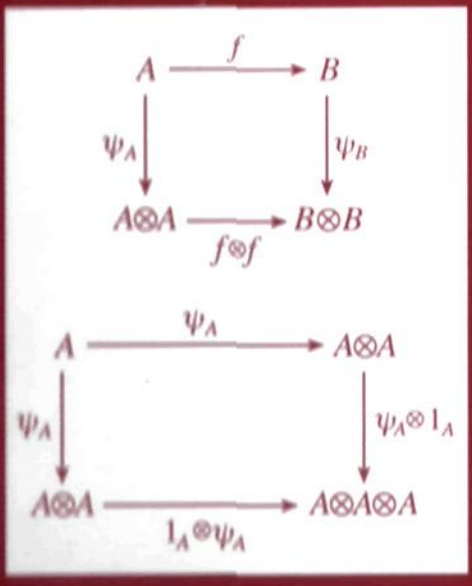
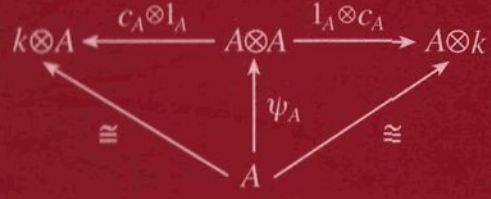
S. A. Selesnick

# QUANTA, LOGIC AND SPACETIME

Second Edition




World Scientific

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*Suppose that calculating machines arose in nature, but that people could not break open their cases. Now suppose that these people use these devices in the manner that we use calculation, say, of which they know nothing. Thus, for example, they make predictions with the help of these calculating machines, though for them manipulating these strange devices amounts to experimentation.*

*These people lack the concepts we have; but what takes their place?* —

Ludwig Wittgenstein
*Remarks on the Foundations of Mathematics*
(From IV, §4, 1942-43, translated by the author.)

*Quantum mechanical calculations follow a simple two-step pattern:* you write down the answer, *and then you* do the computation.

Marvin Chester
*Primer of Quantum Mechanics*
(From the Epilogue.)

S. A. Selesnick
*University of Missouri-St. Louis, USA*

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For Jenny

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# Preface

The impulse behind this work was the naïve but nagging question of a mathematician upon contemplating—for years—the arcana of modern physics: namely, where do the fundamental Lagrangians come from? Thanks to the profound insights of Dirac and Feynman, these expressions are the starting points for the machinations (known as quantum field theory) by which physical content is extracted from theoretical input. The standard view of them seems to be that, once the correct symmetries associated with an interaction have been surmised from experience (or guessed at on aesthetic grounds), then other criteria, such as a belief in gauge invariance, assumptions concerning the nature of the ambient spacetime, renormalizability, etc., may be invoked to hammer out a particular form for the Lagrangian associated with that interaction. (In the case of arguments from gauge invariance one soon receives the heartbreaking news that the gauge must be fixed, and, a little later, that ghosts must be invoked.) The resulting expressions seem to resonate with distinctly geometric overtones that have mesmerized generations of mathematicians: there seem to be fibre bundles, connections, curvatures and worse.

But all of this surely begs the question of origins. For one thing, these expressions start out *classically,* and then have to be dragged, kicking and screaming, into the quantum domain. Since everything is quantal at bottom, should we not be going in the other direction? And, if we do this, namely start from some quantum origin, should the beautiful geometry of the classical background not be relinquished as a Whiggish classical illusion, or red herring, since spacetime itself must also be quantal at bottom? It would seem that

the bathtub itself must be thrown out, along with its contents. Whatever it is we are left with must then provide a means to make a fresh computational start: a *prior quantal structure,* from which, through the application of easily apprehended primitive rules or axioms, a quantum precursor of at least one familiar Lagrangian should emerge. An algebraic machine should be built, according to some simple rules for combining quantum resources, that spits out Lagrangians as its handle is cranked.

These fancies remained dormant until a chance reading, in 1991, of David Finkelstein's "Space-Time Code" papers (Finkelstein 1969, 1972a, b, 1974, Finkelstein *et al.* 1974), their subsequent refinements and collateral descendants. The extraordinary first paper in this series (Finkelstein 1969) rather shockingly derives the basic elements of special relativity from quantum logical principles. Roughly speaking, it is shown that if the pointlike objects that are presumed to underlie the plenum are treated according to the rules of *quantum* logic—rather than classical (set theoretic) logic—then the basic elements of relativity emerge spontaneously. Quantum theory, far from being in conflict with relativity, is shown actually to underlie it. This compelling insight led Finkelstein and co-workers on a quest for a theory which fuses quantum theory with relativity at a deep level. An account of this work, as well as full references to related literature, is to be found in Finkelstein's monograph "Quantum Relativity" (Springer 1996) which, as our primary source, will be abbreviated to QR in the sequel. Of course, the quest continues, and Finkelstein *et al.* have moved on from the work reported in QR. Nevertheless, for our purposes here, QR is still more than adequate as a general reference.

Of the two major conceptual strands that contribute to the fabric of the Space-Time Code papers, namely spacetime *causal structure* and *quantum logic,* the first was very much on the minds of other relativists in the 1960s, the second less so, if at all. Combinatorial or discrete models of spacetime had also been toyed with since at least the 1940s. A mathematically sophisticated descendant of these was produced by R. Penrose in the late 1960s: namely, the *spin net.*

Among the innovations here was the weaving together of *spinors* and combinatorial structure to produce a spacetime antecedent, and Finkelstein acknowledges the strong influence of this work on his own ideas. (Penrose's spin nets have recently staged a dramatic comeback in the theory of loop quantum gravity. See §12.5 for a very brief discussion of this topic.) On the other hand, some quantum physicists, perhaps working closer to the epistemological edge, had for decades been deeply suspicious of the classical spacetime with which necessity had saddled them. Occasional protests notwithstanding, this union of convenience would not be seriously threatened until extremely high energies (and/or the incorporation of gravity) were contemplated. Nevertheless, at least one visionary had understood early on (by the 1940s) the fundamental rôle that quantum logic should play at this level, and, among other remarkable insights, had explicitly suggested a connection between representations of SU(2) and possible quantum versions of the "binary alternative." This was C. F. von Weizsäcker, another acknowledged influence on Finkelstein's synthesis. (See QR for references.)

It seemed that Finkelstein's insights might be adapted to provide the kind of computational context I had been looking for: the present work is the result of that enterprise.

Starting from a small subset of notions culled primarily from QR I aim to show here that it is possible in principle to reconstruct *ab initio* a fair slice of fundamental physics, as seen by a (relatively) macroscopic observer. Although I set out explicitly from (a subset of) Finkelstein's foundations, ignorance of physics at the scale considered often forces me to supply formal or mathematical arguments: nevertheless, such results as are obtained, are, I think, in striking agreement with the Standard Model, at least for a massless world. Specifically, Finkelstein's suggestion that some sort of quantum set-like structure, dubbed a "quantum net," should underlie our macroscopic perception of spacetime, is taken up in the context of the deductive system developed in the first six chapters. It is used, in conjunction with a fairly lengthily articulated correspondence principle, to deduce Lagrangian densities. The correspondence principle itself is

regarded as an interpretation, translation or *valuation* of a certain set of quantal expressions into a set of continuum-based expressions modeled upon elements of the Schrödinger representation. (Other continuum-based models could have been used to accommodate such a translation, a possibility not explored here.) The Lagrangian densities that emerge take the form of power series in a "net constant" (having the dimension of length) whose lowest order terms reproduce the standard forms for massless Dirac fermions, General Relativity, and Yang–Mills fields for the gauge groups $U(1)\times SU(2)_L$ and $SU(3)$. These Lagrangians emerge replete with (Feynman) gauge-fixing terms; the associated ghost fields are obtained at little extra cost. A comparison with the standard Lagrangians at the "grand-unified" scale then yields an estimate for the net constant of about $19 l_P$, where $l_P (\approx 1.6\times 10^{-33}\text{cm})$ denotes the Planck length.

Neither a candidate for a Higgs-like field nor a mechanism to drive generational differences are immediately apparent, and many quantum numbers seem to be missing. In addition, the gravitational term is accompanied by an extra piece whose explicit computation I have found intractable. Thus, it is possible that I have avoided Pauli's Curse (of not even being wrong). On the other hand, a fairly conspicuous candidate for a chiral symmetry breaking mechanism for the $SU(2)$ component does present itself. (Cf. the discussions following equations (11.2.1.6) and (11.2.1.33).)

Since intuitions—both physical and geometrical—are likely to be unreliable at what turns out to be the Planck scale, some level of formality seems to be mandatory. Indeed, Finkelstein has advised the voyager into these submicroscopic regions to travel light. Ignorance has made this initial unburdening easier for me, while increasing the dangers of the journey. Among the bulkier pieces I have carried with me I should mention:

- a fairly liberal use and interpretation of the algebraic notion of *derivation*, and what amounts to an associated Schwinger-like action principle, though expressed in path integral form;

- certain macroscopic geometrical notions such as "infinitesimal" parallel transport (which is, however, globalized in a rather formal way in Chapter 8), and a little Lie theory, used in Chapters 3 and 11.

(Other macroscopic or continuum-based notions have undoubtedly stowed away, despite earnest efforts to evict them.)

On the other hand, among the baggage to be left behind at the outset I have tried to include:

- prejudices concerning the invariances, gauge or otherwise, of the expressions that will emerge as Lagrangian densities;
- prejudices concerning the rôle, or even the existence, of a spacetime metric.

The first of these is required to avoid an obvious circularity—at the cost of having to derive BRST invariant expressions from first principles. The issues concerning the spacetime metric are perhaps more inflammatory. Briefly stated, macroscopic experimenters become aware of the existence and behavior of the metric in the continuum entirely as a result of the passage through it of physical probes, such as photons. Since I aim to deduce the existence and behavior of such probes from prior structures (in the form of their Lagrangian densities), I cannot make *a priori* assumptions concerning even the *existence* of the metric, let alone the nature of its entry into the Lagrangians. Instead, I must wait patiently for the emergent dynamics to reveal their metrical entanglements. (This conforms with the view adopted in modern canonical quantum gravity theories, such as loop quantum gravity (§12.5), that posit a manifold but do not specify a metric on it at the outset: following Finkelstein *et al.* I have tried to dispense with the manifold itself.)

This second edition differs from the first primarily in the greatly extended treatment of the logical, or, rather, *deductive* under-

pinnings of what was always intended to be merely a computation, and one, moreover, eschewing (as far as possible) philosophy, theory-making and other such pursuits so disheartening to amateurs. There are three new chapters, including a very brief introduction to "natural deduction" (Chapter 4), a fairly comprehensive overview of certain parts of quantum logic (Chapter 5) and a new formalization (into a sequent calculus) of the deductive procedures that were less formal in the first edition (Chapter 6). This formalization has had the surprising benefit of producing an argument that helps to sew up the most embarrassing open seam in the entire structure as presented in the first edition and elsewhere: namely, the adoption at the outset of the "quantum version" of the *classical* logical unit or bit, now known as the *qubit,* which was clearly a retrograde step. For, if quantal things underlie classical things, the bit should appear in the macrocosm *because* it is the degenerate macroscopic limit of the more fundamental qubit, and not *vice versa.* Thus, one should seek a *more* fundamental theory of deduction that yields up the qubit as the basic computational quantum unit, without explicit recourse to specific classical prototypes. This argument is produced in Chapter 6, and involves the surprising but satisfying reappearance, from Chapter 2, of Finkelstein's Grassmannian quantum set theory. The argument is not watertight, since there seem to be no available uniqueness theorems, but it is, I think, logically compelling. (It would become watertight in the presence of a proof of a certain uniqueness theorem, to which there is no known counter-example.) The more refined interpretation of the qubit entailed by this analysis leads to an improved argument in favor of the fundamental SL(2,$\mathbb{C}$) "logic" symmetry, and that portion of the first edition has accordingly been completely re-engineered (§7.1).

The logical analysis also illuminates the process of "quantum duplication" as a kind of entangling operator that reappears later, *via* a different route, to give rise to what we perceive in the macrocosm as path composition (cf. §6.3.4 and the discussion at the end of Chapter 8). This operator arises independently yet again in our discussion of loop states in §12.5. Moreover, the logical formalization

provides in addition the means to specify a reasonable *axiom* (in §6.3.3) that encapsulates a kind of irreducible or atomic dynamical action, and the arguments leading to the fundamental quantum net construction have been recast accordingly (§7.1). Dynamics arises essentially by duplicating, or multiplexing, in the quantum sense, this primitive atomic action.

Many of these arguments address exactly the foundational issues faced by the hotly pursued new discipline known as Quantum Computing, and the first six chapters have been designed to be read as a fairly self-contained introduction to that subject.

Other changes from the first edition include: revisions and corrections permeating much of the text; updates and additions to the set of references; additional algebraic material in Chapters 3 and 8; a new section on loop states in the last chapter (§12.5) and the provision of citation and notational indexes.

Approximate minimal prerequisites for a reading of this work would include those necessary for first year graduate work in mathematics or theoretical physics. For the mathematics: some knowledge of the usual abstract algebra, including a little experience in elementary finite dimensional multilinear algebra (mainly tensor and exterior products), and elementary functional analysis, would be helpful. The language of categories is used from time to time, but no actual technical results in this area are invoked. The merest smattering of Lie group and Lie algebra theory is required in the penultimate chapter. A very slight knowledge of vector bundle theory and cohomology would be helpful in the last chapter, but is not necessary for an understanding of the main arguments. For the logic: a slight acquaintance with the general notion of a lattice, and of Boolean algebra theory up to a statement of the Stone theorem, would be helpful. For the physics: an encounter with elementary quantum mechanics should have taken place, up to the notion of second quantization upon Fock and Fermi–Dirac spaces, though hardly more than the definitions of these last will be required. Since Relativity, in both its Special and General incarnations, is supposed to emerge from the formalism, knowledge of these is not actually required, but would,

needless to say, be helpful. I have tried to supply copious references to all the topics listed above at the relevant points in the text. (For superb modern treatments of general elementary quantum mechanics the reader may consult Isham 1995 and Chester 2003. An extremely clear but more traditional treatment may be found in Peebles 1992, which could be used along with the former pair of references for the purposes of triangulating this still contentious subject.)

An overview of the layout of the work may, I hope, be derived by glancing at the table of contents. I have tried to proceed with the argument in a linear fashion, refraining from the use of footnotes and appendices. Nevertheless, certain nonlinearities and digressions have crept in, most of which may be ignored at a first reading—or forever. These are clearly marked within the text, but, for the record and as an enticement to the reader, the sections that may be skipped are as follows: the digression to prove a result of Finkelstein's in §1.2; §3.1.7, which is used only in §12.5 and has no bearing on the main calculation; §5.1.5, §5.2, §6.3.4, §7.3, §11.2.3 in their entireties. The argument essentially concludes with §12.2, subsequent sections of this last chapter being devoted to an account of superconduction, electroweak unification (§12.3) and other symmetry breaking mechanisms that seem consistent with our model (§12.4). The so-called loop states are taken up briefly in §12.5, as noted earlier. Any or all of these sections may be skipped. It goes without saying that the proofs of theorems, where offered, may also all be skipped.

Incidentally, the handle of the algebraic machine, whose cranking produces the Lagrangians, appears in equation (11.2.2).

It is obvious that I owe a special debt of gratitude to David Finkelstein, not only for having inspired the vacuum in the first place, but also for his patience in attempting to share his deep insights with a tyro, and his unstinting hospitality while doing so. My thanks to him also for cogent criticism of earlier versions, and to him and Springer–Verlag for permission to quote certain passages from QR.

This second edition would not have appeared had it not been for the continued moral support and encouragement of Chris Isham,

Anastasios Mallios, and Ioannis Raptis: to all of them I will be eternally grateful. If any of it is readable at all, the blame must lie with the polymathic Theresa Jeevanjee, who courageously subjected herself to a reading of *every* word and equation, and who did everything she could to heal the fractures in my English, my mathematics, etc.: my heartfelt thanks to her. For crucial exchanges on various issues that arose in the preparation of this edition I offer my grateful thanks also to John L. Bell, Richard Blute, Jeremy Butterfield, Marvin Chester, Maria Luisa Dalla Chiara, Prakash Panangaden, J. Piers Rawling, Ivan Selesnick, Gerald Teschl and Mingsheng Ying.

Over a period of many years my outlook on these and manifold other matters has been broadened and deepened through fortunate encounters with many other remarkable individuals, whose wisdom and generosity it is a pleasure to acknowledge here. These include: Michael Atiyah, the late K. T. Chen, W. Henson Graves, Karl Heinrich Hofmann, Theodore Jacobson, John Madore, Wolfgang Mantke, Christopher Mulvey, Ronald Munson, Prabhakar Rao, Kenneth A. Ross, Subramanian Sankaran, Dana Scott, Rafael Sorkin, Moss Sweedler, Zhong Tang, Morwen Thistlethwaite, Alexander Vlasov, Grant Welland, Josef Wichmann, and the late Eugene P. Wigner. My grateful thanks to all of them, and to the staff of World Scientific, particularly Daniel Cartin and Ms. E. H. Chionh, for allowing me the opportunity to revisit this material.

Finally, but primarily, I thank my wife Jenny, whose love and patience continue to defy rational explanation.

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# QUANTA, LOGIC AND SPACETIME

Second Edition

# Part I

# Preliminaries

# 1

# Foundations

The fundamental attribute of a quantum system, according to the standard elementary picture of quantum theory, is a Hilbert space of "states," or, more archaically, "wave functions." The "pure states" of the system are in one-to-one correspondence with the normalized elements of the Hilbert space (hence in one-to-one correspondence with the rays, or one-dimensional subspaces, of the said Hilbert space). A general "state" of the system is identifiable with a certain convex combination of such normalized elements or pure states (cf. §5.2). In the case of a classical system, the pure states may be (seductively) identified with the elements of a set, or "space," of actual physical states of the system, epitomized by the phase space of a particle. Consequently, the temptation to similarly interpret a normalized vector in the Hilbert space of a quantum system as the representer of some kind of physical state of the system—to which certain attributes of objectivity may be presumed to adhere—seems irresistible. Indeed, it is a temptation not often resisted. The primal nonclassical attribute of quantum states, namely the possibility of their *superposition,* then also acquires a pseudo-classical gloss, in which a superposition of "states" is construed as a kind of classical interference of "wave functions."

Physics, for the most part, is carried out against (or within) a classical backdrop: regardless of the degree to which the foreground fields, etc., are quantized, the ambient space or spacetime remains

classical. This statement embraces even those theories in which spacetime is subjected to mutilation, as in lattice theories, or to augmentation, as in string theories. In none of these theories are the actual points or regions of the spatio-temporal structures themselves quantized: they, and their assemblages into manifolds, etc., are treated as entirely classical *objects* that do not themselves partake of any quantal attributes such as superposition, entanglement, etc.

Under these circumstances, namely when the stage accommodating the play of physics is not itself subjected to microscopic scrutiny, it seems relatively harmless to cling to a quasi-realist interpretation of quantum theory. (One price to be paid for this is the conceptual discomfort experienced when the non-objective nature of quantum entities collides with classical preconceptions concerning the behavior of objects. Although we hope to show that it is possible to ameliorate this discomfort, it seems unlikely that gross macroscopic human thinkers will ever be able to dispel it entirely: evolution did not select for an appreciation of non-objects. We would not be here if it had.) However, a promiscuous maintenance of the conventional classical picture of spacetime as a highly structured continuous object, such as a manifold, is not only in clear conflict with quantum principles, it also seems to preclude the kind of constructive computational procedure we propose to initiate here.

Following the line of reasoning pioneered by Finkelstein, if classical spacetime is first of all a classical *set* (of "events," say), then quantum spacetime should at least be a *quantum* set of some kind. And "quantum sets," whatever they are, will not be objects in the classical sense. Nor, consequently, will the associated logic of propositions concerning such entities be classical logic: rather, it will be some kind of quantum logic, presumably closely related to, if not identical with, that elderly discipline known as Quantum Logic. Moreover, the proposal to found a theory of the Plenum upon so formal a base demands a clear ontological commitment to be made at the outset. We shall follow Finkelstein in positing a thoroughgoing operational view of quantum theory, and present in this first chapter a brief account of this approach, its chief virtue being its strong non-

objective flavor. This account will be continued in the next chapter.

## 1.1 Physics without Objects

Epistemology is destiny, and perhaps the most profound epistemological lesson of the Heisenbergian quantum revolution is that objects do not exist. A quantum entity may act, may have "coordinates" and other attributes, but may not be said to have a *state of being*: that is, among other things, it may not possess a unique set of attributes that are independent of putative *experiments* or *experimenters.* Indeed, physical acts of selection of quanta may not commute: for example, the outcome of an experiment in which light is passed through a pair of polarizers will, in general, depend upon the order of the polarizers. Here a single polarizer effects the *initial (external) act* of injecting a correspondingly polarized photon, and, dually, an analyzer effects a *final (external) act* of detection. A *quantum system* is then identified with a collection of ideal, maximally informative (i.e. "sharp") initial and final acts. These acts are attributes of putative experimenters or *episystems.* However, initial acts do not necessarily commute and, moreover, two such acts performed in sequence do not necessarily yield another such act. This is to be contrasted with the (imagined) behavior of (non-existent) "classical systems": ideal physical acts of selection are infinitely gentle and do not change the things selected—they mutually commute. (For quanta, every effective physical selective act fails to commute with almost every other.) Thus, for a classical mechanical system consisting of a particle, say, an initial act places the particle into a certain state represented by a point in a "space" (or "phase space") of such states. Thus, sharp initial external acts correspond one-to-one with elements of the state space and, since nothing is changed by such an act, these are also in one-to-one correspondence with final acts.

A sharp initial act $\alpha$ paired with a sharp final act $\omega$ constitutes a minimal kind of experiment, and we distinguish two possible outcomes, namely success or failure. We may loosely express these by $\omega\alpha \neq 0$ and $\omega\alpha = 0$ respectively. In the first case the minimal experi-

ment *goes*, and the transition $\alpha \to \omega$ is *allowed*; in the second the experiment doesn't go and the transition $\alpha \to \omega$ is *forbidden*. (In a polarization experiment for example, in which $\alpha$ corresponds to the initial act of polarization, and $\omega$ to the final act of analysis, $\omega\alpha = 0$ when the analyzer is orthogonal to the polarizer.) One may classify and categorize general acts in various ways to reflect experimental situations. For example, for a *closed* system, which is insulated from all others between external acts, one might suppose that, for each initial act $\alpha$, there corresponds a unique final act, denoted $\alpha^\dagger$, the *adjoint* of $\alpha$, such that the transition $\alpha \to \alpha^\dagger$ is allowed: $\alpha^\dagger\alpha \neq 0$. For each act of importation there should correspond an act of exportation. For an *open* system, which is one that is not closed, every initial act need not have an adjoint, or, if it does, the adjoint need not be unique. Various rules may now be laid down for adjoints. (For example, the conventions of QR require that the adjoint be conjugate linear, so that the associated inner product is conjugate linear in the first variable.)

It is a remarkable fact that these simple ingredients, namely a given set of external acts for a closed system, in the presence of six further assumptions (some admittedly rather technical), are enough to produce (roughly speaking) a unique vector space $W$ equipped with an inner product such that the initial acts correspond to the rays in $W$, the final acts to the rays in the dual space $W^*$ of $W$, and the adjoint to the map $W \longrightarrow W^*$ induced in the obvious way by the inner product. The coefficient field is not completely specified by this theorem, the Strong Form of the Fundamental Theorem: QR, §6.9. We shall opt for the minimal choice of $\mathbb{C}$, the complex numbers, for the coefficient field, for no better reasons than those of convenience and familiarity.

Having neglected to precisely state the fundamental principles of quantum kinematics (for a closed system), we can now do so in vectorial form (QR, §2.4.3).

Q0. *The (sharp) initial acts of a system correspond bijectively with the rays in a vector space; the final acts correspond with the rays in the dual space.*

Q1. *A transition is forbidden if and only if the corresponding ray of final functionals annuls the corresponding initial ray.*

If we use Dirac kets $|\alpha\rangle$ ($\in W$, say) to denote initial action vectors, then it does not abuse the notation too much to use Dirac bras, $\langle\omega|$, to denote elements of the dual space $W^*$. Then Q1 is equivalent to the assertion that the transition $|\alpha\rangle \rightarrow \langle\omega|$ is forbidden if and only if $\langle\omega|\alpha\rangle = 0$, where $\langle\omega|\alpha\rangle$ denotes the value of the functional $\langle\omega|$ at $|\alpha\rangle$.

We shall assume also the existence of an adjoint operation (now expressible in the form $|\alpha\rangle^\dagger = \langle\alpha|$, with the complex conjugate linearity of the operation being understood), but cannot afford to be too hidebound about the positive definiteness of the metric it defines, since indefiniteness inevitably encroaches in relativistic theories: cf. the problems with photon polarizations in ordinary QED, which is curable in various ways. In the absence of positive definiteness (and a cure for it), and anticipating the Born Law to be discussed below (from which it emerges that $\langle\alpha|\alpha\rangle$ should be interpretable in terms of the number of quanta in a sequence of trials with initial act $|\alpha\rangle$), one is faced with problems of interpretation for acts $|\alpha\rangle$ for which $\langle\alpha|\alpha\rangle \leq 0$. Finkelstein (QR, §4.2.2, §4.7) makes the compelling suggestion that in these cases $|\alpha\rangle$ represents not an act proper to an experimenter, that could in principle be carried out on the system, but rather an act pertaining to possible relations among experimenters. (The analogy is with spacelike vectors in relativity, that play precisely this rôle.) An act that cannot be carried out by an experimenter upon a system may nevertheless serve to establish relations among experimenters, as do spacelike vectors in relativity, that effect acts of separation, or ghosts in gauge theories, that effect acts of coordination: cf. §11.2.3. (This interpretation seems unavailable if one insists upon

investing kets with the state-like attributes of an underlying object.) Thus we need to qualify the notion of initial act in the presence of an indefinite metric: namely, a *proper* initial act is one performable on the system—these are the ones represented by the rays of vectors of positive norm: $\langle\alpha|\alpha\rangle > 0$. It then becomes necessary to specify a subspace of such proper initial acts within the space of all initial acts.

These two principles, Q0 and Q1, succinctly encapsulate the key non-classical aspects of quantum behavior, as Finkelstein points out (QR, §2.4.3). Namely, the *integrity* of quanta: a transition is made or it is not; and *quantum interference*: for each pair of initial acts $|\alpha\rangle$ and $|\beta\rangle$ that effect a transition to $|\omega\rangle$, say—that is, $\langle\omega|\alpha\rangle \neq 0 \neq \langle\omega|\beta\rangle$—there exists an initial act $|\gamma\rangle$ such that $\langle\omega|\gamma\rangle = 0$. That is, there exists an initial act for which the transition is forbidden. Here, $|\gamma\rangle$ is a linear combination of $|\alpha\rangle$ and $|\beta\rangle$, and represents the interference of alternate possible acts of an experimenter upon the same quantum system. To quote Finkelstein (QR, §2.6):

> "Opening a new path may block a transition. This is quantum interference, the root of all quantum paradoxes, a famous anomaly differentiating any quantum theory from any classical one. It is built into the action vector description and is an essential part of the quantum kinematics."

An initial act $|\gamma\rangle$ is said to be a *coherent superposition* of two others $|\alpha\rangle$ and $|\beta\rangle$ when

$$(\langle\omega|\alpha\rangle = 0 \text{ and } \langle\omega|\beta\rangle = 0) \text{ implies } (\langle\omega|\gamma\rangle = 0). \qquad (1.1.1)$$

Clearly, every ray in the plane spanned by $|\alpha\rangle$ and $|\beta\rangle$ qualifies as a coherent superposition, whereas in classical thought, if these vectors represented states of an object, one would conclude from

$$(\omega \neq \alpha \text{ and } \omega \neq \beta) \text{ implies } (\omega \neq \gamma) \qquad (1.1.2)$$

(which is the precise classical analog of equation (1.1.1)), or its logical equivalent:

$$(\omega=\gamma) \text{ implies } ((\omega=\alpha) \text{ or } (\omega=\beta)), \tag{1.1.3}$$

that $\gamma=\alpha$ or $\gamma=\beta$.

Classically, if ($\omega$ *being* $\gamma$) entails (($\omega$ *being* $\alpha$) or ($\omega$ *being* $\beta$)), then $\gamma$ must *be* $\alpha$ or $\gamma$ must *be* $\beta$. This existential principle does not apply in the action vector picture (i.e. quantum theoretically) and it is this difference, as noted by Finkelstein (QR, §1.3.5), that seems to cause much of the disquiet experienced by the object-intoxicated mind while contemplating quantum phenomena. It seems less disquieting in the action vector picture since the referents in equation (1.1.1) are associated with an experimenter or episystem, rather than with any presumed inherent attributes—such as those associated with the existence of states of being—of the system itself.

## 1.2 Observables

Classically, an "observable" is a value assumed or assumable by each state of a system: one often speaks of a value of the observable "in" a certain state. That is, a classical observable is a (possibly complex) function on the state space. In the quantum case the term "observable" is a misnomer since it carries connotations of non-intervention. Finkelstein (QR, §4.2.5) suggests the use of the terms "variable" or "coordinate" instead.

Given the space of action vectors of some quantum system, what can possibly correspond with this classical notion? In answering this question we follow the more-or-less traditional path of relativizing from the classical case. The most general mathematically amenable model of an algebra of classical observables would seem to be the commutative C*-algebra $C_0(X)$ of continuous complex functions on a locally compact Hausdorff space $X$, vanishing at infinity. That is, the (commutative) algebra of continuous functions $f: X \longrightarrow \mathbb{C}$ having the property that for any $\varepsilon > 0$, there exists a

compact $K_\varepsilon \subset \mathbb{C}$ such that $|f(x)| < \varepsilon$ whenever $x \notin K_\varepsilon$, equipped with the uniform norm. In classical thinking, the act of assigning a value to a state does not change the state, whereas in the quantum case, that has no underlying states, all that can ever be affected are the initial and final acts. Although a classical coordinate, i.e. an element $f \in C_0(X)$, affects no states, there are clearly vector spaces ($C_0(X)$ itself is one) upon which $f$ *does* effect changes. For instance, $f$ acts linearly by multiplication upon $C_0(X)$. A consideration of the most general actions of $C_0(X)$ upon *Hilbert* spaces then leads to the following Spectral Theorem.

THEOREM 1.2.1

THE SPECTRAL THEOREM

*Let $f \mapsto T_f$, $f \in C_0(X)$, denote a *-representation by bounded operators on a Hilbert space $\mathfrak{H}$. Then there exists a function $E \mapsto P(E)$ from the Borel sets of $X$ to projections on $\mathfrak{H}$ such that:*

(i) $P(X) = I$, *the identity operator on* $\mathfrak{H}$,
(ii) *for* $E_1, E_2, \ldots$ *pairwise disjoint,*

$$P(E_1 \cup E_2 \cup E_3 \cup \ldots)\xi = \sum_{n=1}^{\infty} P(E_n)\xi$$

*for all* $\xi \in \mathfrak{H}$,
(iii) $P(E_1 \cap E_2) = P(E_1)P(E_2)$,
(iv) *the projections* $P(E)$ *commute with all bounded operators that commute with all the* $T_f$.

*Moreover, for each* $f \in C_0(X)$ *and any* $\xi, \eta \in \mathfrak{H}$

$$\langle \xi | T_f | \eta \rangle = \int_X f(\lambda)\, d\mu^P_{\xi,\eta}(\lambda) \qquad (1.2.1)$$

*where* $\mu^P_{\xi,\eta}$ *is the complex measure on* $X$ *given, for a Borel set* $E \subseteq \mathbb{R}$, *by* $\mu^P_{\xi,\eta}(E) = \langle \xi | P(E) | \eta \rangle$.

*The last equation may also be written*

$$T_f = \int_X f(\lambda) dP(\lambda). \qquad (1.2.2)$$

*The equation* (1.2.1) *specifies the function* $P$ *uniquely.*

The equations (1.2.1) and (1.2.2) express the fact that any Hilbert space *-representation of a commutative C*-algebra may be decomposed as a direct integral of one-dimensional representations with respect to a *projection-valued measure,* which is a function satisfying (i) – (ii) of the theorem. (Note that $P(\varnothing)=0$ follows from (ii). It can also be shown that (iii) follows from (i) and (ii): see for example Halmos 1957, Theorem 2, §36, where the terminology *spectral measure* is used. We have followed tradition in including (iii) among the hypotheses. We note that Teschl 2002 also contains a very careful treatment of this theorem and some of it ramifications. See also Arveson 2002, or Hewitt and Ross 1979, C. 40.)

It is clear that, given such a measure, one may form a representation of the algebra satisfying equation (1.2.1) and that this association of representations of $C_0(X)$ with projection valued measures is one-to-one. Now, if we identify the Boolean ($\sigma$-) algebra of Borel subsets of $X$ with the *logic* of the classical system whose state space purports to be $X$, then a projection valued measure clearly has the effect of *interpreting* this logic in the set of projections on $\mathfrak{H}$. Moreover, since projections correspond to closed subspaces, this is equivalent to an interpretation of the Boolean algebra of the classical system in a certain family of closed subspaces of $\mathfrak{H}$. But the family of all closed subspaces of a Hilbert space has its own structure as a (non-distributive) lattice, and the family corresponding to the $P(E)$ turns out to constitute a distributive, hence Boolean, sublattice. This is because the $P(E)$ *commute* (cf. (iii) in the theorem statement above), that in turn causes the lattice of corresponding subspaces to be distributive. (The structure of this lattice is described in more detail in Chapter 5. Suffice it to note here that the lattice structure is given as

follows: the join (least upper bound) of a family $E_i$ of closed subspaces of a Hilbert space is the closure of their algebraic sum $\Sigma E_i$ and their meet (greatest lower bound) is their ordinary intersection as sets. The operation corresponding to the Boolean complementation of a closed subspace is given by taking the orthogonal complement of the subspace.) Thus, any representation of the algebra of classical observables (or coordinates) gives rise to a Boolean sublattice of closed subspaces of a Hilbert space, and each operator in the representation has a (diagonal-like) spectral resolution (equation (1.2.2)) in terms of minimal (one-dimensional) commuting projections. The eigenvalues of $T_f$ are exactly the values of $f$—that is, the possible values of an observable appear as eigenvalues of the operator corresponding to it.

Let us now artificially suppose that the Hilbert space carrying such a representation may be taken as the space of initial acts for the supposed underlying classical system. Then each projection essentially performs an act of selection, the subspace it projects upon representing the system(s) selected. Thus, *via* its spectral resolution, each observable (or variable, or coordinate) is a superposition of such *selective acts*, the eigenvalue associated with each projection in the spectral resolution being the *initial value* ascribed by the variable to the systems selected by that projection. Since the selective acts (i.e. the projections) commute, there is no interference among them: think of the special case of projections onto orthogonal subspaces.

This non-interference is characteristic of classical acts, so it is now easy to see how to relativize this situation to the quantum case: where there was no interference among selective acts, we must admit interference. That is, we must now consider families of non-commuting projections. Since there is now no single underlying family of Borel sets or projection-valued measure our burden shifts to the choice of an appropriate family or families of projections. If we wish to maintain the idea of a logic for the underlying system, we should choose a family of projections whose corresponding family of subspaces forms at least a sublattice of the lattice of closed subspaces of a Hilbert space. The chosen lattice will be non-distributive in general, since the projections will be noncommutative in general. Since we

have assumed that the Hilbert space involved represents all the initial acts of the system, it seems that we should choose the entire lattice of closed subspaces. This is the prototypical example of an *orthomodular lattice:* a formal definition will be worked up to in Chapter 5, §5.1.4. (The question of which Hilbert space to choose in the first place is, of course, still begged.)

Finally, with respect to general candidates for quantum variables (or coordinates), we preserve the analog of the spectral resolution so that our quest reduces to a search for those operators admitting such a resolution. The broadest general class of such operators is the class of *normal* operators: namely, those that commute with their (Hilbert space) adjoints. (The smallest C*-algebra generated by such an operator is commutative, hence its good spectral behavior.) The corresponding eigenvalues then have the same interpretation as in the classical case.

Following Finkelstein (QR, §4.3.2, §4.4.3) we can now reasonably specify that any maximal algebra of normal operators on the space of initial vectors of a quantum system is the coordinate algebra of some experimenter. This is in marked contrast to the classical case where the maximal commutative algebra of $T_f$'s is supposed to provide a unique, absolute coordinate *frame,* agreeable to all observers. In the quantum case there may be as many frames as experimenters, and these are generically incompatible (i.e. not necessarily mutually commuting). This is true, *a fortiori,* for open systems, in which adjoints may also vary with the experimenter.

(This non-existence of an absolute frame is the essential relativity built into quantum theory, as emphasized by Finkelstein (QR, §4.3.1, §4.4.3), who was apparently the first to have done so.)

Thus, we arrive at one of the founding principles of quantum theory:

THE EIGENVALUE PRINCIPLE

*The possible experimental values of a variable are exactly its eigenvalues.*

Each of the relevant projections in a spectral resolution therefore functions as an idealized minimal "black box," capable of outputting its associated spectral value. The variable is then a superposition of these "single channel" black boxes. The theory seems not to allow any finer specification of the idea of "experimental value" than this (cf. QR, §4.2.5). Nevertheless, this was enough for Finkelstein (Finkelstein 1963) to deduce a very close approximation (indeed, arbitrarily so) to that holy of holies, the Born Law, from Q0, Q1, the Eigenvalue Principle, and an axiom of "assembly" (cf. Chapter 2). This law states that, for normalized initial and final acts, $|\alpha\rangle$ and $\langle\omega|$ respectively,

$$\text{Prob}(\langle\omega| \leftarrow |\alpha\rangle) = |\langle\omega|\alpha\rangle|^2, \tag{1.2.3}$$

that was (and is) often regarded as an axiom of the theory.

We digress to briefly review this important result. Certain notions used here will themselves be reviewed in Chapter 2. Readers may prefer to delay a reading of the rest of this subsection, or to skip it entirely.

The idea is to design a variable whose appropriate eigenvalue is the appropriate probability. We note first that since projections have only the eigenvalues 0 and 1 they may be regarded as binary variables or *predicates.* Choose and fix a normalized initial vector $|\alpha\rangle$ and a normalized final vector $\langle\omega|$. Then the projection $P_\omega$ onto the ray spanned by $|\omega\rangle$ (that may be expressed in Dirac notation as $P_\omega = |\omega\rangle\langle\omega|$) represents a predicate pertaining to (or a property of) the system. Specifically, the eigenvalue 1 of $P_\omega$ is the initial value of the subsystem(s) selected by $P_\omega$ precisely when a transition from those subsystems to $\langle\omega|$ is *allowed,* and 0 otherwise.

Now consider the experiment determined by these two acts $|\alpha\rangle$ and $\langle\omega|$. If we perform this experiment $n$ times and assume that these trials are independent of each other, then the run can be considered as a single experiment conducted on a sequence of $n$ replicas of the original system. The spaces of acts for such sequences are the $n$-fold tensor products of the spaces of acts for the single system (cf.

§2.2, or Mac Lane 1963, for the algebraic notion of tensor product). The initial vector for the repeated experiment is thus

$$|\alpha_n\rangle \equiv \otimes^n |\alpha\rangle \tag{1.2.4}$$

and similarly the final vector is

$$\langle\omega_n| \equiv \otimes^n \langle\omega|. \tag{1.2.5}$$

The predicate corresponding to $P_\omega$, but now considered as a property of the $k^{\text{th}}$ system in the sequence, is

$$P_k \equiv 1 \otimes \cdots \otimes P_\omega \otimes \cdots \otimes 1 \tag{1.2.6}$$

where the 1s denote identity operators and $P_\omega$ is in the $k^{\text{th}}$ position from the right.

Thus, since $P_\omega$ is the variable whose non-zero eigenvalue represents a successful transition to $\langle\omega|$, $P_k$ is the variable whose non-zero eigenvalue represents a successful transition to $\langle\omega|$ of the appropriate subsystems of the $k^{\text{th}}$ system in the sequence. Then the variable representing the number of successful such transitions in a sequence of $n$ trials is $\Sigma_k P_k$, and the variable representing the average number of successful transitions to $\langle\omega|$ in a sequence of $n$ trials is

$$F_n = \left(\frac{1}{n}\right)\Sigma_k P_k. \tag{1.2.7}$$

If $|\alpha_n\rangle$ were an eigenvector of $F_n$, then the corresponding eigenvalue would represent the initial value ascribed to it by $F_n$: that is, this eigenvalue would be exactly the average number of transitions $|\alpha\rangle \rightarrow \langle\omega|$ in a run of $n$ trials. The probability $\text{Prob}(\langle\omega| \leftarrow |\alpha\rangle)$ for the transition to occur would then be the limit, in some sense, of these eigenvalues as $n \rightarrow \infty$.

The failure of $|\alpha_n\rangle$ to be an eigenvector of $F_n$ is measured by

the quantity (abusing the notation slightly):

$$\Delta(f_n) \equiv \|(F_n - f_n)|\alpha_n\rangle\|^2 \tag{1.2.8}$$

where $f_n$ is some real number.

But

$$\begin{aligned}\Delta(f_n) &= \langle (F_n - f_n)\alpha_n | (F_n - f_n)\alpha_n \rangle \\ &= \langle \alpha_n | (F_n - f_n)^2 | \alpha_n \rangle \\ &= \langle \alpha_n | F_n^{\,2} | \alpha_n \rangle - 2f_n \langle \alpha_n | F_n | \alpha_n \rangle + f_n^{\,2} \langle \alpha_n | \alpha_n \rangle \\ &= \left(\frac{1}{n}\right) |\langle \omega | \alpha \rangle|^2 + \left(\frac{n-1}{n}\right) |\langle \omega | \alpha \rangle|^4 - 2f_n |\langle \omega | \alpha \rangle|^2 + f_n^{\,2}.\end{aligned} \tag{1.2.9}$$

Thus $|\alpha_n\rangle$ approaches the condition of being an eigenvalue over increasingly long runs if

$$\begin{aligned}\lim_{n\to\infty} \Delta(f_n) &= |\langle \omega | \alpha \rangle|^4 - 2(\lim_{n\to\infty} f_n) |\langle \omega | \alpha \rangle|^2 + (\lim_{n\to\infty} f_n)^2 \\ &= 0\end{aligned} \tag{1.2.10}$$

assuming $\lim f_n$ exists. If it does, we then have

$$\lim f_n = |\langle \omega | \alpha \rangle|^2. \tag{1.2.11}$$

So if we take each $f_n$ itself to be this limiting value $f \equiv |\langle \omega | \alpha \rangle|^2$, then equation (1.2.9) yields

$$\Delta(f) = \left(\frac{1}{n}\right) f(1-f). \tag{1.2.12}$$

Thus, over a lengthening number of finite trials, $|\alpha_n\rangle$ moves closer to being an eigenvector of $F_n$ with eigenvalue $|\langle\omega|\alpha\rangle|^2$.

If $|\alpha_n\rangle$ were to become an *exact* eigenvector of $F_n$ in an appropriate limit as $n \to \infty$, then the Born Law (equation (1.2.3)) would follow from the Eigenvalue Principle.

Alternatively, the theory of infinite tensor products of Hilbert spaces may apparently be invoked to define the $F$ operator and perform the analogous calculations for genuinely infinite sequences. The result may then be obtained exactly.

(This result is recounted in Chapter 8 of QR. For another illuminating account of it, and some of its ramifications, see Smolin 1984.)

## 1.3 Finite Dimensional Heuristics

We shall be largely concerned with finite dimensional vector spaces of action vectors (always over $\mathbb{C}$, except where otherwise specified), the idea being that the proliferation of the dimensions of the spaces involved in the quantum physics of the continuum is an artifact of this structure: these spaces (and the continuum itself) should be obtainable ultimately through some sort of limiting process (or condensation) of underlying finite, or finitely generated, substructure.

Notable among the many algebraic simplifications unique to the finite dimensional case is the following. Let $W$ denote a finite dimensional vector space with dual space (of linear maps of $W$ into $\mathbb{C}$) denoted $W^*$. Then an elementary tensor $a \otimes f \in W \otimes W^*$ determines a linear transformation of $W$ into itself in which $w \in W$ is sent to $f(w)a$ (recalling that $\otimes \equiv \otimes_{\mathbb{C}}$). The linear map $W \otimes W^* \longrightarrow \mathrm{End}W$ so induced is easily seen to be an isomorphism, where $\mathrm{End}W$ denotes the space of endomorphisms of (i.e. operators on)

$W$. Moreover, this isomorphism is *natural,* meaning, among other things, that it does not depend upon any choice of bases. This has the interesting consequence that the space $\mathrm{End}W$ is naturally isomorphic with its own dual, an unusual occurrence even for finite dimensional spaces. This follows from the chain of natural isomorphisms:

$$\mathrm{End}W \cong W \otimes W^* \cong W^* \otimes W \cong$$

$$\cong W^* \otimes W^{**} \cong (W \otimes W^*)^* \cong (\mathrm{End}W)^*. \qquad (1.3.1)$$

It is a simple matter to verify the commutativity of the following diagram:

$$\begin{array}{ccc} W \otimes W^* & \xrightarrow{\cong} & \mathrm{End}W \\ & {\scriptstyle c}\searrow \quad \swarrow{\scriptstyle \mathrm{tr}} & \\ & \mathbb{C} & \end{array} \qquad (1.3.2)$$

where $c$ denotes contraction and tr is the ordinary trace.

Also easily seen is the fact that ordinary multiplication of operators, considered as a map $\mathrm{End}W \otimes \mathrm{End}W \longrightarrow \mathrm{End}W$, is equivalent to (i.e. commutes, *via* the isomorphism $\mathrm{End}W \cong W \otimes W^*$, with) the map:

$$1 \otimes c \otimes 1 : W \otimes W^* \otimes W \otimes W^* \longrightarrow W \otimes \mathbb{C} \otimes W^* \cong W \otimes W^*. \qquad (1.3.3)$$

Suppose a positive definite sesquilinear form is given on $W$ making it a finite dimensional Hilbert space. Then, as for any Hilbert space, the conjugate linear map (or *adjoint*) $W \longrightarrow W^*$ induced by this form in the usual way, namely by sending $w \in W$ to $\langle w| \ \rangle : W \longrightarrow \mathbb{C}$, is one-to-one and onto. (Recall that our conventions require $\langle \ | \ \rangle$ to be conjugate linear in the first variable.)

Thus, with the help of the adjoint map $W \longrightarrow W^*$, contrac-

tion, inner product and trace all become equivalent.

Upon resurrecting the Dirac notation, an elementary (or *simple*) tensor in $W\otimes W^*$ may now be written in the form $|v\rangle\otimes\langle u|$ because everything in $W^*$ is an adjoint. The operator in End$W$ corresponding to this tensor is the one sending $|w\rangle$ to $\langle u|w\rangle|v\rangle$. That is, upon identifying the tensor with the operator it corresponds to, we have

$$(|v\rangle\otimes\langle u|)|w\rangle = \langle u|w\rangle|v\rangle. \tag{1.3.4}$$

So $|v\rangle\otimes\langle u|$ is the syntactically correct form of the operator usually written $|v\rangle\langle u|$ in Dirac's notation.

If $W$ were the space of initial vectors for a quantum system, the trace of such an elementary operator, namely

$$\mathrm{tr}(|v\rangle\otimes\langle u|) = \langle u|v\rangle, \tag{1.3.5}$$

has the interpretation given it in the theory sketched above, namely, as the *transition amplitude*, whose modulus squared may be related to the probability that the transition occurs.

A general element $\theta$ of $W\otimes W^*$ is of the form

$$\theta = \sum_{i,j} \lambda_{ij}|u_i\rangle\otimes\langle u_j| \tag{1.3.6}$$

which we may think of as a superposition of minimal experiments, or acts of selection, $|u_i\rangle\otimes\langle u_j|$, each comprising a sequence of one initial action and one final one (cf. §2.2). Contraction then produces a sum of transition amplitudes whose value is independent of the representation of $\theta$ as a linear combination of the type shown in equation (1.3.6). (In the sequel we shall often regard operators as generalized acts of selection in this sense and refer to them as such.)

A general element of $W\otimes W\otimes W^*\otimes W^*$ is similarly a superposition of compound experiments of the form $|u_i\rangle\otimes|u_j\rangle\otimes\langle u_k|\otimes\langle u_l|$ in which the inner pair may be thought to comprise a *medial* or *in-*

*ternal* subexperiment whose contraction yields the amplitude for the relevant internal transition. Performing this inner contraction while holding the outer vectors fixed then yields the elementary experiment

$$\langle u_k|u_j\rangle|u_i\rangle\otimes\langle u_l|. \tag{1.3.7}$$

The isomorphism $W\otimes W\otimes W^*\otimes W^* \cong W\otimes W^*\otimes W\otimes W^*$ allows us to view this process in terms of operators: namely, it yields exactly the product of the pair of operators whose tensor product (in $W\otimes W^*\otimes W\otimes W^*$) corresponds to the original element (equation (1.3.3)). Conversely, the product of any two operators may be interpreted as an inner contraction of this type.

This inner contraction is exactly analogous to a (short!) path integral in that it effectively sums over allowed "internal" transitions that constitute alternate routes, or paths, between sets of outermost initial and final vectors. (See §10.3 for a minimal sketch of the notion of path integral.)

The contraction of the remaining "external" vectors in expressions of the form (1.3.7) yields finally the amplitude for the inner and outer transitions both to occur: this is the transition amplitude associated with the entire compound experiment (or superposition of such experiments) represented by the pattern of initial and final action vectors in a given element of $W\otimes W\otimes W^*\otimes W^*$.

On the other hand, this final contraction gives, as usual, the trace of the corresponding operator, that in this case is the product of a pair of operators. In sum, the general form of these fully contracted amplitudes is that of a trace of a product of operators. Conversely, a trace may be interpreted as a full contraction of this type.

This type of inner-to-outer full contraction of appropriate general tensors is a paradigm of externality: internality-to-externality is established by the order of contractions within the tensor. This association of the order of contraction with notions of externality seems to be in accord with intuition, since, if an experimenter is external in this sense to some compound experiment, all internal transitions must be taken into account before a full transition across that

experiment's interface with the system, represented by the outermost action vectors, can be accounted for.

Thus, maximally external experimenters—for instance macroscopic ones—are frequently confronted with amplitudes in the form of traces (of products) of operators.

These ideas will be revisited in subsequent chapters.

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# 2

# Quantum Sets

It is the main purpose of this chapter to briefly describe the elements of Finkelstein's quantum set theory, which is a component of the enormous extension of quantum logic achieved in QR (Chapters 7, 9, 10 and 14). The quantum set construct will be used later in fundamental ways: to motivate an approach to quantum symmetry (Chapter 3), to motivate an embedding of a fragment of quantum logic into a certain sequent calculus (Chapter 6), and to perform the construction of the quantum net (Chapter 7). Since our own use of it here will be fairly rudimentary, we confine our discussion accordingly, digressing here and there on certain other related matters.

In most respects, our mathematical treatment differs from that given in QR, though we always arrive at the same destination.

## 2.1 Logics and Lattices

An understanding that the correct relativization of the logic of a classical system to that of a quantum system corresponds to the replacement of the Boolean algebra of subsets of a set of states by the lattice of subspaces of a Hilbert space, was the cornerstone of a vast and ongoing program aimed at deriving the whole panoply of quantum mechanics from this basic logical structure, suitably abstracted. This program, initiated by Birkhoff and von Neumann in the 1930s (Birkhoff and von Neumann 1936, and Chapter 3, §5 of von Neu-

mann 1996), has resulted in a hugely impressive and convincing body of results, while having little discernible effect on the conduct of most working physicists. Although the mathematical results are of course unambiguous, the same cannot be said for some of their physical interpretations, and debate continues to rage hotly among the philosophically minded. (For superb accounts of this story see the references listed in the introduction to Chapter 5.)

"Quantum Logic" proper (**QL**), as we shall understand it here, is that logic epitomized by (or *modeled* by—in a sense to be made precise later) the lattice of closed subspaces of a Hilbert space. Thus, these lattices replace in the quantum case the (Boolean) lattices of classes that epitomize (or model) classical Propositional Calculus (**PC**). (Technicalities will be discussed at length later.) It transpires that quantum logic exhibits certain fundamental and non-obvious differences from **PC** that are generally regarded by logicians as being anomalous or pathological. We shall adopt the view that these phenomena are in fact revelatory of an actual deep quantum structure, that leaves its stamp indelibly even upon so denatured a description of it as is available through the use of purely logical constructs. (The attempt to interpret this evidence of deep structure will inform much of our work.)

These so-called anomalies of quantum logic will be discussed later (Chapter 5). The major concern of the present chapter is with another problematical aspect of standard quantum logic that Finkelstein seems to have been the first to recognize and address. Namely, its failure to take account of *extensionality*. In the standard interpretation of **QL** quantum logical *predicates* (that would determine classes as their *extensions* in naïve classical set theory) correspond to *projections*, or equivalently, *closed subspaces* of a Hilbert space (§1.2), but *sets* of quanta apparently do not. Thus, there is an asymmetry between quantum *classes* (i.e. quantum predicates, or closed subspaces of a Hilbert space) and quantum *sets* (represented by rays, not in the original space, but in a certain Fock space based upon it (§2.2.3)). This asymmetry is absent in (naïve, finitary) classical set theory, where every class is a set. Finkelstein noted also that in considering

higher order set-theoretic constructs, such as sets of sets, there arises a concomitant problem: standard **QL** is necessarily only first order, dealing with predicates, but not with predicates whose subjects are predicates, etc. These matters are addressed in §2.3.

Needless to say, both of these principles (namely the existence of higher order constructions such as sets of sets, etc., and the principle that sets may be formed as the "extensions" of predicates) are used with cavalier abandon, if not entirely unconsciously, not only in classical physics, but also in most current classical-quantum hybrid theories such as those coming under the rubric "quantum field theory," in which the points of a perfectly classical set, namely an underlying spacetime manifold, are used, among other things, to index fields of operators. Thus Finkelstein's insistence upon repairing these defects in quantum logic, and restoring the symmetries that should underlie those witnessed by macroscopic wielders of classical (finitary) logic, must be seen as an essential first step in the construction of a properly quantal description of the plenum.

## 2.2 Some First-order Quantum Aggregates

In classical thinking, combinations of units into various kinds of aggregates can be effected by carrying out higher-order set theoretic operations. Thus, for instance, sequences are functions with the integers or the natural numbers as domain, Cartesian products are certain sets of functions from indexing sets into disjoint unions of the sets indexed (families of indexed sets and disjoint unions of sets being themselves describable by functions), etc.

In contrast, the foundations of any version of quantum theory assume at least some version of set theory sufficient to support the mathematics used, and so already constitute in themselves vastly rich structures of high order from the point of view of classical logic. (We seem still to require a classical metalanguage to discuss quantum structure, a systemic cut parallel to the one advocated by Bohr but possibly deeper and more painful. Other classical (meaning non-quantal) logical universes, or *toposes*, exist, that embody, for instance,

different models of set theory: so different metalanguages certainly exist. (See the references given in §5.2.2 and the discussion in §12.6.)

This inherent expressivity of quantum theory makes it possible to specify certain quantum analogs of classical combinations in terms of the relatively "first-order" quantum (i.e. linear) structures already present, which is not to say that these quantum combinations necessarily exist physically, though some of them do.

We now derive some of these descriptions that are, for the most part, based on templates taken directly from QR, Chapter 7. To complement the treatment in QR we shall approach the specification problem a little differently, however.

All vector spaces of initial vectors for the individual systems may be taken to be finite dimensional in what follows, and we temporarily drop the ket notation for reasons to appear.

### *2.2.1 Finite Products*

Given a finite number of Hilbert spaces $W_i$, $i=1,\ldots,n$, of initial action vectors for $n$ quantum systems, the problem is to describe the Hilbert space appropriate to the "product" system.

Let $\mathfrak{H}$ denote a possible solution to this problem: that is, elements of $\mathfrak{H}$ are supposed to be initial vectors for the product system. Then, at the very least, some of the initial acts for the product system should be obtainable by arranging in some manner arrays of initial acts upon the individual systems. That is, $n$-tuples $(\alpha_1,\ldots,\alpha_n)$ should be *interpretable* in $\mathfrak{H}$: namely, we should be able to construct a certain function

$$t : W_1 \times \ldots \times W_n \longrightarrow \mathfrak{H}. \qquad (2.2.1.1)$$

The *interpretation* $t(\alpha_1,\ldots,\alpha_n)$ of $(\alpha_1,\ldots,\alpha_n)$ is then as that act, upon the product system, which injects/creates a quantum from each individual system. If an individual action vector is replaced by a superposition, then this interference among alternative acts should be in-

herited by the interpreted act that should similarly produce a superposition of the correspondingly interpreted acts upon the product system. Thus, $t$ in equation (2.2.1.1) may be taken to be linear. That is, the *acts* corresponding to either side of the (vector) equation expressing linearity should be the same: therefore the corresponding vectors should be projectively related. Since the scalar involved in this projective ambiguity is physically irrelevant, we may choose it so that the two sides of the vector equation expressing linearity are the same.

(In this case the argument above rather belabors the obvious fact that if acts upon the $W$-product system are to be interpreted as acts upon the $\mathfrak{H}$ system, then the map from the $W$-product space to $\mathfrak{H}$ implementing the interpretation should be multilinear. However, the principle that an equivalence of acts *may* be represented, in a hunt for a specimen, by an *equality* of the corresponding vectors, will be invoked below to impose further algebraic structure upon the candidate $\mathfrak{H}$-spaces for other templates.)

Now there is a "universal" solution to our problem. Given a finite number of vector spaces, $W_i$, $i = 1,\ldots,n$, say, there exists a vector space $T$ and an $n$-linear map $t : W_1 \times \ldots \times W_n \longrightarrow T$ such that, for any $n$-linear map $f : W_1 \times \ldots \times W_n \longrightarrow E$, there exists a *unique* linear map $\tilde{f} : T \longrightarrow E$ making commutative the following diagram:

$$\begin{array}{ccc} W_1 \times \ldots \times W_n & \xrightarrow{\ t\ } & T \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle \tilde{f}} & \\ & E & \end{array} \qquad (2.2.1.2)$$

It follows easily from the uniqueness requirement that $T$ is unique with this property up to isomorphisms commuting with the maps corresponding to $t$. In fact, we may take

$$T = \bigotimes_{i=1}^{n} W_i \qquad (2.2.1.3)$$

with

$$t(\alpha_1,\ldots,\alpha_n)=\alpha_1\otimes\ldots\otimes\alpha_n. \qquad (2.2.1.4)$$

This is the "universal" property of the algebraic tensor product—it reduces the study of multilinearity to that of ordinary (uni-)linearity.

For the case in which the $W_i$ are Hilbert spaces, there is a similar universal diagram in the category of Banach spaces in which $T$ is a Hilbert space. Namely,

$$T=\widehat{\bigotimes}_{i=1}^{n}W_i, \qquad (2.2.1.5)$$

the completion of $\otimes^n W_i$ with respect to a canonical pre-Hilbert space inner product given on elementary tensors by

$$\langle\omega\otimes\eta\otimes\ldots|\alpha\otimes\beta\otimes\ldots\rangle=\langle\omega|\alpha\rangle\langle\eta|\beta\rangle\ldots \qquad (2.2.1.6)$$

If the $W_i$ are all finite dimensional, $\otimes^n W_i$, also being finite dimensional, is of course already complete. (There are various ways to see that equation (2.2.1.6) defines an inner product independently of the representation of general tensors as linear combinations of elementary ones. For very careful treatments see Palmer 1994 or Hewitt and Ross 1970. For greater generality see Ryan 2002 or Mallios 1986.)

Returning to equation (2.2.1.1) it is now apparent that there is a unique solution (up to commuting isomorphisms), namely $t$ in diagram (2.2.1.2), that is optimal in the sense that any other such solution (such as the $f$ in diagram (2.2.1.2)) "factors through" it: $f=\tilde{f}\circ t$.

Thus, one obtains (or defines) the space of initial action vectors for the product system by tensoring the individual system spaces in the specified order.

In particular, finite sequences of a single system are obtained by taking the appropriate tensor powers of the space of the individual system.

### *2.2.2 Sequences*

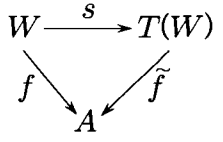
Here we wish to capture the idea of a system whose initial acts inject/create arbitrarily long sequences of the quanta of a given system (with initial space $W$, say). In such sequences the order counts.

As before, let us suppose that a solution, $\mathfrak{H}$, is to hand. Thus elements of $\mathfrak{H}$ are supposed to represent the acts having the property specified. Since any two such elements represent initial acts for a pair of sequences of quanta, there should exist an element in $\mathfrak{H}$ corresponding to the initial act of injecting the sequence obtained by concatenating these two sequences in a given order. Moreover, if either of the two chosen elements in $\mathfrak{H}$ is replaced by a superposition of alternatives, the act corresponding to the concatenated sequence should maintain this interference of alternatives and decompose as a superposition of the appropriate acts for the concatenated sequences. In other words, $\mathfrak{H}$ should admit a *product,* corresponding to concatenation of sequences, that may be taken to be bilinear according to an analog of the superposition argument given above, and clearly also may be taken to be associative. That is, $\mathfrak{H}$ should admit the structure of an associative algebra. The initial action vector corresponding to the act of injecting the null or empty sequence clearly acts as a unit for this algebra structure. Moreover the acts of the original system should be *interpretable* in $\mathfrak{H}$ as sequences of length one, with superpositions maintaining themselves as usual in this interpretation. That is to say, there should exist a linear map

$$s : W \longrightarrow \mathfrak{H} \tag{2.2.2.1}$$

of the space $W$ into the associative unital algebra $\mathfrak{H}$.

It will come as no surprise that this problem, too, has an optimal universal solution: for any vector space $W$ there exists an associative unital algebra $T(W)$ and a linear map $s : W \longrightarrow T(W)$ such that, if $f : W \longrightarrow A$ is any linear map into an associative algebra $A$, there exists a *unique* algebra map $\tilde{f} : T(W) \longrightarrow A$ making commutative the following diagram:




(2.2.2.2)

Again, $T(W)$ is unique with this property up to algebra isomorphisms commuting with the maps corresponding to $s$. In this case, one may take for $T(W)$ the *tensor algebra* over $W$:

$$T(W)=\bigoplus_{k=0}^{\infty}\otimes^k W$$

$$=\mathbb{C}\oplus W\oplus W\otimes W\oplus\ldots \qquad (2.2.2.3)$$

(Note that $\otimes^0 W\equiv\mathbb{C}$, $\otimes^1 W\equiv W$.) The algebra product is given on simple (i.e. homogeneous) tensors by tensoring, the 1 in the first summand being the unit. (See for example Lang 1993, Knapp 1988, p.61, or Mac Lane 1963.) The map $s$ in diagram (2.2.2.2) is the inclusion of $W$ as the first-grade summand $\otimes^1 W\equiv W$.

For $W$ a Hilbert space, a universal diagram similar to (2.2.2.2) exists with $A$ now a Banach algebra, $f$, $\tilde{f}$ and $s$ continuous and $T(W)$ replaced by its Hilbert space completion.

Happily, this interpretation is consistent with our earlier one since we recover, as subspaces of the tensor algebra, finite sequences of arbitrary length.

### *2.2.3 Sets*

Unlike sequences, sets should be insensitive to the order of their constituents. A more fundamental difference is that the repeated membership of an element is disallowed.

For a given system (with initial space $W$) assume as usual that we have found a space $\mathfrak{H}$ whose elements represent acts of injection of sets of the quanta whose space of initial action vectors is $W$. Then exactly as in the previous case, since each element in $\mathfrak{H}$ represents the

act of injecting a set of quanta, we obtain an associative unital algebra structure by supposing that these acts can be combined in such a way as to inject *unions* of the sets of quanta injected by the constituent acts. Again, the unit for this algebra structure corresponds to the act of injecting the null set (of quanta). As before, we should have a linear map

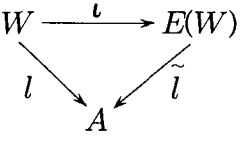
$$\iota : W \longrightarrow \mathfrak{H} \tag{2.2.3.1}$$

that interprets elements of $W$ in $\mathfrak{H}$ as initial action vectors for the injection of singletons. That is, for $\alpha \in W$, $\iota(\alpha)$ represents the act of injecting the *set* whose sole member is the quantum injected by $\alpha$.

The difference now is that a repeated "unioning" of such a singleton must result in that null initial act on the system from which all transitions are forbidden, namely 0, since repeated set membership is not allowed. That is, for all $\alpha \in W$

$$\iota(\alpha)^2 = 0. \tag{2.2.3.2}$$

Once again (Lang 1993, Knapp 1988, p.65, Mac Lane 1963) there is a universal solution in the algebraic category. Specifically, for every vector space $W$, there exists an associative algebra $E(W)$ and a linear map $\iota : W \longrightarrow E(W)$ satisfying equation (2.2.3.2), such that for any linear map $l : W \longrightarrow A$ into an algebra satisfying an analog of equation (2.2.3.2), there exists a unique algebra map $\tilde{l} : E(W) \longrightarrow A$ making commutative the diagram:




(2.2.3.3)

The algebra $E(W)$ is, as usual, unique up to appropriately commuting algebra isomorphisms. It is called the *exterior algebra* over $W$ and given explicitly by:

$$E(W)=\bigoplus_{k=0}^{\infty}\bigwedge^k W$$

$$=\mathbb{C}\oplus W\oplus W\bigwedge W\oplus\ldots \qquad (2.2.3.4)$$

Here $\bigwedge$ denotes the usual exterior product, $\bigwedge^0 W\equiv\mathbb{C}$ and $\bigwedge^1 W\equiv W$, $\iota$ is inclusion of $W$ as the $\bigwedge^1$-summand, and multiplication of homogeneous terms is by $\wedge$-ing them together.

In this case, if $W$ is finite dimensional, of dimension $n$, say, since

$$\dim\bigwedge^k W=\binom{n}{k}, \qquad (2.2.3.5)$$

the series in equation (2.2.3.4) terminates at $k=n$, and

$$\dim E(W)=2^n. \qquad (2.2.3.6)$$

So the exterior product emerges as the analog of the (disjoint) union connective for "quantum" sets.

We note here one further property of the exterior algebra that will be of significance later. Namely, for finite dimensional vector spaces $V$ and $W$, the linear map

$$\bigwedge^m V\otimes\bigwedge^n W\longrightarrow\bigwedge^{m+n}(V\oplus W) \qquad (2.2.3.7)$$

given in an obvious notation by

$$(v_1\wedge\ldots\wedge v_m)\otimes(w_1\wedge\ldots\wedge w_n)\mapsto v_1\wedge\ldots\wedge v_m\wedge w_1\wedge\ldots\wedge w_n$$

$$(2.2.3.8)$$

induces an isomorphism

$$\Lambda^p(V\oplus W)\cong\bigoplus_{k=0}^{p}\Lambda^k V\otimes\Lambda^{p-k}W \qquad (2.2.3.9)$$

whence an isomorphism of vector spaces (not algebras)

$$E(V\oplus W)\cong E(V)\otimes E(W)\,. \qquad (2.2.3.10)$$

(Cf. Fulton and Harris 1991, Appendix B.) We will return to this topic in §3.1.8.

### *2.2.4 Sibs*

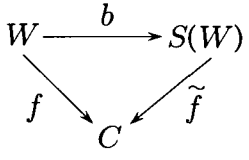
A *sib* is an arrangement, like a sequence, but symmetric under all permutations. Given a system, with space $W$, our task is as usual to find a candidate $\mathfrak{H}$ to play the rôle of space of initial vectors for sibs of $W$-quanta. Assuming $\mathfrak{H}$ exists, each of its elements represents the initial act of injecting a "symmetrical arrangement" of $W$-quanta. Two of these acts determine a third, namely the act of injecting the symmetrical arrangement of the union of their constituents, and the familiar argument yields an associative unital algebra structure for $\mathfrak{H}$. This time, however, the product may be taken to be *commutative*, since interchanging two given elements in $\mathfrak{H}$ has the effect merely of subjecting the arrangement injected by the act represented by the product element to a permutation, to which it is impervious. Therefore, we may use the same vector to represent the product act regardless of the order of the original factors.

So the problem reduces to finding a commutative unital associative algebra $S(W)$ and a linear map

$$b: W\longrightarrow S(W)\,. \qquad (2.2.4.1)$$

Algebraists have once again supplied a universal solution to this problem, at least in the algebraic category (Lang 1993, Knapp

1988, p. 70, Mac Lane 1963). That is to say, if $f: W \longrightarrow C$ is a linear map of $W$ into any commutative algebra $C$, then there is a unique extension $\tilde{f}: S(W) \longrightarrow C$, that is an algebra map rendering commutative the following diagram:




(2.2.4.2)

Up to isomorphism $S(W)$ is the *symmetric algebra* over $W$:

$$S(W) = \bigoplus_{k=0}^{\infty} \textcircled{S}^k W$$

$$= \mathbb{C} \oplus W \oplus W \textcircled{S} W \oplus \ldots, \qquad (2.2.4.3)$$

where Ⓢ denotes the symmetric product (consisting of totally symmetrized tensors).

We note that an isomorphism analogous to (2.2.3.10) obtains also for symmetric algebras:

$$S(V \oplus W) \cong S(V) \otimes S(W) \qquad (2.2.4.4)$$

that is in this case an isomorphism of algebras. This can be proved by showing that $S(V) \otimes S(W)$ satisfies the universal property for the map

$$V \oplus W \longrightarrow S(V) \otimes S(W)$$

given by:

$$(v, w) \mapsto v \otimes 1 + 1 \otimes w. \qquad (2.2.4.5)$$

The unique extension to $S(V)\otimes S(W)$ of any linear map of $V\oplus W$ into a commutative algebra $C$ is easily concocted and depends on the fact that in the case of $C$ being commutative—and only in this case—the algebra product $C\otimes C \longrightarrow C$ is itself a map of algebras (cf. §3.1.1). A naïve attempt to prove (2.2.3.10) to be an algebra map—or an analogous result for $T(\ )$—by using the analogous universal property, would fail for this reason.

## 2.3 Quantum Set Theory

To each of the infinite quantum aggregates described above, namely sequences ($T(W)$), sets ($E(W)$) and sibs ($S(W)$), is associated a different *statistics*: the completed Hilbert space version of $E(W)$ is exactly the Fermi–Dirac Fock space over $W$, that carries a representation of the CAR (canonical anticommutation relations), and the Hilbert space version of $S(W)$ is exactly the Bose–Einstein Fock space, carrying a representation of the CCR (canonical commutation relations). See for example Baez, Segal and Zhou, 1992, Thirring 1980 or Emch 1972. As Finkelstein points out (QR, §7.1) the statistics associated with sequences, or $T(W)$, even though quantized, is essentially the classical statistics of Maxwell–Boltzmann. No actual physical assemblies exhibit these statistics: rather, they exhibit one or other of the first mentioned pair. Roughly and colloquially speaking, quanta of "matter" obey the statistics of Fermi-Dirac, and those of "radiation," that of Bose-Einstein.

Since the extension of quantum logic to a fully fledged quantum set theory (as exposed in QR, and discussed briefly below) proceeds on the basis provided by exterior algebra, Finkelstein argues persuasively that Fermi–Dirac statistics is actually the fundamental one. To quote from one of the most telling and beautiful passages in QR (§7.4)—the term *extensor* refers to an element in an exterior algebra:

> "When we write or speak we produce a sequence of symbols, not merely a set of them . . .

> And whence these classical sequences? For all we know, the medium (say paper) supporting these sequences of symbols is made of fermions, best described by extensors, not tensors. How does Fermi–Dirac statistics at the microscopic level give rise to Maxwell–Boltzmann at the macroscopic surface?
>
> It seems that the underlying fermions have undergone a phase transition, a condensation (say into paper). They have been linked into a persistent pattern of fermions that is unaffected by an interchange of the basic fermions. Our symbols are not fermions themselves but changes in a pattern of linkages among equivalent fermions.
>
> Now one of the media on which we write is spacetime, as when we communicate by electromagnetic signals. If, so to speak, ordinary tensors can condense out of extensors, then spacetime too might be a condensation of fermionic constituents described by extensors, not tensors . . ."

(Later, in Chapter 8, we will conflate this idea—namely that the passage from quantum sets to sequences reflects a "phase change" from microcosm to macrocosm—into a general correspondence principle.)

Returning to the quantum set construction above we note again that the exterior algebra product is to be interpreted in this context as a sort of (disjoint) *union* for which Cartan's symbol $\wedge$ is a rather unfortunate misnomer. Finkelstein advocates a return to the earlier notation of Peano, that Cartan's inverts: $\vee$. This usage risks confusion with the disjunction of ordinary classical (or modal) logic, that we will be using later. However, since the latter operates only upon logical formulae, or elements of a Boolean lattice, confusion may be avoided. We shall in the meantime follow Finkelstein in using this notation for exterior product in quantum logical contexts, backsliding later in geometrical ones. Thus equation (2.2.3.4) is now

written

$$E(W)=\bigoplus_{k=0}^{\infty}\bigvee^{k}W$$

$$=\mathbb{C}\oplus W\oplus W\vee W\oplus\ldots \qquad (2.3.1)$$

Our next observation concerns the map $\iota: W\longrightarrow E(W)$. This map interprets an element $\alpha\in W$ as a (quantum) set $\iota(\alpha)$ in $E(W)$, that is the analog of the classical set $\{\alpha\}$. (This explains the iota, that was Peano's notation for the "unitizing" operation upon sets: $\iota A\equiv\{A\}$.) Since $\iota$ is linear, the ray determined by $\alpha$ is sent to the ray determined by $\iota(\alpha)$. (In ordinary set theory, the set $\{x\}$ is the extension of the proposition "$x=x$," or "$x$ is $x$." Analogously, the ray determined by $\iota(\alpha)$ is a *quantum set,* now interpretable, as in ordinary set theory, as the *extension* of a certain predicate *about* the predicate corresponding to $\alpha$: namely, the predicate "being $\alpha$," roughly speaking.) Moreover, as Grassmann noticed, this sort of association can be extended so as to apply to all subspaces of $W$: merely choose a basis $\{\alpha_1,\ldots,\alpha_k\}$ for the given subspace and form the exterior product $\iota(\alpha_1)\vee\ldots\vee\iota(\alpha_k)$. Then doing this with another basis of the subspace merely replaces this exterior product by a scalar multiple of itself, this scalar being the determinant of the linear transformation induced by the basis change. Thus subspaces of $W$ correspond bijectively with rays of homogeneous elements in $E(W)$, and (finite) extensional symmetry is now restored to quantum logic. We note also that the ray $\mathbb{C}$ appearing as the first summand in $E(W)$ represents the empty set $\varnothing$—this follows from our original construction. It is the extension of no quantum predicate.

Any homogeneous element $\iota(\alpha_1)\vee\ldots\vee\iota(\alpha_k)$, say, in $E(W)$ is a quantum analog of the (disjoint) union $\{\alpha_1\}\cup\ldots\cup\{\alpha_k\}=\{\alpha_1,\ldots,\alpha_k\}$, but superpositions are allowed, which of course have no classical counterpart. In fact, $E(W)$ contains a version of classical set theory—a realization that was not lost on Grassmann and some of his followers.

(The tangled story of Grassmann's famously puzzling "Ausdehnungslehre" (Kannenberg 1995, Barnabei, Brini and Rota 1985) and its influence on the likes of Peano and Clifford, as well as its non-influence on the nearly parallel work of Hamilton, Boole, etc., is well treated in QR. It is surprising but somehow fitting that Grassmann's ideas have eventually achieved their apotheosis in Finkelstein's dramatic extension of quantum logic, that we only touch on here. The deeper significance of Finkelstein's suggestion will emerge later, in Chapter 6, as a result of an entirely independent line of argument.)

