}}}\frac{h^{\mathrm{c}}+(4/3)h^{\mathrm{f}}}{h^{\mathrm{c}}+h^{\mathrm{f}}} + \frac{Q_{ij}^{\mathrm{c}}}{6Q_{ij}^{\mathrm{f}}}\frac{(h^{\mathrm{c}})^2}{h^{\mathrm{f}}(h^{\mathrm{c}}+h^{\mathrm{f}})}\right), \ i,j = 1,2,6 \end{aligned} \tag{4.3.20}$$

$$A_{ij}^{\mathrm{sLa}} = A_{ij}^{\mathrm{sSa}}\left(1 + \frac{h^{\mathrm{f}}C_{ij}^{\mathrm{f}}}{2h^{\mathrm{c}}C_{ij}^{\mathrm{c}}}\right), \ i,j = 4,5 \tag{4.3.21}$$

Generally the core of the sandwich is less stiff than the cover sheets

$$Q_{ij}^{\mathrm{c}} \ll Q_{ij}^{\mathrm{f}}$$

and the relations (4.3.20) can be simplified

$$A_{ij}^{\mathrm{La}} \approx A_{ij}^{\mathrm{Sa}}, \ D_{ij}^{\mathrm{La}} \approx D_{ij}^{\mathrm{Sa}}\left(1 + \frac{h^{\mathrm{f}}}{h^{\mathrm{c}}}\frac{h^{\mathrm{c}}+(4/3)h^{\mathrm{f}}}{h^{\mathrm{c}}+h^{\mathrm{f}}}\right) \tag{4.3.22}$$

Equation (4.3.21) stays unchanged.

The bending stiffness D_{ij} are modified with respect to the theory of sandwiches and can be evaluated by the influence of the sheet thickness. If for example $h^{\mathrm{c}} = 10$ mm and the sheet thickness $h^{\mathrm{f}} = 1$ mm/ 3 mm or 5 mm we find $D_{ij}^{\mathrm{La}} = 1,103 D_{ij}^{\mathrm{Sa}}/1,326 D_{ij}^{\mathrm{Sa}}$ or $1,555 D_{ij}^{\mathrm{Sa}}$, a difference of more than 10%, 30% or 50 %.

4.4 Problems

Exercise 4.11. The reduced stiffness Q_{ij}^{c} and Q_{ij}^{f} of a symmetric sandwich satisfies the relation $Q_{ij}^{\mathrm{c}} \ll Q_{ij}^{\mathrm{f}}$. Evaluate the influence of the sheet thickness on the bending stiffness ratio $D_{ij}^{\mathrm{La}}/D_{ij}^{\mathrm{Sa}}$ if the core thickness is constant ($h^{\mathrm{c}} = 10$ mm) and the sheet thickness vary: $h^{\mathrm{f}} = 0.5/1.0/3.0/5.0/8.0/10.0$ mm.

Solution 4.11. Using the simplified formula (4.3.22) the ratio values are 1.051/1.103/1.323/1.555/1.918/2.167 i.e. the difference

$$\frac{D_{ij}^{\mathrm{La}} - D_{ij}^{\mathrm{Sa}}}{D_{ij}^{\mathrm{Sa}}} 100\%$$

of the stiffness values for D_{ij}^{La} and D_{ij}^{Sa} are more than 5%/10%/32%/55%/91% or 116%.

Conclusion 4.2. The sandwich formulas of Sect. 4.3.2 should be used for thin cover sheets only, i.e. $h^{\mathrm{f}} \ll h^{\mathrm{c}}$.

Exercise 4.12. A sandwich beam has faces of aluminium alloy and a core of polyurethane foam. The geometry of the cross-section is given in Fig. 4.17. Cal-

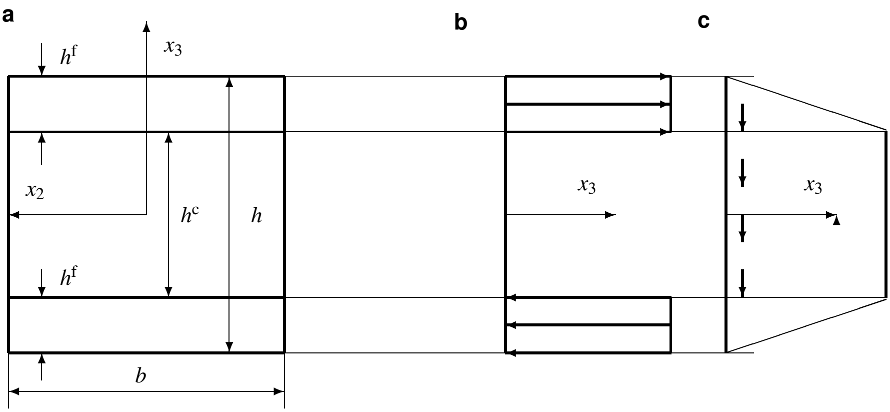


Fig. 4.17 Sandwich beam. **a** Geometry of the cross-section of a sandwich beam, **b** Distribution of the bending stress, if the local stiffness of the faces and the bending stiffness of the core are dropped, **c** Distribution of the shear stress, if only the core transmit shear stresses

culate the bending stiffness D and the distributions of the bending and the shear stress across the faces and the core, if the stress resultants M and Q are given.

Solution 4.12. The bending stiffness D of the sandwich beam is the sum of the flexural rigidities of the faces and the core

$$D = 2E^{\mathrm{f}}\frac{bh^{\mathrm{f}3}}{12} + 2E^{\mathrm{f}}bh^{\mathrm{f}}\left(\frac{h^{\mathrm{c}}+h^{\mathrm{f}}}{2}\right)^2 + E^{\mathrm{c}}\frac{bh^{\mathrm{c}3}}{12}$$

E^{f} and E^{c} are the effective Young's moduli of the face and the core materials. The first term presents the local bending stiffness of the faces about their own axes, the third term represents the bending stiffness of the core. Both terms are generally very small in comparison to the second term. Provided that

$$(h^{\mathrm{c}}+h^{\mathrm{f}})/h^{\mathrm{f}} > 5,77, \quad [(E^{\mathrm{f}}h^{\mathrm{f}})/(E^{\mathrm{c}}h^{\mathrm{c}})][(h^{\mathrm{c}}+h^{\mathrm{f}})/h^{\mathrm{c}}]^2 > 100/6$$

i.e.

$$\frac{E^{\mathrm{f}}bh^{\mathrm{f}3}/6}{E^{\mathrm{f}}[bh^{\mathrm{f}}(h^{\mathrm{c}}+h^{\mathrm{f}})^2]/2} < \frac{1}{100} \quad \frac{E^{\mathrm{c}}bh^{\mathrm{c}3}/12}{E^{\mathrm{f}}[bh^{\mathrm{f}}(h^{\mathrm{c}}+h^{\mathrm{f}})^2]/2} < \frac{1}{100}$$

the first and the third term are less than 1% of the second term and the bending stiffness is approximately

$$D \approx E^{\mathrm{f}}bh^{\mathrm{f}}(h^{\mathrm{c}}+h^{\mathrm{f}})^2/2$$

The bending stress distributions through the faces and the core are

$$\sigma^{\mathrm{f}} = M\frac{E^{\mathrm{f}}}{D}x_3 \approx \pm M\frac{E^{\mathrm{f}}}{D}\frac{h^{\mathrm{c}}+h^{\mathrm{f}}}{2}, \quad \sigma^{\mathrm{c}} = M\frac{E^{\mathrm{c}}}{D}x_3 \approx 0$$

The assumptions of the classical beam theory yield the shear stress equation for the core

$$\begin{aligned}\tau &= \frac{QS(x_3)}{bI} = \frac{Q}{bD}[E^{\mathrm{f}}S^{\mathrm{f}}(x_3) + E^{\mathrm{c}}S^{\mathrm{c}}(x_3)]\\ &\approx \frac{Q}{D}\left[E^{\mathrm{f}}\frac{h^{\mathrm{f}}(h^{\mathrm{c}}+h^{\mathrm{f}})}{2} + \frac{E^{\mathrm{c}}}{2}\left(\frac{h^{\mathrm{c}2}}{4} - x_3^2\right)\right]\end{aligned}$$

The maximum core shear stress will occur at $x_3 = 0$. If

$$\frac{E^{\mathrm{f}}h^{\mathrm{f}}(h^{\mathrm{c}}+h^{\mathrm{f}})}{E^{\mathrm{c}}h^{\mathrm{c}2}/4} > 100$$

the ratio of the maximum core shear stress to the minimum core shear stress is $< 1\%$ and the shear stress distribution across the core can be considered constant

$$\tau \approx \frac{Q}{D}\frac{E^{\mathrm{f}}h^{\mathrm{f}}(h^{\mathrm{c}}+h^{\mathrm{f}})}{2}$$

and with

$$D \approx E^{\mathrm{f}} b h^{\mathrm{f}} (h^{\mathrm{c}} + h^{\mathrm{f}})^2 / 2$$

follow $\tau \approx Q/b(h^{\mathrm{c}} + h^{\mathrm{f}}) \approx Q/bh$. In Fig. 4.17 the distributions of the bending and shear stresses for sandwich beams with thin faces are illustrated. Note that for thicker faces the approximate flexural bending rigidity is

$$D \approx E^{\mathrm{f}} b h^{\mathrm{f}} (h^{\mathrm{c}} + h^{\mathrm{f}}) / + E^{\mathrm{f}} b {h^{\mathrm{f}}}^3 / 12$$

Chapter 5
Classical and Improved Theories

In this chapter, the theoretical background for two commonly used structural theories for the modelling and analysis of laminates and sandwiches is considered, namely the classical laminate theory and the first-order shear deformation theory. The classical laminate theory (CLT) and the first-order shear deformation theory (FSDT) are the most commonly used theories for analyzing laminated or sandwiched beams, plates and shells in engineering applications. The CLT is an extension of Kirchhoff's[1] classical plate theory for homogeneous isotropic plates to laminated composite plates with a reasonable high width-to-thickness ratio. For homogeneous isotropic plates the Kirchhoff's theory is limited to thin plates with ratios of maximum plate deflection w to plate thickness $h < 0.2$ and plate thickness/ minimum in-plane dimensions < 0.1. Unlike homogeneous isotropic structure elements, laminated plates or sandwich structures have a higher ratio of in-plane Young's moduli to the interlaminar shear moduli, i.e. such composite structure elements have a lower transverse shear stiffness and often have significant transverse shear deformations at lower thickness-to span ratios < 0.05. Otherwise the maximum deflections can be considerable larger than predicted by CLT. Furthermore, the CLT cannot yield adequate correct through-the-thickness stresses and failure estimations. As a result of these considerations it is appropriate to develop higher-order laminated and sandwich theories which can be applied to moderate thick structure elements, e.g. the FSDT. CLT and FSDT are so-called equivalent single-layer theories (ESLT). Moreover a short overview of so-called discrete-layer or layerwise theories is given, which shall overcome the drawbacks of equivalent single layer theories.

[1] Gustav Robert Kirchhoff (*12 March 1824 Königsberg - †17 October 1887 Berlin) - physicist who contributed to the fundamental understanding of electrical circuits, spectroscopy, and the emission of black-body radiation by heated objects, in addition, he formulated a plate theory which was an extension of the Euler-Bernoulli beam theory

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0_5

5.1 General Remarks

A classification of the structural theories in composite mechanics illustrates that the following approaches for the modelling and analysis of beams and plates composed of composite materials can be used:

1. So called equivalent single-layer theories: These theories are derived from the three-dimensional elasticity theory by making assumptions concerning the kinematics of deformation and/or the stress distribution through the thickness of a laminate or a sandwich. With the help of these assumptions the modelling can be reduced from a 3D-problem to a 2D-problem. In engineering applications equivalent single-layer theories are mostly used in the form of the classical laminate theory, for very thin laminates, and the first order shear deformation theory, for thicker laminates and sandwiches.
   An equivalent single layer model is developed by assuming continuous displacement and strain functions through the thickness. The stresses jump from ply to ply and therefore the governing equations are derived in terms of thickness averaged resultants. Also second and higher order equivalent single layer theories by using higher order polynomials in the expansion of the displacement components through the thickness of the laminate are developed. Such higher order theories introduce additional unknowns that are often difficult to interpret in mechanical terms. The CLT requires C^1-continuity of the transverse displacement, i.e. the displacement and the derivatives must be continuous, unlike the FSDT requires C^0-continuity only. Higher order theories generally require at least C^1-continuity.
2. Three-dimensional elasticity theories such as the traditional 3D-formulations of anisotropic elasticity or the so-called layerwise theories: In contrast to the equivalent single-layer theories only the displacement components have to be continuous through the thickness of a laminate or a sandwich but the derivatives of the displacements with respect to the thickness coordinate x_3 may be discontinuous at the layer interfaces. We say that the displacement field exhibits only C^0-continuity through the thickness directions.

The basic assumption of modelling structural elements in the framework of the anisotropic elasticity is an approximate expression of the displacement components in the form of polynomials for the thickness coordinate x_3. Usually the polynomials are limited to degree three and can be written in the form

$$
\begin{aligned}
u_1(x_1,x_2,x_3) &= u(x_1,x_2) + \alpha x_3 \frac{\partial w(x_1,x_2)}{\partial x_1} + \beta x_3 \psi_1(x_1,x_2) \\
&\quad + \gamma x_3^2 \phi_1(x_1,x_2) + \delta x_3^3 \chi_1(x_1,x_2), \\
u_2(x_1,x_2,x_3) &= v(x_1,x_2) + \alpha x_3 \frac{\partial w(x_1,x_2)}{\partial x_2} + \beta x_3 \psi_2(x_1,x_2) \\
&\quad + \gamma x_3^2 \phi_2(x_1,x_2) + \delta x_3^3 \chi_2(x_1,x_2), \\
u_3(x_1,x_2,x_3) &= w(x_1,x_2) + \overline{\beta} x_3 \psi_3(x_1,x_2) + \overline{\gamma} x_3^2 \phi_3(x_1,x_2)
\end{aligned}
\qquad (5.1.1)
$$

A displacement field in the form of (5.1.1) satisfies the compatibility conditions for strains, Sect. 2.2.1, and allows possible cross-sectional warping, transverse shear deformations and transverse normal deformations to be taken into account. The displacement components of the middle surface are $u(x_1,x_2), v(x_1,x_2), w(x_1,x_2)$. In the case of dynamic problems the time t must be introduced in all displacement functions.

The polynomial approach (5.1.1) of the real displacement field yields the following equivalent single-layer theories

- Classical laminate theories

$$\alpha = -1, \beta = \gamma = \delta = \overline{\beta} = \overline{\gamma} = 0$$

- First-order shear deformation theory

$$\alpha = 0, \beta = 1, \gamma = \delta = \overline{\beta} = \overline{\gamma} = 0$$

- Second order laminate theory

$$\alpha = 0, \beta = 1, \gamma = 1, \delta = \overline{\beta} = \overline{\gamma} = 0$$

- Third order laminate theory

$$\alpha = 0, \beta = 1, \gamma = 1, \delta = 1, \overline{\beta} = \overline{\gamma} = 0$$

Theories higher than third order are not used because the accuracy gain is so little that the effort required to solve the governing equations is not justified. A third order theory based on the displacement field u_1, u_2, u_3 has 11 unknown functions of the in-plane coordinates x_1, x_2. u, v, w denote displacements and ψ_1, ψ_2 rotations of the transverse normals referred to the plane $x_3 = 0$. ψ_3 has the meaning of extension of a transverse normal and the remaining functions can be interpreted as warping functions that specify the deformed shape of a straight line perpendicular to the reference plane of the undeformed structure. In addition, any plate theory should fulfill some consistency requirements which was first time discussed for the simplest case of a homogeneous isotropic plate in Kienzler (2002) and later extended to other cases by Schneider and Kienzler (2015); Schneider et al (2014). Also implementations of higher order theories into finite element approximations cannot be recommended. If a laminated plate is thick or the 3D stress field must be calculated in local regions, a full 3D analysis should be carried out.

The most widely used approach reduces the polynomial function of degree three to a linear or first order approximation, which includes the classical and the first-order shear deformation theory

$$\begin{aligned}
u_1(x_1,x_2,x_3) &= u(x_1,x_2) + x_3\psi_1(x_1,x_2),\\
u_2(x_1,x_2,x_3) &= v(x_1,x_2) + x_3\psi_2(x_1,x_2),\\
u_3(x_1,x_2,x_3) &= w(x_1,x_2)
\end{aligned} \tag{5.1.2}$$

The classical approximation can be obtained if

$$\psi_1(x_1,x_2) = -\frac{\partial w}{\partial x_1}, \qquad \psi_2(x_1,x_2) = -\frac{\partial w}{\partial x_2}$$

The number of unknown functions reduces to three, that are u,v,w. On the other hand there are five independent unknown functions u,v,w,ψ_1,ψ_2.

The strain-displacement equations (2.2.3) give for the first order displacement approximation a first order strain field model with transverse shear

$$\begin{aligned}
\varepsilon_1 &= \frac{\partial u}{\partial x_1} + x_3\frac{\partial \psi_1}{\partial x_1}, \quad \varepsilon_2 = \frac{\partial v}{\partial x_2} + x_3\frac{\partial \psi_2}{\partial x_2}, \quad \varepsilon_3 = 0,\\
\varepsilon_4 &= \frac{\partial w}{\partial x_2} + \psi_2, \quad \varepsilon_5 = \frac{\partial w}{\partial x_1} + \psi_1,\\
\varepsilon_6 &= \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} + x_3\left(\frac{\partial \psi_2}{\partial x_1} + \frac{\partial \psi_1}{\partial x_2}\right)
\end{aligned} \tag{5.1.3}$$

For the in-plane strains one can write in contracted form

$$\varepsilon_i(x_1,x_2,x_3) = \varepsilon_i(x_1,x_2) + x_3\kappa_i(x_1,x_2), \quad i = 1,2,6,$$

i.e. the in-plane strains $\varepsilon_1,\varepsilon_2$ and ε_6 vary linearly through the thickness h.

The stress-strain relations in on-axis coordinates are

$$\sigma_i' = C_{ij}'\varepsilon_j', \quad i,j = 1,2,\ldots,6$$

From the transformation rule (4.1.27) follow the stiffness coefficients in the off-axis-coordinates

$$\boldsymbol{C} = \overset{3}{\boldsymbol{T}}{}^{\varepsilon \mathrm{T}} \boldsymbol{C}' \overset{3}{\boldsymbol{T}}{}^{\varepsilon}$$

and with (4.1.26) the constitutive equation is

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16}\\ C_{12} & C_{22} & C_{23} & 0 & 0 & C_{26}\\ C_{13} & C_{23} & C_{33} & 0 & 0 & C_{36}\\ 0 & 0 & 0 & C_{44} & C_{45} & 0\\ 0 & 0 & 0 & C_{45} & C_{55} & 0\\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66}\end{bmatrix} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix} \tag{5.1.4}$$

Assuming $\sigma_3 \approx 0$, the stiffness matrix can be rewritten by separating the transverse shear stresses and strains in analogy to (4.2.3) - (4.2.5)

$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\\ \sigma_4\\ \sigma_5\end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{16} & 0 & 0\\ Q_{12} & Q_{22} & Q_{26} & 0 & 0\\ Q_{16} & Q_{26} & Q_{66} & 0 & 0\\ 0 & 0 & 0 & C_{44} & C_{45}\\ 0 & 0 & 0 & C_{45} & C_{55}\end{bmatrix} \begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\\ \varepsilon_4\\ \varepsilon_5\end{bmatrix} \tag{5.1.5}$$

and from

$$\sigma_3 = C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{33}\varepsilon_3 + C_{36}\varepsilon_6 = 0$$

it follows

$$\varepsilon_3 = -\frac{1}{C_{33}}(C_{13}\varepsilon_1 + C_{23}\varepsilon_2 + C_{36}\varepsilon_6)$$

The Q_{ij} are the reduced stiffness in the off-axis reference system

$$Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}},\ i,j = 1,2,6,\ Q_{ij} = C_{ij},\ i,j = 4,5$$

Summarizing, one can say that the first order displacement approach (5.1.2) includes the classical and the shear deformation theory for laminates and sandwiches. In both cases the in-plane displacements and strains vary linearly through the thickness, but the explicit expressions for the curvature vector $\boldsymbol{\kappa}$ differ. The force and moment resultants can be defined for both theories in the usual way, e.g. (4.2.13), (4.2.14), but in the classical theory there are only constitutive equations for the in-plane force and the moment resultants $\boldsymbol{N},\boldsymbol{M}$. It can be proved that a CLT approach is sufficient for very thin laminates and it has been used particularly to determine the global response of thin composite structure elements, i.e. deflections, overall buckling, vibration frequencies, etc. The FSDT approach is sufficient for determining in-plane stresses even if the structure slenderness is not very high.

The CLT neglects all transverse shear and normal effects, i.e. structural deformation is due entirely to bending and in-plane stretching. The FSDT relaxes the kinematic restrictions of CLT by including a constant transverse shear strain. Both first order theories yield a complete understanding of the through-the-thickness laminate response. Transverse normal and shear stresses, however, play an important role in the analysis of beams, plates and shells since they significantly affect characteristic failure modes like, e.g., delamination. The influence of interlaminar transverse stresses are therefore taken into account by several failure criteria. Simple but sufficient accurate methods for determination of the complete state of stress in composite structures are needed to overcome the limitations of the simple first order 2D modelling in the frame of an extended 2D modelling. In Sects. 5.2 and 5.3 a short description of CLT and FSDT is given including some remarks to calculate transverse stress components. In Chap. 11 will be seen that both the CLT and the FSDT yield finite elements with an economical number of degrees of freedom, both have some drawbacks. CLT-models require C^1-continuity which complicates the implementation in commonly used FEM programs. FSDT-models have the advantage of requiring only C^0-continuity but they can exhibit so-called locking effects if laminates becomes thin. Further details are given in Chap. 11.

5.2 Classical Laminate Theory

The classical laminate theory uses the first-order model equations (5.1.2) but makes additional assumptions:

1. All layers are in a state of plane stress, i.e.

$$\sigma_3 = \sigma_4 = \sigma_5 = 0$$

2. Normal distances from the middle surface remain constant, i.e. the transverse normal strain ε_3 is negligible compared with the in-plane strains $\varepsilon_1, \varepsilon_2$.
3. The transverse shear strains $\varepsilon_4, \varepsilon_5$ are negligible. This assumption implies that straight lines normal to the middle surface remain straight and normal to that surface after deformation (Bernoulli/Kirchhoff/Love[2] hypotheses in the theory of beams, plates and shells).

Further we recall the general assumption of linear laminate theory that each layer is quasi-homogeneous, the displacements are continuous through the total thickness h, the displacements are small compared with the thickness h and the constitutive equations are linear.

From assumptions 2. and 3. it follows from (5.1.3) that

$$\psi_1(x_1,x_2) = -\frac{\partial w}{\partial x_1}, \quad \psi_2(x_1,x_2) = -\frac{\partial w}{\partial x_2}, \tag{5.2.1}$$

and the displacement approach (5.1.2) and the strain components (5.1.3) are written by

$$\begin{aligned}
u_1(x_1,x_2,x_3) &= u(x_1,x_2) \quad - x_3\frac{\partial w(x_1,x_2)}{\partial x_1},\\
u_2(x_1,x_2,x_3) &= v(x_1,x_2) \quad - x_3\frac{\partial w(x_1,x_2)}{\partial x_2},\\
u_3(x_1,x_2,x_3) &= w(x_1,x_2),
\end{aligned} \tag{5.2.2}$$

$$\begin{gathered}
\varepsilon_1 = \frac{\partial u}{\partial x_1} - x_3\frac{\partial w^2}{\partial x_1^2}, \quad \varepsilon_2 = \frac{\partial v}{\partial x_2} - x_3\frac{\partial w^2}{\partial x_2^2}, \quad \varepsilon_3 = 0,\\
\varepsilon_4 = 0, \quad \varepsilon_5 = 0, \quad \varepsilon_6 = \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} - 2x_3\frac{\partial w^2}{\partial x_1 \partial x_2}
\end{gathered} \tag{5.2.3}$$

The condensed form for the in-plane strains can be noted as

$$\varepsilon_i(x_1,x_2,x_3) = \varepsilon_i(x_1,x_2) + x_3\kappa_i, \; i = 1,2,6$$

with

[2] Augustus Edward Hough Love (*17 April 1863, Weston-super-Mare – †5 June 1940, Oxford) - mathematician, mathematical theory of elasticity

$$\varepsilon_1 = \frac{\partial u}{\partial x_1}, \quad \varepsilon_2 = \frac{\partial v}{\partial x_2}, \quad \varepsilon_6 = \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1},$$
$$\kappa_1 = -\frac{\partial w^2}{\partial x_1^2}, \quad \kappa_2 = -\frac{\partial w^2}{\partial x_2^2}, \quad \kappa_6 = -2\frac{\partial w^2}{\partial x_1 \partial x_2}$$

$\boldsymbol{\varepsilon}^{\mathrm{T}} = [\varepsilon_1\ \varepsilon_2\ \varepsilon_6]$ is the vector of midplane strains (stretching and shearing) and $\boldsymbol{\kappa}^{\mathrm{T}} = [\kappa_1\ \kappa_2\ \kappa_6]$ the vector of curvature (bending and twisting). For all k layers the stresses are given in condensed form by

$$\sigma_i^{(k)} = Q_{ij}^{(k)}\varepsilon_i + x_3 Q_{ij}^{(k)}\kappa_i, \quad i,j = 1,2,6 \tag{5.2.4}$$

and the stiffness equations for the stress resultants follow from (4.2.13) - (4.2.18).

The classical laminate theory is also called shear rigid theory, the material equations yield zero shear stresses σ_4, σ_5 for zero strains $\varepsilon_4, \varepsilon_5$, in the case that the shear stiffness has finite values. But the equilibrium conditions yield non-zero stresses σ_4, σ_5, if the stresses σ_1, σ_2 and σ_6 are not all constant. This physical contradiction will be accepted in the classical theory and the transverse shear stresses are approximately calculated with the given stresses $\sigma_1, \sigma_2, \sigma_6$ by the equilibrium equations (4.1.56).

The approximate calculation of transverse shear stresses can be simplified if one assumes the case of cylindrical bending, i.e. $N_1 = N_2 = N_6 \approx 0, M_6 \approx 0$. The constitutive equation (4.2.18) or the inverted Eq. (4.2.19) with $\boldsymbol{N} \equiv \boldsymbol{0}$ gives

$$\begin{bmatrix} \boldsymbol{0} \\ \cdots \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \vdots & \boldsymbol{B} \\ \cdots & \cdots & \cdots \\ \boldsymbol{B} & \vdots & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \cdots \\ \boldsymbol{\kappa} \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{\varepsilon} \\ \cdots \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} & \vdots & \boldsymbol{b} \\ \cdots & \cdots & \cdots \\ \boldsymbol{b}^{\mathrm{T}} & \vdots & \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} \\ \cdots \\ \boldsymbol{M} \end{bmatrix} \tag{5.2.5}$$

that is with Eqs. (4.2.20) - (4.2.26)

$$\boldsymbol{\varepsilon} = -\boldsymbol{A}^{-1}\boldsymbol{B}\boldsymbol{\kappa}, \boldsymbol{M} = (\boldsymbol{D} - \boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B})\boldsymbol{\kappa} = \boldsymbol{D}^*\boldsymbol{\kappa}$$
$$\boldsymbol{\varepsilon} = \boldsymbol{b}\boldsymbol{M} = \boldsymbol{B}^*\boldsymbol{D}^{*-1}\boldsymbol{M}, \boldsymbol{\kappa} = \boldsymbol{d}\boldsymbol{M} = \boldsymbol{D}^{*-1}\boldsymbol{M} \tag{5.2.6}$$

For symmetric laminates are $\boldsymbol{B} = \boldsymbol{0}, \boldsymbol{B}^* = \boldsymbol{0}, \boldsymbol{D}^* = \boldsymbol{D}$ and Eqs. (5.2.6) can be replaced by

$$\boldsymbol{\varepsilon} = \boldsymbol{0}, \boldsymbol{\kappa} = \boldsymbol{D}^{-1}\boldsymbol{M} \tag{5.2.7}$$

The partial extensional and coupling stiffness $\tilde{\boldsymbol{A}}(x_3), \tilde{\boldsymbol{B}}(x_3)$, Fig. 5.1, become

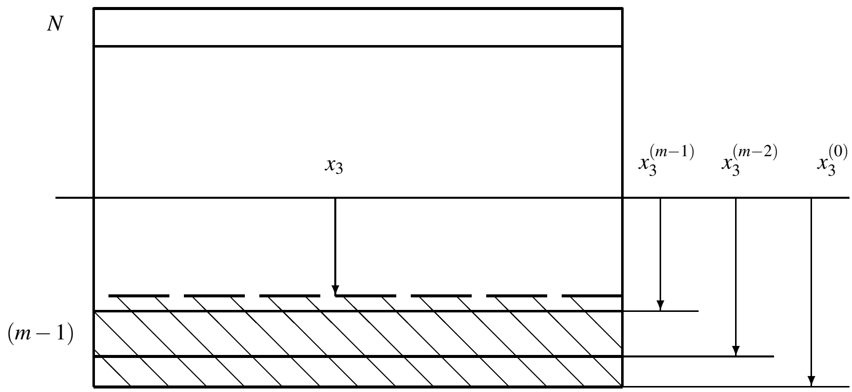


Fig. 5.1 Derivation of partial stiffness $\tilde{\boldsymbol{A}}(x_3)$ and $\tilde{\boldsymbol{B}}(x_3)$ for the shaded part of the cross-section

$$
\begin{aligned}
\tilde{\boldsymbol{A}}(x_3) &= \int\limits_{x_3^{(0)}}^{x_3} \boldsymbol{Q}(x_3)\mathrm{d}x_3 \\
&= \sum_{k=1}^{m-1} \boldsymbol{Q}^{(k)} h^{(k)} + \boldsymbol{Q}^{(m)} \left(x_3 - x_3^{(m-1)} \right), \\
\tilde{\boldsymbol{B}}(x_3) &= \int\limits_{x_3^{(0)}}^{x_3} \boldsymbol{Q}(x_3) x_3 \mathrm{d}x_3 \\
&= \sum_{k=1}^{m-1} \boldsymbol{Q}^{(k)} s^{(k)} + \frac{1}{2}\boldsymbol{Q}^{(m)} \left(x_3^2 - x_3^{(m-1)^2} \right),
\end{aligned}
\tag{5.2.8}
$$

$$
h^{(k)} = x_3^{(k)} - x_3^{(k-1)},\ s^{(k)} = h^{(k)}\overline{x}_3^{(k)} = \frac{1}{2}\left(x_3^{(k)} + x_3^{(k-1)}\right)\left(x_3^{(k)} - x_3^{(k-1)}\right)
$$

Outgoing from the equilibrium equations (2.2.1) the shear stress equations can be written

$$
\begin{aligned}
\sigma_5(x_3) &= -\int\limits_{x_3^{(0)}}^{x_3} \left(\frac{\partial \sigma_1}{\partial x_1} + \frac{\partial \sigma_6}{\partial x_2} \right) \mathrm{d}x_3 \\
&= -\int\limits_{x_3^{(0)}}^{x_3} \left[\frac{\partial}{\partial x_1} \left(Q_{1j}^{(k)} \varepsilon_j + x_3 Q_{1j}^{(k)} \kappa_j \right) + \frac{\partial}{\partial x_2} \left(Q_{6j}^{(k)} \varepsilon_j + x_3 Q_{6j}^{(k)} \kappa_j \right) \right] \mathrm{d}x_3, \\
\sigma_4(x_3) &= -\int\limits_{x_3^{(0)}}^{x_3} \left(\frac{\partial \sigma_6}{\partial x_1} + \frac{\partial \sigma_2}{\partial x_2} \right) \mathrm{d}x_3 \\
&= -\int\limits_{x_3^{(0)}}^{x_3} \left[\frac{\partial}{\partial x_1} \left(Q_{6j}^{(k)} \varepsilon_j + x_3 Q_{6j}^{(k)} \kappa_j \right) + \frac{\partial}{\partial x_2} \left(Q_{2j}^{(k)} \varepsilon_j + x_3 Q_{2j}^{(k)} \kappa_j \right) \right] \mathrm{d}x_3
\end{aligned}
\tag{5.2.9}
$$

or in vector-matrix notation

$$
\begin{aligned}
\begin{bmatrix} \sigma_5(x_3) \\ \sigma_4(x_3) \end{bmatrix} = &-\int\limits_{x_3^{(0)}}^{x_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_{1j}^{(k)}(\varepsilon_{j,x_1} + x_3 \kappa_{j,x_1}) \\ Q_{2j}^{(k)}(\varepsilon_{j,x_1} + x_3 \kappa_{j,x_1}) \\ Q_{6j}^{(k)}(\varepsilon_{j,x_1} + x_3 \kappa_{j,x_1}) \end{bmatrix} \mathrm{d}x_3 \\
&-\int\limits_{x_3^{(0)}}^{x_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Q_{1j}^{(k)}(\varepsilon_{j,x_2} + x_3 \kappa_{j,x_2}) \\ Q_{2j}^{(k)}(\varepsilon_{j,x_2} + x_3 \kappa_{j,x_2}) \\ Q_{6j}^{(k)}(\varepsilon_{j,x_2} + x_3 \kappa_{j,x_2}) \end{bmatrix} \mathrm{d}x_3
\end{aligned}
\tag{5.2.10}
$$

with $(\ldots)_{,x_\alpha} = \partial \ldots / \partial x_\alpha, \alpha = 1,2, j = 1,2,6$. Using Eqs. (5.2.6) - (5.2.8)

$$
\boldsymbol{\sigma}^{\mathrm{s}}(x_3) = -\boldsymbol{B}_1 \boldsymbol{F}(x_3) \boldsymbol{M}_{,x_1} - \boldsymbol{B}_2 \boldsymbol{F}(x_3) \boldsymbol{M}_{,x_2} \tag{5.2.11}
$$

with

$$
\begin{aligned}
&\boldsymbol{\sigma}^{\mathrm{s}}(x_3) = [\sigma_5 \ \sigma_4]^{\mathrm{T}}, \quad \boldsymbol{M}_{,x_i} = [M_{1,x_i} \ M_{2,x_i} \ M_{6,x_i}]^{\mathrm{T}}, \\
&\boldsymbol{F}(x_3) = [\tilde{\boldsymbol{A}}(x_3) \boldsymbol{A}^{-1} \boldsymbol{B} - \tilde{\boldsymbol{B}}(x_3)] \boldsymbol{D}^{*-1} = \begin{bmatrix} F_{11} & F_{12} & F_{16} \\ F_{21} & F_{22} & F_{26} \\ F_{61} & F_{62} & F_{66} \end{bmatrix}
\end{aligned}
\tag{5.2.12}
$$

in the general case if non-symmetrical laminate and

$$
\boldsymbol{F}(x_3) = \tilde{\boldsymbol{B}}(x_3) \boldsymbol{D}^{-1} \tag{5.2.13}
$$

for symmetrical laminates,

$$
\boldsymbol{B}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{B}_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{5.2.14}
$$

are so called Boolean[3] matrices. Equation (5.2.11) can also be written in component notation.

Equations (5.2.11) and (5.2.16) constitute the straight forward equilibrium approach for transverse shear stresses which only neglects the influence of the in-plane force derivatives $\boldsymbol{N}_{,x_i}$, but this is a very minor restriction, since, in most engineering applications, the dominating source for transverse shear stresses are transverse force resultants. To express the bending moment derivatives by transverse shear stress resultants it is necessary to assume special selected displacements modes. If one selects the cylindrical bending around the x_1- and the x_2-axis one obtains $M_6 = 0, M_{1,x_2} = 0, M_{2,x_1} = 0$

$$M_{1,x_1}(x_1) = Q_1^{\mathrm{s}}(x_1), \quad M_{2,x_2}(x_2) = Q_2^{\mathrm{s}}(x_2) \tag{5.2.15}$$

with the transverse forces

$$\begin{aligned} Q_1^{\mathrm{s}}(x_1) &= \int\limits_{(h)} \sigma_5(x_3)\mathrm{d}x_3 = \sum_{k=1}^{n} \int\limits_{(h)} \sigma_5^{(k)}(x_3)\mathrm{d}x_3, \\ Q_2^{\mathrm{s}}(x_2) &= \int\limits_{(h)} \sigma_4(x_3)\mathrm{d}x_3 = \sum_{k=1}^{n} \int\limits_{(h)} \sigma_4^{(k)}(x_3)\mathrm{d}x_3 \end{aligned} \tag{5.2.16}$$

Equation (5.2.11) becomes in matrix notation

$$\begin{aligned} &\boldsymbol{\sigma}^{\mathrm{s}}(x_3) = \boldsymbol{F}(x_3)\boldsymbol{Q}^{\mathrm{s}}, \\ &\boldsymbol{\sigma}^{\mathrm{s}} = [\sigma_5(x_3)\ \sigma_4(x_3)]^{\mathrm{T}}, \quad \boldsymbol{Q}^{\mathrm{s}} = [Q_1^{\mathrm{s}}(x_1)\ Q_2^{\mathrm{s}}(x_2)]^{\mathrm{T}}, \\ &\boldsymbol{F} = \begin{bmatrix} F_{11}(x_3) & F_{62}(x_3) \\ F_{61}(x_3) & F_{22}(x_3) \end{bmatrix} \end{aligned} \tag{5.2.17}$$

Summarizing the derivations of transverse shear stresses we have considered two cases

1. $\boldsymbol{N} \equiv \boldsymbol{0}, \boldsymbol{M} = [M_1\ M_2\ M_6]^{\mathrm{T}}$,
2. $\boldsymbol{N} \equiv \boldsymbol{0}, \boldsymbol{M} = [M_1(x_1)\ M_2(x_2)]^{\mathrm{T}}$

In case 1. follow Eqs. (5.2.18) and in case 2. Eqs. (5.2.19)

$$\begin{aligned} \begin{bmatrix} \sigma_5(x_3) \\ \sigma_4(x_3) \end{bmatrix} = &\begin{bmatrix} F_{11}(x_3) & F_{12}(x_3) & F_{16}(x_3) \\ F_{61}(x_3) & F_{62}(x_3) & F_{66}(x_3) \end{bmatrix} \frac{\partial}{\partial x_1} \begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix} \\ &+ \begin{bmatrix} F_{61}(x_3) & F_{62}(x_3) & F_{66}(x_3) \\ F_{21}(x_3) & F_{22}(x_3) & F_{26}(x_3) \end{bmatrix} \frac{\partial}{\partial x_2} \begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix}, \end{aligned} \tag{5.2.18}$$

[3] George Boole (*2 November 1815 Lincoln - †8 December 1864 Ballintemp) - mathematician, educator, philosopher and logician

$$\begin{bmatrix}\sigma_5(x_3)\\ \sigma_4(x_3)\end{bmatrix} = \begin{bmatrix}F_{11}(x_3) & F_{62}(x_3)\\ F_{61}(x_3) & F_{22}(x_3)\end{bmatrix}\begin{bmatrix}Q_1\\ Q_2\end{bmatrix}, \quad \begin{array}{l}Q_1^{\mathrm{s}} = Q_1^{\mathrm{s}}(x_1)\\ Q_2^{\mathrm{s}} = Q_2^{\mathrm{s}}(x_2)\end{array}, \qquad (5.2.19)$$
$$\frac{\partial M_1(x_1)}{\partial x_1} = Q_1, \ \frac{\partial M_2(x_2)}{\partial x_2} = Q_2$$

Symmetric laminates are preferred in engineering applications. In this case $\boldsymbol{D}^* = \boldsymbol{D}, \boldsymbol{B} \equiv \boldsymbol{0}$ and $\boldsymbol{F}(x_3) = -\tilde{\boldsymbol{B}}(x_3)\boldsymbol{D}^{-1}$. The calculation of the transverse shear stresses is more simple. The approximate solution for transverse shear stresses in the classical laminate theory satisfies the equilibrium condition. The shear stresses are layerwise parabolic functions and there is no stress jump at the layer interfaces.

Also in the frame of the classical laminate theory an approximate constitutive equation can be formulated

$$\boldsymbol{Q}^{\mathrm{s}} = \boldsymbol{A}^{\mathrm{s}}\boldsymbol{\varepsilon}^{\mathrm{s}} \quad \text{or} \quad \begin{bmatrix}Q_1^{\mathrm{s}}\\ Q_2^{\mathrm{s}}\end{bmatrix} = \begin{bmatrix}A_{55} & A_{45}\\ A_{45} & A_{44}\end{bmatrix}\begin{bmatrix}\varepsilon_5\\ \varepsilon_4\end{bmatrix} \qquad (5.2.20)$$

Regarding the complementary transverse shear theory formulated in shear stresses

$$W_1^{*\mathrm{s}} = \frac{1}{2}\int\limits_{(h)} \boldsymbol{\sigma}^{\mathrm{sT}}(\boldsymbol{C})^{-1}\boldsymbol{\sigma}^{\mathrm{s}}\mathrm{d}x_3 \qquad (5.2.21)$$

and in shear forces

$$W_2^{*\mathrm{s}} = \frac{1}{2}\boldsymbol{Q}^{\mathrm{sT}}(\boldsymbol{A}^{\mathrm{s}})^{-1}\boldsymbol{Q}^{\mathrm{s}} \qquad (5.2.22)$$

The stress vector $\boldsymbol{\sigma}^{\mathrm{s}}$ is a function of x_3 only, and therefore the integration is carried out over x_3. In

$$\boldsymbol{C}^{\mathrm{s}} = \begin{bmatrix}C_{55} & C_{45}\\ C_{45} & C_{44}\end{bmatrix}$$

the $C_{ij}, i, j = 4,5$ are the elastic parameters of the Hooke's law. In Eq. (5.2.21) the stress can be replaced by the transverse force resultants, Eq. (5.2.19). The Q_i^{s} do not depend on x_3 and Eq. (5.2.21) yields

$$W_1^{*\mathrm{s}} = \frac{1}{2}\boldsymbol{Q}^{\mathrm{sT}}\left[\int\limits_{(h)} \boldsymbol{F}^{\mathrm{T}}(x_3)(\boldsymbol{C}^{\mathrm{s}})^{-1}\boldsymbol{F}(x_3)\mathrm{d}x_3\right]\boldsymbol{Q}^{\mathrm{s}} \qquad (5.2.23)$$

$\boldsymbol{F}(x_3)$ is the reduced elasticity matrix Eq. (5.2.18) and Eq. (5.2.23) leads to

$$W_1^{*\mathrm{s}} = \frac{1}{2}[Q_1^{\mathrm{s}} \ Q_2^{\mathrm{s}}]\left\{\int\limits_{(h)} \begin{bmatrix}F_{11} & F_{62}\\ F_{61} & F_{22}\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}C_{55} & C_{45}\\ C_{45} & C_{44}\end{bmatrix}^{-1}\begin{bmatrix}F_{11} & F_{62}\\ F_{61} & F_{22}\end{bmatrix}\mathrm{d}x_3\right\}\begin{bmatrix}Q_1^{\mathrm{s}}\\ Q_2^{\mathrm{s}}\end{bmatrix} \qquad (5.2.24)$$

With $W_1^{*\mathrm{s}} = W_2^{*\mathrm{s}}$ follows the approximate shear stiffness

$$\boldsymbol{A}^{\mathrm{s}} = \left[\int\limits_{(h)} \boldsymbol{F}^{\mathrm{T}} (\boldsymbol{C}^{\mathrm{s}})^{-1} \boldsymbol{F} \mathrm{d}x_3 \right]^{-1} \tag{5.2.25}$$

The C_{ij} are layerwise constant. The calculation of $\boldsymbol{A}^{\mathrm{s}}$ demands an integration over layerwise defined polynomials of 4th order and can be just simple carried out by programming. For unsymmetrical laminates $\boldsymbol{F}(x_3)$ is defined by Eq. (5.2.12).

Hygrothermal effects have no influence on the transverse shear stresses. In the classical laminate theory for mechanical and hygrothermal loading as demonstrated in Sect. 4.2.5, the resultants $\boldsymbol{N}$ and $\boldsymbol{M}$ must be substituted by the effective resultants $\tilde{\boldsymbol{N}}$ and $\tilde{\boldsymbol{M}}$.

5.3 Shear Deformation Theory for Laminates and Sandwiches

The classical laminate theory allows us to calculate the stresses and strains with high precision for very thin laminates except in a little extended region near the free edges. The validity of the classical theory has been established by comparing theoretical results with experimental tests and with more exact solutions based on the general equations of the linear anisotropic elasticity theory.

If the width-to-thickness ratio is less about 20, the results derived from the classical theory show significant differences with the actual mechanical behavior and the modelling must be improved.

A first improvement is to include approximately the effect of shear deformation in the framework of a first-order displacement approach. A further improvement is possible by introducing correction factors for the transverse shear moduli.

The model used now has the same general form, as (5.1.2), for the displacements, but contrary to the classical theory, ψ_1 and ψ_2 are independent functions and a normal line to the middle surface of the composite remains straight under deformation, however it is not normal to the deformed middle plane. In the shear deformation theory the actual deformation state is approximated by 5 independent two-dimensional functions u, v, w, ψ_1, ψ_2, in the classical theory by 3 functions u, v, w, respectively. The strains are deduced from the displacements, (5.1.3). The components of the strains

$$\boldsymbol{\varepsilon}(x_1,x_2,x_3) = \boldsymbol{\varepsilon}(x_1,x_2) + x_3 \boldsymbol{\kappa}(x_1,x_2), \quad i = 1,2,6$$

again vary linearly through the thickness h and are given by

$$\begin{aligned} &\varepsilon_1 = \frac{\partial u}{\partial x_1}, \quad \varepsilon_2 = \frac{\partial v}{\partial x_2}, \quad \varepsilon_6 = \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}, \\ &\kappa_1 = \frac{\partial \psi_1}{\partial x_1}, \kappa_2 = \frac{\partial \psi_2}{\partial x_2}, \kappa_6 = \frac{\partial \psi_2}{\partial x_1} + \frac{\partial \psi_1}{\partial x_2} \end{aligned} \tag{5.3.1}$$

The components of the vector $\boldsymbol{\varepsilon}^{\mathrm{T}} = [\varepsilon_1 \ \varepsilon_2 \ \varepsilon_6]$ are not changed, however the components of the curvature vector $\boldsymbol{\kappa}^{\mathrm{T}} = [\kappa_1 \ \kappa_2 \ \kappa_6]$ are now expressed by the derivatives of the functions ψ_1, ψ_2. The stresses in the kth layer can be expressed by

$$\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \\ \sigma_4 \\ \sigma_5 \end{bmatrix}^{(k)} = \boldsymbol{Q}^{(k)} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \varepsilon_4 \\ \varepsilon_5 \end{bmatrix}^{(k)} = \boldsymbol{Q}^{(k)} \begin{bmatrix} \dfrac{\partial u}{\partial x_1} + x_3 \dfrac{\partial \psi_1}{\partial x_1} \\ \dfrac{\partial v}{\partial x_2} + x_3 \dfrac{\partial \psi_2}{\partial x_2} \\ \dfrac{\partial u}{\partial x_2} + \dfrac{\partial v}{\partial x_1} + x_3 \left(\dfrac{\partial \psi_2}{\partial x_1} + \dfrac{\partial \psi_1}{\partial x_2} \right) \\ \dfrac{\partial w}{\partial x_2} + \psi_2 \\ \dfrac{\partial w}{\partial x_1} + \psi_1 \end{bmatrix} \tag{5.3.2}$$

The stresses σ_1, σ_2 and σ_6 are superimposed on the extensional and the flexural stresses and vary linearly through a layer thickness, the stresses σ_4, σ_5 are, in contradiction to the equilibrium equations, constant through $h^{(k)}$. The strains $\varepsilon_1, \varepsilon_2, \varepsilon_6$ vary linearly and the strains $\varepsilon_4, \varepsilon_5$ constant through the laminate thickness h, i.e. they vary continuously through the total thickness. Unlike, the corresponding stresses $\sigma_1, \sigma_2, \sigma_6$ and σ_4, σ_5 vary linearly or remain constant, respectively, through each layer thickness $h^{(k)}$ only. Therefore is no stress continuity through the laminate thickness but stress jumps from ply to ply at their interfaces depending on the reduced stiffness $\boldsymbol{Q}$ and $\boldsymbol{Q}^{\mathrm{s}}$.

With the definition equations for the stress resultants $\boldsymbol{N}, \boldsymbol{M}, \boldsymbol{Q}^{\mathrm{s}}$ and the stiffness coefficients $A_{ij}, B_{ij}, D_{ij}, A^{\mathrm{s}}_{ij}$ for laminates (4.2.13) - (4.2.15) or sandwich (4.3.8) - (4.3.10), (4.3.12) - (4.3.14), respectively, the constitutive equation can be written in the condensed hypermatrix form, Eqs. (4.2.16)

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \\ \boldsymbol{Q}^{\mathrm{s}} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} & \boldsymbol{0} \\ \boldsymbol{B} & \boldsymbol{D} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{A}^{\mathrm{s}} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \\ \boldsymbol{\gamma}^{\mathrm{s}} \end{bmatrix} \tag{5.3.3}$$

The stretching, coupling and bending stiffness A_{ij}, B_{ij}, D_{ij} stay unchanged in comparison to the classical laminate theory. The shear stiffness are approximately given by

$$A^{\mathrm{s}}_{ij} = \sum_{k=1}^{n} C^{(k)}_{ij} h^{(k)}, \quad i, j = 4, 5 \tag{5.3.4}$$

The $C^{(k)}_{ij}$ are the constant shear moduli of the kth lamina. These approximated shear stiffness overestimate the shear stiffness since they are based on the assumption of constant transverse shear strains and also do not satisfy the transverse shear stresses vanishing at the top and bottom boundary layers.

The stiffness values can be improved with help of shear correction factors (Vlachoutsis, 1992; Altenbach, 2000; Gruttmann and Wagner, 2017). In this case the part of the constitutive equation relating to the resultants $\boldsymbol{N}, \boldsymbol{M}$ is not modified. The other part relating to transverse shear resultants $\boldsymbol{Q}^{\mathrm{s}}$ is modified by replacing the stiffness A_{ij}^{s} by $(kA)_{ij}^{\mathrm{s}}$. The constants k_{ij}^{s} are the shear correction factors. A very simple approach is to introduce a weighting function $f(x_3)$ for the distribution of the transverse shear stresses through the thickness h.

Assume a parabolic function $f(x_3)$

$$f(x_3) = \frac{5}{4}\left[1 - \left(\frac{x_3}{h/2}\right)^2\right] \tag{5.3.5}$$

and considering that for the kth layer

$$\sigma_4^{(k)} = Q_{44}^{(k)}\varepsilon_4 + Q_{45}^{(k)}\varepsilon_5, \quad \sigma_5^{(k)} = Q_{45}^{(k)}\varepsilon_4 + Q_{55}^{(k)}\varepsilon_5 \tag{5.3.6}$$

the transverse resultants are:

$$\begin{aligned}
Q_2 &= \sum_{k=1}^{n} \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \sigma_4^{(k)} f(x_3)\mathrm{d}x_3 \\
&= \frac{5}{4}\left\{\sum_{k=1}^{n} Q_{44}^{(k)}\varepsilon_4 \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \left[1 - \left(\frac{x_3}{h/2}\right)^2\right]\mathrm{d}x_3 + \sum_{k=1}^{n} Q_{45}^{(k)}\varepsilon_5 \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \left[1 - \left(\frac{x_3}{h/2}\right)^2\right]\mathrm{d}x_3\right\},
\end{aligned}$$

$$\begin{aligned}
Q_1 &= \sum_{k=1}^{n} \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \sigma_5^{(k)} f(x_3)\mathrm{d}x_3 \\
&= \frac{5}{4}\left\{\sum_{k=1}^{n} Q_{45}^{(k)}\varepsilon_4 \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \left[1 - \left(\frac{x_3}{h/2}\right)^2\right]\mathrm{d}x_3 + \sum_{k=1}^{n} Q_{55}^{(k)}\varepsilon_5 \int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \left[1 - \left(\frac{x_3}{h/2}\right)^2\right]\mathrm{d}x_3\right\}
\end{aligned}$$

The shear stiffness coefficients A_{ij}^{s} of the constitutive equations

$$Q_2 = A_{44}^{\mathrm{s}}\varepsilon_4 + A_{45}^{\mathrm{s}}\varepsilon_5, \quad Q_1 = A_{45}^{\mathrm{s}}\varepsilon_4 + A_{55}^{\mathrm{s}}\varepsilon_5 \tag{5.3.7}$$

are calculated by

$$\begin{aligned}
A_{ij}^{\mathrm{s}} &= \frac{5}{4}\sum_{k=1}^{n} Q_{ij}^{(k)}\left[\left(x_3^{(k)} - x_3^{(k-1)}\right) - \frac{4}{3h^2}\left({x_3^{(k)}}^3 - {x_3^{(k-1)}}^3\right)\right] \\
&= \frac{5}{4}\sum_{k=1}^{n} Q_{ij}^{(k)}\left[h^{(k)} - \frac{4}{h^2}h^{(k)}\left(\frac{{h^{(k)}}^2}{12} + {\bar{x}_3^{(k)}}^2\right)\right], \quad i,j = 4,5
\end{aligned} \tag{5.3.8}$$

This approach yields for the case of single layer with $Q_{44} = Q_{55} = G, Q_{45} = 0$ a shear correction factor $k^{\mathrm{s}} = 5/6$ for the shear stiffness Gh

$$A^{\mathrm{s}} = \frac{5}{4}G\left[h - \frac{4h}{h^2}\left(\frac{h^2}{12} + 0\right)\right] = \frac{5}{6}Gh \tag{5.3.9}$$

The weighting function (5.3.5) resulting in a shear correction factor k^{s} is consistent with the Reissner theory of shear deformable single layer plates (Reissner, 1944) and slightly differ from Mindlin's value (Mindlin, 1951).

A second method to determine shear correction factors consists of considering the strain energy per unit area of the composite. Some remarks on this method are given in Chaps. 7 and 8. However shear correction factors depend on the special loading and stacking conditions of a laminate and not the only factors is generally applicable.

A particularly physical foundation to improve the shear stiffness values $\boldsymbol{A}^{\mathrm{s}}$ is the equilibrium approach, Eq. (5.2.25). The sequence of calculation steps for determining improved transverse shear stresses in the frame of the FSDT are analogous to the CLT and shall be shortly repeated

- firstly, calculate the improved shear stiffness

$$\boldsymbol{A}^{\mathrm{s}} = \left[\int\limits_{(h)} \boldsymbol{F}^{\mathrm{T}}\boldsymbol{C}^{\mathrm{s}-1}\boldsymbol{F}\,\mathrm{d}x_3\right]^{-1} \tag{5.3.10}$$

- secondly, calculate the resultant transverse shear forces

$$\boldsymbol{Q}^{\mathrm{s}} = \boldsymbol{A}^{\mathrm{s}}\boldsymbol{\varepsilon}^{\mathrm{s}} \tag{5.3.11}$$

- thirdly, calculate the improved transverse shear stresses

$$\begin{aligned}
&\boldsymbol{\sigma}^{\mathrm{s}} = \boldsymbol{F}\boldsymbol{Q}^{\mathrm{s}}\\
&\boldsymbol{A}^{\mathrm{s}} = [A_{ij}], i,j = 5,4, \quad \boldsymbol{C}^{\mathrm{s}} = [C_{ij}], i,j = 5,4\\
&\boldsymbol{F} = \begin{bmatrix} F_{11} & F_{62}\\ F_{61} & F_{22}\end{bmatrix}, \quad \boldsymbol{Q}^{\mathrm{s}} = [Q_1^{\mathrm{s}}\; Q_2^{\mathrm{s}}]^{\mathrm{T}}, \quad \boldsymbol{\sigma} = [\sigma_5\; \sigma_4]^{\mathrm{T}}, \quad \boldsymbol{\varepsilon}^{\mathrm{s}} = [\varepsilon_5\; \varepsilon_4]^{\mathrm{T}}
\end{aligned} \tag{5.3.12}$$

Relying on the results of calculation improved transverse shear stresses $\boldsymbol{\sigma}^{\mathrm{s}}$, the transverse normal stress can be evaluated. The following equations explain the principal way. One starts with solving the equilibrium condition for σ_3, Eq. (2.2.1)

$$\sigma_3(x_3) = -\int\limits_{x_3=0}^{x_3}\left(\frac{\partial\sigma_5}{\partial x_1} + \frac{\partial\sigma_4}{\partial x_2}\right)\mathrm{d}x_3 + p_0 \tag{5.3.13}$$

p_0 denotes the transverse load at the starting point of integration.

With

$$\boldsymbol{F}(x_3) = \begin{bmatrix} F_{11} & F_{62} \\ F_{61} & F_{22} \end{bmatrix} = \begin{bmatrix} \boldsymbol{f}_1^{\mathrm{T}} \\ \boldsymbol{f}_2^{\mathrm{T}} \end{bmatrix} \tag{5.3.14}$$

we are able to replace the transverse shear stresses in Eq. (5.3.13) by Eq. (5.3.12)

$$\sigma_3(x_3) = -\left[\int\limits_{x_3=0}^{x_3} \boldsymbol{f}_1^{\mathrm{T}} \mathrm{d}x_3 \boldsymbol{Q}^{\mathrm{s}}_{,x_1} + \int\limits_{x_3=0}^{x_3} \boldsymbol{f}_2^{\mathrm{T}} \mathrm{d}x_3 \boldsymbol{Q}^{\mathrm{s}}_{,x_2}\right] + p_0 \tag{5.3.15}$$

Only the components of $\boldsymbol{f}_1$ and $\boldsymbol{f}_2$ depend on x_3 and therefore the derivatives of $\boldsymbol{Q}^{\mathrm{s}}$ remain unchanged by the integration process. Moreover, Eq. (5.2.12) demonstrates that only the partial stiffness $\tilde{\boldsymbol{A}}(x_3)$ and $\tilde{\boldsymbol{B}}(x_3)$ depend on x_3, but not the matrices $\boldsymbol{A},\boldsymbol{B}$ and $\boldsymbol{D}^*$. Therefore the integration of $\boldsymbol{F}(x_3)$ yields

$$\int\limits_{x_3=0}^{x_3} \boldsymbol{F}(x_3)\mathrm{d}x_3 = \left[\int\limits_{x_3=0}^{x_3} \tilde{\boldsymbol{A}}(x_3)\mathrm{d}x_3 \boldsymbol{A}^{-1}\boldsymbol{B} - \int\limits_{x_3=0}^{x_3} \tilde{\boldsymbol{B}}(x_3)\mathrm{d}x_3\right] \boldsymbol{D}^{*-1} = \hat{\boldsymbol{F}}(x_3) \tag{5.3.16}$$

For symmetrical laminates is the coupling matrix $\boldsymbol{B} \equiv \boldsymbol{0}$ and $\hat{\boldsymbol{F}}(x_3)$ can be simplified to

$$\int\limits_{x_3=0}^{x_3} \boldsymbol{F}(x_3)\mathrm{d}x_3 = -\int\limits_{x_3=0}^{x_3} \tilde{\boldsymbol{B}}(x_3)\mathrm{d}x_3 \boldsymbol{D}^{-1} = \hat{\boldsymbol{F}}(x_3) \tag{5.3.17}$$

Now, Eq. (5.3.15) can be transformed into

$$\sigma_3(x_3) = -\left[\hat{\boldsymbol{f}}_1^{\mathrm{T}} \boldsymbol{Q}^{\mathrm{s}}_{,x_1} + \hat{\boldsymbol{f}}_2^{\mathrm{T}} \mathrm{d}x_3 \boldsymbol{Q}^{\mathrm{s}}_{,x_2}\right] + p_0, \tag{5.3.18}$$

where

$$\hat{\boldsymbol{f}}_1^{\mathrm{T}} = [\hat{F}_{11}\ \hat{F}_{62}], \quad \hat{\boldsymbol{f}}_2^{\mathrm{T}} = [\hat{F}_{61}\ \hat{F}_{22}]$$

and

$$Q^{\mathrm{s}}_{2,x_2} = (\boldsymbol{A}^{\mathrm{s}}\boldsymbol{\varepsilon}^{\mathrm{s}})_{,x_2}$$

The boundary conditions of vanishing transverse shear stresses at both surfaces are fulfilled automatically. The boundary conditions for the transverse normal stresses must be regarded and are taken into account in the integration process.

Summarizing the considerations on single layers or smeared modelling of laminated structures it can be seen that an increasing number of higher order theories particularly for the analysis of laminated plates has been published. The vast majority falls into the class of plate theories known as displacement based ones. All consideration in this textbook are restricted to such theories. The term "higher order theories" indicates that the displacement distribution over the thickness is represented by polynomials of higher than first order. In general, a higher approximation will lead to better results but also requires more expensive computational effort and the accuracy improvement is often so little that the effort required to solve the more complicated equations is not justified. In addition, the mechanical interpretation of the boundary conditions for higher order terms is very difficult. The most used ESLT in engineering applications of composite structure elements is the FSDT. The CLT applications are limited to very thin laminates only, for in comparison to homogeneous isotropic plates, the values of the ratio thickness to minimum in-plane dimension to regard a plate as "thin" or as "moderate thick" must be considerably reduced. Generally, fibre-reinforced material is more susceptible to transverse shear than its homogeneous isotropic counterpart and reduces the range of applicability of CLT. Increasing in-plane stiffness may alternatively be regarded as relevant reduction of its transverse shear strength.

The FSDT yields mostly sufficient accurate results for the displacements and for the in-plane stresses. However, it may be recalled, as an example, that transverse shear and transverse normal stresses are main factors that cause delamination failure of laminates and therefore an accurate determination of the transverse stresses is needed.

In Sect. 5.3 it was demonstrated that one way to calculate the transverse stresses is an equilibrium approach in the frame of an extended 2D-modelling. Another relative simple method is to expand the FSDT from five to six unknown functions or degrees of freedom, respectively, by including an x_3-dependent term into the polynomial representation of the out-of-plane displacement $u_3(x_1,x_2,x_3)$. Several other possibilities can be found in the literature.

5.4 Layerwise Theories

Layerwise theories are developed for laminates or sandwiches with thick single layers. Layerwise displacement approximations provide a more kinematically correct representation of the displacement functions through the thickness including cross-sectional warping associated with the deformation of thick composite structures. So-called partial layerwise theories are mostly used which assume layerwise expansions for the in-plane displacement components only. Otherwise so-called full layerwise theories use expansions for all three displacement components. Compared with equivalent single layer models the partial layerwise model provides a more realistic description of the kinematics of composite laminates and the discrete-layer behavior of the in-plane components.

Assume a linear displacement approximation (5.1.2) for each layer

$$\begin{aligned} u_1^{(k)}(x_1,x_2,x_3) &= u^{(k)}(x_1,x_2) + x_3\psi_1^{(k)}(x_1,x_2), \\ u_2^{(k)}(x_1,x_2,x_3) &= v^{(k)}(x_1,x_2) + x_3\psi_2^{(k)}(x_1,x_2), \\ u_3^{(k)}(x_1,x_2,x_3) &= w(x_1,x_2) \end{aligned} \tag{5.4.1}$$

with $x_3^{(k-1)} \leq x_3 \leq x_3^{(k)}; k = 1,2,\ldots n$. A laminate with n layers is determined by $(4n+1)$ unknown functions $u^{(k)}, v^{(k)}, \psi_1^{(k)}, \psi_2^{(k)}, w; k = 1,2,\ldots,n$. The continuity conditions of the displacements at the layer interfaces yield $2(n-1)$ equations and the equilibrium for the transverse shear stresses yield additional $2(n-1)$ equations. With these $2 \cdot 2(n-1)$ equations the maximum number of the unknown functions can be eliminated and we have independent of the number of layers in all cases $(4n+1)-(4n-4)=5$ unknown trial functions. An equivalent single layer model in the first-order shear deformation theory and the partial layerwise model have the same number of functional degrees of freedom, which are 5. The modelling of laminates or sandwiches on the assumption of the partial layerwise theory is often used in the finite element method. A comparison of equivalent single layer and layerwise theories one can find in Reddy (1993).

Summarizing one can say for the class of partial or discrete layer-wise models that all analytical or numerical equations are two-dimensional and in comparison to a real three-dimensional modelling, their modelling and solution effort, respectively, is less time and cost consuming. The transverse normal displacement does not have a layerwise representation, but compared to the equivalent single layer modelling, the partial layerwise modelling provides more realistic description of the kinematics of composite laminates or sandwiches by introducing discrete layerwise transverse shear effects into the assumed displacement field.

Discrete layerwise theories that neglect transverse normal strain are not capable of accurately determining interlaminar stresses and modelling localized effects such as cutouts, free edges, delamination etc. Full or generalized layerwise theories include in contrast to the partial layerwise transverse shear and transverse normal stress effects.

Displacement based finite element models of partial and full layerwise theories have been developed and can be found in the literature. In Chap. 11 the exemplary consideration of finite beam and plate elements have been restricted to CLT and FSDT.

5.5 Problems

Exercise 5.1. The displacement field of a third order laminate (5.1.1) may defined by $\alpha = -c_0, \beta = 1, \gamma = 0, \delta = -c_1, \bar{\beta} = \bar{\gamma} = 0$.

1. Formulate the displacement equations and recover the displacement equations for the classical and the shear deformation laminate theory.
2. Introduce new variables $\phi_1 = \psi_1 - c_0 \partial w/\partial x_1, \phi_2 = \psi_2 - c_0 \partial w/\partial x_2$ and express the displacement field in terms of ϕ_1 and ϕ_2.
3. Substitute the displacements into the linear strain-displacement relations.
4. Formulate the equations for the transverse shear stresses σ_4, σ_5. Find the equations for c_1 so that the transverse shear stresses vanish at the top and the bottom of the laminate if $c_0 = 1$.

Solution 5.1. In the case of a third order displacement field one obtains the following answers:

1. The starting point is the displacement field

$$\begin{aligned}
u_1(x_1,x_2,x_3) &= u(x_1,x_2)+x_3\left[\psi_1(x_1,x_2)-c_0\frac{\partial w(x_1,x_2)}{\partial x_1}\right]-x_3^3c_1\chi_1(x_1,x_2),\\
u_2(x_1,x_2,x_3) &= v(x_1,x_2)+x_3\left[\psi_2(x_1,x_2)-c_0\frac{\partial w(x_1,x_2)}{\partial x_2}\right]-x_3^3c_1\chi_2(x_1,x_2),\\
u_3(x_1,x_2,x_3) &= w(x_1,x_2)
\end{aligned}$$

Classical laminate theory: $c_1 = 0, \psi_1 = 0, \psi_2 = 0, c_0 = 1$
First shear deformation theory: $c_0 = c_1 = 0$

2. The starting point is now another displacement field

$$\begin{aligned}
u_1(x_1,x_2,x_3) &= u(x_1,x_2)+x_3\phi_1(x_1,x_2)-c_1x_3^3\chi_1(x_1,x_2),\\
u_2(x_1,x_2,x_3) &= v(x_1,x_2)+x_3\phi_2(x_1,x_2)-c_1x_3^3\chi_2(x_1,x_2),\\
u_3(x_1,x_2,x_3) &= w(x_1,x_2)
\end{aligned}$$

3. Using the strain-displacement equation (2.2.3) and substitute equations b) we find

$$\begin{aligned}
\varepsilon_1 &= \frac{\partial u_1}{\partial x_1} = \frac{\partial u}{\partial x_1}+x_3\frac{\partial \phi_1}{\partial x_1}-x_3^3c_1\frac{\partial \chi_1}{\partial x_1}\\
&= \varepsilon_1^0+x_3\varepsilon_1^I+x_3^3\varepsilon_1^{II},\\
\varepsilon_2 &= \frac{\partial u_2}{\partial x_2} = \frac{\partial v}{\partial x_2}+x_3\frac{\partial \phi_2}{\partial x_2}-x_3^3c_1\frac{\partial \chi_2}{\partial x_2}\\
&= \varepsilon_2^0+x_3\varepsilon_2^I+x_3^3\varepsilon_2^{II},\\
\varepsilon_6 &= \frac{\partial u_2}{\partial x_1}+\frac{\partial u_1}{\partial x_2} = \frac{\partial v}{\partial x_1}+\frac{\partial u}{\partial x_2}+x_3\left(\frac{\partial \phi_2}{\partial x_1}+\frac{\partial \phi_1}{\partial x_2}\right)-x_3^3c_1\left(\frac{\partial \chi_2}{\partial x_1}+\frac{\partial \chi_1}{\partial x_2}\right)\\
&= \varepsilon_6^0+x_3\varepsilon_6^I+x_3^3\varepsilon_6^{II},\\
\varepsilon_4 &= \frac{\partial u_3}{\partial x_2}+\frac{\partial u_2}{\partial x_3} = \frac{\partial w}{\partial x_2}+\phi_2-3c_1x_3^2\chi_2\\
&= \varepsilon_4^0+x_3^2\varepsilon_4^{II},\\
\varepsilon_5 &= \frac{\partial u_3}{\partial x_1}+\frac{\partial u_1}{\partial x_3} = \frac{\partial w}{\partial x_1}+\phi_2-3c_1x_3^2\chi_1\\
&= \varepsilon_5^0+x_3^2\varepsilon_5^{II}
\end{aligned}$$

Note $\varepsilon_i^0 \equiv \varepsilon_i$.

4. The transverse shear stress in the kth layer of a laminate follow with (5.3.5) to

$$\begin{aligned}
\sigma_4^{(k)} &= Q_{44}^{(k)}\varepsilon_4+Q_{45}^{(k)}\varepsilon_5 = Q_{44}^{(k)}(\varepsilon_4^0+x_3^2\varepsilon_4^{II})+Q_{45}^{(k)}(\varepsilon_5^0+x_3^2\varepsilon_5^{II}),\\
\sigma_5^{(k)} &= Q_{45}^{(k)}\varepsilon_4+Q_{55}^{(k)}\varepsilon_5 = Q_{55}^{(k)}(\varepsilon_5^0+x_3^2\varepsilon_5^{II})+Q_{45}^{(k)}(\varepsilon_4^0+x_3^2\varepsilon_4^{II})
\end{aligned}$$

The transverse shear stresses shall vanish at the bottom and the top of the laminate, i.e. $\sigma_4^{(k)}(\pm h/2) = \sigma_5^{(k)}(\pm h/2) = 0$ if $k = 1$ or n.

$$
\begin{aligned}
Q_{44}^{(1)}\left(\varepsilon_4^0+\frac{h^2}{4}\varepsilon_4^{II}\right)+Q_{45}^{(1)}\left(\varepsilon_5^0+\frac{h^2}{4}\varepsilon_5^{II}\right)&=0,\\
Q_{44}^{(n)}\left(\varepsilon_4^0+\frac{h^2}{4}\varepsilon_4^{II}\right)+Q_{45}^{(n)}\left(\varepsilon_5^0+\frac{h^2}{4}\varepsilon_5^{II}\right)&=0,\\
Q_{55}^{(1)}\left(\varepsilon_5^0+\frac{h^2}{4}\varepsilon_5^{II}\right)+Q_{45}^{(1)}\left(\varepsilon_4^0+\frac{h^2}{4}\varepsilon_4^{II}\right)&=0,\\
Q_{55}^{(1)}\left(\varepsilon_5^0+\frac{h^2}{4}\varepsilon_5^{II}\right)+Q_{45}^{(1)}\left(\varepsilon_4^0+\frac{h^2}{4}\varepsilon_4^{II}\right)&=0
\end{aligned}
$$

$$
\Longrightarrow \varepsilon_4^0+\frac{h^2}{4}\varepsilon_4^{II}=0,\quad \varepsilon_5^0+\frac{h^2}{4}\varepsilon_5^{II}=0
$$

In view of the fact that for $c_0 = 1$ follows

$$
\begin{aligned}
\phi_2 = \psi_2-\frac{\partial w}{\partial x_2} &\Rightarrow \varepsilon_4=\psi_2-x_3^2 3c_1\chi_2,\\
\varepsilon_4^0+\frac{h^2}{4}\varepsilon_4^{II}=0 &\Rightarrow \psi_2=\frac{h^2}{4}3c_1\chi_2
\end{aligned}
$$

If $3c_1 = 4/h^2 \Rightarrow \chi_2 = \psi_2$. Analogously follow with $3c_1 = 4/h^2$ that $\chi_1 = \psi_1$, i.e

$$
\varepsilon_4^0+\frac{1}{3c_1}\varepsilon_4^{II}=\psi_2-\psi_2=0,\quad \varepsilon_5^0+\frac{1}{3c_1}\varepsilon_5^{II}=\psi_1-\psi_1=0
$$

The condition $1/3c_1 = h^2/4$, i.e. $c_1 = 4/3h^2$ is sufficient to make the transverse shear stresses σ_4 and σ_5 zero at the top and the bottom of the laminate.

Exercise 5.2. A symmetric cross-ply laminate $[0^0/90^0]_S$ has the properties $h = 1$ mm, $E'_1 = 141$ GPa, $E'_2 = 9,4$ GPa, $E'_4 \equiv G'_{23} = 3,2$ GPa, $E'_5 \equiv G'_{13} = E'_6 \equiv G'_{12} = 4,3$ GPa, $\nu'_{12} = 0,3$.

1. Using the simplified equations (5.2.8) to calculate the shear stresses $\sigma_5(x_3), \sigma_4(x_3)$ and sketch their distribution across the laminate thickness h for given transverse force resultants $Q_1 = \partial M_1/\partial x_1, Q_2 = \partial M_2/\partial x_2$ and $M_6 \equiv 0$.
2. Compare the average shear stiffness with the improved corrected stiffness values.

Solution 5.2. The solution can be obtained as follows.

1. The reduced stiffness matrix $\boldsymbol{Q}$ and the shear stiffness matrix $\boldsymbol{C} \equiv \boldsymbol{G}$ must be calculated for the four layers
   0^0-layers, $\nu'_{21} = \nu_{12}E'_2/E'_1$:

$$\boldsymbol{Q}_{[0^0]} \equiv \boldsymbol{Q}' = \begin{bmatrix} \dfrac{E_1'}{(1-\nu_{12}'\nu_{21}')} & \dfrac{\nu_{12}'E_2'}{(1-\nu_{12}'\nu_{21}')} & 0 \\ \dfrac{\nu_{12}'E_2'}{(1-\nu_{12}'\nu_{21}')} & \dfrac{E_2'}{(1-\nu_{12}'\nu_{21}')} & 0 \\ 0 & 0 & E_6' \end{bmatrix}$$

$$= \begin{bmatrix} 141,85 & 2,84 & 0 \\ 2,84 & 9,46 & 0 \\ 0 & 0 & 4,3 \end{bmatrix} \text{GPa},$$

$$\boldsymbol{G}_{[0^0]} \equiv \boldsymbol{G}' = \begin{bmatrix} G_{13}' & 0 \\ 0 & G_{23}' \end{bmatrix} = \begin{bmatrix} 4,3 & 0 \\ 0 & 3,2 \end{bmatrix} \text{GPa}$$

90^0-layers:

$$\boldsymbol{Q}_{[90^0]} = \begin{bmatrix} 9,46 & 2,84 & 0 \\ 2,84 & 141,85 & 0 \\ 0 & 0 & 4,3 \end{bmatrix} \text{GPa},$$

$$\boldsymbol{G}_{[90^0]} = \begin{bmatrix} 3,2 & 0 \\ 0 & 4,3 \end{bmatrix} \text{GPa}$$

The bending stiffness matrix follows with (4.2.15)

$$D_{ij} = \sum_{k=1}^{4} Q_{ij}^{(k)} \left[(\bar{x}_3^{(k)})^2 + (h^{(k)})^2/12 \right] h^{(k)},$$

$$\boldsymbol{D} = \begin{bmatrix} 9,654 & 0,207 & 0 \\ 0,207 & 1,379 & 0 \\ 0 & 0 & 0,314 \end{bmatrix} \text{GPa}\,\text{mm}^3$$

The corrected flexural stiffness matrix $\boldsymbol{D}^*$ (5.2.6) is identical $\boldsymbol{D}$ for symmetric laminates, i.e. $D^* = D$, and the $\boldsymbol{F}(x_3)$-matrix in (5.2.12) can be simplified

$$\boldsymbol{F}(x_3) = -\tilde{\boldsymbol{B}}(x_3)\boldsymbol{D}^{-1}$$

The inversion of the matrix $\boldsymbol{D}$ yields with $\Delta = 22,572$ the elements D_{11}^{-1} of the inverse matrix $\boldsymbol{D}^{-1}$

$$D_{11}^{-1} = D_{22}/\Delta, D_{22}^{-1} = D_{11}/\Delta, D_{12}^{-1} = D_{12}/\Delta, D_{66}^{-1} = (D_{66})^{-1}$$

$$\boldsymbol{D}^{-1} = \begin{bmatrix} 0,104 & -0,016 & 0 \\ -0,016 & 0,727 & 0 \\ 0 & 0 & 3,185 \end{bmatrix} [\text{GPa}\,\text{mm}^3]^{-1}$$

Using (5.2.6) the shearing coupling stiffness $\tilde{\boldsymbol{B}}(x_3)$ for the layers of the laminate can be calculated

$$\tilde{\boldsymbol{B}}_{[0^0]}(x_3) = \begin{bmatrix} 70,93 & 1,42 & 0 \\ 1,42 & 4,73 & 0 \\ 0 & 0 & 2,30 \end{bmatrix} \text{GPa} x_3^2 - \begin{bmatrix} 17,73 & 0,36 & 0 \\ 0,36 & 1,18 & 0 \\ 0 & 0 & 0,58 \end{bmatrix} \text{kN},$$

$$\tilde{\boldsymbol{B}}_{[90^0]}(x_3) = \begin{bmatrix} 4,73 & 1,42 & 0 \\ 1,42 & 70,93 & 0 \\ 0 & 0 & 2,30 \end{bmatrix} \text{GPa} x_3^2 - \begin{bmatrix} 13,60 & 0,36 & 0 \\ 0,36 & 5,32 & 0 \\ 0 & 0 & 0,57 \end{bmatrix} \text{kN}$$

and with $\boldsymbol{F}(x_3) = -\tilde{\boldsymbol{B}}(x_3)\boldsymbol{D}^{-1}$

$$\begin{bmatrix} \sigma_5(x_3) \\ \sigma_4(x_3) \end{bmatrix} = \begin{bmatrix} F_{11} & F_{62} \\ F_{61} & F_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix},$$

$$F_{[0^0]} = -\begin{bmatrix} 6,79 & 0 \\ 0 & 2,17 \end{bmatrix} x_3^2 + \begin{bmatrix} 1,70 & 0 \\ 0 & 0,54 \end{bmatrix} \text{mm}^{-1},$$

$$F_{[90^0]} = -\begin{bmatrix} 0,44 & 0 \\ 0 & 32,80 \end{bmatrix} x_3^2 + \begin{bmatrix} 1,30 & 0 \\ 0 & 2,46 \end{bmatrix} \text{mm}^{-1}$$

$$\begin{aligned} \sigma_{5[0^0]}(x_3) &= F_{11[0^0]}(x_3)Q_1 = (-6,79x_3^2 + 1,70)Q_1, \\ \sigma_{5[90^0]}(x_3) &= F_{11[90^0]}(x_3)Q_1 = (-0,44x_3^2 + 1,30)Q_1, \\ \sigma_{4[0^0]}(x_3) &= F_{22[0^0]}(x_3)Q_2 = (-2,17x_3^2 + 0,54)Q_2, \\ \sigma_{4[90^0]}(x_3) &= F_{22[90^0]}(x_3)Q_2 = (-32,80x_3^2 + 2,46)Q_2 \end{aligned}$$

The distribution of the shear stresses through the laminate thickness h is sketched in Fig. 5.2.

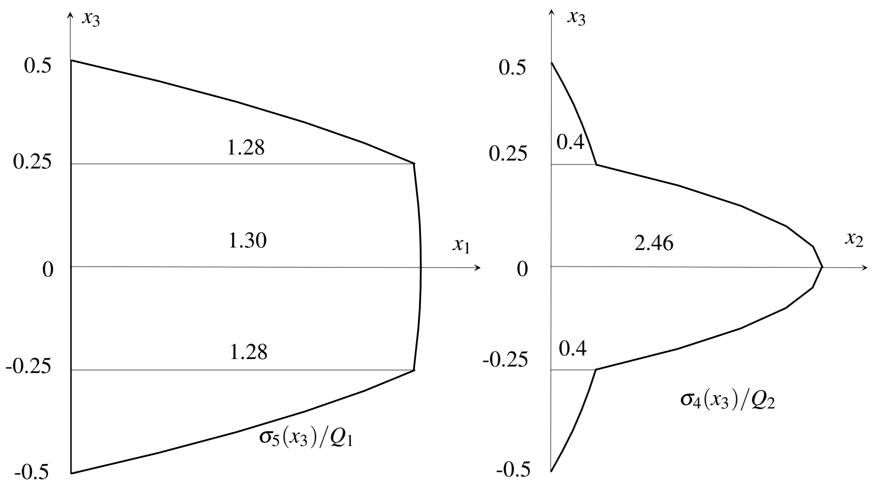


Fig. 5.2 Distribution of the shear stresses $\sigma_5(x_3)/Q_1$ and $\sigma_4(x_3)/Q_2$ across the laminate thickness

2. A simplified calculation of the average shear stiffness $\bar{A}^{\mathrm{s}}_{ij}$ yields (4.2.15)

$$\bar{A}^{\mathrm{s}}_{ij} = \sum_{k=1}^{4} G^{(k)}_{ij} h^{(k)} \Longrightarrow \bar{\boldsymbol{A}}^{\mathrm{s}} = \begin{bmatrix} 3,75 & 0 \\ 0 & 3,75 \end{bmatrix} \mathrm{GPa\,mm}$$

An improved shear stiffness matrix which include the transverse shear stress distribution follows with the help of the complementary strain energy W^*

$$\begin{aligned} W^* &= \frac{1}{2} \int\limits_{(h)} \boldsymbol{\sigma}^{\mathrm{sT}} \boldsymbol{G}'^{-1} \boldsymbol{\sigma}^{\mathrm{s}} \mathrm{d}x_3 \\ &= \frac{1}{2} \boldsymbol{Q}^{\mathrm{T}} \left[\int\limits_{(h)} \boldsymbol{F}^{\mathrm{T}} \boldsymbol{G}'^{-1} \boldsymbol{F} \mathrm{d}x_3 \right] \boldsymbol{Q} = \frac{1}{2} \boldsymbol{Q}^{\mathrm{T}} \boldsymbol{A}^{\mathrm{s}-1} \boldsymbol{Q} \end{aligned}$$

With

$$\begin{aligned} \int\limits_{0,25}^{0,5} \boldsymbol{F}^{\mathrm{T}}_{[0^0]} \boldsymbol{G}'^{-1} \boldsymbol{F}_{[0^0]} \mathrm{d}x_3 &= \left(\begin{bmatrix} 2,000 & 0 \\ 0 & 0,295 \end{bmatrix} x_3^5 \right. \\ &\quad - \begin{bmatrix} 1,667 & 0 \\ 0 & 0,246 \end{bmatrix} x_3^3 \\ &\quad \left. + \begin{bmatrix} 0,625 & 0 \\ 0 & 0,092 \end{bmatrix} x_3 \right) \mathrm{GPa}^{-1}, \\ \int\limits_{0}^{0,25} \boldsymbol{F}^{\mathrm{T}}_{[90^0]} \boldsymbol{G}'^{-1} \boldsymbol{F}_{[90^0]} \mathrm{d}x_3 &= \left(\begin{bmatrix} 0,012 & 0 \\ 0 & 46,69 \end{bmatrix} x_3^5 \right. \\ &\quad - \begin{bmatrix} 0,119 & 0 \\ 0 & 11,68 \end{bmatrix} x_3^3 \\ &\quad \left. + \begin{bmatrix} 0,529 & 0 \\ 0 & 1,314 \end{bmatrix} x_3 \right) \mathrm{GPa}^{-1} \end{aligned}$$

follows by the sum up over the four layers the improved matrix $\boldsymbol{A}^{\mathrm{s}}$

$$\boldsymbol{A}^{\mathrm{s}} = \begin{bmatrix} 3,01 & 0 \\ 0 & 2,54 \end{bmatrix} \mathrm{GPa}$$

The comparison of $\bar{\boldsymbol{A}}^{\mathrm{s}}$ and $\boldsymbol{A}^{\mathrm{s}}$ can be carried out in the form

$$\boldsymbol{k}^{\mathrm{s}} \bar{\boldsymbol{A}}^{\mathrm{s}} = \boldsymbol{A}^{\mathrm{s}}$$

which yields the shear correction vector

$$\boldsymbol{k}^{\mathrm{s}} = \begin{bmatrix} 0,7718 \\ 0,6513 \end{bmatrix}$$

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Chapter 6
Failure Mechanisms and Criteria

Failure of structural elements can be defined in a different manner. As in the case of buckling, a structural element may be considered failure though the material is still intact, but there are excessive deformations. In Chap. 6 failure will be considered to be the loss of integrity of the composite material itself.

The failure analysis procedures for metallic structures were well established a long time ago. In the case of monolithic materials stress concentrations, e.g. around notches and holes, cause localized failures. For brittle materials local failures may lead to fracture and therefore to a total loss of load-carrying capability. For ductile materials local failure may be in the form of yielding and remains localized, i.e., it is tolerated better than brittle failure. The fail-safe philosophy has been employed in the design of metallic structures and is standard in engineering applications. Similar procedures for composite materials are not well defined and are the object of intensive scientific research up to now. Failure of fibre-reinforced materials is a very complex topic. While it is important to understand the principal mechanisms of failure, for many applications it is impossible to detail each step of the failure process. Main causes of failure are design errors, fabrication and processing errors or unexpected service conditions. Design errors can be made in both material and structure. The stress level carried by each lamina in a laminate depends on the elastic moduli. This may cause large stress gradients between laminae which are oriented at considerably large angles to each other (e.g. 90^0). If the stress gradients are close to a limit value, fracture may occur. Such high levels of internal stresses in adjacent laminae may develop a result of external applied loads but also by temperature and moisture changes. Though manufacturing control and material inspection tests are carried out, structural composites with abnormalities can be produced. The mechanical properties of composites may be significantly reduced by high temperature variations, impact damage, etc. Service anomalies can include improper operation, faulty maintenance, overloads or environmental incurred damage.

If structural loadings produce local discontinuities inside the material we speak of a crack. Micro-cracking is considered as the nucleation of micro-cracks at the microscopic level starting from defects and may cause the initiation of material fracture. Macro-cracking is the propagation of a fracture by the creation of new

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0_6

fracture surfaces at the macroscopic level. For composite materials the fraction initiation is generally well developed before a change in the macroscopic behavior can be observed.

If in a laminate macro-cracks occur, it may not be catastrophic, for it is possible that some layers fail first and the composite continues to take more loads until all laminae fail. Failed laminae may still contribute to the stiffness and strength of the laminate. Laminate failure estimations are based on procedures for finding the successive loads between the first and the last ply failure of the laminate. The failure of a single layer plays a central function in failure analysis of laminates.

In this section the elastic behavior of laminae is primarily discussed from a macroscopic point of view. But in the case of failure estimations and strength analysis of a lamina it is important to understand the underlying failure mechanisms within the constituents of the composites and their effect to the ultimate macroscopic behavior. For this reason some considerations on micro-mechanic failure mechanisms are made first and then failure criteria are discussed more in detail.

Summarizing one can say that the ability of failure prediction is a key aspect in design of engineering structures. The first step is to consider what is meant by failure. Material failure of metallic structures is mostly related with material yielding or rupture, but with composites it is more complex. Therefore research is ongoing in developing failure mechanisms and failure criteria for unidirectional fibre laminae and their laminates and in evaluating the accuracy of the failure criteria.

6.1 Fracture Modes of Laminae

Composite fracture mechanisms are rather complex because of their anisotropic nature. The failure modes depend on the applied loads and on the distribution of reinforcements in the composites. In continuous fibre reinforced composites the types of fracture may be classified by these basic forms:

- Intralaminar fracture,
- interlaminar fracture,
- translaminar fracture.

Intralaminar fracture is located inside a lamina, interlaminar fracture shows the failure developed between laminae and translaminar fracture is oriented transverse to the laminate plane. Inter- and intralaminar fractures occur in a plane parallel to that of the fibre reinforcement.

Composite failure is a gradual process. The degradation of a layer results in a redistribution of stresses in the laminate. It is characterized by different local failure modes

- The failure is dominated by fiber degradation, e.g. rupture, microbuckling, etc.
- The failure is dominated by matrix degradation, e.g. crazing.
- The failure is dominated by singularities at the fiber-matrix interface, e.g. crack propagation, delamination, etc.

Failure modes of sandwich material may be characterized by

- Tensile failure of the sandwich faces
- Wrinkling failure of the faces due to compressive stresses. Wrinkling is characterized by the eigenmodes of buckling faces.
- Shear failure of core or adhesive failure between core and face.
- Crushing failure of the face and core at a support or tensile respectively shear failure at fasteners.

The following considerations are restricted to the strength of an unidirectional layer and to the development of reliable criteria for the predicting of the failure of laminae and laminates. The failure criteria in engineering applications are mainly of a phenomenological character, i.e. analytical approximations of experimental results, e.g. by curve fitting.

The fracture of a UD-lamina is the result of the accumulation of various elementary fracture mechanisms:

- Fibre fracture,
- transverse matrix fracture,
- longitudinal matrix fracture
- fracture of the fibre-matrix interface.

Figure 6.1 illustrates various fracture modes of a single layer. In the fibre direction, as a tensile load is applied, Fig. 6.1a, failure is due to fibre tensile fracture.

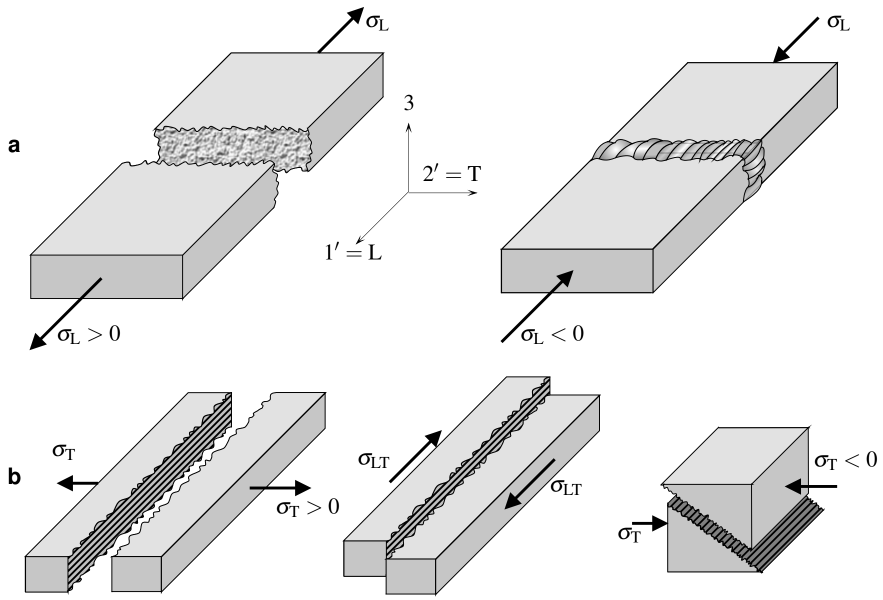

Fig. 6.1 Fracture modes of a single layer in the case of elementary load states. **a** Fibre fracture by pure tension $\sigma_L > 0$ or compression $\sigma_L < 0$ (micro-buckling), **b** Matrix fracture by pure tension $\sigma_T > 0$, pure shearing σ_{LT} and pure compression $\sigma_T < 0$

One fibre breaks and the load is transferred through the matrix to the neighboring fibres which are overloaded and fail too. The failure propagates rapidly with small increasing load. Otherwise a tensile fracture perpendicular to the fibres, Fig. 6.1b, due a combination of different micromechanical failure mechanisms: tensile failure of matrix material, tensile failure of fibres across the diameters, failure of the interface between fibre and matrix. The shear strength, Fig. 6.1b, is limited by the shear strength of the matrix material, the shear strength between the fibre and the matrix, etc. Figure 6.2 shows the basic strength parameters of a unidirectional lamina referred to the principal material axes. For in-plane loading of a lamina 5 strength parameters are necessary, but it is important to have in mind that for composite materials different strength parameters are measured for tensile and for compression tests. If the shear stresses act parallel or transverse to the fibre orientation there is

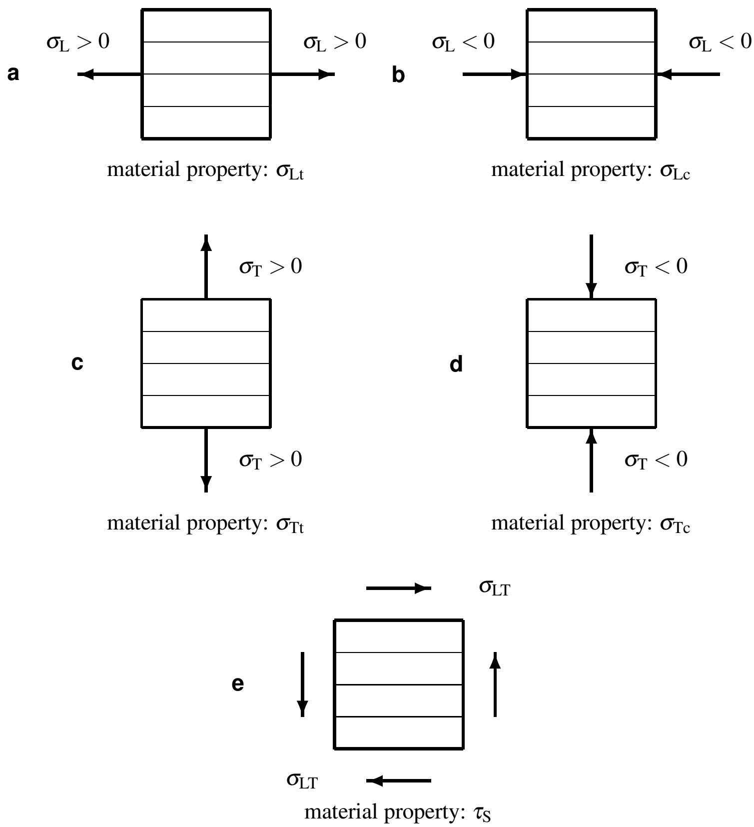

Fig. 6.2 Basic strength parameters. **a** Longitudinal tensile strength σ_{Lt}, **b** Longitudinal compressive strength σ_{Lc}, **c** Transverse tensile strength σ_{Tt}, **d** Transverse compressive strength σ_{Tc}, **e** In-plane (intralaminar) shear strength τ_S

no influence of the load direction (Fig. 6.3a). Otherwise the positive shear stress $\sigma_6 > 0$ causes tensile in L-direction and compression in T-direction and vice versa for $\sigma_6 < 0$ and other strength parameters are standard. The required experimental characterization is relatively simple for the parameters σ_{Lt} and σ_{Tt}, but more complicated for the strength parameters σ_{Lc}, σ_{Tc} and τ_S.

In the case of laminates, besides the basic failure mechanisms for a single layer, such as fibre fracture, longitudinal and transverse matrix fraction, fibre-matrix debonding, etc. described above, another new fracture mode occurs. This mode is called delamination and consists of separation of layers from one another. Through-the-thickness variation of stresses may be caused even if a laminate is loaded by uniform in-plane loads. Generally, the matrix material that holds the laminae of a laminate together has substantially smaller strength than the in-plane strength of the layers. Stresses perpendicular to the interface between laminae may cause breaking of the bond between the layers in mostly localized, small regions. However, even if

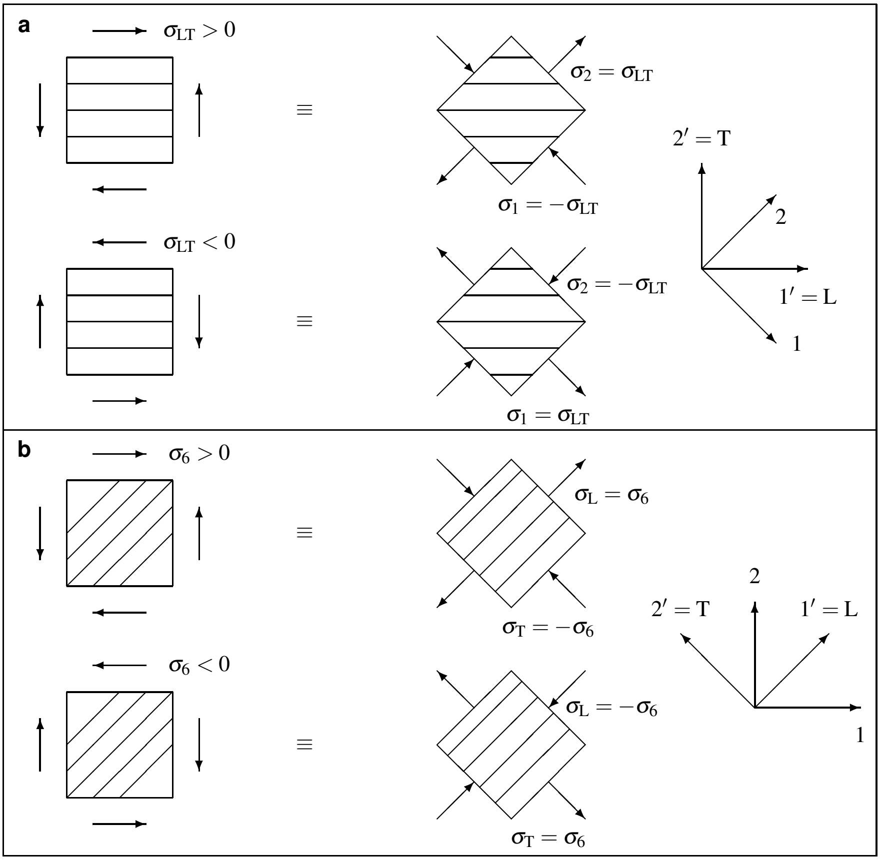

Fig. 6.3 In-plane shear. **a** Positive and negative shear stresses along the principal material axes, **b** Positive and negative shear stresses at 45^0 with the principal material axis

the size of such delaminations is small they may affect the integrity of a laminate and can degrade their in-plane load-carrying capability. Therefore, in practical engineering applications it is important to calculate the interlaminar normal and shear stresses σ_3, σ_4 and σ_5 and to check interlaminar failure too.

The definition of failure may change from case to case and depends on the composite material and the kind of loads. For composite material, such as UD-laminates, the end of the elastic domain is associated with the development of micro-cracking. But in the first stage, the initiated cracks do not propagate and their development changes the stiffness of the material very gradually but the degradation is irreversible. In the following section failure criteria for laminae will be discussed first to allow the designer to have an evaluation of the mechanical strength of laminae. Secondly, concepts for laminate failure are considered.

6.2 Failure Criteria

Failure criteria for composites are many and varied. In their simplest form they are similar, in principle, to those used for isotropic materials, e.g. maximum stress/strain and distortional energy theories. The major difference between isotropic materials and unidirectional fibrous composite materials is the directional dependence of the strength on a macrosopic scale. It is important to realize that failure criteria are purely empirical. Their purpose is to define a failure envelope by using a minimum number of test data. Generally, these experimental data are obtained from relatively simple uniaxial and pure shear tests. Combined stress tests are more difficult to perform and should be, if possible, not included in the determination failure envelopes.

We shall start by considering a single lamina before moving on to discuss failure of laminates. Longitudinal tension or pressure, transverse tension or pressure and shear are the five basic modes of failure of a lamina. Generally the strength in the principal material axes are regarded as the fundamental parameters defining failure. When the lamina is loaded at an angle to the fibres one has to determine the stresses in the principal directions and compare them with the fundamental strength parameters. Failure criteria usually grouped in literature into three different classes: limit criteria, interactive criteria and hybrid criteria which combine selected aspects of limit and interactive methods. In the following we only discuss selected criteria of the first two classes.

Failure criteria for homogeneous isotropic materials are well established. Macromechanical failure theories for composite materials have been developed by extending and adapting isotropic failure theories to account for anisotropy in stiffness and strength of the composite. All theories can be expressed as functions of the basic strength parameters referred to the principal material axes (Fig. 6.2). Some criteria do not account for interaction of stress components while others do. Some interaction criteria require additional strength parameters obtained by more expended biaxial experimental tests.

Laminate failure criteria are applied on a ply-by-ply basis and the load-carrying capability of the entire composite is predicted by the laminate or sandwich theories given in Chaps. 4 and 5. A laminate may be assumed to have failed when the strength criterion of any one of its laminae is reached (first-ply failure). However, the failure of a single layer not necessarily leads to a total fracture of the laminate structure. Criteria of an on-axis lamina can be determined with relative easily. Off-axis criteria can be obtained by coordinate transformations of stresses or strains. Based on the ply-by-ply analysis first-ply failure and last-ply failure concepts can be developed.

Failure criteria have been established in the case of a layer. Of all failure criteria available, the following four are considered representative and more widely used:

- Maximum stress theory
- Maximum strain theory
- Deviatoric or distorsion strain energy criteria of Tsai-Hill[1]
- Interactive tensor polynomial criterion of Tsai-Wu[2]

Maximum stress and maximum strain criteria assume no stress interaction while the other both include full stress interaction. In the maximum stress theory, failure occurs when at least one stress component along one of the principal material axes exceeds the corresponding strength parameter in that direction

$$\begin{aligned} \sigma_{\mathrm{L}} &= \sigma_{\mathrm{Lt}}, \qquad \sigma_{\mathrm{L}} > 0, \\ \sigma_{\mathrm{T}} &= \sigma_{\mathrm{Tt}}, \qquad \sigma_{\mathrm{T}} > 0, \\ \sigma_{\mathrm{L}} &= \sigma_{\mathrm{Lc}}, \qquad \sigma_{\mathrm{L}} < 0, \\ \sigma_{\mathrm{T}} &= \sigma_{\mathrm{Tc}}, \qquad \sigma_{\mathrm{T}} < 0, \\ |\sigma_{\mathrm{LT}}| &= \tau_{\mathrm{S}}, \end{aligned} \tag{6.2.1}$$

Note that failure can occur for more than one reason. A layer failure does not occur if

$$\begin{aligned} -\sigma_{\mathrm{Lc}} &< \sigma_{\mathrm{L}} < \sigma_{\mathrm{Lt}}, \\ -\sigma_{\mathrm{Tc}} &< \sigma_{\mathrm{T}} < \sigma_{\mathrm{Tt}}, \\ -\tau_{\mathrm{S}} &< \sigma_{\mathrm{LT}} < \tau_{\mathrm{S}} \end{aligned} \tag{6.2.2}$$

For a two-dimensional state of normal stresses, i.e. $\sigma_{\mathrm{L}} \neq 0, \sigma_{\mathrm{T}} \neq 0, \sigma_{\mathrm{LT}} = 0$, the failure envelope, Fig. 6.4, takes the form of a rectangle. In the case of off-axis tension or compression of a UD-lamina, Fig. 6.5, the transformed stresses are

$$\begin{array}{rclcl} \sigma_{\mathrm{L}} &=& \sigma_1 \cos^2\theta &=& \sigma_1 c^2 \\ \sigma_{\mathrm{T}} &=& \sigma_1 \sin^2\theta &=& \sigma_1 s^2 \\ \sigma_{\mathrm{LT}} &=& -\sigma_1 \sin\theta\cos\theta &=& -\sigma_1 sc \end{array} \Rightarrow \begin{array}{rcl} \sigma_1 &=& \sigma_{\mathrm{L}}/c^2 \\ \sigma_1 &=& \sigma_{\mathrm{T}}/s^2 \\ \sigma_1 &=& -\sigma_{\mathrm{LT}}/sc \end{array} \tag{6.2.3}$$

and the maximum stress criteria is expressed as follows

[1] Rodney Hill (*11 June 1921, Stourton, Leeds - † 2 February 2011, Yorkshire) - applied mathematician and a former Professor of Mechanics of Solids at Gonville and Caius College, Cambridge, UK

[2] Edward Ming-Chi Wu (*30 September 1938 - †3 June 2009) - US-American engineer

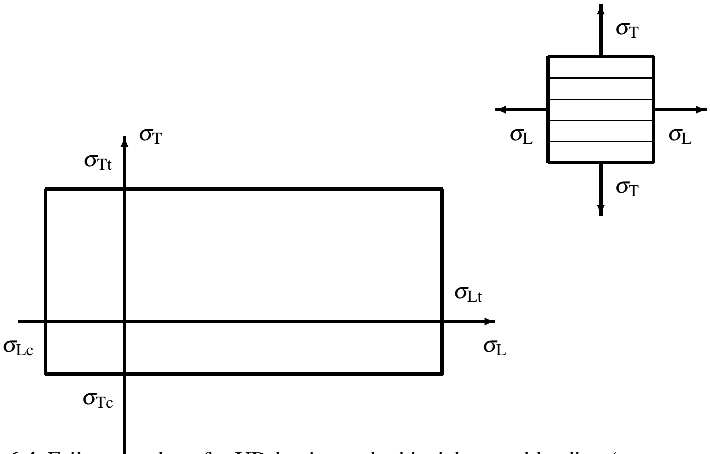

Fig. 6.4 Failure envelope for UD-lamina under biaxial normal loading (max. stress criterion)

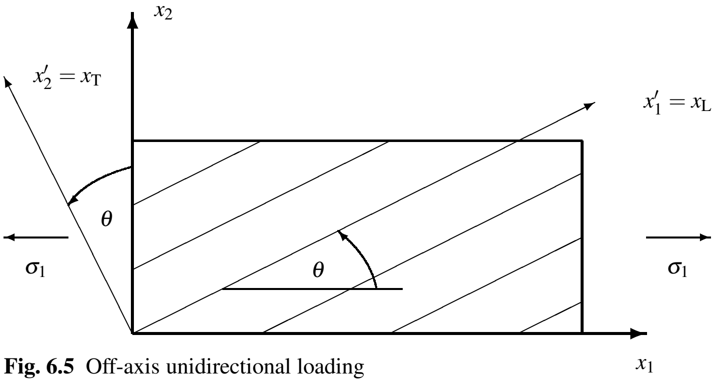

Fig. 6.5 Off-axis unidirectional loading

$$\begin{aligned} -\sigma_{Lc} &< \sigma_1 c^2 < \sigma_{Lt}, \\ -\sigma_{Tc} &< \sigma_1 s^2 < \sigma_{Tt}, \\ -\tau_S &< \sigma_1 sc < \tau_S \end{aligned} \tag{6.2.4}$$

The ultimate strength for σ_1 corresponds to the smallest of the following six values

$$\begin{aligned} &\sigma_{1t} = \sigma_{Lt}/c^2,\ \sigma_{1t} = \sigma_{Tt}/s^2,\ \sigma_{1t} = \tau_S/sc,\ \sigma_1 > 0, \\ &\sigma_{1c} = \sigma_{Lc}/c^2,\ \sigma_{1c} = \sigma_{Tc}/s^2,\ \sigma_{1c} = \tau_S/sc,\ \sigma_1 < 0 \end{aligned} \tag{6.2.5}$$

The failure modes depend on the corresponding ultimate strength σ_{1u}

$$\begin{aligned} &\sigma_{1u} = \sigma_{Lt}/c^2 \text{ fibre failure}, \\ &\sigma_{1u} = \sigma_{Tt}/s^2 \text{ transverse normal stress failure}, \\ &\sigma_{1u} = \tau_S/sc \quad \text{in-plane shear failure} \end{aligned}$$

In the more general case of off-axis loading, the stress transformation rule, Table 4.1, is used

$$\begin{bmatrix} \sigma_1' \equiv \sigma_L \\ \sigma_2' \equiv \sigma_T \\ \sigma_6' \equiv \sigma_{LT} \end{bmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix} \tag{6.2.6}$$

Because of the orthotropic symmetry, shear strength is independent of the sign of σ_{LT} (Fig. 6.3) and there are five independent failure modes in the maximum stress criterion. There is no interaction among the modes although in reality the failure processes are highly interacting. The maximum stress theory may be applicable for brittle modes of failure of material, e.g. follow from transverse or longitudinal tension $(\sigma_L > 0, \sigma_T > 0)$.

The maximum strain theory is quite similar to the maximum stress theory. Now the strains are limited instead of the stresses. Failure of a lamina occurs when at least one of the strain components along the principal material axes exceeds the corresponding ultimate strain in that direction

$$\begin{aligned} \varepsilon_L &= \varepsilon_{Lt} & \varepsilon_L &> 0, \\ \varepsilon_T &= \varepsilon_{Tt} & \varepsilon_T &> 0, \\ \varepsilon_L &= \varepsilon_{Lc} & \varepsilon_L &< 0, \\ \varepsilon_T &= \varepsilon_{Tc} & \varepsilon_T &< 0, \\ |\varepsilon_{LT}| &= \varepsilon_S \end{aligned} \tag{6.2.7}$$

The lamina failure does not occur if

$$\begin{aligned} -\varepsilon_{Lc} &< \varepsilon_L < \varepsilon_{Lt}, \\ -\varepsilon_{Tc} &< \varepsilon_T < \varepsilon_{Tt}, \\ -\varepsilon_S &< \varepsilon_{LT} < \varepsilon_S \end{aligned} \tag{6.2.8}$$

In the case of unidirectional off-axis tension or compression (Fig. 6.5), the stress relations are given by (6.2.3). For the in-plane stress state strains in the principal material axes are

$$\begin{bmatrix} \varepsilon_L \\ \varepsilon_T \\ \varepsilon_{LT} \end{bmatrix} = \begin{bmatrix} S_{11}' & S_{12}' & 0 \\ S_{12}' & S_{22}' & 0 \\ 0 & 0 & S_{66}' \end{bmatrix} \begin{bmatrix} \sigma_L \\ \sigma_T \\ \sigma_{LT} \end{bmatrix} \tag{6.2.9}$$

By associating (6.2.3) and (6.2.6) and expressing the compliance parameters S_{ij}' as functions of the engineering moduli in the principal directions, E_L, E_T, G_{LT}, ν_{LT}, ν_{TL}, it follows that

$$\begin{aligned} \varepsilon_L &= \frac{1}{E_L}(c^2 - \nu_{LT}s^2)\sigma_1, \\ \varepsilon_T &= \frac{1}{E_T}(s^2 - \nu_{TL}c^2)\sigma_1, \\ \varepsilon_{LT} &= -\frac{1}{G_{LT}}sc\sigma_1 \end{aligned} \tag{6.2.10}$$

The maximum strain and the maximum stress criteria must lead to identical values in the cases of longitudinal loading and $\theta = 0^0$ or transverse unidirectional loading and $\theta = 90^0$. The identity of the shear equations is given in both cases. This implies that

$$\varepsilon_{\mathrm{Lt}} = \frac{\sigma_{\mathrm{Lt}}}{E_{\mathrm{L}}},\ \varepsilon_{\mathrm{Lc}} = -\frac{\sigma_{\mathrm{Lc}}}{E_{\mathrm{L}}},\ \varepsilon_{\mathrm{Tt}} = \frac{\sigma_{\mathrm{Tt}}}{E_{\mathrm{T}}},\ \varepsilon_{\mathrm{Tc}} = -\frac{\sigma_{\mathrm{Tc}}}{E_{\mathrm{T}}},\ \varepsilon_{\mathrm{S}} = \frac{\tau_{\mathrm{S}}}{G_{\mathrm{LT}}} \tag{6.2.11}$$

and the maximum strain criterion may be rewritten as follows

$$\begin{array}{lll} -\sigma_{\mathrm{Lc}} & < \sigma_1(c^2 - \nu_{\mathrm{LT}}s^2) & < \sigma_{\mathrm{Lt}}, \\ -\sigma_{\mathrm{Tc}} & < \sigma_1(s^2 - \nu_{TL}c^2) & < \sigma_{\mathrm{Tt}}, \\ -\tau_{\mathrm{S}} & < \sigma_1 sc & < \tau_{\mathrm{S}} \end{array} \tag{6.2.12}$$

By comparing Eqs. (6.2.4) and (6.2.12) we establish that the two criteria differ by the introduction of the Poisson's ratio ν_{LT} in the strain criterion. In practice these terms modify the numerical results slightly. In the special case of a two-dimensional stress state $\sigma_{\mathrm{L}} \neq 0, \sigma_{\mathrm{T}} \neq 0, \sigma_{\mathrm{LT}} = 0$, compare Fig. 6.4, the failure envelope takes the form of a parallelogram for the maximum strain criterion, Fig. 6.6.

One of the first interactive criteria applied to anisotropic materials was introduced by Hill. For a two-dimensional state of stress referred to the principal stress directions, von Mises[3] developed a deviatoric or distortional energy criterion for isotropic ductile metals (von Mises, 1913) which can be presented for the two-dimensional stress state as

$$\sigma_I^2 + \sigma_{II}^2 - \sigma_I\sigma_{II} = \sigma_{\mathrm{eq}}$$

or in a general reference system

$$\sigma_1^2 + \sigma_2^2 - \sigma_1\sigma_2 + 3\sigma_6^2 = \sigma_{\mathrm{eq}}$$

σ_I, σ_{II} are principal stresses, σ_{eq} the equivalent stress. This criterion was extended in von Mises (1928) and modified for the case of orthotropic ductile materials by Hill (1948)

$$A\sigma_1^2 + B\sigma_2^2 + C\sigma_1\sigma_2 + D\sigma_6^2 = 1 \tag{6.2.13}$$

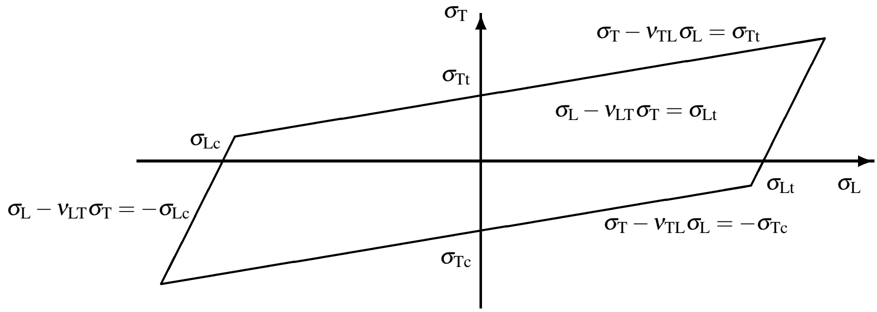

Fig. 6.6 Failure envelope for UD-lamina under biaxial normal loading (max. strain criterion)

[3] Richard Edler von Mises (*19 April 1883 Lemberg, Austria-Hungary (now Lviv, Ukraine) - † 14 July 1953 Boston, Massachusetts) - mathematician who worked on solid mechanics, fluid mechanics, aerodynamics, aeronautics, statistics and probability theory, one of founders of the journal Zeitschrift für Angewandte Mathematik und Mechanik

A, B, C, D are material parameters. Equation (6.2.13) cannot be defined as distorsion energy, since in anisotropy distorsion and dilatation energies are not separated. The criterion (6.2.13) was applied to UD-laminae by Tsai and Wu (1971)

$$A\sigma_{\mathrm{L}}^2 + B\sigma_{\mathrm{T}}^2 + C\sigma_{\mathrm{L}}\sigma_{\mathrm{T}} + D\sigma_{\mathrm{LT}}^2 = 1 \tag{6.2.14}$$

The material parameters A, B, C, D can be identified by tests with acting basic loadings

$$\begin{aligned}
\sigma_{\mathrm{L}} &= \sigma_{\mathrm{LU}}, \ \sigma_{\mathrm{T}} = 0, \qquad \sigma_{\mathrm{LT}} = 0 \ \Rightarrow A = \frac{1}{\sigma_{\mathrm{LU}}^2},\\
\sigma_{\mathrm{L}} &= 0, \qquad \sigma_{\mathrm{T}} = \sigma_{\mathrm{TU}}, \ \sigma_{\mathrm{LT}} = 0 \ \Rightarrow B = \frac{1}{\sigma_{\mathrm{TU}}^2},\\
\sigma_{\mathrm{L}} &= 0, \qquad \sigma_{\mathrm{T}} = 0, \qquad \sigma_{\mathrm{LT}} = \tau_{\mathrm{U}} \Rightarrow D = \frac{1}{\tau_{\mathrm{U}}^2}
\end{aligned} \tag{6.2.15}$$

In dependence on the failure mode, the superscript U must be substituted by t, c or s and denotes the ultimate value of stress at failure.

The remaining parameter C must be determined by a biaxial test. The C-term yields the interaction between the normal stresses. Under equal biaxial normal loading $\sigma_{\mathrm{L}} = \sigma_{\mathrm{T}} \neq 0, \sigma_{\mathrm{LT}} = 0$ it can be assumed that the failure follows the maximum stress criterion, i.e failure will occur when the transverse stress reaches the transverse strength σ_{TU} which is much lower than the longitudinal strength σ_{LU}. Equation (6.2.14) yields

$$\left(\frac{\sigma_{\mathrm{L}}}{\sigma_{\mathrm{LU}}}\right)^2 + \left(\frac{\sigma_{\mathrm{T}}}{\sigma_{\mathrm{TU}}}\right)^2 + C\sigma_{\mathrm{T}}^2 = 1, \quad \sigma_{\mathrm{T}} = \sigma_{\mathrm{TU}} \Longrightarrow C = -\frac{1}{\sigma_{\mathrm{LU}}^2} \tag{6.2.16}$$

The Tsai-Hill criterion in the case of plane stress state and on-axis loading may be written

$$\left(\frac{\sigma_{\mathrm{L}}}{\sigma_{\mathrm{LU}}}\right)^2 + \left(\frac{\sigma_{\mathrm{T}}}{\sigma_{\mathrm{TU}}}\right)^2 - \frac{\sigma_{\mathrm{L}}\sigma_{\mathrm{T}}}{\sigma_{\mathrm{LU}}^2} + \left(\frac{\sigma_{\mathrm{LT}}}{\tau_{\mathrm{U}}}\right)^2 = 1 \tag{6.2.17}$$

In the case of tension or compression off the principal material directions, Fig. 6.5, the Tsai-Hill criterion becomes

$$\left(\frac{\sigma_1 c^2}{\sigma_{\mathrm{LU}}}\right)^2 + \left(\frac{\sigma_1 s^2}{\sigma_{\mathrm{TU}}}\right)^2 - \left(\frac{\sigma_1 cs}{\sigma_{\mathrm{LU}}}\right)^2 + \left(\frac{\sigma_1 sc}{\tau_{\mathrm{U}}}\right)^2 = 1 \tag{6.2.18}$$

and the strength parameter σ_{1U} in x_1-direction is

$$\begin{aligned}
\left(\frac{1}{\sigma_{1\mathrm{U}}}\right)^2 &= \left(\frac{c^2}{\sigma_{\mathrm{LU}}}\right)^2 + \left(\frac{s^2}{\sigma_{\mathrm{TU}}}\right)^2 + \left(\frac{1}{\tau_{\mathrm{U}}^2} - \frac{1}{\sigma_{\mathrm{LU}}^2}\right)c^2 s^2\\
&\approx \left(\frac{c^2}{\sigma_{\mathrm{LU}}}\right)^2 + \left(\frac{s^2}{\sigma_{\mathrm{TU}}}\right)^2 + \left(\frac{cs}{\tau_{\mathrm{U}}^2}\right)^2
\end{aligned} \tag{6.2.19}$$

The approximated form presumes $\sigma_{\mathrm{LU}} \gg \tau_{\mathrm{U}}$.

The Tsai-Hill criterion is a single criterion instead of the three subcriteria required in maximum stress and strain theories. It allows considerable interaction among the strain components and for ductile material the failure estimation agrees well with experimental results.

Gol'denblat and Kopnov (1965) proposed a tensor polynomial criterion. Tsai and Wu modified this criterion by assuming the existence of a failure surface in stress space. They took into account only the first two terms of the polynomial criterion and postulated that fracture of an anisotropic material occurs when the following equation is satisfied

$$a_{ij}\sigma_{ij} + a_{ijkl}\sigma_{ij}\sigma_{kl} = 1 \tag{6.2.20}$$

or in a contracted notation

$$a_i\sigma_i + a_{ij}\sigma_i\sigma_j = 1 \tag{6.2.21}$$

We are interested in the case of an orthotropic composite material, i.e. a unidirectional lamina, subjected to plane stress state, and the Tsai-Wu criterion may be expressed as

$$\begin{aligned} a_{\mathrm{L}}\sigma_{\mathrm{L}} + a_{\mathrm{T}}\sigma_{\mathrm{T}} + a_{\mathrm{S}}\sigma_{\mathrm{S}} + a_{\mathrm{LL}}\sigma_{\mathrm{L}}^2 + a_{\mathrm{TT}}\sigma_{\mathrm{T}}^2 + \\ a_{\mathrm{SS}}\sigma_{\mathrm{S}}^2 + 2a_{\mathrm{LT}}\sigma_{\mathrm{L}}\sigma_{\mathrm{T}} + 2a_{\mathrm{LS}}\sigma_{\mathrm{L}}\sigma_{\mathrm{S}} + 2a_{\mathrm{TS}}\sigma_{\mathrm{T}}\sigma_{\mathrm{S}} = 1 \end{aligned} \tag{6.2.22}$$

Equation (6.2.22) is written in the on-axis system and $\nu_{\mathrm{LT}} \equiv a_{\mathrm{S}}$.