Quantum sets can be made by forming exterior products of "unitized" vectors $\iota(\alpha)$, but not out of the original elements of $W$. Thus we seem to be denied the very ability to form set-like aggregates out of the elements of $W$ that we were seeking in the first place. The way out of this dilemma, as Finkelstein notes (QR, §10), is to realize that we should *start* from $\iota(W)$, not $W$. Or, rather, that we have *already* started from $\iota(W)$. This can be reinforced by noticing that the classical "definite descriptor," namely $\{\ \}$, plays exactly the rôle of Dirac's ket $|\ \rangle$, so that (noting that $|\ \rangle$ is linear) we should *identify* $\iota(\ )$ with $|\ \rangle$. That is, for $\alpha \in W$

$$\iota(\alpha) = |\alpha\rangle. \tag{2.3.2}$$

In other words, we have been doing first-order set theory right from the beginning. All our spaces of initial action vectors may be regarded as being first-order extensions ($|\alpha\rangle$) of zero$^{\text{th}}$-order quantum predicates ($\alpha$).

The map $\iota$ ($=|\ \rangle$) together with its interpretation given above now opens the door to a higher-order quantum set theory (for a given finite dimensional $W$), in which we may form sets of sets, and so on, to any depth of nesting. The idea is to define a structure in which it is possible to iterate $\iota$.

We start the construction with $E(W)$ itself, writing

$$E^1 \equiv E(W) = \mathbb{C} \oplus W \oplus W \bigvee W \oplus \ldots \tag{2.3.3}$$

and recalling that $E^1$ is finite dimensional. Writing the iota map for $E^1$ as $\iota_1$:

$$\iota_1 : E^1 \longrightarrow E(E^1), \tag{2.3.4}$$

we define $E^2$ so as to avoid repeating $\varnothing$, while keeping the old quantum sets ($E^1$) and adding unitized versions of them ($\iota_1(E^1)$):

$$\begin{aligned} E^2 &\equiv \bigoplus\nolimits_{k>0} \bigvee^k (\iota_1(E^1) \oplus E^1) \\ &= (\iota_1(E^1) \oplus E^1) \oplus \bigvee^2 (\iota_1(E^1) \oplus E^1) \oplus \ldots \end{aligned} \tag{2.3.5}$$

Continuing in this way, we define, for $m = 2,3,\ldots$

$$E^m \equiv \bigoplus\nolimits_{k>0} \bigvee^k (\iota_{m-1}(E^{m-1}) \oplus E^{m-1}). \tag{2.3.6}$$

Notice that for each $m$ there are two monomorphisms of $E^{m-1}$ into $E^m$: namely, the inclusion of $E^{m-1}$ as the second summand in equation (2.3.6), and the map $\iota_{m-1}$ followed by the inclusion of its image into $E^m$ as the first summand. Moreover, with the first mentioned unnamed inclusions running vertically, and the iota based maps running horizontally and bearing their corresponding names, we have commutation in the following diagram:

$$\begin{array}{ccc} E^1 & \xrightarrow{\iota_1} & E^2 \\ \downarrow & & \downarrow \\ E^2 & \xrightarrow{\iota_2} & E^3 \\ \downarrow & & \downarrow \\ \vdots & & \vdots \end{array} \tag{2.3.7}$$

A finite dimensional Hilbert space structure on $W$ determines a Hilbert space structure on each exterior product (*via* an analog of equation (2.2.1.6)) and therefore also on each (finite dimensional) $E^m$. It follows that the (infinite dimensional) algebraic direct limit of the vertically nested inclusions of vector spaces in the diagram above admits a canonical pre-Hilbert space structure, and in this limit we obtain a continuous linear map

$$\varinjlim \iota_m : \varinjlim E^m \longrightarrow \varinjlim E^m \tag{2.3.8}$$

commuting with all the $\iota_m$. Completing this map, we obtain a map of Hilbert spaces that we write as

$$\iota : \text{SET}\, W \longrightarrow \text{SET}\, W\,. \tag{2.3.9}$$

(Such a limit, over a directed set of inclusions of Hilbert spaces, is also often referred to as the *union* of the spaces $E^m$.)

Clearly, $\iota$ fulfils the requirement of enabling arbitrary nesting of sets within sets and we shall usually write it as $|\ \rangle$. In this context we shall refer to these kets as quantum sets or *qets*. We shall return to a discussion of the physical significance of the map depicted in equation (2.3.9) in §7.2.

This concludes our presentation of a snippet of the enormously richer structure revealed in QR. We have failed to discuss the analogs of other set theoretic operations such as intersection and complementation; many other symmetries; Hodge duality; operator, lattice, measure and geometric structure; functions; etc. (See QR, Chapters 7, 9, 10, 14.)

We have also failed to pursue the connections with classical set theory. It is shown in QR (§7.2.4 and Chapter 14) how this extended quantum logic contains a version of classical logic as a sort classical limit or degeneration. There would appear to be a version of classical set theory, or *type*, for each initial space $W$. The issue of types will be taken up later.

# 3

# Group Duality, Coherence and Cyclic Actions

In this chapter we touch on the subject of quantum replacements for groups, that, as explained in §3.1.9, are objects more radically quantized than those currently bearing the name "quantum groups." The need to find such replacements, or quantizations, presses from various directions. On purely mathematical grounds, once we have even a rudimentary theory of quantum sets, a concomitant non-objective quantum theory of *symmetries* should follow, relativizing the classical theory.

This need can also be seen on physical grounds. To quote Finkelstein (QR, §15.1, p. 447):

> "When we imagine that a microscopic agent acts on a quantum microsystem—for example when we rotate, boost or translate an electron—it is plausible to describe the agent, and act itself, by classical parameters, which commute and do not change during the action. Whenever we inspect closely what actually acts on a quantum, however, we always find another quantum. Then the parameters of the agent are also non-commuting quantum variables. . . If the actions still form a group at all

> under such high resolution, it is at best a quantum group."

Roughly speaking, groups enter physics *via* external (or physical) parameters, depending, for example, upon classically treated spatio-temporal coordinates. These external parameters should themselves exhibit quantum behavior at sufficiently high resolution and should be treated accordingly.

A complete program to quantize general (finite) groups and semigroups using a paradigm based essentially on a quantized (or atomized) notion of categorial morphism, or arrow, is worked out in QR, Chapter 15. In the simplified model we discuss here, it turns out that quantum replacements are needed only for (finite) cyclic groups. Accordingly, we focus our attention on this special case.

We approach this problem from two directions, one emanating from a consideration of the quantum set theory sketched in the previous chapter (§2.3), the other requiring a certain extrapolation of the classical theory of group duality, which is the subject of §3.1. For the case of cyclic actions, our rather heuristic treatment complements the one given in QR (Chapter 15), to which it is quite close in spirit if not in detail. For the cases considered, our final results will be identical at the complex infinitesimal level (i.e. *qua* complex Lie algebras) with those obtained in QR.

During the course of our treatment the important notion of coherence is found to arise fairly spontaneously (§3.2.2), and its physical significance is discussed briefly. This topic reappears in subsequent chapters.

The quantum replacements for the classical actions found here will play a fundamental rôle in what follows. We shall use them to specify quantum analogs of certain systems exhibiting symmetries with respect to these actions, or to resolve more finely the level of quantization of such systems.

## 3.1 The Duality of Groups and Hopf Algebras

All known classical theories of group duality may be subsumed somewhere or other within the theory of Hopf algebras. Since we shall require some of these notions later in various other contexts (§8.2) we devote this section to a brief outline of the rudiments of the theory, digressing slightly in places. The standard (and by now classical) references are Sweedler 1969 and Abe 1977. However, recent interest in a subclass of Hopf algebras, the so-called "quantum groups," has led in the last couple of decades to a vast and exponentially exploding literature on this subject. For two excellent examples among many, see Klymik and Schmüdgen 1997 and Shnider and Sternberg 1993.

It will be convenient in this section to admit arbitrary base fields, so that unadorned $\otimes$ will, for the duration of this section, stand for $\otimes_k$, where $k$ denotes an arbitrary field. For $k$-vector spaces $A$ and $B$, the *twist isomorphism* $T: A\otimes B \longrightarrow B\otimes A$ is the isomorphism determined by $T(a\otimes b) = b\otimes a$. The identity map on $A$ will be denoted by $1_A$ or just 1 when the context is clear. References to $k$ will also be dropped from time to time.

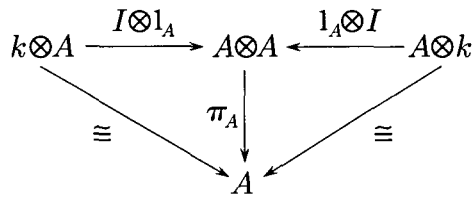
First, we use the universal property of the tensor product to express the definition of an algebra in terms of maps within the category of vector spaces.

### *3.1.1 Algebras*

An *associative k-algebra* is a $k$-vector space $A$ together with a $k$-linear map $\pi_A : A\otimes A \longrightarrow A$, called the *product* (or *multiplication*) making the following diagram commute:

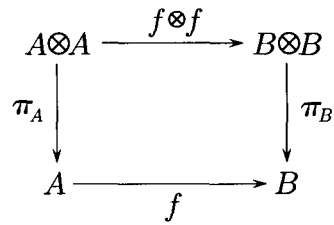
$$\begin{array}{ccc} A\otimes A\otimes A & \xrightarrow{\pi_A\otimes 1_A} & A\otimes A \\ \downarrow{\scriptstyle 1_A\otimes\pi_A} & & \downarrow{\scriptstyle \pi_A} \\ A\otimes A & \xrightarrow[\pi_A]{} & A \end{array} \qquad (3.1.1.1)$$

A *unital* $k$-algebra is an algebra $(A,\pi_A)$ equipped with a *unit* $I$ which is a linear map $I : k \longrightarrow A$ making the following diagrams commute:



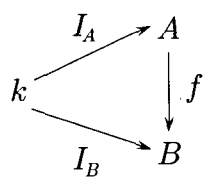

(3.1.1.2)

A linear map $f : A \longrightarrow B$ of algebras $(A,\pi_A)$, $(B,\pi_B)$ is an *algebra map* if the following diagram commutes:




(3.1.1.3)

If these algebras have units $I_A$ and $I_B$, then in order for $f$ to constitute a *unital* algebra map it is required in addition to make the following diagram commute:




(3.1.1.4)

If $(A,\pi_A,I_A)$, $(B,\pi_B,I_B)$ are unital algebras, so also is their tensor product $A\otimes B$ with product $(\pi_A\otimes\pi_B)\circ(1_A\otimes T\otimes 1_B)$ and unit $I_A\otimes I_B$. An algebra is said to be *commutative* if $\pi_A \circ T = \pi_A$. It is important to note that $A$ being commutative is equivalent to $\pi_A : A\otimes A \longrightarrow A$ itself being a map of *algebras* (with the above product on $A\otimes A$).

### *3.1.2 Coalgebras*

A (*coassociative*) *k-coalgebra* is a $k$-vector space $A$ equipped with a $k$-linear map $\psi_A : A \longrightarrow A \otimes A$ called the *coproduct* (or *comultiplication*) making commutative the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\psi_A} & A \otimes A \\ {\scriptstyle \psi_A}\downarrow & & \downarrow{\scriptstyle \psi_A \otimes 1_A} \\ A \otimes A & \xrightarrow[1_A \otimes \psi_A]{} & A \otimes A \otimes A \end{array} \qquad (3.1.2.1)$$

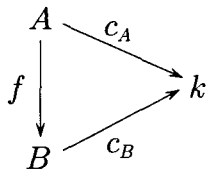
A *counit* is a linear map $c_A : A \longrightarrow k$ making commutative the diagram(s) dual to diagram (3.1.1.2)—that is, the analogous diagram with arrows reversed:

$$\begin{array}{ccccc} k \otimes A & \xleftarrow{c_A \otimes 1_A} & A \otimes A & \xrightarrow{1_A \otimes c_A} & A \otimes k \\ & \nwarrow^{\cong} & \uparrow{\scriptstyle \psi_A} & \nearrow_{\cong} & \\ & & A & & \end{array} \qquad (3.1.2.2)$$

Maps, tensor products, etc., of coalgebras are similarly defined dually to those already described for algebras. Namely, a linear map $f : A \longrightarrow B$ of coalgebras is a *coalgebra map* if the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \psi_A}\downarrow & & \downarrow{\scriptstyle \psi_B} \\ A \otimes A & \xrightarrow[f \otimes f]{} & B \otimes B \end{array} \qquad (3.1.2.3)$$

If these coalgebras have counits $c_A$ and $c_B$, then $f$ is a *counital* map if the following diagram commutes:




(3.1.2.4)

If $(A,\psi_A,c_A)$, $(B,\psi_B,c_B)$ are counital coalgebras so also is their tensor product with coproduct $(1_A\otimes T\otimes 1_B)\circ(\psi_A\otimes\psi_B)$ and counit $c_A\otimes c_B$. A coalgebra is said to be *cocommutative* if $\psi_A\circ T=\psi_B$.

If $(A,\psi)$ is a coalgebra, the dual space $A^*$ may be given an algebra structure by taking the composition:

$$A^*\otimes A^*\longrightarrow(A\otimes A)^*\xrightarrow{\psi^*}A^* \qquad (3.1.2.5)$$

where the first map denotes the act of taking tensor products of linear functionals. (This product on $A^*$ is often called "convolution" for reasons explained below.) A similar construction cannot be used to equip the dual space of an algebra with a coalgebra structure because the first map is not in general (i.e. when the algebra is not finite dimensional) an isomorphism. Nevertheless, a functorial construction is available that assigns to each algebra a corresponding coalgebra, the latter reducing to the dual of equation (3.1.2.5) in case the algebra is finite dimensional.

An important example is afforded by the algebra $\mathrm{End}W$, $\dim W=n$, say. Choose a basis for $\mathrm{End}W$ consisting of matrices of the form $v_{rs}$, whose $(i,j)$ entry is

$$(v_{rs})_{i,j}\equiv\delta_{ri}\delta_{sj},\ i,j=1,\ldots,n. \qquad (3.1.2.6)$$

That is, $v_{rs}$ is the matrix with 1 in the $(r,s)$ position and 0 elsewhere. Then, upon *identifying* this basis with its dual in $(\mathrm{End}W)^*$,

the coalgebra structure induced upon $\mathrm{End}W$ is given by

$$\psi(v_{rs}) = \sum_{k=1}^{n} v_{rk} \otimes v_{ks}. \qquad (3.1.2.7)$$

(For the elementary proof of an equivalent assertion see Selesnick 1973a, Proposition 4.)

We note the interesting fact that the ordinary trace acts as a counit for this coalgebra structure.

It follows from general algebra-coalgebra duality theory that $(\mathrm{End}W, \psi)$ is a simple coalgebra: that is, one having no proper subcoalgebras.

### *3.1.3 Bialgebras and Hopf Algebras*

A *bialgebra* is a vector space that carries both an algebra structure and a coalgebra structure in such a way that either structure respects the other: it then transpires that this respect is mutual.

In detail, a *bialgebra* is a vector space such that:

(i) $(A, \pi, I)$ is a (unital) algebra,
(ii) $(A, \psi, c)$ is a (counital) coalgebra,
(iii) $c : A \longrightarrow k$ is a map of *algebras*,
(iv) $\psi : A \longrightarrow A \otimes A$ is a map of *algebras*.

A glance at the relevant diagrams reveals that (iii) and (iv) can together be replaced by:

(iii)′ $I : k \longrightarrow A$ is a map of *coalgebras*,
(iv)′ $\pi : A \otimes A \longrightarrow A$ is a map *coalgebras*.

If $\pi$ (respectively $\psi$) is commutative (respectively cocommutative) the bialgebra is said to be commutative (respectively cocommutative). A bialgebra *map* is an algebra map that is simultaneously a coalgebra map.

If $A$ is a coalgebra and $B$ is an algebra then the vector space $\mathrm{Hom}(A, B)$ of linear maps from $A$ to $B$ has the structure of an algebra with product given by "convolution":

$$f*g = \pi_B \circ (f \otimes g) \circ \psi_A. \qquad (3.1.3.1)$$

(This coincides with (3.1.2.5) when $B = k$.) The unit element for this algebra is $I_B \circ c_A$. If $A$ is a bialgebra we may use its coalgebra structure in the first position and its algebra structure in the second to give $\mathrm{End}A = \mathrm{Hom}(A,A)$ an algebra structure. A (necessarily unique) two-sided inverse for $1_A$ in this algebra, if it exists, is called the *antipode* of the bialgebra $A$.

A *Hopf* algebra is a bialgebra equipped with an antipode (denoted $\eta_A: A \longrightarrow A$). A *map* of Hopf algebras is a map of the underlying bialgebras that commutes with the antipodes. It can be shown that, among other things, the antipode is an anti-isomorphism for the algebra structure: $\eta_A(ab) = \eta_A(b)\eta_A(a)$.

Now suppose that $B$ is a *commutative* algebra and consider the set of algebra maps $\mathrm{Hom}_{\mathrm{alg}}(A,B)$, with $A$ a Hopf algebra. This set can be shown to be a *group* under convolution, the inverse of any algebra map $f: A \longrightarrow B$ being $f \circ \eta_A$, an algebra map since $B$ is commutative.

We shall describe some pertinent examples below.

### *3.1.4 The Additive Affine Group*

The $k$-polynomial algebra $k[x]$ is a commutative cocommutative Hopf algebra with its usual product and unit ($I(1) = 1$), with coproduct determined by

$$\psi(x) = 1 \otimes x + x \otimes 1, \qquad (3.1.4.1)$$

counit by

$$c(x) = 0, \qquad (3.1.4.2)$$

and antipode by

$$\eta(x) = -x. \tag{3.1.4.3}$$

When $k$ is algebraically closed the group $\mathrm{Hom}_{\mathrm{alg}}(k[x],k)$ may be identified with the additive group of $k$. This example generalizes in the obvious way to any finite number of variables (Abe 1977, p.172).

### *3.1.5 Finite Group Algebras*

Let $F$ denote a finite group and consider $k^F$, the set of all functions from $F$ to $k$. This is a finite dimensional commutative algebra with pointwise (or coordinatewise) operations. It becomes a Hopf algebra with

$$\psi(f)(x,y) = f(xy) \tag{3.1.5.1}$$

for $f \in k^F$ and $x,y \in F$ and noting that $k^F \otimes k^F \cong k^{F\times F}$.

The counit is given by

$$c(f) = f(e), \tag{3.1.5.2}$$

where $e$ is the identity of $F$, and the antipode by

$$\eta(f)(x) = f(x^{-1}). \tag{3.1.5.3}$$

In this case, since $k^F$ is finite dimensional, we have—upon choosing as basis the characteristic functions of the elements of $F$, and associating this basis with its dual—an isomorphism $k^F \cong (k^F)^*$: "convolution" (equation (3.1.2.5)) induced on $k^F$ by $\psi^*$ is exactly the usual convolution. Thus, $k^F$ with this induced product is exactly the usual group algebra $k[F]$ as this is usually defined. The (cocommutative) coproduct on $k[F]$ induced by the (commutative) pointwise multiplication of functions in $k^F$ is just the "diagonal" copro-

duct, given, in the usual representation of a basis in $k[F]$ by elements of $F$, by $x \mapsto x \otimes x$, $x \in F$. The counit is given by $x \mapsto 1$ and the antipode by $x \mapsto x^{-1}$.

As a Hopf algebra $k[F]$ is thus finite dimensional, cocommutative, and in case $k$ has characteristic 0 and is algebraically closed, cosemisimple: meaning that as a coalgebra it is a direct sum of simple subcoalgebras. (This is the dual form of a special case of Maschke's theorem.) Conversely, any such Hopf algebra (namely, a finite dimensional, cocommutative and cosemisimple one) is the group algebra of some finite group. Moreover, this association extends to an equivalence of the relevant categories and constitutes a *duality theory* for finite groups (Abe 1977, p.159).

### *3.1.6 Topological Hopf and Coalgebras*

In this and the subsequent subsections $k = \mathbb{C}$.

For a compact topological group (or semigroup) $G$, the product $G \times G \longrightarrow G$ induces a C*-algebra map $C(G) \longrightarrow C(G \times G)$. On recalling that there exists a canonical "cross-norm" on the algebraic tensor product of a pair of Banach spaces, namely the smallest, or $\varepsilon$-norm, such that $C(G \times G) \cong C(G) \hat{\otimes}_{\varepsilon} C(G)$, where as usual the hat denotes completion, it becomes apparent that the notions of Hopf algebra, coalgebra, and, indeed, algebra, may be formulated in categories of Banach (or more general) spaces, with the ordinary tensor product replaced with a (completed) topological one. For instance, the above example may be refined to yield the classical Tannaka–Krein Duality Theorem, part of which is described in the next subsection (Sankaran and Selesnick 1971). The topological version of the dual algebra product (diagram (3.1.2.5)) gives in this case exactly the convolution of finite Borel measures on $G$. We note also that for $G$ a compact group, the group product induces a $\hat{\otimes}_{\pi}$-coalgebra structure on $L^1(G)$, where the $\pi$ refers to the largest cross-norm: that is, $L^1(G)$ is a *Banach coalgebra.* A complete theory of Banach coalgebras can be found in Wichmann 1975. (For background on topological tensor products of spaces and algebras see the references

given in §2.2.)

At the time of writing, the last word on global duality theories of general locally compact groups *via* topological Hopf algebras seems to be contained in Enock and Schwartz 1992. Despite close similarities, this topic seems somewhat peripheral to the main thrust of the subject known as "quantum groups," that in a sense represents a local version of it: cf. §3.1.9 below. For a pioneering work on Hopf and coalgebras in the category of C*-algebras with applications to compact semigroups—a subject of much greater difficulty than the group case—we refer to Hofmann 1970, and note that this neglected masterpiece appeared nearly two decades before the advent of quantum groups. Other kinds of applications of Hopf and bialgebras in analysis appear in Schürmann 1993 and Selesnick 1973b.

### *3.1.7 The Algebra of Representative Functions on a Compact Group*

For a compact topological group $G$ we denote by $R(G)$ the subspace of the algebra $C(G)$ spanned by matrix entries of continuous irreducible unitary representations of $G$, each such entry being a continuous function on $G$. This space can be shown to be a $*$-subalgebra of $C(G)$. It is, moreover, a Hopf algebra with

$$\psi(f)(g_1, g_2) = f(g_1 g_2) \tag{3.1.7.1}$$

for $f \in R(G)$ and $g_1, g_2 \in G$, exactly as in the last pair of examples. We note that $R(G \times G) \cong R(G) \otimes R(G)$ and that $\psi$ has an explicit form when $f$ is a matrix entry. That is, if

$$u_{ij}^{(\sigma)}(g) = \langle \xi_i | U^{(\sigma)}(g) | \xi_j \rangle, \tag{3.1.7.2}$$

where $\{\xi_i\}$ is an orthonormal basis in the space carrying the irreducible representation $U^{(\sigma)}$, then

$$\psi(u_{ij}^{(\sigma)}) = \sum_{k=1}^{n} u_{ik}^{(\sigma)} \otimes u_{kj}^{(\sigma)} \qquad (3.1.7.3)$$

where $n$ is the dimension of the representation and the index $\sigma$ ranges over the set $\hat{G}$ of equivalence classes of irreducible unitary representations of $G$. The counit and antipode are defined as in §3.1.5. It is worth noting here that $R(G)$ is finitely generated as an algebra if and only if $G$ is a Lie group. (Cf. Bröcker and tom Dieck 1985, Proposition 4.3, and the remark following it.)

The group $G$ can be recovered from $R(G)$ both topologically and algebraically as the set of involution preserving algebra maps from $R(G)$ to $\mathbb{C}$: each such map is given by the evaluation of functions in $R(G)$ at unique elements of $G$. This assertion is one version of the Tannaka–Krein Duality Theorem, which may be reduced to the compact case of the Pontryagin Duality Theorem when $G$ is abelian. (See Bröcker and tom Dieck 1985 for the Lie case, and the references cited below for the general case.)

The structure of $R(G)$, both as an ordinary algebra and as a Hopf algebra, is well understood. See Bröcker and tom Dieck 1985, Selesnick 1973a, Hewitt and Ross 1970, Hofmann 1967 or Hochschild 1965. This is not to say, however, that the relation *between* the equivalent objects $G$ and $R(G)$ is entirely understood, even in the most tractable (non-discrete) case in which $G$ is Lie, which, as noted, is equivalent to $R(G)$ being finitely generated as an algebra. For example, as a coalgebra,

$$R(G) = \bigoplus_{\sigma \in \hat{G}} H^{(\sigma)}, \qquad (3.1.7.4)$$

where $H^{(\sigma)}$ is a simple subcoalgebra isomorphic to the dual of the algebra $M(d_\sigma, \mathbb{C})$ of complex $d_\sigma \times d_\sigma$ matrices: compare equations (3.1.7.3) and (3.1.2.7). Therefore, the algebra structure induced on the algebraic dual is

$$R(G)^* = \prod_{\sigma \in \hat{\hat{G}}} M(d_\sigma, \mathbb{C}). \tag{3.1.7.5}$$

On the other hand, since $R(G)$ is dense in the (Hopf) C*-algebra $C(G)$ by the theorem of Peter–Weyl, it is not hard to show that as algebras:

$$M(G) \cong R(G)' \tag{3.1.7.6}$$

where $M(G)$ denotes the convolution algebra of finite complex Borel measures on $G$, and the prime denotes the topological dual space of bounded linear functionals on $R(G)$ when the latter is equipped with the uniform norm, that inherits an algebra structure from $R(G)^*$. From the inclusion of $R(G)'$ in $R(G)^*$ we obtain from the last two equations an algebra monomorphism

$$M(G) \longrightarrow \prod_{\sigma \in \hat{\hat{G}}} M(d_\sigma, \mathbb{C}). \tag{3.1.7.7}$$

Thus, each measure in $M(G)$ has a unique "decomposition" as a matrix-valued function on $\hat{G}$, that is a section of the bundle over the (discrete) "base space" $\hat{\hat{G}}$, with non-discrete fibre $M(d_\sigma, \mathbb{C})$. This map turns out to be the noncommutative analog of the Fourier transform for measures, to which it reduces in case $G$ is abelian. See Hewitt and Ross 1970—the Fourier transform is not defined in this way in that work, however. Even though this transform allows a refined analysis of elements of $M(G)$ (and some of its ideal theory) to be derived, the extremely complicated algebraic structure of $M(G)$ seems still to defy complete capture, even in simple cases, as does a characterization of its image under the map in equation (3.1.7.7).

If $G$ is discrete, hence finite, then the Hopf algebra $R(G)^* = R(G)'$ reduces to the group algebra $\mathbb{C}[G]$ considered in §3.1.5.

### *3.1.8 Tensor, Symmetric and Exterior Algebras*

For a finite dimensional vector space $W$, the algebras $T(W)$ and $S(W)$ are (cocommutative) Hopf algebras with definitions analogous to those given in §3.1.4, that example being a special case of both of these.

For the tensor case, we note that the linear map

$$W \longrightarrow T(W) \otimes T(W)$$

given by:

$$w \mapsto 1 \otimes w + w \otimes 1 \tag{3.1.8.1}$$

may be extended by the universal property to an algebra map

$$T(W) \longrightarrow T(W) \otimes T(W)$$

that is easily seen to be coassociative, with counit as given. The antipode $\eta_T$, being an algebra anti-isomorphism, is given, for $w_i \in W$, by

$$\eta_T(w_1 \otimes \ldots \otimes w_n) = (-1)^n w_n \otimes \ldots \otimes w_1 \tag{3.1.8.2}$$

A similar argument may be made in the case of $S(W)$.

For the exterior algebra case we must proceed with caution. It was noted in §2.2.3 that the vector space isomorphism equation (2.2.3.10), which we shall now label, namely,

$$\Phi: E(V \oplus W) \cong E(V) \otimes E(W), \tag{3.1.8.3}$$

does not preserve the algebra structure, where the ordinary tensor product multiplication, namely that given by (cf. §3.1.1)

$$(a\otimes b)(c\otimes d) = ac\otimes bd, \tag{3.1.8.4}$$

is understood on the right-hand side of equation (3.1.8.3). This right-hand side admits, however, an algebra product for which $\Phi$ *does* preserve the algebra structure. This is the *graded* product, given by

$$(a\otimes b)(c\otimes d)=(-1)^{\deg(b)\deg(c)}(ac\otimes bd), \tag{3.1.8.5}$$

where the degree $\deg(f)$ of a homogeneous element $f$ is the power of the exterior product it belongs to.

To see this, note that equation (2.2.3.9) applied with $p=1$ gives the effect of the isomorphism $\Phi$ on the degree 1 elements $V\oplus W\cong(k\otimes W)\oplus(V\otimes k)$ of $E(V\oplus W)$: namely,

$$(v,w)\mapsto 1\otimes w+v\otimes 1. \tag{3.1.8.6}$$

With the algebra product on $E(V)\otimes E(W)$ being given as in equation (3.1.8.5) we have

$$\begin{aligned}(1\otimes w+v\otimes 1)^2 &= (v\otimes 1)(1\otimes w)+(1\otimes w)(v\otimes 1)\\ &= (-1)^0(v\otimes w)+(-1)^1(v\otimes w)\\ &= 0.\end{aligned} \tag{3.1.8.7}$$

Thus, by the universal property of the exterior algebra, there is a unique extension of the map of $V\oplus W$ into $E(V)\otimes E(W)$ given by (3.1.8.6) to $E(V\oplus W)$ that is an algebra map (for the graded product). One then observes (after a little work) that this extension coincides with $\Phi$, proving the assertion.

Exterior algebras have a coalgebra structure (similar to that of tensor and symmetric algebras), that is preserved by $\Phi$, with the or-

dinary tensor coproduct on the right-hand side of equation (3.1.8.3) (cf. §3.1.2). To see this, we first define the coproduct on $E(V)$. Note first that any linear map $f:V\longrightarrow W$ composed with $\iota_W:W\longrightarrow E(W)$ (diagram (2.2.3.3)) yields a map $\iota_W\circ f:V\longrightarrow E(W)$. By the universal property this extends to a unique algebra map $E(V)\longrightarrow E(W)$ rendering the obvious diagram commutative. Denoting this map by $E(f)$, we obtain a covariant functor $E(\ )$ from the category of vector spaces (over the relevant field) to a certain category of algebras. (Similar considerations apply in the cases of $T(\ )$ and $S(\ )$ of course.)

It is easily verified that for any appropriate $f$, $g$:

$$E(f)\otimes E(g)\circ\Phi=\Phi\otimes\Phi\circ E(f\oplus g) \qquad (3.1.8.8)$$

for the appropriate $\Phi$s. (The reader is encouraged to draw the diagram.)

For a vector space $V$ the *diagonal* map

$$\Delta_V:V\longrightarrow V\oplus V \qquad (3.1.8.9)$$

is defined by

$$\Delta_V(v)=(v,v). \qquad (3.1.8.10)$$

It induces an algebra map

$$E(\Delta_V):E(V)\longrightarrow E(V\oplus V)\cong E(V)\otimes E(V) \qquad (3.1.8.11)$$

(with the graded product on $E(V)\otimes E(V)$) whose coassociativity follows from the corresponding diagram for the $\oplus$-coassociativity of $\Delta_V$ in the category of vector spaces, and the properties of $E(\ )$. Consequently, with this map, denoted $\psi_V$, and counit induced by the map $V\longrightarrow\{0\}$, $E(V)$ becomes a cocommutative coalgebra. Note that for $v\in V$,

$$\psi_V(v)=1\otimes v+v\otimes 1. \qquad (3.1.8.12)$$

That $\Phi$ is a coalgebra morphism for this coproduct now follows from the identification:

$$\Delta_{V\oplus W}=(1\oplus t\oplus 1)(\Delta_V\oplus\Delta_W), \qquad (3.1.8.13)$$

where $t:V\oplus W\longrightarrow W\oplus V$ interchanges summands, and the properties of $E(\ )$.

One may now observe that the algebra product on $E(V)$ is actually induced by the addition map $V\oplus V\longrightarrow V$, with unit induced by the map $\{0\}\longrightarrow V$. Thus $E(V)$ satisfies the definition of a Hopf algebra, with antipode induced by $v\mapsto -v$, if the algebra structure on tensor products of (graded) algebras is taken in the graded sense of equation (3.1.8.5). That is to say, $E(V)$ is a *graded* Hopf algebra.

This Hopf algebra has an unusual property: it is self-dual. That is to say, the linear duals of the product and unit maps for the algebra structure on $E(V)$ are precisely the coproduct and counit, respectively, for the coalgebra structure on $E(V^*)\cong E(V)^*$, while the linear duals of the coproduct and counit for the coalgebra structure on $E(V)$ are precisely the product and unit for the algebra structure on $E(V^*)\cong E(V)^*$. This follows from the observation that the linear dual of $\Delta_V$ is precisely the addition map $V^*\oplus V^*\longrightarrow V^*$ and that the linear dual of the addition map $V\oplus V\longrightarrow V$ is precisely the diagonal map $\Delta_{V^*}$. Similar remarks apply to the other structural maps defining units, counits, etc.

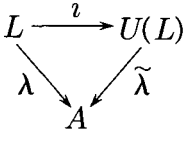
### *3.1.9 The Universal Enveloping Algebra of a Lie Algebra*

Suppose $A$ is an (associative) unital algebra over $\mathbb{C}$. Then $A$ becomes a Lie algebra with respect to the operation of taking the commutator of a pair of elements. That is, for $a,b\in A$,

$$[a,b] \equiv ab - ba \tag{3.1.9.1}$$

gives $A$ the structure of a (complex) Lie algebra.

Now suppose that $L$ is a Lie algebra, always assumed complex until further notice. Then there exists a unital algebra $U(L)$ and a map $i : L \longrightarrow U(L)$ of Lie algebras (with the commutator Lie structure on $U(L)$) such that, if $\lambda : L \longrightarrow A$ is a map of Lie algebras (with the commutator Lie structure on $A$ always assumed), there exists a unique map $\tilde{\lambda} : U(L) \longrightarrow A$ of unital algebras making commutative the diagram:




(3.1.9.2)

Clearly, $U(L)$ is uniquely determined (up to algebra isomorphism) by this universal property and is called the *universal enveloping algebra* of $L$. Its usefulness resides in the fact that representations of the Lie algebra $L$ correspond one-to-one with representations of the associative algebra $U(L)$. (Knapp 1988, Abe 1977.)

A deep theorem of Poincaré, Birkhoff and Witt (PBW) asserts that $i$ is injective and that its image generates $U(L)$ as an algebra.

Note that for Lie algebra maps such as $\lambda$, if $[a,b]=0$, then $\lambda(a)$ *commutes* with $\lambda(b)$.

In fact $U(L)$ is a Hopf algebra. This is deducible entirely from its universal property. One notes first that for two Lie algebras $L$ and $M$, with componentwise Lie product, the map

$$i : L \oplus M \longrightarrow U(L) \otimes U(M)$$

defined by

$$i(l,m) = i_L(l) \otimes 1 + 1 \otimes i_M(m), \tag{3.1.9.3}$$

(where the subscripted "$i$"s are self explanatory) determines an algebra isomorphism:

$$U(L\oplus M)\cong U(L)\otimes U(M). \tag{3.1.9.4}$$

To show this, we observe first that the map (3.1.9.3) is a map of Lie algebras. Now let $F:L\oplus M\longrightarrow A$ be any map of Lie algebras and define Lie algebra maps $F_L:L\longrightarrow A$, $F_M:M\longrightarrow A$, respectively by $F_L(l)=F((l,0))$, and $F_M(m)=F((0,m))$. Then define

$$\widetilde{F}:U(L)\otimes U(M)\longrightarrow A$$

by

$$\widetilde{F}\equiv \pi_A\circ(\widetilde{F}_L\otimes\widetilde{F}_M) \tag{3.1.9.5}$$

where $\pi_A$ denotes the product map $A\otimes A\longrightarrow A$ of $A$. It is easily checked that $\widetilde{F}\circ i = F$. To show that $\widetilde{F}$ is an algebra map, despite the fact that $A$ need not in general be commutative, it suffices to show that it acts multiplicatively upon generators of $U(L)\otimes U(M)$, which are given by the PBW theorem applied to each factor. That $\widetilde{F}$ is indeed multiplicative upon generators follows, since this condition reduces to the requirement that elements of the form $F(l,0)$ commute (in $A$) with elements of the form $F(0,m)$. This they do, since $[(l,0),(0,m)]=0$.

A similar application of the PBW theorem serves to establish the relevant uniqueness of $\widetilde{F}$. In this connection note that $i_L(l)\otimes i_M(m)=i((l,0))i((0,m))$.

That the map (3.1.9.3) induces an algebra isomorphism (3.1.9.4) now follows from the uniqueness up to isomorphy of the universal object.

Noting, again from the universal property, that

$$U(0)\cong\mathbb{C} \tag{3.1.9.6}$$

and that $U(\ )$ is a covariant functor, $U(L)$ acquires a (cocommutative) Hopf algebra structure with coproduct induced from the diagonal $L \longrightarrow L \oplus L$, etc., as in §3.1.8. The antipode is the "principal anti-isomorphism" induced by the map $L \longrightarrow L^{op}$ given by $l \mapsto -l$. (Sweedler 1969, Abe 1977.) In general the antipode is given by the analog of equation (3.1.8.2).

A beautiful characterization of these Hopf algebras is attributed by Sweedler to Kostant (Sweedler 1969, Chapter XIII). Namely, a Hopf algebra is the universal enveloping Hopf algebra of a Lie algebra if and only if it is cocommutative and contains a unique simple subcoalgebra. (Cf. Abe 1977, §5.2, where the result is unattributed.)

If the Lie algebra in question is the Lie algebra $\mathfrak{g}$ of a Lie group $G$, then, unsurprisingly, $U(\mathfrak{g})$ may be related to entities tied to $G$. For example, *very* roughly speaking, let $H$ denote the local algebra of germs of analytic functions at the identity of an analytic group $G$. Then $H$ inherits a sufficiently Hopf-like structure from the global group operations of $G$ that its continuous linear dual in a certain topology carries a legitimate cocommutative Hopf algebra structure. This dual is essentially identifiable with $U(\mathfrak{g})$. (See Serre 1965 for the precise context.) That is, $U(\mathfrak{g})$ may be realized as a sort of localized version of the (measure-like) Hopf algebra dual to a Hopf algebra of functions on the group.

Now these enveloping Hopf algebras admit a deformation theory, *qua* Hopf algebras, yielding a richer structure than the deformation theory of Lie groups, which are rather rigid objects: for instance, one obtains noncocommutative coalgebras. These deformed Hopf algebras have acquired the name "quantum groups," an appellation they seem to share with other similar objects. They are not entirely quantum entities, however, since the compatibility between the coalgebra and the algebra structures (cf. §3.1.3) is still assumed. This bialgebra structure is a vestige of the compatibility between the algebraic and topological (or differentiable) structures in the original underlying classical object, and need not be present in a complete "quantization."

## 3.2 Quantum Versions of Cyclic Groups

As noted above, we shall approach this topic from opposite (and independent) directions, using analogies derived respectively from the q-set theory of Chapter 2 to go in the direction: (certain) q-set symmetries $\longrightarrow$ (classical) cyclic actions (§3.2.1), and from a consideration of Hopf-like structures, to go in the opposite direction (§3.2.3).

### *3.2.1 Quantum Permutations: from $\mathfrak{sl}(n,\mathbb{C})$ to $\mathbb{Z}_n$*

First, some terminology. Let $A$ denote an associative algebra and $M$ an $A$-bimodule. A *derivation* $d : A \longrightarrow M$ is a linear map satisfying:

$$d(ab) = da.b + a.db, \tag{3.2.1.1}$$

where the dot denotes the action of $A$ upon the bimodule $M$. (We shall revisit this notion in a little more detail in §9.2.)

Now let $W$ denote a vector space of dimension $n$. Then the following result is not hard to prove. Any $f \in \mathrm{End}W$ may be extended uniquely to a derivation $d_f : E(W) \longrightarrow E(W)$ that is of degree 0: i.e. $d_f(\bigvee^k W) \subset \bigvee^k W$, $k = 0,\ldots,n$. (See Chevalley 1956, Chapter V, §11.) Thus, in particular, for a homogeneous element of the form $w_1 \vee w_2 \vee w_3 \ldots$, we have

$$d_f(w_1 \vee w_2 \vee w_3 \ldots) = f(w_1) \vee w_2 \vee w_3 \ldots + w_1 \vee f(w_2) \vee w_3 \ldots + \ldots \tag{3.2.1.2}$$

Since $\bigvee^n W$ is one dimensional, $d_f$ reduces on it to multiplication by a constant. Choosing a basis in $W$ and expressing $f$ as a matrix relative to it, this constant is easily seen to be $\mathrm{tr}f$. (In Chevalley 1956, Chapter V, §12, this is taken as the definition of the trace.)

If $W$ is the initial space for a quantum system, then this observation may be given the following q-set interpretation. *Each* act $f \in \mathrm{End}W$ of selection by the experimenter among basic initial qets induces an "infinitesimal increment" or "small fluctuation" $d_f$ upon the *entire* system of qets $E(W)$—a typically quantum phenomenon, or "effect." (Classical selective acts upon classical sets do not necessarily induce changes upon the whole Boolean algebra of subsets.) In particular, the initial value of the small fluctuation induced by a selective act $f$ upon the "whole" qet $\bigvee^n W$, namely the eigenvalue of $d_f : \bigvee^n W \longrightarrow \bigvee^n W$, is precisely $\mathrm{tr}f$. Consequently, those $f$s for which $\mathrm{tr}f = 0$ correspond to selective acts with respect to which the whole qet $\bigvee^n W$ remains in a certain sense *rigid*: its initial "configuration" remains impervious to the small fluctuations induced by $f$.

What is the classical set-theoretic correspondent of this phenomenon? The trace free $f$s represent selective acts that rigidly maintain the structure of the qet $\bigvee^n W$. This latter qet corresponds to the whole classical set $S$, say, of $n$ elements, and the "selective acts" maintaining the structure of $S$ are just the set isomorphisms, or *permutations*, of $S$. So the transformations of $S$ corresponding to the trace free $f$s are the permutations of $S$ with respect to which $S$ remains, in a sense, rigid: the whole set should be permuted by them *as a unit.*

Now it is an elementary result that any permutation may be factored into a product of cyclic permutations, acting respectively upon disjoint subsets of the given set $S$. Thus the subcycles of a given permutation effect a partition of $S$ into disjoint subsets, and each of these subsets is mapped upon itself by a cyclic action. Therefore, a non-trivial permutation maps the whole set upon itself *as a unit* precisely when the number of its disjoint subcycles is as small as possible—namely, when there is only one: $S$ itself. Then the whole set is acted upon cyclically by the permutation. So the permutations we are seeking as classical analogs of trace free endomorphisms of $W$ are precisely the cyclic permutations on $n$ objects. In other words, the cyclic group $\mathbb{Z}_n$ may be regarded as the classical analog of the space $\mathfrak{sl}(n,\mathbb{C})$ of trace free endomorphisms of an $n$ dimensional vector

space, that happens also to be a Lie algebra.

### *3.2.2 Condensation and Coherence*

There is a very general relationship between self-derivations and automorphisms of an algebra. One way of viewing this relationship is to regard such a derivation as the "infinitesimal generator" of a one-parameter family of automorphisms. For an algebra $A$, such a one-parameter family may be expressed as the formal series

$$\Phi_t = 1_A + t\varphi_1 + t^2\varphi_2 + \ldots \qquad (3.2.2.1)$$

Then the condition for $\Phi_t$ to be an algebra map forces $\varphi_1$ to be a derivation. Moreover, given a derivation, $D$ say, $\Phi_t \equiv e^{tD}$ is a one-parameter family of automorphisms with $\varphi_1 = D$.

Alternatively, in case $A$ is finite dimensional, one may note that the set $\mathrm{Der}A$ of derivations of $A$ into itself is a Lie subalgebra of the Lie algebra $\mathrm{End}A$. This Lie algebra is in fact the Lie algebra of a group, namely the group of automorphisms of $A$, the automorphism corresponding to a derivation $D$ being $e^D$ (cf. Jacobson 1962).

Thus, if $D$ is a derivation the automorphism $e^D$ is, in a sense, its globalization.

Specializing to the case at hand, in which $A = E(W)$, we note that for $f \in \mathrm{End}W$,

$$e^{d_f}(\theta \vee \phi) = e^{d_f}(\theta) \vee e^{d_f}(\phi), \qquad (3.2.2.2)$$

since $e^{d_f}$ is an automorphism.

If the unique derivation $d_f$ on $E(W)$ that extends $f \in \mathrm{End}W$ is interpreted, as above, as a small quantum increment induced on each element of $E(W)$, presumably $e^{d_f}$ should be interpreted as a corresponding *large scale* increment induced on qets, that distributes over disjoint union according to equation (3.2.2.2). This accords with intuition, since a large scale increment, or enlargement,

of a disjoint union of sets should be the union of the individually enlarged sets.

Thus, we conclude that operators of the form $e^{d_f}$ are the appropriate ones for *macroscopic* experimenters to implement the quantum permutational effects of selective acts upon the whole qet algebra $E(W)$. In particular, when this operator is restricted to $W$ itself, we obtain $e^f$. In a sense the latter operator represents a *coarsening* of the operator $f$: if microscopic experimenters use $f$ to implement a quantum permutation, then macroscopic experimenters are bound to use $e^{d_f}$, that reduces to $e^f$ on $W$. (Such quantum permutations will be interpreted later as "allowable" changes of the experimenter's frame.)

Thus, in the case at hand, namely $f \in \mathfrak{sl}(n,\mathbb{C})$, we obtain as the appropriate family of operators implementing the quantum analog of $\mathbb{Z}_n$ actions for macroscopic experimenters, the (Lie) *group* $\mathrm{SL}(n,\mathbb{C})$. (The group structure follows from the Lie algebra structure $\mathfrak{sl}(n,\mathbb{C})$ happens to possess.)

The appearance of the exponential of an operator suggests the physical process of coherent state formation in many-particle systems. Often, in such systems, long-range order may be established. Typically, the long-range order parameter is an expectation value of some fundamental field operator: this occurs, for example, in theories of phonon interaction, ferromagnetism, superfluidity and superconductivity. The superconducting transition, for instance, may be regarded as the establishment of a macroscopic expectation value for the Cooper pair field operator. ( The BCS theory of superconduction will be described briefly in §12.3.)

These ordered states, which are often superpositions of many-particle condensates (and therefore do not contain a definite number of particles), do indeed exhibit macroscopic behaviors, thereby justifying the use of the otherwise objectionable word "state." This phenomenon is exemplified most simply by taking as "fundamental field" operators the annihilation operators $a_k$ in the Fock space of a Bose system of quanta of $n$ kinds. Then we seek states $\eta$ that maximize each "long-range order parameter" $\langle \eta | a_k | \eta \rangle$. These are precisely the vectors $\eta$ satisfying

$$a_k \eta = z_k \eta \tag{3.2.2.3}$$

for some complex $z_k$, which are called *coherent states*. They may be given, up to normalization, by

$$\eta = [\exp(\sum_k z_k a_k^\dagger)]|0\rangle \tag{3.2.2.4}$$

where $|0\rangle$ denotes the Fock vacuum. One of their notable properties is that they are the states "closest" to being classical. (There are several different ways of expressing these vectors: see QR, §8.3, Feng, Gilmore and Zhang 1990 and Blaizot and Ripka 1986.) Note the vectorlike character of the exponent.

It is a presumed transition to this sort of long range coherence, giving rise to macroscopic vectorlike states, that Finkelstein refers to in the passage quoted in §2.3, and it is the specific example of superconduction that inspired the original construction of a quantum net to be given in §7.2. The precise dynamical nature of the transition to such a state remains hidden from us at the level of our formulation. However, we shall find other, geometrical and logical routes to this paradigm in Chapters 7 and 8.

Returning to the group context, it now appears that at least certain Lie group actions may be thought to cohere macroscopically, in a similar fashion, out of infinitesimal "creation" operators, namely their Lie algebra generators.

### *3.2.3 Quantizing Cycles: from $\mathbb{Z}_n$ to $\mathfrak{sl}(n,\mathbb{C})$*

In this subsection we shall attempt to produce an argument going in the direction opposite to the one taken in §3.2.1. Namely, starting from a cyclic group of order $n$, we shall attempt to produce a "quantum Lie algebra" analog of it.

We note first that a Lie algebra is recoverable from the Hopf structure of its universal enveloping algebra as the (Lie algebra) of

*primitive* elements: namely, those elements satisfying:

$$\psi(l) = 1 \otimes l + l \otimes 1. \tag{3.2.3.1}$$

After attempting to find a quantum analog of a universal enveloping algebra, this criterion cannot be expected to be useful for extracting a corresponding Lie algebra analog, since a quantum coproduct can no longer be expected to reflect underlying (local) topological or differentiable structure, that does not exist in the quantum case. Instead, we note that a Lie algebra $L$ is contained in the kernel of the counit of its universal enveloping algebra. This follows from the last equation, but may also be seen directly from the following diagram:

$$\begin{array}{ccc} & \overset{i}{} & \\ L & \longrightarrow & U(L) \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & U(0) \cong \mathbb{C} \\ & \underset{i}{} & \end{array} \tag{3.2.3.2}$$

in which the right-most vertical map is the counit and the horizontal maps are one-to-one, by the Poincaré–Birkhoff–Witt theorem. So, once we have concocted a quantum version of the enveloping Hopf algebra, we should look at what Lie subalgebras the kernel of the counit may contain. Since we are aiming to capture all available infinitesimal symmetries of the quantum system, we should presumably choose the largest of these Lie subalgebras.

Returning to a cyclic group $G$ of order $n$, our first task is to find an analog of the local structure "around" a typical element, that is to play the rôle of local (Hopf) algebra of germs of analytic functions at a point (available in the analytic Lie case): then we must quantize it. In order to specify a "local" structure we must first identify a global structure. To this end, we note that any finite group of order $n$ may be embedded as a subgroup of the group of permutations, denoted $S_n$, on its underlying set, simply by allowing each element to act by multiplication. Writing $G = \{g, g^2, \ldots, g^n\}$, the permutation induced in this way by the generator $g$ is, of course, a cycle,

and may be decomposed further in $S_n$: namely, as a product $\tau_n\tau_{n-1}\ldots\tau_1$, where $\tau_k$ is the *transposition* that interchanges $g$ with $g^k$, while leaving all the other elements fixed. Thus the set $T = \{\tau_1,\ldots,\tau_n\}$ may be thought to parametrize the cyclic action $g$, and therefore every other element of $G$ within $S_n$. Note also that $\tau_k = \tau_k^{-1}$ in $S_n$ for each $k$: that is, inversion in $S_n$, when restricted to $T$, coincides with the identity function on $T$.

If these $\tau_k$ are thought of as local parameters of $G$, then the commutative C*-algebra $C(T)$ may be thought to represent a replacement for the algebra of functions in a "neighborhood" of a point. Its dual should then provide an analog of the universal enveloping algebra. But we have a candidate for the quantum version of the algebra $C(T)$, namely End$W$ with its usual multiplication, where $W$ is of dimension $n$ (and may be thought of as $L^2(T)$ taken relative to counting measure on $T$). Therefore we have a candidate for the coalgebra structure of the quantum enveloping algebra, namely, End$W$ with the dual noncocommutative coproduct $\psi$ given by equation (3.1.2.7), a counit for which is just the trace, whose kernel is $\mathfrak{sl}(n,\mathbb{C})$, itself a Lie algebra.

Thus, we would be led again to $\mathfrak{sl}(n,\mathbb{C})$ as an infinitesimal quantum replacement for $\mathbb{Z}_n$, if End$W$ is found to possess an analog of the algebra product reflecting the algebraic structure of the underlying "local" set $T$, in analogy with the enveloping algebra of the Lie algebra of a Lie group, that acquires its product structure from the local algebraic structure of the underlying group. This (unital) product, $\pi$ say, on End$W$, should capture the algebraic property of $T$ mentioned earlier—namely that the identity map on $T$ coincides with inversion (in $S_n$) restricted to $T$—since this condition may be expressed in non-objective terms, and therefore should survive quantization. In fact, the condition is that the identity function on the algebra should act as an antipode:

$$\pi\circ(1\otimes 1)\circ\psi = I\circ\mathrm{tr}, \qquad (3.2.3.3)$$

where $I$ denotes the unit for $\pi$ (cf. equation (3.1.3.1)). As noted

(§3.1.8), the product $\pi$ need not be compatible with the coproduct $\psi$ in the sense of bialgebras since this compatibility in the classical case is a vestige of an underlying topological *object*, that we must relinquish upon quantization.

Now, Finkelstein observed that the *opposite algebra* structure on End$W$, namely $\pi(a\otimes b) = ba$, satisfies this requirement. For, with $v_{ij}$ a matrix basis as in equation (3.1.2.7), we have:

$$\begin{aligned}(\pi \circ (1\otimes 1) \circ \psi)(v_{ij}) &= (\pi \circ \psi)(v_{ij}) \\ &= \pi(\sum_k v_{ik}\otimes v_{kj}) \\ &= \sum_k v_{kj}v_{ik} \\ &= \sum_k \delta_{ji}v_{kk} \\ &= \mathrm{tr}(v_{ij})I(1) \\ &= (I \circ \mathrm{tr})(v_{ij}). \qquad (3.2.3.4)\end{aligned}$$

Thus, End$W$ equipped with $\psi$ is a noncocommutative counital simple coalgebra (having no proper subcoalgebras), and equipped with $\pi$ it is a simple noncommutative unital algebra (having no proper two-sided ideals). These two structures are not compatible in the sense of bialgebras, but meet in equation (3.2.3.3). It is as if some classical proto-Hopf algebra has bifurcated or decoupled into separate algebra and coalgebra structures upon quantization, while the coalgebra and algebra structures have each localized, or become irreducible, in their respective categories. This seems fitting behavior for a quantum grouplike entity.

We note that a product $\mathbb{Z}_m \times \mathbb{Z}_n$, say, of cyclic groups, may be

parametrized as above by the transpositions determined by a pair of generators, one from each group. Thus, the product group is parametrized by the disjoint union $T_m \cup T_n$ of the respective sets of transpositions, leading to the algebra $C(T_m \cup T_n) \cong C(T_m) \oplus C(T_n)$ and thence, upon quantization, to $\mathrm{End}W^{(m)} \oplus \mathrm{End}W^{(n)}$. Thus, the appropriate infinitesimal quantum version of $\mathbb{Z}_m \times \mathbb{Z}_n$ is $\mathfrak{sl}(m,\mathbb{C}) \oplus \mathfrak{sl}(n,\mathbb{C})$. One may also argue, in the spirit of similar discussions in Chapter 2, that the appropriate quantum algebra $\mathfrak{H}$ for the product $\mathbb{Z}_m \times \mathbb{Z}_n$ should accommodate interpretations of the individual algebras $\mathfrak{sl}(m,\mathbb{C})$ and $\mathfrak{sl}(n,\mathbb{C})$. That is, there should exist a diagram of the form:

$$\mathfrak{sl}(m,\mathbb{C}) \longrightarrow \mathfrak{H} \longleftarrow \mathfrak{sl}(n,\mathbb{C}). \qquad (3.2.3.5)$$

Then,

$$\mathfrak{H} \cong \mathfrak{sl}(m,\mathbb{C}) \oplus \mathfrak{sl}(n,\mathbb{C}) \qquad (3.2.3.6)$$

is the optimal universal solution to this diagram, in the sense that any other factors through it.

Finally we note that for the trivial group, $\mathfrak{sl}(1,\mathbb{C}) = \{0\}$, and that the coherent form of this is again the trivial group. Thus, the trivial group coincides with its quantum version, in our sense.

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# Part II

# Computational Paradigms

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# 4

# Natural Deduction

Our ultimate goal is to discover where the fundamental Lagrangians "come from." That is to say, we aim to find some sort of algorithmic or calculational procedure that will eventually produce these expressions from more primitive ones, whose provenance is, ideally, axiomatic, or at least more readily apprehended than the Lagrangians themselves. A minimum requirement for such a program is a set of rules that prescribe how expressions may be legally reduced to other expressions, and a set of expressions with which to start the process. Now, a prescribed set of expressions, together with a set of rules for manipulating or rewriting them, comprises the backbone of what is known as a *deductive system.* The study of such systems has come to occupy a significant sector of the modern theory of computation.

A *deduction* (or *derivation*) in such a system is a sequence of rule-based replacements (or rewrites) of expressions starting from a set specified as *axioms.* One may view such a deduction geometrically in various ways: as tree-like, for example, with axioms as leaves and the concluding expression as the root. (Much mileage is obtained from the examination of such tree-like structures in the formal theory.) Although we have been vague about the nature of the "expressions" involved, it should already be clear that a deduction is very much like a (computer) program, which proceeds in steps to reconfigure patterns of data.

The expressions of interest are, of course, those to be found at the roots of deductions and it is important to remark on the obvious fact that these are produced by entirely *constructive* processes. A derived expression may be specified—in the sense that it may be *constructed*—from the axioms together with the particular deduction tree at whose root it sits. Clearly, this association (of derived expression with deduction tree) is not one-to-one, since a given expression may have many deductions (or, indeed, none). From a constructivist view-point it would be better to associate a derived expression with the *set* of deductions leading to it. This kind of association lies at the heart of Heyting's interpretation of intuitionistic logic as that logic which arises from a wholesale adherence to constructivist principles: cf. §§4.1 and 4.2.

(Insofar as we deal with logic *per se* in this work we shall deal only with *propositional* logic: that is, we ignore quantification ($\forall$, $\exists$) entirely. However, there is no doubt that a full treatment along the lines to be advocated in this work should include quantification: cf. QR.)

In this chapter we briefly discuss a standard minimal deductive system and some of the issues that arise out of it.

## 4.1 Natural Deduction for a Minimal System

In this section we informally explore some of the issues associated with deduction by examining a certain system known as *natural deduction.* Specifically, we shall discuss the natural deduction system for minimal implicational intuitionistic (propositional) logic. This treatment combines elements from the early chapters of both Girard *et al.* 1988 and Troelstra and Schwichtenberg 2000.

The basic object of interest in this system is a *deduction* of a *formula* (or *sentence*) $A$, say, which, after Girard *et al.* 1988, we shall denote by:

$$\begin{array}{c} \vdots \\ A \end{array} \qquad (4.1.1)$$

The dots stand for subdeductions, and the whole structure is to be regarded as a finite tree, or at least as being tree-like, since the tree structure will soon be vitiated.

The first rule of deduction, or *inference,* is that a single formula by itself is a deduction (of itself). Strictly speaking, this axiom should be asserted only for a set of *atomic* formulae: the result then follows for all formulae. We will follow custom in this abbreviated overview by omitting the complication of specifying the atoms at this stage.

There are two other rules of inference, which enable new deductions to be constructed from old ones. One rule *introduces* the implication sign $\Rightarrow$ and the other rule *eliminates* it. The expression of these rules requires some notational preliminaries. Suppose $A$ appears in a *single* top node (or *leaf*) of a deduction whose *conclusion* is $B$. Then we may unambiguously write:

$$\begin{array}{c} A \\ \vdots \\ B \end{array} \tag{4.1.2}$$

In this case, the rule of introduction posits the new deduction:

$$\frac{\begin{array}{c} A \\ \vdots \\ B \end{array}}{A \Rightarrow B}\ \Rightarrow I \tag{4.1.3}$$

(Here, the $\Rightarrow I$ labels the rule being used—namely "$\Rightarrow$ introduction"—to extend the tree: it is frequently dropped when ambiguity does not threaten.)

The *occurrence* of $A$ is said to be *open* (or *live*) in (4.1.2) but considered to be *closed* (or *killed,* or *discharged*) by the application of $\Rightarrow I$ in (4.1.3). The open occurrences of a formula like $A$ in (4.1.3) are said to be *hypotheses* for the deduction.

Now, $A$ may appear and be open in other places, for instance in ambient deductions, and in this case we would wish to keep track of which open occurrence of $A$ is being discharged at the $\Rightarrow I$ inference. This can be accomplished by labelling $A$ and then invoking the label at the point of inference. Thus, in place of (4.1.3) we now write:

$$\frac{\begin{array}{c} A^u \\ \vdots \\ B \end{array}}{A \Rightarrow B}\, u, \Rightarrow I \qquad (4.1.4)$$

This intuitionistic version of implication may be interpreted in light of the so-called *Heyting paradigm,* (Heyting 1956) which gives a semantics for formal intuitionistic logic (**IL**). (Cf. Troelstra and Schwichtenberg 2000, §2.5.1, p.55, where the attribution also includes Brouwer and Kolmogorov.) In this interpretation of **IL**, a formula is intuitionistically *valid* only if a deduction can be explicitly presented or constructed. The interpretation of $A \Rightarrow B$ in (4.1.4) then becomes: if a deduction of $A^u$ can be constructed then a deduction of $B$ can be constructed, *via* the deduction above the inference line in (4.1.4). After this encapsulation of the whole process in the formula $A \Rightarrow B$, the open assumption $A^u$ is no longer needed and may be discharged (or closed), the deduction leading to it being, in a sense, discarded.

Now, as noted, it is possible that open occurrences of $A$ may appear a number of times in the deduction leading to $B$, and we may choose to discharge a collection of these at the inference. The deductions leading to those occurrences of $A$ in the chosen collection are all then discarded simultaneously at the inference. Members of such a collection may be grouped under a single label, since there is no need to distinguish among these discarded deductions. The notation for such a collection of open occurrences of $A$ is $[A]^u$. Of course, there may be other collections of open occurrences of $A$ that are not chosen for discharge at the inference: these remain open after it.

The complete statement of the $\Rightarrow I$ rule now reads:

$$\dfrac{\begin{array}{c}[A]^u \\ \vdots \\ B\end{array}}{A \Rightarrow B}\ u, \Rightarrow I \tag{4.1.5}$$

(Here the degenerate case of $[A]^u$ being empty is allowed. This empty case would still require a label at the inference. Thus,

$$\dfrac{B}{A \Rightarrow B}\ v \tag{4.1.6}$$

is a legal deduction. The $v$ labels the empty class of occurrences, which is discharged at the inference.)

There is some linguistic awkwardness in referring to $[A]$ since it denotes a pattern of *occurrences* of the formula $A$ and is not, strictly speaking, a *set.*

The deduction (4.1.5) should be understood as follows. Given a deduction of $B$ with a (possibly empty) collection $[A]^u$ of open occurrences of $A$ among its set of hypotheses—a set which could include other open occurrences of $A$—a new deduction is formed whose conclusion is $A \Rightarrow B$, and in which the occurrences of $A$ in $[A]^u$ have been discharged (or closed). (Other open occurrences of $A$ are not discharged. That is to say, they remain open at the inference point.)

The other rule of inference in this system, which is a rule for eliminating $\Rightarrow$, is just *modus ponens,* and may be rendered as:

$$\dfrac{\begin{array}{cc}\vdots & \vdots \\ A & A \Rightarrow B\end{array}}{B} \Rightarrow E \tag{4.1.7}$$

Here, two deductions—of $A$ and $A \Rightarrow B$—are combined to produce

a new deduction with conclusion $B$. The hypotheses of the two subdeductions above the inference line, taken together, are the hypotheses of the new deduction (4.1.7).

There are natural ways to simplify certain deductions. For instance, a deduction of the form:

$$\dfrac{\begin{array}{cc} \vdots & \begin{array}{c}[A]^u \\ \vdots \\ B\end{array} \\ A & \dfrac{}{A\Rightarrow B}\,u \end{array}}{B} \tag{4.1.8}$$

may be replaced by the following simpler direct deduction, considered to be equivalent to it:

$$\begin{array}{c}\vdots \\ [A] \\ \vdots \\ B\end{array} \tag{4.1.9}$$

The understanding here is that each (discharged) occurrence of $A$ in $[A]^u$ (in (4.1.8)) has been replaced by a copy of the new deduction of $A$ introduced on the left (in (4.1.8)).

Note the pattern of discharges in the following two deductions of $A\Rightarrow(A\Rightarrow A)$ (from Troelstra and Schwichtenberg 2000, p.25):

$$\dfrac{\dfrac{A^w \quad \dfrac{A^u}{A\Rightarrow A}\,v}{A}}{\dfrac{A\Rightarrow A}{A\Rightarrow(A\Rightarrow A)}\,w}\,u \qquad\qquad \dfrac{\dfrac{A^w \quad \dfrac{A^u}{A\Rightarrow A}\,u}{A}}{\dfrac{A\Rightarrow A}{A\Rightarrow(A\Rightarrow A)}\,v}\,w \tag{4.1.10}$$

(Cf. (4.1.6) for the label $v$ in both cases.)

## 4.2 The Curry–Howard Isomorphism

In the previous section labels were introduced merely to keep track of the flow of closings of collections of open formulae as the $\Rightarrow I$ inference is enacted. (As Girard observes (Girard *et al.* 1988), the link this labelling scheme sets up between formula and inference point effectively destroys the illusion of a tree-like structure.)

The significance of this apparently innocent labelling scheme may be realized by another appeal to the Heyting paradigm. In this interpretation of **IL** a formula is intuitionistically *valid* only if a deduction of it can be produced. Thus, a formula may be *identified* with its set of deductions. In more formal terms, a formula determines a *type,* $A$ say, and a label $u$ of $A$ is considered to be a *variable* of type $A$, for which the standard notation is $u{:}A$. (Formal definitions of types, terms, variables, etc., may be found in the works cited above. For our purposes the informal intuitive notion of a type as being a special kind of set, while variables refer to elements of such sets, etc., will suffice.) Returning to the labelling scheme of the last section, we note that the label $u$ in $A^u$ could be regarded as standing in for a generic deduction of $A$: it is in fact not merely $A$ that is being labelled but a deduction of $A$. In view of the Heyting interpretation, $A^u$ can be rewritten as $u{:}A$. Similarly, the $u$ in $[A]^u$ stands in for generic deductions of the occurrences of $A$ in the collection $[A]$, which are all "discarded" simultaneously at the inference. Consequently, $[A]^u$ can be rewritten as $[u{:}A]$.

Now that $u$ is being regarded as a variable of type $A$, this status should be recorded at the point of inference in (4.1.5). Likewise, the variable of type $B$ corresponding to the deduction of $B$ which appears above the inference line in (4.1.5), and which "depends" upon the deduction of $A$ labelled by $u$, should also be explicitly annotated. Then, (4.1.5) may be rewritten as:

$$\dfrac{\begin{array}{c}[u{:}A]\\ \vdots\\ t{:}B\end{array}}{\lambda u.t{:}A\longrightarrow B} \tag{4.2.1}$$

Here, the symbol $\lambda$ serves to *bind* $u$ within $t$. The type $A\longrightarrow B$ is the indicated "function" type, which, in terms of sets, is the set of *functions* from $A$ into $B$. As noted in the last section, the Heyting paradigm interprets intuitionistic implication $A\Rightarrow B$ as a function from the set of deductions of the formula $A$ to the set of deductions of the formula $B$.

The expression $\lambda u.t$ is the name of the function (of type $A\longrightarrow B$) which produces $t$ upon the "input" of $u$.

Note also that the binding of $u$ within $t$ *via* the symbol $\lambda$ in the expression $\lambda u.t$ recapitulates *exactly* the discharging of the associated formula occurrences.

Similarly, the inference rule (4.1.7), which eliminates $\Rightarrow$, may be rewritten in type theoretic terms as:

$$\dfrac{\begin{array}{cc}\vdots & \vdots\\ s{:}A & t{:}A\longrightarrow B\end{array}}{ts{:}B} \tag{4.2.2}$$

where $ts$ denotes *application* of the function type $t$ to $s$.

Using these translations of the inference rules, any deduction may be used to generate a "$\lambda$-term" which completely describes, or encapsulates, the deduction. For example, the applications of these translations to the left-most deduction in (4.1.10) yields:

$$\dfrac{\dfrac{\dfrac{\begin{array}{cc}w{:}A & \dfrac{u{:}A}{\lambda v.u{:}A\longrightarrow A}\end{array}}{(\lambda v.u)w{:}A}}{\lambda u.(\lambda v.u)w{:}A\longrightarrow A}}{\lambda w.(\lambda u.(\lambda v.u)w){:}A\longrightarrow(A\longrightarrow A)} \tag{4.2.3}$$

The reader may check that the translation of the right-most deduction in (4.1.10) yields the non-equivalent $\lambda$-term:

$$\lambda v.(\lambda w.(\lambda u.u)w):A\longrightarrow(A\longrightarrow A). \tag{4.2.4}$$

The calculus of $\lambda$-terms (without explicit typing) was posited independently by Church in the 1930s as a means of investigating the computational and logical possibilities of pure functionality. Today the theory goes by the name "simply-typed $\lambda$-calculus." The observation, by Curry (Curry and Feys 1958), that the translation given above induces a complete structural isomorphism between the minimal natural deduction system outlined in §4.1 and simply-typed $\lambda$-calculus, apparently came as a surprise to logicians.

Readers familiar with $\lambda$-calculus may note that the contraction of deduction (4.1.8) to deduction (4.1.9) corresponds to the replacement of an expression of the form $(\lambda u.t)s$ by the expression $t[u/s]$, where the notation means that $u$ is to be replaced by $s$ in $t$. This is known as $\beta$-conversion in the $\lambda$-calculus context (modulo many glossed details) and is the basic rule for evaluating functions.

The computational resources of simply-typed $\lambda$-calculus (and other $\lambda$-calculi) have been well studied: see for example Troelstra and Schwichtenberg 2000, Mitchell 1996, Gunter 1992, Asperti and Longo 1991, Girard *et al.* 1988 and Stoy 1977 among many others.

The isomorphism sketched above may be extended to one that obtains between the minimal intuitionistic implicational deductive fragment of §4.1 with inference rules for conjunction ($\wedge$) and disjunction ($\vee$) appended, and an appropriately supplemented version of simply-typed $\lambda$-calculus.

The inference rules to be appended to the natural deduction system are as follows. Conjunction is introduced through three rules of inference (one Introduction and two Eliminations), namely:

$$\frac{A \quad B}{A\wedge B}\wedge I \tag{4.2.5}$$

$$\frac{A \wedge B}{A} \wedge 1E \qquad \frac{A \wedge B}{B} \wedge 2E \tag{4.2.6}$$

There are identifications among certain deductions involving $\wedge$. For example:

$$\frac{\begin{array}{cc}\vdots & \vdots \\ A & B\end{array}}{\dfrac{A \wedge B}{A}} \quad \text{is identified with} \quad \begin{array}{c}\vdots \\ A\end{array} \tag{4.2.7}$$

and similarly for the other elimination rules.

Disjunction in an intuitionistic system is independent of conjunction (since De Morgan duality does not obtain) and is generally contentious, as we shall see. In our system there are two Introduction rules, namely

$$\frac{A}{A \vee B} \vee 1I \quad \text{and} \quad \frac{B}{A \vee B} \vee 2I \tag{4.2.8}$$

and one problematical Elimination rule, namely:

$$\frac{\begin{array}{ccc} & [A] & [B] \\ \vdots & \vdots & \vdots \\ A \vee B & C & C\end{array}}{C} \vee E \tag{4.2.9}$$

The problem here is the extraneous $C$, which introduces an uncontrollable element into the business of deriving general theorems about deductions: see Girard *et al.* 1988, Chapter 10.

To extend the Curry isomorphism to this supplemented natural deduction system, we again appeal to the Heyting paradigm. In order for the conjunction $A \wedge B$ to be intuitionistically valid, we must possess a deduction of $A$ *and* a deduction of $B$, and know which deduction belongs to which formula. That is, we must possess

an ordered pair of deductions. If a formula is identified with its set of deductions, then the set of deductions of $A \wedge B$ should be identified with the product of the set of deductions of $A$ and the set of deductions of $B$.

Thus, the $\wedge$ of formulae should be associated, in the extended correspondence, with the product, $\times$, of the corresponding types.

Similarly, $A \vee B$ is intuitionistically valid only if we have deduction of $A$ *or* a deduction of $B$, and an indication of *which* one of these formulae has been deduced. The collection of such pairs constitutes the disjoint union (or direct sum in the category of sets) of the sets of deductions of the constituent formulae.

Thus, the $\vee$ of formulae should be associated, in the extended correspondence, with the sum, $+$, of the corresponding types.

The Curry correspondence thus extended is part of W. A. Howard's contribution to the full isomorphism, which now bears the name Curry–Howard: cf. Troelstra and Schwichtenberg 2000, p. 59. (The other part of Howard's contribution to the isomorphism involves quantifiers, which we are ignoring here.)

The importance to computational theory of isomorphisms of the Curry–Howard type is that, since formulae may be regarded as types through their use, *deductions* may be concomitantly regarded as *computations* (or *programs*), which transform types (patterns of data) into types in stepwise fashion. Reversing this perspective, such isomorphisms allow us to regard the apparently static program represented by a $\lambda$-term in a dynamical light, since such a term may be unfolded to reveal the underlying deductive structure, with its flow of openings and closings of assumptions. It is this aspect of the Curry–Howard isomorphism that arguably has had the most impact.

## 4.3 The Gentzen Sequent Calculus

The Gentzen sequent calculus may be regarded initially as a metacalculus for handling deductions in natural deduction systems, though it has been developed in various directions as a style of deductive reasoning in its own right. In its guise as a metacalculus for

natural deduction, the sequent calculus delineates certain symmetries and structural aspects of the underlying deductive system which remain hidden, or at least less apparent, if one remains fixed at the natural deduction level. This organizing power of the style has had a major impact on the proof theoretic aspects of deductive logic.

The basic object is the *sequent:*

$$\Gamma \vdash \Delta \tag{4.3.1}$$

in which $\Gamma$ and $\Delta$ stand for (possibly empty) finite sequences of formulae. It is possible—and indeed advisable—to allow more general assemblages of formulae. This becomes apparent when natural deduction is used as the underlying model: then $\Gamma$, etc., would stand for collections of formula occurrences. The use of sequences will suffice for our purposes. (Upper case Greek characters will have this (standard) connotation in our discussions of sequents, risking possible confusion with their later use in other contexts: we beg the reader's indulgence in this matter of syntactic overload.)

The informal reading of (4.3.1) is along the lines of: "$\bigwedge\Gamma \Rightarrow \bigvee\Delta$." This reading can be adduced from the natural deduction model, if (4.3.1) is supposed to describe a deduction with a set $\Gamma$ of hypotheses and conclusion in $\Delta$: it forces the interpretation of $\vdash\Delta$ as asserting the truth of $\bigvee\Delta$ and $\Gamma\vdash$ as asserting the falsity of $\bigwedge\Gamma$. (Here we have adhered to the standard notational convention that the empty set, or null sequence, of formulae on either side of the turnstile is simply omitted.)

In keeping with this model, and noting again the disruptive effects of disjunction in intuitionistic systems, sequents in which $\Delta$ consists of at most a single formula are termed "intuitionistic."

In Gentzen calculi the inference rules are often divided into classes: structural rules, logical rules and an "identity group." A deduction in sequent calculus is usually referred to as a *proof.*

By way of example, we shall briefly describe the rules for a non-intuitionistic minimal propositional sequent calculus. (The horizontal line in a rule represents the inference of the sequent below it

from the sequent or sequents appearing immediately above it.)

### Structural Rules

These refer to the management of formulae within sequents. (The appropriate label appears to the right of the inference line, as in natural deduction: LE for left exchange, etc.)

#### Exchange

$$\frac{\Gamma, A, B, \Gamma' \vdash \Delta}{\Gamma, B, A, \Gamma' \vdash \Delta}\,\mathrm{LE} \qquad \frac{\Gamma \vdash \Delta, A, B, \Delta'}{\Gamma \vdash \Delta, B, A, \Delta'}\,\mathrm{RE} \qquad (4.3.2)$$

#### Weakening

$$\frac{\Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,\mathrm{LW} \qquad \frac{\Gamma \vdash \Delta}{\Gamma \vdash \Delta, A}\,\mathrm{RW} \qquad (4.3.3)$$

#### Contraction

$$\frac{A, A, \Gamma \vdash \Delta}{A, \Gamma \vdash \Delta}\,\mathrm{LC} \qquad \frac{\Gamma \vdash \Delta, A, A}{\Gamma \vdash \Delta, A}\,\mathrm{RC} \qquad (4.3.4)$$

These rules appear quite innocent at first sight: they are what one would expect from the presumed properties of $\wedge$ and $\vee$ in the informal reading of the sequent $\Gamma \vdash \Delta$ as "$\bigwedge\Gamma \Rightarrow \bigvee\Delta$." They appear less innocent in the reading of $\Gamma \vdash \Delta$ as a description of a deduction in a natural deduction system of the type described in the last section. In this reading, Weakening corresponds to the possibility of introducing spurious or null collections of occurrences of a formula $A$, while Contraction corresponds to the possibility of amalgamating certain collections of occurrences of $A$. Further innocence is lost, as Girard points out (Girard *et al.* 1988), in an *operational* reading of

the sequent calculus. In this reading, formulae, considered as types *à la* Curry–Howard, are regarded as *resources*, and $\Gamma \vdash \Delta$ has the informal interpretation: "Use up $\Gamma$ to produce $\Delta$." Then LC (4.3.4), for example, has the connotation that, while two $A$s are required to produce $\Delta$, we can get away with only one use of $A$ to effect the production of $\Delta$. The "resource" $A$ must then be *storable* and can be copied, or cloned, for reuse. One might say that $A$ *admits storage.* Clearly, many real resources, like coins, do not have this convenient property: if an item requires two coins for its purchase, then one will not suffice. (We will always use the term "Contraction" in this context so as not to confuse it with the term "contraction" used earlier: cf. diagram (1.3.2).)

(End structural rules.)

### The Identity Group

This terminology seems to be due to Girard (Girard *et al.* 1988).

#### Axiom

This is the analog of the first rule of inference for natural deduction, namely that a (well-formed) formula is by itself a deduction. The same provisos obtain: the axiom is properly stated only for atomic formula and then can be shown to obtain for general ones. Since we have continued to procrastinate on the issue of atomic formulae, we shall state the axiom in the customary form, to wit:

$$A \vdash A \ \text{Ax} \qquad (4.3.5)$$

CUT

$$\frac{\Gamma \vdash \Delta, A \qquad A, \Gamma' \vdash \Delta'}{\Gamma, \Gamma' \vdash \Delta, \Delta'} \text{CUT} \tag{4.3.6}$$

The CUT rule is an extremely reasonable meta-rule for the handling of natural deductions. Indeed, its natural deduction analog can be deduced from the other rules of natural deduction. The use of $A$ in this rule is akin to the use of a lemma in a mathematical proof, or the use of a subroutine in a computer program. In these forms, the CUT rule would seem to be part and parcel of both of these august disciplines, among others. It is, however, problematical from the point of proof theory itself, since the appearance and disappearance of the possibly extraneous and uncontrollable $A$ greatly complicates tree handling techniques. It may therefore come as a bitter-sweet surprise to learn that, even in very general Gentzen calculi, cuts can be removed from any proof. That is to say, any proof involving uses of CUT may be recast without using CUT. This is the gist of Gentzen's justly famous "Hauptsatz": cf. references already cited. This centrally important result is rather counterintuitive at face value since it seems to imply that the usual modes of proof—for instance in mathematics—are somehow redundant. In the programming analogy the removability of cuts seems more plausible: to "remove" the cuts—i.e. subroutine calls—from a program, compile it into runnable object code. Or, to put it more dynamically, *run* the program. This is, of course, simplistic, but encapsulates the main idea behind the proof.

Girard notes (*op. cit.*) that CUT is another way of expressing identity. To paraphrase him, the AXIOM asserts that $A$ on the left of the turnstile ($\vdash$) is "less than" $A$ on the right: resource $A$ on the left is "consumed by" the $A$ on the right. On the other hand, CUT asserts that if $A$ has been produced—appears on the right of the turnstile—then it may be consumed by the $A$ on the left of the second turnstile: $A$ on the right is less than $A$ on the left. That this is a *redundant* expression of identity is demonstrated ultimately—i.e.

after the proof of a difficult theorem—by the fact of its eliminability.

(End identity group.)

## LOGICAL RULES

These rules *introduce* the logical operators (*via* the right rules) and *eliminate* them (*via* the left rules).

$$\frac{\Gamma, A_i \vdash \Delta}{\Gamma, A_0 \wedge A_1 \vdash \Delta} \mathrm{L}i\wedge,\ i=0,1. \qquad \frac{\Gamma \vdash A, \Delta \quad \Gamma' \vdash B, \Delta'}{\Gamma, \Gamma' \vdash A \wedge B, \Delta, \Delta'} \mathrm{R}\wedge \tag{4.3.7}$$

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \vee B \vdash \Delta, \Delta'} \mathrm{L}\vee \qquad \frac{\Gamma \vdash A_i, \Delta}{\Gamma \vdash A_0 \vee A_1, \Delta} \mathrm{R}i\vee,\ i=0,1. \tag{4.3.8}$$

$$\frac{\Gamma \vdash A, \Delta \quad \Gamma', B \vdash \Delta'}{\Gamma, \Gamma', A \Rightarrow B \vdash \Delta, \Delta'} \mathrm{L}\Rightarrow \qquad \frac{\Gamma, A \vdash B, \Delta}{\Gamma \vdash A \Rightarrow B, \Delta} \mathrm{R}\Rightarrow \tag{4.3.9}$$

For intuitionistic systems, all of these logical rules—with the exception of L$\vee$—are restricted merely by allowing at most one formula to the right of turnstiles. Only in the case of L$\vee$ is the intuitionistic version not just a restriction of this kind, since the $\Delta, \Delta'$ is disallowed. Instead, the rule is replaced by:

$$\frac{\Gamma, A \vdash \Delta \quad \Gamma', B \vdash \Delta}{\Gamma, \Gamma', A \vee B \vdash \Delta} \tag{4.3.10}$$

where $\Delta$ contains at most one formula.

(End logical rules.)

For an *intuitionistic* Gentzen sequent calculus it is generally possible to produce a natural deduction system that might be presumed to underlie it. This is done by judiciously (and recursively) as-

signing terms to sequents, and then regarding these terms as $\lambda$-calculus-like descriptors of underlying deductions. (The correspondence sending a sequent proof to its associated $\lambda$-term is generally not one-to-one.)

For instance, for the intuitionistic version of the CUT rule, which reads:

$$\frac{\Gamma \vdash A \quad A, \Delta \vdash B}{\Gamma, \Delta \vdash B} \tag{4.3.11}$$

the term assignment takes the form:

$$\frac{\Gamma \vdash t{:}A \quad x{:}A, \Delta \vdash u{:}B}{\Gamma, \Delta \vdash u[x/t]{:}B} \tag{4.3.12}$$

This is a formalized version of an obvious replacement of deductions in natural deduction: $t$ labels the deduction of $A$ from $\Gamma$, and $u$ labels the deduction of $B$ from the deduction $x$ of $A$ and $\Delta$. Thus, from $\Gamma$, $\Delta$ we may deduce $B$ by using the deduction $t$ in place of $x$, thereby cutting $A$ out of the lower sequent.

The term $u[x/t]{:}B$ may be reducible. It turns out that the other sequent rules do not produce $\lambda$-terms that are reducible. (For instance, the term assignment for intuitionistic $L\Rightarrow$ is:

$$\frac{\Gamma \vdash t{:}A \quad \Gamma', x{:}B \vdash u{:}C}{\Gamma, f{:}A \rightarrow B, \Gamma' \vdash u[x/ft]{:}B} \tag{4.3.13}$$

Here, the term produced is not reducible since $f$ is a variable.)

Readers familiar with $\lambda$-calculus will recognize that sequent proofs conducted without the use of CUT will produce *normal* $\lambda$-terms: i.e. terms that are not reducible.

The eliminability of CUT is essentially equivalent to the fact that simply-typed λ-calculus is (strongly) normalizable: every λ-term is reducible to a unique normal form. (Cf. works already cited.)

More to the point for our purposes is the observation (cf. Abramsky 1993) that the *computational* aspects of such deductive systems are seen to reside precisely in the process of cut elimination.

The Gentzen sequent formalism reveals structural and behavioral attributes of the underlying—or associated—natural deduction system and the equivalent term calculus. Among its lessons, we emphasize:

- the critical importance of the structural rules, and their sensitivity to different semantic readings of the associated natural deduction system;
- the fact, just noted, that all computation resides in the process of cut elimination;
- the value—much appreciated by computer scientists—of the explicit typing of terms and the careful maintenance of such typing through the course of deductions.

Soon we shall attempt to carry these lessons into the quantum domain.

# 5

# Quantum Logic

To say that quantum logic was developed without any intuitionistic or constructivist paradigm in mind would be an understatement: indeed, in their seminal paper Birkhoff and von Neumann 1936, the authors go out of their way to launch an attack on intuitionistic logic, as it was then constituted, ostensibly on the issue of of the distributive law. Their archetype was ordinary classical logic, and even in this static arena classically conditioned students of quantum logic are confronted at almost every turn with affronts to their thinking. This appropriately parallels the experiences of classically minded students of quantum physics itself.

In this chapter we rehearse certain aspects of quantum logic with an eye towards its possible exploitation as a computational resource. In this respect it will be found wanting, though its apparent failures are instructive, and will inform our attempts in the next chapter to extend its intuitionistic core to a wider logic.

For the sake of completeness we have included a brief account of how certain physical notions may be couched in quantum logical terms. These sections, namely §5.1.5, §5.2.1, and the discussion of the Kochen–Specker Theorem in §5.2.2, may be safely skipped.

(The difficulty of the subject matter of this chapter is compensated for by the excellence of the available texts. A list of these should include: Dalla Chiara *et al.* 2002, Rédei 1998, Svozil 1998, Bub 1997, Isham 1995, Hughes 1989, Gibbins 1987, Varadarajan

1985, Kalmbach 1983 and its sequel Kalmbach 1985, Beltrametti and Cassinelli 1981, Holland 1970 and 1995. In particular, the encyclopedic Kalmbach 1983 remains the standard reference to the astonishingly rich mathematical theory.)

Sections of this chapter began as joint work with J. P. Rawling (cf. Rawling and Selesnick 2000).

## 5.1 Orthologic and its Model Theory

The minimal core of quantum logic is known as *orthologic* (**OL**). This is simply the weakening of classical logic which results when one does not insist that AND distributes over OR: it is the logic which might have replaced classical logic had classical logicians failed to notice this distributivity in their ambient world of macroscopic objects. Indeed, they might have viewed this principle with more skepticism had they been capable of a cognitive resolution sufficiently fine to apprehend the kind of quantum interference that is revealed by the double slit experiment.

In this experiment a source of quanta, electrons say, is placed behind a screen in which two slits have been made. With one slit covered up, a detector placed on the other side of the screen reveals a pattern of arrivals entirely consistent with the idea that electrons are "particles" in the classical sense: namely, there is a peak of intensity in front of the slit, tailing off smoothly in all directions. If these particles obeyed "classical" rules, then the effect of uncovering both slits should be to produce two peaks of intensity in front of the slits, tailing off smoothly. But this is not at all what experiment reveals: instead a wave-form appears, indicating that the electrons have interfered with each other to produce cancellations at some places and reinforcements at others. This pattern persists, even when one reduces the intensity of the beam so that a single electron is emitted. Somehow, the presence of the other slit has affected the behavior of even a single electron. A naïve "quantum logicist's" view of this might proceed along the following lines. The way we think—our *logic*—has been conditioned by our necessarily macroscopic experience. When

we try to apply these rules to microscopic entities like electrons—to deduce the outcome of the double slit experiment for example—we should not expect them to work. Instead, we should *infer* their logic from experiment. In the experiment just described, for example, the statement:

(*electron A leaves the source*) AND
[(*electron A goes through slit 1*) OR (*electron A goes through slit 2*)]

is verified—or at least not obviously falsified—while the statement:

[(*electron A leaves the source*) AND (*electron A goes through slit 1*)]
OR
[(*electron A leaves the source*) AND (*electron A goes through slit 2*)]

is *not* verified, for if it was verified, the arrival pattern would meet classical expectations. So the fundamental classical "distributive law," namely

p AND (q OR r) = (p AND q) OR (p AND r)

does *not* hold in the logic of these new entities called quanta.

This argument against distributivity may be disputed by those with a prior knowledge of quantum physics, who will be quick to question the very meaning of certain constituent propositions: "*electron A goes through slit 1,*" for example, which may not be interpretable in light of the Uncertainty Principle. Regardless of the logical status of such refutations, they are surely not arguments in *favor* of distributivity, so that deep skepticism toward the validity of this principle in the quantum domain would still seem to be in order.

In classical propositional calculus (**PC**) the material implication connective ($\rightarrow$) is expressed in terms of other connectives. Namely, $p \rightarrow q \equiv \neg p \vee q$, a problematic interpretation entailing certain anomalies of great antiquity. (Namely, $p \rightarrow q$ is true for *any* $q$ if $p$ is false: round squares are round *and* square, as Lewis and Langford 1959 puts it.) In the absence of the distributive law, we might

expect further problems for an implication connective cobbled together out of other connectives. This expectation is maximally realized, as we shall see later (§5.1.4). In fact no viable implication for orthologic can be manufactured out of the other connectives at all.

### *5.1.1 Orthologic as a Deductive System*

The realization of (first-order) orthologic as a (non intuitionistic) deductive system seems first to have been achieved by R. Goldblatt (Goldblatt 1974; see also Dalla Chiara *et al.* 2002). The atoms or primitive symbols are:

(i) a denumerable collection $\Phi_0$ of propositional variables $a_1, a_2, \ldots$;
(ii) the connectives $\sim$ ("negation") and $\sqcap$ ("conjunction"); and
(iii) parentheses.

The set $\Phi$ of (well-formed) *orthoformulae* (or just *formulae*, until this designation becomes ambiguous) is constructed from these in the usual way. Elements of $\Phi$ will be denoted by lower case Greek characters $\alpha, \beta, \ldots$, taken usually from the beginning of the alphabet. (We shall try to reserve characters at the end of the alphabet for elements of sets of various kinds.)

Since there is no implication sign in $\Phi$ a formal deductive calculus is based on *sequents* involving at most single formulae and written in the form:

$$\alpha \vdash \beta \qquad (5.1.1.1)$$

for $\alpha, \beta \in \Phi$, the intended reading of which is that $\beta$ may be inferred from $\alpha$. Certain sequents are designated as *axioms*, and there are three *rules of inference*, namely, for any formulae $\alpha, \beta$:

AXIOMS

O1. $\alpha \vdash \alpha$

O2. $\alpha \sqcap \beta \vdash \alpha$

O3. $\alpha \sqcap \beta \vdash \beta$

O4. $\alpha \vdash \sim\sim\alpha$

O5. $\sim\sim\alpha \vdash \alpha$

O6. $\alpha \sqcap \sim\alpha \vdash \beta$

INFERENCE RULES

O7. $$\frac{\alpha \vdash \beta \quad \beta \vdash \gamma}{\alpha \vdash \gamma}$$

O8. $$\frac{\alpha \vdash \beta \quad \alpha \vdash \gamma}{\alpha \vdash \beta \sqcap \gamma}$$

O9. $$\frac{\alpha \vdash \beta}{\sim\beta \vdash \sim\alpha}$$

A conjunctive connective may be introduced according to the definition

$$\alpha \sqcup \beta \equiv \sim((\sim\alpha) \sqcap (\sim\beta)) \qquad (5.1.1.2)$$

and dual forms of O2, O3, O6 and O8 follow.

A string $s_1; s_2; \ldots; s_n$ of sequents is called a *proof* of its last member $s_n$ if each $s_i$ is either an axiom or follows from some preceding sequent through the use of one of the rules of inference.

If there exists a proof of a sequent $\alpha \vdash \beta$ we write

$$\alpha \vdash_{\mathrm{O}} \beta \qquad (5.1.1.3)$$

and say that $\beta$ is *deducible from* $\alpha$ *in orthologic.*

If $\alpha \vdash_{\mathrm{O}} \beta$ for *any* formula $\alpha$, we say that $\beta$ is a *theorem of or-*

*thologic* or an *orthotheorem,* and we write

$$\vdash_{\mathrm{O}} \beta. \tag{5.1.1.4}$$

(Note that this condition is equivalent to $\alpha \sqcup {\sim}\alpha \vdash_{\mathrm{O}} \beta$.)

We recall that there are completeness theorems for ordinary propositional logic (**PC**) and intuitionistic logic (**IL**) which assert connections between the analogous forms of deducibility in these logics and the behavior of morphisms, or valuations, of formulae into certain classes of lattices: Boolean algebras in the case of **PC** and Heyting algebras in the case of **IL**. (Cf. Bell and Slomson 1969, Hughes and Cresswell 1968, *inter alia.*) There is an analogous characterization of orthologic, involving a class of lattices called *ortholattices.*

An *ortholattice* is a bounded lattice $\langle L, \sqcup, \sqcap, 0_L, 1_L, {}' \rangle$ where $(\ )'$ is a unary operation called *orthocomplementation* satisfying:

| | |
|---|---|
| complementarity: | $\forall a \in L,\ a \sqcap a' = 0_L,\ a \sqcup a' = 1_L$ |
| unitarity: | $a'' = a$ |
| antitonicity: | $a \sqsubseteq b$ iff $b' \sqsubseteq a'$ |

It is easily shown that any ortholattice satisfies De Morgan's laws, e. g.

$$a \sqcup b = (a' \sqcap b')'. \tag{5.1.1.5}$$

An ortholattice is said to be *complete* if arbitrary subsets have meets and joins: a complete ortholattice satisfies the complete generalizations of the De Morgan laws.

Examples of ortholattices include all Boolean algebras and lattices of closed subspaces of Hilbert spaces, with operations given in §1.2.

Given an ortholattice $L$, a function $v_L : \Phi_0 \longrightarrow L$ determines a *valuation* upon $\Phi$ *via* the recursive definitions:

$$v_L(\alpha \sqcap \beta) = v_L(\alpha) \sqcap v_L(\beta) \qquad (5.1.1.6)$$

$$v_L(\sim\alpha) = v_L(\alpha)' \qquad (5.1.1.7)$$

The algebraic characterization theorem for orthologic may be stated as follows.

THEOREM 5.1.1.1 (Goldblatt 1974, Bell 1983)

$\gamma \vdash_{\mathbf{O}} \alpha$ *iff* $v_L(\gamma) \sqsubseteq v_L(\alpha)$ *for all ortholattices* $L$ *and all valuations* $v_L$.

COROLLARY 5.1.1.1

$\vdash_{\mathbf{O}} \alpha$ *iff* $v_L(\alpha) = 1_L$ *for all ortholattices* $L$ *and all valuations* $v_L$.

A pair $\mathcal{A} \equiv \langle L, v_L \rangle$ may be called an *algebraic orthomodel* (for orthologic), and a formula $\alpha$ deemed *valid* in this model if $v_L(\alpha) = 1_L$. Then the above corollary may be stated as:

$$\vdash_{\mathbf{O}} \alpha \text{ iff } \alpha \text{ is valid in every algebraic orthomodel.} \qquad (5.1.1.8)$$

The following corollary follows immediately, since Boolean algebras are ortholattices.

COROLLARY 5.1.1.2

*If* $\vdash_{\mathbf{O}} \alpha$ *then* $\alpha$ *is a theorem of* **PC**.

(The converse is clearly false.)

A canonical class of examples of ortholattices is obtained as follows. An *orthogonality space* $F = \langle W, \perp \rangle$ comprises a set $W$ and a binary relation $\perp \subseteq W \times W$ which is an *orthogonality*: namely, it is

*irreflexive* (not $x \perp x$) and *symmetric* ($x \perp y$ iff $y \perp x$).

For $x \in W$, $Y \subseteq W$ we write

$$x \perp Y \text{ iff } x \perp y \ \forall y \in Y \tag{5.1.1.9}$$

and define

$$Y^{\perp} \equiv \{x : x \perp Y\}. \tag{5.1.1.10}$$

In Goldblatt's terminology (Goldblatt 1973) $Y \subseteq W$ is said to be *regular* if

$$Y^{\perp\perp} = Y. \tag{5.1.1.11}$$

Then the class $R(F)$ of $\perp$-regular subsets of $W$ is a complete ortholattice under the partial order given by set inclusion, with the lattice meet given by set intersection and $\perp$ as orthocomplement. (A proof of this is contained in Lemma 5.1.2.1 given below.) This class of examples is canonical in light of Goldblatt's fundamental theorem *à la* Stone for ortholattices:

THEOREM 5.1.1.2 (Goldblatt 1973, Bell 1983)

GOLDBLATT'S STONEAN THEOREM

*Any ortholattice $L$ is (completely) isomorphic to a sublattice of $R(F_L)$ for some orthogonality space $F_L$.*

Since this theorem is the cornerstone of all the models we shall discuss, it is worth describing the construction. For a given ortholattice $L$, let $F_L = \langle W_L, \perp \rangle$ where $W_L$ is the class of *proper filters* of $L$, and, for $x, y \in W_L$,

$$x \perp y \text{ iff } \exists a \in L \text{ such that } a' \in x \text{ and } a \in y.$$

(A proper filter of $L$ is an upward closed subset of $L$, closed also under finite meets, which does not contain $0_L$.)

The map

$$\phi: L \longrightarrow \mathbf{2}^{W_L}, \tag{5.1.1.12}$$

where $\mathbf{2}$ denotes the two-element set (so that $\mathbf{2}^X$ for a set $X$ may be, as usual, identified with the set of subsets of $X$) given, for $a \in L$, by

$$\phi(a) = \{x \in W_L : a \in x\}, \tag{5.1.1.13}$$

embeds $L$ into $R(F_L)$.

Goldblatt also characterizes the image of $L$ under $\phi$: it coincides with the family of those $\perp$-regular subsets of $W_L$ which are *clopen* (closed and open) in that topology on $W_L$ that has as a subbase the sets $\phi(a)$, $a \in L$, and their complements.

In case $L$ is complete $\phi$, is onto $R(F_L)$.

Clearly, we may restrict to the class of complete ortholattices in the algebraic characterization theorem (Theorem 5.1.1.1).

Now a proper filter is like a "possible world" which "validates" each of its members: if the membership of $a$ in $x$ is construed as "$a$ is true in world $x$," then everything that can be inferred from $a$ should also "be true in world $x$," which is the case if $x$ is a filter, and $\sqsubseteq$ is read as a form of implication. Under this reading, each $\phi(a)$ (equation (5.1.1.12)), being a set of worlds, could be interpreted as a proposition asserting the truth of $a$ in a certain set of possible worlds. Indeed, Goldblatt's Stonean theorem (Theorem 5.1.1.2), taken together with the characterization theorem (Theorem 5.1.1.1), yields a class of models that also characterize orthologic, but that carry a rather more interesting semantic interpretation.

The discussion of these will be facilitated by the introduction of some formal results that are the business of the next section.

(As for the classical Stone Theorem, many accounts of it are available. A short, elegant one, requiring some assembly, may be

found in Atiyah and MacDonald 1969, p. 14, exercise 25.)

### *5.1.2 Modal Logic and Kripke Models*

The problems with classical material implication, as defined by $p \rightarrow q \equiv \neg p \vee q$, were known to the ancients, and the idea that there could be a stronger kind of truth, *necessary* truth, which might validate implications in such a way that the anomalies were avoided, appears in the writings of Aristotle. In this approach, the truth of propositions may be modified, or *modalized,* and this can be done formally by introducing a unary operator into the usual sentential language of **PC**. This operator is often denoted by a *box:* $\Box$. Thus $\Box p$ is informally read as "necessarily $p$." There is another modality of "truth," namely *possible* truth, which it is convenient to have a separate symbol for, namely $\Diamond$. Thus $\Diamond p$ may be read informally as "possibly $p$." Since "possibly $p$" is synonymous with "not necessarily not $p$," we may express $\Diamond$ in terms of $\Box$ as:

$$\Diamond = \neg \Box \neg \qquad (5.1.2.1)$$

and concomitantly we have:

$$\Box = \neg \Diamond \neg. \qquad (5.1.2.2)$$

Thus, we may regard either modality as primitive and derive the other. Various rules regarding the interaction of these new operators with the old ones may be laid down to produce a large array of new logical systems.

This topic has had a tortuous (and tortured) history, having drifted in and out of fashion over the centuries. Interest in it intensified in the 1930s after Gödel found an embedding of intuitionistic logic (**IL**) into the modal system that has come to be known as S4, following the appearance in 1932 of the first edition of Lewis and Langford 1959. (For references on modal logic see, among others, Goldblatt 1992, Chellas 1980, Hughes and Cresswell 1968 and

1984.) Needless to say, the modal formalism admits more general interpretations than those pertaining merely to modifications of notions of truth and has found wide application, for instance in computer science: cf. Goldblatt 1992.

A compellingly intuitive model theory for a broad class of modal logics—namely the so-called *normal* ones—was developed by the teenaged Saul Kripke and announced in Kripke 1959, 1963. These models consist of sets $W$ equipped with binary relations $R$ ($\subseteq W \times W$) and other structures to be described later for the particular modal logic we are interested in here. The pair $\langle W, R\rangle$ is often called a *Kripke frame* or just a *frame.* The elements of $W$ are called "worlds" or "possible worlds," a terminology derived from Leibniz, and something of a misnomer for the application we have in mind. The relation $R$—and we write as usual $vRw$ for $(v,w)\in R$—is construed as an accessibility relation among worlds. The intuitive basis for these models is the notion that if $p$ is *necessarily* true at a world $w$ then it will be true at all worlds accessible from $w$. This assumes that we have a means of testing the validity of modal formulae at each world. The technical means of doing this is exactly the part of the model we have not yet discussed. The completeness result for a typical modal logic states roughly that a formula may be deduced in the logic if and only if it is true at all worlds in all frames having accessibility relations of a certain type, which depends upon the specifics of the modal logic involved.

The frames of interest to us are of the type known as *proximity spaces.* A *proximity space* is a pair $\langle W, \approx\rangle$ in which the relation "$\approx$" is *reflexive* ($w \approx w$) and *symmetric* ($v \approx w$ iff $w \approx v$). Clearly each proximity space $\langle W, \approx\rangle$ determines an orthogonality space $\langle W, \perp\rangle$ where $x \perp y$ iff $x \not\approx y$, and, conversely, each orthogonality space $\langle W, \perp\rangle$ determines a proximity space $\langle W, \approx\rangle$ where $x \approx y$ iff not $x \perp y$.

Given a proximity space $\langle W, \approx\rangle$ we define for each $E \subseteq W$:

$$\Diamond E \equiv \{w \in W\colon \exists v \in E \text{ such that } w \approx v\}, \qquad (5.1.2.3)$$

and note that $E \subseteq \Diamond E$. It will prove convenient to employ the notation

$$S_v \equiv \Diamond\{v\} = \{w \in W:\ w \approx v\} \tag{5.1.2.4}$$

(the "sphere" around $v$, in the terminology of D. Lewis). Then one may also write

$$\Diamond E = \bigcup_{v \in E} S_v . \tag{5.1.2.5}$$

Dually, we write

$$\Box E \equiv \left(\Diamond E^c\right)^c , \tag{5.1.2.6}$$

where the superscript $c$ denotes set complementation relative to $W$.

PROPOSITION 5.1.2.1

*For $\langle W, \approx \rangle$ a proximity space and $E, F \subseteq W$:*

M1) $\Diamond(E \cup F) = \Diamond E \cup \Diamond F$

M2) *For a family $\mathcal{F}$ of subsets of $W$:*
$$\Diamond\left(\bigcap_{F \in \mathcal{F}} F\right) \subseteq \bigcap_{F \in \mathcal{F}} (\Diamond F)$$

M3) $\Diamond E = (E^{\perp})^c \equiv E^{\perp c}$

M4) $\Box E = E^{c\perp} = \{w \in W:\ S_w \subseteq E\}$

M5) $E \subseteq E^{\perp\perp}$

M6) $E \subseteq F$ *implies* $F^{\perp} \subseteq E^{\perp}$

M7) $E^{\perp\perp\perp} = E^{\perp}$

M8) $\Box\Diamond\Box E = \Box E$

M9) $\Diamond\Box\Diamond E = \Diamond E$

M10) $(E \cup F)^{\perp} = E^{\perp} \cap F^{\perp}$

M11) $E^{\perp} \cup F^{\perp} \subseteq (E \cap F)^{\perp}$

PROOF

The proofs of these elementary identities may be safely left to the interested reader. For instance, to prove (M3) we note that $w \notin \Diamond E$ implies $w \not\approx x$ for all $x \in E$ so that $w \perp E$ showing that $(\Diamond E)^c \subseteq E^{\perp}$ from which it follows that $(E^{\perp})^c \subseteq \Diamond E$. The converse inclusion is obvious.

The proof of (M4) now follows, since

$$\Box E \equiv (\Diamond E^c)^c$$

$$= ((E^c)^{\perp c})^c$$

$$= E^{c\perp}$$

$$= \{w : w \perp E^c\}$$

$$= \{w : v \notin E \text{ implies } v \not\approx w\}$$

$$= \{w : v \approx w \text{ implies } v \in E\}$$

$$= \{w : S_w \subseteq E\}. \qquad (5.1.2.7)$$

From (M6) applied to (M5), we obtain $(E^{\perp\perp})^{\perp} \subseteq E^{\perp}$. But from (M5) we also have $E^{\perp} \subseteq (E^{\perp})^{\perp\perp}$, and (M7) follows.

(M8), (M9) and (M10) follow immediately from (M3) etc., and (M11) follows from (M3) and (M2). ∎

The *Kripke models* for orthologic seem to have appeared first in Goldblatt 1974 and have been extensively elaborated upon by Dalla Chiara and others (cf. Dalla Chiara *et al.* 2002).

Given a proximity space $P = \langle W, \approx \rangle$ we follow Dalla Chiara *et al.* 2002 in defining a *proposition of* $P$ (or just a *proposition,* if the context is clear) to be a subset $X \subseteq W$ satisfying:

> if $x$ has the property that for each $y$ with $y \approx x$ there exists a $z \in X$ such that $y \approx z$, then $x \in X$,

or, equivalently, $X$ is a proposition of $P$ iff

$$S_x \subseteq \Diamond X \text{ implies } x \in X. \qquad (5.1.2.8)$$

(The reverse implication holds for any $X \subseteq W$ since $\Diamond X = \bigcup_{x \in X} S_x$.)

LEMMA 5.1.2.1

1. *For all* $X \subseteq W$, $\Box\Diamond X = X^{\perp\perp}$ *and* $X$ *is a proposition iff* $\Box\Diamond X = X^{\perp\perp} = X$.
2. $X$ *is a proposition iff* $x \notin X$ *implies* $\exists y \approx x$ *with* $y \perp X$.
3. *For any* $Y \subseteq W$, $Y^{\perp}$ *is a proposition.*
4. *If* $\mathscr{C}$ *is a family of propositions, then* $\bigcap \mathscr{C}$ *is a proposition.*
5. *If* $Y$ *is a proposition, then* $X \subseteq Y$ *iff* $\Diamond X \subseteq \Diamond Y$.

PROOF

1. The first assertion follows from (M3) and (M4). For the second, note that (M4) and the condition labelled (5.1.2.8) imply that $\Box\Diamond X \subseteq X$, while (M5) gives the reverse inclusion.
2. This follows immediately from (5.1.2.8) which is equivalent to: $x \notin X$ implies $S_x \not\subseteq \Diamond X = X^{\perp c}$.
3. Immediate from (M7).

4. $x \notin \bigcap_{C \in \mathcal{C}} C$ implies $\exists C_0 \in \mathcal{C}$ such that $x \notin C_0$. Thus $S_x \not\subseteq \Diamond C_0$, since $C_0$ is a proposition (cf. the proof of (2) above). So $S_x \not\subseteq \bigcap_{C \in \mathcal{C}} (\Diamond C)$ from which it follows that $S_x \not\subseteq \Diamond(\bigcap_{C \in \mathcal{C}} C)$ in view of (M2).
5. Implication in one direction is immediate; implication in the other follows from (M3), (M5) and (M6). ∎

Thus, in view of Lemma 5.1.2.1(1), the family of propositions of a proximity space is exactly the family $R(\langle W, \perp \rangle)$ of regular sets of the associated orthogonality space and the existence of the complete ortholattice structure possessed by this family of subsets (already alluded to) may be read off from the result established above. (Note that $W$ and $\varnothing$ are also propositions.)

A *Kripke orthomodel* $\mathcal{M} = \langle W, \approx, \varrho \rangle$ is a proximity space $P = \langle W, \approx \rangle$ and a function (called a *valuation*) $\varrho : \Phi \longrightarrow R(\langle W, \perp \rangle)$ satisfying:

$$\varrho(\sim\alpha) = \varrho(\alpha)^{\perp} \tag{5.1.2.9}$$

$$\varrho(\alpha \sqcap \beta) = \varrho(\alpha) \cap \varrho(\beta). \tag{5.1.2.10}$$

We will say that a formula $\alpha$ is:

*true at the "world"* $w \in W$, and write $w \models_{\mathcal{M}} \alpha$,
iff $w \in \varrho(\alpha)$;
*true on a set* $E \subseteq W$, and write $E \models_{\mathcal{M}} \alpha$,
iff $w \models_{\mathcal{M}} \alpha$ for all $w \in E$—that is, iff $E \subseteq \varrho(\alpha)$;
*true in the Kripke orthomodel* $\mathcal{M}$
iff it is true at every world in $\mathcal{M}$;
*Kripke valid,* and write $\models \alpha$,
iff it is true in *all* Kripke orthomodels.

In the sequel we shall abbreviate $R(\langle W, \perp \rangle)$ to $R(W)$, since the proximity and orthogonality relations will never be ambiguous.

THEOREM 5.1.2.1

$$\vdash_0 \alpha \text{ iff } \models \alpha.$$

PROOF

Suppose $\vdash_0 \alpha$, and let $\mathcal{M} = \langle W, \approx, \varrho \rangle$ denote an arbitrary Kripke orthomodel. Then $\langle R(W), \varrho \rangle$ constitutes an algebraic orthomodel and Corollary 5.1.1.1 (cf. statement (5.1.1.8)) implies that $\varrho(\alpha) = W$ so that $\alpha$ is valid in the Kripke orthomodel $\mathcal{M}$. But $\mathcal{M}$ was arbitrary, so $\alpha$ is Kripke valid.

Conversely, suppose $\models \alpha$, and let $\mathcal{A} \equiv \langle L, v_L \rangle$ denote an arbitrary algebraic orthomodel. Then $\mathcal{M} \equiv \langle W_L, \approx_L, \phi_L \circ v_L \rangle$ (where $W_L, \phi_L$ and $\perp_L$ are the entities appearing in the discussion of Goldblatt's Stonean theorem (Theorem 5.1.1.2)) constitutes a Kripke orthomodel. Since $\alpha$ is assumed Kripke valid, it is true in $\mathcal{M}$, so that $\phi_L(v_L(\alpha)) = W_L$. But then $v_L(\alpha) = 1_L$ since $\phi_L$ is injective. Thus $\alpha$ is true in the arbitrarily chosen algebraic orthomodel $\mathcal{A}$, so $\vdash_0 \alpha$ by (5.1.1.8). ∎

A question that now interposes itself (and that will occupy us in the next chapter) concerns the semantics of disjunction. In a Kripke orthomodel we have, for formulae $\alpha$ and $\beta$, and $E \subseteq W$ as above:

$$E \models_{\mathcal{M}} \alpha \sqcup \beta \text{ iff } E \subseteq \varrho(\alpha \sqcup \beta)$$

$$= \varrho(\sim(\sim\alpha \sqcap \sim\beta))$$

$$= (\varrho(\alpha)^{\perp} \cap \varrho(\beta)^{\perp})^{\perp}$$

$$= (\varrho(\alpha) \cup \varrho(\beta))^{\perp\perp} \qquad \text{by (M10)}$$

$$\supseteq \varrho(\alpha) \cup \varrho(\beta). \qquad (5.1.2.11)$$

Thus, the interpretations of orthodisjuncts are, in a sense, double negations of ordinary disjuncts of propositions. These are not necessarily themselves ordinary disjuncts: there are generally more worlds in $\varrho(\alpha \sqcup \beta)$ than there are in $\varrho(\alpha) \cup \varrho(\beta)$. That is, one could have $w \models_{\mathcal{M}} \alpha \sqcup \beta$ while neither $w \models_{\mathcal{M}} \alpha$ nor $w \models_{\mathcal{M}} \beta$ holds. This circumstance embodies much of what seems anomalous to classical minds when confronted with "quantum" phenomena, and should be compared to a similar remark made at the end of §1.1.

### *5.1.3 A Modal Translation Theorem*

It is evident, in light of the Kripke semantics for normal modal logics, that so-called proximity spaces are frames for some such logic. In fact, the particular modal logic supported by these frames has a history longer than that of quantum logic itself, having been introduced by O. Becker in 1930.

A *system of modal logic* has as formulae the usual formulae of **PC**, with the ordinary Boolean connectives $\neg$, $\wedge$, $\vee$, supplemented with the unary (necessity) operator $\Box$. Material implication is defined in the usual way and $\Diamond$ is defined as in equation (5.1.2.1). We will continue the use of lower case Greek characters to denote formulae in these systems.

The axioms and inference rules for a system of modal logic include the usual ones for **PC** with some specifically modal additions. For instance, those systems which include the so-called *Kripke axiom* **K**:

$$\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta);$$

the rule of "necessitation":

if $\alpha$ is deducible then $\Box\alpha$ is deducible;

and *modus ponens:*

if $\alpha$ and $\alpha\rightarrow\beta$ are deducible then $\beta$ is deducible,

are said to be *normal* systems of modal logic.

The system of Becker's of interest to us has two additional axioms, namely:

$$\Box\alpha\rightarrow\alpha$$

and the "Brouwersche" axiom:

$$\alpha\rightarrow\Box\Diamond\alpha.$$

The origin of the peculiar appellation for this last axiom—adopted by tradition directly from Becker's original German—is worth discussing here. Accordingly, we return to the constructivist interpretation of **IL** to discuss intuitionistic negation. In order for a formula $\neg q$ to be **IL**-valid in the constructivist interpretation, we must constructively prove that $q$ is *false.* That is, we must produce a proof that $q$ possesses *no* proof. If $q$ is $\neg p$, then in order for $\neg\neg p\equiv\neg q$ to be **IL**-valid we must produce a proof that $\neg p$ possesses no proof: that is, we must produce a proof that no proof exists demonstrating the non-existence of proofs of $p$. Clearly, such a proof may be producible in the absence of any proof of $p$ itself being in hand. That is to say, $\neg\neg p$ could be **IL**-valid without $p$ being **IL**-valid. Thus, $\neg\neg p\Rightarrow p$ is **IL**-*invalid.* However, if we *do* have a proof of $p$, then no proof can exist demonstrating the non-existence of proofs of $p$. Thus, we have demonstrated the **IL**-validity of:

$$p\Rightarrow\neg\neg p. \qquad (5.1.3.1)$$

Now the Brouwersche axiom can be written:

$$p \rightarrow \Box \neg \Box \neg p$$

and if $\Box\neg$ is read as some kind of strong form of negation ("necessarily not") then the axiom looks like the corresponding **IL**-valid, or Brouwerian, axiom (5.1.3.1), hence the terminology.

The set of modal formulae—i.e. the set of well-formed formulae in the usual language of **PC**, with $\Box$ and $\Diamond$ now included —will be denoted by $\Phi_M$ . A modal formula $\alpha$ which can be deduced from the axioms and inference rules listed above is called a *theorem of the modal B-system,* and we write:

$$\vdash_{\mathbf{B}} \alpha$$

if $\alpha$ is a theorem of the modal B-system.

A *(Kripke) B-model* is a triple $\mathfrak{B} = \langle W, \approx, \mathrm{v} \rangle$, where $W$ is a set (of "worlds"), $\approx$ a proximity relation on $W$ and $\mathrm{v}: \Phi_M \times W \longrightarrow \mathbf{2}$ is a function satisfying:

V1. For each $w \in W$, $\mathrm{v}(\ ,w): \Phi_M \longrightarrow \mathbf{2}$ is a Boolean valuation with respect to $\neg$ and $\wedge$. That is:
$\mathrm{v}(\neg\alpha,w) = \neg\mathrm{v}(\alpha,w)$, and
$\mathrm{v}(\alpha \wedge \beta, w) = \mathrm{v}(\alpha,w) \wedge \mathrm{v}(\beta,w)$;

V2. For any modal formula $\alpha$, $\mathrm{v}(\Box\alpha,w) = 1$ iff $\mathrm{v}(\alpha,v) = 1$ for every $v$ such that $v \approx w$.

It follows that:

V3. $\mathrm{v}(\alpha \vee \beta,w) = \mathrm{v}(\alpha,w) \vee \mathrm{v}(\beta,w)$, where $\vee$ denotes the classical disjunct, and

V4. For any modal formula $\alpha$, $\mathrm{v}(\Diamond\alpha,w) = 1$ iff $\exists v \approx w$ (i.e. $\exists v$ with $v \approx w$) such that $\mathrm{v}(\alpha,v) = 1$.

(In general a *Kripke model* is a frame $\langle W, R\rangle$ together with a set of valuations—indexed by worlds—of the type specified above: namely, they are Boolean valued, and act on modal formulae in the manner specified. Since each such valuation $\mathrm{v}(\ ,w)$ is determined by its behavior on atoms, a completely equivalent way of specifying the conditions listed above could be expressed in terms of a countable family of subsets of $W$, to be construed as the *truth sets* of atoms, the truth set of the atom $a_i$ being the set of worlds at which—please see below—$a_i$ is declared true. Some authors prefer this formulation: cf. Chellas 1980.)

A modal formula $\alpha$ is said to be:

*true at the world* $w$ *in the B-model* $\mathfrak{B}$, written $w \models_{\mathfrak{B}} \alpha$,
iff $\mathrm{v}(\alpha, w) = 1$;

*true on the set* $E \subseteq W$, written $E \models_{\mathfrak{B}} \alpha$,
iff $w \models_{\mathfrak{B}} \alpha$ for all $w \in E$;

*true in the B-model* $\mathfrak{B}$ iff $W \models_{\mathfrak{B}} \alpha$;

*B-valid,* written $\models \alpha$, if it is true in *all* B-models.

These models characterize the B-system:

THEOREM 5.1.3.1

$$\vdash_{\mathbf{B}} \alpha \textit{ iff } \models \alpha.$$

(This is proved in the references cited above.)

That this resemblance between Kripke models for **OL** and models for the modal B-system goes deeper, seems to have been noticed almost simultaneously by R. Goldblatt (Goldblatt 1974) and H. Dishkant (Dishkant 1977). Since the means are at our disposal, and since we wish to exploit this result as a springboard to a more constructive externalization of **OL**, we shall sketch a proof of a simplified version of their result that **OL** may be interpreted within the

modal B-system. Specifically, we shall sketch below a proof that theorems of **OL** admit a translation into theorems of the B-system.

These results have provided important insights into the peculiarities of **OL** when compared with classical logic, since properties of the ambient B-system were well-known. Extensive elaborations of these results were obtained by Dalla Chiara and others. Cf. references in Dalla Chiara *et al.* 2002 and Burghardt 1984.

The translation recursively assigns to each orthoformula $\alpha \in \Phi$ a modal formula $\alpha^\circ \in \Phi_M$ as follows:

T1) For atomic formulae $a_i$:
$a_i^\circ = \Box\Diamond a_i$
T2) $(\alpha \sqcap \beta)^\circ = \alpha^\circ \wedge \beta^\circ$
T3) $(\sim\alpha)^\circ = \Box\neg\alpha^\circ$

(Note that the "quantization" of $a_i$, namely $\Box\Diamond a_i$, is really a kind of double negation of $a_i$ since $\Box\Diamond \equiv \Box\neg\Box\neg$, which amounts to a double $\perp$ on sets of worlds: cf. the remark following equation (5.1.2.11).)

We will prove that $\alpha$ is an orthotheorem iff $\alpha^\circ$ is a theorem of the B-system.

Suppose a Kripke orthomodel $\mathcal{M} = \langle W, \approx, \varrho \rangle$ is given. Then a B-model

$$\mathcal{B}_{\mathcal{M}} = \langle W, \approx, \mathrm{v}_\varrho \rangle \tag{5.1.3.2}$$

may be constructed by defining, for $w \in W$ and atomic formulae $a_i$:

$$\mathrm{v}_\varrho(a_i, w) = \begin{cases} 1 & \text{if } w \in \varrho(a_i) \\ 0 & \text{if } w \notin \varrho(a_i) \end{cases} \tag{5.1.3.3}$$

which may then be inductively extended to $\Phi_M$ according to the rules (V1)–(V4) given above.

Let us write, for each modal formula $\alpha$ and B-model $\mathcal{B} = \langle W, \approx, \mathrm{v} \rangle$,

$$\|\alpha\|_{\mathcal{B}} = \{w \in W: w \models_{\mathcal{B}} \alpha\}. \tag{5.1.3.4}$$

Then it is clear that:

$$\|\alpha \wedge \beta\|_{\mathcal{B}} = \|\alpha\|_{\mathcal{B}} \cap \|\beta\|_{\mathcal{B}}, \tag{5.1.3.5}$$

$$\|\neg\alpha\|_{\mathcal{B}} = \|\alpha\|_{\mathcal{B}}^{c}, \tag{5.1.3.6}$$

$$\|\Box\alpha\|_{\mathcal{B}} = \Box\|\alpha\|_{\mathcal{B}} \tag{5.1.3.7}$$

and

$$\|\Diamond\alpha\|_{\mathcal{B}} = \Diamond\|\alpha\|_{\mathcal{B}}, \tag{5.1.3.8}$$

where the right hand sides are as in equations (5.1.2.3), etc. (We will often drop the subscript when the context is clear.)

Returning to the B-model $\mathcal{B}_{\mathcal{M}}$ we note that

$$\|a_i\|_{\mathcal{B}_{\mathcal{M}}} = \varrho(a_i). \tag{5.1.3.9}$$

LEMMA 5.1.3.1

*For atomic* $a_i \in \Phi$

1. $w \models_{\mathcal{B}_{\mathcal{M}}} \Box\Diamond a_i$ *iff* $w \in \varrho(a_i)$
2. $w \models_{\mathcal{B}_{\mathcal{M}}} \Box\neg a_i$ *iff* $w \in \varrho(\sim a_i)$

PROOF

1. $\mathrm{v}_{\varrho}(\Box\Diamond a_i, w) = 1$ iff

$$w \in \|\Box\Diamond a_i\| = \Box\Diamond\|a_i\| = \Box\Diamond\varrho(a_i) = \varrho(a_i)^{\perp\perp} = \varrho(a_i).$$

2. $\mathrm{v}_{\varrho}(\Box\neg a_i, w) = 1$ iff

$$w \in \|\Box\neg a_i\| = \Box\neg\|a_i\| = \Box\|a_i\|^c = \Box\varrho(a_i)^c = \varrho(a_i)^{\perp} = \varrho(\sim a_i).$$ ∎

It follows easily from (T3) and (M8) that

$$\|(\sim a_i)^{\circ}\| = \|\Box\neg a_i\|.$$

In particular,

$$\mathrm{v}_{\varrho}((\sim a_i)^{\circ}, w) = 1 \text{ iff } \mathrm{v}_{\varrho}(\Box\neg a_i, w) = 1 \qquad (5.1.3.10)$$

so the last lemma can be restated as:

COROLLARY 5.1.3.1

1. $w \models_{\mathcal{B}_M} a_i^{\circ}$ *iff* $w \in \varrho(a_i)$
2. $w \models_{\mathcal{B}_M} (\sim a_i)^{\circ}$ *iff* $w \in \varrho(\sim a_i)$

PROPOSITION 5.1.3.1

*For any* $\alpha \in \Phi$

$$w \models_{\mathcal{B}_M} \alpha^{\circ} \text{ iff } w \in \varrho(\alpha).$$

The proof is an easy induction on the complexity (i.e. length) of $\alpha$, the base case being covered by the last corollary.

PROPOSITION 5.1.3.2

*For* $\alpha \in \Phi$

$$\vdash_{\mathbf{B}} \alpha^{\circ} \textit{ implies } \vdash_{\mathbf{O}} \alpha.$$

PROOF

Choose a Kripke orthomodel $\mathcal{M} = \langle W, \approx, \varrho \rangle$. Then, $\vdash_{\mathbf{B}} \alpha^{\circ}$ implies that $\alpha^{\circ}$ is true in the B-model associated with $\mathcal{M}$, namely, $\mathcal{B}_{\mathcal{M}} = \langle W, \approx, \mathrm{v}_{\varrho} \rangle$ (equation (5.1.3.2)): that is, $w \models_{\mathcal{B}_{\mathcal{M}}} \alpha^{\circ}$ for all $w \in W$. Then, by the last proposition, $\varrho(\alpha) = W$, so $\alpha$ is true in the arbitrarily chosen Kripke orthomodel $\mathcal{M}$, hence in every such orthomodel, so $\vdash_{\mathbf{O}} \alpha$ (Theorem 5.1.2.1). ∎

Now, given a B-model $\mathcal{B} = \langle W, \approx, \mathrm{v} \rangle$ we may construct a Kripke orthomodel $\mathcal{M}^{\mathcal{B}} \equiv \langle W, \approx, \varrho_{\mathrm{v}} \rangle$, where $\varrho_{\mathrm{v}}$ is given on atoms by

$$\varrho_{\mathrm{v}}(a_i) \equiv \| a_i^{\circ} \|_{\mathcal{B}} = \| \Box \Diamond a_i \|_{\mathcal{B}} = \| a_i \|_{\mathcal{B}}^{\perp\perp} \qquad (5.1.3.11)$$

(which is a proposition by Lemma 5.1.2.1 (3)), and defined inductively on $\Phi$.

The following lemma is again an easy induction on complexity.

LEMMA 5.1.3.2

*For any* $\alpha \in \Phi$

$$\varrho_{\mathrm{v}}(\alpha) = \| \alpha^{\circ} \|_{\mathcal{B}}.$$

PROPOSITION 5.1.3.3

*For* $\alpha \in \Phi$

$$\vdash_{\mathbf{O}} \alpha \textit{ implies } \vdash_{\mathbf{B}} \alpha^{\circ}.$$

PROOF

Choose any B-model $\mathcal{B} = \langle W, \approx, \mathrm{v} \rangle$. Then if $\vdash_{\mathbf{O}} \alpha$, $\alpha$ is true in the associated Kripke orthomodel $\mathcal{M}^{\mathcal{B}}$ (Theorem 5.1.2.1): that is, $\varrho_{\mathrm{v}}(\alpha) = W$. So, by the last lemma, $\|\alpha^{\circ}\|_{\mathcal{B}} = W$ showing that $\alpha^{\circ}$ is true in the arbitrary B-model $\mathcal{B}$ and hence B-valid. So $\vdash_{\mathbf{B}} \alpha^{\circ}$. ∎

Thus, from the last two propositions, we obtain the modal translation theorem, namely:

THEOREM 5.1.3.2

*For* $\alpha \in \Phi$

$$\vdash_{\mathbf{O}} \alpha \textit{ iff } \vdash_{\mathbf{B}} \alpha^{\circ}.$$

We shall return to the modal interpretation of **OL** in the next chapter.

It is to be remarked at this point that the Kripke frames that arise in the physics of quantum systems are of the following type. Let $\mathfrak{H}$ denote a (complex) Hilbert space with inner product $\langle \ | \ \rangle$. Then, with $\mathfrak{h}$ denoting the set $\mathfrak{H} - \{0\}$ of non-zero vectors in $\mathfrak{H}$, and with $\xi \perp \eta$ iff $\langle \xi | \eta \rangle = 0$ for $\xi, \eta \in \mathfrak{h}$, $\langle \mathfrak{h}, \perp \rangle$ is an orthogonality space.

Recall that if $E$ is a subset of $\mathfrak{h}$ then the smallest closed subspace of $\mathfrak{H}$ containing $E$, which we shall denote by $[E]$, is just $E^{\perp\perp} \cup \{0\}$, with terminology as in equation (5.1.1.10). Thus if $E$ is a subset of $\mathfrak{h}$ we have

$$E^{\perp\perp} = \{\eta \in \mathfrak{H} : \eta \perp E^{\perp}\} - \{0\}$$

$$= [E] - \{0\} \qquad (5.1.3.12)$$

so if $E$ is a proposition (of $\mathfrak{h}$)

$$E = [E] - \{0\}. \qquad (5.1.3.13)$$

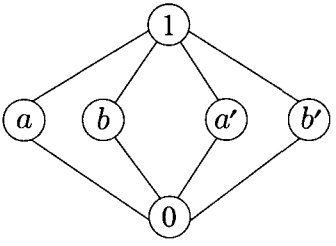
Thus, a proposition of $\mathfrak{h}$ is a closed subspace of $\mathfrak{H}$ with the zero element removed. Conversely if $F$ is a closed subspace of $\mathfrak{H}$ it is quickly seen that $F - \{0\}$ is a proposition of $\mathfrak{h}$. So the propositions of $\mathfrak{h}$ correspond with the closed subspaces of $\mathfrak{H}$ *via* the assignment $E \mapsto [E]$, a correspondence immediately seen to be bijective. This bijection is also easily seen to preserve the respective ortholattice structures. (Note that in this correspondence $\varnothing \mapsto [\varnothing] = \{0\}$.) That is, the ortholattice $R(\mathfrak{h})$ is isomorphic with the ortholattice of closed subspaces of $\mathfrak{H}$.

### *5.1.4 The Implication Problem and Orthomodular Logic*

In ordinary classical **PC** the interpretation of material implication, $p \rightarrow q$, as $\neg p \vee q$ has the consequence that for any Boolean algebra valued valuation $v$,

$$v(p \rightarrow q) = v(p)' \vee v(q) = 1 \text{ iff } v(p) \leq v(q). \qquad (5.1.4.1)$$

This situation fails to hold in **OL**, however, as the following example shows:




(5.1.4.2)

In this (nondistributive) ortholattice, known as the *Chinese lantern,* we have $a' \sqcup b = 1$ but $a \not\sqsubseteq b$. Thus (5.1.4.1) would fail for certain valuations into this lattice of certain orthoformulae, showing that $\sim\alpha \sqcup \beta$ would not be a viable interpretation of a deduction $\alpha \vdash_0 \beta$ in **OL**, in view of Theorem 5.1.1.1.

There is another characterization of classical implication. In any Boolean algebra the element $p \rightarrow q$, defined as above, is characterized by the property:

$$r \leq p \rightarrow q \text{ iff } r \wedge p \leq q \qquad (5.1.4.3)$$

from which it follows that $r = p \rightarrow q$ is the largest element satisfying $r \wedge p \leq q$. Such elements need not exist in nondistributive lattices, so this avenue of generalization seems to be closed to us: it will re-open later.

The condition (5.1.4.3) is an expression of the fact that a Boolean algebra, when considered as a category whose objects are its elements and with morphisms given by $\leq$, is *cartesian closed,* $p \rightarrow q$ being the exponential object usually denoted by $q^p$: cf. Mac Lane and Moerdijk 1991, p. 48.

What we seek is an orthoformula (or orthopolynomial) in $\alpha$, $\beta$—denote it by $\alpha \underset{*}{\longrightarrow} \beta$—for which $\alpha \vdash_0 \beta$ iff $\vdash_0 \alpha \underset{*}{\longrightarrow} \beta$. For any orthovaluation $v$ we would then have:

$$v(\alpha \underset{*}{\longrightarrow} \beta) = v(\alpha) \underset{*}{\longrightarrow} v(\beta) = 1 \text{ iff } v(\alpha) \sqsubseteq v(\beta). \qquad (5.1.4.4)$$

This is a problem involving only one pair of elements in the target lattice at a time. If these elements themselves lay inside a Boolean subalgebra of the target lattice then the condition $v(\alpha) \sqsubseteq v(\beta)$ would be equivalent to the condition $v(\alpha)' \sqcup v(\beta) = 1$ and the hunt for $\underset{*}{\longrightarrow}$ (with the hope that, at least in this case, we would have $v(\alpha \underset{*}{\longrightarrow} \beta) = v(\alpha)' \sqcup v(\beta)$) might be greatly simplified, albeit at the cost of specializing the logic itself.

Let us then confine our choice of algebraic models to the subclass of ortholattices $L$ satisfying the condition:

> For $a,b \in L$, if $a \sqsubseteq b$ then the subortholattice of $L$ generated by $a$ and $b$ is distributive, hence Boolean.

(The subortholattice of $L$ generated by a subset $M \subseteq L$ is the smallest subset of $L$ containing $M$ that is an ortholattice with respect to the operations of $L$. It is possible for a subset to be a sublattice with respect to $\sqcap$ and $\sqcup$ but not closed with respect to orthocomplementation.)

Ortholattices satisfying this condition are precisely the *orthomodular* ones. The following result collects a few of the known equivalent conditions.

THEOREM 5.1.4.1 (Kalmbach 1983, Dalla Chiara *et al.* 2002)

*Let $L$ denote an ortholattice. The following are equivalent, for $a,b \in L$:*

1. *if $a \sqsubseteq b$ then the subortholattice of $L$ generated by $a$ and $b$ is distributive, hence Boolean;*
2. *if $a \sqsubseteq b$ then $b = a \sqcup (a' \sqcap b)$—the* orthomodular *condition;*
3. *if $a \sqsubseteq b$ and $b \sqcap a' = 0$ then $a = b$;*
4. *$a \sqsubseteq b$ iff $a \sqcap (a \sqcap b)' = 0$;*
5. *for any $a$, $b$: $a \sqcap (a' \sqcup (a \sqcap b)) \sqsubseteq b$.*

Examples of orthomodular lattices include Boolean algebras,

which are just the distributive ones, and the lattice of projections in a $W^*$-algebra, an example which includes the case of *Hilbert lattices:* namely, the lattices of closed subspaces of Hilbert spaces. The Chinese lantern, depicted above, is also orthomodular, as may be easily checked *via* the first condition listed in the theorem.

There is an important notion of *compatibility* among elements in an ortholattice, an appellation having a physical origin: cf. §5.1.5. Namely, an element $a$ is said to be *compatible* with an element $b$, written $aCb$, iff:

$$a=(a\sqcap b)\sqcup(a\sqcap b') \tag{5.1.4.5}$$

It turns out that in an orthomodular lattice:

$$aCb \text{ iff } bCa \tag{5.1.4.6}$$

and that this condition also characterizes orthomodularity: it could be added to the list in Theorem 5.1.4.1.

In a Hilbert lattice this condition is equivalent to the commutativity of the corresponding projections and for this reason the compatibility relation is often called *commutativity,* and written more symmetrically as $a\leftrightarrow b$. This symmetry is justified in an orthomodular lattice in view of (5.1.4.6). Note that $a\leftrightarrow b$ iff $a\leftrightarrow b'$ and that if $a\sqsubseteq b$ then $a\leftrightarrow b$. We can also define *orthogonality* ($\perp$) in an ortholattice: namely, $a\perp b$ iff $a\sqsubseteq b'$. Thus, if $a\perp b$ in an orthomodular lattice we also have $a\leftrightarrow b$.

An important result in the general theory of orthomodular lattices asserts that families of mutually commuting elements generate Boolean subalgebras (Varadarajan 1985). Thus, orthomodular lattices have a plentiful supply of Boolean subalgebras.

Now we return to the search for an implicative connective in the subclass of orthomodular ortholattices. It can be shown (Kalmbach 1983, Dalla Chiara *et al.* 2002) that in an orthomodular lattice there are exactly *five* candidates for an implication $\underset{*}{\longrightarrow}$ satisfying condition (5.1.4.4). Of these, only one satisfies the following "weak

cartesian closure" property (cf. (5.1.4.3)), also called the "weak import-export" property:

$$\text{if } a \leftrightarrow b, \text{ then } c \sqsubseteq a \underset{*}{\longrightarrow} b \text{ iff } c \sqcap a \sqsubseteq b \qquad (5.1.4.7)$$

and is given by:

$$a \underset{q}{\longrightarrow} b \equiv a' \sqcup (a \sqcap b). \qquad (5.1.4.8)$$

This connective has come to be called the *Sasaki hook,* though the list of names of other pioneering toilers in this field include those of Finch, Mittelstaedt and Hardegree: please see the references already cited, particularly Dalla Chiara *et al.* 2002. By reason of (5.1.4.7) the Sasaki hook is often the implicative connective of choice for the logic that is characterized by algebraic models consisting of orthomodular lattices and valuations into them. As Goldblatt has shown (Goldblatt 1974, Dalla Chiara *et al.* 2002) this logic may be axiomatized by adding a single axiom (labelled OM) to the list O1–O6 in §5.1.1, namely:

$$\text{OM. } \alpha \sqcap (\sim\alpha \sqcup (\alpha \sqcap \beta)) \vdash \beta,$$

which should be compared with Theorem 5.1.4.1 (5).

Deducibility in this logic is defined as in **OL**, and will be denoted by $\vdash_{\mathbf{OM}}$. We will refer to this *orthomodular* logic as **OML**. (Warning: Dalla Chiara *et al.* 2002 labels it **OQL**.)

Thus, we have the following theorem.

THEOREM 5.1.4.2

$\alpha \vdash_{\mathbf{OM}} \beta$ *iff* $\vdash_{\mathbf{OM}} \alpha \underset{q}{\longrightarrow} \beta$ *iff* $v_L(\alpha) \sqsubseteq v_L(\beta)$

*for all orthomodular lattice valued valuations* $v_L$.

Now it happens that the Sasaki hook, optimal though it may be, is, nevertheless, rather anomalous: it can be shown for instance that

$$\alpha \underset{q}{\longrightarrow} (\beta \underset{q}{\longrightarrow} \alpha) \qquad (5.1.4.9)$$

is not always true. Insofar as $\underset{q}{\longrightarrow}$ reflects deducibility in **OML**, it would appear from the invalidity of (5.1.4.9) that this type of deducibility is far from being constructive in the sense of natural deduction: cf. equation (4.1.6) for instance. This intractability, unsurprisingly, shows up also in Gentzen calculi for **OML**. Here, CUT is generally not eliminable.

Our conclusion is that standard **OML** is even less suited to the purpose of constructive deduction than is ordinary classical **PC**, over and above the obviously non-constructive axioms O5 and O6 (§5.1.1). In the next chapter we attempt to redress this by extending the intuitionistic "formulae-as-types" paradigm into the quantum domain.

The logic we have termed **OML** was almost universally adopted as *the* "quantum" logic until 1981, since it had been thought to embody the logic characterized by algebraic models based on Hilbert lattices. In Greechie 1981 an example of an orthomodular lattice was produced with logical properties distinctly different from those of Hilbert type. For this reason we will in this work reserve the term Quantum Logic (**QL**) for that logic determined by the class of Hilbert lattices. (Warning: this logic is termed **HQL** in Dalla Chiara *et al.* 2002.) We will return to a consideration of this logic in the next section.

The presence of the orthomodular condition, though apparently innocent, inflicts a surprising amount of damage upon logical preconceptions. Indeed, even the notion of orthomodularity itself is, in a certain sense, *intractable,* as Goldblatt 1984a stunningly demonstrated. Specifically, although Kripkean models for **OL** can be extended to accommodate a characterization of **OML**, there is no first-order characterization of the orthomodularity condition for or-

thoframes. This has the consequence that the traditional method of proving that the relevant class of Kripke frames characterizes the logic—namely by showing that the canonical frame belongs to the class—fails for **OML**. That is, this method fails to determine a class of orthoframes which characterize **OML**. This characterization problem seems to remain open at the time of this writing. The orthomodular condition somehow causes **OML** to escape the confines of this classically designed trap.

### *5.1.5 Orthomodular Foundations for Quantum Mechanics*

It is possible to axiomatize the fundamentals of traditional quantum theory entirely within the context of orthomodular lattices—a conceptually fruitful enterprise, which is the work of many hands. A mathematically rigorous, virtually seamless and physically convincing development of these ideas was set out in Mackey 1963. (See Varadarajan 1985 for an exquisitely detailed account of this program.) A basic tenet of this approach is that the "logic" of a quantum system—i.e. the lattice of "propositions" concerning its "phase space"—should constitute an orthomodular lattice, in contradistinction to the Boolean logic of propositions that determine subsets of the phase space of a classical system. (This orthomodularity may be adduced from various assumptions concerning the measurement process, *inter alia*.) We will content ourselves here with a very brief recital of some of the basic definitions, most of which do not appear to have purely logically interpretations.

In the following definitions, $L$ will denote a countably complete orthomodular lattice: that is, $L$ contains meets and joins of countable families of elements. (As Theorem 5.1.1.2 shows, any ortholattice may be embedded into a complete one. The problem of whether or not an analogous result obtains for orthomodular lattices seems to be open at the time of this writing.)

DEFINITION 5.1.5.1 (Varadarajan 1985)

*An* observable associated with $L$ *is a mapping*

$$x:\mathcal{B}(\mathbb{R})\longrightarrow L, \tag{5.1.5.1}$$

*where* $\mathcal{B}(\mathbb{R})$ *denotes the* $\sigma$*-algebra of Borel sets of the real line, satisfying:*

1. $x(\varnothing)=0$, $x(\mathbb{R})=1$;
2. *for Borel sets* $E$, $F$ *if* $E\cap F=\varnothing$, *then* $x(E)\perp x(F)$;
3. *for a sequence* $E_i$ *of mutually disjoint Borel sets:*

$$x(\bigcup E_i)=\bigsqcup x(E_i). \tag{5.1.5.2}$$

This generalizes the classical case, for which an observable is a real-valued function, $f:X\longrightarrow\mathbb{R}$, measurable at least, on the space $X$ of states of the system, with

$$f^{-1}:\mathcal{B}(\mathbb{R})\longrightarrow\mathbf{2}^X \tag{5.1.5.3}$$

obeying properties exactly analogous to those listed in the definition.

It is shown in Varadarajan 1985, Lemma 3.4, p. 46, that for any sequence $E_i$ of Borel sets:

$$x(\bigcup E_i)=\bigsqcup x(E_i), \tag{5.1.5.4}$$

$$x(\bigcap E_i)=\bigsqcap x(E_i), \tag{5.1.5.5}$$

and for any Borel set $E$:

$$x(E^c)=x(E)'. \tag{5.1.5.6}$$

It follows from this that for any Borel sets $E$, $F$, $x(E) \leftrightarrow x(F)$.

There is a remarkable connection between the commutativity relation and the notion of an observable. Namely, it is shown in Varadarajan 1985, Lemma 3.7, that for $a,b \in L$, $a \leftrightarrow b$ iff there exists an observable $x$ and Borel sets $E$, $F$ such that $x(E)=a$ and $x(F)=b$. If we interpret $x(E)$ as the quantum version of the proposition that "a value of $x$ lies in $E$," then the proposition $x(E)$ would be verifiable "by $x$." Consequently, by the result above, if $a \leftrightarrow b$ then the "propositions" $a$ and $b$ are verifiable by the *same* observable and are therefore said to be *simultaneously verifiable,* or—to avoid spurious and misleading temporal references—*jointly verifiable,* or *co-verifiable.*

The notion of commutativity may be extended to observables in the obvious way: observables $x$ and $y$ are said to be *simultaneously observable,* or *co-measurable,* or *compatible,* if, for any Borel sets $E$, $F$, $x(E) \leftrightarrow y(F)$. Thus, two observables are compatible if, roughly speaking, they can be verified to *jointly have values.*

Now we turn to the notion of *states.* In the standard formulation of quantum mechanics a quantum system can only be ascertained to be "in" a certain "state" probabilistically. Speaking roughly again, propositions concerning the system can only be assigned probabilities of holding in a given "state." That is to say, a "state" induces a probability measure on $L$, whence the following definition.

DEFINITION 5.1.5.2

*A* state of $L$ *is a function* $p: L \longrightarrow [0,1]$ *such that:*

1. $p(0)=0$, $p(1)=1$;
2. *if* $a_i$ *is a sequence of mutually orthogonal elements of* $L$, *then* $p(\bigsqcup a_i)=\Sigma p(a_i)$.

Clearly, if $p_k$ is a sequence of states of $L$ and $c_k$ is a sequence of real numbers with $c_k \geq 0$ for all $k$ and $\Sigma c_k = 1$, then

$$p \equiv \Sigma c_k p_k \tag{5.1.5.7}$$

is also a state of $L$, representing a statistical mixture of states $p_k$, with $c_k$ playing the rôle of a probability—namely, the probability that the system will be "in" the state $p_k$ when it is "in" the mixed state $p$.

Thus, the set of states of $L$ is *convex*. The *extreme points* of this set—i.e. the states not expressible in the form shown in equation (5.1.5.7)—are called the *pure* states of $L$.

(In a simplified version of the classical case, in which $L$ is replaced by $\mathbf{2}^X$ for a set of classical states $X$, the measures corresponding to the pure states are just the Dirac measures on points of $X$. That is, for $x \in X$, the corresponding pure state is the measure on $\mathbf{2}^X$ given, for $Y \subseteq X$, by:

$$\delta_x(Y) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y \end{cases} \tag{5.1.5.8}$$

Thus, elements of $X$ may be identified with the pure states in the sense of the definition.)

Given an observable $x$ and a state $p$ of $L$, we obtain a probability measure on $\mathbb{R}$ given, for $E \in \mathcal{B}(\mathbb{R})$, by:

$$P_x^p(E) \equiv p(x(E)). \tag{5.1.5.9}$$

This quantity may be interpreted, roughly speaking, as the probability that the value of the observable $x$, when the system is "in" the state $p$, lies in $E$.

In the next section we will identify these entities in the case when $L$ is a Hilbert lattice.

## 5.2 Quantum Logic Proper: Hilbert Models

Finally we come to the logic we have dubbed **QL**, namely the logic characterized by algebraic orthomodels consisting of Hilbert lat-

tices and valuations into them. (As noted, Dalla Chiara *et al.* 2002 calls this logic **HQL**.) Hilbert lattices being orthomodular, we have:

$$\alpha \vdash_{\mathrm{O}} \beta \text{ implies } \alpha \vdash_{\mathrm{OM}} \beta \text{ implies } \alpha \vdash_{\mathrm{Q}} \beta,$$

where $\vdash_{\mathrm{Q}}$ denotes deducibility in **QL**. Unfortunately, little seems to be known about deducibility in **QL**: even an axiomatization of it seems to be lacking. What is known is that the last implication cannot be reversed, and proof of this fact had to wait until Greechie 1981, in which it is shown that there is a **QL** deduction that is invalid in **OML**.

Of course, the nonconstructive nature of **OML** is shared by **QL**, and is compounded with another highly non-classical aspect of Hilbert lattice models. This is the concern of the Kochen–Specker Theorem, which demonstrates the non-objective nature of **QL**—and by extension, of quantum theory itself—in its most brutal form. We shall briefly outline a compelling recent reformulation of this result in §5.2.2, but first we return to the definitions of *observable* and *state* given in the last subsection to see how they manifest themselves in the Hilbertian case.

### *5.2.1 Observables and States for Hilbert Lattices*

For a Hilbert space $\mathfrak{H}$ we shall habitually identify the lattice $L(\mathfrak{H})$ of closed subspaces with the lattice of projections on $\mathfrak{H}$. The corresponding operations in this lattice of projections are given explicitly—for the indicated special cases—by:

$$P_M \sqcup P_N = P_M + P_N - P_M P_N, \tag{5.2.1.1}$$

where $P_M$ denotes the projection upon the closed subspace $M$.

If $P_M P_N = P_N P_M$ then

$$P_{M \sqcap N} = P_M P_N , \tag{5.2.1.2}$$

where recall that here $M \sqcap N = M \cap N$.

Finally,

$$P'_M = P_{M^\perp} = I - P_M . \tag{5.2.1.3}$$

The Spectral Theorem (Theorem 1.2.1) specializes in various ways. For instance, if $A$ denotes a bounded self-adjoint operator on the Hilbert space $\mathfrak{H}$ the unital C*-algebra generated by $A$—denoted by $C^*(A)$—is commutative, and therefore isomorphic, *via* the Gelfand transform, with the algebra $C(\sigma(A))$ of continuous complex functions on the maximal ideal space $\sigma(A)$ of $C^*(A)$. This compact space is exactly the classical spectrum of the operator $A$, which is a subset of $\mathbb{R}$. Consequently, the naturally arising *-representation of $C^*(A) \cong C(\sigma(A))$ upon $\mathfrak{H}$ lifts to a *-representation of $C_0(\mathbb{R})$ upon $\mathfrak{H}$, and the Spectral Theorem yields a projection-valued measure $P^A$ for which:

$$\langle \xi | A | \eta \rangle = \int_{\mathbb{R}} \lambda \, d\mu_{\xi,\eta}^{P^A}(\lambda) \tag{5.2.1.4}$$

for all $\xi, \eta \in \mathfrak{H}$, where $\mu_{\xi,\eta}^{P^A}$ is defined as in Theorem 1.2.1.

The last equation may also be expressed in the form:

$$A = \int_{\mathbb{R}} \lambda \, dP^A(\lambda). \tag{5.2.1.5}$$

This result generalizes to the case in which $A$ is unbounded, and therefore definable only upon a dense subspace of $\mathfrak{H}$: in this case $\eta$ in equation (5.2.1.4) must be confined to the dense subspace in question.

Conversely, given a projection-valued measure $P$, equation

(5.2.1.5) defines a unique, not necessarily bounded self-adjoint operator on a dense subspace $\mathcal{D} \equiv \{\eta \in \mathfrak{H}: \int_{\mathbf{R}} \lambda^2 d\mu^P_{\eta,\eta}(\lambda) < \infty\}$ of $\mathfrak{H}$.

Now, it follows from equations (5.1.5.5), (5.2.1.1), (5.2.1.2) and the mutual commutation of the lattice elements in the range of an observable, that an observable associated with $L(\mathfrak{H})$ in the sense of Definition 5.1.5.1 is exactly a projection valued measure. Thus, there is a one-to-one correspondence between observables associated with $L(\mathfrak{H})$ and not necessarily bounded self-adjoint operators on $\mathfrak{H}$.

The characterization of states on Hilbert lattices is the subject of a profound theorem of Gleason. Before stating it, we recall that a bounded operator $A$ on the Hilbert space $\mathfrak{H}$ is said to be *of trace class* if:

$$\sum_i |\langle \xi_i | A | \xi_i \rangle| < \infty \tag{5.2.1.6}$$

for any orthonormal basis $\{\xi_i\}$. The *trace* of $A$ is then defined as

$$\mathrm{tr}(A) = \sum_i |\langle \xi_i | A | \xi_i \rangle|, \tag{5.2.1.7}$$

a quantity independent of the choice of orthonormal basis $\{\xi_i\}$.

A bounded self-adjoint operator $\rho$ on $\mathfrak{H}$ is a *von Neumann operator* (or *density matrix*) if:

1. $\rho$ is positive semi-definite: that is, for all $\xi \in \mathfrak{H}$, $\langle \xi | \rho | \xi \rangle \geq 0$;
2. $\rho$ is of trace class and $\mathrm{tr}(\rho) = 1$.