The linear terms take account the actual differences between composite material behavior under tension and compression. The term $a_{\mathrm{LT}}\sigma_{\mathrm{L}}\sigma_{\mathrm{T}}$ represents independent interaction among the stresses σ_{L} and σ_{T} and the remaining quadratic terms describe an ellipsoid in stress space. Since the strength of a lamina loaded under pure shear stress τ_{S} in the on-axis system is independent of the sign of the shear stress, all linear terms in σ_{S} must vanish

$$a_{\mathrm{S}} = a_{\mathrm{LS}} = a_{\mathrm{TS}} = 0 \tag{6.2.23}$$

Then the Tsai-Wu criterion for a single layer in on-axis system has the form

$$a_{\mathrm{L}}\sigma_{\mathrm{L}} + a_{\mathrm{T}}\sigma_{\mathrm{T}} + a_{\mathrm{LL}}\sigma_{\mathrm{L}}^2 + a_{\mathrm{TT}}\sigma_{\mathrm{T}}^2 + a_{\mathrm{SS}}\sigma_{\mathrm{S}}^2 + 2a_{\mathrm{LT}}\sigma_{\mathrm{L}}\sigma_{\mathrm{T}} = 1 \tag{6.2.24}$$

The four quadratic terms in (6.2.24) correspond to the four independent elastic characteristics of orthotropic materials, the linear terms allow the distinction between tensile and compressive strength. The coefficients of the quadratic Tsai-Wu criterion are obtained by applying elementary basic loading conditions to the lamina

$$\begin{matrix} \sigma_{\mathrm{L}} = \sigma_{\mathrm{Lt}}, \quad \sigma_{\mathrm{T}} = \sigma_{\mathrm{S}} = 0 \\ \sigma_{\mathrm{L}} = -\sigma_{\mathrm{Lc}}, \; \sigma_{\mathrm{T}} = \sigma_{\mathrm{S}} = 0 \end{matrix} \Rightarrow \begin{matrix} a_{\mathrm{L}}\sigma_{\mathrm{Lt}} + a_{\mathrm{LL}}\sigma_{\mathrm{Lt}}^2 = 1 \\ -a_{\mathrm{L}}\sigma_{\mathrm{Lc}} + a_{\mathrm{LL}}\sigma_{\mathrm{Lc}}^2 = 1 \end{matrix} \Rightarrow \begin{matrix} a_{\mathrm{L}} = \dfrac{1}{\sigma_{\mathrm{Lt}}} - \dfrac{1}{\sigma_{\mathrm{Lc}}} \\ a_{\mathrm{LL}} = \dfrac{1}{\sigma_{\mathrm{Lt}}\sigma_{\mathrm{Lc}}} \end{matrix}$$

$$\begin{matrix} \sigma_{\mathrm{T}} = \sigma_{\mathrm{Tt}}, \quad \sigma_{\mathrm{L}} = \sigma_{\mathrm{S}} = 0 \\ \sigma_{\mathrm{T}} = -\sigma_{\mathrm{Tc}}, \; \sigma_{\mathrm{L}} = \sigma_{\mathrm{S}} = 0 \end{matrix} \Rightarrow \begin{matrix} a_{\mathrm{T}}\sigma_{\mathrm{Tt}} + a_{\mathrm{TT}}\sigma_{\mathrm{Tt}}^2 = 1 \\ -a_{\mathrm{T}}\sigma_{\mathrm{Tc}} + a_{\mathrm{TT}}\sigma_{\mathrm{Tc}}^2 = 1 \end{matrix} \Rightarrow \begin{matrix} a_{\mathrm{T}} = \dfrac{1}{\sigma_{\mathrm{Tt}}} - \dfrac{1}{\sigma_{\mathrm{Tc}}} \\ a_{\mathrm{TT}} = \dfrac{1}{\sigma_{\mathrm{Tt}}\sigma_{\mathrm{Tc}}} \end{matrix}$$

$$\sigma_S = \tau_S,\ \sigma_L = \sigma_T = 0 \ \Rightarrow\ a_{SS}\tau_S^2 = 1 \ \Rightarrow\ a_{SS} = \frac{1}{\tau_S^2} \tag{6.2.25}$$

The remaining coefficient a_{LT} must be obtained by biaxial testing

$$\begin{aligned} &\sigma_L = \sigma_T = \sigma_U, \quad \sigma_S = 0 \Rightarrow \\ &(a_L + a_T)\sigma_U + (a_{LL} + a_{TT} + 2a_{LT})\sigma_U^2 = 1 \end{aligned} \tag{6.2.26}$$

σ_U is the experimentally measured strength under equal biaxial tensile loading $\sigma_L = \sigma_T$.

In many cases the interaction coefficient is not critical and is given approximately. A sufficient approximation is in this case

$$a_{LT} \approx -\frac{1}{2}\sqrt{a_{LL}a_{TT}} \tag{6.2.27}$$

The Tsai-Wu criterion may also be formulated in strain space.

Summarizing the considerations on interactive failure criteria lead: The Tsai-Hill and the Tsai-Wu failure criteria are quadratic interaction criteria which have the general form

$$F_{ij}\sigma_i\sigma_j + F_i\sigma_i = 1, \quad i, j = \mathrm{L}, \mathrm{T}, \mathrm{S} \tag{6.2.28}$$

F_{ij} and F_i are strength parameters and σ_i, σ_j the on axis stress components.

For plane stress state six strength parameters $F_{LL}, F_{TT}, F_{SS}, F_{LT}, F_L, F_T$ are required for implementation of the failure criterion, $F_{LS} = F_{TS} = F_S = 0$, see Eq. (6.2.23). Five of these strength parameters are conventional tensile, compressive or shear strength terms which can be measured in a conventional experimental test programme. The strength parameter F_{LT} is more difficult to obtain, since a biaxial test is necessary and such test is not easy to perform. The two-dimensional representation of the general quadratic criterion (6.2.28) in the stress space can be given in the equation below

$$\frac{\sigma_L^2}{\sigma_{Lt}\sigma_{Lc}} + \frac{\sigma_T^2}{\sigma_{Tt}\sigma_{Tc}} + \frac{\sigma_{LT}^2}{\tau_S^2} + 2F_{LT}\sigma_L\sigma_T + \left(\frac{1}{\sigma_{Lt}} - \frac{1}{\sigma_{Lc}}\right)\sigma_L + \left(\frac{1}{\sigma_{Tt}} - \frac{1}{\sigma_{Tc}}\right)\sigma_T = 1 \tag{6.2.29}$$

Equation (6.2.29) reduces, e.g., for

$$\sigma_{Lt} = \sigma_{Lc}, \sigma_{Tt} = \sigma_{Tc}, F_{LT} = -\frac{1}{2\sigma_{Lt}^2}$$

to the Tsai-Hill criterion, for

$$\sigma_{Lt} \neq \sigma_{Lc}, \sigma_{Tt} \neq \sigma_{Tc}, F_{LT} = -\frac{1}{2\sigma_{Lt}\sigma_{Lc}}$$

to the Hoffman criterion, and for

$$\sigma_{Lt} \neq \sigma_{Lc}, \sigma_{Tt} \neq \sigma_{Tc}, F_{LT} = -\frac{1}{2\sqrt{\sigma_{Lt}\sigma_{Lc}\sigma_{Tt}\sigma_{Tc}}}$$

to the Tsai-Wu criterion. Hoffman's criterion is a simple generalization of the Hill criterion that allows different tensile and compressive strength parameters (Hoffman, 1967).

If one defines dimensionless stresses as

$$\sigma_{\mathrm{L}}^{*} = \sqrt{F_{\mathrm{LL}}}\sigma_{\mathrm{L}}, \sigma_{\mathrm{T}}^{*} = \sqrt{F_{\mathrm{TT}}}\sigma_{\mathrm{T}}, \sigma_{\mathrm{LT}}^{*} = \sqrt{F_{\mathrm{SS}}}\sigma_{\mathrm{LT}}$$

and normalized strength coefficients as

$$F_{\mathrm{L}}^{*} = F_{\mathrm{L}}/\sqrt{F_{\mathrm{LL}}}, F_{\mathrm{T}}^{*} = F_{\mathrm{T}}/\sqrt{F_{\mathrm{T}}}, F_{\mathrm{LT}}^{*} = F_{\mathrm{LT}}/\sqrt{F_{\mathrm{LL}}F_{\mathrm{TT}}}$$

Equation (6.2.29) can be rewritten as

$$\sigma_{\mathrm{L}}^{*2} + \sigma_{\mathrm{T}}^{*2} + \sigma_{\mathrm{LT}}^{*2} + 2F_{\mathrm{LT}}^{*}\sigma_{\mathrm{L}}^{*}\sigma_{\mathrm{T}}^{*2} + F_{\mathrm{L}}^{*}\sigma_{\mathrm{L}}^{*2} + F_{\mathrm{T}}^{*}\sigma_{\mathrm{T}}^{*2} = 1 \tag{6.2.30}$$

Note that in the case of isotropic materials with $\sigma_{\mathrm{Lt}} = \sigma_{\mathrm{Lc}} = \sigma_{\mathrm{Tt}} = \sigma_{\mathrm{Tc}}$ follow $F_{\mathrm{L}}^{*} = F_{\mathrm{T}}^{*} = 0$. There the principal stress state will have $\sigma_{\mathrm{LT}}^{*} = \sigma_{\mathrm{LT}} = 0$. Equation (6.2.30) reduces with $F_{\mathrm{LT}}^{*} = -\frac{1}{2}$ to the known von Mises criterion.

Using the above failure criteria the possibility of a lamina failing can be determined, for example. In the maximum stress criterion, the lamina failes if any of the inequalities (6.2.4) are violated. However, the criterion does not give information about how much the load can be increased by if the lamina is safe or how much it can be decreased if the lamina has failed. To overcome this problem, strength ratios are defined as

$$R = \frac{\text{maximum load which can be applied}}{\text{load applied}} \tag{6.2.31}$$

This definition is applicable to all failure criteria. If $R > 1$, then the lamina is safe and the applied load can be increased by a factor of R. If $R < 1$ the lamina is unsafe and the applied load needs to be reduced. A value of $R = 1$ implies the failure load. The stress ratio factor assumes that the material is linear elastic, for each state of stress there is a corresponding state of strain and all components of stress and strain increase by the same proportion.

Summarizing the discussion above, the strength ratio for the four criteria can be formulated:

Maximum stress criterion

$$\begin{aligned}
R_{\mathrm{Lt}}\sigma &= \sigma_{\mathrm{Lt}}/\sigma_{\mathrm{L}}, && \sigma_{\mathrm{L}} > 0 \text{ Strength factor fibre fracture,}\\
R_{\mathrm{Tt}}\sigma &= \sigma_{\mathrm{Tt}}/\sigma_{\mathrm{T}}, && \sigma_{\mathrm{T}} > 0 \text{ Strength factor matrix fracture,}\\
R_{\mathrm{Lc}}\sigma &= \sigma_{\mathrm{Lc}}/|\sigma_{\mathrm{L}}|, && \sigma_{\mathrm{L}} < 0 \text{ Strength factor micro-buckling,}\\
R_{\mathrm{Tc}}\sigma &= \sigma_{\mathrm{Tc}}/|\sigma_{\mathrm{T}}|, && \sigma_{\mathrm{T}} < 0 \text{ Strength factor matrix fracture,}\\
R_{\mathrm{S}}\sigma &= \tau_{\mathrm{S}}/|\sigma_{\mathrm{LT}}|, && \text{Strength factor matrix fracture}
\end{aligned} \tag{6.2.32}$$

Maximum strain criterion

$$\begin{aligned}
R_{\mathrm{Lt}}\varepsilon &= \varepsilon_{\mathrm{Lt}}/\varepsilon_{\mathrm{L}}, & \varepsilon_{\mathrm{L}} &> 0,\\
R_{\mathrm{Tt}}\varepsilon &= \varepsilon_{\mathrm{Tt}}/\varepsilon_{\mathrm{T}}, & \varepsilon_{\mathrm{T}} &> 0,\\
R_{\mathrm{Lc}}\varepsilon &= \varepsilon_{\mathrm{Lc}}/|\varepsilon_{\mathrm{L}}|, & \varepsilon_{\mathrm{L}} &< 0,\\
R_{\mathrm{Tc}}\varepsilon &= \varepsilon_{\mathrm{Tc}}/|\varepsilon_{\mathrm{T}}|, & \varepsilon_{\mathrm{T}} &< 0,\\
R_{\mathrm{S}}\varepsilon &= \varepsilon_{\mathrm{S}}/|\varepsilon_{\mathrm{LT}}|
\end{aligned} \qquad (6.2.33)$$

Tsai-Hill-criterion
Only one strength ratio can be introduced

$$\left(\frac{R^{\mathrm{TH}}\sigma_{\mathrm{L}}}{\sigma_{\mathrm{LU}}}\right)^2 + \left(\frac{R^{\mathrm{TH}}\sigma_{\mathrm{T}}}{\sigma_{\mathrm{TU}}}\right)^2 - \frac{R^{\mathrm{TH}}\sigma_{\mathrm{L}}R^{\mathrm{TH}}\sigma_{\mathrm{T}}}{\sigma_{\mathrm{LU}}^2} + \left(\frac{R^{\mathrm{TH}}\sigma_{\mathrm{LT}}}{\tau_{\mathrm{U}}}\right)^2 = 1 \qquad (6.2.34)$$

With the ultimate strength $\sigma_{\mathrm{LU}}, \sigma_{\mathrm{TU}}$ for tension and compression the strength ratio R^{TH} follows from

$$\frac{1}{(R^{\mathrm{TH}})^2} = \left(\frac{\sigma_{\mathrm{L}}}{\sigma_{\mathrm{LU}}}\right)^2 + \left(\frac{\sigma_{\mathrm{T}}}{\sigma_{\mathrm{TU}}}\right)^2 - \frac{\sigma_{\mathrm{L}}\sigma_{\mathrm{T}}}{\sigma_{\mathrm{LU}}^2} + \left(\frac{\sigma_{\mathrm{LT}}}{\tau_{\mathrm{U}}}\right)^2$$

Tsai-Wu-criterion
The Tsai-Hill and the Tsai-Wu criterion define only one strength ratio R^{TW}

$$(a_{\mathrm{L}}\sigma_{\mathrm{L}} + a_{\mathrm{T}}\sigma_{\mathrm{T}})R^{\mathrm{TW}} + (a_{\mathrm{LL}}\sigma_{\mathrm{L}}^2 + a_{\mathrm{TT}}\sigma_{\mathrm{T}}^2 + a_{\mathrm{SS}}\sigma_{\mathrm{S}}^2 + 2a_{\mathrm{LT}}\sigma_{\mathrm{L}}\sigma_{\mathrm{T}})R^{\mathrm{TW}^2} = 1$$

or in symbolic notation

$$AR^{\mathrm{TW}} + B(R^{\mathrm{TW}})^2 = 1 \Rightarrow (R^{\mathrm{TW}})^2 + \frac{A}{B}R^{\mathrm{TW}} = \frac{1}{B}$$

with the solutions

$$R_{1/2}^{\mathrm{TW}} = -\frac{1}{2}\frac{A}{B} \pm \sqrt{\frac{1}{4}\frac{A^2}{B^2} + \frac{1}{B}} = \frac{1}{2B}\left(-A \pm \sqrt{A^2 + 4B}\right)$$

R^{TW} must be positive

$$R^{\mathrm{TW}} = \sqrt{A^2 + 4B} - A/2B \qquad (6.2.35)$$

The procedure for laminate failure estimation on the concept of first ply and last ply failure is given as follows:

1. Use laminate analysis to find the midplane strains and curvatures depending on the applied mechanical and hygrothermical loads.
2. Calculate the local stresses and strains in each lamina under the assumed load.
3. Use the ply-by-ply stresses and strains in lamina failure theory to find the strength ratios. Multiplying the strength ratio to the applied load gives the load level of the failure of the first lamina. This load may be called the first ply failure load. Using the conservative first-ply-failure concept stop here, otherwise go to step 4.

4. Degrade approximately fully the stiffness of damaged plies. Apply the actual load level of previous failure.
5. Start again with step 3. to find the strength ratios in the undamaged laminae. If $R > 1$ multiply the applied load by the strength ratio to find the load level of the next ply failure. If $R < 1$, degrade the stiffness and strength characteristics of all damaged lamina.
6. Repeat the steps above until all plies have failed. That is the last-ply-failure concept.

The laminate failure analysis can be subdivided into the following four parts. The first-ply-failure concept demands only one run through, the last-ply-failure requires several iterations with degradation of lamina stiffness.

Failure analysis of laminates in stress space:

Step 1

Calculate the stiffnesses

$$\boldsymbol{Q}'^{(k)} = \begin{bmatrix} Q'_{11} & Q'_{12} & 0 \\ Q'_{12} & Q'_{22} & 0 \\ 0 & 0 & Q'_{66} \end{bmatrix} \equiv \begin{bmatrix} Q_{\mathrm{LL}} & Q_{\mathrm{LT}} & 0 \\ Q_{\mathrm{LT}} & Q_{\mathrm{TT}} & 0 \\ 0 & 0 & Q_{\mathrm{SS}} \end{bmatrix}$$

of all k single layers in on-axis system with help of the layer moduli $E_{\mathrm{L}}^{(k)}, E_{\mathrm{T}}^{(k)}, \nu_{\mathrm{TL}}^{(k)}, G_{\mathrm{TL}}^{(k)}$ and the layer thicknesses $h^{(k)}$

Transformation of the reduced stiffnesses $\boldsymbol{Q}'^{(k)}$ of single layers in on-axis system to the reduced stiffnesses $\boldsymbol{Q}^{(k)}$ of single layers in off-axis system

$$\boldsymbol{Q}^{(k)} = (\boldsymbol{T}^{\varepsilon'})^{\mathrm{T}} \boldsymbol{Q}'^{(k)} \boldsymbol{T}^{\varepsilon'}$$

Calculate the laminate stiffnesses $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{D}$

$$A_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)},\ B_{ij} = \frac{1}{2} \sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)^2} - x_3^{(k-1)^2} \right) = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)} \overline{x}_3^{(k)},$$

$$D_{ij} = \frac{1}{3} \sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)^3} - x_3^{(k-1)^3} \right) = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)} \left(\overline{x}_3^{(k)^2} + \frac{1}{12} h^{(k)^2} \right),$$

$$\overline{x}_3^{(k)} = \frac{1}{2} \left(x_3^{(k)} + x_3^{(k-1)} \right)$$

Inversion of the matrices $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{D}$

Calculate the compliance matrices $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{d}$ of the laminate

$$\boldsymbol{a} = \boldsymbol{A}^* - \boldsymbol{B}^* \boldsymbol{D}^{*-1} \boldsymbol{C}^*,\ \boldsymbol{b} = \boldsymbol{B}^* \boldsymbol{D}^{*-1},\ \boldsymbol{c} = -\boldsymbol{D}^{*-1} \boldsymbol{C}^*,\ \boldsymbol{d} = \boldsymbol{D}^{*-1},$$

$$\boldsymbol{A}^* = \boldsymbol{A}^{-1},\ \boldsymbol{B}^* = -\boldsymbol{A}^{-1} \boldsymbol{B},\ \boldsymbol{C}^* = \boldsymbol{B} \boldsymbol{A}^{-1},\ \boldsymbol{D}^* = \boldsymbol{D} - \boldsymbol{B} \boldsymbol{A}^{-1} \boldsymbol{B}$$

Step 2

Calculation of the laminate stress resultants $\boldsymbol{N}$ and $\boldsymbol{M}$ by structural analysis of beam or plate structures

Step 3.

Calculate the laminate strains $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} + x_3\boldsymbol{\kappa}$

$$\begin{bmatrix} \boldsymbol{\varepsilon} \\ \cdots \\ \boldsymbol{\kappa} \end{bmatrix} = \begin{bmatrix} \boldsymbol{a} \,\vdots\, \boldsymbol{b} \\ \cdots\cdot \\ \boldsymbol{c} \,\vdots\, \boldsymbol{d} \end{bmatrix} \begin{bmatrix} \boldsymbol{N} \\ \cdots \\ \boldsymbol{M} \end{bmatrix}$$

and the strains for all laminae at lamina interfaces

$\boldsymbol{\varepsilon}^{(k)} = \boldsymbol{\varepsilon}_0 + x_3^{(k)}\boldsymbol{\kappa}, k = 0, 1, 2, \dots, n$

Calculate the stresses for all interface surfaces of single layers

$$\left.\begin{array}{ll} \boldsymbol{\sigma}^{(k)-} = \boldsymbol{Q}^{(k)}\boldsymbol{\varepsilon}^{(k-1)}, & \text{bottom surface of lamina } k \\ \boldsymbol{\sigma}^{(k)+} = \boldsymbol{Q}^{(k)}\boldsymbol{\varepsilon}^{k}, & \text{top surface of lamina } k \end{array}\right\}, \; k = 0, 1, 2, \dots, n$$

Transformation of the interface stresses $\boldsymbol{\sigma}^{(k)-}, \boldsymbol{\sigma}^{(k)+}$

to the on-axis system of layer k k = 0, 1, 2, ..., n

Step 4

Failure analysis based on a selected failure criterion in stress space

Summarizing the strength ratios concept to the general quadratic interaction criteria Eq. (6.2.28) we formulate with the maximum values of stresses

$$F_{ij}\sigma_i^{\prime\max}\sigma_j^{\prime\max} + F_i\sigma_i^{\prime\max} = 1$$

Substituting $R\sigma_i^{\prime\text{applied}}$ for $\sigma_i^{\prime\max}$ yield the quadratic equation for the strength ratio R

$$(F_{ij}\sigma_i\sigma_j)R^2 + (F_i\sigma_i)R - 1 = 0$$

or

$$aR^2 + bR - 1 = 0, \quad a = F_{ij}\sigma_i\sigma_j, b = F_i\sigma_i \tag{6.2.36}$$

The strength ratio R is equal to the positive quadratic root

$$R = -\frac{b}{2a} + \sqrt{\left(\frac{b}{2a}\right)^2 + \frac{1}{a}}$$

As considered above this approach is easy to use because the resulting ratio provides a linear scaling factor, i.e.
if $R \leq 1$ failure occurs,
if $R > 1$, e.g. $R = 2$, the safety factor is 2 and the load can be doubled or the laminate thickness reduced by 0.5 before failure occurs.

The same strength ratio can be determined from the equivalent quadratic criterion in the strain space. With $\boldsymbol{\sigma} = \boldsymbol{Q\varepsilon}$ follows, e.g. with Eqs. (6.2.24) - (6.2.27) the Tsai-Wu criterion in the strain space as

$$b_{\mathrm{L}}\varepsilon_{\mathrm{L}} + b_{\mathrm{T}}\varepsilon_{\mathrm{T}} + b_{\mathrm{LL}}\varepsilon_{\mathrm{L}}^2 + b_{\mathrm{TT}}\varepsilon_{\mathrm{T}}^2 + b_{\mathrm{SS}}\varepsilon_{\mathrm{S}}^2 + 2b_{\mathrm{LT}}\varepsilon_{\mathrm{L}}\varepsilon_{\mathrm{T}} = 1 \tag{6.2.37}$$

with

$$
\begin{aligned}
b_{\mathrm{L}} &= a_{\mathrm{L}}Q_{\mathrm{LL}} + a_{\mathrm{T}}Q_{\mathrm{LT}},\\
b_{\mathrm{T}} &= a_{\mathrm{T}}Q_{\mathrm{TT}} + a_{\mathrm{L}}Q_{\mathrm{LT}},\\
b_{\mathrm{LL}} &= a_{\mathrm{LL}}Q_{\mathrm{LL}}^2 + a_{\mathrm{TT}}Q_{\mathrm{LT}}^2 + 2a_{\mathrm{LT}}Q_{\mathrm{LL}}Q_{\mathrm{LT}},\\
b_{\mathrm{TT}} &= a_{\mathrm{TT}}Q_{\mathrm{TT}}^2 + a_{\mathrm{LL}}Q_{\mathrm{LT}}^2 + 2a_{\mathrm{LT}}Q_{\mathrm{TT}}Q_{\mathrm{LT}},\\
b_{\mathrm{LT}} &= a_{\mathrm{LL}}Q_{\mathrm{LL}}Q_{\mathrm{LT}} + a_{\mathrm{TT}}Q_{\mathrm{TT}}Q_{\mathrm{LT}} + a_{\mathrm{LT}}(Q_{\mathrm{LT}}^2 + Q_{\mathrm{LL}}Q_{\mathrm{TT}})
\end{aligned}
\tag{6.2.38}
$$

In the more general form analogous to the strength ratio equation is

$$
\begin{aligned}
(G_{ij}\varepsilon_i\varepsilon_j)R^2 + (G_i\varepsilon_i)R - 1 &= 0,\\
cR^2 + dR - 1 &= 0,\\
R = -\frac{d}{c} + \sqrt{\left(\frac{d}{2c}\right) + \frac{1}{c}}
\end{aligned}
\tag{6.2.39}
$$

To determine R from this equivalent quadratic criterion the strain space may be preferred, because laminae strains are either uniform or vary linearly across each lamina thickness.

As considered above, the most widely used interlaminar failure criteria are the maximum stress criterion, the maximum strain criterion and the quadratic failure criteria as a generalization of the von Mises yield criterion, in particular the Tsai-Hill and the Tsai-Wu criterion. The interlaminar failure modes can be fibre breaking, fibre buckling, fibre pullout, fibre-matrix debonding or matrix cracking. The prediction of the First-Ply Failure with one of the above mentioned criteria is included in nearly all available analysis tools for layered fibre reinforced composites.

Interlaminar failure, i.e. failure of the interface between adjacent plies, is a delamination mode. Delamination failure can have different causes. Weakly bonded areas impact initial delamination in the inner region of a laminate, whereas delamination along free edges is a result of high interlaminar stresses. Free edges delamination is one of the most important failure modes in layered composite structures. Along a free edge a tri-axial stress state is present and must be considered. Free edge delamination is subject of actual intensive research.

The strength analysis of laminate presupposes experimental measured ultimate stresses or strains for the laminae and realistic or approximate assumptions for stiffness degradation of damaged layers. Strength under longitudinal tensile and compression stresses is usually determined with unidirectional plane specimen, strength under transverse tension and compression is measured with plane specimen or circumferentially reinforced tubes and shear strength is determined in torsion test of such tubes. Note that compression testing is much more difficult than tension testing since there is a tendency of premature failure due to crushing or buckling.

Summarizing the discussion above on failure analysis one can say that for determination of safety factors of fibre reinforced laminated structural elements there is a strong need for fracture criteria and degradation models which are simple enough for engineering applications but being also in sufficient agreement with the physical reality. In spite of many efforts were made during recent years strength analysis of laminates is still underdeveloped in comparison to the stress and strain analysis.

Essential for recent success in failure analysis was to distinguish between fibre failure and inter-fibre failure by separate failure criteria introduced by Puck[4]. The theory and application of Puck's criterion are detailed described in special literature (Knops, 2008) and are not considered here. In addition, Christensen[5] has presented some arguments concerning the best choice of failure criteria - stress or strain based (Christensen, 2013). On some actual problems and the state of the art is reported in Talreja (2016).

6.3 Problems

Exercise 6.1.
A UD lamina is loaded by biaxial tension $\sigma_{\mathrm{L}} = 13\sigma_{\mathrm{T}}, \sigma_{\mathrm{LT}} = 0$. The material is a glass-fibre epoxy composite with $E_{\mathrm{L}} = 46$ GPa, $E_{\mathrm{T}} = 10$ GPa, $G_{\mathrm{LT}} = 4{,}6$ GPa, $\nu_{\mathrm{LT}} = 0{,}31$. The basic strength parameters are $\sigma_{\mathrm{Lt}} = 1400$ MPa, $\sigma_{\mathrm{Tt}} = 35$ MPa, $\tau_{\mathrm{S}} = 70$ MPa. Compare the maximum stress and the maximum strain criteria.

Solution 6.1. Maximum stress criterion ($\sigma_{\mathrm{L}} < \sigma_{\mathrm{Lt}}, \sigma_{\mathrm{T}} < \sigma_{\mathrm{Tt}}$)

$$\begin{array}{r} 13\sigma_{\mathrm{T}} = \sigma_{\mathrm{L}} < \sigma_{\mathrm{Lt}} \\ \sigma_{\mathrm{T}} = \sigma_{\mathrm{Tt}} < \sigma_{\mathrm{Lt}} \end{array} \Longrightarrow \begin{array}{l} \sigma_{\mathrm{T}} < 107{,}69\ \mathrm{MPa} \\ \sigma_{\mathrm{T}} < 35\ \mathrm{MPa} \end{array}$$

The ultimate stress is determined by the smallest of the two values, i.e. failure occurs by transverse fracture. The stress state is then

$$\sigma_{\mathrm{T}} = 35\ \mathrm{MPa}, \sigma_{\mathrm{L}} = 13 \cdot 35 = 455\ \mathrm{MPa} < 1400\ \mathrm{MPa}$$

Maximum strain criterion ($\varepsilon_{\mathrm{L}} < \varepsilon_{\mathrm{Lt}}, \varepsilon_{\mathrm{T}} < \varepsilon_{\mathrm{Tt}}$)
To determine the ultimate strains we assume approximately a linear stress-strain relation up to fracture. Then follows the ultimate strains

$$\varepsilon_{\mathrm{Lt}} = \sigma_{\mathrm{Lt}}/E_{\mathrm{L}}, \quad \varepsilon_{\mathrm{Tt}} = \sigma_{\mathrm{Tt}}/E_{\mathrm{T}}$$

The strains caused by the biaxial tension state are

$$\begin{aligned} \varepsilon_{\mathrm{L}} &= S_{\mathrm{LL}}\sigma_{\mathrm{L}} + S_{\mathrm{LT}}\sigma_{\mathrm{T}} = \frac{1}{E_{\mathrm{L}}}\sigma_{\mathrm{L}} - \frac{\nu_{\mathrm{LT}}}{E_{\mathrm{L}}}\sigma_{\mathrm{T}} = \frac{1}{E_{\mathrm{L}}}(\sigma_{\mathrm{L}} - \nu_{\mathrm{LT}}\sigma_{\mathrm{T}}) < \varepsilon_{\mathrm{Lt}}, \\ \varepsilon_{\mathrm{T}} &= S_{\mathrm{LT}}\sigma_{\mathrm{L}} + S_{\mathrm{TT}}\sigma_{\mathrm{T}} = -\frac{\nu_{\mathrm{TL}}}{E_{\mathrm{T}}}\sigma_{\mathrm{L}} + \frac{1}{E_{\mathrm{T}}}\sigma_{\mathrm{T}} = \frac{1}{E_{\mathrm{T}}}(\sigma_{\mathrm{T}} - \nu_{\mathrm{TL}}\sigma_{\mathrm{L}}) < \varepsilon_{\mathrm{Tt}} \end{aligned}$$

The maximum strain criterion can be written

---

[4] Alfred Puck (* 1927) - engineer and professor, development of a physical-based strength criterion for UD reinforced laminates

[5] Richard M. Christensen (*3 July 1932 Idaho Falls, Idaho) - specialist in mechanics of materials

$$\sigma_L - \nu_{LT}\sigma_T < \sigma_{Lt}, \sigma_T - \nu_{TL}\sigma_L < \sigma_{Tt}, \nu_{TL} = \frac{E_T}{E_L}\nu_{LT}$$

Since $\sigma_L = 13\sigma_T$ follows

$$\sigma_T < \sigma_{Lt}/(13 - \nu_{LT}) = 110,32\,\text{MPa},$$

$$\sigma_T < \sigma_{Tt}/(1 - 13\nu_{LT}E_T/E_L) = 282,72\,\text{MPa}$$

The ultimate stress is given by the lowest of both values, i.e. failure occurs by longitudinal fracture and the stress state is then

$$\sigma_L = 13 \cdot 110,32 = 1434,16\,\text{MPa}, \sigma_T = 110,32\,\text{MPa}$$

The values of both criteria differ significantly and the fracture mode is reversed from transverse to longitudinal fracture. Because linear elastic response is assumed to fail, the criterion can predict strength also in terms of stresses. In reality the relation between ultimate stress and strain is more complex.

Exercise 6.2. Consider an off-axis unidirectional tension of a glass fibre/polyster resin laminate (Fig. 6.5), $\sigma_1 = 3,5$ MPa, $\theta = 60^0$. Estimate the state of stress with the help of the maximum stress, the maximum strain and the Tsai-Hill failure criterion. The lamina properties are $E_1' = 30$ GPa, $E_2' = 4$ GPa, $G_{12}' = 1,2$ GPa, $\nu_{12}' = 0,28, \nu_{21}' = 0,037, \sigma_{Lt} = 1200$ MPa, $\sigma_{Tt} = 45$ MPa, $\tau_S = 35$ MPa, $\varepsilon_{Lt} = 0,033, \varepsilon_{Tt} = 0,002, \varepsilon_S = 0,0078$.

Solution 6.2. The solution is split with respect to different criteria.

1. Maximum stress criterion
   Using (6.2.6) the stresses in the principal material axes can be calculated

$$\begin{aligned} \sigma_1' &= \sigma_1 \cos^2\theta = 0,875\,\text{MPa} && < \sigma_{Lt}, \\ \sigma_2' &= \sigma_1 \sin^2\theta = 2,625\,\text{MPa} && < \sigma_{Tt}, \\ \sigma_6' &= \sigma_1 \sin\theta\cos\theta = -1,515\,\text{MPa} && < \tau_S \end{aligned}$$

   The off-axis ultimate tensile strength σ_{1t} is the smallest of the following stresses

$$\begin{aligned} \sigma_1 &= \sigma_{Lt}/\cos^2\theta = 4800\,\text{MPa}, \\ \sigma_1 &= \sigma_{Tt}/\sin^2\theta = 60\,\text{MPa}, \\ \sigma_1 &= \tau_S/\sin\theta\cos\theta = 80,8\,\text{MPa} \end{aligned}$$

   i.e. $\sigma_{1t} = 60$ MPa. All stresses σ_i' are allowable, the lamina does not fail.
2. Maximum strain criterion
   From the Hooke's law for orthotropic materials follows

$$\begin{aligned} \varepsilon_1' &= \sigma_1'/E_1' - \nu_{21}'\sigma_2'/E_2' = \sigma_1'/E_1' - \nu_{12}'\sigma_2'/E_1', \\ \varepsilon_2' &= -\nu_{12}'\sigma_1'/E_1' + \sigma_2'/E_2', \\ \varepsilon_6' &= \sigma_6'/E_6' \end{aligned}$$

   The transformation for σ_i' yields

$$\varepsilon_1' = \frac{1}{E_1'}[\cos^2\theta - \nu_{12}'\sin^2\theta]\sigma_1 = 0,0000047 < \varepsilon_{\mathrm{Lt}},$$
$$\varepsilon_2' = \frac{1}{E_2'}[\sin^2\theta - \nu_{12}'\frac{E_2'}{E_1'}\cos^2\theta]\sigma_1 = 0,0006 < \varepsilon_{\mathrm{Tt}},$$
$$\varepsilon_6' = \frac{1}{G_{12}'}\sin\theta\cos\theta\sigma_1 = 0,0013 < \varepsilon_{\mathrm{S}}$$

All strains are allowed. The composite does not fail.

3. Tsai-Hill criterion
Using (6.2.18) the criterion can be written

$$\left(\frac{\cos^2\theta}{\sigma_{\mathrm{Lt}}}\right)^2 + \left(\frac{\sin^2\theta}{\sigma_{\mathrm{Tt}}}\right)^2 - \left(\frac{\sin\theta\cos\theta}{\sigma_{\mathrm{Lt}}}\right)^2 + \left(\frac{\sin\theta\cos\theta}{\tau_{\mathrm{S}}}\right)^2 < \frac{1}{\sigma_1^2},$$

$$\left[\left(\frac{0,25}{1200}\right)^2 + \left(\frac{0,75}{45}\right)^2 - \left(\frac{0,433}{1200}\right)^2 + \left(\frac{0,433}{35}\right)^2\right]\mathrm{MPa}^{-2} < 0,00043\mathrm{MPa}^{-2},$$

$$\frac{1}{\sigma_1^2} = 0,0816\mathrm{MPa}^{-2}$$

The left-hand side is smaller than the right-hand side, therefore the composite does not fail.

Exercise 6.3. The plane stress state of a UD-lamina is defined by

$$\sigma_1 = 2\sigma, \sigma_2 = -3\sigma, \sigma_6 = 4\sigma, \sigma > 0$$

The material properties are

$$E_1' = 181\mathrm{GPa}, E_2' = 10,3\mathrm{GPa}, \nu_{12}' = 0,28, G_{12}' = 7,17\mathrm{GPa}, \nu_{21}' = 0,01593,$$
$$\sigma_{\mathrm{Lt}} = 1500\mathrm{MPa}, \sigma_{\mathrm{Lc}} = 1500\mathrm{MPa}, \sigma_{\mathrm{Tt}} = 40\mathrm{MPa}, \sigma_{\mathrm{Tc}} = 246\mathrm{MPa}, \tau_{\mathrm{S}} = 68\mathrm{MPa}$$

The fibre angle is $\theta = 60^0$. Calculate the maximum value for σ by using the different failure criteria.

Solution 6.3. The solution is given for the four cases separately.

1. Maximum stress criterion
Transformation of the stresses from the off-axis to on-axis reference system yields (Table 4.1)

$$\begin{bmatrix}\sigma_1'\\ \sigma_2'\\ \sigma_6'\end{bmatrix} = \begin{bmatrix}0,250 & 0,750 & 0,866\\ 0,750 & 0,250 & -0,866\\ -0,433 & 0,433 & -0,500\end{bmatrix}\begin{bmatrix}2\sigma\\ -3\sigma\\ 4\sigma\end{bmatrix} = \begin{bmatrix}1,714\\ -2,714\\ -4,165\end{bmatrix}\sigma$$

Using (6.2.2) we find the inequalities

$$\begin{aligned} -1500\,\text{MPa} &< \ \ 1,714\sigma \ < 1500\,\text{MPa}, \\ -246\,\text{MPa} &< -2,714\sigma < 40\,\text{MPa}, \\ -68\,\text{MPa} &< -4,165\sigma < 68\,\text{MPa}, \end{aligned}$$

$$\Longrightarrow \begin{aligned} -875,1\,\text{MPa} &< \sigma < 875,1\,\text{MPa}, \\ -14,73\,\text{MPa} &< \sigma < 90,64\,\text{MPa}, \\ -16,33\,\text{MPa} &< \sigma < 16,33\,\text{MPa} \end{aligned}$$

The three inequalities are satisfied if $0 < \sigma < 16.33$ MPa. The maximum stress state which can be applied before failure is

$$\sigma_1 = 32,66 \text{ MPa}, \sigma_2 = 48,99 \text{ MPa}, \sigma_6 = 65,32 \text{ MPa}$$

The mode of failure is shear.

2. Maximum strain criterion
Using the transformation rule (4.1.5) for strains ε_i' follows with $S_{11}' = 1/E_1' = 0,5525\ 10^{-11}\ \text{Pa}^{-1}$, $S_{22}' = 1/E_2' = 9,709\ 10^{-11}\ \text{Pa}^{-1}$, $S_{66}' = 1/G_{12}' = 13,95\ 10^{-11}\ \text{Pa}^{-1}$, $S_{12}' = -\nu_{12}'/E_1' = -0,1547\ 10^{-11}\ \text{Pa}^{-1}$

$$\begin{bmatrix} \varepsilon_1' \\ \varepsilon_2' \\ \varepsilon_6' \end{bmatrix} = \begin{bmatrix} S_{11}' & S_{12}' & 0 \\ S_{12}' & S_{22}' & 0 \\ 0 & 0 & S_{66}' \end{bmatrix} \begin{bmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_6' \end{bmatrix} = \begin{bmatrix} 0,1367 \\ -2,662 \\ -5,809 \end{bmatrix} 10^{-10}\sigma \left[\frac{\text{MPa}}{\text{MPa}}\right]$$

Assuming a linear relationship between the stresses and the strains until failure, we can calculate the ultimate strains in a simple way

$$\begin{aligned} &\varepsilon_{\text{Lt}} = \sigma_{\text{Lt}}/E_1' = 8,287\ 10^{-3},\ \varepsilon_{\text{Lc}} = \sigma_{\text{Lc}}/E_1' = 8,287\ 10^{-3}, \\ &\varepsilon_{\text{Tt}} = \sigma_{\text{Tt}}/E_2' = 3,883\ 10^{-3},\ \varepsilon_{\text{Tc}} = \sigma_{\text{Tc}}/E_2' = 23,88\ 10^{-3}, \\ &\varepsilon_{\text{S}} = \tau_{\text{S}}/G_{12}' = 9,483\ 10^{-3} \end{aligned}$$

and the inequalities (6.2.8) yield

$$\begin{aligned} -8,287\ 10^{-3} &< \ 0,1367\ 10^{-10}\sigma \ < 8,287\ 10^{-3}, \\ -23,88\ 10^{-3} &< -2,662\ 10^{-10}\sigma < 3,883\ 10^{-3}, \\ -9,483\ 10^{-3} &< -5,809\ 10^{-10}\sigma < 9,483\ 10^{-3} \end{aligned}$$

or

$$\begin{aligned} -606,2\ 10^6 &< \sigma < 606,2\ 10^6, \\ -14,58\ 10^6 &< \sigma < 89,71\ 10^6, \\ -16,33\ 10^6 &< \sigma < \ 16,33\ 10^6 \end{aligned}$$

The inequalities are satisfied if $0 < \sigma < 16,33$ MPa, i.e. there is the same maximum value like using the maximum stress criterion, because the mode of failure is shear. For other failure modes there can be significant differences, see Exercise 6.1.

3. Tsai-Hill criterion
Using (6.2.17) we have

$$\left[\left(\frac{1,714}{1500}\right)^2+\left(\frac{-2,714}{40}\right)^2-\left(\frac{1,714}{1500}\right)\left(\frac{-2,714}{1500}\right)+\left(\frac{-4,165}{68}\right)^2\right]\frac{\sigma^2}{10^{12}}<1$$

i.e. $\sigma < 10,94$.
The Tsai-Hill criterion is an interactive criterion which cannot distinguish the failure modes. In the form used above it also does not distinguish between compression and tensile strength which can result in an underestimation of the allowable loading in compression with other failure criteria. Generally the transverse tensile strength of a UD-lamina is much less than the transverse compressive strength. Therefore the criteria can be modified. In dependence of the sign of the σ_i' the corresponding tensile or compressive strength is substituted. For our example follows

$$\left[\left(\frac{1,714}{1500}\right)^2+\left(\frac{-2,714}{246}\right)^2-\left(\frac{1,714}{1500}\right)\left(\frac{-2,714}{1500}\right)+\left(\frac{-4,165}{68}\right)^2\right]\frac{\sigma^2}{10^{12}}<1$$

i.e. $\sigma < 16,06$ MPa.

4. Tsai-Wu criterion
Now (6.2.24) must be applied. The coefficients can be calculated

$$\begin{aligned}
a_{\mathrm{L}} &= \left(\frac{1}{\sigma_{\mathrm{Lt}}}-\frac{1}{\sigma_{\mathrm{Lc}}}\right)=0,\\
a_{\mathrm{TT}} &= \frac{1}{\sigma_{\mathrm{Tt}}\sigma_{\mathrm{Tc}}}=1,0162\;10^{-16}\mathrm{Pa}^{-2},\\
a_{\mathrm{T}} &= \left(\frac{1}{\sigma_{\mathrm{Tt}}}-\frac{1}{\sigma_{\mathrm{Tc}}}\right)=2,093\;10^{-8}\mathrm{Pa}^{-1},\\
a_{\mathrm{SS}} &= \frac{1}{\tau_{\mathrm{Tt}}^2}=2,1626\;10^{-16}\mathrm{Pa}^{-2},\\
a_{\mathrm{LL}} &= \left(\frac{1}{\sigma_{\mathrm{Lt}}\sigma_{\mathrm{Lc}}}\right)=4,44441\;10^{-19}\mathrm{Pa}^{-2},\\
a_{\mathrm{LT}} &\approx -\frac{1}{2}\sqrt{a_{\mathrm{LL}}a_{\mathrm{TT}}}=-3,360\;10^{-18}\mathrm{Pa}^{-2}
\end{aligned}$$

Substituting the values of the coefficients in the criterion it yields the following equation

$$\begin{aligned}
&0\cdot(1,714)\sigma+2,093(10^{-8})(-2,714)\sigma+4,4444(10^{-19})(1,714\sigma)^2\\
&+1,0162(10^{-16})(-2,714\sigma)^2+2,1626(10^{-16})(-4,165\sigma)^2\\
&+2(-3,360)(10^{-18})(1,714)(-2,714)\sigma^2<1
\end{aligned}$$

The solution of the quadratic equation for σ yields $\sigma < 22,39$ MPa.

Summarizing the results of the four failure criteria we have

Max. stress criterion:	$\sigma = 16,33\bar{\sigma}\ (\bar{\sigma} \equiv R_{\mathrm{S}\sigma})$
Max. strain criterion:	$\sigma = 16,33\bar{\sigma}\ (\bar{\sigma} \equiv R_{\mathrm{S}\varepsilon})$
Tsai-Hill criterion:	$\sigma = 10,94\bar{\sigma}\ (\bar{\sigma} \equiv R_{\mathrm{TH}})$
Mod. Tsai-Hill criterion:	$\sigma = 16,06\bar{\sigma}\ (\bar{\sigma} \equiv R_{\mathrm{THm}})$
Tsai-Wu criterion:	$\sigma = 22,39\bar{\sigma}\ (\bar{\sigma} \equiv R_{\mathrm{TW}})$

The values $\bar{\sigma} \equiv \sigma$ are identical with the strength ratios (6.2.32) - (6.2.35).

Remark 6.1. A summary of the examples demonstrates that different failure criteria can lead to different results. Unfortunately, there is no one universal criterion which works well for all situations of loading and all materials. For each special class of problems a careful proof of test data and predicted failure limits must be conducted before generalizations can be made. In general, it may be recommended that more than one criterion is used and the results are compared.

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Part III
Analysis of Structural Elements

The third part (Chaps. 7–9) is devoted to the analysis of structural elements (beams, plates and shells) composed of laminates and sandwiches. The modelling of laminated and sandwich plates and shells is limited to rectangular plates and circular cylindrical shells. The individual fiber reinforced laminae of laminated structured elements are considered to be homogeneous and orthotropic, but the laminate is heterogeneous through the thickness and generally anisotropic. An equivalent single layer theory using the classical lamination theory, and the first order shear deformation theory are considered. Multilayered theories or laminate theories of higher order are not discussed in detail.

Chapter 7
Modelling and Analysis of Beams

In Chap. 1 the classification of composite materials, the significance, advantages and limitations of composite materials and structures and the material characteristics of the constituents of composite materials were considered. Chapter 2 gave a short introduction to the governing equations of the linear theory of anisotropic material behavior. Chapter 3 defined effective material moduli of composites including elementary mixture rules and improved formulas. Chapter 4 developed in detail the modelling of the mechanical behavior of laminates and sandwiches in the frame of classical theories including thermal and hygroscopic effects. The constitutive equations, describing the relationships between stress resultants and in-plane strains and mid-surface curvatures were developed for unidirectional laminae, laminates and sandwiches with the assumptions of the classical laminate theory. Further the calculation of in-plane and through-the-thickness stresses was considered. Chapter 5 gave an introduction to classical and refined laminate theories. In Chap. 6 selected failure mechanisms and criteria were briefly discussed. These parts of the book give the basic knowledge, how the design engineer can tailor composite materials to obtain the desired properties by the appropriate choice of the fibre and matrix constituents, a laminate or a sandwich material, the stacking sequence of layers, etc. This basic knowledge can be utilized to develop the modelling and analysis of structural elements and structures composed of composite materials.

7.1 Introduction

The analysis of structural elements can be performed by analytical and semi-analytical approaches or by numerical methods. The advantage of analytical solutions is their generality allowing the designer to take into account various design parameters. Analytical solutions may be either closed form solutions or infinite series and may be exact solutions of the governing equations or variational approaches. However, analytical solutions are restricted to the analysis of simple structural elements such as beams, plates and shells with simple geometry. Otherwise numer-

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0_7

ical methods have to be applied more general for structural analysis. Chapter 7 to 10 describe analytical solutions for one- and two-dimensional structural elements. Chapter 11 gives an insight into numerical solutions based on the finite element method.

In the following sections of Chap. 7 we consider rods, columns and beams. These are one-dimensional structural elements with a thickness h and a width b which are small relative to the element length l, i.e. $h, b \ll l$. When this element is loaded by an axial force only one speaks of a rod if the loading is tensile, and of columns if the load is compressive. One calls this element a beam when it is acted upon by lateral loads. In general a combination of lateral and axial loadings is possible and so we shall speak of beams under lateral and axial loadings. The other type of one-dimensional structural elements, so called plate strips under cylindrical bending, are discussed in Chap. 8. The modelling and analysis of generalized beams based a thin-walled folded structure are considered in Chap. 10 (generalized Vlasov beam theory).

The elementary or classical beam theory assumes that the transverse shear strains are negligible and plane cross-sections before bending remain plane and normal to the axis of the beam after bending (Bernoulli-Euler beam theory, Sect. 7.2). The assumption of neglecting shear strains is valid if the thickness h is small relative to the length l ($h/l < 1/20$). In the Bernoulli-Euler beam theory the transverse deflection u_3 is assumed to be independent of coordinates x_2, x_3 of the cross-section (Fig. 7.1), i.e. $u_3 \equiv w = w(x_1)$. In Sect. 7.2 the governing equations of the classical beam theory for composite beams are considered. The differential equations and variational formulations will be developed in detail for bending only, the equations for vibration and buckling are briefly summarized.

In the case of sandwich beams or moderately thick laminate beams, the results derived from the Bernoulli-Euler theory can show significant differences with the actual mechanical behavior, i.e. the deflection, stress distribution, etc. An improvement is possible by introducing the effect of transverse shear deformation, i.e. we apply Timoshenko beam theory (Sect. 7.3). The assumptions of the classical theory

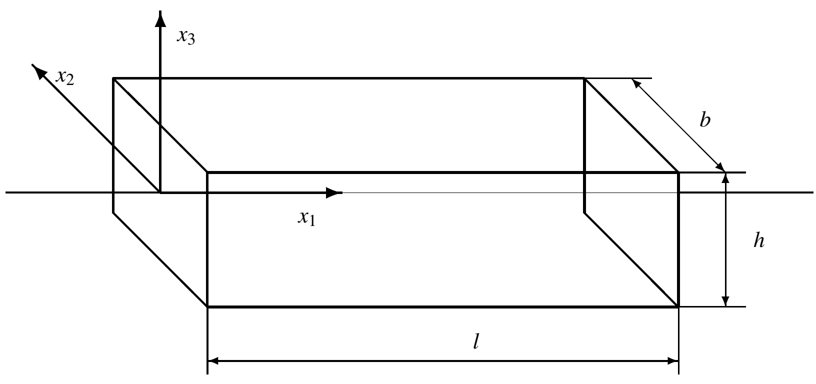

Fig. 7.1 Rod/column/beam

have then to be relaxed in the following way: the transverse normals do not remain perpendicular to the deformed axis of the beam after straining. Section 7.4 discuss some special aspects of sandwich beams.

Laminate or sandwich beams with simple or double symmetric cross-sections are most important in engineering applications. The derivations in Sects. 7.2 - 7.5 are therefore limited to straight beams with simple or double symmetric constant cross-sections which are predominantly rectangular. The bending moments act in a plane of symmetry. Also cross-sections consisting of partition walls in and orthogonal to the plane of bending, e.g. I- or box beams, are considered.

7.2 Classical Beam Theory

Frequently, as engineers try to optimize the use of materials, they design composite beams made from two or more materials. The design rationale is quite straight forward. For bending loading, stiff, strong, heavy or expensive material must be far away from the neutral axis at places where its effect will be greatest. The weaker, lighter or less expensive material will be placed in the central part of the beam. At one extreme is a steel-reinforced concrete beam, where weight is not a major concern, but strength and cost are. At the other extreme is a sandwich beam used e.g. in an aircraft with fibre-reinforced laminate cover sheets and a foam core. In that case, stiffness and weight are essential but cost not.

First we consider elementary beam equations: The cross-section area A can have various geometries but must be symmetric to the x_3-axis. The fibre reinforcement of the beam is parallel to the x_1-axis and the volume fraction is a function of the cross-sectional coordinates x_2, x_3, i.e. $v_\mathrm{f} = v_\mathrm{f}(x_2, x_3)$. The symmetry condition yields $v_\mathrm{f}(x_2, x_3) = v_\mathrm{f}(-x_2, x_3)$ and $E_1(x_2, x_3) = E_1(-x_2, x_3)$.

With the known equations for the strain ε_1 and the stress σ_1 at $x_1 = \mathrm{const}$

$$\varepsilon_1(x_3) = \varepsilon_1 + x_3\kappa_1, \quad \sigma_1(x_2, x_3) = \varepsilon_1 E_1(x_2, x_3) + x_3\kappa_1 E_1(x_2, x_3) \tag{7.2.1}$$

follow the stress resultants $N(x_1), M(x_1)$ of a beam

$$\begin{aligned} N &= \varepsilon_1 \int\limits_{(A)} E_1(x_2, x_3)\mathrm{d}A + \kappa_1 \int\limits_{(A)} x_3 E_1(x_2, x_3)\mathrm{d}A, \\ M &= \varepsilon_1 \int\limits_{(A)} x_3 E_1(x_2, x_3)\mathrm{d}A + \kappa_1 \int\limits_{(A)} x_3^2 E_1(x_2, x_3)\mathrm{d}A \end{aligned} \tag{7.2.2}$$

The effective longitudinal modulus of elasticity is (3.1.8)

$$E_1 = E_\mathrm{f} v_\mathrm{f} + E_\mathrm{m} v_\mathrm{m} = E_\mathrm{m} + \phi(x_2, x_3)(E_\mathrm{f} - E_\mathrm{m}) \tag{7.2.3}$$

and with $E_\mathrm{f} = \mathrm{const}$, $E_\mathrm{m} = \mathrm{const}$, $\phi(x_2, x_3) = \phi(-x_2, x_3)$ it follows that

$$
\begin{aligned}
N &= a\varepsilon_1 + b\kappa_1, \quad a = E_\mathrm{m}A + (E_\mathrm{f} - E_\mathrm{m}) \int\limits_{(A)} \phi(x_2,x_3)\mathrm{d}A, \\
M &= b\varepsilon_1 + d\kappa_1, \quad b = (E_\mathrm{f} - E_\mathrm{m}) \int\limits_{(A)} \phi(x_2,x_3)x_3\mathrm{d}A, \\
I &= \int\limits_{(A)} x_3^2\mathrm{d}A, \qquad d = E_\mathrm{m}I + (E_\mathrm{f} - E_\mathrm{m}) \int\limits_{(A)} \phi(x_2,x_3)x_3^2\mathrm{d}A
\end{aligned}
\tag{7.2.4}
$$

The inverse of the stress resultants, (7.2.4), are

$$
\varepsilon_1 = \frac{dN - bM}{ad - b^2}, \quad \kappa_1 = \frac{aM - bN}{ad - b^2} \tag{7.2.5}
$$

and the stress equation (7.2.1) has the form

$$
\sigma_1(x_2,x_3) = \frac{dN - bM + (aM - bN)x_3}{ad - b^2} E_1(x_2,x_3) \tag{7.2.6}
$$

Taking into consideration the different moduli E_f and E_m, the fibre and matrix stresses are

$$
\begin{aligned}
\sigma_\mathrm{f}(x_3) &= \frac{dN - bM + (aM - bN)x_3}{ad - b^2} E_\mathrm{f}, \\
\sigma_\mathrm{m}(x_3) &= \frac{dN - bM + (aM - bN)x_3}{ad - b^2} E_\mathrm{m}
\end{aligned}
\tag{7.2.7}
$$

In the case of a double symmetric geometry and fibre volume fraction function ϕ, $b = 0$ and the equations can be simplified

$$
\begin{aligned}
&\varepsilon_1 = N/a, \quad \kappa_1 = M/d, \\
&\sigma_\mathrm{f}(x_3) = (N/a + x_3 M/d)E_\mathrm{f}, \quad \sigma_\mathrm{m}(x_3) = (N/a + x_3 M/d)E_\mathrm{m}
\end{aligned}
\tag{7.2.8}
$$

For a uniform fibre distribution, $\phi(x_2,x_3) = \mathrm{const}$, (7.2.3) – (7.2.4) give

$$
\begin{aligned}
a &= E_\mathrm{m}A + (E_\mathrm{f} - E_\mathrm{m})\phi A = E_1 A, \\
b &= 0, \\
d &= E_\mathrm{m}I + (E_\mathrm{f} - E_\mathrm{m})\phi I = E_1 I
\end{aligned}
\tag{7.2.9}
$$

and the stress relations for fibre and matrix, (7.2.8), are transformed to

$$
\sigma_\mathrm{f}(x_3) = \left(\frac{N}{A} + x_3\frac{M}{I}\right)\left(\frac{E_\mathrm{f}}{E_1}\right), \quad \sigma_\mathrm{m}(x_3) = \left(\frac{N}{A} + x_3\frac{M}{I}\right)\left(\frac{E_\mathrm{m}}{E_1}\right) \tag{7.2.10}
$$

If $E_\mathrm{f} = E_\mathrm{m}$ and $E_1 = E$, (7.2.10) becomes the classical stress formula for isotropic beam with axial and lateral loadings

$$
\sigma(x_3) = \frac{N}{A} + \frac{M}{I}x_3 \tag{7.2.11}
$$

Now we consider laminate beams loaded by axial and lateral loading. For simplicity, thermal and hygrothermal effects are ignored. The derivation of the beam equations presume the classical laminate theory (Sects. 4.1 and 4.2). There are two different cases of simple laminated beams with rectangular cross-section:

1. The beam is loaded orthogonally to the plane of lamination.
2. The beam is loaded in the plane of lamination.

In the first case, we start from the constitutive equations (4.2.18).

> The beam theory makes the assumption that in the case of bending and stretching in the $(x_1 - x_3)$-plane of symmetry, i.e. no unsymmetrical or skew bending, $N_2 = N_6 = 0$, $M_2 = M_6 = 0$ and that all Poisson's effects are neglected.

With these assumptions (4.2.18) is reduced to

$$\begin{bmatrix} N_1 \\ M_1 \end{bmatrix} = \begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \kappa_1 \end{bmatrix} \tag{7.2.12}$$

and from (4.2.14)

$$Q_1 = A_{55}\varepsilon_5 \tag{7.2.13}$$

If the beam has a midplane symmetry, there is no bending-stretching coupling, so that $B_{11} = 0$ and (7.2.12) becomes

$$N_1 = A_{11}\varepsilon_1,\ M_1 = D_{11}\kappa_1 \tag{7.2.14}$$

Note that in the classical theory, the transverse shear strain will be ignored, i.e $\varepsilon_5 = 0$, and there is no constitutive equation for resultant shear forces.

The starting point for derivation of structural equations for beams is the equilibrium equations for stress resultants N, M and Q at the undeformed beam element, Fig. 7.2. The in-plane and transverse stress resultants N_1, Q_1 and the resultant mo-

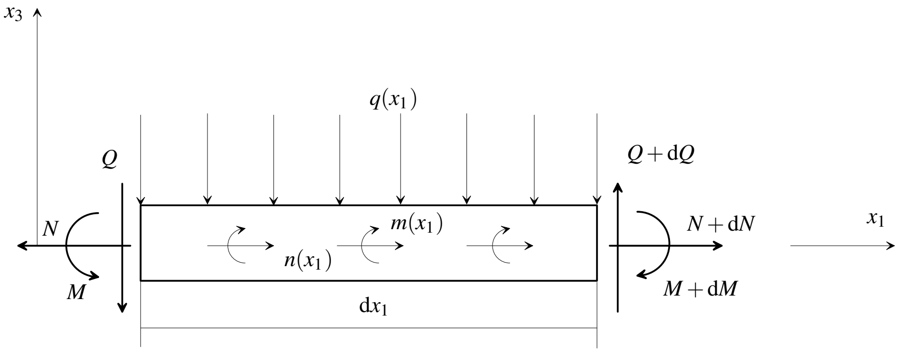

Fig. 7.2 Stress resultants N, Q and M of the infinite beam element, $q(x_1), n(x_1)$ are line forces, $m(x_1)$ is a line moment

Table 7.1 Differential relations for laminate beams based on the classical beam theory ($\varepsilon_1 = u'(x_1), \kappa_1 = -w''(x_1)$)

Relations between stress resultants and loading

$$N'(x_1) = -n(x_1), \quad Q'(x_1) = -q(x_1),$$

$$M'(x_1) = Q(x_1) - m(x_1), \quad M''(x_1) = -q(x_1) - m'(x_1)$$

Relations between stress resultants and strains

$$\begin{aligned} N &= bA_{11}u'(x_1) - bB_{11}w''(x_1) \\ M &= bB_{11}u'(x_1) - bD_{11}w''(x_1) \end{aligned} \iff \begin{bmatrix} N(x_1) \\ M(x_1) \end{bmatrix} = \begin{bmatrix} bA_{11} & bB_{11} \\ bB_{11} & bD_{11} \end{bmatrix} \begin{bmatrix} u'(x_1) \\ -w''(x_1) \end{bmatrix}$$

Differential equations for the displacements

General case

$$\begin{aligned} (bA_{11}u')'' - (bB_{11}w'')'' &= -n' \\ (bB_{11}u')'' - (bD_{11}w'')'' &= -q - m' \end{aligned} \qquad (\ldots)' = \frac{\mathrm{d}}{\mathrm{d}x_1}$$

Constant stiffness

$$\begin{bmatrix} bA_{11} & bB_{11} \\ bB_{11} & bD_{11} \end{bmatrix} \begin{bmatrix} u''' \\ -w'''' \end{bmatrix} = \begin{bmatrix} -n' \\ -q - m' \end{bmatrix}$$

Midplane symmetric laminates ($B_{11} = 0$)

$$\begin{aligned} (bA_{11}u')' &= -n & bA_{11}u'' &= -n \\ (bD_{11}w'')'' &= q + m' & bD_{11}w'''' &= q + m' \end{aligned}$$

Special cases

$m'(x_1) = 0:$

$$u''(x_1) = -\frac{n(x_1)}{bA_{11}}, \quad w''''(x_1) = \frac{q(x_1)}{bD_{11}}$$

$$N'(x_1) = -n(x_1), \quad Q'(x_1) = -q(x_1),$$
$$M'(x_1) = Q(x_1) - m(x_1)$$

$m(x_1) = 0, n(x_1) = 0:$

$$u'(x_1) = -\frac{N}{bA_{11}} = \text{const}, \quad w''''(x_1) = \frac{q(x_1)}{bD_{11}}$$

$$Q'(x_1) = -q(x_1), \quad M'(x_1) = Q(x_1)$$

ment M_1 in (7.2.12)–(7.2.13) are loads per unit length and must be multiplied by the beam width b, i.e. the beam resultants are $N = bN_1, Q = bQ_1, M = bM_1$.

The differential relations for laminate beams loaded orthogonally to the plane of lamination are summarized in Table 7.1. Note that when N is a compressive load, we have to consider additional stability conditions.

The stresses $\sigma_1^{(k)}(x_1, x_3)$ in the kth layer are given by

$$\sigma_1^{(k)} = Q_{11}^{(k)}\varepsilon_1^{(k)} = Q_{11}^{(k)}(\varepsilon_1 + x_3\kappa_1) = Q_{11}^{(k)}\left[\frac{\mathrm{d}u(x_1)}{\mathrm{d}x_1} - x_3\frac{\mathrm{d}^2w(x_1)}{\mathrm{d}x_1^2}\right] \tag{7.2.15}$$

or with

$$\varepsilon_1 = \frac{D_{11}N - B_{11}M}{A_{11}D_{11} - B_{11}^2}, \quad \kappa_1 = \frac{A_{11}M - B_{11}N}{A_{11}D_{11} - B_{11}^2},$$

one get

$$\sigma_1^{(k)} = Q_{11}^{(k)}\frac{1}{b}\left(\frac{D_{11}N - B_{11}M}{A_{11}D_{11} - B_{11}^2} + x_3\frac{A_{11}M - B_{11}N}{A_{11}D_{11} - B_{11}^2}\right) \tag{7.2.16}$$

In the most usual case of midplane symmetric beams the stress equations (7.2.15), (7.2.16) can be simplified to

$$\begin{aligned} \sigma_{1M}^{(k)}(x_1) &= Q_{11}^{(k)}\frac{\mathrm{d}u(x_1)}{\mathrm{d}x_1} &&= Q_{11}^{(k)}\frac{N(x_1)}{bA_{11}}, \\ \sigma_{1B}^{(k)}(x_1) &= Q_{11}^{(k)}\left[-x_3\frac{\mathrm{d}^2w(x_1)}{\mathrm{d}x_1^2}\right] &&= Q_{11}^{(k)}x_3\frac{M(x_1)}{bD_{11}} \end{aligned} \tag{7.2.17}$$

$\sigma_{1M}^{(k)}$ are the layerwise constant stretching or membrane stresses produced by $N(x_1)$ and $\sigma_{1B}^{(k)}$ the layerwise linear distributed flexural or bending stresses produced by $M(x_1)$. The strain $\varepsilon_1 = \varepsilon_1 + x_3\kappa_1$ is continuous and linear through the total beam thickness h. The stresses $\sigma_1^{(k)}$ are continuous and linear through each single layer and have stress jumps at the layer interfaces (Fig. 7.3) With the help of effective moduli E_{eff}^N and E_{eff}^M for stretching and flexural loading we can compare the stress equations of a laminate beam with the stress equation of a single layer beam.

With $N \neq 0, M = 0$ Eq. (7.2.12) becomes

$$\begin{bmatrix} N_1 \\ 0 \end{bmatrix} = b\begin{bmatrix} A_{11} & B_{11} \\ B_{11} & D_{11} \end{bmatrix}\begin{bmatrix} \varepsilon_1 \\ \kappa_1 \end{bmatrix} \tag{7.2.18}$$

and with

$$0 = bB_{11}\varepsilon_1 + bD_{11}\kappa_1, \quad \kappa_1 = -\frac{B_{11}}{D_{11}}\varepsilon_1 \tag{7.2.19}$$

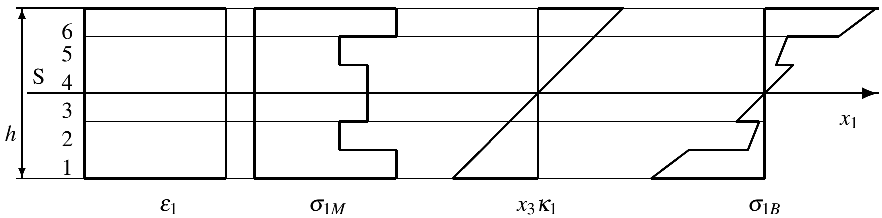

Fig. 7.3 Qualitative distribution of the stresses and strains through the beam thickness h assuming $Q_{11}^{(1)} = Q_{11}^{(6)} > Q_{11}^{(3)} = Q_{11}^{(4)} > Q_{11}^{(2)} = Q_{11}^{(5)}$

follows the equations for $N = bN_1$ and ε_1

$$N = bA_{11}\varepsilon_1 + bB_{11}\kappa_1 = \varepsilon_1 b\frac{A_{11}D_{11} - B_{11}^2}{D_{11}},$$

$$\varepsilon_1 = \frac{D_{11}}{(A_{11}D_{11} - B_{11}^2)b}N = \frac{D_{11}h}{(A_{11}D_{11} - B_{11}^2)}\frac{N}{bh} = \frac{N}{E_{\text{eff}}^N A} \tag{7.2.20}$$

with

$$E_{\text{eff}}^N = \frac{A_{11}D_{11} - B_{11}^2}{D_{11}h}, \quad A = bh$$

The strain ε_1 on the beam axis of a single layer isotropic, homogeneous beam is $\varepsilon_1 = N/EA$. Replacing E by E_{eff}^N gives the strain equation for the laminate beam.

The stresses $\sigma_1^{(k)}$ in the k layers are then

$$\begin{aligned}
\sigma_1^{(k)} &= Q_{11}^{(k)}(\varepsilon_1 + x_3\kappa_1) = Q_{11}^{(k)}\varepsilon_1\left(1 - x_3\frac{B_{11}}{D_{11}}\right) \\
&= Q_{11}^{(k)}\left(1 - x_3\frac{B_{11}}{D_{11}}\right)\frac{\mathrm{d}u}{\mathrm{d}x_1}, \\
\sigma_1^{(k)} &= \frac{Q_{11}^{(k)}}{E_{\text{eff}}^N}\frac{N}{bh}\left(1 - x_3\frac{B_{11}}{D_{11}}\right) = \frac{E_1^{(k)}}{E_{\text{eff}}^N}\frac{N}{bh}\left(1 - x_3\frac{B_{11}}{D_{11}}\right)
\end{aligned} \tag{7.2.21}$$

or for midplane symmetric beams with $B_{11} = 0$

$$\sigma_1^{(k)} = \frac{E_1^{(k)}}{E_{\text{eff}}^N}\frac{N}{bh} = \frac{E_1^{(k)}}{E_{\text{eff}}^N}\frac{N}{A}, \quad E_{\text{eff}}^N = \frac{A_{11}}{h} \tag{7.2.22}$$

In an analogous manner it follows from (7.2.12) with $N = 0, M \neq 0$ that

$$\begin{aligned}
M &= bB_{11}\varepsilon_1 + bD_{11}\kappa_1 = \kappa_1 b\frac{A_{11}D_{11} - B_{11}^2}{A_{11}}, \\
\kappa_1 &= \frac{A_{11}}{(A_{11}D_{11} - B_{11}^2)b}M = \frac{A_{11}h^3}{12(A_{11}D_{11} - B_{11}^2)}\frac{M}{\frac{bh^3}{12}} = \frac{M}{E_{\text{eff}}^M I},
\end{aligned} \tag{7.2.23}$$

with

$$E_{\text{eff}}^M = \frac{12(A_{11}D_{11} - B_{11}^2)}{A_{11}h^3}, \quad I = \frac{bh^3}{12} \tag{7.2.24}$$

For an isotropic homogeneous single layer beam of width b and thickness h one get $\kappa_1 = M/EI = 12M/bh^3$, $I = bh^3/12$. Replacing now E by E_{eff}^M, the stress equations are

$$
\begin{aligned}
\sigma_1^{(k)} &= Q_{11}^{(k)}(\varepsilon_1 + x_3\kappa_1) = Q_{11}^{(k)}\kappa_1\left(-\frac{B_{11}}{A_{11}} + x_3\right)\\
&= Q_{11}^{(k)}\left(\frac{B_{11}}{A_{11}} - x_3\right)\frac{\mathrm{d}^2 w}{\mathrm{d}x_1^2},\\
&= \frac{Q_{11}^{(k)}}{E_{\mathrm{eff}}}\frac{M}{I}\left(x_3 - \frac{B_{11}}{A_{11}}\right) = \frac{E_1^{(k)}}{E_{\mathrm{eff}}}\frac{M}{I}\left(x_3 - \frac{B_{11}}{A_{11}}\right)
\end{aligned}
\tag{7.2.25}
$$

or with $B_{11} = 0$ for the symmetric case

$$
\sigma_1^{(k)} = \frac{E_1^{(k)}}{E_{\mathrm{eff}}^M}\frac{M}{I}x_3, \quad E_{\mathrm{eff}}^M = \frac{12D_{11}}{h^3} \tag{7.2.26}
$$

If both in-plane and lateral loads occur simultaneously, the stress in each lamina of the beam is as for symmetric case

$$
\sigma_1^{(k)}(x_1, x_3) = Q_{11}^{(k)}\left(\frac{\mathrm{d}u}{\mathrm{d}x_1} - x_3\frac{\mathrm{d}^2 w}{\mathrm{d}x_1^2}\right) = E_1^{(k)}\left(\frac{N}{E_{\mathrm{eff}}^N A} + \frac{M}{E_{\mathrm{eff}}^M I}x_3\right), \tag{7.2.27}
$$

$$
A = bh, \quad I = bh^3/12, \quad E_{\mathrm{eff}}^N = A_{11}/h, \quad E_{\mathrm{eff}}^M = 12D_{11}/h^3
$$

Conclusion 7.1. Summarizing the equations for symmetric laminated beams, one can say that the equations for $u(x_1)$ and $w(x_1)$ are identical in form to those of elementary theory for homogeneous, isotropic beams. Hence all solutions available, e.g. for deflections of isotropic beams under various boundary conditions, can be used by replacing the modulus E with E_{eff}^N or E_{eff}^M, respectively. The calculation of the stresses illustrates that constant in-plane layer stresses produced by N are proportional to the layer modulus $E_1^{(k)}$ (7.2.22). N/AE_{eff}^N is for a cross-section $x_1 = \mathrm{const}$ a constant value. Analogous are the flexural layer stresses proportional to $E_1^{(k)}x_3$ (7.2.26). In general, the maximum stress does not occur at the top or the bottom of a laminated beam, but the maximum stress location through the thickness depends on the lamination scheme.

From the bending moment-curvature relation ($N = 0$)

$$
M = bD_{11}\kappa_1 \tag{7.2.28}
$$

it follows that

$$
\kappa_{1\max} = \frac{M_{\max}}{bD_{11}} = -\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)_{\max},
$$

and the maximum stress can be calculated for each lamina

$$
\sigma_{1\max}^{(k)} = Q_{11}^{(k)}\kappa_{1\max}x_3 = -Q_{11}^{(k)}\left(\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right)_{\max} x_3 = \frac{E_1^{(k)}}{E_{\mathrm{eff}}^M}\frac{M_{\max}}{I}x_3 \tag{7.2.29}
$$

$\sigma_{1\max}^{(k)}$ must be compared with the allowable strength value.

The calculation of the transverse shear stress $\sigma_5^{(k)}(x_1,x_3)$ is analogous as in the elementary beam theory. We restrict our calculation to midplane symmetric beams and therefore all derivations can be given for the upper part of the beam element ($x_3 \geq 0$). The equilibrium equations in the x_1-direction lead with

$$\sigma_1^{(j)} + \mathrm{d}\sigma_1^{(j)} \approx \sigma_1^{(j)} + \frac{\mathrm{d}\sigma_1^{(j)}}{\mathrm{d}x_1}\mathrm{d}x_1$$

(Fig. 7.4) and no edge shear stresses on the upper and lower faces

$$\sigma_5^{(k)}\mathrm{d}x_1 - \sum_{j=k+1}^{N} \int\limits_{x_3^{(j-1)}}^{x_3^{(j)}} \left[\left(\sigma_1^{(j)} + \frac{\mathrm{d}\sigma_1^{(j)}}{\mathrm{d}x_1}\mathrm{d}x_1\right) - \sigma_1^{(j)}\right]\mathrm{d}x_3 = 0 \tag{7.2.30}$$

or

$$\sigma_5^{(k)} = \sum_{j=k+1}^{N} \int\limits_{x_3^{(j-1)}}^{x_3^{(j)}} \frac{\mathrm{d}\sigma_1^{(j)}}{\mathrm{d}x_1}\mathrm{d}x_3 = \sum_{j=k+1}^{N} \int\limits_{x_3^{(j-1)}}^{x_3^{(j)}} \frac{E_1^{(j)}}{E_{\mathrm{eff}}^M I}\frac{\mathrm{d}M}{\mathrm{d}x_1}x_3\mathrm{d}x_3$$

With $Q = dM/\mathrm{d}x_1$ it follows that

$$\begin{aligned}\sigma_5^{(k)}(x_1,x_3) &= \frac{Q(x_1)}{E_{\mathrm{eff}}^M I}\sum_{j=k+1}^{N} \int\limits_{x_3^{(j-1)}}^{x_3^{(j)}} E_1^{(j)}x_3\mathrm{d}x_3 = \frac{Q(x_1)}{E_{\mathrm{eff}}^M I}\sum_{j=k+1}^{N} E_1^{(j)}\frac{1}{2}\left(x_3^{(j)2} - x_3^{(j-1)2}\right)\\ &= \frac{Q(x_1)}{E_{\mathrm{eff}}^M I}\sum_{j=k+1}^{N} E_1^{(j)}h^{(j)}\overline{x}_3^{(j)}\end{aligned} \tag{7.2.31}$$

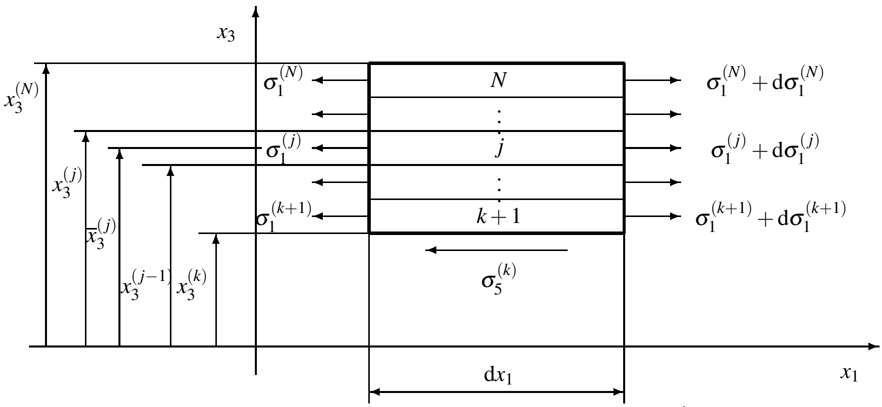

Fig. 7.4 Beam element $b(x_3^{(N)} - x_3^{(k)})\mathrm{d}x_1$ with flexural normal stresses $\sigma_1^{(j)}$ and the interlaminar stress $\sigma_5^{(k)}$

For a single layer homogeneous, isotropic beam (7.2.31) yields the known parabolic shear stress distribution through h

$$\sigma_5(x_1,x_3) = \frac{Q}{I}\int_{x_3}^{h/2} x_3 \mathrm{d}x_3 = \frac{12Q}{bh^3}\frac{1}{2}\left(\frac{h^2}{4} - x_3^2\right) = \frac{3Q}{2bh}\left[1 - 4\left(\frac{x_3}{h}\right)^2\right] \tag{7.2.32}$$

With an increasing number of equal thickness layers, the transverse shear stress distribution (7.2.31) approaches the parabolic function of the single layer beam.

All stress equations presume that the Poisson's effects can be completely neglected, i.e. $Q_{ij}^{(k)} = D_{ij} = 0, i \neq j, i, j = 1,2,6$. They are summarized for symmetric laminated beams ($N \neq 0, M \neq 0$) in Table 7.2. For symmetric laminated beams loaded orthogonally to the plane of lamination, the classical laminate theory yields identical differential equations for $u(x_1)$ and $w(x_1)$ with to Bernoulli's beam theory of single layer homogeneous isotropic beams, if one substitutes bA_{11} by $EA = Ebh$ and bD_{11} by $EI = Ebh^3/12$. An equal state is valid for beam vibration and beam buckling.

The following equations are given without a special derivation (b, A, D_{11}, ρ are constant values):

- *Differential equation of flexure* ($N = 0$)

$$\frac{\mathrm{d}^2 w(x_1)}{\mathrm{d}x_1^2} = -\frac{M(x_1)}{bD_{11}}, \quad bD_{11}\frac{\mathrm{d}^4 w(x_1)}{\mathrm{d}x_1^4} = q(x_1) \tag{7.2.33}$$

Table 7.2 Stress formulas for symmetric laminated beams, classical theory

$$\begin{aligned}
\sigma_1^{(k)}(x_1,x_3) &= \sigma_{1M}^{(k)}(x_1) + \sigma_{1B}^{(k)}(x_1,x_3)\\
&= Q_{11}^{(k)}\frac{\mathrm{d}u(x_1)}{\mathrm{d}x_1} - Q_{11}^{(k)}\frac{\mathrm{d}^2 w(x_1)}{\mathrm{d}x_1^2}x_3\\
&= Q_{11}^{(k)}\frac{N(x_1)}{bA_{11}} + Q_{11}^{(k)}\frac{M(x_1)}{bD_{11}}x_3\\
&= \frac{E_1^{(k)}}{E_{\mathrm{eff}}^N}\frac{N(x_1)}{A_{11}} + \frac{E_1^{(k)}}{E_{\mathrm{eff}}^M}\frac{Mx_1)}{I}x_3,\\
\sigma_5^{(k)}(x_1,x_3) &= \frac{Q(x_1)}{E_{\mathrm{eff}}^M I}\sum_{j=k+1}^{N} E_1^{(j)} h^{(j)} \bar{x}_3^{(j)},
\end{aligned}$$

$$A = bh, \quad I = bh^3/12,$$

$$E_{\mathrm{eff}}^N = A_{11}/h, \quad E_{\mathrm{eff}}^M = 12D_{11}/h^3, \quad \bar{x}^{(j)} = \frac{1}{2}(x_3^{(j)} + x_3^{(j-1)})$$

- *Forced or free vibrations*

$$bD_{11}\frac{\mathrm{d}^4 w(x_1,t)}{\mathrm{d}x_1^4}+\rho A\frac{\mathrm{d}^2 w(x_1,t)}{\mathrm{d}t^2}=q(x_1,t),\quad \rho=\frac{1}{h}\sum_{k=1}^{N}\rho^{(k)}h^{(k)} \tag{7.2.34}$$

Rotational inertia terms are neglected. For free vibration with $q=0$ the solution is assumed periodic: $w(x_1,t)=W(x_1)\exp(i\omega t)$.
- *Buckling equation*

$$\frac{\mathrm{d}^2 M(x_1)}{\mathrm{d}x_1^2}+N_1(x_1)\frac{\mathrm{d}^2 w(x_1)}{\mathrm{d}x_1^2}=0,\quad bD_{11}\frac{\mathrm{d}^4 w(x_1)}{\mathrm{d}x_1^4}-N_1(x_1)\frac{\mathrm{d}^2 w(x_1)}{\mathrm{d}x_1^2}=0 \tag{7.2.35}$$

or with $N(x_1)=-F$

$$\frac{\mathrm{d}^2 M(x_1)}{\mathrm{d}x_1^2}-F\frac{\mathrm{d}^2 w(x_1)}{\mathrm{d}x_1^2}=0,\quad bD_{11}\frac{\mathrm{d}^4 w(x_1)}{\mathrm{d}x_1^4}+F\frac{\mathrm{d}^2 w(x_1)}{\mathrm{d}x_1^2}=0$$

All solutions of the elementary beam theory for single layer isotropic beams can transferred to laminate beams. Note that laminate composites are stronger shear deformable than metallic materials and the classical beam theory is only acceptable when the ratio $l/h>20$.

The equations for flexure, vibration and buckling can also be given in a variational formulation (Sect. 2.2.2). With the elastic potential for a flexural beam $(N=0,M\neq 0)$

$$\Pi(w)=\frac{1}{2}\int_0^l bD_{11}\left(\frac{\mathrm{d}^2 w(x_1)}{\mathrm{d}x_1^2}\right)^2\mathrm{d}x_1-\int_0^l q\mathrm{d}x_1 \tag{7.2.36}$$

and the kinetic energy

$$T(w)=\frac{1}{2}\int_0^l \rho\left(\frac{\partial^2 w}{\partial t}\right)^2\mathrm{d}x_1 \tag{7.2.37}$$

the Lagrange function is given by $L(w)=T(w)-\Pi(w)$ (Sect. 2.2.2).

The variational formulation for a symmetric laminated beam without bending-stretching coupling subjected to a lateral load q in x_3-direction ($N=0,M\neq 0$, $\varepsilon_5\ \approx\ 0,\nu_{ij}\approx 0$) based on the theorem of minimum of total potential energy is given in the form

$$\begin{aligned}\Pi[w(x_1)]\ &=\frac{1}{2}\int_0^l bD_{11}\left(\frac{\mathrm{d}^2 w(x_1)}{\mathrm{d}x_1^2}\right)^2\mathrm{d}x_1-\int_0^l q\mathrm{d}x_1,\\ \delta\Pi[w(x_1)]&=0\end{aligned} \tag{7.2.38}$$

The variational formulation for the buckling of a symmetric laminate beam with $N(x_1)=-F$ is

$$\begin{aligned}\Pi[w(x_1)] &= \frac{1}{2}\int_0^l bD_{11}\left(\frac{\mathrm{d}^2 w(x_1)}{\mathrm{d}x_1^2}\right)^2 \mathrm{d}x_1 - \frac{1}{2}\int_0^l F\left(\frac{\mathrm{d}w(x_1)}{\mathrm{d}x_1}\right)^2 \mathrm{d}x_1, \\ \delta\Pi[w(x_1)] &= 0\end{aligned} \qquad (7.2.39)$$

The variational formulation for free flexural beam vibration (additional to approaches noted above the rotatory inertia effects are neglected) can be given by the Hamilton's principle

$$H[w(x_1,t)] = \int_{t_1}^{t_2} L[W(x_1,t)]\mathrm{d}t, \quad \delta H[w(x_1,t)] = 0 \qquad (7.2.40)$$

The variational formulations can be used for approximate analytical solution with the Rayleigh-Ritz procedure or numerical solutions.

In the second case of laminate beams, the loading is in the plane of lamination. We restrict our considerations to symmetric layered beams and neglect all Poisson's ratio effects. The beam is illustrated in Fig. 7.5. For a symmetric layer stacking there is no bending-stretching coupling and we have the constitutive equations

$$N_1 = A_{11}\varepsilon_1, \quad M_1 = \frac{b^3}{12h}A_{11}\kappa_1$$

or for the beam resultants $N = hN_1, M = hM_1$

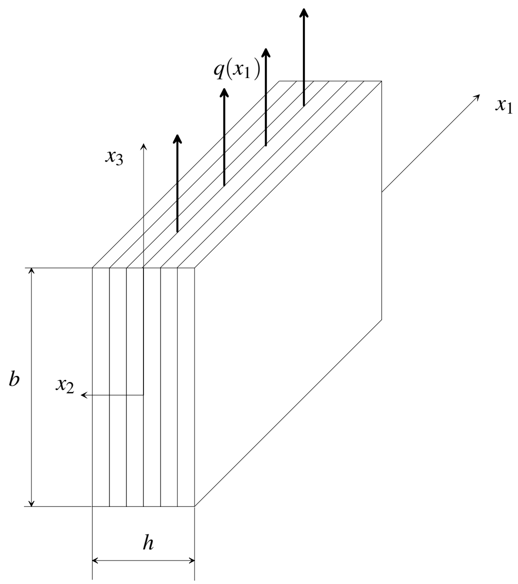

Fig. 7.5 Laminate beam loaded in the plane of lamination

$$N = hA_{11}\varepsilon_1, \quad M = \frac{b^3}{12}A_{11}\kappa_1, \quad A_{11} = \sum_{k=1}^{N} Q_{11}^{(k)}h^{(k)} \tag{7.2.41}$$

The differential equations of flexure ($N = 0$) are

$$\frac{\mathrm{d}^2 w(x_1)}{\mathrm{d}x_1^2} = -\frac{M(x_1)}{A_{11}}\frac{b^3}{12}, \quad \frac{12A_{11}}{b^3}\frac{\mathrm{d}^4 w(x_1)}{\mathrm{d}x_1^4} = q(x_1) \tag{7.2.42}$$

and the additional equation for the case $N \neq 0$ is

$$hA_{11}\frac{\mathrm{d}u(x_1)}{\mathrm{d}x_1} = N(x_1) \tag{7.2.43}$$

The calculation of stresses is analogous to case 1 of layered beams.

When beam profiles consist of partition-walls in the plane of loading and orthogonal to the plane of loading, e.g. I-profiles or box-beams, the bending differential equations can be written in the form, given above. The bending stiffness is obtained by combining the results of orthogonal to plane loading and in-plane loading.

As an example for a box-beam, we consider the beam as shown in Fig. 7.6, which may be subjected to axial loads in x_1-direction, a bending moment with respect to the x_2-axis and a twisting moment with respect to the x_1-axis. For an isotropic beam the stiffness needed are the extensional stiffness, EA, the flexural stiffness, EI, and the torsional stiffness, GI_t.

The axial force resultant (per unit width) in x_1-direction is $N_1 = A_{11}\varepsilon_1$ and the axial load carried by the whole section is then

$$N(x_1) = 2N_1^I b + 2N_1^{II} h = 2[(A_{11})_I b + (A_{11})_{II} h]\varepsilon_1 \tag{7.2.44}$$

The extensional stiffness for the box cross-section is given by

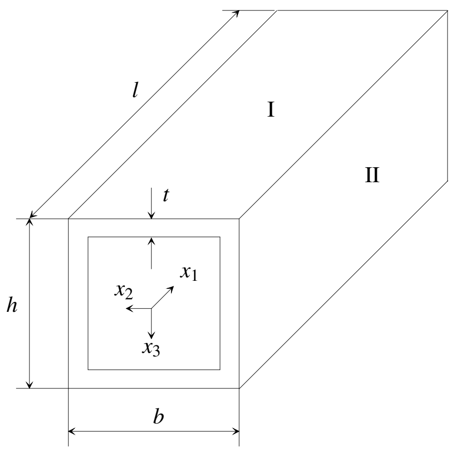

Fig. 7.6 Laminated box-beam with identical top and bottom panels I and vertical walls II

$$(EA)_{\text{eff}} = 2(A_{11})_I b + 2(A_{11})_{II} h \tag{7.2.45}$$

The box beam is bent in the $(x_1 - x_3)$ plane, and the moment curvature relation is

$$\begin{aligned} M &= \left[2(D_{11})_I b + 2(A_{11})_I b \left(\frac{h}{2}\right)^2 + 2(A_{11})_{II} \frac{h^3}{12} \right] \kappa_1 \\ &\approx \left[2(A_{11})_I b \left(\frac{h}{2}\right)^2 + \frac{1}{6}(A_{11})_{II} h^3 \right] \kappa_1 \end{aligned} \tag{7.2.46}$$

Since the top and bottom panels are thin relative to the height of the box profile, i.e. $t \ll h$, $(D_{11})_I$ can be neglected and the bending stiffness of the box cross-section is

$$(EI)_{\text{eff}} \approx 2(A_1 1)_I b \left(\frac{h}{2}\right)^2 + \frac{1}{6}(A_1 1)_{II} h^3 \tag{7.2.47}$$

If the box-beam is acted by a torsional moment M_{T} this is equivalent to the moment of the shear flows with respect to the x_1-axis and we have

$$M_{\text{T}} = 2N_6^I b(h/2) + 2N_6^{II} h(b/2), \quad N_6^I = A_{66}^I \varepsilon_6^I, \quad N_6^{II} = A_{66}^{II} \varepsilon_6^{II} \tag{7.2.48}$$

In the elementary theory of strength of materials the equation for the angle of twist of a box-beam is given by

$$\theta = \frac{1}{2A} \oint \frac{q(s)}{Gt} \mathrm{d}s \tag{7.2.49}$$

$q(s)$ is the shear flow. In our case, Fig. 7.6, the displacements of the contours of the walls of the box beam are denoted by δ_I and δ_{II} and the angle of twist becomes

$$\theta = \frac{\delta_I}{(h/2)} = \frac{\delta_{II}}{(b/2)} \quad \text{with} \quad \frac{\delta_I}{l} = \varepsilon_6^I, \quad \frac{\delta_{II}}{l} = \varepsilon_6^{II} \tag{7.2.50}$$

From (7.2.48) - (7.2.50) we have

$$M_{\text{T}} = \frac{bh}{l} [(A_{66}^I) h + (A_{66}^{II}) b] \theta \tag{7.2.51}$$

and the torsional stiffness of the cross-section is

$$(GI_{\text{t}})_{\text{eff}} = \frac{bh}{l} [(A_{66}^I) h + (A_{66}^{II}) b] \tag{7.2.52}$$

For the I-profile in Fig. 7.7 the calculation for bending is analogous. The bending stiffness is

$$\begin{aligned} (EI)_{\text{eff}} &= \left[2(D_{11}) b + 2(A_{11}) b \left(\frac{h}{2}\right)^2 + (A_{11}) \frac{h^3}{12} \right] \\ &\approx A_{11} \frac{6h^2 b + h^3}{12} \end{aligned} \tag{7.2.53}$$

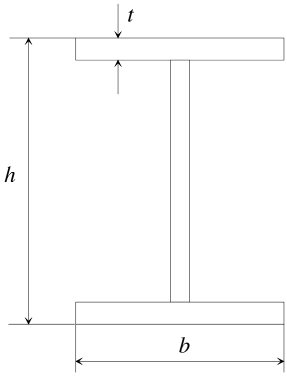

Fig. 7.7 I-profile with uniform thickness t

if $D_{11} \approx 0$. Note that for a one-dimensional thin structural element which is symmetric with respect to all mid-planes and Poisson's effect is neglected, we have the simple relationships

$$Q_{11} = E_1, \quad A_{11} = E_1 t, \quad \kappa_1 = \frac{M}{(EI)_{\text{eff}}}$$

Summarizing the classical beam equations it must be noted that the effect of Poisson's ratio is negligible only if the length-to-with ratio l/b is large ($l \gg b$), otherwise the structure behavior is more like a plate strip than a beam (Sect. 8.2). This is of particular importance for angle-ply laminates, i.e. orthotropic axes of material symmetry in each ply are not parallel to the beam edges and anisotropic shear coupling is displayed.

7.3 Shear Deformation Theory

The structural behavior of many usual beams may be satisfactorily approximated by the classical Euler-Bernoulli theory. But short and moderately thick beams or laminated composite beams which l/h ratios are not rather large cannot be well treated in the frame of the classical theory. To overcome this shortcoming Timoshenko extended the classical theory by including the effect of transverse shear deformation. However, since Timoshenko's beam theory assumed constant shear strains through the thickness h a shear correction factor is required to correct the shear strain energy.

In this section we study the influence of transverse shear deformation upon the bending of laminated beams. The similarity of elastic behavior of laminate and sandwich beams with transverse shear effects included allows us generally to transpose the results from laminate to sandwich beams. When applied to beams, the first order shear deformation theory is known as Timoshenko's beam theory. Figure 7.8 illustrates the cross-section kinematics for the Bernoulli's and the Timoshenko's bending beam.

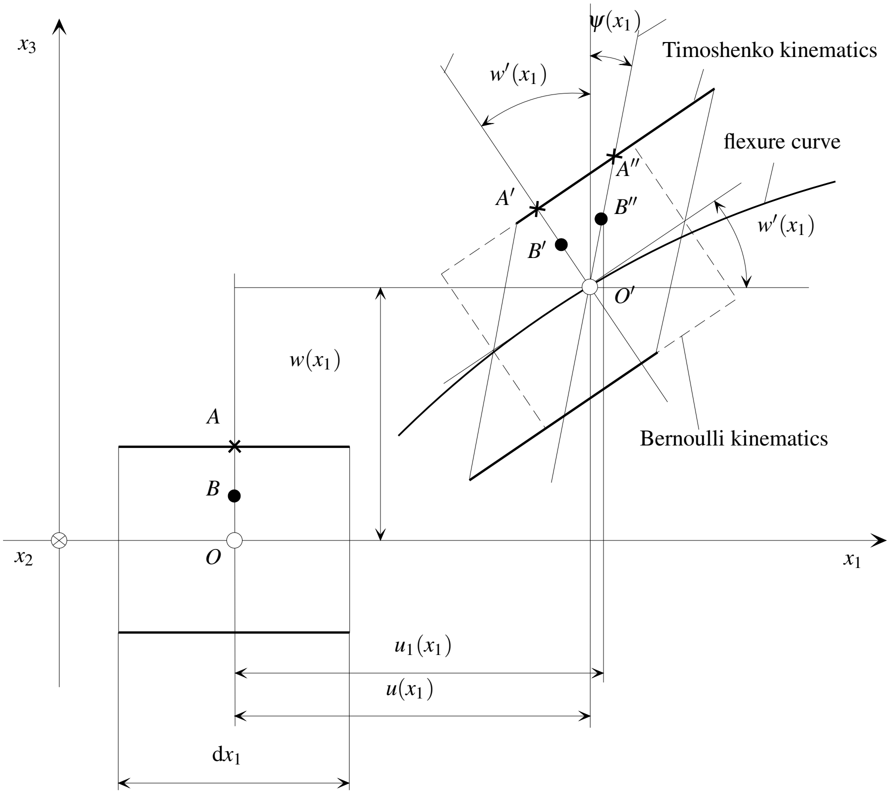

Fig. 7.8 Kinematics of a bent Timoshenko- and Bernoulli-beam in the $(x_1 - x_3)$ plane

When all Poisson's effects are neglected the constitutive equations are identical with (7.2.12) - (7.2.13), but from Sect. 5.1 the strains of the Timoshenko's beam are

$$
\begin{aligned}
\varepsilon_1 &= \frac{\partial u_1}{\partial x_1} = \frac{\mathrm{d}u}{\mathrm{d}x_1} + x_3 \frac{\mathrm{d}\psi_1}{\mathrm{d}x_1}, \quad \varepsilon_2 \equiv 0,\ \varepsilon_3 \equiv 0,\\
\varepsilon_5 &= \frac{\partial u_1}{\partial x_3} + \frac{\partial w}{\partial x_1} = \psi_1 + \frac{\mathrm{d}w}{\mathrm{d}x_1}, \quad \varepsilon_4 \equiv 0,\ \varepsilon_6 \equiv 0
\end{aligned}
\tag{7.3.1}
$$

i.e. we only have one longitudinal and one shear strain

$$
\begin{aligned}
&\varepsilon_1(x_1,x_3) = \varepsilon_1(x_1) + x_3 \kappa_1(x_1), \quad \varepsilon_5(x_1,x_3) = \psi_1(x_1) + w'(x_1),\\
&\varepsilon_1(x_1) = \frac{\mathrm{d}u}{\mathrm{d}x_1}, \quad \kappa_1(x_1) = \frac{\mathrm{d}\psi_1(x_1)}{\mathrm{d}x_1}
\end{aligned}
\tag{7.3.2}
$$

When the transverse shear strain are neglected it follows with $\varepsilon_5 \approx 0$ that the relationship is $\psi_1(x_1) = -w'(x_1)$ and that is the Bernoulli's kinematics.

In the general case of an unsymmetric laminated Timoshenko's beam loaded orthogonally to the laminated plane and $N \neq 0, M \neq 0$, the constitutive equations (stress resultants - strain relations) are given by

$$N = \overline{A}_{11}\varepsilon_1 + \overline{B}_{11}\kappa_1, \quad M = \overline{B}_{11}\varepsilon_1 + \overline{D}_{11}\kappa_1, \quad Q = k^{\mathrm{s}}\,\overline{A}_{55}\gamma^{\mathrm{s}} \tag{7.3.3}$$

with $\overline{A}_{11} = bA_{11}, \overline{B}_{11} = bB_{11}, \overline{D}_{11} = bD_{11}, \gamma^{\mathrm{s}} = \varepsilon_5$ and the stiffness equations are from (4.2.15) and $Q_{11}^{(k)} \equiv C_{11}^{(k)} = E_1^{(k)}, Q_{55}^{(k)} \equiv C_{55}^{(k)} = G_{13}^{(k)}$

$$\begin{aligned}
\overline{A}_{11} &= b\sum_{k=1}^{n} C_{11}^{(k)}\left(x_3^{(k)} - x_3^{(k-1)}\right) = b\sum_{k=1}^{n} C_{11}^{(k)} h^{(k)}, \\
\overline{A}_{55} &= b\sum_{k=1}^{n} C_{55}^{(k)}\left(x_3^{(k)} - x_3^{(k-1)}\right) = b\sum_{k=1}^{n} C_{55}^{k} h^{(k)}, \\
\overline{B}_{11} &= b\frac{1}{2}\sum_{k=1}^{n} C_{11}^{(k)}\left(x_3^{(k)^2} - x_3^{(k-1)^2}\right), \\
\overline{D}_{11} &= b\frac{1}{3}\sum_{k=1}^{n} C_{11}^{(k)}\left(x_3^{(k)^3} - x_3^{(k-1)^3}\right)
\end{aligned}$$

k^{s} is the shear correction factor (Sect. 5.3).

In the static case, the equilibrium equations for the undeformed beam element (Fig. 7.2) yield again for lateral loading $q \neq 0$

$$\frac{\mathrm{d}M}{\mathrm{d}x_1} - Q = 0, \quad \frac{\mathrm{d}Q}{\mathrm{d}x_1} + q = 0 \tag{7.3.4}$$

When considerations are limited to symmetric laminated beams the coupling stiffness B_{11} is zero and from (7.3.3) it follows that

$$M = \overline{D}_{11}\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1}, \quad Q = k^{\mathrm{s}}\,\overline{A}_{55}\left(\psi_1 + \frac{\mathrm{d}w}{\mathrm{d}x_1}\right) \tag{7.3.5}$$

Substituting the relations (7.3.5) into (7.3.4) leads to the differential equations of flexure

$$\begin{aligned}
[\overline{D}_{11}\psi_1'(x_1)]' - k^{\mathrm{s}}\,\overline{A}_{55}[\psi_1(x_1) + w'(x_1)] &= 0, \\
k^{\mathrm{s}}\,\overline{A}_{55}\,[\psi_1(x_1) + w'(x_1)]' + q(x_1) &= 0
\end{aligned} \tag{7.3.6}$$

Derivation of the first equation of (7.3.6) and setting in the second equation yields a differential equation of 3rd order for $\psi_1(x_1)$

$$[\overline{D}_{11}\psi'(x)]'' = -q(x) \tag{7.3.7}$$

and with

$$Q = \frac{\mathrm{d}M}{\mathrm{d}x_1} = [\overline{D}_{11}\psi_1'(x_1)]' \tag{7.3.8}$$

and

$$Q = k^{\mathrm{s}}\overline{A}_{55}[\psi_1(x_1) + w'(x_1)]$$

follows an equation for $w'(x_1)$

$$w'(x_1) = -\psi_1(x_1) + \frac{[\overline{D}_{11}\psi_1'(x_1)]'}{k^{\mathrm{s}}\overline{A}_{55}} \tag{7.3.9}$$

Summarizing the derivations above, the equations for a bent Timoshenko's beam are:

$$\begin{aligned}
[\overline{D}_{11}\psi_1'(x_1)]'' &= -q(x_1), \\
M(x_1) &= \overline{D}_{11}\psi_1'(x_1), \\
Q(x_1) &= [\overline{D}_{11}\psi_1'(x_1)]', \\
w'(x_1) &= -\psi_1(x_1) + \frac{[\overline{D}_{11}\psi_1'(x_1)]'}{k^{\mathrm{s}}\overline{A}_{55}}
\end{aligned} \tag{7.3.10}$$

When the laminated beam problem allows to write the bending moment M and the transverse force Q in terms of the known applied lateral loads q, like in statically determined beam problems, (7.3.5) can be utilized to determine first $\psi_1(x_1)$ and then $w(x_1)$. Otherwise (7.3.6) or (7.3.10) are used to determine $w(x_1)$ and $\psi_1(x_1)$.

Integrating the second Eq. (7.3.6) with respect to x_1, we obtain

$$k^{\mathrm{s}}\overline{A}_{55}[w'(x_1) + \psi_1(x_1)] = -\int q(x_1)\mathrm{d}x_1 + c_1$$

Substituting the result into the first equation of (7.3.6) and integrating again with respect to x_1 yields

$$\begin{aligned}
\overline{D}_{11}\psi_1'(x_1) &= -\iint q(x_1)\mathrm{d}x_1\mathrm{d}x_1 + c_1x_1 + c_2, \\
\overline{D}_{11}\psi_1(x_1) &= -\iiint q(x_1)\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1 + c_1\frac{x_1^2}{2} + c_2x_1 + c_3
\end{aligned} \tag{7.3.11}$$

Substituting $\psi_1(x_1)$ and $\psi_1'(x_1)$ in (7.3.9), considering (7.3.7) and integrating once more with respect to x_1 we obtain

$$\begin{aligned}
w(x_1) &= \frac{1}{\overline{D}_{11}}\left[\iiiint q(x_1)\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1 + c_1\frac{x_1^3}{6} + c_2\frac{x_1^2}{2} + c_3x_1 + c_4\right] \\
&\quad - \frac{1}{k^{\mathrm{s}}\overline{A}_{55}}\left[\iint q(x_1)\mathrm{d}x_1\mathrm{d}x_1 + c_1x_1\right] \\
&= w^{\mathrm{B}}(x_1) + w^{\mathrm{S}}(x_1)
\end{aligned} \tag{7.3.12}$$

The transverse deflection consists of two parts. The bending part $w^{\mathrm{B}}(x_1)$ is the same as derived in the classical theory. When the transverse stiffness goes to infinity, the shear deflection $w^{\mathrm{S}}(x_1)$ goes to zero, $\psi_1(x_1)$ goes to $-w'(x_1)$ and the Timoshenko's beam theory reduces to the classical Bernoulli's beam theory.

The relations for the stresses σ_1 are the same as in the classical theory. The transverse shear stress can be computed via a constitutive equation in the Timoshenko theory

$$\sigma_5^{(k)}(x_1,x_3) = Q_{55}^{(k)} \frac{Q(x_1)}{k^{\mathrm{s}}\overline{A}_{55}} \tag{7.3.13}$$

The variational formulation for a lateral loaded symmetric laminated beam is given by

$$\Pi(w,\psi_1) = \Pi_{\mathrm{i}} + \Pi_{\mathrm{a}} \tag{7.3.14}$$

with

$$\begin{aligned}
\Pi_{\mathrm{i}} &= \frac{1}{2}\int_0^l \left[\overline{D}_{11}\left(\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1}\right)^2 + k^{\mathrm{s}}\overline{A}_{55}\left(\psi_1 + \frac{\mathrm{d}w}{\mathrm{d}x_1}\right)^2\right]\mathrm{d}x_1,\\
\Pi_{\mathrm{a}} &= -\int_0^l q(x_1)w\,\mathrm{d}x_1
\end{aligned} \tag{7.3.15}$$

In the more general case of unsymmetric laminated beams and axial and lateral loadings we have $\Pi(u,w,\psi_1)$. The Π_{i} expression can be expanded to

$$\Pi_{\mathrm{i}} = \frac{1}{2}\int_0^L \left[\overline{A}_{11}\left(\frac{\mathrm{d}u}{\mathrm{d}x_1}\right)^2 + 2\overline{B}_{11}\frac{\mathrm{d}u}{\mathrm{d}x_1}\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1} + \overline{D}_{11}\left(\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1}\right)^2 + k^{\mathrm{s}}\overline{A}_{55}\left(\psi + \frac{\mathrm{d}w}{\mathrm{d}x_1}\right)^2\right]\mathrm{d}x_1 \tag{7.3.16}$$

and Π_{a} has to include axial and lateral loads.

Since the transverse shear strains are represented as constant through the laminate thickness, it follows that the transverse stresses will also be constant. In the elementary beam theory of homogeneous beams, the transverse shear stress varies parabolically through the beam thickness and in the classical laminate theory the transverse shear stresses vary quadratically through layer thickness. This discrepancy between the stress state compatible with the equilibrium equations and the constant stress state of the first order shear deformation theory can be overcome approximately by introducing a shear correction factor (Sect. 5.3).

The shear correction factor k^{s} can be computed such that the strain energy W_1 due to the classical transverse shear stress equals the strain energy W_2 due to the first order shear deformation theory. Consider, for example, a homogeneous beam with a rectangular cross-section $A = bh$. The classical shear stress distribution following from the course of elementary strength of materials is given by

$$\sigma_{13} = \tau_1 = \frac{3}{2}\frac{Q}{bh}\left[1 - \left(\frac{2x_3}{h}\right)^2\right], \quad -\frac{h}{2} \leq x_3 \leq +\frac{h}{2} \tag{7.3.17}$$

The transverse stress in the first order shear deformation theory is constant through the thickness h

$$\sigma_{13} = \tau_2 = \frac{Q}{bh}, \quad \gamma_2 = \frac{Q}{k^{\mathrm{s}}G} \tag{7.3.18}$$

With $W_1 = W_2$ it follows that

$$\frac{1}{2}\int\limits_{(A)} \frac{\tau_1^2}{G}\mathrm{d}A = \frac{1}{2}\int\limits_{(A)} \frac{\tau_2^2}{k^{\mathrm{s}}G}\mathrm{d}A,$$

$$\frac{3}{5}\frac{Q^2}{Gbh} = \frac{1}{k^{\mathrm{s}}}\frac{Q^2}{2Gbh} \Longrightarrow k^{\mathrm{s}} = \frac{5}{6} \tag{7.3.19}$$

The shear correction factor for a general laminate depends on lamina properties and lamina stacking and is given here without a special derivation by

$$\frac{1}{k^{\mathrm{s}}} = A_{55}b\sum_{k=1}^{N}\int\limits_{x_3^{(k-1)}}^{x_3^{(k)}} \frac{g^{(k)^2}(x_3)}{G^{(k)}}\mathrm{d}x_3, \tag{7.3.20}$$

$$g^{(k)}(z) = \overline{d}_{11}\left\{-C_{11}^{(k)}\frac{z^2}{2} + \sum_{j=1}^{k}\left[C_{11}^{(j)} - C_{11}^{(j-1)}\right]\frac{z^{(j-1)^2}}{2}\right\}, \quad C_{11}^{(0)} \equiv 0$$

$\overline{d}_{11} = 1/\overline{D}_{11}$ is the beam compliance, $C_{11}^{(k)} = E_1^{(k)}$.

Summarizing the beam equations for the first order shear deformation theory for symmetrically laminated cross-sections, including vibration and buckling, the following relations are valid for constant values of $h, b, A, \overline{D}_{11}, \rho$:

- *Flexure equations* ($N = 0, M \neq 0$)

$$\begin{aligned} k^{\mathrm{s}}\,\overline{A}_{55}\,[\psi_1(x_1) + w'(x_1)]' + q(x_1) &= 0,\\ [\overline{D}_{11}\psi_1'(x_1)]' - k^{\mathrm{s}}\,\overline{A}_{55}[\psi_1(x_1) + w'(x_1)] &= 0 \end{aligned} \tag{7.3.21}$$

  or Eqs. (7.3.10).

- *Forced or free vibrations equations*

$$\begin{aligned} k^{\mathrm{s}}\,\overline{A}_{55}\,[\psi_1(x_1,t) + w'(x_1,t)]' - \rho_0\ddot{w}(x_1,t) + q(x_1,t) &= 0,\\ [\overline{D}_{11}\psi_1'(x_1,t)]' - k^{\mathrm{s}}\,\overline{A}_{55}[\psi_1(x_1,t) + w'(x_1,t)] - \rho_2\ddot{\psi}_1(x_1,t) &= 0, \end{aligned} \tag{7.3.22}$$

$$\rho_0 = b\sum_{k=1}^{n}\rho^{(k)}\left(x_3^{(k)} - x_3^{(k-1)}\right), \quad \rho_2 = b\sum_{k=1}^{n}\frac{1}{3}\rho^{(k)}\left(x_3^{(k)^3} - x_3^{(k-1)^3}\right)$$

  The terms involving ρ_0 and ρ_2 are called translatory or rotatory inertia terms. For free vibrations we assume that the transverse load q is zero and the motion is periodic:

$$w(x_1,t) = W(x_1)\exp(i\omega t), \quad \psi_1(x_1,t) = \Psi_1(x_1)\exp(i\omega t)$$

- *Buckling equations*

$$\begin{aligned} k^{\mathrm{s}}\,\overline{A}_{55}\,[\psi_1(x_1) + w'(x_1)]' - N(x_1)w''(x_1) &= 0,\\ [\overline{D}_{11}\psi_1'(x_1)]' - k^{\mathrm{s}}\,\overline{A}_{55}[\psi_1(x_1) + w'(x_1)] &= 0 \end{aligned} \tag{7.3.23}$$

or with $N(x_1) = -F$

$$\overline{D}_{11}\left[1 - \frac{F}{k^s\overline{A}_{55}}\right] w''''(x_1) + Fw''(x_1) = 0$$

The variational formulation for the symmetric bending beam is given by Eq. (7.3.15).

For vibrations the Lagrange function $L(w,\psi_1) = T(w,\psi_1) - \Pi(w,\psi_1)$ yields the Hamilton's principle

$$\begin{aligned} H[w(x_1,t),\psi_1(x_1,t)] &= \int\limits_{t_1}^{t_2} L[w(x_1,t),\psi_1(x_1,t)]\mathrm{d}t,\\ \delta H[w(x_1,t),\psi_1(x_1,t)] &= 0 \end{aligned}$$

with

$$\begin{aligned} \Pi[w(x_1,t),\psi_1(x_1,t)] &= \frac{1}{2}\int\limits_0^l [\overline{D}_{11}\psi_1'^2 + k^s A_{55}(\psi_1 + w')^2]\mathrm{d}x_1\\ &\quad - \int\limits_0^l q(x_1,t)w\mathrm{d}x_1, \qquad (7.3.24)\\ T[w(x_1,t),\psi_1(x_1,t)] &= \frac{1}{2}\int\limits_0^l [\rho_0\dot{w}^2 + \rho_2\dot{\psi}_1^2]\mathrm{d}x_1 \end{aligned}$$

For buckling problems with $N(x_1) = -F$ it follows that

$$\begin{aligned} \Pi[w(x_1,t),\psi_1(x_1,t)] &= \frac{1}{2}\int\limits_0^l [\overline{D}_{11}\psi_1'^2 + k^s A_{55}(\psi_1 + w')^2]\mathrm{d}x_1\\ &\quad - \frac{1}{2}\int\limits_0^l Fw'^2\mathrm{d}x_1 \end{aligned} \qquad (7.3.25)$$

Equations (7.3.21) to (7.3.25) summarize the bending, buckling and vibration differential and variational statements for laminated beams based on the shear deformation theory.

7.4 Sandwich Beams

The similarity of the elastic behavior between symmetric laminates and symmetric sandwich beams in the first order shear deformation theory (Sects. 4.3 and 5.3) allows us to transpose the results derived above to the bending of sandwich beams.

In addition to the differences between the expressions for the flexural and transverse shear stiffness D_{11} and A_{55} the essential difference is at the level of stress distribution. The model assumptions for sandwich composites with thin and thick cover sheets are considered in detail in Sects. 4.3.1 to 4.3.3. There one can find the stiffness values A_{11}, D_{11}, A_{55}. With these values, all differential and variational formulation of the theory of laminated beams including transverse shear deformation can be transposed.

In the case of a symmetric sandwich beam with thin cover sheets we have, for example, the stiffness values

$$\begin{aligned}
A_{11} &= 2A^{\mathrm{f}}_{11} &&= 2\sum_{k=1}^{n} Q^{(k)}_{11} h^{(k)}, \\
D_{11} &= 2h^{\mathrm{c}} C^{\mathrm{f}}_{11} &&= h^{\mathrm{c}} \sum_{k=1}^{n} Q^{(k)}_{11} h^{(k)} \overline{x}^{(k)}, \\
A^{\mathrm{s}}_{55} &= h^{\mathrm{c}} G^{\mathrm{c}}_{13}
\end{aligned} \tag{7.4.1}$$

n is the number of the face layers.

The coefficient A^{s}_{55} can be corrected by a shear correction factor k^{s}. In addition to the calculation of k^{s}, derived for a laminated beam an approximate formula was developed by Reuss, for sandwich beams with thin cover sheets. With the inverse effective shear stiffness G_{R}^{-1}, given by the Reuss model, and the effective shear stiffness G_{V}, given by the Voigt model, we have

$$G_{\mathrm{V}} = \sum_{k=1}^{n} G^{(k)} \frac{h^{(k)}}{h} = \frac{\overline{A}_{55}}{bh}, \quad G_{\mathrm{R}}^{-1} = \sum_{k=1}^{n} \frac{1}{G^{(k)}} \frac{h^{(k)}}{h}, \quad k^{\mathrm{s}} = \frac{G_{\mathrm{R}}}{G_{\mathrm{V}}} \tag{7.4.2}$$

The use of sandwich structures is growing very rapidly. Sandwich beams has a high ratio of flexural stiffness to weight and in comparison to other beam structures they have lower lateral deformations, higher buckling resistance and higher natural frequencies. As a result sandwich constructions quite often provide a lower structural weight than other structural elements for a given set of mechanical and environmental loads.

The elastic behavior of sandwich beams was modelled by the laminate theory, Sect. 4.3, but it is appropriately to distinguish thin and thick sandwich faces. The differential equations or variational statements describing the structural behavior of sandwich beams generally based in the first order shear deformation theory Sect. 5.3, and only if very flexible cores are used a higher order theory may be needed.

Because of the continuing popularity of sandwich structures Sect. 7.4 intends to recall and summarizes the results of Sects. 4.3 and 5.3 to cover some of the most important aspects of sandwich beam applications.

7.4.1 Stresses and Strains for Symmetrical Cross-Sections

Figure 7.9 shows a sandwich beam with a symmetrical lay up, i.e. the faces have the same thickness h^{f} and are of the same material. As derived in Sect. 4.3 and 4.4 the flexural rigidy is $(D_{11} \equiv D_{11}^{\mathrm{La}}, Q_{11} \equiv E_1 = E)$

$$bD_{11} = D = b\left[\frac{E^{\mathrm{f}}(h^{\mathrm{f}})^3}{6} + \frac{E^{\mathrm{f}}h^{\mathrm{f}}d^2}{2} + \frac{E^{\mathrm{c}}(h^{\mathrm{c}})^3}{12}\right] = 2D^{\mathrm{f}} + D_{\mathrm{o}} + D^{\mathrm{c}} \tag{7.4.3}$$

and both, $2D^{\mathrm{f}}$ and D^{c} are less than 1% of D_{o} if $d/h^{\mathrm{f}} > 5,77$ and $(6E^{\mathrm{f}}h^{\mathrm{f}}d^2)/E^{\mathrm{c}}(h^{\mathrm{c}})^3 > 100$. Thus, for a sandwich with thin faces $h^{\mathrm{f}} \ll h^{\mathrm{c}}$ and a weak core, $E^{\mathrm{c}} \ll E^{\mathrm{f}}$, the flexural rigidity is approximately

$$D \approx D_{\mathrm{o}} = b\frac{E^{\mathrm{f}}h^{\mathrm{f}}d^2}{2} \tag{7.4.4}$$

It can be noted that in most engineering applications using structural sandwich beam elements, the dominating term in flexural rigidity is that of the faces bending about the neutral axes of the beam, i.e. the dominating part D_{o} of the total rigidity D originating from a direct tension-compression of the cover sheets. But is there no monolithic bonding between the faces and the core the flexural rigidity will be nearly lost.

The following derivations assume in-plane-, bending- and shear stiffness for all layers, i.e. for the faces and the core. Therefore we use the laminate theory including transverse shear, Sects. 4.3.3 and 5.3. All calculations are first restricted to midplane symmetric beams.

The bending strains vary linearly with x_3 over the cross-section:

$$\varepsilon_1^M = \frac{M}{D}x_3 \tag{7.4.5}$$

Unlike the bending strains, which vary linearly with x_3 over the whole cross-section, the bending stresses vary linearly within each material constituent, but there is a jump in the stresses at the face/core interfaces:

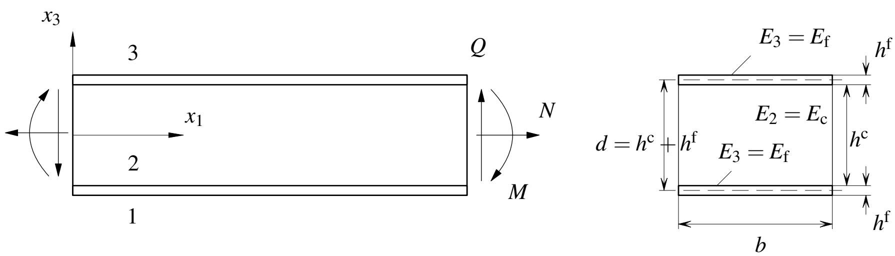

Fig. 7.9 Symmetrical sandwich beam: $N = bN_1, Q = bQ_1, M = bM_1$ are the beam stress resultants

$$\sigma_1^{(k)} = M\frac{E^{(k)}}{D}x_3 \quad (k=1,2,3) \quad \Rightarrow \quad \begin{aligned} \sigma^{\mathrm{f}} &= M\frac{E^{\mathrm{f}}}{D}x_3 \\ \sigma^{\mathrm{c}} &= M\frac{E^{\mathrm{c}}}{D}x_3 \end{aligned} \tag{7.4.6}$$

With Eq. (7.2.12) follows

$$M = bD_{11}\kappa_1, \quad \kappa_1 = \frac{M}{bD_{11}} = \frac{h^3}{12D_{11}}\frac{M}{bh^3/12} = \frac{M}{E_{\mathrm{eff}}^M I}, \tag{7.4.7}$$

with

$$E_{\mathrm{eff}} = \frac{12}{h^3}D_{11}, \quad I = \frac{bh^3}{12}$$

and the stress equations can be written as

$$\sigma_1^{(k)} = \frac{E^{(k)}}{E_{\mathrm{eff}}^M}\frac{M}{I}x_3 \quad (k=1,2,3) \tag{7.4.8}$$

The strains due to in-plane loading are:

$$\varepsilon_1^N = \frac{N}{\sum\limits_{k=1}^{3} Q^{(k)}h^{(k)}} = \frac{N}{\sum\limits_{k=1}^{3} E^{(k)}h^{(k)}} = \frac{N}{bh}\frac{h}{A_{11}} = \frac{N}{E_{\mathrm{eff}}^N A} \tag{7.4.9}$$

with

$$E_{\mathrm{eff}}^N = \frac{A_{11}}{h}, \quad A_{11}\sum_{k=1}^{3} Q_{11}^{(k)}h^{(k)} = \sum_{k=1}^{3} E^{(k)}h^{(k)}, \quad A = bh$$

ε_1^N is the strain of the neutral axis. The in-plane stresses follow to

$$\sigma_1^{(k)} = \frac{E^{(k)}}{E_{\mathrm{eff}}^N}\frac{N}{A} \quad (k=1,2,3) \quad \Rightarrow \quad \begin{aligned} \sigma^{\mathrm{f}} &= \frac{E^{(1)}}{E_{\mathrm{eff}}^N}\frac{N}{A} = \frac{E^{(3)}}{E_{\mathrm{eff}}^N}\frac{N}{A} \\ \sigma^{\mathrm{c}} &= \frac{E^{(2)}}{E_{\mathrm{eff}}^N}\frac{N}{A} \end{aligned} \tag{7.4.10}$$

The strains and stresses due to in-plane loads and bending can be superimposed.

In the same manner as outlined above a general definition can be found for shear strains and shear stresses. Consider the beam element $b(x_3^{(3)} - x_3)\mathrm{d}x_1$, Fig. 7.10. The upper edge of the sandwich, i.e. $x_3 = (d + h^{\mathrm{f}})/2$, Fig. 7.9, is stress free and we have $\tau[(d + h^{\mathrm{f}})/2] = 0$.