For a closed subspace $M$ of $\mathfrak{H}$ and von Neumann operator $\rho$ define:

$$p_\rho(M) \equiv \mathrm{tr}(P_M \rho). \tag{5.2.1.8}$$

Then it is easily verified that $p_\rho$ is a state of $L(\mathfrak{H})$. (Cf. Definition 5.1.5.2.)

THEOREM 5.2.1.1 (Gleason 1957)

*Let $\mathfrak{H}$ denote a (separable) Hilbert space of dimension at least 3. Then the map $\rho \mapsto p_\rho$ defined as in equation (5.2.2.8) is a convex isomorphism of the convex set of von Neumann operators on $\mathfrak{H}$ onto the convex set of states of $L(\mathfrak{H})$.*

*If $\rho = |\xi\rangle\otimes\langle\xi|$ for $\xi \in \mathfrak{H}$ with $\|\xi\| = 1$, then $p_\rho$ is a pure state and all pure states are of this form. Moreover, the pure states determined by two such unit vectors, $\xi, \eta$ say, are equal iff there exists some complex number $z$ with $|z| = 1$ such that $\eta = z\xi$.*

(For a detailed proof see Varadarajan 1985, and, for insight into the physical interpretation of density matrices, see Isham 1995.)

It follows from the last part of the theorem that the pure states of $L(\mathfrak{H})$ are in one-to-one correspondence with the rays—or one-dimensional subspaces—of $\mathfrak{H}$. We note also the significant fact that pure states can be superpositions of other pure states, in marked contradistinction to the case of a classical system.

Given a von Neumann operator $\rho$ and an observable $A$, the interpretation of equation (5.1.5.9) as giving the probability that the "value of $A$" lies in the Borel set $E$ when the system is "in" the state $\rho$, reads, in the Hilbert case:

$$\text{prob}(A \in E; \rho) \equiv \text{tr}(P^A(E)\rho), \qquad (5.2.1.9)$$

where $P^A$ is the projection valued measure given by the Spectral Theorem. In particular, if $\rho$ is pure, $\rho = |\xi\rangle\otimes\langle\xi|$ for some unit vector $\xi$ and we have:

$$\text{prob}(A \in E; \rho) = \text{tr}(P^A(E)|\xi\rangle\otimes\langle\xi|)$$

$$= \langle\xi|P^A(E)|\xi\rangle$$

$$= \| P^A(E)\xi \|^2. \qquad (5.2.1.10)$$

The expectation value of $A$ "in" the state $\rho$ may now be computed *via* equations (5.2.1.9) and (5.2.1.5) as:

$$\langle A \rangle_\rho \equiv \sum_{\lambda \in \mathbb{R}} \lambda \, \mathrm{prob}(A \in \{\lambda\}; \rho)$$

$$= \int_{\mathbb{R}} \lambda \, d\,\mathrm{tr}(P^A(\lambda)\rho)$$

$$= \mathrm{tr}\left( \int_{\mathbb{R}} \lambda \, dP^A(\lambda)\rho \right)$$

$$= \mathrm{tr}(A\rho). \qquad (5.2.1.11)$$

If $\rho = |\xi\rangle\otimes\langle\xi|$ is pure this yields

$$\langle A \rangle_\rho = \mathrm{tr}(A|\xi\rangle\otimes\langle\xi|)$$

$$= \langle \xi | A | \xi \rangle. \qquad (5.2.1.12)$$

The equations (5.2.1.9) – (5.2.1.12) are fundamental to the standard interpretation of quantum mechanics.

### *5.2.2 The Kochen–Specker Theorem* à les *Isham and Butterfield*

Classical logic is non-contextual in the sense that **PC** is complete with respect to valuations into the *constant* Boolean algebra **2**, which plays a special rôle among Boolean algebras thanks to the Stone Theorem. This theorem may be interpreted as expressing any Boolean algebra as a (continuous) product of **2**s indexed over the Stone space of the algebra or, equivalently, as the Boolean algebra of

continuous sections of the constant sheaf of **2**s over the Stone space of the algebra. (For a generalization of this version of the theorem to orthomodular lattices, see Graves and Selesnick 1973, or Kalmbach 1983, p. 82.)

Although orthomodular lattices have a copious supply of Boolean subalgebras, these subalgebras do not in general comport themselves in a classical manner within the ambient lattice. In fact, there is a sort of relative twisting among Boolean subalgebras of an orthomodular lattice which has the effect of confuting any attempt to impose a naïve realist, objective, or non-contextual interpretation upon this collection of apparently classical sublogics. This is compellingly revealed by a recent reformulation of the Kochen–Specker Theorem (Kochen and Specker 1967) by Isham and Butterfield *et al.* (Isham and Butterfield *et al.* 1998–2000) in terms of the theory of *toposes.* (References can be found below.) Before giving a very brief account of their important insight, we return to the issue of compatibility in Hilbert lattices.

It is not hard to prove that for two elements $M$, $N$ of the lattice $L(\mathfrak{H})$, $M \leftrightarrow N$ iff the corresponding projections commute (Varadarajan 1985, Lemma 4.9). Moreover, two observables are compatible (as in §5.1.5) iff the corresponding operators commute, in the case in which these operators are bounded (*ibid.*, Theorem 4.11): thus they have a common set of eigenvectors. The exact physical import of this relation of compatibility among observables is, needless to say, a vexed question. For a lucid discussion of it, see Isham 1995, §6.3.

As we have noted, Boolean subalgebras of the lattice $L(\mathfrak{H})$ play a crucial rôle in the interpretation of the quantum theory of the system associated to the Hilbert space $\mathfrak{H}$. Every family of mutually compatible elements of $L(\mathfrak{H})$—i.e. jointly verifiable propositions about the system—generates a Boolean subalgebra, as we have noted, and, moreover, each observable gives rise to one *via* its spectral resolution, as does every orthonormal basis. Each such subalgebra in a sense represents a "classical window" through which "states" of the system may be viewed. These "windows" are partially ordered by in-

clusion, a relation that may be interpreted as a coarsening of the resolving power of the "view" afforded by the ambient algebra to that afforded by the included algebra. The classical-like "states" that can be viewed "through" a given Boolean subalgebra $A$ of $L(\mathfrak{H})$ may be identified with the elements of that algebra's Stone space $\mathrm{Spec}\,A$, which may in turn be identified with the set of valuations of $A$ into **2**. Thus each "state" viewable through the classical window $A$ corresponds to a classical truth-value assignment to the elements of $A$. If such a "state" is to have any kind of objective reality, then it should be viewable through each classical window, and furthermore such views should be *consistent* with respect to common windows.

Such a state will then give rise to a choice, for each Boolean subalgebra $A \subseteq L(\mathfrak{H})$, of an element $\chi_A \in \mathrm{Spec}\,A$, such that if $A \subseteq B$, where $B$ is another Boolean subalgebra of $L(\mathfrak{H})$, then

$$\chi_A = \chi_B \,|\, A, \tag{5.2.2.1}$$

the vertical line denoting restriction.

As Isham and Butterfield (Isham and Butterfield 1998) point out, it is a consequence of the Kochen–Specker theorem that no such assignment exists, if the dimension of $\mathfrak{H}$ is greater than two. That is to say, for the dimensions specified, that there are *no* states, or Kripkean worlds, at which classical truth values may be consistently assigned to the propositions represented by the elements of each Boolean subalgebra: there are *no* consistent classical views through all classical windows.

Isham and Butterfield observe, furthermore, that an assignment of the above type, namely $A \mapsto \chi_A$, is *precisely* a global section of a certain *presheaf*, defined as follows. First, regard the set $\mathcal{W}$ of Boolean subalgebras of $L(\mathfrak{H})$, partially ordered by inclusion, as a category in the usual way. Then the assignment

$$\mathbf{Spec}: \mathcal{W} \longrightarrow \mathcal{S}et, \tag{5.2.2.2}$$

where $\mathcal{S}et$ denotes the category of sets, given for each $A$ in $\mathcal{V}$ by $\mathbf{Spec}(A) = \mathrm{Spec}\, A$, with morphisms in $\mathcal{V}$ being sent to restrictions in $\mathcal{S}et$, constitutes a contravariant functor, or *presheaf*, on $\mathcal{V}$. Now the category $\mathcal{S}et^{\mathcal{V}^{\mathrm{op}}}$ of such presheaves is a simple example of the structure known as a *topos* (MacLarty 1992, Asperti and Longo 1991, Bell 1988, Lambek and Scott 1986, Goldblatt 1984b), and contains, among other objects, a terminal object, denoted 1: this is the presheaf that assigns to each object in $\mathcal{V}$ the singleton set. For a presheaf $\mathbf{S}$ in this topos, a morphism $1 \longrightarrow \mathbf{S}$ is called a *global section* of $\mathbf{S}$. It is clear that the global sections of **Spec** are exactly the assignments of the form $A \mapsto \chi_A$ that tie up consistently relative to restrictions, whose existence is thwarted by the Kochen–Specker theorem.

Isham and Butterfield *et al.* exploit this insight to reevaluate realism. Although global sections of **Spec** do not exist, *local* sections certainly do, and in abundance: these are sections over subobjects of the terminal object, and correspond to consistent choices of morphisms $\chi_A$ for *some* objects $A$ of $\mathcal{V}$. Now in every topos there is an object $\mathbf{\Omega}$, the subobject classifier, with the property that subobjects of any object are classified by morphisms from that object into $\mathbf{\Omega}$. (In the topos $\mathcal{S}et$ it is the set **2**.) Local sections of **Spec** give rise to subobjects of **Spec** and hence to ("global") presheaf morphisms

$$\mathbf{Spec} \longrightarrow \mathbf{\Omega}. \tag{5.2.2.3}$$

To interpret this in more mundane terms, let $A$ now denote some arbitrary Boolean algebra. If we think of subsets of $\mathrm{Spec}\, A$ as corresponding to, or *being*, Kripkean "$A$-propositions," then $A$ is embedded by the Stone theorem as a subalgebra of the algebra of all possible $A$-propositions, namely $\mathbf{2}^{\mathrm{Spec}A}$. Functions of the form

$$\mathrm{Spec}\, A \longrightarrow \mathbf{2} \tag{5.2.2.4}$$

correspond one-to-one with $A$-propositions. In particular, the singletons $\{\chi\}$, $\chi \in \mathrm{Spec}\, A$, correspond to certain $A$-propositions, namely the atoms in the lattice of $A$-propositions. If we were now to allow $A$

to vary over the category $\mathcal{W}$, then we would not be able to select even an atomic proposition $\{\chi_A\}$ for each object $A$ of $\mathcal{W}$ in a consistent manner if $\dim \mathcal{S} > 2$, by the Kochen–Specker theorem (*à les* Isham and Butterfield). One interpretation of this negative result might be that our notion of $A$-proposition, with truth-values in **2**, is inappropriate to the context in which $A$ varies over the category $\mathcal{W}$. Indeed, since we are working in a topos there *is* a more appropriate interpretation of the notion of a (varying per object) proposition, namely as a morphism of the type displayed in equation (5.2.2.3). Thus, for each object $A$ of $\mathcal{W}$, propositions of the form equation (5.2.2.4) are replaced by those of the form

$$\mathrm{Spec}\, A \longrightarrow \mathbf{\Omega}(A) \qquad (5.2.2.5)$$

so that the static set of classical truth-values **2** in (5.2.2.4) is replaced by the context dependent set $\mathbf{\Omega}(A)$.

It is a general result that the subobject classifier in a topos is an (internal) Heyting algebra (so that each $\mathbf{\Omega}(A)$ is a Heyting algebra) and that the logic modelled by propositions of the form (5.2.2.3) is intuitionistic.

Thus, Kochen–Specker prohibitions notwithstanding, certain aspects of realism—namely the existence of global valuations—may be reinstated in quantum theory, at the expense of requiring these valuations to lie in contextually dependent Heyting algebras. This is highly suggestive, since it shows that logical consistency among the classical sublogics can be maintained if the static Boolean propositional structure is traded, per subalgebra, for an essentially *constructive* (i.e. intuitionistic) one. (In contrast, the logic of the ambient Hilbert lattice is highly non-constructive, as we note below.)

In Isham and Butterfield *et al.* 1998–2000 a finely detailed theory of Heyting algebra valued valuations is developed and interpreted (very roughly along the lines of (5.2.2.3)) for various presheaf toposes associated with Hilbert spaces.

## 5.3 Critique of Quantum Logic

Since its inception quantum logic has been roundly criticized—rather unfairly, in this writer's opinion—for its various shortcomings *vis-à-vis* a chauvinistic classicism. Its status as a logic has been hotly debated by certain philosophers, with debatable results. Their problems are compounded for those physicists who gamely strive for an "interpretation" of quantum theory, and who often find pure quantum logic lacking certain attributes they would prefer it to have: for instance, it seems to be impossible to distinguish, on quantum logical grounds, between composite systems and monolithic ones. (The category of orthomodular lattices does not have coproducts: see Svozil 1998 and Beltrametti and Cassinelli 1981.) At the same time, it must be said that quantum logical considerations have greatly clarified such subtle issues as the status of hidden variable theories, theories of measurement, etc. (Bub 1997).

It may be noted that certain criticisms leveled at quantum logic *qua* logic—such as the unpleasant behavior of the best available recipe for implication, namely the Sasaki hook—apply also to classical logic. Notable exceptions to this assertion include the intractability of the orthomodular condition, the failure of CUT elimination in Gentzen calculi for quantum logic, and the failure of other classical principles involving deducibility (Dalla Chiara *et al.* 2002, §4). These failures compound the already obvious underlying non-constructive aspects of the logic; thus it would appear that quantum logic is even more non-constructive than classical logic. This deeper difference should not be too surprising, given the ephemeral and non-objective nature of quantum "resources."

The need of computational theorists to expand the expressiveness of ordinary logic led, *via* correspondences of the Curry–Howard type, to an appreciation that the logics most appropriate to the business of computation are *intuitionistic* ones. With this model in mind, *pace* Birkhoff and von Neumann, we turn in the next chapter to an investigation of the possible computational resources available in the quantum case.

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# 6

# The Computational Resources of Quantum Logic

Standard quantum logic has been found wanting as a deductive system since deducibility in it is intrinsically non-constructive, a failing it shares with classical **PC**. In the classical case the path to a more expressive deductive logic led, through (intuitionistic) proof theoretic systems, to type theories like simply typed $\lambda$-calculus and beyond.

Since the introduction of orthomodularity apparently did nothing to ameliorate these non-constructive failings, we jettison this condition and return to a consideration of the core logic **OL** from a proof theoretic perspective as a step along the path—paralleling the route taken in the classical case—to a more expressive resource-sensitive version of quantum logic.

In detail, we first revisit the modal translation of **OL** in order to investigate possible relations between classical and quantum deductions. Here we find a crucial difference, embodied in the quantum phenomenon of superposition or "quantum parallelism," which is faithfully parodied in the Kripkean framework. It also becomes apparent from these models that the classical Heyting paradigm would fail in the orthological case, even in principle.

In the next section (§6.2) we introduce a purely syntactic minimal intuitionistic Gentzen sequent calculus based upon pre-

sumed properties of quantum types, or *resources,* and show (§6.3) that a translation of the intuitionistic fragment of **OL** into this calculus may be effected simply by invoking a *quantum* version of the Heyting paradigm (§6.3.1).

The calculus is then interpreted in the category of finite dimensional Hilbert spaces, which provides a model for it (§6.3.2). A one-step "quantum computation" is specified syntactically in the sequent calculus through the introduction of a single extra axiom. When this axiom is realized in the category of finite dimensional Hilbert spaces, the familiar *qubit* arises spontaneously as the irreducible storage capable quantum computational unit (§6.3.3).

The notion of quantum storage, accompanied by the concomitant dual notion of quantum copying or duplication, emerges directly from a consideration of the rule of Contraction as it is realized in our sequent calculus, and the need to invoke quantum entanglement in the course of implementing it is immediately apparent. This is discussed briefly in §6.3.4, which may be skipped.

In §6.4 we subvert our constructivist quantum principles in an attempt to accommodate classical time as the multiplexed storage capable version of the symbolic time quantum, or step, used in the newly added axiom. Although they are rather formal, these maneuvers reproduce (in a fairly natural manner) the standard picture of a quantum computation as being a one-parameter unitary dynamical group acting in the Schrödinger manner upon a tensor product of qubits. (We do not pursue the standard, classically timed, theory of quantum computation any further in this work.)

With this chapter we close Part II and with it our purely logical considerations. In the final part of this work we will apply some of its lessons to our primary goal of computing the fundamental physical Lagrangians.

## 6.1 An Orthological Toy

The modal translation theorem (Theorem 5.1.3.2)—even in the restricted form in which we have presented it—can be used to re-

veal an essential point of difference between (non-constructive) classical and quantum (or, rather, **OL**) deduction.

To effect such a comparison we note that **PC**, which we may regard as the case of a modal system in which the modalities are collapsed, is complete with respect to the class of degenerate Kripke orthomodels $\langle S,=,v\rangle$, in which every subset of $S$ is a proposition. (For, Boolean lattices of subsets suffice to model **PC**, in view of the Stone Theorem.) We shall interpret the elements of $S$ as the states of some classical "physical system" and, every theorem of **PC** being valid in this model, we have, for $s\in S$ and formula $a$ (chosen to be atomic for simplicity):

$$s\models a\vee\neg a \tag{6.1.1}$$

(omitting the turnstile subscript). The "semantics" of this assertion reads:

$$s\models a \text{ or } s\models\neg a. \tag{6.1.2}$$

We may think of $a$ as a variable describing some aspect of the system, such as a dial being lit on the front of the system's "case." Then equation (6.1.1) asserts the classical triviality that in each state of the system the dial light can be either on or off, and this limited repertoire exhausts the informational content provided in each state by the given theorem.

This trivial classical example exhibits, upon quantization, decidedly non-trivial behaviors, however. By "quantization" we mean that the trivial orthomodel $\langle S,=,v\rangle$ is suitably replaced by one in which the trivial proximity relation "=" is replaced by a more general one, "$\approx$," to obtain a non-trivial B-model, $\mathcal{B}$ say, while the **OL** version of the theorem in equation (6.1.1) is translated into its modal form. (This notion of quantization may be formalized: cf. Rawling and Selesnick 2000.) With $\Leftrightarrow$ denoting logical equivalence in the B-modal system, the modal translation (T1–T3, §5.1.3) of the or-

thotheorem $a \sqcup {\sim}a$ yields:

$$
\begin{aligned}
(a \sqcup {\sim}a)^{\circ} &\equiv ({\sim}({\sim}a \sqcap a))^{\circ} \\
&\equiv \Box\neg({\sim}a \sqcap a)^{\circ} \\
&\equiv \Box\neg(\Box\neg a^{\circ} \wedge a^{\circ}) \\
&\equiv \Box\neg(\Box\neg\Box\Diamond a \wedge \Box\Diamond a) \\
&\Leftrightarrow \Box\neg(\Box\Diamond\Box\neg a \wedge \Box\Diamond a) \\
&\Leftrightarrow \Box\neg(\Box\neg a \wedge \Box\Diamond a) \\
&\qquad \text{(cf. (M8), Proposition 5.1.2.1)} \\
&\Leftrightarrow \Box(\neg\Box\neg a \vee \neg\Box\Diamond a) \\
&\Leftrightarrow \Box(\Diamond a \vee \Diamond\Box\neg a) \\
&\Leftrightarrow \Box\Diamond(a \vee \Box\neg a) \qquad (6.1.3) \\
&\qquad \text{(cf. (M1), Proposition 5.1.2.1).}
\end{aligned}
$$

Thus, at each "quantum state" $w$ of the B-model we have

$$w \models_{\mathcal{B}} \Box\Diamond(a \vee \Box\neg a), \qquad (6.1.4)$$

which unfolds to read

$$\forall v \approx w \; \exists x \approx v \text{ such that } x \models_{\mathcal{B}} a \text{ or } x \models_{\mathcal{B}} \Box\neg a. \qquad (6.1.5)$$

In contrast to the classical case, a commitment to the truth of either $a$ or $\neg a$ is *not* made at each $w$. Rather, it is made at certain granddaughter states, and both outcomes are possible: there may be some states accessible from states accessible from $w$ at which $a$ is true and others at which $a$ is false. Thus, generic states, in a sense, carry attributes pertaining to *both* possible outcomes simultaneously. This property persists through any imagined evolution of states into other states, and is an expression of "quantum parallelism"—or superposition—in the context of this model. It is precisely this notion that lies at the heart of the current interest in "quantum computing." Proponents of this idea hope to harness this massive and instantaneous quantum parallelism to computational ends.

In the classical case the modalities are collapsed: there is no branching of accessibilities and therefore no parallelism.

It may be noted also that there is a qualitative difference, apparent even in this crude logical model, between the two possible outcomes at states like $x$ in (6.1.5). Namely, in the case that

$$x \models_{\mathcal{B}} \Box\neg a \tag{6.1.6}$$

obtains, we would have

$$z \models_{\mathcal{B}} \neg a \tag{6.1.7}$$

for *all* $z \approx x$. This might not be the case if only the other outcome transpires: in that case there could be states accessible from $x$ at which either condition obtains. If we were to extend the classical analogy we used earlier, then we might conclude that, if case (6.1.6) obtained, the dial light would be off, and stay off, whereas had only $x \models_{\mathcal{B}} a$ obtained, the condition of the light would be in a sense indeterminate. Or, rather, in the absence of extraneous probabilistic assumptions, its condition in the latter case would not be interpretable in classical terms, which are in any case inappropriate here.

The case of an orthotheorem of the general form $\alpha \sqcup \beta$ may

be treated similarly. Namely,

$$(\alpha \sqcup \beta)^\circ \equiv (\sim(\sim\alpha \sqcap \sim\beta))^\circ$$

$$\equiv \Box\neg(\sim\alpha \sqcap \sim\beta)^\circ$$

$$\equiv \Box\neg(\Box\neg\alpha^\circ \wedge \Box\neg\beta^\circ)$$

$$\Leftrightarrow \Box(\Diamond\alpha^\circ \vee \Diamond\beta^\circ)$$

$$\Leftrightarrow \Box\Diamond(\alpha^\circ \vee \beta^\circ). \qquad (6.1.8)$$

Thus, we have at each $w$ (in each B-model $\mathcal{B}$):

$$w \models_{\mathcal{B}} \Box\Diamond(\alpha^\circ \vee \beta^\circ), \qquad (6.1.9)$$

which is equivalent to

$$\forall v \approx w \; \exists x \approx v \text{ such that } x \models_{\mathcal{B}} \alpha^\circ \text{ or } x \models_{\mathcal{B}} \beta^\circ. \qquad (6.1.10)$$

Consequently, similar conclusions regarding quantum parallelism apply to theorems which can be written as orthodisjuncts.

This property is an aspect of the circumstance noted earlier (equation (5.1.2.11)) that, at a world $w$ in a Kripke orthomodel, one could have $w \models \alpha \sqcup \beta$ without having either $w \models \alpha$ or $w \models \beta$. With $\varrho$ as before denoting a valuation in such a model,

$$w \models \alpha \sqcup \beta \text{ iff } w \in \varrho(\alpha \sqcup \beta) \text{ iff } \{w\}^{\perp\perp} \subseteq \varrho(\alpha \sqcup \beta) = \varrho(\alpha) \sqcup \varrho(\beta)$$

so that $\{w\}^{\perp\perp}$ may be regarded as representing a *deduction* of $\varrho(\alpha) \sqcup \varrho(\beta)$ while *not* representing a deduction of either $\varrho(\alpha)$ or $\varrho(\beta)$. Deductions of orthodisjuncts are not necessarily determined by

deductions of either component. Herein lies one of the highly non-constructive aspects of quantum logic and one that stands in the way of a direct application of the standard Heyting paradigm (§4.2) to effect a transition to an intuitionistic "quantum" type theory, since, in this case, the classical set of deductions of a "quantum" disjunct cannot be identified with the sum of the classical sets of deductions of the individual components. Rather, some quantum version of the paradigm is called for.

## 6.2 GQ: A Minimal Intuitionistic Propositional Sequent Calculus for Quantum Resources

In this section we initiate an entirely syntactic attempt to specify a "quantum" type theory in formal imitation of the Curry-Howard correspondence. As we have learnt, the ordinary set theoretic type combinators are inadequate as intuitionistic models here, so new ones must be introduced: this will be done by means of an intuitionistic Gentzen calculus that we shall dub **GQ**. Upper case Latin characters, $A$, $B$, ... shall be used to denote formulae (or, synonymously, types) in **GQ** and we leave the choice of atoms in abeyance.

The multiplicative operation on types that is supposed to correspond intuitionistically to the $\sqcap$ of **OL** ($\times$ corresponds to $\wedge$ in the ordinary Curry–Howard correspondence) will, for obvious reasons, be denoted by $\otimes$. Similarly, the operation on types corresponding intuitionistically to the $\sqcup$ of **OL** will be denoted by $\oplus$, and that corresponding to $\sim(\,)$ by $(\,)^*$. These symbols ($\otimes$, $\oplus$, $(\,)^*$) should not (yet!) be confused with their linear algebra counterparts: their use here is purely syntactic, the purpose being to bring to the fore the *logical* connections between the intuitionistic fragment of **OL** to be discussed below, and the Gentzen system at hand. Different symbols could (and probably should) be used, but this option seems specious.

We shall discuss the rules of **GQ** informally first, collecting them together later in an official tabulation. Recall that an intuitionistic sequent calculus is one which is supposed to be a metacalculus for some (notional or derivable) underlying natural deduction sys-

tem, so that only single formulae—or none at all—are allowed on the right hand sides of sequents. We introduce the notation $D$ to represent either a single formula or the absence of a formula (i.e. the null sequence). Otherwise, upper case Greeks will denote (possibly empty) sequences of formulae.

In constructing these rules, we have taken seriously the notion of *discharging* hypotheses in natural deduction: cf. §4.2. The turnstile $\vdash$ will be read as a kinematical interface through which formulae (quantum resources) may be discharged, this process being registered by the production of the starred version of the formula on the other side of the turnstile. The idea is that a deduction

$$\begin{array}{c} A \\ \vdots \\ B \end{array} \tag{6.2.1}$$

in the notional underlying natural deduction system results in the discharge of $A$ while $B$ is produced. Put another way, $A$ is discharged *in the presence of* $B$, resulting in the inference:

$$\frac{\begin{array}{c} A \\ \vdots \\ B \end{array}}{A^* \otimes B} \tag{6.2.2}$$

Here, $A^*$ may be regarded informally as the *discharged version* of $A$ and read "$A$ discharged." In sequent language, this is expressed as

$$\frac{A \vdash B}{\vdash A^* \otimes B} \tag{6.2.3}$$

which may be read: if $A$ produces $B$ then it is the case that $A$ is discharged in the presence of $B$.

If $A$ produces nothing, as in $A \vdash$, then it may discharge by

itself:

$$\frac{A\vdash}{\vdash A^*} \tag{6.2.4}$$

Similarly, from the presumed behavior of the quantum interface, a sequent of the form $\Gamma, A\vdash B$ has the reading that $A$ in the presence of $\Gamma$ produces $B$, and may be discharged independently of $\Gamma$. This yields the rule:

$$\frac{\Gamma, A\vdash B}{\Gamma\vdash A^*\otimes B} \tag{6.2.5}$$

We presume in addition that this interface is symmetrical to the extent allowed by the structural constraints of an intuitionistic calculus. For instance, if it is the case that $A$ and $B$ are "present," namely $\vdash A\otimes B$, then $A$ may be discharged to produce $B$ and this process is also a deduction. This yields the rule:

$$\frac{\vdash A\otimes B}{A^*\vdash B} \tag{6.2.6}$$

in which the lower sequent may be read: $A$ discharged produces $B$. (Here, $B$ may be absent.)

Similarly, this may be done in the case in which $\Gamma\vdash A\otimes B$. $A$ discharged in the presence of $\Gamma$ then produces $B$:

$$\frac{\Gamma\vdash A\otimes B}{\Gamma, A^*\vdash B}. \tag{6.2.7}$$

The rules (6.2.5) and (6.2.7) are the rules of *negation* in our

calculus. Here, negation is seen as a form of discharge, absorption or annihilation, and should not be confused with negation in **OL**, just as negation in **PC** should not be confused with **IL**-negation. (In fact, negation in **IL** may also be seen as a form of annihilation: to **IL**-negate a formula one must deduce falsity from any purported proof of it. That is, identifying the formula $A$ with its set of proofs, and denoting by **f** the *falsum*, or logical constant for falsity, the **IL**-negation of $A$ may be written $A \Rightarrow \mathbf{f}$. Thus $A$ is consigned to the logical vacuum, or annihilated.)

We now consider the structural rules. We shall retain the Exchange rules (cf. (4.3.2)), insofar as they may be applied under intuitionistic constraints, since there is no implicit logical ordering of the component formulae in **OL** conjuncts. (Issue could certainly be taken with this point, but we shall adopt this option here, if only for reasons of simplicity.)

The other structural rules are more problematical. We will adopt as our informal guide in these considerations a *quantum* version of the Heyting paradigm. Thus, we will think of the resource (or type) $A$ as behaving *like* a "quantum set" of deductions of some underlying **OL** formula. Thus, the *terms* of type $A$ will be like deductions of some **OL** formula, but subject to quantum operations such as superposition. This should not be taken in any literal sense, since our purpose here is merely to arrive at a collection of logical or syntactical rules. (Later, in §6.3, we will indeed take this quantum version of the Heyting paradigm more literally.)

Consider the rule of Contraction (cf. §4.3.4), which has only a left form in the intuitionistic calculus, namely:

$$\frac{A, A, \Gamma \vdash D}{A, \Gamma \vdash D}. \qquad (6.2.8)$$

In the classical case of a notional underlying natural deduction system this is justified, since the sets of labels for the two collections of deductions of the formula corresponding to the type $A$ can be amal-

gamated into a single set, while remaining intact, and discharged simultaneously: only a single invocation of $A$ is therefore required. These resources are storage capable. Deductions associated with one occurrence of $A$ can be copied, or duplicated for the benefit of other occurrences of $A$. In our quantum case certain terms associated with one of the occurrences of the resource $A$ may be annihilated in the course of a deduction while some of those associated with the other occurrence may remain. Thus, amalgamation into single collection may not be possible, owing to the evanescence of quantum processes, and we must jettison Contraction as a general rule. This has the consequence that general quantum resources of this type are *not* storage capable.

The meaning of Weakening (cf. (4.3.3)), which it will prove convenient to express in the form

$$\frac{\Gamma \vdash D}{\Gamma, A \vdash D}, \tag{6.2.9}$$

may be interpreted analogously, in the natural deduction model, as the capability of introducing spurious, or null, collections of $A$ occurrences which have no contextual side effects. This seems contrary to the general behavior of actual quantum resources: the introduction of new quantum acts into extant arrangements of acts may interfere with the behavior of those arrangements. (Consider, for example, the interposition of a filter between orthogonal polarizers on an optical bench. Photons previously blocked may now pass through the array.) Unless $A$ is somehow insulated, its introduction might affect the context $\Gamma$ by mixing or superposition so that $\Gamma, A \vdash D$ is not guaranteed. Thus, we must also relinquish Weakening as a general rule.

If this were all that could be said about the structural rules, our investigation would end here. For, in the absence of storage capable elements, no useful computations could be carried out, even in principle: iterative processes would be blocked and the calculus would be useless. Consequently, inspired by Girard in a similar con-

text (Girard *et al.* 1988), we will institute a search for possible special instances of quantum resources for which the structural rules might be reinstated: or, rather, we search for the logical rules which specify such resources. Before embarking on this, we note that, in light of the discussion above, if a storage capable resource could be found for which Contraction holds, then the annihilation or discharge of the terms belonging to separate instances of it in any sequent in which it appears more than once, must, in a sense, be coordinated. Thus quantum duplication would necessarily be associated with some kind of coordination or correlation of terms distributed over separate instances of the resource. This observation will be confirmed in detail later, a circumstance that has profound consequences (cf. §6.3.4).

We continue to adopt as our informal guide in this search a quantum version of the Heyting paradigm, which, curiously, will turn out to work formally as well, revealing better behavior than its non-quantum counterpart. Let us suppose, then, that a quantum type $A$ is in fact the "quantum set" of deductions of some orthoformula $\alpha$. Then the terms of $A$ denote deductions of $\alpha$, so certain terms of $A$ may be annihilated or discharged without affecting $\alpha$ itself. Thus, $\alpha$ may be used to *regenerate* $A$. So, in this case, reuses of the resource—and its concomitant storage capability—could be envisaged. Of course, $A$ cannot generally be regarded as being of this type, but we could try this idea at the next level. Namely, let us assume that the (quantum) set of proofs of $A$ could itself be assembled into a quantum type, denoted $!A$ (pronounced *of course A*: name and notation due to Girard). Then identical considerations apply to $A$ rather than to $\alpha$, with $!A$ now being storage capable and, presumably, subject to the rule of Contraction.

Moreover, reverting to the case in which $A$ models the quantum set of deductions of some orthoformula $\alpha$, there is a collection of deductions of $\alpha$ corresponding to instances of axiom O6 of §5.1.1. Each such deduction corresponds to the inclusion of the proposition $\varnothing$ into any proposition in any Kripke orthomodel. We could introduce a spurious **OL** deduction of the form $\varpi \vdash_0 \alpha$, where $\varpi$ denotes the *quantum falsum*, which is a logical constant sent to 0

in any algebraic orthomodel. This spurious deduction of $\alpha$ would then give rise to a term of type $A$ denoting in a sense the generic spurious quantum collection associated with $A$. Recapitulating this at the higher level in which $A$ replaces $\alpha$ and $!A$ replaces $A$, the existence of such a spurious quantum collection—associated now with $!A$—could presumably be used to implement Weakening for $!A$ in place of $A$ in (6.2.9).

In addition to Contraction and Weakening—which we now posit for formulae of the form $!A$—we require two more rules pertaining to the operator !. The first,

$$\frac{\Gamma, A \vdash D}{\Gamma, !A \vdash D}, \tag{6.2.10}$$

reads informally (continuing the strain of wishful thinking of the previous paragraph): if $\Gamma$ in the presence of $A$ can produce the resource $D$, then $\Gamma$ in the presence of the type representing all proofs of $A$ can also produce $D$. This would be reasonable if we were to adopt the axiom $A \vdash A$, which we shall be doing.

The second rule asserts the basic defining property of the operator ! : namely, in informal terms, it specifies the explicit circumstances under which a formula $A$ may determine $!A$. To wit:

$$\frac{!\Gamma \vdash A}{!\Gamma \vdash !A}. \tag{6.2.11}$$

(Here $!\Gamma \equiv !A_1, !A_2, \ldots, !A_n$ if $\Gamma \equiv A_1, A_2, \ldots, A_n$.) If $A$ has been produced through the possibly repeated use of the storage capable resources $!\Gamma$, then these resources may also be used to produce the multiplexed or repeatable version of $A$, namely $!A$. In this rule a formula $A$ must actually be present.

We can now dismantle the preceding verbal scaffolding and formally display the basic sequents of our calculus. Recall that $D$ stands for a single formula or no formula (the empty sequence).

When it appears in the form $\otimes D$, the $\otimes$ sign is understood to be absent when $D$ is empty.

## GQ

### Structural Rules

#### Exchange

$$\frac{\Gamma, A, B, \Gamma' \vdash D}{\Gamma, B, A, \Gamma' \vdash D}\,\mathrm{LE} \qquad \frac{\Gamma \vdash A \otimes B}{\Gamma \vdash B \otimes A}\,\mathrm{RE} \qquad (6.2.12)$$

#### Weakening

$$\frac{\Gamma \vdash D}{\Gamma, !A \vdash D}\,\mathrm{LW} \qquad \text{No RW} \qquad (6.2.13)$$

#### Contraction

$$\frac{!A, !A, \Gamma \vdash D}{!A, \Gamma \vdash D}\,\mathrm{LC} \qquad \text{No RC} \qquad (6.2.14)$$

### The Identity Group

#### Axiom

$$A \vdash A \;\; \mathrm{Ax} \qquad (6.2.15)$$

#### Cut

$$\frac{\Gamma \vdash A \qquad A, \Gamma' \vdash D}{\Gamma, \Gamma' \vdash D}\,\mathrm{CUT} \qquad (6.2.16)$$

**LOGICAL RULES**

CONJUNCTIVE (MULTIPLICATIVE) CONNECTIVE

$$\frac{\Gamma, A, B \vdash D}{\Gamma, A \otimes B \vdash D} \mathrm{L}\otimes \qquad \frac{\Gamma \vdash A \quad \Gamma' \vdash B}{\Gamma, \Gamma' \vdash A \otimes B} \mathrm{R}\otimes \tag{6.2.17}$$

DISJUNCTIVE (ADDITIVE) CONNECTIVE

$$\frac{\Gamma, A \vdash D \quad \Gamma, B \vdash D}{\Gamma, A \oplus B \vdash D} \mathrm{L}\oplus \qquad \frac{\Gamma \vdash A}{\Gamma \vdash A \oplus B} \mathrm{R}\oplus_1 \tag{6.2.18a}$$

$$\frac{\Gamma \vdash B}{\Gamma \vdash A \oplus B} \mathrm{R}\oplus_2 \tag{6.2.18b}$$

NEGATION

$$\frac{\Gamma \vdash A \otimes D}{\Gamma, A^* \vdash B} \mathrm{L}^* \qquad \frac{\Gamma, A \vdash D}{\Gamma \vdash A^* \otimes D} \mathrm{R}^* \tag{6.2.19}$$

!

$$\frac{\Gamma, A \vdash D}{\Gamma, !A \vdash D} \mathrm{L}! \qquad \frac{!\Gamma \vdash A}{!\Gamma \vdash !A} \mathrm{R}! \tag{6.2.20}$$

The rule L⊗ is a *formation* rule, while R⊗ is inherited from O8, etc. (§5.1.1.) It is apparent that **GQ** bears a close resemblance to a fragment of Linear Logic (**LL**) (cf.: Troelstra and Schwichtenberg 2000, Abramsky 1993, Blute *et al.* 1993, Asperti and Longo 1991, Seely 1989, Girard *et al.* 1988 among many other references to this

vast subject.) It is in fact equivalent to a degenerate form of a fragment of this logic: namely a version of **LL** in which the operators $\otimes$ and $\oplus$ coincide with their dual forms. (**LL** often flirts with misconstruals by using the sign $\perp$ for negation.) Various formal connections between versions of **LL** and versions of quantum logic have already been proposed: cf. the last part of Dalla Chiara *et al.* 2002 for an account of some of these and references to others. None of these seem to be obviously identical to what we have proposed here.

Denoting proof in **GQ** by $\vdash_{GQ}$ we note the following.

LEMMA 6.2.1

1. $A \vdash_{GQ} A^{**}$ *and* $A^{**} \vdash_{GQ} A$ *for any* $A$.

2. $$\frac{\Gamma, A \vdash_{GQ} B}{\Gamma, B^* \vdash_{GQ} A^*}$$

PROOF

1. For the first assertion apply Ax, then R* with $\Gamma$ empty, then L*. Similarly for the second.
2. Apply R*, then RE, then L*. ∎

## 6.3 Intuitionistic Orthologic and GQ

If the formal calculus we have posited above is to properly reflect deductions in the underlying deductive system it purports to describe, then we should be able to reproduce this underlying system within the calculus itself. Of course, the best that we could hope for would be to recover an intuitionistic version of this underlying system, namely **OL**.

We obtain an intuitionistic version of **OL** by discarding the **IL** invalid axioms—namely O5 and O6 (cf. §5.1.1)—and adding new ones for the disjunctive connective, since the De Morgan Law does not hold intuitionistically. We denote these connectives by the

same symbols as before. The rules for the resulting system—which we shall call **IOL**—are displayed below.

AXIOMS

IO1. $\alpha \vdash \alpha$

IO2. $\alpha \sqcap \beta \vdash \alpha$

IO3. $\alpha \sqcap \beta \vdash \beta$

IO4. $\alpha \vdash \alpha \sqcup \beta$

IO5. $\beta \vdash \alpha \sqcup \beta$

IO6. $\alpha \vdash \sim\sim\alpha$

INFERENCE RULES

IO7. $$\frac{\alpha \vdash \beta \quad \beta \vdash \gamma}{\alpha \vdash \gamma}$$

IO8. $$\frac{\alpha \vdash \beta \quad \alpha \vdash \gamma}{\alpha \vdash \beta \sqcap \gamma}$$

IO9. $$\frac{\beta \vdash \alpha \quad \gamma \vdash \alpha}{\beta \sqcup \gamma \vdash \alpha}$$

IO10. $$\frac{\alpha \vdash \beta}{\sim\beta \vdash \sim\alpha}$$

Deduction in **IOL** will be defined as it is in **OL** and denoted by $\vdash_{IO}$.

### *6.3.1 A Translation Theorem*

We now attempt to translate **IOL** formulae into **GQ** formulae (assuming some common set of atoms) by reinstating some of the scaffolding used to arrive at the **GQ** rules. Specifically, we return to the informal reading of $!A$ as the "quantum set of proofs of $A$." Then we try a translation that is simply the (quantum) Heyting paradigm applied recursively to the logical operators. That is to say, if

$\alpha^e$ denotes the **GQ** formula that is the translated version of the **IOL** formula $\alpha$ with

$$\alpha^e \equiv \alpha \qquad \text{for } \alpha \text{ atomic,} \qquad (6.3.1.1)$$

then the (quantum) Heyting paradigm yields exactly:

$$(\alpha \sqcap \beta)^e = !\alpha^e \otimes !\beta^e, \qquad (6.3.1.2)$$

$$(\alpha \sqcup \beta)^e = !\alpha^e \oplus !\beta^e, \qquad (6.3.1.3)$$

$$(\sim\alpha)^e = (!\alpha^e)^*. \qquad (6.3.1.4)$$

Thus, equation (6.3.1.2) in this reading states: the **GQ** translation of $\alpha \sqcap \beta$ is as (the quantum set of proofs of $\alpha^e$)$\otimes$(the quantum set of proofs of $\beta^e$). Equation (6.3.1.4) in this reading states: the translation of $\sim\alpha$ is as the annihilator of all proofs of $\alpha^e$. This is the correct intuitionistic interpretation of falsity: every possible proof is refuted.

This translation will be recognized immediately by readers conversant with **LL** as being almost identical with the Girard embedding of **IL** into **LL**. It is worth noting that the Heyting paradigm in its simple pristine non-quantum form is not usually invoked to motivate the Girard embedding, though presumably it lurks somewhere in the background. In the quantum case it seems to work perfectly and in explicit detail, and will soon restore us to the comforting arms of Finkelstein's extensor calculus, thus closing the circle and broadening certain perspectives in the process.

Specifically, with the translation rules given above, we have the following.

THEOREM 6.3.1.1

$$\textit{If } \alpha \vdash_{\mathbf{IO}} \beta \textit{ then } !\alpha^e \vdash_{\mathbf{GQ}} \beta^e.$$

PROOF

Before embarking upon the proof, some motivational remarks are in order. We note first that the presence of the **GQ** formula $\alpha^e$ may be fleeting, whereas the **IOL** formula $\alpha$ is static and repeatable. Consequently, to have any expectation that the deduction $\alpha \vdash_{\mathbf{IO}} \beta$ may be translatable into a proof in **GQ**, we should render the "producer" $\alpha^e$ repeatable in **GQ**. Only then may deductions in **IOL**, which may require repeated uses of $\alpha$, be done also in **GQ**: this explains the presence of $!\alpha^e$ in the translated version of the deduction.

The proof of the theorem is by induction on the length of a deduction: that is, the number $n$ of steps in a deduction $s_1; s_2; \ldots; s_n$ of the sequent $s_n$, where the axioms and inference rules used are those of **IOL**.

A deduction with one step must be an axiom, and we first prove the theorem for each axiom in turn.

The proof for (IO1):

For any $\alpha$,

$$\frac{\alpha^e \vdash_{\mathbf{GQ}} \alpha^e}{!\alpha^e \vdash_{\mathbf{GQ}} \alpha^e} \quad \text{L!}$$

For (IO2):

For any $\alpha$,

$$\cfrac{\cfrac{\cfrac{!\alpha^e \vdash_{\mathbf{GQ}} \alpha^e}{!\alpha^e, !\beta^e \vdash_{\mathbf{GQ}} \alpha^e}\ \text{above and W!}}{!\alpha^e \otimes !\beta^e \vdash_{\mathbf{GQ}} \alpha^e}\ \text{L}\otimes}{!(!\alpha^e \otimes !\beta^e) \vdash_{\mathbf{GQ}} \alpha^e}\ \text{L!}$$

or $$!(\alpha \sqcap \beta)^e \vdash_{GQ} \alpha^e .$$

For (IO3): Similar to (IO2), but using LE to interchange $!\alpha^e$ and $!\beta^e$ after the second step.

For (IO4):
$$\frac{!\alpha^e \vdash_{GQ} !\alpha^e}{!\alpha^e \vdash_{GQ} !\alpha^e \oplus !\beta^e} \quad R\oplus_1$$

For (IO5): Similar, using $R\oplus_2$.
For (IO6):

For any $\alpha$,
$$\frac{(!\alpha^e)^* \vdash_{GQ} (!\alpha^e)^*}{!(!\alpha^e)^* \vdash_{GQ} (!\alpha^e)^*} \quad L!$$
$$\frac{!(!\alpha^e)^* \vdash_{GQ} (!\alpha^e)^*}{(!\alpha^e)^{**} \vdash_{GQ} (!(!\alpha^e)^*)^*} \quad \text{Lemma 6.2.1(2)}$$
$$\frac{(!\alpha^e)^{**} \vdash_{GQ} (!(!\alpha^e)^*)^*}{!\alpha^e \vdash_{GQ} (!(!\alpha^e)^*)^*} \quad \text{Lemma 6.2.1(1) and CUT.}$$

But
$$(\sim\sim\alpha)^e = (!(\sim\alpha)^e)^*$$
$$= (!(!\alpha^e)^*)^*$$

which proves the theorem for (IO6).

The inductive hypothesis for $n$ is that the theorem holds for the last sequent in all **IOL** deductions of length less than $n$. (The base case has been covered above.)

Consider a deduction $s_1; s_2; \ldots; s_n$ of length $n$. If $s_n$ is an axiom then we are done, as above. If $s_n$ is not an axiom, then it follows from a rule of inference applied to preceding sequents. Each preceding sequent is itself the result of a shorter deduction, so the theorem holds for each of these, by the induction hypothesis.

We consider each possible rule of inference in turn.

(IO7): We suppose that $s_n$ is of the form $\alpha \vdash_{IO} \gamma$ and follows, *via*

(IO7), from preceding deductions $\alpha \vdash_{IO} \beta$ and $\beta \vdash_{IO} \gamma$. Since, as remarked, these latter deductions are shorter than $n$, the theorem holds for them, namely $!\alpha^e \vdash_{GQ} \beta^e$ and $!\beta^e \vdash_{GQ} \gamma^e$. It follows from R! that $!\alpha^e \vdash_{GQ} !\beta^e$ and then from CUT that $!\alpha^e \vdash_{LL} \gamma^e$, so the theorem holds for this $s_n$.

(IO8): If $s_n$ is of the form $\alpha \vdash_{IO} \beta \sqcap \gamma$ and follows, *via* (IO8), from prior deductions $\alpha \vdash_{IO} \beta$ and $\alpha \vdash_{IO} \gamma$, then, as above, $!\alpha^e \vdash_{GQ} !\beta^e$ and $!\alpha^e \vdash_{GQ} !\gamma^e$. So

$$\dfrac{\dfrac{!\alpha^e \vdash_{GQ} !\beta^e \quad !\alpha^e \vdash_{GQ} !\gamma^e}{!\alpha^e, !\alpha^e \vdash_{GQ} !\beta^e \otimes !\gamma^e}\ R\otimes}{!\alpha^e \vdash_{GQ} !\beta^e \otimes !\gamma^e}\ LC$$

or $$!\alpha^e \vdash_{GQ} (\beta \sqcap \gamma)^e$$

so the theorem holds for this $s_n$.

(IO9): If $s_n$ is of the form $\beta \sqcup \gamma \vdash_{IO} \alpha$, etc., then we have

$$\dfrac{\dfrac{!\beta^e \vdash_{GQ} \alpha^e \quad !\gamma^e \vdash_{GQ} \alpha^e}{!\beta^e \oplus !\gamma^e \vdash_{GQ} \alpha^e}\ L\oplus}{!(!\beta^e \oplus !\gamma^e) \vdash_{GQ} \alpha^e}\ L!$$

or $$!(\beta \sqcup \gamma)^e \vdash_{GQ} \alpha^e,$$

so the theorem holds for this $s_n$.

(IO10): If $s_n$ is of the form $\sim\beta \vdash_{IO} \sim\alpha$ and follows, *via* (IO10), from the shorter deduction $\alpha \vdash_{IO} \beta$, then

$$\dfrac{\dfrac{\dfrac{!\alpha^e \vdash_{\mathbf{GQ}} \beta^e}{!\alpha^e \vdash_{\mathbf{GQ}} !\beta^e}\ \text{R!}}{(!\beta^e)^* \vdash_{\mathbf{GQ}} (!\alpha^e)^*}\ \text{Lemma 6.2.1(2)}}{!(!\beta^e)^* \vdash_{\mathbf{GQ}} (!\alpha^e)^*}\ \text{L!}$$

which is $!(\sim\beta)^e \vdash_{\mathbf{GQ}} (\sim\alpha)^e$ so the theorem holds for this $s_n$. ∎

### *6.3.2 The Model Category*

The pioneering efforts of J. Lambek (Lambek and Scott 1986 and references therein)—who demonstrated a perfect correspondence between certain categories (namely the closed cartesian ones) and certain typed $\lambda$- calculi (namely the $\lambda\beta\eta$-calculi with surjective pairing)—have led to a general appreciation that certain categories provide good models for certain type theories. In such a model, the types (or formulae) are interpreted as objects in an appropriate category, and deductions are interpreted as morphisms going between the appropriate objects.

In the case of our system **GQ** the choice of category in which to carry out such an interpretation would be clear on physical and constructive grounds, even if we had used a different notation: namely, the category $\mathcal{H}_F$ of finite dimensional (complex) Hilbert spaces. To carry out this interpretation, we need to specify, for each unnamed atomic **GQ** formula, a corresponding object in $\mathcal{H}_F$. Supposing this to be done, we then obtain for each **GQ** formula $A$ an object of $\mathcal{H}_F$ merely by interpreting the occurrences of $\otimes$, $\oplus$, $(\,)^*$ in $A$ as carrying their usual meaning in the category $\mathcal{H}_F$. We could now proceed informally by considering **GQ** formulae to be finite dimensional Hilbert spaces, and, leaving aside for a moment the interpretation of the operator !, we could replace each comma in a non-empty sequence $\Gamma$ by $\otimes$ and each empty sequence by $\mathbb{C}$. **GQ** sequents $A \vdash_{\mathbf{GQ}} B$ are then interpreted inductively as elements of $\mathrm{Hom}(A,B)$

according to the interpretations specified for the inference rules. For instance, $A \vdash_{\mathbf{GQ}} A$ (Ax) shall be interpreted as (or by) the identity map $1_A \in \mathrm{Hom}(A,A)$: the other rules hold in the category $\mathcal{H}_F$ and linear maps may be built up which interpret **GQ** proofs in an obvious way. Thus, for example, in the case of CUT, if we have a proof interpreted as an element of $\mathrm{Hom}(\Gamma,A) \cong \Gamma^* \otimes A$ and a proof interpreted as an element of

$$\mathrm{Hom}(A \otimes \Gamma', D) \cong (A \otimes \Gamma')^* \otimes D \cong A^* \otimes (\Gamma')^* \otimes D,$$

then the tensor product of these two elements lies in

$$\Gamma^* \otimes A \otimes A^* \otimes (\Gamma')^* \otimes D.$$

The $A \otimes A^*$ component may now be contracted (small c!) to yield an element in $(\Gamma \otimes \Gamma')^* \otimes D$. We specify this element as the interpretation in $\mathcal{H}_F$ of the proof $\Gamma, \Gamma' \vdash_{\mathbf{GQ}} D$ given by the CUT rule applied to the original proofs. The other rules not involving ! can be treated similarly, using the properties of the connectives in $\mathcal{H}_F$, an exercise we leave to the interested reader.

Now we turn to the question of how to model $!A$ for a given finite dimensional Hilbert space $A$. To do this we resurrect the relevant part of the scaffolding used to derive the rules of **GQ**: namely, we now take seriously the interpretation of $!A$ as the "quantum set of proofs of $A$." Now, the lattice $L(A)$ constitutes a model of **OL** and equivalence classes of **OL** deductions of $A$ in the model $L(A)$ correspond with subspaces of $A$, by Theorem 5.1.1.1. These subspaces can be organized into a "quantum set," namely the exterior algebra $E(A)$—the quantum version of the set of subsets of the "set" $A$—which is an object in $\mathcal{H}_F$: this is exactly the substance of Finkelstein's extensorial calculus of quantum sets discussed in §2.3. Happily, this structure does, in fact, satisfy all the properties required of it by **GQ**, namely:

LW: Consider the counit $c_A: E(A) \longrightarrow \mathbb{C}$, given by projection upon the first component of $E(A)$ (cf. §3.1.8). Then an interpretation of a proof $\Gamma \vdash_{\mathbf{GQ}} D$—namely, an element of $\mathrm{Hom}(\Gamma, D)$—may be composed with the map $1 \otimes c_A: \Gamma \otimes E(A) \longrightarrow \Gamma \otimes \mathbb{C} \cong \Gamma$ to obtain an element in $\mathrm{Hom}(\Gamma \otimes E(A), D)$.

This element is declared to be the interpretation of the proof obtained *via* LW of the original proof.

LC: A similar argument using the coproduct $E(A) \xrightarrow{\psi_A} E(A) \otimes E(A)$ (equation (3.1.8.12)). We shall discuss the interpretation of this rule in a little more detail since it embodies the important notion of quantum copying—or *duplication*—of storage capable quantum resources.

For the purposes of this discussion let us introduce labels (or *terms)* for **GQ** sequents. Thus, a sequent $\Gamma \vdash D$ may be labelled on the left as in $f: \Gamma \vdash D$. (This is equivalent to the notationally more standard expression $\vdash f: \Gamma^* \otimes D$.) Rules should now be introduced for the correct formation of terms as **GQ** proofs are constructed. We shall illustrate only a single short proof, in which these assignments are self-evident: namely

$$\dfrac{\dfrac{f: !A \vdash_{\mathbf{GQ}} B \quad g: !A \vdash_{\mathbf{GQ}} C}{\langle f, g \rangle: !A, !A \vdash_{\mathbf{GQ}} B \otimes C}\ \mathrm{R}\otimes}{\mathbf{dup}_{!A}\langle f, g \rangle: !A \vdash_{\mathbf{GQ}} B \otimes C}\ \mathrm{LC}$$

Read operationally, $\mathbf{dup}_{!A}\langle f, g \rangle$ labels the deduction obtained by "quantum duplicating" the storage capable resource $!A$ in the preceding sequent, and then performing the deduction labelled by $\langle f, g \rangle$. When interpreted in $\mathscr{H}_F$, $f$ and $g$ may be regarded as the appropriate linear maps, and we have:

$$\langle f, g \rangle \text{ is interpreted as } f \otimes g$$

and

$$\mathbf{dup}_{!A}\langle f,g\rangle \text{ is interpreted as } f\otimes g\circ\psi_A\,. \qquad (6.3.2.1)$$

L!: A similar argument using the projection $E(A)\longrightarrow A$ upon first grade elements.

R!: It suffices to show this for $\Gamma$ containing at most a single formula, since, if $\Gamma\equiv A_1,A_2,\ldots,A_n$, $!\Gamma$ is interpreted as $!A_1\otimes !A_2\otimes\ldots\otimes !A_n$ which is isomorphic as a coalgebra with $!(A_1\oplus A_2\oplus\ldots\oplus A_n)$ (§3.1.8). Then $!\Gamma\vdash A$ is interpreted as a map $E(\Gamma)\longrightarrow A$. Dualizing this we obtain a map $A^*\longrightarrow E(\Gamma)^*\cong E(\Gamma^*)$. From the universal property of $E(\ )$ this map lifts to a map $E(A^*)\longrightarrow E(\Gamma^*)$, and, dualizing again, we obtain a map $E(\Gamma)\longrightarrow E(A)$. This is the interpretation of $!\Gamma\vdash !A$ in the conclusion of R!.

All of this could be done much more formally, with little gain in transparency as far as our endeavors in this work are concerned. That the category of finite dimensional vector spaces models full **LL**, with $!A$ taken to be $E(A^*)^*$, was shown in Blute *et al.* 1993. (In view of the remarks at the end §3.1.8 our use of $E(A)$ is equivalent.) See also Seely 1989 for a clear discussion of more general categorical interpretations of **LL**.

Although we have fetched up essentially where we started, some valuable new perspectives have been revealed:

- We have shown that, by means of the translation equations (6.3.1.1) – (6.3.1.4) and the interpretation described above, **IOL** may be realized within the familiar category $\mathcal{H}_F$ *via* a literal use of a quantum version of the Heyting paradigm. This circumstance vindicates Finkelstein's earlier proposal to restore extensionality to quantum logic in a rather unexpected way. Moreover, the logic of $\mathcal{H}_F$, as specified by the rules of **GQ**, is seen to be an externalization of the intuitionistic

fragment of the logic of each of its object's "inner" subspace-lattice models of **OL**.

- The correct notion of (intuitionistic) "quantum" implication is now seen to be interpretable in terms of morphisms in $\mathscr{H}_F$: that is, in terms of ordinary linear transformations between the underlying vector spaces, all of which are necessarily continuous for any chosen inner products.

- These logical considerations have thrown up a formal specification of the notion of *storage capable quantum resource.* Such resources would be fundamental to any "quantum computational" endeavor, and the exploration of this notion in one form or another will occupy us for the remainder of this work.

### *6.3.3 A Model of Computation and the Emergence of the Qubit*

The systems **GQ**, **OL**, etc., are empty of physical content, embodying, rather, minimal rules for making certain deductions about abstract quantum "resources." The task before us is to supply physical input in the form of additional axioms, and additional rules pertaining to the "post-processing" of certain ensuing deductions. A system obtained by adding axioms to an existing system (such as **GQ**) is called by logicians a *theory* (or a **GQ**-*theory*). (Often, extra technical constraints are put upon these added axioms to ensure desirable deductive behavior, but we shall not so constrain our (few) axioms here.)

As a test of the formalism we shall (temporarily!) add an **IOL** axiom meant to simulate a single "time"-stepped deduction or computation which preserves each type. Here we consider "time"-steps to be resources—necessarily constrained by our formalism to be "quantum" resources—which are produced to accompany, or label, such a

transformation. This may be expressed in the static, resource insensitive language of **IOL** by the axiom:

$$\alpha \vdash_{\mathbf{IO}} \mathbf{t} \sqcap \alpha. \tag{6.3.3.1}$$

Here $\alpha$ is any **IOL** formula, and $\mathbf{t}$ is an atom. This is meant to capture the idea that $\alpha$ (re)produces $\alpha$ to the accompaniment (or production) of a single time-step, time-quantum, or clock-tick. It is a crude attempt to force some preconceived notion of "time" upon the logical *tabula rasa.*

The translation of this into **GQ** then yields the axiom to be added to **GQ**, namely:

$$!\alpha^e \vdash_{\mathbf{GQ}} !\mathbf{t} \otimes !\alpha^e, \tag{6.3.3.2}$$

or, equivalently (in view of L*, LE and R*):

$$(!\mathbf{t})^* \vdash_{\mathbf{GQ}} (!\alpha^e)^* \otimes !\alpha^e. \tag{6.3.3.3}$$

Thus, the axiom amounts to the specification of a deduction from $(!\mathbf{t})^*$ to $(!\alpha^e)^* \otimes !\alpha^e$ for each $\alpha$.

When realized in the category $\mathcal{H}_F$, the interpretation of $!\mathbf{t}$ is somewhat problematical, but, whatever interpretation is given to it, $\alpha^e$ will be interpreted as a finite dimensional Hilbert space, $W$ say, of dimension $n$, say, and $(!\alpha^e)^* \otimes !\alpha^e$ will be interpreted as

$$E(W)^* \otimes E(W) \cong \mathrm{End}E(W).$$

In view of equation (2.2.3.10) we have

$$E(W) \cong E(\oplus^n \mathbb{C})$$

$$\cong \otimes^n E(\mathbb{C}) \qquad (6.3.3.4)$$

where $E(\mathbb{C})=\mathbb{C}\oplus\mathbb{C}$, the two-dimensional Hilbert space. This space, the *irreducible quantum storage capable unit* in $\mathcal{H}_F$, has come to be called (in the quantum computing literature) the *qubit.* In view of Finkelstein's Grassmannian interpretation of the functor $E(\ )$, the first $\mathbb{C}$ represents the empty quantum set (or the zero subspace of $\mathbb{C}$) while the second $\mathbb{C}$ represents the subspace $\mathbb{C}$ of $\mathbb{C}$, or the whole quantum set. If quantum superpositions were suppressed, we would have discovered the ordinary classical bit. Note that bit-based notions were not explicit in any of the considerations leading up to **QG**. Thus, the classical bit emerges, quite appropriately, as a classical degeneration of the spontaneously arising qubit: quantum notions should indeed underlie classical ones.

Equation (6.3.3.3) thus characterizes a "quantum computation," taking place in some version of "quantum time," as a map from a representer of the dual of the multiplexed quantum time-step resource **t**, namely $(!\mathbf{t})^*$, to a space of the form $\mathrm{End}(\otimes^n\mathfrak{H}^{(2)})$, where $\mathfrak{H}^{(n)}$ denotes a Hilbert space of dimension $n$ $(<\infty)$. That is: the annihilators or absorbers of finite quantum sets of time-steps are mapped to endomorphisms of tensor products of qubits.

The problem here is that the formalism seems to have worked too well in that "time" is also necessarily finitely or constructively quantized when forced into the picture, whereas the exigencies of macroscopic existence might require us to adopt a model of time that is infinite and classical. In order to attempt to redress this problem, and arrive at the standard notion of a quantum computation, we will need to step outside the categorical confines of $\mathcal{H}_F$. This will be done in §6.4. First, we are required to interpret rather more fully the notion of quantum duplication.

### *6.3.4 Quantum Duplication as Entanglement*

As we have noted, the general storage capable object in $\mathcal{H}_F$ is of the form $E(\mathfrak{H}^{(n)}) = \otimes^n\mathfrak{H}^{(2)}$: such a tensor product of qubits has

come to be called a *quantum register.*

The *quantum duplication operator* that interprets the **GQ** Contraction rule, namely

$$\frac{!\mathfrak{H}^{(n)}, !\mathfrak{H}^{(n)}, \Gamma \vdash D}{!\mathfrak{H}^{(n)}, \Gamma \vdash D}, \qquad (6.3.4.1)$$

is the coproduct $E(\mathfrak{H}^{(n)}) \longrightarrow E(\mathfrak{H}^{(n)}) \otimes E(\mathfrak{H}^{(n)})$. Moreover, from the discussion at the end of §3.1.8, the isomorphism equation (6.3.3.4), which now reads $E(\mathfrak{H}^{(n)}) \equiv \otimes^n \mathfrak{H}^{(2)}$, is a coalgebra isomorphism. Thus, it will be sufficient for our purposes to discuss the quantum duplication operator for the case of a single qubit $\mathfrak{H}^{(2)}$.

At this point there arises an unfortunate clash of notations. When the qubit is realized as the (graded) Hopf algebra $E(\mathbb{C}) = \mathbb{C} \oplus \mathbb{C}$, the first component is generated by the unit of this algebra which is usually denoted by 1, and, since the coproduct $\psi: E(\mathbb{C}) \longrightarrow E(\mathbb{C}) \otimes E(\mathbb{C})$ preserves units, we have:

$$\psi(1) = 1 \otimes 1. \qquad (6.3.4.2)$$

For an element $x$ of the other $\mathbb{C}$ component we have (cf. equation (3.1.8.12))

$$\psi(x) = 1 \otimes x + x \otimes 1. \qquad (6.3.4.3)$$

In the quantum computational context, a basis $\{1, x\}$ of the qubit would be written, when normalized, as $\{|0\rangle, |1\rangle\}$, a notation that is quite consistent with the earlier quantum set interpretation of the Dirac $|\ \rangle$: as noted, the first element corresponds to the empty quantum set and the second to the whole quantum set. The duplication operations expressed by the above equations become

$$\psi(|0\rangle)=|0\rangle\otimes|0\rangle \tag{6.3.4.4}$$

$$\psi(|1\rangle)=|0\rangle\otimes|1\rangle+|1\rangle\otimes|0\rangle \tag{6.3.4.5}$$

relative to the chosen so-called *computational basis* $\{|0\rangle,|1\rangle\}$.

Thus, quantum duplication applied to the "off" computational basis element $|0\rangle$ produces a simple homogeneous pure state of the combined system $\mathfrak{H}^{(2)}\otimes\mathfrak{H}^{(2)}$, whereas duplication applied to the "on" basis element emphatically does not. Indeed, the state corresponding to the right hand side of equation (6.3.4.5) is a *maximally entangled* state. We digress briefly to introduce the notion of entanglement.

Let $\mathfrak{H}_A$ and $\mathfrak{H}_B$ denote the Hilbert spaces—assumed finite dimensional for simplicity—associated with "systems" $A$ and $B$. Then the "combined" system has $\mathfrak{H}_A\otimes\mathfrak{H}_B$ as its associated Hilbert space. Consider a pure state of the combined system corresponding to a unit vector $|\theta\rangle\in\mathfrak{H}_A\otimes\mathfrak{H}_B$. This vector may be expressed as

$$|\theta\rangle=\sum_{i,j}c_{ij}|\xi_i\rangle\otimes|\eta_j\rangle, \tag{6.3.4.6}$$

where $\{|\xi_i\rangle\}$and $\{|\eta_j\rangle\}$ are orthonormal bases for $\mathfrak{H}_A$ and $\mathfrak{H}_B$ respectively: then $\Sigma|c_{ij}|^2=1$.

The following question arises: what is the state of a component system corresponding to this pure state of the combined system? Or, in reist terms, what state is the component system *in* when the combined system is *in* the pure state $|\theta\rangle$? To answer this question we consider first a pure state of a general system with Hilbert space $\mathfrak{H}$ of dimension greater than two (§5.2.1). This is of the form $|\xi\rangle\otimes\langle\xi|$ for a unit vector $|\xi\rangle$ in $\mathfrak{H}$. In terms of the action vector picture of Chapter 1, the operator $|\xi\rangle\otimes\langle\xi|\in\mathfrak{H}^*\otimes\mathfrak{H}$ may be thought to describe the primitive experiment in which a quantum is injected by the initial act $|\xi\rangle$ and absorbed by the final act $\langle\xi|$.

For $|\theta\rangle$ the associated density matrix $|\theta\rangle\otimes\langle\theta|$ is an element of

$$(\mathfrak{H}_A\otimes\mathfrak{H}_B)^*\otimes\mathfrak{H}_A\otimes\mathfrak{H}_B\cong\mathfrak{H}_A^*\otimes\mathfrak{H}_B^*\otimes\mathfrak{H}_B\otimes\mathfrak{H}_A, \qquad (6.3.4.7)$$

and, as such, may be considered to describe a bipartite experiment along the lines of the discussion in §1.3. Namely, we may consider the possible acts upon the system $B$ to be *intermediary,* or *hidden,* relative to those pertaining to system $A$ (and *vice versa*). (For instance, the two component systems could be spatially separated by arbitrarily large distances, though they are still combined in the manner prescribed.) Then, the effect of the experiment $|\theta\rangle\otimes\langle\theta|$ upon the subsystem $A$ may be obtained by taking into account all possible transitions associated with the $B$ system, now considered to be "internal" (or hidden) relative to system $A$. That is to say, the effect upon $A$ is obtained by performing the $B$-contraction in equation (6.3.4.7) to obtain an element $\rho_A$ in $\mathfrak{H}_A^*\otimes\mathfrak{H}_A$. This is called the trace *over* $B$ (or the *partial,* or *relative* trace) and is denoted by:

$$\rho_A\equiv \mathrm{tr}_B(|\theta\rangle\otimes\langle\theta|)$$

$$=\sum_{i,j,k} c_{ik}\bar{c}_{jk}|\xi_i\rangle\otimes\langle\xi_j|. \qquad (6.3.4.8)$$

Now, it is easily checked that $\rho_A$ is in fact a density matrix for the $A$ system (cf. §5.2.1) and so represents a mixed state thereof. A similar argument applies in the case of the $B$ system. Thus, a pure state of the combined system entails, or induces, *mixed* states of the component systems. This highly non-classical behavior is the essence of the phenomenon known as *quantum entanglement.*

Returning to equation (6.3.4.5), the pure state in $\mathfrak{H}^{(2)}\otimes\mathfrak{H}^{(2)}$ $(\cong\mathfrak{H}_A\otimes\mathfrak{H}_B)$ determined by $\psi(|1\rangle)$ is represented by the unit vector

$$|\theta\rangle\equiv\tfrac{1}{\sqrt{2}}(|0\rangle\otimes|1\rangle+|1\rangle\otimes|0\rangle) \qquad (6.3.4.9)$$

and one quickly finds that

$$\rho_A = \mathrm{tr}_B(|\theta\rangle\otimes\langle\theta|) = \tfrac{1}{2} I_A \qquad (6.3.4.10a)$$

$$\rho_B = \mathrm{tr}_A(|\theta\rangle\otimes\langle\theta|) = \tfrac{1}{2} I_B \qquad (6.3.4.10b)$$

(where the $I$s denote identity operators on the respective Hilbert spaces). Consequently, by equation (5.2.1.9), experiments to determine whether system $A$ is in either state $|0\rangle$ or $|1\rangle$ will generate completely random outcomes, and similarly for system $B$. For this reason, whenever the relative trace yields a multiple of the identity operator, as in this case, the state corresponding to $|\theta\rangle$ is said to be *maximally entangled:* no information about the internal constitution of the state—its *preparation*—can be obtained by separately performing "local" experiments upon the component subsystems.

The duplication map $\psi$ applied to any vector in $\mathfrak{H}^{(2)}$ will be a linear combination of the right hand sides of equations (6.3.4.4) and (6.3.4.5), and one of the upshots of our logical machinations is that quantum duplication—namely, that quantum process that corresponds to the classical possibility of freely copying a resource—*must* in general entail quantum entanglement. This is borne out in the standard theory of quantum computing, where quantum entanglement has been recognized as a fundamental resource and must be used in subtle ways—for instance to implement the transmission of quantum states by "teleportation." Naïve attempts to copy such states would be confounded in view of the so-called "No Cloning" Theorem: see for example, Nielsen and Chuang 2000.

It seems rather remarkable that "merely" logical considerations have led directly to this subtlety regarding quantum duplication.

## 6.4 Quantum Computing in Classical Time: An Algebraic Model

The interpretation of coproducts as quantum duplication-*via*-entanglement may be extended to other relevant coalgebras that arise in similar contexts, such as those associated with $T(\ )$, $S(\ )$, $U(\ )$ and End( ) (cf. Chapter 3). Regarded as functors on appropriate categories, some of these coalgebras realize certain of the **GQ** axioms for !, etc., but generally not all of them: cf. Blute *et al.* 1993. We close this chapter with a treatment of quantum computation as embodied in our axiom (6.3.3.2) but with the multiplexed time-step type !**t** now interpreted "classically." We do this for two reasons: firstly, as a limbering up exercise in preparation for what is to come in Part III—namely a less macroscopically chauvinistic treatment, in which classical "time" is replaced by something more intrinsic (which will turn out to be a quantum version of relativistic spacetime), and, secondly, as a means of arriving at the standard view of quantum computation as being performed in classical time.

To render classical the type !**t** we need to interpret it in classical terms. This can be done by modelling ! not by the exterior algebra but by the commutative algebra $S(\ )$. Thus, we now regard the "formal parameter" **t** as the generator of a one-dimensional vector space $V$ so that $S(V)$ may be identified with the one-dimensional affine algebraic group $\mathbb{C}[\mathbf{t}]$: this is just the usual complex polynomial algebra in the indeterminate **t**, equipped with the Hopf algebra structure described in §3.1.4. Since it is only time that is being treated classically here, we maintain the quantum interpretation of ! in the other parts of axiom (6.3.3.3).

Thus, we are required to specify a map

$$\phi: \mathbb{C}[\mathbf{t}]^* \longrightarrow \mathrm{End}(\otimes^n \mathfrak{H}^{(2)}). \tag{6.4.1}$$

We shall make two assumptions concerning this map which together will yield the *Schrödinger option* for describing the classically timed dynamics of a quantum register. The first of these concerns the notion of duplication. The *quantum duplication* of a resource corre-

sponds to the classical operation of copying or repeating the resource. Our first requirement on $\phi$ is that it should respect this type of repetitive behavior: in other words, $\phi$ should be required to match the repetitive behavior of the resource "time" to that of the target resource—a sort of synchronization assumption. Thus, $\phi$ is required to *respect quantum duplication:* for $f, g \in \mathbb{C}[\mathbf{t}]^*$, we should have

$$\phi(\mathbf{dup}_{\mathbb{C}[\mathbf{t}]}\langle f, g\rangle) = \mathbf{dup}_{\mathrm{End}(\otimes^n \mathfrak{H}^{(2)})}\langle \phi(f), \phi(g)\rangle \qquad (6.4.2)$$

or, from equation (6.3.2.1),

$$\phi(f \otimes g \circ \psi_A) = \phi(f) \otimes \phi(g) \circ \psi_E, \qquad (6.4.3)$$

where $\psi$ and $\psi_E$ denote the coproducts respectively of $\mathbb{C}[\mathbf{t}]$ and $\mathrm{End}(\otimes^n \mathfrak{H}^{(2)})$.

Equation (6.4.3) is exactly the requirement that $\phi$ be an algebra map for the algebra structures dual to the respective coalgebra structures. The dual algebra product on $\mathrm{End}(\otimes^n \mathfrak{H}^{(2)})$ is, by design, just the usual one, while the commutative algebra product on $\mathbb{C}[\mathbf{t}]^*$ is easily described.

First, we denote by $\delta_m$ the element in $\mathbb{C}[\mathbf{t}]^*$ dual to the basis element $\mathbf{t}^m$ of the vector space $\mathbb{C}[\mathbf{t}]$, $m = 0, 1, \ldots$, so that

$$\delta_m(\mathbf{t}^n) = \delta_{m,n}, \qquad (6.4.4)$$

where $\delta_{m,n}$ denotes the usual Kronecker delta. Then elements of $\mathbb{C}[\mathbf{t}]^*$ may be conveniently written as formal sums of the form $\Sigma c_n \delta_n$.

PROPOSITION 6.4.1

*The commutative algebra product, denoted* $*$, *induced upon* $\mathbb{C}[\mathbf{t}]^*$ *by the dual of the coproduct* $\psi$ *of the Hopf algebra* $\mathbb{C}[\mathbf{t}]$ *is given by*

$$\delta_m * \delta_n = \binom{m+n}{m}\delta_{m+n}$$

$$= \frac{(m+n)!}{m!n!}\,\delta_{m+n}. \qquad (6.4.5)$$

PROOF

For any $m$, $n$, $k$:

$$(\delta_m * \delta_n)(\mathbf{t}^k) = (\delta_m \otimes \delta_n)(\psi(\mathbf{t}^k))$$

$$= (\delta_m \otimes \delta_n)(\psi(\mathbf{t})^k)$$

$$= (\delta_m \otimes \delta_n)(1 \otimes \mathbf{t} + \mathbf{t} \otimes 1)^k$$

$$= (\delta_m \otimes \delta_n)\left(\sum_{l=0}\binom{k}{l}\mathbf{t}^l \otimes \mathbf{t}^{k-l}\right)$$

$$= \sum_{l=0}\binom{k}{l}\delta_{m,l}\delta_{n,k-l}. \qquad (6.4.6)$$

This sum can be non-zero only if $m+n=k$, and, when this is the case, the single surviving term occurs when $m=l$.

Thus,

$$(\delta_m * \delta_n)(\mathbf{t}^k) = \binom{m+n}{m}\delta_{m+n,k}$$

$$= \left(\binom{m+n}{m}\delta_{m+n}\right)(\mathbf{t}^k). \blacksquare \qquad (6.4.7)$$

It follows immediately from equation (6.4.5) that

$$\delta_{m+n} = \frac{m!n!}{(m+n)!}\delta_m * \delta_n, \tag{6.4.8}$$

so that, for $n>0$,

$$\begin{aligned}\delta_n &= \frac{1}{n}\,\delta_{n-1} * \delta_1 \\ &= \frac{1}{n!}\overbrace{\delta_1 * \ldots * \delta_1}^{n} \\ &= \frac{1}{n!}\,\delta_1^n. \end{aligned} \tag{6.4.9}$$

Thus, general elements of $\mathbb{C}[\mathbf{t}]^*$ may be expressed in the form

$$\sum \frac{c_n}{n!}\delta_1^n \tag{6.4.10}$$

and $\phi$, being an algebra map, will be specified once $\phi(\delta_1)$ is assigned.

The second Schrödinger-like assumption on $\phi$ concerns the interpretation of $\delta_1$. The logical atom $\mathbf{t}$ was introduced to represent the notional generic "time-step." Let us now take it more literally to represent the generic infinitesimal time differential $dt$. Then, its linear dual $\delta_1$ should be interpreted as the dual of $dt$, which is the tangent $\partial/\partial t$. As an operator, densely defined upon $L^2(\mathbb{R})$, it has the property that

$$\left(\frac{\partial}{\partial t}\right)^{\dagger} = -\frac{\partial}{\partial t}, \tag{6.4.11}$$

where the dagger denotes the Hilbert space adjoint.

Our second assumption on $\phi$ is that it should be chosen to preserve this (virtual) property of $\delta_1$. That is,

$$\phi(\delta_1)^{\dagger} = -\phi(\delta_1). \tag{6.4.12}$$

Then we may choose

$$\phi(\delta_1) = -iH \tag{6.4.13}$$

for some Hermitian matrix $H$.

Now, it is classical that the group $\mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[\mathbf{t}], \mathbb{C})$, with group operation obtained as in equation (3.1.3.1), may be identified with the additive group of $\mathbb{C}$. This identification is obtained by noting that every element of $\mathrm{Hom}_{\mathrm{alg}}(\mathbb{C}[\mathbf{t}], \mathbb{C})$ is given by

$$h_z(\mathbf{t}^n) = z^n, \tag{6.4.14}$$

for some $z \in \mathbb{C}$. That the association $h_z \mapsto z$ is a group morphism is immediate (cf. Abe 1977, Chapter 4).

These $h_z$ are also elements of $\mathbb{C}[\mathbf{t}]^*$, forming a subgroup with respect to the algebra product, and may be written (for each $z \in \mathbb{C}$) in the form

$$\begin{aligned} h_z &= \delta_0 + z\delta_1 + z^2\delta_2 + z^3\delta_3 + \ldots \\ &= \delta_0 + z\delta_1 + \frac{z^2}{2!}\delta_1^2 + \frac{z^3}{3!}\delta_1^3 + \ldots \end{aligned} \tag{6.4.15}$$

from equation (6.4.9). Thus we obtain a map

$$\mathbb{C} \longrightarrow \mathrm{End}(\otimes^n \mathfrak{H}^{(2)}) \tag{6.4.16}$$

given formally by

$$z \mapsto \phi(h_z) = I + z\phi(\delta_1) + \frac{z^2}{2!}\phi(\delta_1)^2 + \ldots, \tag{6.4.17}$$

since $\delta_0$ is the unit for $*$.

Supposing time to be real, we restrict to the additive subgroup $\mathbb{R}$ of $\mathbb{C}$ to obtain the map

$$\mathbb{R} \longrightarrow \mathrm{End}(\otimes^n \mathfrak{H}^{(2)}) \tag{6.4.18}$$

given by $t \mapsto \exp(t\phi(\delta_1))$. Though defined formally, this series will always converge.

Thus, the map realizing the action of time (or, rather, the action of time *intervals*) that constitutes a "quantum computation" may be written, in view of equation (6.4.13), in the form $t \mapsto e^{-iHt}$. (The physical interpretation of $H$ is, up to an additive real constant, as the operator Hamiltonian of the system.)

Despite its formality, this model seems to have revealed the major qualitative aspects of those processes called quantum computations. To wit:

- the unitarity and time reversibility of the processes;
- the structure of the underlying Hilbert space as a quantum register, or tensor product of qubits;
- the primary rôle of quantum entanglement as a resource in the implementation of quantum duplication.

We note also that the unitarity of the action of the dynamical

operator entails the preservation, through the computation, of the associated Kripke orthomodel and subspace lattice structures.

## 6.5 Conclusions

We have arrived at the point at which the current treatments of the embryonic theory of quantum computation start, and interested readers could consult the vast and burgeoning list of works devoted to this fascinating subject. We will leave it here, though, and, in the remainder of this work, will attempt a more general investigation into the nature of what it is that should replace that figment "classical time" as a universal labeller, and what it is that is labelled. In doing this, we will carry with us the qubit, and certain other quantum computational resources culled from our logical considerations.

In closing, we note some questions raised by the material in this chapter but not addressed here. Namely,

1. Are there lattice characterizations of **IOL**? Such lattices might stand in relation to ortholattices as Heyting algebras do to Boolean algebras.
2. Does the translation theorem (Theorem 6.3.1.1) have a converse?
3. Is CUT eliminable from proofs in **GQ**?
4. Do other categories exist in which **GQ** is realizable?

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# Part III

# The Plenum

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# 7

# A Quantum Net

In this chapter we introduce an example of what Finkelstein has dubbed a *quantum net.* In general, this term refers to a quantum set-based construct exhibiting the kind of connective or causal structure that might be expected to yield a version of macroscopic spacetime in some kind of "classical" limit. This expectation is fed by the foreknowledge that the causal structure of a (classical) spacetime is its fundamental attribute, and essentially determines it completely. (For definitive results in a long line of development in this direction see Malament 1977, excellent background for which may be found in Naber 1992. For a sample of recent work on causal structure, that also addresses the quantization problem, see Raptis 2000a.) Finkelstein's innovation here is the introduction of *quantum* sets, and concomitant versions of causal notions.

There is, inevitably, a profound disjunction between the terms of any spacetime theory founded upon a thorough-going non-objective action vector interpretation of quantum theory, and the terms of cosmological theories of a more traditional stripe, in which the entire universe, or parts of it, are objectified, and may support collapsible wave-functions, C*-algebras, etc. In the non-objective action vector view to which we aspire here, the net is not a "model" of spacetime or any such *thing,* but rather a collection of acts performable by an experimenter—albeit an ideal one. It is the pattern of transitions associated with this act-structure that is supposed to coa-

lesce ultimately into those amplitudes, etc., that a macroscopic observer might associate with the plenum of her or his experience. Our problem is to devise a candidate for such a collection of acts.

The particular net we adopt here is based on a somewhat phenomenological "superconducting" model introduced in Finkelstein 1988b. Our approach to the central pairing phenomenon will be through a logical analysis of qubit symmetries and the adoption of an axiom along the lines of §6.3.3. There are, however, intrinsic *post hoc* mathematical reasons for believing in the Cooper pair paradigm in the form used here. These are sketched in a final section (§7.3), which may be safely skipped.

## 7.1 Symmetries of the Qubit

For an $n$-dimensional Hilbert space—denoted for simplicity here by $W$—the one-dimensional $n^{\text{th}}$-grade component $\bigvee^n W$ of $E(W)$ represents the quantum set whose classical counterpart is the "whole set" regarded as an element of the set of subsets of a finite set. If we allow a quantum-classical correspondence of the form $E(W) \leftrightarrow \mathbf{2}^X$, where $X$ is a finite set, then

$$\bigvee^n W \leftrightarrow X \in \mathbf{2}^X. \tag{7.1.1}$$

In the Boolean model of ordinary **PC** that $\mathbf{2}^X$ (with its lattice structure) represents, $X$ is the *local truth*—i.e. the value assigned to all **PC** theorems by valuations into the Boolean algebra $\mathbf{2}^X$: namely, the upper bound 1—while $\varnothing$ represents the *local falsity*, or lower bound 0.

Now the choice of an isomorphism $\varphi: \bigvee^n W \longrightarrow \mathbb{C}$ is entirely determined by a choice of generator for $\bigvee^n W$ and so corresponds, in the rough sense of equation (7.1.1), to a choice of "local truth" for the corresponding quantum set theory.

Suppose now that $n = 2$. For any choice of basis, $\{e_1, e_2\}$ say, of $W$, an experimenter may realize $W$ as the qubit $E(\mathbb{C})$ in two

ways: either $e_1$ is taken to determine the ray corresponding to the zero-th grade subspace generated by $|0\rangle$ and $e_2$ as the first-grade ray (or local truth) $|1\rangle$, or *vice versa.* We can operationalize these choices as follows: first, choose and fix an isomorphism

$$\varphi: \bigvee^2 W \longrightarrow \mathbb{C} \tag{7.1.2}$$

(which fixes a local truth, or generator of $\bigvee^2 W$, as that element sent to 1 by $\varphi$). Then we have a map

$$W \otimes W \xrightarrow{\vee} W \vee W \xrightarrow{\varphi} \mathbb{C} \tag{7.1.3}$$

which is non-degenerate in the sense that if $\varphi(x \vee y) = 0$ for all $y$, then $x = 0$. Consequently, there exists an isomorphism

$$\varepsilon: W \longrightarrow W^* \tag{7.1.4}$$

given by

$$\varepsilon(x)(y) = \varphi(x \vee y). \tag{7.1.5}$$

(The dependence of $\varepsilon$ upon $\varphi$ is suppressed in our notation, but should be borne in mind. Note that any two such $\varphi$s are projectively related and so give rise to projectively related $\varepsilon$s.)

Note that for any basis $\{e_1, e_2\} \subset W$:

$$\varepsilon(e_1)(e_1) = 0 \tag{7.1.6}$$

$$\varepsilon(e_1)(e_2) = \varphi(e_1 \vee e_2) \tag{7.1.7}$$

so that

$$\varepsilon(e_1) = \varphi(e_1 \vee e_2) e_2^*, \tag{7.1.8}$$

where $e_2^*$ denotes the basis element in $W^*$ dual to $e_2$.

Similarly, it is seen that

$$\varepsilon(e_2) = \varphi(e_2 \vee e_1)e_1^*. \tag{7.1.9}$$

First, we note that

$$\varepsilon(e_1): W \longrightarrow \mathbb{C} \tag{7.1.10}$$

induces the splitting $W \cong \mathbb{C} \oplus \mathbb{C}$ corresponding to the *operation* of choosing $e_2$ to act as the $|1\rangle$ component: it is mapped to the representer $\varphi(e_1 \vee e_2)$ in $\mathbb{C}$ of the local truth $e_1 \vee e_2$. Similarly, $\varepsilon(e_2)$ may be interpreted as the splitter corresponding to the choice of $e_1$ as the $|1\rangle$ component, etc. Indeed, all elements of $W^*$ are thus realized as superpositions of these two choices of splittings.

The second observation is that equations (7.1.8) and (7.1.9) are completely symmetrical with respect to $e_1$ and $e_2$. Interchanging them leaves the pair of equations unchanged. This gives rise to the "logical" symmetry of the classical degeneration represented by the two-element set $\{e_1, e_2\}$. We could take one element to represent 0 and the other to represent 1, or *vice versa.* Valuations of **PC** formulae into this set are intertranslatable among these two choices by means of Boolean duality, in which 0 and 1 are interchanged. Thus we recover $\mathbb{Z}_2$ as the appropriate group of logic symmetries in the degenerate classical case.

In the quantum case there is an infinite number of different choices of bases (thanks to the presence of superposition) and concomitantly a wider choice of "allowable" logical translations among them. Since a choice of basis corresponds to the choice of a family of commuting observables, or classical window, the basis changes we seek to identify as allowable should correspond to allowable translations among different choices of realizations of $W$ as "the" qubit. The set of such transformations should form a group whose elements are akin to dictionary entries, translating among different basis choices.

It is expected to be a wordier dictionary than the simple classical one, which merely translates 0 as 1 and 1 as 0.

A change from one basis to another is equivalent to having an invertible linear transformation, $L: W \longrightarrow W$ say. We shall deem such a basis change "allowable" if it preserves the operational choice structure ($\varepsilon$) associated with the transformed basis. This amounts to the requirement that $L$ renders commutative the following diagram:

$$\begin{array}{ccc} W & \xrightarrow{\varepsilon} & W^* \\ {\scriptstyle L}\downarrow & & \uparrow{\scriptstyle L^*} \\ W & \xrightarrow[\varepsilon]{} & W^* \end{array} \tag{7.1.11}$$

That is, we require:

$$L^* \circ \varepsilon \circ L = \varepsilon \tag{7.1.12}$$

or

$$\varepsilon \circ L = (L^*)^{-1} \circ \varepsilon. \tag{7.1.13}$$

Now, for $x, y \in W$

$$\begin{aligned} ((\varepsilon \circ L)(x))(y) &= \varepsilon(Lx)(y) \\ &= \varphi(Lx \vee y), \end{aligned} \tag{7.1.14}$$

while

$$\begin{aligned} (((L^*)^{-1} \circ \varepsilon)(x))(y) &= ((L^*)^{-1}(\varepsilon(x)))(y) \\ &= ((L^{-1})^*(\varepsilon(x)))(y) \end{aligned}$$

$$= \varepsilon(x)(L^{-1}y)$$

$$= \varphi(x \vee L^{-1}y)$$

$$= \varphi(L^{-1}Lx \vee L^{-1}y)$$

$$= (\det L^{-1})\varphi(Lx \vee y)$$

$$= (\det L)^{-1}((\varepsilon \circ L)(x))(y) \qquad (7.1.15)$$

from equation (7.1.14).

Thus, the diagram commutes only if $\det L = 1$: that is to say, $L \in \mathrm{SL}(2,\mathbb{C})$ and the latter group emerges as the group of allowable "logical" transformations of the qubit among macroscopic experimenters. This symmetry corresponds to the classical $\mathbb{Z}_2$-symmetry of Boolean duality (to which it degenerates) and is a globalized version of the $n = 2$ case of the quantization of $\mathbb{Z}_n$ as $\mathfrak{sl}(n,\mathbb{C})$ found in §3.2.3.

Insofar as the qubit represents the non-objective quantum analog of the classical binary alternative, or bit system $\{0,1\}$, we have found that, as compared to classical experimenters, quantum experimenters are faced with a different and wordier dictionary—namely $\mathrm{SL}(2,\mathbb{C})$—to effect logically allowable transformations, since, roughly speaking, the firm ground of a common logical frame is no longer underfoot.

A classical experimenter may ultimately boil down the result of any act of interrogation to a sequence of bits. The framing of such an atomic or irreducible act of interrogation corresponds to a certain initial act for a quantum theorist, and the classical registration of the resulting bit corresponds to a certain final act for the quantum theorist. The space or spaces of such quantum acts are thus qubits admit-

ting non-trivial representations of $SL(2,\mathbb{C})$.

It is appropriate at this juncture to recall a peculiar fact concerning representations of the groups $SL(n,\mathbb{C})$, which may be inconvenient mathematically but is of special import to the (macroscopic) quantum relativist. Namely, these groups admit no non-trivial irreducible finite dimensional *unitary* representations (cf. Hewitt and Ross 1979). Therefore, no candidate initial space for the "quantum binary alternative" chosen as above can admit a *unitary* representation of $SL(2,\mathbb{C})$. Suppose an experimenter were to choose a candidate representation upon a space $\mathfrak{H}$, whose Hilbert space inner product $\langle\ |\ \rangle$ is given, or determined, by the experimenter's choice of *frame*: i.e. a choice of basis assumed orthonormal. Then, if $g \mapsto T_g$ for $g \in SL(2,\mathbb{C})$ denotes the representation, for an initial vector $\xi$, $T_g(\xi)$ will represent the corresponding initial vector for—or *relative to*—an allowable transformation of the frame. Since the representation cannot be unitary for any choice of inner product, there will exist for some $g \in SL(2,\mathbb{C})$ an initial vector $\xi$ such that $\|T_g(\xi)\| \neq \|\xi\|$. Thus, the outcome of the simple ideal experiment $|\xi\rangle\otimes\langle\xi|$ may depend upon the experimenter's choice of allowably transformed frame: "Quantum transformations relate determinations which may express inequivalent, indeed incompatible, information." (QR, §4.3.1.) This seems to demonstrate an essential feature of Finkelstein's notion of quantum relativity. The passage just quoted continues:

> "Classical transformations relate experimenters who assign different names to equivalent actions. Thus a classical transformation is like a bilingual dictionary that has a unique equivalent in each of its two languages for every word in the other.
>
> Quantum transformations proper relate ideal experimenters who generically choose to perform incompatible actions. They change not merely our description but our conception of nature . . ."

An analogous situation arises in classical relativity: an experi-

ment—to determine the relative positions of spacetime events, say—may yield different results when an experimenter transforms to another (local) Lorentz frame. In this classical theory there are assumed to be independent external *objects*, namely spacetime events, and an infinite multiplicity of "other" experimenters, or observers. The quantum experimenter, in contrast, may assume neither *objects* nor "other" experimenters, who would presumably also be objects, but may perform "allowable" $SL(2,\mathbb{C})$ transformations between frames (and other acts of selection among initial vectors). Indeed, these allowable transformations, though as yet necessarily somewhat uninterpretable, will ultimately disclose their dynamical import in relativistic terms.

To return to our quest for a suitable class of qubits, we recall that, up to equivalence, the irreducible finite dimensional representations of $SL(2,\mathbb{C})$ are tensor products of the form $H_m \otimes \tilde{H}_n$ where $H_m$ is a certain irreducible representation of dimension $m$ and the representation *conjugate* (Hewitt and Ross 1970) to the representation $H_n$ is denoted by superscripting with a tilde: $\tilde{H}_n$. One may think of the matrices implementing the conjugate representation as being the complex conjugates of those implementing the representation upon $H_n$. (See Knapp 1986, Chapter 2.) There is a unique such tensor product for each pair $(m,n)$ of dimensions. In particular, there are exactly two inequivalent two-dimensional irreducible representations, $S$, say, and $\tilde{S}$. In $S$'s equivalence class one finds the natural, or fundamental, representation of $SL(2,\mathbb{C})$ upon $\mathbb{C}^2$. Thus, it appears that there are *two* different "flavors" of quantum binary alternative, namely the qubits having $S$ and $\tilde{S}$, respectively, as initial spaces. Since these two representations are inequivalent we shall regard the underlying spaces as *orthogonal*: only the null transition between acts pertaining to different truth-flavors should be allowed.

It is worth digressing briefly to remark on a significant distinction between these two "truth-flavors." The representation whose representation space we have denoted by $S$, is, as a map $SL(2,\mathbb{C}) \longrightarrow GL(n,\mathbb{C})$, *holomorphic*, while that associated with $\tilde{S}$ is *antiholomorphic*. This has the implication that $S$ is determined by a complex

linear representation of $\mathfrak{sl}(2,\mathbb{C})$ whereas $\tilde{S}$ is not. Thus $S$ might be thought of as arising as the coherent form of the infinitesimal or "microscopic" quantization of $\mathbb{Z}_2$, whereas $\tilde{S}$ is entirely an artifact of the global or macroscopic construction performed to arrive at equation (7.1.15), etc. This may have implications for time asymmetry at the deepest quantum level: see the remark at the end of §12.4.

If we interpret the elements of $S$ and $\tilde{S}$ as the basic interrogative acts available to experimenters, then they may be used to label certain transformations of quantum resources, just as we tried to introduce classical time as a label in §6.3.3. That is to say, an act of interrogation may be used to label the transformation it induces upon a quantum resource. In this case, two different labels are available and we posit a naïve static **IOL** axiom in the from

$$\alpha \vdash_{\mathrm{IO}} (\mathbf{t} \sqcap \bar{\mathbf{t}}) \sqcap \alpha \tag{7.1.16}$$

in imitation of equation (6.3.3.1). Here, upon "interrogation," $\alpha$ (re)produces itself accompanied by the registration of a label of each of the two possible types, regarded as logical atoms. This yields upon translation into **GQ** the axiom

$$!\alpha^e \vdash_{\mathrm{GQ}} !(!\mathbf{t} \otimes !\bar{\mathbf{t}}) \otimes !\alpha^e \tag{7.1.17}$$

or

$$(!(!\mathbf{t} \otimes !\bar{\mathbf{t}}))^* \vdash_{\mathrm{GQ}} (!\alpha^e)^* \otimes !\alpha^e. \tag{7.1.18}$$

We may realize this axiom in $\mathscr{H}_F$ by assigning finite dimensional Hilbert spaces to the atoms in the usual way. To the atom $\mathbf{t}$ we assign the one-dimensional first-grade component of $S$ for some choice of splitting which realizes $S$ as the exterior algebra of this component, and to $\bar{\mathbf{t}}$ we similarly assign the other truth-flavor's first-grade component in some splitting of $\tilde{S}$. In such a realization,

(7.1.18) takes the form of a map

$$E(S\otimes\tilde{S})^* \longrightarrow \mathrm{End}\,\mathfrak{H}^{(2n)}. \tag{7.1.19}$$

Thus, annihilators or absorbers of (finite) quantum sets of $S$, $\tilde{S}$ pairs are *used up* to produce transformations upon whatever system is involved: cf. §6.3.3. The basic sets of pairs are like lists of Boolean questions—i.e. *questionnaires*—whose application induces transformations upon the system examined, as the instrument of interrogation is itself used up. Unlike classical questionnaires, these admit superposition.

As matters stand, the lengths of the questionnaires are severely restricted by the finite dimensionality of the spaces involved. An actual experimenter should be able to apply questionnaires of arbitrary length. Consequently, there is again a need to break out of the confines of the category $\mathcal{H}_F$, at least as far as the multiplexing of $S\otimes\tilde{S}$ in the antecedents of the last pair of equations is concerned.

Just such a structure was proposed by Finkelstein in Finkelstein 1988b.

## 7.2 A Superconducting Quantum Net

For historical reasons—a history that we will find ourselves recapitulating in miniature in later chapters—these representations $S$ and $\tilde{S}$ are known as *spinor representations* of $\mathrm{SL}(2,\mathbb{C})$ and their elements are referred to as *spinors*. It seems convenient to maintain this standard usage even though an explanation of its physical significance must be postponed.

The appearance of the space of spinor–conjugate spinor pairs $S\otimes\tilde{S}$ as representers of the fundamental ideal (or "pure") interrogative act available to a macroscopic experimenter is irresistibly reminiscent of Cooper pair formation in the BCS theory of superconductivity. In this theory, as a result of the complex dynamics within a crystalline solid such as a metal (or more general systems), electrons may

pair up in a subtle way at a low enough temperature to form a macroscopic boson-like ground state. (This new vacuum spontaneously breaks the gauge symmetry of any ambient electromagnetic field and gives rise to the characteristic peculiarities of superconductors, such as the Meißner effect, etc. A toy geometric model of this phenomenon is discussed in §12.3.) Thus, we might suspect that the space $S\otimes\tilde{S}$ of spinor pairs may similarly represent an intermediate Maxwell–Boltzmann stage in the progression (cf. §2.3):

$$\begin{array}{ccccc} \text{Underlying Fermi–Dirac} & \longrightarrow & S\otimes\tilde{S} & \longrightarrow & \text{macroscopic} \\ \text{structure} & & \text{(Maxwell–Boltzmann} & & \text{boson-like} \\ & & \text{phase)} & & \text{"ground state"} \end{array} \tag{7.2.1}$$

The arrows represent, possibly, collective effects such as phase changes and/or spontaneous breakages of symmetry, etc., whose details lie below the resolution we are able to address in this work. (There is another path to the Cooper pair analogy that goes in the other direction along the right-most arrow in the above diagram. This is discussed in §7.3.)

For $g\in \mathrm{SL}(2,\mathbb{C})$, let $g\mapsto L_g$ denote the natural representation whose representation space we have denoted by $S$. Then $g\mapsto \tilde{L_g}$ denotes the corresponding operator in the conjugate representation of $\mathrm{SL}(2,\mathbb{C})$ upon $\tilde{S}$. Now, the representation denoted by

$$\mathfrak{D}(g)\equiv L_g\otimes\tilde{L_g} \tag{7.2.2}$$

is irreducible, and a candidate for the right-most entry in the diagram begins to heave into view when it is realized (by a macroscopic experimenter) that the representation $\mathfrak{D}$ has the property

$$\mathfrak{D}(\pm 1) = I_S\otimes I_{\tilde{S}} \tag{7.2.3}$$

and so extends to a representation of the quotient $\mathrm{SL}(2,\mathbb{C})/\{-1,+1\}$ —identifiable as the group of Lorentz transformations $\Lambda^i_j$ with

$\det\Lambda = 1$ and $\Lambda^0_0 \geq 1$ and known as the *restricted* or *orthochronous* Lorentz group $L^{\uparrow}_{+}$: see, for example, Bogolubov, Logunov and Todorov 1975. (In this connection recall that $\mathrm{SL}(2,\mathbb{C})$ is the universal covering group of $L^{\uparrow}_{+}$ and that we have an exact sequence

$$0 \longrightarrow \mathbb{Z}_2 \cong \{-1,+1\} \longrightarrow \mathrm{SL}(2,\mathbb{C}) \longrightarrow L^{\uparrow}_{+} \longrightarrow 1, \qquad (7.2.4)$$

where the map $\mathrm{SL}(2,\mathbb{C}) \longrightarrow L^{\uparrow}_{+}$ is a local isomorphism.)

Thus the elements of $S \otimes \tilde{S}$ "transform like" complex Lorentz vectors with respect to the representation $\mathfrak{D}$. Consequently we may entertain the possibility that the macroscopic boson-like "ground state" figuring in diagram (7.2.1) will be an initial act (or aggregate of acts) pertaining to the continuous spacetime structure apprehended by macroscopic experimenters.

The remainder of this work will be occupied mainly with the task of postulating and/or interpreting the various elements of the diagram (7.2.1), and certain associated structures.

In Finkelstein 1988b a candidate for the left-most element in the diagram was put forward. (See also Finkelstein 1988a, 1989a and b, and 1991.) To describe it, we note first that, according to the discussion in §2.3, a first step toward a deeper level underlying the "Maxwell–Boltzmann phase" represented by the Cooper-like pairs $S \otimes \tilde{S}$ could be effected by replacing these pairs by q-sets. Since we require q-sets based on both $S$ and $\tilde{S}$, we should base the qet structure on $S \oplus \tilde{S}$, obtaining $\mathrm{SET}(S \oplus \tilde{S})$. Now, as suggested earlier, in classical thinking the most general form of "interrogation" would seem to take the form of a *questionnaire*. This is an *ordered* set of propositions or deductions, the traversal of which may involve tree-like branching. At each stage of a traversal the results up to that point are compiled, or "unitized," or "definitely described." In set theoretic terms we may see this as the placement of set-making braces { } around the result obtained so far. The next step would entail the insertion (*via* set union) of another logical element, and this would be followed by bracing again, and so on. Thus, a hierarchical order is

imposed upon certain elements in the appropriate higher-order set algebra.

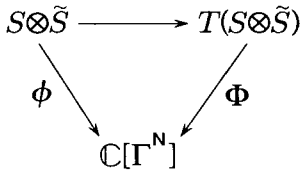
A similar construction may be carried out by a quantum experimenter using q-sets in the algebra $\text{SET}(S \oplus \tilde{S})$ based upon initial (quantum) interrogative acts represented by $S$ and $\tilde{S}$. Finkelstein argues that the hierarchical order imposed upon the qets involved, that breaks the permutational symmetry of the $\vee$ operation appearing therein, should give rise ultimately to the local causal structure of macroscopic spacetime. The operation of $\vee$-ing an interrogative act, followed by unitizing the result by applying $\iota = |\ \rangle$, then constitutes the basic microscopic synchronic step, creating the causally adjacent act. The pairing hypothesis then descends to the requirement that the causal insertions of interrogative acts *via* the $\vee$ operation should occur in pairs to produce, from qets $\xi$ already unitized, causally adjacent acts of the form

$$\Gamma^{N}_{\Sigma\tilde{\Sigma}}(\xi) = \langle |\Sigma\tilde{\ }\rangle \vee \langle |\Sigma\rangle \vee \xi\rangle\rangle, \tag{7.2.5}$$

where $\Sigma \in S$ and $\Sigma\tilde{\ } \in \tilde{S}$.

Starting with $\xi = |\varnothing\rangle$, the space of qets generated by repeated applications of the $\Gamma^{N}$-operators is an example of a *quantum net.* (For a discussion of the general notion see QR, Chapter 16.) These are the qets obtained by operating upon $|\varnothing\rangle$ with the complex algebra generated by the $\Gamma^{N}$-operators. This latter algebra (of operators upon $\text{SET}(S \oplus \tilde{S})$) shall be denoted by $\mathbb{C}[\Gamma^{N}]$.

The assignment $(\Sigma, \Sigma\tilde{\ }) \mapsto \Gamma^{N}_{\Sigma\tilde{\Sigma}}$, considered as a map from $S \times \tilde{S}$ into the algebra $\mathbb{C}[\Gamma^{N}]$, is bilinear, and therefore lifts uniquely to a linear map $\phi$ of $S \otimes \tilde{S}$ into this algebra (§2.2.1). This latter map then itself lifts uniquely to a map $\Phi : T(S \otimes \tilde{S}) \longrightarrow \mathbb{C}[\Gamma^{N}]$ of unital algebras (§2.2.2). It is then only a mild abuse of the notation to write $\phi(\theta) = \Gamma^{N}_{\theta}$ for $\theta \in S \otimes \tilde{S}$. The horizontal map in the following commutative diagram is the canonical inclusion of $S \otimes \tilde{S}$ into its tensor algebra:




(7.2.6)

Now, elements of $S\otimes\tilde{S}$ ($\subset T(S\otimes\tilde{S})$) act on $T(S\otimes\tilde{S})$ *via* the "left regular" representation, which we denote by $\Gamma_\theta$, for $\theta\in S\otimes\tilde{S}$. Namely, for $\theta_1,\ldots,\theta_n\in S\otimes\tilde{S}$ we have

$$\begin{aligned}\Gamma_{\theta_1}(\theta_2\otimes\ldots\otimes\theta_n) &= \theta_1\otimes\theta_2\otimes\ldots\otimes\theta_n \\ &= \Gamma_{\theta_1}\Gamma_{\theta_2}\ldots\Gamma_{\theta_n}(1),\end{aligned} \tag{7.2.7}$$

where 1 is the unit in $T(S\otimes\tilde{S})$.

Thus, from (7.2.6),

$$\begin{aligned}\Phi(\Gamma_{\theta_1}\Gamma_{\theta_2}\ldots\Gamma_{\theta_n}(1)) &= \Phi(\theta_1\otimes\theta_2\otimes\ldots\otimes\theta_n) \\ &= \Phi(\theta_1)\Phi(\theta_2)\ldots\Phi(\theta_n) \\ &= \phi(\theta_1)\phi(\theta_2)\ldots\phi(\theta_n) \\ &= \Gamma^{\mathsf{N}}_{\theta_1}\Gamma^{\mathsf{N}}_{\theta_2}\ldots\Gamma^{\mathsf{N}}_{\theta_n}.\end{aligned} \tag{7.2.8}$$

We shall denote the complex algebra generated by the operators $\Gamma_\theta$, $\theta\in S\otimes\tilde{S}$ by $\mathbb{C}[\Gamma]$. Then, if $\{\theta_k\}_{k=1}^4$ denotes a basis for $S\otimes\tilde{S}$, it is easy to see that $\mathbb{C}[\Gamma]$ is freely generated by $\{\Gamma_{\theta_k}\}_{k=1}^4$ and that the algebra map $\mathbb{C}[\Gamma]\longrightarrow T(S\otimes\tilde{S})$ determined by the assignments $\Gamma_{\theta_{k_1}}\ldots\Gamma_{\theta_{k_n}}\mapsto\Gamma_{\theta_{k_1}}\ldots\Gamma_{\theta_{k_n}}(1)=\theta_1\otimes\theta_2\otimes\ldots\otimes\theta_n$ (equation (7.2.7))

is an isomorphism that is, in fact, independent of the choice of basis. (For: it is easily checked that the inverse of this isomorphism is the analog of $\Phi$ for the linear map $S \otimes \tilde{S} \longrightarrow \mathbb{C}[\Gamma]$ sending $\theta$ to $\Gamma_\theta$. This analog of $\Phi$ does not depend upon the basis chosen, and so its inverse doesn't either.)

Composing this isomorphism $\mathbb{C}[\Gamma] \longrightarrow T(S \otimes \tilde{S})$ with $\Phi$ yields a surjective algebra map $\mathbb{C}[\Gamma] \longrightarrow \mathbb{C}[\Gamma^{\mathbf{N}}]$ in which $\Gamma_{\theta_{k_1}} \ldots \Gamma_{\theta_{k_n}}$ is mapped to $\Gamma^{\mathbf{N}}_{\theta_{k_1}} \ldots \Gamma^{\mathbf{N}}_{\theta_{k_n}}$. (Although unproven, it seems quite likely that this map is also one-to-one.)

As we have remarked, the hierarchical structure that results from the unitizations in equation (7.2.5) breaks the symmetry of the $\vee$ operation, and this breaking constitutes, as it were, a partial traversal of the left-most arrow in diagram (7.2.1). That is to say, the adoption of the pairing hypothesis, taken in conjunction with the q-set construction, effects a partial "phase change" from a Grassmannian substructure to a Maxwell–Boltzmann aggregate, as postulated in §2.3: in a sense, $\vee$ may be replaced by $\otimes$. Viewed in this light, the algebra $\mathbb{C}[\Gamma]$ represents a lifting of the "net" algebra $\mathbb{C}[\Gamma^{\mathbf{N}}]$ into the Maxwell–Boltzmann phase, i.e. along the first arrow in (7.2.1).

(In the context of our sequent calculus considerations, net elements correspond to *quantum* sets of terms. The corresponding classical sets of terms may be obtained by reversing the quantization process and replacing each $|\ \rangle$ by $\{\ \}$ and each $\vee$ by a comma (while suppressing superposition). Then the "phase transition" in which $\vee$ is replaced by $\otimes$ may be viewed also as part of a quantum version of our earlier interpretation of **GQ** in $\mathscr{H}_F$ (§6.3.2), in which ordinary sequences of formulae are realized as tensor products of the corresponding objects in $\mathscr{H}_F$. The idea of viewing such a phase change (namely, the one associated with the right-most arrow in (7.2.1)) as a translation between expressions lying in different categories, will be used again in the subsequent chapters.)

If the net and its algebra represent acts performable upon a "discontinuous" Fermi–Dirac-like quantum system by an ideal microscopic experimenter, then we may interpret the corresponding Maxwell–Boltzmann (tensor algebra) phase in terms of acts per-

formable by an experimenter upon the *macroscopic surface* of this discontinuum: see §2.3. (Indeed, the Maxwell–Boltzmann operators $\Gamma_\theta$ reflect the (causal) linkages in the underlying Fermi–Dirac structure, which is the import of equation (7.2.8).) Such ideal macroscopic experimenters would, of course, be capable of vastly greater resolving power than is currently (or would conceivably ever be) available to human physicists—even those still funded.

The second arrow in diagram (7.2.1) should then somehow interpret the pairs $\Sigma\otimes\Sigma^\sim$ in terms of (Lorentz) vectors in the continuum, which combine commutatively. Thus, the reticular operators $\Gamma_\theta^{\mathsf{N}}$—or their intermediate Maxwell–Boltzmann versions $\Gamma_\theta$—should somehow acquire a continuum interpretation in terms of commutative operators in passing from the left of the diagram to the right, perhaps by undergoing some sort of phase change or condensation. We will begin to form such an association, or "correspondence principle," in the next chapter.

### *7.2.1 Some Remarks on Types and Reductions*

We shall continue to regard Hilbert spaces of action vectors as *types* in the sense of **GQ** (despite the fact that we will sometimes have to step beyond the category $\mathscr{H}_F$), and will attempt to maintain the integrity of such typing as we proceed. Though the value of such attention to typing is well known to computer scientists, it is usually absent from physical computations (except perhaps in the realm of dimensional analysis). We hope to show that it has surprising benefits in this arena also.

So far, our fundamental types are $S$ and $\tilde{S}$. Later it will prove convenient to incorporate (or insert) into the net structure other types besides $S$ and $\tilde{S}$. (These insertions will arise out of an attempt to explicitly realize spaces of defects in the net.) In general, the understanding will be that, when circumstances require it, a new type may be amalgamated into the SET algebra as above, and made available for insertion into the net. This will have the effect, in the Maxwell–Boltzmann phase, in which $\vee$ is replaced by $\otimes$, of intro-

ducing tensors of mixed types. In keeping with the discussion in §1.3, contractions among matching types within multi-indexed tensors of this kind (which yield amplitudes, traces, operator products, etc.) may be used to model externalized views of such compound acts or experiments. For example, the contraction of the inner pair in a segment of type $\ldots V^*\otimes W\otimes W^*\otimes D\ldots$ yields an operator composition, contraction of the left-most pair in a segment of type $W\otimes W^*\otimes D$ yields the image (in $D$) of the left-most element *via* the operator corresponding to the right-most-pair, etc.

Both of these reductions (or deductions) are interpretations of **GQ** CUT executions or eliminations, the first being the interpretation of an inference of the form

$$\frac{V\vdash_{\mathbf{GQ}} W \quad W\vdash_{\mathbf{GQ}} D}{V\vdash_{\mathbf{GQ}} D}, \tag{7.2.1.1}$$

while the second is an interpretation of the special case

$$\frac{\vdash_{\mathbf{GQ}} W \quad W\vdash_{\mathbf{GQ}} D}{\vdash_{\mathbf{GQ}} D}, \tag{7.2.1.2}$$

cf. §6.3.2. (Note that our rules interpret $\vdash_{\mathbf{GQ}} D$ as a map $\mathbb{C}\longrightarrow D$, which amounts to specifying an element of $D$.)

As another special case, the contraction of $W\otimes W^*$ may be regarded as an interpretation of a CUT of the form

$$\frac{\vdash_{\mathbf{GQ}} W \quad W\vdash_{\mathbf{GQ}}}{\vdash_{\mathbf{GQ}}}. \tag{7.2.1.3}$$

Here, the naked $\vdash_{\mathbf{GQ}}$ is interpreted according to the rules as an element in $\mathrm{Hom}(\mathbb{C},\mathbb{C})\cong\mathbb{C}$: namely, the trace of the operator corresponding to that element in $W\otimes W^*$ which is the interpretation of

the line above.

Segments of mixed type, such as $V^*\otimes W$, may be viewed as spaces of compound experiments whose constituent acts pertain to different systems. These may be reduced if a transitional *relationship* between $V$ and $W$ is implicit or has been specified by the experimenter—for instance, by specifying a mapping from $V^*$ to $W^*$ (or equivalently from $W$ to $V$) and then using the map to define the transitional structure in an obvious way. As a special case we could map $V^*$ to 0 in $W^*$, or equivalently assume that $V$ has been declared orthogonal to $W$, meaning that transitions $w \longrightarrow v$ are expressly forbidden, as in the case of $S$ and $\tilde{S}$. Then the contraction is defined, and yields the zero amplitude.

This amounts to supplying a sequent of the form $W \vdash_{GQ} V$ in the presence of $V^*\otimes W$ so that CUT may be applied to arrive at an element in $V \vdash_{GQ} V$ interpreted as a linear operator on $V$. After such a deduction we arrive at an amplitude by taking the trace.

Elements of type $V^*\otimes V\otimes W^*\otimes W$ may be considered to be of the form $P\otimes Q$ for operators $P$ on $V$ and $Q$ on $W$. Inner-to-outer contraction of the variety considered in §1.3 may not be defined for such elements: this would reflect the circumstance that the two subexperiments do not interface (since in this case internal transitions are not defined), and should be viewed externally as a pair of experiments upon two separate systems. Then amplitudes can be obtained only by contracting like pairs, producing values of the form $\mathrm{tr}P\,\mathrm{tr}Q$, which we note is exactly $\mathrm{tr}(P\otimes Q)$: thus, for such compound experiments upon such systems, transition amplitudes seen by maximally external experimenters must be of this type.

The corresponding sequent deduction is of the form

$$\dfrac{\dfrac{V \vdash_{GQ} V \quad W \vdash_{GQ} W}{V, W \vdash_{GQ} V \otimes W}\, R\otimes}{V \otimes W \vdash_{GQ} V \otimes W}\, L\otimes \qquad (7.2.1.4)$$

with the trace of the resulting operator being taken as a final, or *post processing,* step.

In summary, if a certain compound experiment is represented in the form $P \otimes Q$, where $P$ and $Q$ are operators on not already suitably related spaces—or, if they act upon already suitably related spaces but are assumed to represent independent or non-interfacing experimental acts—then the fully contracted amplitude, as seen by a maximally external experimenter, is

$$\mathrm{tr}(P \otimes Q) = \mathrm{tr}P\,\mathrm{tr}Q. \qquad (7.2.1.5)$$

On the other hand, for an act of the form $P \otimes Q$, where $P$ and $Q$ act on the *same* space (or suitably related spaces), then (§1.3) the fully contracted amplitude, as seen by an external experimenter, may assume the form

$$\mathrm{tr}(PQ) \qquad (7.2.1.6)$$

if the (now possible) internal transitions are assumed to take precedence. Effectively, in the latter case, the internal contraction across the $\otimes$ in equation (7.2.1.5) is possible because $P$ and $Q$ act on the same space. (This corresponds to the first case considered (7.2.1.1) with $V = D = W$, followed by the taking of the trace.)

Thus, reductions of mixed tensors follow the pattern of CUT eliminations and the taking of traces, the latter process being itself a degenerate form of CUT elimination: cf. the remarks following equation (4.3.13) and those at the end of that chapter.

Later, in Chapter 11, we will find that the presence of just such a mix of types serves, among other things, to distinguish the form of the gauge field interactions—that is, Yang-Mills—from that

of gravity—i.e. Einstein–Hilbert. (Cf. end of §11.2 and the discussion preceding equation (11.2.2.47).)

## 7.3 A Geometrical Approach to the Net

In this section (which may be safely skipped) we sketch an approach to the derivation of a semi-quantum net-like structure that in essence predates the author's encounter with Finkelstein's ideas. It starts in the continuum and goes in the other direction: that is, we attempt to pull certain continuum notions backwards along the right-most arrow in diagram (7.2.1). We arrive at a (commutative) structure that is very similar locally to the (noncommutative) one obtained by pushing the superconducting net forward along the left-most arrow in the diagram (replacing $\vee$ by $\otimes$).

The idea is to suppose that an experimenter wishes to design a quantum mechanical initial act that injects an infinitesimal "quantum" step in some direction away from some point, $x$ say, in a model of classical spacetime, which we take to be a manifold $M$. Ultimately, the experimenter will want to build local path-like "questionnaire," or *history,* injectors by forming sequences of such initial acts. ("Local" here means *tangential*: these path-like acts will pertain to a quantum version of the tangent space at the point in question.) Now the "directions" through the point $x$ may be identified with rays in the tangent space $\mathcal{T}_x$ at $x$, and an obvious candidate for the initial space of acts "injecting" such a ray would surely be the space $\mathcal{T}_x$ itself, assuming an appropriate Hilbert space structure could be found for it. Such a structure could be imposed if $M$ were actually a *complex* manifold, an assumption we now make.

Thus, starting with a complex manifold $M$ we have interpreted the fibres of the tangent bundle as spaces of initial vectors representing acts of injection of the corresponding *directions* through point in the manifold over which the fibre sits. Now the directions through a point $x$ are parametrized by the projective space $\mathbb{P}(\mathcal{T}_x)$ of the tangent space $\mathcal{T}_x$. So effectively we have attached to each point $x \in M$ the projective space $\mathbb{P}(\mathcal{T}_x)$ and then associated with each ele-

ment of $\mathbb{P}(\mathcal{T}_x)$ the corresponding *ray*, which is interpreted as representing the initial act of injection of that *direction* (i.e. point of $\mathbb{P}(\mathcal{T}_x)$). This collection of projective spaces assembles into a bundle $\mathbb{P}(\mathcal{T})$ over $M$ called the *projectivization* of $\mathcal{T}$. The collection of rays now associated with each element of this bundle space $\mathbb{P}(\mathcal{T})$ constitutes a line bundle over $\mathbb{P}(\mathcal{T})$ called the *tautological* bundle (cf. Bott and Tu 1982). Each point of the manifold has in a sense been replaced by the projective space of directions through it to produce a new manifold $\mathbb{P}(\mathcal{T})$, which is in fact a bundle over the original manifold, with bundle projection map we shall denote by $\pi : \mathbb{P}(\mathcal{T}) \longrightarrow M$. Over the manifold $\mathbb{P}(\mathcal{T})$ sits the tautological line bundle, which, following geometrical usage, we shall denote by $\mathcal{O}(-1)$. The fibre in this latter bundle over an element of $\mathbb{P}(\mathcal{T})$, which is a direction through a point of $M$, is the ray corresponding to that direction:

$$\begin{array}{ccc} \mathcal{O}(-1) & & \mathcal{T} \\ \downarrow & & \downarrow \\ \mathbb{P}(\mathcal{T}) & \xrightarrow[\pi]{} & M \end{array} \qquad (7.3.1)$$

The vectors in this ray are interpreted as initial vectors for the injection of a generic "step" in the corresponding direction.

For geometrical reasons it will be convenient to consider the dual of the tautological bundle, denoted $\mathcal{O}(1)$. (This is the line bundle that generates the group of line bundles on the projectivization: its Chern class is 1.) Since bundles are determined by their sheaves of germs of sections, it will be an additional convenience to identify a bundle with its equivalent (locally free) sheaf of germs of sections. In this language $\mathcal{O}(1)$ is known as the *twisting sheaf* of Serre. For us, the fibres of this bundle represent final acts of registration of the local steps alluded to above. Then the fibres of the bundle $T(\mathcal{O}(1))$ (cf. §2.2.2) represent final acts of registration of sequences of such tangential steps: that is, final acts for *paths* in the tangent

spaces. Thus we have effected a partial quantization of the tangent bundle of $M$. In fact, tensor products of line bundles are *symmetric,* since the fibres are one-dimensional and bundles are determined by their sections which, in this case, are locally just ordinary functions into $\mathbb{C}$. (See Selesnick 1983 for an algebraic view of this property.) Consequently, $T(\mathcal{O}(1))$ is a bundle of *commutative* algebras. This commutativity is, as usual, a consequence of the initial positing of an underlying *object,* namely the manifold $M$. Thus the commutative algebra bundle $T(\mathcal{O}(1))$ is a sort of semi-quantum bosonic approximation to a quantum version of the tangent bundle.

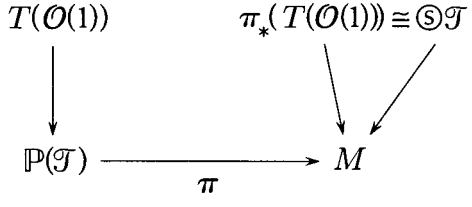
Now sheaves, unlike bundles, always have pushouts (or "direct images") which are also sheaves, and pushing out along $\pi$ yields in this case sheaf isomorphisms (over $M$) for $n \geq 0$:

$$\pi_*(\otimes^n \mathcal{O}(1)) \cong \textcircled{S}^n \mathcal{T}\,, \tag{7.3.2}$$

so that

$$\pi_*(T(\mathcal{O}(1))) \cong \textcircled{S}\mathcal{T}\,, \tag{7.3.3}$$

as in the following diagram:




(7.3.4)

(For a proof in the algebraic category, see Hartshorne 1977.)

The pushout $\pi_*(T(\mathcal{O}(1))) \cong \textcircled{S}\mathcal{T}$ in a sense represents our semi-quantum tangent bundle as viewed in the continuum. A fibre of the right hand side of equation (7.3.3) is isomorphic with the polynomial algebra in the generators of the corresponding tangent space

(considered as a set of commuting indeterminates). This latter algebra serves as a model for a Bargmann representation of the CCR (in which it is dense) corresponding to the dimension of $M$. Here, the boson creation operators are implemented by multiplication by the appropriate generator: thus the tangent vectors are realized as boson creators. The peculiar relation between such Bargmannesque power series-like representations and the intuitive notion of a curve will be taken up in §8.2.

Now there is a likely, indeed canonical candidate for $M$, namely the conformal compactification of Minkowski space, realized as the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^4)$ of two-dimensional subspaces of $\mathbb{C}^4$ (Manin 1988). This manifold has the interesting property that its tangent bundle admits a decomposition of the form:

$$\mathcal{T} \cong \tilde{S}^* \otimes S^*, \tag{7.3.5}$$

where $S$ and $\tilde{S}$ here denote bundles of "Grassmannian" spinors. Thus $T(\mathcal{O}(1))$ is in this case a (bundle of) *commutative* algebras generated locally by Cooper-like pairs of (dual) spinors. (Duals appear because we chose to work with the dual of the tautological bundle.)

Our algebra $\mathbb{C}[\Gamma^{\mathsf{N}}]$ of $\Gamma^{\mathsf{N}}$-operators, when pushed along the left-most arrow of diagram (7.2.1) and realized as the algebra $\mathbb{C}[\Gamma]$ of multiplications in the tensor algebra, yields an exact *noncommutative* tensorial analog of the (dual form of) the typical fibre of $T(\mathcal{O}(1))$. (An attempt—not pursued further in these notes—to interpret this chimerical model in reticular terms may be found in Selesnick 1991.)

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# 8

# Towards a Correspondence Principle for the Quantum Net

Three levels (or degrees) of resolution seem to have emerged from our considerations so far. At the deepest Fermi–Dirac quantum set level lies the quantum net itself, corresponding to hierarchical arrangements of interrogative acts performable by highly idealized experimenters capable of such resolving power that they can detect the discontinuous underlying Fermi–Dirac structure of the system. For want of a better name we have referred to such experimenters as "microscopic," although even they have some macroscopic contamination, since we have chosen spaces of initial vectors that explicitly incorporate those global symmetries appropriate to "macroscopic" experimenters who implement the coherent, or Lie group, form of the transformations. (We remark again that the adoption of these coherent forms entails the appearance of the antiholomorphic two-dimensional representation of $SL(2,\mathbb{C})$.) This incongruity can do no harm until we actually start investigating the consequences of adopting such transformations, and this we shall do mainly at the next and coarser level of resolution, which we consider to be macroscopic in any case. At this next level we find the Maxwell–Boltzmann "surface" phase, consisting of acts appropriate to somewhat less ideal experimenters with somewhat lower (but still unimaginably high) resolving power: a power insufficient, for example, to detect the underlying

Fermi–Dirac structure of the system though capable of course of detecting its Maxwell–Boltzmann granularity. Finally, we have the macroscopic continuum of experience, idealized and reified as a manifold of some kind.

We may arrange these levels in a progression paralleling the one in diagram (7.2.1):

| Quantum Net: $\mathbb{C}[\Gamma^N]$ (discontinuous; microscopic) | ⟶ | Maxwell–Boltzmann surface: $\mathbb{C}[\Gamma]$ (discontinuous; macroscopic) | ⟶ | Spacetime Manifold: ? (continuous; macroscopic) | (8.0.1) |
|---|---|---|---|---|---|

In this chapter we prepare the ground for a later attempt to fling ourselves across the enormous gulf represented by the right-most arrows in diagrams (7.2.1) and (8.0.1), and to fill in the ? appearing in the latter one.

It is apparent that our algebras $\mathbb{C}[\Gamma^N]$ and $\mathbb{C}[\Gamma]$ are noncommutative precursors of some commutative algebra of creation-like operators in some Fock-like representation. On the other hand, some version of ordinary spacetime is supposed to emerge on the right-hand sides of the diagrams. Thus, in attempting to mathematically bridge the huge physical chasm represented by the right-most arrows, we confront the peculiar problem of having to find interpretations within a Fock-like formalism of certain aspects of ordinary geometry in a continuum. This is the task we apply ourselves to in the present chapter, and it is here that our classical continuum-based prejudices are most blatantly on display. Despite the weight of this classical baggage it is comforting to see the fairly spontaneous reemergence of coherent states (as vector descriptors), this time coming out of two different attempts to marry geometry to many-body theory.

In §8.1 we address what amounts to the local theory, and argue in favor of the main operators (along with their continuum correspondents) to be used in what follows. A certain globalization of one class of these operators is obtained in §8.2 by formulating a reticular notion of "intrinsic parallel transport." An abstract algebraic

specification of the macroscopic operators which implement this kind of transport is found to include those arising from ordinary transport along classical curves, in an intrinsic form first investigated by Chen. These operators reduce to the coherent forms already found if the curve is a line segment: this closes a circle, since vectors are again, and by a different route, realized explicitly as coherent operators implementing parallel transport along the corresponding line segment.

The connection is established between this geometrical approach to the specificaton of extended path-like entities and our earlier attempt to build a logic-based dynamics around classical time, thereby closing another circle.

## 8.1 Vectors

Our ultimate aim is to associate elements of the discontinuous structures to the left and center of the diagrams (7.2.1) and (8.0.1) (which are associated with the "highly resolving" acts at the level of the net or its macroscopic Maxwell–Boltzmann surface) with corresponding low resolution continuum elements. This reduces to a passage from a noncommutative Fock-like exterior or tensor algebra to a corresponding representation of the usual CCRs for bosons of four kinds, which may be realized, for instance, on the usual kinds of Fock space, which are commutative as algebras (§2.2.4).

At the same time, such a correspondence should mediate between the more-or-less familiar local geometry of the continuum and some sort of "pre-geometry" in the discrete phases. We shall view such a correspondence as an *interpretation*—in the sense of type theory—of reticular expressions in terms of expressions pertaining to coarse, or low resolution macroscopic continuous entities. Presumably, a coarse macroscopic experimenter immersed in an apparent continuum will be forced to design acts of injection, registration and selection which differ markedly from those that would be available to our ideal highly resolving experimenters. Indeed, such coarse macroscopic experimenters are likely to have certain designs thrust upon

them by the vagaries of local evolution. For example, corneas, retinal rods and cones, etc., have evolved in such a way that they perform certain specific acts of injection and registration and only them: acts of injection of gamma rays, for instance, effect null transitions. The sort of correspondence principle we have in mind will eventually postulate rules to interpret certain reticular acts in terms of certain continuum ones. We proffer this as (literally) a stop-gap measure until more refined methods of analyzing the arrows in those diagrams become available. In this chapter we hope to make plausible the correspondences we will formally adopt in the next chapter.

Recall that the net comprises elements of the form $\rho|\varnothing\rangle$, with $\rho \in \mathbb{C}[\Gamma^N]$. We seek macroscopic continuum interpretations of certain elements of this type within the terms of a Fock representation of the CCR for quanta of four kinds, for reasons given. It would seem reasonable to start by associating the vacuum qet $|\varnothing\rangle$ with the Fock vacuum, which, to avoid syntactically overloading the ket notation, we shall denote by $\varpi$. That is to say, our continuum "interpretation" of $|\varnothing\rangle$ shall be as $\varpi$. It will now prove convenient to borrow (and abuse) the logical notation for a *semantic function*, a procedure we glossed over when interpreting **GQ** in $\mathcal{H}_F$. Accordingly, we denote the macroscopic continuum interpretation of some expression by enclosing that expression in hollow brackets: $[\![\ ]\!]$. Thus, we may write:

$$[\![\, |\varnothing\rangle \,]\!] = \varpi \,. \tag{8.1.1}$$

In this attempt to extract vectorial geometry from the net, it is clear that $|\varnothing\rangle$, and therefore $\varpi$, should correspond to the zero vector, or origin. (This should be apparent also to readers of §7.3. There the tangent vectors to a classical model of the continuum were realized as the creators of bosonic "steps" away from the point in question. The vacuum must then be interpreted as the final act—in that dualized picture—of detection or registration of *no* steps: i.e. the tangent space origin detector.)

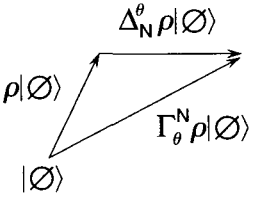
We now seek elements $\rho$ (in $\mathbb{C}[\Gamma^N]$ or some extension of it)

such that the net element $\rho|\varnothing\rangle$ may be interpreted as a "geometrical" vector in the continuum. That is, with

$$[\![\rho|\varnothing\rangle]\!] \equiv [\![\rho]\!][\![|\varnothing\rangle]\!] \tag{8.1.2}$$

we are required to find simultaneously: an interpretation of the Fock space element $[\![\rho]\!]\varpi$ as a geometrical vector (in view of equation (8.1.1)), and candidates for $[\![\rho]\!]$ and $\rho$.

Now, for $\theta \in S \otimes \tilde{S}$, the operator $\Gamma_\theta^{\mathsf{N}}$ $(= \phi(\theta)$, §7.2) may be thought of as an incrementation operator which extends its argument by (or along) the vector $\theta$. Then, if $\rho|\varnothing\rangle$ really were a "geometrical" vector, there would exist a corresponding vector difference in the $\theta$ direction, written $\Delta_{\mathsf{N}}^{\theta}\rho|\varnothing\rangle$, such that the following *virtual* diagram of would-be vectors obtains:




(8.1.3)

By *virtuality* we mean that this diagram is not to be construed as an actual diagram at the reticular level but rather in the continuum, since true vectorial behavior can be expected only there, where reticular acts are reinterpreted at low resolution (or "from afar"). Then, one way of expressing the virtual vector relation depicted above in the continuum is by asserting that the (low resolution) continuum version of the operator $\Gamma_\theta^{\mathsf{N}} - \Delta_{\mathsf{N}}^{\theta}$, when applied to the continuum version of $\rho|\varnothing\rangle$, should yield this same act. That is, the associated Fock space vectors should be projectively related, or:

$$[\![\Gamma_\theta^{\mathsf{N}} - \Delta_{\mathsf{N}}^{\theta}]\!][\![\rho]\!]\varpi = z_\theta(\rho)[\![\rho]\!]\varpi\,, \tag{8.1.4}$$

for some complex number $z_\theta(\rho)$.

If $\rho = 1$ the arrow labelled $\rho|\varnothing\rangle$ in diagram (8.1.3) disappears, leaving the *virtual* (not actual) identity $\Gamma_\theta^{\mathsf{N}}|\varnothing\rangle = \Delta_{\mathsf{N}}^\theta|\varnothing\rangle$, or, in the continuum:

$$[\![\Gamma_\theta^{\mathsf{N}} - \Delta_{\mathsf{N}}^\theta]\!]\varpi = 0\,, \tag{8.1.5}$$

whence, upon comparison with equation (8.1.4), we conclude that $z_\theta(1) = 0$ and $[\![1]\!] = 1$ so that we may set

$$[\![\Gamma_\theta^{\mathsf{N}} - \Delta_{\mathsf{N}}^\theta]\!] = \kappa a(\theta) \tag{8.1.6}$$

where $a(\theta)$ denotes the appropriate Fock annihilation operator and $\kappa$ is an arbitrary non-zero complex constant. Moreover, equation (8.1.4) now specifies $[\![\rho]\!]\varpi$ as a coherent state independently of any explicit physical many-body phenomenology.

Choosing a basis $\{\theta_k\}$ in $S\otimes\tilde{S}$, and putting $a_k \equiv a(\theta_k)$, $z_k \equiv z_{\theta_k}(\rho)$, this state may be expressed in a form in which

$$[\![\rho]\!] = \exp\sum_k\left[\kappa^{-1}z_k a_k^\dagger - \overline{\kappa^{-1}z_k}a_k\right]. \tag{8.1.7}$$

(See for example Feng, Gilmore and Zhang 1990, §II.1.)

Note that if we insist upon the coefficients $\kappa^{-1}z_k$ being *imaginary*,

$$\kappa^{-1}z_k = ir_k\,, \tag{8.1.8}$$

say, with $r_k$ real, then equation (8.1.7) assumes the form

$$[\![\rho]\!] = \exp\left[i\sum_k r_k(a_k^\dagger + a_k)\right]. \tag{8.1.9}$$

For a certain real number $\tau$, whose significance is to be deter-

mined, let us define self-adjoint operators

$$q_k = \frac{\tau}{\sqrt{2}}(a_k + a_k^\dagger) \tag{8.1.10}$$

and

$$p_k = \frac{1}{i\tau\sqrt{2}}(a_k - a_k^\dagger) \tag{8.1.11}$$

so that

$$a_k = \frac{1}{\sqrt{2}}\left(\frac{1}{\tau}q_k + i\tau p_k\right) \tag{8.1.12}$$

and

$$a_k^\dagger = \frac{1}{\sqrt{2}}\left(\frac{1}{\tau}q_k - i\tau p_k\right). \tag{8.1.13}$$

Note that if the chosen basis in $S\otimes\tilde{S}$ were orthogonal with respect to some Hilbert space inner product so that the usual commutation relations obtained for the operators $a_k$, $a_k^\dagger$, then:

$$[p_j, q_k] = -i\delta_{jk}. \tag{8.1.14}$$

So, if these operators bore their usual significance—in the Schrödinger representation for example—then the normalization chosen in equations (8.1.10) and (8.1.11) would be tantamount among other things to a choice of units in which $\hbar = 1$.

Equation (8.1.9) now reads

$$[\![\rho]\!] = \exp\left[\left(\frac{i\sqrt{2}}{\tau}\right)\sum_k r_k q_k\right]. \tag{8.1.15}$$

Assuming $[\![\ ]\!]$ to be at least additive, and in an obvious notation, equations (8.1.6) and (8.1.12) yield

$$[\![\Gamma^{\mathsf{N}}_k]\!] - [\![\Delta^k_{\mathsf{N}}]\!] = \frac{\kappa}{\sqrt{2}}\left(\frac{1}{\tau}q_k + i\tau p_k\right). \tag{8.1.16}$$

Thus, although we have not yet found candidates for the reticular operators $\Delta^k_{\mathsf{N}}$, we may nonetheless adopt the interpretations expressed by:

$$[\![\Gamma^{\mathsf{N}}_k]\!] = \frac{\kappa}{\tau\sqrt{2}}q_k \tag{8.1.17}$$

and

$$[\![\Delta^k_{\mathsf{N}}]\!] = \frac{-i\tau\kappa}{\sqrt{2}}p_k, \tag{8.1.18}$$

where we recall that $\kappa$ is an arbitrary complex number and $\tau$ is real.

We have arrived at this point on the basis of the presumed coarse behavior in the continuum of certain interpretations of reticular elements, but have not yet put forth an explicit connection between the algebraic operations of the net and the geometrical (vector) operations we believe to underlie those apparent to coarse experimenters.

To this end we fix an element $\theta \in S \otimes \tilde{S}$ and return to the net to try to form a net element $\rho_\theta|\varnothing\rangle$ which could be interpreted as a vector *parallel* to $\theta$. A reticular expression for this may be written:

$$\rho_\theta|\varnothing\rangle - |\varnothing\rangle = \alpha_\theta \Gamma^{\mathsf{N}}_\theta \rho_\theta|\varnothing\rangle, \tag{8.1.19}$$

where $\alpha_\theta$ is a complex number.

This equation explicitly asserts certain relations between reticular acts and the low resolution geometry apparent to coarse experimenters. It defines the relation, for a would-be vector $\rho_\theta|\varnothing\rangle$, of being parallel to $\theta$: namely, that *relative* to the "vector" origin $|\varnothing\rangle$ (the left hand side of the equation) this element specifies the same initial act as does the $\Gamma_\theta^{\mathbf{N}}$ increment of itself. The choice of constant $\alpha_\theta$ is, as usual, not fixed.

From this equation one obtains

$$\begin{aligned}\rho_\theta|\varnothing\rangle &= |\varnothing\rangle + \alpha_\theta\Gamma_\theta^{\mathbf{N}}\rho_\theta|\varnothing\rangle \\ &= |\varnothing\rangle + \alpha_\theta\Gamma_\theta^{\mathbf{N}}(|\varnothing\rangle + \alpha_\theta\Gamma_\theta^{\mathbf{N}}(|\varnothing\rangle + \ldots \\ &= |\varnothing\rangle + \alpha_\theta\Gamma_\theta^{\mathbf{N}}|\varnothing\rangle + (\alpha_\theta\Gamma_\theta^{\mathbf{N}})^2|\varnothing\rangle + \ldots \\ &= (1 + \alpha_\theta\Gamma_\theta^{\mathbf{N}} + (\alpha_\theta\Gamma_\theta^{\mathbf{N}})^2 + \ldots)|\varnothing\rangle. \qquad (8.1.20)\end{aligned}$$

If we put $\theta = \sum\alpha_k\theta_k$ and now extend our algebra $\mathbb{C}[\Gamma^{\mathbf{N}}]$ to the associated algebra $\mathbb{C}[[\Gamma^{\mathbf{N}}]]$ of formal series in the $\Gamma^{\mathbf{N}}$s this equation will be satisfied with

$$\rho_\theta = 1 + \alpha_\theta\sum\alpha_k\Gamma_k^{\mathbf{N}} + \alpha_\theta^2\sum\alpha_k\alpha_m\Gamma_k^{\mathbf{N}}\Gamma_m^{\mathbf{N}} + \ldots \qquad (8.1.21)$$

Since $\mathbb{C}[\Gamma^{\mathbf{N}}]$ has not been shown to be free, our grip upon $\mathbb{C}[[\Gamma^{\mathbf{N}}]]$ is somewhat tenuous. The corresponding algebra $\mathbb{C}[\Gamma]$ *is* free on the $\Gamma$s, and so the algebra $\mathbb{C}[[\Gamma]]$ of formal $\Gamma$-series may be identified as a vector space with the linear dual of $\mathbb{C}[\Gamma]$. Moreover, the original algebra $\mathbb{C}[\Gamma]$ may be embedded into $\mathbb{C}[[\Gamma]]$ in the obvious way (and is dense in it in a certain topology). These relations will be of significance later.

Proceeding formally from equation (8.1.21), and assuming $[\![\ ]\!]$ now to be linear in the $\Gamma^{\mathbf{N}}$-monomials, we obtain

$$[\![\rho_\theta]\!] = 1 + \alpha_\theta \sum \alpha_k [\![\Gamma_k^N]\!] + \alpha_\theta^2 \sum \alpha_k \alpha_m [\![\Gamma_k^N \Gamma_m^N]\!] + \dots \quad (8.1.22)$$

On the other hand, rewriting equation (8.1.15) with $\rho = \rho_\theta$ and $r_k$ chosen accordingly, we have

$$[\![\rho_\theta]\!] = 1 + \left(\frac{i\sqrt{2}}{\tau}\right) \sum r_k q_k + \frac{1}{2!}\left(\frac{i\sqrt{2}}{\tau}\right)^2 \sum r_k r_m q_k q_m + \dots \quad (8.1.23)$$

Comparing first order terms in the last two equations, and using equation (8.1.17), we find satisfaction if

$$\kappa \alpha_\theta \alpha_k = 2ir_k. \quad (8.1.24)$$

The constant $\kappa$, being arbitrary (equation (8.1.6)), may now be fixed by choosing

$$\kappa = i\sqrt{2}, \quad (8.1.25)$$

so that from equation (8.1.24)

$$r_k = \frac{1}{\sqrt{2}} \alpha_\theta \alpha_k. \quad (8.1.26)$$

Then equation (8.1.17) becomes

$$[\![\Gamma_k^N]\!] = \frac{i}{\tau} q_k \quad (8.1.27)$$

while equation (8.1.18) becomes

$$[\![\Delta_{\mathsf{N}}^{k}]\!] = \tau p_k . \tag{8.1.28}$$

A comparison of higher order terms in equations (8.1.22) and (8.1.23), and the use of equation (8.1.26), now gives satisfaction if:

$$\alpha_\theta^n \alpha_{k_1} \ldots \alpha_{k_n} [\![\Gamma_{k_1}^{\mathsf{N}} \ldots \Gamma_{k_n}^{\mathsf{N}}]\!] = \frac{1}{n!}\left(\frac{i\sqrt{2}}{\tau}\right)^n r_{k_1} \ldots r_{k_n} q_{k_1} \ldots q_{k_n}$$

$$= \frac{1}{n!}\left(\frac{i\sqrt{2}}{\tau}\right)^n \left(\frac{\alpha_\theta}{\sqrt{2}}\right)^n \alpha_{k_1} \ldots \alpha_{k_n} q_{k_1} \ldots q_{k_n}.$$

(8.1.29)

So, finally, we conclude that we may take

$$[\![\Gamma_{k_1}^{\mathsf{N}} \ldots \Gamma_{k_n}^{\mathsf{N}}]\!] = \frac{1}{n!}\left(\frac{i}{\tau}\right)^n q_{k_1} \ldots q_{k_n} \tag{8.1.30}$$

for $n \geq 0$, noting that for $n = 0$ we recover

$$[\![1]\!] = 1 . \tag{8.1.31}$$

Our assumptions concerning various constants, namely that $\kappa$ and $\kappa^{-1} z_k$ are to be chosen imaginary (equations (8.1.8) and (8.1.25)), carry the implication that the "coordinates" $z_k$ should be real, and therefore that the vector space underlying the apparent geometry of the continuum should be considered to be a real vector space. This assumption is part of the experiential burden we carry with us from the continuum, and the problem of focusing this assumed spontaneous breakdown to a real subspace will be taken up in the next chapter.

Likewise, an analysis of the operators $\Delta_{\mathsf{N}}^{k}$ will be postponed until the next chapter. It may suffice to say here that these operators carry *dynamical* connotations, in the sense that they destructively

change, or "use up," the experimenter's net qets, which, as hierarchical arrangements of initial acts, may be thought of as idealized "measuring" devices, or "instruments." (Cf. §10.1.) The $\Gamma^{\mathsf{N}}_k$ perform the dual or complementary acts of forming new "instruments." If the $\Delta^k_{\mathsf{N}}$ correspond to dynamical acts, then the $\Gamma^{\mathsf{N}}_k$ correspond to kinematical ones: their algebra provides a means of describing the possible reticular acts the experimenter may choose to perform, regardless of the pattern of ensuing transitions.

In the next section we attempt to specify a basic class of such kinematical entities, namely those associated with the idea of parallel transport. When interpreted in the continuum this class turns out to contain the classical notion of path, or *curve*.

## 8.2 Transport, Curves and a little Chenism

In this concluding section we attempt to formulate an intrinsic notion of parallel transport, for simplicity in the algebraic tensorial language of the Maxwell–Boltzmann phase. (This phase is slightly more tractable mathematically than the net phase, entailing little or no loss of information. Some of our conclusions will be pulled back to the net later with the help of the map $\Phi$ of §7.2.)

First we consider the infinitesimal notion. In an ordinary affine space the "infinitesimal" transport of a point consists in translating the point through an "infinitesimal" distance. Once an origin is chosen (so that we have a vector space) such a translation may be effected by the vector addition of an "infinitesimal" vector to the ("radius") vector emanating from the origin and ending at the point in question.

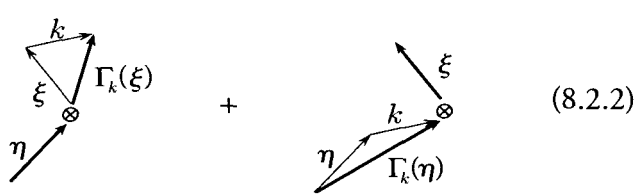
We now try to imitate this in the context of the algebra $T(S\otimes\tilde{S})$. We fix some $\eta$ in this algebra and regard it as a choice of "origin." Then any other element, $\xi$ say, may be "attached" to $\eta$ *via* $\otimes$ (which is standing in for $\vee$), producing the *sequence* $\xi\otimes\eta$, in analogy with the way in which a radius vector is attached to the origin in an ordinary vector space. Thus the family of elements of the form $\xi\otimes\eta$, with $\eta$ fixed and $\xi$ varying over a set of generators in $T(S\otimes\tilde{S})$,

functions rather like a basis in a vector space, albeit a "noncommutative" one in the sense that the acts of attachment comprising the substructure of would-be vectors do not commute. (Varying $\eta$ then corresponds to varying the origin of the virtual space: that is, varying the space itself. One might also regard a variation in $\eta$ as variation of the *frame* defined by the family $\xi \otimes \eta$.)

Pursuing the analogy in which $\xi$ corresponds with a radius vector (with "origin" $\eta$), and retaining the basis chosen for $S \otimes \tilde{S}$ in §8.1, we may advance its end-point by one chrononic step in the $k^{\text{th}}$ direction, say, by applying the operator $\Gamma_k$, and attaching the resulting virtual vector $\Gamma_k(\xi)$ to the virtual origin $\eta$, to obtain the *sequence* $\Gamma_k(\xi) \otimes \eta$. But there is another virtual vectorial route, or mode of advancement, for this end-point: namely, advance the origin (end-point of $\eta$) through one unit in the $k^{\text{th}}$ direction, and attach $\xi$ to the result. We then obtain the sequence $\xi \otimes \Gamma_k(\eta)$. As proper quantum experimenters we should now superpose these two alternatives to obtain

$$\Gamma_k(\xi) \otimes \eta + \xi \otimes \Gamma_k(\eta). \tag{8.2.1}$$

This (superposition of sequences of acts) is a representation of the act resulting from the "infinitesimal" transport of $\xi$ as a virtual vector in the $k^{\text{th}}$ direction. Note that the first term encodes the advancement of the "head" of $\xi$, while its "foot" remains rooted at the head of $\eta$, while the second term encodes the parallel advancement of this foot, as depicted in the following diagrammatic rendition of the expression (8.2.1), in which $k$ labels the "infinitesimal" vector in the $k^{\text{th}}$ direction:




(8.2.2)

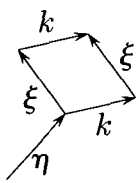
Suppose now that we wished to translate $\xi$ through two chrononic steps: $\Gamma_{k_1}$ and $\Gamma_{k_2}$, say. In this case more virtual vectorial routes are available and should be included in the superposition analogous to the one in expression (8.2.1):

$$(\Gamma_{k_1}\Gamma_{k_2}+\Gamma_{k_2}\Gamma_{k_1})(\xi)\otimes\eta+\Gamma_{k_1}(\xi)\otimes\Gamma_{k_2}(\eta)+$$

$$+\Gamma_{k_2}(\xi)\otimes\Gamma_{k_1}(\eta)+\xi\otimes(\Gamma_{k_1}\Gamma_{k_2}+\Gamma_{k_2}\Gamma_{k_1})(\eta). \quad (8.2.3)$$

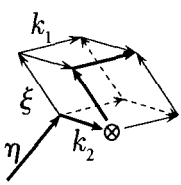
The inner terms encode the possible virtual vectorial routes obtained by advancing the head and foot of $\xi$ through substeps of $\Gamma_{k_1}\Gamma_{k_2}$.

Thus, if we conflate the two diagrams in (8.2.2) so as to depict expression (8.2.1) in the vector form




(8.2.4)

then the corresponding diagram for expression (8.2.3) is




(8.2.5)

with the term $\Gamma_{k_1}(\xi)\otimes\Gamma_{k_2}(\eta)$, for example, corresponding to the pattern of arrows drawn with heavier lines, and with the $\otimes$ inserted appropriately. (The left-most term in expression (8.2.3) corresponds to the advancement of the head of $\xi$ around the top face of the paral-

lelepiped and the attachment of the result to $\eta$, while the right-most corresponds to the attachment of $\xi$ to the result of advancing its foot—which is the head of $\eta$—around the bottom face.)