Since we restrict on calculations to midplane symmetric beams all derivation can be given for the upper part of the beam element ($x_3 \geq 0$). The equilibrium equation in the x_1-direction yield with $\sigma_1 d\sigma_1 \approx \sigma_1 + (\partial\sigma_1/\partial x_1)\mathrm{d}x_1$ and no edge shear stresses on the upper face

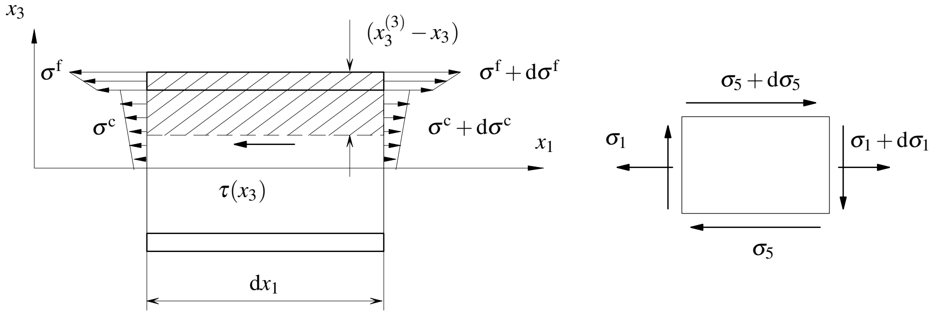

Fig. 7.10 Sandwich beam element $b(x_3^{(3)} - x_3){\rm d}x_1$: $\sigma^{(3)}(x_3) \equiv \sigma^{\rm f}(x_3)$, $\sigma^{(2)}(x_3) \equiv \sigma^{\rm c}(x_3)$, $\sigma_5(x_3) \equiv \tau(x_3)$

$$\tau(x_3)b{\rm d}x_1 - \int\limits_{x_3}^{(d+h^{\rm f})/2} \left[\left(\sigma_1 + \frac{\partial \sigma_1}{\partial x_1}{\rm d}x_1\right) - \sigma_1\right] b{\rm d}x_3 = 0$$

$$\Rightarrow \tau(x_3) = \int\limits_{x_3}^{(d+h^{\rm f})/2} \frac{\partial \sigma_1}{\partial x_1}{\rm d}x_3 = 0 \qquad (7.4.11)$$

Using the relations $dM(x_1)/{\rm d}x_1 = Q(x_1)$ and $\sigma_1 = M(E(x_3)/D)x_3$ we have

$$\frac{{\rm d}\sigma_1}{{\rm d}x_1} = \frac{Q(x_1)}{D}E(x_3)x_3, \quad \tau(x_3) = \frac{Q}{bD} \int\limits_{x_3}^{(d+h^{\rm f}/2)} E(x_3)x_3 b{\rm d}x_3 = \frac{Q}{bD}S(x_3) \qquad (7.4.12)$$

$S(x_3)$ is the first moment of the area $(x_3^{(3)} - x_3)b$. For a single layer homogeneous and isotropic beam we have the well-known formula

$$S(x_3) = \frac{b}{2}\left(\frac{h^2}{4} - x_3^2\right); \quad h = (d + h^{\rm f})$$

For sandwich beams we have a more generalized definition for the first moment of area:

$$\int\limits_{x_3}^{(d+h^{\rm f})/2} E(x_3)x_3 b{\rm d}x_3 = \begin{cases} b\left[\dfrac{E^{\rm f}h^{\rm f}d}{2} + \dfrac{E^{\rm c}}{2}\left(\dfrac{h^{\rm c}}{2} - x_3\right)\left(\dfrac{h^{\rm c}}{2} + x_3\right)\right], \\ |x_3| \leq \dfrac{h^{\rm c}}{2} \\ b\left[\dfrac{E^{\rm f}}{2}\left(\dfrac{h^{\rm c}}{2} + h^{\rm f} - x_3\right)\left(\dfrac{h^{\rm c}}{2} + h^{\rm f} + x_3\right)\right], \\ \dfrac{h^{\rm c}}{2} \leq |x_3| \leq \dfrac{h^{\rm c}}{2} + h^{\rm f} \end{cases} \qquad (7.4.13)$$

The shear stresses for the core and the faces are

$$\begin{aligned}\tau^{\mathrm{c}}(x_3) &= \frac{Q}{D}\left[\frac{E^{\mathrm{f}}h^{\mathrm{f}}d}{2} + \frac{E^{\mathrm{c}}}{2}\left(\frac{(h^{\mathrm{c}})^2}{4} - x_3^2\right)\right],\\ \tau^{\mathrm{f}}(x_3) &= \frac{Q}{D}\frac{E^{\mathrm{f}}}{2}\left(\frac{(h^{\mathrm{c}})^2}{4} + h^{\mathrm{c}}h^{\mathrm{f}} + (h^{\mathrm{f}})^2 - x_3^2\right)\end{aligned} \tag{7.4.14}$$

The maximum shear stress appears at the neutral axes:

$$\tau_{\max} = \tau^{\mathrm{c}}(x_3 = 0) = \frac{Q}{D}\left(\frac{E^{\mathrm{f}}h^{\mathrm{f}}d}{2} + \frac{E^{\mathrm{c}}(h^{\mathrm{c}})^2}{8}\right) \tag{7.4.15}$$

The shear stress in the core/face interface is

$$\tau^{\mathrm{c}}_{\min} \equiv \tau^{\mathrm{f}}_{\max} = \tau\left(\frac{h^{\mathrm{c}}}{2}\right) = \frac{Q}{D}\left(\frac{E^{\mathrm{f}}h^{\mathrm{f}}d}{2}\right) \tag{7.4.16}$$

There is no jump in the shear stresses at the interfaces and the shear stresses are zero at the outer fibres of the faces. If we have

$$\frac{4E^{\mathrm{f}}h^{\mathrm{f}}d}{E^{\mathrm{c}}(h^{\mathrm{c}})^2} > 100 \tag{7.4.17}$$

the shear stresses in the core are nearly constant. The difference between $\tau^{\mathrm{c}}_{\max}$ and $\tau^{\mathrm{c}}_{\min}$ is less than 1%. As it was outlined in Sect. 4.3, the stress equation in sandwich beams very often can be simplified.

Summarizing the stress estimations due to bending and shear for symmetrical sandwich beams we have the following equations:

1. The core is weak, $E^{\mathrm{c}} \ll E^{\mathrm{f}}$, but the faces can be thick

$$\begin{aligned}\sigma^{\mathrm{f}}(x_3) &\approx M\frac{E^{\mathrm{f}}}{D_{\mathrm{o}} + 2D^{\mathrm{f}}}x_3,\\ \sigma^{\mathrm{c}}(x_3) &\approx 0,\\ \tau^{\mathrm{f}}(x_3) &\approx \frac{Q}{D_{\mathrm{o}} + 2D^{\mathrm{f}}}\frac{E^{\mathrm{f}}}{2}\left(\frac{(h^{\mathrm{c}})^2}{4} + h^{\mathrm{c}}h^{\mathrm{f}} + (h^{\mathrm{f}})^2 - x_3^2\right),\\ \tau^{\mathrm{c}}(x_3) &\approx \frac{QE^{\mathrm{f}}h^{\mathrm{f}}d}{2(D_{\mathrm{o}} + 2D^{\mathrm{f}})}\end{aligned} \tag{7.4.18}$$

2. The core is weak, $E^{\mathrm{c}} \ll E^{\mathrm{f}}$, and the faces are thin, $h^{\mathrm{f}} \ll h^{\mathrm{c}}$

$$\sigma^{\mathrm{f}}(x_3) \approx \pm\frac{M}{bh^{\mathrm{f}}d}, \quad \sigma^{\mathrm{c}}(x_3) \approx 0, \quad \tau^{\mathrm{f}}(x_3) \approx 0, \quad \tau^{\mathrm{c}}(x_3) \approx \frac{Q}{bd} \tag{7.4.19}$$

This approximation can be formulated as: The faces of the sandwich beam carry bending moments as constant tensile and compressive stresses and the core carries the transverse forces as constant shear stresses.

7.4.2 Stresses and Strains for Non-Symmetrical Cross-Sections

In engineering applications also sandwich beams with dissimilar faces are used, Fig. 7.11. The first moment of area is zero when integrated over the entire cross-section and x_3 is the coordinate from the neutral axes

$$\int E(x_3)x_3 b\mathrm{d}x_3 = 0 \tag{7.4.20}$$

The location of neutral axis is unknown. With the coordinate transformation $x_3^* = x_3 - e$ from a known axis of the cross-section the equation above becomes

$$S(x_3) = \int E(x_3)x_3 b\mathrm{d}x_3 = \int E(x_3^* + e)b\mathrm{d}x_3^* = 0, \quad e\int E\mathrm{d}x_3^* = -\int E x_3^*\mathrm{d}x_3^*$$

For the sandwich cross-section, Fig. 7.11, follows

$$\begin{aligned} &e\left(E^{(1)}h^{(1)} + E^{(2)}h^{(2)} + E^{(3)}h^{(3)}\right) \\ &= E^{(1)}h^{(1)}\left(\frac{1}{2}h^{(1)} + h^{(2)} + \frac{1}{2}h^{(3)}\right) + \frac{1}{2}E^{(2)}h^{(2)}\left(h^{(2)} + h^{(3)}\right) \end{aligned}$$

and we get an equation for the unknown value e

$$e = \frac{E^{(1)}h^{(1)}\left(h^{(1)} + 2h^{(2)} + h^{(3)}\right) + E^{(2)}h^{(2)}\left(h^{(2)} + h^{(3)}\right)}{2\left(E^{(1)}h^{(1)} + E^{(2)}h^{(2)} + E^{(3)}h^{(3)}\right)} \tag{7.4.21}$$

If the core is weak, $E^{(2)} \ll (E^{(1)}, E^{(3)})$ we have approximately

$$e = \frac{E^{(1)}h^{(1)}d}{E^{(1)}h^{(1)} + E^{(3)}h^{(3)}} \quad \text{or} \quad d - e = \frac{E^{(3)}h^{(3)}d}{E^{(1)}h^{(1)} + E^{(3)}h^{(3)}}, \tag{7.4.22}$$

where $d = \frac{1}{2}h^{(1)} + h^{(2)} + \frac{1}{2}h^{(3)}$.

The bending stiffness $D = \int E(x_3)x_3^2 b\mathrm{d}x_3$ yields in the general case

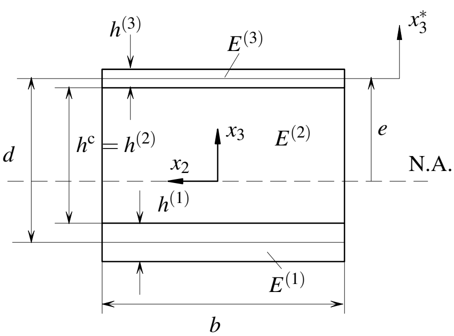

Fig. 7.11 Definition of the neutral axis (N.A.) of an unsymmetrical sandwich: $x_3 = x_3^* + e$

$$
\begin{aligned}
D = \frac{1}{12}\left[E^{(1)}(h^{(1)})^3 + E^{(2)}(h^{(2)})^3 + E^{(3)}(h^{(3)})^3\right] \\
+ E^{(1)}h^{(1)}(d-e)^2 + E^{(3)}h^{(3)}e^2 + E^{(2)}h^{(2)}\left[\frac{1}{2}(h^{(2)}+h^{(3)}) - e\right]^2
\end{aligned}
\qquad (7.4.23)
$$

and can be simplified for $E^{(2)} \ll (E^{(1)}, E^{(3)})$ but thick faces as

$$
D \approx \frac{E^{(1)}(h^{(1)})^3}{12} + \frac{E^{(3)}(h^{(3)})^3}{12} + \frac{E^{(1)}h^{(1)}E^{(3)}h^{(3)}d^2}{E^{(1)}h^{(1)} + E^{(3)}h^{(3)}} \qquad (7.4.24)
$$

For thin faces the first two terms vanish

$$
D \approx D_{\mathrm{o}} = \frac{E^{(1)}h^{(1)}E^{(3)}h^{(3)}d^2}{E^{(1)}h^{(1)} + E^{(3)}h^{(3)}} \qquad (7.4.25)
$$

Now the bending and shearing stresses can be calculated in the usual way

$$
\sigma_1(k)(x_3) = M\frac{E^{(k)}}{D}x_3, \quad \tau(k)(x_3) = \frac{Q}{bD}S(x_3) \qquad (7.4.26)
$$

For sandwich beams with weak core and thin but dissimilar faces the stress formulas are approximately

$$
\begin{aligned}
\sigma_1^{(3)} &\equiv \sigma_1^{\mathrm{f}_1} \approx M\frac{E^{\mathrm{f}_1}}{D}e = \frac{M}{bh^{\mathrm{f}_1}d}, \\
\sigma_1^{(1)} &\equiv \sigma_1^{\mathrm{f}_2} \approx -M\frac{E^{\mathrm{f}_2}}{D}(d-e) = -\frac{M}{bh^{\mathrm{f}_2}d}, \\
\tau^{(2)} &\equiv \tau^{\mathrm{c}} \;\; \approx \frac{Q}{bd}, \qquad \tau^{(3)} = \tau^{(1)} \approx 0
\end{aligned}
\qquad (7.4.27)
$$

7.4.3 Governing Sandwich Beam Equations

The following derivations assumed, as generally in Chap. 7, straight beams with at least single symmetric constant cross-sections which are rectangular, i.e we consider single core sandwich beams. The faces can be thin or thick and symmetrical or non-symmetrical. The bending moments and axial forces act in the plane of symmetry $(x_1 - x_3)$. The influence of transverse shear deformation is included, because the core of sandwich beams has a low transverse modulus of rigidity G_{13}. The shear correction factor k^{s} is determined approximately for sandwich beams with thin cover sheets with the Reuss formula (7.4.1), or more generally using equivalent shear strain energy, i.e the potential energy of the applied load equals the strain energy of the beam to account for the nonuniform shear distribution through the thickness. The shear deformation theory (Sect. 7.3) is valid and we can adapt the equations of this section to the special case of sandwich beams.

The strains ε_1 and $\varepsilon_5 \equiv \gamma$ are given, (7.3.2), as

$$\varepsilon_1 = \frac{\mathrm{d}u}{\mathrm{d}x_1} + x_3 \frac{\mathrm{d}\psi}{\mathrm{d}x_1}, \quad \gamma = \frac{\mathrm{d}w}{\mathrm{d}x_1} + \psi$$

With

$$\overline{A}_{11} = A, \quad \overline{B}_{11} = B, \quad \overline{D}_{11} = D, \quad k^{\mathrm{s}}\overline{A}_{55} = S \tag{7.4.28}$$

the constitutive equations (7.2.12), (7.2.13) yield

$$\begin{bmatrix} N \\ M \\ Q \end{bmatrix} = \begin{bmatrix} A & B & 0 \\ B & D & 0 \\ 0 & 0 & S \end{bmatrix} \begin{bmatrix} u' \\ \psi' \\ w' + \psi \end{bmatrix} \tag{7.4.29}$$

For static loading $q(x_1) \neq 0, n(x_1) \equiv 0$ the equilibrium equations are as in the classical beam theory, Table 7.1,

$$N' = 0, \quad Q' + q = 0, \quad M' - Q = 0 \tag{7.4.30}$$

If $N \equiv 0$ the neutral axes position $x_3^{\text{N.A.}}$ is constant along the length of the beam and is given by

$$\varepsilon_1(x_3^{\text{N.A.}}) = u' + x_3^{\text{N.A.}}\psi' = 0, \quad N = Au' + B\psi' = 0$$

$$x_3^{\text{N.A.}} = -\frac{u'}{\psi'} = \frac{B}{A} \tag{7.4.31}$$

Thus, if the stiffness A, B, D, S are constant, the substitution of Eq. (7.4.29) into (7.4.30) yields the following two governing differential equations for sandwich beams

$$\begin{aligned} D_{\mathrm{R}}\psi''(x_1) - S[w'(x_1) + \psi(x_1)] &= 0, \\ S[w''(x_1) + \psi'(x_1)] &= -q(x_1) \end{aligned} \tag{7.4.32}$$

with $D_{\mathrm{R}} = D - (B^2/A)$. Derivation of the first equation and setting in the second equation yield

$$D_{\mathrm{R}}\psi'''(x_1) = -q(x_1) \tag{7.4.33}$$

and with

$$M' = Q = S(w' + \psi), \quad M = Bu' + D_{\mathrm{R}}\psi', \quad M' = Bu'' + D_{\mathrm{R}}\psi''$$

follow

$$w'(x_1) = -\psi(x_1) + \frac{1}{S}(Bu'' + D_{\mathrm{R}}\psi'')$$

For symmetrical cross-sections is $B = 0, D_{\mathrm{R}} = D$. In the general case, if all stiffness are constant and unequal zero the substitution of (7.4.4) into (7.4.5) yields the governing simultaneous differential equations for unsymmetrical sandwich beams

$$\begin{aligned} D_{\mathrm{R}}\psi''(x_1) - S[w'(x_1) + \psi(x_1)] &= 0, \quad D_{\mathrm{R}} = D - \left(\frac{B^2}{A}\right) \\ S(w''(x_1) + \psi'(x_1)) &= -q(x_1) \\ u'(x_1) &= -\frac{B}{A}\psi'(x_1) \end{aligned} \tag{7.4.34}$$

Derivation of the first equation and setting in the second equation yield one uncoupled equation for $\psi(x_1)$

$$D_{\mathrm{R}}\psi'''(x_1) = -q(x_1) \tag{7.4.35}$$

The constitutive equations (7.4.4) give the relations

$$M(x_1) = Bu'(x_1) + D\psi'(x_1), \quad Q(x_1) = S[w'(x_1) + \psi(x_1)] \tag{7.4.36}$$

and with

$$M'(x_1) = Q(x_1) = Bu''(x_1) + D\psi''(x_1), \quad u'(x_1) = -\frac{B}{A}\psi'(x_1)$$

follow

$$w'(x_1) = -\psi(x_1) + \frac{1}{S}[Bu''(x_1) + D_{\mathrm{R}}\psi''(x_1)] = -\psi(x_1) + \frac{D_{\mathrm{R}}}{S}\psi''(x_1) \tag{7.4.37}$$

Thus we have three uncoupled differential equations:

$$\begin{aligned} D_{\mathrm{R}}\psi'''(x_1) &= -q(x_1), \\ w'(x_1) &= -\psi(x_1) + \frac{D_{\mathrm{R}}}{S}\psi''(x_1), \\ u'(x_1) &= \frac{B}{A}\psi'(x_1) \end{aligned} \tag{7.4.38}$$

For symmetrically cross-sections is $B \equiv 0$, $D_{\mathrm{R}} \equiv D$ and the differential equations reduce to

$$\begin{aligned} D\psi'''(x_1) &= -q(x_1), \\ w'(x_1) &= -\psi(x_1) + \frac{D}{S}\psi''(x_1)), \\ M(x_1) &= D\psi'(x_1) \quad \text{or} \quad \psi'(x_1) = \frac{M(x_1)}{E_{\mathrm{eff}}^{M} I} \\ Q(x_1) &= D\psi''(x_1) \quad \text{or} \quad M'(x_1) = S(w'(x_1) + \psi(x_1)) \end{aligned} \tag{7.4.39}$$

The equations (7.4.39) correspond to the equations (7.3.10) of the laminated beam and the analytical solutions (7.3.11), (7.3.12) can be transposed with $\overline{D}_{11} = D$, $k^{\mathrm{s}}\overline{A}_{55} = S$. In dependence of the calculation of the stiffness D and S the equation are valid for sandwich beams with thin or thick faces.

The stresses σ_1 and τ can be calculated with the help of the stress formulas derived in Sects. 7.4.1 and 7.4.2. For statically determinate structures, $M(x_1)$ and $Q(x_1)$ can be calculated with the equilibrium equations and the last two equations (7.4.39) can directly used for static calculations.

We consider as an example the cantilever beam, Fig. 7.12, then:

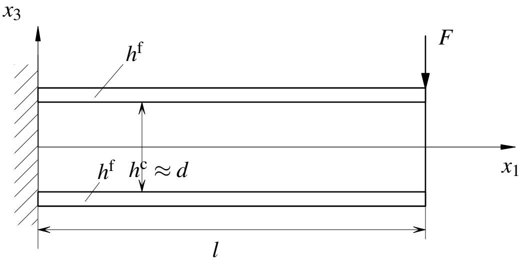

Fig. 7.12 Symmetrical cantilever beam with thin faces

$$
\begin{aligned}
&M(x_1) = F(l-x_1), \quad Q(x_1) = -F,\\
&\psi'(x_1) = \frac{M(x_1)}{D} = \frac{1}{D}F(l-x_1),\\
&\psi(x_1) = \frac{F}{D}\left(lx_1 - \frac{1}{2}x_1^2\right) + C_1,\\
&\psi(0) = 0 \quad \Rightarrow \quad C_1 = 0,\\
&w'(x_1) = -\frac{F}{D}\left(lx_1 - \frac{1}{2}x_1^2\right) + \frac{Q}{S} = -\frac{F}{D}\left(lx_1 - \frac{1}{2}x_1^2\right) - \frac{F}{S},\\
&w(x_1) = -\frac{F}{D}\left(\frac{1}{2}lx_1^2 - \frac{1}{6}x_1^3\right) - \frac{Fx_1}{S} + C_2,\\
&w(0) = 0 \quad \Rightarrow \quad C_2 = 0,\\
&w(x_1) = -\frac{F}{D}\left(\frac{1}{2}lx_1^2 - \frac{1}{6}x_1^3\right) - \frac{Fx_1}{S} = w^{\mathrm{B}}(x_1) + w^{\mathrm{S}}(x_1),\\
&w(l) = -\frac{Fl^3}{3D} - \frac{Fl}{S}
\end{aligned}
$$

Assume thin faces and weak core, i.e

$$h^{\mathrm{f}} \ll h^{\mathrm{c}}, \quad E^{\mathrm{c}} \ll E^{\mathrm{f}}$$

we have

$$D = D_{\mathrm{o}} = b\frac{E^{\mathrm{f}}h^{\mathrm{f}}d^2}{2}, \quad S = k^{\mathrm{s}}G^{\mathrm{c}}bd$$

and the stresses are

$$\sigma^{\mathrm{f}} = \pm M\frac{E^{\mathrm{f}}}{D}\frac{\mathrm{d}}{2} = \pm\frac{M}{bdh^{\mathrm{f}}} = \pm\frac{F(l-x_1)}{bdh^{\mathrm{f}}}, \quad \tau^{\mathrm{c}} = \frac{Q}{bd} = -\frac{F}{bd}, \quad \sigma^{\mathrm{c}} = \tau^{\mathrm{f}} = 0$$

Consider $w(l) = w^{\mathrm{B}}(l) + w^{\mathrm{S}}(l)$ it can be seen that the shear deformation strongly depends on l and S. It is important for short and shear weak beams and negligible for slender shear stiff beams.

Summarizing the aspects of sandwich beams it could be demonstrated in the static case that the shear deformation theory for laminated beams is valid for sandwich beams, if the stiffness $\overline{A}_{11},\overline{B}_{11}$ and $\overline{D}_{11}$ of a laminated beam are replaced by the stiffness A,B,D and S, Eq. (7.4.3), of the sandwich beam. The same conclusion is valid not only for bending but also for buckling and vibration and for differential and variational formulations. In this way all formulas (7.3.19) to (7.3.23) can easy transposed to symmetrically sandwich beams.

In the considerations above we have assumed that the effect of core transverse deformability is negligible on the bending, vibration and the overall buckling of sandwich beams. But in a special case of buckling, called face wrinkling the transverse normal stiffness of the core has an important influence. Wrinkling is a form of local instability of thin faces associated with short buckling waves. This phenomenon was not discussed here.

7.5 Hygrothermo-Elastic Effects on Beams

In Sects. 7.2 and 7.4 the effect of mechanical loads acting upon fibre reinforced beams with $E_1(x_2,x_3) = E_1(-x_2,x_3)$ and laminated or sandwich beams was considered. The considerations for laminated beams as derived are valid in the framework of the classical laminate theory, Sect. 7.2, and of the first order shear deformation theory, Sect. 7.3. Section 7.4 considered some special aspects of sandwich beams with thin or thick cover sheets and different stiffness of the core.

In the present section the effects of hygrothermally induced strains, stresses and displacements are examined. We assume a moderate hygrothermal loading such that the mechanical properties remain unchanged for the temperature and moisture differences considered.

With Eqs. (4.2.63) to (4.2.68) the beam equations (7.2.1) have additional terms

$$\begin{aligned}\varepsilon_1(x_3) = \varepsilon_1 + x_3\kappa_1 &= \frac{\sigma_1(x_2,x_3)}{E_1(x_2,x_3)}\\ &+ [\alpha^{\mathrm{th}}(x_2,x_3)T(x_2,x_3) + \alpha^{\mathrm{mo}}(x_2,x_3)M^*(x_2,x_3)],\end{aligned} \tag{7.5.1}$$

$$\begin{aligned}\sigma_1(x_2,x_3) &= E_1(x_2,x_3)[\varepsilon_1 + x_3\kappa_1\\ &- \alpha^{\mathrm{th}}(x_2,x_3)T(x_2,x_3) - \alpha^{\mathrm{mo}}(x_2,x_3)M^*(x_2,x_3)]\end{aligned} \tag{7.5.2}$$

$\alpha^{\mathrm{th}}, \alpha^{\mathrm{mo}}$ are the thermal and moisture expansion coefficients, T the temperature change and M^* the weight of moisture absorption per unit weight. Equations (7.2.4) have now additional terms $N^{\mathrm{th}}, N^{\mathrm{mo}}, M^{\mathrm{th}}, M^{\mathrm{mo}}$, the so-called fictitious hygrothermal resultants, (4.2.67), and with

$$\tilde{N} = N + N^{\mathrm{th}} + N^{\mathrm{mo}}, \quad \tilde{M} = M + M^{\mathrm{th}} + M^{\mathrm{mo}} \tag{7.5.3}$$

the extended hygrothermal constitutive equation for the composite beam are

$$\begin{bmatrix}\tilde{N}\\ \tilde{M}\end{bmatrix} = \begin{bmatrix}a & b\\ b & d\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \kappa_1\end{bmatrix}, \quad \begin{bmatrix}\varepsilon_1\\ \kappa_1\end{bmatrix} = \begin{bmatrix}a & b\\ b & d\end{bmatrix}^{-1}\begin{bmatrix}\tilde{N}\\ \tilde{M}\end{bmatrix} \tag{7.5.4}$$

The stress formula (7.2.6) yields with Eq. (7.2.5)

$$\sigma_1(x_2,x_3) = \frac{(dN-bM)+(aM-bN)x_3}{ad-b^2}E_1(x_2,x_3) \\ - E_1(x_2,x_3)[\alpha^{\mathrm{th}}T+\alpha^{\mathrm{mo}}M^*] \tag{7.5.5}$$

For double symmetric cross-sectional geometry the coupling coefficient is zero and the stress equation can be simplified.

For uniform fibre distribution, i.e. $\phi = \text{const}$, (7.2.9) follow for a,b,d and the stress relations for fibres and matrices material are

$$\begin{aligned} \sigma_{\mathrm{f}}(x_3) &= (\tilde{N}/A+x_3\tilde{M}/I)(E_{\mathrm{f}}/E_1)-E_{\mathrm{f}}(\alpha^{\mathrm{th}}T+\alpha^{\mathrm{mo}}M^*), \\ \sigma_{\mathrm{m}}(x_3) &= (\tilde{N}/A+x_3\tilde{M}/I)(E_{\mathrm{m}}/E_1)-E_{\mathrm{m}}(\alpha^{\mathrm{th}}T+\alpha^{\mathrm{mo}}M^*) \end{aligned} \tag{7.5.6}$$

With $E_{\mathrm{f}} = E_{\mathrm{m}} = E_1 = E$ comes the stress equation for isotropic beams with mechanical and hygrothermal loadings

$$\sigma(x_3) = \frac{\tilde{N}}{A}+x_3\frac{\tilde{M}}{I}-E(\alpha^{\mathrm{th}}T+\alpha^{\mathrm{mo}}M^*) \tag{7.5.7}$$

For laminate or sandwich beams the developments are similar. All problems are linear and the principle of superposition is valid and can be used to calculate the hygrothermal effects. Consider for example a symmetric laminate beam in the frame of the classical laminate theory and include hygrothermal loads, (7.2.27) yield

$$\sigma_1^{(k)} = E_1^{(k)}\frac{\tilde{N}}{E_{\mathrm{eff}}^{N}A}+x_3\frac{\tilde{M}}{E_{\mathrm{eff}}^{M}I}-E_1^{(k)}(\alpha^{th(k)}T^{(k)}+\alpha^{mo(k)}M^{*(k)}) \tag{7.5.8}$$

The differential equations for deflection and midplane displacement of a symmetric laminated beam are

$$\begin{aligned} [\overline{D}_{11}w''(x_1)]'' &= q(x_1)-M^{\mathrm{th}}(x_1)''-M^{\mathrm{mo}}(x_1)'', \\ [\overline{A}_{11}u'x_1)]' &= -n(x_1)+N^{\mathrm{th}}(x_1)'-N^{\mathrm{mo}}(x_1)' \end{aligned} \tag{7.5.9}$$

In an analogous manner the differential equations including shear deformation can be found. The differential equation for a symmetric Timoshenko's beam with lateral loading and hygrothermal effects follows with (7.3.10)

$$[\overline{D}_{11}\psi_1'(x_1)]'' = -q(x_1)+M^{\mathrm{th}}(x_1)''+M^{\mathrm{mo}}(x_1)'' \tag{7.5.10}$$

The relation for the layer stresses $\sigma_1^{(k)}$ are identical to the classical theory. The transverse shear stresses $\sigma_5^{(k)}$ are not changed by hygrothermal effects.

7.6 Analytical Solutions

The differential equations for bending, vibration and buckling of symmetric laminated beams loaded orthogonally to the plane of lamination are summarized by

(7.2.33) to (7.2.35) for the classical Bernoulli's beam theory and by (7.3.21) to (7.3.23) for the Timoshenko's beam theory including transverse shear deformation. All stiffness and material parameters are constant values.

The simplest problem is the analysis of bending. The general solution of the differential equation of 4th order (7.2.33) for any load $q(x_1)$ is given by

$$\begin{aligned} bD_{11}w(x_1) &\equiv bD_{11}w^{\mathrm{B}}(x_1) \\ &= C_1\frac{x_1^3}{6} + C_2\frac{x_1^2}{2} + C_3x_1 + C_4 + \iiiint q(x_1)\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1 \end{aligned} \tag{7.6.1}$$

The general solution of the Timoshenko's beam is given by (7.3.12) in the form $w(x_1) = w^{\mathrm{B}}(x_1) + w^{\mathrm{S}}(x_1)$. The correction term $w^{\mathrm{S}}(x_1)$ describes the influence of the shear deformation and it decreases with increasing shear stiffness $k^{\mathrm{s}}A_{55}$.

The free vibration of Bernoulli's beams is modelled by (7.2.34), rotatory inertia terms are neglected. The partial differential equation

$$\frac{\partial^4 w(x_1,t)}{\partial x_1^4} + \frac{\rho A}{bD_{11}}\frac{\partial^2 w(x_1,t)}{\partial t^2} = 0 \tag{7.6.2}$$

can be separated with $w(x_1,t) = W(x_1)T(t)$ and yields

$$W''''(x_1)T(t) = -\frac{\rho A}{bD_{11}}W(x_1)\ddot{T}(t) \tag{7.6.3}$$

or

$$\frac{\ddot{T}(t)}{T(t)} = -\frac{\rho A}{bD_{11}}\frac{W''''(x_1)}{W(x_1)} = -\omega^2 \tag{7.6.4}$$

We get two differential equations

$$\ddot{T}(t) + \omega T(t) = 0, \quad W''''(x_1) - \frac{\rho A}{bD_{11}}\omega^2 W(x_1) = 0 \tag{7.6.5}$$

with the solutions

$$\begin{aligned} T(t) &= A\cos\omega t + B\sin\omega t, \\ W(x_1) &= C_1\cos\frac{\lambda}{l}x_1 + C_2\sin\frac{\lambda}{l}x_1 + C_3\cosh\frac{\lambda}{l}x_1 + C_4\sinh\frac{\lambda}{l}x_1, \\ \left(\frac{\lambda}{l}\right)^4 &= \frac{\rho A}{bD_{11}}\omega^2 \end{aligned} \tag{7.6.6}$$

The vibration mode is periodic, and ω is called the natural circular frequency. The mode shapes depend on the boundary conditions of the beam. Consider, for example, a simply supported beam, we have $W(0) = W''(0) = W(l) = W''(l) = 0$ and therefore $C_1 = C_3 = C_4 = 0$ and $C_2\sin(\lambda/l)l = C_2\sin\lambda = 0$, which implies that

$$\lambda = n\pi, \quad \omega_n^2 = \frac{n^4\pi^4}{\rho A}\frac{bD_{11}}{l^4}, \quad \omega_n = \left(\frac{n\pi}{l}\right)^2\sqrt{\frac{bD_{11}}{\rho A}} \tag{7.6.7}$$

For each n there is a different natural frequency and a different mode shape. The lowest natural frequency, corresponding to $n = 1$, is termed the fundamental frequency. If the laminate beam is unsymmetric to the middle surface, i.e. $B_{11} \neq 0$, then D_{11} in Eq. (7.6.7) can be approximately replaced by $(A_{11}D_{11} - B_{11}^2)/A_{11}$, the so called reduced or apparent flexural stiffness.

Including shear deformation effects, i.e. using the Timoshenko vibration equation (7.3.22), involves considerable analytical complications. To prove whether the transverse shear deformation can be important for the natural frequencies, we compare the natural frequencies for a simply supported Bernoulli and Timoshenko beam. Using (7.3.22) the boundary conditions for the Timoshenko beam are $w(0,t) = w(l,t) = \psi_1(0,t) = \psi_1(l,t) = 0$ and by introducing

$$w = A\sin\omega t\sin\frac{n\pi x_1}{l}, \quad \psi_1 = B\sin\omega t\cos\frac{n\pi x_1}{l},$$

in Eqs. (7.3.22) we can calculate the natural frequencies

$$\omega_n^2 = \frac{n^4\pi^4}{\rho A}\frac{bD_{11}}{l^4}\Big/\left(1 + \frac{\pi^2 D_{11}n^2}{l^2k^{\mathrm{s}}A_{55}}\right), \quad \rho A \equiv \rho_0 \tag{7.6.8}$$

i.e.

$$\omega_n = \omega_n^{\mathrm{Bernoulli}}\sqrt{\frac{1}{1 + (n^2\pi^2D_{11})/(l^2k^{\mathrm{s}}A_{55})}}$$

Transverse shear deformation reduces the values of vibration frequencies. As in the case of static bending the influence of shear on the values of vibration frequencies depends on the ratio $E_1/G_{13} \equiv E_1/E_5$ and the ratio l/h, i.e the span length between the supports to the total thickness of the laminate. For more general boundary conditions we can develop a mode shape function similar to (7.6.6).

In an analogous way, one can show that the buckling loads for a simply supported Bernoulli and Timoshenko beam with a compression load F follow from (7.2.35) and (7.3.23) and are

$$\begin{aligned} F^{\mathrm{cr}} &= \frac{\pi^2 bD_{11}}{l^2} && \text{(Minimum Euler load, Bernoulli beam)} \\ F^{\mathrm{cr}} &= \frac{\pi^2 bD_{11}}{l^2}\frac{1}{1 + \dfrac{\pi^2 D_{11}}{l^2k^{\mathrm{s}}A_{55}}} && \text{(Minimum buckling load, Timoshenko beam)} \end{aligned} \tag{7.6.9}$$

The buckling loads for clamped beams or beams with more general boundary conditions can be calculated analytically analogous to eigenfrequencies of vibration problems.

For non constant cross-section, stiffness or material parameters there are no exact analytical solutions. Approximate analytical solutions can be found with the help of the Rayleigh-Ritz procedure. In Sect. 7.7 exact and approximate analytical solution procedures are illustrated for selected beam problems.

7.7 Problems

Exercise 7.1. A reinforced concrete beam is loaded by a bending moment M (Fig. 7.13). It is assumed that the concrete has zero strength in tension so that the entire tensile load associated with the bending moment is carried by the steel reinforcement. Calculate the stresses $\sigma_{\mathrm{m}}(x_3)$ and $\sigma_{\mathrm{f}}(x_3)$ in the concrete part $(A_{\mathrm{m}}, E_{\mathrm{m}})$ and the steel reinforcements $(A_{\mathrm{f}}, E_{\mathrm{f}})$.

Solution 7.1. The neutral axis x_1 of the beam is in an unknown distance αh from the top, $A_{\mathrm{c}} \equiv A_{\mathrm{m}}$ is the effective area of the concrete above the x_1-axis. The strains will vary linearly from the x_1-axis and the stresses will equal strain times the respective moduli. The stress resultant $N(x_1)$ must be zero

$$\sigma_{\mathrm{f}} A_{\mathrm{f}} - \frac{1}{2}\sigma_{\mathrm{m}}(\alpha h) b \alpha h = 0, \quad \sigma_{\mathrm{m}}(\alpha h) = \sigma_{\mathrm{m}}^{\max}$$

With (7.2.1) follows

$$\sigma_{\mathrm{f}} = (h - \alpha h)\kappa_1 E_{\mathrm{f}}, \quad \sigma_{\mathrm{m}}(\alpha h) = \alpha h \kappa_1 E_{\mathrm{m}}$$

i.e.

$$(h - \alpha h) E_{\mathrm{f}} A_{\mathrm{f}} - \frac{1}{2}(\alpha h)^2 b E_{\mathrm{m}} = 0,$$

$$\alpha = \frac{E_{\mathrm{f}} A_{\mathrm{f}}}{E_{\mathrm{m}} b h}\left(-1 + \sqrt{1 + 2\frac{bh}{A_{\mathrm{f}}}\frac{E_{\mathrm{m}}}{E_{\mathrm{f}}}}\right)$$

or

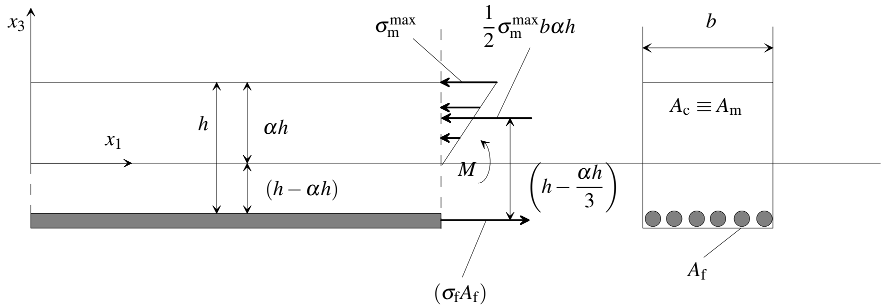

Fig. 7.13 Reinforced concrete beam loaded by pure bending

$$\alpha = \frac{1}{m}(-1+\sqrt{1+2m}), \quad m = \frac{E_{\mathrm{m}}bh}{E_{\mathrm{f}}A_{\mathrm{f}}}$$

Now the bending moment is with

$$\sigma_{\mathrm{f}}A_{\mathrm{f}} = \frac{1}{2}\sigma_{\mathrm{m}}bh^2\alpha^2$$

$$\begin{aligned} M = (\sigma_{\mathrm{f}}A_{\mathrm{f}})\left(h-\frac{\alpha h}{3}\right) &= \sigma_{\mathrm{f}}A_{\mathrm{f}}h\left(1-\frac{\alpha}{3}\right) \\ &= \kappa_1 E_{\mathrm{f}}A_{\mathrm{f}}(h-\alpha h)\left(h-\frac{\alpha}{3}\right) \end{aligned}$$

The maximal stress in the concrete is

$$\sigma_{\mathrm{m}}(\alpha h) = -\kappa_1\alpha h E_{\mathrm{m}} = M\frac{E_{\mathrm{m}}\alpha h}{E_{\mathrm{f}}A_{\mathrm{f}}(h-\alpha h)(h-\alpha h/3)}$$

and the reinforcement stress is

$$\sigma_{\mathrm{f}} = \kappa_1(h-\alpha h)E_{\mathrm{f}} = \frac{M}{A_{\mathrm{f}}(h-\alpha h/3)}$$

Exercise 7.2. A symmetric cross-ply laminate beam is shown in Fig. 7.14. The material properties and the geometry are defined by

$$\begin{aligned} &E_1' = 17,24\ 10^4\text{MPa}, E_2' = 0,6895\ 10^4\text{MPa}, \\ &G_{12}' = G_{13}' = 0,3448\ 10^4\text{MPa}, G_{23}' = 0,1379\ 10^4\text{MPa}, \nu_{12}' = 0,25, \\ &L = 240\ \text{mm}, b = 10\ \text{mm}, h^{(1)} = h^{(2)} = h^{(3)} = 8\ \text{mm}, h = 24\ \text{mm}, \\ &q_0 = 0,6895\ \text{N/mm} \end{aligned}$$

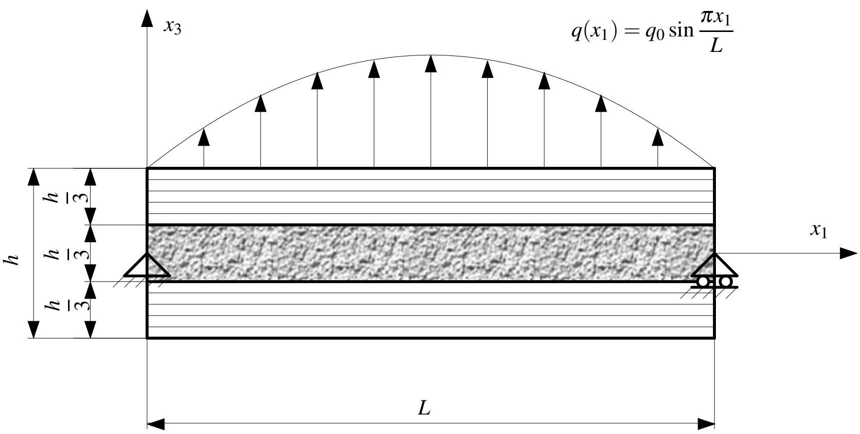

Fig. 7.14 Simply supported cross-ply laminated beam [0/90/0]

Calculate a approximative solution using the Timoshenko beam model and two one-term Ritz procedures.

Solution 7.2. Let us introduce the cross-section geometry

$$x_3^{[0]} = -12 \text{ mm}, x_3^{[1]} = -4 \text{ mm}, x_3^{[2]} = 4 \text{ mm}, x_3^{[3]} = 12 \text{ mm}$$

The shear correction factor can be calculated with Eq. (7.3.20) to $k^{\mathrm{s}} = 0,569$. The bending stiffness $\bar{D}_{11}$ and the shear stiffness $k^{\mathrm{s}}\bar{A}_{55}$ follow with Eq. (7.3.3)

$$\begin{aligned}
\bar{D}_{11} &= \frac{b}{3}[E_1((-4)^3 - (-12)^3) + E_2(4^3 - (-4)^3) + E_1(12^3 - 4^3)] \\
&= 1,92\ 10^9 \text{Nmm}^2, \\
k^{\mathrm{s}}\bar{A}_{55} &= k^{\mathrm{s}}b[G_{12}8 + G_{23}8 + G_{12}8] = 3,76\ 10^5 \text{N}
\end{aligned}$$

The variational formulation for a lateral loaded symmetric laminate beam is given with (7.3.15)

$$\Pi(w,\psi) = \frac{1}{2}\int_0^L \left[\bar{D}_{11}\left(\frac{\mathrm{d}\psi}{dx_1}\right)^2 + k^{\mathrm{s}}\bar{A}_{55}\left(\psi + \frac{\mathrm{d}w}{dx_1}\right)^2\right]\mathrm{d}x_1 - \int_0^L q_0 \sin\left(\frac{\pi x_1}{L}\right) w\, \mathrm{d}x_1$$

The essential boundary conditions are

$$w(x_1 = 0) = 0,\ w(x_1 = L) = 0,\ \psi'(x_1 = 0) = 0,\ \psi'(x_1 = L) = 0$$

The approximate functions are

$$\tilde{w}(x_1) = a_1 \sin\left(\frac{\pi x_1}{L}\right), \quad \tilde{\psi}(x_1) = b_1 \cos\left(\frac{\pi x_1}{L}\right)$$

and it follows

$$\begin{aligned}
\tilde{\Pi}(\tilde{w},\tilde{\psi}) = &\frac{1}{2}\int_0^L \left[\bar{D}_{11}\left(b_1\frac{\pi}{L}\right)^2 \sin^2\frac{\pi x_1}{L}\right. \\
&+ \left. k^{\mathrm{s}}\bar{A}_{55}\left(b_1 \cos\frac{\pi x_1}{L} + a_1\frac{\pi}{L}\cos\frac{\pi x_1}{L}\right)^2\right]\mathrm{d}x_1 \\
&- \int_0^L q_0 \sin\left(\frac{\pi x_1}{L}\right)\left(a_1 \sin\frac{\pi x_1}{L}\right)\mathrm{d}x_1 = \tilde{\Pi}(a_1,b_1)
\end{aligned}$$

With (2.2.41) must be $\partial\tilde{\Pi}/\partial a_1 = 0, \partial\tilde{\Pi}/\partial b_1 = 0$ which yields the two equations

$$\begin{aligned}
\left(\frac{\bar{D}_{11}}{k^{\mathrm{s}}\bar{A}_{55}}\frac{\pi}{L} + \frac{L}{\pi}\right)b_1 + \quad a_1 &= 0, \\
b_1 + \frac{\pi}{L}a_1 &= \frac{q_0 L}{k^{\mathrm{s}}\bar{A}_{55}\pi}
\end{aligned}$$

and the solution for the unknown constants a_1, b_1

$$
\begin{aligned}
a_1 &= \frac{q_0 L^4}{\bar{D}_{11}\pi^4}\left(1 + \frac{\bar{D}_{11}}{k^s \bar{A}_{55}}\frac{\pi^2}{L^2}\right),\\
b_1 &= -\frac{q_0 L^3}{\bar{D}_{11}\pi^3}
\end{aligned}
$$

The approximate solutions are now

$$
\begin{aligned}
\tilde{w}(x_1) &= \frac{q_0 L^4}{\bar{D}_{11}\pi^4}\left(1 + \frac{\bar{D}_{11}}{k^s \bar{A}_{55}}\frac{\pi^2}{L^2}\right)\sin\frac{\pi x_1}{L},\\
\tilde{\psi}(x_1) &= -\frac{q_0 L^3}{\bar{D}_{11}\pi^3}\cos\frac{\pi x_1}{L}
\end{aligned}
$$

Note that

$$
\tilde{w}_{\max} = \tilde{w}\left(x_1 = \frac{L}{2}\right) = \frac{q_0 L^4}{\bar{D}_{11}\pi^4}\left(1 + \frac{\bar{D}_{11}}{k^s \bar{A}_{55}}\frac{\pi^2}{L^2}\right)
$$

The transverse deflection consists of two parts

$$
\begin{aligned}
\tilde{w}^{\mathrm{B}}(x_1) &= \frac{q_0 L^4}{\bar{D}_{11}\pi^4}\sin\frac{\pi x_1}{L} \quad \text{(bending deflections)},\\
\tilde{w}^{\mathrm{S}}(x_1) &= \frac{q_0 L^4}{\bar{D}_{11}\pi^4}\frac{\bar{D}_{11}}{k^s \bar{A}_{55}}\frac{\pi^2}{L^2}\sin\frac{\pi x_1}{L}
\end{aligned}
$$

For $k^s \bar{A}_{55} \to \infty$ follows $\tilde{w}^{\mathrm{S}} \to 0$, i.e. $\tilde{w}^{\mathrm{B}}(x_1)$ is the solution of the Bernoulli beam model and we found

$$
w_{\text{Timoshenko}} = k_1 w_{\text{Bernoulli}}
$$

with

$$
k_1 = 1 + \frac{\bar{D}_{11}}{k^s \bar{A}_{55}}\frac{\pi^2}{L^2} = 1{,}875
$$

For the laminate beam with $h/L = 1/10$ the Bernoulli model cannot be accepted, the relative error for the maximum value of the deflection is 46,7 %. Equations (7.3.10) lead

$$
\begin{aligned}
\tilde{M}_{\max} &= \tilde{M}\left(x_1 = \frac{L}{2}\right) = \frac{q_0 L^2}{\pi^2} = 12{,}64\,\mathrm{Nm},\\
\tilde{Q}_{\max} &= \tilde{Q}(x_1 = 0) = \frac{q_0 L}{\pi} = 52{,}67\,\mathrm{N}
\end{aligned}
$$

The strains ε_1 follow from Eqs. (7.3.1) or (7.3.2) and (7.3.10)

$$
\varepsilon_1(x_1) = x_3\psi'(x_1) = x_3\kappa_1 = x_3 M(x_1)/\bar{D}_{11}
$$

$\varepsilon_1(x_1)$ is linear distributed across h and we calculate the following values for the cross-section $x_1 = L/2$

$$\tilde{\varepsilon}_1^{(3)}\left(x_1=\frac{L}{2},x_3^{[3]}\right)=2,51\ 10^{-5},\ \varepsilon_1^{(3)}\left(x_1=\frac{L}{2},x_3^{[2]}\right)=0,84\ 10^{-5}$$

The bending stresses $\sigma_1(x_1,x_3)$ in the 3 layers are for $x_1=L/2$ and $x_3=x_3^{(k)}$

$$\begin{aligned}
\tilde{\sigma}_1^{(3)}(x_3^{(3)}) &= \tilde{\varepsilon}_1(x_3^{(3)})E_1' = 4,327\ \text{MPa},\\
\tilde{\sigma}_1^{(3)}(x_3^{(2)}) &= \tilde{\varepsilon}_1(x_3^{(2)})E_1' = 1,448\ \text{MPa},\\
\tilde{\sigma}_1^{(2)}(x_3^{(2)}) &= \tilde{\varepsilon}_1(x_3^{(2)})E_2' = 0,579\ \text{MPa},\\
\tilde{\sigma}_1^{(2)}(0) &= \tilde{\varepsilon}_1(0)E_2' = 0\ \text{MPa}
\end{aligned}$$

Exercise 7.3. Find the analytical solution for the natural vibrations of a simply supported symmetric laminate or sandwich beam. Test the influence of the transverse shear deformation and the rotatory inertia upon the natural frequencies.

Solution 7.3. Starting point are the (7.3.22) with $q(x_1,t)\equiv 0$ and the boundary conditions

$$w(0,t)=w(l,t)=0,\quad \psi'(0,t)=\psi'(l,t)=0$$

For a simply supported beam we can assume the periodic motion in the form

$$w(x_1,t)=W(x_1)\sin\omega t,\quad \psi_1(x_1,t)=\Psi(x_1)\sin\omega t$$

These functions are substituted in (7.3.22)

$$\begin{aligned}
k^{\mathrm{s}}\bar{A}_{55}[W''(x_1)+\Psi'(x_1)]+\rho_0\omega^2W(x_1) &= 0,\\
\bar{D}_{11}\Psi''(x_1)-k^{\mathrm{s}}\bar{A}_{55}[W'(x_1)+\Psi(x_1)]+\rho_2\omega^2\Psi(x_1) &= 0
\end{aligned}$$

Now we can substitute

$$k^{\mathrm{s}}\bar{A}_{55}\Psi'(x_1)=-\rho_0\omega^2W(x_1)-k^{\mathrm{s}}\bar{A}_{55}W''(x_1),$$

i.e.

$$\Psi'(x_1)=-\frac{\rho_0\omega^2}{k^{\mathrm{s}}\bar{A}_{55}}W(x_1)-W''(x_1)$$

into the derivative of the second equation and we find

$$\begin{aligned}
&\bar{D}_{11}W''''(x_1)+\left(\frac{\bar{D}_{11}\rho_0}{k^{\mathrm{s}}\bar{A}_{55}}+\rho_2\right)\omega^2W''(x_1)\\
&-\left(1-\frac{\omega^2\rho_2}{k^{\mathrm{s}}\bar{A}_{55}}\right)\rho_0\omega^2W(x_1)=0
\end{aligned}$$

or

$$aW''''(x_1)+bW''(x_1)-cW(x_1)=0$$

with the coefficients

$$a = \bar{D}_{11}, \quad b = \left(\frac{\bar{D}_{11}}{k^s\bar{A}_{55}} + \frac{\rho_2}{\rho_0}\right)\rho_0\omega^2, \quad c = \left(1 - \frac{\omega^2\rho_2}{k^s\bar{A}_{55}}\right)\rho_0\omega^2$$

The linear differential equation of 4th order has constant coefficients and the general solutions follow with the solutions λ_i of the bi-quadratic characteristic equation

$$a\lambda^4 - b\lambda^2 - c = 0$$

i.e. $(2a\lambda^2 - b)^2 = b^2 + 4ac$

$$\lambda_{1-4} = \pm\sqrt{\frac{1}{2a}(b \pm \sqrt{b^2 + 4ac})}$$

$$W(x_1) = C_1 \sin\lambda_1 x_1 + C_2 \cos\lambda_2 x_1 + C_3 \sinh\lambda_3 x_1 + C_4 \cosh\lambda_4 x_1$$

For a simply supported beam the boundary conditions are

$$W(0) = 0, \quad W(L) = 0, \quad \Psi'(0) = 0, \quad \Psi'(L) = 0$$

or the equivalent equations

$$W(0) = 0, \quad W(L) = 0, W''(0) = 0, \quad W''(L) = 0$$

The boundary conditions lead to the result $C_2 = C_3 = C_4 = 0$ and $C_1 \sin\lambda_1 L = 0$ which implies

$$\lambda_{1n} = \frac{n\pi}{L} = \lambda_n$$

The bi-quadratic equation can be written alternatively in terms of ω

$$A\omega^4 - B\omega^2 + C = 0$$

with

$$A = \frac{\rho_2}{k^s\bar{A}_{55}}, \quad B = \left[1 + \left(\frac{\bar{D}_{11}}{k^s\bar{A}_{55}} + \frac{\rho_2}{\rho_0}\right)\lambda^2\right], \quad C = \frac{\bar{D}_{11}}{\rho_0}\lambda^4$$

i.e. the roots of the equation are

$$(\omega^2)_{1/2} = \frac{1}{2A}(B \pm \sqrt{B^2 - 4AC})$$

It can be shown that $B^2 - 4AC > 0$. Therefore the frequency given by $-\sqrt{B^2 - 4AC}$ is the smaller of the two roots.

When the rotatory inertia is neglected follows $A \equiv 0$ and the frequency is given by

$$\omega^2 = \frac{C}{B}$$

with

$$B = 1 + \frac{\bar{D}_{11}}{k^s\bar{A}_{55}}\lambda^2, \quad C = \frac{\bar{D}_{11}}{\rho_0}\lambda^4$$

and for $k^s\bar{A}_{55} \to \infty$ follow with $\tilde{B} \to 1$ the natural frequency for the Bernoulli beam model. Substitute $\lambda_n = n\pi/L$ for the simply supported beam we obtain the results:
General case

$$\omega_n^2 = \frac{k^s\bar{A}_{55}}{2\rho_2}\left[1 + \left(\frac{\bar{D}_{11}}{k^s\bar{A}_{55}} + \frac{\rho_2}{\rho_0}\right)\left(\frac{n\pi}{L}\right)^2 \right.$$
$$\left. - \sqrt{\left[1 + \left(\frac{\bar{D}_{11}}{k^s\bar{A}_{55}} + \frac{\rho_2}{\rho_0}\right)\left(\frac{n\pi}{L}\right)^2\right]^2 - 4\frac{\rho_2}{k^s\bar{A}_{55}}\frac{\bar{D}_{11}}{\rho_0}\left(\frac{n\pi}{L}\right)^4}\,\right]$$

Rotatory inertia neglected ($\rho_2 \equiv 0$)

$$\begin{aligned}\omega_n^2 &= \left(\frac{n\pi}{L}\right)^4 \frac{\bar{D}_{11}}{\rho_0}\left(1 - \frac{\left(\frac{n\pi}{L}\right)^2 \bar{D}_{11}}{k^s\bar{A}_{55} + \bar{D}_{11}\left(\frac{n\pi}{L}\right)^2}\right) \\ &= \left(\frac{n\pi}{L}\right)^4 \frac{\bar{D}_{11}}{\rho_0}\left(\frac{1}{1 + \bar{D}_{11}\left(\frac{n\pi}{L}\right)^2 / k^s\bar{A}_{55}}\right)\end{aligned}$$

Classical beam theory ($k^s\bar{A}_{55} \to \infty, \rho_2 = 0$)

$$\omega_n^2 = \left(\frac{n\pi}{L}\right)^4 \frac{\bar{D}_{11}}{\rho_0}$$

Conclusion 7.2. $(\omega_n^{\text{Timoshenko}})^2 < (\omega_n^{\text{Bernoulli}})^2$, i.e. the shear deformation decreases the frequencies of natural vibration. In the case of classical beam theory with rotatory inertia ($k^s\bar{A}_{55} \to 0, \rho_2 \neq 0$) we have $A = 0, B = 1 + \lambda^2\rho_2/\rho_0, C = \lambda^4\bar{D}_{11}/\rho_0$, i.e.

$$\omega_n^2 = \left(\frac{n\pi}{L}\right)^4 \frac{\bar{D}_{11}}{\rho_0}\left(\frac{1}{1 + \left(\frac{n\pi}{L}\right)^2 \frac{\rho_2}{\rho_0}}\right)$$

and we see that also the rotatory inertia decreases the eigenfrequencies. All formulas can be used for computing natural frequencies for all symmetric laminate and sandwich beams. The values for $L, \rho_0, \rho_2, k^s, \bar{D}_{11}, \bar{A}_{55}$ correspond to the special beam model. Note that the classical laminate theory and the neglecting of rotatory inertia lead to a overestimation of the natural frequencies.

Exercise 7.4. Calculate the buckling load of a simply supported and a clamped symmetric laminate or sandwich beam. Compare the results for the classical beam theory and the beam theory including shear deformation.

Solution 7.4. Staring point are Eqs. (7.3.23) with $N(x_1) = -F$, i.e.

$$\bar{D}_{11}\left(1 - \frac{F}{k^s\bar{A}_{55}}\right) w''''(x_1) + Fw''(x_1) = 0$$

or

$$w''''(x_1) + k^2 w''(x_1) = 0, \quad k^2 = \frac{F}{\bar{D}_{11}\left(1 - \dfrac{F}{k^s \bar{A}_{55}}\right)}, \quad F \frac{k^2 \bar{D}_{11}}{1 + k^2 \bar{D}_{11}/k^s \bar{A}_{55}}$$

The linear differential equation with constant coefficients has the characteristic equation

$$\lambda^4 + k^2 \lambda^2 = 0 \Rightarrow \lambda^2(\lambda^2 + k^2) = 0$$

with the four roots

$$\lambda_{1/2} = 0, \quad \lambda_{3/4} = \pm \mathrm{i}k$$

and the general solution is

$$w(x_1) = C_1 \sin kx_1 + C_2 \cos kx_1 + C_3 x_1 + C_4$$

1. Simply supported beam
   Boundary conditions are $w(0) = w(L) = 0, w''(0) = w''(L) = 0$, which leads the constants $C_2 = C_3 = C_4 = 0$ and for $C_1 \neq 0$ follows $\sin kL = 0$ implies $kL = n\pi, k = n\pi/L$. Substituting k into the equation for F we obtain

$$F = \left(\frac{n\pi}{L}\right)^2 \bar{D}_{11} \left[\frac{k^s \bar{A}_{55}}{k^s \bar{A}_{55} + \left(\dfrac{n\pi}{L}\right)^2 \bar{D}_{11}}\right]$$

$$= \left(\frac{n\pi}{L}\right)^2 \bar{D}_{11} \left[1 - \frac{\bar{D}_{11}\left(\dfrac{n\pi}{L}\right)^2 / k^s \bar{A}_{55}}{1 + \bar{D}_{11}\left(\dfrac{n\pi}{L}\right)^2 / k^s \bar{A}_{55}}\right]$$

   The critical buckling load F^{cr} is given for the minimum $(n = 1)$

$$F = \left(\frac{\pi}{L}\right)^2 \bar{D}_{11} \left[\frac{1}{1 + \left(\dfrac{\pi}{L}\right)^2 \bar{D}_{11}/k^s \bar{A}_{55}}\right]$$

   For the classical beam model is $k^s \bar{A}_{55} \rightarrow \infty$ and we obtain

$$F = \left(\frac{\pi}{L}\right)^2 \bar{D}_{11}$$

2. At both ends fixed beam (clamped beam)
   Now we have the boundary conditions

$$w(0) = w(L) = 0, \quad \psi(0) = \psi(L) = 0$$

   From (7.3.23) follows

$$\begin{aligned} k^{\mathrm{s}}\bar{A}_{55}[w''(x_1)+\psi'(x_1)]-Fw''(x_1) &= 0,\\ \bar{D}_{11}\psi''(x_1)-k^{\mathrm{s}}\bar{A}_{55}[w'(x_1)+\psi(x_1)] &= 0 \end{aligned}$$

The first equation yields

$$\begin{aligned} k^{\mathrm{s}}\bar{A}_{55}\psi'(x_1) &= -(k^{\mathrm{s}}\bar{A}_{55}-F)w''(x_1),\\ k^{\mathrm{s}}\bar{A}_{55}\psi(x_1) &= -(k^{\mathrm{s}}\bar{A}_{55}-F)w'(x_1)+K_1 \end{aligned}$$

The boundary conditions lead the equations

$$C_2+C_4=0, \quad C_1\sin kL+C_2\cos kL+C_3L+C_4=0,$$

$$-\left(1-\frac{F}{k^{\mathrm{s}}\bar{A}_{55}}\right)kC_1-C_3=0,$$

$$-\left(1-\frac{F}{k^{\mathrm{s}}\bar{A}_{55}}\right)(kC_1\cos kL-kC_2\sin kL)-C_3=0$$

Note that

$$\frac{F}{k^2}=\frac{\bar{D}_{11}}{1+k^2\dfrac{\bar{D}_{11}}{k^{\mathrm{s}}\bar{A}_{55}}}=\left(1-\frac{F}{k^{\mathrm{s}}\bar{A}_{55}}\right)\bar{D}_{11},$$

i.e.

$$\left(1-\frac{F}{k^{\mathrm{s}}\bar{A}_{55}}\right)=\frac{1}{1+k^2\dfrac{\bar{D}_{11}}{k^{\mathrm{s}}\bar{A}_{55}}},$$

expressing C_4 and C_3 in terms of C_1 and C_2 and setting the determinant of the remaining homogeneous algebraic equations zero we obtain the buckling equation

$$2(\cos kL-1)\left(1+\frac{k^2\bar{D}_{11}}{k^{\mathrm{s}}\bar{A}_{55}}\right)+kL\sin kL=0$$

With $k^{\mathrm{s}}\bar{A}_{55}\to\infty$ follows the buckling equation for the classical beam

$$kL\sin kL+2\cos kL-2=0$$

Conclusion 7.3. Transverse shear deformation has the effect of decreasing the buckling loads, i.e. the classical laminate theory overestimates buckling loads. The buckling equations can be applied to all symmetric laminate and sandwich beams if the corresponded material and stiffness values are calculated and substituted.

Exercise 7.5. A sandwich beam is modelled by the laminated beam version and the shear deformation theory. Consider the variational formulation for applied distributed transverse loading and calculate the Euler differential equation and the boundary conditions.

Solution 7.5. The elastic potential $\Pi(u,w,\psi)$ for unsymmetrical laminated beams is given by Eq. (7.3.16). Using the notations for the stiffness of sandwich beams, Sect. 7.4, we have

$$\Pi(u,w,\psi) = \frac{1}{2}\int_0^l [Au'^2 + 2Bu'\psi' + D\psi'^2 + S(w'+\psi)^2]\mathrm{d}x_1 - \int_0^l qw\mathrm{d}x_1$$

Taking the variation $\delta\Pi = 0$ one can write the following equation

$$\begin{aligned}\delta\Pi = &\int_0^l \{Au'\delta u' + B(u'\delta\psi' + \psi'\delta u') + D\psi'\delta\psi' \\ &+ S[(w'+\psi)\delta w' + (w'+\psi)\delta\psi]\}\mathrm{d}x_1 - \int_0^l q\delta w\mathrm{d}x_1 = 0\end{aligned}$$

Integrating by parts, i.e

$$\int_0^l f'g'\mathrm{d}x = [f'g]_0^l - \int_0^l f''g\mathrm{d}x$$

yield

$$\begin{aligned}&\int_0^l \{(Au''+B\psi'')\delta u + [Bu''+D\psi''-S(w'+\psi)]\delta\psi + S[(w''+\psi')+q]\delta w\}\mathrm{d}x_1 \\ &\qquad -[(Au'+B\psi')\delta u]_0^l - [(Bu'+D\psi')\delta\psi]_0^l - S[(w'+\psi)\delta w]_0^l = 0\end{aligned}$$

and the associated differential equations and boundary conditions are

$$\begin{aligned}Au''+B\psi'' &= 0,\\ Bu''+D\psi''-S(w'+\psi) &= 0,\\ S(w''+\psi')+q &= 0\end{aligned}$$

Putting in $u'' = -(B/A)\psi''$ into the second equation yield

$$\left(D - \frac{B^2}{A}\right)\psi'' - S(w'+\psi) = 0$$

and we obtain the differential equations (7.4.32)

$$\begin{aligned}D_\mathrm{R}\psi'' - S(w'+\psi) &= 0,\\ S(w''+\psi) &= -q\end{aligned}$$

The boundary conditions for $x_1 = 0, l$ are

$$
\begin{array}{lll}
u=0 & \text{or} & Au'+B\psi'=N=0\\
\psi=0 & \text{or} & Bu'+D\psi'=M=0\\
w=0 & \text{or} & S(w'+\psi)=Q=0
\end{array}
$$

u,ψ,w represent the essential and N,M,Q the natural boundary conditions of the problem. For symmetric sandwich beams the equations can be simplified: $B=0, D_{\mathrm{R}}=D$.

Exercise 7.6. A sandwich beam is modelled by the differential equations and boundary conditions of Excercise 7.5. Calculate the exact solution for a simply supported beam with $q(x_1)=q_0, N(x_1)=0$.

Solution 7.6. The boundary conditions are:

$$
w(0)=w(l)=0, \quad M(0)=M(l)=0
$$

Using the equations (7.4.38)

$$
\begin{aligned}
\psi'''(x_1) &= -\frac{q_0}{D_{\mathrm{R}}},\\
\psi''(x_1) &= -\frac{q_0 x_1}{D_{\mathrm{R}}}+C_1,\\
\psi'(x_1) &= -\frac{q_0 x_1^2}{2D_{\mathrm{R}}}+C_1 x_1+C_2,\\
\psi(x_1) &= -\frac{q_0 x_1^3}{6D_{\mathrm{R}}}+C_1\frac{x_1^2}{2}+C_2 x_1+C_3,
\end{aligned}
$$

$$
\psi'(0)=0 \Rightarrow C_2=0,
$$

$$
\psi'(l)=0 \Rightarrow C_1=\frac{q_0 l}{2D_{\mathrm{R}}},
$$

$$
\begin{aligned}
w'(x_1) &= -\psi(x_1)+\frac{D_{\mathrm{R}}}{S}\psi''=\frac{q_0 x_1^3}{6D_{\mathrm{R}}}-\frac{q_0 l x_1^2}{4D_{\mathrm{R}}}+C_3-\frac{q_0}{S}\left(x_1-\frac{l}{2}\right),\\
w(x_1) &= \frac{q_0}{24D_{\mathrm{R}}}(x_1^4-2lx_1^3)+C_3x_1-\frac{q_0}{2S}\left(x_1^2-lx_1\right),
\end{aligned}
$$

$$
w(l)=0 \Rightarrow C_3=\frac{q_0 l^3}{24D_{\mathrm{R}}},
$$

$$
\begin{aligned}
u'(x_1) &= -\frac{B}{A}\psi'(x_1)=\frac{B}{A}\frac{q_0}{2D_{\mathrm{R}}}(x_1^2-lx_1),\\
u(x_1) &= \frac{B}{A}\frac{q_0}{12D_{\mathrm{R}}}(2x_1^3-3lx_1)+C_4,
\end{aligned}
$$

$$
u(0)=0 \Rightarrow C_4=0
$$

Finally we get

$$\begin{aligned}
\psi(x_1) &= -\frac{q_0 x_1^3}{6D_{\mathrm{R}}} + \frac{q_0 l}{2D_{\mathrm{R}}}\frac{x_1^2}{2} + \frac{q_0 l^3}{24D_{\mathrm{R}}},\\
w(x_1) &= \frac{q_0}{24D_{\mathrm{R}}}(x_1^4 - 2lx_1^3 + l^3 x_1) + \frac{q_0}{2S}\left(lx_1 - x_1^2\right),\\
u(x_1) &= \frac{B}{A}\frac{q_0}{12D_{\mathrm{R}}}(2x_1^3 - 3lx_1)
\end{aligned}$$

For symmetrical beams the solution simplified with $B = 0, D_{\mathrm{R}} = D$ to

$$\begin{aligned}
w(x_1) &= \frac{q_0 l^4}{24D}(\overline{x}_1^4 - 2\overline{x}_1^3 + \overline{x}_1) + \frac{q_0 l^2}{2S}(\overline{x}_1 - \overline{x}_1^2)\\
&= w^{\mathrm{B}}(\overline{x}_1) + w^{\mathrm{S}}(\overline{x}_1),\\
\psi(x_1) &= \frac{q_0 l^3}{24D}(-1 + 6\overline{x}_1^2 - 4\overline{x}_1^3)\\
u(x_1) &\equiv 0, \qquad \overline{x}_1 = x/l
\end{aligned}$$

Chapter 8
Modelling and Analysis of Plates

The modelling and analysis of plates constituted of laminate or sandwich material is a problem of more complexity than that of beams, considered in Chap. 7. Generally, plates are two-dimensional thin structure elements with a plane middle surface. The thickness h is small relatively to the two other dimensions a, b (Fig. 8.1).

In Chap. 8 all derivatives are as a matter of priority restricted to rectangular plates including the special case of a plate strip, i.e. a rectangular plate element which is very long, for instance in the x_2-direction and has finite dimension in the x_1-direction. When the transverse plate loading, the plate stiffness, and the boundary conditions for the plate edges $x_1 = const$ are independent of the coordinate x_2, the plate strip modelling can be reduced to a one-dimensional problem. The analysis is

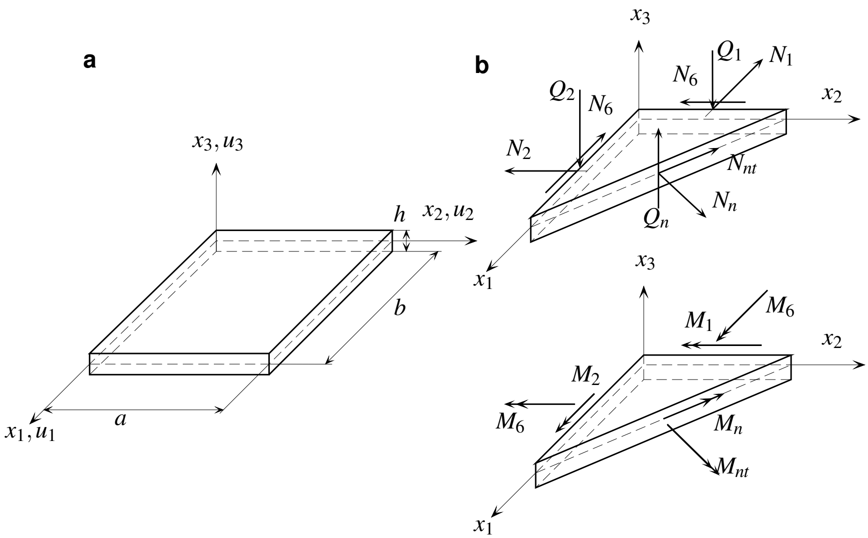

Fig. 8.1 Rectangular plate. **a** Geometry, **b** force resultants N_1, N_2, N_6, Q_1, Q_2 and moment resultants M_1, M_2, M_6. N_n, N_{nt}, Q_n and M_n, M_{nt} are force and moment resultants for an oblique edge

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0_8

nearly the same as in the beam theory. Chapter 8 gives a first introduction to the classical plate theory and the plate theory including transverse shear deformations. The derivations of the principal equations for plates relies upon the basic considerations of Chap. 5.

8.1 Introduction

In the theory of plate bending the most complex problem is the modelling and analysis of laminate plates with an arbitrary stacking of the layers. These plates present couplings of stretching and bending, stretching and twisting and bending and twisting and the design engineer has to look for simplifications.

The first and most important simplification is to design symmetric laminates for which no coupling exists between in-plane forces and flexural moments. The coupling terms B_{ij} of the constitutive equations vanish. An additional simplification occurs when no bending-twisting coupling exist, i.e the terms D_{16} and D_{26} are zero. As we discussed in Sect. 4.2, in some cases of layer stacking these coupling terms decrease with an increasing number of layers. Symmetric laminates for which no bending-twisting coupling exists are referred to as specially orthotropic laminates. These laminates are considered in detail in this chapter, because analytical solutions exist for various loadings and boundary conditions. Specially orthotropic plates are obtained for single layer plates with orthotropic material behavior or symmetric cross-ply laminates. Symmetric balanced laminates with a great number of layers have approximately a specially orthotropic behavior. This class of laminates is greatly simplified and will be used to gain a basic understanding of laminate plate response. Like in Chap. 7 for beams, we consider the plates in the framework of the classical and the first order shear deformation theory. For a better understanding the assumptions of both plate theories given in Sects. 5.1 and 5.2 are reviewed.

The first order shear deformation theory accounted for a constant state of transverse shear stresses, but the transverse normal stress is often neglected. In the framework of this plate theory, the computation of interlaminar shear stresses through constitutive equations is possible, which is simpler than deriving them through equilibrium equations.

The most significant difference between the classical and first-order shear deformation theory is the effect of including transverse shear deformation in the prediction of deflections, frequencies or buckling loads. It can be noted that the classical laminate theory underestimates deflections and overestimates frequencies as well as buckling loads when the plate side-to-thickness ratio is of the order 20 or less. For this reason it is necessary to include shear deformation for moderately thick plates. In general, moderately thick plates must be computed by numerical methods, application of analytical methods are much more restricted than in the classical plate theory.

8.2 Classical Laminate Theory

In the classical laminate theory one presumes that the Kirchhoff hypotheses of the classical plate theory remains valid:

- Transverse normals before deformation remain straight after deformation and rotate such that they remain normal to the middle surface.
- Transverse normals are inextensible, i.e. they have no elongation.

These assumptions imply that the transverse displacement w is independent of the thickness coordinate x_3, the strains $\varepsilon_3, \varepsilon_4$ and ε_5 are zero and the curvatures κ_i are given by

$$[\kappa_1\ \kappa_2\ \kappa_6] = \left[-\frac{\partial^2 w}{\partial x_1^2}\ -\frac{\partial^2 w}{\partial x_2^2}\ -2\frac{\partial^2 w}{\partial x_1 \partial x_2}\right] \tag{8.2.1}$$

Figure 8.1 shows the plate geometry and the plate stress resultants. The equilibrium equations will be formulated for a plate element $\mathrm{d}x_1\mathrm{d}x_2$ (Fig. 8.2) and yield three force and two moments equations

$$\begin{aligned}
\swarrow\ & \frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = -p_1,\\
\rightarrow\ & \frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} = -p_2,\\
\uparrow\ & \frac{\partial Q_1}{\partial x_1} + \frac{\partial Q_2}{\partial x_2} = -p_3,\\
\rightarrow\ & \frac{\partial M_1}{\partial x_1} + \frac{\partial M_6}{\partial x_2} = Q_1,\\
\swarrow\ & \frac{\partial M_6}{\partial x_1} + \frac{\partial M_2}{\partial x_2} = Q_2
\end{aligned} \tag{8.2.2}$$

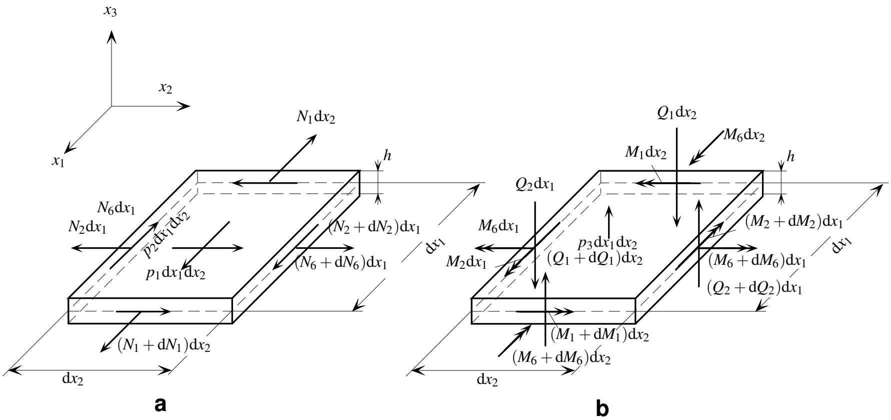

Fig. 8.2 Stress resultants applied to a plate element

The transverse shear force resultants Q_1, Q_2 can be eliminated and the five equations (8.2.2) reduce to three equations. The in-plane force resultants N_1, N_2 and N_6 are uncoupled with the moment resultants M_1, M_2 and M_6

$$\begin{aligned}&\frac{\partial N_1}{\partial x_1}+\frac{\partial N_6}{\partial x_2}=-p_1, \quad \frac{\partial N_6}{\partial x_1}+\frac{\partial N_2}{\partial x_2}=-p_2,\\&\frac{\partial^2 M_1}{\partial x_1^2}+2\frac{\partial^2 M_6}{\partial x_1 \partial x_2}+\frac{\partial^2 M_2}{\partial x_2^2}=-p_3\end{aligned} \tag{8.2.3}$$

The equations are independent of material laws and present the static equations for the undeformed plate element. The further considerations neglect the in-plane plate loads p_1 and p_2, i.e. $p_1 = p_2 = 0$, $p_3 \neq 0$. In-plane reactions can be caused by coupling effects of unsymmetric laminates or sandwich plates.

Putting the constitutive equations

$$\begin{bmatrix}\boldsymbol{N}\\ \cdots \\ \boldsymbol{M}\end{bmatrix}=\begin{bmatrix}\boldsymbol{A} \vdots \boldsymbol{B}\\ \cdots\cdots \\ \boldsymbol{B} \vdots \boldsymbol{D}\end{bmatrix}\begin{bmatrix}\boldsymbol{\varepsilon}\\ \cdots \\ \boldsymbol{\kappa}\end{bmatrix} \tag{8.2.4}$$

into the equilibrium (8.2.3) and replacing using Eqs. (5.2.3) the in-plane strains ε_i and the curvatures κ_i by

$$\begin{aligned}\boldsymbol{\varepsilon}^{\mathrm{T}} &= [\varepsilon_1 \ \varepsilon_2 \ \varepsilon_6] = \left[\frac{\partial u}{\partial x_1} \quad \frac{\partial v}{\partial x_2} \quad \left(\frac{\partial u}{\partial x_2}+\frac{\partial v}{\partial x_1}\right)\right],\\ \boldsymbol{\kappa}^{\mathrm{T}} &= [\kappa_1 \ \kappa_2 \ \kappa_6] = \left[-\frac{\partial^2 w}{\partial x_1^2} \quad -\frac{\partial^2 w}{\partial x_2^2} \quad -2\frac{\partial^2 w}{\partial x_1 \partial x_2}\right]\end{aligned} \tag{8.2.5}$$

gives the differential equations for general laminated plates

$$\begin{aligned}&A_{11}\frac{\partial^2 u}{\partial x_1^2}+2A_{16}\frac{\partial^2 u}{\partial x_1 \partial x_2}+A_{66}\frac{\partial^2 u}{\partial x_2^2}+A_{16}\frac{\partial^2 v}{\partial x_1^2}+(A_{12}+A_{66})\frac{\partial^2 v}{\partial x_1 \partial x_2}\\&+A_{26}\frac{\partial^2 v}{\partial x_2^2}-B_{11}\frac{\partial^3 w}{\partial x_1^3}-3B_{16}\frac{\partial^3 w}{\partial x_1^2 \partial x_2}-(B_{12}+2B_{66})\frac{\partial^3 w}{\partial x_1 \partial x_2^2}-B_{26}\frac{\partial^3 w}{\partial x_2^3}=0,\\&A_{16}\frac{\partial^2 u}{\partial x_1^2}+(A_{12}+A_{66})\frac{\partial^2 u}{\partial x_1 \partial x_2}+A_{26}\frac{\partial^2 u}{\partial x_2^2}+A_{66}\frac{\partial^2 v}{\partial x_1^2}+2A_{26}\frac{\partial^2 v}{\partial x_1 \partial x_2}\\&+A_{22}\frac{\partial^2 v}{\partial x_2^2}-B_{16}\frac{\partial^3 w}{\partial x_1^3}-(B_{12}+2B_{66})\frac{\partial^3 w}{\partial x_1^2 \partial x_2}-3B_{26}\frac{\partial^3 w}{\partial x_1 \partial x_2^2}-B_{22}\frac{\partial^3 w}{\partial x_2^3}=0,\\&D_{11}\frac{\partial^4 w}{\partial x_1^4}+4D_{16}\frac{\partial^4 w}{\partial x_1^3 \partial x_2}+2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2}+4D_{26}\frac{\partial^4 w}{\partial x_1 \partial x_2^3}\\&+D_{22}\frac{\partial^4 w}{\partial x_2^4}-B_{11}\frac{\partial^3 u}{\partial x_1^3}-3B_{16}\frac{\partial^3 u}{\partial x_1^2 \partial x_2}-(B_{12}+2B_{66})\frac{\partial^3 u}{\partial x_1 \partial x_2^2}-B_{26}\frac{\partial^3 u}{\partial x_2^3}\\&-B_{16}\frac{\partial^3 v}{\partial x_1^3}-(B_{12}+2B_{66})\frac{\partial^3 v}{\partial x_1^2 \partial x_2}-3B_{26}\frac{\partial^3 v}{\partial x_1 \partial x_2^2}-B_{22}\frac{\partial^3 v}{\partial x_2^3}=p_3\end{aligned} \tag{8.2.6}$$

Equations (8.2.6) are three coupled partial differential equations for the displacements $u(x_1,x_2), v(x_1,x_2), w(x_1,x_2)$. Equation (8.2.6) can be formulated in matrix form as

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{31} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}, \quad L_{ij} = L_{ji} \tag{8.2.7}$$

The differential operators are given in App. C.

The differential operators L_{11}, L_{12} and L_{22} are of second order, L_{13} and L_{23} of third order and L_{33} of fourth order. The homogeneous part of the coupled partial differential equations (8.2.7) can be reduced to one partial equation of eight order

$$[(L_{11}L_{22} - L_{12}^2)L_{33} - (L_{11}L_{23}^2 - 2L_{12}L_{13}L_{23} + L_{13}^2 L_{22})]w = 0 \tag{8.2.8}$$

Consistent with the eight order set of differential equations four boundary conditions must be prescribed for each edge of the plate. The classical boundary conditions are: Either

$$N_n \quad \text{or} \quad u, \quad N_{nt} \quad \text{or} \quad v, \quad M_n \quad \text{or} \quad \frac{\partial w}{\partial n}, \quad V_n \equiv Q_n + \frac{\partial M_{nt}}{\partial t} \quad \text{or} \quad w \tag{8.2.9}$$

must be specified. The subscripts n and t in the boundary conditions above denote the coordinates normal and tangential to the boundary. It is well known that in the classical plate theory the boundary cannot responded separately to the shear force resultant Q_n and the twisting moment M_{nt} but only to the effective or Kirchhoff shear force resultant

$$V_n \equiv Q_n + \frac{\partial M_{nt}}{\partial t} \tag{8.2.10}$$

Equations (8.2.9) may be used to represent any form of simple edge conditions, e.g. clamped, simply supported and free.

The boundary conditions (8.2.9) represent pairs of response variables. One component of these pairs involve a force or a moment resultant, the other a displacement or a rotation. Take into account that in addition to the edge conditions it can be necessary to fulfil the point corner conditions, e.g. for a free corner. Sometimes more general boundary conditions, which are applicable to edges having elastic constraints, are used, e.g. the transverse and/or rotatory plate conditions

$$V_n \pm c_{\mathrm{T}} w = 0; \quad M_n \pm c_{\mathrm{R}} \frac{\partial w}{\partial n} = 0 \tag{8.2.11}$$

c_{T} and c_{R} denote the spring stiffness of the constraints.

In applying the boundary conditions (8.2.9) it is useful to have explicit expressions for the stress resultants in a displacement formulation. According to Eqs. (8.2.5) and (8.2.4) the stress resultants can be written as

$$N_1 = A_{11}\frac{\partial u}{\partial x_1} + A_{12}\frac{\partial v}{\partial x_2} + A_{16}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right)$$

$$
\begin{aligned}
&\quad - B_{11}\frac{\partial^2 w}{\partial x_1^2} - B_{12}\frac{\partial^2 w}{\partial x_2^2} - 2B_{16}\frac{\partial^2 w}{\partial x_1 \partial x_2},\\
N_2 &= A_{12}\frac{\partial u}{\partial x_1} + A_{22}\frac{\partial v}{\partial x_2} + A_{26}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right)\\
&\quad - B_{12}\frac{\partial^2 w}{\partial x_1^2} - B_{22}\frac{\partial^2 w}{\partial x_2^2} - 2B_{26}\frac{\partial^2 w}{\partial x_1 \partial x_2},\\
N_6 &= A_{16}\frac{\partial u}{\partial x_1} + A_{26}\frac{\partial v}{\partial x_2} + A_{66}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right)\\
&\quad - B_{16}\frac{\partial^2 w}{\partial x_1^2} - B_{26}\frac{\partial^2 w}{\partial x_2^2} - 2B_{66}\frac{\partial^2 w}{\partial x_1 \partial x_2},\\
M_1 &= B_{11}\frac{\partial u}{\partial x_1} + B_{12}\frac{\partial v}{\partial x_2} + B_{16}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right)\\
&\quad - D_{11}\frac{\partial^2 w}{\partial x_1^2} - D_{12}\frac{\partial^2 w}{\partial x_2^2} - 2D_{16}\frac{\partial^2 w}{\partial x_1 \partial x_2},\\
M_2 &= B_{12}\frac{\partial u}{\partial x_1} + B_{22}\frac{\partial v}{\partial x_2} + B_{26}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right)\\
&\quad - D_{12}\frac{\partial^2 w}{\partial x_1^2} - D_{22}\frac{\partial^2 w}{\partial x_2^2} - 2D_{26}\frac{\partial^2 w}{\partial x_1 \partial x_2},\\
M_6 &= B_{16}\frac{\partial u}{\partial x_1} + B_{26}\frac{\partial v}{\partial x_2} + B_{66}\left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right)\\
&\quad - D_{16}\frac{\partial^2 w}{\partial x_1^2} - D_{26}\frac{\partial^2 w}{\partial x_2^2} - 2D_{66}\frac{\partial^2 w}{\partial x_1 \partial x_2},\\
Q_1 &= B_{11}\frac{\partial^2 u}{\partial x_1^2} + 2B_{16}\frac{\partial^2 u}{\partial x_1 \partial x_2} + B_{66}\frac{\partial^2 u}{\partial x_2^2} + B_{16}\frac{\partial^2 v}{\partial x_1^2} + (B_{12}+B_{66})\frac{\partial^2 v}{\partial x_1 \partial x_2}\\
&\quad + B_{26}\frac{\partial^2 v}{\partial^2 x_2} - D_{11}\frac{\partial^3 w}{\partial x_1^3} - 3D_{16}\frac{\partial^3 w}{\partial x_1^2 \partial x_2} - (D_{12}+2D_{66})\frac{\partial^3 w}{\partial x_1 \partial x_2^2} - D_{26}\frac{\partial^3 w}{\partial x_2^3},\\
Q_2 &= B_{16}\frac{\partial^2 u}{\partial x_1^2} + (B_{12}+B_{66})\frac{\partial^2 u}{\partial x_1 \partial x_2} + B_{26}\frac{\partial^2 u}{\partial x_2^2} + B_{66}\frac{\partial^2 v}{\partial x_1^2} + 2B_{26}\frac{\partial^2 v}{\partial x_1 \partial x_2}\\
&\quad + B_{22}\frac{\partial^2 v}{\partial^2 x_2} - D_{16}\frac{\partial^3 w}{\partial x_1^3} - (D_{12}+2D_{66})\frac{\partial^3 w}{\partial x_1^2 \partial x_2} - 3D_{26}\frac{\partial^3 w}{\partial x_1 \partial x_2^2} - D_{22}\frac{\partial^3 w}{\partial x_2^3},\\
V_1 &= B_{11}\frac{\partial^2 u}{\partial x_1^2} + 3B_{16}\frac{\partial^2 u}{\partial x_1 \partial x_2} + 2B_{66}\frac{\partial^2 u}{\partial x_2^2} + B_{16}\frac{\partial^2 v}{\partial x_1^2} + (B_{12}+2B_{66})\frac{\partial^2 v}{\partial x_1 \partial x_2}\\
&\quad + 2B_{26}\frac{\partial^2 v}{\partial^2 x_2} - D_{11}\frac{\partial^3 w}{\partial x_1^3} - 4D_{16}\frac{\partial^3 w}{\partial x_1^2 \partial x_2} - (D_{12}+D_{66})\frac{\partial^3 w}{\partial x_1 \partial x_2^2} - 2D_{26}\frac{\partial^3 w}{\partial x_2^3},
\end{aligned}
\tag{8.2.12}
$$

$$
\begin{aligned}
V_2 = {} & 2B_{16}\frac{\partial^2 u}{\partial x_1^2} + (B_{12}+2B_{66})\frac{\partial^2 u}{\partial x_1 \partial x_2} + B_{26}\frac{\partial^2 u}{\partial x_2^2} + 2B_{66}\frac{\partial^2 v}{\partial x_1^2} + 3B_{26}\frac{\partial^2 v}{\partial x_1 \partial x_2} \\
& + B_{22}\frac{\partial^2 v}{\partial x_2^2} - 2D_{16}\frac{\partial^3 w}{\partial x_1^3} - (D_{12}+4D_{66})\frac{\partial^3 w}{\partial x_1^2 \partial x_2} - 4D_{26}\frac{\partial^3 w}{\partial x_1 \partial x_2^2} - D_{22}\frac{\partial^3 w}{\partial x_2^3}
\end{aligned}
$$

The coupled system of three partial differential equations (8.2.6) or (8.2.7), respectively, can be simplified for special layer stacking, Sect. 4.2.3. The differential operators L_{ij} for some special cases are given in App. C.

1. *Symmetric laminates*
   Because all coupling stiffness B_{ij} are zero the in-plane and the out-of-plane displacement response are uncoupled. With $L_{13} = L_{31} = 0, L_{23} = L_{32} = 0$ Eq. (8.2.7) simplifies to
$$
\begin{bmatrix} L_{11} & L_{12} & 0 \\ L_{12} & L_{22} & 0 \\ 0 & 0 & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p_3 \end{bmatrix} \qquad (8.2.13)
$$
   The plate equation reduces to $L_{33}w = p_3$ and corresponds to the plate equation of an anisotropic homogeneous plate.
2. *Antisymmetric laminates*
   The in-plane and the transverse part of Eq. (8.2.7) are coupled, but with $A_{16} = A_{26} = 0, D_{16} = D_{26} = 0$ the differential operators L_{11}, L_{22}, L_{33} and L_{12} are reduced. It is no in-plane tension/shearing coupling and no bending/twisting coupling.
3. *Balanced laminates*
   For general balanced laminates with $A_{16} = A_{26}$ only the in-plane tension/shearing coupling is zero, for an antisymmetric balanced laminate we have $A_{16} = A_{26} = 0, D_{16} = D_{26} = 0$ and for symmetric balanced laminates follow $A_{16} = A_{26} = 0$, $B_{ij} = 0$. The last case yields the equations
$$
\begin{bmatrix} L_{11} & L_{12} & 0 \\ L_{12} & L_{22} & 0 \\ 0 & 0 & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p_3 \end{bmatrix}
$$
   with simplified differential operators L_{11} and L_{22}. Only the in-plane equations correspond to an orthotropic stiffness behavior.
4. *Cross-ply laminates*
   The stacking can be unsymmetrical, i.e. $A_{16} = A_{26} = 0, D_{16} = D_{26} = 0$, $B_{16} = B_{26} = 0$, antisymmetrical, i.e. $A_{16} = A_{26} = 0, D_{16} = D_{26} = 0$, $B_{12} = B_{16} = B_{26} = B_{66} = 0, B_{22} = -B_{11}$ or symmetrical with $A_{16} = A_{26} = 0$, $D_{16} = D_{26} = 0, B_{ij} = 0$. Cross-ply laminates have an orthotropic response to both in-plane and bending and no in-plane/bending coupling. The plate equation $L_{33}w = p_3$ corresponds to the equation of an homogeneous orthotropic plate.

Summarizing the mathematical structures of the differential equations in dependence on the layer stacking the following conclusions can be drawn:

- The mathematical structure of a general balanced laminate is not much simpler as for a general unsymmetric, unbalanced laminate
- Compared to the general case the mathematical structure of the symmetric cross-ply laminate is nearly trivial. A symmetric cross-ply is orthotropic with respect to both in-plane and bending behavior, and both are uncoupled.
- The most simple mathematical structure yields the laminate with symmetrical arranged isotropic layers. With $A_{11} = A_{22}, A_{16} = A_{26} = 0, B_{ij} = 0$, $D_{11} = D_{22}$, $D_{16} = D_{26} = 0$, it corresponds to a single layer isotropic plate with in-plane and transverse loading.
- For special layer stacking also the force and moment resultant Eqs. (8.2.12) are reduced to more simple equations.

The following developments are restricted to general symmetric plates and plates with specially orthotropic behavior. The equations will be significant simplified, for example in the general case all $B_{ij} = 0$ and for specially orthotropic plates there are additional $D_{16} = D_{26} = 0$. The in-plane and the flexural equations are uncoupled. Table 8.1 summarizes the most important plate equations. In Table 8.1 standard boundary conditions are also expressed. The necessary and sufficient number of boundary conditions for plates considered here are two at each of the boundaries. The standard conditions for the free edge reduce the three static conditions $M_n = 0$, $Q_n = 0$ and $M_{nt} = 0$ to two conditions $M_n = 0, V_n = 0$, where $V_n = Q_n + \partial M_{nt}/\partial t = 0$ is as discussed above the Kirchhoff effective shear resultant. In order to avoid mistakes in the application the equations of Table 8.1, a summary of plate stiffness is given. Table 8.2 contains the plate stiffness for single layer plates. The plate stiffness for symmetric laminates are given in Table 8.3. In all equations the hygrothermal effects are neglected, but it is no problem to include thermal or moisture changes. In this case (4.2.63), (4.2.64) must be used instead of (8.2.4) to put into the equilibrium equations. This will be considered in Sect. 8.5.

The classical laminate theory can be used also for modelling and analysis of vibration and buckling of laminated plates. We restrict the consideration to symmetric plates. In the case of forced transversal vibration the momentum equilibrium equation (8.2.3) has an additional inertial term

$$\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} = -p_3 + \rho h \frac{\partial^2 w}{\partial t^2} \tag{8.2.14}$$

M_1, M_2, M_6, w and p_3 are functions of x_1, x_2 and the time t, h is the total thickness of the plate and ρ the mass density

$$h = \sum_{k=1}^{n} h^{(k)}, \quad \rho = \frac{1}{h}\sum_{k=1}^{n} \rho^{(k)} \left(x_3^{(k)} - x_3^{(k-1)} \right) = \frac{1}{h}\sum_{k=1}^{n} \rho^{(k)} h^{(k)} \tag{8.2.15}$$

The rotatory inertia is neglected. The Eqs. (8.2.14), (8.2.4) and (8.2.5) yield the plate equations for force vibration. For the both layer stacking discussed above we obtain:

Table 8.1 Plate equation, boundary conditions and stress resultants of symmetric laminates

1. General case: $B_{ij} = 0, D_{ij} \neq 0\ i, j = 1,2,6$ $D_{11}\dfrac{\partial^4 w}{\partial x_1^4} + 4D_{16}\dfrac{\partial^4 w}{\partial x_1^3 \partial x_2} + 2(D_{12} + 2D_{66})\dfrac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + 4D_{26}\dfrac{\partial^4 w}{\partial x_1 \partial x_2^3} + D_{22}\dfrac{\partial^4 w}{\partial x_2^4} = p_3$
2. Specially orthotropic laminates: $B_{ij} = 0, D_{16} = D_{26} = 0$ $D_{11}\dfrac{\partial^4 w}{\partial x_1^4} + 2(D_{12} + 2D_{66})\dfrac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + D_{22}\dfrac{\partial^4 w}{\partial x_2^4} = p_3$ or with $D_{11} = D_1, D_{22} = D_2, D_{12} + 2D_{66} = D_3$ $D_1\dfrac{\partial^4 w}{\partial x_1^4} + 2D_3\dfrac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + D_2\dfrac{\partial^4 w}{\partial x_2^4} = p_3$
3. Laminates with isotropic layers: $D_{11} = D_{22} = D_1, (D_{12} + 2D_{66}) = D_3$ $D_1\dfrac{\partial^4 w}{\partial x_1^4} + 2D_3\dfrac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + D_1\dfrac{\partial^4 w}{\partial x_2^4} = p_3$
Typical boundary conditions: 1. Simply supported edge: $w = 0, M_n = 0$ 2. Clamped edge: $w = 0, \partial w/\partial n = 0$ 3. Free edge: $M_n = 0, V_n = Q_n + \partial M_{nt}/\partial t = 0$
Stress resultants: 1. General case $\begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{bmatrix} -\partial^2 w/\partial x_1^2 \\ -\partial^2 w/\partial x_2^2 \\ -2\partial^2 w/\partial x_1 \partial x_2 \end{bmatrix}, \quad Q_1 = \dfrac{\partial M_1}{\partial x_1} + \dfrac{\partial M_6}{\partial x_2}, \ Q_2 = \dfrac{\partial M_6}{\partial x_1} + \dfrac{\partial M_2}{\partial x_2}$ 2. Specially orthotropic $\begin{bmatrix} M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix} \begin{bmatrix} -\partial^2 w/\partial x_1^2 \\ -\partial^2 w/\partial x_2^2 \\ -2\partial^2 w/\partial x_1 \partial x_2 \end{bmatrix}, \quad Q_1 = \dfrac{\partial M_1}{\partial x_1} + \dfrac{\partial M_6}{\partial x_2}, \ Q_2 = \dfrac{\partial M_6}{\partial x_1} + \dfrac{\partial M_2}{\partial x_2}$ 3. Isotropic layers (like 2. with $D_{11} = D_{22}$)

1. *General case of symmetric plates*

$$\left(L_{33} + \rho h \frac{\partial}{\partial t}\right) w = p_3$$

or explicitly

Table 8.2 Plate stiffness for single layer

Anisotropic single layer $D_{ij} = Q_{ij}^{(k)} \frac{h^3}{12}$
Specially orthotropic single layer (on-axis) $D_{11} = Q_{11}\frac{h^3}{12},\ D_{12} = Q_{12}\frac{h^3}{12},\ D_{22} = Q_{22}\frac{h^3}{12},\ D_{66} = Q_{66}\frac{h^3}{12},$ $Q'_{11} = \left(\frac{E'_1}{1-\nu'_{12}\nu'_{21}}\right),\ Q'_{22} = \left(\frac{E'_2}{1-\nu'_{12}\nu'_{21}}\right),\ Q'_{12} = \left(\frac{\nu'_{12}E_1}{1-\nu'_{12}\nu'_{21}}\right),$ $Q'_{66} = G'_{12} = E'_6$
Isotropic single layer $D_{11} = D_{22} = \frac{Eh^3}{12(1-\nu^2)} = D,\ D_{12} = \nu D = \frac{\nu E h^3}{1-\nu^2},$ $D_{66} = \frac{1-\nu}{2}D = \frac{Eh^3}{24(1+\nu)},$

$$D_{11}\frac{\partial^4 w}{\partial x_1^4} + 4D_{16}\frac{\partial^4 w}{\partial x_1^3 \partial x_2} + 2(D_{12}+2D_{66})\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + 4D_{26}\frac{\partial^4 w}{\partial x_1 \partial x_2^3} + D_{22}\frac{\partial^4 w}{\partial x_2^4} = p_3 - \rho h \frac{\partial^2 w}{\partial t^2}$$

with $w = w(x_1,x_2,t), p = (x_1,x_2,t)$.

2. *Specially orthotropic plates*

$$D_1\frac{\partial^4 w}{\partial x_1^4} + 2D_3\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + D_2\frac{\partial^4 w}{\partial x_2^4} = p_3 - \rho h \frac{\partial^2 w}{\partial t^2} \tag{8.2.16}$$

The equation of symmetric laminate plates with isotropic layers follows from (8.2.16) with $D_1 = D_2$, the plate stiffness are taken from Table 8.2 (single layer plates) or Table 8.3 (laminates). In the case of the computation of natural or eigenvibrations, the forcing function $p_3(x_1,x_2,t)$ is taken to be zero and the time dependent motion is a harmonic oscillation. The differential equation is homogeneous, leading an eigenvalue problem for the eigenvalues (natural frequencies) and the eigenfunctions (mode shapes).

To predict the buckling for plates, in-plane force resultants must be included. For a coupling of in-plane loads and lateral deflection, the equilibrium (8.2.2) will be formulated for the deformed plate element with $p_1 = p_2 = p_3 = 0$ and modified to

Table 8.3 Plate stiffness for symmetric laminates

Symmetric angle ply laminate $D_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)^3} - x_3^{(k-1)^3} \right) = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)} \left(\overline{x}_3^{(k)} + \frac{h^{(k)^2}}{12} \right)$, $\overline{x}_3^{(k)} = \frac{1}{2} \left(x_3^{(k)} + x_3^{(k-1)} \right)$, the $Q_{ij}^{(k)}$ follow from Table 4.2.
Symmetric balanced laminates $D_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)} \left(\overline{x}_3^{(k)} + \frac{h^{(k)^2}}{12} \right)$ The $Q_{ij}^{(k)}$ follow from Table 4.2.
Symmetric cross-ply laminate (specially orthotropic) $D_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)} \left(\overline{x}_3^{(k)} + \frac{h^{(k)^2}}{12} \right)$, $D_{16} = D_{26} = 0$ $Q_{11}^{(k)} = \left(\frac{E_1}{1 - \nu_{12}\nu_{21}} \right)^{(k)}, \ Q_{22}^{(k)} = \left(\frac{E_2}{1 - \nu_{12}\nu_{21}} \right)^{(k)}$, $Q_{12}^{(k)} = \left(\frac{\nu_{12} E_1}{1 - \nu_{12}\nu_{21}} \right)^{(k)}, \ Q_{66}^{(k)} = G_{12}^{(k)}$
Symmetric laminate with isotropic layers (x_1-direction equal fibre direction) $D_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)} \left(\overline{x}_3^{(k)} + \frac{h^{(k)^2}}{12} \right)$, $D_{16} = D_{26} = 0, D_{11} = D_{22}$, $Q_{11}^{(k)} = Q_{22}^{(k)} = \left(\frac{E}{1 - \nu^2} \right)^{(k)}, \ Q_{12}^{(k)} = \left(\frac{\nu E}{1 - \nu^2} \right)^{(k)}$, $Q_{66}^{(k)} = \left(\frac{E}{2(1 + \nu)} \right)^{(k)}$

$$
\begin{aligned}
&\frac{\partial^2 M_1}{\partial x_1^2} + 2\frac{\partial^2 M_6}{\partial x_1 \partial x_2} + \frac{\partial^2 M_2}{\partial x_2^2} = N_1 \frac{\partial^2 w}{\partial x_1^2} + N_2 \frac{\partial^2 w}{\partial x_2^2} + 2N_6 \frac{\partial^2 w}{\partial x_1 \partial x_2}, \\
&\frac{\partial N_1}{\partial x_1} + \frac{\partial N_6}{\partial x_2} = 0, \qquad \frac{\partial N_6}{\partial x_1} + \frac{\partial N_2}{\partial x_2} = 0
\end{aligned}
\tag{8.2.17}
$$

In the *general case of a symmetric laminate*, the plate equation can be expressed by

$$
\begin{aligned}
&D_{11} \frac{\partial^4 w}{\partial x_1^4} + 4D_{16} \frac{\partial^4 w}{\partial x_1^3 \partial x_2} + 2(D_{12} + 2D_{66}) \frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} \\
&+ 4D_{26} \frac{\partial^4 w}{\partial x_1 \partial x_2^3} + D_{22} \frac{\partial^4 w}{\partial x_2^4} = N_1 \frac{\partial^2 w}{\partial x_1^2} + N_2 \frac{\partial^2 w}{\partial x_2^2} + 2N_6 \frac{\partial^2 w}{\partial x_1 \partial x_2}
\end{aligned}
\tag{8.2.18}
$$

and for *specially orthotropic laminates*

$$D_1\frac{\partial^4 w}{\partial x_1^4}+2D_3\frac{\partial^4 w}{\partial x_1^2\partial x_2^2}+D_2\frac{\partial^4 w}{\partial x_2^4}=N_1\frac{\partial^2 w}{\partial x_1^2}+N_2\frac{\partial^2 w}{\partial x_2^2}+2N_6\frac{\partial^2 w}{\partial x_1\partial x_2} \qquad (8.2.19)$$

The *special case of symmetric laminates with isotropic layers* follows from (8.2.19) with $D_1 = D_2$. The buckling load is like the natural vibration independent of the lateral load and p_3 is taken to be zero. The classical bifurcation buckling requires to satisfy the governing differential equations derived above and the boundary equations. Both sets of equations are again homogeneous and represent an eigenvalue problem for the buckling modes (eigenvalues) and the mode shapes (eigenfunctions).

To calculate the in-plane stress resultants N_1,N_2,N_6 it is usually convenient to represent they by the *Airy stress function* $F(x_1,x_2)$

$$N_1=\frac{\partial^2 F}{\partial x_2^2},\quad N_2=\frac{\partial^2 F}{\partial x_1^2},\quad N_6=-\frac{\partial^2 F}{\partial x_1\partial x_2} \qquad (8.2.20)$$

If Eqs. (8.2.19) are substituted into the first two equilibrium equations (8.2.3) it is seen that these equations are identically satisfied. Using Eq. (4.2.22)

$$\boldsymbol{M}=\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{N}-(\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}-\boldsymbol{D})\boldsymbol{\kappa}$$

and substitute $\boldsymbol{N}$ with help of the Airy's stress function and $\boldsymbol{\kappa}$ by the derivatives of w the third equilibrium equation (8.2.3) yields one coupled partial differential equation for F and w. The necessary second equation yields the in-plane compatibility condition (Sect. 2.2)

$$\frac{\partial^2\varepsilon_1}{\partial x_2^2}+\frac{\partial^2\varepsilon_2}{\partial x_1^2}=\frac{\partial^2\varepsilon_6}{\partial x_1\partial x_2}$$

together with Eq. (4.2.25) to substitute the strains by the stress resultants. Suppressing the derivations and restricting to symmetric problems yield the following in-plane equations which are summarized in Table 8.4. The stiffness $\boldsymbol{A}^*,\boldsymbol{B}^*,\boldsymbol{C}^*,\boldsymbol{D}^*$ follow with Eq. (4.2.23) as $\boldsymbol{A}^*=\boldsymbol{A}^{-1}$, $\boldsymbol{B}^*=-\boldsymbol{A}^{-1}\boldsymbol{B}$, $\boldsymbol{C}^*=\boldsymbol{B}\boldsymbol{A}^{-1}$, $\boldsymbol{D}^*=\boldsymbol{D}-\boldsymbol{B}\boldsymbol{A}^{-1}\boldsymbol{B}$. One can see from Table 8.4 that in the general case the mathematical structure of the partial differential equation corresponds to an anisotropic and in the special orthotropic case to an orthotropic in-plane behavior of a single layer homogeneous anisotropic or orthotropic plate. A summary of the in-plane stiffness is given in Table 8.5. The $Q_{ij}^{(k)}$ for angle-ply laminates are calculated in Table 4.2.

Similar to the beam theory the plate equations for flexure, vibration and buckling can be given in a variational formulation (Sect. 2.2). This formulation provides the basis for the development of approximate solutions. We restrict the variational formulation to symmetric laminated plates and to the classical energy principles. From (2.2.24) it follows with $\varepsilon_3=\varepsilon_4=\varepsilon_5\approx 0$ that the elastic potential Π is

Table 8.4 In-plane equations, boundary conditions and stress resultants for symmetric laminates

1. Angle-ply laminates $B^*_{ij}=0, C^*_{ij}=0, i,j=1,2,6$ $A^*_{22}\dfrac{\partial^4 F}{\partial x_1^4}-2A^*_{26}\dfrac{\partial^4 F}{\partial x_1^3\partial x_2}+(2A^*_{12}+A^*_{66})\dfrac{\partial^4 F}{\partial x_1^2\partial x_2^2}-2A^*_{16}\dfrac{\partial^4 F}{\partial x_1\partial x_2^3}+A^*_{11}\dfrac{\partial^4 F}{\partial x_2^4}=0$
2. Cross-ply laminates $B^*_{ij}=0, A^*_{16}=A^*_{26}=0,$ $A^*_{22}\dfrac{\partial^4 F}{\partial x_1^4}+(2A^*_{12}+A^*_{66})\dfrac{\partial^4 F}{\partial x_1^2\partial x_2^2}+A^*_{11}\dfrac{\partial^4 F}{\partial x_2^4}=0$ or with $A^*_{11}=A^*_1, A^*_{22}=A^*_2, (2A^*_{12}+A^*_{66})=A^*_3,$ $A^*_2\dfrac{\partial^4 F}{\partial x_1^4}+2A^*_3\dfrac{\partial^4 F}{\partial x_1^2\partial x_2^2}+A^*_1\dfrac{\partial^4 F}{\partial x_2^4}=0$
3. Laminates with isotropic layers $A^*_1=A^*_2=A^*_3=1,$ $\dfrac{\partial^4 F}{\partial x_1^4}+2\dfrac{\partial^4 F}{\partial x_1^2\partial x_2^2}+\dfrac{\partial^4 F}{\partial x_2^4}=0$
Typical boundary conditions Edge $x_1=\text{const}$ $\dfrac{\partial^2 F}{\partial x_2^2}=N_1(x_1=\text{const},x_2), -\dfrac{\partial^2 F}{\partial x_1\partial x_2}=N_6(x_1=\text{const},x_2),$ For an unloaded edge follow $N_1=0, N_6=0$
Stress resultants $N_1=\dfrac{\partial^2 F}{\partial x_2^2}, N_2=\dfrac{\partial^2 F}{\partial x_1^2}, N_6=-\dfrac{\partial^2 F}{\partial x_1\partial x_2}$

$$
\begin{aligned}
\Pi &= \frac{1}{2}\int\limits_V(\sigma_1\varepsilon_1+\sigma_2\varepsilon_2+\sigma_6\varepsilon_6)\mathrm{d}V-\int\limits_A p_3(x_1,x_2)w(x_1,x_2)\mathrm{d}A \\
&= \frac{1}{2}\sum_{k=1}^{n}\int\limits_A\int\limits_{x_3^{(k-1)}}^{x_3^{(k)}}(\sigma_1^{(k)}\varepsilon_1+\sigma_2^{(k)}\varepsilon_2+\sigma_6^{(k)}\varepsilon_6)\mathrm{d}x_3\mathrm{d}A-\int\limits_A p_3(x_1,x_2)w(x_1,x_2)\mathrm{d}A
\end{aligned}
\tag{8.2.21}
$$

With

$$\boldsymbol{\varepsilon}(x_1,x_2,x_3)=\boldsymbol{\varepsilon}(x_1,x_2)+x_3\boldsymbol{\kappa}(x_1,x_2) \tag{8.2.22}$$

Table 8.5 In-plane stiffness for symmetric laminates

1. Angle-ply laminates $A_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)} - x_3^{(k-1)} \right) = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)}, i, j = 1, 2, 6$
2. Cross-ply laminates $A_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)}, i, j = 1, 2, 6, A_{16} = A_{26} = 0,$ $Q_{11}^{(k)} = \left(\frac{E_1}{1 - \nu_{12}\nu_{21}} \right)^{(k)}, Q_{22}^{(k)} = \left(\frac{E_2}{1 - \nu_{12}\nu_{21}} \right)^{(k)}$ $Q_{12}^{(k)} = \left(\frac{\nu_{12}E_1}{1 - \nu_{12}\nu_{21}} \right)^{(k)}, Q_{66}^{(k)} = G_{12}^{(k)}$
3. Laminates with isotropic layers $A_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)}, i, j = 1, 2, 6, A_{16} = A_{26} = 0, A_{11} = A_{22}$ $Q_{11}^{(k)} = Q_{22}^{(k)} = \left(\frac{E}{1 - \nu^2} \right)^{(k)}, Q_{12}^{(k)} = \left(\frac{\nu E}{1 - \nu^2} \right)^{(k)},$ $Q_{66}^{(k)} = \left(\frac{E}{2(1 + \nu)} \right)^{(k)} = G^{(k)}$
4. Single layer For anisotropic and orthotropic single layers the A_{ij} followed by 1. and 2. For an isotropic single layer is $A_{11} = A_{22} = A = \frac{Eh}{1 - \nu^2}, A_{12} = \nu A = \frac{\nu Eh}{1 - \nu^2},$ $A_{66} = \frac{(1 - \nu)}{2} A = \frac{Eh}{2(1 + \nu)} = G$

$$
\begin{aligned}
\boldsymbol{\varepsilon}^{\mathrm{T}} &= \left[\frac{\partial u}{\partial x_1} \quad \frac{\partial v}{\partial x_2} \quad \left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} \right) \right], \\
\boldsymbol{\kappa}^{\mathrm{T}} &= \left[-\frac{\partial^2 w}{\partial x_1^2} \quad -\frac{\partial^2 w}{\partial x_2^2} \quad -2\frac{\partial^2 w}{\partial x_1 \partial x_2} \right]
\end{aligned}
$$

and the constitutive equations for the strains and the stress resultants

$$
\boldsymbol{\sigma}^{(k)} = \boldsymbol{Q}^{(k)}(\boldsymbol{\varepsilon} + x_3\boldsymbol{\kappa}), \quad \begin{bmatrix} \boldsymbol{N} \\ \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \vdots & \boldsymbol{0} \\ \cdots & & \cdots \\ \boldsymbol{0} & \vdots & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \\ \boldsymbol{\kappa} \end{bmatrix} \tag{8.2.23}
$$

one obtains the elastic potential for the general case of symmetric plates and for the special cases of orthotropic or isotropic structure behavior.

Bending of plates, classical laminate theory:
Angle-ply laminates

$$\begin{aligned}\Pi(w) = \frac{1}{2}\int\limits_A \Bigg[& D_{11}\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 + D_{22}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} + 4D_{66}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 \\ & + 4\left(D_{16}\frac{\partial^2 w}{\partial x_1^2} + D_{26}\frac{\partial^2 w}{\partial x_2^2}\right)\frac{\partial^2 w}{\partial x_1 x_2}\Bigg] \mathrm{d}A - \int\limits_A p_3 w \mathrm{d}A\end{aligned} \tag{8.2.24}$$

Cross-ply laminates

$$D_{16} = D_{26} = 0$$

Laminates with isotropic layers

$$D_{16} = D_{26} = 0, D_{11} = D_{22}$$

The principle of minimum of the total potential yields

$$\delta \Pi[w(x_1, x_2)] = 0$$

as the basis to derive the differential equation and boundary conditions or to apply the direct variational methods of Ritz, Galerkin or Kantorovich for approximate solutions.

Vibration of plates, classical laminate theory:
The kinetic energy of a plate is (rotatory energy is neglected)

$$T = \frac{1}{2}\int\limits_A \rho h \left(\frac{\partial w}{\partial t}\right)^2 \mathrm{d}A, \qquad \rho = \frac{1}{h}\sum_{k=1}^{n} \rho^{(k)} h^{(k)} \tag{8.2.25}$$

The Hamilton principle for vibrations yields

$$\delta H(x_1, x_2, t) = 0$$

with

$$H = \int\limits_{t_1}^{t_2} (T - \Pi) dt = \int\limits_{t_1}^{t_2} L \mathrm{d}t \tag{8.2.26}$$

Buckling of plates, classical laminate theory:
To calculate buckling loads, the in-plane stress resultants N_1, N_2, N_6 must be included into the potential Π. These in-plane stress resultants are computed in a first step or are known a priori. With the known N_1, N_2, N_6 the potential Π can be formulated for angle-ply laminates with bending in-plane forces

$$
\begin{aligned}
\Pi(w) = \frac{1}{2}\int\limits_A \Bigg\{ & \left[D_{11}\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 + D_{22}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 + 2D_{12}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} \right. \\
& \left. + 4D_{66}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 + 4\left(D_{16}\frac{\partial^2 w}{\partial x_1^2} + D_{26}\frac{\partial^2 w}{\partial x_2^2}\right)\frac{\partial^2 w}{\partial x_1 x_2} \right] \\
& - \left[N_1\left(\frac{\partial w}{\partial x_1}\right)^2 + N_2\left(\frac{\partial w}{\partial x_2}\right)^2 + 2N_6\left(\frac{\partial w}{\partial x_1}\frac{\partial w}{\partial x_2}\right) \right] - 2p_3 w \Bigg\} \mathrm{d}A
\end{aligned}
\tag{8.2.27}
$$

The buckling formulation one get with $p_3 \equiv 0$. With $D_{16} = D_{26} = 0$ or $D_{16} = D_{26} = 0, D_{11} = D_{22}$ follows the equations for cross-ply laminates plates and for plates with isotropic layers. The plate stiffness can be taken from Tables 8.2 or 8.3.

The variational principle $\delta\Pi = 0$ applied to (8.2.24) and (8.2.27) yield solutions for bending and bending with in-plane forces. Hamilton's principle and $p_3 \neq 0$ is valid to calculate forced vibrations. With $p_3(x_1,x_2,t) = 0$ in vibration equations or $p_3(x_1,x_2) = 0$ in (8.2.27), we have formulated eigenvalue problems to compute natural frequencies or buckling loads.

Summarizing the derivations of governing plate equations in the frame of classical laminate theory there are varying degrees of complexity:

- An important simplification of the classical two-dimensional plate equations is the behavior of cylindrical bending. In this case one considers a laminated plate strip with a very high length-to-width ratio. The transverse load and all displacements are functions of only x_1 and all derivatives with respect to x_2 are zero. The laminated beams, Chap. 7, and the laminated strips under cylindrical bending are the two cases of laminated plates that can be treated as one-dimensional problems. In Sect. 8.6 we discuss some applications of cylindrical bending
- In the case of two-dimensional plate equations the first degree of simplification for plates is to be symmetric. Symmetric laminates can be broken into cross-ply laminates (specially orthotropic plates) with uncoupling in-plane and bending response ($B_{ij} = 0$) and vanishing bending-twisting terms ($D_{16} = D_{26} = 0$) and angle-ply laminates (only $B_{ij} = 0$). The governing equations of symmetric cross-ply laminates have the mathematical structure of homogeneous orthotropic plates, symmetric angle-ply laminates of homogeneous anisotropic plates. For special boundary conditions symmetric cross-ply laminated rectangular plates can be solved analytically. The solutions were obtained in the same manner as for homogeneous isotropic plates, Sect. 8.6.
- Laminates with all coupling effects are more complicated to analyze. Generally, approximate analytical or numerical methods are used.

8.3 Shear Deformation Theory

In Sect. 8.2 we have neglected the transverse shear deformations effects. The analysis and results of the classical laminate theory are sufficiently accurate for thin plates, i.e. $a/h, b/h > 20$. Such plates are often used in civil engineering. For moderately thick plates we have to take into account the shear deformation effects, at least approximately. The theory of laminate or sandwich plates corresponds then with the Reissner or Mindlin[1] plate theory. In the Reissner-Mindlin theory the assumptions of the Kirchhoff's plate theory are relaxed only in one point. The transverse normals do not remain perpendicular to the middle surface after deformation, i.e. a linear element extending through the thickness of the plate and perpendicular to the mid-surface prior to loading, upon the load application undergoes at most a translation and a rotation. Plate theories based upon this assumption are called first order shear deformation theories and are most used in the analysis of moderate thick laminated plates and of sandwich plates. Higher order theories which do not require normals to remain straight are considerably more complicated.

Based upon that kinematical assumption of the first order shear deformation theory the displacements of the plate have the form (5.1.2)

$$\begin{aligned} u_1(x_1,x_2,x_3) &= u(x_1,x_2) + x_3\psi_1(x_1,x_2), \\ u_2(x_1,x_2,x_3) &= v(x_1,x_2) + x_3\psi_2(x_1,x_2), \\ u_3(x_1,x_2,x_3) &= w(x_1,x_2) \end{aligned} \tag{8.3.1}$$

and with (5.1.3) are the strains

$$\begin{aligned} &\varepsilon_i(x_1,x_2,x_3) = \varepsilon_i(x_1,x_2) + x_3\kappa_i(x_1,x_2), \quad i = 1,2,6, \\ &\boldsymbol{\varepsilon}^{\mathrm{T}} = \left[\frac{\partial u}{\partial x_1}\ \frac{\partial v}{\partial x_2}\ \frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1}\right], \quad \boldsymbol{\kappa}^{\mathrm{T}} = \left[\frac{\partial \psi_1}{\partial x_1}\ \frac{\partial \psi_2}{\partial x_2}\ \frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1}\right], \\ &\varepsilon_4(x_1,x_2) = \frac{\partial w}{\partial x_2} + \psi_2, \quad \varepsilon_5(x_1,x_2) = \frac{\partial w}{\partial x_1} + \psi_1 \end{aligned} \tag{8.3.2}$$

One can see that a constant state of transverse shear stresses is accounted for. The stresses for the kth layer are formulated in (5.3.2) to

$$\boldsymbol{\sigma}^{(k)} = \boldsymbol{Q}^{(k)}\boldsymbol{\varepsilon}^{(k)}, \quad \sigma^{\mathrm{T}} = [\sigma_1\ \sigma_2\ \sigma_6\ \sigma_4\ \sigma_5], \quad \boldsymbol{\varepsilon}^{\mathrm{T}} = [\varepsilon_1\ \varepsilon_2\ \varepsilon_6\ \varepsilon_4\ \varepsilon_5] \tag{8.3.3}$$

$\sigma_1, \sigma_2, \sigma_6$ vary linearly and σ_4, σ_5 constant through the thickness h of the plate. With the stress resultants $\boldsymbol{N}, \boldsymbol{M}, \boldsymbol{Q}^{\mathrm{s}}$ and stiffness coefficients $A_{ij}, B_{ij}, D_{ij}, A^{\mathrm{s}}_{ij}$ for laminates or sandwiches given in Eqs. (4.2.13) - (4.2.15) or (4.3.8) - (4.3.22), respectively, the constitutive equation can be formulated in a hypermatrix form, (4.2.16). The stiffness coefficients A_{ij}, B_{ij}, D_{ij} stay unchanged in comparison to the classical theory and the A^{s}_{ij} are defined in (5.3.4) and can be improved with the help of shear correc-

[1] Raymond David Mindlin (*17 September 1906 New York - †22 November 1987 Hanover, New Hempshire) - mechanician, seminal contributions to many branches of applied mechanics, applied physics, and engineering sciences

tion factors k^s_{ij} of plates similar to beams (7.3.19) – (7.3.20). The definition of the positive rotations ψ_1, ψ_2 is illustrated in Fig. 8.3. The equilibrium equations (8.2.2) - (8.2.3) stay unchanged.

Substituting the kinematic relations (5.3.1) into the constitutive equations (5.3.3) and then these equations into the five equilibrium equations (8.2.2) one obtains the governing plate equations for the shear deformation theory in a matrix form as

$$\begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} & \tilde{L}_{13} & \tilde{L}_{14} & 0 \\ \tilde{L}_{21} & \tilde{L}_{22} & \tilde{L}_{23} & \tilde{L}_{24} & 0 \\ \tilde{L}_{31} & \tilde{L}_{32} & \tilde{L}_{33} & \tilde{L}_{34} & \tilde{L}_{35} \\ \tilde{L}_{41} & \tilde{L}_{42} & \tilde{L}_{43} & \tilde{L}_{44} & \tilde{L}_{45} \\ 0 & 0 & \tilde{L}_{53} & \tilde{L}_{54} & \tilde{L}_{55} \end{bmatrix} \begin{bmatrix} u \\ v \\ \psi_1 \\ \psi_2 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ p \end{bmatrix} \tag{8.3.4}$$

The differential operators $\tilde{L}_{ij}$ are given in App. C.2 for unsymmetric angle-ply, symmetric angle-ply and symmetric cross-ply laminates. Symmetric laminates leading, additional to (8.3.4), the uncoupled plate equations

$$\begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} \\ \tilde{L}_{21} & \tilde{L}_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} \tilde{L}_{33} & \tilde{L}_{34} & \tilde{L}_{35} \\ \tilde{L}_{43} & \tilde{L}_{44} & \tilde{L}_{45} \\ \tilde{L}_{53} & \tilde{L}_{54} & \tilde{L}_{55} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix} \tag{8.3.5}$$

Equation (8.3.4) can also formulated in a compact matrix form

$$\tilde{\boldsymbol{L}}\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{p}}$$

$\tilde{\boldsymbol{L}}$ is a (5×5) matrix and $\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{p}}$ are (5×1) matrices.

The governing plate equations including transverse shear deformations are a set of three coupled partial equations of second order, i.e. the problem is of sixth order an for each edge of the plate three boundary conditions must be prescribed. The most usual boundary conditions are:

- fixed boundary

$$w = 0, \quad \psi_n = 0, \quad \psi_t = 0$$

- free boundary

$$M_n = 0, \quad M_{nt} = 0, \quad Q_n = 0$$

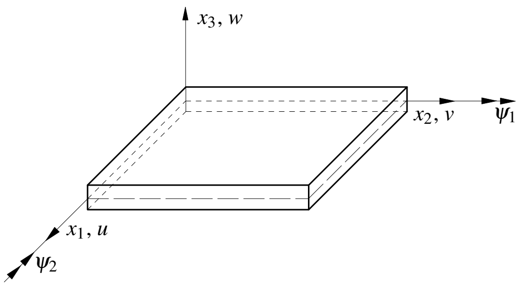

Fig. 8.3 Positive definition of rotations ψ_i

- free edge

$$M_n = 0, \quad \psi_t = 0, \quad w = 0$$

- simply supported boundary

a) $w = 0$, $M_n = 0$, $\psi_t = 0$ or $w = 0$, $\partial\psi_n/\partial n = 0$, $\psi_t = 0$ (hard hinged support)
b) $w = 0$, $M_n = 0$, $M_{nt} = 0$

Case b) is more complicated for analytical or semianalytical solutions. Generally, boundary conditions require prescribing for each edge one value of each of the following five pairs: (u or N_n), (v or N_{nt}), (ψ_n or M_n), (ψ_t or M_{nt}), (w or Q_n).

With $\psi_1 = -\partial w/\partial x_1$ and $\psi_2 = -\partial w/\partial x_2$ Eq. (8.3.5) can be reduced to the classical plate equation.

In the following we restrict our development to plates that are midplane symmetric ($B_{ij} = 0$), and additional all coupling coefficients $(\ldots)_{16}, (\ldots)_{26}, (\ldots)_{45}$ are zero. The constitutive equations are then simplified to

$$\begin{array}{lll} N_1 = A_{11}\varepsilon_1 + A_{12}\varepsilon_2, & N_2 = A_{12}\varepsilon_1 + A_{22}\varepsilon_1, & N_6 = A_{66}\varepsilon_6, \\ M_1 = D_{11}\kappa_1 + D_{12}\kappa_2, & M_2 = D_{12}\kappa_1 + D_{22}\kappa_2, & M_6 = D_{66}\kappa_6, \\ Q_1 = k^{\mathrm{s}}_{55}A_{55}\varepsilon_5, & Q_2 = k^{\mathrm{s}}_{44}A_{44}\varepsilon_4 & \end{array} \tag{8.3.6}$$

or in a contracted notation

$$\begin{gathered} \begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix}, \qquad \boldsymbol{Q}^{\mathrm{s}} = \boldsymbol{A}^{\mathrm{s}}\boldsymbol{\varepsilon}^{\mathrm{s}}, \\ \boldsymbol{N}^{\mathrm{T}} = [N_1\ N_2\ N_6], \quad \boldsymbol{M}^{\mathrm{T}} = [M_1\ M_2\ M_6], \quad \boldsymbol{Q}^{\mathrm{s}^{\mathrm{T}}} = [Q_1\ Q_2], \\ \boldsymbol{\varepsilon}^{\mathrm{T}} = [\varepsilon_1\ \varepsilon_2\ \varepsilon_6], \quad \boldsymbol{\kappa}^{\mathrm{T}} = [\kappa_1\ \kappa_2\ \kappa_6], \quad \boldsymbol{\varepsilon}^{\mathrm{s}^{\mathrm{T}}} = [\varepsilon_5\ \varepsilon_4], \\ \boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}, \quad \boldsymbol{A}^{\mathrm{s}} = \begin{bmatrix} k^{\mathrm{s}}_{55}A_{55} & 0 \\ 0 & k^{\mathrm{s}}_{44}A_{44} \end{bmatrix} \end{gathered} \tag{8.3.7}$$

Substituting the constitutive equations for M_1, M_2, M_6, Q_1, Q_2 into the three equilibrium equations (8.2.2) of the moments and transverse force resultants results in the following set of governing differential equations for a laminated composite plate subjected to a lateral load $p_3(x_1, x_2)$ and including transverse shear deformation

$$\begin{gathered} D_{11}\frac{\partial^2\psi_1}{\partial x_1^2} + (D_{12} + D_{66})\frac{\partial^2\psi_2}{\partial x_1\partial x_2} + D_{66}\frac{\partial^2\psi_1}{\partial x_2^2} - k^{\mathrm{s}}_{55}A_{55}\left(\psi_1 + \frac{\partial w}{\partial x_1}\right) = 0, \\ D_{66}\frac{\partial^2\psi_2}{\partial x_1^2} + (D_{12} + D_{66})\frac{\partial^2\psi_1}{\partial x_1\partial x_2} + D_{22}\frac{\partial^2\psi_2}{\partial x_2^2} - k^{\mathrm{s}}_{44}A_{44}\left(\psi_2 + \frac{\partial w}{\partial x_2}\right) = 0, \\ k^{\mathrm{s}}_{55}A_{55}\left(\frac{\partial\psi_1}{\partial x_1} + \frac{\partial^2 w}{\partial x_1^2}\right) + k^{\mathrm{s}}_{44}A_{44}\left(\frac{\partial\psi_2}{\partial x_2} + \frac{\partial^2 w}{\partial x_2^2}\right) + p_3(x_1, x_2) = 0 \end{gathered} \tag{8.3.8}$$

Analogous to the classical plate equations the shear deformation theory can be used for modelling and analysis of forced vibrations and buckling of laminate plates. In the general case of forced vibrations the displacements u, v, w, the rotations ψ_1, ψ_2 and the transverse load p in Eq. (8.3.4) are functions of x_1, x_2 and t. In-plane loading is not considered but in-plane displacements, rotary and coupling inertia terms have to take into account. Therefore, generalized mass densities must be defined

$$\begin{aligned} \rho_0 &= \sum_{k=1}^{n} \rho^{(k)} \left(x_3^{(k)} - x_3^{(k-1)} \right) = \sum_{k=1}^{n} \rho^{(k)} h^{(k)}, \\ \rho_1 &= \sum_{k=1}^{n} \rho^{(k)} \left(x_3^{(k)^2} - x_3^{(k-1)^2} \right), \\ \rho_2 &= \sum_{k=1}^{n} \rho^{(k)} \left(x_3^{(k)^3} - x_3^{(k-1)^3} \right) \end{aligned} \tag{8.3.9}$$

Coupling inertia terms ρ_1 are only contained in unsymmetric plate problems.

If one wishes to determine the natural frequencies of the rectangular plate considered above, then in (8.3.8) $p_3(x_1, x_2)$ must be set zero but a term $-\rho_0 \partial^2 w / \partial t^2$ must be added on the right hand side. In addition, because ψ_1 and ψ_2 are both independent variables which are independent of the transverse displacement w, there will be an oscillatory motion of a line element through the plate thickness which results in rotary inertia terms $\rho_2 \partial^2 \psi_1 / \partial t^2$ and $\rho_2 \partial^2 \psi_2 / \partial t^2$, respectively, on the right hand side of the first two equations of (8.3.8).

The governing equations for the calculation of natural frequencies of specially orthotropic plates with $A_{45} = 0$ are

$$\begin{aligned} & D_{11} \frac{\partial^2 \psi_1}{\partial x_1^2} + (D_{12} + D_{66}) \frac{\partial^2 \psi_2}{\partial x_1 \partial x_2} + D_{66} \frac{\partial^2 \psi_1}{\partial x_2^2} \\ & \qquad - k_{55}^{\mathrm{s}} A_{55} \left(\psi_1 + \frac{\partial w}{\partial x_1} \right) = \rho_2 \frac{\partial^2 \psi_1}{\partial t^2}, \\ & D_{66} \frac{\partial^2 \psi_2}{\partial x_1^2} + (D_{12} + D_{66}) \frac{\partial^2 \psi_1}{\partial x_1 \partial x_2} + D_{22} \frac{\partial^2 \psi_2}{\partial x_2^2} \\ & \qquad - k_{44}^{\mathrm{s}} A_{44} \left(\psi_2 + \frac{\partial w}{\partial x_2} \right) = \rho_2 \frac{\partial^2 \psi_2}{\partial t^2}, \\ & k_{55}^{\mathrm{s}} A_{55} \left(\frac{\partial \psi_1}{\partial x_2} + \frac{\partial^2 w}{\partial x_1^2} \right) + k_{44}^{\mathrm{s}} A_{44} \left(\frac{\partial \psi_2}{\partial x_2} + \frac{\partial^2 w}{\partial x_2^2} \right) = \rho_0 \frac{\partial^2 w}{\partial t^2}, \end{aligned} \tag{8.3.10}$$

$$\begin{aligned} \rho_0 &= \sum_{k=1}^{n} \rho^{(k)} (x_3^{(k)} - x_3^{(k-1)}) = \sum_{k=1}^{n} \rho^{(k)} h^{(k)}, \\ \rho_2 &= \frac{1}{3} \sum_{k=1}^{n} \rho^{(k)} (x_3^{(k)^3} - x_3^{(k-1)^3}) \end{aligned}$$

w, ψ_1 and ψ_2 are functions of x_1, x_2 and t.

In a similar way the governing equations for buckling problems can be derived. In the matrix equations (8.3.4) and (8.3.5) only the differential operator $\tilde{L}_{55}$ is substituted by

$$\tilde{L}_{55} - \left(N_1 \frac{\partial^2}{\partial x_1^2} + 2N_6 \frac{\partial^2}{\partial x_1 \partial x_2} + N_2 \frac{\partial^2}{\partial x_2^2} \right) \tag{8.3.11}$$

For a cross-ply symmetrically laminated plate is with $B_{ij} = 0, D_{16} = 0$, $D_{26} = 0$, $A_{45} = 0$

$$\begin{aligned}
&D_{11} \frac{\partial^2 \psi_1}{\partial x_1^2} + (D_{12} + D_{66}) \frac{\partial^2 \psi_2}{\partial x_1 \partial x_2} + D_{66} \frac{\partial^2 \psi_1}{\partial x_2^2} - k_{55}^{\mathrm{s}} A_{55} \left(\psi_1 + \frac{\partial w}{\partial x_1} \right) = 0, \\
&D_{66} \frac{\partial^2 \psi_2}{\partial x_1^2} + (D_{12} + D_{66}) \frac{\partial^2 \psi_1}{\partial x_1 \partial x_2} + D_{22} \frac{\partial^2 \psi_2}{\partial x_2^2} - k_{44}^{\mathrm{s}} A_{44} \left(\psi_2 + \frac{\partial w}{\partial x_2} \right) = 0, \\
&k_{55}^{\mathrm{s}} A_{55} \left(\frac{\partial \psi_1}{\partial x_2} + \frac{\partial^2 w}{\partial x_1^2} \right) + k_{44}^{\mathrm{s}} A_{44} \left(\frac{\partial \psi_2}{\partial x_2} + \frac{\partial^2 w}{\partial x_2^2} \right) \\
&= N_1 \frac{\partial^2 w}{\partial x_1^2} + 2N_6 \frac{\partial^2 w}{\partial x_1 \partial x_2} + N_2 \frac{\partial^2 w}{\partial x_2^2}
\end{aligned} \tag{8.3.12}$$

The variational formulation of laminated plates including shear deformation may be based for example upon the principle of minimum potential energy for static problems and the Hamilton's principle for dynamic problems. Formulating the elastic potential Π we have to consider that in the general case of unsymmetric laminate plates, including shear deformation, $\Pi = \Pi(u, v, w, \psi_1, \psi_2)$ is a potential function of five independent variables and that the strain energy Π_{i} has a membrane, a bending and a transverse shearing term, i.e.

$$\begin{aligned}
\Pi_{\mathrm{i}} &= \frac{1}{2} \int_V (\sigma_1 \varepsilon_1 + \sigma_2 \varepsilon_2 + \sigma_6 \varepsilon_6 + \sigma_5 \varepsilon_5 + \sigma_4 \varepsilon_4)\, \mathrm{d}V \\
&= \Pi_{\mathrm{i}}^{\mathrm{m}} + \Pi_{\mathrm{i}}^{\mathrm{b}} + \Pi_{\mathrm{i}}^{\mathrm{s}}
\end{aligned} \tag{8.3.13}$$

with

$$\begin{aligned}
\Pi_{\mathrm{i}}^{\mathrm{m}} &= \frac{1}{2} \int_V (N_1 \varepsilon_1 + N_2 \varepsilon_2 + N_6 \varepsilon_6) \mathrm{d}A, \\
\Pi_{\mathrm{i}}^{\mathrm{b}} &= \frac{1}{2} \int_V (M_1 \kappa_1 + M_2 \kappa_2 + M_6 \kappa_6) \mathrm{d}A, \\
\Pi_{\mathrm{i}}^{\mathrm{s}} &= \frac{1}{2} \int_V (Q_1^{\mathrm{s}} \varepsilon_5 + Q_2^{\mathrm{s}} \varepsilon_4) \mathrm{d}A
\end{aligned} \tag{8.3.14}$$

The stress resultants, stiffness and constitutive equations are formulated in Sect. 4.2, e.g. (4.2.10) - (4.2.17). The elastic potential Π is then given by

$$\begin{aligned}
\Pi(u, v, w, \psi_1, \psi_2) = \frac{1}{2} \int_A &(\boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{A} \boldsymbol{\varepsilon} + \boldsymbol{\kappa}^{\mathrm{T}} \boldsymbol{B} \boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^{\mathrm{T}} \boldsymbol{B} \boldsymbol{\kappa} + \boldsymbol{\kappa}^{\mathrm{T}} \boldsymbol{D} \boldsymbol{\kappa} \\
&+ \boldsymbol{\varepsilon}^{\mathrm{sT}} \boldsymbol{A}^{\mathrm{s}} \boldsymbol{\varepsilon}^{\mathrm{s}}) \mathrm{d}x_1 \mathrm{d}x_2 - \int_A p_3 w \, \mathrm{d}x_1 \mathrm{d}x_2
\end{aligned} \tag{8.3.15}$$

In (8.3.15) the in-plane loads p_1, p_2 are not included and must be added in general loading cases. Shear correction coefficients can be developed for plates quite similar to beams. Approximately one considers a laminate strip of the width "1" orthogonal to the x_1-direction and independently another laminate strip orthogonal to the x_2-direction and calculates the correction factors k^{s}_{55} and k^{s}_{44} like in Chap. 7 for beams. Sometimes the shear correction factors were used approximately equal to homogeneous plates, i.e, $k^{\mathrm{s}}_{44} = k^{\mathrm{s}}_{45} = k^{\mathrm{s}}_{55} = k^{\mathrm{s}} = 5/6$.

Mostly we have symmetric laminates and the variational formulation for bending Mindlin's plates can be simplified

$$\Pi(w, \psi_1, \psi_2) = \frac{1}{2}\int\limits_A (\boldsymbol{\kappa}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{\kappa} + \boldsymbol{\varepsilon}^{\mathrm{sT}}\boldsymbol{A}^{\mathrm{s}}\boldsymbol{\varepsilon}^{\mathrm{s}})\mathrm{d}x_1\mathrm{d}x_2 - \int\limits_A p_3 w \mathrm{d}x_1\mathrm{d}x_2 \tag{8.3.16}$$

If we restricted the Hamilton's principle to vibration of symmetric plates, the variational formulation yields

$$\begin{aligned} L(w, \psi_1, \psi_2) &= T(w, \psi_1, \psi_2) - \Pi(w, \psi_1, \psi_2), \\ T(w, \psi_1, \psi_2) &= \frac{1}{2}\int\limits_A \left[\rho_0\left(\frac{\partial w}{\partial t}\right)^2 + \rho_2\left(\frac{\partial \psi_1}{\partial t}\right)^2 + \rho_2\left(\frac{\partial \psi_2}{\partial t}\right)^2\right]\mathrm{d}x_1\mathrm{d}x_2 \end{aligned} \tag{8.3.17}$$

Π is given by (8.3.16) and ρ_0, ρ_2 by Eqs. (8.3.10), T is the kinetic energy.

For a symmetric and specially orthotropic Mindlin's plate assuming $A_{45} = 0$ it follows from (8.3.16) for bending problems that

$$\begin{aligned} \Pi(w, \psi_1, \psi_2) = \frac{1}{2}\int\limits_A \Bigg[& D_{11}\left(\frac{\partial \psi_1}{\partial x_1}\right)^2 + 2D_{12}\left(\frac{\partial \psi_1}{\partial x_1}\frac{\partial \psi_2}{\partial x_2}\right) + D_{22}\left(\frac{\partial \psi_2}{\partial x_2}\right)^2 \\ & + D_{66}\left(\frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1}\right)^2 + k^{\mathrm{s}}_{55}A_{55}\left(\psi_1 + \frac{\partial w}{\partial x_1}\right)^2 \\ & + k^{\mathrm{s}}_{44}A_{44}\left(\psi_2 + \frac{\partial w}{\partial x_2}\right)^2\Bigg]\mathrm{d}x_1\mathrm{d}x_2 - \int\limits_A p_3 w \mathrm{d}x_1\mathrm{d}x_2, \\ & \delta\Pi(w, \psi_1, \psi_2) = 0 \end{aligned} \tag{8.3.18}$$

For natural vibration the variational formulation for that plate is

$$L(w, \psi_1, \psi_2) = T(w, \psi_1, \psi_2) - \Pi(w, \psi_1, \psi_2), \quad \delta\int\limits_{t_1}^{t_2} L(w, \psi_1, \psi_2)\mathrm{d}t = 0 \tag{8.3.19}$$

To calculate buckling loads the in-plane stress resultants must be, like in the Kirchhoff's plate theory, included into the part Π_{a} of Π. Consider a plate with a constant in-plane force N_1 it follows

$$\Pi(w,\psi_1,\psi_2) = \Pi_\mathrm{i}(w,\psi_1,\psi_2) - \frac{1}{2}\int N_1 \left(\frac{\partial w}{\partial x_1}\right)^2 \mathrm{d}x_1 \mathrm{d}x_2 \qquad (8.3.20)$$

The case of a more general in-plane loading can be transposed from (8.2.27). The term Π_i stay unchanged.

8.4 Sandwich Plates

To formulate the governing differential equations or tht variational statement for sandwich plates we draw the conclusion from the similarity of the elastic behavior between laminates and sandwiches in the first order shear deformation theory that all results derived above for laminates can be applied to sandwich plates. We restrict our considerations to symmetric sandwich plates with thin or thick cover sheets. Like in the beam theory, there are differences in the expressions for the flexural stiffness $D_{11}, D_{12}, D_{22}, D_{66}$ and the transverse shear stiffness A_{55}, A_{44} of laminates and sandwiches (Sects. 4.3.2 and 4.3.3). Furthermore there are essential differences in the stress distributions. The elastic behavior of sandwiches and the general model assumptions are considered in detail in Sect. 4.3. The stiffness relations for sandwiches with thin and thick skins are also given there:

- *Symmetric sandwiches with thin cover sheets* (4.3.12) - (4.3.14)

$$\begin{aligned}
&A_{ij} = 2A_{ij}^\mathrm{f} = 2\sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)},\\
&D_{ij} = h^\mathrm{c} C_{ij}^\mathrm{f} = h^\mathrm{c} \sum_{k=1}^{n} Q_{ij}^{(k)} h^{(k)} \bar{x}_3^{(k)}, \quad \bar{x}_3^{(k)} = \frac{1}{2}\left(x_3^{(k)} + x_3^{(k-1)}\right)\\
&(ij) = (11), (12), (22), (66)\\
&A_{ij}^\mathrm{s} = h^\mathrm{c} C_{ij}^\mathrm{s}, (ij) = (44), (55), \quad C_{44}^\mathrm{s} = G_{23}^\mathrm{s}, C_{55}^\mathrm{s} = G_{13}^\mathrm{s}
\end{aligned} \qquad (8.4.1)$$

  h^c is the thickness of the core, n is the number of faces layers and G_{13}, G_{23} are the core shear stiffness moduli. Shear correction factors can be calculated similarly to the beams approximately with the help of (7.4.2).

- *Symmetric sandwiches with thick cover sheets* (4.3.16) - (4.3.17) of one lamina

$$
\begin{aligned}
&A_{ij} = A_{ij}^{\mathrm{Sa}} = 2h^{\mathrm{f}}Q_{ij}^{\mathrm{f}} \quad \text{or} \quad A_{ij} = A_{ij}^{\mathrm{La}} = 2h^{\mathrm{f}}Q_{ij}^{\mathrm{f}} + h^{\mathrm{c}}Q_{ij}^{\mathrm{c}}, \quad i,j = 1,2,6\\
&D_{ij} = D_{ij}^{\mathrm{Sa}} = \frac{1}{2}Q_{ij}^{\mathrm{f}}\left(h^{\mathrm{f}} + h^{\mathrm{c}}\right)h^{\mathrm{f}}h^{\mathrm{c}} \quad \text{or}\\
&D_{ij} = D_{ij}^{\mathrm{La}} = \frac{1}{2}h^{\mathrm{f}}Q_{ij}^{\mathrm{f}}\left[\left(h^{\mathrm{f}} + h^{\mathrm{c}}\right)^2 + \frac{1}{3}h^{\mathrm{f}2}\right] + \frac{1}{12}h^{\mathrm{c}3}Q_{ij}^{\mathrm{c}}, \quad i,j = 1,2,6 \qquad (8.4.2)\\
&(ij) = (11),(12),(22),(66)\\
&A_{ij}^{\mathrm{s}} = A_{ij}^{\mathrm{s\ Sa}} = h^{\mathrm{c}}C_{ij}^{\mathrm{c}}, \quad A_{ij}^{\mathrm{s}} = A_{ij}^{\mathrm{s\ La}} = 2h^{\mathrm{f}}C_{ij}^{\mathrm{f}} + h^{\mathrm{c}}C_{ij}^{\mathrm{c}}, (ij) = (44),(55)\\
&C_{44}^{\mathrm{f}} = G_{23}^{\mathrm{f}}, \quad C_{44}^{\mathrm{c}} = G_{23}^{\mathrm{c}}, \quad C_{55}^{\mathrm{f}} = G_{13}^{\mathrm{f}}, \quad C_{55}^{\mathrm{c}} = G_{13}^{\mathrm{c}}
\end{aligned}
$$

With these stiffness values for the two types of sandwich plates the differential equations (8.3.8), (8.3.10) or the variational formulations (8.3.18) - (8.3.20) of the theory of laminate plates including transverse shear deformation can be transposed to sandwich plates.

Equation (4.3.22) demonstrated that for sandwich plates with thick faces the stiffness $A_{ij}^{\mathrm{La}}, D_{ij}^{\mathrm{La}}, (ij) = (11),(22),(66)$ and $A_{ij}^{\mathrm{La}}, (ij) = (44),(55)$ should be used. Because generally $Q_{ij}^{\mathrm{c}} \ll Q_{ij}^{\mathrm{f}}$ usually the simplified stiffness

$$
A_{ij}^{\mathrm{La}} \approx A_{ij}^{\mathrm{Sa}} = 2A_{ij}^{\mathrm{f}}, \quad D_{ij}^{\mathrm{La}} \approx D_{ij}^{\mathrm{Sa}}\left(1 + \frac{h^{\mathrm{f}}}{h^{\mathrm{c}}}\frac{h^{\mathrm{c}} + (4/3)h^{\mathrm{f}}}{h^{\mathrm{c}} + h^{\mathrm{f}}}\right)
$$

yield satisfying results in engineering applications. Thus is valid for isotropic-facing sandwich plates and for sandwich plates having orthotropic composite material facings (cross-ply laminates).

In Sect. 4.3 generally and in Sect. 7.4 for beams the continuing popularity of sandwich structures was underlined. Sect. 7.4 also recalled and summarized the main aspects of modelling and analysis of sandwich structures. Engineering applications to sandwich beams were discussed in detail. Keeping this in mind, the derivations to sandwich plates can be restricted here to few conclusions:

- Most sandwich structures can be modelled and analyzed using the shear deformation theory for laminated plates.
- Generally, the stiffness matrices $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{D}$ of laminated plates are employed.
- Consider the lower face as lamina 1, the core as lamina 2 and the upper face as lamina 3 one can include or ignore the effect of the core on the response to bending and in-plane loads and the effect of transverse shear deformation on the response of the facings.
- The shear deformation theory of laminated plates can be not only transposed to sandwich plates for bending, vibration and buckling induced by mechanical loads but include also other loading, e.g. hydrothermal effects.

With the special sandwich stiffness including or ignore in-plane, bending and transverse shear deformation response all differential equations and variational formulations of Sect. 8.3 stay valid. Some examples for sandwich plates are considered in Sect. 8.7.

8.5 Hygrothermo-Elastic Effects on Plates

Elevated temperature and absorbed moisture can alter significantly the structural response of fibre-reinforced laminated composites. In Sects. 8.2 to 8.4 the structural response of laminated plates as result of mechanical loading was considered and thermal or hygrosgopic loadings were neglected.

This section focuses on hygrothermally induced strains, stresses and displacements of thin or moderate thick laminated plates. We assume as in Sect. 7.5 moderate hygrothermal loadings such that the mechanical properties remain approximately unchanged for the temperature and moisture differences considered. Because the mathematical formulations governing thermal and hygroscopic loadings are analogous, a unified derivation is straightforward and will be considered in the frame of the classical laminate theory and the shear deformation theory.

The following derivations use the basic equations, Sect. 4.2.5, on thermal and hygroscopic effects in individual laminae and in general laminates. The matrix formulations for force and moment resultants, Eq. (4.2.75), can be written explicitly as

$$\begin{bmatrix} N_1 \\ N_2 \\ N_6 \\ M_1 \\ M_2 \\ M_6 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ & & A_{66} & B_{16} & B_{26} & B_{66} \\ \mathrm{S} & & & D_{11} & D_{12} & D_{16} \\ & \mathrm{Y} & & & D_{22} & D_{26} \\ & & \mathrm{M} & & & D_{66} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_6 \\ \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix} - \begin{bmatrix} N_1^{\mathrm{th}} \\ N_2^{\mathrm{th}} \\ N_6^{\mathrm{th}} \\ M_1^{\mathrm{th}} \\ M_2^{\mathrm{th}} \\ M_6^{\mathrm{th}} \end{bmatrix} - \begin{bmatrix} N_1^{\mathrm{mo}} \\ N_2^{\mathrm{mo}} \\ N_6^{\mathrm{mo}} \\ M_1^{\mathrm{mo}} \\ M_2^{\mathrm{mo}} \\ M_6^{\mathrm{mo}} \end{bmatrix} \tag{8.5.1}$$

with the known matrix elements, Eq. (4.2.15),

$$\begin{aligned} A_{ij} &= \sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)} - x_3^{(k-1)} \right), \\ B_{ij} &= \frac{1}{2} \sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)^2} - x_3^{(k-1)^2} \right), \\ D_{ij} &= \frac{1}{3} \sum_{k=1}^{n} Q_{ij}^{(k)} \left(x_3^{(k)^3} - x_3^{(k-1)^3} \right) \end{aligned} \tag{8.5.2}$$

The thermal and moisture stress resultants $\boldsymbol{N}^{\mathrm{th}}, \boldsymbol{N}^{\mathrm{mo}}, \boldsymbol{M}^{\mathrm{th}}, \boldsymbol{M}^{\mathrm{mo}}$ are resultants per unit temperature or moisture change, Eqs. (4.2.67).

Substituting the hygrothermal constitutive equation (8.5.1) into the equilibrium equations (8.2.3) and replacing the in-plane strains ε_i and the curvatures κ_i by the displacements u, v, w, Eq. (8.2.5), yield the following matrix differential equation for the classical laminate theory

$$\boldsymbol{L}\boldsymbol{u} = \boldsymbol{p} - \boldsymbol{\partial} \begin{bmatrix} \boldsymbol{N}^* \\ \boldsymbol{M}^* \end{bmatrix} \tag{8.5.3}$$

$\boldsymbol{Lu} = \boldsymbol{p}$ is identically to Eq. (8.2.7) with L_{ij} given in App. C. $\boldsymbol{N}^* = \boldsymbol{N}^{\mathrm{th}} + \boldsymbol{N}^{\mathrm{mo}}$, $\boldsymbol{M}^* = \boldsymbol{M}^{\mathrm{th}} + \boldsymbol{M}^{\mathrm{mo}}$ are the hygrothermal stress results and $\boldsymbol{\partial}$ is a special (3×6) differential matrix

$$\boldsymbol{\partial} = \begin{bmatrix} -\frac{\partial}{\partial x_1} & -\frac{\partial}{\partial x_2} & 0 & 0 & 0 & 0 \\ 0 & -\frac{\partial}{\partial x_2} & -\frac{\partial}{\partial x_1} & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{\partial^2}{\partial x_1^2} & -\frac{\partial^2}{\partial x_2^2} & -\frac{2\partial^2}{\partial x_1 \partial x_2} \end{bmatrix} \tag{8.5.4}$$

For selected layer stacking Eq. (8.5.3) can be simplified. The matrix $\boldsymbol{L}$ and the differential operators L_{ij} are summarized for the most important special laminates in App. C.

Hygrothermal induced buckling can be modelled as

$$\boldsymbol{Lu} + \boldsymbol{\partial} \begin{bmatrix} \boldsymbol{N}^* \\ \boldsymbol{M}^* \end{bmatrix} = \left(N_1 \frac{\partial^2}{\partial x_1^2} + 2N_6 \frac{\partial^2}{\partial x_1 \partial x_2} + N_2 \frac{\partial^2}{\partial x_2^2} \right) \boldsymbol{u}^* \tag{8.5.5}$$

with $\boldsymbol{u}^{*\mathrm{T}} = [0\ 0\ w]$. Prebuckling displacements and stress resultants are determined by solving Eq. (8.5.5) with $\boldsymbol{N} \equiv \boldsymbol{0}$. For the corresponding buckling problem, N_1, N_2 and N_6 are taken to be the stress resultant functions corresponding to the prebuckling state. The buckling loads are found by solving the eigenvalue problem associated with (8.5.5), i.e. with $\boldsymbol{N}^* = \boldsymbol{0}$ and $\boldsymbol{M}^* = \boldsymbol{0}$.

Because energy methods are useful to obtain approximate analytical solutions for hygrothermal problems the total potential energy Π is formulated. Restricting to symmetrical problems with $A_{16} = A_{26} = 0$ and $D_{16} = D_{26} = 0$, i.e to cross-ply laminates, we have

$$\begin{aligned} \Pi(u,v,w) = \frac{1}{2} \int_A \Bigg\{ & A_{11} \left(\frac{\partial u}{\partial x_1} \right)^2 + 2A_{12} \left(\frac{\partial u}{\partial x_1} \right) \left(\frac{\partial v}{\partial x_2} \right) + A_{22} \left(\frac{\partial v}{\partial x_2} \right)^2 \\ & + A_{66} \left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} \right)^2 + D_{11} \left(\frac{\partial^2 w}{\partial x_1^2} \right)^2 + 2D_{12} \left(\frac{\partial^2 w}{\partial x_1^2} \right) \left(\frac{\partial^2 w}{\partial x_2^2} \right) \\ & + D_{22} \left(\frac{\partial^2 w}{\partial x_2^2} \right)^2 + 4D_{66} \left(\frac{\partial^2 w}{\partial x_1 \partial x_2} \right)^2 \\ & - 2N_1^* \frac{\partial u}{\partial x_1} - 2N_2^* \frac{\partial v}{\partial x_2} - 2N_6^* \left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} \right) \\ & + 2M_1^* \frac{\partial^2 w}{\partial x_1^2} + 2M_2^* \frac{\partial^2 w}{\partial x_2^2} - 4M_6^* \left(\frac{\partial^2 w}{\partial x_1 \partial x_2} \right) \\ & - \left[N_1 \left(\frac{\partial^2 w}{\partial x_1^2} \right)^2 + 2N_6 \left(\frac{\partial w}{\partial x_1} \right) \left(\frac{\partial w}{\partial x_2} \right) + N_2 \left(\frac{\partial^2 w}{\partial x_2^2} \right)^2 \right] \Bigg\} \mathrm{d}A \end{aligned} \tag{8.5.6}$$

The classical laminate theory which neglect transverse shear deformations can lead to significant errors for moderately thick plates and hygrothermal loadings. Using the shear deformation theory, Sect. 8.3, we can formulate corresponding to Eq. (8.3.4).

$$\tilde{\boldsymbol{L}}\tilde{\boldsymbol{u}} = \tilde{\boldsymbol{p}} - \tilde{\boldsymbol{\partial}} \begin{bmatrix} \boldsymbol{N}^* \\ \boldsymbol{M}^* \end{bmatrix} \tag{8.5.7}$$

The matrix $\tilde{\boldsymbol{\partial}}$ is identically to Eq. (8.5.4). For hygrothermal induced buckling we have analogous to Eq. (8.5.5)

$$\tilde{\boldsymbol{L}}\tilde{\boldsymbol{u}} + \tilde{\boldsymbol{\partial}} \begin{bmatrix} \boldsymbol{N}^* \\ \boldsymbol{M}^* \end{bmatrix} = \left(N_1 \frac{\partial^2}{\partial x_1^2} + 2N_6 \frac{\partial^2}{\partial x_1 \partial x_2} + N_2 \frac{\partial^2}{\partial x_2^2} \right) \boldsymbol{u}^* \tag{8.5.8}$$

For prebuckling analysis the terms involving N_1, N_2 and N_6 are ignored. Then buckling loads are calculated by substituting the values N_1, N_2, N_6 determined for the prebuckling state into Eq. (8.5.8) dropping now the hygrothermal stress resultants $\boldsymbol{N}^*$ and $\boldsymbol{M}^*$. For special laminate stacking the differential operators are summarized in App. C.

The elastic potential Π is now a function of five independent functions u, v, w, ψ_1, ψ_2. Restricting again to cross-ply laminates the elastic total potential Π can be formulated as

$$\begin{aligned}
\Pi = \frac{1}{2} \int\limits_A \Bigg\{ & A_{11} \left(\frac{\partial u}{\partial x_1} \right)^2 + 2A_{12} \left(\frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_2} \right) + A_{22} \left(\frac{\partial v}{\partial x_2} \right)^2 + A_{66} \left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} \right)^2 \\
& + D_{11} \left(\frac{\partial \psi_1}{\partial x_1} \right)^2 + 2D_{12} \left(\frac{\partial \psi_1}{\partial x_1} \frac{\partial \psi_2}{\partial x_2} \right) + D_{22} \left(\frac{\partial \psi_2}{\partial x_2} \right)^2 + D_{66} \left(\frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1} \right) \\
& + k_{44}^{\mathrm{s}} A_{44} \left(\frac{dw}{\mathrm{d}x_2} + \psi_2 \right)^2 + k_{55}^{\mathrm{s}} A_{55} \left(\frac{dw}{\mathrm{d}x_1} + \psi_1 \right)^2 \\
& - 2N_1^* \frac{\partial u}{\partial x_1} - 2N_2^* \frac{\partial v}{\partial x_2} - 2N_6^* \left(\frac{\partial u}{\partial x_2} + \frac{\partial v}{\partial x_1} \right) \\
& - 2M_1^* \frac{\partial \psi_1}{\partial x_1} - 2M_2^* \frac{\partial \psi_2}{\partial x_2} - 2M_6^* \left(\frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1} \right) \\
& - \left[N_1 \left(\frac{\partial^2 w}{\partial x_1^2} \right)^2 + 2N_6 \left(\frac{\partial w}{\partial x_1} \right) \left(\frac{\partial w}{\partial x_2} \right) + N_2 \left(\frac{\partial^2 w}{\partial x_2^2} \right)^2 \right] \Bigg\} \mathrm{d}A
\end{aligned} \tag{8.5.9}$$

Equations (8.5.6) and (8.5.9) are the starting point for solving hygrothermal induced buckling problems e.g. with the Ritz- or Galerkin approximation or the finite element method. Analytical solutions are in general not possible. As considered above, the force resultants N_1, N_2 and N_6 have to be calculated in the prebuckling state, i.e. for $\boldsymbol{N} \equiv \boldsymbol{0}$ and the calculation force resultants are substituted into Eqs. (8.5.6) or (8.5.9), respectively, with $\boldsymbol{N}^* = \boldsymbol{0}, \boldsymbol{M}^* = \boldsymbol{0}$ to calculate the buckling loads. If there are transverse loads p, Eqs. (8.5.3) or (8.5.7) the bending problem follows from (8.5.6) or (8.5.7) by setting $\boldsymbol{N} = \boldsymbol{0}$ and substitute an additional term

$$\int_A pw\mathrm{d}A$$

8.6 Analytical Solutions

The analysis of rectangular plates with selected layer stacking and boundary conditions can be carried out analytically in a similar manner to homogeneous isotropic and orthotropic plates. The analytical methods of homogeneous isotropic plates, e.g. the double series solutions of Navier[2] or the single series solutions of Nádai[3]-Lévy[4] can be applied to laminated plates with special layer stacking and analogous boundary conditions. In Sect. 8.6 possibilities of analytical solutions in the frame of classical laminate theory and shear deformation theory are demonstrated for bending, buckling and vibration problems.

8.6.1 Classical Laminate Theory

There are varying degrees of complexity in laminated plate analysis. The least complicated problems are one-dimensional formulations of cylindrical plate bending. For cylindrical bending both, symmetric and unsymmetric laminates, are handled in a unique manner assuming all deformations are one-dimensional.

In the case of two-dimensional plate equations the most important degree of simplification is for plates being midplane symmetric because of their uncoupling in-plane and out-of-plane response. The mathematical structure of symmetric angle-ply plate equations corresponds to homogeneous anisotropic plate equations and that of symmetric cross-ply plate equations to homogeneous orthotropic plate equations. To illustrate analytical solutions for rectangular plates in the framework of the classical laminate theory we restrict our developments to specially orthotropic, i.e. to symmetric cross-ply plates. For this type of laminated plates the Navier solution method can be applied to rectangular plates with all four edges simply supported. The Nádai-Lévy solution (Nádai, 1925) method can be applied to rectangular plates with two opposite edges have any possible kind of boundary conditions. For more general boundary conditions of special orthotropic plates or other symmetric or unsymmetric rectangular plates approximate analytical solutions are possible, e.g. using the Ritz-, the Galerkin- or the Kantorovich methods, Sect. 2.2.3, or numerical methods are applied, Chap. 11.

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2 Claude Louis Marie Henri Navier (*10 February 1785 Dijon - †21 August 1836 Paris) - engineer and physicist

3 Árpád Nádai (*3 April 1883 Budapest - †18 July 1963 Pittsburgh) - professor of mechanics, contributions to the plate theory and theory of plasticity

4 Maurice Lévy (*28 February 1838 Ribeauvillé - †30 September 1910 Paris) - engineer, total strain theory

As considered above the simplest problem of plate bending is the so-called cylindrical bending for a plate strip i.e. a very long plate in one direction with such a lateral load and edge support in this direction that the plate problem may be reduced to a one-dimensional problem and a quasi-beam solution can be used. In the following we demonstrate analytical solutions for various selected examples.

8.6.1.1 Plate Strip

The model "plate strip" (Fig. 8.4) describes approximately the behavior of a rectangular plate with $a/b \ll 1$. The plate dimension a in x_1-direction is considered finite, the other dimension b in x_2-direction approximately infinite. The boundary conditions for the edges $x_1 = 0, x_1 = a$ may be quite general, but independent of x_2 and the lateral load is $p_3 = p_3(x_1)$. All derivatives with respect to x_2 are zero and the plate equation reduces to a one-dimensional equation. For symmetric laminated strips Eqs. (8.2.6) and (8.2.9) reduce to

$$\begin{aligned}
D_{11}w''''(x_1) &= p_3(x_1),\\
M_1(x_1) &= -D_{11}w''(x_1),\\
M_2(x_1) &= -D_{12}w''(x_1),\\
M_6(x_1) &= -D_{16}w''(x_1) \quad \text{(general case)},\\
M_6(x_1) &= 0 \quad \text{(specially orthotropic case}, D_{16} = 0),\\
Q_1(x_1) &= M_1'(x_1) = -D_{11}w'''(x_1),\\
Q_2(x_1) &= M_6'(x_1) = -D_{16}w'''(x_1) \quad \text{(general case)},\\
Q_2(x_1) &= 0 \quad \text{(specially orthotropic case)}
\end{aligned} \tag{8.6.1}$$

When one compares the differential equation of the strip with the differential equation $bD_{11}w''''(x_1) = q(x_1)$ of a beam it can be stated that all solutions of the beam equation can be used for the strip.

For the normal stresses in the layer k the equations are

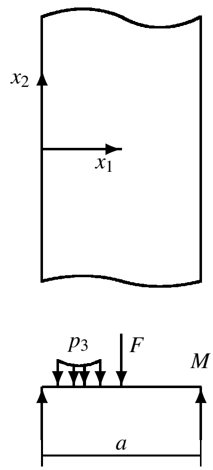


Fig. 8.4 Plate strip

$$\begin{aligned}
\sigma_1^{(k)}(x_1,x_3) &= -Q_{11}^{(k)} x_3 \frac{d^2w}{dx_1^2},\\
\sigma_2^{(k)}(x_1,x_3) &= -Q_{12}^{(k)} x_3 \frac{d^2w}{dx_1^2},\\
\sigma_6^{(k)}(x_1,x_3) &= 0
\end{aligned}$$

or

$$\begin{aligned}
\sigma_1^{(k)}(x_1,x_3) &= -\frac{Q_{11}^{(k)}}{D_{11}} M_1(x_1)x_3,\\
\sigma_2^{(k)}(x_1,x_3) &= -\frac{Q_{12}^{(k)}}{D_{12}} M_2(x_1)x_3,\\
\sigma_6^{(k)}(x_1,x_3) &= 0
\end{aligned} \tag{8.6.2}$$

and the transverse shear stresses follow from (5.2.19) to

$$\sigma_4(x_1,x_3) = F_{61}Q_1(x_1), \quad \sigma_5(x_1,x_3) = F_{11}Q_1(x_1)$$

and

$$\boldsymbol{F}(x_3) = \tilde{\boldsymbol{B}}(x_3)\boldsymbol{D}^{-1}$$

i.e

$$\begin{bmatrix} \sigma_5 \\ \sigma_4 \end{bmatrix} = \begin{bmatrix} F_{11} & F_{62} \\ F_{61} & F_{22} \end{bmatrix} \begin{bmatrix} Q_1 \\ 0 \end{bmatrix}, \tag{8.6.3}$$

Considering the solutions of the symmetrical laminated plate strip, we have following conclusions:

- The solutions for laminate beams and plate strips are very similar, but the calculation of the strip bending stiffness D_{ij} has to include Poisson's effects.
- Because of including of Poisson's effect we have the relation

  $$w(x_1)_{\text{strip}} < w(x_1)_{\text{beam}}$$

  and $M_2(x_1) \neq 0$.
- $M_1(x_1)$ and $Q_1(x_1)$ of the strip and the beam are identical. If $M_6(x_1) = 0$ then $V_1 \equiv Q_1$, i.e there is no special effective Kirchhoff transverse force.

The solutions for plate strips with cylindrical bending can be transposed to lateral loads $p_3(x_1,x_2) = x_2 p(x_1)$. From $w(x_1,x_2) = x_2 w(x_1)$ it follows that $x_2 w''''(x_1) = x_2 p(x_1)/D_{11}$. The displacement $w(x_1)$ and the stress resultants $M_1(x_1), Q_1(x)$ of the plate strip with the lateral load $p(x_1)$ have to be multiplied by the coordinate x_2 to get the solution for the lateral load $x_2 p(x_1)$. Note that in contrast to the case above, here $M_6 = -2D_{66}w'(x_1) - D_{16}w''(x_1)$ in the general case and $M_6 = -2D_{66}w'(x_1)$ for specially orthotropic strips.

For unsymmetric laminated plate strips the system of three one-dimensional differential equations for the displacements $u(x_1), v(x_1)$ and $w(x_1)$ follow with (8.2.6) as

$$
\begin{aligned}
A_{11}\frac{\mathrm{d}^2u}{\mathrm{d}x_1^2}+A_{16}\frac{\mathrm{d}^2v}{\mathrm{d}x_1^2}-B_{11}\frac{\mathrm{d}^3w}{\mathrm{d}x_1^3} &= 0\\
A_{16}\frac{\mathrm{d}^2u}{\mathrm{d}x_1^2}+A_{66}\frac{\mathrm{d}^2v}{\mathrm{d}x_1^2}-B_{16}\frac{\mathrm{d}^3w}{\mathrm{d}x_1^3} &= 0\\
D_{11}\frac{\mathrm{d}^4w}{\mathrm{d}x_1^4}-B_{11}\frac{\mathrm{d}^3u}{\mathrm{d}x_1^3}-B_{16}\frac{\mathrm{d}^3v}{\mathrm{d}x_1^3} &= p_3
\end{aligned}
\tag{8.6.4}
$$

These equations can be uncoupled and analytically solved. The first and the second equation yield

$$
\frac{\mathrm{d}^2u}{\mathrm{d}x_1^2}=\frac{B}{A}\frac{\mathrm{d}^3w}{\mathrm{d}x_1^3},\qquad \frac{\mathrm{d}^2v}{\mathrm{d}x_1^2}=\frac{C}{A}\frac{\mathrm{d}^3w}{\mathrm{d}x_1^3}
\tag{8.6.5}
$$

with $A=A_{11}A_{66}-A_{16}^2, B=A_{66}B_{11}-A_{16}B_{16}, C=A_{11}B_{16}-A_{16}B_{11}$. Differentiating both equations and substituting the results in the third equation of (8.6.4) we obtain one differential equation of fourth order in $w(x_1)$

$$
\frac{\mathrm{d}^4w}{\mathrm{d}x_1^4}=\frac{A}{D}p_3,\qquad D=D_{11}A-B_{11}B-B_{16}C
\tag{8.6.6}
$$

Equation (8.6.6) can be integrated to obtain $w(x_1)$ and than follow with (8.6.5)

$$
\frac{\mathrm{d}^3u}{\mathrm{d}x_1^3}=\frac{B}{D}p_3,\qquad \frac{\mathrm{d}^3v}{\mathrm{d}x_1^3}=\frac{C}{D}p_3
\tag{8.6.7}
$$

For a transverse load $p_3=p_3(x_1)$ we obtain the analytical solutions for the displacements $u(x_1), v(x_1)$ and $w(x_1)$ as

$$
\begin{aligned}
w(x_1) &= \frac{A}{D}\iiiint p_3\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1+C_1\frac{x_1^3}{6}+C_2\frac{x_1^2}{2}+C_3x_1+C_4,\\
u(x_1) &= \frac{B}{D}\iiint p_3\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1+B_1\frac{x_1^2}{2}+B_2x_1+B_3,\\
v(x_1) &= \frac{C}{D}\iiint p_3\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1+A_1\frac{x_1^2}{2}+A_2x_1+B_3
\end{aligned}
\tag{8.6.8}
$$

With Eqs. (8.6.5) follows $A_1=B_1=C_1$ and we have eight boundary conditions to calculate 8 unknown constants, e.g. for clamped supports

$$
u(0)=u(a)=v(0)=v(a)=w(b)=w(a)=0, w'(0)=w'(a)=0,
$$

Eqs. (8.2.12) yield the one-dimensional equations for the forces and moments resultants

$$
\begin{aligned}
N_1 &= A_{11}\frac{\mathrm{d}u}{\mathrm{d}x_1}+A_{16}\frac{\mathrm{d}v}{\mathrm{d}x_1}-B_{11}\frac{\mathrm{d}^2w}{\mathrm{d}x_1^2},\\
N_2 &= A_{12}\frac{\mathrm{d}u}{\mathrm{d}x_1}+A_{26}\frac{\mathrm{d}v}{\mathrm{d}x_1}-B_{12}\frac{\mathrm{d}^2w}{\mathrm{d}x_1^2},
\end{aligned}
$$

$$
\begin{aligned}
N_6 &= A_{16}\frac{\mathrm{d}u}{\mathrm{d}x_1} + A_{66}\frac{\mathrm{d}v}{\mathrm{d}x_1} - B_{16}\frac{\mathrm{d}^2w}{\mathrm{d}x_1^2},\\
M_1 &= B_{11}\frac{\mathrm{d}u}{\mathrm{d}x_1} + B_{16}\frac{\mathrm{d}v}{\mathrm{d}x_1} - D_{11}\frac{\mathrm{d}^2w}{\mathrm{d}x_1^2},\\
M_2 &= B_{12}\frac{\mathrm{d}u}{\mathrm{d}x_1} + B_{26}\frac{\mathrm{d}v}{\mathrm{d}x_1} - D_{12}\frac{\mathrm{d}^2w}{\mathrm{d}x_1^2},\\
M_6 &= B_{16}\frac{\mathrm{d}u}{\mathrm{d}x_1} + B_{66}\frac{\mathrm{d}v}{\mathrm{d}x_1} - D_{16}\frac{\mathrm{d}^2w}{\mathrm{d}x_1^2},\\
Q_1 &= B_{11}\frac{\mathrm{d}^2u}{\mathrm{d}x_1^2} + B_{16}\frac{\mathrm{d}^2v}{\mathrm{d}x_1^2} - D_{11}\frac{\mathrm{d}^3w}{\mathrm{d}x_1^3},\\
Q_2 &= B_{16}\frac{\mathrm{d}^2u}{\mathrm{d}x_1^2} + B_{66}\frac{\mathrm{d}^2v}{\mathrm{d}x_1^2} - D_{16}\frac{\mathrm{d}^3w}{\mathrm{d}x_1^3},\\
V_1 &= B_{11}\frac{\mathrm{d}^2u}{\mathrm{d}x_1^2} + B_{16}\frac{\mathrm{d}^2v}{\mathrm{d}x_1^2} - D_{11}\frac{\mathrm{d}^3w}{\mathrm{d}x_1^3} = Q_1,\\
V_2 &= 2B_{16}\frac{\mathrm{d}^2u}{\mathrm{d}x_1^2} + 2B_{66}\frac{\mathrm{d}^2v}{\mathrm{d}x_1^2} - 2D_{16}\frac{\mathrm{d}^3w}{\mathrm{d}x_1^3} = 2Q_2
\end{aligned}
\tag{8.6.9}
$$

The general symmetric case follows with $B_{ij} = 0$ and for a symmetric cross-ply strip are $B_{ij} = 0$ and $A_{16} = 0, D_{16} = 0$.

Analytical solutions can also be formulated for vibration and buckling of strips with one-dimensional deformations. The eigen-vibrations of unsymmetrical plate strips taking account of $u(x_1,t) = u(x_1)\mathrm{e}^{i\omega t}, v(x_1,t) = v(x_1)\mathrm{e}^{i\omega t}, w(x_1,t) = w(x_1)\mathrm{e}^{i\omega t}$ are mathematically modelled as

$$
\begin{aligned}
A_{11}\frac{\mathrm{d}^2u}{\mathrm{d}x_1^2} + A_{16}\frac{\mathrm{d}^2v}{\mathrm{d}x_1^2} - B_{11}\frac{\mathrm{d}^3w}{\mathrm{d}x_1^3} - \rho h\omega^2 w &= 0,\\
A_{16}\frac{\mathrm{d}^2u}{\mathrm{d}x_1^2} + A_{66}\frac{\mathrm{d}^2v}{\mathrm{d}x_1^2} - B_{16}\frac{\mathrm{d}^3w}{\mathrm{d}x_1^3} - \rho h\omega^2 v &= 0,\\
D_{11}\frac{\mathrm{d}^4w}{\mathrm{d}x_1^4} - B_{11}\frac{\mathrm{d}^3u}{\mathrm{d}x_1^3} - B_{16}\frac{\mathrm{d}^3w}{\mathrm{d}x_1^3} - \rho h\omega^2 w &= 0
\end{aligned}
\tag{8.6.10}
$$

u, v and w are now functions of x_1 and t.

If the in-plane inertia effects are neglected the Eqs. (8.6.5) are valid. Differentiating these equations and substituting the result in the third Eq. (8.6.10) lead to the vibration equation

$$
\frac{\mathrm{d}^4w}{\mathrm{d}x_1} - \frac{A}{D}\rho h\omega^2 w = 0 \tag{8.6.11}
$$

For a symmetrically laminated cross-ply strip we obtain with $A/D = 1/D_{11}$

$$
\frac{\mathrm{d}^4w}{\mathrm{d}x_1} - \frac{\rho h\omega^2}{D_{11}} w = 0, \quad \rho h = \sum_{k=1}^{N} \rho^{(k)} h^{(k)} \tag{8.6.12}
$$

The analytical solutions correspond to the beam solutions in Sect. 7.6

$$
\begin{aligned}
&w(x_1) = C_1 \cos\frac{\lambda}{a}x_1 + C_2 \sin\frac{\lambda}{a}x_1 + C_3 \cosh\frac{\lambda}{a}x_1 + C_4 \sinh\frac{\lambda}{a}x_1,\\
&\left(\frac{\lambda}{a}\right)^4 = \frac{\rho h}{D_{11}}\omega^2 \quad \text{(symmetric cross-ply strip)},\\
&\left(\frac{\lambda}{a}\right)^4 = \rho h\frac{A}{D}\omega^2 \quad \text{(general unsymmetric strip)}
\end{aligned}
\tag{8.6.13}
$$

For a simply supported strip we have $w(0) = w(a) = w''(0) = w''(a) = 0$ and therefore $C_1 = C_3 = C_4 = 0$ and $C_2 \sin\left(\frac{\lambda}{a}\right) a = 0$, i.e. with $\lambda = n\pi$ follow

$$
\omega^2 = \frac{n^4\pi^4}{a^4\rho h}\frac{D}{A}, \quad \omega = \left(\frac{n\pi}{a}\right)^2\sqrt{\frac{D}{\rho hA}} \tag{8.6.14}
$$

Analytical solutions can be calculated for all boundary conditions of the strip.

In analogous manner analytical solutions follow for the buckling behavior of strips which are subjected to an initial compressive load $N_1 = -N_0$. The third equation of (8.6.4) is formulated with $p_3 = 0$ as

$$
D_{11}\frac{\mathrm{d}^4 w}{\mathrm{d}x_1^4} - B_{11}\frac{\mathrm{d}^3 u}{\mathrm{d}x_1^3} - B_{16}\frac{\mathrm{d}^3 v}{\mathrm{d}x_1^3} - N_1\frac{\mathrm{d}^2 w}{\mathrm{d}x_1^2} = 0 \tag{8.6.15}
$$

and with Eq. (8.6.5) follows

$$
\begin{aligned}
&\frac{\mathrm{d}^4 w}{\mathrm{d}x_1^4} - \frac{A}{D}N_1\frac{\mathrm{d}^2 w}{\mathrm{d}x_1^2} = 0, \quad \text{(general case)}\\
&\frac{\mathrm{d}^4 w}{\mathrm{d}x_1^4} - \frac{1}{D_{11}}N_1\frac{\mathrm{d}^2 w}{\mathrm{d}x_1^2} = 0, \quad \text{(symmetrical cross-ply case)}
\end{aligned}
\tag{8.6.16}
$$

with $N_1(x_1) = -N_0$. The buckling equations correspond again to the beam equation (7.2.35) and can be solved for all boundary conditions of the strip

$$
\begin{aligned}
&w(x_1) = C_1 \cos\frac{\lambda}{a}x_1 + C_2 \sin\frac{\lambda}{a}x_1 + C_3 \cosh\frac{\lambda}{a}x_1 + C_4 \sinh\frac{\lambda}{a}x_1,\\
&\left(\frac{\lambda}{a}\right)^2 = \frac{A}{D}N_0
\end{aligned}
\tag{8.6.17}
$$

For a simply supported strip we have with $w(0) = w(a) = w''(0) = w''(a) = 0$

$$
C_2 \sin\lambda = 0, \quad \lambda = n\pi
$$

A nonzero solution is obtained if

$$
N_0 = \frac{n^2\pi^2}{a^2}\frac{D}{A} \tag{8.6.18}
$$

Thus the critical buckling load follows with to

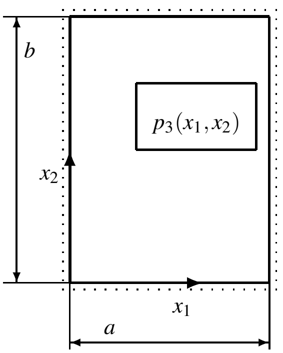
$$N_{0cr} = \frac{\pi^2}{a^2}\frac{D}{A} \tag{8.6.19}$$

Summarizing the developments of analytical solutions for unsymmetrical laminated plate strips we have the following conclusions:

- The system of three coupled differential equations for the displacements $u(x_1), v(x_1)$ and $w(x_1)$ can be uncoupled and reduced to one differential equation of fourth order for $w(x_1)$ and two differential equations of third order for $u(x_1)$ and $v(x_1)$, respectively.
- Analytical solutions for bending of unsymmetrical laminated plate strips can be simple derived for all possible boundary conditions. In the general case all stress resultants (8.6.4) are not equal to zero. The general symmetric case and symmetrical cross-ply strips are included as special solutions.
- The derivations of bending equations can be expanded to buckling and vibration problems.
- The derivation of analytical solutions for unsymmetrical laminated strips can, like for the symmetrical case, expanded to lateral loads $p_3(x_1,x_2) = x_2 p(x_1)$.

8.6.1.2 Navier Solution

Figure 8.5 shows a specially orthotropic rectangular plate simply supported at all edges with arbitrary lateral load $p_3(x_1,x_2)$. In the Navier solution one expands the deflection $w(x_1,x_2)$ and the applied lateral load $p(x_1,x_2)$, respectively, into double infinite Fourier sine series because that series satisfies all boundary conditions



Boundary conditions:

$w(0,x_2) = w(a,x_2) = w(x_1,0) = w(x_1,b) = 0$

$M_1(0,x_2) = M_1(a,x_2) = M_2(x_1,0) = M_2(x_1,b) = 0$

Fig. 8.5 Rectangular plate, all edges are simply supported, specially orthotropic

$$
\begin{aligned}
p_3(x_1,x_2) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} p_{rs}\sin\alpha_r x_1 \sin\beta_s x_2,\\
p_{rs} &= \frac{4}{ab}\int_0^a\int_0^b p_3(x_1,x_2)\sin\alpha_r x_1 \sin\beta_s x_2 \mathrm{d}x_1\mathrm{d}x_2,\\
w(x_1,x_2) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} w_{rs}\sin\alpha_r x_1 \sin\beta_s x_2
\end{aligned}
\tag{8.6.20}
$$

with $\alpha_r = r\pi/a, \beta_s = s\pi/b$. The coefficients w_{rs} are to be determined such that the plate equation (Table 8.1) is satisfied.

Substituting Eqs. (8.6.20) into the plate equation yields

$$
\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} w_{rs}\left(D_1\alpha_r^4 + 2D_3\alpha_r^2\beta_s^2 + D_2\beta_s^4\right)\sin\alpha_r x_1 \sin\beta_s x_2 = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} p_{rs}\sin\alpha_r x_1 \sin\beta_s x_2
\tag{8.6.21}
$$

and we obtain the coefficients w_{rs}

$$
w_{rs} = \frac{p_{rs}}{D_1\alpha_r^4 + 2D_3\alpha_r^2\beta_s^2 + D_2\beta_s^4} = \frac{p_{rs}}{d_{rs}}
\tag{8.6.22}
$$

The solution becomes

$$
w(x_1,x_2) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{p_{rs}}{d_{rs}}\sin\alpha_r x_1 \sin\beta_s x_2
\tag{8.6.23}
$$

The load coefficients p_{rs} one obtains by integrating (8.6.20) for the given lateral loading $p_3(x_1,x_2)$. For a uniform distributed load $p_3(x_1,x_2) = p = \text{const}$ we obtain, for instance,

$$
p_{rs} = \frac{16p}{\pi^2 rs}, \quad r,s = 1,3,5,\ldots
\tag{8.6.24}
$$

From Table 8.1, the equations for the moment resultants are:

$$
\begin{aligned}
M_1(x_1,x_2) &= -D_{11}\frac{\partial^2 w}{\partial x_1^2} - D_{12}\frac{\partial^2 w}{\partial x_2^2}\\
&= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(D_{11}\alpha_r^2 + D_{12}\beta_s^2)w_{rs}\sin\alpha_r x_1 \sin\beta_s x_2,\\
M_2(x_1,x_2) &= -D_{12}\frac{\partial^2 w}{\partial x_1^2} - D_{22}\frac{\partial^2 w}{\partial x_2^2}\\
&= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}(D_{12}\alpha_r^2 + D_{22}\beta_s^2)w_{rs}\sin\alpha_r x_1 \sin\beta_s x_2,
\end{aligned}
\tag{8.6.25}
$$

$$
\begin{aligned}
M_6(x_1,x_2) &= -2D_{66}\frac{\partial^2 w}{\partial x_1 \partial x_2} = -2D_{66}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\alpha_r\beta_s w_{rs}\cos\alpha_r x_1\cos\beta_s x_2,\\
Q_1(x_1,x_2) &= \frac{\partial M_1}{\partial x_1}+\frac{\partial M_6}{\partial x_2},\quad V_1(x_1,x_2)=Q_1+\frac{\partial M_6}{\partial x_2},\\
Q_2(x_1,x_2) &= \frac{\partial M_6}{\partial x_1}+\frac{\partial M_2}{\partial x_2},\quad V_1(x_1,x_2)=Q_2+\frac{\partial M_6}{\partial x_2},
\end{aligned}
$$

and with the stress relation for the k layers

$$
\boldsymbol{\sigma}^{(k)}(x_1,x_2,x_3)=\boldsymbol{Q}^{(k)}x_3\boldsymbol{\kappa}=-x_3\begin{bmatrix}Q_{11}^{(k)} & Q_{12}^{(k)} & 0\\ Q_{12}^{(k)} & Q_{22}^{(k)} & 0\\ 0 & 0 & Q_{66}^{(k)}\end{bmatrix}\begin{bmatrix}\partial^2 w/\partial x_1^2\\ \partial^2 w/\partial x_2^2\\ 2\partial^2 w/\partial x_1\partial x_2\end{bmatrix} \tag{8.6.26}
$$

one obtains the solutions for the in-plane stresses $\sigma_1^{(k)},\sigma_2^{(k)},\sigma_6^{(k)}$

$$
\begin{bmatrix}\sigma_1^{(k)}\\ \sigma_2^{(k)}\\ \sigma_6^{(k)}\end{bmatrix}=\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{p_{rs}}{D_1\alpha_r^4+2D_3\alpha_r^2\beta_s^2+D_2\beta_s^4}\begin{bmatrix}(Q_{11}^{(k)}\alpha_r^2+Q_{12}^{(k)}\beta_s^2)\sin\alpha_r x_1\sin\beta_s x_2\\ (Q_{12}^{(k)}\alpha_r^2+Q_{22}^{(k)}\beta_s^2)\sin\alpha_r x_1\sin\beta_s x_2\\ -2Q_{66}^{(k)}\alpha_r\beta_s\cos\alpha_r x_1\cos\beta_s x_2\end{bmatrix} \tag{8.6.27}
$$

With the simplified formula (5.2.19) follows the transverse shear stresses $\sigma_4^{(k)},\sigma_5^{(k)}$

$$
\begin{aligned}
\begin{bmatrix}\sigma_5^{(k)}(x_1,x_3)\\ \sigma_4^{(k)}(x_1,x_3)\end{bmatrix} &= \begin{bmatrix}F_{11}x_3 & F_{62}x_3\\ F_{61}x_3 & F_{22}x_3\end{bmatrix}\\
&\quad\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\begin{bmatrix}(D_{11}\alpha_r^3+D_{12}\beta_s^2\alpha_r)w_{rs}\cos\alpha_r x_1\sin\beta_s x_2\\ (D_{12}\alpha_r^2\beta_s+D_{22}\beta_s^3)w_{rs}\sin\alpha_r x_1\cos\beta_s x_2\end{bmatrix}
\end{aligned} \tag{8.6.28}
$$

The Navier solution method can be applied to all simply supported specially orthotropic laminated rectangular plates in the same way. For a given lateral load $p_3(x_1,x_2)$ one can obtain the load coefficients p_{rs} by integrating (8.6.20), and by substituting p_{rs} in (8.6.3) follows the w_{rs}. Some conclusions can be drawn from the application of the Navier solution:

- The solution convergence is rapid for the lateral deflection $w(x_1,x_2)$ and uniform loaded plates. The convergence decreases for the stress resultants and the stresses and in general with the concentration of lateral loads in partial regions.
- The solution convergence is more rapid for the stresses $\sigma_1^{(k)}$ in the fibre direction but is not as rapid in calculating $\sigma_2^{(k)}$.

The Navier solutions can be also developed for antisymmetric cross-ply laminate and for symmetric and antisymmetric angle-ply laminates. For these laminates the plate equations (8.2.6) are not uncoupled and we have to prescribe in-plane and out-of-plane boundary conditions. It is easy to review that the Naviers double series solutions Type 1 and Type 2, i.e.

Type 1:

$$
\begin{aligned}
u(x_1,x_2) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} u_{rs}\cos\alpha_r x_1 \sin\beta_s x_2,\\
v(x_1,x_2) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} v_{rs}\sin\alpha_r x_1 \cos\beta_s x_2,\\
w(x_1,x_2) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} w_{rs}\sin\alpha_r x_1 \sin\beta_s x_2,
\end{aligned}
$$

Type 2:

$$
\begin{aligned}
u(x_1,x_2) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} u_{rs}\sin\alpha_r x_1 \cos\beta_s x_2,\\
v(x_1,x_2) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} v_{rs}\cos\alpha_r x_1 \sin\beta_s x_2,\\
w(x_1,x_2) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} w_{rs}\sin\alpha_r x_1 \sin\beta_s x_2,
\end{aligned}
$$

$\alpha_r = \pi r/a, \beta_s = \pi s/a$ satisfy the following alternative boundary conditions for selected laminated plates:

- Simply supported boundary conditions, Type 1
  $x_1 = 0$ and $x_1 = a$

  $$w = 0, \quad M_1 = 0, \quad v = 0, \quad N_1 = 0$$

  $x_2 = 0$ and $x_2 = b$

  $$w = 0, \quad M_2 = 0, \quad u = 0, \quad N_2 = 0$$

  The Naviers double series Type 1 for u, v and w can be used only for laminates, whose stiffness $A_{16}, A_{26}, B_{16}, B_{26}, D_{16}, D_{26}$ are zero, i.e for symmetric or antisymmetric cross-ply laminates
- Simple supported boundary conditions, Type 2
  $x_1 = 0$ and $x_1 = a$

  $$w = 0, \quad M_1 = 0, \quad u = 0, \quad N_6 = 0$$

  $x_2 = 0$ and $x_2 = b$

  $$w = 0, \quad M_2 = 0, \quad v = 0, \quad N_6 = 0$$

  The Navier double series solution Type 2 for u, v, w can be used only for laminate stacking sequences with $A_{16}, A_{26}, B_{11}, B_{12}, B_{22}, B_{66}, D_{16}, D_{26}$ equal zero, i.e for symmetric or antisymmetric angle ply laminates.

The Navier solutions can be used for calculating bending, buckling and vibration. For buckling the edge shear force N_6 and, respectively, for vibration the in-plane inertia terms must be necessarily zero.

8.6.1.3 Nádai-Lévy Solution

For computing the bending of specially orthotropic rectangular plates with two opposite edges simply supported, a single infinite series method can be used. The two other opposite edges may have arbitrary boundary conditions (Fig. 8.6). Nádai introduced for isotropic plates the solution of the plate equation in the form

$$w(x_1,x_2) = w_p(x_1) + w_h(x_1,x_2), \quad p_3 = p_3(x_1), \tag{8.6.29}$$

where $w_p(x_1)$ represents the deflection of a plate strip and $w_h(x_1,x_2)$ is the solution of the homogeneous plate equation ($p_3 = 0$). w_h must be chosen such that $w(x_1,x_2)$ in (8.6.29) satisfy all boundary conditions of the plate. With the solutions for w_h, suggested by Lévy, and w_p, suggested by Nádai,

$$w_h(x_1,x_2) = \sum_{r=1}^{\infty} f_r(x_2)\sin\alpha_r x_1, \quad w_p(x_1) = \sum_{r=1}^{\infty} \frac{p_r \sin\alpha_r x_1}{D_1\alpha_r^4} \tag{8.6.30}$$

with $\alpha_r = r\pi/a$ and

$$p_3(x_1) = \sum_{r=1}^{\infty} p_r \sin\alpha_r x_1, \quad p_r = \frac{2}{a}\int_0^a p_3(x_1)\sin\alpha_r x_1 \mathrm{d}x_1$$

the boundary conditions for $x_1 = 0$ and $x_1 = a$ are satisfied.

Substituting (8.6.30) into the plate equation for specially orthotropic plates, Table 8.1, it follow for each term $f_r(x_2)$ a differential equation of 4th order with constant coefficients

$$D_2\frac{\mathrm{d}^4 f_r(x_2)}{\mathrm{d}x_2^4} - 2D_3\alpha_r^2\frac{\mathrm{d}^2 f_r(x_2)}{\mathrm{d}x_2^2} + D_1\alpha_r^4 f_r(x_2) = p_r \tag{8.6.31}$$

or

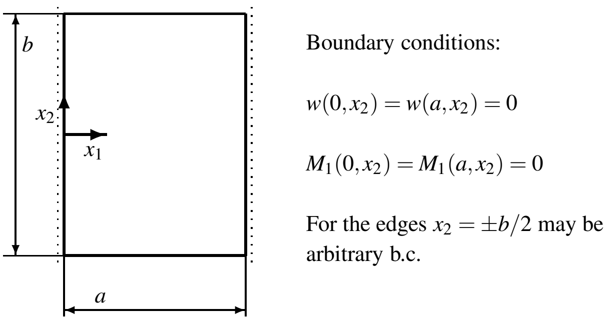


Fig. 8.6 Rectangular specially orthotropic rectangular plate with two opposite edges simply supported

$$\frac{\mathrm{d}^4 f_r(x_2)}{\mathrm{d}x_2^4} - \frac{2D_3\alpha_r^2}{D_2}\frac{\mathrm{d}^2 f_r(x_2)}{\mathrm{d}x_2^2} + \frac{D_1}{D_2}\alpha_r^4 f_r(x_2) = \frac{p_r}{D_2} \tag{8.6.32}$$

The homogeneous differential equation, i.e. $p_r = 0$, can be solved with

$$f_{rh}(x_2) = C_r \exp(\lambda_r \alpha_r x_2) \tag{8.6.33}$$

and yields the characteristic equation for the four roots

$$\lambda_r^4 - \frac{2D_3}{D_2}\lambda_r^2 + \frac{D_1}{D_2} = 0 \Longrightarrow \lambda_r^2 = \frac{D_3}{D_2} \pm \sqrt{\left(\frac{D_3}{D_2}\right)^2 - \frac{D_1}{D_2}} \tag{8.6.34}$$

In the case of isotropic plates it follows with $D_1 = D_2 = D_3 = D$ there are repeated roots ± 1.

For specially orthotropic laminated plates the form of $f_{rh}(x_2)$ depends on the character of the roots of the algebraic equation of 4th order. There are three different sets of roots:

1. $(D_3/D_2)^2 > (D_1/D_2)$: In this case (8.6.34) leads to four real and different roots

$$\lambda_{1/2} = \pm\delta_1, \lambda_{3/4} = \pm\delta_2, \delta_1, \delta_2 > 0,$$

$$\begin{aligned} f_{rh}(x_2) &= A_r \cosh\delta_1\alpha_r x_2 + B_r \sinh\delta_1\alpha_r x_2 \\ &+ C_r \cosh\delta_2\alpha_r x_2 + D_r \sinh\delta_2\alpha_r x_2 \end{aligned} \tag{8.6.35}$$

2. $(D_3/D_2)^2 = (D_1/D_2)$: In this case (8.6.34) leads to four real and equal roots

$$\lambda_{1/2} = +\delta, \lambda_{3/4} = -\delta, \delta > 0,$$

$$f_{rh}(x_2) = (A_r + B_r x_2)\cosh\delta\alpha_r x_2 + (C_r + D_r x_2)\sinh\delta\alpha_r x_2 \tag{8.6.36}$$

3. $(D_3/D_2)^2 < (D_1/D_2)$: In this case the roots are complex

$$\lambda_{1/2} = \delta_1 \pm i\delta_2, \lambda_{3/4} = -\delta_1 \pm i\delta_2, \delta_1, \delta_2 > 0,$$

$$\begin{aligned} f_{rh}(x_2) &= (A_r \cos\delta_2\alpha_r x_2 + B_r \sinh\delta_2\alpha_r x_2)\cosh\delta_1\alpha_r x_2 \\ &+ (C_r \cos\delta_1\alpha_r x_2 + D_r \sin\delta_1\alpha_r x_2)\sinh\delta_1\alpha_r x_2 \end{aligned} \tag{8.6.37}$$

For a given plate for which materials and fibre orientations have been specified only, one of the three cases exists. However in the design problem, trying to find the best variant, more than one case may be involved with the consequence of determining not just four constants A_r, B_r, C_r, D_r, but eight or all twelve to calculate which construction is optimal for the design.

Concerning the particular solution, it is noted that the lateral load may be at most linear in x_2 too, i.e $p_3(x_1,x_2) = p_3(x_1)q(x_2)$ with q at most linear in x_2. The solution w_p in (8.6.29) is then replaced by

$$w_p(x_1,x_2) = q(x_2)\sum_{r=1}^{\infty}\frac{p_r \sin\alpha_r x_1}{D_1\alpha_r^4}\sin\alpha_r x_1 \tag{8.6.38}$$

With the solution $w(x_1,x_2) = w_h(x_1,x_2) + w_p(x_1,x_2)$ the stress resultants and stresses can be calculated in the usual way.

The Navier and Nádai-Lévy solution method can be also applied to eigenvalue problems. We assume, for instance, that the vibration mode shapes of a laminated plate with specially orthotropic behavior, which is simply supported at all four edges, is identical to an isotropic plate. We choose

$$w(x_1,x_2,t) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} w_{rs} \sin \alpha_r x_1 \sin \alpha_s x_2 \sin \omega t \tag{8.6.39}$$

to represent the expected harmonic oscillation and to satisfy all boundary conditions. Substituting the expression (8.6.39) into (8.2.16) with $p_3 \equiv 0$ yields

$$[D_1\alpha_r^4 + 2D_3\alpha_r^2\alpha_s^2 + D_2\alpha_s^4 - \rho\omega^2]w_{rs} = 0 \tag{8.6.40}$$

A non-zero value of w_{rs}, i.e. a non-trivial solution, is obtained only if the expression in the brackets is zero, hence we can find the equation for the natural frequencies

$$\omega_{rs}^2 = \frac{\pi^4}{\rho h}\left[D_1\left(\frac{r}{a}\right)^4 + 2D_3\left(\frac{r}{a}\right)^2\left(\frac{s}{a}\right)^2 + D_2\left(\frac{s}{a}\right)^4\right] \tag{8.6.41}$$

The fundamental frequency corresponds to $r = s = 1$ and is given by

$$\omega_{11}^2 = \frac{\pi^4}{\rho h a^4}\left[D_1 + 2D_3\left(\frac{a}{b}\right)^2 + D_2\left(\frac{a}{b}\right)^4\right] \tag{8.6.42}$$

Note that the maximum amplitude w_{rs} cannot be determined, only the vibration mode shapes are given by (8.6.39). In the case of an isotropic plate the natural frequencies are with $D_1 = D_2 = D_3 = D$

$$\omega_{rs}^2 = k_{rs}\frac{\pi^2}{a^2}\sqrt{\frac{D}{\rho h}}, \quad k_{rs} = \left[r^2 + s^2\left(\frac{a}{b}\right)^2\right] \tag{8.6.43}$$

If we consider a buckling problem, e.g. a specially orthotropic laminated plate simply supported at all edges with a biaxial compression N_1 and N_2, it follows from (8.2.19) that

$$D_1\frac{\partial^4 w}{\partial x_1^4} + 2D_3\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + D_2\frac{\partial^4 w}{\partial x_2^4} = N_1\frac{\partial^2 w}{\partial x_1^2} + N_2\frac{\partial^2 w}{\partial x_2^2} \tag{8.6.44}$$

The Navier solution method yields with (8.6.39)

$$\pi^2 w_{rs}[D_1 r^4 + 2D_3 r^2 s^2\gamma^2 + D_2 s^4\gamma^4] = -w_{rs}[N_1 r^2 + N_2 s^2\gamma^2]a^2 \tag{8.6.45}$$

with $\gamma = a/b$. A non-zero solution of the buckling problem ($w_{rs} \neq 0$) leads to

$$N_1 r^2 + N_2 s^2 \gamma^2 = -\frac{\pi^2}{a^2}[D_1 r^4 + 2D_3 r^2 s^2 \gamma^2 + D_2 s^4 \gamma^4] \tag{8.6.46}$$

We consider the example of uniform compression $N_1 = -N$ and $N_2 = -\kappa N$, where the boundary force N is positive. Equation (8.6.46) yields

$$N = \frac{\pi^2 (D_1 r^4 + 2D_3 r^2 s^2 \gamma^2 + D_2 s^4 \gamma^4)}{a^2 (r^2 + \kappa s^2 \gamma^2)}$$

The critical buckling load N_{cr} corresponds to the lowest value of N. If $\kappa = 0$ we have the case of uniaxial compression and the buckling equation simplifies to

$$N = \frac{\pi^2}{a^2 r^2}(D_1 r^4 + 2D_3 r^2 s^2 \gamma^2 + D_2 s^4 \gamma^4)$$

For a given r, the smallest value of N is obtained for $s = 1$, because s appears only in the numerator. To determine which r provides the smallest value N_{cr} is not simple and depends on the stiffness D_1, D_2, D_3, the length-to-width ratio $\gamma = a/b$ and r. However, for a given plate it can be easily determined numerically. Summarizing the discussion of the classical laminate theory applied to laminate plates we can formulate the following conclusions:

- Specially orthotropic laminate plates can be analyzed with the help of the Navier solution or the Nádai-Lévy solution of the theory of isotropic Kirchhoff's plates, if all or two opposite plate edges are simply supported. These solution methods can be applied to plate bending, buckling and vibration.
- For more general boundary conditions specially orthotropic plates may be solved analytically with the help of the variational approximate solutions method of Rayleigh-Ritz or in a more generalized way based on a variational method of Kantorovich.
- Plates with extensional-bending couplings should be solved numerically, e.g. with the help of the finite element method, Chap. 11. Note that in special cases antisymmetric cross-ply respectively symmetric and antisymmetric angle-ply laminates can be analyzed analytically with Navier's solution method.

In this section we illustrated detailed analytical solutions for specially orthotropic laminates which can predict "exact" values of deflections, natural frequencies of vibration and critical buckling loads. But even the "exact" solutions become approximate because of the truncation of the infinite series solutions or round-off errors in the solution of nonlinear algebraic equations, etc. However these solutions help one to understand, at least qualitatively, the mechanical behavior of laminates. Many laminates with certain fibre orientations have decreasing values of the coefficients D_{16}, D_{26} for bending-torsion coupling and they can be analyzed with the help of the solution methods for specially orthotropic plates.

8.6.2 Shear Deformation Laminate Theory

The analysis of laminated rectangular plates including transverse shear deformations is much more complicated than in the frame of classical laminate theory. Also for plate analysis including shear deformations the at least complicated problem is cylindrical bending, i.e one-dimensional formulations for plate strips.

Unlike to classical plate strips equations only symmetric and unsymmetric cross-ply laminates can be handled in a unique manner. In the case of two-dimensional plate equations we restrict the developments of analytical solutions for bending, buckling and vibrations analogous to Eqs. (8.3.6) - (8.3.8) to midplane symmetric cross-ply plates with all $B_{ij} = 0$ and additional $A_{16} = A_{26} = D_{16} = D_{26} = 0, A_{45} = 0$.

8.6.2.1 Plate Strip

Consider first the cylindrical bending for the plate strip with an infinite length in the x_2-direction and uniformly supported edges $x_1 = 0, x_1 = a$, subjected to a load $p_3 = p(x_1)$. If we restrict the considerations to cross-ply laminated strips the governing strip equations follow with $A_{16} = A_{26} = 0, B_{16} = B_{26} = 0$, $D_{16} = D_{26} = 0$, $A_{45} = 0$ and result in a cylindrical deflected middle surface with $v = 0, \psi_2 = 0$, $u = u(x_1), \psi_1 = \psi_1(x_1), w = w(x_1)$ from (8.3.4) as

$$\begin{aligned} A_{11}\frac{\mathrm{d}^2 u}{\mathrm{d}x_1^2} + B_{11}\frac{\mathrm{d}^2 \psi_1}{\mathrm{d}x_1^2} &= 0, \\ B_{11}\frac{\mathrm{d}^2 u}{\mathrm{d}x_1^2} + D_{11}\frac{\mathrm{d}^2 \psi_1}{\mathrm{d}x_1^2} - k_{55}^{\mathrm{s}} A_{55}\left(\psi_1 + \frac{\mathrm{d}w}{\mathrm{d}x_1}\right) &= 0, \\ k_{55}^{\mathrm{s}} A_{55}\left(\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1} + \frac{\mathrm{d}^2 w}{\mathrm{d}x_1^2}\right) + p_3(x_1) &= 0 \end{aligned} \tag{8.6.47}$$

The stress resultants $N_i(x)_1, M_i(x_1), i = 1,2,6$ and $Q_j, j = 1,2$ are with (8.3.2) and (8.3.6)

$$\begin{aligned} N_1(x_1) &= A_{11}\frac{\mathrm{d}u}{\mathrm{d}x_1} + B_{11}\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1}, \\ N_2(x_1) &= A_{12}\frac{\mathrm{d}u}{\mathrm{d}x_1} + B_{12}\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1}, \\ N_6(x_1) &= 0, \\ M_1(x_1) &= B_{11}\frac{\mathrm{d}u}{\mathrm{d}x_1} + D_{11}\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1}, \\ M_2(x_1) &= B_{12}\frac{\mathrm{d}u}{\mathrm{d}x_1} + D_{12}\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1}, \\ M_6(x_1) &= 0, \\ Q_1(x_1) &= k_{55}^{\mathrm{s}} A_{55}\left(\psi_1 + \frac{\mathrm{d}w}{\mathrm{d}x_1}\right), \\ Q_2(x_1) &= 0 \end{aligned} \tag{8.6.48}$$

The three coupled differential equations for u, w and ψ_1 can be reduced to one uncoupled differential equation for ψ_1. The first equation yields

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x_1^2} = -\frac{B_{11}}{A_{11}}\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x_1^2}, \qquad \frac{\mathrm{d}^3 u}{\mathrm{d}x_1^3} = -\frac{B_{11}}{A_{11}}\frac{\mathrm{d}^3\psi_1}{\mathrm{d}x_1^3} \tag{8.6.49}$$

Differentiating the second equation and substituting the equation above result in

$$-\frac{B_{11}^2}{A_{11}}\frac{\mathrm{d}^3\psi_1}{\mathrm{d}x_1^3} + D_{11}\frac{\mathrm{d}^3\psi_1}{\mathrm{d}x_1^3} - k_{55}^{\mathrm{s}}A_{55}\left(\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1} + \frac{\mathrm{d}^2 w}{\mathrm{d}x_1^2}\right) = 0$$

or with $\left(D_{11} - \dfrac{B_{11}^2}{A_{11}}\right) = D_{11}^R$

$$k_{55}^{\mathrm{s}}A_{55}\left(\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1} + \frac{\mathrm{d}^2 w}{\mathrm{d}x_1^2}\right) = D_{11}^R\frac{\mathrm{d}^3\psi_1}{\mathrm{d}x_1^3} \tag{8.6.50}$$

Substituting Eq. (8.6.50) in the third equation (8.6.47) yield an uncoupled equation for $\psi_1(x_1)$

$$D_{11}^R\frac{\mathrm{d}^3\psi_1}{\mathrm{d}x_1^3} = -p_3 \tag{8.6.51}$$

The uncoupled equations for $u(x_1)$ and $w(x_1)$ follow then as

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x_1^2} = -\frac{A_{11}}{B_{11}}\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x_1^2}, \qquad \frac{\mathrm{d}w}{\mathrm{d}x_1} = -\psi_1 + \frac{D_{11}^R}{k_{55}^{\mathrm{s}}A_{55}}\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x_1^2} \tag{8.6.52}$$

The three uncoupled equations can be simple integrated

$$\begin{aligned}
D_{11}^R\psi_1(x_1) &= -\iiint p_3(x_1)\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1 + C_1\frac{x_1^2}{2} + C_2x_1 + C_3,\\
w(x_1) &= \frac{1}{D_{11}^R}\left[\iiiint p_3(x_1)\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1 + C_1\frac{x_1^3}{6} + C_2\frac{x_1^2}{2}\right.\\
&\quad \left. + C_3x_1 + C_4\right] - \frac{1}{k_{55}^{\mathrm{s}}A_{55}}\left[\iint p_3(x_1)\mathrm{d}x_1\mathrm{d}x_1 + C_1x_1\right]\\
&= w^B(x_1) + w^S(x_1),\\
u(x_1) &= -\frac{A_{11}}{B_{11}}\frac{1}{D_{11}^R}\left[\iiint p_3(x_1)\mathrm{d}x_1\mathrm{d}x_1\mathrm{d}x_1 + C_1x_1 + C_5\right]
\end{aligned} \tag{8.6.53}$$

Thus the general analytical solutions for unsymmetric cross-ply laminated strips are calculated. For symmetrical cross-ply laminated strips the equations yield $D_{11}^R = D_{11}$ and $A_{11}u''(x_1) = 0$. Restricting to symmetrical cross-ply laminated strips analytical solutions for buckling or vibrations can be developed analogous to Timoshenko's beams or to the classical strip problems.

For a buckling load $N_1(x_1) = -N_0$ follow with $p_3 = 0$

$$\begin{aligned}
&D_{11}\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x_1^2} - k_{55}^{\mathrm{s}}A_{55}\left(\psi_1 + \frac{\mathrm{d}w}{\mathrm{d}x_1}\right) = 0,\\
&k_{55}^{\mathrm{s}}A_{55}\left(\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1} + \frac{\mathrm{d}^2w}{\mathrm{d}x_1^2}\right) + N_0\frac{\mathrm{d}^2w}{\mathrm{d}x_1} = 0
\end{aligned} \tag{8.6.54}$$

The equations can be uncoupled. With

$$\left(\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1} + \frac{\mathrm{d}^2w}{\mathrm{d}x_1^2}\right) = \frac{D_{11}}{k_{55}^{\mathrm{s}}A_{55}}\frac{\mathrm{d}^3\psi_1}{\mathrm{d}x_1^3}, \quad k_{55}^{\mathrm{s}}A_{55}\frac{\mathrm{d}^3\psi_1}{\mathrm{d}x_1^3} = -k_{55}^{\mathrm{s}}A_{55}\frac{\mathrm{d}^4w}{\mathrm{d}x_1^4} - N_0\frac{\mathrm{d}^2w}{\mathrm{d}x_1}$$

one obtains analogous to Eq. (7.3.23)

$$D_{11}\left(1 - \frac{N_0}{k_{55}^{\mathrm{s}}A_{55}}\right)\frac{\mathrm{d}^4w}{\mathrm{d}x_1^4} + N_0\frac{\mathrm{d}^2w}{\mathrm{d}x_1} = 0 \tag{8.6.55}$$

The general solution for the eigenvalue problem (8.6.55) follows with

$$w(x_1) = C\mathrm{e}^{\lambda x_1} \tag{8.6.56}$$

and the characteristic equation

$$D_{11}\left(1 - \frac{N_0}{k_{55}^{\mathrm{s}}A_{55}}\right)\lambda^4 + N_0\lambda^2 = 0 \quad \text{or} \quad D_{11}\lambda^4 + k^2\lambda^2 = 0 \tag{8.6.57}$$

with the solutions

$$\lambda_{1/2} = \pm\mathrm{i}k, \quad \lambda_{3/4} = 0$$

as

$$w(x_1) = C_1\sin kx_1 + C_2\cos kx_1 + C_3x_1 + C_4 \tag{8.6.58}$$

If we assume, e.g. simply supported edges $x_1(0) = 0, x_1(a)$, follow with $w(0) = w(a) = w''(0) = w''(a) = 0$ the free coefficients $C_2 = C_3 = C_4 = 0$ and $C_1\sin ka = 0$. If $C_1 \neq 0$ follow with $\sin ka = 0$ the solution $k = m\pi/a = \alpha_m$ $(m = 1,2,\ldots)$ and $k^2 = \alpha_m^2$ and thus

$$\frac{N_0}{D_{11}\left(1 - \dfrac{N_0}{k_{55}^{\mathrm{s}}A_{55}}\right)} = \alpha_m^2, \quad N_0 = \frac{D_{11}k_{55}^{\mathrm{s}}A_{55}\alpha_m^2}{D_{11}\alpha_m^2 + k_{55}^{\mathrm{s}}A_{55}}$$

The critical buckling load corresponds to the smallest value of N_0 which is obtained for $m = 1$

$$N_{cr} = \frac{D_{11}k_{55}^{\mathrm{s}}A_{55}\pi^2}{D_{11}\pi^2 + k_{55}^{\mathrm{s}}A_{55}a^2} = \frac{\pi^2D_{11}}{a^2}\frac{1}{1 + \dfrac{\pi^2D_{11}}{a^2k_{55}^{\mathrm{s}}A_{55}}} \tag{8.6.59}$$

It can be seen that analogous to the Timoshenko's beam, Sect. 7.3, the including of shear deformations decreases the buckling loads.

The free vibrations equations of the Timoshenko's beams were also considered in Sect. 7.3. For symmetric cross-ply laminated plate strips we obtain comparable equations

$$\begin{aligned} D_{11}\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x_1^2} - k_{55}^{\mathrm{s}}A_{55}\left(\psi_1 + \frac{\mathrm{d}w}{\mathrm{d}x_1}\right) &= \rho_2\frac{\partial^2\psi}{\partial t^2}, \\ k_{55}^{\mathrm{s}}A_{55}\left(\frac{\mathrm{d}\psi_1}{\mathrm{d}x_1} + \frac{\mathrm{d}^2 w}{\mathrm{d}x_1^2}\right) &= \rho_0\frac{\partial^2 w}{\partial t^2} \end{aligned} \tag{8.6.60}$$

ρ_0 and ρ_2 were defined as

$$\rho_0 = \sum_{k=1}^{n}\rho^{(k)}h^{(k)}, \quad \rho_2 = \frac{1}{3}\sum_{k=1}^{n}\rho^{(k)}\left(x_3^{(k)^3} - x_3^{(k-1)^3}\right)$$

and the terms involving ρ_0 and ρ_2 are the translatory and the rotatory inertia terms. ψ_1 and w are functions of x_1 and t and thus we have partial derivatives. If we assume again both strip edges simply supported the analytical solution follow with

$$\begin{aligned} w(x_1,t) &= C_{1m}\mathrm{e}^{-i\omega_m t}\sin\frac{m\pi x_1}{a}, \quad w(0,t) = w(a,t) = 0, \\ \psi_1(x_1,t) &= C_{2m}\mathrm{e}^{-i\omega_m t}\cos\frac{m\pi x_1}{a}, \quad \frac{\partial\psi_1(0,t)}{\partial x_1} = \frac{\partial\psi_1(a,t)}{\partial x_1} = 0 \end{aligned} \tag{8.6.61}$$

Substituting these solution functions into the vibration equations (8.6.60) follow

$$\begin{bmatrix} D_{11}\alpha_m^2 + k_{55}^{\mathrm{s}}A_{55} - \rho_2\omega_m^2 & k_{55}^{\mathrm{s}}A_{55}\alpha_m \\ k_{55}^{\mathrm{s}}A_{55}\alpha_m & k_{55}^{\mathrm{s}}A_{55}\alpha_m^2 - \rho_0\omega_m^2 \end{bmatrix}\begin{bmatrix} C_{2m} \\ C_{1m} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The nontrivial solution of the homogeneous algebraic equation yields the eigenfrequencies ω_m

$$\begin{vmatrix} D_{11}\alpha_m^2 + k_{55}^{\mathrm{s}}A_{55} - \rho_2\omega_m^2 & k_{55}^{\mathrm{s}}A_{55}\alpha_m \\ k_{55}^{\mathrm{s}}A_{55}\alpha_m & k_{55}^{\mathrm{s}}A_{55}\alpha_m^2 - \rho_0\omega_m^2 \end{vmatrix} = 0 \tag{8.6.62}$$

or

$$\rho_0\rho_2\omega_m^4 - (D_{11}\rho_0\alpha_m + k_{55}^{\mathrm{s}}A_{55}\rho_0 + k_{55}^{\mathrm{s}}A_{55}\rho_2\alpha_m^2)^2\omega_m^2 + D_{11}k_{55}^{\mathrm{s}}A_{55}\alpha_m^4 = 0$$

$$A\omega_m^4 - B\omega_m^2 + C = 0, \quad \omega_m^2 = \frac{B}{2A} \pm \frac{1}{2A}\sqrt{B^2 - 4A^2C}$$

The general solution for the vibration equations can be formulated for arbitrary boundary conditions. For harmonic oscillations we write

$$w(x_1,t) = w(x_1)\mathrm{e}^{i\omega t}, \quad \psi_1(x_1,t) = \psi_1(x_1)\mathrm{e}^{i\omega t} \tag{8.6.63}$$

Substituting $w(x_1,t)$ and $\psi_1(x_1,t)$ in the coupled partial differential equations (8.6.60) yield

$$
\begin{aligned}
D_{11}\frac{\mathrm{d}^2\psi_1(x_1)}{\mathrm{d}x_1^2} - k_{55}^{\mathrm{s}}A_{55}\left(\psi_1 + \frac{\mathrm{d}w(x_1)}{\mathrm{d}x_1}\right) + \rho_2\omega^2\psi(x_1) &= 0,\\
k_{55}^{\mathrm{s}}A_{55}\left(\frac{\mathrm{d}\psi_1(x_1)}{\mathrm{d}x_1} + \frac{\mathrm{d}^2w(x_1)}{\mathrm{d}x_1^2}\right) + \rho_0\omega^2 w(x_1) &= 0
\end{aligned}
\qquad (8.6.64)
$$

These both equations can be uncoupled. With

$$
k_{55}^{\mathrm{s}}A_{55}\frac{\mathrm{d}\psi_1(x_1)}{\mathrm{d}x_1} = -\rho_0\omega^2 w(x_1) - k_{55}^{\mathrm{s}}A_{55}\frac{\mathrm{d}^2w(x_1)}{\mathrm{d}x_1}
$$

and

$$
D_{11}\frac{\mathrm{d}^3\psi_1(x_1)}{\mathrm{d}x_1^3} - k_{55}^{\mathrm{s}}A_{55}\frac{\mathrm{d}^2w(x_1)}{\mathrm{d}x_1^2} - k_{55}^{\mathrm{s}}A_{55}\frac{\mathrm{d}\psi_1(x_1)}{\mathrm{d}x_1} - \rho_2\omega^2\frac{\mathrm{d}\psi_1(x_1)}{\mathrm{d}x_1} = 0
$$

follow

$$
D_{11}\frac{\mathrm{d}^4w(x_1)}{\mathrm{d}x_1^4} + \left(\frac{D_{11}\rho_0}{k_{55}^{\mathrm{s}}A_{55}} + \rho_2\right)\omega^2\frac{\mathrm{d}^2w(x_1)}{\mathrm{d}x_1^2} - \left(1 - \frac{\rho_2\omega^2}{k_{55}^{\mathrm{s}}A_{55}}\right)\rho_0\omega^2 w(x_1) = 0
$$

or

$$
a\frac{\mathrm{d}^4w(x_1)}{\mathrm{d}x_1^4} + b\frac{\mathrm{d}^2w(x_1)}{\mathrm{d}x_1^2} - cw(x_1) = 0 \qquad (8.6.65)
$$

The general solution can be derived as

$$
w(x_1) = C_1\sin\lambda_1x_1 + C_2\cos\lambda_2x_1 + C_3\sinh\lambda_3x_1 + C_4\cosh\lambda_4x_1 \qquad (8.6.66)
$$

The λ_i are the roots of the characteristic algebraic equation of (8.6.65). The derivations above demonstrated that for any boundary conditions an analytical solution is possible. Unlike to the classical theory we restricted the considerations in the frame of the shear deformation theory to cross-ply laminated strips. Summarizing the derivations we can draw the following conclusions:

- Cylindrical bending yields simple analytical solutions for unsymmetrical and symmetrical cross-ply laminated plate strips.
- Restricting to symmetrical laminated cross-ply plate strips we can obtain analytical solutions for buckling and vibrations problems, but for general boundary conditions the analytical solution can be with difficulty.

8.6.2.2 Navier Solution

Navier's double series solution can be used also in the frame of the shear deformations plate theory. Analogous to Sect. 8.6.1 double series solutions can be obtain for symmetric and antisymmetric cross-ply and angle-ply laminates with special types of simply supported boundary conditions. In the interest of brevity the discussion is limited here to symmetrical laminated cross-ply plates, i.e. specially orthotropic plates. The in-plane and out-of-plane displacements are then uncoupled.

Rectangular specially orthotropic plates may be simply supported (hard hinged support) on all four edges.

$$
\begin{aligned}
&x_1 = 0, \quad x_1 = a : w = 0, \quad M_1 = 0 \quad \text{respectively} \quad \frac{\partial \psi_1}{\partial x_1} = 0, \quad \psi_2 = 0,\\
&x_2 = 0, \quad x_2 = b : w = 0, \quad M_2 = 0 \quad \text{respectively} \quad \frac{\partial \psi_2}{\partial x_2} = 0, \quad \psi_1 = 0
\end{aligned}
\tag{8.6.67}
$$

The boundary conditions can be satisfied by the following expressions:

$$
\begin{aligned}
w(x_1,x_2) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} w_{rs} \sin\alpha_r x_1 \sin\beta_s x_2,\\
\psi_1(x_1,x_2) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \psi_{1rs} \cos\alpha_r x_1 \sin\beta_s x_2, \quad \alpha_r = \frac{r\pi}{a}, \quad \beta_s = \frac{s\pi}{b},\\
\psi_2(x_1,x_2) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \psi_{2rs} \sin\alpha_r x_1 \cos\beta_s x_2
\end{aligned}
\tag{8.6.68}
$$

The mechanical loading $p_3(x_1,x_2)$ can be also expanded in double Fourier sine series

$$
\begin{aligned}
p_3(x_1,x_2) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} p_{rs} \sin\alpha_r x_1 \sin\beta_s x_2,\\
p_{rs} &= \frac{4}{ab}\int_0^a\int_0^b p_3(x_1,x_2) \sin\alpha_r x_1 \sin\beta_s x_2 \mathrm{d}x_1 \mathrm{d}x_2
\end{aligned}
\tag{8.6.69}
$$

Now the Navier solution method can be extended to Mindlin's plates with all edges simply supported, but the solution is more complex than for Kirchhoff's plates. Substituting the expression (8.6.68) and (8.6.69) into the plate differential equations (8.3.8) gives

$$
\begin{bmatrix} L_{11} & L_{12} & L_{13}\\ L_{12} & L_{22} & L_{23}\\ L_{13} & L_{23} & L_{33} \end{bmatrix}
\begin{bmatrix} \psi_{1rs}\\ \psi_{2rs}\\ w_{rs} \end{bmatrix}
=
\begin{bmatrix} 0\\ 0\\ p_{rs} \end{bmatrix}
\tag{8.6.70}
$$

with

$$
\begin{aligned}
L_{11} &= D_{11}\alpha_r^2 + D_{66}\beta_s^2 + k_{55}^{\mathrm{s}}A_{55}, \quad L_{12} = (D_{12}+D_{66})\alpha_r\beta_s, \quad L_{13} = k_{55}^{\mathrm{s}}A_{55}\alpha_r,\\
L_{22} &= D_{66}\alpha_r^2 + D_{22}\beta_s^2 + k_{44}^{\mathrm{s}}A_{44}, \quad L_{33} = k_{55}^{\mathrm{s}}A_{55}\alpha_r^2 + k_{44}^{\mathrm{s}}A_{44}\beta_s^2, \quad L_{33} = k_{44}^{\mathrm{s}}A_{44}\beta_s
\end{aligned}
\tag{8.6.71}
$$

Solving the Eqs. (8.6.65), one obtains

$$
\psi_{1rs} = \frac{L_{12}L_{23} - L_{22}L_{13}}{\mathrm{Det}(L_{ij})}p_{rs}, \ \psi_{2rs} = \frac{L_{12}L_{13} - L_{11}L_{23}}{\mathrm{Det}(L_{ij})}p_{rs}, \ w_{rs} = \frac{L_{11}L_{22} - L_{12}^2}{\mathrm{Det}(L_{ij})}p_{rs}
\tag{8.6.72}
$$

$\mathrm{Det}(L_{ij})$ is the determinant of the matrix in (8.6.65).

If the three kinematic values $w(x_1,x_2), \psi_1(x_1,x_2), \psi_2(x_1,x_2)$ are calculated the curvatures κ_1, κ_2 and κ_6 may be obtained and the stresses in each lamina follow from (8.3.3) to

$$\begin{aligned}
\begin{bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_6 \end{bmatrix}^{(k)} &= x_3 \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{66} \end{bmatrix}^{(k)} \begin{bmatrix} \kappa_1 \\ \kappa_2 \\ \kappa_6 \end{bmatrix}, \\
\begin{bmatrix} \sigma_5 \\ \sigma_4 \end{bmatrix}^{(k)} &= \begin{bmatrix} C_{55} & 0 \\ 0 & C_{44} \end{bmatrix}^{(k)} \begin{bmatrix} \psi_1 + \dfrac{\partial w}{\partial x_1} \\ \psi_2 + \dfrac{\partial w}{\partial x_2} \end{bmatrix}
\end{aligned} \tag{8.6.73}$$

In a analogous manner natural vibrations and buckling loads can be calculated for rectangular plates with all edges hard hinged supported.

8.6.2.3 Nádai-Lévy Solution

The Nádai-Lévy solution method can also be used to develop analytical solutions for rectangular plates with special layer stacking and boundary conditions, respectively, but the solution procedure is more complicated than in the frame of classical plate theory. We do without detailed considerations and recommend approximate analytical solutions or numerical methods to analyze the behavior of general laminated rectangular plates including shear deformations and supported by any combination of clamped, hinged or free edges.

Summarizing the discussion of analytical solutions for plates including transverse shear deformations one can formulate following conclusion

- Analytical solutions for symmetrical and unsymmetrical laminated plates can be derived for cylindrical bending, buckling and vibration.
- Navier's double series solutions can be simple derived for specially orthotropic plates. Navier's solution method can be also applied to symmetric or antisymmetric cross-ply and angle-ply laminates, but the solution time needed is rather high.
- Ritz's, Galerkin's or Kantorovich's methods are suited to analyze general laminated rectangular plates with general boundary conditions.
- Plates with general geometry or with cut outs etc. should be analyzed by numerical methods

8.7 Problems

Exercise 8.1. A plate strip has the width a in x_1-direction and is infinitely long in the x_2-direction. The strip is loaded transversely by a uniformly distributed load p_0 and simply supported at $x_1 = 0, x_1 = a$. Calculate the deflection w, the resultant moments M_1, M_2, M_6 and the stresses $\sigma_1, \sigma_2, \sigma_6$

1. for a symmetrical four layer plate [0/90/90/0],
2. for a unsymmetrical four layer plate [0/0/90/90]

Solution 8.1. The solutions are presented for both stacking sequences separately.

1. The plate strip is a symmetric cross-ply laminate, i.e. $B_{ij} = 0, D_{16} = D_{26} = 0$. The governing differential equations are

$$\begin{aligned}
&\frac{\mathrm{d}^2 M_1}{\mathrm{d}x_1^2} = -p_0, \quad \frac{\mathrm{d}M_1}{\mathrm{d}x_1} = Q_1,\\
&D_{11}\frac{\mathrm{d}^2 w}{\mathrm{d}x_1^2} = -M_1, \quad D_{12}\frac{\mathrm{d}^2 w}{\mathrm{d}x_1^2} = -M_2, \quad M_6 = 0,\\
&D_{11}\frac{\mathrm{d}^4 w}{\mathrm{d}x_1^4} = p_0
\end{aligned}$$

The vertical deflection $w = w(x_1)$ is

$$w(x_1) = \frac{1}{D_{11}}\left[p_0\frac{x_1^4}{24} + C_1\frac{x_1^3}{6} + C_2\frac{x_1^2}{2} + C_3 x_1 + C_4\right]$$

Satisfying the boundary conditions

$$w(0) = 0, \quad w(a) = 0, \quad M_1(0) = 0, \quad M_1(a) = 0$$

yield the unknown constants C_1 - C_4

$$C_1 = -\frac{q_0 a}{2}, \quad C_2 = 0, C_3 = \frac{q_0 a^3}{24}, \quad C_4 = 0$$

and as result the complete solution for the deflection $w(x_1)$

$$w(x_1) = \frac{p_0 a^4}{24 D_{11}}\left[\left(\frac{x_1}{a}\right)^4 - 2\left(\frac{x_1}{a}\right)^3 + \left(\frac{x_1}{a}\right)\right]$$

The moment resultants follow as

$$\begin{aligned}
M_1(x_1) &= -\frac{p_0 a^2}{2}\left[\left(\frac{x_1}{a}\right)^2 - \frac{x_1}{a}\right],\\
M_2(x_1) &= \frac{D_{12}}{D_{11}} M_1(x_1) = \frac{D_{12}}{D_{11}}\frac{p_0 a^2}{2}\left[\left(\frac{x_1}{a}\right)^2 - \frac{x_1}{a}\right],\\
M_6(x_1) &= 0
\end{aligned}$$

The strains and stresses at any point can be determined as follow

$$\begin{aligned}
&\kappa_1 = -\frac{\mathrm{d}^2 w}{\mathrm{d}x_1} = \frac{p_0 a^2}{2}\left[\frac{x_1}{a} - \left(\frac{x_1}{a}\right)^2\right], \quad \kappa_2 = 0, \quad \kappa_6 = 0,\\
&\varepsilon_1 = x_3\kappa_1 = x_3\frac{p_0 a^2}{2}\left[\frac{x_1}{a} - \left(\frac{x_1}{a}\right)^2\right], \quad \varepsilon_2 = 0, \quad \varepsilon_6 = 0
\end{aligned}$$

The stresses in each layer are

$$\begin{aligned}
0^0-\text{layers}:\quad & \sigma_1=\sigma_1'=x_3\frac{Q_{11}}{D_{11}}\frac{p_0a^2}{8}\left[\frac{x_1}{a}-\left(\frac{x_1}{a}\right)^2\right],\\
& \sigma_2=\sigma_2'=x_3\frac{Q_{12}}{D_{11}}\frac{p_0a^2}{8}\left[\frac{x_1}{a}-\left(\frac{x_1}{a}\right)^2\right],\\
& \sigma_6=0,\\
90^0-\text{layers}:\quad & \sigma_2=\sigma_1'=x_3\frac{Q_{11}}{D_{11}}\frac{p_0a^2}{8}\left[\frac{x_1}{a}-\left(\frac{x_1}{a}\right)^2\right],\\
& \sigma_1=\sigma_2'=x_3\frac{Q_{12}}{D_{11}}\frac{p_0a^2}{8}\left[\frac{x_1}{a}-\left(\frac{x_1}{a}\right)^2\right],\\
& \sigma_6=0
\end{aligned}$$

2. The plate strip is an unsymmetric cross-ply laminate, i.e. $A_{16}=A_{26}=0, B_{16}=B_{26}=0, D_{16}=D_{26}=0$. The governing equations follow from Eqs. (8.2.6) and (8.2.12)

$$\begin{aligned}
&A_{11}\frac{\mathrm{d}^2u}{\mathrm{d}x_1^2}-B_{11}\frac{\mathrm{d}^3w}{\mathrm{d}x_1^3}=0,\quad A_{66}\frac{\mathrm{d}^2v}{\mathrm{d}x_1^2}=0,\quad D_{11}\frac{\mathrm{d}^4w}{\mathrm{d}x_1^4}-B_{11}\frac{\mathrm{d}^3u}{\mathrm{d}x_1^3}=p_3,\\
&N_1=A_{11}\frac{\mathrm{d}u}{\mathrm{d}x_1}-B_{11}\frac{\mathrm{d}^2w}{\mathrm{d}x_1^2},\quad N_2=A_{12}\frac{\mathrm{d}u}{\mathrm{d}x_1}-B_{12}\frac{\mathrm{d}^2w}{\mathrm{d}x_1^2},\quad N_6=0,\\
&M_1=B_{11}\frac{\mathrm{d}u}{\mathrm{d}x_1}-D_{11}\frac{\mathrm{d}^2w}{\mathrm{d}x_1^2},\quad M_2=B_{12}\frac{\mathrm{d}u}{\mathrm{d}x_1}-D_{12}\frac{\mathrm{d}^2w}{\mathrm{d}x_1^2},\quad M_6=0
\end{aligned}$$

The equilibrium equations for the stress resultants are

$$\frac{\mathrm{d}N_1}{\mathrm{d}x_1}=0,\quad \frac{\mathrm{d}^2M_1}{\mathrm{d}x_1^2}=-p_3$$

The displacement $u(x_1)$ and $w(x_1)$ are coupled. Substitute

$$A_{11}\frac{\mathrm{d}^3u}{\mathrm{d}x_1^3}=B_{11}\frac{\mathrm{d}^4w}{\mathrm{d}x_1^4}$$

into the second differential equation yield

$$\left(D_{11}-\frac{B_{11}^2}{A_{11}}\right)\frac{\mathrm{d}^4w}{\mathrm{d}x_4}=p_3$$

or with

$$D_{11}\left(1-\frac{B_{11}^2}{A_{11}D_{11}}\right)=D_{11}^R,\quad p_3=p_0\quad\Rightarrow\quad\frac{\mathrm{d}^4w}{\mathrm{d}x_1^4}=\frac{p_0}{D_{11}^R}$$

For the displacement $u(x_1)$ follows

$$\frac{\mathrm{d}^3 u}{\mathrm{d}x_1^3} = \frac{B_{11}}{A_{11}} \frac{\mathrm{d}^4 w}{\mathrm{d}x_1^4}$$

or with

$$A_{11}\left(1 - \frac{B_{11}^2}{A_{11}D_{11}}\right) = A_{11}^R, \quad \frac{\mathrm{d}^4 w}{\mathrm{d}x_1^4} = \frac{p_0}{D_{11}^R} \quad \Rightarrow \quad \frac{\mathrm{d}^3 u}{\mathrm{d}x_1^3} = \frac{B_{11}}{D_{11}} \frac{p_0}{A_{11}^R}$$

These differential equations can be simple integrated

$$\begin{aligned} w(x_1) &= \frac{1}{D_{11}^R}\left[\frac{q_0 x_1^4}{24} + C_1\frac{x_1^3}{6} + C_2\frac{x_1^2}{2} + C_3 x_1 + C_4\right] \\ u(x_1) &= \frac{B_{11}}{D_{11}A_{11}^R}\left[\frac{q_0 x_1^3}{6} + C_1\frac{x_1^2}{2} + C_5 x_1 + C_6\right] \end{aligned}$$

Note that with

$$\frac{\mathrm{d}^2 u}{\mathrm{d}x_1^2} = \frac{B_{11}}{A_{11}} \frac{\mathrm{d}^3 w}{\mathrm{d}x_1^3}$$

in both equations there are equal constants C_1. The boundary conditions for w and M_1 are identically to case 1.
The in-plane boundary conditions are formulated for a fixed-free support, i.e. $u(0) = 0, N_1(a) = 0$. The boundary conditions lead to the six unknown constants C_1 - C_6 and the solution functions are

$$\begin{aligned} u(x_1) &= \frac{B_{11}}{D_{11}A_{11}^R}\frac{p_0 a^3}{12}\left[2\left(\frac{x_1}{a}\right)^3 - \left(\frac{x_1}{a}\right)^2\right] \\ w(x_1) &= \frac{1}{D_{11}^R}\frac{p_0 a^4}{24}\left[\left(\frac{x_1}{a}\right)^4 - 2\left(\frac{x_1}{a}\right)^3 + \frac{x_1}{a}\right] \end{aligned}$$

The stress and moment resultants follow as

$$\begin{aligned} &N_1(x_1) = N_6(x_1) = 0, \\ &N_2(x_1) = \left(\frac{A_{12}B_{11}}{D_{11}A_{11}^R} - \frac{B_{12}}{D_{11}^R}\right)\frac{p_0 a^2}{2}\left[\left(\frac{x_1}{a}\right)^2 - \frac{x_1}{a}\right], \\ &M_1(x_1) = -\frac{p_0 a^2}{2}\left[\left(\frac{x_1}{a}\right)^2 - \frac{x_1}{a}\right], \\ &M_2(x_1) = \left(\frac{B_{11}^2 - D_{12}A_{11}}{A_{11}}\right)\frac{p_0 a^2}{2D_{11}^R}\left[\left(\frac{x_1}{a}\right)^2 - \frac{x_1}{a}\right], \\ &M_6(x_1) = 0 \end{aligned}$$

It is interesting to compare the results of case 1. and case 2. The forms of $w(x_1)$ are for the two cases identical except for the magnitude. With

$$\frac{1}{D_{11}^R} = \frac{1}{D_{11}\left(1 - \dfrac{B_{11}^2}{A_{11}D_{11}}\right)} > \frac{1}{D_{11}}$$

the deflection of the unsymmetric laminate strip will be greater than the deflection of the symmetric laminate. Note that there is no force resultant $N_1(x_1)$ in the unsymmetric case but it is very interesting that there is a force resultant N_2 as a function of x_1, but $N_2(0) = N_2(a) = 0$. With

$$\begin{aligned}
\varepsilon_1 &= \frac{\mathrm{d}u}{\mathrm{d}x_1} = \frac{B_{11}}{D_{11}A_{11}^R}\frac{p_0a^2}{2}\left[\left(\frac{x_1}{a}\right)^2 - \frac{x_1}{a}\right], \qquad \varepsilon_2 = \varepsilon_6 = 0,\\
\kappa_1 &= -\frac{\mathrm{d}^2}{\mathrm{d}x_1^2} = -\frac{p_0a^2}{2D_{11}^R}\left[\left(\frac{x_1}{a}\right)^2 - \frac{x_1}{a}\right], \qquad \kappa_2 = \kappa_6 = 0
\end{aligned}$$

follow the strains $\varepsilon_1, \varepsilon_2$ and the stresses σ_1, σ_2 for the 0^0 and 90^0-layers in a similar manner like case 1. With $B_{11} = B_{12} = 0$ case 2. yields the symmetrical case 1.

Exercise 8.2. A plate strip of the width a with a symmetrical cross-ply stacking is subjected a downward line load q_0 at $x_1 = a/2$. Both edges of the strip are fixed. Calculate the maximum deflection $w_{\max}$ using the shear deformation theory.

Solution 8.2. With (8.6.51) and (8.6.52) follow

$$D_{11}^R \equiv D_{11}, \quad D_{11}\frac{\mathrm{d}^3\psi_1}{\mathrm{d}x_1^3} = q_0\delta\left(x_1 - \frac{1}{2}a\right), \quad \frac{\mathrm{d}w}{\mathrm{d}x_1} = -\psi_1 + \frac{D_{11}}{k_{55}^s A_{55}}\frac{\mathrm{d}^2\psi_1}{\mathrm{d}x_1^2}$$

with

$$\delta\left(x_1 - \frac{1}{2}a\right) = \begin{cases} 0 & x_1 \neq a/2 \\ 1 & x_1 = a/2 \end{cases}, \quad \int \delta\left(x_1 - \frac{1}{2}a\right)\mathrm{d}x_1 = \left\langle x_1 - \frac{1}{2}a\right\rangle^0$$

$< x_1 - e >$ is Föppel's[5] bracket symbol:

$$\begin{aligned}
&< x_1 - e >^n = \begin{cases} 0 & x_1 < e \\ (x_1 - e)^n & x_1 > e \end{cases},\\
&\frac{\mathrm{d}}{\mathrm{d}x_1} < x_1 - e >^n = n < x_1 - e >^{n-1},\\
&\int < x_1 - e >^n \mathrm{d}x_1 = \frac{1}{1+n} < x_1 - e >^{n+1} + C
\end{aligned}$$

With (8.6.53) the analytical solutions for ψ_1 and w are given

[5] August Otto Föppl (*25 January 1854 Groß-Umstadt – †12 August 1924 Ammerlan) - professor of engineering mechanics and graphical statics

$$\begin{aligned}
D_{11}\psi_1(x_1) &= \frac{1}{2}q_0 < x_1 - \frac{1}{2}a >^2 + C_1\frac{x_1^2}{2} + C_2x_1 + C_3, \\
w(x_1) &= \frac{1}{D_{11}}\left[\frac{1}{6}q_0 < x_1 - \frac{1}{2}a >^3 + C_1\frac{x_1^3}{6} + C_2\frac{x_1^2}{2} + C_3x_1 + C_4\right] \\
&- \frac{1}{k_{55}^{\mathrm{s}}A_{55}}\left[q_0 < x_1 - \frac{1}{2}a > + C_1x_1\right]
\end{aligned}$$

$$\begin{aligned}
&\psi_1(0) = 0 : C_3 = 0, \\
&w(0) = 0 : \ C_4 = 0, \\
&\psi_1(a) = 0 : \frac{1}{2}q_0\left(\frac{1}{2}a\right)^2 + C_1\left(\frac{a^2}{2}\right) + C_2a = 0, \\
&w(a) = 0 : \ \frac{1}{6}q_0\left(\frac{1}{2}a\right)^3 + C_1\left(\frac{a^3}{6}\right) + C_2\frac{a^2}{2} - \frac{D_{11}}{k_{55}^{\mathrm{s}}A_{55}}\left(q_0\frac{a}{2} + C_1a\right) = 0, \\
&C_1 = \frac{q_0}{2}, \quad C_2 = -\frac{q_0a}{8},
\end{aligned}$$

$$\begin{aligned}
\psi_1(x_1) &= \frac{q_0a^2}{8D_{11}}\left[\left(\frac{x_1}{a}\right)^2 - \frac{x_1}{a}\right], \\
w(x_1) &= -\frac{q_0a^3}{48D_{11}}\left[3\left(\frac{x_1}{a}\right)^2 - 4\left(\frac{x_1}{a}\right)^3\right] - \frac{q_0a}{2k_{55}^{\mathrm{s}}A_{55}}\frac{x_1}{a} = w^B + w^S, \\
w_{\max} &= \frac{q_0a^3}{192D_{11}} + \frac{q_0a}{4k_{55}^{\mathrm{s}}A_{55}}
\end{aligned}$$

The classical plate theory yields with $k_{55}^{\mathrm{s}}A_{55} \to \infty$ the known value

$$w_{\max} = \frac{q_0a^3}{192D_{11}}$$

Exercise 8.3 (Bending of a quadratic sandwich plate). A quadratic sandwich plate has a symmetric cross-section. The plate properties are $a = b = 1$ m, $h^{\mathrm{f}} = 0{,}2875\ 10^{-3}$ m, $h^{\mathrm{c}} = 24{,}71\ 10^{-3}$ m, $E^{\mathrm{f}} = 1{,}42\ 10^5$ MPa, $\nu^{\mathrm{f}} = 0{,}3$, $G^{\mathrm{f}} = E^{\mathrm{f}}/2(1+\nu^{\mathrm{f}})$, $G^{\mathrm{c}} = 22$ MPa. The cover sheet and the core material are isotropic, $h^{\mathrm{f}} \ll h^{\mathrm{c}}$. The transverse uniform distributed load is $p = 0{,}05$ MPa. The boundary conditions are hard hinged support for all boundaries. Calculate the maximum flexural displacement $w_{\max}$ with the help of a one-term Ritz approximation.