For a general transportation through $n$ steps, $\Gamma_{k_1},\ldots,\Gamma_{k_n}$ say, we obtain the superposition corresponding to all partitions of all permutations of $\{k_1,\ldots,k_n\}$ into two subsets (corresponding to the placement of the $\otimes$). With $\mathrm{symm}(\Gamma_{k_1}\ldots\Gamma_{k_n})$ denoting the sum of products of all permutations of $\Gamma_{k_1},\ldots,\Gamma_{k_n}$ we may write this general transportation in the symbolic form:

$$\Psi(\Gamma_{k_1},\ldots,\Gamma_{k_n})(\xi\otimes\eta)\equiv \mathrm{symm}(\Gamma_{k_1}\ldots\Gamma_{k_n})(\xi)\otimes\eta+$$

$$+\sum\mathrm{symm}(\overbrace{\Gamma_{\cdot}\ldots\Gamma_{\cdot}}^{n-1})(\xi)\otimes\Gamma_{\cdot}(\eta)+\sum\mathrm{symm}(\overbrace{\Gamma_{\cdot}\ldots\Gamma_{\cdot}}^{n-2})(\xi)\otimes\mathrm{symm}(\Gamma_{\cdot}\Gamma_{\cdot})(\eta)+$$

$$\ldots+\xi\otimes\mathrm{symm}(\Gamma_{k_1}\ldots\Gamma_{k_n})(\eta), \qquad (8.2.6)$$

where the summations are over those permutations of the subscripts that maintain the original order within each partition induced by the presence of the tensor sign.

For example, with $n=3$ we obtain:

$$\Psi(\Gamma_{k_1},\Gamma_{k_2},\Gamma_{k_3})=$$

$$\mathrm{symm}(\Gamma_{k_1}\Gamma_{k_2}\Gamma_{k_3})\otimes 1+\mathrm{symm}(\Gamma_{k_1}\Gamma_{k_2})\otimes\Gamma_{k_3}+\mathrm{symm}(\Gamma_{k_1}\Gamma_{k_3})\otimes\Gamma_{k_2}+$$

$$+\mathrm{symm}(\Gamma_{k_2}\Gamma_{k_3})\otimes\Gamma_{k_1}+\Gamma_{k_1}\otimes\mathrm{symm}(\Gamma_{k_2}\Gamma_{k_3})+\Gamma_{k_2}\otimes\mathrm{symm}(\Gamma_{k_1}\Gamma_{k_3})+$$

$$+\Gamma_{k_3}\otimes\mathrm{symm}(\Gamma_{k_1}\Gamma_{k_2})+1\otimes\mathrm{symm}(\Gamma_{k_1}\Gamma_{k_2}\Gamma_{k_3}). \qquad (8.2.7)$$

To interpret this in the continuum we apply the version of equation (8.1.30) lifted up to the (Maxwell–Boltzmann phase) tensor

algebra by replacing the $\Gamma^N$s by $\Gamma$s. With $\sigma$ denoting a permutation of $\{k_1,\ldots,k_n\}$ we note first that:

$$\begin{aligned}
[\![\mathrm{symm}(\Gamma_{k_1}\ldots\Gamma_{k_n})]\!] &= \left[\!\!\left[\sum_\sigma \Gamma_{\sigma(k_1)}\ldots\Gamma_{\sigma(k_n)}\right]\!\!\right] \\
&= \sum_\sigma [\![\Gamma_{\sigma(k_1)}\ldots\Gamma_{\sigma(k_n)}]\!] \\
&= \frac{1}{n!}\left(\frac{i}{\tau}\right)^n \sum_\sigma q_{\sigma(k_1)}\ldots q_{\sigma(k_n)} \\
&= \left(\frac{i}{\tau}\right)^n q_{k_1}\ldots q_{k_n} \\
&= [\![\Gamma_{k_1}]\!]\ldots[\![\Gamma_{k_n}]\!], \qquad (8.2.8)
\end{aligned}$$

the last line following upon equation (8.1.27).

It will prove convenient to write

$$q_k \equiv [\![\Gamma_k]\!]. \qquad (8.2.9)$$

Then, from equations (8.2.6), (8.2.8) and (8.2.9) we obtain

$$\begin{aligned}
[\![\Psi(\Gamma_{k_1},\ldots,\Gamma_{k_n})]\!] &= q_{k_1}\ldots q_{k_n}\otimes 1 + \\
&+\sum \overbrace{(q_\cdot\ldots q_\cdot)}^{n-1}\otimes q_\cdot + \sum \overbrace{(q_\cdot\ldots q_\cdot)}^{n-2}\otimes q_\cdot q_\cdot + \ldots \\
&\ldots + 1\otimes q_{k_1}\ldots q_{k_n} \\
&= (q_{k_1}\otimes 1 + 1\otimes q_{k_1})(q_{k_2}\otimes 1 + 1\otimes q_{k_2})\ldots(q_{k_n}\otimes 1 + 1\otimes q_{k_n}). \qquad (8.2.10)
\end{aligned}$$

But the right hand side of the last equation defines exactly the coproduct specifying the additive affine algebraic (vector) group structure of the algebra generated by the $\mathfrak{q}$s: see §3.1.4. That is to say, the map defined by

$$\psi(\mathfrak{q}_{k_1}\ldots\mathfrak{q}_{k_n}) = \psi([\![\,\mathrm{symm}(\Gamma_{k_1}\ldots\Gamma_{k_n})\,]\!])$$

$$\equiv [\![\,\Psi(\Gamma_{k_1},\ldots,\Gamma_{k_n})\,]\!] \qquad (8.2.11)$$

induces the additive vector structure. In this way we recover the *operation* of vector addition in the continuum from an underlying quantal structure, though not yet the acts corresponding to the (injectors of) the vectors themselves.

The interpretation of the coproduct $\psi$ implied by equation (8.2.11), namely, as the continuum version of an act of discrete infinitesimal transport, may itself be carried into the continuum. The act of transport shown in equation (8.2.11), insofar as it is such an act, is *along* the segmented infinitesimal "path" whose steps are encoded by the product $\mathfrak{q}_{k_1}\ldots\mathfrak{q}_{k_n}$. This product is the continuum version of the path-like entity specified by superposing all possible discrete "paths" of the form $\Gamma_{k_1}\ldots\Gamma_{k_n}$: this is the import of equation (8.2.8).

Now we turn to the problem of specifying vectors and other classical *objects* in the continuum. (That dreaded word may be used in the continuum with perfect consistency: fictions may be embedded within the larger fiction that is the continuum, and a correspondence principle should be able to account for them.) Note first that we may regard a (geometrical, continuum) vector as an *act* of transport. Specifically, we carry an object (a point, frame or another vector) along a line segment. Moreover, we may regard such transport to be done *parallelly*: lengths are preserved during the course of the transport.

In terms of selective acts, or operations, this macroscopic parallel transport—of a vector, say—along a *track* in the continuum is

tantamount to performing the pair of acts comprising the act of advancing the head of the vector along the track and the act of advancing the foot of the vector along an identical track. That is to say, the act of transporting a vector parallelly along a macroscopic track in the continuum should be equivalent to the act of performing, *in sequence*, the act of advancing the head of the vector along the track and the act of advancing the foot of the vector along an identical parallel track.

If $P$ denotes an element in the algebra of operators (upon the space accommodating our chosen representation of the CCR) specifying the structure of such a track—namely one admitting parallel transport along itself—then $\psi(P)(f\otimes g)$ represents the result of transporting the "virtual" vector $f$ "along" $P$, and $P(f)$ (respectively $P(g)$) represents the result of advancing the head (respectively foot) of the virtual vector $f$ "along" $P$. Then the condition for $P$ to admit parallel transport along itself may be expressed as

$$\psi(P)(f\otimes g) = P(f)\otimes P(g) \tag{8.2.12}$$

or

$$\psi(P) = P\otimes P. \tag{8.2.13}$$

(Note that this may be interpreted as asserting that $P$ is characterized by the property that, for it, quantum duplication—the left-hand side of the last equation—coincides with "macroscopic" duplication, namely the right-hand side.)

Thus, the elementary "infinitesimal" act of transport represented by $\psi(\mathfrak{q}_k)$ is not macroscopically parallel in this sense: it reflects the underlying (reticular, quantum) superposition of the two discrete acts, $\mathfrak{q}_k\otimes 1$ and $1\otimes\mathfrak{q}_k$, of head and tail advancement, rather than their (continuum, macroscopic) *sequencing.*

Solutions to equation (8.2.13) will then in a sense represent the selective acts whose performance will construct injectors for those macroscopic "tracks" through the continuum which admit the paral-

lel transport of vectors along themselves in the above sense.

Now we note that if $S$ satisfies

$$\psi(S) = S \otimes 1 + 1 \otimes S \tag{8.2.14}$$

then $e^S$ satisfies equation (8.2.13). For

$$\begin{aligned} \psi(e^S) &= e^{\psi(S)} \\ &= e^{S \otimes 1 + 1 \otimes S} && \text{from equation (8.2.14)} \\ &= e^{S \otimes 1} e^{1 \otimes S} && \text{since the exponents commute} \\ &= (e^S \otimes 1)(1 \otimes e^S) \\ &= e^S \otimes e^S. \end{aligned} \tag{8.2.15}$$

In particular, for any linear combination $\Sigma \alpha_m \mathfrak{q}_m$, $\exp(\Sigma \alpha_m \mathfrak{q}_m)$ is a solution to equation (8.2.13). With the $\alpha_m$ *real*, we recover our earlier vector representation (equation (8.1.15)) and at the same time guarantee that, in case the Schrödinger representation is chosen, $\exp(\Sigma \alpha_m \mathfrak{q}_m)$, which is $\exp((i/\tau)\Sigma \alpha_m q_m)$, is a sufficiently well-defined operator on $L^2(\mathbb{R}^4)$. If $\kappa$ had had a real part, the operator appearing instead of the last named one would carry a non-oscillatory factor which would send certain $L^2$ functions to non-$L^2$ functions.

There are more general solutions to equation (8.2.13). Indeed, it follows from work of Chen (Chen 1958) that unique descriptors for curves in manifolds may be found among such solutions. (See also Hain and Tondeur 1990 and Tavares 1994.) We briefly describe the mechanism involved, tailoring the discussion to

the case at hand.

A *curve* $C$ is defined by specifying a continuous function $f:[a,b]\longrightarrow\mathbb{R}^4$. Writing $f(t)=(f_1(t),\ldots,f_4(t))$, we assume each $f_k$ to be of bounded variation as a function on the interval. For each integer $m\geq 0$ and $t\in[a,b]$ we define an *iterated integral* (over the curve $C$) recursively by

$$\int_a^t 1 = 1\ , \tag{8.2.16}$$

$$\int_a^t dx_{k_1}\ldots dx_{k_m} = \int_a^t \left(\int_a^s dx_{k_1}\ldots dx_{k_{m-1}}\right) df_{k_m}(s)\ . \tag{8.2.17}$$

The associated *Chen series* is then the formal series

$$\Theta(C) = 1 + \sum_{p=1}^{\infty}\sum\left(\int_C dx_{k_1}\ldots dx_{k_p}\right)q_{k_1}\ldots q_{k_p}\ , \tag{8.2.18}$$

where $\int_C$ stands for $\int_a^b$.

As first noted by Ree (Ree 1958) these iterated integrals have the property that they preserve—that is, are $\mathbb{R}$-valued homomorphisms for—the *shuffle product* of their arguments. This commutative product, of a pair of multi-indexed symbols say, is obtained by summing over all shuffles of the pair of sets of indices, a shuffle being a permutation of the union of the two sets of indices which preserves the initial order of each set. Thus for instance,

$$\int_C (dx_1 dx_2 + dx_2 dx_1) = \int_C dx_1 \int_C dx_2\ , \tag{8.2.19}$$

$$\int_C (dx_1 dx_2 dx_3 + dx_2 dx_1 dx_3 + dx_2 dx_3 dx_1) = \int_C dx_1 \int_C dx_2 dx_3,$$

(8.2.20)

etc.

As the reader may check, this immediately implies that Chen series are formal solutions to equation (8.2.13). (By way of illustration, let us compare third-order terms in $\psi(\Theta)$ and $\Theta \otimes \Theta$, where we write

$$\Theta = 1 + \sum_{p=1}^{\infty} \sum c_{k_1 \cdots k_p} q_{k_1} \ldots q_{k_p} \tag{8.2.21}$$

$$= 1 + \sum c_{k_1} q_{k_1} + \sum c_{k_1 k_2} q_{k_1} q_{k_2} + \ldots \tag{8.2.22}$$

and the coefficients preserve the shuffle product, as above. The third-order term in $\psi(\Theta)$ is:

$$\sum c_{k_1 k_2 k_3} (q_{k_1} q_{k_2} q_{k_3} \otimes 1 + q_{k_1} q_{k_2} \otimes q_{k_3} + q_{k_1} q_{k_3} \otimes q_{k_2} + q_{k_2} q_{k_3} \otimes q_{k_1} +$$

$$+ q_{k_1} \otimes q_{k_2} q_{k_3} + q_{k_2} \otimes q_{k_1} q_{k_3} + q_{k_3} \otimes q_{k_1} q_{k_2} + 1 \otimes q_{k_1} q_{k_2} q_{k_3})$$

$$= \sum (c_{k_1 k_2 k_3} q_{k_1} q_{k_2} q_{k_3} \otimes 1 + c_{k_1 k_2 k_3} q_{k_1} q_{k_2} \otimes q_{k_3} + c_{k_1 k_3 k_2} q_{k_1} q_{k_2} \otimes q_{k_3} +$$

$$+ c_{k_3 k_1 k_2} q_{k_1} q_{k_2} \otimes q_{k_3} + \ldots + c_{k_1 k_2 k_3} 1 \otimes q_{k_1} q_{k_2} q_{k_3})$$

$$= \sum (c_{k_1 k_2 k_3} q_{k_1} q_{k_2} q_{k_3} \otimes 1 + (c_{k_1 k_2 k_3} + c_{k_1 k_3 k_2} + c_{k_3 k_1 k_2}) q_{k_1} q_{k_2} \otimes q_{k_3} + \ldots)$$

$$= \sum (c_{k_1 k_2 k_3} q_{k_1} q_{k_2} q_{k_3} \otimes 1 + c_{k_1 k_2} c_{k_3} q_{k_1} q_{k_2} \otimes q_{k_3} + \ldots) \tag{8.2.23}$$

where we have used the shuffle product preservation property of the *c*s on the internal terms in the last step. The right hand side is now easily seen to be exactly the third-order term in $\Theta\otimes\Theta$.)

Note that if $C$ is the line segment from the origin to $(\alpha_1,\ldots,\alpha_4)$ given by $f_k(t)=\alpha_k t,\ t\in[0,1]$, then

$$\int_C dx_{k_1}\ldots dx_{k_p}=\frac{1}{p!}\,\alpha_{k_1}\ldots\alpha_{k_p} \tag{8.2.24}$$

so that in this case

$$\begin{aligned}\Theta(C)&=1+\sum\alpha_{k_1}q_{k_1}+\frac{1}{2!}\sum\alpha_{k_1}\alpha_{k_2}q_{k_1}q_{k_2}+\ldots\\&=\exp(\sum\alpha_k q_k)\\&=\exp(\frac{i}{\tau}\sum\alpha_k q_k)\end{aligned} \tag{8.2.25}$$

and we recover our elementary coherent operator solution.

Our version of passive or intrinsic parallel transportability—that is, the property of admitting parallel transport—in its continuum interpretation thus encompasses at least the notion of a curve, *à la* Chen. (In the full version of Chen's theory noncommutative series of this type are found to completely specify sufficiently nice curves, and therefore contain all the geometrical information inherent in such a curve: cf. Chen 1958. In a sense, the iterated integrals, though constrained by the shuffle product preservation property, function as *coordinates* for curves, and as such have reappeared in the modern theory of loop representations of gauge theories: see Gambini and Pullin 1996, Chapter 2, Tavares 1994 and our §12.5.)

The significance of the Chen series solutions to equation (8.2.13) is presumably that they describe extended macroscopic classical geometrical *objects* in the continuum as infinite superpositions

of finitely-occupied ("$n$-body") quantum condensates: this is one of the ways in which quantum behavior gives rise to classical, and bears out the idea that spacetime itself should be realizable as such a condensate (cf. §2.3). When the curve is a line segment we recover a familiar form of coherence. In general, this coherence is lost: presumably some attribute related to the curvature of a curve is a measure of the departure from coherence of its Chen series.

Using equation (8.1.30) we may now pull a Chen series back to formal series in the algebras $\mathbb{C}[[\Gamma]]$ and $\mathbb{C}[[\Gamma^{\mathsf{N}}]]$, obtaining in the latter case a formal series of the form

$$1 + \sum_{p=1}^{\infty} \sum p! \left( \int_C dx_{k_1} \ldots dx_{k_p} \right) \Gamma^{\mathsf{N}}_{k_1} \ldots \Gamma^{\mathsf{N}}_{k_p} . \qquad (8.2.26)$$

(In case the curve is the line segment considered earlier, this reduces to

$$1 + \sum \alpha_{k_1} \Gamma^{\mathsf{N}}_{k_1} + \sum \alpha_{k_1} \alpha_{k_2} \Gamma^{\mathsf{N}}_{k_1} \Gamma^{\mathsf{N}}_{k_2} + \ldots ) \qquad (8.2.27)$$

A formal series of this type embodies instructions for building the sequences of selective acts whose macroscopic continuum correspondents tend, upon superposition, toward the operator effecting "intrinsic" parallel transport along the curve.

The terms of such a series may be parsed on the net and interpreted in terms of our original "infinitesimal transport" paradigm in its $\Gamma^{\mathsf{N}}$ or reticular form. Thus, suppose we restrict ourselves first to (all) infinitesimal transports *along* the curve which consist of only one step. That is, partition the curve into a large number of very small segments corresponding to a partition of the parameter interval $[a,b]$ into subintervals of typical length $\Delta t_j$. Then a typical one-step (infinitesimal) transport *along* the curve is approximately effected by an operator of the form $\Delta x_k(j)\Gamma^{\mathsf{N}}_k$, where $\Delta x_k(j)$ is the appropriate tangential component at $t = t_j$:

$$\Delta x_k(j) = f_k'(t_j)\Delta t_j. \tag{8.2.28}$$

A superposition over $k$ and all curve segments yields

$$\sum_{k,j} \Delta x_k(j)\Gamma_k^{\mathsf{N}}, \tag{8.2.29}$$

which approaches

$$\sum_k (\int_C dx_k)\Gamma_k^{\mathsf{N}}, \tag{8.2.30}$$

the first degree term in the series (8.2.26), as we refine the partition of the curve: we regard such an integral as just another superposition, as in the ordinary theory of (Feynman) path integrals, for example.

Note that the factors $p!$ multiplying the iterated integral (which integral amounts to an ordinary integral over a simplex of the form $\{(t_{k_1},\ldots t_{k_p}): a \le t_{k_1} \le \ldots \le t_{k_p} \le b\}$) make it possible to regard the coefficients in (8.2.26) as products of the individual contour integrals. Thus, the degree $p$ term in the series may similarly be interpreted as a superposition over all available infinitesimal transports taken locally along the curve that consist of $p$ elementary steps. Maintaining the interpretation of integration-as-superposition we may obtain the series (8.2.26) by superposing *all* possible outputs from the following simple program:

**begin**

*Effect a finite-step local infinitesimal reticular transport along the curve C*

**end**

Note that the null "finite-step infinitesimal transport," comprising *no* $\Gamma^{\mathsf{N}}$ transports along the curve, leaves any argument un-

touched and so yields 1, the first term of the series.

The one-line "program" exhibited above appears to embody a remnant of the original classical object, while the reticular Chen series (8.2.26) itself describes the selective act resulting from quantum interference among the program's outputs.

We note once again that the line segment special case, namely equation (8.2.25), reproduces the coherent operator that was posited earlier and independently as the operator with which to implement *vector* transport in the continuum: that is, as transport along a directed line segment. So it would appear legitimate to adopt this type of solution as fundamental, and note that products of solutions to equation (8.2.13) are also solutions. In this way one might justify the ordered building up of exponentials of path integrals and invest them with dynamical significance (cf. Chapter 10).

The pattern that has emerged above is in fact consistent with our earlier naïve attempt, in §6.4, to incorporate classical time *via* the axiom (6.3.3.1) and to implement quantum multiplexing *via* the coproduct of the one-dimensional additive affine group. To see this, consider the coproduct $\psi$ on $\mathbb{C}[q]$, the complex polynomial algebra in the $q$s, namely that given by

$$\psi(q_k) = 1 \otimes q_k + q_k \otimes 1 \tag{8.2.31}$$

and extended to be an algebra map (equation (8.2.11)).

PROPOSITION 8.2.1

*The dual of $\psi$ is the shuffle product on $\mathbb{C}[q]^* \cong \mathbb{C}[[q]]$.*

PROOF

Monomials $q_{k_1} \ldots q_{k_n}$ in $\mathbb{C}[[q]]$ act on monomials $q_{p_1} \ldots q_{p_m}$ in $\mathbb{C}[q]$ in the obvious way, expressed by the dual pairing:

$$(q_{k_1}\ldots q_{k_n}, q_{p_1}\ldots q_{p_m}) = \begin{cases} 1 & \text{if } q_{k_1}\ldots q_{k_n} = q_{p_1}\ldots q_{p_m} \\ 0 & \text{otherwise} \end{cases} \qquad (8.2.32)$$

So, with the asterisk denoting the dual product,

$$(q_{k_1}\ldots q_{k_n} * q_{j_1}\ldots q_{j_l}, q_{p_1}\ldots q_{p_m})$$

$$= (q_{k_1}\ldots q_{k_n} \otimes q_{j_1}\ldots q_{j_l}, \psi(q_{p_1})\psi(q_{p_2})\ldots\psi(q_{p_m}))$$

$$= (q_{k_1}\ldots q_{k_n} \otimes q_{j_1}\ldots q_{j_l}, (1\otimes q_{p_1} + q_{p_1}\otimes 1)\ldots(1\otimes q_{p_m} + q_{p_m}\otimes 1))$$

$$= (q_{k_1}\ldots q_{k_n} \otimes q_{j_1}\ldots q_{j_l}, 1\otimes q_{p_1}\ldots q_{p_m} + q_{p_1}\otimes q_{p_2}\ldots q_{p_m} + \cdots$$

$$\cdots + q_{p_1}\ldots q_{p_m}\otimes 1)$$

$$= (q_{k_1}\ldots q_{k_n}, 1)(q_{j_1}\ldots q_{j_l}, q_{p_1}\ldots q_{p_m}) +$$

$$+ (q_{k_1}\ldots q_{k_n}, q_{p_1})(q_{j_1}\ldots q_{j_l}, q_{p_2}\ldots q_{p_m}) + \cdots \qquad (8.2.33)$$

Clearly, only one term in this sum can survive (to give the value 1) and this occurs when and if the $k$ monomial equals the $p$ monomial to the left of the tensor sign, while the $j$ monomial equals the $p$ monomial to the right of it. This can only happen when the number of $p$ indices equals the sum of the number of $k$ indices and the number of $j$ indices. It will certainly happen if

$$q_{p_1}\ldots q_{p_m} = q_{k_1}\ldots q_{k_n} q_{j_1}\ldots q_{j_l} \qquad (8.2.34)$$

for then the term

$$q_{k_1}\ldots q_{k_n}\otimes q_{j_1}\ldots q_{j_l}, \tag{8.2.35}$$

will occur exactly once in the expansion of $\psi(q_{p_1}\ldots q_{p_m})$ giving

$$(q_{k_1}\ldots q_{k_n} * q_{j_1}\ldots q_{j_l},\ q_{k_1}\ldots q_{k_n}q_{j_1}\ldots q_{j_l}) = 1. \tag{8.2.36}$$

Now, the expansion of $\psi(q_{p_1}\ldots q_{p_m})$ consists of the sum of all tensors of the following form: to the left of the tensor sign appears products of a certain number (including 0, corresponding to the appearance of 1) of the $q$s taken strictly in the order of their appearance in the original monomial, while to the right of the tensor sign in each of these tensors the remaining $q$s appear, also in the order of their occurrence in the original monomial. For example

$$\psi(q_1q_2q_3) = 1\otimes q_1q_2q_3 + q_1\otimes q_2q_3 + q_2\otimes q_1q_3 + q_3\otimes q_1q_2 +$$

$$+ q_1q_2\otimes q_3 + q_2q_3\otimes q_1 + q_1q_3\otimes q_2 + q_1q_2q_3\otimes 1. \tag{8.2.37}$$

The term $q_{k_1}\ldots q_{k_n}\otimes q_{j_1}\ldots q_{j_l}$ will thus appear (once) in the $\psi$ expansion of a monomial exactly when that monomial is a permutation of $q_{k_1}\ldots q_{k_n}q_{j_1}\ldots q_{j_l}$ within which the original orderings, respectively of the $k$-tuple and the $j$-tuple, are maintained. For, the expansion of any such monomial produces (uniquely) all such pairs of orderings, and so must contain the expression (8.2.35).

These permutations are precisely the shuffles. Thus, the monomial $q_{k_1}\ldots q_{k_n}q_{j_1}\ldots q_{j_l}$ may be replaced in equation (8.2.36) by monomials in which $(k_1,\ldots,k_n)$ is shuffled with $(j_1,\ldots,j_l)$. This shows that $q_{k_1}\ldots q_{k_n} * q_{j_1}\ldots q_{j_l}$, considered as an element in the algebra of formal series, is exactly the sum of all the shuffled versions of $q_{k_1}\ldots q_{k_n}q_{j_1}\ldots q_{j_l}$. ∎

Since the shuffle product of two monomials is a finite sum of other monomials, it is clear that $\mathbb{C}[q]$, realized as the subspace of

finite formal series, is a subalgebra of the shuffle algebra, which, we recall, is commutative. Then, the "path-like" elements $P$ of $\mathbb{C}[[q]](\cong \mathbb{C}[q]^*)$ satisfying equation (8.2.13) are exactly the complex valued algebra homomorphisms of $\mathbb{C}[q]$ when it is equipped with the shuffle product. To see this, note that for any monomial $q_{k_1}\ldots q_{k_n}$ in $\mathbb{C}[[q]]$ and $f,g\in\mathbb{C}[q]$, where $\mathbb{C}[q]$ is regarded as a subalgebra of $\mathbb{C}[[q]]$ with the shuffle product, we have

$$\begin{aligned}(q_{k_1}\ldots q_{k_n}, f*g) &= (f*g, q_{k_1}\ldots q_{k_n})\\ &= (f\otimes g, \psi(q_{k_1}\ldots q_{k_n}))\\ &= \psi(q_{k_1}\ldots q_{k_n})(f\otimes g)\end{aligned} \tag{8.2.38}$$

from which it follows that for any $F\in\mathbb{C}[[q]]$

$$(F, f*g) = \psi(F)(f\otimes g). \tag{8.2.39}$$

Then, for $P$ satisfying equation (8.2.13),

$$\begin{aligned}(P, f*g) &= \psi(P)(f\otimes g)\\ &= (P\otimes P)(f\otimes g)\\ &= (P,f)(P,g)\end{aligned} \tag{8.2.40}$$

showing that $P$ is a homomorphism for the shuffle product.

Conversely, if $P$ is such a homomorphism, then, from the last set of equations, we would have

$$\psi(P)(f\otimes g) = (P\otimes P)(f\otimes g) \tag{8.2.41}$$

for all $f, g \in \mathbb{C}[q]$, and equation (8.2.13) would follow.

Thus, we arrive at a higher dimensional generalization of the structure that emerged in our attempt to simulate the multiplexing of the classical time step in §6.4. The product given in equation (6.4.5) is exactly the shuffle product for the case of a single variable, the $h_t$ in equation (6.4.12), for real $t$, being the Chen series corresponding to curves in $\mathbb{R}$, which are just intervals.

Interestingly, the structure demanded by our original axiom (6.3.3.3) to sit on the left of the turnstile—namely, the dual of what we interpreted as a tensor algebra—seems to have emerged here independently through a consideration of geometrical paradigms. This perception is presumably a Whiggish illusion, since it is the logical structure itself that must underlie these geometrical paradigms.

In closing this chapter, we note the fundamental rôle played by the quantum duplication operator: it is this form of quantum entanglement that presumably gives rise to what we perceive macroscopically as vector addition, or, more generally, as path composition. (A version of the operator appears again, independently, in §12.5.)

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# 9

# A Correspondence Principle for the Quantum Net

We continue and complete the search for a correspondence principle of the kind envisaged in the last chapter. The first section is devoted to an index-free account of familiar material on spinor duality, which is now amenable to a perhaps less familiar quantum interpretation.

In the following section (§9.2) we exploit the algebraic expressivity of the various structures inherent in our basic quantum types to introduce a notion of *infinitesimal variation* for action vectors and operators belonging to these types. When applied to the basic reticular variables, this brings in its wake the appearance of a new type—which we interpret in terms of *defects*—and certain associated selective acts. It is perhaps surprising that these latter operators turn out to be exactly the Dirac matrices (in a complex, basis independent form), which thereby acquire an apparently new derivation and a new interpretation. The expressions obtained for these maps reveal their internal structure in a rather transparent manner, and various properties are easily obtained using this new representation.

In §9.3 we return to the $\Delta$ operators of §8.1 and complete the search for them. We also discuss some related matters concerning the defect type, Dirac maps, and their continuum interpretations.

In §9.4 we give vent once again to our continuum prejudices

in directing the choice of spinorial frames so as to make matters of interpretation easier for macroscopic experimenters. (In fact, these choices, and the "correspondence principle" itself, are all merely matters of convenience. Other frame choices would merely render a macroscopic interpretation—at least by means of *this* correspondence principle—more difficult.)

At last, in §9.5, we put forth our final choice of correspondence principle.

During the course of this chapter we discover how various "collective" quantum effects—traceable ultimately to the ubiquitous phenomenon of coherence—might conspire to tempt a macroscopic experimenter into a belief in the classical fiction of an *objective* reality.

## 9.1 Spinor Duality

The association (§7.1) of two-dimensional spaces of initial action vectors with quantum "binary alternatives" has further interesting epistemological consequences. We return to the isomorphism $\varphi: W \vee W \longrightarrow \mathbb{C}$ chosen in §7.1 and its concomitant $\varepsilon: W \longrightarrow W^*$. The isomorphism $\varphi$ determines a similar one for $W^*$, namely

$$\varphi^{\#}: W^* \vee W^* \longrightarrow \mathbb{C}, \tag{9.1.1}$$

as follows. Note that the map defined by $(f,h) \mapsto f \otimes h - h \otimes f$ of $W^* \times W^*$ into $(W \otimes W)^*$ is bilinear and alternating. Consequently, it extends uniquely to a map of $W^* \vee W^*$ into $(W \otimes W)^*$ by the universal property of exterior products. This lifted map sends $f \vee h$ in $W^* \vee W^*$ to $(f \otimes h - h \otimes f)$ in $(W \otimes W)^*$. This last named map is itself alternating on $W \times W$ and so it now lifts to a map of $W \vee W$ into $\mathbb{C}$, under which $x \vee y$ is sent to $f(x)h(y) - h(x)f(y)$. Thus we obtain finally a map

$$a: W^* \vee W^* \longrightarrow (W \vee W)^* \tag{9.1.2}$$

given by

$$a(f \vee h)(x \vee y) = f(x)h(y) - h(x)f(y). \tag{9.1.3}$$

The map $\varphi^{\#}$ is now defined by the following composition:

$$\varphi^{\#} : W^* \vee W^* \xrightarrow{a} (W \vee W)^* \xrightarrow{(\varphi^{-1})^*} \mathbb{C}^* \cong \mathbb{C}. \tag{9.1.4}$$

Using this $\varphi^{\#}$ as in §7.1, there is a corresponding isomorphism

$$\varepsilon^{\#}: W^* \longrightarrow W^{**}. \tag{9.1.5}$$

SPINOR DUALITY THEOREM

*With notation as above:*

(i) *Let* $\{e_1, e_2\} \subset W$ *be a basis and suppose that* $\varphi(e_1 \vee e_2) = \lambda$. *Then*

$$\varepsilon(e_1) = \lambda e_2^* \tag{9.1.6}$$

*and*

$$\varepsilon(e_2) = -\lambda e_1^*, \tag{9.1.7}$$

*where* $e_i^*$ *is the element dual to* $e_i$, $i = 1,2$.

(ii) *Let* $(\ )^{\wedge}: W \longrightarrow W^{**}$ *denote the natural isomorphism given by* $x^{\wedge}(f) = f(x)$. *Then*

$$\varepsilon^{\#} \circ \varepsilon = -(\ )^{\wedge}. \tag{9.1.8}$$

(iii) *For all* $g \in \mathrm{SL}(2,\mathbb{C})$, *and with* $L_g$ *denoting as before the natural action of* $g$ *upon* $W$:

$$\varepsilon \circ L_g = L^*_{g^{-1}} \circ \varepsilon . \tag{9.1.9}$$

PROOF

Parts (i) and (iii) have already been proven in §7.1, so it remains to prove (ii).

To this end, we first show that

$$\varphi^{\#}(e_1^* \vee e_2^*) = \varphi(e_1 \vee e_2)^{-1} . \tag{9.1.10}$$

With $\varphi(e_1 \vee e_2) = \lambda$, and for $\alpha \in \mathbb{C}$, (9.1.4) gives:

$$\begin{aligned}
\varphi^{\#}(e_1^* \vee e_2^*)(\alpha) &= (((\varphi^{-1})^* \circ a)(e_1^* \vee e_2^*))(\alpha) \\
&= ((\varphi^{-1})^*(a(e_1^* \vee e_2^*)))(\alpha) \\
&= a(e_1^* \vee e_2^*)(\varphi^{-1}(\alpha)) \\
&= a(e_1^* \vee e_2^*)(\alpha\lambda^{-1} e_1 \vee e_2) \\
&= \alpha\lambda^{-1}(e_1^*(e_1)e_2^*(e_2) - e_2^*(e_1)e_1^*(e_2)) \\
&= \alpha\lambda^{-1}, \qquad (9.1.11)
\end{aligned}$$

which proves our assertion. Then, from (i),

$$(\varepsilon^{\#} \circ \varepsilon)(e_1) = \varepsilon^{\#}(\lambda e_2^*)$$

$$= \lambda \varepsilon^{\#}(e_2^*)$$

$$= \lambda(-\lambda^{-1} e_1^{**})$$

from equation (9.1.7) applied to $W^*$,

$$= -\hat{e}_1 \tag{9.1.12}$$

and similarly

$$(\varepsilon^{\#} \circ \varepsilon)(e_2) = \varepsilon^{\#}(-\lambda e_1^*)$$

$$= -\lambda \varepsilon^{\#}(e_1^*)$$

$$= -\lambda(\lambda^{-1} e_2^{**})$$

$$= -\hat{e}_2, \tag{9.1.13}$$

as required. ∎

The third assertion expresses the invariance of $\varepsilon$ with respect to SL(2,$\mathbb{C}$) transformations, and admits an "epistemological" interpretation. First we recall that the natural allowable transformation upon $W^*$ associated with the transformation $L_g$ upon $W$ is exactly $L^*_{g^{-1}}$. In light of our interpretation of $W^*$ as a space of final action vectors, we may now view $\varepsilon$ as a one-to-one association of initial vectors $x$ with final vectors $\varepsilon(x)$. The corresponding transition $x \to \varepsilon(x)$ is forbidden, so $\varepsilon$ establishes a *pattern* of null transitions among the acts performable upon the system, namely the set of pairs of acts $(x, \varepsilon(x))$. (Clearly, this pattern is independent of the choice of $\varphi$, by the remark following equation (7.1.5).) If we now apply a SL(2,$\mathbb{C}$) transformation $L_g$ to a pair $(x, \varepsilon(x))$, we obtain the pair

$(L_g(x), L^*_{g^{-1}}(\varepsilon(x)))$, which is $(L_g(x), \varepsilon(L_g(x)))$ from equation (9.1.9). That is,

$$L_g(x, \varepsilon(x)) = (L_g(x), \varepsilon(L_g(x))), \tag{9.1.14}$$

so that the pattern of null transitions established by $\varepsilon$ is respected by SL(2,$\mathbb{C}$) actions. This is one of the ways our hitherto rather conventional term "allowable" may be imbued with meaning. Indeed, since a certain pattern of null transitions remains invariant with respect to these transformations, it begins to seem possible to forgive a macroscopic experimenter for believing in the existence of some kind of *objective* reality. Such an experimenter might be tempted to associate allowably transformed frames with equivalent views of the same *object*—a pattern of null transitions, in this instance. We note that since SL(2,$\mathbb{C}$) is the *coherent* version of the microscopic algebra $\mathfrak{sl}(2,\mathbb{C})$ of allowable transformations, the emergence of this aspect of classical epistemology bears out once again the fundamental rôle this type of coherence seems to play in mediating the transition from the objectless quantum microcosm to the object-laden macrocosm of experience (cf. Chapter 8).

## 9.2 Variation, Derivation and the Dirac Maps

We come now to a major conceit, or piece of baggage to be dragged back from the macrocosm: namely, the idea that experimenters should be able to subject their various action vectors, selective acts, etc., to *variations* within the parameters allowed. We have already assumed a form of this in our use—under the rubric "allowable"—of the transformations induced by quantum symmetries or permutations (Chapter 3). As in that case, we shall find that the (rather reasonable) granting of this freedom to experimenters leads to the introduction of certain selective acts to be interpreted as dynamical; it also leads to the appearance of certain amalgams of spinorial types, whose significance will emerge later.

To see how this comes about, let us consider the general question of how a structure comprising elements of the same type may be varied. One may vary the structure itself with respect to its constituent elemental type (by deletions or additions of these elements, for instance), and/or one may vary the elemental type itself, to the extent that it admits variation. The most general variation is obtained by carrying out these variations—namely, the variation of the structure with respect to its elements, and the variation of the elemental type itself—in sequence. For "small" variations this is reminiscent of the expression

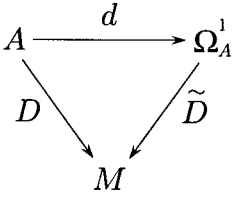
$$\Delta f \approx \sum_k \frac{\partial f}{\partial x_k} \Delta x_k . \tag{9.2.1}$$

Here, the $x_k$ correspond to the "elemental subtypes" of $f$, the partial derivatives to the variations of $f$ with respect to these elements, and the $\Delta x_k$ to the variations of the elements themselves.

Now there is a far-reaching exact algebraic analog of this last expression. Namely, suppose $A$ is an algebra and $M$ an $A$-bimodule. Recall from equation (3.2.1.1) that a linear map $D : A \longrightarrow M$ is called a *derivation* if

$$D(ab) = Da.b + a.Db . \tag{9.2.2}$$

For any $A$ there exists a so-called universal derivation $d : A \longrightarrow \Omega^1_A$ with the following property. If $D : A \longrightarrow M$ is any derivation, there exists a unique bimodule map $\tilde{D} : \Omega^1_A \longrightarrow M$ making commutative the following diagram:




(9.2.3)

Thus for each $M$ there is an $A$-bimodule isomorphism:

$$\mathrm{Der}(A, M) \cong \mathrm{Hom}_A(\Omega^1_A, M), \tag{9.2.4}$$

where Der(?,?) denotes the $A$-bimodule of derivations, and $\mathrm{Hom}_A(?,?)$ the bimodule of bimodule maps, from the first argument into the second in both cases. (If $A$ is commutative $\Omega^1_A$ is the familiar module of Kähler differentials, whereas if $A$ is noncommutative, $\Omega^1_A$ is entirely different. This difference is highly significant for the development of "noncommutative geometry": see Gracia-Bondía *et al.* 2001, Loday 1998, Landi 1997, Madore 1995, Connes 1994, Karoubi and Leruste 1989, Karoubi 1987.)

If $A$ is finitely generated, with $\{x_k\}$ a set of commuting generators, then $\Omega^1_A$ is generated as an $A$-bimodule by $\{dx_k\}$. Moreover, noting that we may write $d : A \longrightarrow A \otimes_A \Omega^1_A \cong \Omega^1_A$, we have, in an obvious notation,

$$df = \sum_k \frac{\partial f}{\partial x_k} \otimes dx_k. \tag{9.2.5}$$

This is the advertised algebraic analog of equation (9.2.1).

Usually, the generators of an algebra (such as the $x_k$ in the last example) are regarded as indecomposable monoliths, and the algebraic pseudo-variation represented by $dx_k$ cannot be carried further. This is not the case, however, for our algebras $\mathbb{C}[\Gamma^{\mathsf{N}}]$ and $\mathbb{C}[\Gamma]$, whose generators contain (or hide) a very significant substructure: namely, the spinors and conjugate-spinors comprising the underlying chrononic pairs. Only by carrying forward and including the variability of these fundamental constituent interrogative acts can an experimenter hope to capture the possible underlying dynamics of the net.

For an arbitrary pair, $(\Sigma_i, \Sigma_j^{\sim})$ say, in $S \times \tilde{S}$, we consider the operators $\Gamma^{\mathsf{N}}_{\Sigma_i \Sigma_j^{\sim}}$; similar considerations will always apply to the operators $\Gamma_{\Sigma_i \Sigma_j^{\sim}}$, often without comment: cf. §7.2. Since these operators depend bilinearly upon their spinor arguments we may for-

mally pursue their variability by writing, for instance (see diagram (7.2.6)):

$$d\phi(\Sigma_i \otimes \Sigma_j^{\sim}) \equiv d\Gamma^{\mathsf{N}}_{\Sigma_i \otimes \Sigma_j}$$

$$= \Gamma^{\mathsf{N}}_{d(\Sigma_i \otimes \Sigma_j)}\,, \tag{9.2.6}$$

with

$$d(\Sigma_i \otimes \Sigma_j^{\sim}) = d\Sigma_i.\Sigma_j^{\sim} + \Sigma_i.d\Sigma_j^{\sim}\,. \tag{9.2.7}$$

This last equation may be considered to hold in the algebraic category by noting that $\Sigma_i$, $\Sigma_j^{\sim}$ and $\Sigma_i \otimes \Sigma_j^{\sim}$ are all elements of the noncommutative tensor algebra $T(S \oplus \tilde{S})$. Then $d$ may be taken as the universal derivation for this algebra, with the dots on the right hand side denoting the appropriate bimodule action upon the universal module of differentials for this tensor algebra. For typographical reasons we shall denote it with an unadorned $\Omega^1$.

To open up the underlying spinor variables to variation beyond this formal algebraic level, we must interpret the right hand side of equation (9.2.7) in terms of selective acts. It will prove convenient to start with the second term, namely $\Sigma_i.d\Sigma_j^{\sim}$. Then we observe that each $d\Sigma_j^{\sim}$ already admits an interpretation as a differential, since the function $(\Sigma_j^{\sim})^{\wedge} : \tilde{S}^* \longrightarrow \mathbb{C}$ is holomorphic with respect to the underlying (trivial) complex manifold structure of $\tilde{S}^*$. Thus its ordinary differential $d(\Sigma_j^{\sim})^{\wedge}$ is an element of the dual space of the (holomorphic) tangent space, at 0, say, in $\tilde{S}^*$. But this tangent space may be identified with $\tilde{S}^*$ itself, upon which the differential, by its definition, acts *via* duality. Thus, we identify $d\Sigma_j^{\sim}$ first with $d(\Sigma_j^{\sim})^{\wedge}$, and then the latter with $(\Sigma_j^{\sim})^{\wedge}$ itself, upon identifying $\tilde{S}^*$ with its (holomorphic) tangent space at 0. Consequently, $\Sigma_i.d\Sigma_j^{\sim}$ determines a linear map

$$\sigma_{\Sigma_i\Sigma_j^{\sim}}\colon \tilde{S}^* \longrightarrow S, \tag{9.2.8}$$

given by

$$\begin{aligned}\sigma_{\Sigma_i\Sigma_j^{\sim}}(\ ) &= (\Sigma_i . d\Sigma_j^{\sim})(\ ) \\ &= (\Sigma_j^{\sim})^{\wedge}(\ )\Sigma_i \\ &= (\ )(\Sigma_j^{\sim})\Sigma_i . \end{aligned} \tag{9.2.9}$$

Alternatively, one may note that the association $(\Sigma_i, \Sigma_j^{\sim}) \mapsto \Sigma_i . d\Sigma_j^{\sim}$ is bilinear and so lifts to a linear map $S \otimes \tilde{S} \longrightarrow \Omega^1$, in which $\Sigma_i \otimes \Sigma_j^{\sim} \mapsto \Sigma_i . d\Sigma_j^{\sim}$. Since the map $\Sigma_j^{\sim} \mapsto (\Sigma_j^{\sim})^{\wedge}$ is a natural isomorphism, we may lift $\Sigma_i . d\Sigma_j^{\sim}$ from $\Omega^1$ up to $S \otimes \tilde{S} \cong S \otimes \tilde{S}^{**} \cong \mathrm{Hom}(\tilde{S}^*, S)$ (*via* the map just described) to obtain $\Sigma_i \otimes \Sigma_j^{\sim}$, which may then be identified with $\Sigma_i \otimes (\Sigma_j^{\sim})^{\wedge}$. In this way, $\Sigma_i . d\Sigma_j^{\sim}$ may be realized as the same map given in equation (9.2.9), namely:

$$\sigma_{\Sigma_i\Sigma_j^{\sim}} = \Sigma_i \otimes (\Sigma_j^{\sim})^{\wedge}. \tag{9.2.10}$$

If we were now to proceed similarly with the other term in equation (9.2.7), we would obtain an analogous map $S^* \longrightarrow \tilde{S}$. We could then interpret the + in equation (9.2.7) as the additive **GQ** connective $\oplus$ (§6.2.17) to obtain a map $\tilde{S}^* \oplus S^* \longrightarrow S \oplus \tilde{S} \cong \tilde{S} \oplus S$. A selective act, or operator, would then result from a *careful* choice of identifications of certain spaces with their duals. Various choices of such identifications are available and perhaps they yield physically equivalent results. There is an essentially canonical choice, however, obtained as follows. Arguing as above with the first term on the right hand side of equation (9.2.7), we obtain the association

$$d\Sigma_i . \Sigma_j^{\sim} \mapsto (\Sigma_i)^{\wedge} \otimes \Sigma_j^{\sim}, \tag{9.2.11}$$

the latter being realizable as a map $S^* \longrightarrow \tilde{S}$, as noted above. However, we may embed spinors into their dual spaces in such a way that allowable SL(2,$\mathbb{C}$) transformations are respected: cf. equation (9.1.9). Here we may assume that $\varphi$ and $\tilde{\varphi}$ have been supplied, with concomitant $\varphi^{\#}$, $\tilde{\varphi}^{\#}$, $\varepsilon$, $\tilde{\varepsilon}$ etc., now being defined (§9.1), the tildes having an obvious connotation. Then we may carry the assignment (9.2.11) further, obtaining

$$(\Sigma_i)^{\wedge} \otimes \Sigma_j^{\sim} \mapsto (\varepsilon(\Sigma_i))^{\wedge} \otimes \tilde{\varepsilon}(\Sigma_j^{\sim}) \mapsto \varepsilon(\Sigma_i) \otimes \tilde{\varepsilon}(\Sigma_j^{\sim}) \qquad (9.2.12)$$

which is realizable as a map denoted

$$\sigma^{\Sigma_i \Sigma_j^{\sim}} : S \longrightarrow \tilde{S}^* \qquad (9.2.13)$$

and given by

$$\sigma^{\Sigma_i \Sigma_j^{\sim}}( \ ) = \varepsilon(\Sigma_i)( \ ) \tilde{\varepsilon}(\Sigma_j^{\sim}) \qquad (9.2.14a)$$

$$= \varphi(\Sigma_i \vee ( \ )) \tilde{\varepsilon}(\Sigma_j^{\sim}). \qquad (9.2.14b)$$

Here we should emphasize that the purpose of this "raising" of $(\Sigma_i)^{\wedge} \otimes \Sigma_j^{\sim}$ through the use of the $\varepsilon$-isomorphisms is to respect a *quantum* symmetry—a symmetry among choices of *acts*. This distinction is important since classically conditioned experimenters may have become habituated to viewing this SL(2,$\mathbb{C}$) symmetry as classical, pertaining perhaps to underlying *states*.

It is interesting to note that $\sigma_{\Sigma_i \Sigma_j^{\sim}}$ has a similar expression. By the second assertion of the spinor duality theorem (equation (9.1.8)):

$$( \ )(\Sigma_j^{\sim}) = (\Sigma_j^{\sim})^{\wedge}( \ )$$

$$= ((-\tilde{\varepsilon}^{\#} \circ \tilde{\varepsilon})(\Sigma_j^{\sim}))( \ )$$

$$= -\tilde{\varphi}^{\#}(\tilde{\varepsilon}(\Sigma_j^{\sim}) \vee (\ )) \tag{9.2.15}$$

so that (equation (9.2.8) repeated here for convenience)

$$\sigma_{\Sigma_i \Sigma_j^{\sim}} : \tilde{S}^* \longrightarrow S$$

is given by (equation (9.2.9))

$$\sigma_{\Sigma_i \Sigma_j^{\sim}}(\ ) = (\ )(\Sigma_j^{\sim})\Sigma_i \tag{9.2.16a}$$

$$= -\tilde{\varphi}^{\#}(\tilde{\varepsilon}(\Sigma_j^{\sim}) \vee (\ ))\Sigma_i \tag{9.2.16b}$$

from equation (9.2.15).

The symmetry exhibited by the equation sets (9.2.14) and (9.2.16) may be noted.

We now obtain, without further ado, the family of linear maps

$$\gamma_{\Sigma_i \Sigma_j^{\sim}} : S \oplus \tilde{S}^* \longrightarrow S \oplus \tilde{S}^*, \tag{9.2.17}$$

given by

$$\gamma_{\Sigma_i \Sigma_j^{\sim}} = t \circ (\sigma^{\Sigma_i \Sigma_j^{\sim}} \oplus \sigma_{\Sigma_i \Sigma_j^{\sim}}), \tag{9.2.18}$$

where $t$ is the isomorphism that interchanges direct summands. We shall refer to them as *Dirac* maps. (This invocation will be justified shortly: cf. §9.4.)

It is a simple matter to check that these maps depend bilinearly upon their index pairs: therefore, the assignment $(\Sigma_i, \Sigma_j^{\sim}) \mapsto \gamma_{\Sigma_i \Sigma_j^{\sim}}$ lifts to a unique linear map $S \otimes \tilde{S} \longrightarrow \mathrm{End}V$ by the universal property

of tensor product. As usual, we shall write this map as $\theta \mapsto \gamma_\theta$, for $\theta \in S \otimes \tilde{S}$.

To interpret these maps, let us return to equation (9.2.6) and write formally,

$$d\Gamma^{\mathsf{N}}_{\Sigma_i \otimes \Sigma_j} = \Gamma^{\mathsf{N}}_{\gamma_{\Sigma_i \Sigma_j^\sim}} \ . \tag{9.2.19}$$

If $\Gamma^{\mathsf{N}}_\theta$ is thought of as the operator that inserts $\theta$ into the net, then the right hand side of the last equation should be interpreted as the operator effecting the insertion into the net of the act $\gamma_{\Sigma_i \Sigma_j^\sim}$, belonging to a new *type*.

Writing

$$V \equiv S \oplus \tilde{S}^* \tag{9.2.20}$$

for this space of *bispinors*, we note that the Dirac maps are of type $V \otimes V^*$. Moreover, since the implication is that the "increment" in $\Gamma^{\mathsf{N}}_{\Sigma_i \Sigma_j^\sim}$ occasioned by the variation is $\Gamma^{\mathsf{N}}_{\Sigma_i \Sigma_j^\sim} + d\Gamma^{\mathsf{N}}_{\Sigma_i \Sigma_j^\sim}$, we find that the effect of the variation is to introduce alternative net insertions of type $V \otimes V^*$. Roughly speaking, the selective acts $\Gamma^{\mathsf{N}}_{\Sigma_i \Sigma_j^\sim}$ of chrononic pair insertion may undergo minimal changes only through quantum interference with alternative insertions of certain acts of type $V \otimes V^*$. These latter pairs represent the minimal selective acts for quanta whose initial action vectors are of type $V$ (§1.3). Since the net insertions of the form $\Gamma^{\mathsf{N}}_\gamma$ are *alternatives* to the "normal" acts $\Gamma^{\mathsf{N}}_{\Sigma\Sigma^\sim}$ of insertion of chrononic pairs, we presume that the new selective acts (of type $V \otimes V^*$) are to be used *instead* of these normal pairs when such a pair is varied. Since single spinors represent absences of pairs, gaps, or *defects* in the reticular pair structure, the initial acts represented by the elements of $V$, if inserted into the net, may be interpreted as *reticular defect* injectors. The Dirac maps then represent certain selective acts upon this defect type which arise as a result of varying chrononic pairs as above. (Cf. the discussion following equation (9.3.15).)

Dirac bispinors arise here as experimental *acts*, not as "wave-

functions" of particles. To paraphrase a remark of Finkelstein's: it is physicists, not electrons, who emit wave-functions. Bispinors are interpreted here as initial action vectors for a new type of quantum, which, in a sense, encodes the possible response of this particular collection of "instruments," or experimental arrangements—namely, the net—to small changes in its basic constituent chrononic acts. We will return to this notion in subsequent chapters.

We return to the Dirac maps and note some of their elementary properties. First, we note the trivial fact, which follows immediately from the definition (equation (9.2.18)), that

$$\mathrm{tr}(\gamma_{\Sigma_i\Sigma_j^\sim})=0. \tag{9.2.21}$$

Then, we notice the only slightly less trivial fact that $\gamma_{\Sigma_i\Sigma_j^\sim}\gamma_{\Sigma_k\Sigma_l^\sim}$ may be expressed as a direct sum of maps in $\mathrm{End}S$ and $\mathrm{End}\tilde{S}^*$ respectively. Namely, for $\Sigma_1\oplus\Sigma_2^{\sim *}\in V$, (using variously the equation sets (9.2.14) and (9.2.16)):

$$\gamma_{\Sigma_i\Sigma_j^\sim}\gamma_{\Sigma_k\Sigma_l^\sim}(\Sigma_1\oplus\Sigma_2^{\sim *})$$

$$=\gamma_{\Sigma_i\Sigma_j^\sim}(t(\varphi(\Sigma_k\vee\Sigma_1)\tilde{\varepsilon}(\Sigma_l^\sim)\oplus\Sigma_2^{\sim *}(\Sigma_l^\sim)\Sigma_k))$$

$$=t(\Sigma_2^{\sim *}(\Sigma_l^\sim)\sigma^{\Sigma_i\Sigma_j^\sim}(\Sigma_k)\oplus\varphi(\Sigma_k\vee\Sigma_1)\sigma_{\Sigma_i\Sigma_j^\sim}(\tilde{\varepsilon}(\Sigma_l^\sim)))$$

$$=\varphi(\Sigma_k\vee\Sigma_1)\tilde{\varphi}(\Sigma_l^\sim\vee\Sigma_j^\sim)\Sigma_i\oplus\Sigma_2^{\sim *}(\Sigma_l^\sim)\varphi(\Sigma_i\vee\Sigma_k)\tilde{\varepsilon}(\Sigma_j^\sim)$$

$$=[\tilde{\varphi}(\Sigma_l^\sim\vee\Sigma_j^\sim)\varepsilon(\Sigma_k)(\ )\Sigma_i\oplus\varphi(\Sigma_i\vee\Sigma_k)(\Sigma_l^\sim)\hat{}(\ )\tilde{\varepsilon}(\Sigma_j^\sim)](\Sigma_1\oplus\Sigma_2^{\sim *}). \tag{9.2.22}$$

That is,

$$\gamma_{\Sigma_i\Sigma_j^\sim}\gamma_{\Sigma_k\Sigma_l^\sim} = \tilde{\varphi}(\Sigma_l^\sim \vee \Sigma_j^\sim)(\varepsilon(\Sigma_k)\otimes\Sigma_i) \oplus \varphi(\Sigma_i \vee \Sigma_k)((\Sigma_l^\sim)^\wedge \otimes \tilde{\varepsilon}(\Sigma_j^\sim)),$$

(9.2.23)

so that the trace is obtained simply by adding the contractions of the summands on the right hand side of the last equation:

$$\begin{aligned} \mathrm{tr}(\gamma_{\Sigma_i\Sigma_j^\sim}\gamma_{\Sigma_k\Sigma_l^\sim}) &= \tilde{\varphi}(\Sigma_l^\sim \vee \Sigma_j^\sim)\varepsilon(\Sigma_k)(\Sigma_i) + \varphi(\Sigma_i \vee \Sigma_k)\tilde{\varepsilon}(\Sigma_j^\sim)(\Sigma_l^\sim) \\ &= \tilde{\varphi}(\Sigma_l^\sim \vee \Sigma_j^\sim)\varphi(\Sigma_k \vee \Sigma_i) + \varphi(\Sigma_i \vee \Sigma_k)\tilde{\varphi}(\Sigma_j^\sim \vee \Sigma_l^\sim) \\ &= 2\varphi(\Sigma_i \vee \Sigma_k)\tilde{\varphi}(\Sigma_j^\sim \vee \Sigma_l^\sim). \end{aligned} \tag{9.2.24}$$

The trace of any product of these Dirac maps may now be obtained in fairly short order. For example, the presence of an odd number of twists $t$ in the product of an odd number of these maps will nullify the trace of such a product, whereas the product of an even number of them may be obtained, from equation (9.2.23), by contracting tensorial expressions: the trace may then be obtained, as in equation (9.2.24), by summing further contractions (§1.3).

We have sought (§1.3) to interpret such traces as amplitudes associated with certain compound experiments of "maximally external" experimenters, so we are forced to similarly interpret the traces of these so-called Dirac maps. The system that is the subject of these experiments must then be some sort of reticular defect structure, according to our earlier discussion. Only the transitions associated with an even number of Dirac maps then survive to be detected by maximally external experimenters. For such an experimenter to be able to detect the transitions associated with an odd number of these maps, more refined acts of amplification must be interposed among them: cf. §10.2.

Since we may also vary spinorial parameters *via* (logically) al-

lowable SL(2,$\mathbb{C}$) transformations, it is clearly of acute interest to investigate how the Dirac maps themselves behave with respect to such transformations. Recall (from the discussion following equation (9.1.23)) that the natural action of $g \in \mathrm{SL}(2,\mathbb{C})$ upon $\widetilde{S}^*$ is $L^{\sim *}_{g^{-1}}$.

Therefore, the naturally associated action of $g$ upon $V$ is:

$$\mathcal{S}(g) \equiv L_g \oplus L^{\sim *}_{g^{-1}}. \tag{9.2.25}$$

Noting that we may write (in an obvious notation) $L^{\sim}_g = L_{g^\sim}$ and $L^{\sim *}_{g^{-1}} = L^*_{g^{\sim -1}}$, and that the latter map reduces to the complex conjugate transpose of $L_g^{-1}$ when expressed in matricial form, we find that $\mathcal{S}(g)$ is exactly the standard *bispinor representation* of SL(2,$\mathbb{C}$) (Bogolubov, Logunov and Todorov 1975, equation (7.3), p.170, Bjorken and Drell 1964, §2.2.) Then we have an independent proof of the following fundamental result.

DIRAC MAP TRANSFORMATION THEOREM

$$\mathcal{S}(g)^{-1} \gamma_{g\Sigma_i g^\sim \Sigma_j^\sim} \mathcal{S}(g) = \gamma_{\Sigma_i \Sigma_j^\sim}. \tag{9.2.26}$$

PROOF

We have

$$\sigma_g(\,) \equiv \sigma_{g\Sigma_i g^\sim \Sigma_j^\sim}(\,)$$

$$= (\,)(g^\sim \Sigma_j^\sim) g\Sigma_i \qquad \text{from equation (9.2.16a)}$$

$$= ((L^*_{g^\sim}(\,))(\Sigma_j^\sim)) g\Sigma_i, \tag{9.2.27}$$

and

$$\sigma^g \equiv \sigma^{g\Sigma_i\, g\sim\Sigma_j^\sim}(\,)$$

$$= \varepsilon(g\Sigma_i)(\,)\tilde{\varepsilon}(g\sim\Sigma_j^\sim) \quad \text{from equation (9.2.14a)}$$

$$= ((\varepsilon \circ L_g)(\Sigma_i))(\,)(\tilde{\varepsilon} \circ L_{g\sim})(\Sigma_j^\sim)$$

$$= ((L^*_{g-1} \circ \varepsilon)(\Sigma_i))(\,)(L^*_{g\sim -1} \circ \tilde{\varepsilon})(\Sigma_j^\sim)$$

from equation (9.1.13)

$$= (L^*_{g-1}(\varepsilon(\Sigma_i)))(\,)L^*_{g\sim -1}(\tilde{\varepsilon}(\Sigma_j^\sim)), \qquad (9.2.28)$$

whence

$$(L_g^{-1}\,\sigma_g\,L^*_{g\sim -1})(\,) = L_g^{-1}(\,\sigma_g(L^*_{g\sim -1}(\,)))$$

$$= L_g^{-1}((L^*_{g\sim}\,(L^*_{g\sim -1}(\,))(\Sigma_j^\sim))g\Sigma_i)$$

from equation (9.2.27)

$$= L_g^{-1}(((\,)(\Sigma_j^\sim))g\Sigma_i)$$

$$= (\,)(\Sigma_j^\sim)\Sigma_i$$

$$= \sigma_{\Sigma_i\Sigma_j^\sim}(\,), \qquad \text{from equation (9.2.16a)}$$

(9.2.29)

and

$$(L^{*-1}_{g^{\sim}-1}\sigma^{g}L_{g})(\ ) = (L^{*-1}_{g^{\sim}-1}(\sigma^{g}(L_{g}(\ ))))$$

$$= L^{*}_{g^{\sim}}([L^{*}_{g^{-1}}(\varepsilon(\Sigma_{i}))(g(\ ))]\,L^{*}_{g^{\sim}-1}(\tilde{\varepsilon}(\Sigma^{\sim}_{j})))$$

from equation (9.2.28)

$$= L^{*}_{g^{\sim}}([\varepsilon(\Sigma_{i})(\ )]\,L^{*}_{g^{\sim}-1}(\tilde{\varepsilon}(\Sigma^{\sim}_{j})))$$

$$= \varepsilon(\Sigma_{i})(\ )\tilde{\varepsilon}(\Sigma^{\sim}_{j})$$

$$= \sigma^{\Sigma_{i}\Sigma^{\sim}_{j}}(\ ) \qquad \text{from equation (9.2.14a).}$$

(9.2.30)

Equation (9.2.26) now follows immediately from equations (9.2.29) and (9.2.30) and the definition of the Dirac maps, equation (9.2.18). ∎

One immediate consequence of this result is that the traces of products of Dirac maps are seen to be invariant with respect to allowable transformations of the basic chrononic $\Sigma\Sigma^{\sim}$-pairs, which, in view of §7.2 (equation (7.2.4)), we need not be shy to identify with Lorentz transformations. In fact, equation (9.2.26) shows that the Dirac maps behave with respect to these transformations exactly as complex Lorentz (cotangent) vectors should. (That is, when these differential forms are realized, *via* Clifford algebra, as automorphisms of the spinor bundle over a Lorentz spin manifold: see for example Lawson and Michelsohn 1989.) As a result of this behavior the conclusions arrived at by macroscopic experimenters through their use—conclusions concerning, for instance, matters of Lorentz covariance or invariance, as with the Dirac traces—would tend to confirm

their belief in independent eventlike *objects*. For this is exactly the behavior to be expected if there actually was an independent set of eventlike objects which *each* putative experimenter may make a "state-space" valued map of, these maps being locally transformable among themselves *via* Lorentz transformations. Then a macroscopic experimenter would be tempted to interpret these transformations as taking place between equivalent *observers* of an *objective* plenum having the aspect of a (Lorentz, spin) manifold. This view could provide a conceptual starting point for the promulgation of classical relativity: needless to say, this was not the path followed by history.

We note here that if we had started out from the first term in equation (9.2.7), and had "$\varepsilon$-raised" the map coming from the second, the physical conclusions we eventually reach will remain unaltered. The choice we have made above will however lead to an immediately recognizable representation of familiar items, namely the Dirac matrices: see §9.4.

## 9.3 The Δ Operators

We now return to the context of §8.1 to resume our search for the reticular precursors of the operators $\tau p_k$ (equation (8.1.28)). Note first that in the Schrödinger representation

$$p_k = \frac{1}{i}\frac{\partial}{\partial q_k} \tag{9.3.1}$$

so that equation (8.1.28) would read

$$[\![\Delta_{\mathsf{N}}^k]\!] = \frac{\tau}{i}\frac{\partial}{\partial q_k}. \tag{9.3.2}$$

Now the partial derivative appearing on the right hand side of the last equation, which acts upon certain elements of $L^2(\mathbb{R}^4)$, is realizable as—or is extendible to—one of the basic self-derivations of

the algebra of formal series in the $q$s. Thus it seems reasonable to assume that $\Delta_{\mathsf{N}}^{k}$ is similarly realizable as an operator upon the algebra of formal series in the $\Gamma^{\mathsf{N}}$s in such a way that when interpreted macroscopically it acts in the specified manner upon the $q$-algebra. That is to say, for a formal series $\xi$ in the $\Gamma^{\mathsf{N}}$s:

$$[\![\Delta_{\mathsf{N}}^{k}\xi]\!]=[\![\Delta_{\mathsf{N}}^{k}]\!][\![\xi]\!]\,. \tag{9.3.3}$$

In particular, with

$$\xi=1+\sum\alpha_k\Gamma_k^{\mathsf{N}}+\sum\alpha_k\alpha_m\Gamma_k^{\mathsf{N}}\Gamma_m^{\mathsf{N}}+\ldots, \tag{9.3.4}$$

so that (equation (8.1.30))

$$[\![\xi]\!]=\exp\left[\frac{i}{\tau}\sum_m\alpha_m q_m\right], \tag{9.3.5}$$

and from equation (9.3.2), we obtain, for the right hand side of equation (9.3.3):

$$\begin{aligned}[\![\Delta_{\mathsf{N}}^{k}]\!][\![\xi]\!]&=\frac{\tau}{i}\frac{\partial}{\partial q_k}\left(\exp\left[\frac{i}{\tau}\sum_m\alpha_m q_m\right]\right)\\&=\alpha_k[\![\xi]\!]\\&=[\![\alpha_k\xi]\!]\,.\end{aligned} \tag{9.3.6}$$

So equation (9.3.3) may be satisfied if

$$\Delta_{\mathsf{N}}^{k}\xi=\alpha_k\xi\,. \tag{9.3.7}$$

It is now a simple matter to check that the last equation is satisfied by the linear operator defined by

$$\Delta^k_{\mathsf{N}}(\Gamma^{\mathsf{N}}_{m_1}\dots\Gamma^{\mathsf{N}}_{m_p}) \equiv \frac{1}{p}\frac{\partial}{\partial\Gamma^{\mathsf{N}}_k}(\Gamma^{\mathsf{N}}_{m_1}\dots\Gamma^{\mathsf{N}}_{m_p}), \tag{9.3.8}$$

and that with this definition of $\Delta^k_{\mathsf{N}}$ equation (9.3.3) is satisfied in general. (The partial differentiation in this definition is to be carried out on its noncommuting arguments according to the product rule, while the order of the factors is maintained.)

As usual, analogous remarks apply when the Maxwell-Boltzmann phase is considered and the Ns are left off.

Now the new bispinor type $V$ was introduced in the last section in order to implement "small" variations in the basic chrononic $\Sigma\tilde{\Sigma}$-pairs: elements of $V$ were then found to be interpretable as acts of injection of reticular "defects." How should experimenters insert such defectlike acts into their nets so as to implement these variations?

Moving for simplicity to the Maxwell–Boltzmann phase, we note that for an arbitrary $\xi$ in $\mathbb{C}[\Gamma]$, $[\![\xi]\!]$ may be thought of as a function of the $[\![\Gamma_k]\!]$. Adopting the convention that repeated indices are to be summed over except where the contrary is specified, we may then write

$$\begin{aligned} d[\![\xi]\!] &= \frac{\partial[\![\xi]\!]}{\partial[\![\Gamma_k]\!]}\otimes d[\![\Gamma_k]\!] \\ &= \left(\frac{\tau}{i}\right)\frac{\partial[\![\xi]\!]}{\partial q_k}\otimes d[\![\Gamma_k]\!] \end{aligned} \tag{9.3.9}$$

from equation (8.1.27). But

$$d[\![\Gamma_k]\!] = \left(\frac{\tau}{i}\right)dq_k \tag{9.3.10}$$

is, strictly speaking, not defined as a recognizable type in the coarse domain of the Schrödinger representation, whereas $d\Gamma_k$ has been defined in a fairly canonical manner in the last section in terms of the corresponding Dirac map $\gamma_k$.

Moreover, since bispinorial acts are interpretable as defect injectors at the lowest level of the net itself, this type would seem not to be an artifact of the various condensation processes presumed to contribute to the types apparent to macroscopic experimenters, but is, rather, available to them in pristine condition. That is to say, bispinorial acts should be interpretable in the *same way* by all experimenters, whatever their presumed level of resolution. (Later, we will compromise this principle slightly. As remarked earlier, macroscopic experimenters will need to "amplify" these acts in order to register certain transitions. See §10.2.)

Consequently, we set

$$\begin{aligned} d[\![\Gamma_k]\!] &= [\![\gamma_k]\!] \\ &= \gamma_k \,. \end{aligned} \tag{9.3.11}$$

Then, in view of equations (9.3.9), (9.3.11) and (9.3.3):

$$\begin{aligned} d[\![\xi]\!] &= \left(\frac{\tau}{i}\right)\frac{\partial[\![\xi]\!]}{\partial q_k}\otimes d[\![\Gamma_k]\!] \\ &= [\![\Delta^k]\!][\![\xi]\!]\otimes\gamma_k \\ &= [\![\Delta^k\xi]\!]\otimes[\![\gamma_k]\!] \\ &= [\![\Delta^k\xi\otimes\gamma_k]\!] \,. \end{aligned} \tag{9.3.12}$$

We distribute $[\![\ ]\!]$ over $\otimes$ in the last step because the notion of *sequence* is the same at all resolutions (§2.2): the macroscopic in-

terpretation of a sequence should be as the sequence of individual interpretations.

The upshot of the last equation is that the expression

$$d\xi \equiv \Delta^k \xi \otimes \gamma_k \tag{9.3.13}$$

should be regarded as the precursor (in the Maxwell–Boltzmann phase) of the continuum expression

$$d[\![\xi]\!] = \left(\frac{\tau}{i}\right)\partial^k [\![\xi]\!] \otimes \gamma_k , \tag{9.3.14}$$

where $\partial^k \equiv \dfrac{\partial}{\partial q_k}$, for then we would have

$$[\![d\xi]\!] = d[\![\xi]\!] . \tag{9.3.15}$$

(It may be noted that the adoption of equation (9.3.13) bears out our earlier interpretation of "quantum" variation of reticular elements as entailing the *replacement* of the varied $\Sigma\tilde{\Sigma}$-pair by a $\gamma_{\Sigma\tilde{\Sigma}}$: see the discussion following equation (9.2.20). Thus, the equation (9.3.13) may be interpreted as asserting the identification of the selective act associated with a (small) variation of $\xi$ with a superposition of pairs of acts of the following type: each pair comprises an act of destruction of a $\Sigma\tilde{\Sigma}$-pair (*via* the action of $\Delta^{\Sigma\tilde{\Sigma}}$) and an insertion of a $\gamma_{\Sigma\tilde{\Sigma}}$ in its stead. Readers who find the argument following equation (9.2.20) convincing may presumably use it to motivate the choice of an equation like (9.3.13): then they will discover its continuum correspondent (equation (9.3.14)), which will turn out to be the usual Dirac operator. Such readers will then have derived this operator entirely at the reticular level, without contaminating the argument, as we have, by pulling back from the continuum equation (9.3.9).)

Applying $d$ to an element of the form $\xi \otimes s \in \mathbb{C}[\Gamma] \otimes V$, and as-

suming for the time being that the "defect" injector $s$ does not *itself* partake of that (small) variation its insertion is supposed to implement, we obtain:

$$\begin{aligned} d(\xi\otimes s) &= d\xi\otimes s \\ &= \Delta^k\xi\otimes\gamma_k\otimes s. \end{aligned} \tag{9.3.16}$$

Now, since $\gamma_k\in V\otimes V^*$ and $s\in V$, a macroscopic external experimenter would presumably have access only to the contracted form of this act so, effectively,

$$\begin{aligned} [\![ d(\xi\otimes s)]\!] &= [\![\Delta^k\xi\otimes\gamma_k\otimes s]\!] \\ &= [\![\Delta^k]\!][\![\xi]\!]\otimes[\![\gamma_k(s)]\!] \\ &= \left(\frac{\tau}{i}\right)\partial^k[\![\xi]\!]\otimes\gamma_k(s) \\ &= \left(\frac{\tau}{i}\right)(\partial^k\otimes\gamma_k)([\![\xi]\!]\otimes s) \\ &\equiv d[\![\xi\otimes s]\!]. \end{aligned} \tag{9.3.17}$$

In this way, we arrive at a Diraclike operator which will soon be shown to be the usual one. Its reticular precursor, namely the reticular analog of the $d$ in equation (9.3.13), has a quite specific interpretation in terms of infinitesimal reverse parallel transport of defects through the net. We shall return to these matters in the next chapter.

For later use we remark that a second variation applied to equation (9.3.16) yields

$$d(d\xi) = d(\Delta^k \xi \otimes \gamma_k)$$

$$= d(\Delta^k \xi) \otimes \gamma_k$$

$$= \Delta^l \Delta^k \xi \otimes \gamma_l \otimes \gamma_k . \qquad (9.3.18)$$

In this case, reductions to the forms appropriate to macroscopic or external experimenters may proceed, for the spinorial factor $\gamma_l \otimes \gamma_k$, through two contractive steps. The first yields

$$[\![ d(d\xi) ]\!] = [\![ \Delta^l \Delta^k \xi ]\!] \otimes [\![ \gamma_l \otimes \gamma_k ]\!]$$

$$= [\![ \Delta^l \Delta^k \xi ]\!] \otimes [\![ \gamma_l \gamma_k ]\!]$$

$$= [\![ \Delta^l \Delta^k \xi ]\!] \otimes \gamma_l \gamma_k . \qquad (9.3.19)$$

A further contraction is possible, yielding $\mathrm{tr}(\gamma_l \gamma_k)$ instead of $\gamma_l \gamma_k$ on the right hand side of the last equation. Macroscopic experimenters who perform this final contraction increase to the maximum their degree of externality with respect to the defect structure.

We note in passing that some care is required in the handling of the $\Delta^k$ operators (equation (9.3.8))—they are not derivations on their domain algebras, for example.