Solution 8.3. The elastic potential $\Pi(w,\psi_1,\psi_2)$ of a symmetric and special orthotropic Mindlin's plate is given by (8.3.18). For stiff thin cover sheets and a core which transmits only transverse shear stresses the bending and shear stiffness for isotropic face and core materials are (8.4.1)

$$D_{ij} = h^{\mathrm{c}}C_{ij}^{(f)} = h^{\mathrm{c}}\left[Q_{ij}^{\mathrm{f}}h^{(f)}\bar{x}_3^{(f)}\right] = h^{\mathrm{c}}h^{\mathrm{f}}\frac{1}{2}\left(h^{\mathrm{c}} + h^{\mathrm{f}}\right)Q_{ij}^{(f)},$$

$((ij) = (11),(22),(66),(12))$ with

$$Q_{11} = \frac{E^{\mathrm{f}}}{1-(\nu^{\mathrm{f}})^2} = Q_{22}, \quad Q_{12} = \frac{\nu^{\mathrm{f}}E^{\mathrm{f}}}{1-(\nu^{\mathrm{f}})^2}, \quad Q_{66} = G^{\mathrm{f}}$$

and

$$A_{ij}^{S} = h^{\mathrm{c}} C_{ij}^{\mathrm{c}} = h^{\mathrm{c}} G^{\mathrm{c}}, (ij) = (44), (55)$$

$$\begin{aligned}
\Pi(w,\psi_1,\psi_2) = \int_A \Bigg\{ & \frac{1}{2} h^{\mathrm{c}} h^{\mathrm{f}} (h^{\mathrm{c}} + h^{\mathrm{f}}) \left[Q_{11} \left(\frac{\partial \psi_1}{\partial x_1} \right)^2 + 2Q_{12} \left(\frac{\partial \psi_1}{\partial x_1} \frac{\partial \psi_2}{\partial x_2} \right) \right. \\
& \left. + Q_{22} \left(\frac{\partial \psi_2}{\partial x_2} \right)^2 + Q_{66} \left(\frac{\partial \psi_1}{\partial x_2} + \frac{\partial \psi_2}{\partial x_1} \right)^2 \right] \\
& + k^{\mathrm{s}} h^{\mathrm{c}} G^{\mathrm{c}} \left[\left(\psi_1 + \frac{\partial w}{\partial x_1} \right)^2 + \left(\psi_2 + \frac{\partial w}{\partial x_2} \right)^2 \right] \Bigg\} \mathrm{d}x_1 \mathrm{d}x_2 \\
& - \int p w \mathrm{d}x_1 \mathrm{d}x_2
\end{aligned}$$

The one-term approximations

$$\begin{aligned}
w(x_1,x_2) &= a_1 \sin\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right), \\
\psi_1(x_1,x_2) &= a_2 \cos\left(\frac{\pi x_1}{a}\right) \sin\left(\frac{\pi x_2}{a}\right), \\
\psi_2(x_1,x_2) &= a_3 \sin\left(\frac{\pi x_1}{a}\right) \cos\left(\frac{\pi x_2}{a}\right)
\end{aligned}$$

satisfy the boundary conditions. Substituting these approximative functions into Π follow $\tilde{\Pi} = \tilde{\Pi}(a_1,a_2,a_3)$ and the conditions for a minimum of Π, i.e. $\partial\tilde{\Pi}/\partial a_1 = 0$, $i = 1,2,3$ yield the equations for the undetermined coefficients a_i

$$\boldsymbol{Ka} = \boldsymbol{q}$$

with

$$\boldsymbol{a}^{\mathrm{T}} = [a_1 \ a_2 \ a_3], \quad \boldsymbol{q}^{\mathrm{T}} = [16p/\pi^2 \ 0 \ 0]$$

and

$$\boldsymbol{K} = \begin{bmatrix} 2h^{\mathrm{c}}G^{\mathrm{c}}\lambda^2 & h^{\mathrm{c}}G^{\mathrm{c}}\lambda & h^{\mathrm{c}}G^{\mathrm{c}}\lambda \\ h^{\mathrm{c}}G^{\mathrm{c}}\lambda & h^{\mathrm{c}}h^{\mathrm{f}}\bar{x}_3^{\mathrm{f}}(Q_{11}+Q_{66})\lambda^2 + h^{\mathrm{c}}G^{\mathrm{c}} & h^{\mathrm{c}}h^{\mathrm{f}}\bar{x}_3^{\mathrm{f}}(Q_{12}+Q_{66})\lambda^2 \\ h^{\mathrm{c}}G^{\mathrm{c}}\lambda & h^{\mathrm{c}}h^{\mathrm{f}}\bar{x}_3^{\mathrm{f}}(Q_{12}+Q_{66})\lambda^2 & h^{\mathrm{c}}h^{\mathrm{f}}\bar{x}_3^{\mathrm{f}}(Q_{22}+Q_{66})\lambda^2 + h^{\mathrm{c}}G^{\mathrm{c}} \end{bmatrix}$$

with $\lambda = \pi/a$. The solution of the system of three linear equations leads to $a_1 = 0{,}0222, a_2 = a_3 = -0{,}046$ and the maximum displacement follows to $w_{\max} = w(x_1 = a/2, x_2 = a/2) = a_1 = 2{,}22$ cm.

Exercise 8.4. A simply supported laminate plate $[0^0/90^0/0^0]$ has the following material properties: $E_m = 3.4$ GPa, $E_{\mathrm{f}} = 110$ GPa, $\nu_{\mathrm{m}} = 0.35, \nu_{\mathrm{f}} = 0.22$, $v_{\mathrm{m}} = 0.4$, $v_{\mathrm{f}} \equiv \phi = 0.6$, $G_{\mathrm{m}} = E_{\mathrm{m}}/2(1+\nu_{\mathrm{m}}) = 1.2593$ GPa, $G_{\mathrm{f}} = E_{\mathrm{f}}/2(1+\nu_{\mathrm{f}}) = 45.0820$ GPa, $h^{(1)} = h^{(2)} = h^{(3)} = 5$ mm, $a = b = 1$ m.

1. Formulate the equation for the bending surface for a lateral unit load $F = 1$ N at $x_1 = \xi_1, x_2 = \xi_2$ using the classical laminate theory.
2. Formulate the equation for the natural frequencies of the laminate plate using the classical plate theory and neglecting the rotatory inertia.

Solution 8.4. The solutions for both cases can be presented as follows.

1. The stacking sequence of the layers yields a symmetric cross-ply plate which is specially orthotropic (Table 8.1) $B_{ij} = 0, D_{16} = D_{26} = 0$

$$D_{11}\frac{\partial^4 w}{\partial x_1^4} + 2(D_{12} + 2D_{66})\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + D_{22}\frac{\partial^4 w}{\partial x_2^2} = p_3(x_1, x_2)$$

The boundary conditions are (Fig. 8.5)

$$w(0,x_2) = w(a,x_2) = w(x_1,0) = w(x_1,b) = 0,$$

$$M_1(0,x_2) = M_1(a,x_2) = M_2(x_1,0) = M_2(x_1,b) = 0$$

The Navier's double infinite series solution (8.6.21) - (8.6.23) leads to

$$w(x_1,x_2) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\frac{p_{rs}}{d_{rs}}\sin\alpha_r x_1 \sin\beta_s x_2$$

with

$$d_{rs} = [D_{11}\alpha_r^4 + 2(D_{12} + 2D_{66})\alpha_r^2\beta_s^2 + D_{22}\beta_s^4], \quad \alpha_r = \frac{r\pi}{a},\ \beta_s = \frac{s\pi}{b},$$
$$p_{rs} = \frac{4F}{ab}\sin\alpha_r\xi_1\sin\beta_s\xi_2$$

With (section 2.2.1)

$$E_1' = E_\mathrm{f}v_\mathrm{f} + E_\mathrm{m}v_\mathrm{m} = 67,36\ \mathrm{GPa}, \quad E_2' = \frac{E_\mathrm{f}E_\mathrm{m}}{E_\mathrm{f}v_\mathrm{m} + E_\mathrm{m}v_\mathrm{f}} = 8,12\ \mathrm{GPa},$$

$$G_{12}' = \frac{G_\mathrm{f}G_\mathrm{f}}{G_\mathrm{f}v_\mathrm{m} + G_\mathrm{m}v_\mathrm{f}} = 3,0217\ \mathrm{GPa}$$

$$\nu_{12}' = \nu_\mathrm{f}v_\mathrm{f} + \nu_\mathrm{m}v_\mathrm{m} = 0,272, \quad \nu_{21}' = \nu_{12}'E_2'/E_1' = 0,0328$$

and (4.1.3)

$$\begin{aligned}
Q_{11}' &= E_1'/(1 - \nu_{12}'\nu_{21}') = 67,97\ \mathrm{GPa},\\
Q_{22}' &= E_2'/(1 - \nu_{12}'\nu_{21}') = 8,194\ \mathrm{GPa},\\
Q_{12}' &= Q_{21}' = \nu_{12}'Q_{22}' = 2,229\ \mathrm{GPa},\\
Q_{66}' &= G_{12}' = 3,02\ \mathrm{GPa}
\end{aligned}$$

follow (4.2.15) the stiffness

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{3} Q_{ij}^{(k)} \left((x_3^{(k)})^3 - (x_3^{(k-1)})^3\right),$$

$$Q_{ij}^{(1)} = Q_{ij}^{(3)} = Q_{ij}^{[0^0]} = Q'_{ij}, \quad Q_{ij}^{(2)} = Q_{ij}^{[90^0]}, \quad Q_{11}^{(2)} = Q_{22}^{(1)},$$

$$Q_{22}^{(2)} = Q_{11}^{(1)}, \quad Q_{66}^{(2)} = Q_{66}^{(1)},$$

$$x_3^{(0)} = -7,5 \text{ mm}, \quad x_3^{(1)} = -2,5 \text{ mm},$$

$$x_3^{(2)} = 2,5 \text{ mm}, \quad x_3^{(3)} = 7,5 \text{ mm},$$

$$D_{11} = 18492 \text{ Nm}, D_{22} = 2927 \text{ Nm},$$

$$D_{12} = D_{21} = 627 \text{ Nm}, D_{66} = 849 \text{ Nm}$$

The equation for the bending surface is

$$w(x_1,x_2) = \frac{Fa^2}{\pi^4}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{\sin\alpha_r\xi_1 \sin\beta_s\xi_2}{18492r^4 + 4650r^2s^2 + 2927s^4} \sin\alpha_r x_1 \sin\beta_s x_2$$

If $F = 1$ N then $w(x_1,x_2)$ represents the influence surface, i.e. the deflection at (x_1,x_2) due to a unit load at (ξ_1,ξ_2). This influence function $w(x_1,x_2;\xi_1,\xi_2)$ is sometimes called Green's function of the plate with all boundaries simply supported. In the more general case of a rectangular plate $a \neq b$ the Green's function is

$$w(x_1,x_2;\xi_1,\xi_2) = \frac{F}{\pi^4 ab}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \frac{\sin\alpha_r\xi_1 \sin\beta_s\xi_2}{d_{rs}} \sin\alpha_r x_1 \sin\beta_s x_2$$

The Green's function can be used to calculate the bending surfaces of simply supported rectangular plates with any transverse loading. With the solution $w(x_1,x_2)$ we can calculate the stress resultants M_1,M_2,M_6,Q_1,Q_2 and the stresses $\sigma_1,\sigma_2,\sigma_6,\sigma_5$ and σ_4 using (8.6.27) and (8.6.28).

2. Using (8.6.41) the equation for the natural frequencies of a simply supported rectangular plate is

$$\omega_{rs}^2 = \frac{\pi^4}{\rho h}\left[D_{11}\alpha_r^4 + 2(D_{12}+2D_{66})\alpha_r^2\beta_s^2 + D_{22}\beta_s^4\right]$$

with

$$h = \sum_{(k)} h^{(k)}, \quad \rho = \frac{1}{h}\sum_{(k)} \rho^{(k)}\left(x_3^{(k)} - x_3^{(k-1)}\right)$$

The fundamental frequency corresponds to $r = s = 1$ and is given by

$$\omega_{11}^2 = \frac{\pi^4}{\rho h a^4}\left[D_{11} + 2(D_{12}+2D_{66})\left(\frac{a}{b}\right)^2 + D_{22}\left(\frac{a}{b}\right)^4\right]$$

For $a = b = 1$ m and the given material properties we find the fundamental natural frequency

$$\omega_{11} = \frac{1593,5}{\sqrt{\rho h}}$$

Exercise 8.5. Consider a cylindrically orthotropic circular plate with a midplane symmetric layer stacking under the conditions of axisymmetric loading and displacements.

1. Develop the differential equations for in-plane loading. Calculate the stress resultants for a solid disk $(R, h, E_r, E_\theta, \nu_{r\theta})$ loaded $\alpha)$ with a radial boundary force $N_r(R) = -N_{ro}$ and $\beta)$ with a body force $hp_r = h\rho\omega^2 r$ caused by spinning the disk about the axis with an angular velocity ω.
2. Develop the differential equations for transverse loading under the condition of the first order shear deformation theory. Calculate the stress resultants for a solid plate $(R, h, E_r, E_\theta, \nu_{r\theta})$ loaded by a uniform constant pressure $p_3(r) \equiv -p_0$ and $\alpha)$ clamped, respectively, $\beta)$ simply supported at the boundary $r = R$.

Solution 8.5. With Sect. 2.1.6 we obtain $x_1 = x_r, x_2 = \theta, x_3 = z$, $\sigma_1 = \sigma_r, \sigma_2 = \sigma_\theta$, $\sigma_6 = \sigma_{r\theta}$, $\varepsilon_1 = \varepsilon_r, \varepsilon_2 = \varepsilon_\theta, \varepsilon_6 = \varepsilon_{r\theta}$. For axisymmetric deformations of circular disks and plates all stresses, strains and displacements are independent of θ, i.e. they are functions of r alone and $\sigma_6 = 0, \varepsilon_6 = 0$.

1. For an in-plane loaded cylindrical orthotropic circular disk under the condition of axisymmetric deformations the equilibrium, constitutive and geometric equations are:

   - Equilibrium Equations (Fig. 8.7)
     With $\cos(\pi/2 - \mathrm{d}\theta/2) = \sin(\mathrm{d}\theta/2) \approx \mathrm{d}\theta/2$ follow

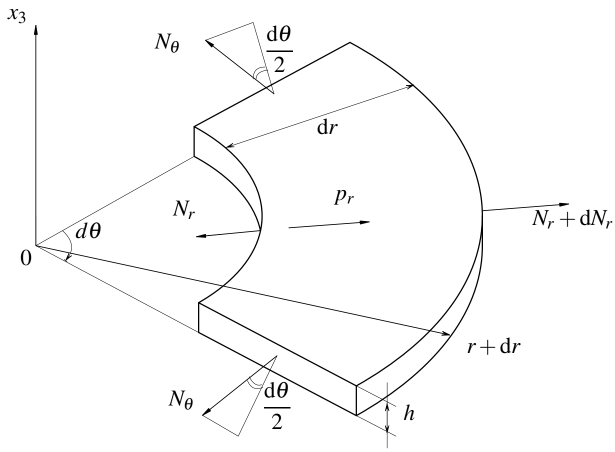


Fig. 8.7 Disc element $(r\mathrm{d}r\mathrm{d}\theta)h$

$$\frac{\mathrm{d}(rN_r)}{dr} - N_\theta + p_r r = 0$$

- Constitutive Equations

$$N_r = A_{11}\varepsilon_r + A_{12}\varepsilon_\theta, \quad N_\theta = A_{12}\varepsilon_r + A_{22}\varepsilon_\theta, \quad N_{r\theta} = 0$$

- Geometric Equations

$$\varepsilon_r = \frac{\mathrm{d}u}{\mathrm{d}r}, \quad \varepsilon_\theta = \frac{u}{r}, \quad \gamma_{r\theta} = 0$$

These equations forming the following system of three ordinary differential equations

$$\begin{aligned} &\frac{\mathrm{d}(rN_r)}{dr} - N_\theta = -p_0 r, \\ &N_r = A_{11}\frac{\mathrm{d}u}{\mathrm{d}r} + A_{12}\frac{u}{r}, \quad N_\theta = A_{12}\frac{\mathrm{d}u}{\mathrm{d}r} + A_{22}\frac{u}{r} \end{aligned}$$

involving three unknown quantities N_r, N_θ and u. Substituting the stress resultants in the equilibrium equations yield one uncoupled differential equation for $u(r)$

$$r\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{\mathrm{d}u}{\mathrm{d}r} - \frac{1}{r}\delta^2 u = -\frac{rp_r}{A_{11}}$$

with $\delta^2 = A_{22}/A_{11}$ or

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \frac{\delta^2}{r^2}u = -\frac{p_r}{A_{11}}$$

α) Radial boundary force

$$-\frac{p_r}{A_{11}} = 0, \quad N_r(R) = -N_{r0}, \quad R_1 = 0, \quad R_2 = R$$

The general solution of the differential equations follow with

$$u(r) = Cr^\lambda, \quad \lambda = \pm\delta$$

as

$$u(r) = C_1 r^{+\delta} + C_2 r^{-\delta}$$

With $R_1 = 0, R_2 = R$ we obtain $C_2 = 0, C_1 = -N_{r0}/[(A_{11}\delta + A_{12})R^{\delta-1}]$ and such

$$N_r(r) = -N_{r0}\left(\frac{r}{R}\right)^{\delta-1}, \quad N_\theta(r) = -N_{r0}\delta\left(\frac{r}{R}\right)^{\delta-1}$$

Conclusion 8.1. For $\delta = 1$ we have an isotropic disk with the well-known solution $N_r = N_\theta = -N_{r0}$. For $\delta > 0$, i.e. the circumferential stiffness exceeds the radial stiffness, at $r = 0$ we have $N_r = N_\theta = 0$, otherwise for $\delta < 0$, i.e. the radial stiffness exceeds the circumferential, at $r = 0$ we have infinitely high stress resultants or stresses, respectively.

β) Body force caused by rotation With $p_r = \rho\omega^2 r$ we obtain the solution of the inhomogeneous differential equations as

$$u(r) = C_1 r^\delta - \frac{1}{A_{11}} \frac{\rho\omega^2}{9-\delta^2} r^3 = C_1 r^\delta + \frac{1}{A_{11}} \frac{\rho\omega^2}{\delta^2 - 9} r^3$$

For $\delta = 1$ follow the well-known solution

$$u(r) = Cr - \frac{1-\nu^2}{E} \frac{\rho\omega^2}{8} r^3$$

2. With Fig. 8.8 we obtain:

- Equilibrium Equations

$$\frac{\mathrm{d}(rM_r)}{\mathrm{d}r} - M_\theta - rQ_r = 0, \quad \frac{\mathrm{d}(rQ_r)}{\mathrm{d}r} + rp_3 = 0$$

- Constitutive Equations

$$\begin{aligned} M_r &= D_{11}\kappa_r + D_{12}\kappa_\theta, \quad M_\theta = D_{12}\kappa_r + D_{22}\kappa_\theta, \\ Q_r &= k_{55}^{\mathrm{s}} A_{55} \left(\psi_r + \frac{\mathrm{d}w}{\mathrm{d}r} \right) \end{aligned}$$

- Geometric Equations

$$\kappa_r = \frac{\mathrm{d}\psi_r}{\mathrm{d}r}, \quad \kappa_\theta = \frac{\psi_r}{r}, \quad \gamma_{rz} = \left(\psi_r + \frac{\mathrm{d}w}{\mathrm{d}r} \right)$$

Integrating the second equilibrium equation

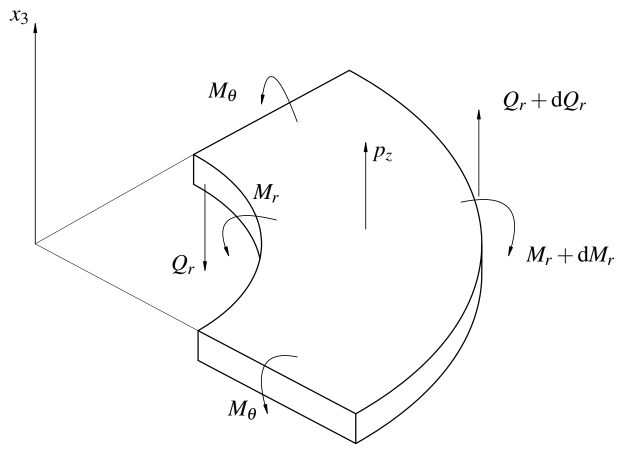


Fig. 8.8 Plate element $(r\mathrm{d}r\mathrm{d}\theta)h$

$$Q_r(r) = \frac{1}{r}\left(C_1 - \int p_3(r) r \mathrm{d}r\right)$$

and substituting M_r, M_θ and Q_r in the first equilibrium equation yield

$$r\frac{\mathrm{d}^2\psi_r}{\mathrm{d}r^2} + \frac{\mathrm{d}\psi_r}{\mathrm{d}r} - \frac{1}{r}\delta_p^2\psi_r = \frac{1}{D_{11}}\left(C_1 - \int p_3 r \mathrm{d}r\right), \quad \delta_p^2 = \frac{D_{22}}{D_{11}}$$

The general solution has again the form

$$\psi_r(r) = C_2 r^{\delta_p} + C_3 r^{-\delta_p} + \psi_p(r)$$

$\psi_p(r)$ is the particular solution of the inhomogeneous differential equation depending on the form of the loading functions $p_3(r)$. The differential equation for the plate deflection $w(r)$ follows with

$$\begin{aligned}
k^s A_{55}\frac{\mathrm{d}w}{\mathrm{d}r} &= Q_r - k^s A_{55}\psi_r,\\
\frac{\mathrm{d}w}{\mathrm{d}r} &= \frac{1}{k^s A_{55}}\left(C_1\frac{1}{r} - \frac{1}{r}\int p_3(r) r \mathrm{d}r\right) - C_2 r^{\delta_p} - C_3 r^{-\delta_p} + \psi_0,\\
w(r) &= \frac{1}{k^s A_{55}}\left(C_1 \ln r - \int \frac{1}{r}\int p_3(r) r \mathrm{d}r \mathrm{d}r\right)\\
&\quad - C_2\frac{r^{\delta_p+1}}{\delta_p+1} - C_3\frac{r^{-\delta_p+1}}{1-\delta_p} + C_4 - \int \psi_p \mathrm{d}r
\end{aligned}$$

For a constant pressure $p_3(r) = -p_0$ we obtain

$$\begin{aligned}
w(r) &= \frac{1}{k^s A_{55}}\left(C_1 \ln r + \frac{p_0 r^2}{4}\right) - \frac{C_2}{1+\delta_p} r^{1+\delta_p} - \frac{C_3}{1-\delta_p} r^{1-\delta_p}\\
&\quad + C_4 + \frac{C_1 r^2}{2D_{11}(\delta_p^2 - 1)} + \frac{p_0 r^4}{8D_{11}(\delta_p^2 - 9)},\\
\psi_r(r) &= C_2 r^{\delta_p} + C_3 r^{-\delta_p} - \frac{C_1 r}{D_{11}(\delta_p^2 - 1)} - \frac{p_0 r^3}{2D_{11}(\delta_p^2 - 9)}
\end{aligned}$$

This general solution is not valid for $\delta_p = 1$ and $\delta_p = 3$ because the particular solutions ψ_p for theses δ_p-values include terms coinciding with the fundamental solutions r and r^3. Therefore, the particular solutions must be determined in another form. For $\delta_p = 1$, i.e. for the isotropic case, one can use $\psi_p = Ar\ln r + Br^3$ and for $\delta_p = 3$ $\psi_p = Ar + Br^3$ and one obtains the general solutions

$$\begin{aligned}
&\delta_p = 1\\
w(r) &= \frac{1}{k^s A_{55}}\left(C_1 \ln r + \frac{p_0 r^2}{4}\right) - \frac{1}{2}C_2 r^2 - C_3 \ln r\\
&\quad + C_4 - \frac{C_1 r^4}{4D_{11}}\left(\ln r - \frac{1}{2}\right) + \frac{p_0 r^4}{64 D_{11}},\\
\psi_r(r) &= C_2 r + C_3\frac{1}{r} + C_1\frac{r\ln r}{2D_{11}} + \frac{p_0 r^3}{16 D_{11}}
\end{aligned}$$

$$
\begin{aligned}
&\delta_p = 3\\
&w(r) \;=\; \frac{1}{k^s A_{55}}\left(C_1 \ln r + \frac{p_0 r^2}{4}\right) - \frac{1}{4}C_2 r^4 + C_3 \frac{1}{2r^2}\\
&\qquad + C_4 - C_1 \frac{r^2}{16D_{11}} + \frac{p_0 r^4}{48D_{11}}\left(\ln r - \frac{1}{4}\right),\\
&\psi_r(r) = C_2 r^3 + C_3 \frac{1}{r^3} - C_1 \frac{r \ln r}{8D_{11}} + \frac{p_0}{12D_{11}} r^3 \ln r
\end{aligned}
$$

The constants C_1, C_2, C_3 and C_4 are determined from the boundary conditions at the inner and outer plate edge. For solid plates with $R_1 = 0, R_2 = R$ the constants C_1 and C_3 must be zero, otherwise ψ_r and w tend to infinity at the plate center. For $\delta \neq 3$ the general solution for solid plates is

$$
\begin{aligned}
w(r) \;&=\; \frac{p_0 r^2}{4}\left[\frac{1}{k_{55}^s A_{55}} + \frac{r^2}{2D_{11}(\delta_p^2 - 9)}\right] - \frac{C_2}{1+\delta_p} r^{1+\delta_p} + C_4,\\
\psi_r(r) &= C_2 r^{\delta_p} - \frac{p_0 r^3}{2D_{11}(\delta_p^2 - 9)}
\end{aligned}
$$

α) *Clamped solid circular plate* ($\delta_p \neq 3$)
The boundary conditions are $\psi_1(R) = 0, w(R) = 0$ yield the constants C_2 and C_4 and the solution as

$$
w(r) = -\frac{p_0}{4k^s A_{55}}(R^2 - r^2) + \frac{p_0}{2D_{11}(\delta_p^2 - 9)}\left[\frac{R^{3-\delta_p} r^{1+\delta_p}}{1+\delta_p} - \frac{r^4}{4} + \frac{R^4(\delta_p - 3)}{4(1+\delta_p)}\right]
$$

β) *Simply supported solid circular plate* ($\delta_p \neq 3$)
We take now the boundary conditions $w(R) = 0$ and $M_r(R) = 0$ and have the solution

$$
\begin{aligned}
w(r) &= -\frac{p_0}{4k^s A_{55}}(R^2 - r^2)^2\\
&\quad + \frac{p_0}{2D_{11}(\delta_p^2 - 9)}\left[\frac{(3D_{11} + D_{12})R^{3-\delta_p}}{(\delta_p D_{11} + D_{12})(1+\delta_p)}(r^{1+\delta_p} - R^{1+\delta_p}) + \frac{1}{4}(R^4 - r^4)\right]
\end{aligned}
$$

Note that if the transverse shear deformations are neglected we must put $k_{55}^s A_{55} \to \infty$. In the particular case $k_{55}^s A_{55} \to \infty$ and $\delta_p = 1$ follow the well-known solutions for the classical theory of isotropic plates, i.e.

$$
\begin{aligned}
\alpha) \quad w(r) &= -\frac{p_0}{64D}(R^2 - r^2)^2 = -\frac{p_0 R^4}{64D}\left[1 - \left(\frac{r}{R}\right)\right]^2,\\
\beta) \quad w(r) &= -\frac{p_0}{64D}(R^2 - r^2)\left(\frac{5+\nu}{1+\nu}R^2 - r^2\right)\\
&= -\frac{p_0 R^4}{64D}\left[1 - \left(\frac{r}{R}\right)\right]^2\left[\frac{5+\nu}{1+\nu} - \left(\frac{r}{R}\right)^2\right]
\end{aligned}
$$

Exercise 8.6. A rectangular uniformly loaded symmetric cross-ply plate, Fig. 8.9, is clamped at the edges $x_2 = \pm b$ and can be arbitrary supported at the edges $x_1 = \pm a$. The deflection $w(x_1,x_2)$ may be represented in separated-variables form $w(x_1,x_2) = w_{ij}(x_1,x_2) = f_i(x_1)g_j(x_2)$.

1. Formulate one-term approximate solutions using the Vlasov-Kantorovich method, (2.2.45) - (2.2.47), based on the variation of the potential energy $\Pi(w)$.
2. Demonstrate for the special case of a plate clamped at all edges the extended Kantorovich method using the Galerkin's equations.

Solution 8.6. The differential equation and the elastic potential energy can be formulated, Table 8.1 and Eq. (8.2.24),

$$D_1\frac{\partial^4 w}{\partial x_1^4} + 2D_3\frac{\partial^4 w}{\partial x_1^2 \partial x_2^2} + D_2\frac{\partial^4 w}{\partial x_2^4} = p_0,$$

with $D_1 = D_{11}, D_2 = D_{22}, D_3 = D_{12} + 2D_{66}, p_z = p_0$

$$\begin{aligned}\Pi(w) = \frac{1}{2}\int\limits_{-a}^{a}\int\limits_{-b}^{b}\Bigg[&D_{11}\left(\frac{\partial^2 w}{\partial x_1^2}\right)^2 + D_{22}\left(\frac{\partial^2 w}{\partial x_2^2}\right)^2 \\ &+ 2D_{12}\frac{\partial^2 w}{\partial x_1^2}\frac{\partial^2 w}{\partial x_2^2} + 4D_{66}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)^2 - 2p_0\Bigg]\mathrm{d}x_1\mathrm{d}x_2\end{aligned}$$

The one-term approximate solution $\tilde{w}(x_1,x_2) = w_{ij}(x_1,x_2) = f_i(x_1)g_j(x_2)$ has an unknown function $f_i(x_1)$ and a priori chosen trial function $g_j(x_2)$, which satisfy at least the geometric boundary conditions at $x_2 = \pm b$.

1. The variation $\delta\Pi$ of the elastic potential energy $\Pi(w)$ yields

$$\begin{aligned}\delta\Pi(w) = \frac{1}{2}\int\limits_{-a}^{a}\int\limits_{-b}^{b}\Bigg[&\left(D_{11}\frac{\partial^2 w}{\partial x_1^2} + D_{12}\frac{\partial^2 w}{\partial x_2^2}\right)\delta\left(\frac{\partial^2 w}{\partial x_1^2}\right) \\ &+ \left(D_{12}\frac{\partial^2 w}{\partial x_1^2} + D_{22}\frac{\partial^2 w}{\partial x_2^2}\right)\delta\left(\frac{\partial^2 w}{\partial x_2^2}\right)\end{aligned}$$

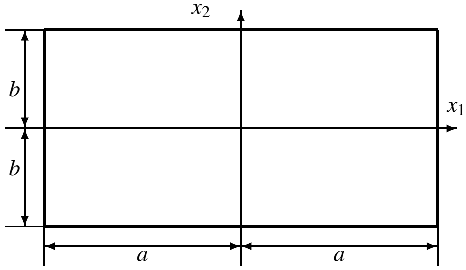


Fig. 8.9 Rectangular uniformly loaded plate, cross-ply symmetrically laminated, clamped at the longitudinal edges $x_2 = \pm b$ and arbitrary boundary conditions at the edges $x_1 = \pm a$

$$+\ 4D_{66}\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right)\delta\left(\frac{\partial^2 w}{\partial x_1 \partial x_2}\right) - p_0 \delta w\Bigg]\,\mathrm{d}x_1 \mathrm{d}x_2$$

Substituting $\tilde{w}(x_1,x_2) = f_i(x_1)g_j(x_2)$ one obtains

$$\delta \Pi(w) = \int\limits_{-a}^{a} \left[D_{11}Af_i''\delta f_i'' + D_{12}B(f_i''\delta f_i + f_i\delta f_i'') + D_{22}Cf_i\delta f_i\right.$$
$$\left. +\ 4D_{66}Df_i'\delta f_i' - \bar{p}_0\delta f_i\right]\mathrm{d}x_1$$

where

$$A = \int\limits_{-b}^{b} g_j^2 \mathrm{d}x_2, B = \int\limits_{-b}^{b} g_j'' g_j \mathrm{d}x_2, C = \int\limits_{-b}^{b} (g_j'')^2 \mathrm{d}x_2, D = \int\limits_{-b}^{b} (g_j')^2 \mathrm{d}x_2, \bar{p}_0 = \int\limits_{-b}^{b} p_0 g_j \mathrm{d}x_2$$

Integrating B by parts yield

$$B = g_j' g_j\Big|_{-b}^{b} - \int\limits_{-b}^{b} (g_j')^2 \mathrm{d}x_2 = -D$$

because $g_j(\pm b) = 0$ for plates with clamped or simply supported edges $x_2 = \pm b$. Now we integrated by parts the term

$$\int\limits_{-a}^{a} f_i''\delta f_i'' \mathrm{d}x_1$$

of $\delta \Pi$

$$\int\limits_{-a}^{a} f_i''\delta f_i'' \mathrm{d}x_1 = \int\limits_{-a}^{a} f_i''(\delta f_i)'' \mathrm{d}x_1 = f_i''\delta f_i'\Big|_{-a}^{a} - \int\limits_{-a}^{a} f_i'''(\delta f_i)' \mathrm{d}x_1$$
$$= f_i''\delta f_i'\Big|_{-a}^{a} - f_i'''\delta f_i\Big|_{-a}^{a} + \int\limits_{-a}^{a} f_i''''\delta f_i \mathrm{d}x_1$$

and the condition $\delta \Pi = 0$ yields the ordinary differential equations and the natural boundary conditions for $f_i(x_1)$

$$\begin{aligned}
&D_1Af_i''''(x_1) - 2D_3Df_i''(x_1) + D_2Cf_i(x_1) = \bar{p}_0,\\
&\text{at}\quad x_1 = \pm a: [D_{11}Af_i''(x_1) - D_{12}Df_i(x_1)]\delta f_i'(x_1) = 0,\\
&\qquad\qquad\qquad [D_{11}Af_i'''(x_1) - D_{12}Df_i'(x_1) + 4D_{66}Df_i'(x_1)]\delta f_i(x_1) = 0
\end{aligned}$$

If a plate edge is clamped, we have $f = 0, f' = 0$, if it is simply supported, we have $f = 0, f'' = 0$ and if it is free, we have

$$(D_{11}Af_i'' + D_{12}Bf_i) = 0, \quad (D_{11}Af_i''' + D_{12}Bf_i' - 4D_{66}Bf_i') = 0$$

The differential equation for $f(x_1)$ can be written in the form

$$f''''(x_1) - 2k_1^2 f''(x_1) + k_2^4 f(x_1) = k_p$$

with

$$k_1^2 = \frac{DD_3}{AD_1}, \quad k_2^4 = \frac{CD_2}{AD_1}, \quad k_p = \frac{\bar{p}_0}{AD_1}$$

The solutions of the differential equation are given in App. E in dependence on $k_2^2 < k_1^2$, $k_2^2 = k_1^2$ or $k_2^2 > k_1^2$ in the form

$$f(x_1) = \sum_{l=1}^{4} C_l \Phi_l(x_1) + f_p$$

with $f_p = \bar{p}_0/D_2C$. The solutions can be simplified if the problem is symmetric or antisymmetric. The constants C_l can be calculated with the boundary conditions at $x_1 = \pm a$.

2. In the special case of all plate edges are clamped the corresponding boundary conditions are

$$x_1 = \pm a : w = 0, \frac{\partial w}{\partial x_1} = 0, \quad x_2 = \pm b : w = 0, \frac{\partial w}{\partial x_2} = 0$$

The one-term deflection approximation is assumed again in the form $\tilde{w}(x_1,x_2) = w_{ij}(x_1,x_2) = f_i(x_1)g_j(x_2)$. The Galerkin's procedure yields

$$\int_{-a}^{a}\int_{-b}^{b} \left(D_1 \frac{\partial^4 w_{ij}}{\partial x_1^4} + 2D_3 \frac{\partial 4 w_{ij}}{\partial x_1^2 \partial x_2^2} + D_2 \frac{\partial 4 w_{ij}}{\partial x_2^4} - p_0 \right) g_j \mathrm{d}x_2 = 0$$

and we obtain

$$D_1 \left(\int_{-b}^{b} g_j^2 \mathrm{d}x_2 \right) \frac{\mathrm{d}^4 f_i}{\mathrm{d}x_1^4} + 2D_3 \left(\int_{-b}^{b} \frac{\mathrm{d}^2 g_j}{\mathrm{d}x_2^2} g_j \mathrm{d}x_2 \right) \frac{\mathrm{d}^2 f_i}{\mathrm{d}x_1^2} + \\ D_2 \left(\int_{-b}^{b} \frac{\mathrm{d}^4 g_j}{\mathrm{d}x_2^4} g_j \mathrm{d}x_2 \right) f_i = \int_{-b}^{b} p_0 g_j \mathrm{d}x_2$$

Two of the integral coefficients must be integrated by parts

$$\int_{-b}^{b} \frac{\mathrm{d}^2 g_j}{\mathrm{d}x_2^2} g_j \mathrm{d}x_2 = \frac{\mathrm{d}g_j}{\mathrm{d}x_2} g_j \Big|_{-b}^{b} - \int_{-b}^{b} \left(\frac{\mathrm{d}g_j}{\mathrm{d}x_2}\right) \mathrm{d}x_2,$$

$$\int_{-b}^{b} \frac{\mathrm{d}^4 g_j}{\mathrm{d}x_2^4} g_j \mathrm{d}x_2 = \frac{\mathrm{d}^3 g_j}{\mathrm{d}x_2^3} g_j \Big|_{-b}^{b} - \frac{\mathrm{d}^2 g_j}{\mathrm{d}x_2^2} \frac{\mathrm{d}g_j}{\mathrm{d}x_2} \Big|_{-b}^{b} + \int_{-b}^{b} \left(\frac{\mathrm{d}^2 g_j}{\mathrm{d}x_2^2}\right)^2 \mathrm{d}x_2$$

The results can be simplified because for the clamped edges follow

$$g_j(\pm b) = 0, \quad \frac{\mathrm{d}g_j}{\mathrm{d}x_2}\Big|_{\pm b} = 0$$

and we obtain the same differential equation as in 1.

$$D_1 \left(\int_{-b}^{b} g_j^2 \mathrm{d}x_2\right) \frac{\mathrm{d}^4 f_i}{\mathrm{d}x_1^4} - 2D_3 \left[\int_{-b}^{b} \left(\frac{\mathrm{d}g_j}{\mathrm{d}x_2}\right)^2 \mathrm{d}x_2\right] \frac{\mathrm{d}^2 f_i}{\mathrm{d}x_1^2} + D_2 \left(\int_{-b}^{b} \frac{\mathrm{d}^2 g_j}{\mathrm{d}x_2^2} \mathrm{d}x_2\right) f_i = \int_{-b}^{b} p_0 g_j \mathrm{d}x_2$$

To improve the one-term approximative plate solution we present in a second step now $f_i(x_1)$ a priori and obtain in a similar manner a differential equation for an unknown function $g_j(x_2)$

$$D_2 \left(\int_{-a}^{a} f_i^2 \mathrm{d}x_1\right) \frac{\mathrm{d}^4 g_j}{\mathrm{d}x_2^4} - 2D_3 \left[\int_{-a}^{a} \left(\frac{\mathrm{d}f_i}{\mathrm{d}x_1}\right)^2 \mathrm{d}x_2\right] \frac{\mathrm{d}^2 g_j}{\mathrm{d}x_2^2} + D_1 \left(\int_{-a}^{a} \left(\frac{\mathrm{d}^2 f_i}{\mathrm{d}x_2^2}\right)^2 \mathrm{d}x_1\right) g_j = \int_{-a}^{a} p_0 f_i \mathrm{d}x_1$$

In this way we have two ordinary differential equations of the iterative solution procedure which can be written

$$\begin{aligned} D_1 A_g \frac{\mathrm{d}^4 f_i}{\mathrm{d}x_1^4} - 2D_3 D_g \frac{\mathrm{d}^2 f_i}{\mathrm{d}x_1^2} + D_2 C_g f_i &= \bar{p}_{0g}, \\ D_2 A_\mathrm{f} \frac{\mathrm{d}^4 g_j}{\mathrm{d}x_2^4} - 2D_3 D_\mathrm{f} \frac{\mathrm{d}^2 g_j}{\mathrm{d}x_2^2} + D_1 C_\mathrm{f} g_j &= \bar{p}_{0f} \end{aligned}$$

Both equations can be rearranged in the standard form, App. E

$$\frac{\mathrm{d}^4 f_i}{\mathrm{d}x_1^4} - 2k_{1g}^2 \frac{\mathrm{d}^2 f_i}{\mathrm{d}x_1^2} + k_{2g}^4 f_i = k_{pg}, \quad \frac{\mathrm{d}^4 g_j}{\mathrm{d}x_2^4} - 2k_{1f}^2 \frac{\mathrm{d}^2 g_j}{\mathrm{d}x_2^2} + k_{2f}^4 g_j = k_{pf}$$

with

$$k_{1g}^2 = \frac{D_g D_3}{A_g D_1}, k_{2g}^4 = \frac{C_g D_2}{A_g D_1}, k_{pg} = \frac{\bar{p}_{0g}}{A_g D_1}, k_{1f}^2 = \frac{D_\mathrm{f} D_3}{A_\mathrm{f} D_1}, k_{2f}^4 = \frac{C_\mathrm{f} D_2}{A_\mathrm{f} D_1}, k_{pf} = \frac{\bar{p}_{0f}}{A_\mathrm{f} D_1}$$

The solutions of both equations are summarized in App. E and depend on the relation between k_{2g}^2 and k_{1g}^2 or k_{2f}^2 and k_{1f}^2, respectively.
The iterations start by choosing the first approximation as

$$w_{10}^{[1]} = f_1(x_1)g_1(x_2), \quad w_{21}^{[2]} = f_2(x_1)g_1(x_2), \quad w_{22}^{[3]} = f_2(x_1)g_2(x_2), \ldots$$

In the special case under consideration the first approximation is

$$w_{10}(x_1,x_2) = f_1(x_1)(x_2^2 - b^2)^2$$

and satisfy the boundary conditions $w = 0, \partial w/\partial x_2 = 0, x_2 = \pm b$. For a number of widely used composite material we have $k_2^2 > k_1^2$. Because the problem is symmetric we have then the simplified solution

$$f_1(x_1) = C_1 \cosh ax_1 \cos bx_1 + C_2 \sinh ax_1 \sin bx_1 + \frac{k_{pg}}{k_{2g}^4}$$

The constants C_1, C_2 can be calculated with

$$f_1(\pm a) = 0, \quad \left.\frac{\mathrm{d}f_1}{\mathrm{d}x_1}\right|_{\pm a} = 0$$

and $w_{10}^{[0]}(x_1,x_2)$ is determined. Now one can start the next step

$$w_{11}^{[1]}(x_1,x_2) = f_1(x_1)g_1(x_2)$$

with the function $f_1(x_1)$ as the a priori trial function. The iteration steps can be repeated until the convergence is satisfying. In the most engineering applications

$$w_{11}^{[0]}(x_1,x_2) = f_1(x_1)g_1(x_2)$$

can be used as satisfying closed analytical solution, i.e. $w_{11}^{[1]}(x_1,x_2)$ is suitable for engineering analysis of deflection and stresses in a clamped rectangular special orthotropic plate with uniform lateral load and different aspect ratios.

References

Nádai A (1925) Die elastischen Platten: die Grundlagen und Verfahren zur Berechnung ihrer Formänderungen und Spannungen, sowie die Anwendungen der Theorie der ebenen zweidimensionalen elastischen Systeme auf praktische Aufgaben. Springer, Berlin

Chapter 9
Modelling and Analysis of Circular Cylindrical Shells

In the previous Chaps. 7 and 8 we have considered beams and plates, i.e. one- and two-dimensional structural elements with straight axes and plane reference surfaces. Thin-walled laminated or sandwich shells can be also modelled as two-dimensional structural elements but with single or double curved reference surfaces. To cover shells of general shape a special book is necessary, because a general treatment of shells of any geometry demands a detailed application of differential geometry relations.

To give a brief insight into the modelling of shells only the simplest shell geometry will be selected and the following considerations are restricted to circular cylindrical shells. The modelling and analysis of circular cylindrical shells fabricated from fibre composite material, i.e. its structural theory, depends on the radius/thickness ratio R/h. For thin-walled shells, i.e. for $R/h \gg 1$ ($R/h > 10$), either the classical or the first order shear deformation shell theory is capable of accurately predicting the shell behavior. For thick-walled shells, say $R/h < 10$, a three-dimensional modelling must be used.

Each single lamina of a filamentary composite material behaves again macroscopically as if it were a homogeneous orthotropic material. If the material axes of all laminae are lined up with the shell-surface principal coordinates, i.e., the axial and circumferential directions, the shell is said to be special orthotropic or circumferential cross-ply circular cylindrical shell. Since the often used cylindrical shells with closely spaced ring and/or stringer stiffeners also can be approximated by considering them to be specially orthotropic, a greater number of analysis have been carried out for such shell type. If the material-symmetry axes are not lined up with the shell principal axes, the shell is said to be anisotropic, but since there is no structural advantage for shells constructed in this way it has been not often subjected to analysis.

In Chap. 9 there are only a short summarizing section on sandwich shells and no special section considering hygrothermo-elastic effects. Both problems can be simple retransmitted from the corresponding sections in Chaps. 7 and 8. Also a special discussion of analytical solution methods will be neglected, because no general shell problems are considered.

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0_9

9.1 Introduction

Chapter 9 gives a short introduction to the theory of circular cylindrical shells in the frame of the classical shell theory and the shell theory including transverse shear deformations. Figure 9.1 shows a laminated circular cylindrical shell with general layer stacking, the global coordinates $x_1 = x, x_2 = s = R\varphi, x_3 = z$, and the principal material coordinates $1 = x'_1, 2 = x'_2$. In the theory of circular cylindrical shells the most complex problem is the modelling and analysis of laminated shells with an arbitrary stacking of the layers and arbitrary loading. The at least complex problem is a mid-plane symmetric cross-ply laminated shell with axially symmetric loads using the classical shell theory. The mathematically modelling leads in this case to

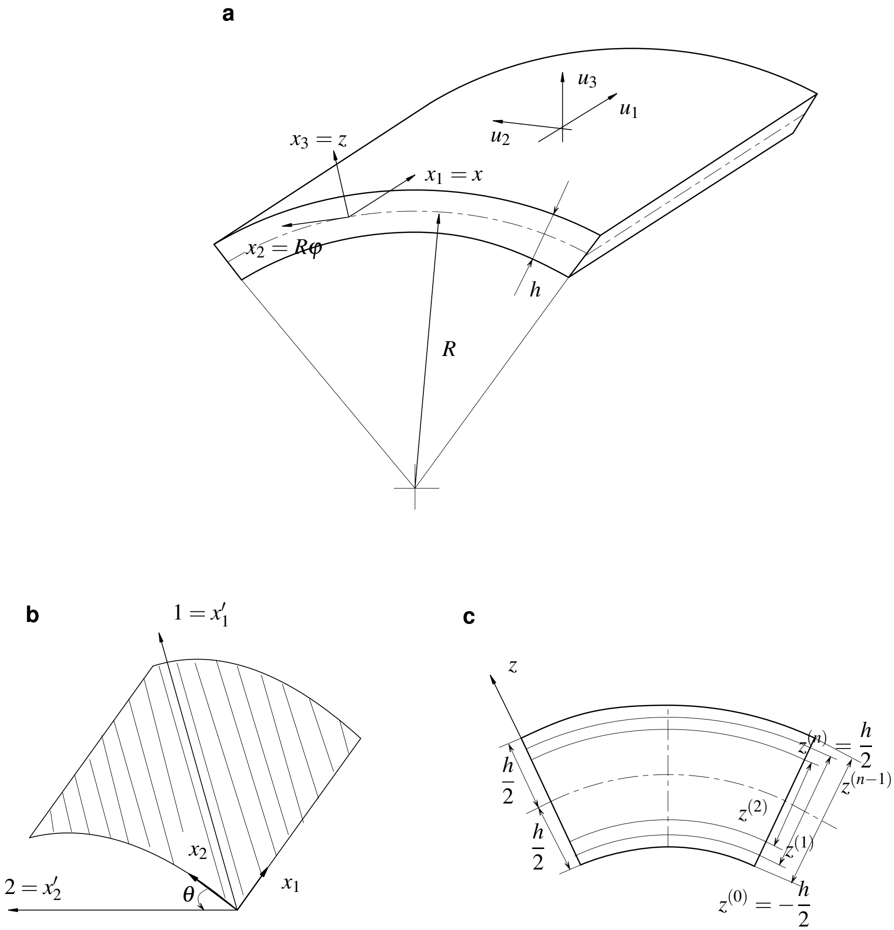


Fig. 9.1 Circular cylindrical shell. **a** Geometry, global coordinates $x_1 = x, x_2 = s = R\varphi$, **b** shell middle surface, principal material coordinates $x'_1 = 1, x'_2 = 2$, fibre angle θ, **c** laminate structure, n layers, layer coordinates $z^{(k)}$, layer thickness $h^{(k)} = z^{(k)} - z^{(k-1)}$

an ordinary differential equation. This type of stacking and loading will be primary considered in Chap. 9, because analytical solutions can be derived. Generally assumed is that each layer having a constant angle of wrap, constant volume ratio of fibre to resin, and the fibre and resin are both isotropic and homogeneous within themselves. The ply material axes, Fig. 9.1 b, will be rotated away from the global axes by an angle θ, positive in the counterclockwise direction.

9.2 Classical Shell Theory

The following hypotheses are the basis to derivative the equations of the classical shell theory:

- Displacements are small compared to the shell thickness, all strain-displacement relations may be assumed to be linear.
- The Kirchhoff hypothesis is applicable, i.e. line elements normal to the middle surface before deformation remain straight, normal to the deformed middle surface, and unchanged in length after deformation.
- All components of translational inertia are included in modelling vibration problems, but all components of rotatory inertia are neglected.
- The ratio of the shell thickness h to the radius R of the middle surface is small as compared with unity and Love's first-approximation shell theory is used which define a thin or classical shell theory: $h/R \ll 1$ and all terms $1+(z/R) \approx 1$. It can be shown that this relationship is consistent with the neglect of transverse shear deformation and transverse normal stress.

In addition we assume that each individual layer is considered to behave macroscopically as a homogeneous, anisotropic, linear-elastic material, that all layers are assumed to be bonded together with a perfect bond and that each layer may be of arbitrary thickness and may be arranged either symmetrically or unsymmetrically with respect to the middle surface.

9.2.1 General Case

The governing differential equations are formulated in terms of the three middle-surface displacement components ($u_1 \equiv u_x, u_2 \equiv u_s, u_3 \equiv u_z$)

$$u_1(x_1,x_2,0) = u(x,s), \quad u_2(x_1,x_2,0) = v(x,s), \quad u_3(x_1,x_2,0) = w(x,s) \qquad (9.2.1)$$

The strain displacement relations for a circular, cylindrical shell of any material, neglecting the effects of transverse shear deformation and using Love's first approximation are given by

$$\begin{aligned}
\varepsilon_x &= \frac{\partial u}{\partial x}, & \varepsilon_s &= \frac{\partial v}{\partial s} + \frac{w}{R}, & \varepsilon_{xs} &= \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x}, \\
\kappa_x &= -\frac{\partial^2 w}{\partial x^2}, & \kappa_s &= -\frac{\partial^2 w}{\partial s^2} + \frac{1}{R}\frac{\partial v}{\partial s}, & \kappa_{xs} &= -2\frac{\partial^2 w}{\partial x \partial s} + \frac{1}{R}\frac{\partial v}{\partial x}
\end{aligned} \tag{9.2.2}$$

The total strains at a arbitrary distance z of the middle surface are

$$\varepsilon_x = \varepsilon_x + \kappa_x z, \quad \varepsilon_s = \varepsilon_s + \kappa_s z, \quad \varepsilon_{xs} = \varepsilon_{xs} + \kappa_{xs} z$$

or

$$\varepsilon_j = \varepsilon_j + \kappa_j z, \quad j = (1,2,6) \equiv (x,s,xs) \tag{9.2.3}$$

Each individual layer is assumed to be in a state of generalized plane stress, the Hooke's law yields

$$\sigma_i^{(k)} = Q_{ij}^{(k)} \varepsilon_j, \quad i,j = (1,2,6) \tag{9.2.4}$$

and in the general anisotropic case the Q_{ij} matrix is full populated (Table 4.2).

Using again the Love's first approximation $1 + (z/R) \approx 1$), i.e. neglecting the difference in the areas above and below the middle surface $z = 0$, the force and moment resultants, Fig. 9.2, are defined analogous to plates

$$N_i = \int\limits_{-h/2}^{h/2} \sigma_i \mathrm{d}z, \quad M_i = \int\limits_{-h/2}^{h/2} \sigma_i z \mathrm{d}z, \quad i = (1,2,6) \equiv (x,s,xs) \tag{9.2.5}$$

Putting Eq. (9.2.4) into (9.2.5) yields the constitutive equations in the known form

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix} \tag{9.2.6}$$

with

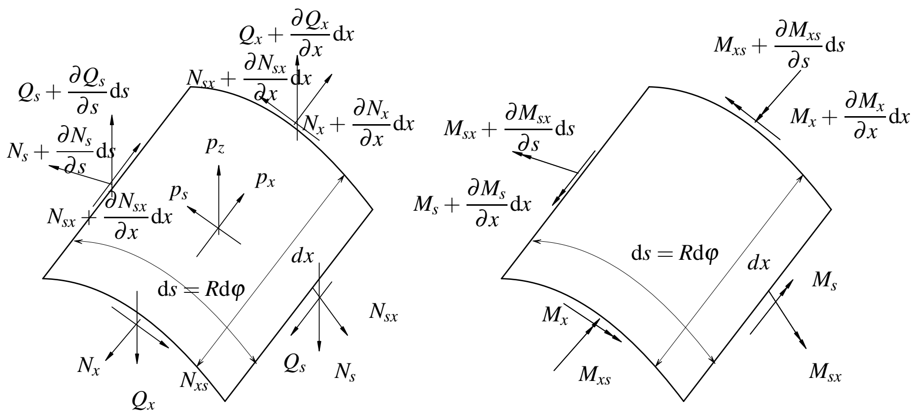


Fig. 9.2 Positive directions for stress resultants

$$(A_{ij}, B_{ij}, D_{ij}) = \int\limits_{-h/2}^{h/2} (1, z, z^2) Q_{ij} \mathrm{d}z$$

i.e. for n laminate layers

$$\begin{aligned}
A_{ij} &= \sum_{k=1}^{n} Q_{ij} \left(z^{(k)} - z^{(k-1)} \right), \\
B_{ij} &= \frac{1}{2} \sum_{k=1}^{n} Q_{ij} \left(z^{(k)^2} - z^{(k-1)^2} \right), \\
D_{ij} &= \frac{1}{3} \sum_{k=1}^{n} Q_{ij} \left(z^{(k)^3} - z^{(k-1)^3} \right)
\end{aligned}$$

$$\begin{aligned}
\boldsymbol{N}^{\mathrm{T}} &= [N_x \, N_s \, N_{xs}], \quad & \boldsymbol{M}^{\mathrm{T}} &= [M_x \, M_s \, M_{xs}], \\
\boldsymbol{\varepsilon}^{\mathrm{T}} &= [\varepsilon_x \, \varepsilon_s \, \varepsilon_{xs}], \quad & \boldsymbol{\kappa}^{\mathrm{T}} &= [\kappa_x \, \kappa_s \, \kappa_{xs}]
\end{aligned} \tag{9.2.7}$$

The equilibrium equations follow with Fig. 9.2 as

$$\begin{aligned}
&\frac{\partial N_x}{\partial x} + \frac{\partial N_{xs}}{\partial s} + p_x = 0, \quad & &\frac{\partial M_x}{\partial x} + \frac{\partial M_{xs}}{\partial s} - Q_x = 0, \\
&\frac{\partial N_{xs}}{\partial x} + \frac{\partial N_s}{\partial s} + \frac{Q_s}{R} + p_s = 0, \quad & &\frac{\partial M_{xs}}{\partial x} + \frac{\partial M_s}{\partial s} - Q_s = 0, \\
&\frac{\partial Q_x}{\partial x} + \frac{\partial Q_s}{\partial s} - \frac{N_s}{R} + p_z = 0 & &
\end{aligned} \tag{9.2.8}$$

The moment equations (9.2.8) can be used to eliminate the transverse shear resultants and one obtains

$$\begin{aligned}
&\frac{\partial N_x}{\partial x} + \frac{\partial N_{xs}}{\partial s} + p_x &&= 0, \\
&\frac{\partial N_{xs}}{\partial x} + \frac{\partial N_s}{\partial s} + \frac{1}{R} \left(\frac{\partial M_s}{\partial s} + \frac{\partial M_{xs}}{\partial x} \right) + p_s &&= 0, \\
&\frac{\partial^2 M_x}{\partial x^2} + 2 \frac{\partial^2 M_{xs}}{\partial x \partial s} + \frac{\partial^2 M_s}{\partial s^2} - \frac{N_s}{R} + p_z &&= 0
\end{aligned} \tag{9.2.9}$$

Substituting Eqs. (9.2.6) into (9.2.9) yields a set of three coupled partial differential equations for the three displacements u, v, w, which can be written in matrix form

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = - \begin{bmatrix} p_x \\ p_s \\ p_z \end{bmatrix} \tag{9.2.10}$$

The linear differential operators L_{ij} are defined in App. D. For symmetrically arranged layers the differential operators can be simplified, but the matrix (9.2.10) stay full populated (App. D).

If we consider natural vibrations of laminated circular cylindrical shells in Eqs. (9.2.9) and (9.2.10) the distributed loads p_x, p_s, p_z are taken zero, i.e.

$p_x = p_s = p_z = 0$, but all components of translatory inertia must be included. Without detailed derivation on obtains

$$\begin{aligned}
&\frac{\partial N_x}{\partial x} + \frac{\partial N_{xs}}{\partial s} &&= \rho_0 \frac{\partial^2 u}{\partial t^2},\\
&\frac{\partial N_{xs}}{\partial x} + \frac{\partial N_s}{\partial s} + \frac{1}{R}\left(\frac{\partial M_s}{\partial s} + \frac{\partial M_{xs}}{\partial x}\right) &&= \rho_0 \frac{\partial^2 v}{\partial t^2},\\
&\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xs}}{\partial x \partial s} + \frac{\partial^2 M_s}{\partial s^2} - \frac{N_s}{R} &&= \rho_0 \frac{\partial^2 w}{\partial t^2}
\end{aligned} \tag{9.2.11}$$

and Eq. (9.2.10) changes to

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \rho_0 \frac{\partial^2 w}{\partial t^2} \begin{bmatrix} u \\ v \\ w \end{bmatrix} \tag{9.2.12}$$

with

$$\rho_0 = \sum_{k=1}^{n} \int_{z^{(k-1)}}^{z^{(k)}} \rho_0^{(k)} \mathrm{d}z = \sum_{k=1}^{n} \rho_0^{(k)} h^{(k)}$$

The stress resultants and the displacement are now functions of x, s and t. $\rho_0^{(k)}$ is the mass density of the kth layer, ρ_0 the mass inertia with respect to the middle surface.

9.2.2 Specially Orthotropic Circular Cylindrical Shells Subjected by Axial Symmetric Loads

Now we consider cross-ply laminated circular cylindrical shells. The laminate stacking may be not middle-surface symmetric, but the fiber angles are $\theta = 0^0$ or $\theta = 90^0$ and the principal material axes $1' - 2' - 3$ coincide with the structural axes x, s, z, i.e. the stiffness $A_{16} = A_{26} = 0, D_{16} = D_{26} = 0$. In the case of axial symmetry loading and deformations there are both, all derivations $\partial/\partial s$ and v, N_{xs}, M_{xs} zero. For the loads per unit of the surface area are the following conditions valid

$$p_x = 0, \quad p_s = 0, \quad p_z = p_z(x)$$

The equilibrium equations (9.2.8) reduce to

$$\frac{\mathrm{d}N_x}{\mathrm{d}x} = 0, \quad \frac{\mathrm{d}Q_x}{\mathrm{d}x} - \frac{N_s}{R} + p_z = 0, \quad \frac{\mathrm{d}M_x}{\mathrm{d}x} - Q_x = 0$$

or eliminating Q_x

$$\frac{\mathrm{d}N_x}{\mathrm{d}x} = 0, \quad \frac{\mathrm{d}^2 M_x}{\mathrm{d}x^2} - \frac{N_s}{R} = -p_z \tag{9.2.13}$$

The strain-displacement relations follow from (9.2.2)

$$\varepsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x}, \quad \varepsilon_s = \frac{w}{R}, \quad \kappa_x = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}, \quad \varepsilon_{xs} = 0, \kappa_s = \kappa_{xs} = 0 \tag{9.2.14}$$

and the stresses from (9.2.4) and (9.2.14) with $Q_{16} = Q_{26} = Q_{66} = 0$

$$\begin{aligned}
&\sigma_x^{(k)} = Q_{11}^{(k)}(\varepsilon_x + z\kappa_x) + Q_{12}^{(k)}\varepsilon_s = Q_{11}^{(k)}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) + Q_{12}^{(k)}\frac{w}{R},\\
&\sigma_s^{(k)} = Q_{12}^{(k)}(\varepsilon_x + z\kappa_x) + Q_{22}^{(k)}\varepsilon_s = Q_{12}^{(k)}\left(\frac{\mathrm{d}u}{\mathrm{d}x} - z\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}\right) + Q_{22}^{(k)}\frac{w}{R},\\
&Q_{11}^{(k)} = \frac{E_x^{(k)}}{1 - \nu_{xs}^{(k)}\nu_{sx}^{(k)}}, \quad Q_{22}^{(k)} = \frac{E_s^{(k)}}{1 - \nu_{xs}^{(k)}\nu_{sx}^{(k)}}, \quad Q_{12}^{(k)} = \frac{\nu_{sx}^{(k)}E_x^{(k)}}{1 - \nu_{xs}^{(k)}\nu_{sx}^{(k)}},\\
&\frac{\nu_{xs}^{(k)}}{E_x^{(k)}} = \frac{\nu_{sx}^{(k)}}{E_s^{(k)}}
\end{aligned} \tag{9.2.15}$$

The constitutive equations (9.2.6) can be written as follow

$$\begin{aligned}
N_x &= A_{11}\varepsilon_x + A_{12}\varepsilon_s + B_{11}\kappa_x,\\
N_s &= A_{12}\varepsilon_x + A_{22}\varepsilon_s + B_{12}\kappa_x,\\
M_x &= B_{11}\varepsilon_x + B_{12}\varepsilon_s + D_{11}\kappa_x,\\
M_s &= B_{12}\varepsilon_x + B_{22}\varepsilon_s + D_{12}\kappa_x,
\end{aligned} \tag{9.2.16}$$

with $\varepsilon_x, \varepsilon_s$ and κ_x from Eq. (9.2.14).

Putting (9.2.12) and (9.2.14) in the equilibrium equations (9.2.13) one obtains after a rearrangement

$$\begin{aligned}
A_{11}\frac{\mathrm{d}^2 u}{\mathrm{d}x^2} + \frac{A_{12}}{R}\frac{\mathrm{d}w}{\mathrm{d}x} - B_{11}\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} &= 0,\\
\left[\frac{A_{11}D_{11} - B_{11}^2}{A_{11}}\right]\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + \frac{2}{R}\left[\frac{A_{12}B_{11}}{A_{11}} - B_{12}\right]\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}&\\
+\frac{1}{R^2}\left[\frac{A_{11}A_{22} - A_{11}^2}{A_{11}}\right] w &= p_z - \frac{A_{12}}{A_{11}}\frac{N_x}{R},
\end{aligned}$$

and with

$$D^R = \frac{A_{11}D_{11} - B_{11}^2}{A_{11}}, \quad 4\lambda^4 = \frac{1}{D^R R^2}\frac{A_{11}A_{22} - A_{11}^2}{A_{11}} \tag{9.2.17}$$

can finally be written

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + \frac{2}{RD^R}\left[\frac{A_{12}B_{11}}{A_{11}} - B_{12}\right]\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + 4\lambda^4 w = \frac{1}{D^R}\left(p_z - \frac{A_{12}}{A_{11}}\frac{N_x}{R}\right) \tag{9.2.18}$$

This is a ordinary differential equation of fourth order with constant coefficients and can be solved by standard methods .

For the most important case of a symmetrical layer stacking Eq. (9.2.18) can be reduced with $B_{11} = B_{12} = 0, D^R = D_{11}$ as

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + 4\lambda^4 w = \frac{1}{D_{11}}\left(p_z - \frac{A_{12}}{A_{11}}\frac{N_x}{R}\right), \quad 4\lambda^4 = \frac{1}{D_{11}R^2}\frac{A_{11}A_{22} - A_{11}^2}{A_{11}} \tag{9.2.19}$$

The inhomogeneous linear differential equation of fourth order has constant coefficient and can be analytically solved (App. E)

$$w(x) = w_h(x) + w_p(x)$$

The homogeneous solution $w_h(x) = C\mathrm{e}^{\alpha x}$ yields the characteristic equation

$$\alpha^4 + 4\lambda^4 = 0$$

with the conjugate complex roots

$$\alpha_{1-4} = \pm\lambda(1 \pm \mathrm{i}), \quad \mathrm{i} = \sqrt{-1} \tag{9.2.20}$$

and with $\mathrm{e}^{\pm\lambda x} = \cosh\lambda x \pm \sinh\lambda x$, $\mathrm{e}^{\pm\mathrm{i}\lambda x} = \cos\lambda x \pm \mathrm{i}\sin\lambda x$ one obtains the solution of the homogeneous differential equation as

$$\begin{aligned} w_h(x) &= C_1 \cosh\lambda x \cos\lambda x + C_2 \cosh\lambda x \sin\lambda x \\ &+ C_3 \sinh\lambda x \cos\lambda x + C_4 \sinh\lambda x \sin\lambda x \end{aligned} \tag{9.2.21}$$

or

$$w_h(x) = \mathrm{e}^{-\lambda x}(C_1 \cos\lambda x + C_2 \sin\lambda x) + \mathrm{e}^{\lambda x}(C_3 \cos\lambda x + C_4 \sin\lambda x) \tag{9.2.22}$$

The particular solution $w_p(x)$ of the inhomogeneous equation depends on the loading term.

In solving (9.2.19), another solution form may be utilized, the so-called bending-layer solution. Note the Eqs. (9.2.16) for the symmetrical case, i.e.

$$M_x = -D_{11}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}, \quad Q_x = \frac{\mathrm{d}M_x}{\mathrm{d}x} = -D_{11}\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}$$

the solution can be written as:

$$\begin{aligned} w(x) = &\frac{M_0}{2\lambda^2 D_{11}}\mathrm{e}^{-\lambda x}(\sin\lambda x - \cos\lambda x) - \frac{Q_0}{2\lambda^3 D_{11}}\mathrm{e}^{-\lambda x}\cos\lambda x \\ &+ \frac{M_L}{2\lambda^2 D_{11}}\mathrm{e}^{-\lambda(L-x)}(\sin\lambda(L-x) - \cos\lambda(L-x)) \\ &+ \frac{Q_L}{2\lambda^3 D_{11}}\mathrm{e}^{-\lambda(L-x)}\cos\lambda(L-x) + w_p(x) \end{aligned} \tag{9.2.23}$$

Instead of the general constants $C_i, i = 1,2,3,4$ the resultant stress moments M_0, M_L and resultant stress forces Q_0, Q_L at $x = 0$ respectively $x = L$ are used as integration constants. To determine the $w_p(x)$ solution one have to consider N_x in Eq. (9.2.19) as a constant value following by boundary condition and $p(x)$ have to be restricted to cases where $\mathrm{d}^4 p(x)/\mathrm{d}x^4 = 0$, what is almost true from view point of practical applications. It can be easy seen that

$$w_p(x) = \frac{1}{4\lambda^2 D_{11}} \left[p(x) - \frac{A_{12}}{A_{11}} \frac{N_x}{R} \right] \tag{9.2.24}$$

is a solution of the inhomogeneous differential equation (9.2.19).

The advantage of the solution from (9.2.23) is easily seen. The trigonometric terms oscillate between ± 1 and are multiplied by exponential terms with a negative exponent which yields to an exponential decay. If we set $\lambda x = 1.5\pi$ or $\lambda(L-x) = 1.5\pi$ then is $\mathrm{e}^{-1.5\pi} \approx 0.009$, i.e. the influence of the boundary values M_0, Q_0 or M_L, Q_L is strong damped to $< 1\%$. With $0 \leq x \leq 1.5\pi/\lambda$ or $0 \leq L - x \leq 1.5\pi/\lambda$ bending boundary layers are defined which depend on the shell stiffness.

The important point is that at each end of the shell a characteristic length L_B can be calculated and the M_0- and Q_0-terms approach zero at the distance $x > L_B$ from $x = 0$ while the M_L- and Q_L-terms approach zero at the same distance L_B from $x = L$. In the boundary layer region bending stresses induced from M_0, Q_0 a M_L, Q_L are superimposed to membrane stresses induced from p_z. Looking at a long shell, Fig. 9.3, with $L > L_B$ in the region A-B only M_0, Q_0 and w_p are non-zero, in the region C-D only M_L, Q_L and w_p and in the region B-C only the particular solution w_p is nonzero, i.e. in this region only a membrane solution exists. With the calculated $w(x)$ the first differential equation (9.2.17) with $B_{11} = 0$ can be solved and yields the displacement function $u(x)$. It should be noted that only some terms of $u(x)$ decay away from the boundary edges.

In the case of axially symmetric loading and deformation the bending stresses in each lamina are given by

$$\begin{bmatrix} \sigma_x \\ \sigma_s \end{bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix}^{(k)} \begin{bmatrix} \varepsilon_x \\ \varepsilon_s \end{bmatrix} + z \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix}^{(k)} \begin{bmatrix} \kappa_x \\ 0 \end{bmatrix} \tag{9.2.25}$$

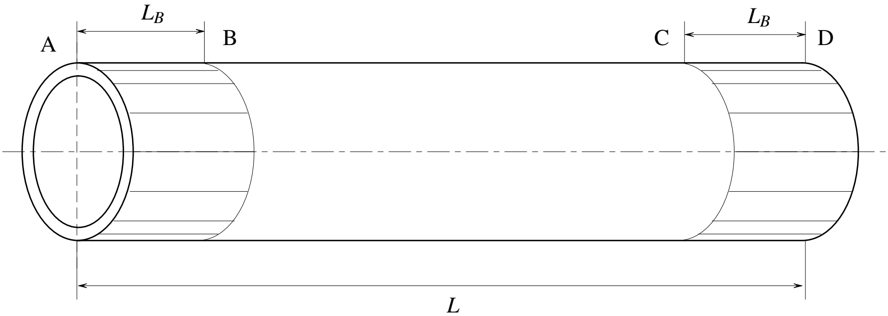


Fig. 9.3 Long circular cylindrical shell: Bending boundary regions (A-B) and (C-D), membrane region (B-C)

The transverse shear stress σ_{xz} follows analogous to the classical beam equations with (7.2.31).

Summarizing the results of the classical shell equations one can draw the following conclusions:

- The most general case of laminated circular cylindrical shells is that of arbitrarily laminated anisotropic layers, i.e. angle-ply layers arbitrarily arranged. The analysis of these shells is based on approximately analytical methods using Ritz-, Galerkin- or Kantorovich method and numerical methods , e.g. FEM.
- Cross-ply laminated shells, i.e shells with orthotropic layers aligned either axially or circumferentially and arranged symmetrically with respect to the shell middle surface have governing shell equations which are the same as those for a single-layer specially orthotropic shell. For axis symmetrical loading the shell equations reduce in the static case to ordinary differential equations of the x-coordinate and can be solved analytically. If the orthotropic layers are arranged to an unsymmetric laminated cross-ply shell then bending-stretching, coupling is induced and the governing equations are more complex.
- When circular cylindrical shells are laminated of more than one isotropic layer with each layer having different elastic properties and thickness and the layers are arranged symmetrically with respect to the middle surface, the governing equations are identical to those of single layer isotropic shells. However, if the isotropic layers are arranged unsymmetrically to the middle surface, there is a coupling between in-surface, i.e stretching and shear, and out-of-surface, i.e bending and twisting, effects.
- Additional to the Kirchhoff's hypotheses all equations of the classical shell theory assumed Love's first approximation, i.e. the ratio h/R is so small compared to 1 that the difference in the areas of shell wall element above and below the middle surface can be neglected.

9.2.3 Membrane and Semi-Membrane Theories

Thin-walled singe layer shells of revolution can be analyzed in the frame of the so-called membrane theory. One neglects all moments and transverse stress resultants, all stresses are considered approximatively constant through the shell thickness i.e. there are no bending stresses and the coupling and bending stiffness are taken to be zero in the constitutive equations. In some cases it is possible to use the membrane theory for structural analysis of laminated shells. The efficient structural behavior of shells based on the shell curvature that yields in wide regions of shells of revolution approximately a membrane response upon loading as the basic state of stresses and strains. The membrane theory is not capable to predict sufficient accurate results in regions with concentrated loads, boundary constraints or curvature changes, i.e. in regions located adjacent to each structural, material or load discontinuity. Restricting the consideration again to circular cylindrical shells with unsymmetric cross-ply stacking we arrive the following equations

$$
\begin{aligned}
&M_x = M_s = M_{xs} = 0, \quad Q_x = Q_s = 0,\\
&\frac{\partial N_x}{\partial s} + \frac{\partial N_{xs}}{\partial s} = -p_x, \quad \frac{\partial N_{xs}}{\partial s} + \frac{\partial N_s}{\partial s} = -p_s, \quad N_s = Rp_z,\\
&\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_s = \frac{\partial v}{\partial s} + \frac{w}{R}, \quad \varepsilon_{xs} = \frac{\partial u}{\partial s} + \frac{\partial v}{\partial x} = \gamma_{xs},\\
&N_x = A_{11}\varepsilon_x + A_{12}\varepsilon_s, \quad N_s = A_{12}\varepsilon_x + A_{22}\varepsilon_s, \quad N_{xs} = A_{66}\varepsilon_{xs}
\end{aligned}
\tag{9.2.26}
$$

The membrane theory yield three equilibrium conditions to calculate three unknown stress resultants, i.e the membrane theory is statically determined. The membrane theory is the simplest approach in shell analysis and admit an approximative analytical solution that is very convenient for a first analysis and design of circular cylindrical shells.

But the problems which can be solved by the membrane theory are unfortunately limited. To avoid generally to use the more complex bending theory we can consider a so-called semi-membrane theory of circular cylindrical shells. The semi-membrane theory is slightly more complicated than the membrane theory but more simpler than the bending theory. The semi-membrane theory was first developed by Vlasov on the basis of statically and kinematically hypotheses.

If one intends to construct a semi-membrane theory of composite circular cylindrical shells bearing in mind the hypotheses underlying the classical single layer shell theory and the characteristics of the composite structure. The semi-membrane theory for composite circular cylindrical shells introduces the following assumptions:

- The shell wall has no stiffness when bended but in axial direction and when twisted, i.e. $D_{11} = D_{66} = 0, B_{11} = B_{66} = 0$.
- The Poisson's effect is neglected, i.e. $A_{12} = 0, B_{12} = 0, D_{12} = 0$.
- With the assumptions above follow $M_x = M_{xs} = 0, Q_x = 0$.
- The cross-section contour is inextensible, i.e.

$$\varepsilon_s = \frac{\partial v}{\partial s} + \frac{w}{R} = 0$$

  The shear stiffness of composite shells can be small. Therefore, the assumption of the classical single layer semi-membrane theory that the shear stiffness is infinitely large, is not used.

Taking into account the assumptions above, one obtains the following set of eleven equations for eleven unknown functions.

$$
\begin{gathered}
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xs}}{\partial s} = 0, \quad \frac{\partial N_{xs}}{\partial x} + \frac{\partial N_s}{\partial s} + \frac{Q_s}{R} + p_s = 0,\\
\frac{\partial M_s}{\partial s} - Q_s = 0, \quad \frac{\partial Q_s}{\partial s} - \frac{N_s}{R} - p_z = 0,\\
\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_{xs} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial s}, \quad \varepsilon_s = \frac{\partial v}{\partial s} + \frac{w}{R},
\end{gathered}
\tag{9.2.27}
$$

$$N_x = A_{11}\varepsilon_x, \quad N_{xs} = A_{66}\varepsilon_{xs}, \quad M_s = D_{22}\kappa_s, \quad Q_s = k^{\mathrm{s}}_{44}A_{44}\left(\psi_s + \frac{\partial w}{\partial s}\right)$$

The system (9.2.27) can be reduced. For the circular cylindrical shell the unknown functions and loads can be represented with trigonometric series and after some manipulations we obtained one uncoupled ordinary differential equation of fourth order for $w_n(x), n = 0,1,2,\ldots$. The detailed derivation of the governing solutions shall not be considered.

9.3 Shear Deformation Theory

Analogous to plates, considered in Chap. 8, the classical shell theory is only sufficiently accurate for thin shells. For moderately thick shells we have to take, at least approximately, the transverse shear deformation effects into account. The Kirchhoff's hypotheses are again relaxed in one point: the transverse normals do not remain perpendicular to the middle-surface after deformation, but a line element through the shell thickness perpendicular to the middle-surface prior loading, undergoes at most a translation and rotation upon the load applications, no stretching or curvature.

The following considerations are restricted to axial symmetrical problems of symmetrical laminated cross-ply circular cylindrical shells including transverse shear deformation. We start with a variational formulation including the trapeze effect, i.e. Love's first approximation is not valid. For axial symmetrical problems we have the following simplifications of the shell equations:
All derivatives $\partial/\partial s(\ldots)$ are zero and for the strains, stress resultants and loads we assume

$$\begin{array}{l}\varepsilon_{xs} = 0, \ \kappa_s = 0, \quad \kappa_{xs} = 0,\\ N_{xs} = 0, \ M_{xs} = 0,\\ p_s = 0, \ \ p_x = 0, \quad p_z = p_z(x)\end{array} \tag{9.3.1}$$

The kinematical assumptions yield with (5.1.2) the shell displacements

$$\begin{aligned} u_x(x,z) &= u(x) + z\psi_x(x),\\ u_s(x,z) &= 0,\\ u_z(x,z) &= w(x)\end{aligned} \tag{9.3.2}$$

The strain-displacement relations are

$$\begin{aligned} \varepsilon_x &= \frac{\mathrm{d}u}{\mathrm{d}x} + z\frac{\mathrm{d}\psi_x}{\mathrm{d}x},\\ \varepsilon_s &= \frac{w}{R+z},\\ \varepsilon_{xz} &= \psi_x + \frac{\mathrm{d}w}{\mathrm{d}x}\end{aligned} \tag{9.3.3}$$

and the stresses in the kth layer of the shell are

$$\begin{bmatrix} \sigma_x \\ \sigma_s \\ \sigma_{xz} \end{bmatrix}^{(k)} = \begin{bmatrix} Q_{11} & Q_{12} & 0 \\ Q_{12} & Q_{22} & 0 \\ 0 & 0 & Q_{55} \end{bmatrix}^{(k)} \begin{bmatrix} \dfrac{\mathrm{d}u}{\mathrm{d}x} + z\dfrac{\mathrm{d}\psi_x}{\mathrm{d}x} \\ \dfrac{w}{r+z} \\ \psi_x + \dfrac{\mathrm{d}w}{\mathrm{d}x} \end{bmatrix} \tag{9.3.4}$$

For a special orthotropic shell the Q_{ij} are

$$\begin{aligned} Q_{11} &= \frac{E_x}{1-\nu_{xs}\nu_{sx}}, \quad Q_{22} = \frac{E_s}{1-\nu_{xs}\nu_{sx}}, \\ Q_{12} &= \frac{\nu_{xs}E_x}{1-\nu_{xs}\nu_{sx}}, \quad Q_{55} = G_{xz} \end{aligned} \tag{9.3.5}$$

The stress resultant forces and couples are defined as

$$\begin{aligned} N_x &= \sum_{k=1}^{n} \int_{h^{(k)}} \sigma_x^{(k)} \left(1+\frac{z}{R}\right) \mathrm{d}z, \quad N_s = \sum_{k=1}^{n} \int_{h^{(k)}} \sigma_s^{(k)} \mathrm{d}z, \\ M_x &= \sum_{k=1}^{n} \int_{h^{(k)}} \sigma_x^{(k)} z \left(1+\frac{z}{R}\right) \mathrm{d}z, \quad M_s = \sum_{k=1}^{n} \int_{h^{(k)}} \sigma_s^{(k)} z\mathrm{d}z, \\ Q_x &= \sum_{k=1}^{n} \int_{h^{(k)}} \sigma_{xz}^{(k)} \left(1+\frac{z}{R}\right) \mathrm{d}z, \end{aligned} \tag{9.3.6}$$

and one obtains with (9.3.4) and (9.3.6)

$$\begin{aligned} N_x &= A_{11}\frac{\mathrm{d}u}{\mathrm{d}x} + A_{12}\frac{w}{R} + \frac{1}{R}D_{11}\frac{\mathrm{d}\psi_x}{\mathrm{d}x}, \\ N_s &= A_{12}\frac{\mathrm{d}u}{\mathrm{d}x} + T_{22}w, \\ M_x &= D_{11}\left(\frac{1}{R}\frac{\mathrm{d}u}{\mathrm{d}x} + \frac{\mathrm{d}\psi_x}{\mathrm{d}x}\right), \\ M_s &= D_{12}\frac{\mathrm{d}\psi_x}{\mathrm{d}x} + \tilde{T}_{22}w, \\ Q_x &= A_{55}\left(\psi_x + \frac{\mathrm{d}w}{\mathrm{d}x}\right) \quad \text{or} \quad Q_x = k_{55}^{\mathrm{s}}A_{55}\left(\psi_x + \frac{\mathrm{d}w}{\mathrm{d}x}\right) \end{aligned} \tag{9.3.7}$$

The stiffness coefficients are

$$
\begin{aligned}
(A_{ij},D_{ij}) &= \sum_{k=1}^{n}\int\limits_{h^{(k)}} Q_{ij}^{(k)}(1,z^2)\mathrm{d}z, \quad (ij)=(11),(12),\\
T_{22} &= \sum_{k=1}^{n}\int\limits_{h^{(k)}} Q_{22}^{(k)}\frac{\mathrm{d}z}{R+z} = \sum_{k=1}^{n} Q_{22}^{(k)}\left[\ln\left(1+\frac{z}{R}\right)\right]_{-h/2}^{h/2}\\
&\approx \frac{1}{R}A_{22}+\frac{1}{R^3}D_{22},\\
\tilde{T}_{22} &= \sum_{k=1}^{n}\int\limits_{h^{(k)}} Q_{22}^{(k)}\frac{z\mathrm{d}z}{R+z} = \sum_{k=1}^{n} Q_{22}^{(k)}R\left[\frac{z}{R}-\ln\left(1+\frac{z}{R}\right)\right]_{-h/2}^{h/2}\\
&\approx \frac{1}{R^2}D_{22}
\end{aligned}
\tag{9.3.8}
$$

and k_{55}^{s} is the shear correction factor.

The variational formulation for the axial symmetrically circular cylindrical shell with symmetrically laminated $\theta = 0^0$ and $\theta = 90^0$ laminae and coincided principal material and structural axes is given as

$$
\begin{aligned}
\Pi(u,w,\psi_x) &= \Pi_{\mathrm{i}} - \Pi_{\mathrm{a}},\\
\Pi_{\mathrm{i}}(u,w,\psi_x) &= \frac{1}{2}\int\limits_{-h/2}^{h/2}\int\limits_{0}^{\pi}\int\limits_{0}^{L}(\sigma_x\varepsilon_x+\sigma_s\varepsilon_s+\sigma_{xz}\varepsilon_{xz})\mathrm{d}x(R+z)\mathrm{d}\varphi\mathrm{d}z\\
&= \frac{1}{2}\sum_{k=1}^{n}\int\limits_{h^{(k)}}\int\limits_{0}^{\pi}\int\limits_{0}^{L}\left(Q_{11}^{(k)}\varepsilon_x^2\right.\\
&\quad \left. + \; 2Q_{12}^{(k)}\varepsilon_x\varepsilon_s + Q_{22}^{(k)}\varepsilon_s^2 + Q_{55}^{(k)}\varepsilon_{xz}^2\right)(R+z)\mathrm{d}x\mathrm{d}\varphi\mathrm{d}z
\end{aligned}
\tag{9.3.9}
$$

Using Eq. (9.3.3) one obtains

$$
\begin{aligned}
\Pi_{\mathrm{i}} &= \frac{1}{2}\int\limits_{0}^{2\pi}\int\limits_{0}^{L}\left\{R\left[A_{11}\left(\frac{\mathrm{d}u}{\mathrm{d}x}\right)^2 + D_{11}\left(\frac{\mathrm{d}\psi_x}{\mathrm{d}x}\right)^2 + 2D_{11}\frac{\mathrm{d}u}{\mathrm{d}x}\frac{\mathrm{d}\psi_x}{\mathrm{d}x}\right.\right.\\
&\quad \left.\left. + \; 2A_{12}w\frac{\mathrm{d}u}{\mathrm{d}x} + T_{22}w^2 + Rk_{55}^{\mathrm{s}}A_{55}\left[\psi_x^2 + 2\psi_x\frac{\mathrm{d}w}{\mathrm{d}x} + \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2\right]\right]\right\}\mathrm{d}x\mathrm{d}\varphi, \qquad (9.3.10)\\
\Pi_{\mathrm{a}} &= \int\limits_{0}^{2\pi}\int\limits_{0}^{L} p_z(x)wR\mathrm{d}x\mathrm{d}\varphi
\end{aligned}
$$

Equation (9.3.10) can be used for solving shell problems by the variational methods of Ritz, Galerkin or Kantorovich. It can be also used to derive the differential equations and boundary conditions but this will be done later on the direct way.

Hamilton's principle is formulated to solve vibration problems. The potential energy function Π is given with (9.3.10) but all displacements are now functions of x and the time t. If we analyze natural vibrations the transverse load p_z is taken zero. The kinetic energy follows as

$$
\begin{aligned}
T = \sum_{k=1}^{n} T^{(k)} &= \frac{1}{2}\sum_{k=1}^{n} \int\limits_{h^{(k)}} \int\limits_{0}^{2\pi} \int\limits_{0}^{L} \rho^{(k)} \left[\left(\frac{\partial u}{\partial t}\right)^2 + 2z\frac{\partial u}{\partial t}\frac{\partial \psi_x}{\partial t}\right.\\
&\quad \left. + \, z^2\left(\frac{\partial \psi_x}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right] R\mathrm{d}\varphi\mathrm{d}x\mathrm{d}z\\
&= \frac{1}{2}\int\limits_{0}^{2\pi}\int\limits_{0}^{L}\left\{R\rho_0\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]\right.\\
&\quad \left. + \, 2R\rho_1\frac{\partial u}{\partial t}\frac{\partial \psi_x}{\partial t} + R\rho_2\left(\frac{\partial \psi_x}{\partial t}\right)^2\right\}\mathrm{d}\varphi\mathrm{d}x
\end{aligned}
\tag{9.3.11}
$$

In Eq. (9.3.11) $\rho^{(k)}$ is the mass density of the kth layer, ρ_0 and ρ_2 are the mass and the moment of inertia with respect to the middle surface per unit area and ρ_1 represents the coupling between extensional and rotational motions. ρ_1 does not appear in equations for homogeneous shells.

Now with the Lagrange function $L(u,w,\psi_x) = T(u,w,\psi_x) - \Pi(u,w,\psi_x)$ the Hamilton's principle is obtained as

$$
\delta \int\limits_{t_1}^{t_2} L(u,w,\psi_x)\mathrm{d}t = 0 \tag{9.3.12}
$$

The direct derivation of the differential equations for symmetrical cross-ply circular cylindrical shells follow using the constitutive, kinematics and equilibrium equations. The stiffness matrix is defined as

$$
\begin{aligned}
\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix} &= \begin{bmatrix} \boldsymbol{A} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix}, \quad \boldsymbol{Q}^{\mathrm{s}} = \boldsymbol{A}^{\mathrm{s}}\boldsymbol{\varepsilon}^{\mathrm{s}},\\
\boldsymbol{N}^{\mathrm{T}} &= [N_x \; N_s \; N_{xs}], \quad \boldsymbol{M}^{\mathrm{T}} = [M_x \; M_s \; M_{xs}], \quad \boldsymbol{Q}^{\mathrm{sT}} = [Q_x \; Q_s],\\
\boldsymbol{\varepsilon}^{\mathrm{T}} &= \left[\frac{\partial u}{\partial x} \; \left(\frac{\partial v}{\partial s} + \frac{w}{R}\right) \; \left(\frac{\partial u}{\partial s} + \frac{\partial v}{\partial x}\right)\right],\\
\boldsymbol{\kappa}^{\mathrm{T}} &= \left[\frac{\partial \psi_x}{\partial x} \; \frac{\partial \psi_s}{\partial s} \; \left(\frac{\partial \psi_x}{\partial s} + \frac{\partial \psi_s}{\partial x}\right)\right],\\
\boldsymbol{\varepsilon}^{\mathrm{sT}} &= \left[\left(\psi_x + \frac{\partial w}{\partial x}\right) \; \left(\psi_s + \frac{\partial w}{\partial s} - \frac{v}{R}\right)\right]
\end{aligned}
\tag{9.3.13}
$$

For symmetric cross-ply shells all B_{ij} are zero and also the A_{ij}, D_{ij} with $(ij) = (16), (26)$ and (45). Such we have

$$
\boldsymbol{A} = \begin{bmatrix} A_{11} & A_{12} & 0 \\ A_{12} & A_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad \boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & A_{66} \end{bmatrix}, \quad \boldsymbol{A}^{\mathrm{s}} = \begin{bmatrix} k_{55}^{\mathrm{s}}A_{55} & 0 \\ 0 & k_{44}^{\mathrm{s}}A_{44} \end{bmatrix}
$$

The static equilibrium equations are identical with (9.2.8). For vibration analysis inertia terms have to be added and one can formulate

$$\begin{aligned}
\frac{\partial N_x}{\partial x} + \frac{\partial N_{xs}}{\partial s} \qquad &= -p_x + \rho_0 \frac{\partial^2 u}{\partial t^2} + \rho_1 \frac{\partial^2 \psi_x}{\partial t^2},\\
\frac{\partial N_{xs}}{\partial x} + \frac{\partial N_s}{\partial s} + \frac{Q_s}{R} &= -p_s + \rho_0 \frac{\partial^2 v}{\partial t^2} + \rho_1 \frac{\partial^2 \psi_s}{\partial t^2},\\
\frac{\partial Q_x}{\partial x} + \frac{\partial Q_s}{\partial s} - \frac{N_s}{R} &= -p_z + \rho_0 \frac{\partial^2 w}{\partial t^2},\\
\frac{\partial M_x}{\partial x} + \frac{\partial M_{xs}}{\partial s} - Q_x &= \rho_2 \frac{\partial^2 \psi_x}{\partial t^2} + \rho_1 \frac{\partial^2 u}{\partial t^2},\\
\frac{\partial M_{xs}}{\partial x} + \frac{\partial M_s}{\partial s} - Q_s &= \rho_2 \frac{\partial^2 \psi_s}{\partial t^2} + \rho_1 \frac{\partial^2 v}{\partial t^2}
\end{aligned} \tag{9.3.14}$$

ρ_0, ρ_1, ρ_2 are like in (9.3.11) generalized mass density and are defined in (8.3.9). Putting the constitutive equations (9.3.12) in the equilibrium equations (9.3.14) the equations can be manipulated in similar manner to those of the classical theory and one obtains the simultaneous system of differential equations

$$\begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} & \tilde{L}_{13} & \tilde{L}_{14} & \tilde{L}_{15}\\ \tilde{L}_{21} & \tilde{L}_{22} & \tilde{L}_{23} & \tilde{L}_{24} & \tilde{L}_{25}\\ \tilde{L}_{31} & \tilde{L}_{32} & \tilde{L}_{33} & \tilde{L}_{34} & \tilde{L}_{35}\\ \tilde{L}_{41} & \tilde{L}_{42} & \tilde{L}_{43} & \tilde{L}_{44} & \tilde{L}_{45}\\ \tilde{L}_{51} & \tilde{L}_{52} & \tilde{L}_{53} & \tilde{L}_{54} & \tilde{L}_{55} \end{bmatrix} \begin{bmatrix} u\\ v\\ \psi_x\\ \psi_s\\ w \end{bmatrix} = - \begin{bmatrix} p_x\\ p_s\\ 0\\ 0\\ p_z \end{bmatrix} + \begin{bmatrix} \rho_0 & 0 & \rho_1 & 0 & 0\\ 0 & \rho_0 & 0 & \rho_1 & 0\\ \rho_1 & 0 & \rho_2 & 0 & 0\\ 0 & \rho_1 & 0 & \rho_2 & 0\\ 0 & 0 & 0 & 0 & \rho_0 \end{bmatrix} \frac{\partial^2}{\partial t^2} \begin{bmatrix} u\\ v\\ \psi_x\\ \psi_s\\ w \end{bmatrix} \tag{9.3.15}$$

The linear differential operators are defined in App. D.2.

For free vibrations the loads p_x, p_s, p_z are zero and the shell will perform simple harmonic oscillations with the circular frequency ω. Corresponding to simple supported conditions on both ends of the cylinder, i.e

$$N_x = 0, v = 0, w = 0, M_x = 0, \psi_s = 0,$$

the spatial dependence can be written as products of two trigonometric functions and the complete form of vibrations can be taken as

$$\begin{aligned}
u(x,\varphi,t) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} U_{rs} \mathrm{e}^{\mathrm{i}\omega_{rs}t} \cos\alpha_m x \cos n\varphi,\\
v(x,\varphi,t) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} V_{rs} \mathrm{e}^{\mathrm{i}\omega_{rs}t} \sin\alpha_m x \sin n\varphi,\\
w(x,\varphi,t) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} W_{rs} \mathrm{e}^{\mathrm{i}\omega_{rs}t} \sin\alpha_m x \cos n\varphi,\\
\psi_x(x,\varphi,t) &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \Psi_{rs} \mathrm{e}^{\mathrm{i}\omega_{rs}t} \cos\alpha_m x \cos n\varphi,
\end{aligned} \tag{9.3.16}$$

$$\psi_s(x,\varphi,t) = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} \tilde{\Psi}_{rs} \mathrm{e}^{\mathrm{i}\omega_{rs}t} \sin\alpha_m x \sin n\varphi,$$

where $U_{rs}, V_{rs}, W_{rs}, \Psi_{rs}, \tilde{\Psi}_{rs}$ denote amplitudes, $\alpha_m = m\pi/l$, m,n are the longitudinal and the circumferential wave numbers. Substituting Eqs. (9.3.16) into (9.3.15) results in a homogeneous algebraic system and its solutions for a particular pair (m,n) gives the frequency and amplitude ratio corresponding to these wave numbers. For arbitrary boundary conditions the Ritz' or Galerkin's method can be recommended to obtain the characteristic equations for solving the eigenvalue problem. Then the natural frequencies and the mode shapes can be calculated. The solution process is manageable, but involved.

If one restricts the problem to statics and to axially symmetrical loading $p_z = p_z(x)$ the Eqs. (9.3.13) - (9.3.15) can be simplified:

- Equilibrium equations

$$\frac{\mathrm{d}N_x}{\mathrm{d}x} = 0, \quad \frac{\mathrm{d}M_x}{\mathrm{d}x} - Q_x = 0, \quad \frac{\mathrm{d}Q_x}{\mathrm{d}x} - \frac{N_s}{R} + p_z = 0 \tag{9.3.17}$$

- Strain-displacement equations

$$\varepsilon_x = \frac{\mathrm{d}u}{\mathrm{d}x}, \quad \varepsilon_s = \frac{w}{R}, \quad \kappa_x = \frac{\mathrm{d}\psi_x}{\mathrm{d}x}, \quad \varepsilon_{xz} = \psi_x + \frac{\mathrm{d}w}{\mathrm{d}x} \tag{9.3.18}$$

- Constitutive equations

$$\begin{aligned} N_x &= A_{11}\varepsilon_x + A_{12}\varepsilon_s, \\ N_s &= A_{12}\varepsilon_x + A_{22}\varepsilon_s, \\ M_x &= D_{11}\frac{\mathrm{d}\psi_x}{\mathrm{d}x}, \\ Q_x &= k_{55}^{\mathrm{s}} A_{55}\varepsilon_{xz} \end{aligned} \tag{9.3.19}$$

All derivatives $\partial/\partial s(\ldots)$ and $v, \varepsilon_{xs}, \kappa_s, \varepsilon_{sz}, N_{xs}, M_{xs}, Q_s$ are zero. The stress resultant-displacement relations follow as

$$\begin{aligned} N_x &= A_{11}\frac{\mathrm{d}u}{\mathrm{d}x} + A_{12}\frac{w}{R}, \\ N_s &= A_{12}\frac{\mathrm{d}u}{\mathrm{d}x} + A_{22}\frac{w}{R}, \\ M_x &= D_{11}\frac{\mathrm{d}\psi_x}{\mathrm{d}x}, \\ Q_x &= k_{55}^{\mathrm{s}} A_{55}\left(\psi_x + \frac{\mathrm{d}w}{\mathrm{d}x}\right) \end{aligned} \tag{9.3.20}$$

With $\mathrm{d}N_x/\mathrm{d}x = 0$ we have $N_x = \text{const} = N_0$ and one obtains

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{1}{A_{11}}\left(N_0 - A_{12}\frac{w}{R}\right) \tag{9.3.21}$$

The equilibrium equation (9.3.17) for Q_x yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\left(\psi_x+\frac{\mathrm{d}w}{\mathrm{d}x}\right)&=\frac{1}{k_{55}^{\mathrm{s}}A_{55}}\left(\frac{N_s}{R}-p_z\right)\\&=\frac{1}{k_{55}^{\mathrm{s}}A_{55}}\left[\frac{1}{R}\left(A_{12}\frac{\mathrm{d}u}{\mathrm{d}x}+A_{22}\frac{w}{R}\right)-p_z\right]\end{aligned}\qquad(9.3.22)$$

and the equilibrium equation $\mathrm{d}M_x/\mathrm{d}x-Q_x=0$

$$D_{11}\frac{\mathrm{d}^2\psi_x}{\mathrm{d}x^2}-k_{55}^{\mathrm{s}}A_{55}\left(\psi_x+\frac{\mathrm{d}w}{\mathrm{d}x}\right)=0\qquad(9.3.23)$$

After some manipulations follow with (9.3.22), (9.3.23) two differential equations for ψ_x and w as

$$\begin{aligned}\frac{\mathrm{d}\psi_x}{\mathrm{d}x}&=-\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+\frac{1}{k_{55}^{\mathrm{s}}A_{55}}\left[\frac{1}{R}\left(\frac{A_{12}}{A_{11}}N_0-\frac{A_{12}^2-A_{11}A_{22}}{A_{11}}\frac{w}{R}\right)-p_z\right],\\\frac{\mathrm{d}w}{\mathrm{d}x}&=\frac{D_{11}}{k_{55}^{\mathrm{s}}A_{55}}\frac{\mathrm{d}^2\psi_x}{\mathrm{d}x^2}-\psi_x\end{aligned}\qquad(9.3.24)$$

Differentiating the second equation (9.3.24) and eliminating ψ_x the first equation leads to one uncoupled differential equation of fourth order for $w(x)$

$$\begin{aligned}\frac{\mathrm{d}^4w}{\mathrm{d}x^4}&-\frac{1}{k_{55}^{\mathrm{s}}A_{55}}\frac{1}{R^2}\frac{A_{11}A_{22}-A_{12}^2}{A_{11}}\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+\frac{1}{R^2}\frac{A_{11}A_{22}-A_{12}^2}{D_{11}A_{11}}w\\&=\frac{1}{D_{11}}\left(-\frac{A_{12}}{A_{11}}\frac{N_0}{R}+p_z\right)-\frac{1}{k_{55}^{\mathrm{s}}A_{55}}\frac{\mathrm{d}^2p_z}{\mathrm{d}x^2},\\\frac{\mathrm{d}\psi_x}{\mathrm{d}x}&=-\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+\frac{1}{k_{55}^{\mathrm{s}}A_{55}}\frac{A_{11}A_{22}-A_{12}^2}{R^2A_{11}}w+\frac{1}{k_{55}^{\mathrm{s}}A_{55}}\left(\frac{A_{12}}{A_{11}}\frac{N_0}{R}-p_z\right)\end{aligned}\qquad(9.3.25)$$

With $k_{55}^{\mathrm{s}}A_{55}\to\infty$ Eqs. (9.3.25) simplify to the corresponding equations of the classical shell theory (9.2.19).

The governing equation of the axisymmetric problem for or circular cylindrical shell in the frame of the shear deformation theory can be written as

$$\frac{\mathrm{d}^4w}{\mathrm{d}x^4}-2k_1\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+k_2^4w=k_p\qquad(9.3.26)$$

with

$$\begin{aligned}k_1^2&=\frac{1}{k_{55}^{\mathrm{s}}A_{55}}\frac{1}{R}\frac{A_{11}A_{22}-A_{12}^2}{A_{11}},\quad k_2^4=\frac{1}{k_{55}^{\mathrm{s}}A_{55}}\frac{1}{R^2}\frac{A_{11}A_{22}-A_{12}^2}{D_{11}A_{11}},\\k_p&=\frac{1}{D_{11}}\left(-\frac{A_{12}}{A_{11}}\frac{N_0}{R}+p_z\right)-\frac{1}{D_{11}k_{55}^{\mathrm{s}}A_{55}}\frac{\mathrm{d}^2p_z}{\mathrm{d}x^2}\end{aligned}$$

The differential equation can be analytical solved

$$w(x)=w_h(x)+w_p(x)$$

The particular solution $w_p(x)$ has e.g. with $\mathrm{d}^2 p_z/\mathrm{d}x^2 = 0$, the form

$$w_p(x) = \frac{k_p}{k_2^4} = \frac{R^2 p_z - RA_{12}N_0}{A_{11}A_{22} - A_{12}^2} \tag{9.3.27}$$

The homogeneous solution $w_h(x) = C\mathrm{e}^{\alpha x}$ yield the characteristic equation

$$\alpha^4 - 2k_1^2\alpha^2 + k_2^4 = 0 \tag{9.3.28}$$

with the roots

$$\alpha_{1-4} = \pm\sqrt{k_1^2 \pm \sqrt{k_1^4 - k_1^4}}$$

which can be conjugate complex, real or two double roots depending on the relations of the constants k_1 and k_2.

The general solution can be written as (App. E)

$$w(x) = \sum_{i=1}^{4} C_i \Phi_i(x) + w_p(x)$$

The functions $\Phi_i(x)$ are given in different forms depending on the roots of the characteristic equation (9.3.28). The roots and the functions $\Phi_i(x)$ are summarized in App. E. The most often used solution form in engineering application is given for $k_2^2 > k_1^2$.

For short shells with edges affecting one another, the $\Phi_i(x)$ involving the hyperbolic functions are convenient. If there are symmetry conditions to the middle cross-section $x = L/2$ the solution can be simplified, for we have $\Phi_3 = \Phi_4 = 0$. For long shells with ends not affecting one another applying the $\Phi_i(x)$ that involve exponential functions.

Analogous to the classical shell solution for long shells a bending-layer solution can be applied. Only inside the bending-layer region with the characteristic length L_B the homogeneous part w_h and the particular part w_p of the general solution w have to superimposed. Outside the bending-layer region, i.e for $x > L_B$ or $(L - x) > L_B$ only w_p characterizes the shell behavior.

Summarizing the results of the shear deformation shell theory one can say

- If one restricts the consideration to symmetrical cross-ply circular cylindrical shells subjected to axially symmetric loadings the modelling and analysis is most simplified and correspond to the classical shell theory.
- In more general cases including static loading and vibration and not neglecting the trapeze effect the variational formulation is recommended and approximative analytical or numerical solutions should be applied.
- Circular cylindrical shells are one of the most used thin-walled structures of conventional or composite material. Such shells are used as reservoirs, pressure vessels, chemical containers, pipes, aircraft and ship elements. This is the reason for a long and intensive study to model and analyze circular cylindrical shells and as result efficient theories and solutions methods are given in literature.

9.4 Sandwich Shells

Sandwich shells are widely used in many industrial branches because sandwich constructions often results in designs with lower structural weight then constructions with other materials. But there is not only weight saving interesting, but in several engineering applications the core material of a sandwich construction can be also used as thermal insulator or sound absorber. Therefore one can find numerous literature on modelling and analysis for sandwich shells subjected static, dynamic or environmental loads .

But as written in Sects. 7.4 and 8.4 sandwich constructions are, simply considered, laminated constructions involving three laminae: the lower face, the core, and the upper face. And by doing so, one can employ all methods of modelling and analysis of laminated structural elements.

It was discussed in detail in Sect. 8.4 that, considering sandwich structural elements, we have to keep in mind the assumptions on the elastic behavior of sandwiches. Such there are differences in the expressions for the flexural bending and transverse shear stiffness in comparison with laminated circular cylindrical shells and essential differences in the stress distribution over the thickness of the shell wall. The stiffness parameter for sandwich shells depend on the modelling of sandwiches having thin or thicker faces, in the same manner as for plates, Eqs. (8.4.1) and (8.4.2).

For sandwich constructions generally the ratio of the in-plane moduli of elasticity to the transverse shear moduli is high and transverse shear deformations are mostly included in its structural modelling. For this reason, the first order shear deformation theory of laminated shells is used in priority for sandwich shells. But for thin-walled sandwich shells with a higher shear stiffness approximately the classical sandwich theory can be used.

The correspondence between laminated and sandwich shells is for vibration or buckling problems limited and only using for overall buckling and vibration. There are some special local problems like face wrinkling and core shear instability in buckling or the face must be additional considered as a shell of elastic foundation on the core and also shear mode vibration can occur where each face is vibrating out of phase with the other face. These problems are detailed discussed in a number of special papers and can not considered in this book.

9.5 Problems

Exercise 9.1. A circular cylindrical sandwich shell has two unequal faces with the reduced stiffness $Q_{11}^{\mathrm{f}_1}, Q_{11}^{\mathrm{f}_3}$ and the thicknesses $h^{\mathrm{f}_1}, h^{\mathrm{f}_3}$. The shell has an orthotropic material behavior and the material principal axes shall coincide with the structural axes x, s. The core with the thickness h^{c} does not contribute significantly to the extensional and the flexural shell stiffness. The lateral distributed load is $p_z = p_z(x)$.

Formulate the differential equation using a perturbation constant to characterize the asymmetry of the sandwich and find the perturbation solution way.

Solution 9.1. The shell problem is axially symmetric. In the frame of the classical shell theory one can use the differential equation (9.2.18)

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + \frac{2}{RD^R}\left[\frac{A_{12}B_{11}}{A_{11}} - B_{12}\right]\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + 4\lambda^4 w = \frac{1}{D^R}\left(p_z - \frac{A_{12}}{A_{11}}\frac{N_x}{R}\right)$$

The stiffness parameter are calculated for sandwiches with thin faces, Sect. 4.3.2, $h \approx h^c$

$$\begin{aligned}
A_{11} &= Q_{11}^{\mathrm{f}_1}h^{\mathrm{f}_1} + Q_{11}^{\mathrm{f}_3}h^{\mathrm{f}_3} = Q_{11}^{\mathrm{f}_1}h^{\mathrm{f}_1}\left(1 + \frac{Q_{11}^{\mathrm{f}_3}h^{\mathrm{f}_3}}{Q_{11}^{\mathrm{f}_1}h^{\mathrm{f}_1}}\right),\\
D_{11} &= \left(\frac{h}{2}\right)^2 Q_{11}^{\mathrm{f}_1}h^{\mathrm{f}_1} + \left(\frac{h}{2}\right)^2 Q_{11}^{\mathrm{f}_3}h^{\mathrm{f}_3} = \left(\frac{h}{2}\right)^2 Q_{11}^{\mathrm{f}_1}h^{\mathrm{f}_1}\left(1 + \frac{Q_{11}^{\mathrm{f}_3}h^{\mathrm{f}_3}}{Q_{11}^{\mathrm{f}_1}h^{\mathrm{f}_1}}\right),\\
B_{11} &= -\frac{h}{2}Q_{11}^{\mathrm{f}_1}h^{\mathrm{f}_1} + \frac{h}{2}Q_{11}^{\mathrm{f}_3}h^{\mathrm{f}_3} = \frac{h}{2}Q_{11}^{\mathrm{f}_1}h^{\mathrm{f}_1}\left(-1 + \frac{Q_{11}^{\mathrm{f}_3}h^{\mathrm{f}_3}}{Q_{11}^{\mathrm{f}_1}h^{\mathrm{f}_1}}\right)
\end{aligned}$$

B_{12} and D_{12} can be calculated analogous. A asymmetry constant can be defined as

$$\eta = \frac{B_{11}}{\sqrt{D_{11}A_{11}}} = \frac{-1 + \dfrac{Q_{11}^{\mathrm{f}_3}h^{\mathrm{f}_3}}{Q_{11}^{\mathrm{f}_1}h^{\mathrm{f}_1}}}{1 + \dfrac{Q_{11}^{\mathrm{f}_3}h^{\mathrm{f}_3}}{Q_{11}^{\mathrm{f}_1}h^{\mathrm{f}_1}}}$$

For a symmetric sandwich wall is $B_{11} = 0$ and so $\eta = 0$. For an infinite stiffness of face 1 follows $\eta \to -1$ and of face 3 $\eta \to +1$, i.e. the constant η is for any sandwich construction given as

$$-1 < \eta < +1$$

The differential equation (9.2.18) can be written with the constant η as

$$\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + \frac{2}{RD^R}\left[\frac{A_{12}\sqrt{D_{11}}}{\sqrt{A_{11}}} - \frac{\sqrt{A_{11}}B_{12}\sqrt{D_{11}}}{B_{11}}\right]\eta\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + 4\lambda^4 w = \frac{1}{D^R}\left(p_z - \frac{A_{12}}{A_{11}}\frac{N_x}{R}\right)$$

Since $|\eta| < 1$ one can find $w(x)$ in the form of a perturbation solution

$$w(x) = \sum_{n=0}^{\infty} w_n(x)\eta^n$$

For $n = 0$ follow

$$\frac{\mathrm{d}^4 w_0}{\mathrm{d}x^4} - 4\lambda^4 w_0 = -\frac{1}{D^R}\left(p_z - \frac{A_{12}}{A_{11}}\frac{N_x}{R}\right)$$

and for $n \geq 1$

$$\frac{\mathrm{d}^4 w_n}{\mathrm{d}x^4} + 4\lambda^4 w_n = -\frac{2}{RD^R}\left[\frac{A_{12}\sqrt{D_{11}}}{\sqrt{A_{11}}} - \frac{\sqrt{A_{11}}B_{12}\sqrt{D_{11}}}{B_{11}}\right]\eta\frac{\mathrm{d}^2 w_{n-1}}{\mathrm{d}x^2}$$

The left hand side corresponds to the middle-surface symmetric shell with axially symmetric loading. The right hand side corresponds to the second derivation of the previously obtained w-solution.

Conclusion 9.1. The perturbation solution yield the solution of the differential equation as a successive set of solutions of axially symmetric problems of which many solutions are available. The perturbation solution converges to the exact solution. In many engineering applications $w(x) = w_o(w) + \eta w_1(x)$ will be sufficient accurate.

Exercise 9.2. A symmetrical cross-ply circular cylindrical shell is loaded at the boundary $x = 0$ by an axially symmetric line pressure Q_0 and line moment M_0. Calculate the ratio M_0/Q_0 that the boundary shell radius does not change if the shell is very long.

Solution 9.2. We use the solution (9.2.23) with $w_p = 0$ and neglect for the long shell the influence of M_L and Q_L

$$w(x) = \frac{M_0}{2\lambda^2 D_{11}}\mathrm{e}^{-\lambda x}(\sin\lambda x - \cos\lambda x) - \frac{Q_0}{2\lambda^3 D_{11}}\cos\lambda x$$

The condition of no radius changing yields

$$w(x=0) = 0 \Rightarrow -\frac{M_0}{2\lambda^2 D_{11}} - \frac{Q_0}{2\lambda^3 D_{11}} = 0 \Rightarrow \frac{M_0}{Q_0} = -\frac{1}{\lambda}$$

Exercise 9.3. For a long fluid container, Fig. 9.4 determinate the displacement $w(x)$ and the stress resultants $N_s(x)$ and $M_x(x)$. The container has a symmetrical cross-ply layer stacking and can be analyzed in the frame of the classical laminate theory.

Solution 9.3. For a long circular cylindrical shell the solution for $w_h(x)$, Eq. (9.2.22), can be reduced to the first term with the negative exponent

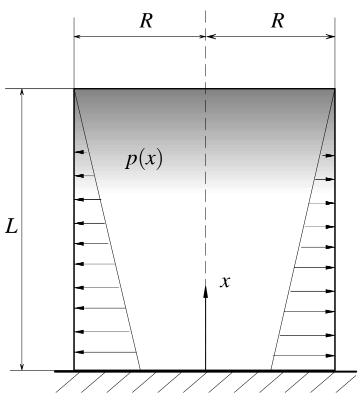


Fig. 9.4 Long fluid container, $L > L_B$

$$w_h(x) = \mathrm{e}^{-\lambda x}(C_1 \sin\lambda x + C_2 \cos\lambda x)$$

The particular solution $w_p(x)$ follow with

$$p(x) = p_0\left(1 - \frac{x}{L}\right)$$

and Eq. (9.2.24) as

$$w_p(x) = \frac{p_0}{4\lambda^4 D_{11}}\left(1 - \frac{x}{L}\right)$$

The boundary constraints are

$$w(0) = 0 \Rightarrow C_2 = -\frac{p_0}{4\lambda^4 D_{11}}, \quad \frac{\mathrm{d}w(0)}{\mathrm{d}x} = 0 \Rightarrow C_1 = -\frac{p_0}{4\lambda^4 D_{11}}\left(1 - \frac{x}{L}\right)$$

and we obtain the solutions

$$w(x) = \frac{p_0}{4\lambda^4 D_{11}}\left\{1 - \frac{x}{L} - \left[\cos\lambda x + \left(1 - \frac{x}{L}\right)\sin\lambda x\right]\mathrm{e}^{-\lambda x}\right\}$$

In addition,

$$N_s = A_{11}\varepsilon_x + A_{22}\varepsilon_s = A_{11}\frac{\mathrm{d}u}{\mathrm{d}x} + A_{22}\frac{w}{R}$$

i.e. with $\varepsilon_x = 0$

$$N_s = \frac{A_{22}}{R}w(x), \quad M_x = D_{11}\kappa_x = -D_{11}\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}, \quad Q_x = \frac{\mathrm{d}M_x}{\mathrm{d}x} = -D_{11}\frac{\mathrm{d}^3 w}{\mathrm{d}x^3}$$

Exercise 9.4. Consider a cantilever circular cylindrical shell, Fig. 9.5. The normal and shear forces N_x and N_{xs} as are distributed along the contour of the cross-section $x = L$ that they can reduced to the axial force F_{H}, the transverse force F_{V}, the bending moment M_{B} and the torsion moment M_{T}. Calculate the resultant membrane stress forces with the membrane theory.

Solution 9.4. With (9.2.26) we have the following equations

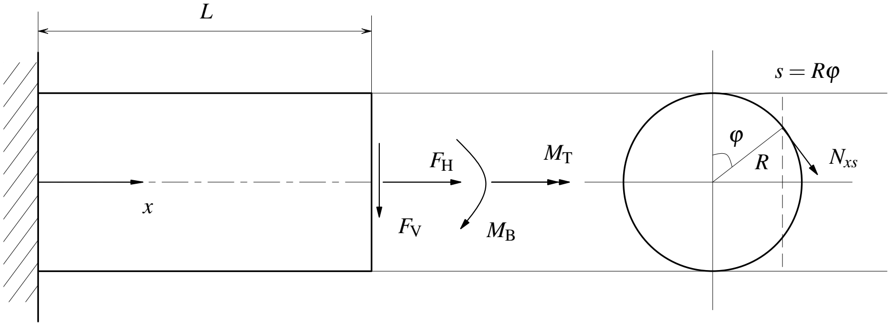


Fig. 9.5 Tension, bending and torsion of a cantilever circular cylindrical shell

$$N_s = Rp_z, \quad \frac{\partial N_{xs}}{\partial x} = -p_s, \quad \frac{\partial N_x}{\partial x} = -p_x - \frac{\partial N_{xs}}{\partial s}$$

and $p_z = p_s = p_x = 0$ yield

$$N_s = 0, \quad N_{xs} = \text{const}, \quad N_x = \text{const}$$

The distributions of $F_{\text{H}}, F_{\text{V}}, M_{\text{B}}$ and M_{T} over the cross-section contour $x = L$ can be represented as

$$\begin{aligned} N_x(x=L) &= \frac{1}{2\pi R}\left(F_{\text{H}} + \frac{2M_{\text{B}}}{R}\cos\varphi\right), \\ N_{rs}(x=L) &= \frac{1}{2\pi R}\left(\frac{M_{\text{T}}}{R} + 2F_{\text{V}}\sin\varphi\right) \end{aligned}$$

and yield the reduced forces $F_{\text{V}}, F_{\text{M}}$ and moments $M_{\text{B}}, M_{\text{T}}$

$$\begin{aligned} 2\int\limits_0^{\pi} N_x(x=L)R\mathrm{d}\varphi &= F_H, \\ 2\int\limits_0^{\pi} N_{xs}(x=L)\sin\varphi R\mathrm{d}\varphi &= \frac{1}{\pi R}\int\limits_0^{\pi} 2F_V\sin^2\varphi R\mathrm{d}\varphi = F_V, \\ 4\int\limits_0^{\pi/2} N_x(x=L)R\cos\varphi\mathrm{d}\varphi &= \frac{4}{\pi}M_B\int\limits_0^{\pi/2}\cos^2\varphi\mathrm{d}\varphi = M_B, \\ 2\int\limits_0^{\pi/2} RN_{xs}(x=L)R\mathrm{d}\varphi &= \frac{1}{R\pi}\int\limits_0^{\pi/2} M_T R\mathrm{d}\varphi = M_T \end{aligned}$$

The equilibrium equations yield

$$\begin{aligned} N_x(x) &= \frac{F_{\text{H}}}{2\pi R} - [M_{\text{B}} + F_{\text{V}}(L-x)]\frac{\cos\varphi}{\pi R}, \\ N_s(x) &= 0, \\ N_{sx} &= \frac{M_{\text{T}}}{2\pi R12} + \frac{F_{\text{V}}}{\pi R}\sin\varphi \end{aligned}$$

Part IV
Modelling and Analysis of Thin-Walled Folded Plate Structures

The fourth part (Chap. 10) includes the modelling and analysis of thin-walled folded plate structures or generalized beams. This topic is not normally considered in standard textbooks on structural analysis of laminates and sandwiches, but it is included here because it demonstrates the possible application of Vlasov's theory of thin-walled beams and semi-membrane shells on laminated structural elements.

Chapter 10
Modelling and Analysis of Thin-walled Folded Structures

The analysis of real structures always is based on a structural and mathematical modelling. It is indispensable for obtaining realistic results that the structural model represents sufficiently accurate the characteristic structure behavior.

Generally the structural modelling can be divided into three structure levels

- *Three-dimensional modelling*. It means structural elements, their dimensions in all three directions are of the same order, we have no preferable direction.
- *Two-dimensional modelling*. One dimension of a structural element is significant smaller in comparison with the other both, so that we can regard it as a quasi two-dimensional element. We have to distinguish plane and curved elements e.g. discs, plates and shells.
- *One-dimensional modelling*. Here we have two dimensions (the cross-section) in the same order and the third one (the length) is significant larger in comparison with them, so that we can regard such a structural element as quasi one-dimensional. We call it rod, column,bar, beam or arch and can distinguish straight and curved forms also.

The attachment of structural elements to one of these classes is not well defined rather it must be seen in correlation with the given problem.

Many practical problems, e.g., in mechanical or civil engineering lead to the modelling and analysis of complex structures containing so-called thin-walled elements. As a result of the consideration of such structures a fourth modelling class was developed, the modelling class of thin-walled beams and so-called beam shaped shells including also folded plate structures. In this fourth modelling class it is typical that we have structures with a significant larger dimension in one direction (the length) in comparison with the dimensions in transverse directions (the cross-section) and moreover a significant smaller thickness of the walls in comparison with the transverse dimensions.

In Chap. 7 the modelling of laminate beams is given in the frame of the Bernoulli's and Timoshenko's beam theory which cannot applied generally to thin-walled beams. The modelling of two-dimensional laminate structures as plates and shells was the subject of the Chaps. 8 and 9. In the present Chap. 10 the investi-

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0_10

gation of beams with thin-walled cross-sections and beam shaped shells especially folded structures is carried out. Chapter 10 starting in Sect. 10.1 from a short recall of the classical beam models. In Sect. 10.2 a generalized beam model for prismatic thin-walled folded plate structures is introduced, including all known beam models. Section 10.3 discusses some solution procedures and in Sect. 10.4 selected problems are demonstrated.

10.1 Introduction

Analyzing thin-walled structures it can be useful to distinguish their global and local structural behavior. Global bending, vibration or buckling is the response of the whole structure to external loading and is formulated in a global coordinate system. A typical example for global structure behavior is the deflection of a ship hull on the waves. But the deflections and stresses in a special domain of the ship e.g. in the region of structure loading or deck openings or the vibrations or buckling of single deck plates represent typical local effects.

A necessary condition for a global analysis is that the geometry of the structure allows its description in a global co-ordinate system, i.e. the thin-walled structure is sufficient long how it is given in case of a quasi one-dimensional structure.

Of course there are interactions between global and local effects, and in the most cases these interactions are nonlinear. Usually the global analysis is taken as the basic analysis and its results are the boundary conditions for local considerations by using special local co-ordinate systems. The reactions of local to global effects whereas are neglected.

From this point of view the global analysis of thin-walled beams and beam shaped shells can be done approximately by describing them as one-dimensional structures with one-dimensional model equations. For such problems the classical beam model of J. Bernoulli was used. This model is based on three fundamental hypotheses:

- There are no deformations of the cross-sectional contour.
- The cross-section is plane also in case of deformed structures.
- The cross-section remain orthogonally to the deformed system axis.

As a result of bending without torsion we have normal stresses σ and strains ε only in longitudinal direction. Shear deformations are neglected. The shear stresses τ caused by the transverse stress resultants are calculated with the help of the equilibrium equations, but they are kinematically incompatible.

The Bernoulli's beam model can be used for beams with compact and sufficient stiff thin-walled cross-sections. In case of thin-walled cross-section it is supposed that the bending stresses σ_{b} and the shear stresses τ_{q} are distributed constantly over the thickness t. If we have closed thin-walled cross-sections a statically indeterminate shear flow must be considered.

A very important supplement to Bernoulli's beam model was given by Saint-Venant[1] for considering the torsional stress. Under torsional stresses the cross-sections have out-of-plane warping, but assuming that these are the same in all cross-sections and they are not constrained we have no resulting longitudinal strains and normal stresses. In this way we have also no additional shear stresses. The distribution of the so-called Saint-Venant torsional shear stresses is based on a closed shear flow in the cross-section. For closed thin-walled cross-sections the well-known elementary formulae of Bredt[2] can be used.

The Timoshenko's beam model is an extension of the Bernoulli's beam model. It enables to consider the shear deformations approximately. The first two basic hypotheses of the Bernoulli's model are remained. The plane cross-section stays plane in this case but is not orthogonally to the system axes in the deformed structure. For the torsional stress also the relationships of Saint-Venant are used.

Rather soon the disadvantages of this both classical beam models were evident for modelling and analysis of general thin-walled beam shaped structures. Especially structures with open cross-section have the endeavor for warping, and because the warping generally is not the same in all cross-sections, there are additional normal stresses, so-called warping normal stresses and they lead to warping shear stresses too. Therefore the torsional moment must be divided into two parts, the Saint-Venant part and the second part caused by the warping shear stresses.

Very fundamental and general works on this problem were done by Vlasov. Because his publications are given in Russian language they stayed unknown in western countries for a long time. In 1958 a translation of Vlasov's book "General Shell Theory and its Application in Technical Sciences" into German language was published (Wlassow, 1958) and some years later his book on thin-walled elastic beams was published in English (Vlasov, 1961). By Vlasov a general and systematic terminology was founded, which is used now in the most present papers.

The Vlasov's beam model for thin-walled beams with open cross-sections is based on the assumption of a rigid cross-section contour too, but the warping effects are considered. Neglecting the shear strains of the mid-planes of the walls the warping of the beam cross-section are given by the so-called law of sectorial areas. The application of this Vlasov beam model to thin-walled beams with closed cross-sections leads to nonsatisfying results, because the influences of the cross-sectional contour deformations and of the mid-plane shear strains in the walls are significant in such cases.

Therefore a further special structural model was developed by Vlasov in form of the so-called semi-moment shell theory, in which the longitudinal bending moments and the torsional moments in the plates of folded structures with closed cross-sections are neglected. By this way we have in longitudinal direction only membrane stresses and in transversal direction a mixture of membrane and bending stresses.

This two-dimensional structural model can be reduced to a one-dimensional one by taking into account the Kantorovich relationships in form of products of two

[1] Adhémar Jean Claude Barré de Saint-Venant (*23 August 1797 Villiers-en-Bière – †6 January 1886 Saint-Ouen) - mechanician and mathematician

[2] Rudolf Bredt (*17 April 1842 Barmen - †18 May 1900 Wetter) - mechanical engineer

functions. One of them describes a given deformation state of the cross-section, considered as a plane frame structure and the other is an unknown function of the longitudinal co-ordinate.

In 1994 the authors of this textbook published a monograph on thin-walled folded plate structures in German (Altenbach et al, 1994). Starting from a general structural model for isotropic structures also a short outlook to anisotropic structures was given. The general model equations including the semi-moment shell model and all classical and generalized linear beam models could be derived by neglecting special terms in the elastic energy potential function or by assuming special conditions for the contour deformation states. In Sect. 10.2 the derivation of generalized folded structural model is given for anisotropic plates, e.g. off-axis loaded laminates. The derivations are restricted to prismatic systems with straight system axes only.

Summarizing one can conclude from the above discussion there are several reasons why for thin-walled structures must be given special consideration in design and analysis. In thin-walled beams the shear stresses and strains are relatively much larger than those in beams with solid, e.g. rectangular, cross-sections. The assumptions of Bernoulli's or Timoshenko's beam theory can be violated e.g. by so-called shear lag effects, which result in a non-constant distribution of normal bending stresses which are different from that predicted by the Bernoulli hypotheses for beams carrying only bending loads. When twisting also occurs warping effects, e.g. warping normal and shear stresses, have to add to those arising from bending loads. The warping of the cross-section is defined as its out-of-plane distorsion in the direction of the beam axis and violated the Bernoulli's hypotheses and the Timoshenko's hypotheses too.

Because of their obvious advantages fibre reinforced laminated composite beam structures are likely to play an increasing role in design of the present and, especially, of future constructions in the aeronautical and aerospace, naval or automotive industry. In addition to the known advantages of high strength or high stiffness to weight ratio, the various elastic and structural couplings, which are the result of the directional nature of composite materials and of laminae-stacking sequence, can be successfully exploited to enhance the response characteristics of aerospace or naval vehicles.

In order to be able to determine the behavior of these composite beam structures, consistent mechanical theories and analytical tools are required. So a Vlasov type theory for fiber-reinforced beams with thin-walled open cross-sections made from mid-plane symmetric fiber reinforced laminates was developed but in the last 30 years many improved or simplified theories were published.

Because primary or secondary structural configurations such as aircraft wings, helicopter rotor blades, robot arms, bridges and other structural elements in civil engineering can be idealized as thin- or thick-walled beams, especially as box beams, beam models appropriate for both thin- and thick-walled geometries which include the coupled stiffness effects of general angle-ply laminates, transverse shear deformation of the cross-section and the beam walls, primary and secondary warping, etc. were developed. But nearly all governing equations of thin- and thick-walled composite beams adopt the basic Vlasov assumption:

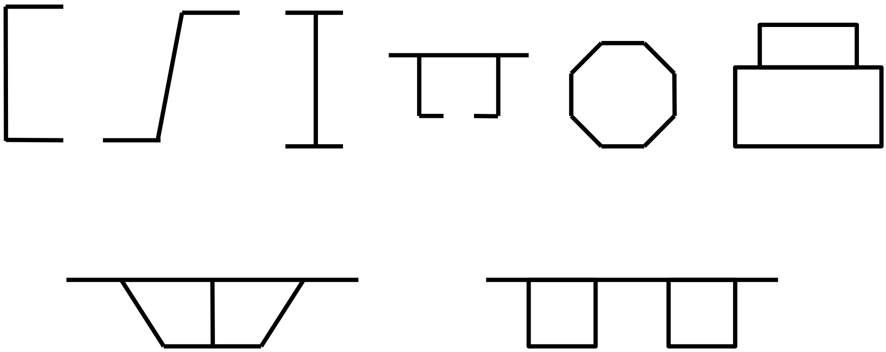


Fig. 10.1 Thin-walled prismatic folded plate structures with open or closed cross-sections

> The contours of the original beam cross-section do not deform in their own planes.

This assumption implies that the normal strain ε_s in the contour direction is small compared to the normal strain ε_z parallel to the beam axis. This is particular valid for thin-walled open cross-sections, for thin-walled closed cross-section with stiffeners (transverse sheets) and as the wall thickness of closed cross-sections increase. Chapter 10 focuses the considerations to a more general model of composite thin-walled beams which may be include the classical Vlasov assumptions or may be relax these assumptions, e.g. by including the possibility of a deformation of the cross-section in its own plane, etc.

In the following a special generalized class of thin-walled structures is considered, so-called folded plate structures. A folded plate structure shall be defined as a prismatic thin-walled structure which can be formed by folding a flat rectangular plate or joining thin plate strips along lines parallel to their length. Figure 10.1 demonstrates thin-walled structures of the type defined above. The plate strips can be laminates.

10.2 Generalized Beam Models

Section 10.2 defines the outline of modelling beam shaped, thin-walled prismatic folded plate structures with open, one or multi-cell closed or mixed open-closed cross-sections. The considerations are limited to global structural response. Assuming the classical laminate theory for all laminated plate strips of the beam shaped structure the elastic energy potential function is formulated. The energy potential is a two-dimensional functional of the coordinate x of the structure axis and the cross-section contour coordinate s.

Following the way of Vlasov-Kantorovich the two-dimensional functional is reduced to an approximate one-dimensional one. A priori fixed generalized coordinate functions describing the cross-section kinematics are introduced. Generalized displacement functions which depend on the system coordinate x only are the independent functions of the reduced variational statement which leads to a system of matrix differential equations, the Euler equations of the variational statement, and to the possible boundary equations.

The general structural model can be simplified by neglecting selected terms in the energy formulation or by restricting the number of the generalized coordinate functions, i.e. the cross-section kinematics. All results are discussed under the viewpoint of a sufficient general structural model for engineering applications. A general structural model is recommended which includes all above noted forms of cross-sections and enables to formulate efficient numerical solution procedures.

10.2.1 Basic Assumptions

A prismatic system is considered, its dimensions are significant larger in one direction (the length) in comparison with these in transverse directions. The system consists of n plane thin-walled strip elements; it means their thickness is significant smaller than the strip width, i.e. $t_i \ll d_i$. Rigid connections of the plate strips along their length lines are supposed. Closed cross-sections as well as open cross-sections and combined forms are possible. In Fig. 10.2 a general thin-walled folded structure is shown. There is a global co-ordinate system x, y, z with any position. In each strip we have a local co-ordinate system x, s_i, n_i, the displacements are u_i, v_i, w_i. We restrict our considerations to prismatic structures only and neglect the transverse shear strains in the strips normal to their mid-planes, it means the validity of the Kirchhoff hypotheses is supposed or we use the classical laminate theory only. All constants of each strip are constant in x-direction. For the displacements we can write

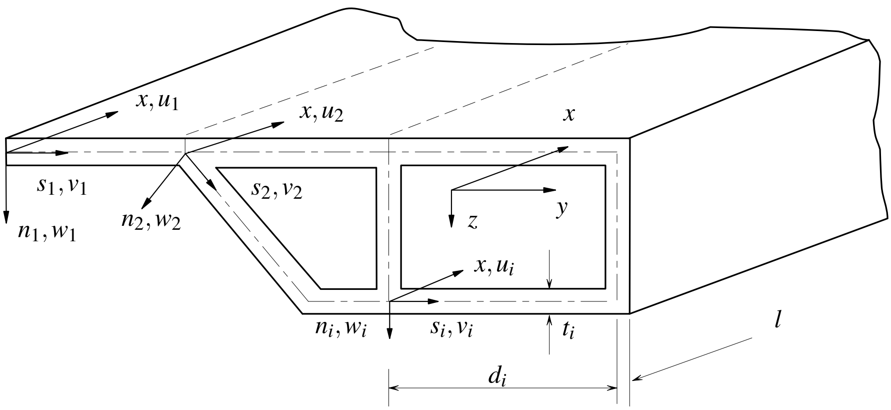


Fig. 10.2 Thin-walled folded structure geometry and co-ordinate systems

$$u_i = u_i(x, s_i), \quad v_i = v_i(x, s_i), \quad w_i = w_i(x, s_i) \tag{10.2.1}$$

u_i and v_i are the displacements in the mid-plane and w_i is the deflection normal to the mid-plane of the ith strip. As loads are considered:

- surface forces, distributed on the unit of the mid-plane

$$p_{x_i} = p_{x_i}(x, s_i), \quad p_{s_i} = p_{s_i}(x, s_i), \quad p_{n_i} = p_{n_i}(x, s_i) \tag{10.2.2}$$

- line forces, distributed on the length unit of the boundaries of the structure

$$\begin{aligned} q_{x_i|x=0} &= q_{x_i}(0, s_i),\ q_{s_i|x=0} = q_{s_i}(0, s_i),\ q_{n_i|x=0} = q_{n_i}(0, s_i) \\ q_{x_i|x=l} &= q_{x_i}(l, s_i),\ q_{s_i|x=l} = q_{s_i}(l, s_i),\ q_{n_i|x=l} = q_{n_i}(l, s_i) \end{aligned} \tag{10.2.3}$$

If a linear anisotropic material behavior is supposed, for each strip we can use the constitutive relationship given as

$$\begin{bmatrix} \boldsymbol{N} \\ \boldsymbol{M} \end{bmatrix}_i = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix}_i \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix}_i \tag{10.2.4}$$

or

$$\begin{bmatrix} N_{x_i} \\ N_{s_i} \\ N_{xs_i} \\ M_{x_i} \\ M_{s_i} \\ M_{xs_i} \end{bmatrix} = \begin{bmatrix} A_{11_i} & A_{12_i} & A_{16_i} & B_{11_i} & B_{12_i} & B_{16_i} \\ A_{12_i} & A_{22_i} & A_{26_i} & B_{12_i} & B_{22_i} & B_{26_i} \\ A_{16_i} & A_{26_i} & A_{66_i} & B_{16_i} & B_{26_i} & B_{66_i} \\ B_{11_i} & B_{12_i} & B_{16_i} & D_{11_i} & D_{12_i} & D_{16_i} \\ B_{12_i} & B_{22_i} & B_{26_i} & D_{12_i} & D_{22_i} & D_{26_i} \\ B_{16_i} & B_{26_i} & B_{66_i} & D_{16_i} & D_{26_i} & D_{66_i} \end{bmatrix} \begin{bmatrix} \varepsilon_{x_i} \\ \varepsilon_{s_i} \\ \varepsilon_{xs_i} \\ \kappa_{x_i} \\ \kappa_{s_i} \\ \kappa_{xs_i} \end{bmatrix}$$

The following steps are necessary for calculating the elements of the matrices $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{D}$ for the ith strip:

- Calculate the reduced stiffness matrix $\boldsymbol{Q}'$ for each lamina (k) of the strip (i) by using the four elastic moduli $E_\mathrm{L}, E_\mathrm{T}, \nu_\mathrm{LT}, G_\mathrm{LT}$, Eqs. (4.1.2) and (4.1.3).
- Calculate the values of the transformed reduced stiffness matrix $\boldsymbol{Q}$ for each lamina (k) of the strip (i) (Table 4.2).
- Considering the stacking structure, it means, considering the positions of all laminae in the ith strip calculate the matrix elements $A_{kl_i}, B_{kl_i}, D_{kl_i}$, (4.2.15).

It must be noted that the co-ordinates x_1, x_2, x_3 used in Sect. 4.1.3 are corresponding to the coordinates x, s_i, n_i in the present chapter and the stresses $\sigma_1, \sigma_2, \sigma_6$ here are $\sigma_{x_i}, \sigma_{s_i}, \sigma_{xs_i}$. For the force and moment resultants also the corresponding notations $N_{x_i}, N_{s_i}, N_{xs_i}, M_{x_i}, M_{s_i}, M_{xs_i}$ are used and we have to take here:

$$N_{x_i} = \int\limits_{-t_i/2}^{t_i/2} \sigma_{x_i} \mathrm{d}n_i, \quad N_{s_i} = \int\limits_{-t_i/2}^{t_i/2} \sigma_{s_i} \mathrm{d}n_i, \quad N_{xs_i} = \int\limits_{-t_i/2}^{t_i/2} \sigma_{xs_i} \mathrm{d}n_i, \tag{10.2.5}$$

$$M_{x_i} = \int\limits_{-t_i/2}^{t_i/2} \sigma_{x_i} n_i \mathrm{d}n_i,\ M_{s_i} = \int\limits_{-t_i/2}^{t_i/2} \sigma_{s_i} n_i \mathrm{d}n_i,\ M_{xs_i} = \int\limits_{-t_i/2}^{t_i/2} \sigma_{xs_i} n_i \mathrm{d}n_i$$

In Fig. 10.3 the orientations of the loads, see Eqs. (10.2.2) and (10.2.3), and the resultant forces and moments in the ith wall are shown. In the frame of the classical laminate theory the transverse force resultants N_{sn_i} and N_{xn_i} follow with the help of the equilibrium conditions for a strip element.

In the same way here we have the following definitions for the elements of the deformation vector $[\varepsilon_1\ \varepsilon_2\ \varepsilon_6\ \kappa_1\ \kappa_2\ \kappa_6]^{\mathrm{T}} \equiv [\varepsilon_{x_i}\ \varepsilon_{s_i}\ \varepsilon_{xs_i}\ \kappa_{x_i}\ \kappa_{s_i}\ \kappa_{xs_i}]^{\mathrm{T}}$ with

$$\begin{aligned} \varepsilon_{x_i} &= \frac{\partial u_i}{\partial x} = u_i', & \varepsilon_{s_i} &= \frac{\partial v_i}{\partial s_i} = v_i^{\bullet}, \\ \varepsilon_{xs_i} &= \frac{\partial u_i}{\partial s_i} + \frac{\partial v_i}{\partial x} = u_i^{\bullet} + v_i', & \kappa_{x_i} &= -\frac{\partial^2 w_i}{\partial x^2} = -w_i'', \end{aligned} \tag{10.2.6}$$

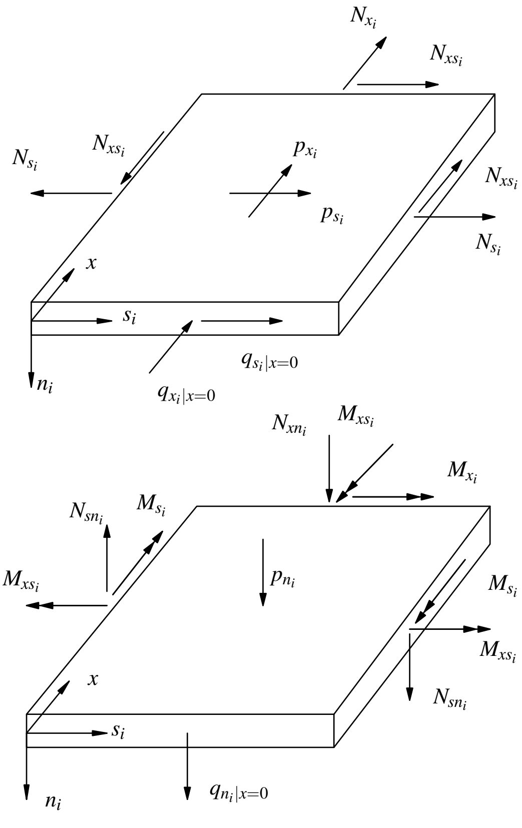


Fig. 10.3 Loads and resultant forces and moments in the ith strip

$$\kappa_{s_i} = -\frac{\partial^2 w_i}{\partial s_i^2} = -w_i^{\bullet\bullet}, \qquad \kappa_{xs_i} = -2\frac{\partial^2 w_i}{\partial x \partial s_i} = -2w_i'^{\bullet}$$

10.2.2 Potential Energy of the Folded Structure

The potential energy of the whole folded structure can be obtained by summarizing the energy of all the n strips

$$\Pi = \frac{1}{2}\sum_{(i)} \int_0^l \int_0^{d_i} \left[\boldsymbol{N}^{\mathrm{T}}\, \boldsymbol{M}^{\mathrm{T}}\right]_i \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix}_i \mathrm{d}s_i \mathrm{d}x - W_{\mathrm{a}} \tag{10.2.7}$$

With equation (10.2.4) the vectors of the resultant forces and moments can be expressed and we obtain

$$\Pi = \frac{1}{2}\sum_{(i)} \int_0^l \int_0^{d_i} \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix}_i^{\mathrm{T}} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{B} & \boldsymbol{D} \end{bmatrix}_i \begin{bmatrix} \boldsymbol{\varepsilon} \\ \boldsymbol{\kappa} \end{bmatrix}_i \mathrm{d}s_i \mathrm{d}x - W_{\mathrm{a}} \tag{10.2.8}$$

The external work of the loads is also the sum of all the n strips

$$\begin{aligned} W_{\mathrm{a}} = \sum_i \Bigg\{ & \frac{1}{2} \int_0^l \int_0^{d_i} 2(p_{x_i}u_i + p_{s_i}v_i + p_{n_i}w_i)\, \mathrm{d}s_i\, \mathrm{d}x \\ & + \int_0^{d_i} \left[(q_{x_i}u_i + q_{s_i}v_i + q_{n_i}w_i)\Big|_{x=0} + (q_{x_i}u_i + q_{s_i}v_i + q_{n_i}w_i)\Big|_{x=l} \right] \mathrm{d}s_i \Bigg\} \end{aligned} \tag{10.2.9}$$

After some steps considering the Eqs. (10.2.4) and (10.2.6) Eq. (10.2.9) leads to

$$\begin{aligned} \Pi = \sum_{(i)} \Bigg\{ \frac{1}{2} \int_0^l \int_0^{d_i} & \Big[A_{11_i}u_i'^2 + 2A_{12_i}u_i'v_i^{\bullet} + 2A_{16_i}u_i'(u_i^{\bullet} + v_i') \\ & + A_{22_i}v_i^{\bullet 2} + 2A_{26_i}v_i^{\bullet}(u_i^{\bullet} + v_i') + A_{66_i}\left(u_i^{\bullet} + v_i'\right)^2 \\ & - 2B_{11_i}u_i'w_i'' - 2B_{12_i}u_i'w_i^{\bullet\bullet} - 2B_{12_i}v_i^{\bullet}w_i'' \\ & - 4B_{16_i}u_i'w_i'^{\bullet} - 2B_{16_i}(u_i^{\bullet} + v_i')w_i'' - 2B_{22_i}v_i^{\bullet}w_i^{\bullet\bullet} - 4B_{26_i}v_i^{\bullet}w_i'^{\bullet} \\ & - 2B_{26_i}(u_i^{\bullet} + v_i')w_i^{\bullet\bullet} - 4B_{66_i}(u_i^{\bullet} + v_i')w_i'^{\bullet} \\ & + D_{11_i}w_i''^2 + 2D_{12_i}w_i''w_i^{\bullet\bullet} + 4D_{16_i}w_i''w_i'^{\bullet} \\ & + D_{22_i}w_i^{\bullet\bullet 2} + 4D_{26_i}w_i^{\bullet\bullet}w_i'^{\bullet} + 4D_{66_i}w_i'^{\bullet 2} \\ & - 2(p_{x_i}u_i + p_{s_i}v_i + p_{n_i}w_i)\Big]\, \mathrm{d}s_i\, \mathrm{d}x \end{aligned} \tag{10.2.10}$$

$$- \int_0^{d_i} [(q_{x_i}u_i + q_{s_i}v_i + q_{n_i}w_i)|_{x=0} + (q_{x_i}u_i + q_{s_i}v_i + q_{n_i}w_i)|_{x=l}]\, \mathrm{d}s_i \Bigg\}$$

10.2.3 Reduction of the Two-dimensional Problem

Equation (10.2.10) represents the complete folded structure model, because it contains all the energy terms of the membrane stress state and of the bending/torsional stress state under the validity of the Kirchhoff hypotheses. An analytical solution of this model equations is really impossible with the exception of some very simple cases. Therefore here we will take another way. As the main object of this section we will find approximate solutions by reducing the two-dimensional problem to an one-dimensional one taking into account the so-called Kantorovich separation relationships (Sect. 2.2).

For the displacements u_i, v_i, w_i in the ith strip we write the approximative series solutions

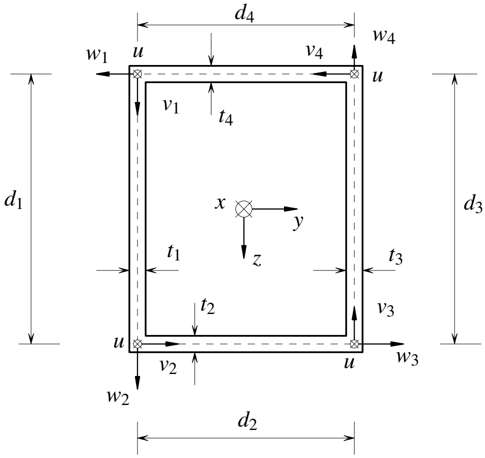
$$\begin{aligned} u_i(x,s_i) &= \sum_{(j)} U_j(x)\varphi_{ij}(s_i) = \boldsymbol{U}^{\mathrm{T}}\boldsymbol{\varphi} = \boldsymbol{\varphi}^{\mathrm{T}}\boldsymbol{U}, \\ v_i(x,s_i) &= \sum_{(k)} V_k(x)\psi_{ik}(s_i) = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{\psi} = \boldsymbol{\psi}^{\mathrm{T}}\boldsymbol{V}, \\ w_i(x,s_i) &= \sum_{(k)} V_k(x)\xi_{ik}(s_i) = \boldsymbol{V}^{\mathrm{T}}\boldsymbol{\xi} = \boldsymbol{\xi}^{\mathrm{T}}\boldsymbol{V} \end{aligned} \tag{10.2.11}$$

Here the $\varphi_{ij}(s_i), \psi_{ik}(s_i), \xi_{ik}(s_i)$ are a priori given trial functions of the co-ordinates s_i and $U_j(x), V_k(x)$ unknown coefficient functions of the longitudinal co-ordinate x. Vlasov defined the $\varphi_{ij}(s_i), \psi_{ik}(s_i), \xi_{ik}(s_i)$ as the generalized co-ordinate functions and the $U_j(x), V_k(x)$ as the generalized displacement functions. Of course it is very important for the quality of the approximate solution, what kind and which number of generalized co-ordinates $\varphi_{ij}(s_i), \psi_{ik}(s_i), \xi_{ik}(s_i)$ are used.

Now we consider a closed thin-walled cross-section, e.g. the cross-section of a box-girder, and follow the Vlasov's hypotheses:

- The out-of-plane displacements $u_i(s_i)$ are approximately linear functions of s_i. In this case there are n^* linear independent trial functions φ_{ij}. n^* is the number of parallel strip edge lines of cross-section.
- The strains $\varepsilon_{s_i}(s_i)$ can be neglected, i.e. $\varepsilon_{s_i} \approx 0$. The trial functions $\psi_{ik}(s_i)$ are then constant functions in all strips and we have n^{**} linear independent $\psi_{ik}(s_i)$ and $\xi_{ik}(s_i)$ with $n^{**} = 2n^* - m^*$. m^* is the number of strips of the thin-walled structure and n^* is defined above.

The generalized co-ordinate functions can be obtained as unit displacement states in longitudinal direction (φ) and in transversal directions (ψ, ξ). Usually however generalized co-ordinate functions are used, which allow mechanical interpretations. In Fig. 10.4, e.g., the generalized co-ordinate functions for a one-cellular rectangular cross-section are shown. φ_1 characterizes the longitudinal displacement of the



$$d_1 = d_3 = d_S$$
$$d_2 = d_4 = d_G$$

$$a = d_S/2, \quad b = d_G/2$$
$$c = d_S d_G/4$$

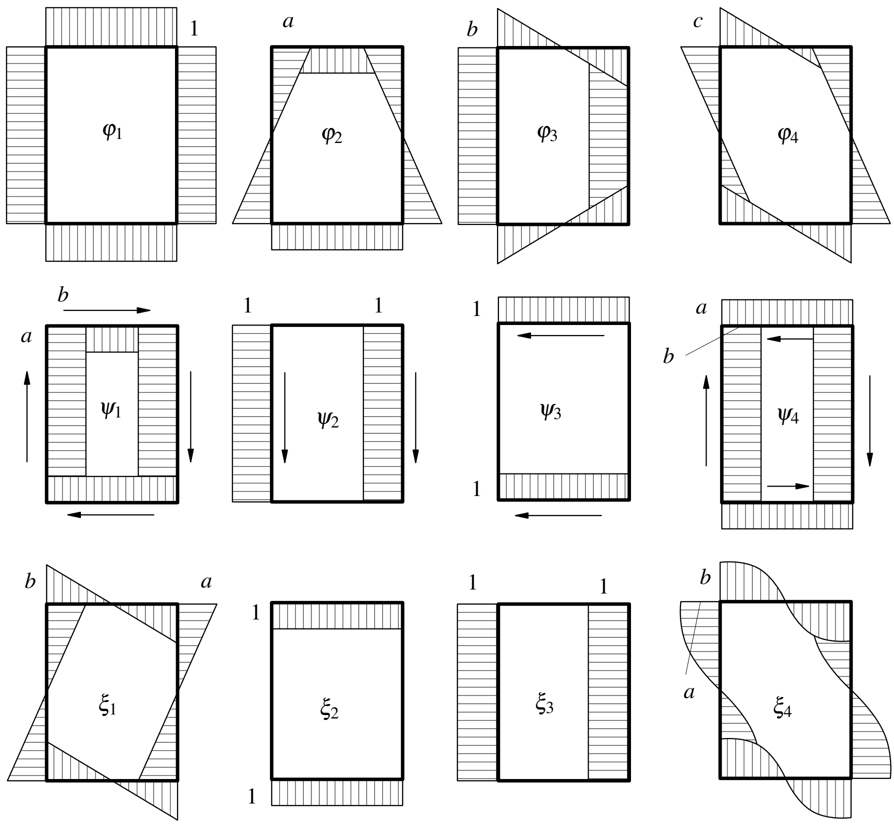


Fig. 10.4 Generalized coordinate functions of an one-cellular rectangular cross-section

whole cross-section, φ_2 and φ_3 its rotations about the global y- and z-axes. φ_1, φ_2, φ_3 represent the plane cross-section displacements, while φ_4 shows its warping. ψ_2 and ψ_3 characterize the plan cross-section displacements in z- and y-direction and ψ_1 the rotation of the rigid cross-section about the system axis x. ψ_4 defines a cross-sectional contour deformation, e.g. a distorsion. The generalized co-ordinate functions ξ_1, ξ_2, ξ_3, ξ_4 represent displacements of the strips corresponding to ψ_1, ψ_2, ψ_3, ψ_4. For the example of a box-girder cross-section there is $n^* = 4$, $m^* = 4$ and $n^{**} = 8 - 4 = 4$.

In the following more general derivations the strains ε_{s_i} will be included, we will take into account more complicated forms of warping functions and therefore there are no restrictions for the number of generalized co-ordinate functions. After the input of Eq. (10.2.11) into (10.2.10) and with the definition of the 28 matrices

$$
\begin{aligned}
\hat{\boldsymbol{A}}_1 &= \sum_{(i)} \int_0^{d_i} A_{11_i} \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathrm{T}} \mathrm{d}s_i, &
\hat{\boldsymbol{A}}_2 &= \sum_{(i)} \int_0^{d_i} A_{16_i} \boldsymbol{\varphi}^{\bullet}\boldsymbol{\varphi}^{\mathrm{T}} \mathrm{d}s_i, \\
\hat{\boldsymbol{A}}_3 &= \sum_{(i)} \int_0^{d_i} A_{66_i} \boldsymbol{\varphi}^{\bullet}\boldsymbol{\varphi}^{\bullet\mathrm{T}} \mathrm{d}s_i, &
\hat{\boldsymbol{A}}_4 &= \sum_{(i)} \int_0^{d_i} A_{66_i} \boldsymbol{\psi}\boldsymbol{\psi}^{\mathrm{T}} \mathrm{d}s_i, \\
\hat{\boldsymbol{A}}_5 &= \sum_{(i)} \int_0^{d_i} A_{26_i} \boldsymbol{\psi}^{\bullet}\boldsymbol{\psi}^{\mathrm{T}} \mathrm{d}s_i, &
\hat{\boldsymbol{A}}_6 &= \sum_{(i)} \int_0^{d_i} A_{22_i} \boldsymbol{\psi}^{\bullet}\boldsymbol{\psi}^{\bullet\mathrm{T}} \mathrm{d}s_i, \\
\hat{\boldsymbol{A}}_7 &= \sum_{(i)} \int_0^{d_i} D_{11_i} \boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}} \mathrm{d}s_i, &
\hat{\boldsymbol{A}}_8 &= \sum_{(i)} \int_0^{d_i} D_{16_i} \boldsymbol{\xi}^{\bullet}\boldsymbol{\xi}^{\mathrm{T}} \mathrm{d}s_i, \\
\hat{\boldsymbol{A}}_9 &= \sum_{(i)} \int_0^{d_i} D_{66_i} \boldsymbol{\xi}^{\bullet}\boldsymbol{\xi}^{\bullet\mathrm{T}} \mathrm{d}s_i, &
\hat{\boldsymbol{A}}_{10} &= \sum_{(i)} \int_0^{d_i} D_{12_i} \boldsymbol{\xi}^{\bullet\bullet}\boldsymbol{\xi}^{\mathrm{T}} \mathrm{d}s_i, \\
\hat{\boldsymbol{A}}_{11} &= \sum_{(i)} \int_0^{d_i} D_{26_i} \boldsymbol{\xi}^{\bullet\bullet}\boldsymbol{\xi}^{\bullet\mathrm{T}} \mathrm{d}s_i, &
\hat{\boldsymbol{A}}_{12} &= \sum_{(i)} \int_0^{d_i} D_{22_i} \boldsymbol{\xi}^{\bullet\bullet}\boldsymbol{\xi}^{\bullet\bullet\mathrm{T}} \mathrm{d}s_i, \\
\hat{\boldsymbol{A}}_{13} &= \sum_{(i)} \int_0^{d_i} A_{16_i} \boldsymbol{\varphi}\boldsymbol{\psi}^{\mathrm{T}} \mathrm{d}s_i, &
\hat{\boldsymbol{A}}_{14} &= \sum_{(i)} \int_0^{d_i} A_{12_i} \boldsymbol{\varphi}\boldsymbol{\psi}^{\bullet\mathrm{T}} \mathrm{d}s_i, \\
\hat{\boldsymbol{A}}_{15} &= \sum_{(i)} \int_0^{d_i} A_{66_i} \boldsymbol{\varphi}^{\bullet}\boldsymbol{\psi}^{\mathrm{T}} \mathrm{d}s_i &
\hat{\boldsymbol{A}}_{16} &= \sum_{(i)} \int_0^{d_i} A_{26_i} \boldsymbol{\varphi}^{\bullet}\boldsymbol{\psi}^{\bullet\mathrm{T}} \mathrm{d}s_i, \\
\hat{\boldsymbol{A}}_{17} &= \sum_{(i)} \int_0^{d_i} B_{11_i} \boldsymbol{\varphi}\boldsymbol{\xi}^{\mathrm{T}} \mathrm{d}s_i, &
\hat{\boldsymbol{A}}_{18} &= \sum_{(i)} \int_0^{d_i} B_{16_i} \boldsymbol{\varphi}\boldsymbol{\xi}^{\bullet\mathrm{T}} \mathrm{d}s_i,
\end{aligned}
\tag{10.2.12}
$$

$$\hat{\boldsymbol{A}}_{19} = \sum_{(i)} \int_0^{d_i} B_{12_i} \boldsymbol{\varphi} \boldsymbol{\xi}^{\bullet\bullet\mathrm{T}} \mathrm{d}s_i, \ \hat{\boldsymbol{A}}_{20} = \sum_{(i)} \int_0^{d_i} B_{16_i} \boldsymbol{\varphi}^{\bullet} \boldsymbol{\xi}^{\mathrm{T}} \mathrm{d}s_i,$$

$$\hat{\boldsymbol{A}}_{21} = \sum_{(i)} \int_0^{d_i} B_{66_i} \boldsymbol{\varphi}^{\bullet} \boldsymbol{\xi}^{\bullet\mathrm{T}} \mathrm{d}s_i, \ \hat{\boldsymbol{A}}_{22} = \sum_{(i)} \int_0^{d_i} B_{26_i} \boldsymbol{\varphi}^{\bullet} \boldsymbol{\xi}^{\bullet\bullet\mathrm{T}} \mathrm{d}s_i,$$

$$\hat{\boldsymbol{A}}_{23} = \sum_{(i)} \int_0^{d_i} B_{16_i} \boldsymbol{\psi} \boldsymbol{\xi}^{\mathrm{T}} \mathrm{d}s_i, \quad \hat{\boldsymbol{A}}_{24} = \sum_{(i)} \int_0^{d_i} B_{66_i} \boldsymbol{\psi} \boldsymbol{\xi}^{\bullet\mathrm{T}} \mathrm{d}s_i,$$

$$\hat{\boldsymbol{A}}_{25} = \sum_{(i)} \int_0^{d_i} B_{26_i} \boldsymbol{\psi} \boldsymbol{\xi}^{\bullet\bullet\mathrm{T}} \mathrm{d}s_i, \ \hat{\boldsymbol{A}}_{26} = \sum_{(i)} \int_0^{d_i} B_{12_i} \boldsymbol{\psi}^{\bullet} \boldsymbol{\xi}^{\mathrm{T}} \mathrm{d}s_i,$$

$$\hat{\boldsymbol{A}}_{27} = \sum_{(i)} \int_0^{d_i} B_{26_i} \boldsymbol{\psi}^{\bullet} \boldsymbol{\xi}^{\bullet\mathrm{T}} \mathrm{d}s_i, \ \hat{\boldsymbol{A}}_{28} = \sum_{(i)} \int_0^{d_i} B_{22_i} \boldsymbol{\psi}^{\bullet} \boldsymbol{\xi}^{\bullet\bullet\mathrm{T}} \mathrm{d}s_i$$

and the load vectors

$$\begin{aligned}
\boldsymbol{f}_x &= \sum_{(i)} \int_0^{d_i} p_{x_i} \boldsymbol{\varphi} \mathrm{d}s_i, \quad & \boldsymbol{r}_x &= \sum_{(i)} \int_0^{d_i} q_{x_i} \boldsymbol{\varphi} \mathrm{d}s_i, \\
\boldsymbol{f}_s &= \sum_{(i)} \int_0^{d_i} p_{s_i} \boldsymbol{\psi} \mathrm{d}s_i, \quad & \boldsymbol{r}_s &= \sum_{(i)} \int_0^{d_i} q_{s_i} \boldsymbol{\psi} \mathrm{d}s_i, \\
\boldsymbol{f}_n &= \sum_{(i)} \int_0^{d_i} p_{n_i} \boldsymbol{\xi} \mathrm{d}s_i, \quad & \boldsymbol{r}_n &= \sum_{(i)} \int_0^{d_i} q_{n_i} \boldsymbol{\xi} \mathrm{d}s_i
\end{aligned} \tag{10.2.13}$$

the potential energy in matrix form is obtained as follows

$$\begin{aligned}
\varPi = \frac{1}{2} \int_0^l \Big[& \boldsymbol{U}'^{\mathrm{T}} \hat{\boldsymbol{A}}_1 \boldsymbol{U}' + \boldsymbol{V}^{\mathrm{T}} \hat{\boldsymbol{A}}_6 \boldsymbol{V} + \boldsymbol{U}^{\mathrm{T}} \hat{\boldsymbol{A}}_3 \boldsymbol{U} + 2\boldsymbol{U}^{\mathrm{T}} \hat{\boldsymbol{A}}_{15} \boldsymbol{V}' \\
& + \boldsymbol{V}'^{\mathrm{T}} \hat{\boldsymbol{A}}_4 \boldsymbol{V}' + \boldsymbol{V}''^{\mathrm{T}} \hat{\boldsymbol{A}}_7 \boldsymbol{V}'' + \boldsymbol{V}^{\mathrm{T}} \hat{\boldsymbol{A}}_{12} \boldsymbol{V} + 4\boldsymbol{V}'^{\mathrm{T}} \hat{\boldsymbol{A}}_9 \boldsymbol{V}' \\
& + 2\boldsymbol{U}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{14} \boldsymbol{V} + 2\boldsymbol{U}^{\mathrm{T}} \hat{\boldsymbol{A}}_2 \boldsymbol{U}' + 2\boldsymbol{U}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{13} \boldsymbol{V}' - 2\boldsymbol{U}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{17} \boldsymbol{V}'' \\
& - 2\boldsymbol{U}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{19} \boldsymbol{V} - 4\boldsymbol{U}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{18} \boldsymbol{V}' + 2\boldsymbol{U}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{16} \boldsymbol{V} + 2\boldsymbol{V}^{\mathrm{T}} \hat{\boldsymbol{A}}_5 \boldsymbol{V}' \\
& - 2\boldsymbol{V}^{\mathrm{T}} \hat{\boldsymbol{A}}_{26} \boldsymbol{V}'' - 2\boldsymbol{V}^{\mathrm{T}} \hat{\boldsymbol{A}}_{28} \boldsymbol{V} - 4\boldsymbol{V}^{\mathrm{T}} \hat{\boldsymbol{A}}_{27} \boldsymbol{V}' - 2\boldsymbol{U}^{\mathrm{T}} \hat{\boldsymbol{A}}_{20} \boldsymbol{V}'' \\
& - 2\boldsymbol{V}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{23} \boldsymbol{V}'' - 2\boldsymbol{U}^{\mathrm{T}} \hat{\boldsymbol{A}}_{22} \boldsymbol{V} - 2\boldsymbol{V}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{25} \boldsymbol{V} - 4\boldsymbol{U}^{\mathrm{T}} \hat{\boldsymbol{A}}_{21} \boldsymbol{V}' \\
& - 4\boldsymbol{V}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{24} \boldsymbol{V}' + 2\boldsymbol{V}^{\mathrm{T}} \hat{\boldsymbol{A}}_{10} \boldsymbol{V}'' + 4\boldsymbol{V}'^{\mathrm{T}} \hat{\boldsymbol{A}}_8 \boldsymbol{V}'' + 4\boldsymbol{V}^{\mathrm{T}} \hat{\boldsymbol{A}}_{11} \boldsymbol{V}' \\
& - 2(\boldsymbol{U}^{\mathrm{T}} \boldsymbol{f}_x + \boldsymbol{V}^{\mathrm{T}} \boldsymbol{f}_s + \boldsymbol{V}^{\mathrm{T}} \boldsymbol{f}_n) \Big] \mathrm{d}x \\
& - (\boldsymbol{U}^{\mathrm{T}} \boldsymbol{r}_x + \boldsymbol{V}^{\mathrm{T}} \boldsymbol{r}_s + \boldsymbol{V}^{\mathrm{T}} \boldsymbol{r}_n) \big|_{x=0} - (\boldsymbol{U}^{\mathrm{T}} \boldsymbol{r}_x + \boldsymbol{V}^{\mathrm{T}} \boldsymbol{r}_s + \boldsymbol{V}^{\mathrm{T}} \boldsymbol{r}_n) \big|_{x=l}
\end{aligned} \tag{10.2.14}$$

The variation of the potential energy function (10.2.14) and using

$$
\begin{aligned}
&\frac{\partial \Pi}{\partial \boldsymbol{U}} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \Pi}{\partial \boldsymbol{U}'}\right) = \boldsymbol{0}, \quad \frac{\partial \Pi}{\partial \boldsymbol{V}} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \Pi}{\partial \boldsymbol{V}'}\right) + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{\partial \Pi}{\partial \boldsymbol{V}''}\right) = \boldsymbol{0},\\
&\delta \boldsymbol{U}^{\mathrm{T}}\left[\frac{\partial \Pi}{\partial \boldsymbol{U}'}\right]_{x=0,l} = 0, \quad \delta \boldsymbol{V}^{\mathrm{T}}\left[\frac{\partial \Pi}{\partial \boldsymbol{V}'} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial \Pi}{\partial \boldsymbol{V}''}\right)\right]_{x=0,l} = 0,\\
&\delta \boldsymbol{V}^{'\mathrm{T}}\left[\frac{\partial \Pi}{\partial \boldsymbol{V}''}\right]_{x=0,l} = 0
\end{aligned}
\tag{10.2.15}
$$

leads to a system of matrix differential equations and matrix boundary conditions of the complete thin-walled folded plate structure

$$
\begin{aligned}
&-\hat{\boldsymbol{A}}_1\boldsymbol{U}'' + (\hat{\boldsymbol{A}}_2 - \hat{\boldsymbol{A}}_2^{\mathrm{T}})\boldsymbol{U}' + \hat{\boldsymbol{A}}_3\boldsymbol{U}\\
&+\hat{\boldsymbol{A}}_{17}\boldsymbol{V}''' - (\hat{\boldsymbol{A}}_{13} - 2\hat{\boldsymbol{A}}_{18} + \hat{\boldsymbol{A}}_{20})\boldsymbol{V}''\\
&+(-\hat{\boldsymbol{A}}_{14} + \hat{\boldsymbol{A}}_{15} + \hat{\boldsymbol{A}}_{19} - 2\hat{\boldsymbol{A}}_{21})\boldsymbol{V}' + (\hat{\boldsymbol{A}}_{16} - \hat{\boldsymbol{A}}_{22})\boldsymbol{V} \qquad = \boldsymbol{f}_x\\
&-\hat{\boldsymbol{A}}_{17}^{\mathrm{T}}\boldsymbol{U}''' - (\hat{\boldsymbol{A}}_{13}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{18}^{\mathrm{T}} + \hat{\boldsymbol{A}}_{20}^{\mathrm{T}})\boldsymbol{U}''\\
&+(\hat{\boldsymbol{A}}_{14}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{15}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{19}^{\mathrm{T}} + 2\hat{\boldsymbol{A}}_{21}^{\mathrm{T}})\boldsymbol{U}' + (\hat{\boldsymbol{A}}_{16}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{22}^{\mathrm{T}})\boldsymbol{U}\\
&+\hat{\boldsymbol{A}}_7\boldsymbol{V}'''' + (2\hat{\boldsymbol{A}}_8^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_8 + \hat{\boldsymbol{A}}_{23} - \boldsymbol{A}_{23}^{\mathrm{T}})\boldsymbol{V}'''\\
&-(\hat{\boldsymbol{A}}_4 + 4\hat{\boldsymbol{A}}_9 - \hat{\boldsymbol{A}}_{10} - \hat{\boldsymbol{A}}_{10}^{\mathrm{T}} - 4\hat{\boldsymbol{A}}_{24} + \hat{\boldsymbol{A}}_{26} + \hat{\boldsymbol{A}}_{26}^{\mathrm{T}})\boldsymbol{V}''\\
&+(\hat{\boldsymbol{A}}_5 - \hat{\boldsymbol{A}}_5^{\mathrm{T}} + 2\hat{\boldsymbol{A}}_{11} - 2\hat{\boldsymbol{A}}_{11}^{\mathrm{T}} + \hat{\boldsymbol{A}}_{25} - \hat{\boldsymbol{A}}_{25}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{27} + 2\hat{\boldsymbol{A}}_{27}^{\mathrm{T}})\boldsymbol{V}'\\
&+(\hat{\boldsymbol{A}}_6 + \hat{\boldsymbol{A}}_{12} - 2\hat{\boldsymbol{A}}_{28})\boldsymbol{V} \qquad = \boldsymbol{f}_s + \boldsymbol{f}_n
\end{aligned}
\tag{10.2.16}
$$

$$
\begin{aligned}
\delta\boldsymbol{U}^{\mathrm{T}}\ &[\hat{\boldsymbol{A}}_1\boldsymbol{U}' + \hat{\boldsymbol{A}}_2^{\mathrm{T}}\boldsymbol{U} - \hat{\boldsymbol{A}}_{17}\boldsymbol{V}'' + (\hat{\boldsymbol{A}}_{13} - 2\hat{\boldsymbol{A}}_{18})\boldsymbol{V}'\\
&+(\hat{\boldsymbol{A}}_{14} - \hat{\boldsymbol{A}}_{19})\boldsymbol{V} \pm \boldsymbol{r}_x]_{x=0,l} \qquad = 0\\
\delta\boldsymbol{V}^{\mathrm{T}}\ &[\hat{\boldsymbol{A}}_{17}^{\mathrm{T}}\boldsymbol{U}'' + (\hat{\boldsymbol{A}}_{13}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{18}^{\mathrm{T}} + \hat{\boldsymbol{A}}_{20}^{\mathrm{T}})\boldsymbol{U}' + (\hat{\boldsymbol{A}}_{15}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{21}^{\mathrm{T}})\boldsymbol{U}\\
&-\hat{\boldsymbol{A}}_7\boldsymbol{V}''' + (2\hat{\boldsymbol{A}}_8 - 2\hat{\boldsymbol{A}}_8^{\mathrm{T}} - \hat{\boldsymbol{A}}_{23} + \hat{\boldsymbol{A}}_{23}^{\mathrm{T}})\boldsymbol{V}''\\
&+(\hat{\boldsymbol{A}}_4 + 4\hat{\boldsymbol{A}}_9 - \hat{\boldsymbol{A}}_{10}^{\mathrm{T}} - 4\hat{\boldsymbol{A}}_{24} + \hat{\boldsymbol{A}}_{26}^{\mathrm{T}})\boldsymbol{V}'\\
&+(\hat{\boldsymbol{A}}_5^{\mathrm{T}} + 2\hat{\boldsymbol{A}}_{11}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{25} - 2\hat{\boldsymbol{A}}_{27}^{\mathrm{T}})\boldsymbol{V} \pm \boldsymbol{r}_s \pm \boldsymbol{r}_n]_{x=0,l} \qquad = 0\\
\delta\boldsymbol{V}^{'T}\ &[-\hat{\boldsymbol{A}}_{17}^{\mathrm{T}}\boldsymbol{U}' - \hat{\boldsymbol{A}}_{20}^{\mathrm{T}}\boldsymbol{U}\\
&+\hat{\boldsymbol{A}}_7\boldsymbol{V}'' + (2\hat{\boldsymbol{A}}_8^{\mathrm{T}} - \hat{\boldsymbol{A}}_{23}^{\mathrm{T}})\boldsymbol{V}' + (\hat{\boldsymbol{A}}_{10}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{26}^{\mathrm{T}})\boldsymbol{V}]_{x=0,l} \qquad = 0
\end{aligned}
\tag{10.2.17}
$$

In Eqs. (10.2.17) the upper sign (+) is valid for the boundary $x = 0$ of the structure and the lower one (−) for the boundary $x = l$. This convention is also valid for all following simplified models.

10.2.4 Simplified Structural Models

Starting from the complete folded structure model two ways of derivation simplified models are usual:

- Neglecting of special terms in the potential energy function of the complete folded plate structure.
- Restrictions of the cross-section kinematics by selection of special generalized co-ordinate functions.

For the first way we will consider the energy terms caused by

- the longitudinal curvatures κ_{x_i},
- the transversal strains ε_{s_i},
- the shear deformations of the mid-planes ε_{xs_i} and
- the torsional curvatures κ_{xs_i}

in the strips. But not all possibilities for simplified models shall be taken into account. We will be restricted the considerations to:

A a structure model with neglected longitudinal curvatures κ_{x_i} only,
B a structure model with neglected longitudinal curvatures κ_{x_i} and neglected torsional curvatures κ_{xs_i},
C a structure model with neglected longitudinal curvatures κ_{x_i} and neglected transversal strains ε_{s_i},
D a structure model with neglected longitudinal curvatures κ_{x_i}, neglected transversal strains ε_{s_i} and torsional curvatures κ_{xs_i}, and
E a structure model with neglected longitudinal curvatures κ_{x_i}, neglected transversal strains ε_{s_i} and neglected shear strain ε_{xs_i} of the mid-planes of the strips.

In Fig. 10.5 is given an overview on the development of structural simplified models.

10.2.4.1 Structural Model A

The starting point is the potential energy equation (10.2.10), in which all terms containing w_i'' have to vanish. Together with (10.2.11) and (10.2.12) we find that in this case

$$\hat{\boldsymbol{A}}_7 = \boldsymbol{0}, \hat{\boldsymbol{A}}_8 = \boldsymbol{0}, \hat{\boldsymbol{A}}_{10} = \boldsymbol{0}, \hat{\boldsymbol{A}}_{17} = \boldsymbol{0}, \hat{\boldsymbol{A}}_{20} = \boldsymbol{0}, \hat{\boldsymbol{A}}_{23} = \boldsymbol{0}, \hat{\boldsymbol{A}}_{26} = \boldsymbol{0}$$

The matrix differential equations (10.2.16) and the boundary conditions (10.2.17) change then into

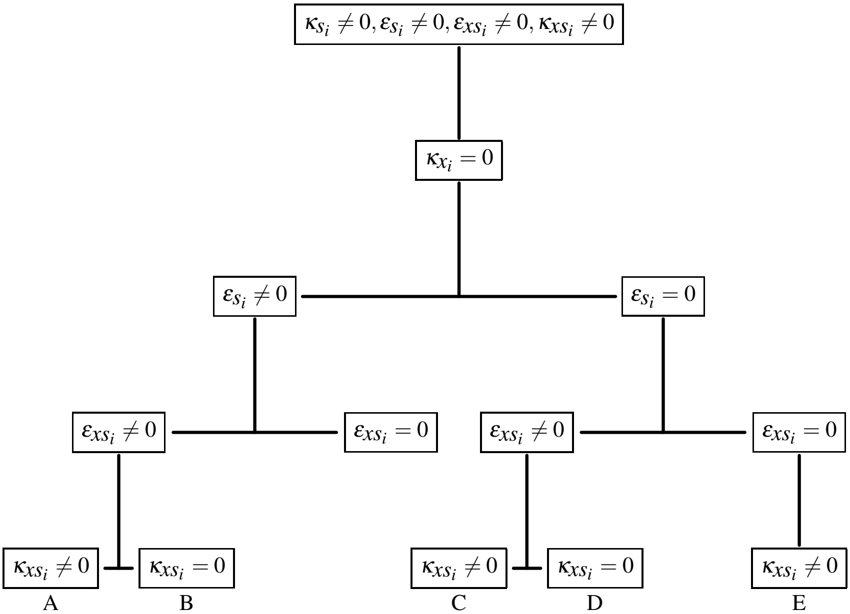


Fig. 10.5 Overview to the derivation of usual simplified models for thin-walled folded plate structures

$$
\begin{aligned}
&-\hat{\boldsymbol{A}}_1\boldsymbol{U}'' + (\hat{\boldsymbol{A}}_2 - \hat{\boldsymbol{A}}_2^{\mathrm{T}})\boldsymbol{U}' + \hat{\boldsymbol{A}}_3\boldsymbol{U} - (\hat{\boldsymbol{A}}_{13} - 2\hat{\boldsymbol{A}}_{18})\boldsymbol{V}'' \\
&+(-\hat{\boldsymbol{A}}_{14} + \hat{\boldsymbol{A}}_{15} + \hat{\boldsymbol{A}}_{19} - 2\hat{\boldsymbol{A}}_{21})\boldsymbol{V}' + (\hat{\boldsymbol{A}}_{16} - \hat{\boldsymbol{A}}_{22})\boldsymbol{V} &&= \boldsymbol{f}_x, \\
&-(\hat{\boldsymbol{A}}_{13}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{18}^{\mathrm{T}})\boldsymbol{U}'' + (\hat{\boldsymbol{A}}_{14}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{15}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{19}^{\mathrm{T}} + 2\hat{\boldsymbol{A}}_{21}^{\mathrm{T}})\boldsymbol{U}' \\
&+(\hat{\boldsymbol{A}}_{16}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{22}^{\mathrm{T}})\boldsymbol{U} - (\hat{\boldsymbol{A}}_4 + 4\hat{\boldsymbol{A}}_9 - 4\hat{\boldsymbol{A}}_{24})\boldsymbol{V}'' \\
&+(\hat{\boldsymbol{A}}_5 - \hat{\boldsymbol{A}}_5^{\mathrm{T}} + 2\hat{\boldsymbol{A}}_{11} - 2\hat{\boldsymbol{A}}_{11}^{\mathrm{T}} + \hat{\boldsymbol{A}}_{25} - \hat{\boldsymbol{A}}_{25}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{27} + 2\hat{\boldsymbol{A}}_{27}^{\mathrm{T}})\boldsymbol{V}' \\
&+(\hat{\boldsymbol{A}}_6 + \hat{\boldsymbol{A}}_{12} - 2\hat{\boldsymbol{A}}_{28})\boldsymbol{V} &&= \boldsymbol{f}_s + \boldsymbol{f}_n,
\end{aligned}
\tag{10.2.18}
$$

$$
\begin{aligned}
&\delta\boldsymbol{U}^{\mathrm{T}} \Big[\hat{\boldsymbol{A}}_1\boldsymbol{U}' + \hat{\boldsymbol{A}}_2^{\mathrm{T}}\boldsymbol{U} + (\hat{\boldsymbol{A}}_{13} - 2\hat{\boldsymbol{A}}_{18})\boldsymbol{V}' \\
&\qquad + (\hat{\boldsymbol{A}}_{14} - \hat{\boldsymbol{A}}_{19})\boldsymbol{V} \pm \boldsymbol{r}_x\Big]_{x=0,l} &&= 0, \\
&\delta\boldsymbol{V}^{\mathrm{T}} \Big[(\hat{\boldsymbol{A}}_{13}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{18}^{\mathrm{T}})\boldsymbol{U}' + (\hat{\boldsymbol{A}}_{15}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{21}^{\mathrm{T}})\boldsymbol{U} \\
&\qquad + (\hat{\boldsymbol{A}}_4 + 4\hat{\boldsymbol{A}}_9 - 4\hat{\boldsymbol{A}}_{24})\boldsymbol{V}' \\
&\qquad + (\hat{\boldsymbol{A}}_5^{\mathrm{T}} + 2\hat{\boldsymbol{A}}_{11}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{25} - 2\hat{\boldsymbol{A}}_{27}^{\mathrm{T}})\boldsymbol{V} \pm \boldsymbol{r}_s \pm \boldsymbol{r}_n\Big]_{x=0,l} &&= 0
\end{aligned}
\tag{10.2.19}
$$

10.2.4.2 Structural Model B

Here the longitudinal curvatures κ_{x_i} and the torsional curvatures κ_{xs_i} are neglected and therefore in the potential energy additionally to $w_i'' \approx 0$ in model A all terms containing $w_i'^{\bullet}$ have to vanish. Additionally to the case of model A now also the matrices $\hat{\boldsymbol{A}}_9, \hat{\boldsymbol{A}}_{11}, \hat{\boldsymbol{A}}_{18}, \hat{\boldsymbol{A}}_{21}, \hat{\boldsymbol{A}}_{24}, \hat{\boldsymbol{A}}_{27}$ are null-matrices. This leads to the following matrix differential equations and the corresponding boundary conditions:

$$
\begin{aligned}
&-\hat{\boldsymbol{A}}_1\boldsymbol{U}'' + (\hat{\boldsymbol{A}}_2 - \hat{\boldsymbol{A}}_2^{\mathrm{T}})\boldsymbol{U}' + \hat{\boldsymbol{A}}_3\boldsymbol{U} - \hat{\boldsymbol{A}}_{13}\boldsymbol{V}'' \\
&+(-\hat{\boldsymbol{A}}_{14} + \hat{\boldsymbol{A}}_{15} + \hat{\boldsymbol{A}}_{19})\boldsymbol{V}' + (\hat{\boldsymbol{A}}_{16} - \hat{\boldsymbol{A}}_{22})\boldsymbol{V} &&= \boldsymbol{f}_x, \\
&-\hat{\boldsymbol{A}}_{13}^{\mathrm{T}}\boldsymbol{U}'' + (\hat{\boldsymbol{A}}_{14}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{15}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{19}^{\mathrm{T}})\boldsymbol{U}' + (\hat{\boldsymbol{A}}_{16}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{22}^{\mathrm{T}})\boldsymbol{U} - \hat{\boldsymbol{A}}_4\boldsymbol{V}'' \\
&+(\hat{\boldsymbol{A}}_5 - \hat{\boldsymbol{A}}_5^{\mathrm{T}} + \hat{\boldsymbol{A}}_{25} - \hat{\boldsymbol{A}}_{25}^{\mathrm{T}})\boldsymbol{V}' + (\hat{\boldsymbol{A}}_6 + \hat{\boldsymbol{A}}_{12} - \hat{\boldsymbol{A}}_{28})\boldsymbol{V} &&= \boldsymbol{f}_s + \boldsymbol{f}_n,
\end{aligned}
\tag{10.2.20}
$$

$$
\begin{aligned}
\delta\boldsymbol{U}^{\mathrm{T}} &\left[\hat{\boldsymbol{A}}_1\boldsymbol{U}' + \hat{\boldsymbol{A}}_2^{\mathrm{T}}\boldsymbol{U} + \hat{\boldsymbol{A}}_{13}\boldsymbol{V}' + (\hat{\boldsymbol{A}}_{14} - \hat{\boldsymbol{A}}_{19})\boldsymbol{V} \pm \boldsymbol{r}_x\right]_{x=0,l} &&= 0, \\
\delta\boldsymbol{V}^{\mathrm{T}} &\left[\hat{\boldsymbol{A}}_{13}^{\mathrm{T}}\boldsymbol{U}' + \hat{\boldsymbol{A}}_{15}^{\mathrm{T}}\boldsymbol{U} + \hat{\boldsymbol{A}}_4\boldsymbol{V}' + (\hat{\boldsymbol{A}}_5^{\mathrm{T}} - \hat{\boldsymbol{A}}_{25})\boldsymbol{V} \pm \boldsymbol{r}_s \pm \boldsymbol{r}_n\right]_{x=0,l} &&= 0
\end{aligned}
\tag{10.2.21}
$$

10.2.4.3 Structural Model C

In this structure model the longitudinal curvatures κ_{x_i} and the strains ε_{s_i} are neglected. Therefore in this case in the potential energy function (10.2.10) all terms containing w_i'' and $v_i^{\bullet}$ have to vanish. Considering the equations (10.2.11) and (10.2.12) we find, that additionally to the case of the structure model A here the matrices $\hat{\boldsymbol{A}}_5, \hat{\boldsymbol{A}}_6, \hat{\boldsymbol{A}}_{14}, \hat{\boldsymbol{A}}_{16}, \hat{\boldsymbol{A}}_{27}, \hat{\boldsymbol{A}}_{28}$ are null-matrices and in this way we obtain the following matrix differential equations and the corresponding boundary conditions:

$$
\begin{aligned}
&-\hat{\boldsymbol{A}}_1\boldsymbol{U}'' + (\hat{\boldsymbol{A}}_2 - \hat{\boldsymbol{A}}_2^{\mathrm{T}})\boldsymbol{U}' + \hat{\boldsymbol{A}}_3\boldsymbol{U} - (\hat{\boldsymbol{A}}_{13} - 2\hat{\boldsymbol{A}}_{18})\boldsymbol{V}'' \\
&+(\hat{\boldsymbol{A}}_{15} + \hat{\boldsymbol{A}}_{19} - 2\hat{\boldsymbol{A}}_{21})\boldsymbol{V}' - \hat{\boldsymbol{A}}_{22}\boldsymbol{V} &&= \boldsymbol{f}_x, \\
&-(\hat{\boldsymbol{A}}_{13}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{18}^{\mathrm{T}})\boldsymbol{U}'' + (-\hat{\boldsymbol{A}}_{15}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{19}^{\mathrm{T}} + 2\hat{\boldsymbol{A}}_{21}^{\mathrm{T}})\boldsymbol{U}' - \hat{\boldsymbol{A}}_{22}^{\mathrm{T}}\boldsymbol{U} \\
&-(\hat{\boldsymbol{A}}_4 + 4\hat{\boldsymbol{A}}_9 - 4\hat{\boldsymbol{A}}_{24})\boldsymbol{V}'' \\
&+(2\hat{\boldsymbol{A}}_{11} - 2\hat{\boldsymbol{A}}_{11}^{\mathrm{T}} + \hat{\boldsymbol{A}}_{25} - \hat{\boldsymbol{A}}_{25}^{\mathrm{T}})\boldsymbol{V}' + \hat{\boldsymbol{A}}_{12}\boldsymbol{V} &&= \boldsymbol{f}_s + \boldsymbol{f}_n,
\end{aligned}
\tag{10.2.22}
$$

$$
\begin{aligned}
\delta\boldsymbol{U}^{\mathrm{T}} &\left[\hat{\boldsymbol{A}}_1\boldsymbol{U}' + \hat{\boldsymbol{A}}_2^{\mathrm{T}}\boldsymbol{U} + (\hat{\boldsymbol{A}}_{13} - 2\hat{\boldsymbol{A}}_{18})\boldsymbol{V}' - \hat{\boldsymbol{A}}_{19}\boldsymbol{V} \pm \boldsymbol{r}_x\right]_{x=0,l} &&= 0, \\
\delta\boldsymbol{V}^{\mathrm{T}} &\Big[(\hat{\boldsymbol{A}}_{13}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{18}^{\mathrm{T}})\boldsymbol{U}' + (\hat{\boldsymbol{A}}_{15}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{21}^{\mathrm{T}})\boldsymbol{U} \\
&+(\hat{\boldsymbol{A}}_4 + 4\hat{\boldsymbol{A}}_9 - 4\hat{\boldsymbol{A}}_{24})\boldsymbol{V}' + (2\hat{\boldsymbol{A}}_{11}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{25})\boldsymbol{V} \pm \boldsymbol{r}_s \pm \boldsymbol{r}_n\Big]_{x=0,l} &&= 0
\end{aligned}
\tag{10.2.23}
$$

10.2.4.4 Structural Model D

This structure model neglects the longitudinal curvatures κ_{x_i}, the strains ε_{s_i} and the torsional curvatures κ_{xs_i}. In the potential energy function all terms containing $w_i'', v_i^\bullet, w_i'^\bullet$ have to vanish and we find together with (10.2.11) and (10.2.12) that additionally to the structure model C the matrices $\hat{\boldsymbol{A}}_9, \hat{\boldsymbol{A}}_{11}, \hat{\boldsymbol{A}}_{18}, \hat{\boldsymbol{A}}_{21}, \hat{\boldsymbol{A}}_{24}$ are null-matrices. We obtain the matrix differential equations and the boundary conditions in the following form:

$$\begin{aligned}
&-\hat{\boldsymbol{A}}_1\boldsymbol{U}'' + (\hat{\boldsymbol{A}}_2 - \hat{\boldsymbol{A}}_2^{\mathrm{T}})\boldsymbol{U}' + \hat{\boldsymbol{A}}_3\boldsymbol{U} \\
&-\hat{\boldsymbol{A}}_{13}\boldsymbol{V}'' + (\hat{\boldsymbol{A}}_{15} + \hat{\boldsymbol{A}}_{19})\boldsymbol{V}' - \hat{\boldsymbol{A}}_{22}\boldsymbol{V} \quad &= \boldsymbol{f}_x, \\
&-\hat{\boldsymbol{A}}_{13}^{\mathrm{T}}\boldsymbol{U}'' + (-\hat{\boldsymbol{A}}_{15}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{19}^{\mathrm{T}})\boldsymbol{U}' - \hat{\boldsymbol{A}}_{22}^{\mathrm{T}}\boldsymbol{U} \\
&-\hat{\boldsymbol{A}}_4\boldsymbol{V}'' + (\hat{\boldsymbol{A}}_{25} - \hat{\boldsymbol{A}}_{25}^{\mathrm{T}})\boldsymbol{V}' + \hat{\boldsymbol{A}}_{12}\boldsymbol{V} \quad &= \boldsymbol{f}_s + \boldsymbol{f}_n,
\end{aligned} \tag{10.2.24}$$

$$\begin{aligned}
\delta\boldsymbol{U}^{\mathrm{T}} \left[\hat{\boldsymbol{A}}_1\boldsymbol{U}' + \hat{\boldsymbol{A}}_2^{\mathrm{T}}\boldsymbol{U} + \hat{\boldsymbol{A}}_{13}\boldsymbol{V}' - \hat{\boldsymbol{A}}_{19}\boldsymbol{V} \pm \boldsymbol{r}_x\right]_{x=0,l} &= 0, \\
\delta\boldsymbol{V}^{\mathrm{T}} \left[\hat{\boldsymbol{A}}_{13}^{\mathrm{T}}\boldsymbol{U}' + \hat{\boldsymbol{A}}_{15}^{\mathrm{T}}\boldsymbol{U} + \hat{\boldsymbol{A}}_4\boldsymbol{V}' - \hat{\boldsymbol{A}}_{25}\boldsymbol{V} \pm \boldsymbol{r}_s \pm \boldsymbol{r}_n\right]_{x=0,l} &= 0
\end{aligned} \tag{10.2.25}$$

10.2.4.5 Structural Model E

Now the longitudinal curvatures κ_{x_i}, the transversal strains ε_{s_i} and the shear strains ε_{xs_i} of the mid-planes shall be neglected. Therefore in the potential energy function all the terms containing w_i'' and $v_i^\bullet$ vanish again. The neglecting of the shear strains of the mid-planes leads with Eq. (10.2.6) to

$$\varepsilon_{xs_i} = \frac{\partial u_i}{\partial s_i} + \frac{\partial v_i}{\partial x} = u_i^\bullet + v_i' = 0$$

and we can see that the generalized displacement functions $\boldsymbol{U}(x)$ and $\boldsymbol{V}(x)$ and also the generalized co-ordinate functions $\boldsymbol{\varphi}(s_i)$ and $\boldsymbol{\psi}(s_i)$ are no more independent from each other

$$u_i^\bullet = -v_i', \boldsymbol{U}^{\mathrm{T}}\boldsymbol{\varphi}^\bullet = -\boldsymbol{V}'^{\mathrm{T}}\boldsymbol{\psi}, \boldsymbol{\varphi}^\bullet = \boldsymbol{\psi}, \boldsymbol{U} = -\boldsymbol{V}', \boldsymbol{U}' = -\boldsymbol{V}'', \boldsymbol{U}'' = -\boldsymbol{V}''' \tag{10.2.26}$$

Therefore the potential energy function must be reformulated before the variation of $\boldsymbol{U}$ and $\boldsymbol{V}$. Considering the vanishing terms $w_i'', v_i^\bullet$ and (10.2.26) we obtain the potential energy function in the following form:

$$
\begin{aligned}
\Pi = \frac{1}{2}\int_0^l \Big[& \boldsymbol{V}''^{\mathrm{T}}\hat{\boldsymbol{A}}_1\boldsymbol{V}'' + \boldsymbol{V}^{\mathrm{T}}\hat{\boldsymbol{A}}_{12}\boldsymbol{V} + 2\boldsymbol{V}''^{\mathrm{T}}\hat{\boldsymbol{A}}_{19}\boldsymbol{V} + 4\boldsymbol{V}''^{\mathrm{T}}\hat{\boldsymbol{A}}_{18}\boldsymbol{V}' \\
& + 4\boldsymbol{V}'^{\mathrm{T}}\hat{\boldsymbol{A}}_9\boldsymbol{V}' + 4\boldsymbol{V}^{\mathrm{T}}\hat{\boldsymbol{A}}_{11}\boldsymbol{V}' - 2(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{f}_x + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{f}_s + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{f}_n)\Big]\,\mathrm{d}x \\
& - (\boldsymbol{U}^{\mathrm{T}}\boldsymbol{r}_x + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{r}_s + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{r}_n)\,|_{x=0} - (\boldsymbol{U}^{\mathrm{T}}\boldsymbol{r}_x + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{r}_s + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{r}_n)\,|_{x=l}
\end{aligned}
\tag{10.2.27}
$$

The variation of the potential energy, see also (10.2.15), leads to the matrix differential equations and the boundary conditions for the structural model E

$$
\begin{aligned}
&\hat{\boldsymbol{A}}_1\boldsymbol{V}'''' + 2(\hat{\boldsymbol{A}}_{18} - \hat{\boldsymbol{A}}_{18}^{\mathrm{T}})\boldsymbol{V}''' + (-4\hat{\boldsymbol{A}}_9 + \hat{\boldsymbol{A}}_{19} + \hat{\boldsymbol{A}}_{19}^{\mathrm{T}})\boldsymbol{V}'' \\
&+2(\hat{\boldsymbol{A}}_{11} - \hat{\boldsymbol{A}}_{11}^{\mathrm{T}})\boldsymbol{V}' + \hat{\boldsymbol{A}}_{12}\boldsymbol{V} = \boldsymbol{f}_x' + \boldsymbol{f}_s + \boldsymbol{f}_n,
\end{aligned}
\tag{10.2.28}
$$

$$
\begin{aligned}
\delta\boldsymbol{V}^{\mathrm{T}}\ & \Big[-\hat{\boldsymbol{A}}_1\boldsymbol{V}''' + 2(-\hat{\boldsymbol{A}}_{18} + \hat{\boldsymbol{A}}_{18}^{\mathrm{T}})\boldsymbol{V}'' + (4\hat{\boldsymbol{A}}_9 - \hat{\boldsymbol{A}}_{19})\boldsymbol{V}' \\
& +2\hat{\boldsymbol{A}}_{11}^{\mathrm{T}}\boldsymbol{V} + \boldsymbol{f}_x \pm \boldsymbol{r}_x \pm \boldsymbol{r}_n\Big]_{x=0,l} = 0, \\
\delta\boldsymbol{V}'^{\mathrm{T}}\ & \Big[\hat{\boldsymbol{A}}_1\boldsymbol{V}'' + 2\hat{\boldsymbol{A}}_{18}\boldsymbol{V}' + \hat{\boldsymbol{A}}_{19}\boldsymbol{V} \mp \boldsymbol{r}_x\Big]_{x=0,l} = 0
\end{aligned}
\tag{10.2.29}
$$

10.2.4.6 Further Special Models by Restrictions of the Cross-Section Kinematics

All the five given simplified structure models include the neglecting of the longitudinal curvatures κ_{x_i} in the strips. Because in the case of a beam shaped thin-walled structure the influence κ_{x_i} on the deformation state and the stresses of the whole structure can be seen as very small, its neglecting is vindicated here. The main advantage of the given five simplified structure models however is that by neglecting the longitudinal curvatures in the strips we have a decreasing of the order of derivations of the generalized displacement functions $\boldsymbol{U}$ and $\boldsymbol{V}$ in the potential energy. This is an important effect for practical solution strategies of the model equations.

The structure models A and C can be used for the analysis of thin-walled beam shaped structures with open or closed cross-sections. The difference exists only in the including or neglecting of the strains ε_{s_i} in the strips. Usually they can be neglected, if we have not temperature loading or concentrated transversal stiffeners in the analyzed structure. The structure models B and D are valid only for structures with closed cross-sections, because there the torsional curvatures and the torsional moments M_{xs_i} are very small. The use of the structure model E is vindicated only for beam shaped structures with open cross-sections. There the shear strains of the mid-planes of the plate strips have only small influence on the displacements and the stress state of the structure how in opposite to the case of a closed cross-section.

Further for each of the five considered models we can develop model variants restricting the cross-section kinematics by selection of special sets of generalized co-ordinate functions $\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\xi}$. For example, in the structure model B the number of generalized co-ordinate functions is unlimited. In model D in contrast the number of $\boldsymbol{\psi}, \boldsymbol{\xi}$-co-ordinates is limited to n^{**}, see Sect. 10.2.3. Restricting in this model additionally the $\boldsymbol{\varphi}$-co-ordinates to n^*, the semi-moment shell theory for an anisotropic behavior of the strips is obtained. The Eqs. (10.2.24) and (10.2.25) stay unchanged.

A symmetric stacking sequence in all the strips leads for this model to a further simplification, because then we have no coupling between stretching and bending in the strips, all the elements of the coupling matrix $\boldsymbol{B}$ vanish and therefore all the matrices $\hat{\boldsymbol{A}}_{17} - \hat{\boldsymbol{A}}_{28}$ are null-matrices. In this special case the following matrix differential equations and boundary conditions are valid

$$\begin{aligned} -\hat{\boldsymbol{A}}_1\boldsymbol{U}'' + (\hat{\boldsymbol{A}}_2 - \hat{\boldsymbol{A}}_2^{\mathrm{T}})\boldsymbol{U}' + \hat{\boldsymbol{A}}_3\boldsymbol{U} - \hat{\boldsymbol{A}}_{13}\boldsymbol{V}'' + \hat{\boldsymbol{A}}_{15}\boldsymbol{V}' &= \boldsymbol{f}_x, \\ -\hat{\boldsymbol{A}}_{13}^{\mathrm{T}}\boldsymbol{U}'' - \hat{\boldsymbol{A}}_{15}^{\mathrm{T}}\boldsymbol{U}' - \hat{\boldsymbol{A}}_4\boldsymbol{V}'' + \hat{\boldsymbol{A}}_{12}\boldsymbol{V} &= \boldsymbol{f}_s + \boldsymbol{f}_n, \end{aligned} \tag{10.2.30}$$

$$\begin{aligned} \delta\boldsymbol{U}^{\mathrm{T}} \left[\hat{\boldsymbol{A}}_1\boldsymbol{U}' + \hat{\boldsymbol{A}}_2^{\mathrm{T}}\boldsymbol{U} + \hat{\boldsymbol{A}}_{13}\boldsymbol{V}' \pm \boldsymbol{r}_x\right]_{x=0,l} &= 0, \\ \delta\boldsymbol{V}^{\mathrm{T}} \left[\hat{\boldsymbol{A}}_{13}^{\mathrm{T}}\boldsymbol{U}' + \hat{\boldsymbol{A}}_{15}^{\mathrm{T}}\boldsymbol{U} + \hat{\boldsymbol{A}}_4\boldsymbol{V}' \pm \boldsymbol{r}_s \pm \boldsymbol{r}_n\right]_{x=0,l} &= 0 \end{aligned} \tag{10.2.31}$$

If we have symmetric cross-ply laminates in all the plate strips and one of the main axes of them is identical with the global x-axis, there is no stretching/shearing or bending/twisting coupling and therefore additionally to the $\boldsymbol{B}$-matrix the elements $A_{16} = A_{26} = 0, D_{16} = D_{26} = 0$. With these the matrices $\hat{\boldsymbol{A}}_2, \hat{\boldsymbol{A}}_{13}$, are null-matrices, Eq. (10.2.12). In this case the Eqs. (10.2.30) and (10.2.31) lead to

$$\begin{aligned} -\hat{\boldsymbol{A}}_1\boldsymbol{U}'' + \hat{\boldsymbol{A}}_3\boldsymbol{U} + \hat{\boldsymbol{A}}_{15}\boldsymbol{V}' &= \boldsymbol{f}_x, \\ -\hat{\boldsymbol{A}}_{15}\boldsymbol{U}' - \hat{\boldsymbol{A}}_4\boldsymbol{V}'' + \hat{\boldsymbol{A}}_{12}\boldsymbol{V} &= \boldsymbol{f}_s + \boldsymbol{f}_n, \end{aligned} \tag{10.2.32}$$

$$\begin{aligned} \delta\boldsymbol{U}^{\mathrm{T}} \left[\hat{\boldsymbol{A}}_1\boldsymbol{U}' \pm \boldsymbol{r}_x\right]_{x=0,l} &= 0, \\ \delta\boldsymbol{V}^{\mathrm{T}} \left[\hat{\boldsymbol{A}}_{15}^{\mathrm{T}}\boldsymbol{U} + \hat{\boldsymbol{A}}_4\boldsymbol{V}' \pm \boldsymbol{r}_s \pm \boldsymbol{r}_n\right]_{x=0,l} &= 0 \end{aligned} \tag{10.2.33}$$

and we find that we have equations of the same type how in case of the classical semi-moment shell theory of Vlasov. Here however the matrices $\hat{\boldsymbol{A}}$ consider the anisotropic behavior of the strips.

Otherwise if starting from the structure model E then the generalized co-ordinate functions $\boldsymbol{\psi}, \boldsymbol{\xi}$ are restricted to three co-ordinates, representing the rigid cross-section, i.e. (ψ_1, ξ_1), the rotation of the cross-section about the x-axis, (ψ_2, ξ_2) and (ψ_3, ξ_3) the displacements in the global y- and z-direction. If we additionally assume only four φ-functions, then the first three representing the plane cross-section, i.e. φ_1 the displacement in x-direction, φ_2, φ_3 the rotations about the y- and z-axes and φ_4 is a linear warping function, the so-called unit warping function according to the sectorial areas law. Therefore we have a structural model similar the classical Vlasov beam model. The difference is only that in the classical Vlasov beam model isotropic material behavior is assumed and here the anisotropic behavior of the plate strips is considered. Note that for comparison of these both models usually the following correlations should be taken into account:

$U_1(x) = u(x)$ displacements of the plane cross-section in x-direction,
$U_2(x) = \varphi_y = w'(x)$ rotation of the plane cross-section about the y-axis,
$U_3(x) = \varphi_z = v'(x)$ rotation of the plane cross-section about the z-axis,
$U_4(x) = \omega\theta'(x)$ the warping with ω as the unit warping function,
$V_1(x) = \theta(x)$ rotation of the rigid cross-section about the x-axis,
$V_2(x) = v(x)$ displacement of the rigid cross-section in y-direction,
$V_3(x) = w(x)$ displacement of the rigid cross-section in z-direction

More details about the equations shall not be given here.

If we in this anisotropic Vlasov beam model suppress the warping of the cross-section and use only the functions $\varphi_1, \varphi_2, \varphi_3$ representing the plane cross-section kinematics and do not take into account the torsion, we obtain a specialized Bernoulli beam model for laminated beams with thin-walled cross-sections and anisotropic material behavior. In a similar way a specialized Timoshenko beam model can be obtained, if we restrict the generalized co-ordinate functions φ_i to the three functions for the plane cross-section kinematics and take into account only the two ψ-co-ordinates for the displacements in y- and z-direction, here however starting from the structural model D.

The both above discussed quasi beam models are specialized for structures with cross-sections consisting of single thin plate strips without any rule of their arrangement in the cross-section. The curvatures κ_{x_i} are generally neglected. The specialized beam equations described above cannot be compared directly with the beam equations in Chap. 7 because the derivation there is not restricted to thin-walled folded plate cross-sections.

10.2.5 An Efficient Structure Model for the Analysis of General Prismatic Beam Shaped Thin-walled Plate Structures

Because in the following only beam shaped thin-walled structures are analyzed the neglecting of the influence of the longitudinal curvatures κ_{x_i} in the single plate strips is vindicated. How it was mentioned above, we have in this case a decreasing of the order of derivations of the generalized displacement functions in the potential energy, and this is very important for the solution procedures.

The selected structure model shall enable the analysis of thin-walled structures with open, closed and mixed open/closed cross-sections. Because the influence of κ_{xs_i} is small only for closed cross-sections but the influence of ε_{xs_i} can be neglected for open cross-sections only, the selected structure model for general cross-sections have to include the torsional curvatures and the shear strains of the mid-planes.

The strains ε_{s_i} have in the most cases only a small influence and could be neglected generally. But we shall see in Chap. 11 that including ε_{s_i} in the model equations leads an effective way to define the shape functions for special finite elements and therefore also the ε_{s_i} are included in the selected structure model.

Summarizing the above discussion the structure model A is selected as an universal model for the modelling and analysis of beam shaped thin-walled plate structures. An extension of the equations to eigen-vibration problems is given in Sect. 10.2.6.

10.2.6 Free Eigen-Vibration Analysis, Structural Model A

Analogous to the static analysis, the eigen-vibration analysis also shall be restricted to global vibration response. Local vibrations, e.g. vibration of single plates, are excluded. A structure model neglecting the longitudinal curvatures cannot describe local plate strip vibrations. Further only free undamped vibrations are considered.

The starting point is the potential energy function, Eq. (10.2.10), but all terms including κ_{x_i} are neglected. With the potential energy $\Pi(u_i, v_i, w_i)$

$$\begin{aligned}\Pi = \sum_{(i)} \frac{1}{2} \int\limits_0^l \int\limits_0^{d_i} \Big[& A_{11_i} u_i'^2 + 2A_{12_i} u_i' v_i^{\bullet} + 2A_{16_i} u_i' (u_i^{\bullet} + v_i') \\ & + A_{22_i} v_i^{\bullet 2} + 2A_{26_i} v_i^{\bullet} (u_i^{\bullet} + v_i') + A_{66_i} (u_i^{\bullet} + v_i')^2 \\ & - 2B_{12_i} u_i' w_i^{\bullet\bullet} - 4B_{16_i} u_i' w_i'^{\bullet} - 2B_{22_i} v_i^{\bullet} w_i^{\bullet\bullet} - 4B_{26_i} v_i^{\bullet} w_i'^{\bullet} \\ & - 2B_{26_i} (u_i^{\bullet} + v_i') w_i^{\bullet\bullet} - 4B_{66_i} (u_i^{\bullet} + v_i') w_i'^{\bullet} \\ & + D_{22_i} w_i^{\bullet\bullet 2} + 4D_{26_i} w_i^{\bullet\bullet} w_i'^{\bullet} + 4D_{66_i} w_i'^{\bullet 2} \Big] \mathrm{d}s_i \, \mathrm{d}x \end{aligned} \tag{10.2.34}$$

and the kinetic energy $T(u_i, v_i, w_i)$

$$T(\boldsymbol{u}) = \frac{1}{2} \sum_{(i)} \int\limits_0^l \int\limits_0^{d_i} \rho_i t_i \left[\left(\frac{\partial u_i}{\partial t} \right)^2 + \left(\frac{\partial v_i}{\partial t} \right)^2 + \left(\frac{\partial w_i}{\partial t} \right)^2 \right] \mathrm{d}s_i \, \mathrm{d}x, \tag{10.2.35}$$

where ρ_i is the average density of the ith plate strip

$$\rho_i = \frac{1}{t_i} \sum_{k=1}^{n} \rho_i^{(k)} t_i^{(k)} \tag{10.2.36}$$

Because we have thin plate strips only, rotational terms of the kinetic energy can be neglected.

The reduction of the two-dimensional problem is carried out again with the generalized co-ordinate functions $\boldsymbol{\varphi}(s_i), \boldsymbol{\psi}(s_i), \boldsymbol{\xi}(s_i)$, but we must remark that the generalized displacement functions $\boldsymbol{U}, \boldsymbol{V}$ are time-dependent and therefore they are written in the following with a tilde. The reduction relationships are

$$
\begin{aligned}
u_i(x,s_i,t) &= \sum_{(j)} \tilde{U}_j(x,t)\varphi_{ij}(s_i) = \tilde{\boldsymbol{U}}^{\mathrm{T}}\boldsymbol{\varphi} = \boldsymbol{\varphi}^{\mathrm{T}}\tilde{\boldsymbol{U}},\\
v_i(x,s_i,t) &= \sum_{(k)} \tilde{V}_k(x,t)\psi_{ik}(s_i) = \tilde{\boldsymbol{V}}^{\mathrm{T}}\boldsymbol{\psi} = \boldsymbol{\psi}^{\mathrm{T}}\tilde{\boldsymbol{V}},\\
w_i(x,s_i,t) &= \sum_{(k)} \tilde{V}_k(x,t)\xi_{ik}(s_i) = \tilde{\boldsymbol{V}}^{\mathrm{T}}\boldsymbol{\xi} = \boldsymbol{\xi}^{\mathrm{T}}\tilde{\boldsymbol{V}}
\end{aligned}
\tag{10.2.37}
$$

Additionally to the $\hat{\boldsymbol{A}}$ matrices equation (10.2.12) the following matrices are defined:

$$
\hat{\boldsymbol{B}}_1 = \sum_{(i)} \int_0^{d_i} \rho_i t_i \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathrm{T}} \mathrm{d}s_i,\ \hat{\boldsymbol{B}}_2 = \sum_{(i)} \int_0^{d_i} \rho_i t_i \boldsymbol{\psi}\boldsymbol{\psi}^{\mathrm{T}} \mathrm{d}s_i,\ \hat{\boldsymbol{B}}_3 = \sum_{(i)} \int_0^{d_i} \rho_i t_i \boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}} \mathrm{d}s_i \tag{10.2.38}
$$

and we obtain the so-called Lagrange function $L = T - \Pi$

$$
\begin{aligned}
L = \sum_{(i)} \frac{1}{2} \int_0^l \Big[& \dot{\tilde{\boldsymbol{U}}}^{\mathrm{T}} \hat{\boldsymbol{B}}_1 \dot{\tilde{\boldsymbol{U}}} + \dot{\tilde{\boldsymbol{V}}}^{\mathrm{T}} \hat{\boldsymbol{B}}_2 \dot{\tilde{\boldsymbol{V}}} + \dot{\tilde{\boldsymbol{V}}}^{\mathrm{T}} \hat{\boldsymbol{B}}_3 \dot{\tilde{\boldsymbol{V}}}\\
&- (\tilde{\boldsymbol{U}}'^{\mathrm{T}} \hat{\boldsymbol{A}}_1 \tilde{\boldsymbol{U}}' + 2\tilde{\boldsymbol{U}}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{14} \tilde{\boldsymbol{V}} + 2\tilde{\boldsymbol{U}}^{\mathrm{T}} \hat{\boldsymbol{A}}_2 \tilde{\boldsymbol{U}}' + 2\tilde{\boldsymbol{U}}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{13} \tilde{\boldsymbol{V}}'\\
&+ \tilde{\boldsymbol{V}}^{\mathrm{T}} \hat{\boldsymbol{A}}_6 \tilde{\boldsymbol{V}} + 2\tilde{\boldsymbol{U}}^{\mathrm{T}} \hat{\boldsymbol{A}}_{16} \tilde{\boldsymbol{V}} + 2\tilde{\boldsymbol{V}}^{\mathrm{T}} \hat{\boldsymbol{A}}_5 \tilde{\boldsymbol{V}}' + \tilde{\boldsymbol{U}}^{\mathrm{T}} \hat{\boldsymbol{A}}_3 \tilde{\boldsymbol{U}}\\
&+ 2\tilde{\boldsymbol{U}}^{\mathrm{T}} \hat{\boldsymbol{A}}_{15} \tilde{\boldsymbol{V}}' + \tilde{\boldsymbol{V}}'^{\mathrm{T}} \hat{\boldsymbol{A}}_4 \tilde{\boldsymbol{V}}' - 2\tilde{\boldsymbol{U}}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{19} \tilde{\boldsymbol{V}} - 4\tilde{\boldsymbol{U}}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{18} \tilde{\boldsymbol{V}}'\\
&- 2\tilde{\boldsymbol{V}}^{\mathrm{T}} \hat{\boldsymbol{A}}_{28} \tilde{\boldsymbol{V}} - 4\tilde{\boldsymbol{V}}^{\mathrm{T}} \hat{\boldsymbol{A}}_{27} \tilde{\boldsymbol{V}}' - 2\tilde{\boldsymbol{U}}^{\mathrm{T}} \hat{\boldsymbol{A}}_{22} \tilde{\boldsymbol{V}} - 2\tilde{\boldsymbol{V}}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{25} \tilde{\boldsymbol{V}}\\
&- 4\tilde{\boldsymbol{U}}^{\mathrm{T}} \hat{\boldsymbol{A}}_{21} \tilde{\boldsymbol{V}}' - 4\tilde{\boldsymbol{V}}'^{\mathrm{T}} \hat{\boldsymbol{A}}_{24} \tilde{\boldsymbol{V}}'\\
&+ \tilde{\boldsymbol{V}}^{\mathrm{T}} \hat{\boldsymbol{A}}_{12} \tilde{\boldsymbol{V}} + 4\tilde{\boldsymbol{V}}^{\mathrm{T}} \hat{\boldsymbol{A}}_{11} \tilde{\boldsymbol{V}}' + 4\tilde{\boldsymbol{V}}'^{\mathrm{T}} \hat{\boldsymbol{A}}_9 \tilde{\boldsymbol{V}}') \Big] \mathrm{d}x
\end{aligned}
\tag{10.2.39}
$$

The time derivations of the generalized displacement functions are written with the point symbol

$$
\dot{\tilde{\boldsymbol{U}}} = \frac{\partial \tilde{\boldsymbol{U}}}{\partial t}, \qquad \dot{\tilde{\boldsymbol{V}}} = \frac{\partial \tilde{\boldsymbol{V}}}{\partial t} \tag{10.2.40}
$$

The Hamilton principle yields the variational statement

$$
\delta \int_{t_1}^{t_2} L \mathrm{d}t = \int_{t_1}^{t_2} \delta L \mathrm{d}t = 0, \quad L = L(x, t, \tilde{\boldsymbol{U}}, \tilde{\boldsymbol{V}}, \tilde{\boldsymbol{U}}', \tilde{\boldsymbol{V}}', \dot{\tilde{\boldsymbol{U}}}, \dot{\tilde{\boldsymbol{V}}}) \tag{10.2.41}
$$

and we obtain two differential equations

$$
\begin{aligned}
\frac{\partial L}{\partial \tilde{\boldsymbol{U}}} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial \tilde{\boldsymbol{U}}'}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\tilde{\boldsymbol{U}}}}\right) &= \boldsymbol{0},\\
\frac{\partial L}{\partial \tilde{\boldsymbol{V}}} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial L}{\partial \tilde{\boldsymbol{V}}'}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\tilde{\boldsymbol{V}}}}\right) &= \boldsymbol{0}
\end{aligned}
\tag{10.2.42}
$$

If further harmonic relationships for the generalized displacement functions are assumed

$$\tilde{\boldsymbol{U}}(x,t) = \boldsymbol{U}(x)\sin\omega_0 t, \quad \tilde{\boldsymbol{V}}(x,t) = \boldsymbol{V}(x)\sin\omega_0 t \tag{10.2.43}$$

and we obtain after some steps the following matrix differential equations:

$$\begin{aligned}
&-\hat{\boldsymbol{A}}_1\boldsymbol{U}'' + (\hat{\boldsymbol{A}}_2 - \hat{\boldsymbol{A}}_2^{\mathrm{T}})\boldsymbol{U}' + (\hat{\boldsymbol{A}}_3 - \omega_0^2\hat{\boldsymbol{B}}_1)\boldsymbol{U} - (\hat{\boldsymbol{A}}_{13} - 2\hat{\boldsymbol{A}}_{18})\boldsymbol{V}'' \\
&+(-\hat{\boldsymbol{A}}_{14} + \hat{\boldsymbol{A}}_{15} + \hat{\boldsymbol{A}}_{19} - 2\hat{\boldsymbol{A}}_{21})\boldsymbol{V}' + (\hat{\boldsymbol{A}}_{16} - \hat{\boldsymbol{A}}_{22})\boldsymbol{V} &&= \boldsymbol{0}, \\
&-(\hat{\boldsymbol{A}}_{13}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{18}^{\mathrm{T}})\boldsymbol{U}'' + (\hat{\boldsymbol{A}}_{14}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{15}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{19}^{\mathrm{T}} + 2\hat{\boldsymbol{A}}_{21}^{\mathrm{T}})\boldsymbol{U}' \\
&+(\hat{\boldsymbol{A}}_{16}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{22}^{\mathrm{T}})\boldsymbol{U} - (\hat{\boldsymbol{A}}_4 + 4\hat{\boldsymbol{A}}_9 - 4\hat{\boldsymbol{A}}_{24})\boldsymbol{V}'' \\
&+(\hat{\boldsymbol{A}}_5 - \hat{\boldsymbol{A}}_5^{\mathrm{T}} + 2\hat{\boldsymbol{A}}_{11} - 2\hat{\boldsymbol{A}}_{11}^{\mathrm{T}} + \hat{\boldsymbol{A}}_{25} - \hat{\boldsymbol{A}}_{25}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{27} + 2\hat{\boldsymbol{A}}_{27}^{\mathrm{T}})\boldsymbol{V}' \\
&+(\hat{\boldsymbol{A}}_6 + \hat{\boldsymbol{A}}_{12} - 2\hat{\boldsymbol{A}}_{28} - \omega_0^2\hat{\boldsymbol{B}}_2 - \omega_0^2\hat{\boldsymbol{B}}_3)\boldsymbol{V} &&= \boldsymbol{0},
\end{aligned} \tag{10.2.44}$$

$$\begin{aligned}
&\delta\boldsymbol{U}^{\mathrm{T}}\ [\hat{\boldsymbol{A}}_1\boldsymbol{U}' + \hat{\boldsymbol{A}}_2^{\mathrm{T}}\boldsymbol{U} + (\hat{\boldsymbol{A}}_{13} - 2\hat{\boldsymbol{A}}_{18})\boldsymbol{V}' + (\hat{\boldsymbol{A}}_{14} - \hat{\boldsymbol{A}}_{19})\boldsymbol{V}]_{x=0,l} &&= 0 \\
&\delta\boldsymbol{V}^{\mathrm{T}}\ [(\hat{\boldsymbol{A}}_{13}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{18}^{\mathrm{T}})\boldsymbol{U}' + (\hat{\boldsymbol{A}}_{15}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{21}^{\mathrm{T}})\boldsymbol{U} + (\hat{\boldsymbol{A}}_4 + 4\hat{\boldsymbol{A}}_9 - 4\hat{\boldsymbol{A}}_{24})\boldsymbol{V}' \\
&\quad +(\hat{\boldsymbol{A}}_5^{\mathrm{T}} + 2\hat{\boldsymbol{A}}_{11}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{25} - 2\hat{\boldsymbol{A}}_{27}^{\mathrm{T}})\boldsymbol{V}]_{x=0,l} &&= 0
\end{aligned} \tag{10.2.45}$$

With these equations given above the global free vibration analysis of prismatic beam shaped thin-walled plate structures can be done sufficient exactly.

10.3 Solution Procedures

Two general kinds of solution procedures may be taken into account

- analytic solutions and
- numerical solutions

The consideration below distinguish exact and approximate analytical solution procedures. In the first case an exact solution of the differential equations is carried out. In the other case, the variational statement of the problem is, e.g., solved by the Ritz or Galerkin method, and in general, the procedures yield in an approximate analytical series solution.

Numerical solution procedures essentially consist of methods outgoing from the differential equation or from the corresponding variational problem. The numerical solutions of differential equations may include such methods as finite difference methods, Runge[3]-Kutta[4] methods and transfer matrix methods. The main representative for the second way is the finite element method (FEM). After a few remarks

[3] Carl David Tolmé Runge (*30 August 1856 Bremen – †3 January 1927 Göttingen) - mathematician and physicist

[4] Martin Wilhelm Kutta (*3 November 1867 Pitschen – †25 December 1944 Fürstenfeldbruck) - mathematician

in Sect. 10.3.1 about analytic solution possibilities for the here considered problems, the numerical solution procedure using the transfer matrix method is considered in detail in Sect. 10.3.2. The application of the FEM and the development of special one-dimensional finite elements for beam shaped thin-walled structures are discussed in Chap. 11.

10.3.1 Analytical Solutions

For the generalized beam models given in Sect. 10.2 only for simplified special cases analytical solutions are possible. If we use, for example, the structure model D in connection with a symmetric cross-ply stacking in all plates, what means that the differential equations are from the same type as in case of the isotropic semi-moment shell theory of Vlasov, analytical solutions can be developed for special cross-sections geometry. It is very useful to choose orthogonal generalized co-ordinate functions $\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\xi}$, because it yields the possibility of decomposition of the system of differential equations into some uncoupled partial systems. For example, the generalized co-ordinate functions $\boldsymbol{\varphi}$ in Fig. 10.4 are completely orthogonal and in this way the matrix $\hat{\boldsymbol{A}}_1$ is a diagonal matrix. Therefore some couplings between the single differential equations vanish.

A suitable method for construction an exact solution is the Krylow[5] method or the so-called method of initial parameters. The first step for the application of this method is to convert the system of differential equations into an equivalent differential equation of n-th order.

$$L[y(x)] = \sum_{\nu=0}^{n} a_\nu y^{(\nu)}(x) = r(x) \tag{10.3.1}$$

Its homogeneous solution shall be written as

$$y_h(x) = y(0)K_1(x) + y'(0)K_2(x) + \ldots + y^{(n-1)}(0)K_n(x) \tag{10.3.2}$$

The free constants of the solution are expressed by the initial constants, i.e. the function $y(x)$ and its derivatives till the $(n-1)$th order at $x = 0$. A particular solution can be obtained with

$$y_p(x) = \int_0^x K_n(x-t)r(t)\,\mathrm{d}t \tag{10.3.3}$$

or in case that $r_0(x)$ is not defined for $x \geq 0$ but for $x \geq x_0$

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$$y_p(x) = \left\| {}_{x>x_0} \int_{x_0}^{x} K_n(x-t) r_0(t)\, \mathrm{d}t \right. \tag{10.3.4}$$

Equation (10.3.4) is a quasi closed analytical solution for the differential equation of the structure model D and different functions $r_i(x)$ for respectively $x > x_i, i = 0,1,\ldots,n$

$$\begin{aligned} y(x) &= y(0)K_1(x) + y'(0)K_2(x) + \ldots + y^{(n-1)}(0)K_n(x) \\ &+ \left\| {}_{x>x_0} \int_{x_0}^{x} K_n(x-t) r_0(t)\, \mathrm{d}t + \right\| {}_{x>x_1} \int_{x_1}^{x} K_n(x-t) r_1(t)\, \mathrm{d}t \\ &+ \left\| {}_{x>x_2} \int_{x_2}^{x} K_n(x-t) r_2(t)\, \mathrm{d}t + \ldots \right. \end{aligned} \tag{10.3.5}$$

Complete closed analytical solutions for isotropic double symmetric thin-walled box-girders and general loads one can find in (Altenbach et al, 1994). Also analytical solution for a two-cellular box-girder including shear lag effects is given there. But in the majority of engineering applications refer to general laminated thin-walled structures, an analytical solution has to be ruled out.

10.3.2 Transfer Matrix Method

The differential equations and their boundary conditions are the starting point of a numerical solution by transfer matrix method. At first the system of higher order differential equations has to transfer into a system of differential equations of first order using the natural boundary conditions as definitions of generalized cross-sectional forces.

For sake of simplicity this solution method shall be demonstrated for the structure model D and for a symmetric cross-ply stacking in all plates of the structure. Then the following system of differential equations and boundary conditions are valid, see also Sect. 10.2.4.6 and Eqs. (10.2.32) and (10.2.33)

$$\begin{aligned} \hat{\boldsymbol{A}}_1\boldsymbol{U}'' - \hat{\boldsymbol{A}}_3\boldsymbol{U} - \hat{\boldsymbol{A}}_{15}\boldsymbol{V}' + \boldsymbol{f}_x &= \boldsymbol{0}, \\ \hat{\boldsymbol{A}}_{15}^{\mathrm{T}}\boldsymbol{U}' + \hat{\boldsymbol{A}}_4\boldsymbol{V}'' - \hat{\boldsymbol{A}}_{12}\boldsymbol{V} + \boldsymbol{f}_s + \boldsymbol{f}_n &= \boldsymbol{0}, \end{aligned} \tag{10.3.6}$$

$$\begin{aligned} \delta\boldsymbol{U}^{\mathrm{T}} \left[\hat{\boldsymbol{A}}_1\boldsymbol{U}' \pm \boldsymbol{r}_x\right]_{x=0,l} &= 0, \\ \delta\boldsymbol{V}^{\mathrm{T}} \left[\hat{\boldsymbol{A}}_{15}^{\mathrm{T}}\boldsymbol{U} + \hat{\boldsymbol{A}}_4\boldsymbol{V}' \pm \boldsymbol{r}_s \pm \boldsymbol{r}_n\right]_{x=0,l} &= 0 \end{aligned} \tag{10.3.7}$$

Equations (10.3.7) leads to the definitions of the generalized cross-sectional forces, i.e. generalized longitudinal forces and transverse forces.

$$\boldsymbol{P} = \hat{\boldsymbol{A}}_1 \boldsymbol{U}', \tag{10.3.8}$$

$$\boldsymbol{Q} = \hat{\boldsymbol{A}}_{15}^{\mathrm{T}} \boldsymbol{U} + \hat{\boldsymbol{A}}_4 \boldsymbol{V}', \tag{10.3.9}$$

It can be shown that we have with Eqs. (10.3.8), (10.3.9) really the definitions of generalized forces

$$\begin{aligned}
\boldsymbol{P} &= \sum_{(i)} \int_0^{d_i} \boldsymbol{\varphi} \sigma_{x_i} t_i \mathrm{d}s_i = \sum_{(i)} \int_0^{d_i} \boldsymbol{\varphi} N_{x_i} \mathrm{d}s_i, \\
\boldsymbol{Q} &= \sum_{(i)} \int_0^{d_i} \boldsymbol{\psi} \tau_{xs_i} t_i \mathrm{d}s_i = \sum_{(i)} \int_0^{d_i} \boldsymbol{\psi} N_{xs_i} \mathrm{d}s_i
\end{aligned}$$

In the here considered structure model we have only membrane stresses σ_{x_i} and τ_{xs_i} because the longitudinal curvatures and longitudinal bending moments are neglected in all plates. Additional with cross-ply stacking are $A_{16} = A_{26} = 0$. Therefore and with Eqs. (10.2.4), (10.2.6), (10.2.11) we can write

$$\begin{aligned}
N_{x_i} &= A_{11_i} \varepsilon_{x_i} = A_{11_i} u_i' &&= A_{11_i} \boldsymbol{\varphi}^{\mathrm{T}} \boldsymbol{U}', \\
N_{xs_i} &= A_{66_i} \varepsilon_{xs_i} = A_{66_i} (u_i^{\bullet} + v_i') &&= A_{66_i} (\boldsymbol{\varphi}^{\bullet \mathrm{T}} \boldsymbol{U} + \boldsymbol{\psi}^{\mathrm{T}} \boldsymbol{V}')
\end{aligned}$$

Considering (10.2.12) we obtain again the definitions of the generalized forces given in (10.3.8) and (10.3.9)

$$\begin{aligned}
\boldsymbol{P} &= \sum_{(i)} \int_0^{d_i} A_{11_i} \boldsymbol{\varphi} \boldsymbol{\varphi}^{\mathrm{T}} \mathrm{d}s_i \boldsymbol{U}' &&= \hat{\boldsymbol{A}}_1 \boldsymbol{U}', \\
\boldsymbol{Q} &= \sum_{(i)} \int_0^{d_i} A_{66_i} (\boldsymbol{\psi} \boldsymbol{\varphi}^{\bullet \mathrm{T}} \boldsymbol{U} + \boldsymbol{\psi} \boldsymbol{\psi}^{\mathrm{T}} \boldsymbol{V}') \mathrm{d}s_i &&= \hat{\boldsymbol{A}}_{15}^{\mathrm{T}} \boldsymbol{U} + \hat{\boldsymbol{A}}_4 \boldsymbol{V}'
\end{aligned}$$

The inversion of the Eqs. (10.3.8) and (10.3.9) leads to

$$\boldsymbol{U}' = \hat{\boldsymbol{A}}_1^{-1} \boldsymbol{P}, \tag{10.3.10}$$

$$\boldsymbol{V}' = -\hat{\boldsymbol{A}}_4^{-1} \hat{\boldsymbol{A}}_{15}^{\mathrm{T}} \boldsymbol{U} + \hat{\boldsymbol{A}}_4^{-1} \boldsymbol{Q} \tag{10.3.11}$$

With the first derivatives of Eqs. (10.3.8) and (10.3.9) and after the input of (10.3.10) and (10.3.11) into (10.3.6) we obtain the following system of differential equations of first order

$$\begin{aligned}
\boldsymbol{U}' &= \hat{\boldsymbol{A}}_1^{-1} \boldsymbol{P}, \\
\boldsymbol{V}' &= -\hat{\boldsymbol{A}}_4^{-1} \hat{\boldsymbol{A}}_{15}^{\mathrm{T}} \boldsymbol{U} + \hat{\boldsymbol{A}}_4^{-1} \boldsymbol{Q}, \\
\boldsymbol{P}' &= (\hat{\boldsymbol{A}}_3 - \hat{\boldsymbol{A}}_{15} \hat{\boldsymbol{A}}_4^{-1} \hat{\boldsymbol{A}}_{15}^{\mathrm{T}}) \boldsymbol{U} + \hat{\boldsymbol{A}}_{15} \hat{\boldsymbol{A}}_4^{-1} \boldsymbol{Q} - \boldsymbol{f}_x, \\
\boldsymbol{Q}' &= \hat{\boldsymbol{A}}_{12} \boldsymbol{V} - \boldsymbol{f}_s - \boldsymbol{f}_n
\end{aligned} \tag{10.3.12}$$

respectively written in matrix notation

$$\begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{P} \\ \boldsymbol{Q} \\ 1 \end{bmatrix}' = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \hat{\boldsymbol{A}}_1^{-1} & \boldsymbol{0} & \boldsymbol{o} \\ -\hat{\boldsymbol{A}}_4^{-1}\hat{\boldsymbol{A}}_{15}^{\mathrm{T}} & \boldsymbol{0} & \boldsymbol{0} & \hat{\boldsymbol{A}}_4^{-1} & \boldsymbol{o} \\ (\hat{\boldsymbol{A}}_3 - \hat{\boldsymbol{A}}_{15}\hat{\boldsymbol{A}}_4^{-1}\hat{\boldsymbol{A}}_{15}^{\mathrm{T}}) & \boldsymbol{0} & \boldsymbol{0} & \hat{\boldsymbol{A}}_{15}\hat{\boldsymbol{A}}_4^{-1} & -\boldsymbol{f}_x \\ \boldsymbol{0} & \hat{\boldsymbol{A}}_{12} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{f}_s - \boldsymbol{f}_n \\ \boldsymbol{o}^{\mathrm{T}} & \boldsymbol{o}^{\mathrm{T}} & \boldsymbol{o}^{\mathrm{T}} & \boldsymbol{o}^{\mathrm{T}} & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{P} \\ \boldsymbol{Q} \\ 1 \end{bmatrix}$$

$$\boldsymbol{y}' = \boldsymbol{B}\boldsymbol{y} \tag{10.3.13}$$

$\boldsymbol{B}$ is the system matrix and $\boldsymbol{y}$ the so-called state vector containing all the generalized displacement functions $\boldsymbol{U}$ and $\boldsymbol{V}$ and all the generalized forces $\boldsymbol{P}$ and $\boldsymbol{Q}$. $\boldsymbol{0}$ and $\boldsymbol{o}$ in the $\boldsymbol{B}$-matrix are null matrices and vectors.

The next step is a discretization of the one-dimensional problem, see Fig. 10.6. Between the state vectors at the point $j+1$ and the point j we have generally the relationship

$$\boldsymbol{y}_{j+1} = \boldsymbol{W}_j \boldsymbol{y}_j \tag{10.3.14}$$

where $\boldsymbol{W}_j$ is the transfer matrix for the structure section $j-(j+1)$.

A first order differential equation

$$y'(x) = b\, y(x), \; b = \mathrm{const}$$

has the solution

$$y(x) = C\mathrm{e}^{bx}$$

and with

$$y(x_0) = C\mathrm{e}^{bx_0} \rightarrow C = y(x_0)\mathrm{e}^{-bx_0},$$

we obtain

$$y(x) = y(x_0)\mathrm{e}^{b(x-x_0)}$$

In the same way the solution of the matrix differential equations is

$$\boldsymbol{y}'(x) = \boldsymbol{B}\boldsymbol{y}(x),$$

$$\boldsymbol{y}(x) = \boldsymbol{y}(x_0)\mathrm{e}^{\boldsymbol{B}(x-x_0)}$$

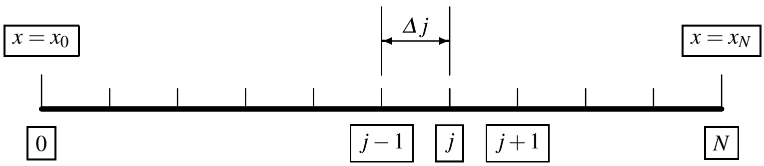

Fig. 10.6 Discretization of the one-dimensional structure

and we find that $e^{\boldsymbol{B}(x-x_0)}$ can be defined as the transfer matrix from the point x_0 to x. Therefore the transfer matrix between two points x_j and x_{j+1} generally is obtained as

$$\boldsymbol{W}_j = e^{\boldsymbol{B}(x_{j+1}-x_j)} \tag{10.3.15}$$

The numerical calculation of transfer matrices can be carried out by series development of the exponential function

$$\boldsymbol{W}_j = \boldsymbol{I} + \triangle_j \boldsymbol{B} + \frac{\triangle_j^2}{2!}\boldsymbol{B}^2 + \frac{\triangle_j^3}{3!}\boldsymbol{B}^3 + \dots \tag{10.3.16}$$

and also by using a Runge-Kutta method

$$\begin{aligned}
\boldsymbol{W}_j &= \boldsymbol{I} + \frac{\triangle_j}{6}(\boldsymbol{M}_{1j} + 2\boldsymbol{M}_{2j} + 2\boldsymbol{M}_{3j} + \boldsymbol{M}_{4j}), \qquad (10.3.17)\\
\boldsymbol{M}_{1j} &= \boldsymbol{B}(x_j),\\
\boldsymbol{M}_{2j} &= \boldsymbol{B}(x_j + \frac{1}{2}\triangle_j)(\boldsymbol{I} + \frac{1}{2}\triangle_j \boldsymbol{M}_{1j}),\\
\boldsymbol{M}_{3j} &= \boldsymbol{B}(x_j + \frac{1}{2}\triangle_j)(\boldsymbol{I} + \frac{1}{2}\triangle_j \boldsymbol{M}_{2j}),\\
\boldsymbol{M}_{4j} &= \boldsymbol{B}(x_j + \triangle_j)(\boldsymbol{I} + \triangle_j \boldsymbol{M}_{3j})
\end{aligned}$$

In both equations $\boldsymbol{I}$ are unit matrices of the same rank as the system matrix. The boundary conditions of the problem can be expressed by a matrix equation

$$\boldsymbol{y}_0 = \boldsymbol{A}\boldsymbol{x}^* \tag{10.3.18}$$

Here $\boldsymbol{A}$ is the so-called start matrix containing the boundary conditions at $x = x_0$ and $\boldsymbol{x}^*$ is the vector of the unknown boundary values there. In the last column of the start matrix the known boundary values are included. For the unknown boundary values the last column elements are zero, and by a unit in the corresponding row the unknown value is associated with an element of the unknown vector $\boldsymbol{x}^*$. For example, in Eq. (10.3.19) a start matrix is shown in case of a free structure end, it means all the displacements are unknown and all forces are given.

$$\begin{bmatrix} U_1 \\ \dots \\ U_m \\ V_1 \\ \dots \\ V_n \\ P_1 \\ \dots \\ P_m \\ Q_1 \\ \dots \\ Q_n \\ 1 \end{bmatrix}_0 = \begin{bmatrix} 1 & & & & & & 0 \\ & \dots & & & & & 0 \\ & & 1 & & & & 0 \\ & & & 1 & & & 0 \\ & & & & \dots & & 0 \\ & & & & & 1 & 0 \\ & & & & & & P_{10} \\ & & & & & & \dots \\ & & & & & & P_{m0} \\ & & & & & & Q_{10} \\ & & & & & & \dots \\ & & & & & & Q_{n0} \\ & & & & & & 1 \end{bmatrix} \begin{bmatrix} x_1^* \\ \dots \\ \dots \\ \dots \\ \dots \\ x_{m+n}^* \\ 1 \end{bmatrix}, \tag{10.3.19}$$

$$\boldsymbol{y}_0 = \boldsymbol{A}_0 \quad \boldsymbol{x}^*$$

Now the multiplications with the transfer matrices can be carried out over all sections (x_j, x_{j+1}) until $x = x_N$. With the equation

$$\boldsymbol{S}\boldsymbol{y}_N = \boldsymbol{0} \tag{10.3.20}$$

the boundary conditions are formulated at $x = x_N$. $\boldsymbol{S}$ is the so-called end matrix containing in its last column the negative values of the given displacements or forces. A unit in an other column of each row yields the association to an element of the state vector $\boldsymbol{y}_N$. Equation (10.3.21) shows the end matrix for a clamped end, where all displacements are given. This matrix equation leads to a system of linear equations for the unknowns in the vector $\boldsymbol{x}^*$

$$\begin{bmatrix} 1 & & & & & \dots & -U_{1N} \\ & \dots & & & & \dots & \dots \\ & & 1 & & & \dots & -U_{mN} \\ & & & 1 & & \dots & -V_{1N} \\ & & & & \dots & \dots & \dots \\ & & & & 1 & \dots & -V_{nN} \end{bmatrix} \begin{bmatrix} U_1 \\ \dots \\ U_m \\ V_1 \\ \dots \\ V_n \\ P_1 \\ \dots \\ P_m \\ Q_1 \\ \dots \\ Q_n \\ 1 \end{bmatrix}_N = \begin{bmatrix} 0 \\ \dots \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix} \tag{10.3.21}$$

The real parts of the eigenvalues of the system matrix $\boldsymbol{B}$ lead to numerical instable solutions especially for long beam structures. From the mechanical point of view it means that the influence of the boundary conditions of both structure ends to each other are very low and with this we have a nearly singular system of linear equations. For the consolidation of this problem intermediate changes of the unknowns are carried out, by formulation of a new start matrix $\boldsymbol{A}$ at such an intermediate point. Usually the generalized displacements are chosen as the new unknowns. The following equations show the general procedure schedule

$$
\begin{aligned}
\boldsymbol{y}_0 &= \boldsymbol{A}_0\boldsymbol{x}_0^* \\
\boldsymbol{y}_1 &= \boldsymbol{W}_0\boldsymbol{y}_0 = \boldsymbol{W}_0\boldsymbol{A}_0\boldsymbol{x}_0^* \\
&\dots \\
\boldsymbol{y}_i &= \boldsymbol{W}_{i-1}\boldsymbol{W}_{i-2}\dots\boldsymbol{W}_0\boldsymbol{A}_0\boldsymbol{x}_0^* = \boldsymbol{F}_i\boldsymbol{x}_0^*
\end{aligned}
$$

$\boxed{\boldsymbol{y}_i = \boldsymbol{A}_1\boldsymbol{x}_1^*}$ first change of unknowns

$$
\begin{aligned}
\boldsymbol{y}_{i+1} &= \boldsymbol{W}_i\boldsymbol{y}_i = \boldsymbol{W}_i\boldsymbol{A}_1\boldsymbol{x}_1^* \\
&\dots \\
\boldsymbol{y}_j &= \boldsymbol{F}_j\boldsymbol{x}_{l-1}^*
\end{aligned}
$$

$\boxed{\boldsymbol{y}_j = \boldsymbol{A}_l\boldsymbol{x}_l^*}$ lth change of unknowns

$$
\begin{aligned}
\boldsymbol{y}_{j+1} &= \boldsymbol{W}_j\boldsymbol{A}_l\boldsymbol{x}_l^* \\
&\dots \\
\boldsymbol{y}_k &= \boldsymbol{F}_k\boldsymbol{x}_{n-1}^*
\end{aligned}
$$

$\boxed{\boldsymbol{y}_k = \boldsymbol{A}_n\boldsymbol{x}_n^*}$ nth change of unknowns

$$
\begin{aligned}
\boldsymbol{y}_{k+1} &= \boldsymbol{W}_k\boldsymbol{A}_n\boldsymbol{x}_n^* \\
&\dots \\
\boldsymbol{y}_N &= \boldsymbol{F}_N\boldsymbol{x}_n^*
\end{aligned}
\tag{10.3.22}
$$

$\boxed{\boldsymbol{S}\boldsymbol{y}_N = \boldsymbol{0}}$ system of linear equations for the solution of the unknowns x_n^*

The multiplications of the state vector $\boldsymbol{y}_0$ with transfer matrices are carried out until the first intermediate change of unknowns. The product of the transfer matrices and the start matrix makes the matrix $\boldsymbol{F}_i$. A new unknown vector $\boldsymbol{x}_1^*$ is defined by the new start matrix $\boldsymbol{A}_1$ and this procedure is repeated at the following intermediate points. General for the l-th intermediate change of unknowns the Eq. (10.3.23) is current

$$\boldsymbol{F}_j\boldsymbol{x}_{l-1}^* = \boldsymbol{A}_l\boldsymbol{x}_l^* \tag{10.3.23}$$

With a segmentation of the state vector $\boldsymbol{y}_j$ into the sub-vectors $\boldsymbol{y}_v$ for the displacements and $\boldsymbol{y}_k$ for the forces

$$\boldsymbol{y}_j = \begin{bmatrix} \boldsymbol{U} \\ \boldsymbol{V} \\ \boldsymbol{P} \\ \boldsymbol{Q} \\ 1 \end{bmatrix}_j = \begin{bmatrix} \boldsymbol{y}_v \\ \boldsymbol{y}_k \\ 1 \end{bmatrix}_j , \tag{10.3.24}$$

we obtain a separated form of Eq. (10.3.23)

$$\begin{bmatrix} \boldsymbol{F}_{1j} & \boldsymbol{f}_{1j} \\ \boldsymbol{F}_{2j} & \boldsymbol{f}_{2j} \\ \boldsymbol{o}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_{l-1}^{*} \\ 1 \end{bmatrix} = \begin{bmatrix} \boldsymbol{A}_{1l} & \boldsymbol{a}_{1l} \\ \boldsymbol{A}_{2l} & \boldsymbol{a}_{2l} \\ \boldsymbol{o}^{\mathrm{T}} & 1 \end{bmatrix} \begin{bmatrix} \tilde{\boldsymbol{x}}_{l}^{*} \\ 1 \end{bmatrix} \tag{10.3.25}$$

With the assumption that the displacements are the new unknowns we find that the sub-matrix $\boldsymbol{A}_{1l}$ is a unit matrix and the sub-vector $\boldsymbol{a}_{1l}$ is a null vector

$$\boldsymbol{A}_{1l} = \boldsymbol{I}, \boldsymbol{a}_{1l} = \boldsymbol{o} \tag{10.3.26}$$

This leads to

$$\boldsymbol{F}_{1j}\tilde{\boldsymbol{x}}_{l-1}^{*} + \boldsymbol{f}_{1j} = \tilde{\boldsymbol{x}}_{l}^{*}, \tag{10.3.27}$$

$$\tilde{\boldsymbol{x}}_{l-1}^{*} = \boldsymbol{F}_{1j}^{-1}(\tilde{\boldsymbol{x}}_{l}^{*} - \boldsymbol{f}_{1j}), \quad \boldsymbol{x}_{l-1}^{*} = \begin{bmatrix} \boldsymbol{F}_{1j}^{-1}(\tilde{\boldsymbol{x}}_{l}^{*} - \boldsymbol{f}_{1j}) \\ 1 \end{bmatrix} \tag{10.3.28}$$

and than the second equation of (10.3.25) yields the structure of the new start matrix

$$\begin{aligned} &\boldsymbol{F}_{2j}\boldsymbol{F}_{1j}^{-1}(\tilde{\boldsymbol{x}}_{l}^{*} - \boldsymbol{f}_{1j}) + \boldsymbol{f}_{2j} = \boldsymbol{A}_{2l}\tilde{\boldsymbol{x}}_{l}^{*} + \boldsymbol{a}_{2l}, \\ &\boldsymbol{A}_{2l} = \boldsymbol{F}_{2j}\boldsymbol{F}_{1j}^{-1}, \ \boldsymbol{a}_{2l} = \boldsymbol{f}_{2j} - \boldsymbol{F}_{2j}\boldsymbol{F}_{1j}^{-1}\boldsymbol{f}_{1j}, \end{aligned} \tag{10.3.29}$$

$$\boldsymbol{A}_{l} = \begin{bmatrix} \boldsymbol{I} & \boldsymbol{o} \\ \boldsymbol{F}_{2j}\boldsymbol{F}_{1j}^{-1} & \boldsymbol{f}_{2j} - \boldsymbol{F}_{2j}\boldsymbol{F}_{1j}^{-1}\boldsymbol{f}_{1j} \\ \boldsymbol{o}^{\mathrm{T}} & 1 \end{bmatrix} \tag{10.3.30}$$

At such an intermediate change point it is also possible to consider the introduction of concentrated generalized forces or the disposition of supports with given generalized displacements. Than the new start matrix must be modified additionally, in the first case by a modification of the sub-vector $\boldsymbol{a}_{2l}$ and in the second case by consideration of the jump behavior of the forces at this point. But more details about this shall not be given here.