## 9.4 The Real Subspace, Frame Choices and Dirac Matrices

We return to the quantum-logical context of our original choice of the space $S \otimes \tilde{S}$ to represent pairs of interrogative acts (§7.1) to ask the following question: what should be the effect of changing the "flavor" of each factor in each pair? Noting that these flavors are changed by the action of the conjugate linear involutions

$$(\,)^{\sim}: S \longrightarrow \tilde{S}, \tag{9.4.1}$$

and

$$(\,)^{\sim}: \tilde{S} \longrightarrow S \tag{9.4.2}$$

(which amount to complex conjugations, once a basis is fixed) we find that the flavor-changing operation for pairs is

$$(\,)^{\sim}\otimes(\,)^{\sim}: S\otimes\tilde{S} \longrightarrow \tilde{S}\otimes S, \tag{9.4.3}$$

an operation that also has the effect of changing the flavor order, or *sequence*, of the constituent acts within a pair. So, to discuss properties of the flavor-changing operation without the side effect of sequence reversal, we should restore the order of pairs by following the map in equation (9.4.3) by a twist. The result is a conjugate linear involution on $S\otimes\tilde{S}$, which we shall also call $(\,)^{\sim}$:

$$(\,)^{\sim}: S\otimes\tilde{S} \longrightarrow S\otimes\tilde{S}, \tag{9.4.4}$$

given by

$$(\alpha\otimes\tilde{\beta})^{\sim} = \beta\otimes\tilde{\alpha}\,. \tag{9.4.5}$$

This involution changes constituent truth-flavors while maintaining the sequential order of flavors within a pair.

Since the pair substructure is apparently macroscopically invisible, being presumably far below the resolution available to habitues of the continuum, it would seem that macroscopic interpretations $[\![\Gamma_k^{\mathbf{N}}]\!]$ of the basic microscopic chrononic selective acts $\Gamma_k^{\mathbf{N}}$ should be *impervious* to this truth-flavor interchange operation and therefore taken to be truth-flavor neutral. Consequently, it will be assumed that macroscopic continuum experimenters may interpret only those net generators $\Gamma_k^{\mathbf{N}} \equiv \Gamma_{\theta_k}^{\mathbf{N}}$ having the property

$$\Gamma^{\mathsf{N}\sim}_{\theta_k} \equiv \Gamma^{\mathsf{N}}_{\theta\tilde{k}}$$

$$= \Gamma^{\mathsf{N}}_{\theta_k}. \tag{9.4.6}$$

For such a basis $\{\theta_k\}$ of $S\otimes\tilde{S}$ the set $\{\Gamma^{\mathsf{N}}_k\}$ will of course still generate the algebra $\mathbb{C}[\Gamma^{\mathsf{N}}]$ and all our considerations concerning it remain unaffected. The above assumption, in a sense, serves merely to restrict the availability to macroscopic continuum experimenters of net generators. Although this assumption may be regarded as an intervention at the macroscopic level, it will inform our choice (below) of a convenient basis $\{\theta_k\}$. This basis will then determine a correspondence principle of the type contemplated in the last chapter: namely, as a certain linear map from the algebra $\mathbb{C}[\Gamma^{\mathsf{N}}]$ (or $\mathbb{C}[\Gamma]$) into the algebra generated by the position operators of a certain Schrödinger representation (with concomitant interpretations for the $\Delta$ operators).

Note that the set

$$\mathsf{M} = \{\Theta \in S\otimes\tilde{S} : \Theta^{\sim} = \Theta\} \tag{9.4.7}$$

is a *real* space, easily seen to be of dimension four, and that it is stable under the action of the representation $\mathcal{D}$ of $\mathrm{SL}(2,\mathbb{C})$ (equation (7.2.2)), which therefore lifts to a representation of $L^{\uparrow}_{+}$ upon it (equation (7.2.3)). This means that any element of $L^{\uparrow}_{+}$ acts upon $\mathsf{M}$ as follows. Choose an element $g \in \mathrm{SL}(2,\mathbb{C})$ that maps to the chosen element in $L^{\uparrow}_{+}$ *via* the surjection in (7.2.3). Then define the action of the chosen element in $L^{\uparrow}_{+}$ as $\mathcal{D}(g)$. This choice will be independent of the choice of $g$. We shall abuse the notation slightly (in the usual way) by denoting both a general element of $L^{\uparrow}_{+}$, and its action upon $\mathsf{M}$, by $\Lambda(g)$. Thus $\Lambda(g)$ will denote the Lorentz transformation corresponding to $g \in \mathrm{SL}(2,\mathbb{C})$ acting upon $\mathsf{M}$ *via* $\mathcal{D}$.

We are now ready to allow an experimenter to choose an *ini-*

*tial frame* in $S$, which, we recall, will be determined by a pair of vectors orthogonal relative to a Hilbert space inner product, and may itself be frame (and experimenter) dependent. Once a frame (and Hilbert space structure) is chosen for $S$ however, we shall assume that the following symmetry obtains between the $S$- and the $\tilde{S}$-amplitudes—here $\langle\ |\ \rangle$ denotes the inner product in either space $S$ or $\tilde{S}$:

$$\langle\alpha|\beta\rangle = \langle\beta^{\sim}|\alpha^{\sim}\rangle. \tag{9.4.8}$$

This assumption embodies the notion that $\alpha$-importers (exporters) are, in a sense, equivalent to $\alpha^{\sim}$-exporters (importers), and that corresponding amplitudes are the same: this reversal of rôles between truth-flavors seems to be a reasonable interpretation of the $\sim$-operation.

The experimenter establishes an initial frame in $S$, then, by choosing a pair of algebraically independent initial action vectors, $\uparrow$ and $\downarrow$, say, to serve as a (computational) basis for $S$, as in §7.1, and declaring them to be orthonormal. Then the Hilbert space structure, and consequently also the adjoint operation, will depend upon this initial choice. The adjoint is then not unique, and these systems are *open* (§1.1). That is to say, initial acts may now have different allowed final acts, depending upon the choice of frame (or experimenter). This openness seems not inappropriate to these smallest nontrivial quantum systems.

Supposing now that some isomorphism $\psi: S \vee S \longrightarrow \mathbb{C}$ has been supplied, we may choose a new isomorphism

$$\varphi \equiv \psi(\uparrow \vee \downarrow)^{-1}\psi \tag{9.4.9}$$

so that

$$\varphi(\uparrow \vee \downarrow) = 1. \tag{9.4.10}$$

Then $\varepsilon$, $\varphi^{\#}$, $\varepsilon^{\#}$ (§9.1) are determined.

Having chosen a frame for $S$, there seems no reason, *a priori*, for an experimenter *not* to choose a similar but unrelated frame for $\tilde{S}$. From here on, however, we shall make such choices explicitly to ameliorate the macroscopic experimenter's task of interpretation. As macroscopic experimenters (or at least theorists), prone to reification, we may be tempted to assume that other choices would not affect the resulting physical conclusions. Whether or not this is true, it is certain that other choices would make their macroscopic interpretation much more difficult, if not impossible. (In making these choices, we undoubtedly ride roughshod over many subtle and difficult issues whose resolution must await the advent of a more refined principle of correspondence.) Consequently, we note from equation (9.4.8) that a choice of orthonormal basis $\{\uparrow,\downarrow\}$ in $S$ determines one in $\tilde{S}$, namely $\{\uparrow^{\sim},\downarrow^{\sim}\}$, the tilde, as usual in this context, denoting the conjugating involution (equation (9.4.2)). Likewise, an isomorphism $\tilde{\varphi}$: $\tilde{S}\vee\tilde{S}$ $\longrightarrow \mathbb{C}$ is determined by $\varphi$ as follows:

$$\tilde{\varphi}(\alpha\vee\beta)\equiv\overline{\varphi(\alpha^{\sim}\vee\beta^{\sim})}\,. \tag{9.4.11}$$

Thus

$$\begin{aligned}\tilde{\varphi}(\uparrow^{\sim}\vee\downarrow^{\sim}) &= \overline{\varphi(\uparrow\vee\downarrow)}\\ &= 1.\end{aligned} \tag{9.4.12}$$

The maps $\tilde{\varepsilon}$, $\tilde{\varphi}^{\#}$, $\tilde{\varepsilon}^{\#}$ are now also determined. (Similarly, we could have started with the latter space.)

A Hilbert space structure is now determined for $S\otimes\tilde{S}$ (§2.2) as is the orthonormal basis

$$\{\uparrow\otimes\uparrow^{\sim},\uparrow\otimes\downarrow^{\sim},\downarrow\otimes\uparrow^{\sim},\downarrow\otimes\downarrow^{\sim}\}, \tag{9.4.13}$$

two elements of which are not in M. To choose a *convenient* basis for $S \otimes \tilde{S}$ all of whose elements lie in M, "convenient" here meaning "convenient for the purposes of macroscopic interpretation," we proceed as follows. First, consider an ordinary numerical matrix of the form

$$A = \begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix} \tag{9.4.14}$$

in which $a$ and $b$ are real and $z$ is complex. Then the matrix

$$B = \begin{pmatrix} a & z \\ \bar{z} & b \end{pmatrix} - \frac{1}{2}(a+b)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9.4.15}$$

is clearly trace free and Hermitian: that is, $iB$ is in $\mathfrak{su}(2)$. Now there is a convenient basis $(i\sigma_1, i\sigma_2, i\sigma_3)$ for $\mathfrak{su}(2)$, the $\sigma_k$ being known as the *Pauli* matrices, namely:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{9.4.16}$$

Thus there exist constants $c_k$, $k = 1,2,3$, such that

$$B = \sum_{k=1}^{3} c_k \sigma_k . \tag{9.4.17}$$

From the (well-known) tracial properties of the Pauli matrices it is easy to see that

$$c_k = \tfrac{1}{2}\,\mathrm{tr}(\sigma_k A), \tag{9.4.18}$$

$k = 1,2,3$, which are real numbers. One may then express the origi-

nal matrix as

$$A = \frac{1}{2}(a+b)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + B$$

$$\equiv \sum_{k=0}^{3} c_k \sigma_k , \tag{9.4.19}$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{9.4.20}$$

and now

$$c_k = \tfrac{1}{2}\,\mathrm{tr}(\sigma_k A), \tag{9.4.21}$$

for $k = 0,1,2,3$.

It is immediate that these four matrices $\sigma_k$ are linearly independent over $\mathbb{C}$ and hence span the space of $2\times 2$ complex matrices. Consequently our maps $\sigma^{\Sigma_i \Sigma_j^{\sim}}$, $\sigma_{\Sigma_i \Sigma_j^{\sim}}$ will be expressible in terms of them, once bases have been chosen in $S$ and $\tilde{S}^*$. Moreover, pseudo-matricial arrays of the form

$$\begin{pmatrix} \sigma_{\uparrow\uparrow^{\sim}} & \sigma_{\downarrow\uparrow^{\sim}} \\ \sigma_{\uparrow\downarrow^{\sim}} & \sigma_{\downarrow\downarrow^{\sim}} \end{pmatrix} \tag{9.4.22}$$

are formally similar in structure to the matrix $A$ in equation (9.4.14). Thus, by expressing each $\sigma^{\Sigma_i \Sigma_j^{\sim}}$, $\sigma_{\Sigma_i \Sigma_j^{\sim}}$ in terms of the matrices $\sigma_k$, one should be able to formally express the pseudo-matrix in (9.4.22), and its companion having raised indices, as linear combinations of the Paulis. The coefficients in such an expansion, being linear combi-

nations of the pseudo-matrix entries, will be $2\times 2$ matrices, and, moreover, will be "formally real" with respect to their arrow index structure. Thence, one may express the corresponding pseudo-matrix of Dirac maps similarly in terms of the Paulis, likewise with "formally real" coefficients, which will now be $4\times 4$ matrices. The arrow index structure of these coefficients should then provide a template for a basis for $S\otimes\tilde{S}$ whose elements reside in $\mathbf{M}$, which will correspond in a one-to-one fashion with the basis just found for the Dirac maps. That is to say, the conventional and macroscopically conditioned choice of the Pauli basis, together with our frame choices, conspires to determine a "formally self-adjoint" and linearly independent set of Dirac maps. Since these maps represent a cotangentlike structure in the sense of equation (9.2.5) an appropriate matching basis for $S\otimes\tilde{S}$ can be chosen by lifting to this latter space the index structure appearing in the set of Dirac maps just obtained.

To implement this plan we note that a basis for $V\equiv S\oplus\tilde{S}^*$ is determined by our frame choices, namely $\{\uparrow, \downarrow, \tilde{\uparrow}^*, \tilde{\downarrow}^*\}$. Then, using equation (9.2.16a) we obtain

$$\sigma_{\uparrow\tilde{\uparrow}}(\tilde{\uparrow}^*) = (\tilde{\uparrow}^*)(\tilde{\uparrow})\uparrow = \uparrow \tag{9.4.23a}$$

$$\sigma_{\uparrow\tilde{\uparrow}}(\tilde{\downarrow}^*) = (\tilde{\downarrow}^*)(\tilde{\uparrow})\uparrow = 0 \tag{9.4.23b}$$

$$\sigma_{\downarrow\tilde{\uparrow}}(\tilde{\uparrow}^*) = (\tilde{\uparrow}^*)(\tilde{\uparrow})\downarrow = \downarrow \tag{9.4.23c}$$

$$\sigma_{\downarrow\tilde{\uparrow}}(\tilde{\downarrow}^*) = (\tilde{\downarrow}^*)(\tilde{\uparrow})\downarrow = 0 \tag{9.4.23d}$$

$$\sigma_{\uparrow\tilde{\downarrow}}(\tilde{\uparrow}^*) = (\tilde{\uparrow}^*)(\tilde{\downarrow})\uparrow = 0 \tag{9.4.23e}$$

$$\sigma_{\uparrow\tilde{\downarrow}}(\tilde{\downarrow}^*) = (\tilde{\downarrow}^*)(\tilde{\downarrow})\uparrow = \uparrow \tag{9.4.23f}$$

$$\sigma_{\downarrow\downarrow^{\sim}}(\uparrow^{\sim^*}) = (\uparrow^{\sim^*})(\downarrow^{\sim})\downarrow = 0 \tag{9.4.23g}$$

$$\sigma_{\downarrow\downarrow^{\sim}}(\downarrow^{\sim^*}) = (\downarrow^{\sim^*})(\downarrow^{\sim})\downarrow = \downarrow. \tag{9.4.23h}$$

So, using column vectors, these maps acquire the following matricial representations relative to the bases chosen for $\tilde{S}^*$ and $S$:

$$\sigma_{\uparrow\uparrow^{\sim}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \tfrac{1}{2}(\sigma_0 + \sigma_3) \tag{9.4.24a}$$

$$\sigma_{\downarrow\uparrow^{\sim}} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \tfrac{1}{2}(\sigma_1 - i\sigma_2) \tag{9.4.24b}$$

$$\sigma_{\uparrow\downarrow^{\sim}} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \tfrac{1}{2}(\sigma_1 + i\sigma_2) \tag{9.4.24c}$$

$$\sigma_{\downarrow\downarrow^{\sim}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \tfrac{1}{2}(\sigma_0 - \sigma_3) \tag{9.4.24d}$$

or

$$\begin{pmatrix} \sigma_{\uparrow\uparrow^{\sim}} & \sigma_{\downarrow\uparrow^{\sim}} \\ \sigma_{\uparrow\downarrow^{\sim}} & \sigma_{\downarrow\downarrow^{\sim}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma_0 + \sigma_3 & \sigma_1 - i\sigma_2 \\ \sigma_1 + i\sigma_2 & \sigma_0 - \sigma_3 \end{pmatrix}. \tag{9.4.25}$$

For the raised maps, we use equations (9.2.14b), (9.4.11) and the first assertion of the spinor duality theorem, equations (9.1.10) and (9.1.11), to obtain:

$$\sigma^{\uparrow\uparrow^{\sim}}(\uparrow) = \varphi(\uparrow \vee \uparrow)\tilde{\varepsilon}(\uparrow^{\sim}) = 0 \tag{9.4.26a}$$

$$\sigma^{\uparrow\tilde{\uparrow}}(\downarrow) = \varphi(\uparrow \vee \downarrow)\tilde{\varepsilon}(\uparrow^{\sim}) = \downarrow^{\sim *}. \qquad (9.4.26b)$$

For the others, in short, we obtain:

$$\sigma^{\downarrow\tilde{\uparrow}}(\uparrow) = -\downarrow^{\sim *} \qquad (9.4.26c)$$

$$\sigma^{\downarrow\tilde{\uparrow}}(\downarrow) = 0 \qquad (9.4.26d)$$

$$\sigma^{\uparrow\tilde{\downarrow}}(\uparrow) = 0 \qquad (9.4.26e)$$

$$\sigma^{\uparrow\tilde{\downarrow}}(\downarrow) = -\uparrow^{\sim *} \qquad (9.4.26f)$$

$$\sigma^{\downarrow\tilde{\downarrow}}(\uparrow) = \uparrow^{\sim *} \qquad (9.4.26g)$$

$$\sigma^{\downarrow\tilde{\downarrow}}(\downarrow) = 0 \qquad (9.4.26h)$$

so

$$\sigma^{\uparrow\tilde{\uparrow}} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \tfrac{1}{2}(\sigma_0 - \sigma_3) \qquad (9.4.27a)$$

$$\sigma^{\downarrow\tilde{\uparrow}} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = -\tfrac{1}{2}(\sigma_1 - i\sigma_2) \qquad (9.4.27b)$$

$$\sigma^{\uparrow\tilde{\downarrow}} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} = -\tfrac{1}{2}(\sigma_1 + i\sigma_2) \qquad (9.4.27c)$$

$$\sigma^{\downarrow\downarrow^{\sim}} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \tfrac{1}{2}(\sigma_0 + \sigma_3). \tag{9.4.27d}$$

That is,

$$\begin{pmatrix} \sigma^{\uparrow\uparrow^{\sim}} & \sigma^{\downarrow\uparrow^{\sim}} \\ \sigma^{\uparrow\downarrow^{\sim}} & \sigma^{\downarrow\downarrow^{\sim}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \sigma_0 - \sigma_3 & -(\sigma_1 - i\sigma_2) \\ -(\sigma_1 + i\sigma_2) & \sigma_0 + \sigma_3 \end{pmatrix}. \tag{9.4.28}$$

Using the definition of the Dirac maps (equation (9.2.18)) and equations (9.4.25) and (9.4.28) one finds

$$\gamma_{\uparrow\uparrow^{\sim}} = \begin{pmatrix} 0 & \sigma_{\uparrow\uparrow^{\sim}} \\ \sigma^{\uparrow\uparrow^{\sim}} & 0 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \sigma_3 \\ -\sigma_3 & 0 \end{pmatrix}\right] \tag{9.4.29a}$$

$$\gamma_{\downarrow\uparrow^{\sim}} = \begin{pmatrix} 0 & \sigma_{\downarrow\uparrow^{\sim}} \\ \sigma^{\downarrow\uparrow^{\sim}} & 0 \end{pmatrix} = \frac{1}{2}\left[\begin{pmatrix} 0 & \sigma_1 \\ -\sigma_1 & 0 \end{pmatrix} - i\begin{pmatrix} 0 & \sigma_2 \\ -\sigma_2 & 0 \end{pmatrix}\right] \tag{9.4.29b}$$

etc.

Thus, with

$$\gamma_0 \equiv \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix} \text{ and } \gamma_k \equiv \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}, \tag{9.4.30}$$

for $k = 1,2,3$, we obtain

$$\gamma \equiv \begin{pmatrix} \gamma_{\uparrow\uparrow^{\sim}} & \gamma_{\downarrow\uparrow^{\sim}} \\ \gamma_{\uparrow\downarrow^{\sim}} & \gamma_{\downarrow\downarrow^{\sim}} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} \gamma_0 + \gamma_3 & \gamma_1 - i\gamma_2 \\ \gamma_1 + i\gamma_2 & \gamma_0 - \gamma_3 \end{pmatrix} \tag{9.4.31}$$

from which it immediately follows that

$$\gamma_0 = \gamma_{\uparrow\uparrow^{\sim}} + \gamma_{\downarrow\downarrow^{\sim}} \tag{9.4.32a}$$

$$\gamma_1 = \gamma_{\uparrow\downarrow^\sim} + \gamma_{\downarrow\uparrow^\sim} \tag{9.4.32b}$$

$$\gamma_2 = i(\gamma_{\downarrow\uparrow^\sim} - \gamma_{\uparrow\downarrow^\sim}) \tag{9.4.32c}$$

$$\gamma_3 = \gamma_{\uparrow\uparrow^\sim} - \gamma_{\downarrow\downarrow^\sim} \tag{9.4.32d}$$

or

$$\gamma_k = \mathrm{tr}(\sigma_k \gamma), \tag{9.4.33}$$

for $k = 0,1,2,3$.

The matrices $\gamma_k$ (equation (9.4.30)) constitute a representation of the usual Dirac matrices in a *chirally diagonalized* form. (See for instance Pokorski 1987, equation (C.4), Appendix C.)

We note that, as anticipated, the arrow index structures of the expressions on the right hand sides of the equations (9.4.32) do indeed render them formally self-adjoint. The set of vectors in **M** so determined is

$$\theta_0 = \uparrow\otimes\uparrow^\sim + \downarrow\otimes\downarrow^\sim \tag{9.4.34a}$$

$$\theta_1 = \uparrow\otimes\downarrow^\sim + \downarrow\otimes\uparrow^\sim \tag{9.4.34b}$$

$$\theta_2 = i(\downarrow\otimes\uparrow^\sim - \uparrow\otimes\downarrow^\sim) \tag{9.4.34c}$$

$$\theta_3 = \uparrow\otimes\uparrow^\sim - \downarrow\otimes\downarrow^\sim, \tag{9.4.34d}$$

or, with

$$\theta \equiv \begin{pmatrix} \uparrow\otimes\uparrow^\sim & \downarrow\otimes\uparrow^\sim \\ \uparrow\otimes\downarrow^\sim & \downarrow\otimes\downarrow^\sim \end{pmatrix}, \tag{9.4.35}$$

we have

$$\theta_k = \mathrm{tr}(\sigma_k \theta), \tag{9.4.36}$$

and

$$\gamma_k = \gamma_{\theta_k}. \tag{9.4.37}$$

It is immediate that the $\theta_k$ are linearly independent (indeed orthogonal) over $\mathbb{C}$ and hence constitute a basis both for $S \otimes \tilde{S}$ and for the real space $\mathbf{M}$.

We leave as an exercise the calculation of the quantities $\mathrm{tr}(\gamma_k \gamma_l)$ (*via* equation (9.2.24)), whose significance will emerge later.

Note also that in terms of the basis $\{\theta_k\}$ and the lifted representation of $L_+^{\uparrow}$ discussed above, equation (9.2.26) now reads

$$\begin{aligned} \gamma_k &= \gamma_{\theta_k} \\ &= \mathcal{S}(g)^{-1} \gamma_{\Lambda_k^m(g)\theta_m} \mathcal{S}(g) \end{aligned} \tag{9.4.38}$$

or

$$\mathcal{S}(g)\, \gamma_k \mathcal{S}(g)^{-1} = \Lambda_k^m(g)\, \gamma_m, \tag{9.4.39}$$

where $\Lambda_k^m(g)$ denotes the matrix representation of $\Lambda(g) : \mathbf{M} \longrightarrow \mathbf{M}$ relative to the basis $\{\theta_k\}$.

### *9.4.1 General Transformations*

Our discussion has been predicated upon an initial choice by the experimenter of a frame in $S$. This initial choice of frame was *arbitrary*, and so we should allow the experimenter a more general form of transformation that takes this initial freedom of choice into ac-

count. Suppose, for instance, that the experimenter had chosen another initial frame in $S$, as above. Denoting the transformation induced on $S$ by mapping the "old" initial frame to this new one by

$$u : S \longrightarrow S, \tag{9.4.1.1}$$

we note that the corresponding transformation on $\widetilde{S}$ is not arbitrary, being given, according to the construction, by

$$\widetilde{u} : \widetilde{S} \longrightarrow \widetilde{S}, \tag{9.4.1.2}$$

where

$$\widetilde{u}(s^{\sim}) = u(s)^{\sim}. \tag{9.4.1.3}$$

Similarly, we have associated transformations

$$\widetilde{u}^* : \widetilde{S}^* \longrightarrow \widetilde{S}^*, \tag{9.4.1.4}$$

$$u \otimes \widetilde{u} : S \otimes \widetilde{S} \longrightarrow S \otimes \widetilde{S}, \tag{9.4.1.5}$$

and

$$u \oplus \widetilde{u}^* : S \oplus \widetilde{S}^* \longrightarrow S \oplus \widetilde{S}^*. \tag{9.4.1.6}$$

It is an elementary matter (cf. equation (9.4.8)) to show that the inner product on $S \otimes \widetilde{S}$ is real-valued when restricted to $\mathsf{M}$, and it is quickly seen that $u \otimes \widetilde{u}$ preserves $\mathsf{M}$. We denote the transformation it induces by

$$F : \mathsf{M} \longrightarrow \mathsf{M}. \tag{9.4.1.7}$$

The experimenter is free to change the new initial frame by applying another such map, or by subjecting the new frame to an al-

lowable transformation indexed by some element of SL(2,ℂ). However, once such an allowable SL(2,ℂ) transformation has been performed, the resulting associated dual frames are no longer candidates for "initial frame"-hood since, for instance, the natural SL(2,ℂ) action upon $\widetilde{S}^*$ is $L^*_{\widetilde{g^{-1}}}$, which is not $L^*_{\widetilde{g}}$ unless $g$ is the identity: this conflicts in general with the starred version of equation (9.4.1.3). Thus, since the resulting "new" collective frame does not give rise to the form we are allowing for an initial dual frame, only further SL(2,ℂ) transformations should be permitted. Then, on **M**, the most general such transformation takes, the form:

$$E = \Lambda(g_0)F\,, \tag{9.4.1.8}$$

for some $F$ induced as above, and some $g_0$ in SL(2,ℂ). This has the consequence that a new $E$, $E'$ say, must be of the form

$$E' = \Lambda(g)E\,, \tag{9.4.1.9}$$

for some $g$ in SL(2,ℂ), which is the "transformation law" for the matrices $E$. Matrices obeying such a transformation law arise classically and for historical reasons are called *vierbeins.*

The new basis for **M** may be expressed in the form

$$E(\theta_\mu) \equiv {}^E\theta_\mu = e^k_\mu \theta_k, \tag{9.4.1.10}$$

for real numbers $e^k_\mu$. The horrible and non-standard notation appearing in the middle of this equation is intended to effect a compromise with physics usage that, under similar circumstances, usually omits reference to $E$, relying instead on the Greekness of the subscript to identify the transformed entity, thus:

$$\theta_\mu = e^k_\mu \theta_k. \tag{9.4.1.11}$$

This notation, though highly efficient, flirts with disaster when

specific values are assigned to $\mu$ (and also under other circumstances); nevertheless, we shall generally adopt it here. (Please see the remark at the end of this subsection.) Thus it follows from the last equation that

$$\gamma_\mu = \gamma_{\theta_\mu}$$

$$= e^k_\mu \gamma_k, \qquad (9.4.1.12)$$

and, denoting the transposed matrix inverse of $E$ by $e^\mu_k$, we have in addition

$$\Delta^\mu = e^\mu_k \Delta^k. \qquad (9.4.1.13)$$

The "transformation law" for the vierbein $e^k_\mu$ (equation (9.4.1.9)) now takes the form

$$(e^k_\mu)' = \Lambda^k_m(g) e^m_\mu. \qquad (9.4.1.14)$$

In summary, the most general change of initial frame is encoded as a matrix of the form shown in equation (9.4.1.8), that transforms as a vierbein (equation (9.4.1.9)). By allowing such transformations (as $E$) among the experimenter's *own* collection of acts—and please note that pure Lorentz transformations are included as special cases—we are able to resist the temptation to interpret them in terms of the views of "other" putative experimenters or observers, whose existence remains in any case unacknowledgeable.

The following remark may prove helpful to readers unused to the compact index notation often used in physics to express linear transformations. We adopt the usual mathematical conventions in which matrices act on the left upon column vectors, and $M_{jk}$ refers to the element in row $j$ and column $k$ of the matrix $M$. Then

$$e_{\mu}^{k} \equiv E_{k\mu}, \tag{9.4.1.15}$$

$$e_{k}^{\mu} \equiv E_{\mu k}^{-1}, \tag{9.4.1.16}$$

where the right hand side of the last equation denotes the $(\mu, k)$ entry in the matrix $E^{-1}$.

Thus, for instance,

$$\begin{aligned} e_{k}^{\mu} e_{\mu}^{l} &= E_{\mu k}^{-1} E_{l\mu} \\ &= E_{l\mu} E_{\mu k}^{-1} \\ &= \delta_{k}^{l}, \end{aligned} \tag{9.4.1.17}$$

(Kronecker delta) a concatenation which arises when showing that combinations of the form $\partial^{\mu} \otimes dx_{\mu}$ transform as scalars.

## 9.5 The Correspondence Principle

Evidence in favor of a correspondence principle of the sort contemplated in §8.1 has accumulated over the last pair of chapters. For the sake of simplicity (and mathematical safety) we shall frame it in terms of the Maxwell-Boltzmann phase: that is, in terms of the algebra $\mathbb{C}[\Gamma]$.

A macroscopic experimenter, then, is assumed merely to choose an arbitrary pair of orthonormal vectors in $S$ (or $\widetilde{S}$). With a basis for $S \otimes \widetilde{S}$ chosen as in equations (9.4.34), and with operators $q_k$ and $\partial^k$ having their usual connotations in the Schrödinger representation upon $L^2(\mathbb{R}^4)$, we could posit a correspondence principle comprising the two equations (8.1.28) and (8.1.30), with the $\mathsf{N}$ superscripts removed:

$$[\![\Gamma_{k_1}\ldots\Gamma_{k_n}]\!]=\frac{1}{n!}\left(\frac{i}{\tau}\right)^n q_{k_1}\ldots q_{k_n}, \tag{9.5.1}$$

$$[\![\Delta^k]\!]= -i\tau\partial^k. \tag{9.5.2}$$

The first of these equations specifies a linear transformation from the (complex) algebra of formal series in the $\Gamma_k$s to the (complex) algebra of formal series in the $q_k$s, which depends upon the choice of basis $\{\theta_k\}$.

The operators $\Delta^k$ (which act upon the algebra of formal series in the $\Gamma_k$) are defined in equation (9.3.8), and satisfy an equation like (9.3.3).

Having chosen an initial frame, however, a macroscopic experimenter may parametrize the general change of that frame by choosing a vierbein as in equation (9.4.1.8) and defining the corresponding general frame as in equation (9.4.1.11). From the linearity of $[\![\ ]\!]$, and with the choices just made, we then arrive at a more general

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$$[\![\Gamma_{\mu_1}\ldots\Gamma_{\mu_n}]\!]=\frac{1}{n!}\left(\frac{i}{\tau}\right)^n q_{\mu_1}\ldots q_{\mu_n}, \tag{9.5.3}$$

$$[\![\Delta^\mu]\!]= -i\tau\partial^\mu. \tag{9.5.4}$$

(We have briefly discussed the correspondence principle for the defect structure in §9.3, and will return to this topic in subsequent chapters. Note that as far as the correspondence principle itself is concerned, the sole purpose served by the introduction of the Dirac maps, and the entire discussion of the defect structure in preceding sections, was to motivate the choice of the basis $\{\theta_k\}$. Since the defect structure in a sense encodes the response of the net to small changes, it must be deeply implicated in the way in which "dy-

namics" enters the experimenter's lexicon. These considerations will occupy us for most of the remainder of this work.)

As for the real constant $\tau$, we note that the basic spinorial interrogative acts, and the pairs of them that comprise the net, apparently have no inherent dimensional content: such content must come rather from macroscopic *use*, once some kind of dynamics, and standards of comparison, have been developed. That is to say, dimensional attributes are part of the macroscopic continuum experimenter's interpretative burden. Consequently, we shall assume that the right hand sides of the equations appearing in the correspondences given above remain *dimensionless*. Since the $q$ eigenvalues correspond to (continuous) determinations of position, we assume that these operators themselves carry the dimensional attribute of *length*. It then follows that $\tau$ also carries this dimensional attribute.

To interpret $\tau$, we note that the value assigned by a macroscopic experimenter to $|\langle \Gamma_\mu \rangle|$, the modulus of the expectation value of $\Gamma_\mu$ in some macroscopic *state*, is presumably

$$|\langle [\![ \Gamma_\mu ]\!] \rangle| = \frac{|\langle q_\mu \rangle|}{|\tau|} \tag{9.5.5}$$

in that state. So, in a state in which the latter value is unity, a macroscopic experimenter finds

$$|\langle q_\mu \rangle| = |\tau|. \tag{9.5.6}$$

That is, the "unit chronon"—i.e. the value assigned to $|\langle q_\mu \rangle|$ by a macroscopic experimenter in a state in which $|\langle [\![ \Gamma_\mu ]\!] \rangle|$ is unity—is $|\tau|$. Consequently, $|\tau|$ is interpretable as the size ascribed by a macroscopic experimenter to one reticular unit, or chronon. Thus $|\tau|$ coincides with a version of Finkelstein's *net constant* (Finkelstein 1988b–1991). As a consequence, the expression "$|\tau| \to 0$" signifies a passage to the "continuum limit" in the same sense that "$\hbar \to 0$" signifies a passage to the "classical limit" in ordinary quantum me-

chanics. Thus, relative to "macroscopic" scales, we expect $|\tau|$ to be very small indeed.

The vexing issue of what sign a macroscopic experimenter should ascribe to $\tau$ will be taken up in §10.1.

There is a further step to be taken towards the right-most sides of the diagrams (7.2.1) and (8.0.1): namely, the replacement $q_\mu \mapsto x_\mu$, where $x_\mu$ has its usual connotation as the "generic" value of a coordinate on the *classical* manifold, which must then be given a physical interpretation. In ordinary quantum mechanics (and field theory), in which a classical manifold is assumed, an algebraic expression involving the classical coordinates $x_\mu$ is quantized into an operator by replacing each $x_\mu$ by $q_\mu$. The idea is that the classical value of $x_\mu$—the position of a classical particle, say—is approximated by the expectation values of $q_\mu$ in certain *states*, namely those sharply peaked around the value of $x_\mu$. Thus the reverse assignment $q_\mu \mapsto x_\mu$ may be left ambiguous, in which case $x_\mu$ may be regarded as a variable or "indeterminate," or it may be assigned an expectation value of $q_\mu$ depending upon the context it inherits from the latter. In the general (ambiguous) case, in which the context is not specified, so that the $x_\mu$ may remain generic variables or indeterminates, a candidate for the **?** in diagram (8.0.1) becomes available: namely the algebra $\mathbb{C}[x_\mu]$ of complex polynomials in the indeterminates $x_\mu$. This algebra may be considered, if necessary, to be a subalgebra of its associated algebra of formal series. We shall generally denote the classical version of an expression embraced by $[\![\ ]\!]$, in which $q_\mu$ has been replaced by $x_\mu$ or by a particular value thereof, by $[\![\ ]\!]_C$.

The very last step, in which the $x_\mu$ are associated with specific physical arrangements, must await the availability of some physical notions (cf. §10.3.1).

We note finally that, as in §8.1 (cf. equation (8.1.14)), our assumptions already incorporate a choice of macroscopic units in which $\hbar = 1$. In the next chapter we shall develop a dynamics sufficient to justify a choice of units in which $c = 1$ in addition.

(For the theory of spinors in classical relativity upon manifolds, and a lot more besides, the reader may consult the epic Penrose

and Rindler 1984, 1989. For related material see also Ward and Wells 1990, Manin 1988, and for terser but still illuminating treatments see de Felice and Clarke 1990 and Stewart 1990. A definitive account of the mathematical theory appears in Lawson and Michelsohn 1989.)

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# 10

# Dynamics I

The $d$ operator introduced in §9.3 effects changes in the selective acts represented by elements of the algebras $\mathbb{C}[\Gamma^N], \mathbb{C}[\Gamma]$. Acting on monomials—interpreted as the creators of "questionnaires," or lists of interrogative acts, or "history" injectors—the $\Delta$ components of $d$ perform certain *truncations* of the list. The operator $d$ then pairs each such truncation with a corresponding Dirac map. This change to net element creators—which, according to the analysis in the last chapter, is the generic small variation admitted by the structure—is presumably in some sense an infinitesimal precursor to what macroscopic experimenters understand by the term "dynamical change": that which legislates changes to histories is what we, as macroscopic experimenters, interpret as "dynamics," or even "dynamical law." For instance, in classical mechanics the coordinates of a point particle represent a kind of history since these values must be determined (or be determinable) by some tacitly assumed procedure, or sequence of actions. The term "dynamics" in this classical context then pertains to possible changes in these determinations.

We may realize the operator $d$ as an operator upon $\mathbb{C}[\Gamma]\otimes V$ if we supplement each element $\xi$ in $\mathbb{C}[\Gamma]$, say, by pairing it (*via* $\otimes$) with an element of $V$, as in equation (9.3.16)—which, for a general choice of basis as in §9.4.1, reads

$$d(\xi \otimes s) = \Delta^{\mu}\xi \otimes \gamma_{\mu} \otimes s \qquad (10.0.1)$$

—and carrying out the right-most contraction at this level to obtain an operator upon $\mathbb{C}[\Gamma]\otimes V$, namely

$$\begin{aligned} d(\xi\otimes s) &= \Delta^{\mu}\xi\otimes\gamma_{\mu}(s) \\ &= (\Delta^{\mu}\otimes\gamma_{\mu})(\xi\otimes s) \\ &\equiv \not{\Delta}(\xi\otimes s), \end{aligned} \qquad (10.0.2)$$

abusing the notation only slightly by maintaining the same symbol, $d$, on the left hand side. We note that if we were to proceed to the continuum by means of the correspondence rules from this point—namely from the right hand side of the last equation—we would arrive at the same result as before: cf. equation (9.3.17).

As remarked earlier, the bispinor $s$ in a pair of the form $\xi\otimes s$ functions rather like a place-holding act, which is not itself an interrogative-pair creator: rather, it implements a kind of gap or absence of basic pairs. The fundamental reticular operator $\not{\Delta}$ is the operator (or selective act, or deduction in the sense of **GQ**) implementing generic "quantum" variation of net elements, and at the same time performs an act of generic parallel transport of defect injectors through one chrononic unit. Thus, the right hand side of equation (10.0.2) is a superposition over all possible pairs of acts comprising a truncation of the net element and a corresponding Dirac map applied to $s$. This latter Dirac map is an operator version of the cotangent corresponding to—that is, "parallel" to—the atomic piece of net removed by its accompanying $\Delta$. We will argue that, at the reticular level, the application of this operator is somewhat akin to the tick of a clock in that it implements the fundamental generic "expenditure" of one chrononic unit. This operator is in a sense the dynamical "quantum," or generic causal synchronic step, which induces the minimal "dynamical" change in the experimenter's repertoire of reticular acts.

Our interpretation of this operator in terms of infinitesimal

variations has been predicated upon the assumption that the "gaps" of type $V$, which have been invoked in order to *implement* such a variation, do not themselves participate in the variation: see the remark preceding equation (9.3.16). We shall continue to maintain this assumption during the course of this chapter, our conclusions here being accordingly provisional or "first-order." The attempt to include possible variations of the $V$-defect itself would seem to require the invocation of an infinite hierarchy of new "internal" types to implement first the gaps *between* the $s$-gaps engendered by a small variation of the latter, then to implement the gaps between *these* gaps engendered by a small variation of these, which must now also be taken into account, and so on. The result is a "clothing," or renormalization, of the original $V$-defect, in a sense to be made explicit in the next chapter. In these terms the first-order defect we deal with in this chapter is "bare," and we shall sometimes refer to it as such.

In §10.1 we attempt to introduce dynamics into the net by "dynamically activating" each net element creator: that is, each monomial in $\mathbb{C}[\Gamma]$ is paired with an appropriate sequence of selective acts, each implementing, in a sense, the most fundamental generic dynamical change conceivable. In a dangerously reist analogy such concatenations of selective acts correspond with the result of carrying the generic measuring device—a clock, say—along the "path" specified by the monomial. *Actual* paths correspond to reticular Chen series, however, and extending this notion of dynamic activation to such series in an obvious way leads to the activation of the associated transport operator (§8.2). The structure of the amplitude associated with the corresponding macroscopic selective act leads to the identification of a candidate for the macroscopic Lagrangian in this bare defect case.

Unfortunately, this quantity is not well defined, the reason being, ultimately, that there *is* no classical theory of a single defect. Attempts to come to terms with this problem lead, in §10.2, to a second quantized operator form for the Lagrangian we are seeking which reproduces exactly the usual Weyl Lagrangian describing the dynamics of a free neutral massless fermion.

In §10.3 we give a thumbnail sketch of the sequence: Feynman path integral ⟶ Schwinger action principle ⟶ field equations, so as to produce the Weyl equations from the Lagrangian just found. Then, in §10.3.1, we pull these equations back to the net and derive solutions for them there by quantum-logically exploiting the representation of the Dirac matrices derived in the last chapter. These solutions then necessarily give rise, *via* the correspondence principle, to solutions of the usual equations with an added interpretation in terms of transport operators. This subsection also contains a discussion of chirality that will be needed in the sequel.

## 10.1 Dynamic Transport and the Lagrangian

The questionnaire (or history)-*cum*-defect injector $\xi \otimes s$ can undergo a generic small variation by superposing the results of removing certain "questions" (or interrogative acts) from the list $\xi$ while simultaneously changing the appended bispinorial act $s$ through the action of the appropriate Dirac map. These alternative selective acts (namely $\Delta^{\mu}\xi \otimes \gamma_{\mu}$, not summed) each shorten the questionnaire by one interrogative act, or chrononic unit, though the net grows in the direction of *lengthening* questionnaires. If the net were a model of any *thing*, it would be a model of the resource, or medium, used by an ideal measuring device (or set of such devices) whose substance gets used up as the fundamental dynamical act, $\not{\Delta}$, is performed upon it: the effect of such an act—regarded as an act of measurement—upon the resource is, in a sense, complementary to the effect upon the stuff measured. (This *using up* of the "instrument of interrogation" in the course of an "interrogation" is actually implicit in our original axiom: cf. equation (7.1.19) and the discussion following it.) A dangerous but irresistible classical analogy is to the operation of a *clock*. Most of these operate by using up some kind of physical reservoir: potential-, kinetic-, chemical- or radiative energy, for instance. We consider time to move forward as the reservoir decreases, the direction *depleted reservoir* ⟶ *full reservoir* signifying the transition *later* ⟶ *earlier*. The reservoir, or some attribute of it, consti-

tutes a collection of finite "histories" of the process, and corresponds to our collection of net elements. (Tape-measures act similarly: distances may be measured by the depletion of the tape inside the device.) For net elements also, *short* $\longrightarrow$ *long* signifies a transition from *fewer* interrogative acts being available to *more* of them being available, and therefore corresponds to the transition (for a macroscopic experimenter) *later* $\longrightarrow$ *earlier*, relative to the contrary effect of the fundamental dynamical destructive act $\not\Delta$, which transports defects *causally*, or in a "forward" direction.

This view of the net and its operators has the important consequence that a macroscopic experimenter should ascribe a *negative value* to the net constant $\tau$, since the macroscopic sense of causal increase is *opposite* to the direction of reticular growth: macroscopic experimenters should measure *backwards* from *more stuff measured* to *less stuff measured*, or, in the case of time, from *later* to *earlier*.

Another classical analogy to the operation of $\not\Delta$ upon net elements, in the same dangerous vein, is afforded by the device known as an hour-glass: as usual, causal progress, or *increase*, is measured by the *decrease* of the amount of sand in the upper bulb. A history injector $\xi$ corresponds to some sequence of sand grains embedded within the volume of sand in the upper bulb, and $s$ to an interstitial gap between the grains. As the sequence of grains falls through the neck of the device the gap moves with it. Of course, net elements are experimental *acts*, not objects, and the operator $\not\Delta$ prescribes how these acts may be changed: we are still very far from being able to describe a macroscopic physical measuring device such as a clock. Moreover, we do not claim that this analogy holds for macroscopic *interpretations* of $\not\Delta$.

One way of introducing "dynamics" into the net is by attempting to insinuate the fundamental causal selective act, $\not\Delta$ (which may also be regarded as a compound experiment) in such a way that net creators are (causally) "timed." Thus (in the Maxwell–Boltzmann phase) a questionnaire consisting of one "question," $\Gamma_\mu$ say, would now be paired with a single synchronic step implementor, or *stamp*, to yield $\not\Delta \otimes \Gamma_\mu$. Similarly, a questionnaire consisting of the $p$ ques-

tions $\Gamma_{\mu_1} \ldots \Gamma_{\mu_p}$ would acquire a stamp consisting of a sequence of $p$ synchronic step implementors, namely

$$\overbrace{\not\Delta \otimes \cdots \otimes \not\Delta}^{p} \otimes \Gamma_{\mu_1} \ldots \Gamma_{\mu_p} = (\otimes^p \not\Delta) \otimes \Gamma_{\mu_1} \ldots \Gamma_{\mu_p} . \qquad (10.1.1)$$

If $\not\Delta$ is analogous to the operation of a generic measuring device (such as a clock) marking off one unit of "time"—this marking off being implemented by the shift through one chrononic unit of a defect injector—then the expression (10.1.1) is similarly analogous to the operation of a sequence of independent clocks—one for each $\Gamma_\mu$—paired with the monomial $\Gamma_{\mu_1} \ldots \Gamma_{\mu_p}$ of net element creators. Such a sequence may also be thought of as analogous to the operation of a single clock (cf. §1.2), each operation, or tick, being independent of its neighbors. In this sense the associated pathlike monomial is "timed." We remark here that this presumed mutual independence of the members of the sequence $\otimes^p \not\Delta$ carries the implication that the members of the sequence do not interface as experimental acts (cf. §1.3) so that possible transitions occur only within each compound act $\not\Delta$ and not between adjacent members of the sequence. This in turn implies that the contractions invoked in coarsening the resolution of such acts for use by external experimenters should proceed term-wise as in equation (7.2.1.5) without first composing the operators.

To explore the consequences of this dynamical *activation* of net elements, let us consider its effect on the (pulled-back Maxwell–Boltzmann phase) Chen series for transport along a curve, $C$, in $\mathbb{R}^4$, which may be written (cf. equation (8.2.26))

$$1 + \sum_{p=1}^{\infty} \Sigma \left( \int_C dx_{k_1} \ldots \int_C dx_{k_p} \right) \Gamma_{k_1} \ldots \Gamma_{k_p} . \qquad (10.1.2)$$

Dynamically "activating" such a formal series according to equation (10.1.1) amounts to replacing the assignments

$$\Gamma_{k_1}\cdots\Gamma_{k_p}\mapsto\int_C dx_{k_1}\cdots\int_C dx_{k_p} \tag{10.1.3}$$

with those of the form

$$\Gamma_{k_1}\cdots\Gamma_{k_p}\mapsto\int_C \not{\Delta}\, dx_{k_1}\otimes\cdots\otimes\int_C \not{\Delta}\, dx_{k_p} \tag{10.1.4}$$

in which each corresponding infinitesimal step along the curve is "timed" by the presence of a $\not{\Delta}$ and the resulting contributions are sequenced by tensoring them together.

For the case in which $C$ is a line segment from the origin to $(\alpha_0,\ldots,\alpha_3)$ the activated series is (cf. equation (8.2.27))

$$P(C)\equiv 1+\not{\Delta}\otimes\sum\alpha_{\mu_1}\Gamma_{\mu_1}+\not{\Delta}\otimes\not{\Delta}\otimes\sum\alpha_{\mu_1}\alpha_{\mu_2}\Gamma_{\mu_1}\Gamma_{\mu_2}+\cdots. \tag{10.1.5}$$

This series, we recall (§8.2), represents an operation of transport in the sense in which the net *grows*. As noted, this sense is *opposite* to the corresponding macroscopic ones, the direction *short net element* ⟶ *long net element* corresponding, for instance, to the macroscopic sense *later* ⟶ *earlier*. This mismatch has been compensated for by taking $\tau$ to be negative.

The right hand side of the last equation is the superposition over all possible outputs of the program obtained from the program appearing after equation (8.2.30) by replacing the expression ". . . *finite-step* . . ." by the expression ". . . *'timed' finite-step* . . ." A suggestive analogy for the act represented by equation (10.1.5) is that it embodies instructions for effecting the transport of the generic measuring device—or clock, or chronon-expender—along $C$. (Perhaps a better analogy, though again dangerous, would be to the operation of a measuring wheel: this is a monocycle-like device that measures distances over bumpy terrain, or maps thereof, by keeping a count of

the revolutions of the wheel as it is rolled along a curve.)

Now we try to determine the amplitude of the macroscopic continuum correspondent (or interpretation) of the selective act $P(C)$. A first step in this direction would be to apply the correspondence principle (§9.5) to obtain the operator

$$[\![P(C)]\!]_C = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{\tau}\right)^n \left(\otimes^n [\![\not{A}]\!]_C\right)\left(\Sigma\alpha_\mu \langle q_\mu \rangle\right)^n . \quad (10.1.6)$$

Then we should "fully contract" $[\![P(C)]\!]_C$ keeping in mind that the presumed mutual independence of the activators entails taking traces as in equation (7.2.1.5):

$$\mathrm{tr}[\![P(C)]\!]_C = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{i}{\tau}\right)^n \mathrm{tr}[\![\not{A}]\!]_C^n \left(\Sigma\alpha_\mu \langle q_\mu \rangle\right)^n$$

$$= \exp\left[\left(\frac{i}{\tau}\right)\mathrm{tr}[\![\not{A}]\!]_C \left(\Sigma\alpha_\mu \langle q_\mu \rangle\right)\right]. \quad (10.1.7)$$

Here we encounter rather profound difficulties, which expose the essential crudity of our Schrödinger operator based correspondence principle: namely, the quantity on the right hand side of the last equation is not well-defined since, firstly, the trace is to be taken over an infinite dimensional space, and, secondly, the values of the $\langle q_\mu \rangle$ have not been specified.

Temporarily setting aside the first problem, let us address the second. In the context of the Schrödinger operator precursor to equation (10.1.7) the operators $q_\mu$ appearing in the expression $\Sigma\alpha_\mu q_\mu$ play the rôle of infinitesimal generators for the vectorial component of the transport operator that the whole expression represents. They may be associated with a unit basis for the abelian Lie algebra $\mathbb{R}^4$ in the appropriate units. That is to say, we shall interpret $\langle q_\mu \rangle$ as the expectation value in a state in which the "microscopic" quantity $|\langle \Gamma_\mu \rangle|$ is unity. The absolute value of this quantity $\langle q_\mu \rangle$ is determined in §9.5 (equations (9.5.5) and (9.5.6)), leaving us with an ambiguity in

its sign. To resolve it, we note that the macroscopic directional mismatch has already been taken into account in the choice of sign for $\tau$ so that we may select the $q_\mu$ to reference the usual "forward" pointing basis, and therefore to be positive:

$$\langle q_\mu \rangle = |\tau| = -\tau . \tag{10.1.8}$$

Now we specialize $C$ so that it lies along one axis, the 0-axis, say, and is as short as possible in the "chrononic" units just chosen for the basis, namely one: that is, $C$ is the line segment from the origin to $(1,0,0,0)$. With these choices, equation (10.1.7) becomes

$$\mathrm{tr}[\![ P(C) ]\!]_C = \exp\left(-i\,\mathrm{tr}[\![ \not{A} ]\!]_C\right) . \tag{10.1.9}$$

Now the original operator $P(C)$ (equation (10.1.5)) acts on sets of defect injectors paired with reticular elements and effects the joint action of parallel transportation of vectorlike elements—this action coming from the sequences of $\Gamma_\mu$s *à la* Chen—while causally "timing" these infinitesimal transports *via* the accompanying selective acts of the form $\otimes^p \not{A}$. Macroscopically, then, the effect of $[\![ P(C) ]\!]$, for the choices just made, is to causally transport through one chrononic unit in the $x_0$ direction, *states* of the form $|\psi(\sigma_t)\rangle$, where $\psi$ labels sets (or *fields*) of defect injectors, and, for $0 \le t \le 1$, $\sigma_t \equiv \sigma(x_0 = t|\tau|)$ denotes a *3-surface* (corresponding to $x_0 = t|\tau|$) upon which $\psi$ is supposedly defined.

Thus we may put

$$[\![ P(C) ]\!]_C \propto |\psi(\sigma_1)\rangle \otimes \langle \psi(\sigma_0)|, \tag{10.1.10}$$

the right hand side denoting the operator sending $|\psi(\sigma_0)\rangle$ $|\psi(\sigma_1)\rangle$. Thus, taking traces,

$$\langle \psi(\sigma_0) | \psi(\sigma_1) \rangle \propto \exp\left(-i\,\mathrm{tr}[\![ \not{A} ]\!]_C\right) , \tag{10.1.11}$$

or

$$\langle \psi(\sigma_1) | \psi(\sigma_0) \rangle \propto \exp\left( i \operatorname{tr} [\![ \not{\Delta} ]\!]_C^\dagger \right) . \tag{10.1.12}$$

This may be compared with the famous Dirac-Feynman expression for an exactly similar amplitude, namely

$$\langle \psi(\sigma_{t+\Delta t}) | \psi(\sigma_t) \rangle \propto \exp(i L_W \Delta t), \tag{10.1.13}$$

where $L_W$ denotes an appropriate function of the fields, etc., known for historical reasons as the *Lagrangian*. (This relation is the cornerstone of the edifice known as the Feynman Path Integral. For brief treatments see, among a vast number of others, Ryder 1999, Ramond 1989, Pokorski 1987 and Mandelstam and Yourgrau 1979. For extended treatments the reader may consult Weinberg 1995, Roepstorff 1994, Swanson 1992, Schulman 1981 and Feynman and Hibbs 1965. A very brief sketch also appears below in §10.3. For a very interesting non-standard treatment of path integrals involving spinors, quite close in spirit to our efforts here, see Jacobson 1984.)

It is a useful convention at this point to introduce in place of $L_W$, a *localized density* $\mathcal{L}_W$, such that

$$L_W \Delta t = \int_{\Delta V} \mathcal{L}_W \, dv \ , \tag{10.1.14}$$

where $dv$ denotes an appropriate volume form and $\Delta V$ is an "element" of 4-volume. To choose $dv$ we note that the integration in the last equation is to be carried out in the macroscopic domain, whose coordinate frame has been chosen *via* the correspondence principle. That is, the volume form should be expressed in Greek coordinates (backsliding notationally by temporarily using the old exterior product wedge):

$$dv = \bigwedge_\mu dx_\mu$$

$$\equiv d^4x_\mu . \tag{10.1.15}$$

Then, since

$$dx_\mu = e_\mu^k dx_k , \tag{10.1.16}$$

we have

$$d^4x_\mu = (\det E)\, d^4x_k . \tag{10.1.17}$$

So, with $\Delta V$ sufficiently small (in $d^4x_\mu$-measure),

$$\begin{aligned}\int_{\Delta V} \mathcal{L}_W dv &= \int_{\Delta V} \mathcal{L}_W \, d^4x_\mu \\ &= \mathcal{L}_W \int_{\Delta V} d^4x_\mu \\ &= \mathcal{L}_W \int_{\Delta V} (\det E) d^4x_k \\ &= \mathcal{L}_W (\det E) \int_{\Delta V} d^4x_k . \end{aligned} \tag{10.1.18}$$

Now the smallest $d^4x_k$-volume an experimenter can ascribe to $\Delta V$ is presumably $\tau^4$: but, putting

$$\int_{\Delta V} d^4x_k = \tau^4 \tag{10.1.19}$$

in the last line of equation (10.1.18) would be consistent with the extraction of $\mathcal{L}_W$ from under the integral further up only if $(\det E)\tau^4$

were still sufficiently small for the classical quantity $\mathscr{L}_W$ not to vary appreciably within a region of that size. Since we expect $\tau$ to be very small indeed compared with classical length scales, the adoption of equation (10.1.19) would seem to place a restriction upon the choice of vierbein $E$: namely, its determinant should not be too large compared with $\tau^{-4}$ at macroscopic scales. This restriction is an artifact and measure of the coarseness of our method, and signals a circumstance under which our conclusions might be expected to be unreliable, for what we are then interpreting as $\mathscr{L}_W$ will only be an average. It is a not unexpected penalty for fraternizing too closely with the classical continuum structure that must break down under extreme circumstances. (The attempt to physically resolve distances below $|\tau|$ would presumably require so much energy that some disruption, such as the formation of black holes or singularities, would destroy the manifold structure.) However, since we are in fact aiming for a macroscopic description, we shall adopt equation (10.1.19) here, obtaining from equations (10.1.14) and (10.1.18):

$$L_W \Delta t = \mathscr{L}_W (\det E) \tau^4 \,. \tag{10.1.20}$$

Now, comparing equations (10.1.12) and (10.1.13), we may identify the corresponding terms to obtain

$$\mathscr{L}_W = \tau^{-4} (\det E)^{-1} \mathrm{tr} [\![ \mathbb{A} ]\!]_C^\dagger \,. \tag{10.1.21}$$

Thus, up to a measure-related factor, $\mathrm{tr}[\![ \mathbb{A} ]\!]_C^\dagger$ appears to function as a localized Lagrangian density describing the dynamic behavior—or "motion," perhaps—of a defect, as seen by a (maximally external) macroscopic observer in the continuum.

In arriving at this correspondence, we have explicitly assumed that what we have been calling $x_0$ should be identified with the macroscopic experimenter's (local) *time*. This is not easy to justify on intrinsic grounds except to point out that, at least in Latin coordinates, this direction corresponds to the identity map in the system of

Pauli matrices (§9.4): it is the direction *not* associated with any spin matrix.

Implicit in our identification of a Lagrangian density (equation (10.1.21)) is the adoption of units in which $c=1$, which we now append to our earlier choice, in which $\hbar=1$, leaving $\tau$ as our remaining dimensional parameter.

We turn now to the problem of interpreting the right hand side of this equation.

## 10.2 Problems with the Dirac Operator

The emergence of $\not{\Delta}$ as our fundamental dynamical operator somewhat vitiates the simplicity of our interpretation of selective acts as "compound experiments" made by superposing sequences of initial and final acts represented by vectors in finite dimensional spaces. As a vector space $\mathbb{C}[\Gamma]$ is infinite dimensional so that both the interpretation of contraction *via* the trace, etc., and the contracted value in this case, are called into question. (The macroscopic continuum version of $\Delta^\mu$, namely $-i\tau\partial^\mu$, is in fact defined only on a dense subspace of $L^2(\mathbb{R}^4)$ and is unbounded there, so these remarks apply to it *a fortiori*.)

A way out of this quandary may be found if the differentiation operators could somehow be lifted, or deflected, so as to operate in a finite dimensional space. Thus, in an expression of the form $\xi \otimes s$, if $\xi$ were to be regarded as a *variable index*, then the expression may be regarded as a bispinor-valued *function* of the index $\xi$. A formal algebraic way of accomplishing this is by regarding the domain of $\not{\Delta}$, namely

$$V_\Gamma \equiv \mathbb{C}[\Gamma]\otimes V\,, \tag{10.2.1}$$

as a $\mathbb{C}[\Gamma]$-*module*, the module action of $\mathbb{C}[\Gamma]$ upon $V_\Gamma$ being given simply by multiplication of the left-most factor. This gives $V_\Gamma(\cong \oplus^4\mathbb{C}[\Gamma])$ the structure of a free $\mathbb{C}[\Gamma]$-module of rank four. Given that the classical continuum interpretation of $\mathbb{C}[\Gamma]$ is $\mathbb{C}[x_\mu]$

(§9.5), these considerations yield, as the continuum version of $V_\Gamma$, the free $\mathbb{C}[x_\mu]$-module $\mathbb{C}[x_\mu]\otimes V$. Now this module may be (trivially) realized as the module of sections of the trivial algebraic bundle with fibre $V$ over $\mathbb{C}^4(\cong \mathbb{R}^4\otimes_{\mathbb{R}}\mathbb{C}\cong \mathbf{M}\otimes_{\mathbb{R}}\mathbb{C})$ regarded as an affine algebraic variety. (The points of this variety correspond with maximal ideals of $\mathbb{C}[x_\mu]$ and the fibre of the bundle at a point is the direct sum of four copies of the quotient of $\mathbb{C}[x_\mu]$ by the corresponding maximal ideal. This quotient field is always isomorphic with $\mathbb{C}$. We remark in passing that this bundle structure may be intuited at the reticular level: think of $V$ defects inserted between every chrononic pair. As we coarsen the resolution the pairs condense into what we see as the spacetime manifold while the defects remain attached fibre-like at each "point.")

From this perspective it seems appropriate to define the operator $[\![\not{\Delta}]\!]_C$ as a *bundle* endomorphism—that is, one that acts fibre-wise—of the trivial bundle (over the "local model" $\mathbf{M}$) whose fibre is $V$. Then the trace of such an endomorphism may in principle be defined fibre-wise—that is, for the fibre over each point—and gives rise to a numerical *function* defined on the manifold. But such classical functions should arise as the expectation values of quantum operators, so the upshot of these considerations is that we may regard $V$ as being "indexed" by $\mathbb{C}[\Gamma]$ if we then realize $\mathrm{tr}[\![\not{\Delta}]\!]_C$ as an *operator.*

To be specific, we have

$$\begin{aligned}[\![\not{\Delta}]\!] &= [\![\Delta^\mu]\!]\otimes\gamma_\mu \\ &= -i\tau\partial^\mu\otimes\gamma_\mu\ , \end{aligned} \tag{10.2.2}$$

and, in view of the discussion above, we should put

$$\begin{aligned}[\![\not{\Delta}]\!]_C &= -i\tau\partial^\mu\gamma_\mu \\ &= -i\tau\not{\partial}. \end{aligned} \tag{10.2.3}$$

That is, $[\![\not{\Delta}]\!]_C$ is interpreted as the *usual* Dirac operator regarded locally as an operator upon sections of the (bi)spinor bundle up to the factor $-i\tau$.

In this case the classical function involved when the trace is taken seems to be identically zero, since each Dirac map has vanishing trace. This we interpret as an indication that the transitions pertaining to the dynamic activity of single defects are *invisible* to classical observers: there *is* no classical, or maximally external theory of single reticular defects. In fact, as we have noted, the bundle structure entails a multiplexing of the single space $V$, over a manifold, and concomitant replacements: constant $\longrightarrow$ function, function $\longrightarrow$ operator, as above. These in turn entail the replacement "wavefunction" $\longrightarrow$ operator which characterizes the transition from first to "second" quantization, and the appearance of a *field* theory (cf. Selesnick 1983).

Despite the non-existence of a classical theory of single defects we shall find, rather remarkably, that a natural second quantization of the bispinor space $V$ undoes the twist in the Dirac maps, and yields a sensible replacement for the trace we are seeking. To accomplish this we note first that for a general operator $P$ acting on some $n$-dimensional Hilbert space $\mathfrak{H}$,

$$\begin{aligned} \mathrm{tr}P &= \sum_{k=1}^{n} \langle \zeta_k | P | \zeta_k \rangle \\ &= \sum_{k=1}^{n} \zeta_k^{\dagger} P \zeta_k, \end{aligned} \tag{10.2.4}$$

where the $k$ labels an orthonormal basis. Upon "second quantization" each basis element $\zeta_k$ is replaced by an operator

$$\psi(\zeta_k) \equiv \psi_{(k)} \tag{10.2.5}$$

so that the right hand side of equation (10.2.5) is replaced by the op-

erator

$$\sum_{k=1}^{n} \psi_{(k)}^{\dagger} P \psi_{(k)}. \tag{10.2.6}$$

The operator $P$ is supposed to act in imitation of its action upon $\mathfrak{H}$: namely $P\psi_{(k)}$ is the same linear combination of the $\psi$s that $P\zeta_k$ is of the $\zeta$s. If $\psi$ is linear, this is equivalent to

$$P\psi(\zeta) \equiv \psi(P\zeta). \tag{10.2.7}$$

If we regard the operators $\psi_{(k)}$ as annihilation operators (for quanta, or "particles") in the standard sense, then an expression of the form

$$\psi_{(k)}^{\dagger} P \psi_{(k)} \tag{10.2.8}$$

parses as the sequence of operations:

*annihilate a k-particle*;
*apply P to the result*;
*create a k-particle*

In the action vector language, which refers to experimental arrangements, this assumes the form:

*remove a k-particle injector*;
*perform the selective act P*;
*insert a k-particle injector*

(This second sequence of operations, when interpreted on the net with $P = \not{\Delta}$, would yield something like

*remove a defect injector*;
*transport the result (which now has a hole in it)*
*through one generic step*;
*insert a defect injector*

These are instructions for moving a defect injector one step through the net.)

This observation is rather suggestive when we turn to the problem at hand, namely, to find an operator interpretation of $\mathrm{tr}\not{\partial}$. Since we have been led, *via* second quantization, to "particle" interpretations, we are required to examine $V$ with a view to determining its particle content. This defect space emerged as an $\oplus$-amalgam of the two types $S$ and $\widetilde{S}^*$, a structure that is preserved under allowed transformations. Consequently, it would seem that we have two defect subtypes, corresponding to the two summands, and defects are amalgams of these two unpaired spinor types: thus we should decide on independent second quantizations for $S$ and $\widetilde{S}^*$. Moreover, since these components originate within the deepest level of the net, they should maintain their pristine Fermi–Dirac character. This means that the elements of $S$, say, should be represented as operators obeying the CAR (canonical anticommutation relations), and, since such representations are essentially unique, it is unnecessary to specify a choice. Similarly, the elements of $\widetilde{S}^*$ should also be represented in this way.

These choices now entail a conflict with our plans to use the Dirac operator as in equation (10.2.6). Namely, since the $\sigma$-matrices (equations (9.2.8) and (9.2.13)) will map operators belonging to one of these representations of the CAR to operators belonging to the other, concatenations of the form (10.2.8) will not be defined since they will involve operators belonging to different representations.

There is a rather natural way out of this dilemma, apparently available only upon second quantization. Let us denote by $\Psi_1$ the representation of the CAR chosen for $S$. That is to say, $\Psi_1$ is a faithful map of the space $S$ into an algebra of operators which satisfies the CAR (and certain other requirements). Now, usually, the *linear* map

that implements this representation sends elements of $S$ to *creation* operators for "$S$-quanta." Our interpretation of $S$ in terms of defects, however, compels us to associate the *creation* of $S$-quanta with the *annihilation* of "physical" particle injectors, and *vice versa.* Therefore we shall regard $\Psi_1$ as being *linear*, the operators $\Psi_1(s)$ representing *creators* of defect injectors, and interpreted therefore as *annihilators* of *particle* injectors. Similarly, we denote the representation of $\tilde{S}^*$ by $\Psi_2$.

Now whichever representations of the CAR are chosen (respectively) for the spaces $S$, $\tilde{S}^*$, it is possible to associate the adjoints of operators in either one of them with operators in the other. Thus, for any $s$ in $S$ we may define

$$\Phi(\Psi_1^\dagger(s)) = \Psi_2(s^{\sim *}), \tag{10.2.9}$$

where the basis choice already made in $\tilde{S}$ may be used to determine the dual element. A similar definition applies with the subscripts interchanged. Both of these maps are immediately seen to be linear ($\Psi_1^\dagger$ and $\Psi_2^\dagger$ being conjugate linear) and reversible, with

$$\Phi^{-1}(\Psi_2(t)) = \Psi_1^\dagger(t^{*\sim}). \tag{10.2.10}$$

(It is an elementary exercise to check that they are also compatible with the appropriate $SL(2,\mathbb{C})$ transformations.)

Any offending expressions arising from the "Dirac-like" mismatch described above may now be replaced by well-defined ones, if we replace adjoint operators by their correspondents under $\Phi$. To effect this, we drop the references to the variables $s \in S$, $t \in \tilde{S}^*$ and write a general element in the space $\Psi_1(S) \oplus \Psi_2(\tilde{S}^*)$, the operator version of $V$, as

$$\Psi \equiv \psi_1 \oplus \psi_2. \tag{10.2.11}$$

This, then, is the second quantized version of the bare defect injec-

tor.

In view of the discussion above, $\psi_1^\dagger$ may be interpreted as an element of the *second* component, and should be so interpreted when used in conjunction with the Dirac maps in the manner contemplated. Similarly, $\psi_2^\dagger$ may be interpreted as an element of the *first* component, and should be so interpreted under similar circumstances. Consequently, this pair of replacement (or identification) operations may be accomplished simply by defining a new adjoint operation on pairs of the form $\Psi$ for use in this context (the $\Phi$ identifications being understood), namely:

$$\overline{\Psi} \equiv \psi_2^\dagger \oplus \psi_1^\dagger . \tag{10.2.12}$$

This operation—namely, ordinary adjoint plus flip—is known in the literature variously as the *Pauli* or *Dirac adjoint.* (Needless to say, this is not exactly the way it is usually introduced.)

Returning now to equation (10.2.3) and using the Pauli–Dirac adjoint for the defect type in the form analogous to equation (10.2.8), we obtain an operator interpretation for $\mathrm{tr}[\![\not{A}]\!]_C$ in the form

$$-i\tau \overline{\Psi}_C \not{\partial} \Psi_C \ , \tag{10.2.13}$$

where $\Psi_C$ denotes the "classical interpretation" of the *field* $\Psi$, which merely supplies a dimensional factor to be chosen so that the density acquires the correct dimension, namely that of inverse four-volume. Thus, a candidate for the Lagrangian density we are seeking emerges from equation (10.1.21):

$$\begin{aligned} \mathscr{L}_W &= \tau^{-4}(\det E)^{-1}(-i\tau \overline{\Psi}_C \not{\partial} \Psi_C)^\dagger \\ &= \tau^{-3}(\det E)^{-1}(-i \overline{\Psi}_C \not{\partial} \Psi_C)^\dagger . \end{aligned} \tag{10.2.14}$$

Recalling that $\tau$ is negative, and writing

$$\Psi_C \equiv (-\tau)^{3/2}\Psi, \tag{10.2.15}$$

this assumes the form

$$\mathcal{L}_W = (i\overline{\Psi}\not{\partial}\Psi)^{\dagger}(\det E)^{-1}. \tag{10.2.16}$$

Noting, in an obvious and fairly standard notation, that $\gamma_\mu$ may be written in the form (equation (9.4.30)):

$$\gamma_\mu = t \circ (\overline{\sigma}_\mu \oplus \sigma_\mu) \tag{10.2.17}$$

and that $i\partial^\mu$ and $\sigma_\mu$ are Hermitian, we obtain

$$\begin{aligned}(\det E)\mathcal{L}_W &= \left[\overline{i(\psi_1 \oplus \psi_2)}(t \circ (\overline{\sigma}_\mu \oplus \sigma_\mu))(\partial^\mu\psi_1 \oplus \partial^\mu\psi_2)\right]^{\dagger} \\ &= \left[i(\psi_2^{\dagger} \oplus \psi_1^{\dagger})(\sigma_\mu\partial^\mu\psi_2 \oplus \overline{\sigma}_\mu\partial^\mu\psi_1)\right]^{\dagger} \\ &= \left[i(\psi_2^{\dagger}\sigma_\mu\partial^\mu\psi_2 \oplus \psi_1^{\dagger}\overline{\sigma}_\mu\partial^\mu\psi_1)\right]^{\dagger} \\ &= i(\psi_2^{\dagger}\sigma_\mu\partial^\mu\psi_2 \oplus \psi_1^{\dagger}\overline{\sigma}_\mu\partial^\mu\psi_1) \\ &= i\overline{\Psi}\not{\partial}\Psi. \end{aligned} \tag{10.2.18}$$

(Note that, with the choices made above, the components of the right hand side of the last equation parse as:

*annihilate a fermion:* $\psi_2$ *for example;*
*causally transport the result infinitesimally:* $\sigma_\mu\partial^\mu$,
*create an identical fermion:* $\psi_2^{\dagger}$.

Thus is the infinitesimal transport of free fermions effected in the continuum.)

In standard treatments $\Psi$ etc. are invariably written in vector form with

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0 \end{pmatrix}. \tag{10.2.19}$$

Then equation (10.2.18) comes out with $+$ in place of $\oplus$, a natural result of the use of vector notation, as exemplified by the following sequence of identifications:

$$P \oplus Q \leftrightarrow \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & Q \end{pmatrix} \leftrightarrow P + Q \tag{10.2.20}$$

This vector notation, whose blandishments are apparently virtually impossible to resist, tends to obscure the provenance (domains, ranges, etc.)—that is to say, the *types*—of the various maps involved.

$\mathcal{L}_W$ (equation (10.2.18)) is precisely the *Weyl* Lagrangian which ordinarily describes the dynamics of neutral massless fermions (cf. §10.3). It is incomplete in our formulation since we have not properly clothed or renormalized the bare defect. The clothed defect admits further symmetry whose presence will generate additional terms, as we shall see in the next chapter.

The Lagrangian $\mathcal{L}_W$ has an important property that can best be seen in our context by explicitly writing the vierbein—tacitly assumed in our use of the correspondence principle—in the form:

$$E = \Lambda(g)F, \tag{10.2.21}$$

where $g$ is some element in $\mathrm{SL}(2,\mathbb{C})$ and $F$ is as in equation (9.4.1.7): recall that underlying $F$ are the maps $u$, $\tilde{u}$, etc. In particular, the initial-frame changing map $u \oplus \tilde{u}^*$ acting on $V$ should now be assumed to have been applied. These maps have the effect merely of

relabelling the spinor variables underlying the CAR representations employed in each component of $V$, in a manner consistent with the definition of the $\Phi$ maps. Consequently we may ignore the action of $F$ upon the spinor fields $\Psi$.

The map on $V$ associated with the $g$ in equation (10.2.21), namely $\mathcal{S}(g)$ (equation (9.2.25)), is not ignorable however, since, as noted in §9.4.1, it does not qualify as an initial frame changer and consequently does not merely relabel the spinor variables underlying $\Psi$. Thus, making explicit the reference to $E$, and ignoring the effect of $F$ upon $\Psi$, we obtain

$$\Psi_E = \mathcal{S}(g)\Psi, \tag{10.2.22}$$

so, succumbing to vector notation,

$$\begin{aligned}\overline{\Psi}_E &= \overline{\mathcal{S}(g)\Psi} \\ &= \overline{\begin{pmatrix} L_g & 0 \\ 0 & L_{g^{-1}}^{\dagger} \end{pmatrix}\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}} \\ &= (\psi_2^{\dagger} L_{g^{-1}}, \psi_1^{\dagger} L_g^{\dagger}) \\ &= \overline{\Psi}\,\mathcal{S}(g^{-1}).\end{aligned} \tag{10.2.23}$$

Noting that since $\not{\partial}$ transforms as a scalar we may put

$$\begin{aligned}\not{\partial}^E &= \gamma_k \partial^k \\ &= \not{\partial}^I,\end{aligned} \tag{10.2.24}$$

where $I$ denotes the identity map or vierbein, obtaining

$$
\begin{aligned}
(\det E)\,\mathcal{L}_W^E &\equiv i\,\overline{\Psi}_E \not{\partial}^E \Psi_E \\
&= i\,\overline{\Psi}\mathcal{S}(g^{-1})\gamma_k \partial^k \mathcal{S}(g)\Psi \\
&= i\,\overline{\Psi}\,\mathcal{S}(g^{-1})\gamma_k \mathcal{S}(g)\,\partial^k \Psi \\
&= i\,\overline{\Psi}\,\Lambda_k^m(g^{-1})\gamma_m \partial^k \Psi \\
&\qquad\qquad \text{from equation (9.4.39)} \\
&= i\,\overline{\Psi}\gamma_m \Lambda_k^m(g^{-1})\,\partial^k \Psi \\
&= i\,\overline{\Psi}\gamma_m \partial^m \Psi \\
&= i\,\overline{\Psi}\not{\partial}^I \Psi \\
&= \mathcal{L}_W^I, \qquad (10.2.25)
\end{aligned}
$$

or

$$
\mathcal{L}_W^E = \mathcal{L}_W^I (\det E)^{-1}. \qquad (10.2.26)
$$

Thus, a macroscopic experimenter would have obtained the same Lagrangian, up to a factor of $(\det E)^{-1}$, if she or he had used the original choice of frame, an observation which will be exploited in the next section.