With the end matrix $\boldsymbol{S}$ and the end state vector $\boldsymbol{y}_N$ the relationship $\boldsymbol{S}\boldsymbol{y}_N$ yields a system of linear equations for the last unknown vector $\boldsymbol{x}_n^{*}$ and after this all the unknown vectors can be calculated by repeatedly using Eq. (10.3.28).

The transfer matrix method with intermediate changes of the unknown state vectors yields in contrast to the classical transfer method a numerical stable procedure also for long beam structures. From the mechanical point of view correspond each intermediate change $x = x_l^{*}$ a substitution of the structure section $0 \leq x \leq x_l^{*}$ by generalized elastic springs.

The transfer matrix procedure is also applicable to the analysis of eigenvibrations. There we have a modified system matrix $\boldsymbol{B}$ containing frequency dependent terms

$$\boldsymbol{y}' = \boldsymbol{B}(\omega_0)\boldsymbol{y} \qquad (10.3.31)$$

Therefore, the transfer matrices can be calculated only with assumed values for the frequencies. The end matrix leads to a homogenous system of linear equations. Its coefficient determinant must be zero. The assumed frequencies are to vary until this condition is fulfilled sufficiently.

The transfer matrix method with numerical stabilization was applied successfully to several isotropic thin-walled box-beam structures. The structure model D considered above has for a symmetrical cross-ply stacking of all plates an analogous mathematical model structure as isotropic semi-moment shell structures. Therefore, the procedure can be simply transferred to such laminated thin-walled beam structures. An application to other structure models, Sect. 10.2, is in principle possible but rather expansive and not efficient.

The development and application of special finite elements and their implementation in a FEM-program system is more generally and more efficiently. FEM will be discussed in detail in Chap. 11.

10.4 Problems

Exercise 10.1. Establish the system of differential equations for the box-girder with a rectangular cross-section, which is shown in Fig. 10.4. It shall be supposed that its dimensions are symmetric to both axes and therefore we have here

$$\begin{aligned} t_1 &= t_3 = t_{\mathrm{S}}, \\ t_2 &= t_4 = t_{\mathrm{G}}, \\ d_1 &= d_3 = d_{\mathrm{S}}, \\ d_2 &= d_4 = d_{\mathrm{G}}, \end{aligned}$$

Further we have a cross-ply stacking in all the plate strips. The stiffness of both horizontally arranged strips (index G) are the same, but they are different from the stiffness of the vertically arranged strips (index S), what means that

$$\begin{aligned} A_{11\,1} &= A_{11\,3} = A_{11\,\mathrm{S}}, \\ A_{11\,2} &= A_{11\,4} = A_{11\,\mathrm{G}}, \\ A_{66\,1} &= A_{66\,3} = A_{66\,\mathrm{S}}, \\ A_{66\,2} &= A_{66\,4} = A_{66\,\mathrm{G}}, \\ D_{22\,1} &= D_{22\,3} = D_{22\,\mathrm{S}}, \\ D_{22\,2} &= D_{22\,4} = D_{22\,\mathrm{G}} \end{aligned}$$

For the calculation of this box girder the simplified structure model D shall be used and because we have cross-ply stacking, the Eqs. (10.2.32) are valid

$$\begin{aligned}
\hat{\boldsymbol{A}}_1\boldsymbol{U}'' - \hat{\boldsymbol{A}}_3\boldsymbol{U} - \hat{\boldsymbol{A}}_{15}\boldsymbol{V}' + \boldsymbol{f}_x &= \boldsymbol{0},\\
\hat{\boldsymbol{A}}_{15}^{\mathrm{T}}\boldsymbol{U}' + \hat{\boldsymbol{A}}_4\boldsymbol{V}'' - \hat{\boldsymbol{A}}_{12}\boldsymbol{V} + \boldsymbol{f}_s + \boldsymbol{f}_n &= \boldsymbol{0}
\end{aligned}$$

Solution 10.1. At first we have to calculate the matrices $\hat{\boldsymbol{A}}_1, \hat{\boldsymbol{A}}_3, \hat{\boldsymbol{A}}_4, \hat{\boldsymbol{A}}_{12}$ and $\hat{\boldsymbol{A}}_{15}$, their definitions are given in Eq. (10.2.12)

$$\begin{aligned}
\hat{\boldsymbol{A}}_1 &= \sum_{(i)} \int_0^{d_i} A_{11\,i} \boldsymbol{\varphi}\boldsymbol{\varphi}^{\mathrm{T}} \mathrm{d}s_i,\\
\hat{\boldsymbol{A}}_3 &= \sum_{(i)} \int_0^{d_i} A_{66\,i} \boldsymbol{\varphi}^{\bullet}\boldsymbol{\varphi}^{\bullet\mathrm{T}} \mathrm{d}s_i,\\
\hat{\boldsymbol{A}}_4 &= \sum_{(i)} \int_0^{d_i} A_{66\,i} \boldsymbol{\psi}\boldsymbol{\psi}^{\mathrm{T}} \mathrm{d}s_i,\\
\hat{\boldsymbol{A}}_{12} &= \sum_{(i)} \int_0^{d_i} D_{22\,i} \boldsymbol{\xi}^{\bullet\bullet}\boldsymbol{\xi}^{\bullet\bullet\mathrm{T}} \mathrm{d}s_i,\\
\hat{\boldsymbol{A}}_{15} &= \sum_{(i)} \int_0^{d_i} A_{66\,i} \boldsymbol{\varphi}^{\bullet}\boldsymbol{\psi}^{\mathrm{T}} \mathrm{d}s_i,
\end{aligned}$$

The co-ordinate functions $\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\xi}$ are also shown in Fig. 10.4. For solving the integrals to obtain the $\hat{\boldsymbol{A}}$-matrices, the functions $\boldsymbol{\varphi}, \boldsymbol{\psi}, \boldsymbol{\xi}$ must be written as functions of the co-ordinates s_i of each strip. In accordance with Fig. 10.4 we find

$$\begin{aligned}
\varphi_1(s_i) &= +1, i = 1,2,3,4; \quad \varphi_1^{\bullet}(s_i) = 0, i = 1,2,3,4;\\
\varphi_2(s_1) &= -\frac{d_{\mathrm{S}}}{2}\left[1 - 2\left(\frac{s_1}{d_{\mathrm{S}}}\right)\right]; \; \varphi_2^{\bullet}(s_i) = \psi_2(s_i), i = 1,2,3,4;\\
\varphi_2(s_2) &= \frac{d_{\mathrm{S}}}{2};\\
\varphi_2(s_3) &= +\frac{d_{\mathrm{S}}}{2}\left[1 - 2\left(\frac{s_3}{d_{\mathrm{S}}}\right)\right];\\
\varphi_2(s_4) &= -\frac{d_{\mathrm{S}}}{2};\\
\varphi_3(s_1) &= \frac{d_{\mathrm{G}}}{2}; \; \varphi_3^{\bullet}(s_i) = \psi_3(s_i), i = 1,2,3,4;\\
\varphi_3(s_2) &= +\frac{d_{\mathrm{G}}}{2}\left[1 - 2\left(\frac{s_2}{d_{\mathrm{G}}}\right)\right];\\
\varphi_3(s_3) &= -\frac{d_{\mathrm{G}}}{2};\\
\varphi_3(s_4) &= -\frac{d_{\mathrm{G}}}{2}\left[1 - 2\left(\frac{s_4}{d_{\mathrm{G}}}\right)\right];
\end{aligned}$$

$$\varphi_4(s_1) = +\frac{d_S d_G}{4}\left[1-2\left(\frac{s_1}{d_S}\right)\right];\ \varphi_4^{\bullet}(s_i) = \psi_4(s_i), i=1,2,3,4;$$

$$\varphi_4(s_2) = -\frac{d_S d_G}{4}\left[1-2\left(\frac{s_2}{d_G}\right)\right];$$

$$\varphi_4(s_3) = +\frac{d_S d_G}{4}\left[1-2\left(\frac{s_3}{d_S}\right)\right];$$

$$\varphi_4(s_4) = -\frac{d_S d_G}{4}\left[1-2\left(\frac{s_4}{d_G}\right)\right];$$

$$\psi_1(s_1) = -\frac{d_G}{2}; \qquad \psi_2(s_1) = +1;$$

$$\psi_1(s_2) = -\frac{d_S}{2}; \qquad \psi_2(s_2) = 0;$$

$$\psi_1(s_3) = -\frac{d_G}{2}; \qquad \psi_2(s_3) = -1;$$

$$\psi_1(s_4) = -\frac{d_S}{2}; \qquad \psi_2(s_4) = 0;$$

$$\psi_3(s_1) = 0; \qquad \psi_4(s_1) = -\frac{d_G}{2};$$

$$\psi_3(s_2) = -1; \qquad \psi_4(s_2) = +\frac{d_S}{2};$$

$$\psi_3(s_3) = 0; \qquad \psi_4(s_3) = -\frac{d_G}{2};$$

$$\psi_3(s_4) = +1; \qquad \psi_4(s_4) = +\frac{d_S}{2};$$

$$\xi_1(s_1) = -\frac{d_S}{2}\left[1-2\left(\frac{s_1}{d_S}\right)\right]; \quad \xi_1^{\bullet\bullet}(s_i) = 0, i=1,2,3,4;$$

$$\xi_1(s_2) = -\frac{d_G}{2}\left[1-2\left(\frac{s_2}{d_G}\right)\right];$$

$$\xi_1(s_3) = -\frac{d_S}{2}\left[1-2\left(\frac{s_3}{d_S}\right)\right];$$

$$\xi_1(s_4) = -\frac{d_G}{2}\left[1-2\left(\frac{s_4}{d_G}\right)\right];$$

$$\begin{aligned}
\xi_2(s_1) &= 0; \quad \xi_2^{\bullet\bullet}(s_i) = 0, i = 1,2,3,4;\\
\xi_2(s_2) &= +1;\\
\xi_2(s_3) &= 0;\\
\xi_2(s_4) &= -1;\\
\xi_3(s_1) &= +1; \quad \xi_3^{\bullet\bullet}(s_i) = 0, i = 1,2,3,4;\\
\xi_3(s_2) &= 0;\\
\xi_3(s_3) &= -1;\\
\xi_3(s_4) &= 0;
\end{aligned}$$

Some additional considerations are necessary to determine the functions $\xi_4(s_i)$. The generalized co-ordinate function ξ_4 is corresponding to ψ_4 and represents therefore a double antisymmetric deflection state of the cross-section. The cross-section is double symmetric in its geometry and in the elastic behavior. Therefore we must have an antisymmetric function $\xi_4(s_i)$ in each strip. It means that the following conditions are valid

$$\begin{array}{llll}
\xi_4(s_i = 0) & = \xi_{i0}, & \xi_4(s_i = d_i) & = -\xi_{i0},\\
\xi_4^{\bullet}(s_i = 0) & = \alpha_0, & \xi_4^{\bullet}(s_i = d_i) & = \alpha_0,\\
\xi_4^{\bullet\bullet}(s_i = 0) & = \kappa_{s_i0}, & \xi_4^{\bullet\bullet}(s_i = d_i) & = -\kappa_{s_i0}
\end{array}$$

Supposing a polynomial function of the third order, we can write

$$\begin{aligned}
\xi_4^{\bullet\bullet}(s_i) &= \kappa_{s_i0}\left(1-2\frac{s_i}{d_i}\right),\\
\xi_4^{\bullet}(s_i) &= \kappa_{s_i0}d_i\left[\left(\frac{s_i}{d_i}\right)-\left(\frac{s_i}{d_i}\right)^2\right]+\alpha_0,\\
\xi_4(s_i) &= \frac{\kappa_{s_i0}d_i^2}{6}\left[3\left(\frac{s_i}{d_i}\right)^2-2\left(\frac{s_i}{d_i}\right)^3\right]+\alpha_0 d_i\frac{s_i}{d_i}+\xi_{i0}
\end{aligned}$$

The condition $\xi_4(s_i = d_i) = -\xi_{i0}$ leads to

$$\alpha_0 d_i = -\left(\frac{\kappa_{s_i0}d_i^2}{6}+2\xi_{i0}\right)$$

and than we obtain

$$\xi_4(s_i) = \frac{\kappa_{s_i0}d_i^2}{6}\left[3\left(\frac{s_i}{d_i}\right)^2-2\left(\frac{s_i}{d_i}\right)^3-\frac{s_i}{d_i}\right]+\xi_{i0}\left[1-2\left(\frac{s_i}{d_i}\right)\right]$$

With the antisymmetric properties mentioned above we find

$$\kappa_{s_1 0} = \kappa_{s_3 0} = \kappa_{S0},$$
$$\kappa_{s_2 0} = \kappa_{s_4 0} = \kappa_{G0}$$

The continuity of the rotation angles at the corners and the equilibrium equation

$$\begin{aligned}
\xi_4^{\bullet}(s_1 = d_S) &= \xi_4^{\bullet}(s_2 = 0),\\
\xi_4^{\bullet}(s_2 = d_G) &= \xi_4^{\bullet}(s_3 = 0),\\
\xi_4^{\bullet}(s_3 = d_S) &= \xi_4^{\bullet}(s_4 = 0),\\
\xi_4^{\bullet}(s_4 = d_G) &= \xi_4^{\bullet}(s_1 = 0),\\
D_{22S}\kappa_{S0} &= -D_{22G}\kappa_{G0}
\end{aligned}$$

lead to the unknown curvatures

$$\kappa_{S0} = -\frac{12D_{22G}}{d_G D_{22S} + d_S D_{22G}}, \quad \kappa_{G0} = +\frac{12D_{22S}}{d_G D_{22S} + d_S D_{22G}}$$

and we obtain

$$\begin{aligned}
\xi_4(s_1) &= -\frac{2D_{22G}d_S^2}{d_G D_{22S} + d_S D_{22G}}\left[3\left(\frac{s_1}{d_S}\right)^2 - 2\left(\frac{s_1}{d_S}\right)^3 - \frac{s_1}{d_S}\right]\\
&\quad + \frac{d_S}{2}\left[1 - 2\frac{s_1}{d_S}\right],\\
\xi_4(s_2) &= \frac{2D_{22S}d_G^2}{d_G D_{22S} + d_S D_{22G}}\left[3\left(\frac{s_2}{d_G}\right)^2 - 2\left(\frac{s_2}{d_G}\right)^3 - \frac{s_2}{d_G}\right]\\
&\quad - \frac{d_G}{2}\left[1 - 2\frac{s_2}{d_G}\right],\\
\xi_4(s_3) &= -\frac{2D_{22G}d_S^2}{d_G D_{22S} + d_S D_{22G}}\left[3\left(\frac{s_3}{d_S}\right)^2 - 2\left(\frac{s_3}{d_S}\right)^3 - \frac{s_3}{d_S}\right]\\
&\quad + \frac{d_S}{2}\left[1 - 2\frac{s_3}{d_S}\right],\\
\xi_4(s_4) &= \frac{2D_{22S}d_G^2}{d_G D_{22S} + d_S D_{22G}}\left[3\left(\frac{s_4}{d_G}\right)^2 - 2\left(\frac{s_4}{d_G}\right)^3 - \frac{s_4}{d_G}\right]\\
&\quad - \frac{d_G}{2}\left[1 - 2\frac{s_4}{d_G}\right],
\end{aligned}$$

$$\begin{aligned}
\xi_4^{\bullet\bullet}(s_1) &= -\frac{12D_{22G}}{d_G D_{22S} + d_S D_{22G}}\left[1 - 2\frac{s_1}{d_S}\right],\\
\xi_4^{\bullet\bullet}(s_2) &= +\frac{12D_{22S}}{d_G D_{22S} + d_S D_{22G}}\left[1 - 2\frac{s_2}{d_G}\right],\\
\xi_4^{\bullet\bullet}(s_3) &= -\frac{12D_{22G}}{d_G D_{22S} + d_S D_{22G}}\left[1 - 2\frac{s_3}{d_S}\right],\\
\xi_4^{\bullet\bullet}(s_4) &= +\frac{12D_{22S}}{d_G D_{22S} + d_S D_{22G}}\left[1 - 2\frac{s_4}{d_G}\right]
\end{aligned}$$

Now all elements of the matrices can be calculated. Here only the calculation of the element $\hat{A}_{1_{22}}$ of the matrix $\hat{\boldsymbol{A}}_1$ shall be derived in a detailed manner

$$
\begin{aligned}
\hat{A}_{1_{11}} &= \sum_{i=1}^{4} \int_0^{d_i} A_{11_i} \varphi_1(s_i)\varphi_1(s_i)\mathrm{d}s_i = 2(A_{11_S}d_\mathrm{S} + A_{11_G}d_\mathrm{G}),\\
\hat{A}_{1_{12}} &= \sum_{i=1}^{4} \int_0^{d_i} A_{11_i} \varphi_1(s_i)\varphi_2(s_i)\mathrm{d}s_i = 0,\\
\hat{A}_{1_{13}} &= \sum_{i=1}^{4} \int_0^{d_i} A_{11_i} \varphi_1(s_i)\varphi_3(s_i)\mathrm{d}s_i = 0,\\
\hat{A}_{1_{14}} &= \sum_{i=1}^{4} \int_0^{d_i} A_{11_i} \varphi_1(s_i)\varphi_4(s_i)\mathrm{d}s_i = 0,\\
\hat{A}_{1_{22}} &= \sum_{i=1}^{4} \int_0^{d_i} A_{11_i} \varphi_2(s_i)\varphi_2(s_i)\mathrm{d}s_i\\
&= A_{11_S}\frac{d_\mathrm{S}^2}{4}\int_0^{d_\mathrm{S}}\left(1-2\frac{s_1}{d_\mathrm{S}}\right)^2\mathrm{d}s_1 + A_{11_G}\frac{d_\mathrm{S}^2}{4}\int_0^{d_\mathrm{G}}\mathrm{d}s_2\\
&+ A_{11_S}\frac{d_\mathrm{S}^2}{4}\int_0^{d_\mathrm{S}}\left(1-2\frac{s_3}{d_\mathrm{S}}\right)^2\mathrm{d}s_3 + A_{11_G}\frac{d_\mathrm{S}^2}{4}\int_0^{d_\mathrm{G}}\mathrm{d}s_4\\
&= A_{11_S}\frac{d_\mathrm{S}^3}{12} + A_{11_G}\frac{d_\mathrm{S}^2}{4}d_\mathrm{G} + A_{11_S}\frac{d_\mathrm{S}^3}{12} + A_{11_G}\frac{d_\mathrm{S}^2}{4}d_\mathrm{G}\\
&= \frac{d_\mathrm{S}^2}{6}(A_{11_S}d_\mathrm{S} + 3A_{11_G}d_\mathrm{G}),\\
\hat{A}_{1_{23}} &= \sum_{i=1}^{4} \int_0^{d_i} A_{11_i} \varphi_2(s_i)\varphi_3(s_i)\mathrm{d}s_i = 0,\\
\hat{A}_{1_{24}} &= \sum_{i=1}^{4} \int_0^{d_i} A_{11_i} \varphi_2(s_i)\varphi_4(s_i)\mathrm{d}s_i = 0,\\
\hat{A}_{1_{33}} &= \sum_{i=1}^{4} \int_0^{d_i} A_{11_i} \varphi_3(s_i)\varphi_3(s_i)\mathrm{d}s_i = \frac{d_\mathrm{G}^2}{6}(3A_{11_S}d_\mathrm{S} + A_{11_G}d_\mathrm{G}),\\
\hat{A}_{1_{34}} &= \sum_{i=1}^{4} \int_0^{d_i} A_{11_i} \varphi_3(s_i)\varphi_4(s_i)\mathrm{d}s_i = 0,
\end{aligned}
$$

$$
\hat{A}_{1_{44}} = \sum_{i=1}^{4} \int_0^{d_i} A_{11_i} \varphi_4(s_i)\varphi_4(s_i)\mathrm{d}s_i = \frac{d_\mathrm{S}^2 d_\mathrm{G}^2}{24}(A_{11_S}d_\mathrm{S} + A_{11_G}d_\mathrm{G})
$$

With all elements $\hat{A}_{1_{ij}}$ we obtain the matrix $\hat{\boldsymbol{A}}_1$ to

$$\hat{\boldsymbol{A}}_1 = \begin{bmatrix} A_{1_{11}} & 0 & 0 & 0 \\ 0 & A_{1_{22}} & 0 & 0 \\ 0 & 0 & A_{1_{33}} & 0 \\ 0 & 0 & 0 & A_{1_{44}} \end{bmatrix}$$

with

$$\begin{aligned} A_{1_{11}} &= 2(A_{11_S} d_{\mathrm{S}} + A_{11_G} d_{\mathrm{G}}), \\ A_{1_{22}} &= \frac{d_{\mathrm{S}}^2}{6}(A_{11_S} d_{\mathrm{S}} + 3A_{11_G} d_{\mathrm{G}}), \\ A_{1_{33}} &= \frac{d_{\mathrm{G}}^2}{6}(3A_{11_S} d_{\mathrm{S}} + 3A_{11_G} d_{\mathrm{G}}), \\ A_{1_{44}} &= \frac{d_{\mathrm{S}}^2 d_{\mathrm{G}}^2}{24}(A_{11_S} d_{\mathrm{S}} + 3A_{11_G} d_{\mathrm{G}}) \end{aligned}$$

One can see that the generalized co-ordinate functions φ_i are orthogonal to each other and therefore the matrix $\hat{\boldsymbol{A}}_1$ is a diagonal matrix

In the same way the other $\hat{\boldsymbol{A}}_i$ matrices are obtained

$$\hat{\boldsymbol{A}}_3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2A_{66_S} d_{\mathrm{S}} & 0 & 0 \\ 0 & 0 & 2A_{66_G} d_{\mathrm{G}} & 0 \\ 0 & 0 & 0 & \frac{d_{\mathrm{S}} d_{\mathrm{G}}}{2}(A_{66_S} d_{\mathrm{G}} + 3A_{66_G} d_{\mathrm{S}}) \end{bmatrix},$$

$$\hat{\boldsymbol{A}}_4 = \begin{bmatrix} \mathscr{A} & 0 & 0 & 0 \\ 0 & 2A_{66_S} d_{\mathrm{S}} & 0 & 0 \\ 0 & 0 & 2A_{66_G} d_{\mathrm{G}} & 0 \\ \mathscr{B} & 0 & 0 & \mathscr{A} \end{bmatrix},$$

$$\hat{\boldsymbol{A}}_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{d_{\mathrm{S}} d_{\mathrm{G}}}{2}(A_{66_S} d_{\mathrm{G}} - A_{66_G} d_{\mathrm{S}}) & 0 & 0 & \frac{96 D_{22_G} D_{22_S}}{d_{\mathrm{G}} D_{22_S} + d_{\mathrm{S}} D_{22_G}} \end{bmatrix},$$

$$\hat{\boldsymbol{A}}_{15} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 2A_{66_S} d_{\mathrm{S}} & 0 & 0 \\ 0 & 0 & 2A_{66_G} d_{\mathrm{G}} & 0 \\ \mathscr{B} & 0 & 0 & \mathscr{A} \end{bmatrix}$$

with

$$\mathscr{A} = \frac{d_{\mathrm{S}} d_{\mathrm{G}}}{2}(A_{66_S} d_{\mathrm{G}} + A_{66_G} d_{\mathrm{S}}), \quad \mathscr{B} = \frac{d_{\mathrm{S}} d_{\mathrm{G}}}{2}(A_{66_S} d_{\mathrm{G}} - A_{66_G} d_{\mathrm{S}})$$

Now the system of differential equations can be developed with the help of Eq. (10.2.32).

$$
\begin{aligned}
&2(A_{11_S}d_\mathrm{S}+A_{11_G}d_\mathrm{G})U_1'' &&= -f_{x_1},\\
&\frac{d_\mathrm{S}^2}{6}(A_{11_S}d_\mathrm{S}+3A_{11_G}d_\mathrm{G})U_2''-2A_{66_S}d_\mathrm{S}(U_2+V_2') &&= -f_{x_2},\\
&2A_{66_S}d_\mathrm{S}(U_2'+V_2'') &&= -(f_{s_2}+f_{n_2}),\\
&\frac{d_\mathrm{G}^2}{6}(3A_{11_S}d_\mathrm{S}+A_{11_G}d_\mathrm{G})U_3''-2A_{66_G}d_\mathrm{G}(U_3+V_3') &&= -f_{x_3},\\
&2A_{66_G}d_\mathrm{G}(U_3'+V_3'') &&= -(f_{s_3}+f_{n_3}),\\
&\frac{d_\mathrm{S}^2d_\mathrm{G}^2}{24}(A_{11_S}d_\mathrm{S}+A_{11_G}d_\mathrm{G})U_4'' &&\\
&-\frac{d_\mathrm{S}d_\mathrm{G}}{2}(A_{66_S}d_\mathrm{G}+A_{66_G}d_\mathrm{S})(U_4+V_4') &&\\
&-\frac{d_\mathrm{S}d_\mathrm{G}}{2}(A_{66_S}d_\mathrm{G}-A_{66_G}d_\mathrm{S})V_1' &&= -f_{x_4},\\
&\frac{d_\mathrm{S}d_\mathrm{G}}{2}(A_{66_S}d_\mathrm{G}-A_{66_G}d_\mathrm{S})(U_4'+V_4'') &&\\
&+\frac{d_\mathrm{S}d_\mathrm{G}}{2}(A_{66_S}d_\mathrm{G}+A_{66_G}d_\mathrm{S})V_1'' &&= -(f_{s_1}+f_{n_1}),\\
&\frac{d_\mathrm{S}d_\mathrm{G}}{2}(A_{66_S}d_\mathrm{G}+A_{66_G}d_\mathrm{S})(U_4'+V_4'') &&\\
&+\frac{d_\mathrm{S}d_\mathrm{G}}{2}(A_{66_S}d_\mathrm{G}-A_{66_G}d_\mathrm{S})V_1'' &&\\
&-\frac{96D_{22_G}D_{22_S}}{d_\mathrm{G}D_{22_S}+d_\mathrm{S}D_{22_G}}V_4 &&= -(f_{s_4}+f_{n_4})
\end{aligned}
$$

We can see, that the system of differential equations is divided into four decoupled partial systems. The first equation describes the longitudinal displacement, the second and the third partial systems represent the bending about the global y- and z-axes and the fourth - the torsion, the warping and the contour deformation of the cross-section. An analytic solution of the fourth partial system is more difficult like the solutions of the first three partial systems but it is possible too. The analytical solution of an analogous system for an isotropic box girder is given in detail by Vlasov and by the authors of this book, see also the remarks in 10.3.1.

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Part V
Finite Classical and Generalized Beam Elements, Finite Plate Elements

The fifth part (Chap. 11) presents a short introduction into the finite element procedures and developed finite classical and generalized beam elements and finite plate elements in the frame of classical and first order shear deformation theory. Selected examples demonstrate the possibilities of finite element analysis.

Chapter 11
Finite Element Analysis

The Finite Element Method (FEM) is one of the most effective methods for the numerical solution of field problems formulated in partial differential equations. The basic idea of the FEM is a discretization of the continuous structure into substructures. This is equivalent to replacing a domain having an infinite number of degrees of freedom by a system having a finite number of degrees of freedom. The actual continuum or structure is represented as an assembly of subdivisions called finite elements. These elements are considered to be interconnected at specified joints which are called nodes. The discretization is defined by the so-called finite element mesh made up of elements and nodes.

We assume one-dimensional elements, when one dimension is very large in comparison with the others, e.g. truss or beam elements, two-dimensional elements, when one dimension is very small in comparison with the others, e.g. plate or shell elements, and volume elements. From the mechanical point of view the nodes are coupling points of the elements, where the displacements of the coupled elements are compatible. On the other hand from the mathematical point of view the nodes are the basic points for the approximate functions of the displacements inside a finite element and so at these nodes the displacements are compatible. It must be noted here that all considerations are restricted to the displacement method. The force method or hybrid methods are not considered in this book.

An important characteristic of the discretization of a structure is the number of degrees of freedom. To every node, a number of degrees of freedom will be assigned. These are nodal constants which usually (but not necessarily) have a mechanical or more general physical meaning. The number of degrees of freedom per element is defined by the product of the number of nodes per element and degrees of freedom per node. The number of degrees of freedom in the structure is the product of the number of nodes and the number of degrees of freedom per node.

Chapter 11 contains an introduction to the general procedure of finite element analysis in a condensed form (Sect. 11.1). For more detailed information see the vast amount of literature. In Sects. 11.2 and 11.3 the development of finite beam elements and finite plate elements for the analysis of laminate structures is given. Section 11.4 contains the development of generalized finite beam elements based on

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0_11

a generalized structure model for beam shaped thin-walled folded structures given in Sect. 10.2. In Sect. 11.5 the results of some numerical applications show the influences of chosen parameters on the behavior of laminate structures.

11.1 Introduction

The principle of the total minimum potential energy and the Hamilton's principle are given in Sect. 2.2.2 in connection with analytical variational approaches, they are also the theoretical basis of the FEM solutions of elastostatic and of dynamic problems. In this way we have variational problems. For such problems the Ritz method may be used as a so-called direct solution method (see Sect. 2.2.3). In the classical Ritz method the approximation functions are defined for the whole structure, and so for complex geometries it is difficult to realize the requirements of satisfying the boundary conditions and of the linear independence and completeness of these functions.

One way to overcome these difficulties is by the discretization of the structure into a number of substructures, if possible of the same kind (finite elements). Then the approximation functions can be defined for the elements only and they must satisfy the conditions of geometrical compatibility at the element boundaries. Because it is usual to define different types of finite elements, we have special types of approximation functions for each element type. Here the approximation functions are denoted N_i, the so-called shape functions. They are arranged in a matrix $\boldsymbol{N}$, the matrix of the shape functions of the particular element type. The following introduction to the FEM procedure is given in a general but condensed form and illustrates that the step-by-step finite element procedure can be stated as follows:

- Discretization of the structure,
- Selection of a suitable element displacement model,
- Derivation of element stiffness matrices and load vectors,
- Assembly of element equations to obtain the system equations,
- Calculation of the system equations for the unknown nodal displacements,
- Computation of element strains and stresses

11.1.1 FEM Procedure

The starting point for elastostatic problems is the total potential energy given in Eq. (2.2.28). In accordance with the Ritz method the approximation

$$\tilde{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{N}(\boldsymbol{x})\boldsymbol{v} \tag{11.1.1}$$

is used for the displacement field vector $\boldsymbol{u}$. Here $\boldsymbol{N}$ is the matrix of the shape functions, they are functions of the position vector $\boldsymbol{x}$, and $\boldsymbol{v}$ is the element displacement vector. The matrix $\boldsymbol{N}$ has the same number of rows as the displacement vector $\boldsymbol{u}$ has components and the same number of columns as the element displacement vector $\boldsymbol{v}$ has components. If the element has n_{KE} nodes and the degree of freedom for each node is n_F, the element displacement vector $\boldsymbol{v}$ contains n_{KE} subvectors $\boldsymbol{v}_i, i = 1, \ldots, n_{KE}$ with n_F components in each, and so $\boldsymbol{v}$ has $n_{KE} n_F$ components. The number of components of the displacement field vector $\boldsymbol{u}$ is n_u. Then the structure of the matrix $\boldsymbol{N}$ is generally

$$\boldsymbol{N} = [N_1 \boldsymbol{I}_{n_u} \ N_2 \boldsymbol{I}_{n_u} \ \ldots N_{\tilde{n}} \boldsymbol{I}_{n_u}], \quad \tilde{n} = \frac{n_{KE} n_F}{n_u} \tag{11.1.2}$$

with $\boldsymbol{I}_{n_u}$ as unit matrices of the size (n_u, n_u). Therefore the size of $\boldsymbol{N}$ is generally $(n_u, n_{KE} n_F)$. In dependence on the kind of continuity at the element boundaries, the so-called $C^{(0)}$- or $C^{(1)}$-continuity, see below, two cases can be distinguished. In the case of $C^{(0)}$-continuity n_F equals n_u and therefore $\tilde{n}$ is equal n_{K_E}, we have only n_{KE} shape functions N_i, whereas we can have up to $n_{KE} n_F$ shape functions in the case of $C^{(1)}$-continuity.

For the stresses and the strains we obtain from (11.1.1)

$$\begin{aligned} \boldsymbol{\sigma}(\boldsymbol{x}) &= \boldsymbol{C}\boldsymbol{\varepsilon}(\boldsymbol{x}) = \boldsymbol{C}\boldsymbol{D}\boldsymbol{N}(\boldsymbol{x})\boldsymbol{v}, \\ \boldsymbol{\varepsilon}(\boldsymbol{x}) &= \boldsymbol{D}\boldsymbol{u}(\boldsymbol{x}) = \boldsymbol{D}\boldsymbol{N}(\boldsymbol{x})\boldsymbol{v} = \boldsymbol{B}(\boldsymbol{x})\boldsymbol{v} \end{aligned} \tag{11.1.3}$$

With the approximation (11.1.1) the total potential energy is a function of all the nodal displacement components arranged in the element displacement vector $\boldsymbol{v}$, e.g. $\Pi = \Pi(\boldsymbol{v})$. The variation of the total potential energy

$$\delta \Pi = \delta \boldsymbol{v}^{\mathrm{T}} \left(\int\limits_V \boldsymbol{B}^{\mathrm{T}} \boldsymbol{C} \boldsymbol{B} \boldsymbol{v} \mathrm{d}V - \int\limits_V \boldsymbol{N}^{\mathrm{T}} \boldsymbol{p} \mathrm{d}V - \int\limits_{A_q} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{q} \mathrm{d}A \right) \tag{11.1.4}$$

leads with $\delta \Pi = 0$ to

$$\delta \boldsymbol{v}^{\mathrm{T}} (\boldsymbol{K}\boldsymbol{v} - \boldsymbol{f}_p - \boldsymbol{f}_q) = 0 \tag{11.1.5}$$

$\boldsymbol{K}$ is the symmetric stiffness matrix with the size $(n_{KE} n_F, n_{KE} n_F)$

$$\boldsymbol{K} = \int\limits_V \boldsymbol{B}^{\mathrm{T}} \boldsymbol{C} \boldsymbol{B} \mathrm{d}V \tag{11.1.6}$$

and $\boldsymbol{f}_p$ and $\boldsymbol{f}_q$ are the vectors of the volume forces and the surface forces

$$\boldsymbol{f}_p = \int\limits_V \boldsymbol{N}^{\mathrm{T}} \boldsymbol{p} \mathrm{d}V, \quad \boldsymbol{f}_q = \int\limits_{A_q} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{q} \mathrm{d}A \tag{11.1.7}$$

If the components of $\delta \boldsymbol{v}$ are independent of each other, we obtain from (11.1.5) a system of linear equations

$$\boldsymbol{K}\boldsymbol{v} = \boldsymbol{f}, \quad \boldsymbol{f} = \boldsymbol{f}_p + \boldsymbol{f}_q \tag{11.1.8}$$

For elastodynamic problems, we have to consider that forces and displacements are also dependent on time and the Hamilton‘s principle is the starting point for the FEM procedure. Assuming again the independence of the components of $\boldsymbol{\delta v}$ the matrix equation is

$$\boldsymbol{M}\ddot{\boldsymbol{v}}(t)+\boldsymbol{K}\boldsymbol{v}(t)=\boldsymbol{f}(t) \tag{11.1.9}$$

for elastic systems without damping effects. $\boldsymbol{M}$ is symmetric mass matrix

$$\boldsymbol{M}=\int\limits_V \rho\boldsymbol{N}^{\mathrm{T}}\boldsymbol{N}\mathrm{d}V \tag{11.1.10}$$

and $\boldsymbol{f}(t)$ the vector of the time dependent nodal forces. Assuming the damping proportional to the relative velocities, an additional term $\boldsymbol{C}_{\mathrm{D}}\dot{\boldsymbol{v}}(t)$ can be supplemented formally in Eq. (11.1.9)

$$\boldsymbol{M}\ddot{\boldsymbol{v}}(t)+\boldsymbol{C}_{\mathrm{D}}\dot{\boldsymbol{v}}(t)+\boldsymbol{K}\boldsymbol{v}(t)=\boldsymbol{f}(t), \tag{11.1.11}$$

where $\boldsymbol{C}_{\mathrm{D}}$ is the damping matrix. $\boldsymbol{C}_{\mathrm{D}}$ has the same size as the matrices $\boldsymbol{K}$ and $\boldsymbol{M}$ and usually it is formulated approximately as a linear combination of $\boldsymbol{K}$ and $\boldsymbol{M}$. The factors α and β can be chosen to give the correct damping at two frequencies

$$\boldsymbol{C}_{\mathrm{D}}\approx\alpha\boldsymbol{M}+\beta\boldsymbol{K} \tag{11.1.12}$$

In selecting the shape functions $\boldsymbol{N}_i(\boldsymbol{x})$ it must be remembered that these functions must be continuous up to the $(n-1)$th derivative, if we have derivatives of the nth order in the variational problem, i.e. in the total potential energy or in the Hamilton's function. In this case only the results of FEM approximations converge to the real solutions by increasing the number of elements. For more-dimensional finite elements in this way it is to realize that the displacements are compatible up to the $(n-1)$th derivative at the boundaries of adjacent elements, if they are compatible at the nodes.

In plane stress or plane strain problems and in general three-dimensional problems the vector $\boldsymbol{u}$ contains displacements only (no rotations) and the differential operator $\boldsymbol{D}$ is of the 1st order. In this way we must only satisfy the displacements compatibility at the element boundaries that means the so-called $C^{(0)}$-continuity.

By using beam or plate models especially of the classical Bernoulli beam model or the classical Kirchhoff plate model, the rotation angles are expressed by derivatives of the displacements of the midline or the midplane and the differential operator $\boldsymbol{D}$ is of the second order. Then we have to satisfy the compatibility of displacements and rotations at the element boundaries. In such cases we speak about a $C^{(1)}$-continuity and finding the shape functions N_i is more difficult.

Because we have no differential operator in connection with the mass matrix $\boldsymbol{M}$, it would be possible to use other, more simple functions N_i^* for it. In such a case the mass matrix would have another population, e.g. a diagonal matrix structure is possible. Then we speak about a so-called condensed mass matrix, otherwise we have a consistent mass matrix. By using the condensed mass matrix we have less computational expense than by using the consistent mass matrix, but a decreasing convergence to the real results is possible.

All equations considered above are only valid for a single element and strictly they should have an additional index E. For example, we have the inner element energy

$$U_E = \frac{1}{2}\boldsymbol{v}_E^{\mathrm{T}} \int\limits_{V_E} \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{B}\mathrm{d}V \boldsymbol{v}_E = \frac{1}{2}\boldsymbol{v}_E^{\mathrm{T}}\boldsymbol{K}_E\boldsymbol{v}_E \tag{11.1.13}$$

with the element stiffness matrix

$$\boldsymbol{K}_E = \int\limits_{V_E} \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{B}\mathrm{d}V \tag{11.1.14}$$

Since the energy is a scalar quantity, the potential energy of the whole structure can be obtained by summing up the energies of the single elements. Previously a system displacement vector containing the displacements of all nodes of the whole system must be defined. By a so-called coincidence matrix $\boldsymbol{L}_E$ the correct position of each single element is determined. $\boldsymbol{L}_E$ is a Boolean matrix of the size $(n_{KE}n_F, n_K n_F)$ with n_K as the number of nodes of the whole structure.

The element displacement vector $\boldsymbol{v}_E$ is positioned into the system displacement vector $\boldsymbol{v}$ by the equation

$$\boldsymbol{v}_E = \boldsymbol{L}_E\boldsymbol{v} \tag{11.1.15}$$

and we obtain the system equation by summing up over all elements

$$\begin{gathered}\left(\sum_i \boldsymbol{L}_{iE}^{\mathrm{T}}\boldsymbol{K}_{iE}\boldsymbol{L}_{iE}\right)\boldsymbol{v} = \left[\sum_i \boldsymbol{L}_{iE}(\boldsymbol{f}_{iEp} + \boldsymbol{f}_{iEq})\right] \\ \boldsymbol{K}\boldsymbol{v} = \boldsymbol{f}\end{gathered} \tag{11.1.16}$$

The system stiffness matrix is also symmetric, but it is a singular matrix, if the system is not fixed kinematically, i.e., we have no boundary conditions constraining the rigid body motion. After consideration of the boundary conditions of the whole system, $\boldsymbol{K}$ becomes a positive definite matrix and the system equation can be solved. Then with the known displacements $\boldsymbol{v}$ the stresses and deformations are calculated using the element equations (11.1.1) and (11.1.3).

For elastodynamic problems, the system stiffness matrix and the system mass matrix are obtained in the same manner and we have the system equation

$$\boldsymbol{M}\ddot{\boldsymbol{v}}(t) + \boldsymbol{C}_{\mathrm{D}}\dot{\boldsymbol{v}}(t) + \boldsymbol{K}\boldsymbol{v}(t) = \boldsymbol{f}(t) \tag{11.1.17}$$

For investigation of the eigen-frequencies of a system without damping harmonic vibrations are assumed and with

$$\boldsymbol{v}(t) = \hat{\boldsymbol{v}}\cos(\omega t + \varphi) \tag{11.1.18}$$

and $\boldsymbol{C}_{\mathrm{D}} = \boldsymbol{0}, \boldsymbol{f}(t) = \boldsymbol{o}$ the matrix eigen-value problem follows

$$(\boldsymbol{K} - \omega^2 \boldsymbol{M})\boldsymbol{v} = \boldsymbol{o} \tag{11.1.19}$$

and the eigen-frequencies and the eigen-vectors characterizing the mode shapes can be calculated.

11.1.2 Problems

Exercise 11.1. A plane beam problem is given. The beam is divided into three plane two-node beam elements. The number of nodal degrees of freedom is three (u, w, φ):

1. What size are the element stiffness matrix and the system stiffness matrix before the consideration of the boundary conditions?
2. Show the coincidence matrix $\boldsymbol{L}_2$ of the second element lying between the nodes 2 and 3!
3. Show the population of the system stiffness matrix and the boundary conditions, if the beam is fixed at node 1 (cantilever beam)! Do the same as in the previous case but consider that the beam is simply supported (node 1 is constrained for the deflections u and w and node 4 only for the deflection w)!

Solution 11.1. For the plane beam problem one gets

1. With $n_{KE} = 2$ and $n_F = 3$ the element stiffness matrix has the size $(6,6)$. Because we have 4 nodes ($n_K = 4$) the size of the system stiffness matrix before the consideration of the boundary conditions is $(12,12)$.
2. The coincidence matrices in this case have the size $(6,12)$. Because it must be

$$\boldsymbol{v}_2 = \boldsymbol{L}_2 \boldsymbol{v}$$

we obtain the coincidence matrix for the element Nr. 2

$$\boldsymbol{L}_2 = \begin{bmatrix} 0&0&0&1&0&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0&1&0&0&0 \end{bmatrix}$$

3. The system stiffness matrix without consideration of the boundary conditions is defined by

$$\boldsymbol{K} = \sum_{i=1}^{4} \boldsymbol{L}_i^{\mathrm{T}} \boldsymbol{K}_i \boldsymbol{L}_i$$

In this case we obtain the following population of the matrix $\boldsymbol{K}$

$$\begin{array}{cc}
\begin{array}{cccccccccccc} u_1 & w_1 & \varphi_1 & u_2 & w_2 & \varphi_2 & u_3 & w_3 & \varphi_3 & u_4 & w_4 & \varphi_4 \end{array} & \\
\left[\begin{array}{cccccccccccc}
[(v)] & [(v)] & [(v)] & [(v)] & [(v)] & [(v)] & [(0)] & [(0)] & [(0)] & [(0)] & [(0)] & [(0)] \\
[(v)] & [(v)] & [(v)] & [(v)] & [(v)] & [(v)] & [(0)] & [(0)] & [(0)] & [(0)] & [(0)] & [(0)] \\
[(v)] & [(v)] & (v) & (v) & (v) & (v) & (0) & (0) & (0) & (0) & (0) & (0) \\
[(v)] & [(v)] & (v) & v+x & v+x & v+x & x & x & x & 0 & [(0)] & 0 \\
[(v)] & [(v)] & (v) & v+x & v+x & v+x & x & x & x & 0 & [(0)] & 0 \\
[(v)] & [(v)] & (v) & v+x & v+x & v+x & x & x & x & 0 & [(0)] & 0 \\
[(0)] & [(0)] & (0) & x & x & x & x+z & x+z & x+z & z & [z] & z \\
[(0)] & [(0)] & (0) & x & x & x & x+z & x+z & x+z & z & [z] & z \\
[(0)] & [(0)] & (0) & x & x & x & x+z & x+z & x+z & z & [z] & z \\
[(0)] & [(0)] & (0) & 0 & 0 & 0 & z & z & z & z & [z] & z \\
[(0)] & [(0)] & [(0)] & [(0)] & [(0)] & [(0)] & [z] & [z] & [z] & [z] & [z] & [z] \\
[(0)] & [(0)] & [(0)] & 0 & 0 & 0 & z & z & z & z & [z] & z
\end{array}\right] &
\begin{array}{c} u_1 \\ w_1 \\ \varphi_1 \\ u_2 \\ w_2 \\ \varphi_2 \\ u_3 \\ w_3 \\ \varphi_3 \\ u_4 \\ w_4 \\ \varphi_4 \end{array}
\end{array}$$

v - components of the stiffness matrix of element No. 1, x - components of the stiffness matrix of element No. 2, z - components of the stiffness matrix of element No. 3.

Considering the boundary conditions for a cantilever beam clamped at node 1 ($u_1 = 0, w_1 = 0, \varphi_1 = 0$) we have to cancel the first three rows and the first three columns in the obtained matrix - characterized by brackets $(\ldots)$. If we have a simply supported beam with $u_1 = 0, w_1 = 0$ and $w_4 = 0$, the first two rows and columns and the row and the column No. 11 must be deleted - characterized by square brackets $[\ldots]$.

11.2 Finite Beam Elements

A beam is a quasi one-dimensional structure, the dimensions of the cross-section of it are very small in comparison to its length. The connection of the centers of the cross-sectional areas is called the midline of the beam. We distinguish between straight beams and beams with an in-plane or spatial curved midline, respectively. Here we consider beams with a straight midline only.

Generally such a beam can be loaded by tension/compression, one- or two-axial bending and torsion. Especially with respect to the use of laminate beams the following investigations are restricted to tension/compression and one-axial bending. For two-axial bending and torsion, laminate beams are not so predestined.

Laminate beams consist of UD-laminae mostly have a rectangular cross-section of the dimension b (width) and h (hight) and very often the laminae are arranged symmetrically to the midline. We will assume this special case for the following development of finite laminate beam elements. In this way we have no coupling of tension and bending and we can divide our considerations into the development of laminate elements for tension/compression, so-called laminate truss elements, and laminate beam elements for bending only.

11.2.1 Laminate Truss Elements

The laminate truss element is a very simple element. It is assumed to be a straight structure of the length l with a constant cross-sectional area A. The nodal degree of freedom is one - the displacement u in axial direction (Fig. 11.1). In the potential energy we have only the first derivative and so we can use a two-node truss element with linear shape functions $N_i(x_1)$ and $N_j(x_1)$, which satisfies $C^{(0)}$-continuity

$$u(x_1) = \boldsymbol{N}\boldsymbol{v}_E, \quad \boldsymbol{v}_E^{\mathrm{T}} = [u_i\ u_j], \quad \boldsymbol{N} = [N_i(x_1)\ N_j(x_1)] \tag{11.2.1}$$

The two shape functions (see also Fig. 11.2) are

$$N_i(x_1) = 1 - \frac{x_1}{l}, \quad N_j(x_1) = \frac{x_1}{l} \tag{11.2.2}$$

With the stress resultant

$$N(x_1) = \int_A \sigma \mathrm{d}A = \overline{A}_{11}\varepsilon_1(x_1) = \overline{A}_{11}\frac{\mathrm{d}u}{\mathrm{d}x_1}, \quad \overline{A}_{11} = b\sum_{k=1}^{n} C_{11}^{(k)} h_k \tag{11.2.3}$$

and the longitudinal load per length $n(x_1)$ the total potential energy can be written as

$$\Pi(u) = \frac{1}{2}\int_0^l \overline{A}_{11}u'^2 \mathrm{d}x_1 - \int_0^l n(x_1)u\mathrm{d}x_1 \tag{11.2.4}$$

and for the element stiffness matrix we obtain

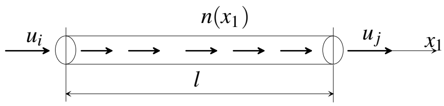


Fig. 11.1 Laminate truss element

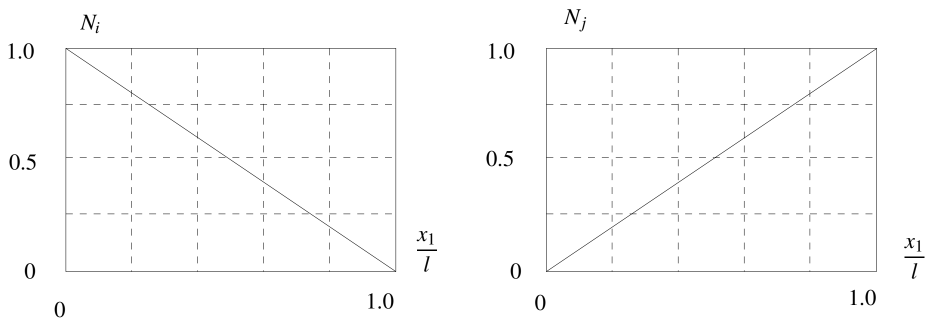


Fig. 11.2 Shape functions of the two-node truss element

$$\boldsymbol{K}_E = \overline{A}_{11} \int\limits_0^l \boldsymbol{N}'^{\mathrm{T}} \boldsymbol{N}' \mathrm{d}x_1 = \frac{\overline{A}_{11}}{l} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad (11.2.5)$$

The element force vector is defined as

$$\boldsymbol{f}_{nE} = \int\limits_0^l \boldsymbol{N}^{\mathrm{T}} n(x_1) \mathrm{d}x_1$$

If we assume that $n(x_1)$ is a linear function with n_i and n_j as the intensities at the nodes

$$n(x_1) = \boldsymbol{N} \begin{bmatrix} n_i \\ n_j \end{bmatrix}$$

then

$$\boldsymbol{f}_{nE} = \int\limits_0^l \boldsymbol{N}^{\mathrm{T}} \boldsymbol{N} \mathrm{d}x_1 \begin{bmatrix} n_i \\ n_j \end{bmatrix} = \frac{l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} n_i \\ n_j \end{bmatrix} \qquad (11.2.6)$$

In case of nodal forces $\boldsymbol{f}_{PE} = [F_i \; F_j]^{\mathrm{T}}$, the vector $\boldsymbol{f}_{PE}$ must be added to the vector $\boldsymbol{f}_{nE}$

$$\boldsymbol{f}_E = \boldsymbol{f}_{nE} + \boldsymbol{f}_{PE} \qquad (11.2.7)$$

The system equation can be obtained in dependence on the structure of the whole system, defined by a coincidence matrix together with the transformation of all element equations into a global coordinate system. Considering the boundary conditions, the system equation can be solved and with the known displacements the stresses can be calculated for each element.

For vibration analysis, the element mass matrix (11.1.10) has to be used

$$\boldsymbol{M}_E = \int\limits_V \rho \boldsymbol{N}^{\mathrm{T}} \boldsymbol{N} \mathrm{d}V = \int\limits_0^l \rho \boldsymbol{N}^{\mathrm{T}} \boldsymbol{N} \mathrm{d}x_1, \quad \rho = \frac{1}{h} \sum_{k=1}^{n} \rho^{(k)} h^{(k)}$$

All parts of the cross-section have the same translation u and the corresponding acceleration $\ddot{u}$ multiplied by the distributed mass produces a distributed axial inertia force. Instead of handling the distributed mass directly, we generate fictitious nodal masses contained in the consistent mass matrix

$$\boldsymbol{M}_E = \frac{\rho A l}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad (11.2.8)$$

With the system equation, obtained in the same manner as for elastostatic problems, the eigen-frequencies and mode shapes can be calculated.

11.2.2 Laminate Beam Elements

For the analysis of laminate beams in this book two theories are considered, the classical laminate theory and the shear deformation theory. The classical laminate theory is based on the Bernoulli beam model and the shear deformation theory on the Timoshenko beam model. The Bernoulli beam model neglects the shear strains in the bending plane and so it seems to be less realistic for the calculation of laminate beams. Therefore it is better to use the Timoshenko beam model, which includes the shear strains in a simple form (Chap. 7).

In the following discussion, only the shear deformation theory is used and we assume a simple rectangular cross-section with a symmetric arrangement of the UD-laminae. This means that we have no coupling of tension and bending. The main advantage of the shear deformation theory in comparison with the Bernoulli theory is that the cross-sectional rotation angle ψ is independent of the displacement w and therefore the differential operator $\boldsymbol{D}$ in the strain energy is of the 1st order. In this way we can use elements with $C^{(0)}$-continuity, and a two-node element with linear shape functions is possible. The nodal degrees of freedom are 2 (w, ψ). In Fig. 11.3 such a two-node beam element is shown. The element displacement vector is

$$\boldsymbol{v}_E^{\mathrm{T}} = [w_i \;\; \psi_i \;\; w_j \;\; \psi_j] \tag{11.2.9}$$

For the displacement vector $\boldsymbol{u}$ the approximation (11.2.11) is used

$$\boldsymbol{u}(x_1) = \begin{bmatrix} w(x_1) \\ \psi(x_1) \end{bmatrix} = \boldsymbol{N}\boldsymbol{v}_E, \tag{11.2.10}$$

where the matrix of the shape functions is

$$\boldsymbol{N} = \left[N_i(x_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad N_j(x_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \tag{11.2.11}$$

with the shape functions (11.2.2), see also Fig. 11.2.

A better element accuracy can be expected, if we consider a three-node element, as shown in Fig. 11.4. Then the element displacement vector is

$$\boldsymbol{v}_E^{\mathrm{T}} = [w_i \;\; \psi_i \;\; w_j \;\; \psi_j \;\; w_k \;\; \psi_k] \tag{11.2.12}$$

and for the matrix $\boldsymbol{N}$ we obtain

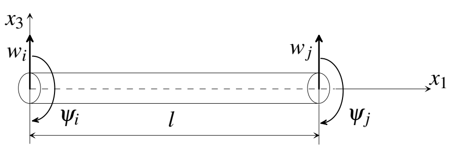


Fig. 11.3 Two-node beam element

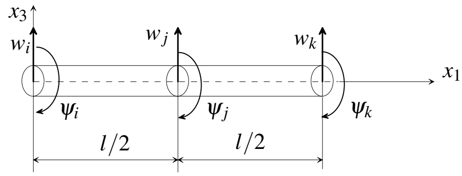


Fig. 11.4 Three-node beam element

$$\boldsymbol{N} = \left[N_i(x_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad N_j(x_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad N_k(x_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \tag{11.2.13}$$

with the shape functions

$$N_i(x_1) = 1 - 3\frac{x_1}{l} + 2\frac{x_1^2}{l^2}, N_j(x_1) = 4\frac{x_1}{l} - 4\frac{x_1^2}{l^2}, N_k(x_1) = -\frac{x_1}{l} + 2\frac{x_1^2}{l^2}, \tag{11.2.14}$$

which are shown in Fig. 11.5. A further increase in the element accuracy can be achieved with a four-node element, see Fig. 11.6. Here the element displacement vector and the matrix of the shape functions are

$$\boldsymbol{v}_E^{\mathrm{T}} = [w_i \ \psi_i \ w_j \ \psi_j \ w_k \ \psi_k \ w_l \ \psi_l] \tag{11.2.15}$$

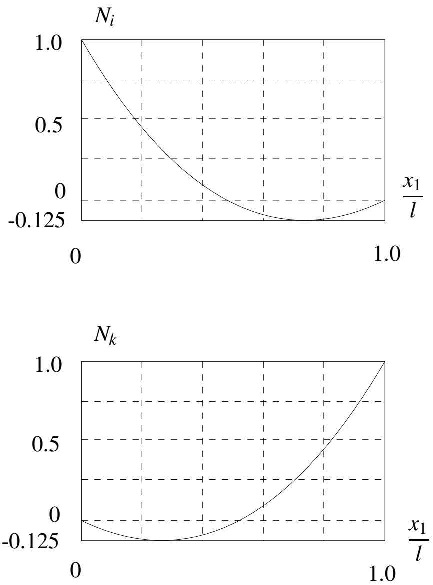


Fig. 11.5 Shape functions of the three-node element

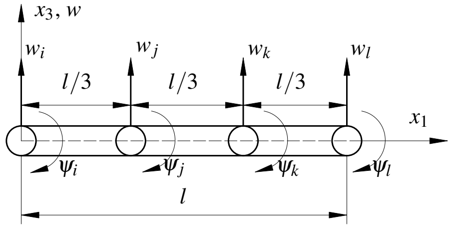


Fig. 11.6 Four-node beam element

$$\boldsymbol{N} = \left[N_i(x_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; N_j(x_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; N_k(x_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \; N_l(x_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right] \tag{11.2.16}$$

with the shape functions

$$\begin{aligned} N_i &= 1 - \frac{11}{2}\frac{x_1}{l} + 9\left(\frac{x_1}{l}\right)^2 - \frac{9}{2}\left(\frac{x_1}{l}\right)^3, & N_j &= 9\frac{x_1}{l} - \frac{45}{2}\left(\frac{x_1}{l}\right)^2 + \frac{27}{2}\left(\frac{x_1}{l}\right)^3, \\ N_k &= -\frac{9}{2}\frac{x_1}{l} + 18\left(\frac{x_1}{l}\right)^2 - \frac{27}{2}\left(\frac{x_1}{l}\right)^3, & N_l &= \frac{x_1}{l} - \frac{9}{2}\left(\frac{x_1}{l}\right)^2 + \frac{9}{2}\left(\frac{x_1}{l}\right)^3 \end{aligned} \tag{11.2.17}$$

which are shown in Fig. 11.7. The three types of beam elements given above show the possibility of using elements of different accuracy. Of course, using the element with higher number of nodes means that less elements and a more coarse mesh can be used, but the calculations of the element stiffness matrices will be more computationally expensive.

The further relationships are developed formally independent of the chosen number of element nodes. The element stiffness matrix is obtained with (11.1.6)

$$\boldsymbol{K}_E = \int_0^l \boldsymbol{B}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{B}\,\mathrm{d}x_1 = \int_0^l \boldsymbol{N}^{\mathrm{T}}\boldsymbol{D}^{\mathrm{T}}\boldsymbol{C}\boldsymbol{D}\boldsymbol{N}\,\mathrm{d}x_1 \tag{11.2.18}$$

Here

$$\boldsymbol{D} = \begin{bmatrix} 0 & \dfrac{\mathrm{d}}{\mathrm{d}x_1} \\ \dfrac{\mathrm{d}}{\mathrm{d}x_1} & 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} \overline{D}_{11} & 0 \\ 0 & k^{\mathrm{s}}\overline{A}_{55} \end{bmatrix} \tag{11.2.19}$$

with the stiffness

$$\overline{D}_{11} = \frac{b}{3}\sum_{k=1}^{n} C_{11}^{(k)} \left(x_3^{(k)^3} - x_3^{(k-1)^3} \right), \quad \overline{A}_{55} = b\sum_{k=1}^{n} C_{55}^{(k)} h^{(k)} \tag{11.2.20}$$

and the shear correction factor k^{s} given in (7.3.20).

For calculation of the element force vector, we assume that the element is loaded by a distributed transverse load per length $q(x_1)$ and we can write the external work as

$$W_E = \int_0^l q(x_1)w(x_1)\mathrm{d}x_1 = \boldsymbol{v}_E^{\mathrm{T}}\boldsymbol{f}_E$$

and with

$$w(x_1) = [w\ \psi] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \boldsymbol{u}^{\mathrm{T}}\boldsymbol{R} = \boldsymbol{v}_E^{\mathrm{T}}\boldsymbol{N}^{\mathrm{T}}\boldsymbol{R} \tag{11.2.21}$$

the element force vector $\boldsymbol{f}_E$ is obtained

$$\boldsymbol{f}_E = \int_0^l \boldsymbol{N}^{\mathrm{T}}\boldsymbol{R}q(x_1)\mathrm{d}x_1 \tag{11.2.22}$$

If single nodal forces or moments are acting, they must be added.

The system equation can be obtained in dependence on the structure of the whole system defined by a coincidence matrix together with the transformation of all element equations into a global coordinate system. After considering the boundary conditions, the system equation can be solved. After this the stress resultants are obtained for all elements

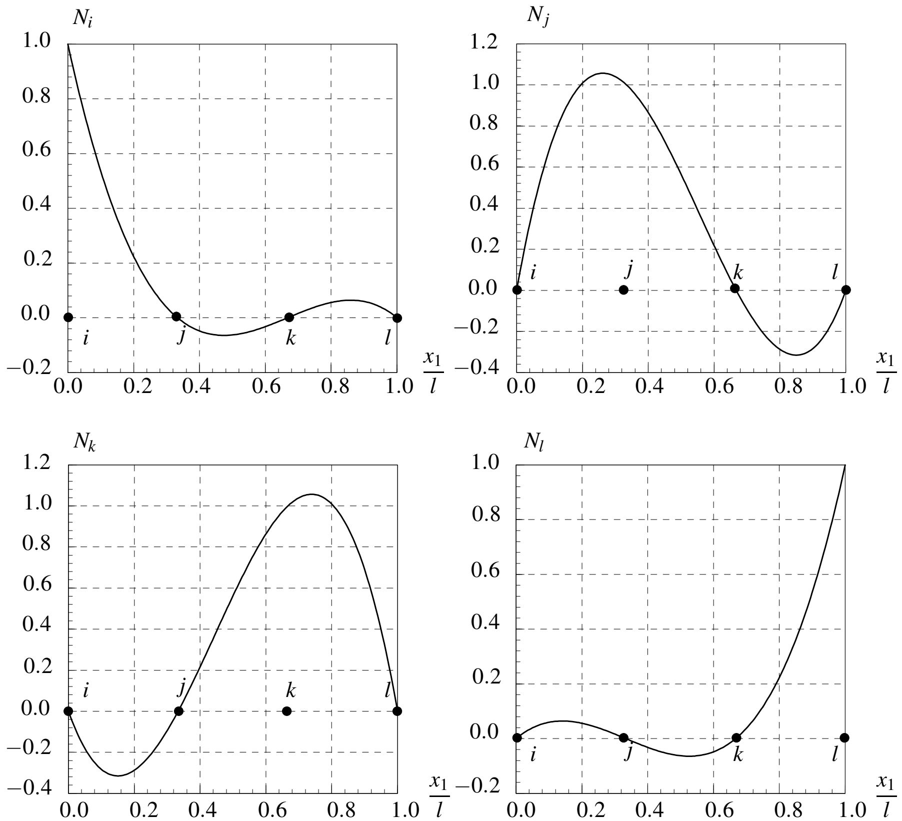


Fig. 11.7 Shape functions of the four-node element

$$\boldsymbol{\sigma}_E = \begin{bmatrix} M \\ Q \end{bmatrix}_E = \boldsymbol{CDN}\boldsymbol{v}_E \tag{11.2.23}$$

For elastodynamic problems the mass matrix must be calculated. The Timoshenko beam model includes in the general case axial, transversal and rotational inertia forces and moments. So it must be noted that the laminae of the beam have different velocities in x_1-direction

$$T_E = \frac{1}{2}\int\limits_0^l \int\limits_{-b/2}^{b/2} \int\limits_{x_3^{(0)}}^{x_3^{(n)}} \rho(\dot{u}^2 + \dot{w}^2)\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3, \quad \dot{u} = \dot{u}_0 + x_3\dot{\psi} \tag{11.2.24}$$

After integration with respect to $\mathrm{d}x_2$ and $\mathrm{d}x_3$ follow

$$T_E = \frac{1}{2}\int\limits_0^l \left[\rho_0(\dot{u}_0^2 + \dot{w}^2) + 2\rho_1\dot{u}_0\dot{\psi} + \rho_2\dot{\psi}^2\right]\mathrm{d}x_1 \tag{11.2.25}$$

with the so-called generalized densities

$$\begin{aligned}
\rho_0 &= b\sum_{k=1}^{n}\rho^{(k)}\left(x_3^{(k)} - x_3^{(k-1)}\right),\\
\rho_1 &= b\frac{1}{2}\sum_{k=1}^{n}\rho^{(k)}\left(x_3^{(k)^2} - x_3^{(k-1)^2}\right),\\
\rho_2 &= b\frac{1}{3}\sum_{k=1}^{n}\rho^{(k)}\left(x_3^{(k)^3} - x_3^{(k-1)^3}\right)
\end{aligned} \tag{11.2.26}$$

$\rho^{(k)}$ is the density of the kth lamina.

Because we assumed a symmetric arrangement of the laminae in the cross-section it follows that

$$\rho_1 = 0, \dot{u}_0 = 0$$

and therefore

$$T_E = \frac{1}{2}\int\limits_0^l (\rho_0\dot{w}^2 + \rho_2\dot{\psi}^2)\mathrm{d}x_1 = \frac{1}{2}\int\limits_0^l \dot{\boldsymbol{u}}^{\mathrm{T}}\boldsymbol{R}_0\dot{\boldsymbol{u}}\mathrm{d}x_1 \tag{11.2.27}$$

with the matrix $\boldsymbol{R}_0$

$$\boldsymbol{R}_0 = \begin{bmatrix} \rho_0 & 0 \\ 0 & \rho_2 \end{bmatrix} \tag{11.2.28}$$

Using (11.1.1)

$$T_E = \frac{1}{2}\int\limits_0^l \dot{\boldsymbol{v}}_E^{\mathrm{T}}\boldsymbol{N}^{\mathrm{T}}\boldsymbol{R}_0\boldsymbol{N}\dot{\boldsymbol{v}}_E\mathrm{d}x_1$$

the element mass matrix $\boldsymbol{M}_E$ is obtained

$$\boldsymbol{M}_E = \int_0^l \boldsymbol{N}^{\mathrm{T}} \boldsymbol{R}_0 \boldsymbol{N} \mathrm{d}x_1 \tag{11.2.29}$$

The system equation is established in the same manner as for elastostatic problems, and with the assumption of harmonic vibrations, the eigen-frequencies and the mode shapes can be calculated.

11.2.3 Problems

Exercise 11.2. Let us assume a two-node beam element.

1. Calculate the element stiffness matrix for a two-node beam element by analytical integration!
2. Calculate the element force vector for a two-node beam element, loaded by a linear distributed transverse load per length $q(x_1)$. The intensities at the nodes are q_i and q_j!
3. Calculate the element mass matrix for a two-node beam element!

Solution 11.2. The three solutions are;

1. In the case of a two-node beam element the matrix of the shape functions is

$$\boldsymbol{N} = \left[N_i(x_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad N_j(x_1) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right]$$

with $N_i(x_1) = 1 - (x_1/l), N_j(x_1) = x_1/l$. The element stiffness matrix is defined by (11.2.18)

$$\boldsymbol{K} = \int_0^l \boldsymbol{N}^{\mathrm{T}} \boldsymbol{D}^{\mathrm{T}} \boldsymbol{C} \boldsymbol{D} \boldsymbol{N} \mathrm{d}x_1,$$

where in (11.2.19) are given

$$\boldsymbol{D} = \begin{bmatrix} 0 & \dfrac{\mathrm{d}}{\mathrm{d}x_1} \\ \dfrac{\mathrm{d}}{\mathrm{d}x_1} & 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} \bar{D}_{11} & 0 \\ 0 & k^{\mathrm{s}} \bar{A}_{55} \end{bmatrix}$$

with $\bar{D}_{11}$ and $k^{\mathrm{s}} \bar{A}_{55}$ in according to (11.2.20). After execution the matrix operations we obtain for the stiffness matrix

$$
\boldsymbol{K}_E = \int_0^l \begin{bmatrix} k^{\mathrm{s}}\bar{A}_{55}\left(\frac{\mathrm{d}N_i}{\mathrm{d}x_1}\right)^2 & 0 & k^{\mathrm{s}}\bar{A}_{55}\frac{\mathrm{d}N_i}{\mathrm{d}x_1}\frac{\mathrm{d}N_j}{\mathrm{d}x_1} & 0 \\ 0 & \bar{D}_{11}\left(\frac{\mathrm{d}N_i}{\mathrm{d}x_1}\right)^2 & 0 & \bar{D}_{11}\frac{\mathrm{d}N_i}{\mathrm{d}x_1}\frac{\mathrm{d}N_j}{\mathrm{d}x_1} \\ k^{\mathrm{s}}\bar{A}_{55}\frac{\mathrm{d}N_i}{\mathrm{d}x_1}\frac{\mathrm{d}N_j}{\mathrm{d}x_1} & 0 & k^{\mathrm{s}}\bar{A}_{55}\left(\frac{\mathrm{d}N_j}{\mathrm{d}x_1}\right)^2 & 0 \\ 0 & \bar{D}_{11}\frac{\mathrm{d}N_i}{\mathrm{d}x_1}\frac{\mathrm{d}N_j}{\mathrm{d}x_1} & 0 & \bar{D}_{11}\left(\frac{\mathrm{d}N_j}{\mathrm{d}x_1}\right)^2 \end{bmatrix} \mathrm{d}x_1
$$

and finally

$$
\boldsymbol{K}_E = \frac{1}{l}\begin{bmatrix} k^{\mathrm{s}}\bar{A}_{55} & 0 & -k^{\mathrm{s}}\bar{A}_{55} & 0 \\ 0 & \bar{D}_{11} & 0 & -\bar{D}_{11} \\ -k^{\mathrm{s}}\bar{A}_{55} & 0 & k^{\mathrm{s}}\bar{A}_{55} & 0 \\ 0 & -\bar{D}_{11} & 0 & \bar{D}_{11} \end{bmatrix}
$$

2. The element force vector to calculate with respect to (11.2.22)

$$
\boldsymbol{f}_E = \int_0^l \boldsymbol{N}^{\mathrm{T}}\boldsymbol{R}q(x_1)\mathrm{d}x_1 \quad \text{with} \quad \boldsymbol{R} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}
$$

For the loading function $q(x_1)$ we can write

$$
q(x_1) = [N_i(x_1)\ N_j(x_1)]\begin{bmatrix} q_i \\ q_j \end{bmatrix}
$$

and then we find

$$
\boldsymbol{f}_E = \int_0^l \begin{bmatrix} N_i(x_1)^2 & N_i(x_1)N_j(x_1) \\ 0 & 0 \\ N_i(x_1)N_j(x_1) & N_j(x_1)^2 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} q_i \\ q_j \end{bmatrix}\mathrm{d}x_1 = \frac{l}{6}\begin{bmatrix} 2q_i + q_j \\ 0 \\ q_i + 2q_j \\ 0 \end{bmatrix}
$$

3. The element mass matrix for such a two-node beam element we find in according to (11.2.29)

$$
\boldsymbol{M}_E = \int_0^l \boldsymbol{N}^{\mathrm{T}}\boldsymbol{R}_0\boldsymbol{N}\mathrm{d}x_1, \quad \boldsymbol{R}_0 = \begin{bmatrix} \rho_0 & 0 \\ 0 & \rho_2 \end{bmatrix}
$$

with the generalized densities ρ_0 and ρ_2 (11.2.26).
Inserting the matrix of the shape functions given above and executing the matrix operations we obtain

$$
\int_0^l \begin{bmatrix} N_i(x_1)^2\rho_0 & 0 & N_i(x_1)N_j(x_1)\rho_0 & 0 \\ 0 & N_i(x_1)^2\rho_2 & 0 & N_i(x_1)N_j(x_1)\rho_2 \\ N_i(x_1)N_j(x_1)\rho_0 & 0 & N_j(x_1)^2\rho_0 & 0 \\ 0 & N_i(x_1)N_j(x_1)\rho_2 & 0 & N_j(x_1)^2\rho_2 \end{bmatrix}\mathrm{d}x_1
$$

and after integration the element mass matrix is in this case

$$\boldsymbol{M}_E = \frac{l}{6}\begin{bmatrix} 2\rho_0 & 0 & \rho_0 & 0 \\ 0 & 2\rho_2 & 0 & \rho_2 \\ \rho_0 & 0 & 2\rho_0 & 0 \\ 0 & \rho_2 & 0 & 2\rho_2 \end{bmatrix}$$

11.3 Finite Plate Elements

Plates are two-dimensional structures that means that one dimension, the thickness, is very small in comparison to the others and in the unloaded state they are plane. Usually, the midplane between the top and the bottom plate surfaces is defined as the reference plane and is taken as the plane of $x-y$. The z-direction corresponds to the thickness direction. To avoid double indexes in the following relationships in this section we will use the coordinates x,y,z instead of x_1,x_2,x_3. Laminate plates consist of a number of bonded single layers. We assume that the single layer as quasi-homogeneous and orthotropic. In each layer we can have different materials, different thicknesses and especially different angle orientations of the fibres. The whole plate is assumed to be a continuous structure. The stacking sequence of the single layers has a great influence on the deformation behavior of the plate. Plates can be loaded by distributed and concentrated loads in all directions, so called in-plane and out of plane loading. In a special case of laminate plates, if we have an arrangement of the single layers symmetric to the midplane, the in-plane and out of plane states are decoupled.

In Chap. 8 the modelling of laminate plates is given and it distinguishes between the classical laminate theory and the shear deformation theory like the modelling of beams. The plate model based on the classical laminate theory usually is called Kirchhoff plate with its main assumption that points lying on a line orthogonal to the midplane before deformation are lying on such a normal line after deformation. This assumption is an extended Bernoulli hypothesis of the beam model to two-dimensional structures.

The application of the classical laminate theory should be restricted to the analysis of very thin plates only. For moderate thick plates it is better to use the shear deformation theory. The plate model based on this theory is called the Mindlin plate model. The following development of finite laminate plate elements will be carried out for both models. Here we will be restricted to symmetric laminate plates in both cases, it means that we have no coupling of membrane and bending/twisting states and we will consider bending only.

In both cases we consider a triangular finite plate element. The approximation of complicated geometric forms, especially of curved boundaries, can be done easily with triangular elements. Usually special coordinates are used for triangular elements. The triangle is defined by the coordinates of the three corner points $P_i(x_i,y_i), i=1,2,3$. A point $P(x,y)$ within the triangle is also defined by the natural triangle coordinates $L_1, L_2, L_3, P(L_1, L_2, L_3)$, see Fig. 11.8. There

$$L_1 = \frac{A_1}{A_\triangle}, \quad L_2 = \frac{A_2}{A_\triangle}, \quad L_3 = \frac{A_3}{A_\triangle} \tag{11.3.1}$$

with the triangle area $A_\triangle$ and the partial areas $A_1, A_2, A_3, A_\triangle = A_1 + A_2 + A_3$. Therefore

$$L_1 + L_2 + L_3 = 1 \tag{11.3.2}$$

The areas $A_1, A_2, A_3, A_\triangle$ can be expressed by determinants

$$\begin{aligned} A_\triangle &= \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \quad A_1 = \begin{vmatrix} 1 & x & y \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}, \\ A_2 &= \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x & y \\ 1 & x_3 & y_3 \end{vmatrix}, \quad A_3 = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{vmatrix} \end{aligned} \tag{11.3.3}$$

and for the coordinates L_1, L_2, L_3 of the point $P(x, y)$

$$\begin{aligned} L_1 &= \frac{1}{2A_\triangle}[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2)y], \\ L_2 &= \frac{1}{2A_\triangle}[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3)y], \\ L_3 &= \frac{1}{2A_\triangle}[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1)y] \end{aligned} \tag{11.3.4}$$

and the coordinates x, y can be expressed by

$$x = x_1 L_1 + x_2 L_2 + x_3 L_3, \quad y = y_1 L_1 + y_2 L_2 + y_3 L_3 \tag{11.3.5}$$

Considering Eq. (11.3.2) we obtain for cartesian coordinates

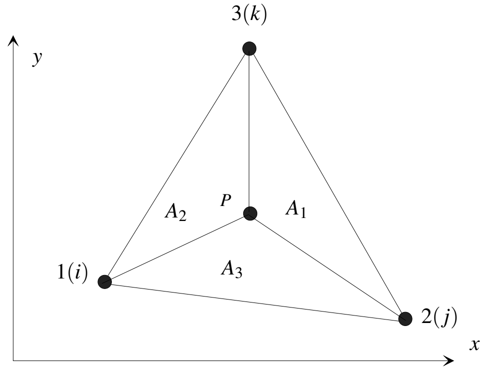


Fig. 11.8 Natural triangle coordinates

$$x = L_1(x_1 - x_3) + L_2(x_2 - x_3) + x_3, \quad y = L_1(y_1 - y_3) + L_2(y_2 - y_3) + y_3 \quad (11.3.6)$$

In Fig. 11.9 the natural triangle coordinates L_1, L_2, L_3 are illustrated for some special points: the corner points and the points in the middle of the sides.

Because the shape functions N_i used for the approximation of the deformation field in a triangular plate element are usually written as functions of the natural element coordinates, it is necessary to find relationships for the derivatives of the shape functions with respect to the global cartesian coordinates. At first the derivatives of the shape functions N_i are given by the natural triangle coordinates L_1 and L_2. L_3 depends from L_1 and L_2, see (11.3.2). So we consider only two independent coordinates. Here we have

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial L_1} \\ \\ \dfrac{\partial N_i}{\partial L_2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial L_1} & \dfrac{\partial y}{\partial L_1} \\ \\ \dfrac{\partial x}{\partial L_2} & \dfrac{\partial y}{\partial L_2} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \boldsymbol{J} \begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} \quad (11.3.7)$$

$\boldsymbol{J}$ is the Jacobi matrix of the coordinate transformation

$$\boldsymbol{J} = \begin{bmatrix} \dfrac{\partial x}{\partial L_1} & \dfrac{\partial y}{\partial L_1} \\ \\ \dfrac{\partial x}{\partial L_2} & \dfrac{\partial y}{\partial L_2} \end{bmatrix} = \begin{bmatrix} c_j & -b_j \\ -c_i & b_i \end{bmatrix} \quad (11.3.8)$$

and the expressions b_i, b_j, c_i, c_j are

$$b_i = y_2 - y_3, \quad b_j = y_3 - y_1, \quad c_i = x_3 - x_2, \quad c_j = x_1 - x_3 \quad (11.3.9)$$

With

$$\mathrm{Det}\boldsymbol{J} = c_j b_i - b_j c_i = \Delta \quad (11.3.10)$$

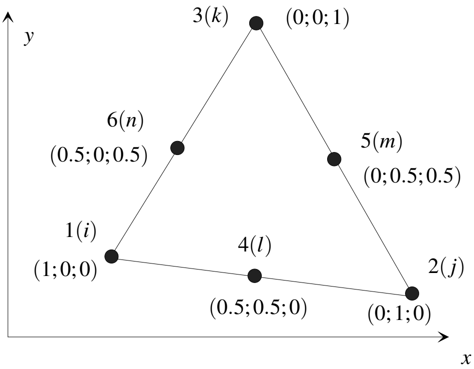


Fig. 11.9 Natural triangle coordinates of special points

it follows that

$$\boldsymbol{J}^{-1} = \frac{1}{\Delta}\begin{bmatrix} b_i & b_j \\ c_i & c_j \end{bmatrix} \tag{11.3.11}$$

and then we obtain for the derivatives of the shape functions N_i with respect to the cartesian coordinates x and y

$$\begin{bmatrix} \dfrac{\partial N_i}{\partial x} \\ \\ \dfrac{\partial N_i}{\partial y} \end{bmatrix} = \boldsymbol{J}^{-1} \begin{bmatrix} \dfrac{\partial N_i}{\partial L_1} \\ \\ \dfrac{\partial N_i}{\partial L_2} \end{bmatrix} \tag{11.3.12}$$

In case of the classical laminate theory, the second partial derivatives

$$\frac{\partial^2 N_i}{\partial L_1^2}, \quad \frac{\partial^2 N_i}{\partial L_1 L_2}, \quad \frac{\partial^2 N_i}{\partial L_2^2}$$

are also required. For this we must put the result for $\partial N_i/\partial L_1$ instead of N_i into the first row of Eq. (11.3.7), and we obtain

$$\frac{\partial^2 N_i}{\partial L_1^2} = \left(\frac{\partial x}{\partial L_1}\right)^2 \frac{\partial^2 N_i}{\partial x^2} + 2\frac{\partial x}{\partial L_1}\frac{\partial y}{\partial L_1}\frac{\partial^2 N_i}{\partial x \partial y} + \left(\frac{\partial y}{\partial L_1}\right)^2 \frac{\partial^2 N_i}{\partial y^2} \tag{11.3.13}$$

In the same manner we can do so with $\partial N_i/\partial L_2$ and the second row of Eq. (11.3.7) and for the mixed second partially derivative with $\partial N_i/\partial L_2$ and the first row or vice versa. The three relationships obtained can be written in matrix form

$$\boldsymbol{J}^* \begin{bmatrix} \dfrac{\partial^2 N_i}{\partial x^2} \\ \dfrac{\partial^2 N_i}{\partial x \partial y} \\ \dfrac{\partial^2 N_i}{\partial y^2} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial^2 N_i}{\partial L_1^2} \\ \dfrac{\partial^2 N_i}{\partial L_1 \partial L_2} \\ \dfrac{\partial^2 N_i}{\partial L_2^2} \end{bmatrix} \tag{11.3.14}$$

where $\boldsymbol{J}^*$ is a modified or extended Jacobi matrix

$$\boldsymbol{J}^* = \begin{bmatrix} \left(\dfrac{\partial x}{\partial L_1}\right)^2 & 2\dfrac{\partial x}{\partial L_1}\dfrac{\partial y}{\partial L_1} & \left(\dfrac{\partial y}{\partial L_1}\right)^2 \\ \dfrac{\partial x}{\partial L_1}\dfrac{\partial x}{\partial L_2} & \dfrac{\partial x}{\partial L_2}\dfrac{\partial y}{\partial L_1} + \dfrac{\partial x}{\partial L_1}\dfrac{\partial y}{\partial L_2} & \dfrac{\partial y}{\partial L_1}\dfrac{\partial y}{\partial L_2} \\ \left(\dfrac{\partial x}{\partial L_2}\right)^2 & 2\dfrac{\partial x}{\partial L_2}\dfrac{\partial y}{\partial L_2} & \left(\dfrac{\partial y}{\partial L_2}\right)^2 \end{bmatrix} \tag{11.3.15}$$

Now the second partial derivatives of the shape functions by the cartesian coordinates can be calculated

$$\begin{bmatrix} \dfrac{\partial^2 N_i}{\partial x^2} \\ \dfrac{\partial^2 N_i}{\partial x \partial y} \\ \dfrac{\partial^2 N_i}{\partial y^2} \end{bmatrix} = \boldsymbol{J}^{*-1} \begin{bmatrix} \dfrac{\partial^2 N_i}{\partial L_1^2} \\ \dfrac{\partial^2 N_i}{\partial L_1 \partial L_2} \\ \dfrac{\partial^2 N_i}{\partial L_2^2} \end{bmatrix} \tag{11.3.16}$$

Of course, by consequently using the natural triangle coordinates it follows that the integrands in the energy terms are functions of these coordinates. Therefore we have to consider for the variables of integration the relationship

$$\mathrm{d}A = \mathrm{d}x\mathrm{d}y = \mathrm{Det}\boldsymbol{J}\mathrm{d}L_1\mathrm{d}L_2 = \Delta\mathrm{d}L_1\mathrm{d}L_2 \tag{11.3.17}$$

In Sects. 11.3.1 and 11.3.2 the development of triangular finite plate elements will be shown in a condensed way for the classical laminate theory and for the shear deformation theory, respectively.

11.3.1 Classical Laminate Theory

The starting point is the total potential energy of an symmetric laminate plate, see also (8.2.24)

$$\begin{aligned} \Pi(w) = \frac{1}{2}\int\limits_A \Bigg[& D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 \\ & + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 4D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 \\ & + 4\left(D_{16}\frac{\partial^2 w}{\partial x^2} + D_{26}\frac{\partial^2 w}{\partial y^2}\right)\frac{\partial^2 w}{\partial x \partial y}\Bigg]\mathrm{d}A - \int\limits_A p_z w \mathrm{d}A \end{aligned} \tag{11.3.18}$$

with the stiffness $D_{ij}, i, j = 1,2,6$, see Table 8.3. The strain energy simplifies the couplings, if we assume special orthotropic laminates (e.g. cross-ply-laminates). We have no bending-twisting coupling, i.e. $D_{16} = D_{26} = 0$. Supposing in other cases these coupling terms as very small, especially if we have a great number of very thin layers, we use the following simplified strain energy approximately

$$\begin{aligned}\Pi(w) = \frac{1}{2}\int_A \Bigg[& D_{11}\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_{22}\left(\frac{\partial^2 w}{\partial y^2}\right)^2 \\ & + 2D_{12}\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} + 4D_{66}\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 - 2p_z w\Bigg] \mathrm{d}A\end{aligned} \tag{11.3.19}$$

The total potential energy of the classical plate model contains second derivatives and so we have to realize $C^{(1)}$-continuity at the element boundaries. This means, continuity of the deflections and the derivatives in normal direction to the boundaries. It must be noted that we do not have $C^{(1)}$-continuity, if the first derivatives at the corner points of adjacent elements are equal because we have to guarantee the continuity of the derivatives in the normal direction at all boundary points of adjacent elements.

It can be shown that we have to use a polynomial with minimum of 18 coefficients, and because we want to have a complete polynomial, we choose a polynomial of fifth order with 21 coefficients. Therefore we define a triangular finite plate element with 6 nodes as shown in Fig. 11.10. At the corner nodes $1,2,3$ (i,j,k) we have 6 degrees of freedom, the deflection, the first derivatives in both directions and the three curvatures, but at the mid-side nodes the first derivatives in normal direction only.

It is a disadvantage when using this element in a general program system that we have a different number of degrees of freedom at the nodes. Therefore an elimination, a so-called static condensation of the nodal constants of the mid-side nodes, can be done and then we have only 18 degrees of freedom for the element. The element is converted into a three-node element, the nodes $4,5,6$ (l,m,n) vanish. The polynomial approximation of the displacement field in the finite element is given by a special 5th order polynomial, it contains however a complete polynomial of 4th order. In this way we obtain 18 shape functions $N_i(L_1,L_2,L_3), i = 1,2,\ldots,18$ which are not illustrated here. Because the coordinates L_1,L_2,L_3 are not independent, see (11.3.2), L_3 usually is eliminated by

$$L_3 = 1 - L_1 - L_2 \tag{11.3.20}$$

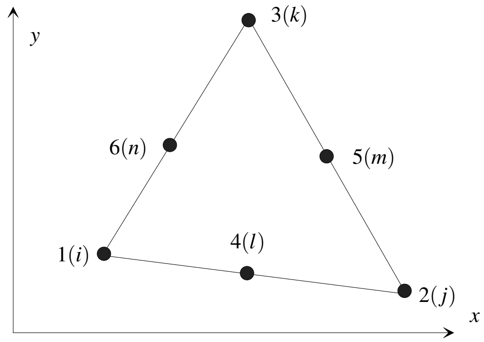


Fig. 11.10 Six-node plate element

According to (11.1.1) we have the approximation

$$w(x,y) = \boldsymbol{N}(L_1, L_2)\boldsymbol{v} \tag{11.3.21}$$

with $\boldsymbol{N}$ as the matrix of the 18 shape functions (here it has only one row) and the element displacement vector $\boldsymbol{v}$ including 18 components. For the differential operator $\boldsymbol{D}^{OP}$ must be written

$$\boldsymbol{D}^{OP} = \left[\frac{\partial^2}{\partial x^2}\ \frac{\partial^2}{\partial y^2}\ 2\frac{\partial^2}{\partial x \partial y}\right]^{\mathrm{T}} \tag{11.3.22}$$

and after this, see also Eq. (11.1.3), the matrix $\boldsymbol{B}$ leads to

$$\boldsymbol{B} = \boldsymbol{D}^{OP}\boldsymbol{N} \tag{11.3.23}$$

Since the shape functions are functions of the natural triangle coordinates L_1 and L_2, for the derivatives by the cartesian coordinates we have to take into consideration (11.3.16). The element stiffness matrix follows according to (11.1.6)

$$\boldsymbol{K}_E = \int\limits_{A_E} \boldsymbol{B}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{B}\mathrm{d}A$$

and with the substitution of the integration variable Eq. (11.3.17)

$$\boldsymbol{K}_E = \int\limits_0^1 \int\limits_0^{1-L_1} \boldsymbol{B}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{B} \triangle \mathrm{d}L_2 \mathrm{d}L_1 \tag{11.3.24}$$

Here $\boldsymbol{D}$ is the matrix of the plate stiffness, the coupling of bending and twisting is neglected ($D_{16} = D_{26} = 0$)

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}$$

According to (11.1.7) we obtain the element force vector

$$\boldsymbol{f}_E = \int\limits_{A_E} \boldsymbol{N}^{\mathrm{T}} p \mathrm{d}A = \int\limits_0^1 \int\limits_0^{1-L_1} \boldsymbol{N}^{\mathrm{T}} p \triangle \mathrm{d}L_2 \mathrm{d}L_1 \tag{11.3.25}$$

where $p(x,y) = p(L_1, L_2)$ is the element surface load.

For the flexural vibration analysis of plates the element mass matrix must be calculated. According to (11.1.10), the element mass matrix reduces to

$$\boldsymbol{M}_E = \int\limits_{V_E} \rho \boldsymbol{N}^{\mathrm{T}}\boldsymbol{N} \mathrm{d}V = \int\limits_0^1 \int\limits_0^{1-L_1} \rho \boldsymbol{N}^{\mathrm{T}}\boldsymbol{N} h \triangle \mathrm{d}L_2 \mathrm{d}L_1 \tag{11.3.26}$$

with ρ as an average density

$$\rho = \frac{1}{h}\sum_{k=1}^{n} \rho^{(k)} h^{(k)} \tag{11.3.27}$$

Note that the classical laminate theory does not consider the rotary kinetic energy.

The integrations in the (11.3.24) for the element stiffness matrix $\boldsymbol{K}_E$, (11.3.25) for the element force vector $\boldsymbol{f}_E$ and (11.3.26) for the element mass matrix must be carried out numerically. Only the force vector $\boldsymbol{f}_E$ can be calculated analytically, if we have a constant surface loading $p(x,y) = \text{const}$. For the numerical solutions it is recommended that integration formulae of the same order are used like the polynomials for the shape functions, in this case of the fifth order.

11.3.2 Shear Deformation Theory

The Mindlin plate model, which is based on the first order shear deformation theory, considers the shear deformation in a simplified form. In the Mindlin plate model the Kirchhoff's hypotheses are relaxed. Transverse normals to the midplane do not remain perpendicular to the middle surface after deformation. In Sect. 8.3 the basic equations are given for this plate model.

Here the starting point is the total potential energy, and if we restrict ourselves to symmetric and special orthotropic laminates, we have

$$\begin{aligned}
\Pi(w,\psi_1,\psi_2) &= \frac{1}{2}\int\limits_A \left[D_{11}\left(\frac{\partial\psi_1}{\partial x}\right)^2 + 2D_{12}\left(\frac{\partial\psi_1}{\partial x}\frac{\partial\psi_2}{\partial y}\right) + D_{22}\left(\frac{\partial\psi_2}{\partial y}\right)^2\right.\\
&\quad + D_{66}\left(\frac{\partial\psi_1}{\partial y} + \frac{\partial\psi_2}{\partial x}\right)^2 + k_{55}^{\mathrm{s}}A_{55}\left(\psi_1 + \frac{\partial w}{\partial x}\right)^2\\
&\quad \left. + k_{44}^{\mathrm{s}}A_{44}\left(\psi_2 + \frac{\partial w}{\partial y}\right)^2\right]\mathrm{d}x\mathrm{d}y - \int\limits_A p_z w\,\mathrm{d}x\mathrm{d}y\\
\delta\Pi(w,\psi_1,\psi_2) &= 0
\end{aligned} \tag{11.3.28}$$

or written in matrix form

$$\Pi(w,\psi_1,\psi_2) = \frac{1}{2}\int\limits_A (\boldsymbol{\kappa}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{\kappa} + \boldsymbol{\varepsilon}^{\mathrm{s}\mathrm{T}}\boldsymbol{A}^{\mathrm{s}}\boldsymbol{\varepsilon}^{\mathrm{s}})\mathrm{d}x\mathrm{d}y - \int\limits_A p_3 w\,\mathrm{d}x\mathrm{d}y \tag{11.3.29}$$

The matrices of the plate stiffness for this case ($D_{16} = D_{26} = 0$) and the shear stiffness with $A_{45} = 0$ are, see also (8.3.7),

$$\boldsymbol{D} = \begin{bmatrix} D_{11} & D_{12} & 0 \\ D_{12} & D_{22} & 0 \\ 0 & 0 & D_{66} \end{bmatrix}, \quad \boldsymbol{A}^{\mathrm{s}} = \begin{bmatrix} k^{\mathrm{s}}_{55}A_{55} & 0 \\ 0 & k^{\mathrm{s}}_{44}A_{44} \end{bmatrix} \tag{11.3.30}$$

The stiffness are given in detail in (4.2.15) and for the shear correction factor see Sect. 8.3. Note that we have, in the elastic potential three independent deformation components (the deflection w and the rotations ψ_1 and ψ_2), so the displacement field vector $\boldsymbol{u}$ has three components ($n_u = 3$), see also (11.1.2).

For the curvatures and the shear strains we have

$$\boldsymbol{\kappa} = \boldsymbol{D}^{\mathrm{b}}\boldsymbol{u}, \quad \boldsymbol{\varepsilon}^{\mathrm{s}} = \boldsymbol{D}^{\mathrm{s}}\boldsymbol{u} \tag{11.3.31}$$

where $\boldsymbol{D}^{\mathrm{b}}$ and $\boldsymbol{D}^{\mathrm{s}}$ are the matrices of the differential operators

$$\boldsymbol{D}^{\mathrm{b}} = \begin{bmatrix} 0 & \dfrac{\partial}{\partial x} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}, \quad \boldsymbol{D}^{\mathrm{s}} = \begin{bmatrix} \dfrac{\partial}{\partial x} & 1 & 0 \\ \dfrac{\partial}{\partial y} & 0 & 1 \end{bmatrix} \tag{11.3.32}$$

The most important property of the elastic potential however is that it contains first derivatives only. Therefore, we have to guarantee only $C^{(0)}$-continuity at the element boundaries and it will be possible to take a three-node finite element with linear shape functions, but it shall be not done here.

Due to the better approximation properties we will choose a six-node element with polynomials of the second order as shape functions. The six-node element with its nodal degrees of freedom is shown in Fig. 11.11. Then we have the nodal and the element displacement vectors

$$\boldsymbol{v}_i^{\mathrm{T}} = [w_i \ \psi_{xi} \ \psi_{yi}], \quad \boldsymbol{v}_E^{\mathrm{T}} = [\boldsymbol{v}_i^{\mathrm{T}} \ \boldsymbol{v}_j^{\mathrm{T}} \ \boldsymbol{v}_k^{\mathrm{T}} \ \boldsymbol{v}_l^{\mathrm{T}} \ \boldsymbol{v}_m^{\mathrm{T}} \ \boldsymbol{v}_n^{\mathrm{T}}] \tag{11.3.33}$$

and according to (11.1.2) with $n_u = n_F, \tilde{n} = n_{K_E}$ the matrix of the shape functions is given by

$$\boldsymbol{N} = [N_i\boldsymbol{I}_3 \ N_j\boldsymbol{I}_3 \ N_k\boldsymbol{I}_3 \ N_l\boldsymbol{I}_3 \ N_m\boldsymbol{I}_3 \ N_n\boldsymbol{I}_3], \tag{11.3.34}$$

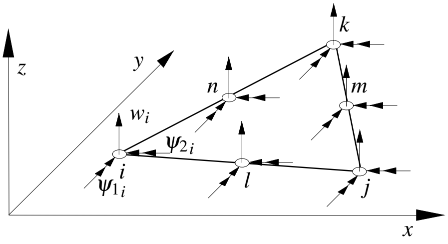


Fig. 11.11 Six-node finite plate element with nodal degrees of freedom

where $\boldsymbol{I}_3$ are unit matrices of the size (3,3). The shape functions are

$$\begin{aligned} N_i &= (2L_1-1)L_1, & N_j &= (2L_2-1)L_2, & N_k &= (2L_3-1)L_3, \\ N_l &= 4L_1L_2, & N_m &= 4L_2L_3, & N_n &= 4L_1L_3 \end{aligned} \qquad (11.3.35)$$

They are functions of the natural triangle co-ordinates L_1, L_2, L_3, see (11.3.1) - (11.3.4).

The curvatures and the shear strains in (11.3.29) can be expressed by

$$\begin{aligned} \boldsymbol{\kappa} &= \boldsymbol{D}^{\mathrm{b}}\boldsymbol{u} = \boldsymbol{D}^{\mathrm{b}}\boldsymbol{N}\boldsymbol{v}_E = \boldsymbol{B}^{\mathrm{b}}\boldsymbol{v}_E, & \boldsymbol{B}^{\mathrm{b}} &= \boldsymbol{D}^{\mathrm{b}}\boldsymbol{N} \\ \boldsymbol{\varepsilon}^{\mathrm{s}} &= \boldsymbol{D}^{\mathrm{s}}\boldsymbol{u} = \boldsymbol{D}^{\mathrm{s}}\boldsymbol{N}\boldsymbol{v}_E = \boldsymbol{B}^{\mathrm{s}}\boldsymbol{v}_E, & \boldsymbol{B}^{\mathrm{s}} &= \boldsymbol{D}^{\mathrm{s}}\boldsymbol{N} \end{aligned} \qquad (11.3.36)$$

and consideration of (11.3.12) leads to the element stiffness matrix, see also (11.1.6) consisting of two parts

$$\boldsymbol{K}_E = \boldsymbol{K}_E^{\mathrm{b}} + \boldsymbol{K}_E^{\mathrm{s}}, \quad \boldsymbol{K}_E^{\mathrm{b}} = \int\limits_{A_E} \boldsymbol{B}^{\mathrm{b}^{\mathrm{T}}}\boldsymbol{D}\boldsymbol{B}^{\mathrm{b}}\mathrm{d}x\mathrm{d}y, \quad \boldsymbol{K}_E^{\mathrm{s}} = \int\limits_{A_E} \boldsymbol{B}^{\mathrm{s}^{\mathrm{T}}}\boldsymbol{A}^{\mathrm{s}}\boldsymbol{B}^{\mathrm{s}}\mathrm{d}x\mathrm{d}y \qquad (11.3.37)$$

Because the shape functions in $\boldsymbol{N}$ are functions of the natural triangle co-ordinates, the integration variables must be substituted by (11.3.17), and then we find

$$\boldsymbol{K}_E^{\mathrm{b}} = \int\limits_0^1 \int\limits_0^{1-L_1} \boldsymbol{B}^{\mathrm{b}^{\mathrm{T}}}\boldsymbol{D}\boldsymbol{B}^{\mathrm{b}}\, \Delta \mathrm{d}L_2\mathrm{d}L_1, \quad \boldsymbol{K}_E^{\mathrm{s}} = \int\limits_0^1 \int\limits_0^{1-L_1} \boldsymbol{B}^{\mathrm{s}^{\mathrm{T}}}\boldsymbol{A}^{\mathrm{s}}\boldsymbol{B}^{\mathrm{s}}\, \Delta \mathrm{d}L_2\mathrm{d}L_1 \qquad (11.3.38)$$

To obtain the element force vector $\boldsymbol{f}_E$ a load vector $\boldsymbol{q}$ must be defined with the same number of components as the displacement field vector $\boldsymbol{u}$. Because only surface loading $p(x,y)$ is considered here, it leads to

$$\boldsymbol{q}^{\mathrm{T}} = [p\;0\;0]$$

and then the element force vector is

$$\boldsymbol{f}_E = \int\limits_{A_E} \boldsymbol{N}^{\mathrm{T}}\boldsymbol{q}\mathrm{d}x\mathrm{d}y, \quad \boldsymbol{f}_E = \int\limits_0^1 \int\limits_0^{1-L_1} \boldsymbol{N}^{\mathrm{T}}\boldsymbol{q}\Delta \mathrm{d}L_2\mathrm{d}L_1 \qquad (11.3.39)$$

with the substitution of integration variables.

The integrations in (11.3.39) can be done analytically only in the case of constant surface loading $p = \mathrm{const}$. In the other cases it must be calculated numerically. For the numerical integration it is recommended to apply integration formulae of the same order as used for shape polynomials, here of the second order. It must be done in this manner for the first part $\boldsymbol{K}_E^{\mathrm{b}}$ of the stiffness matrix, for the second part of $\boldsymbol{K}_E^{\mathrm{s}}$ a lower order can be used. Such a different kind of integration for the two parts of the stiffness matrix is called selective integration.

For dynamic analysis the element mass matrix $\boldsymbol{M}_E$ must also be calculated. For the shear deformation theory the rotatory kinetic energy is usually taken into consideration. The kinetic energy of an element is then

$$T_E = \frac{1}{2}\int\limits_{V_E} \rho \dot{\boldsymbol{u}}^{\mathrm{T}} \dot{\boldsymbol{u}} \mathrm{d}V = \frac{1}{2}\int\limits_{A_E}\int\limits_{-\frac{h}{2}}^{\frac{h}{2}} \rho(\dot{w}^2 + \dot{\psi}_1^2 + \dot{\psi}_2^2)\mathrm{d}z\mathrm{d}A \tag{11.3.40}$$

If the so called generalized densities are used

$$\begin{aligned}
\rho_0 &= \sum_{k=1}^{n} \rho^{(k)}[z^{(k)} - z^{(k-1)}] = \sum_{k=1}^{n} \rho^{(k)} h^{(k)},\\
\rho_1 &= \frac{1}{2}\sum_{k=1}^{n} \rho^{(k)}[z^{(k)^2} - z^{(k-1)^2}],\\
\rho_2 &= \frac{1}{3}\sum_{k=1}^{n} \rho^{(k)}[z^{(k)^3} - z^{(k-1)^3}]
\end{aligned} \tag{11.3.41}$$

and it is noted here that $\rho_1 = 0$, because we have assumed symmetric laminates only, then for the kinetic energy we obtain

$$T_E = \frac{1}{2}\int\limits_{A_E} \dot{\boldsymbol{v}}^{\mathrm{T}} \boldsymbol{R}_0 \dot{\boldsymbol{v}} \mathrm{d}A \tag{11.3.42}$$

$\boldsymbol{R}_0$ is a matrix of the generalized densities

$$\boldsymbol{R}_0 = \begin{bmatrix} \rho_0 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_2 \end{bmatrix} \tag{11.3.43}$$

Using the approximation for the displacement field vector according to Eq. (11.1.1) we obtain

$$T_E = \frac{1}{2}\boldsymbol{v}_E^{\mathrm{T}} \int\limits_{A_E} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{R}_0 \boldsymbol{N} \mathrm{d}A \boldsymbol{v}_E \tag{11.3.44}$$

and the element mass matrix is

$$\boldsymbol{M}_E = \int\limits_{A_E} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{R}_0 \boldsymbol{N} \mathrm{d}A, \quad \boldsymbol{M}_E = \int\limits_0^1 \int\limits_0^{1-L_1} \boldsymbol{N}^{\mathrm{T}} \boldsymbol{R}_0 \boldsymbol{N} \Delta \mathrm{d}L_1 \mathrm{d}L_2 \tag{11.3.45}$$

with substitution of the integration variables.

The finite laminate plate element developed above is called PL18, where the number 18 gives the degrees of freedom of all element nodes. This element can be used only for laminate plates with laminae arranged symmetrically to the midplane, where we have no coupling of membrane and bending/twisting states and we have no in-plane loading.

In many cases we have nonsymmetric laminates and we have a coupling of membrane and bending/twisting states or there are in-plane and out-of-plane loadings. Then an element is necessary where the nodal degrees of freedom also include the deflections in x- and y-direction u, v. For such an element, assuming six nodes again, the nodal and the element displacement vectors are

$$\boldsymbol{v}_i^{\mathrm{T}} = [u_i\ v_i\ w_i\ \psi_{xi}\ \psi_{yi}], \quad \boldsymbol{v}_E^{\mathrm{T}} = [\boldsymbol{v}_i^{\mathrm{T}}\ \boldsymbol{v}_j^{\mathrm{T}}\ \boldsymbol{v}_k^{\mathrm{T}}\ \boldsymbol{v}_l^{\mathrm{T}}\ \boldsymbol{v}_m^{\mathrm{T}}\ \boldsymbol{v}_n^{\mathrm{T}}] \tag{11.3.46}$$

The structure of the matrix of the shape functions $\boldsymbol{N}$ is in this more general case

$$\boldsymbol{N} = [N_i\boldsymbol{I}_5\ N_j\boldsymbol{I}_5\ N_k\boldsymbol{I}_5\ N_l\boldsymbol{I}_5\ N_m\boldsymbol{I}_5\ N_n\boldsymbol{I}_5] \tag{11.3.47}$$

with $\boldsymbol{I}_5$ as unit matrices of the size (5,5), the shape functions remain unchanged.

The total potential energy for this case is, see also (8.3.15),

$$\begin{aligned}\Pi(u,v,w,\psi_1,\psi_2) = \frac{1}{2}\int\limits_A (&\boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{A}\boldsymbol{\varepsilon} + \boldsymbol{\kappa}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{\varepsilon} + \boldsymbol{\varepsilon}^{\mathrm{T}}\boldsymbol{B}\boldsymbol{\kappa} + \boldsymbol{\kappa}^{\mathrm{T}}\boldsymbol{D}\boldsymbol{\kappa}\\ &+ \boldsymbol{\varepsilon}^{\mathrm{s}\mathrm{T}}\boldsymbol{A}^{\mathrm{s}}\boldsymbol{\varepsilon}^{\mathrm{s}})\mathrm{d}x_1\mathrm{d}x_2 - \int\limits_A p_z w\,\mathrm{d}x\mathrm{d}y\end{aligned} \tag{11.3.48}$$

and we have to take into consideration the membrane stiffness matrix $\boldsymbol{A}$ and the coupling matrix $\boldsymbol{B}$ additionally, the element stiffness matrix consists of four parts

$$\boldsymbol{K}_E = \boldsymbol{K}_E^{\mathrm{m}} + \boldsymbol{K}_E^{\mathrm{mb}} + \boldsymbol{K}_E^{\mathrm{b}} + \boldsymbol{K}_E^{\mathrm{s}} \tag{11.3.49}$$

representing the membrane state ($\boldsymbol{K}_E^{\mathrm{m}}$), the coupling of membrane and bending states (K_E^{mb}), the bending state ($\boldsymbol{K}_E^{\mathrm{b}}$) and the transverse shear state ($\boldsymbol{K}_E^{\mathrm{s}}$).

The general form for the element force vector (11.3.39) is unchanged, it must be noted that the loading vector $\boldsymbol{q}$ here has another structure containing loads in three directions

$$\boldsymbol{q}^{\mathrm{T}} = [p_x\ p_y\ p_z\ 0\ 0] \tag{11.3.50}$$

the general form for the element mass matrix is the same as in (11.3.45), but here the matrix of the generalized densities $\boldsymbol{R}_0$ is

$$\boldsymbol{R}_0 = \begin{bmatrix} \rho_0 & 0 & 0 & \rho_1 & 0\\ 0 & \rho_0 & 0 & 0 & \rho_1\\ 0 & 0 & \rho_0 & 0 & 0\\ \rho_1 & 0 & 0 & \rho_2 & 0\\ 0 & \rho_1 & 0 & 0 & \rho_2 \end{bmatrix} \tag{11.3.51}$$

The remarks about the realization of the integrations remains unchanged here. Of course they are all more complicated for this element. Such an extended element would be called PL30, because the degree of freedom of all nodal displacements is 30. Further details about this extended element are not given here.

11.4 Generalized Finite Beam Elements

In civil engineering and also in mechanical engineering a special kind of structures are used very often structures consisting of thin-walled elements with significant larger dimensions in one direction (length) in comparison with the dimensions in the transverse direction. They are called beam shaped shell structures. Beam shaped shell structures include folded plate structures as the most important class of such structures. In Chap. 10 the modelling of folded plate structures was considered and there a generalized beam model was developed by the reduction of the two-dimensional problem to an one-dimensional one following the way of Vlasov-Kantorovich. This folded structure model contains all the energy terms of the membrane stress state and of the bending/twisting stress state under the validity of the Kirchhoff hypotheses. Outgoing from this complete folded structure model some simplified structure models were developed (see Sect. 10.2.4) by neglecting of selected energy terms in the potential function e.g. the terms caused by the longitudinal curvatures κ_{x_i}, the shear strains ε_{xs_i}, the torsional curvatures κ_{xs_i} or the transversal strains ε_{s_i} of the strips. Because the influence of the longitudinal curvatures κ_{x_i} of the single strips to the deformation state and the stress state of the whole structure is very small for beam shaped structures, they are neglected generally. The shear strains ε_{xs_i} of the strips can be neglected for structures with open cross-sections, but not in the case of closed cross-sections. In opposite to this the torsional curvatures κ_{xs_i} can be neglected for closed cross-sections, but not for open cross-sections. Therefore, because we had in mind to find a generalized beam model as well the shear strains as the torsional curvatures are considered. Although the influence of the transversal strains in most cases is very small, they are considered too, because with this we have a possibility to define the generalized co-ordinate functions for a general cross-section systematically. Therefore as a generalized structure model for beam shaped thin-walled folded plate structures the structure model A (see Sects. 10.2.4 and 10.2.5) is chosen, in which only the longitudinal curvatures κ_{x_i} of the strips are neglected.

11.4.1 Foundations

The starting point for the development of generalized finite beam elements is the potential energy, see equation (10.2.10). Because in all strips the longitudinal curvatures κ_{x_i} are neglected all terms containing w_i'' have to vanish. It leads together with equations (10.2.11), (10.2.12) and with $\hat{\boldsymbol{A}}_7 = \boldsymbol{0}, \hat{\boldsymbol{A}}_8 = \boldsymbol{0}, \hat{\boldsymbol{A}}_{10} = \boldsymbol{0}, \hat{\boldsymbol{A}}_{17} = \boldsymbol{0}, \hat{\boldsymbol{A}}_{20} = \boldsymbol{0}$, $\hat{\boldsymbol{A}}_{23} = \boldsymbol{0}, \hat{\boldsymbol{A}}_{26} = \boldsymbol{0}$ to a simplification of the potential energy equation (10.2.10)

$$
\begin{aligned}
\Pi = \frac{1}{2}\int_0^l \Big[& \boldsymbol{U}'^{\mathrm{T}}\hat{\boldsymbol{A}}_1\boldsymbol{U}' + \boldsymbol{V}^{\mathrm{T}}\hat{\boldsymbol{A}}_6\boldsymbol{V} + \boldsymbol{U}^{\mathrm{T}}\hat{\boldsymbol{A}}_3\boldsymbol{U} + 2\boldsymbol{U}^{\mathrm{T}}\hat{\boldsymbol{A}}_{25}\boldsymbol{V}' \\
& + \boldsymbol{V}'^{\mathrm{T}}\hat{\boldsymbol{A}}_4\boldsymbol{V}' + \boldsymbol{V}^{\mathrm{T}}\hat{\boldsymbol{A}}_{12}\boldsymbol{V} + 4\boldsymbol{V}'^{\mathrm{T}}\hat{\boldsymbol{A}}_9\boldsymbol{V}' \\
& + 2\boldsymbol{U}'^{\mathrm{T}}\hat{\boldsymbol{A}}_{14}\boldsymbol{V} + 2\boldsymbol{U}^{\mathrm{T}}\hat{\boldsymbol{A}}_2\boldsymbol{U}' + 2\boldsymbol{U}'^{\mathrm{T}}\hat{\boldsymbol{A}}_{13}\boldsymbol{V}' \\
& - 2\boldsymbol{U}'^{\mathrm{T}}\hat{\boldsymbol{A}}_{19}\boldsymbol{V} - 4\boldsymbol{U}'^{\mathrm{T}}\hat{\boldsymbol{A}}_{18}\boldsymbol{V}' + 2\boldsymbol{U}'^{\mathrm{T}}\hat{\boldsymbol{A}}_{16}\boldsymbol{V} \\
& + 2\boldsymbol{V}^{\mathrm{T}}\hat{\boldsymbol{A}}_5\boldsymbol{V}' - 2\boldsymbol{V}^{\mathrm{T}}\hat{\boldsymbol{A}}_{28}\boldsymbol{V} - 4\boldsymbol{V}^{\mathrm{T}}\hat{\boldsymbol{A}}_{27}\boldsymbol{V}' \\
& - 2\boldsymbol{U}^{\mathrm{T}}\hat{\boldsymbol{A}}_{22}\boldsymbol{V} - 2\boldsymbol{V}'^{\mathrm{T}}\hat{\boldsymbol{A}}_{25}\boldsymbol{V} - 4\boldsymbol{U}^{\mathrm{T}}\hat{\boldsymbol{A}}_{21}\boldsymbol{V}' \\
& - 4\boldsymbol{V}'^{\mathrm{T}}\hat{\boldsymbol{A}}_{24}\boldsymbol{V}' + 4\boldsymbol{V}^{\mathrm{T}}\hat{\boldsymbol{A}}_{11}\boldsymbol{V}' - 2(\boldsymbol{U}^{\mathrm{T}}\boldsymbol{f}_x + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{f}_s + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{f}_n)\Big]\,\mathrm{d}x \\
& - (\boldsymbol{U}^{\mathrm{T}}\boldsymbol{r}_x + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{r}_s + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{r}_n)\,|_{x=0} - (\boldsymbol{U}^{\mathrm{T}}\boldsymbol{r}_x + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{r}_s + \boldsymbol{V}^{\mathrm{T}}\boldsymbol{r}_n)\,|_{x=l}
\end{aligned}
\tag{11.4.1}
$$

We can see that the one-dimensional energy function contains only derivatives of the first order.

11.4.2 Element Definitions

Outgoing from Eq. (11.4.1) a one-dimensional finite element can be defined. Because we have no higher derivatives than of the first order in the potential energy only a $C^{(0)}$ continuity is to satisfy at the element boundaries and therefore it would be possible to use a two-node element with linear shape functions. To have a better accuracy here we will take a three-node element using second order polynomials as shape functions, Fig. 11.12. The shape functions are again like (11.2.15)

$$
N_1(x) = 1 - 3\frac{x}{l} + 2\frac{x^2}{l^2}, \quad N_2(x) = 4\frac{x}{l} - 4\frac{x^2}{l^2}, \quad N_3(x) = -\frac{x}{l} + 2\frac{x^2}{l^2} \tag{11.4.2}
$$

They are shown in Fig. 11.5.

Because a generalized finite beam element with a general cross-section shall be developed at first we must find a rule to define the cross-section topology. We will use for it the profile node concept. For this we will see the midlines of all strips as the cross-sections profile line. The start- and the endpoints of each strip on this profile line are defined as the so-called main profile nodes. In the middle of each strip there

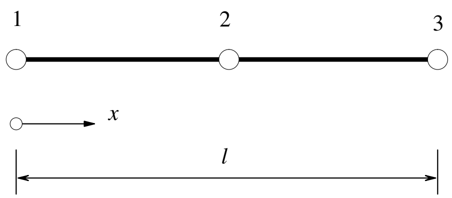


Fig. 11.12 Three-node generalized beam element

are additional profile nodes, they are called secondary profile nodes. Figure 11.13 shows an example for it. The topology of the thin-walled cross-section is described sufficiently by the co-ordinates of the main profile nodes. Additionally the stiffness parameters of each strip must be given. The connections of the strips in the main profile nodes are supposed as rigid.

For the generation of the generalized deflection co-ordinate functions φ, ψ, ξ is assumed that a main profile node has four degrees of freedom, the displacements in the directions of the global co-ordinate axes x, y, z and the rotation about the global x axis, see Fig. 11.14. The displacements of the main profile nodes lead linear generalized co-ordinate functions φ, ψ and cubic functions ξ between the adjacent nodes. For an increasing the accuracy the activation of the secondary profile node degrees of freedom is optional, they are shown in Fig. 11.15. In this case φ and ψ are quadratic and ξ polynomials of 4th and 5th order between the adjacent main nodes. Therefore a more complex deformation kinematics of the cross-section is considerable. The generalized co-ordinate functions for any thin-walled cross-section are here defined as follows:

1. Main node displacements or rotations result in non-zero co-ordinate functions only in the adjacent intervals of the profile line

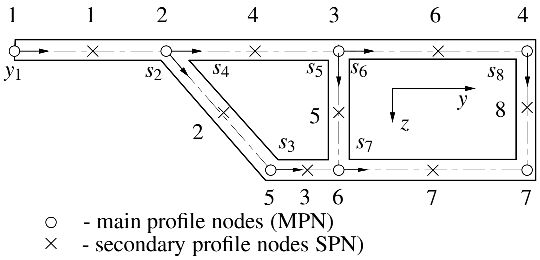


Fig. 11.13 Description of a general cross-section

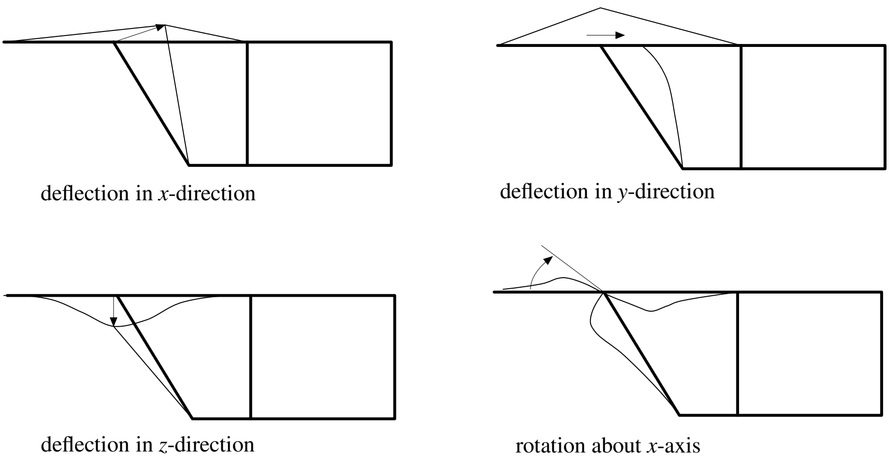


Fig. 11.14 Main profile node degrees of freedom

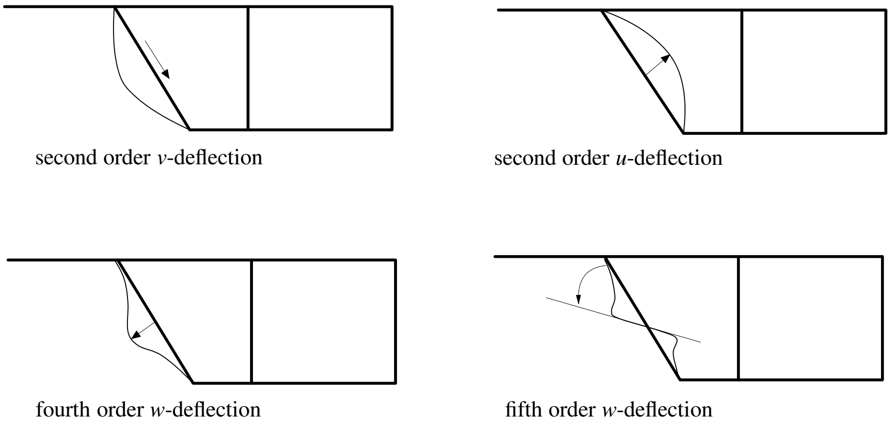


Fig. 11.15 Secondary profile node degrees of freedom

2. Secondary node displacements or rotations result in non-zero co-ordinate functions only in the interval between the adjacent main nodes.

In Fig. 11.16 the generalized coordinate functions for axial parallel arranged strips are shown. Figure 11.17 gives the supplements for slanting arranged strips.

11.4.3 Element Equations

In the case of non-activated degrees of freedom of the secondary profile nodes we have a degree of freedom of an element node of four times the number of main profile nodes (4 n_{MPN}) and the element displacement vector consists of 12 n_{MPN} components

$$\boldsymbol{v}^{\mathrm{T}} = [\boldsymbol{v}_1\ \boldsymbol{v}_2\ \boldsymbol{v}_3], \quad \boldsymbol{v}_j = \begin{bmatrix} \bar{\boldsymbol{u}}_j \\ \bar{\boldsymbol{v}}_j \end{bmatrix} \tag{11.4.3}$$

$\bar{\boldsymbol{u}}_j, \bar{\boldsymbol{v}}_j$ contain the values of the generalized displacement functions at the node j. The displacement vector $\boldsymbol{u}(x)$ contains here the generalized displacement functions $\boldsymbol{U}(x)$ and $\boldsymbol{V}(x)$ and in accordance with Eq. (11.1.1) we obtain for the interpolation

$$\boldsymbol{u}(x) = \begin{bmatrix} \boldsymbol{U}(x) \\ \boldsymbol{V}(x) \end{bmatrix} = \boldsymbol{N}\boldsymbol{v} \tag{11.4.4}$$

The matrix of the shape functions for the chosen three-node element is

$$\boldsymbol{N} = [N_1(x)\boldsymbol{I}\ N_2(x)\boldsymbol{I}\ N_3(x)\boldsymbol{I}], \tag{11.4.5}$$

where $\boldsymbol{I}$ are the unit matrices of the size 4 n_{MPN} and with this the matrix $\boldsymbol{N}$ has the format (4 n_{MPN}, 12 n_{MPN}). Following the equation (11.4.4) for the generalized

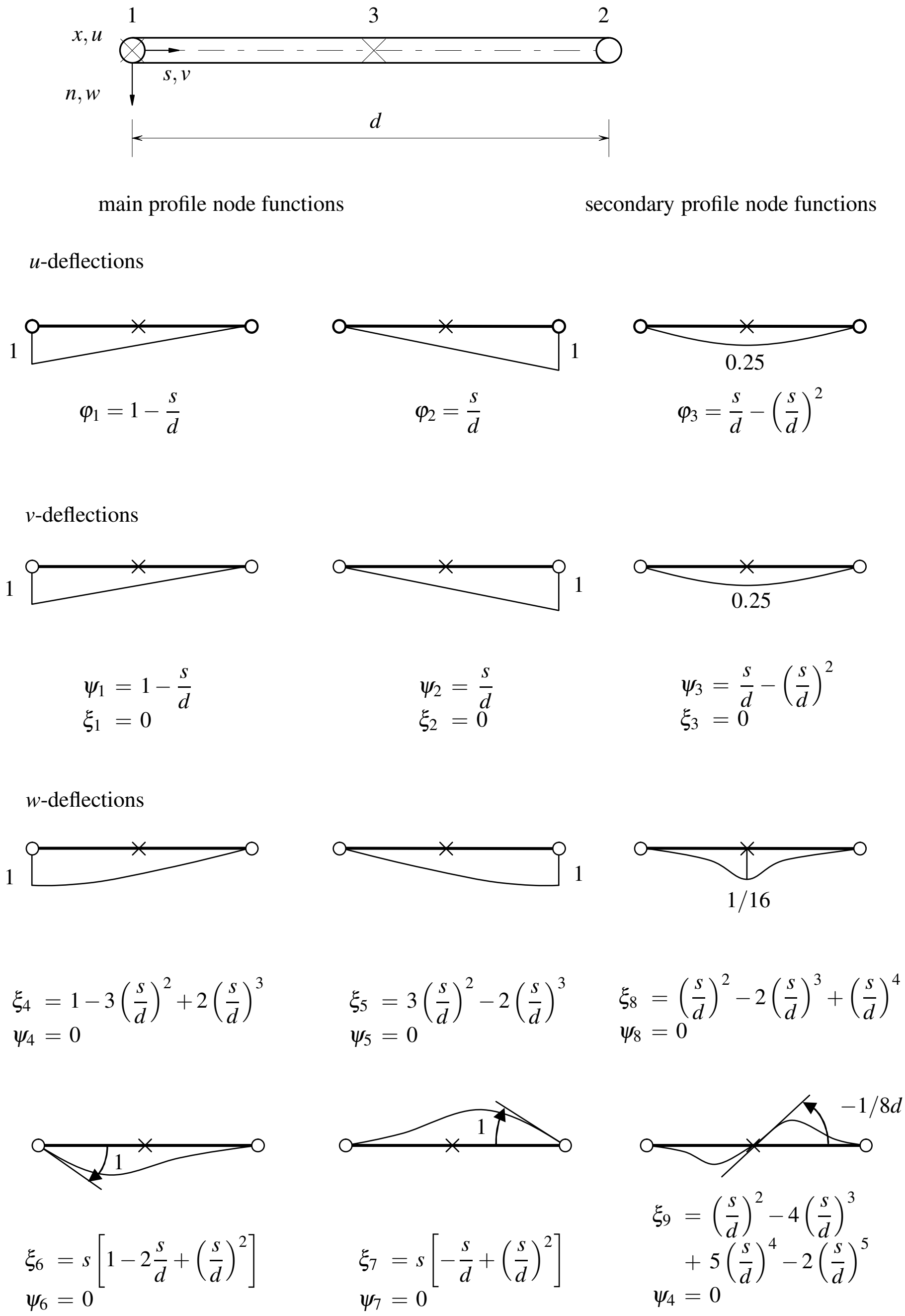
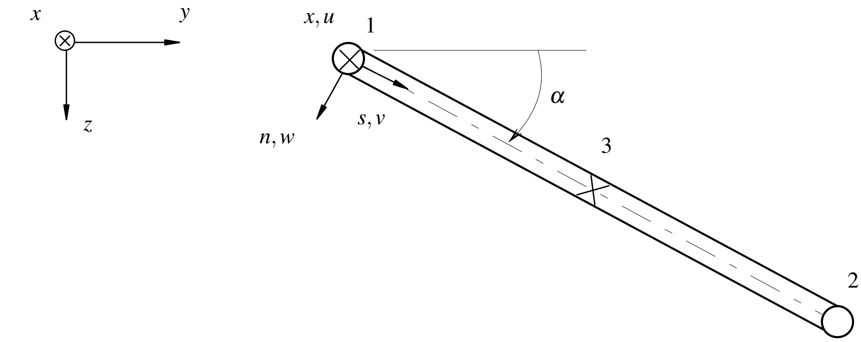
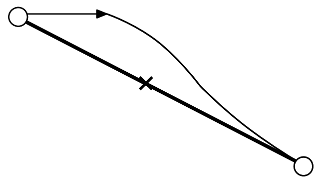
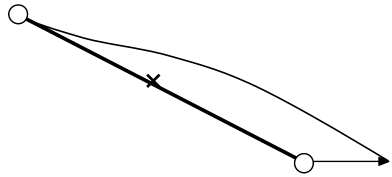


Fig. 11.16 Generalized co-ordinate functions for axial parallel arranged strips

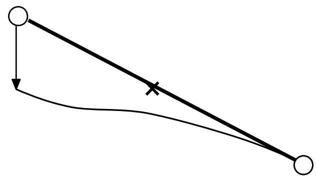




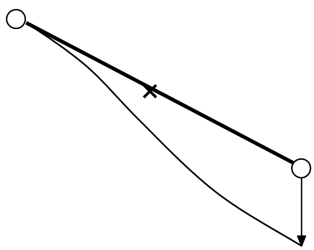
$$\psi_1 = \left(1 - \frac{s}{d}\right)\cos\alpha$$
$$\xi_1 = -\left[1 - 3\left(\frac{s}{d}\right)^2 + 2\left(\frac{s}{d}\right)^3\right]\sin\alpha$$



$$\psi_2 = \frac{s}{d}\cos\alpha$$
$$\xi_2 = -\left[3\left(\frac{s}{d}\right)^2 - 2\left(\frac{s}{d}\right)^3\right]\sin\alpha$$



$$\psi_4 = \left(1 - \frac{s}{d}\right)\sin\alpha$$
$$\xi_4 = \left[1 - 3\left(\frac{s}{d}\right)^2 + 2\left(\frac{s}{d}\right)^3\right]\cos\alpha$$



$$\psi_5 = \frac{s}{d}\sin\alpha$$
$$\xi_5 = \left[3\left(\frac{s}{d}\right)^2 - 2\left(\frac{s}{d}\right)^3\right]\cos\alpha$$

Fig. 11.17 Supplements for slanting arranged strips displacement functions we have to write

$$\boldsymbol{U}(x) = \boldsymbol{L}_{10}^{\mathrm{T}}\boldsymbol{u}(x) = \boldsymbol{L}_{10}^{\mathrm{T}}\boldsymbol{N}\boldsymbol{v}, \quad \boldsymbol{V}(x) = \boldsymbol{L}_{01}^{\mathrm{T}}\boldsymbol{u}(x) = \boldsymbol{L}_{01}^{\mathrm{T}}\boldsymbol{N}\boldsymbol{v} \tag{11.4.6}$$

with the matrices

$$\boldsymbol{L}_{10}^{\mathrm{T}} = [\boldsymbol{I}\,\boldsymbol{0}], \quad \boldsymbol{L}_{01}^{\mathrm{T}} = [\boldsymbol{0}\,\boldsymbol{I}] \tag{11.4.7}$$

In the first case ($\boldsymbol{L}_{10}$) $\boldsymbol{I}$ is a unit matrix of the size n_{MNP} and the null matrix has the format (n_{MPN}, 3 n_{MPN}), in the second case ($\boldsymbol{L}_{01}$) $\boldsymbol{I}$ is a unit matrix of the size 3 n_{MPN} and the null matrix has the format (3 n_{MPN}, n_{MPN}).

Of course in the case of activated degrees of freedom of the secondary profile nodes all the dimensions given above are increased correspondingly. Inserting the generalized displacement functions (11.4.6) into the potential energy equation (11.4.1) we obtain

$$\Pi = \frac{1}{2}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{K}\boldsymbol{v} - \boldsymbol{f}^{\mathrm{T}}\boldsymbol{v} \tag{11.4.8}$$

The condition

$$\frac{\partial \Pi}{\partial \boldsymbol{v}} = \boldsymbol{0} \tag{11.4.9}$$

leads to the element equation

$$\boldsymbol{K}\boldsymbol{v} = \boldsymbol{f} \tag{11.4.10}$$

with the symmetric element stiffness matrix

$$\boldsymbol{K} = \begin{bmatrix} \boldsymbol{K}_{11} & \boldsymbol{K}_{12} & \boldsymbol{K}_{13} \\ \boldsymbol{K}_{12}^{\mathrm{T}} & \boldsymbol{K}_{22} & \boldsymbol{K}_{23} \\ \boldsymbol{K}_{13}^{\mathrm{T}} & \boldsymbol{K}_{23}^{\mathrm{T}} & \boldsymbol{K}_{33} \end{bmatrix} \tag{11.4.11}$$

For the sub-matrices K_{mn} we find the general equation

$$K_{mn} = \sum_{h=1}^{3} \begin{bmatrix} \begin{matrix} \hat{\boldsymbol{A}}_{1h}\boldsymbol{I}_{mn1} \\ +\hat{\boldsymbol{A}}_{2h}\boldsymbol{I}_{mn2} \\ +\hat{\boldsymbol{A}}_{2h}^{\mathrm{T}}\boldsymbol{I}_{mn3} \\ +\hat{\boldsymbol{A}}_{3h}\boldsymbol{I}_{mn4} \end{matrix} & \begin{matrix} (\hat{\boldsymbol{A}}_{13h} - 2\hat{\boldsymbol{A}}_{18h})\boldsymbol{I}_{mn1} \\ +(\hat{\boldsymbol{A}}_{15h} - 2\hat{\boldsymbol{A}}_{21h}\boldsymbol{I}_{mn2} \\ +(\hat{\boldsymbol{A}}_{14h} - \hat{\boldsymbol{A}}_{19h})\boldsymbol{I}_{mn3} \\ +(\hat{\boldsymbol{A}}_{16h} - \hat{\boldsymbol{A}}_{22h})\boldsymbol{I}_{mn4} \end{matrix} \\ \\ \begin{matrix} (\hat{\boldsymbol{A}}_{13h}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{18h}^{\mathrm{T}})\boldsymbol{I}_{mn1} \\ +(\hat{\boldsymbol{A}}_{15h}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{21h}^{\mathrm{T}})\boldsymbol{I}_{mn2} \\ +(\hat{\boldsymbol{A}}_{14h}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{19h}^{\mathrm{T}})\boldsymbol{I}_{mn3} \\ +(\hat{\boldsymbol{A}}_{16h}^{\mathrm{T}} - \hat{\boldsymbol{A}}_{22h}^{\mathrm{T}})\boldsymbol{I}_{mn4} \end{matrix} & \begin{matrix} (\hat{\boldsymbol{A}}_{4h} + \hat{\boldsymbol{A}}_{9h} - 2\hat{\boldsymbol{A}}_{24h} - 2\hat{\boldsymbol{A}}_{24h}^{\mathrm{T}})\boldsymbol{I}_{mn1} \\ +(\hat{\boldsymbol{A}}_{5h} + 2\hat{\boldsymbol{A}}_{11h} - \hat{\boldsymbol{A}}_{25h}^{\mathrm{T}} - 2\hat{\boldsymbol{A}}_{27h})\boldsymbol{I}_{mn2} \\ +(\hat{\boldsymbol{A}}_{5h}^{\mathrm{T}} + 2\hat{\boldsymbol{A}}_{11h} - \hat{\boldsymbol{A}}_{25h} - 2\hat{\boldsymbol{A}}_{27h}^{\mathrm{T}})\boldsymbol{I}_{mn3} \\ +(\hat{\boldsymbol{A}}_{6h} + \hat{\boldsymbol{A}}_{12h} - \hat{\boldsymbol{A}}_{28h} - \hat{\boldsymbol{A}}_{28h}^{\mathrm{T}})\boldsymbol{I}_{mn4} \end{matrix} \end{bmatrix} \tag{11.4.12}$$

with

$$I_{mn1} = \int_0^l N_h N'_m N'_n \mathrm{d}x, \quad I_{mn2} = \int_0^l N_h N_m N'_n \mathrm{d}x,$$

$$I_{mn3} = \int_0^l N_h N'_m N_n \mathrm{d}x, \quad I_{mn4} = \int_0^l N_h N_m N_n \mathrm{d}x$$

To include approximately slight non-prismatic structures the matrices of the stiffness parameters $\hat{\boldsymbol{A}}_i$, see Eq. (10.2.12), are interpolated in the element in the same manner as the displacements

$$\hat{\boldsymbol{A}}_i = \sum_{h=1}^{3} \hat{\boldsymbol{A}}_{ih} N_h \tag{11.4.13}$$

$\hat{\boldsymbol{A}}_{ih}$ are the matrices at the nodes $h = 1,2,3$.

The element force vector is obtained as

$$\boldsymbol{f} = \begin{bmatrix} \boldsymbol{f}_1 \\ \boldsymbol{f}_2 \\ \boldsymbol{f}_3 \end{bmatrix} \tag{11.4.14}$$

with the sub-vectors

$$\boldsymbol{f}_m = \sum_{h=1}^{3} \begin{bmatrix} \boldsymbol{f}_{xh} \int_0^l N_h N_m \mathrm{d}x \\ (f_{sh} + f_{nh}) \int_0^l N_h N_m \mathrm{d}x \end{bmatrix} \tag{11.4.15}$$

Here $\boldsymbol{f}_{xh}, \boldsymbol{f}_{sh}, \boldsymbol{f}_{nh}$ are the generalized load vectors, see Eq. (10.2.13), at the nodes $h = 1,2,3$.

11.4.4 System Equations and Solution

The system equations can be obtained by using the Eqs. (11.1.15) and (11.1.16) with the coincidence matrices, determining the position of each element in the whole structure. In the so founded system stiffness matrix the boundary conditions of the whole structure are to consider, otherwise this matrix is singular, if the structure is not fixed kinematically. The solution of the system equations lead to the nodal displacements and with them the strains and curvatures in the single strips of each element can be calculated, see Eqs. (10.2.6) and (10.2.11),

$$
\begin{aligned}
\varepsilon_x(x,s_i) &= \sum_{h=1}^{3} N'_h \bar{\boldsymbol{u}}_h^{\mathrm{T}} \boldsymbol{\varphi}, & \varepsilon_s(x,s_i) &= \sum_{h=1}^{3} N_h \bar{\boldsymbol{v}}_h^{\mathrm{T}} \boldsymbol{\psi}^{\bullet}, \\
\varepsilon_{xs}(x,s_i) &= \sum_{h=1}^{3} (N_h \bar{\boldsymbol{u}}_h^{\mathrm{T}} \boldsymbol{\varphi}^{\bullet} + N'_h \boldsymbol{v}_h^{\mathrm{T}} \boldsymbol{\psi}), \\
\kappa_s(x,s_i) &= -\sum_{h=1}^{3} N_h \bar{\boldsymbol{v}}_h^{\mathrm{T}} \boldsymbol{\xi}^{\bullet\bullet}, & \kappa_{xs}(x,s_i) &= -2\sum_{h=1}^{3} N'_h \bar{\boldsymbol{v}}_h^{\mathrm{T}} \boldsymbol{\xi}^{\bullet}
\end{aligned}
\tag{11.4.16}
$$

Now we can obtain the stress resultants in the kth lamina, which has the distance n_k from the mid plane of the strip

$$
\begin{bmatrix} N_{xk} \\ N_{sk} \\ N_{xsk} \end{bmatrix} = \begin{bmatrix} A_{11k} & A_{12k} & A_{16k} \\ A_{12k} & A_{22k} & A_{26k} \\ A_{16k} & A_{26k} & A_{66k} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_s + n_k \kappa_s \\ \varepsilon_{xs} + n_k \kappa_{xs} \end{bmatrix}
\tag{11.4.17}
$$

These stress resultants are related on the strip co-ordinate axes x and s_i. Therefore, it is necessary to transform them into the material co-ordinate system of the kth lamina (for the transformation relationship see Table 4.1)

$$
\begin{bmatrix} N_{\mathrm{L}k} \\ N_{\mathrm{T}k} \\ N_{\mathrm{LT}k} \end{bmatrix} = \begin{bmatrix} \cos^2\alpha_k & \sin^2\alpha_k & 2\sin\alpha_k\cos\alpha_k \\ \sin^2\alpha_k & \cos^2\alpha_k & -2\sin\alpha_k\cos\alpha_k \\ -\sin\alpha_k\cos\alpha_k & \sin\alpha_k\cos\alpha_k & \cos^2\alpha_k - \sin^2\alpha_k \end{bmatrix} \begin{bmatrix} N_{xk} \\ N_{sk} \\ N_{xsk} \end{bmatrix}
\tag{11.4.18}
$$

Than the stresses of the kth lamina are obtained

$$
\sigma_{\mathrm{L}k} = \frac{N_{\mathrm{L}k}}{t_k}, \quad \sigma_{\mathrm{T}k} = \frac{N_{\mathrm{T}k}}{t_k}, \quad \tau_{\mathrm{LT}k} = \frac{N_{\mathrm{LT}k}}{t_k}
\tag{11.4.19}
$$

In some cases the strains in the kth lamina related to the material co-ordinate system are important for the failure assessment of the lamina. Then they can be calculated with help of the following matrix equation

$$
\begin{bmatrix} \varepsilon_{\mathrm{L}k} \\ \varepsilon_{\mathrm{T}k} \\ \varepsilon_{\mathrm{LT}k} \end{bmatrix} = \boldsymbol{Q}'^{-1}_k \begin{bmatrix} N_{\mathrm{L}k} \\ N_{\mathrm{T}k} \\ N_{\mathrm{LT}k} \end{bmatrix}
\tag{11.4.20}
$$

There $\boldsymbol{Q}'$ is the reduced stiffness matrix of the kth lamina.

11.4.5 Equations for the Free Vibration Analysis

The variation statement given by the Hamilton's principle, see Eq. (10.2.41) leads with the Lagrange function (10.2.39) and the assumption of harmonic vibrations for the considered generalized beam element to the element equation

$$
(\boldsymbol{K} - \omega^2 \boldsymbol{M})\boldsymbol{v} = \boldsymbol{0}
\tag{11.4.21}
$$

$\boldsymbol{K}$ is the element stiffness matrix, see Eqs. (11.4.11) and (11.4.12), and $\boldsymbol{M}$ is the element mass matrix. The element mass matrix is obtained with the matrices $\hat{\boldsymbol{B}}_1, \hat{\boldsymbol{B}}_2, \hat{\boldsymbol{B}}_3$, see Eqs. (10.2.38)

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{M}_{11} & \boldsymbol{M}_{12} & \boldsymbol{M}_{13} \\ \boldsymbol{M}_{12}^{\mathrm{T}} & \boldsymbol{M}_{22} & \boldsymbol{M}_{23} \\ \boldsymbol{M}_{13}^{\mathrm{T}} & \boldsymbol{M}_{23}^{\mathrm{T}} & \boldsymbol{M}_{33} \end{bmatrix} \tag{11.4.22}$$

with

$$\boldsymbol{M}_{mn} = \sum_{h=1}^{3} \begin{bmatrix} \hat{\boldsymbol{B}}_{1h} I_{mn4} & \boldsymbol{0} \\ \boldsymbol{0} & (\hat{\boldsymbol{B}}_{2h} + \hat{\boldsymbol{B}}_{3h}) I_{mn4} \end{bmatrix} \tag{11.4.23}$$

There the $\hat{\boldsymbol{B}}$ matrices are also interpolated in the element by using the shape functions. In this way slight non-prismatic structures are considerable too. The system equations can be developed in a similar way as it was done for a static analysis. Here we have to find a system stiffness matrix and a system mass matrix. After consideration the boundary conditions the eigen-value problem can be solved and the mode shapes can be estimated.

11.5 Numerical Results

Additional to a great number of special FEM programs general purpose FEM program systems are available. The significance of universal FEM program packages is increasing. In universal FEM program systems we have generally the possibility to consider anisotropic material properties, e.g. in the program system COSMOS/M we can use volume elements with general anisotropic material behavior and plane stress elements can have orthotropic properties.

Laminate shell elements are available e.g. in the universal FEM program systems ANSYS, NASTRAN or COSMOS/M. In many program systems we have no special laminate plate elements, the laminate shell elements are used also for the analysis of laminate and sandwich plates. Perhaps, because of the higher significance of two-dimensional laminate structures in comparison with beam shaped structures laminate beam elements are missing in nearly all universal FEM program systems. The generalized beam elements, Sect. 11.4, are e.g. implemented only in the FEM program system COSAR.

For the following numerical examples the program system COSMOS/M is used.

In COSMOS/M a three node and a four node thin laminate shell element are available (SHELL3L; SHELL4L). Each node has 6 degrees of freedom. The element can consist of up to 50 layers. Each layer can have different material parameters, different thicknesses and especially different angles of fibre directions. We have no restrictions in the stacking structure, symmetric, antisymmetric and nonsymmetric structures are possible. The four nodes of the SHELL4L element must not arranged in-plane. By the program in such case a separation is done into two or four triangular partial elements. Further there is a SHELL9L element available. It has additional nodes at the middles of the four boundaries and in the middle of the element. For the following examples only the element SHELL4L is used.

11.5.1 Examples for the Use of Laminated Shell Elements

By the following four examples the application of the laminate shell element SHELL4L shall be demonstrated. At first a thin-walled beam shaped laminate structure with L-cross-section under a concentrated force loading is considered, and the second example is a thin-walled laminate pipe under torsional loading. In both cases the influence of the fibre angles in the layers is tested. The use of the laminate shell element for the static and dynamic analysis of a sandwich plate is shown in the third example. A buckling analysis of a laminate plate is demonstrated by the fourth example. In all 4 cases a selection of results is given.

11.5.1.1 Cantilever Beam

A cantilever beam with L-cross-section consists of 3 layers with the given material parameters $E_x, E_y, \nu_{xy}, \nu_{yx}, G_{xy}$. It is loaded by a concentrated force F, see Fig. 11.18. The material parameters are

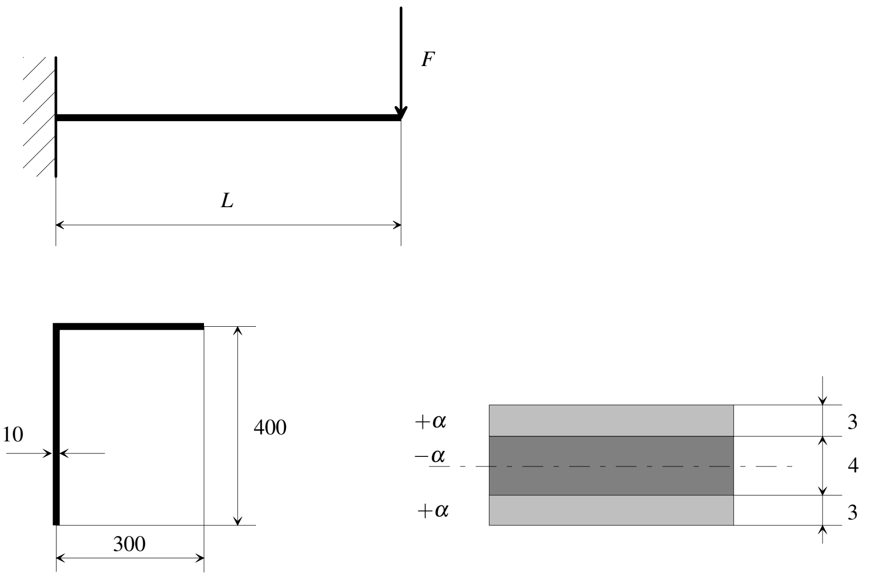


Fig. 11.18 Cantilever beam: cross-section and stacking structure (F = 4.5 kN, L = 4 m, all other geometrical values in mm)

$E_x = 1.53 \cdot 10^4$kN/cm^2, $E_y = 1.09 \cdot 10^3$kN/cm^2, $G_{xy} = 560$kN/cm^2,
$\nu_{xy} = 0.30, \nu_{yx} = 0.021$
The fibre angle α shall be varied: $\alpha = 0°, 10°, 20°, 30°, 40°$.

The FEM model after the input of all properties into COSMOS/M is illustrated in Fig. 11.19. The computing yields a lot of results. In Fig. 11.20, e.g., is shown the deformed shape for a fibre angle of $\alpha = 30°$. Here should be selected only the displacements of the corner node at the free edge (node No. 306 in our FE model) in y- and z-direction and the maximal stresses in fibre direction (σ_x) and perpendicular to it (σ_y) for the left side of the vertical part of the cross-section (layer No. 1, bottom):
$v_{306,y}$ = -2,204 cm, $v_{306,z}$ = -1,805 cm,
$\sigma_{\text{lay1,max},x}$ = 7,487 kN/cm^2, $\sigma_{\text{lay1,max},y}$ = 0,824 kN/cm^2
Similar the displacements and stresses for the fibre angles $\alpha = 0°, 10°, 20°, 40°$ are calculated, and the results are shown in Figs. 11.21 and 11.22. The results show that for such a beam shaped structure the main stresses are lying in the longitudinal direction and therefore the fibre angle 0° leads to the most effective solution.

11.5.1.2 Laminate Pipe

A laminate pipe consisting of 2 layers with the given material parameters E_x, E_y, $\nu_{xy}, \nu_{yx}, G_{xy}$ is fixed at left end and loaded by a torsional moment, see Fig. 11.23. The material parameters are the same as in the previous example:
$E_x = 1{,}53\ 10^4$kN/cm^2, $E_y = 1{,}09\ 10^3$kN/cm^2, $G_{xy} = 560$ kN/cm^2,
$\nu_{xy} = 0{,}30$, $\nu_{yx} = 0{,}021$
The fibre angle α shall be varied: $\alpha = 0°, 15°, 30°, 45°$. After the input of all parame-

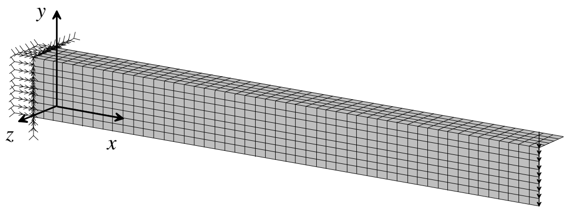


Fig. 11.19 FE-model of cantilever beam in COSMOS/M (650 elements, 714 nodes)

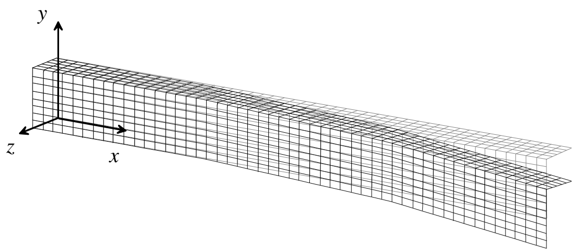


Fig. 11.20 Cantilever beam deformed shape

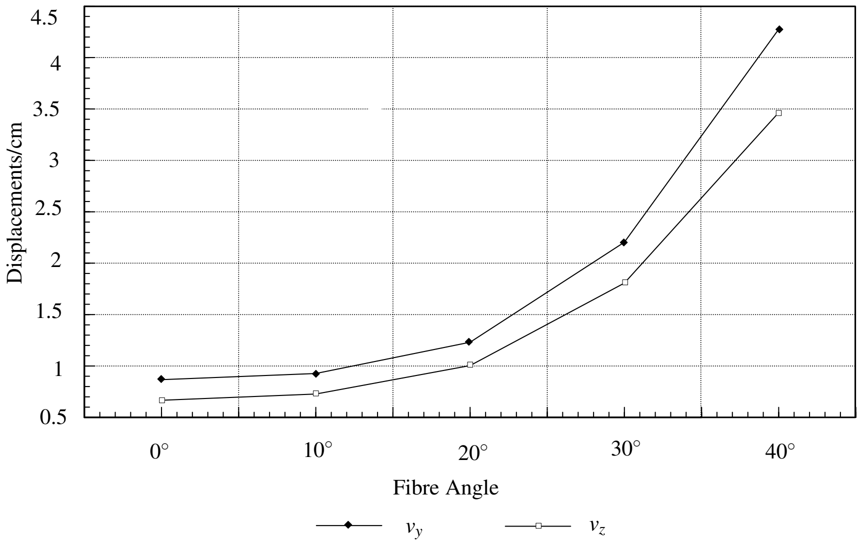


Fig. 11.21 Displacements of the corner point at the free edge

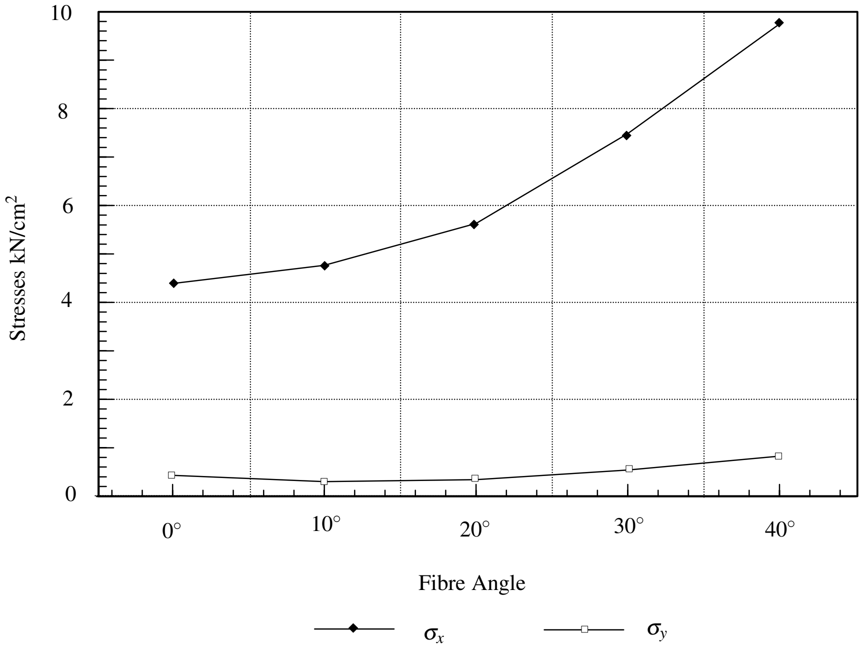


Fig. 11.22 Maximal stresses at the bottom of layer No. 1

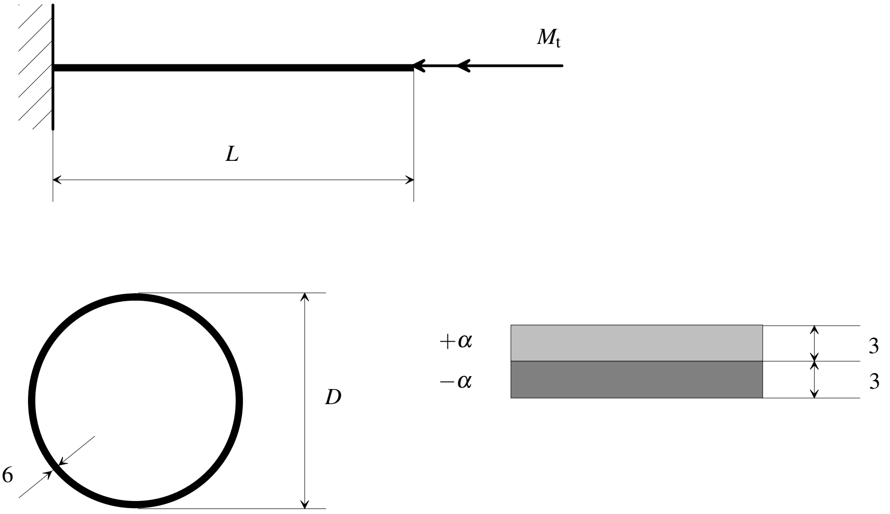


Fig. 11.23 Laminate pipe: geometry, cross-section and stacking sequence ($M_t = 1200$ kNcm, $L = 2$ m, $D = 200$ mm)

ters and properties into COSMOS/M the FEM model can be illustrated (Fig. 11.24).

From the results of the analysis only the twisting angle of the free edge shall be considered here. For this we have to list the results for the displacements in y-direction of two nodes at the free edge, lying in opposite to each other, e.g. the nodes 255 and 663 in our FE model. The twisting angle is calculated by

$$\varphi = (v_{y,255} - v_{y,663})/D$$

Carrying out the analysis for all fibre angles we obtain the results, given in Fig. 11.25. The diagram demonstrates the well known fact that in case of pure shear loading the main normal stresses are lying in a direction with an angle of 45° to the shear stresses. Therefore here the fibre angels of $+45°/-45°$ to the longitudinal axis are the most effective arrangements, because these fibre angels yield the greatest shear rigidity.

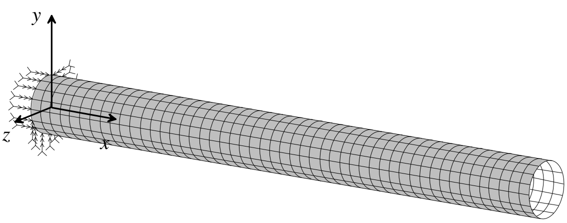


Fig. 11.24 FE-model of Laminate Pipe in COSMOS/M (800 elements, 816 nodes)

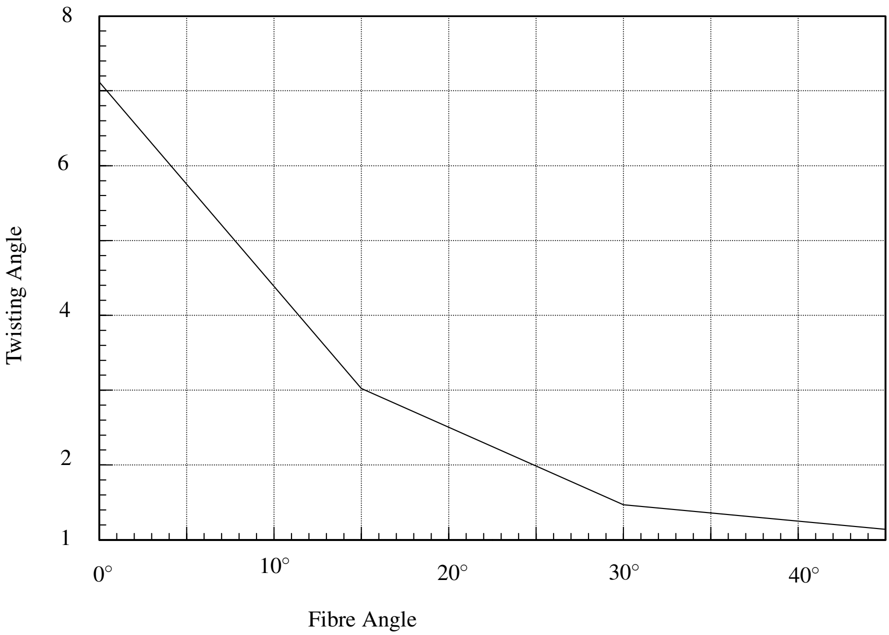


Fig. 11.25 Twisting angle of the free edge

11.5.1.3 Sandwich Plate

The sandwich plate (Fig. 11.26) is clamped at both short boundaries and simply supported at one of the long boundaries. The cover sheets consist of an aluminium alloy and the core of foam of polyurethan. The material parameters are:
AlZnMgCu0.5F450:
$\rho = 2.7 \cdot 10^3$ kg/m^3, $E = 7.0 \cdot 10^{10}$ N/m^2, $\nu = 0.34$
polyurethan foam:
$\rho = 150$ kg/m^3, $E = 4.2 \cdot 10^7$ N/m^2, $\nu = 0.30$

Additional to a stress analysis of the plate under constant pressure loading p, a vibration analysis will be performed is asked. We have to calculate the 4 lowest eigenfrequencies and the mode shapes, respectively. Note that we use in this example only the basic units of the SI-system, so we avoid the calculation of correction factors for the obtained eigenfrequencies.

The FE-model is given in Fig. 11.27. The static analysis leads the displacements and stresses. We consider only the stresses of the bottom of the lower cover sheet (layer 3, top). The Figs. 11.28 and 11.29 show the plots of stress distributions for the flexural stresses σ_x and σ_z, Fig. 11.30 the distribution of the von Mises equivalent stress. The lowest 4 eigenfrequencies and their 4 mode shapes are shown in the following Fig. 11.31. The static and frequency computations confirm the successful application of the SHELL4L element for a sandwich plate.

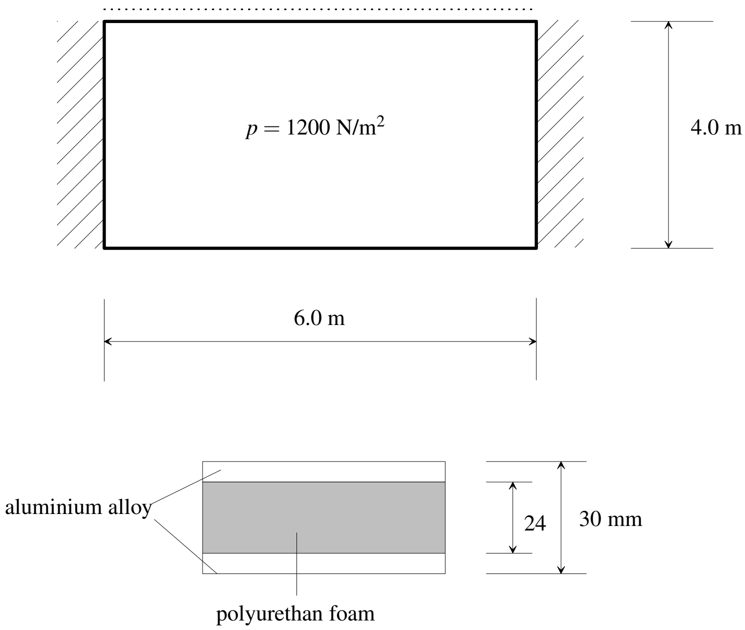


Fig. 11.26 Sandwich plate

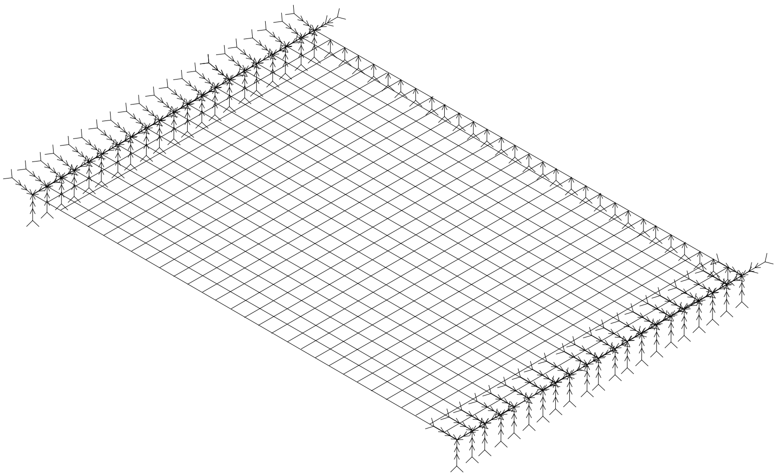


Fig. 11.27 FE-model of Sandwich Plate in COSMOS/M (600 elements, 651 nodes)

11.5.1.4 Buckling Analysis of a Laminate Plate

For a rectangular laminate plate consisting of 4 layers with the given material parameters a buckling analysis shall be carried out. The plate is simply supported at

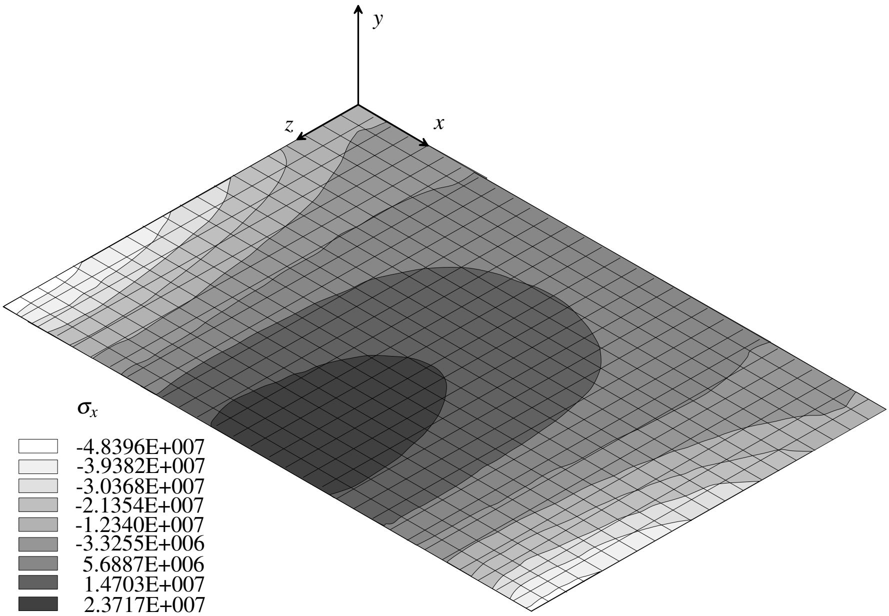


Fig. 11.28 Stresses in x-direction for the bottom of the lower cover sheet

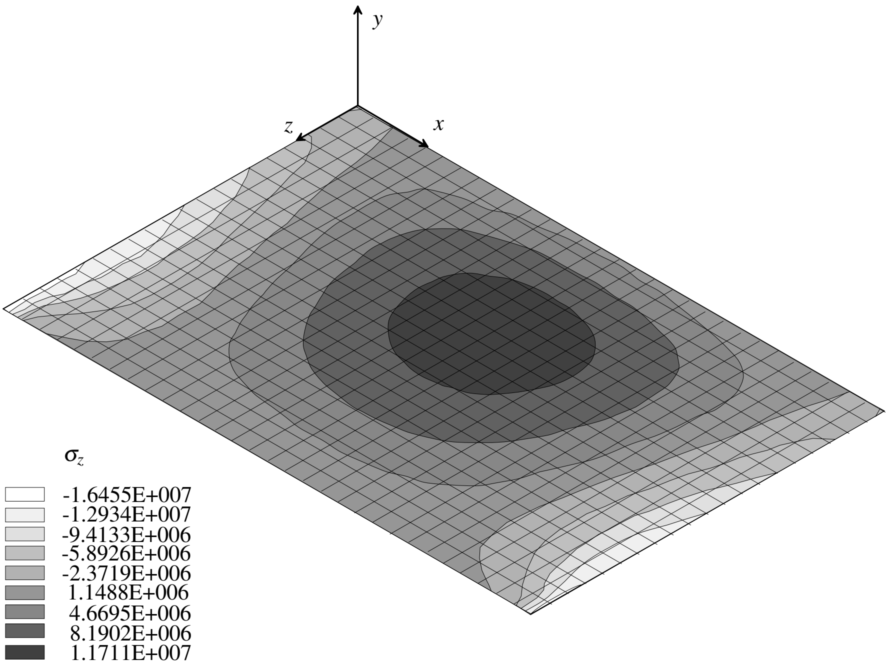


Fig. 11.29 Stresses in z-direction for the bottom of the lower cover sheet

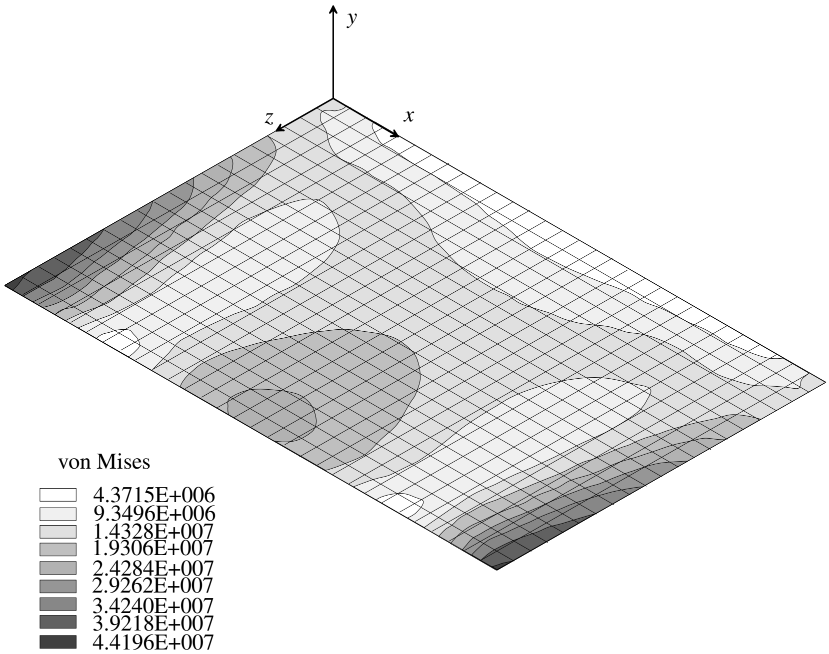


Fig. 11.30 Von Mises stress for the bottom of the lower cover sheet

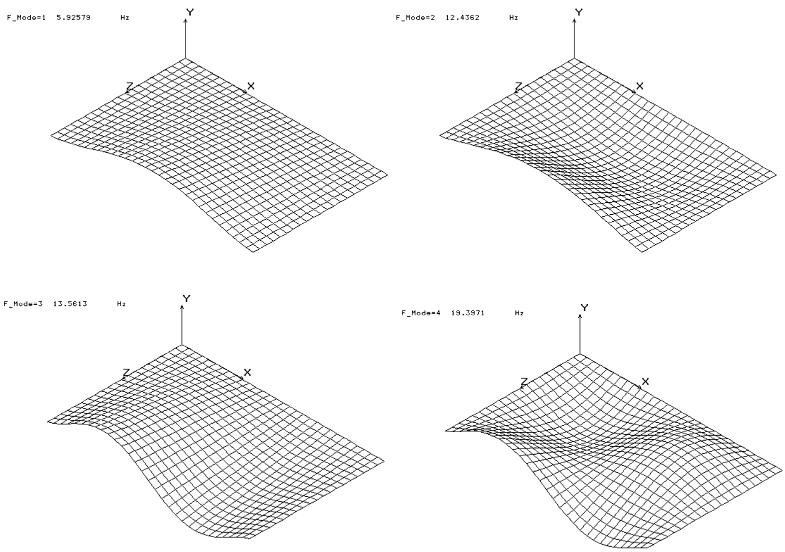


Fig. 11.31 Mode shapes for the lowest four eigenfrequencies: $f_1 = 5,926$ Hz (top-left), $f_2 = 12,438$ Hz (top-right), $f_3 = 13,561$ Hz (bottom-left), $f_4 = 19,397$ Hz (bottom-right)

all boundaries and loaded by a uniaxial uniform load, see Fig. 11.32. Material parameters are again the same as in the previous examples
E_x = 1.53 10^4 kN/cm^2, E_y = 1.09 10^3 kN/cm^2, G_{xy} = 560 kN/cm^2,
$\nu_{xy} = 0.30, \nu_{yx} = 0.021$
For the stacking structure two cases shall be considered, a symmetric (case I) and a antisymmetric (case II) laminate structure (Fig. 11.32). The fibre angle is to vary: $\alpha = 0°, 15°, 30°, 45°, 60°, 75°, 90°$. For the buckling analysis in COSMOS/M a unit pressure loading must be created, and the program calculates a buckling factor ν_B to multiply the unit loading for obtaining the buckling load.

The FE-model created in COSMOS/M by the input of all properties and parameters is shown in Fig. 11.33. The calculation for $\alpha = 30°$ leads to a buckling factor $\nu_B = 1,647$ and to the buckling mode shown in Fig. 11.34. In the same manner the calculations for the other fibre angels and for the antisymmetric laminate were performed. The results for the buckling factors are shown in a diagram in Fig. 11.35. The buckling modes are symmetric to the symmetric axis in loading direction. For the symmetric laminates the buckling modes for $\alpha = 0°, 15°, 30°$ are nearly the same, see Fig. 11.34. For fibre angles $45°, 60°, 75°, 90°$ the buckling modes have different shapes, they are shown in the following figures. The buckling modes for the antisymmetric laminate are very similar but not identical to the buckling modes

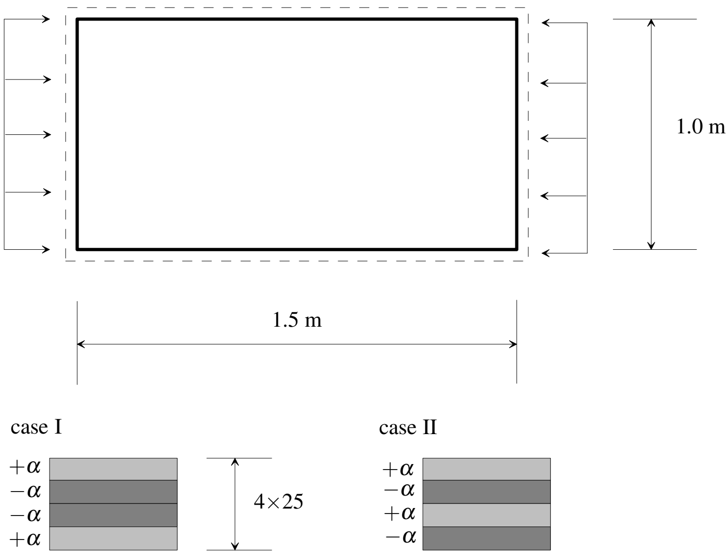


Fig. 11.32 Rectangular laminate plate

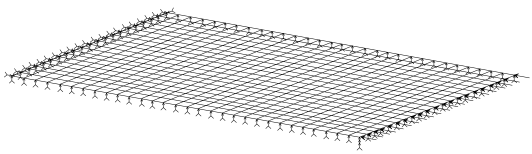


Fig. 11.33 FE-model of the laminate plate in COSMOS/M (600 elements, 651 nodes)

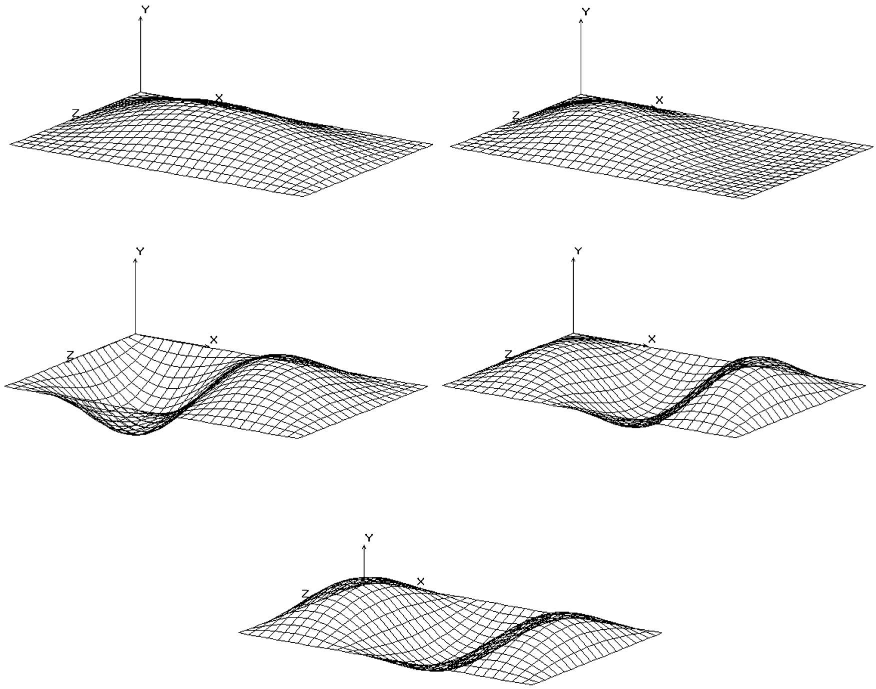


Fig. 11.34 Buckling modes for symmetric laminates $\alpha = 30^\circ$ (top-left), $\alpha = 45^\circ$ (top-right), $\alpha = 60^\circ$ (middle-left), $\alpha = 75^\circ$ (middle-right), $\alpha = 90^\circ$ (bottom)

of the symmetric laminates. They are not given here. A fibre angle near 45° leads to the highest buckling load for a quadratic plate. It shall be noted that the antimetric stacking sequence of the laminate improved the buckling stability.

11.5.2 Examples of the Use of Generalized Beam Elements

Generalized finite elements for the analysis of thin-walled beam shaped plate structures, Sect. 11.4, were implemented and tested in the frame of the general purpose FEM-program system COSAR. The real handling of the FEM-procedures are not given here, but two simple examples shall demonstrate the possibilities of these elements for global static or dynamic structure analysis.

Figure 11.36 shows thin-walled cantilever beams with open or closed cross-sections and different loadings. All these beam structure models have equal length, hight and width and also the total thicknesses of all laminate strips are equal, independent of the number of the layers.

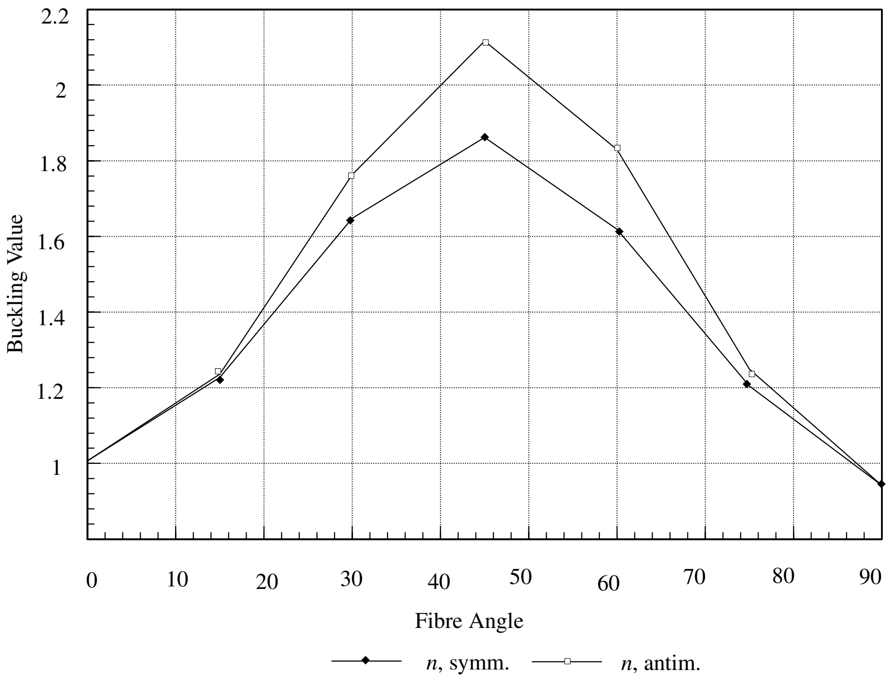


Fig. 11.35 Results of the Buckling Analysis

The stacking structure may be symmetric or antisymmetric. Figure 11.37 shows the two considered variants: case A with three laminae and symmetric stacking and case B with two laminae and antisymmetric stacking. The fibre reinforced material is characterized again by the following effective moduli

$$\begin{aligned} E_{\mathrm{L}} &= 153000\ \mathrm{N/mm^2}, & \nu_{\mathrm{LT}} &= 0.30, \\ E_{\mathrm{T}} &= 10900\ \mathrm{N/mm^2}, & \nu_{\mathrm{TL}} &= 0.021, \\ G_{\mathrm{LT}} &= 5600\ \mathrm{N/mm^2}, & \rho &= 2\ \mathrm{g/cm^3} \end{aligned}$$

The fibre angles shall be varied.

Figure 11.38 shows the profile nodes. There are four main profile nodes for both cross-sections but three secondary profile nodes for the open and four for the closed cross-section. The numerical analysis shall demonstrate the influence of the stacking structure. Figure 11.39 illustrates the relative changes of the cantilever beam in the loaded point, if the symmetric stacking structure (w_A) is change to the antisymmetric one (w_B). The antisymmetric layer stacking leads to higher values of the vertical deflections w_B in comparison to the w_A values in the case of symmetric stacking. Generally, only two degrees of freedom of secondary profile nodes were activated.

In a separate analysis the influence of a higher degrees of freedom in the secondary profile nodes was considered. As a result it can be recommended that for antisymmetric layer structures and for open profiles more than two degrees of free-

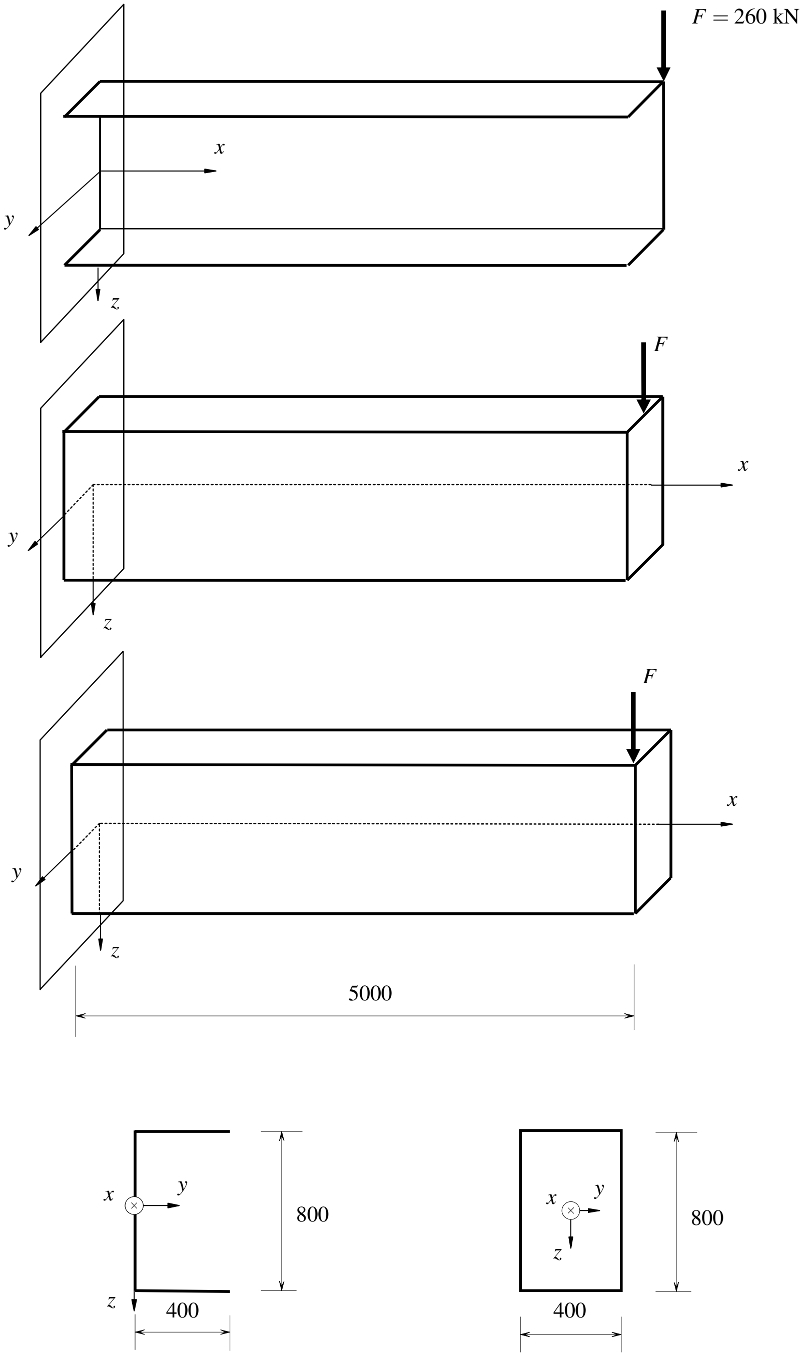


Fig. 11.36 Cantilever beams, geometry and loading

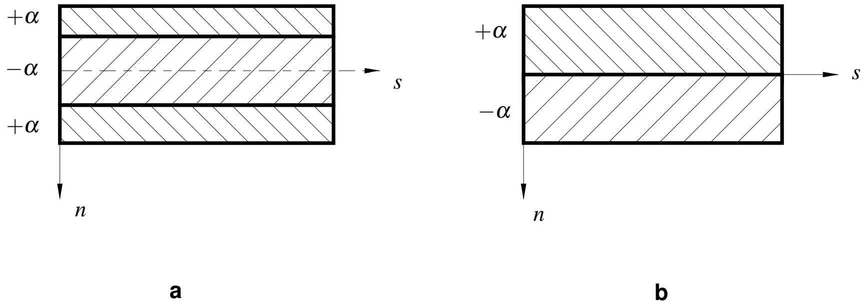


Fig. 11.37 Stacking structure of the laminates. **a** Symmetric sandwich, **b** two-layer

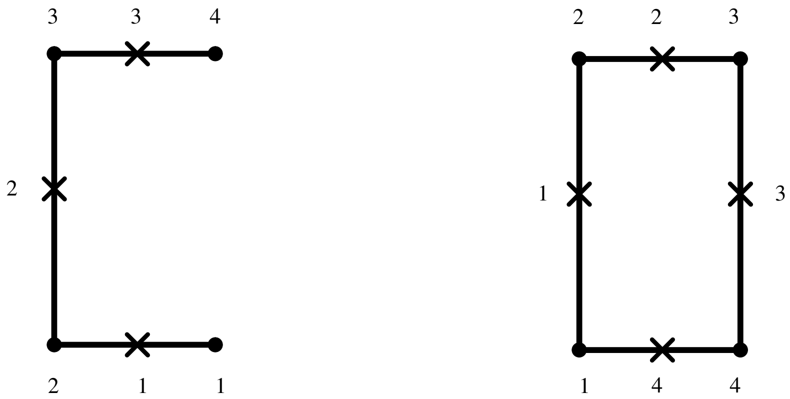


Fig. 11.38 Cross-sections with main profile nodes (•) and secondary profile nodes (×)

dom should be activated. Ignoring the activation of secondary profile node degrees of freedom leads to nonrealistic structure stiffness. The structure model is to stiff and therefore the deflections are to small.

The second example concerned the eigen-vibration analysis. For the closed cross-section with symmetric layer stacking the influences of the degree of freedom of secondary profile nodes and of the variation of the fibre angles were considered. As a result it can be stated that the influence of higher degrees of freedom of the secondary profile nodes is negligible but the influence of the fibre angles is significant. Figure 11.40 illustrates the influence of the fibre angle variations on the eigenfrequencies of the beam, which can be used for structure optimization.

Summarizing Sect. 11.5 one have to say that only a small selection of one- and two-dimensional finite elements was considered. Many finite plate and shell elements were developed using equivalent single layer theories for laminated structures but also multi-layered theories are used. Recent review articles give a detailed overview on the development, implementation and testing of different finite laminate and sandwich elements.

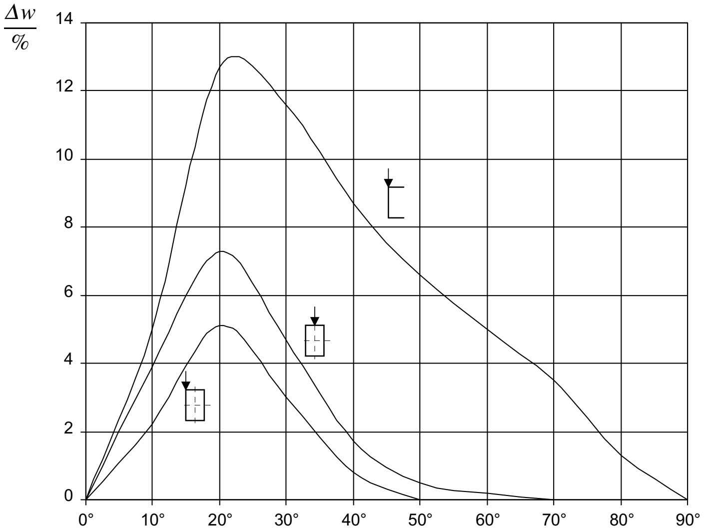


Fig. 11.39 Relative changes $(w_B - w_A)/w_A = \Delta w(\alpha)$ of the vertical deflections w_A and w_B

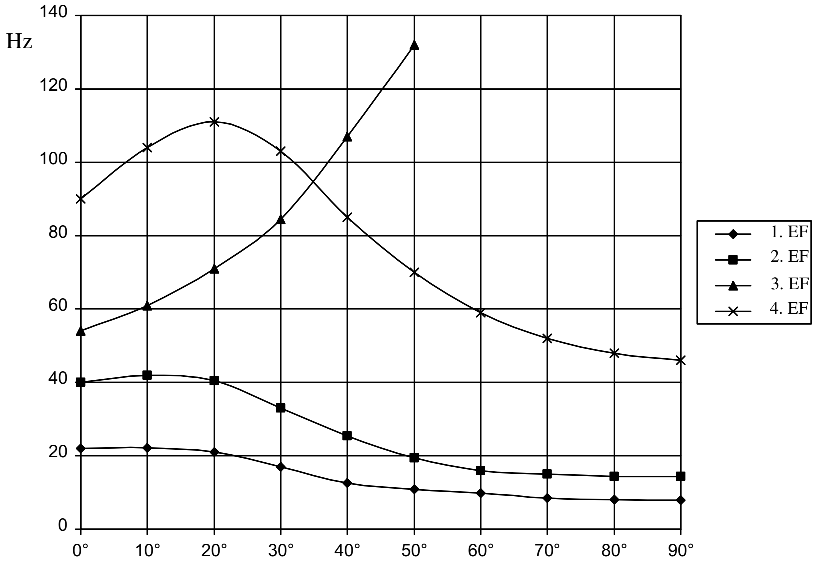


Fig. 11.40 Influence of the fibre angle on the first four eigen-frequencies of the cantilever box-beam

Part VI
Appendices

This part is focussed on some basics of mathematics and mechanics like matrix operations (App. A), stress and strain transformations (App. B), differential operators for rectangular plates (App. C) and differential operators for circular cylindrical shells (App. D). In addition, the Krylow functions as solution forms of a special fourth order ordinary differential equation are discussed (App. E) and some material's properties are given in App. F. In the last one section like always in this book material or constitutive parameters are used as usual. Note that any material parameter is a parameter since there are dependencies on temperature, time, etc. Last but not least there are given some references for further reading (App. G).

Appendix A
Matrix Operations

The following short review of the basic matrix definitions and operations will provide a quick reference and ensure that the particular use of vector-matrix notations in this textbook is correct understood.

A.1 Definitions

1. *Rectangular matrix*

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & \cdots & a_{mn} \end{bmatrix} = [a_{ij}]$$

Rectangular matrix with $i = 1,2,\ldots,m$ rows and $j = 1,2,\ldots,n$ columns, is a rectangular-ordered array of quantities with m rows and n columns. $m \times n$ or often (m,n) is the order of the matrix, a_{ij} is called the (i,j)-element of $\boldsymbol{A}$. There are two important special cases

$$\boldsymbol{a} = \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = [a_i]$$

is a $m \times 1$ matrix or column vector, while

$$\boldsymbol{a}^{\mathrm{T}} = \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} = [a_i]^{\mathrm{T}}$$

is a $1 \times n$ matrix or row vector.
With $\boldsymbol{a}$ respectively $\boldsymbol{a}^{\mathrm{T}}$ a matrix $\boldsymbol{A}$ can be written

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1 \cdots \boldsymbol{a}_n \end{bmatrix}, \quad \boldsymbol{a}_j = \begin{bmatrix} a_{1j} \\ \vdots \\ a_{mj} \end{bmatrix}, \quad j = 1,\ldots,n$$

or

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{a}_1^{\mathrm{T}} \\ \cdots \\ \boldsymbol{a}_m^{\mathrm{T}} \end{bmatrix}, \quad \boldsymbol{a}_j^{\mathrm{T}} = \begin{bmatrix} a_{j1} \cdots a_{jn} \end{bmatrix}, \quad j = 1,\ldots,m$$

If $n = m$ the matrix is square of the order $n \times n$. For a square matrix the elements a_{ij} with $i = j$ define the principal matrix diagonal and are located on it.

2. *Determinant of a square matrix* $\boldsymbol{A}$

$$|\boldsymbol{A}| = \begin{vmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{vmatrix} = |a_{ij}| = \det\boldsymbol{A}$$

The determinant of a matrix $\boldsymbol{A}$ with elements a_{ij} is given by

$$|\boldsymbol{A}| = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} - \ldots (-1)^{1+n} a_{1n}M_{1n},$$

where the minor M_{ij} is the determinant of the matrix $|\boldsymbol{A}|$ with missing row i and column j. Note the following properties of determinants:

- Interchanging two rows or two columns changing the sign of $|\boldsymbol{A}|$.
- If all elements in a row or a column of $\boldsymbol{A}$ are zero then $|\boldsymbol{A}| = 0$.
- Multiplication by a constant factor c of all elements in a row or column of $\boldsymbol{A}$ multiplies $|\boldsymbol{A}|$ by c.
- Adding a constant multiple of row or column k to row or column l does not change the determinant.
- If one row k is a linear combination of the rows l and m then the determinant must be zero.

3. *Regular matrix*
   A square matrix $\boldsymbol{A}$ is regular if $|\boldsymbol{A}| \neq 0$.
4. *Singular matrix*
   A square matrix $\boldsymbol{A}$ is singular if $|\boldsymbol{A}| = 0$.
5. *Trace of a matrix*
   The trace of a square matrix $\boldsymbol{A}$ is the sum of all elements of the principal diagonal, i.e.

$$\mathrm{tr}\boldsymbol{A} = \sum_{k=1}^{m} a_{kk}$$

6. *Rank of a matrix*
   The rank $\mathrm{rk}(\boldsymbol{A})$ of a $m \times n$ matrix $\boldsymbol{A}$ is the largest value of r for which there exist a

$r \times r$ submatrix of $\boldsymbol{A}$ that is non-singular. Submatrices are smaller arrays of $k \times k$ elements a_{ij} of the matrix $\boldsymbol{A}$, i.e. $k \le m$ if $m \le n$ or $k \le n$ if $n \le m$.

A.2 Special Matrices

In the following the δ_{ij} denotes the Kronecker symbol

$$\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

1. *Null matrix* $\mathbf{0}$
   All elements a_{ij} of a $m \times n$ matrix are identically equal zero
   $$a_{ij} \equiv 0, i = 1, \ldots, m, j = 1, \ldots, n$$
2. *Diagonal matrix* $\boldsymbol{D} = \mathrm{diag}[a_{ii}] = \mathrm{diag}[a_{ij}\delta_{ij}]$
   A diagonal matrix is a square matrix in which all elements are zero except those on the principal diagonal
   $$a_{ij} = 0, i \neq j, \quad a_{ij} \neq 0, i = j$$
3. *Unit matrix* $\boldsymbol{I} = [\delta_{ij}]$
   A unit or identity matrix is a special case of the diagonal matrix for which $a_{ij} = 1$ when $i = j$ and $a_{ij} = 0$ when $i \neq j$.
4. *Transpose* $\boldsymbol{A}^{\mathrm{T}}$ *of a matrix* $\boldsymbol{A}$
   The transpose of a matrix $\boldsymbol{A}$ is found by interchanging rows and columns. If $\boldsymbol{A} = [a_{ij}]$ follow $\boldsymbol{A}^{\mathrm{T}} = [a_{ij}^{\mathrm{T}}]$ with $a_{ij}^{\mathrm{T}} = a_{ji}$. A transposed matrix is denoted by a superscript T. Note $(\boldsymbol{A}^{\mathrm{T}})^{\mathrm{T}} = \boldsymbol{A}$
5. *Symmetric matrix* $\boldsymbol{A}^{\mathrm{S}}$
   A square matrix $\boldsymbol{A}$ is said to be symmetric if for all $i \neq j$ $a_{ij} = a_{ji}$, i.e. $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$. A symmetric matrix is denoted by a superscript S.
6. *Skew-symmetric matrix* $\boldsymbol{A}^{\mathrm{A}}$
   A square matrix $\boldsymbol{A}$ is said to be skew-symmetric if all principal diagonal elements are equal zero and for all $i \neq j$ $a_{ij} = -a_{ji}$, i.e. $\boldsymbol{A} = -\boldsymbol{A}^{\mathrm{T}}$. A skew-symmetric matrix is denoted by a superscript A.
7. Any matrix can be decomposed in a symmetric and a skew-symmetric part in a unique manner
   $$\boldsymbol{A} = \boldsymbol{A}^{\mathrm{S}} + \boldsymbol{A}^{\mathrm{A}}$$
   Proof. Since
   $$\boldsymbol{A}^{\mathrm{S}} = \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}})$$
   and
   $$\boldsymbol{A}^{\mathrm{A}} = \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^{\mathrm{T}})$$

the sum $\boldsymbol{A}^{\mathrm{S}} + \boldsymbol{A}^{\mathrm{A}}$ is equal to $\boldsymbol{A}$.

A.3 Matrix Algebra and Analysis

1. *Addition and subtraction*
   A $m \times n$ matrix $\boldsymbol{A}$ can be added or subtracted to a $m \times n$ matrix $\boldsymbol{B}$ to form a $m \times n$ matrix $\boldsymbol{C}$

$$\boldsymbol{A} \pm \boldsymbol{B} = \boldsymbol{C}, \quad a_{ij} \pm b_{ij} = c_{ij}, \quad i = 1, \dots, m, \quad j = 1, \dots, n$$

   Note $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$,$\boldsymbol{A} - \boldsymbol{B} = -(\boldsymbol{B} - \boldsymbol{A}) = -\boldsymbol{B} + \boldsymbol{A}$, $(\boldsymbol{A} \pm \boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} \pm \boldsymbol{B}^{\mathrm{T}}$.
2. *Multiplication*
   - Multiplication the matrix $\boldsymbol{A}$ by a scalar α involves the multiplication of all elements of the matrix by the scalar

$$\begin{aligned} \alpha\boldsymbol{A} = \boldsymbol{A}\alpha &= [\alpha a_{ij}], \\ (\alpha \pm \beta)\boldsymbol{A} &= \alpha\boldsymbol{A} \pm \beta\boldsymbol{A}, \\ \alpha(\boldsymbol{A} \pm \boldsymbol{B}) &= \alpha\boldsymbol{A} \pm \alpha\boldsymbol{B} \end{aligned}$$

   - The product of a $(1 \times n)$ matrix (row vector $\boldsymbol{a}^{\mathrm{T}}$) and a $(n \times 1)$ matrix (column vector $\boldsymbol{b}$) forms a (1×1) matrix, i.e. a scalar α

$$\boldsymbol{a}^{\mathrm{T}}\boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}}\boldsymbol{a} = \alpha, \quad \alpha = \sum_{k=1}^{n} a_k b_k$$

   - The product of a $(m \times n)$ matrix $\boldsymbol{A}$ and a $(n \times 1)$ column vector $\boldsymbol{b}$ forms a $(m \times 1)$ column vector $\boldsymbol{c}$

$$\boldsymbol{A}\boldsymbol{b} = \boldsymbol{c}, \quad c_i = \sum_{j=1}^{n} a_{ij} b_j = a_{i1} b_1 + a_{i2} b_2 + \ldots + a_{in} b_n, \quad i = 1, 2, \dots, m$$

     The forgoing product is only possible if the number of columns of $\boldsymbol{A}$ is equal the number of rows of $\boldsymbol{b}$.

     Note A.1. $\boldsymbol{b}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{c}^{\mathrm{T}}$

   - If $\boldsymbol{A}$ is a $(m \times n)$ matrix and $\boldsymbol{B}$ a $(p \times q)$ matrix the product $\boldsymbol{A}\boldsymbol{B} = \boldsymbol{C}$ exists if $n = p$, in which case $\boldsymbol{C}$ is a $(m \times q)$ matrix. For $n = p$ the matrix $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be conformable for multiplication. The elements of the matrix $\boldsymbol{C}$ are

$$c_{ij} = \sum_{k=1}^{n=p} a_{ik} b_{kj}, \quad i = 1, 2, \dots, m, \quad j = 1, 2, \dots, q$$

     Note A.2. $\boldsymbol{A}\boldsymbol{B} \neq \boldsymbol{B}\boldsymbol{A}, \quad \boldsymbol{A}(\boldsymbol{B} \pm \boldsymbol{C}) = \boldsymbol{A}\boldsymbol{B} \pm \boldsymbol{A}\boldsymbol{C}, \quad (\boldsymbol{A}\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$

$$\overset{(l\times m)}{\boldsymbol{A}}\ \overset{(m\times n)}{\boldsymbol{B}}\ \overset{(n\times p)}{\boldsymbol{C}} = \overset{(l\times p)}{\boldsymbol{D}}$$

3. *Inversion and division*

$$\boldsymbol{AI} = \boldsymbol{IA} = \boldsymbol{A}, \quad \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{AA}^{-1} = \boldsymbol{I}, \quad \left(\boldsymbol{A}^{-1}\right)^{-1} = \boldsymbol{A}$$

The matrix inversion is based on the existence of a $n\times n$ unit matrix $\boldsymbol{I}$ and a square $n\times n$ matrix $\boldsymbol{A}$. $\boldsymbol{A}^{-1}$ is the inverse of $\boldsymbol{A}$ with respect to the matrix multiplication $\boldsymbol{AA}^{-1} = \boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$. If $\boldsymbol{A}^{-1}$ exist, the matrix $\boldsymbol{A}$ is invertible or regular, otherwise non-invertible or singular. Matrix division is not defined.

Note A.3. $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}, (\boldsymbol{ABC})^{-1} = \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1},\dots$

- Cofactor matrix
  With the minors M_{ij} introduced above to define the determinant $|\boldsymbol{A}|$ of a matrix $\boldsymbol{A}$ a so-called cofactor matrix $\boldsymbol{A}^{\mathrm{c}} = [A_{ij}]$ can be defined, where

  $$A_{ij} = (-1)^{i+j}M_{ij}$$

  The cofactor matrix is denoted by the superscript c.
- Adjoint or adjugate matrix
  The adjoint matrix of the square matrix $\boldsymbol{A}$ is the transpose of the cofactor matrix

  $$\mathrm{adj}\boldsymbol{A} = (\boldsymbol{A}^{\mathrm{c}})^{\mathrm{T}}$$

  Note A.4. Because symmetric matrices possess symmetric cofactor matrices the adjoint of a symmetric matrix is the cofactor matrix itself

  $$\mathrm{adj}\boldsymbol{A}^{\mathrm{S}} = \left(\boldsymbol{A}^{\mathrm{S}}\right)^{\mathrm{c}}$$

  It can be shown that

  $$\boldsymbol{A}(\mathrm{adj}\boldsymbol{A}) = |\boldsymbol{A}|\boldsymbol{I}$$

  i.e.

  $$\frac{\boldsymbol{A}(\mathrm{adj}\boldsymbol{A})}{|\boldsymbol{A}|} = \boldsymbol{I} = \boldsymbol{AA}^{-1} \Rightarrow \boldsymbol{A}^{-1} = \frac{\mathrm{adj}\boldsymbol{A}}{|\boldsymbol{A}|}$$

Inverse matrices have some important properties

$$\left(\boldsymbol{A}^{\mathrm{T}}\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^{\mathrm{T}}$$

and if $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}$

$$\boldsymbol{A}^{-1} = \left(\boldsymbol{A}^{-1}\right)^{\mathrm{T}}$$

i.e. the inverse matrix of a symmetric matrix $\boldsymbol{A}$ is also symmetric.

Note A.5. Symmetric matrices posses symmetric transposes, symmetric cofactors, symmetric adjoints and symmetric inverses.

4. *Powers and roots of square matrices*
   If $n \times n$ matrix $\boldsymbol{A}$ is conformable with itself for multiplication, one may define its powers
$$\boldsymbol{A}^n = \boldsymbol{A}\boldsymbol{A}\ldots\boldsymbol{A},$$
   and for symmetric positive semidefinite matrices
$$\boldsymbol{A}^{\frac{1}{n}} = \sqrt[n]{\boldsymbol{A}}, \quad \boldsymbol{A}^{-n} = \left(\boldsymbol{A}^{-1}\right)^n$$
   and if $\boldsymbol{A}$ is regular
$$(\boldsymbol{A}^m)^n = \boldsymbol{A}^{mn}, \quad \boldsymbol{A}^m\boldsymbol{A}^n = \boldsymbol{A}^{m+n},$$
5. *Matrix eigenvalue problems*
   The standard eigenvalue problem of a quadratic $n \times n$ matrix $\boldsymbol{A}$ is of the form: find $(\lambda, \boldsymbol{x})$ with $\neq \boldsymbol{0}$ such that
$$\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x} \quad \text{or} \quad (\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$
   $\boldsymbol{K} = [\boldsymbol{A} - \lambda\boldsymbol{I}]$ is called the characteristic matrix of $\boldsymbol{A}$, $\det\boldsymbol{K} = 0$ is called the characteristic determinant or equation of $\boldsymbol{A}$. The characteristic determinant produces a characteristic polynomial with powers of λ up to λ^n and therefore when it set equal zero having n roots which are called the eigenvalues. If the characteristic equation has n distinct roots, the polynomial can be factorized in the form
$$(\lambda - \lambda_1)(\lambda - \lambda_2)\ldots(\lambda - \lambda_n) = 0$$
   If we put $\lambda = 0$ in the characteristic equation we get
$$\det\boldsymbol{A} = \lambda_1\lambda_2\ldots\lambda_n$$
   Inserting any root λ_i into the standard eigenvalue equation leads to
$$[\boldsymbol{A} - \lambda_i\boldsymbol{I}]\boldsymbol{x}_i = \boldsymbol{0}, \quad i = 1, 2, \ldots, n$$
   $\boldsymbol{x}_i$ are the eigendirections (eigenvectors) which can be computed from the last equation considering the orthogonality condition. A nontrivial solution exists if and only if
$$\det[\boldsymbol{A} - \lambda_i\boldsymbol{I}] = 0$$
   Note A.6. If we have the 3x3 symmetric matrix the eigendirection $\boldsymbol{x}_i$ can be computed for each λ_i from the polynomial of third order. Three different solutions are possible:

   - all solutions λ_i are distinct - three orthogonal eigendirections can be computed (but their magnitudes are arbitrary),
   - one double solution and one distinct solutions - only one eigendirection can be computed (its magnitudes is arbitrary), and
   - all solutions are identically - no eigendirections can be computed.

Anyway, the orthogonality condition $\boldsymbol{x}^{\mathrm{T}} \cdot \boldsymbol{x} = 1$ should be taken into account.

The general eigenvalue problem is given in the form

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{B}\boldsymbol{x}$$

which can be premultiplied by $\boldsymbol{B}^{-1}$ to produce the standard form

$$\boldsymbol{B}^{-1}\boldsymbol{A}\boldsymbol{x} = \boldsymbol{B}^{-1}\lambda\boldsymbol{B}\boldsymbol{x} = (\boldsymbol{B}^{-1}\boldsymbol{A})\boldsymbol{x} = \lambda\boldsymbol{I}\boldsymbol{x} = \lambda\boldsymbol{x}$$

resulting in

$$(\boldsymbol{B}^{-1}\boldsymbol{A} - \lambda\boldsymbol{I})\boldsymbol{x} = \boldsymbol{0}$$

Note A.7. In the case of non-symmetric matrix $\boldsymbol{A}$ the eigenvalue can be complex.

6. *Differentiating and integrating*
   - To differentiate a matrix one differentiates each matrix element a_{ij} in the conventual manner.
   - To integrate a matrix one integrates each matrix element a_{ij} in the conventual manner. For definite integrals, each term is evaluated for the limits of integration.

7. *Partitioning of matrices*
   A useful operation with matrices is partitioning into submatrices. These submatrices may be treated as elements of the parent matrix and manipulated by the standard matrix rules reviewed above. The partitioning is usually indicated by dashed partitioning lines entirely through the matrix

$$\boldsymbol{M} = [m_{ij}] = \begin{bmatrix} \boldsymbol{A} & \vdots & \boldsymbol{B} \\ \cdots & \cdots & \cdots \\ \boldsymbol{C} & \vdots & \boldsymbol{D} \end{bmatrix}$$

For a $m \times n$ matrix $\boldsymbol{M}$ we may have submatrices $\boldsymbol{A}(i \times j)$, $\boldsymbol{B}(i \times p)$, $\boldsymbol{C}(k \times j)$, $\boldsymbol{D}(k \times p)$ with $i + k = m, j + p = n$, i.e.

$$\boldsymbol{M}_{m\times n} = \begin{bmatrix} \boldsymbol{A}_{i\times j} & \vdots & \boldsymbol{B}_{i\times(n-j)} \\ \cdots & \cdots & \cdots \\ \boldsymbol{C}_{(m-i)\times j} & \vdots & \boldsymbol{D}_{(m-i)\times(n-j)} \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{A} & \vdots & \boldsymbol{B} \\ \cdots & \cdots & \cdots \\ \boldsymbol{C} & \vdots & \boldsymbol{D} \end{bmatrix} \pm \begin{bmatrix} \boldsymbol{E} & \vdots & \boldsymbol{F} \\ \cdots & \cdots & \cdots \\ \boldsymbol{G} & \vdots & \boldsymbol{H} \end{bmatrix} = \begin{bmatrix} \boldsymbol{A} \pm \boldsymbol{E} & \vdots & \boldsymbol{B} \pm \boldsymbol{F} \\ \cdots & \cdots & \cdots \\ \boldsymbol{C} \pm \boldsymbol{G} & \vdots & \boldsymbol{D} \pm \boldsymbol{H} \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{A} & \vdots & \boldsymbol{B} \\ \cdots & \cdots & \cdots \\ \boldsymbol{C} & \vdots & \boldsymbol{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{E} & \vdots & \boldsymbol{F} \\ \cdots & \cdots & \cdots \\ \boldsymbol{G} & \vdots & \boldsymbol{H} \end{bmatrix} = \begin{bmatrix} \boldsymbol{AE}+\boldsymbol{BG} & \vdots & \boldsymbol{AF}+\boldsymbol{BH} \\ \cdots & \cdots & \cdots \\ \boldsymbol{CE}+\boldsymbol{DG} & \vdots & \boldsymbol{CF}+\boldsymbol{DH} \end{bmatrix}$$

The multiplications are only defined if the correspondent matrices are conformable for multiplication

$$\begin{bmatrix} \boldsymbol{A} & \vdots & \boldsymbol{B} \\ \cdots & \cdots & \cdots \\ \boldsymbol{C} & \vdots & \boldsymbol{D} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} \boldsymbol{A}^{\mathrm{T}} & \vdots & \boldsymbol{C}^{\mathrm{T}} \\ \cdots & \cdots & \cdots \\ \boldsymbol{B}^{\mathrm{T}} & \vdots & \boldsymbol{D}^{\mathrm{T}} \end{bmatrix}$$

If the matrix

$$\boldsymbol{M} = \begin{bmatrix} \boldsymbol{A} & \vdots & \boldsymbol{B} \\ \cdots & \cdots & \cdots \\ \boldsymbol{C} & \vdots & \boldsymbol{D} \end{bmatrix}$$

is symmetric ($\boldsymbol{M} = \boldsymbol{M}^{\mathrm{T}}$), it follows $\boldsymbol{A} = \boldsymbol{A}^{\mathrm{T}}, \boldsymbol{D} = \boldsymbol{D}^{\mathrm{T}}, \boldsymbol{B} = \boldsymbol{C}^{\mathrm{T}}, \boldsymbol{C} = \boldsymbol{B}^{\mathrm{T}}$

Appendix B
Stress and Strain Transformations

Stress and strain transformations under general orthogonal coordinate transformation $\boldsymbol{e}' = \boldsymbol{R}\boldsymbol{e}$ or $e_i' = R_{ij}e_j$:

1. $\sigma_p' = T_{pq}^{\sigma}\sigma_q$. The matrix $[T_{pq}^{\sigma}]$ is defined by

$$\begin{bmatrix} R_{11}^2 & R_{12}^2 & R_{13}^2 & 2R_{12}R_{13} & 2R_{11}R_{13} & 2R_{11}R_{12} \\ R_{21}^2 & R_{22}^2 & R_{23}^2 & 2R_{22}R_{23} & 2R_{21}R_{23} & 2R_{21}R_{22} \\ R_{31}^2 & R_{32}^2 & R_{33}^2 & 2R_{32}R_{33} & 2R_{31}R_{33} & 2R_{31}R_{32} \\ R_{21}R_{31} & R_{22}R_{32} & R_{23}R_{33} & R_{22}R_{33}+R_{23}R_{32} & R_{21}R_{33}+R_{23}R_{31} & R_{21}R_{32}+R_{22}R_{31} \\ R_{11}R_{31} & R_{12}R_{32} & R_{13}R_{33} & R_{12}R_{33}+R_{13}R_{32} & R_{11}R_{33}+R_{13}R_{31} & R_{11}R_{32}+R_{12}R_{31} \\ R_{11}R_{21} & R_{12}R_{22} & R_{13}R_{23} & R_{12}R_{23}+R_{13}R_{22} & R_{11}R_{23}+R_{13}R_{21} & R_{11}R_{22}+R_{12}R_{21} \end{bmatrix}$$

2. $\varepsilon_p' = T_{pq}^{\varepsilon}\varepsilon_q$. The matrix $[T_{pq}^{\varepsilon}]$ is defined by

$$\begin{bmatrix} R_{11}^2 & R_{12}^2 & R_{13}^2 & R_{12}R_{13} & R_{11}R_{13} & R_{11}R_{12} \\ R_{21}^2 & R_{22}^2 & R_{23}^2 & R_{22}R_{23} & R_{21}R_{23} & R_{21}R_{22} \\ R_{31}^2 & R_{32}^2 & R_{33}^2 & R_{32}R_{33} & R_{31}R_{33} & R_{31}R_{32} \\ 2R_{21}R_{31} & 2R_{22}R_{32} & 2R_{23}R_{33} & R_{22}R_{33}+R_{23}R_{32} & R_{21}R_{33}+R_{23}R_{31} & R_{21}R_{32}+R_{22}R_{31} \\ 2R_{11}R_{31} & 2R_{12}R_{32} & 2R_{13}R_{33} & R_{12}R_{33}+R_{13}R_{32} & R_{11}R_{33}+R_{13}R_{31} & R_{11}R_{32}+R_{12}R_{31} \\ 2R_{11}R_{21} & 2R_{12}R_{22} & 2R_{13}R_{23} & R_{12}R_{23}+R_{13}R_{22} & R_{11}R_{23}+R_{13}R_{21} & R_{11}R_{22}+R_{12}R_{21} \end{bmatrix}$$

3. Rotation about the $\boldsymbol{e}_1$-direction, Fig. B.1:

$$[R_{ij}] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & s \\ 0 & s & c \end{bmatrix}, \quad \boldsymbol{e}' = \overset{1}{\boldsymbol{R}}\,\boldsymbol{e}$$

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0

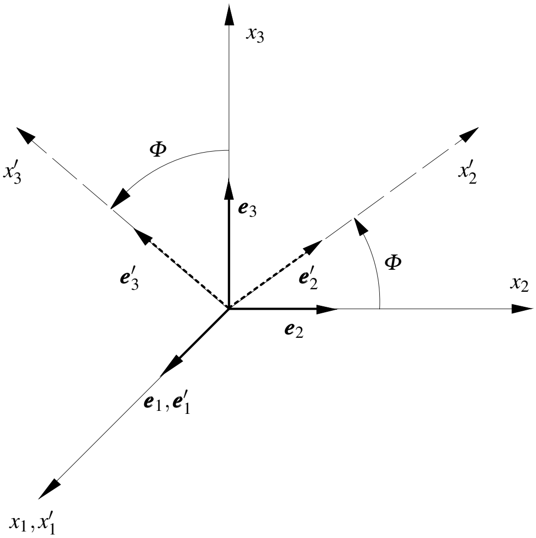


Fig. B.1 Rotation about the $\boldsymbol{e}_1$-direction

$$[\overset{1}{T}{}^{\sigma}_{pq}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c^2 & s^2 & 2cs & 0 & 0 \\ 0 & s^2 & c^2 & -2cs & 0 & 0 \\ 0 & -cs & cs & c^2-s^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & s & c \end{bmatrix}, \quad \boldsymbol{\sigma}' = \overset{1}{\boldsymbol{T}}{}^{\sigma}\, \boldsymbol{\sigma}$$

$$[\overset{1}{T}{}^{\varepsilon}_{pq}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & c^2 & s^2 & cs & 0 & 0 \\ 0 & s^2 & c^2 & -cs & 0 & 0 \\ 0 & -2cs & 2cs & c^2-s^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & c & -s \\ 0 & 0 & 0 & 0 & s & c \end{bmatrix}, \quad \boldsymbol{\varepsilon}' = \overset{1}{\boldsymbol{T}}{}^{\varepsilon}\, \boldsymbol{\varepsilon}$$

Appendix C
Differential Operators for Rectangular Plates

Below two cases will be discussed

- the classical plate theory and
- the shear deformation theory.

C.1 Classical Plate Theory

1. General unsymmetric laminates

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ & L_{22} & L_{23} \\ \text{sym} & & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix},$$

$$\begin{aligned}
L_{11} &= A_{11}\frac{\partial^2}{\partial x_1^2} + 2A_{16}\frac{\partial^2}{\partial x_1 \partial x_2} + A_{66}\frac{\partial^2}{\partial x_2^2},\\
L_{22} &= A_{22}\frac{\partial^2}{\partial x_2^2} + 2A_{26}\frac{\partial^2}{\partial x_1 \partial x_2} + A_{66}\frac{\partial^2}{\partial x_1^2},\\
L_{33} &= D_{11}\frac{\partial^4}{\partial x_1^4} + 4D_{16}\frac{\partial^4}{\partial x_1^3 \partial x_2} + 2(D_{16} + 2D_{66})\frac{\partial^4}{\partial x_1^2 \partial x_2^2}\\
&\quad + 4D_{26}\frac{\partial^4}{\partial x_1 \partial x_2^3} + D_{22}\frac{\partial^4}{\partial x_2^4},\\
L_{12} &= A_{16}\frac{\partial^2}{\partial x_1^2} + (A_{12} + A_{66})\frac{\partial^2}{\partial x_1 \partial x_2} + A_{26}\frac{\partial^2}{\partial x_2^2},\\
L_{13} &= -\left[B_{11}\frac{\partial^3}{\partial x_1^3} + 3B_{16}\frac{\partial^3}{\partial x_1^2 \partial x_2} + (B_{12} + 2B_{66})\frac{\partial^3}{\partial x_1 \partial x_2^2} + B_{26}\frac{\partial^3}{\partial x_2^3}\right],
\end{aligned}$$

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0

$$L_{23} = -\left[B_{22}\frac{\partial^3}{\partial x_2^3} + 3B_{26}\frac{\partial^3}{\partial x_1 \partial x_2^2} + (B_{12}+2B_{66})\frac{\partial^3}{\partial x_1^2 \partial x_2} + B_{16}\frac{\partial^3}{\partial x_1^3}\right]$$

2. General symmetric laminates

$$\begin{bmatrix} L_{11} & L_{12} & 0 \\ & L_{22} & 0 \\ \text{sym} & & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$$

   $B_{ij} = 0$, i.e. $L_{13} = L_{31} = 0, L_{23} = L_{32} = 0$. $L_{11}, L_{22}, L_{33}, L_{12}$ as above in 1.
3. Balanced symmetric laminates

$$\begin{bmatrix} L_{11} & L_{12} & 0 \\ & L_{22} & 0 \\ \text{sym} & & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$$

   In addition to 2. both A_{16} and A_{26} are zero, i.e. $L_{13} = L_{31} = 0$, $L_{23} = L_{32} = 0$ and L_{11}, L_{22} and L_{33} simplify with $A_{16} = A_{26} = 0$.
4. Cross-ply symmetric laminates

$$\begin{bmatrix} L_{11} & L_{12} & 0 \\ & L_{22} & 0 \\ \text{sym} & & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$$

   In addition to 3. both D_{16} and D_{26} are zero, i.e. L_{33} simplifies.
5. Balanced unsymmetric laminates

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ & L_{22} & L_{23} \\ \text{sym} & & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$$

   With $A_{16} = A_{26} = 0$ only the operators L_{11}, L_{22} and L_{12} of 1. can be simplified.
6. Cross-ply unsymmetric laminates

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ & L_{22} & L_{23} \\ \text{sym} & & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$$

   In addition to 5., D_{16}, D_{26}, B_{16} and B_{26} are zero, i.e. all operators of 1. can be simplified.

C.2 Shear Deformation Theory

1. General unsymmetrical laminates

$$\begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} & \tilde{L}_{13} & \tilde{L}_{14} & 0 \\ & \tilde{L}_{22} & \tilde{L}_{23} & \tilde{L}_{24} & 0 \\ & & \tilde{L}_{33} & \tilde{L}_{34} & \tilde{L}_{35} \\ & & & \tilde{L}_{44} & \tilde{L}_{45} \\ \mathrm{S} & \mathrm{Y} & \mathrm{M} & & \tilde{L}_{55} \end{bmatrix} \begin{bmatrix} u \\ v \\ \psi_1 \\ \psi_2 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ p \end{bmatrix}$$

with $\tilde{L}_{11} = L_{11}, \tilde{L}_{22} = L_{22}, \tilde{L}_{12} = L_{12}$ (the L_{ij} can be taken from Appendix C.1) and

$$\begin{aligned}
\tilde{L}_{33} &= D_{11}\frac{\partial^2}{\partial x_1^2} + 2D_{16}\frac{\partial^2}{\partial x_1 \partial x_2} + D_{66}\frac{\partial^2}{\partial x_2^2} - k_{55}^{\mathrm{s}} A_{55}, \\
\tilde{L}_{44} &= D_{66}\frac{\partial^2}{\partial x_1^2} + 2D_{26}\frac{\partial^2}{\partial x_1 \partial x_2} + D_{22}\frac{\partial^2}{\partial x_2^2} - k_{44}^{\mathrm{s}} A_{44}, \\
\tilde{L}_{55} &= -\left(k_{55}^{\mathrm{s}} A_{55}\frac{\partial^2}{\partial x_1^2} + k_{45}^{\mathrm{s}} A_{45}\frac{\partial^2}{\partial x_1 \partial x_2} + k_{44}^{\mathrm{s}} A_{44}\frac{\partial^2}{\partial x_2^2}\right), \\
\tilde{L}_{13} &= \tilde{L}_{31} = B_{11}\frac{\partial^2}{\partial x_1^2} + 2B_{26}\frac{\partial^2}{\partial x_1 \partial x_2} + B_{66}\frac{\partial^2}{\partial x_2^2}, \\
\tilde{L}_{14} &= \tilde{L}_{41} = \tilde{L}_{23} = \tilde{L}_{32} = B_{16}\frac{\partial^2}{\partial x_1^2} + (B_{12} + B_{66})\frac{\partial^2}{\partial x_1 \partial x_2} + B_{26}\frac{\partial^2}{\partial x_2^2}, \\
\tilde{L}_{24} &= \tilde{L}_{42} = B_{66}\frac{\partial^2}{\partial x_1^2} + 2B_{26}\frac{\partial^2}{\partial x_1 \partial x_2} + B_{22}\frac{\partial^2}{\partial x_2^2}, \\
\tilde{L}_{34} &= \tilde{L}_{43} = D_{16}\frac{\partial^2}{\partial x_1^2} + (D_{12} + D_{66})\frac{\partial^2}{\partial x_1 \partial x_2} + D_{26}\frac{\partial^2}{\partial x_2^2}, \\
\tilde{L}_{35} &= \tilde{L}_{53} = -\left(k_{55}^{\mathrm{s}} A_{55}\frac{\partial}{\partial x_1} + k_{45}^{\mathrm{s}} A_{45}\frac{\partial}{\partial x_2}\right), \\
\tilde{L}_{45} &= \tilde{L}_{54} = -\left(k_{45}^{\mathrm{s}} A_{45}\frac{\partial}{\partial x_1} + k_{44}^{\mathrm{s}} A_{44}\frac{\partial}{\partial x_2}\right)
\end{aligned}$$

with $k_{45}^{\mathrm{s}} = \sqrt{k_{44}^{\mathrm{s}} k_{55}^{\mathrm{s}}}$.

2. General symmetric laminates
$B_{ij} = 0$, i.e. $\tilde{L}_{13} = \tilde{L}_{31}$, $\tilde{L}_{14} = \tilde{L}_{41}$, $\tilde{L}_{23} = \tilde{L}_{32}$ and $\tilde{L}_{24} = \tilde{L}_{42}$ are zero

$$\begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} \\ \tilde{L}_{12} & \tilde{L}_{22} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0$$

$$\begin{bmatrix} \tilde{L}_{33} & \tilde{L}_{34} & \tilde{L}_{35} \\ \tilde{L}_{34} & \tilde{L}_{44} & \tilde{L}_{45} \\ \tilde{L}_{53} & \tilde{L}_{54} & \tilde{L}_{55} \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_1 \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ p \end{bmatrix}$$

3. Cross-ply symmetric laminates
   In addition to 2. both D_{16}, D_{26} and A_{16}, A_{26}, A_{45} are zero.

Appendix D
Differential Operators for Circular Cylindrical Shells

Below two cases will be considered

- the classical case and
- the first order shear deformation theory.

D.1 Classical Shell Theory

1. General unsymmetrical laminates

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ & L_{22} & L_{23} \\ \text{SYM} & & L_{33} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = - \begin{bmatrix} p_x \\ p_s \\ p_z \end{bmatrix}$$

$$\begin{aligned}
L_{11} &= A_{11}\frac{\partial^2}{\partial x^2} + 2A_{16}\frac{\partial^2}{\partial x \partial s} + A_{66}\frac{\partial^2}{\partial s^2},\\
L_{12} &= (A_{16} + R^{-1}B_{16})\frac{\partial^2}{\partial x_1^2} + (A_{12} + R^{-1}B_{12} + A_{66} + R^{-1}B_{66})\frac{\partial^2}{\partial x \partial s}\\
&+ (A_{26} + R^{-1}B_{26})\frac{\partial^2}{\partial s^2},\\
L_{13} &= R^{-1}A_{16}\frac{\partial}{\partial x} + R^{-1}A_{26}\frac{\partial}{\partial s} - B_{11}\frac{\partial^3}{\partial x^3} - 3B_{16}\frac{\partial^3}{\partial x^2 \partial s}\\
&- (B_{12} + B_{66})\frac{\partial^3}{\partial x \partial s^2} - B_{26}\frac{\partial^3}{\partial s^3},\\
L_{22} &= (A_{66} + 2R^{-1}B_{66} + R^{-2}D_{66})\frac{\partial^2}{\partial x^2}
\end{aligned}$$

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0

$$
\begin{aligned}
&+ 2(A_{26} + 2R^{-1}B_{26} + 2R^{-2}D_{26})\frac{\partial^2}{\partial x \partial s} \\
&+ (A_{22} + 2R^{-1}B_{22} + R^{-2}D_{22})\frac{\partial^2}{\partial s^2}, \\
L_{23} &= R^{-1}(A_{26} + R^{-1}B_{26})\frac{\partial}{\partial x} + R^{-1}(A_{22} + R^{-1}B_{22})\frac{\partial}{\partial s} \\
&- (B_{16} + R^{-1}D_{16})\frac{\partial^3}{\partial x^3} - [B_{12} + 2B_{66} + R^{-1}(D_{12} + 2D_{66})]\frac{\partial^3}{\partial x^2 \partial s} \\
&- 3(B_{26} + R^{-1}D_{26})\frac{\partial^3}{\partial x \partial s^2} - (B_{22} + R^{-1}D_{22})\frac{\partial^3}{\partial s^3}, \\
L_{33} &= R^{-2}(A_{22} + R^{-1}B_{22}) + 2R^{-1}B_{12}\frac{\partial^2}{\partial x^2} + 4R^{-1}B_{26}\frac{\partial^2}{\partial x \partial s} + 2R^{-1}B_{22}\frac{\partial^2}{\partial s^2} \\
&- D_{11}\frac{\partial^4}{\partial x^4} - 4D_{16}\frac{\partial^4}{\partial x^3 \partial s} - 2(D_{12} + 2D_{66})\frac{\partial^4}{\partial x^2 \partial s^2} \\
&- 4D_{26}\frac{\partial^4}{\partial x \partial s^3} - D_{22}\frac{\partial^4}{\partial s^4}
\end{aligned}
$$

2. General symmetrical laminates
   All $B_{ij} = 0$, but the matrix $[L_{ij}]$ is full populated, i.e. all $[L_{ij}]$ are nonequal zero. Note that for general symmetrically laminated circular cylindrical shells there is a coupling of the in-plane and out-of-plane displacements and stress resultants.
3. Cross-ply symmetrical laminates

$$B_{ij} = 0, \quad A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0$$

4. Cross-ply antisymmetrical laminates

$$B_{22} = -B_{11}, \quad \text{all other} \quad B_{ij} = 0, \quad A_{16} = A_{26} = 0, \quad D_{16} = D_{26} = 0$$

5. Axisymmetric deformations of symmetrical cross-ply laminates
   Additional to 3. all derivative $\partial/\partial s$ and the displacement v are taken zero and yield

$$L_{12} = L_{13} = L_{23} \equiv 0$$

D.2 Shear Deformation Theory

1. General unsymmetrical laminates

$$\begin{bmatrix} \tilde{L}_{11} & \tilde{L}_{12} & \tilde{L}_{13} & \tilde{L}_{14} & \tilde{L}_{15} \\ \tilde{L}_{21} & \tilde{L}_{22} & \tilde{L}_{23} & \tilde{L}_{24} & \tilde{L}_{25} \\ \tilde{L}_{31} & \tilde{L}_{32} & \tilde{L}_{33} & \tilde{L}_{34} & \tilde{L}_{35} \\ \tilde{L}_{41} & \tilde{L}_{42} & \tilde{L}_{43} & \tilde{L}_{44} & \tilde{L}_{45} \\ \tilde{L}_{51} & \tilde{L}_{52} & \tilde{L}_{53} & \tilde{L}_{54} & \tilde{L}_{55} \end{bmatrix} \begin{bmatrix} u \\ v \\ \psi_1 \\ \psi_2 \\ w \end{bmatrix} = - \begin{bmatrix} p_x \\ p_s \\ 0 \\ 0 \\ p_z \end{bmatrix},$$

$$\begin{aligned}
\tilde{L}_{11} &= A_{11}\frac{\partial^2}{\partial x^2} + 2A_{16}\frac{\partial^2}{\partial x \partial s} + A_{66}\frac{\partial^2}{\partial s^2}, \\
\tilde{L}_{12} &= A_{16}\frac{\partial^2}{\partial x^2} + (A_{12} + A_{66})\frac{\partial^2}{\partial x \partial s} + A_{26}\frac{\partial^2}{\partial s^2}, \\
\tilde{L}_{13} &= B_{11}\frac{\partial^2}{\partial x^2} + 2B_{16}\frac{\partial^2}{\partial x \partial s} + B_{66}\frac{\partial^2}{\partial s^2}, \\
\tilde{L}_{14} &= B_{16}\frac{\partial^2}{\partial x^2} + (B_{12} + B_{66})\frac{\partial^2}{\partial x \partial s} + B_{26}\frac{\partial^2}{\partial s^2}, \\
\tilde{L}_{15} &= \frac{1}{R}A_{12}\frac{\partial}{\partial x} + \frac{1}{R}A_{26}\frac{\partial}{\partial s}, \\
\\
\tilde{L}_{22} &= A_{66}\frac{\partial^2}{\partial x^2} + 2A_{26}\frac{\partial^2}{\partial x \partial s} + A_{22}\frac{\partial^2}{\partial s^2} - \frac{1}{R^2}k_{44}^{\mathrm{s}}A_{44}, \\
\tilde{L}_{23} &= B_{16}\frac{\partial^2}{\partial x^2} + (B_{16} + B_{66})\frac{\partial^2}{\partial x \partial s} + B_{26}\frac{\partial^2}{\partial s^2} + \frac{1}{R}k_{45}^{\mathrm{s}}A_{45}, \\
\tilde{L}_{24} &= B_{66}\frac{\partial^2}{\partial x^2} + 2B_{26}\frac{\partial^2}{\partial x \partial s} + B_{22}\frac{\partial^2}{\partial s^2} + \frac{1}{R}k_{44}^{\mathrm{s}}A_{44}, \\
\tilde{L}_{25} &= (A_{12} + k_{55}^{\mathrm{s}}A_{55})\frac{1}{R}\frac{\partial}{\partial x} + (A_{26} + k_{45}^{\mathrm{s}}A_{45})\frac{1}{R}\frac{\partial}{\partial s}, \\
\tilde{L}_{33} &= D_{11}\frac{\partial^2}{\partial x^2} + 2D_{16}\frac{\partial^2}{\partial x \partial s} + D_{66}\frac{\partial^2}{\partial s^2} - k_{55}^{\mathrm{s}}A_{55} \\
\tilde{L}_{34} &= D_{16}\frac{\partial^2}{\partial x^2} + (D_{12} + D_{66})\frac{\partial^2}{\partial x \partial s} + D_{26}\frac{\partial^2}{\partial s^2} - k_{45}^{\mathrm{s}}A_{45}, \\
\tilde{L}_{35} &= \left(B_{12}\frac{1}{R} - A_{55}^{\mathrm{s}}\right)\frac{\partial}{\partial x} + \left(B_{26}\frac{1}{R} - k_{45}^{\mathrm{s}}A_{45}\right)\frac{\partial}{\partial s}, \\
\tilde{L}_{44} &= D_{66}\frac{\partial^2}{\partial x^2} + 2D_{26}\frac{\partial^2}{\partial x \partial s} + D_{22}\frac{\partial^2}{\partial s^2} - k_{44}^{\mathrm{s}}A_{44}, \\
\tilde{L}_{45} &= \left(B_{26}\frac{1}{R} - k_{45}^{\mathrm{s}}A_{45}\right)\frac{\partial}{\partial x} + \left(B_{22}\frac{1}{R} - k_{44}^{\mathrm{s}}A_{44}\right)\frac{\partial}{\partial s}, \\
\tilde{L}_{55} &= A_{55}\frac{\partial^2}{\partial x^2} + 2k_{45}^{\mathrm{s}}A_{45}\frac{\partial^2}{\partial x \partial s} - A_{22})\frac{1}{R^2}
\end{aligned}$$

and

$$\tilde{L}_{51} = -\tilde{L}_{15}, \quad \tilde{L}_{52} = -\tilde{L}_{25}, \quad \tilde{L}_{53} = -\tilde{L}_{35}, \quad \tilde{L}_{54} = -\tilde{L}_{45},$$

$$k_{45}^{s} = \sqrt{k_{44}^{s} k_{55}^{s}}$$

2. Cross-ply symmetrical laminates
   $B_{ij} = 0, \quad i,j = 1,2,6,$
   $A_{16} = A_{26} = A_{45} = 0, \quad D_{16} = D_{26} = 0$
3. Cross-ply antisymmetrical laminates
   $B_{22} = -B_{11}, \quad$ all other $\quad B_{ij} = 0, \quad i,j = 1,2,6,$
   $A_{16} = A_{26} = A_{45} = 0, \quad D_{16} = D_{26} = 0$

Appendix E
Krylow-Functions as Solution Forms of a Fourth Order Ordinary Differential Equation

The solutions of the following fourth order ordinary differential equation

$$w'''' - k_1^2 w'' + k_2^4 w = 0$$

can be presented in the form of so-called Krylow functions (Filonenko-Boroditsch, 1952):

1. $k_2^2 > k_1^2$

$$\alpha_1 = -\alpha_2 = a + \mathrm{i}b, \qquad \alpha_3 = -\alpha_4 = a - \mathrm{i}b,$$
$$a = \sqrt{\frac{1}{2}(k_2^2 + k_1^2)}, \qquad b = \sqrt{\frac{1}{2}(k_2^2 - k_1^2)}$$

   and the solutions are

$$\Phi_1 = \cosh ax \cos bx, \qquad \Phi_2 = \sinh ax \sin bx,$$
$$\Phi_3 = \cosh ax \sin bx, \qquad \Phi_4 = \sinh ax \cos bx$$

   or

$$\Phi_1 = \mathrm{e}^{-ax} \cos bx, \qquad \Phi_2 = \mathrm{e}^{-ax} \sin bx,$$
$$\Phi_3 = \mathrm{e}^{ax} \sin bx, \qquad \Phi_4 = \mathrm{e}^{ax} \cos bx$$

2. $k_2^2 < k_1^2$

$$\alpha_1 = -\alpha_2 = a, \qquad \alpha_3 = -\alpha_4 = b,$$
$$a = \sqrt{k_1^2 - \sqrt{k_1^4 - k_2^4}}, \qquad b = \sqrt{k_1^2 + \sqrt{k_1^4 - k_2^4}}$$

   and the solutions are

$$\Phi_1 = \cosh ax, \qquad \Phi_2 = \cosh bx,$$
$$\Phi_3 = \sinh ax, \qquad \Phi_4 = \sinh bx$$

   or

$$\Phi_1 = \mathrm{e}^{-ax}, \qquad \Phi_2 = \mathrm{e}^{-bx},$$
$$\Phi_3 = \mathrm{e}^{ax}, \qquad \Phi_4 = \mathrm{e}^{bx}$$

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0

3. $k_2^2 = k_1^2$

$$\alpha_1 = \alpha_2 = a, \quad \alpha_3 = \alpha_4 = -a$$

and the solutions are

$$\begin{aligned} \Phi_1 &= \cosh ax, & \Phi_2 &= x \sinh ax, \\ \Phi_3 &= \sinh ax, & \Phi_4 &= x \cosh ax \end{aligned}$$

or

$$\begin{aligned} \Phi_1 &= \mathrm{e}^{-ax}, & \Phi_2 &= x\mathrm{e}^{-ax}, \\ \Phi_3 &= \mathrm{e}^{ax}, & \Phi_4 &= x\mathrm{e}^{ax} \end{aligned}$$

References

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Appendix F
Material's Properties

Below material properties for classical materials, for the constituents of various composites and for unidirectional layers are presented. The information about the properties were taken from different sources (see the *Handbooks, Textbooks* and *Monographs* at the end of this appendix).

Note that the presentation of material data in a unique way is not so easy due to the incompleteness of material data in the original sources. This means that there are some empty places in the above following tables. The authors of this textbook were unable to fill out these places. Another problem is connected with the different unit systems in the original sources. For recalculation approximate relations are used (e.g. 1 kgf $\approx$ 10 N).

With respect to the quick changes in application composite materials all material data are only examples showing the main tendencies. Every year new materials are developed and for the material data one have to contact directly the companies.

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H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0

Table F.1 Material properties of conventional materials at room temperature (bulk form), after Grote and Feldhusen (2014)] and Czichos and Hennecke (2012)

	Density ρ (g/cm^3)	Young's modulus E (GPa)	Maximum specific modulus E/ρ (MNm/kg)	Ultimate strength σ_u (MPa)	Maximum specific strength σ_u/ρ (kNm/kg)	Coefficient of thermal expansion α ($10^{-6}/^\circ$ K)
Steel	7.8-7.85	180-210	27	340-2100	270	13
Gray cast iron	7.1-7.4	64-181	25	140-490	69	9-12
Aluminium	2.7-2.85	69-72	27	140-620	230	23
Titanium	4.4-4.5	110	25	1000-1200	273	11
Magnesium	1.8	40	22	260	144	26
Beryllium	1.8-1.85	300-320	173	620-700	389	
Nickel	8.9	200	22	400-500	56	13
Zirconium	6.5	100	15	390	60	5.9
Tantalum	16.6	180	11	275	17	6.5
Tungsten	19.3	350	18	1100-4100	212	6.5
Glass	2.5	70	28	700-2100	840	3.5-5.5

Table F.2 Material properties of fibre materials, after Hyer (1998)

	Density ρ (g/cm^3)	Young's modulus E (GPa)	Maximum specific modulus E/ρ (MNm/kg)	Ultimate strength σ_u (MPa)	Maximum specific strength σ_u/ρ (kNm/kg)	Coefficient of thermal expansion - fibre direction α ($10^{-6}/^\circ$ K	Diameter d (μm)
E-Glass	2.54	72.4	29	3450	1358	5	8-14
C-Glass	2.49	68.9	28	3160	1269	7.2	
S-Glass	2.49	85.5	34	4600	1847	5.6	10
Carbon							
Intermediate modulus	1.78-1.82	228-276	155	2410-2930	1646	-0.1- -0.5	8-9
High modulus	1.67-1.9	331-400	240	2070-2900	1736	-1- -4	5-7
High strength	1.85	240	130	3500	1892	-1- -4	5-7
Polymeric fibres							
Kevlar-29	1.44	62	43	2760	1917	-2	12
Aramid (Kevlar-49)	1.48	131	89	2800-3792	2562	-2	12
Spectra 900	0.97	117	121	2580	2660		38
Boron	2.63	385	146	2800	1065	4	100-140
Boron Carbide	2.5	480	192	2100-2500	1000		50
Boron Nitride	1.9	90	47	1400	737		7
Titanium Carbide	4.9	450	92	1500	306		280

Table F.3 Material properties of matrix and core materials, after Hyer (1998)

	Density ρ (g/cm^3)	Young's modulus (tension) E^{t} (GPa)	Shear modulus G (GPa)	Young's modulus (compression) E^{c} (GPa)	Ultimate strength σ_u (MPa)	Coefficient of thermal expansion α ($10^{-6}/{}^\circ$ K)
Thermosetting polymers						
Polyester	1.2-1.3	3-4.2	0.7-2	90-250	40-90	80-150
Vinyl ester	1.15	3-4		127	65-90	80-150
Bismaleimide	1.32	3.6	1.8	200	48-78	49
Polyimide	1.43-1.89	3.1-4.9			70-120	90
Epoxy	1.1-1.6	3-6	1.1-1.2	100-200	30-100	45-80
Thermoplastic polymers						
PEEK	1.32	3.6	1.38	140	92-100	47
PPS	1.34	2.5			70-75	54-100
Polysulfone	1.24	2.5			70-75	56-100
Polypropylene	0.9	1-1.4	0.38-0.54		25-38	110
Nylon	1.14	1.4-2.8	0.54-1.08	34	60-75	90
Polcarbonate	1.06-1.2	2.2-2.4		86	45-70	70
Ceramics						
Borosilicate glass	2.3	64	26.4		100	3
Balsa wood	0.1-0.19	2-6			8-18	
Polystyrene	0.03-0.07	0.02-0.03			0.25-1.25	

Table F.4 Material properties of selected unidirectional composites

	E-Glass/ epoxy	S-Glass/ epoxy	Kevlar/ epoxy	Boron/ epoxy	Carbon epoxy
Fibre volume fraction v_f	0.55	0.50	0.6	0.5	0.63
Density (g/cm^3)	2.1	2.0	1.38	2.03	1.58
Longitudinal modulus E_L (GPa)	39	43	87	201	142
Transverse modulus E_T (GPa)	8.6	8.9	5.5	21.7	10.3
In-plane shear modulus G_{LT} (GPa)	3.8	4.5	2.2	5.4	7.2
Major Poisson's ratio ν_{LT}	0.28	0.27	0.34	0.17	0.27
Minor Poisson's ratio ν_{TL}	0.06	0.06	0.02	0.02	0.02
Longitudinal ultimate stress σ_{Lu} (MPa)	1080	1280	1280	1380	2280
Transverse ultimate stress σ_{Tu} (MPa)	39	49	30	56	57
In-plane ultimate shear stress σ_{LTu} (MPa)	89	69	49	62	71
Longitudinal thermal expansion coefficients α_L $(10^{-6}/^\circ$ K)	7	5	-2	6.1	-0.9
Transverse thermal expansion coefficient α_T $(10^{-6}/^\circ$ K)	21	26	60	30	27

Table F.5 Typical properties of unidirectional composites as functions of the fibre volume fraction, after Vasiliev and Morozov (2001)

	Glass/ epoxy	Carbon/ epoxy	Carbon/ PEEK	Aramid/ epoxy	Boron/ epoxy	Boron/ aluminium	Carbon/ carbon	Al_2O_3/ aluminium
Fibre volume fraction v_f	0.65	0.62	0.61	0.6	0.5	0.5	0.6	0.6
Density ρ (g/cm^3)	2.1	1.55	1.6	1.32	2.1	2.65	1.75	3.45
Longitudinal modulus E_L (GPa)	60	140	140	95	210	260	170	260
Transverse modulus E_T (GPa)	13	11	10	5.1	19	140	19	150
In-plane shear modulus G_{LT} (GPa)	3.4	5.5	5.1	1.8	4.8	60	9	60
Major Poisson's ratio ν_{LT}	0.3	0.27	0.3	0.34	0.21	0.3	0.3	0.24
Longitudinal ultimate tensile stress σ^t_{Lu} (GPa)	1.8	2	2.1	2.5	1.3	1.3	0.34	0.7
Longitudinal ultimate compressive stress σ^c_{Lu} (GPa)	0.65	1.2	1.2	0.3	2	2	0.18	3.4
Transverse ultimate tensile stress σ^t_{Tu} (GPa)	0.04	0.05	0.075	0.03	0.07	0.14	0.007	0.19
Transverse ultimate compressive stress σ^c_{Tu} (GPa)	0.09	0.17	0.25	0.13	0.3	0.3	0.05	0.4
In-plane ultimate shear stress σ_{LTu} (GPa)	0.05	0.07	0.16	0.03	0.08	0.09	0.03	0.12

Appendix G
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   - Vol. 7: Testing, Nondestructive Evaluation and Structural Health Monitoring (ed by R. Crane)
   - Vol. 8: Design and Analysis of Composite Structures (ed. by A. Johnson and C. Soutis)

H. Altenbach et al., *Mechanics of Composite Structural Elements*, https://doi.org/10.1007/978-981-10-8935-0

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Index

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