## 10.3 Feynman Path Integral and Field Equations

Let us write equation (10.1.13) for a general $L$ in the form

$$\langle \psi(\sigma_{t+\Delta t}) | \psi(\sigma_t) \rangle = \alpha(\Delta t) \exp(i L(t) \Delta t), \qquad (10.3.1)$$

where $L$ is some numerical, or "c-number" valued function of the "fields" $\psi$, and $\alpha(\Delta t)$ is some number depending upon $\Delta t$. If the field configurations $\psi(\sigma_t)$ may be chosen to be sufficiently *characteristic*, in the sense that eigenvectors are, the last equation may be expressed alternatively in the form

$$\mathrm{tr} R(t + \Delta t, t) = \alpha(\Delta t) \exp(i L \Delta t), \qquad (10.3.2)$$

where $R(t,s)$ is the selective act

$$R(t,s) = |\psi(\sigma_s)\rangle \otimes \langle \psi(\sigma_t)| . \qquad (10.3.3)$$

For each "history" of $\psi$ starting at $t = t_1$ and ending at $t = t_2$ —which consists of a finite sequence $\psi(\sigma_{s_k})$, $k = 1, \ldots, n$, with $s_1 = t_1$, $s_n = t_2$, of configurations of the field $\psi$ on the surface $\sigma_{s_k}$ —we associate the sequence of independent selective acts

$$R(t_2, s_{n-1}) \otimes R(s_{n-1}, s_{n-2}) \otimes \ldots \otimes R(s_2, t_1), \qquad (10.3.4)$$

which effects a sequence of transformations of field configurations along the history. Since the constituent selective acts are assumed to be mutually independent, the "fully contracted" amplitude associated with the act depicted in (10.3.4) is

$$\mathrm{tr} R(t_2, s_{n-1}) \ldots \mathrm{tr} R(s_2, t_1) = \alpha(\Delta s_1) \ldots \alpha(\Delta s_{n-1}) \exp(i \sum L(s_k) \Delta s_k) . \qquad (10.3.5)$$

Now if it were permissible to extend this notion of a history to a more general one, in which arbitrarily fine subdivisions were allowed, then, for one such history, the left-hand side of the last equation would retain its interpretation as the amplitude for a certain "flow" of the fields $\psi$ along the history, while the right-hand side might optimistically be expressed in the form

$$\exp(i\int_{t_1}^{t_2} L\,dt)\,\mathcal{D}\psi \tag{10.3.6}$$

where the $\alpha$s have been collected into a factor $\mathcal{D}\psi$ which depends on the history. (The history involved is a history of the fields hence their appearance in this notation.)

The selective act whose amplitude is $\langle\psi(\sigma_{t_2})|\psi(\sigma_{t_1})\rangle$ is obtained by superposing over all the selective acts analogous to the one shown in equation (10.3.4) but now pertaining to arbitrary histories, or *paths*. Thus the amplitude for this transition is the sum, over all histories, of expressions of the form (10.3.6), and is usually expressed in the symbolic form

$$\langle\psi(\sigma_{t_2})|\psi(\sigma_{t_1})\rangle = \int e^{i\int_{t_1}^{t_2} L\,dt}\,\mathcal{D}\psi\,, \tag{10.3.7}$$

where normalization constants have been absorbed into the history, or path, "measure" $\mathcal{D}\psi$.

Expressions of this form, with all their attendant problems of definition, are known as Feynman Path Integrals. (References already cited.)

Let us apply a general variation, $\delta$, to it, keeping the terminal surfaces fixed:

$$\delta\langle\psi(\sigma_{t_2})|\psi(\sigma_{t_1})\rangle = i\int \delta(\int_{t_1}^{t_2} L\,dt)\,e^{i\int_{t_1}^{t_2} L\,dt}\mathcal{D}\psi\,. \tag{10.3.8}$$

(Here we assume also that the functional form of $L$ remains fixed through this variation.) If we assume that the field configurations on the terminal surfaces remain fixed during this variation, then the left hand side of the last equation vanishes. It can be shown, by making sufficiently characteristic (or eigen-) choices for the $|\psi(\sigma_t)\rangle$, that the vanishing of the right hand side of this equation for variations of this type entails

$$\delta(\int_{t_1}^{t_2} L\, dt) = 0\,, \tag{10.3.9}$$

leading to the conclusion that the appropriate Euler–Lagrange equations hold for $L$. (cf. Mandelstam and Yourgrau 1979.)

Writing the integral in the last equation in terms of a localized density $\mathcal{L}$, for which, following equation (10.1.14),

$$L\,\Delta t = \int_{\Delta V} \mathcal{L}\, dv\,, \tag{10.3.10}$$

we obtain

$$\int_{t_1}^{t_2} L\, dt = \int_V \mathcal{L}\, dv \tag{10.3.11}$$

in expression (10.3.6), where $V$ is some four-dimensional region whose $t = t_k$ surface lies in $\sigma_{t_k}$ for $k = 1,2$. Then equation (10.3.9) becomes

$$\delta(\int_V \mathcal{L}\, dv) = 0 \tag{10.3.12}$$

for variations of the same type vanishing on the boundaries of $V$, leading to the conclusion that $\mathcal{L}$ also satisfies the appropriate Euler–Lagrange equations, dubbed, in this context, the *field equa-*

*tions.*

In cases like the one at hand, namely the Weyl Lagrangian, where $L$ is replaced by matrix elements of an operator Lagrangian taken between sufficiently characteristic field configurations obtaining upon terminal surfaces, all of these matrix elements now satisfy equation (10.3.12). We conclude that the same equation and concomitant Euler–Lagrange field equations hold also for "purely" quantum *operator* Lagrangians.

(In this operator formulation, in which $L$ is replaced by the matrix elements of an operator we shall continue to write as $L$, equation (10.3.8) may be continued a step further to assume the form

$$\delta\langle\psi(\sigma_{t_2})|\psi(\sigma_{t_1})\rangle = i\langle\psi(\sigma_{t_2})|\,\delta(\int_{t_1}^{t_2} L\,dt)\,|\psi(\sigma_{t_1})\rangle, \quad (10.3.13)$$

where the operator appearing on the right hand side is defined appropriately (cf. Mandelstam and Yourgrau 1979). In this "action principle" form the last equation was posited independently by Schwinger. For a very careful treatment of these matters in a context very close to ours, see Mantke 1992 and 1995.)

Varying the fields $\Psi$ and $\overline{\Psi}$ independently in the Weyl Lagrangian (equation (10.2.26)) leads to the field equation(s)

$$\not{\partial}\,\Psi = 0 \quad (10.3.14)$$

and an adjoint version of the same equation(s). This equation for the doublet $\Psi$ is the *Dirac equation* for massless fields. (Variation with respect to the elements of $E$ produces an expression readers might recognize as being formally identical to the *energy-momentum tensor* of the field: its vanishing in this case is a measure of the incompleteness of our picture so far.)

The equations for $\psi_1$ and $\psi_2$ resulting from the last equation, namely (cf. equations (9.4.30) *et al.*)

$$\bar{\sigma}_k \partial^k \psi_1 = (\sigma_0 \partial^0 - \sigma_1 \partial^1 - \sigma_2 \partial^2 - \sigma_3 \partial^3)\psi_1 = 0 \qquad (10.3.15a)$$

$$\sigma_k \partial^k \psi_2 = (\sigma_0 \partial^0 + \sigma_1 \partial^1 + \sigma_2 \partial^2 + \sigma_3 \partial^3)\psi_2 = 0 \qquad (10.3.15b)$$

are named for Weyl.

*10.3.1 Solving the Weyl Equations by (Quantum) Thinking!*

The continuum Weyl operators $\bar{\sigma}_k \partial^k$ and $\sigma_k \partial^k$ appearing on the left-hand sides of the Weyl equations above are the "classical" continuum versions of the operators $\partial^k \otimes \bar{\sigma}_k$ and $\partial^k \otimes \sigma_k$, and may be pulled back to the (Maxwell–Boltzmann phase of the) net, to yield

$$[\![ \Delta^k \otimes \bar{\sigma}_k ]\!] = -i\tau \bar{\sigma}_k \partial^k \qquad (10.3.1.1a)$$

$$[\![ \Delta^k \otimes \sigma_k ]\!] = -i\tau \sigma_k \partial^k . \qquad (10.3.1.1b)$$

Moreover, it is immediate from the relevant formulae in §9.4 that, as operators on $\mathbb{C}[\Gamma] \otimes V$,

$$\Delta^k \otimes \bar{\sigma}_k = \Delta^{\uparrow\uparrow\tilde{}} \otimes \bar{\sigma}_{\uparrow\uparrow\tilde{}} + \ldots = \Delta^{\Sigma\Sigma\tilde{}} \otimes \bar{\sigma}_{\Sigma\Sigma\tilde{}} \equiv \Delta^{\Sigma\Sigma\tilde{}} \otimes \sigma^{\Sigma\Sigma\tilde{}} \qquad (10.3.1.2a)$$

and similarly

$$\Delta^k \otimes \sigma_k = \Delta^{\Sigma\Sigma\tilde{}} \otimes \sigma_{\Sigma\Sigma\tilde{}}, \qquad (10.3.1.2b)$$

the $\Sigma$ sub- and superscripts running through the original arrow choices (cf. Selesnick 1994, equation (7.21)).

Now, (reticular) solutions, $\vartheta_1$ and $\vartheta_2$ say, to the reticular versions of the Weyl equations must yield solutions $[\![ \vartheta_1 ]\!]$ and $[\![ \vartheta_2 ]\!]$

to the continuum Weyl equations, modulo issues of convergence, by virtue of equation (9.3.3).

In fact, we may exploit the reticular interpretation of the Weyl operators arrived at earlier, in conjunction with the representations found in §9.2 for the $\sigma$-maps, to specify reticular solutions for the Weyl equations. These specifications will be done first in terms of utterly trivial algorithms using classical logic. These classical programs are entirely useless, but, upon the "quantization" of their logic, they produce as outputs solutions to the Weyl equations whose continuum correspondents converge: we solve the equations by thinking—albeit quantum logically. In this way, various apparently wildly disparate notions seem to have been harmoniously brought together.

We sketch the argument: further related quantum programs may be found in Selesnick 1994, an otherwise obsolete precursor to some of the material appearing here.

(The computational steps we are about to informally describe refer to manipulations performed upon elements of various spaces of acts and therefore do not fit into the context of **GQ**, which operates only at the level of the ambient spaces, or quantum resources, and does not refer to their elements. What is needed to formally bridge the gap is a full *term calculus* for **GQ**, an issue we do not take up in this work.)

First, we consider the action of the operator (10.3.1.2a) upon an element of the form $\xi \otimes s_1$ where $s_1$ is an element of $S$, and $\xi$ is a monomial in $\mathbb{C}[\Gamma]$:

$$(\Delta^{\Sigma\Sigma'} \otimes \sigma^{\Sigma\Sigma'})(\xi \otimes s_1) = \Delta^{\Sigma\Sigma'}(\xi) \otimes \sigma^{\Sigma\Sigma'}(s_1). \qquad (10.3.1.3)$$

Since this is exactly the result of applying $\mathcal{A}$ to the element $\xi \otimes (s_1 \oplus 0)$, the usual interpretation obtains. Namely, the expression above represents the result of superposing the individual selective acts of removing a $\Gamma_{\Sigma\Sigma'}$ from $\xi$ and pairing the result with the corresponding spinor $\gamma_{\Sigma\Sigma'}(s_1 \oplus 0)$, which is the version of $(s_1 \oplus 0)$ that has been "parallely transported" by the cotangent representer $\gamma_{\Sigma\Sigma'}$ back "along" the $\Gamma_{\Sigma\Sigma'}$ removed. (If $\xi$ contains no $\Gamma_{\Sigma\Sigma'}$, then it is 1 and

the equation is trivially satisfied.)

Moreover, using equation (9.2.14b),

$$\sigma^{\Sigma_i \tilde{\Sigma_j}}(s_1) = \varphi(\Sigma_i \vee s_1)\tilde{\varepsilon}(\tilde{\Sigma_j}). \qquad (10.3.1.4)$$

In order to exploit the inherent quantum set/logical structure during the course of the arguments to follow, we shall freely embed spinors as first grade unitized qets in their associated Grassmannian qet algebra ($s \mapsto |s\rangle$, cf. §2.3) de-embedding them later.

We observe from the last equation that, on being pulled back, $s_1$ is forced to $\vee$ with those elements of similar type. Thus, returning to equation (10.3.1.3), if we could design a qet which somehow represents the creation of all possible $\Sigma$s and inserted this after $\xi$, the application of the Weyl operator in equation (10.3.1.2a) should result in this qet being pulled back to nullify each $\Sigma_i$ in each occurrence of the right hand side of equation (10.3.1.4).

A simple, and useless, c-logic program to construct such a qet and insert it after the monomial $\xi$ might run as follows:

**begin**

*form the set* $\{\Sigma\}$;
*insert appropriately after* $\xi$;
*return the result*

**end**

Quantization of this program entails the appropriate replacement of sets by qets and the requirement that superpositions over alternatives be taken. Thus, we superpose over all the returns from the following "quantum" program:

**begin**

*form the qet* $|\Sigma\rangle$, *superposing alternatives*;
*insert after* $\xi$;
*return the result of this insertion*

**end**

Here $\xi$ ranges over the monomials in $\mathbb{C}[\Gamma]$ and $\Sigma$ can be either $\uparrow$ or $\downarrow$. For any $\xi$ the first statement produces $|\uparrow\rangle+|\downarrow\rangle$ and the second produces $\xi\otimes(|\uparrow\rangle+|\downarrow\rangle)$, which is returned (to the experimenter).

Thus, if the degree of $\xi$ is 0:

$$1\otimes(|\uparrow\rangle+|\downarrow\rangle) \text{ is returned.} \qquad (10.3.1.5)$$

The superposition over all returns when the degree of $\xi$ is 1 is:

$$(\Gamma_{\uparrow\uparrow^\sim}+\ldots+\Gamma_{\downarrow\downarrow^\sim})\otimes(|\uparrow\rangle+|\downarrow\rangle). \qquad (10.3.1.6)$$

The superposition over all returns when the degree of $\xi$ is 2 is:

$$(\Gamma^2_{\uparrow\uparrow^\sim}+\Gamma_{\uparrow\uparrow^\sim}\Gamma_{\uparrow\downarrow^\sim}\ldots+\Gamma^2_{\downarrow\downarrow^\sim})\otimes(|\uparrow\rangle+|\downarrow\rangle)$$

$$=(\Gamma_{\uparrow\uparrow^\sim}+\ldots+\Gamma_{\downarrow\downarrow^\sim})^2\otimes(|\uparrow\rangle+|\downarrow\rangle), \qquad (10.3.1.7)$$

and the superposition over all returns when the degree of $\xi$ is $n$ is

$$(\Gamma_{\uparrow\uparrow^\sim}+\ldots+\Gamma_{\downarrow\downarrow^\sim})^n\otimes(|\uparrow\rangle+|\downarrow\rangle). \qquad (10.3.1.8)$$

It is quickly established that each of these returns does indeed satisfy the Weyl equation corresponding to equation (10.3.1.2a). Moreover, we note from equations (9.4.34) that

$$\Gamma_{\uparrow\uparrow\sim} + \ldots + \Gamma_{\downarrow\downarrow\sim} = \Gamma_0 + \Gamma_1. \qquad (10.3.1.9)$$

Thus, the superposition over all returns may be written formally, using vector notation for the spinor part, as

$$\vartheta_1 = \sum_{n=0} (\Gamma_0 + \Gamma_1)^n \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \qquad (10.3.1.10)$$

Then, from the correspondence principle (equation (9.5.3)),

$$[\![ \vartheta_1 ]\!] = \sum_{n=0} \frac{1}{n!} \left(\frac{i}{\tau}\right)^n (q_0 + q_1)^n \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$= \exp\left(\frac{i}{\tau}(q_0 + q_1)\right) \otimes \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \qquad (10.3.1.11)$$

According to our earlier interpretation (§8.2) this expression represents parallel transport along the vector (1,1,0,0) of the bispinor shown. This vector lies along a generator of the cone in $\mathbb{R}^4$ whose major axis lies along the 0-axis and whose slope is 1. Since $\tau$ is negative, the direction of the macroscopic transport opposes, as before, the direction of the reticular increase, and so, since we have identified the 0-component with time, lies in the direction *earlier* to *later*: i.e. the "motion" is *causal*.

Its *classical* correspondent, which may be written

$$[\![ \vartheta_1 ]\!]_C = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp\left(\frac{i}{\tau}(x_0 + x_1)\right), \qquad (10.3.1.12)$$

is easily checked to be a solution of the standard Weyl equation (10.3.15a). Its standard interpretation, in terms of positive energy ($\tau < 0$) fermion *states* moving with unit speed along one of the spatial axes, seems to accord well with the conclusions we have just ar-

rived at independently. We note also that these particles have sufficient energy to resolve distances down to $|\tau|$.

We turn now to the other Weyl equation corresponding to equation (10.3.1.2b). In this case, with $s^{\sim *}$ in $\tilde{S}^*$, in place of equation (10.3.1.4) we choose equation (9.2.16a):

$$\sigma_{\Sigma_i \tilde{\Sigma_j}}(s^{\sim *}) = (s^{\sim *})(\tilde{\Sigma_j})\Sigma_i. \qquad (10.3.1.13)$$

Here, the pulled back $s^{\sim *}$ meets its match differently: namely, it represents an act of registration or absorption of the argument standing to its right in the right hand side of the last equation. Whereas in the case of equation (10.3.1.4) we sought a representer of the "generic" creator of possible pulled back acts, here we should seek a representer of the "generic" *annihilator* of possible absorbers. For then, pulling back *via* this Weyl operator should result in the failure to absorb any $\tilde{\Sigma_j}$ and produce the null transition.

A classical program to implement this might run as follows:

**begin**

*destroy all* $\Sigma^{\sim *}$: *that is, destroy each*
*element of the set* $\{\uparrow^{\sim *}\} \cup \{\downarrow^{\sim *}\}$;
*form the resulting set*;
*insert after* $\xi$;
*return the result of this insertion*

**end**

Quantization produces the superposition over all $\xi$ of the returns from:

**begin**

*destroy each element of the qet* $|\uparrow^{\sim *}\rangle \vee |\downarrow^{\sim *}\rangle$;
*form the resulting qet, superposing alternatives*;
*insert after* $\xi$;
*return the result of this insertion*

**end**

To execute this program we note that the appropriate destruction operators to use in the first step are the fermion Fock space ones, namely:

$$a(\uparrow^{\sim *})(|\uparrow^{\sim *}\rangle \vee |\downarrow^{\sim *}\rangle) = |\downarrow^{\sim *}\rangle \qquad (10.3.1.14)$$

and

$$a(\downarrow^{\sim *})(|\uparrow^{\sim *}\rangle \vee |\downarrow^{\sim *}\rangle) = -|\uparrow^{\sim *}\rangle, \qquad (10.3.1.15)$$

resulting, at the second step, in $|\downarrow^{\sim *}\rangle - |\uparrow^{\sim *}\rangle$. Proceeding as before, in this case we end up with:

$$[\![ \vartheta_2 ]\!]_C = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \exp\left(\frac{i}{\tau}(x_0 + x_1)\right), \qquad (10.3.1.16)$$

which is a solution to the second Weyl equation.

By inserting these solutions (equations (10.3.1.16) and (10.3.1.12)) into equations (10.3.15), it is immediately seen that they are eigenvectors of the operator

$$\sigma_1 \partial^1 + \sigma_2 \partial^2 + \sigma_3 \partial^3, \qquad (10.3.1.17)$$

having eigenvalues differing only in sign. Eigenvalues of this operator are interpreted "classically"—that is, in the continuum—as the *helic-*

*ity* of the associated particle: roughly speaking, this is the scalar product of the spin 3-vector with the momentum vector. A difference in sign thus implies opposite spin directions relative to the direction of motion. For massless particles the sign of this eigenvalue is a measure also of the *chirality*, or handedness, of the particle. Our solutions therefore represent fermions of opposite chirality.

The question of which chirality is which is easily settled. Since we have carefully chosen the sign of $\tau$ so as to ensure that our macroscopic $x_\mu$-measurements are made in the forward-pointing direction, we can make explicit the identification

$$(x_0, x_1, x_2, x_3) \equiv (t, x, y, z). \tag{10.3.1.18}$$

Then equation (10.3.15a) reads

$$\left(\frac{\partial}{\partial t} - \sigma_1 \frac{\partial}{\partial x} - \sigma_2 \frac{\partial}{\partial y} - \sigma_3 \frac{\partial}{\partial z}\right)\psi_1 = 0, \tag{10.3.1.19}$$

whence

$$\left(i\frac{\partial}{\partial t} + \sigma.\mathbf{p}\right)\psi_1 = 0, \tag{10.3.1.20}$$

where $\mathbf{p}$ denotes the three-dimensional momentum operator

$$\frac{1}{i}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right), \tag{10.3.1.21}$$

and

$$\sigma = (\sigma_1, \sigma_2, \sigma_3). \tag{10.3.1.22}$$

Similarly, equation (10.3.15b) yields

$$\left(i\frac{\partial}{\partial t}-\sigma.\mathbf{p}\right)\psi_2=0. \qquad (10.3.1.23)$$

The chirality associated with the negative (positive) eigenvalue of the helicity operator $\sigma.\mathbf{p}$ in equation (10.3.1.20) (equation (10.3.1.23)) for positive energy solutions is, by convention, called *left (right)* or *left-handed (right-handed)*. For this reason, when used in a Lagrangian, elements of

$S$ may be assigned the *left-handed* chirality;

while elements of

$\tilde{S}^*$ may be assigned the *right-handed* chirality.

In the sequel, therefore, we shall replace the subscript 1(2) we have been using to label the $S(\tilde{S}^*)$ component of bispinors with the subscript $L(R)$.

We note that an earlier notational choice—specifically the decision to subscript rather than superscript the $\Gamma$ operators—has led us into conflict with the current convention regarding the placement of indices, which would have $x^\mu=(t,x,y,z)$. In this, we retroactively and accidentally do honor to R. Feynman, who also employed this now unconventional convention.

# 11

# Dynamics II

In this chapter we clothe the defect and redo the dynamics of the last chapter accordingly (§11.1). The Lagrangian we found there is now replaced (in §11.2) by a power series in the net constant $\tau$ whose lowest order terms yield massless versions of the standard Yang–Mills Lagrangians for the gauge groups SU(2) and SU(3), and the Einstein–Hilbert Lagrangian for gravity (with an added higher order term). These forms emerge with Feynman gauge-fixing terms already included.

The interaction terms are discussed in §11.2.1 where careful attention to the typing of terms highlights certain peculiarities in the SU(2) component caused, essentially, by the indistinguishability of the complexification of its Lie algebra from the $\mathfrak{sl}(2, \mathbb{C})$ we have already used to build the bare defect space (and indeed the whole structure). This results in the turning off of the SU(2) interaction with right-handed spinors, and, at the same time, in the appearance of a U(1) component adhering to the SU(2) one.

The kinetic terms already mentioned are the subject of §11.2.2, the SU(2) component having now become a $U(1) \times SU(2)$ one. In §11.2.3 we find that an additional "unphysical" term must be added if the quantum symmetry of the clothed defect is to be properly maintained when an attempt is made to quantize the fields. This term is identical with the (Faddeev–Popov) ghost term, which thus acquires an apparently new and rather simple interpretation.

Unfortunately, the process by which we clothe the defect leads to the macroscopic contamination of some of our conclusions and produces a mismatch between the levels of resolution of certain terms in the resulting Lagrangian. Although we have taken care to suppress this contamination wherever possible, penalties are nevertheless exacted, among which is included the need to redress the imbalance in the degree of resolution mentioned above, which occurs in the non-gravitational sector of the emerging Lagrangian. This is the business of §11.3.

Throughout this chapter we continue to take pains to maintain the integrity of the emerging type structures throughout the **GQ**-like deductions to which they are subjected in the course of approaching the continuum limit.

## 11.1 The Defect's New Clothes

We aim now to take into account possible variations of the defect itself. Returning to equation (9.3.16) we may naïvely write:

$$d(\xi\otimes s) = \not{\Delta}(\xi)\otimes s + \xi\otimes ds. \tag{11.1.1}$$

Then we are confronted with the problem of interpreting $ds$. On the net, $\xi\otimes s$ represents the result of attaching the act $s$ to the element $\xi$, so $s$ may be regarded as having $\Gamma$-indices (cf. §10.2). Then it would be tempting to write, in analogy with equation (9.3.13),

$$ds = \Delta^{\mu}s\otimes\gamma_{\mu}. \tag{11.1.2}$$

However, the argument in §9.2 leading up to equation (9.3.13) does not seem to apply here. The "subtypes" or parameters of $s$, whatever they may be, are certainly not the $\Gamma_{\mu}$ (cf. the discussion following equation (9.2.1)). Thus $\gamma_{\mu}$ is inappropriate on the right-hand side of equation (11.1.2). As mentioned in §10.0, our plight is even more serious than this since, even if we could find an $s$-subtype to play the

rôle of $\Delta x_\mu$, the variation of this new subtype itself would have to be taken into account (just as we are presently in the act of trying to take the variation of $s$ into account), requiring the introduction of yet another subtype, and so on *ad infinitum.* Since $s$ was reistically interpreted as a defect or gap in the net structure, the first new subtype would be interpreted as a gap between the primary $s$-gaps (or as a defect within the primary defect), and so on.

We shall approach the problem of supplying this infinite hierarchy of subtypes in the spirit of "renormalization." That is, we shall try to infer from the presumed *macroscopic result* of this infinite piling up of defect subtypes the structure of a space of macroscopic candidate acts that subsumes the hierarchy. Specifically, we shall attempt to quantize the classical symmetries of the macroscopic descendant of the defect, using the paradigms of Chapter 3. The crudity of our method has serious drawbacks entailing significant loss of physical information. Nevertheless, enough survives to encourage the view that these methods are worth refining.

To find a candidate for the macroscopic descendant of the defect that bypasses the infinite subtype regress problem, we revert to the original position taken in §8.1. There we macroscopically interpreted the act that injected the empty set, containing no chrononic pairs, namely the qet $|\varnothing\rangle$, as an "appropriate" Fock vacuum $\varpi$ (equation (8.1.1)). Thus the appropriate Fock vacuum—which, in view of the correspondence principle we have adopted, we take to be the one in $L^2(\mathbb{R}^4)$—may be interpreted as a macroscopic *model* of the reticular defect. The idea is that its residual symmetries should have microscopic precursors which we attempt to recover by the group quantization process to follow.

In the Schrödinger representation (which we have chosen to accommodate our correspondence principle) the annihilation operators are given by equation (8.1.12), so the vacuum $\varpi$ satisfies

$$a_\mu \varpi = \frac{1}{\sqrt{2}}\left(\frac{1}{\tau} q_\mu + i\tau p_\mu\right)\varpi = 0 \qquad (11.1.3)$$

or

$$\left(\partial^{\mu} + \frac{1}{\tau^2} q_{\mu}\right) \varpi = 0. \tag{11.1.4}$$

Thus $\varpi$ is of the form

$$\varpi = N \exp\left[-\frac{1}{2\tau^2}\left(x_0^2 + x_1^2 + x_2^2 + x_3^2\right)\right], \tag{11.1.5}$$

where $N$ is a constant, normalized if $N = (4\pi^2\tau^4)^{-1}$.

This vacuum has various "classical" symmetries. First, at the outer-most level, there is symmetry with respect to arbitrary permutations of the $x_{\mu}$. That is to say, it is invariant under the action of $S_4$, the symmetric group (of permutations) upon four elements. This symmetry is ignorable and is ignored at the classical level. Nevertheless, by quantizing this group of symmetries, and seeking representations of the resulting algebra of quantum symmetries, we may gain access to a more finely resolved vacuum structure. Unfortunately, we are able to achieve here only a partial quantization of $S_4$, which admits a well-known semifactorization (see for instance QR, §16.7):

$$S_4 \cong \mathbb{Z}_2 \ltimes (\mathbb{Z}_3 \ltimes (\mathbb{Z}_2 \times \mathbb{Z}_2)). \tag{11.1.6}$$

A partial quantum replacement for this group is obtained by first ignoring the interrelationships implied by the internal actions of the constituent subgroups that give rise to the semidirect factorization, and then replacing each remaining subgroup by the appropriate Lie algebra according to the argument in §3.2. Important attributes of the original classical symmetry group are thereby lost. In particular, if it is supposed that upon quantization, condensation, etc., the cyclic subgroups listed in equation (11.1.6) give rise ultimately to the gauge symmetries of experience, then the lost information might be supposed to pertain to such matters as the mutual interaction of the as-

sociated fields. One may note in this connection that the quantized actions, interpreted as actions of algebra-coalgebra "double algebra" structures of the type considered in §3.2.3 upon other such double algebras, are circumscribed in ways which differ from the ways in which the underlying classical groups interact. (To properly quantize $S_4$ one would require at least an analog of the notion of semidirect product for double algebras, and a concomitant theory of actions of double algebras upon other double algebras.) It will suffice for our purposes to effect the partial quantization of $S_4$ by ignoring these internal semidirect actions: we merely replace each constituent subgroup in (11.1.6) by its "quantum replacement" Lie algebra according to §3.2. Namely, we replace:

$$\mathbb{Z}_2 \text{ by } \mathfrak{sl}(2,\mathbb{C}),$$

$$\mathbb{Z}_3 \text{ by } \mathfrak{sl}(3,\mathbb{C}), \text{ and}$$

$$\mathbb{Z}_2 \times \mathbb{Z}_2 \text{ by } \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{so}(4,\mathbb{C}).$$

Thus, a "refined" vacuum, for this restricted quantization of $S_4$, would be represented by a vector (or ray) in a representation space of the Lie algebra

$$\mathfrak{a} \equiv \mathfrak{sl}(2,\mathbb{C}) \oplus \mathfrak{sl}(3,\mathbb{C}) \oplus \mathfrak{so}(4,\mathbb{C}). \tag{11.1.7}$$

However, the vacuum $\varpi$ (equation (11.1.5)) has a further "internal" symmetry which should also be taken into account, and will serve to restrict further the algebra of quantum symmetries found above. Specifically, $\varpi$ has the internal parameters $x_\mu$, and is invariant with respect to a certain transformation of these parameters: namely, the joint actions $x_\mu \mapsto \overline{x}_\mu$ (complex conjugation of the real $x_\mu$), and $x_\mu \mapsto -x_\mu$. (Note that these two symmetries commute.) The $x_\mu$ are eigenvalues of the operators $q_\mu$ which in turn are macroscopic continuum correspondents of reticular variables. Hence, this

symmetry of $\varpi$ may be attributed ultimately to reticular circumstances: namely, the breakdown to the real subspace $\mathbf{M}$ conjoined with the directional reflection, with respect to which the defect is presumed to be independent. If we assume that a similar internal symmetry is present in the presumed spaces of precursors of $\varpi$ and is preserved by some subalgebra of the algebra $\mathfrak{a}$ (equation (11.1.7)) of "outer" symmetries, then we should seek a pair of corresponding symmetries of $\mathfrak{a}$ whose joint fixed set will constitute the subalgebra we seek.

By this kind of preservation we mean the following. Let us denote by a superscripted $\sigma$ some transformation of variables "internal" to both the elements of a vector space $W$ and to operators upon $W$: for example, the complex conjugation of entries in vectors—i.e. tuples—and matrices of complex numbers. Then, if $A$ is an operator upon $W$, and $w$ in $W$ is *$\sigma$-symmetric*, meaning that $w^\sigma = w$, we would have

$$\begin{aligned}(Aw)^\sigma &= A^\sigma w^\sigma \\ &= A^\sigma w, \end{aligned} \tag{11.1.8}$$

so that if $A$ is to preserve the $\sigma$-symmetric elements of $W$ we should have $A^\sigma = A$.

The algebra $\mathfrak{a}$ is the direct sum of complex semisimple Lie algebras, a class whose structure is well in hand (see for example Fulton and Harris 1991), and we may seek candidates for the two presumed ancestral symmetry operations for each component. Specifically, each complex semisimple Lie algebra $\mathfrak{g}$ has a family of maximal abelian subalgebras each of whose natural (or "adjoint") action upon $\mathfrak{g}$ is diagonalizable. Choosing one such *Cartan* subalgebra $\mathfrak{h}$, say, we then have a so-called *Cartan decomposition*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha, \tag{11.1.9}$$

where $R \subset \mathfrak{h}^*$ and $\mathfrak{g}_\alpha$ is the subspace

$$\mathfrak{g}_\alpha = \{ X \in \mathfrak{g} : [H, X] = \alpha(H)X \text{ for all } H \in \mathfrak{h} \}. \quad (11.1.10)$$

The non-zero $\alpha$s, which constitute the set named $R$, are called the *roots* of $\mathfrak{g}$, and the spaces $\mathfrak{g}_\alpha$ (which turn out to be one-dimensional) are called the *root spaces*. It can be shown that the eigenspace corresponding to zero is precisely $\mathfrak{h}$.

The set of roots $R$ functions in many ways like a set of "internal" parameters for the algebra, which is realized, in the decomposition (11.1.9), as the space of global sections of the "bundle" over $R \cup \{0\}$ with fibre $\mathfrak{g}_\alpha$. In particular, since we are seeking a breakdown to realness in a set of such parameters, our gaze is drawn to those Cartan subalgebras whose roots are all real-valued. It turns out that there is a *unique* real subalgebra $\mathfrak{g}_0$ of $\mathfrak{g}$ such that $\mathfrak{g}$ is the *complexification* of $\mathfrak{g}_0$, that is,

$$\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g} \quad (11.1.11)$$

and $\mathfrak{g}_0$ has a Cartan subalgebra $\mathfrak{h}_0$ acting upon it with real-valued roots. (In addition, it turns out that $\mathfrak{h}_0 \otimes_{\mathbb{R}} \mathbb{C}$ is a Cartan subalgebra of $\mathfrak{g}_0 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}$.) A real subalgebra of $\mathfrak{g}$ having the property that its complexification is $\mathfrak{g}$, as in equation (11.1.11), is called a *real form* of $\mathfrak{g}$. The unique real form $\mathfrak{g}_0$ is called the *split* form.

Using equation (11.1.11) we may now define a conjugate linear involutional Lie morphism, $\sim$, upon $\mathfrak{g}$, determined for $x \in \mathfrak{g}$ and $z \in \mathbb{C}$ by

$$(x \otimes z)^{\sim} \equiv x \otimes \bar{z}. \quad (11.1.12)$$

At the same time there is defined a unique automorphism

$$\varphi : \mathfrak{g} \longrightarrow \mathfrak{g}, \quad (11.1.13)$$

that sends $\mathfrak{h}(=\mathfrak{h}_0\otimes_{\mathbb{R}}\mathbb{C})$ to $-\mathfrak{h}$ and $\mathfrak{g}_\alpha$ to $\mathfrak{g}_{-\alpha}$, and that commutes with $\sim$. Thus, for each component of the Cartan decomposition of $\mathfrak{a}$, we have imitated the two internal symmetries of the Fock vacuum $\varpi$, with the roots playing the rôle of "microscopic" internal parameters. The algebra of quantum symmetries we seek is then the direct sum of the algebras of elements in each component of $\mathfrak{a}$ left fixed by the joint action of $\sim$ and $\varphi$. In the general case of a semisimple Lie algebra $\mathfrak{g}$ it is not hard to show that the subalgebra of $\mathfrak{h}$ left fixed by this joint action is the real subalgebra

$$\mathfrak{h}_c \equiv i\mathfrak{h}_0 \tag{11.1.14}$$

and that the subalgebra of $\mathfrak{g}$ left fixed is a real form, denoted $\mathfrak{g}_c$, having $\mathfrak{h}_c$ as a Cartan subalgebra. The subscript here stands for *compact*, and the terminology stems from the fact that the associated Lie group is indeed compact. (The Killing form on $\mathfrak{h}_c$ is negative definite, from equation (11.1.14).)

Thus we have landed squarely upon the direct sum of the (unique) compact real forms of the complex semisimple Lie algebras that form the components of $\mathfrak{a}$. These are well-known: namely, the compact form of

$$\mathfrak{sl}(n,\mathbb{C}) \text{ is } \mathfrak{su}(n),$$

and the compact form of

$$\mathfrak{so}(4,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}) \text{ is } \mathfrak{so}(4) \cong \mathfrak{su}(2)\oplus\mathfrak{su}(2).$$

Thus,

$$\mathfrak{a}_c \equiv \mathfrak{su}(2)\oplus\mathfrak{su}(3)\oplus\mathfrak{so}(4) \tag{11.1.15}$$

is the algebra we seek.

This reduction to the compact forms has, of course, been

predicated upon a presumed association between the roots of the components of $\mathfrak{a}$ and the precursors of the macroscopic internal parameters $x_\mu$. But these precursors are presumed to be reticular elements: consequently, there should be some sort of association between the root systems of $\mathfrak{a}$ and reticular elements, which we have not followed up here. Such an association, if one exists, is likely to be rather loose, since we have completely ignored essential information inherent in the semidirect product structure in equation (11.1.6). Presumably, the true defect, if it could be found, would reflect the local symmetries of the ambient structure. Consequently, a more refined group quantization scheme, which captures at least the semidirect product information, may impose sufficiently stringent conditions upon the resulting defect that the local symmetries of a presumed ambient structure may be recaptured. (In this connection, the reader may consult Finkelstein *et al.* 1997, and later work of the Finkelstein school, which seems to be aiming at just such a program, though employing methods different from ours.)

As matters stand here, our group quantization procedure, crude as it is, has in fact recaptured an "internal" instance of our earlier bare defect structure, which resides in the choice of representation for the $\mathfrak{so}(4)$ component of $\mathfrak{a}_c$ (equation (11.1.15)). To see this, we note that another real form of $\mathfrak{so}(4,\mathbb{C})$ is given by

$$\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C}) \tag{11.1.16}$$

and that the infinitesimal generators of the bispinor representation $\mathcal{S}$ (equation (9.2.25)) of $\mathfrak{sl}(2,\mathbb{C})$, namely

$$\sigma_{mn} \equiv \frac{i}{2}[\gamma_m, \gamma_n], \tag{11.1.17}$$

constitute a representation of $\mathfrak{so}(3,1)$ upon the bare defect space $V$ of bispinors. In this representation a basis for the real algebra $\mathfrak{so}(3,1)$ is mapped to $i\sigma_{mn}$, where the $\sigma_{mn}$ obey the $\mathfrak{so}(3,1)$ or "angular momentum" commutation relations:

$$[\sigma_{mn},\sigma_{rs}] = -i(\eta_{mr}\sigma_{ns} - \eta_{nr}\sigma_{ms} + \eta_{ns}\sigma_{mr} - \eta_{ms}\sigma_{nr})$$

(11.1.18)

where the diagonal matrix $\eta$ is given by

$$\eta \equiv \mathrm{diag}(1,-1,-1,-1). \qquad (11.1.19)$$

Since the space $V$ accommodating the representation is complex, this representation may be lifted uniquely to the complexification of $\mathfrak{so}(3,1)$, namely $\mathfrak{so}(4,\mathbb{C})$, and then restricted down to $\mathfrak{so}(4)$. Similarly, any complex representation of $\mathfrak{so}(4)$ may be lifted to $\mathfrak{so}(4,\mathbb{C})$ and then restricted to $\mathfrak{so}(3,1)$. This argument applies to any complex representation of any real subalgebra: the complex representations of such subalgebras are essentially equivalent.

At this point it may be expedient to recall that the coherent form of $\mathfrak{so}(3,1)$, namely, the *proper Lorentz* group SO(3,1) is expressible as a union of two connected subsets,

$$\mathrm{SO}(3,1) = L_+^{\uparrow} \cup L_+^{\downarrow}, \qquad (11.1.20)$$

where $L_+^{\uparrow}$, the orthochronous or restricted Lorentz group, has already been met (after equation (7.2.3)) and $L_+^{\downarrow}$ consists of Lorentz transformations $\Lambda_j^i$ with $\det\Lambda = 1$ and $\Lambda_0^0 \leq 1$. The orthochronous Lorentz group emerged earlier essentially as a result of inherent causal structure going back to the pairing on the right-hand side of the turnstile in the axiom (7.1.18). Here, the full proper Lorentz group has emerged as the coherent from of an algebra of quantized symmetries of a vacuum having no underlying directional attributes and therefore a wider repertoire of symmetries.

Since the bispinor representation recaptures the space of bare defect injectors it is clearly the one to choose as the space of acts carry-

ing the quantum symmetry represented by the $\mathfrak{so}(4)$ component of the clothed defect. There is no such unambiguous criterion to decide the choices of representations for the other components, and such choices will remain somewhat in abeyance: another possible mark against the crudity of our group quantization process. However, the simplest possible choice for these representations will lead almost immediately to the conclusion that the interaction associated with the $\mathfrak{su}(2)$ component of $\mathfrak{a}_c$ affects fermions of only one chirality: that is, the $\mathfrak{su}(2)$ interaction breaks chiral symmetry (§11.2.1).

In summary, then, our approximation to the space of clothed defects (i.e. defect injectors, or initial acts) emerges as an appropriate representation space for the Lie algebra $\mathfrak{a}_c$, where we choose the bispinor representation for the $\mathfrak{so}(4)$ component thereby recovering the original bare defect space. The other components could be thought to represent the effect of taking into account a virtual nesting of intermediate defect subtypes all the way up to the continuum level. The effect has been to amalgamate the original bare defect type with new types whose significance will emerge when we redo the dynamics of the last chapter with the clothed defect type. These new types must be expected to carry the degree of macroscopic contamination inherent in our method. (Despite this contamination, the reappearance of our original bare defect encourages us to believe that the process has at least led us back to the correct degree of resolution for this component, if not necessarily for the others.)

Suppose now that an appropriate representation space, $\mathbb{W}$, say, of clothed defects, has been chosen for $\mathfrak{a}_c$. In order to implement small variations in some $\xi$ as before, an experimenter must choose an element, $v$, say, in $\mathbb{W}$, representing an initial act of injection of a clothed defect, and attach it to $\xi$. Now, however, constraints are imposed on the choice of $v$ by the "quantum" symmetry inherent in the clothed defect system. Thus, since the problem of the variability of the unknown internal subtypes has presumably been taken care of by our choice of $\mathbb{W}$, only the "external" reticular positional attributes are left to vary. Thus, the analog of equation (11.1.2) is now justified: namely,

$$dv = \Delta^{\mu} v \otimes \gamma_{\mu}. \tag{11.1.21}$$

Recalling that the effect of $\Delta^{\mu}$ upon reticular variables is to truncate them by removing occurrences of $\Gamma_{\mu}$, we interpret the action of $\Delta^{\mu}$ upon $v$ as the selection of an equivalent defect act for attachment to that net element truncated by the action of $\Delta^{\mu}$. But equivalent acts of this type are connected by elements of $\mathfrak{a}_c$, the algebra of allowable defect symmetries. Thus we may write

$$\Delta^{\mu} v = \mathbb{A}^{\mu} v \tag{11.1.22}$$

for certain choices of $\mathbb{A}^{\mu}$ in $\mathfrak{a}_c$. That is,

$$dv = \Delta^{\mu} v \otimes \gamma_{\mu}$$

$$\equiv \mathbb{A} v. \tag{11.1.23}$$

We may solve this equation, at least formally, by noting that that the $\mathbb{A}$ selected for use in the last equation is being interpreted in an infinitesimal sense. Thus, for instance, when $\mathbb{A}$ is extended as in §3.2.2 to a derivation on the exterior (or Grassmann) algebra over $\mathbb{W} \otimes V$, it correctly reproduces the behavior of the putative derivation $d$ on first grade elements. Then, as in §3.2.2, the exterior algebra automorphism whose differential is $\mathbb{A}$ reduces to the "coarsened" operator $\exp \mathbb{A}$ on the first grade elements $\mathbb{W} \otimes V$. In a sense this operator is an integrated version of $\mathbb{A}$, the latter being regarded as the infinitesimal generator of a quantum symmetry. Then, by adopting a general clothed defect injecting act of the form

$$e^{\mathbb{A}}(w \otimes s) \tag{11.1.24}$$

for $\mathbb{A}^{\mu}$ in $\mathfrak{a}_c$, $w$ in $\mathbb{W}$, and $s$ in $V$, we incorporate the $\mathfrak{a}_c$-, or *gauge*-

symmetry inherent in such a choice. (The expression (11.1.24), involving as it does the coarsened operator $\exp \mathbb{A}$, reflects some of the inherent macroscopic contamination promised earlier.)

The original or first-order defect type ($s$) is still required here to implement the variation with respect to the still variable reticular elements. This $s$ is used in its earlier sense as a *static* place-holder: it is not here subject to the fluctuations allowed to $w$ by the presence of the $\mathfrak{a}_c$ symmetries, which were originally invoked precisely to carry over from $s$ the possible variability of the defect structure. That the invocation of these $\mathfrak{a}_c$ symmetries apparently succeeds in this purpose is shown by its incorporation as a component of $\mathbb{W}$ of the other version, or occurrence, of the $V$ defect space, namely the one acted upon by the $\mathfrak{so}(4)$ component of $\mathfrak{a}_c$, which is accordingly *active*, *movable*, or, slightly more accurately whilst we are succumbing to such reist terminology, *spinnable*.

These contextually different occurrences of the same resource within what we may regard as a **GQ** formula merely reflect a certain sequencing of various experimental acts pertaining to the same system. What will be of significance here is the mode of **GQ** reduction of such formulae. Indeed, as we shall soon discover, it is exactly the transitional structure which results from this particular sequencing (namely *static* $\leftrightarrow$ *spinnable*) according to the **GQ**-based rules derived in §7.2.1, that serves to distinguish that interaction which appears macroscopically as gravitation from the other gauge interactions: cf. §11.2.2.

We note that in the manipulations to follow, it is possible to omit explicit reference to the $s$ in expression (11.1.23) and regard $\exp \mathbb{A}$ as a kind of operator in the earlier sense. A slight problem then arises since, if this is done, the leading term in the expansion

$$e^{\mathbb{A}} w = w + \mathbb{A}^{\mu} w \otimes \gamma_{\mu} + \ldots \qquad (11.1.25)$$

differs in type from the rest of the series, and the interpretation of the first plus sign is called into question. The contortions required to resolve this clash of types hardly justify the slight gain in compactness

that may result.

## 11.2 Dynamic Transport and the Lagrangian, Revisited

We now redo the dynamics of §10.1 using the clothed defect. Elements of the form $\xi \otimes s$ are now replaced by those of the form

$$\xi \otimes e^{\not A}(w \otimes s), \tag{11.2.1}$$

so that

$$d(\xi \otimes e^{\not A}(w \otimes s)) = d\xi \otimes e^{\not A}(w \otimes s) + \xi \otimes (de^{\not A})(w \otimes s) + \xi \otimes e^{\not A}(dw \otimes s) \tag{11.2.2}$$

noting in the last summand that $s$ is *static*: effectively, $ds = 0$.

Before turning to an analysis of the right-hand side of this equation, a few anticipatory remarks are in order. The operator that will eventually emerge will be a replacement for the "dynamical quantum" $\not A$ used in the last chapter. The transport argument used there can be applied to this operator to yield a formula analogous to equation (10.1.21) for a "unified" Lagrangian density, in the form of a power series in $\tau$. The middle summand in the right-hand side of equation (11.2.2), which accounts for variability in the choice of equivalent defect injectors, gives rise to $U(1) \times SU(2)$, $SU(3)$ gauge fields—almost exactly as in the (massless) Standard Model—and gravity, the latter coming from the remaining component of $\mathfrak{a}_c$. The Lagrangians emerge to lowest order in $\tau$ in their familiar forms, namely Yang–Mills for the first two, and Einstein–Hilbert for gravity. Feynman gauge-fixing terms are included free of charge, and the Faddeev–Popov ghost term will be supplied at little extra cost. (No actual values for couplings can be expected in general, since we started from an essentially projective theory: these values, arising ultimately from transition amplitudes—cf. §11.2.1—must presumably

be obtained as the result of some of the experiments the experimenter will actually perform.)

The outer terms must be treated as in §10.2 and give rise to the usual "minimally coupled" interaction terms for fermions living in the chosen representations. (The identification of the two-dimensional portion with the weak interaction of the Standard Model is encouraged by the appearance of a rather conspicuous chiral symmetry breaking mechanism: cf. §11.2.1.)

Returning to the first term on the right-hand side of equation (11.2.2), noting that the first-order static defect type $s$ stands on the right, and denoting by a subscripted 1 the identity map on the subscript, we have

$$d\xi \otimes e^{\mathbb{A}}(w \otimes s) = \Delta^{\mu}\xi \otimes (1_{\mathbb{W}} \otimes \gamma_{\mu})(e^{\mathbb{A}}(w \otimes s))$$

$$= (\Delta^{\mu} \otimes 1_{\mathbb{W}} \otimes \gamma_{\mu})(\xi \otimes e^{\mathbb{A}}(w \otimes s)). \quad (11.2.3)$$

The right-most term (of equation (11.2.2)) yields

$$\xi \otimes e^{\mathbb{A}}(dw \otimes s) = \xi \otimes e^{\mathbb{A}}\mathbb{A}(w \otimes s)$$

$$= \xi \otimes \mathbb{A}e^{\mathbb{A}}(w \otimes s)$$

$$= (1_{\mathbb{C}[\Gamma]} \otimes \mathbb{A})(\xi \otimes e^{\mathbb{A}}(w \otimes s)), \quad (11.2.4)$$

which should be compared with equation (10.2.3).

The $d$ in the middle term on the right-hand side of equation (11.2.2) requires some interpretation since the exponentiated map is a coarsened, or macroscopic, version of the microscopic reticular $\mathbb{A}$.

We shall take the view that this $d$ should act to produce an actual small difference in its macroscopic argument $\exp \mathbb{A}$ of the form: (incremented($\exp \mathbb{A}$)) $-\exp \mathbb{A}$. But how should $\exp \mathbb{A}$ be "incre-

mented?" The experimenter regards $\exp \mathbb{A}$ as a selective act, and a natural way to "increment" such an act of selection is to increment in a uniform way those acts selected by it. This entails following the selective act $\exp \mathbb{A}$ by another equally coarse selective act which we shall write in the form $\exp \delta\mathbb{A}$, with the reticular "increment" $\delta\mathbb{A}$ still to be found. This selective act $\exp \delta\mathbb{A}$ is thus the coarsened version of the small reticular change $\delta\mathbb{A}$. It is supposed to induce in acts selected by $\exp \mathbb{A}$ the increment we seek. Thus, we have shifted the burden of interpretation back to the net, where it properly belongs. Then we have

$$\begin{aligned} de^{\mathbb{A}} &= e^{\delta\mathbb{A}} e^{\mathbb{A}} - e^{\mathbb{A}} \\ &= (e^{\delta\mathbb{A}} - 1_{\mathbb{W}\otimes V}) e^{\mathbb{A}}, \end{aligned} \tag{11.2.5}$$

and the middle term in equation (10.2.2) becomes

$$\begin{aligned} \xi \otimes (de^{\mathbb{A}})(w\otimes s) &= \xi \otimes ((e^{\delta\mathbb{A}} - 1_{\mathbb{W}\otimes V}) e^{\mathbb{A}})(w\otimes s) \\ &= (1_{\mathbb{C}[\Gamma]} \otimes (e^{\delta\mathbb{A}} - 1_{\mathbb{W}\otimes V}))(\xi \otimes e^{\mathbb{A}}(w\otimes s)), \end{aligned} \tag{11.2.6}$$

which should be compared with the right-hand sides of equations (10.2.3) and (10.2.4).

Rewriting equation (11.2.2) with the right-hand side rearranged in the order considered above, we obtain

$$\begin{aligned} &d(\xi\otimes e^{\mathbb{A}}(w\otimes s)) = \\ &\left[\Delta^{\mu}\otimes 1_{\mathbb{W}}\otimes\gamma_{\mu} + 1_{\mathbb{C}[\Gamma]}\otimes \mathbb{A} + 1_{\mathbb{C}[\Gamma]}\otimes(e^{\delta\mathbb{A}} - 1_{\mathbb{W}\otimes V})\right](\xi\otimes e^{\mathbb{A}}(w\otimes s)), \end{aligned} \tag{11.2.7}$$

or

$$d = \Delta^{\mu} \otimes 1_{\mathbb{W}} \otimes \gamma_{\mu} + 1_{\mathbb{C}[\Gamma]} \otimes \not{\mathbb{A}} + 1_{\mathbb{C}[\Gamma]} \otimes (e^{\delta \not{\mathbb{A}}} - 1_{\mathbb{W} \otimes V}). \quad (11.2.8)$$

The operator appearing on the right-hand side of the last equation is thus the replacement for the operator $\not{A}$ appearing in equation (10.0.2). This operator now represents the dynamical quantum, or generic synchronic step, when the defect is clothed. The argument leading to an identification of the appropriate localized continuum Lagrangian density, as given in equation (10.1.21), may now be repeated with this operator in place of $\not{A}$.

### *11.2.1 The Interaction Term*

Before moving on to find the continuum correspondent of this operator, it is worth considering, while still in its pristine form, the second term; specifically the right-most factor of this term, namely

$$\not{\mathbb{A}} \equiv \mathbb{A}^{\mu} \otimes \gamma_{\mu}. \quad (11.2.1.1)$$

This is an element of type

$$\mathbb{W} \otimes \mathbb{W}^{*} \otimes V \otimes V^{*} \quad (11.2.1.2)$$

and represents a compound act comprising superpositions of bispinor injections/ejections paired with $w$-defect injections/ejections. This pairing is replaced, for macroscopic maximally external experimenters, by an associated amplitude whose modulus will yield a measure of the "strength" of the $\alpha_c$ quanta interactions with fermions. Even though (as already remarked) the strengths of these couplings seem not to be calculable within a projective theory such as ours, certain qualitative conclusions may yet be drawn.

To see these, we need to specify the $\mathbb{A}^{\mu}$ and $\mathbb{W}$ a little more

precisely. We have already made a choice for the spin component of $\mathfrak{a}_c$: namely, the bispinor representation acting on $V$. Let us denote the corresponding component of $\mathbb{A}^\mu$ by $A^\mu_S$. For the $\mathfrak{su}(n)$ components we choose the simplest irreducible representations, namely the fundamental ones, given by the defining matrix actions, namely

$$\mathfrak{su}(n) = \{A \in \mathfrak{sl}(n,\mathbb{C}) : A^\dagger + A = 0\}. \qquad (11.2.1.3)$$

We denote the vector spaces accommodating these representations by $W_n$ and the corresponding components of an $\mathbb{A}$ in $\mathfrak{a}_c$ by $A_n$. Thus

$$\mathbb{W} = W_2 \oplus W_3 \oplus V, \qquad (11.2.1.4)$$

and

$$\mathbb{A}^\mu = A^\mu_2 \oplus A^\mu_3 \oplus A^\mu_S \qquad (11.2.1.5)$$

for $A^\mu_n$ in $\mathfrak{su}(n)$, $n = 2,3$, and $A^\mu_S$ in $\mathfrak{so}(3,1) \cong \mathfrak{sl}(2,\mathbb{C})$.

Thus we have

$$\begin{aligned} \not{\mathbb{A}} &= (A^\mu_2 \oplus A^\mu_3 \oplus A^\mu_S) \otimes \gamma_\mu \\ &= (A^\mu_2 \otimes \gamma_\mu) \oplus (A^\mu_3 \otimes \gamma_\mu) \oplus (A^\mu_S \otimes \gamma_\mu) \\ &= \not{A}_2 \oplus \not{A}_3 \oplus \not{A}_S. \end{aligned} \qquad (11.2.1.6)$$

Now we note something immediately striking about $\not{A}_2$. Namely, since the complexification of $\mathfrak{su}(2)$ is $\mathfrak{sl}(2,\mathbb{C})$, the representation of $\mathfrak{su}(2)$ upon $W_2$ lifts to the fundamental complex two-dimensional representation of $\mathfrak{sl}(2,\mathbb{C})$. This representation in turn exponentiates up to that (holomorphic) representation of $\mathrm{SL}(2,\mathbb{C})$

we have been calling $S$. In other words

$$W_2 = S. \tag{11.2.1.7}$$

Thus $\mathcal{A}_2$ is actually of type

$$S \otimes S^* \otimes V \otimes V^* \tag{11.2.1.8}$$

and internal transitions must now be allowed since $V \equiv S \oplus \widetilde{S}^*$. But then surviving interaction amplitudes can only involve $S$, since $\widetilde{S}$ (the anti-holomorphic irreducible two-dimensional representation of $\mathrm{SL}(2,\mathbb{C})$) is orthogonal to it. In other words, an external experimenter can only register couplings between $A_2^\mu$ and fermions of *one chirality* (in view of the remarks at the end of §10.3), namely the left-handed one associated with the $S$ component of $V$, since the transitions giving rise to the interaction with the right-handed $\widetilde{S}^*$ component are null. Thus the $\mathfrak{su}(2)$ interaction breaks the chiral symmetry of the fermions involved. (Cf. equation (11.2.1.34) *et seq.* for more detail.)

There is another peculiarity associated with the fact that the complexification of $\mathfrak{su}(2)$ is $\mathfrak{sl}(2,\mathbb{C})$: namely, the circumstance that the chosen representation for $\mathfrak{su}(2)$ (as a quantum symmetry of $W_2 = S$), when extended *via* complexification to $\mathfrak{sl}(2,\mathbb{C})$, is already *in use* (by the experimenter) as the algebra of symmetries giving rise to the $S$ spinor structure. Moreover, this action of $\mathfrak{su}(2)$ *necessarily* extends to the action of $\mathfrak{sl}(2,\mathbb{C})$ that is in use, since no operational distinction between these two actions could be made, *se ipse,* by the experimenter at the level of the acts represented by the elements of the underlying complex space $S$. Presumably this double use of the underlying space should be taken into account in our dealings with this action of $\mathfrak{su}(2)$. Thus, since the action of $\mathfrak{su}(2)$ upon $W_2$ extends "spontaneously" to the action of $\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{su}(2) \cong \mathfrak{sl}(2,\mathbb{C})$ upon $\mathbb{C} \otimes_{\mathbb{R}} W_2$, such account-taking would entail the replacement of a "seized" defect $w_2$ in $W_2$ by an equivalent one of the form $e^{i\theta} \otimes w_2$, $\theta \in \mathbb{R}$, in which the now explicit parameter $\theta$ must be allowed to vary

independently of the $\mathfrak{su}(2)$-symmetric part $w_2$. Then

$$d(e^{i\theta}\otimes w_2\otimes s) = ie^{i\theta}d\theta\otimes w_2\otimes s + e^{i\theta}\otimes d(w_2\otimes s)$$

$$= (id\theta\otimes 1_{W_2}\otimes 1_V + 1_{\mathbb{C}}\otimes \not{A}_2)(e^{i\theta}\otimes w_2\otimes s)$$

$$= (A_1^{\mu}\otimes 1_{W_2}\otimes \gamma_{\mu} + 1_{\mathbb{C}}\otimes \not{A}_2)(e^{i\theta}\otimes w_2\otimes s), \tag{11.2.1.9}$$

where

$$id\theta\otimes 1_{W_2} = A_1^{\mu}\theta\otimes 1_{W_2}\otimes \gamma_{\mu} \tag{11.2.1.10}$$

for imaginary numbers $A_1^{\mu}$.

Thus, the effect upon the interaction term of spontaneously complexifying the $\mathfrak{su}(2)$ action is that the interaction term $\not{A}_2$ acquires an additive term of the form $\not{A}_1 1_{W_2}$, where the $A_1^{\mu}$ are imaginary numbers. (This $\not{A}_1$ interaction term does not share the chiral breaking, of course.) In going spontaneously from $\mathfrak{su}(2)$ to its complexification we are compelled to incorporate into defect acts the symmetry coming from the $\mathfrak{su}(2)$-independent $\mathbb{C}$ factor. Ordinarily negligible, the independent variation of this factor must nevertheless be taken into account. The $\mathfrak{su}(3)$ action could also be so compromised, but this is not *required* for consistency with any prior or simultaneous uses of $\mathfrak{sl}(3,\mathbb{C})$ since there are none, and therefore no complexification is entailed.

By the same token, there is also something striking about the spin component $\not{A}_S$. This term is of type

$$V^*\otimes V\otimes V^*\otimes V, \tag{11.2.1.11}$$

and internal transitions may now be assumed to take precedence, as

in equation (7.2.1.6). These transitions take place between *static* and *spinnable* bispinorial acts and so presumably register macroscopically as a kind of spin-flow. The ramifications of this will be taken up later.

Now we attempt the passage to the continuum limit. We shall continue to assume that both $\mathfrak{a}_c$ and $\mathbb{W}$ consist of entities that are "reticular" in the sense that their components may ultimately be expressed in terms of reticular variables. Then, from equation (9.3.15), we would have, for instance, with $w$ in $\mathbb{W}$,

$$\begin{aligned}[[\mathbb{A}^{\mu} w \otimes \gamma_{\mu}]] &= [[dw]] \\ &= d[[w]] \end{aligned} \qquad (11.2.1.12)$$

or

$$[[\mathbb{A}^{\mu} w]] \otimes \gamma_{\mu} = -i\tau \partial^{\mu} [[w]] \otimes \gamma_{\mu} \ . \qquad (11.2.1.13)$$

Now $[[\mathbb{A}]]$ is supposed to represent the macroscopic continuum interpretation of the selective act $\mathbb{A}$, so should act to satisfy

$$[[\mathbb{A}]][[w]] = [[\mathbb{A}w]] \qquad (11.2.1.14)$$

and may be so defined. Then it follows immediately that $[[\ ]]$ preserves products of $\mathbb{A}$s.

With $\mathbb{A}_C$ defined so that

$$\partial^{\mu} [[w]] = \mathbb{A}_C^{\mu} [[w]] \qquad (11.2.1.15)$$

we obtain, from equations (11.2.1.13) through (11.2.1.15)

$$[[\mathbb{A}^{\mu}]]_C = -i\tau \mathbb{A}_C^{\mu} \qquad (11.2.1.16)$$

where the $C$ subscript as usual merely connotes the replacement everywhere of $q_\mu$ by one of its classical correspondents such as $x_\mu$.

From equation (11.2.1.15) it would appear that $\mathbb{A}^\mu_C$ is a continuum analog of $\mathbb{A}^\mu$, still acting upon $\mathbb{W}$. In keeping with our earlier assumption that defect acts should maintain their pristine condition even when employed by macroscopic experimenters, we shall assume that an operator of the form $\mathbb{A}_C$ reflects the same symmetries as $\mathbb{A}$, lying therefore in the same Lie algebra, namely $\mathfrak{a}_c$. In particular, considering each component of $\mathbb{A}^\mu$ in turn, we have

$$[\![ A_2^\mu ]\!]_C = -i\tau(A_2^\mu)_C$$

$$\equiv -\frac{\tau}{2}W^\mu, \qquad (11.2.1.17)$$

for some trace free Hermitian $W^\mu$, $(A_2^\mu)_C$ lying in $\mathfrak{su}(2)$ (cf. equation (11.2.1.3)), and with the factor $\frac{1}{2}$ appearing for cosmetic reasons: thus,

$$(A_2^\mu)_C = -\frac{i}{2}W^\mu. \qquad (11.2.1.18)$$

For the three-dimensional component:

$$[\![ A_3^\mu ]\!]_C = -i\tau(A_3^\mu)_C$$

$$\equiv -\frac{\tau}{2}G^\mu \qquad (11.2.1.19)$$

for some trace free Hermitian $G^\mu$, $(A_3^\mu)_C$ lying in $\mathfrak{su}(3)$, etc., giving

$$(A_3^\mu)_C = -\frac{i}{2}G^\mu. \qquad (11.2.1.20)$$

For the spin component

$$[\![A_S^\mu]\!]_C = -i\tau(A_S^\mu)_C, \tag{11.2.1.21}$$

where $(A_S^\mu)_C$ lies in the Lie algebra generated by the (infinitesimal version of) the bispinor representation whose generators are $i\sigma^{mn}$ and may therefore be expressed in the form

$$(A_S^\mu)_C = \frac{1}{2}\left(\frac{1}{2} i\omega_{kl}^\mu \sigma^{kl}\right), \tag{11.2.1.22}$$

where the "spin connection" coefficients $\omega_{kl}^\mu$ are real and antisymmetric in their lower indices. Thus

$$[\![A_S^\mu]\!]_C = \frac{\tau}{4}\omega_{kl}^\mu \sigma^{kl}. \tag{11.2.1.23}$$

Finally, for the $A_1^\mu$ term we have

$$[\![A_1^\mu]\!]_C = -i\tau(A_1^\mu)_C \tag{11.2.1.24}$$

with $(A_1^\mu)_C$ also imaginary,

$$(A_1^\mu)_C \equiv -\frac{i}{2}B^\mu, \tag{11.2.1.25}$$

say, where $B^\mu$ is real. Then

$$[\![A_1^\mu]\!]_C = -\frac{\tau}{2}B^\mu. \tag{11.2.1.26}$$

Returning to the interaction term $\not{A}$, with $\not{A}_1$ appended as in equation (11.2.1.9), we note that its trace vanishes at least because of the presence of the $\gamma_\mu$ s, apparently denying the registration of any

interaction to an external experimenter, as in the treatment of the bare Dirac operator in the last chapter. Also similar to the discussion there is the presence of the $1_{\mathbb{C}[\Gamma]}$ in the second term in equation (11.2.8), whose trace is undefined. Both problems may be resolved in exactly the same way in the continuum. Namely, we multiplex the space $\mathbb{W}\otimes V$ upon which $\mathbb{A}$ acts to obtain a bundle over the manifold with fibre $\mathbb{W}\otimes V$. Now, elements of the latter space may be thought of as tuples of bispinors, with one entry for each dimension of $\mathbb{W}$. Thus, in a sense, each basis element of $\mathbb{W}$ may be replaced by a bispinor, and Lie algebra actions on $\mathbb{W}$ may now be thought of as acting upon tuples of spinors. The next step is to replace each spinor by its operator version as in the last chapter, and proceed to the appropriate operator version of the trace. Ignoring for a moment the complications of the two-dimensional component, the result in the continuum is a contribution to the full Lagrangian density of an expression of the form (cf. equations (10.2.14) through (10.2.16), and (11.2.1.16)):

$$\begin{aligned}\tau^{-4}(\det E)^{-1}\left(\overline{\Psi}_{\mathbb{W}}[\![\mathbb{A}]\!]\Psi_{\mathbb{W}}\right)^{\dagger}_{C} &= \tau^{-4}(\det E)^{-1}(-i\tau(\overline{\Psi}_{\mathbb{W}})_{C}\mathbb{A}_{C}(\Psi_{\mathbb{W}})_{C})^{\dagger}\\ &= (\det E)^{-1}(i\overline{\Psi}_{\mathbb{W}}\mathbb{A}_{C}\Psi_{\mathbb{W}})^{\dagger}\\ &= (\det E)^{-1}(i\overline{\Psi}_{\mathbb{W}}\gamma_{\mu}\mathbb{A}^{\mu}_{C}\Psi_{\mathbb{W}})^{\dagger}\end{aligned} \tag{11.2.1.27}$$

where $\Psi_{\mathbb{W}}$ is a tuple of bispinor operator versions of components of $\mathbb{W}$.

The presence of the $1_{\mathbb{W}}$ in the first term on the right-hand side of equation (11.2.8) similarly gives rise to a continuum expression of the form (ignoring the determinantal factor)

$$(i\overline{\Psi}_{\mathbb{W}}\not{\partial}\Psi_{\mathbb{W}})^{\dagger}, \tag{11.2.1.28}$$

for the same $\Psi_{\mathbb{W}}$s used in equation (11.2.1.27), and may be added to the previous term to yield

$$(i\overline{\Psi}_{\mathbb{W}}(\not{\partial}+\not{\mathbb{A}}_C)\Psi_{\mathbb{W}})^{\dagger}. \qquad (11.2.1.29)$$

This expression may be broken down further into a sum of its individual components. Starting with the apparently less complicated three-dimensional component, its contribution to the expression (11.2.1.29) is, from equation (11.2.1.20), the self-adjoint operator

$$i\overline{\Psi}_G\left(\not{\partial}-\frac{i}{2}\not{G}\right)\Psi_G, \qquad (11.2.1.30)$$

where $\Psi_G$ in this context denotes a triplet of bispinor fields in the fundamental representation of $\mathfrak{su}(3)$.

The spin "connection" (equation (11.2.1.22)) contributes the term

$$i\overline{\Psi}\gamma_\mu\left(\partial^\mu+\frac{i}{4}\omega^\mu_{kl}\sigma^{kl}\right)\Psi, \qquad (11.2.1.31)$$

where $\Psi$ here denotes the basic bispinor operator considered in the last chapter. Since the spin connection operator appearing in the last expression acts in the space of bispinors itself, it should properly be incorporated into all the interaction terms. That is to say, $\partial^\mu$, which acts on individual spinors within tuples, should be, strictly speaking, universally replaced by $\partial^\mu+(i/4)\omega^\mu_{kl}\sigma^{kl}$.

Finally, we turn to the complexified $\mathfrak{su}(2)$ interaction arising from the operator appearing in the right-hand side of equation (11.2.1.9). Its continuum version is, from equations (11.2.1.18) and (11.2.1.25), and writing the identity function on $W_2$ as $I_2$:

$$(\not{A}_1)_C I_2+(\not{A}_2)_C=-\frac{i}{2}(\not{B}I_2+\not{W}). \qquad (11.2.1.32)$$

A further complication here is the non-coupling found earlier of the $\mathfrak{su}(2)$ field with the right-handed ($\tilde{S}^*$) component of each bispinor pair. This has the consequence in the case of $\not{W}$ that it can only appear in the Lagrangian (cf. equation (10.2.18)) acting upon left-handed spinors. The resulting contribution to the Lagrangian thus assumes the form

$$i\left[\psi^{\dagger}_{W,R}\,\sigma_{\mu}(\partial^{\mu}-\frac{i}{2}B^{\mu}I_2)\psi_{W,R}\oplus\psi^{\dagger}_{W,L}\,\bar{\sigma}_{\mu}(\partial^{\mu}-\frac{i}{2}B^{\mu}I_2-\frac{i}{2}W^{\mu})\psi_{W,L}\right].$$

(11.2.1.33)

Here $\psi_{W,k}$, $k=L,R$, denotes an $\mathfrak{su}(2)$ doublet whose entries are the Weyl spinors, which are the appropriate chiral components of $V$.

To see the non-coupling *in situ,* as it were, we note that $W^{\mu}$ is actually of type $S\otimes S^*$ so that its action on a right-handed $W_2$ spinor doublet—i.e. an element of $W_2\otimes\tilde{S}^*=S\otimes\tilde{S}^*$—would entail contractions of the middle pairs in a **GQ** formula, or sequence, of type

$$S\otimes S^*\otimes S\otimes\tilde{S}^* \qquad (11.2.1.34)$$

leading to an element of type

$$S\otimes\tilde{S}^*. \qquad (11.2.1.35)$$

But now a further contraction is possible and gives zero, since $S$ is orthogonal to $\tilde{S}$ (cf. §4.2.1). Thus, external experimenters will detect no interaction of $W^{\mu}$ with fermions of the chirality corresponding with the $\tilde{S}^*$ component of $V$.

*11.2.2 The Kinetic Term*

Now we turn to the problem of macroscopically interpreting the remaining term in the operator on the right-hand side of equation (11.2.8), namely

$$1_{\mathbb{C}[\Gamma]}\otimes(e^{\delta\mathbb{A}}-1_{\mathbb{W}\otimes V}). \qquad (11.2.2.1)$$

Here, again, the appearance of the identity function upon $\mathbb{C}[\Gamma]$ signifies the presence over the macroscopic continuum of the same bundle as before, with fibre $\mathbb{W}\otimes V$, upon which the other factor acts as a bundle endomorphism.

Our first task is to interpret $\delta\mathbb{A}$, which should appear to give rise to an *increment*, when interpreted macroscopically in the continuum. Supposing $\delta$ to act as a derivation on $\mathrm{End}(\mathbb{W}\otimes V)$, we have

$$\begin{aligned}\delta\mathbb{A} &= \delta((\mathbb{A}^{\mu}\otimes 1)(1\otimes\gamma_{\mu}))\\ &= (\delta(\mathbb{A}^{\mu}\otimes 1))(1\otimes\gamma_{\mu})+(\mathbb{A}^{\mu}\otimes 1)(\delta(1\otimes\gamma_{\mu})).\end{aligned} \qquad (11.2.2.2)$$

For the first term on the right-hand side of equation (11.2.2.2) we may put

$$\delta\mathbb{A}^{\mu}=\delta^{\nu}\mathbb{A}^{\mu}\otimes\gamma_{\nu} \qquad (11.2.2.3)$$

so that it remains to interpret the incrementing operator $\delta^{\nu}$. An obvious candidate for $\delta^{\nu}$ is of course $\Delta^{\nu}$. Now, as a consequence of our standing and unverifiable assumption that elements $\mathbb{A}$, say, in the chosen representation of $\mathfrak{a}_c$ may be expressed ultimately in reticular terms—for example by being realizable as matrices with entries in $\mathbb{C}[\Gamma^{N}]$—it follows from equations (9.3.3) and (11.2.1.14) that

$$[\![\Delta^\nu \mathbb{A}]\!] = [\![\Delta^\nu]\!][\![\mathbb{A}]\!]\,. \qquad (11.2.2.4)$$

Then

$$[\![\Delta^\nu \mathbb{A}^\mu]\!]_C = [\![\Delta^\nu]\!][\![\mathbb{A}^\mu]\!]_C$$

$$= -i\tau\partial^\nu(-i\tau\mathbb{A}^\mu_C)$$

from equation (11.2.1.16)

$$= -\tau^2\partial^\nu \mathbb{A}^\mu_C, \qquad (11.2.2.5)$$

which shows that, independently of any presumed directional encoding of $\tau$, $\Delta^\nu$ must be interpreted macroscopically as a *decrementation* operator when acting upon elements of $\mathfrak{a}_c$. Consequently, the correct choice in equation (11.2.2.3) is

$$\delta^\nu = -\Delta^\nu. \qquad (11.2.2.6)$$

For the second term in equation (11.2.2.2) we may take, as usual,

$$\delta w = dw$$

$$= \not{\mathbb{A}} w, \qquad (11.2.2.7)$$

so

$$(\delta\,(1\otimes\gamma_\mu))(w\otimes s) = \delta\,((1\otimes\gamma_\mu)(w\otimes s)) - (1\otimes\gamma_\mu)\delta\,(w\otimes s)$$

$$= \delta\,(w\otimes\gamma_\mu(s)) - (1\otimes\gamma_\mu)(\mathbb{A}^\nu w\otimes\gamma_\nu(s))$$

$$= \mathbb{A}^{\nu} w \otimes (\gamma_{\nu}\gamma_{\mu})(s) - \mathbb{A}^{\nu} w \otimes (\gamma_{\mu}\gamma_{\nu})(s)$$

(cf. equation (9.3.19) for the first term in the last line)

$$= (\mathbb{A}^{\nu} \otimes [\gamma_{\nu}, \gamma_{\mu}])(w \otimes s). \qquad (11.2.2.8)$$

Putting all this together yields, from equations (11.2.2.2) through (11.2.2.8),

$$\delta \not{\mathbb{A}} = (\delta \mathbb{A}^{\mu} \otimes 1)(1 \otimes \gamma_{\mu}) + (\mathbb{A}^{\mu} \otimes 1)(\mathbb{A}^{\nu} \otimes [\gamma_{\nu}, \gamma_{\mu}])$$

$$= -\Delta^{\nu} \mathbb{A}^{\mu} \otimes \gamma_{\nu}\gamma_{\mu} + \mathbb{A}^{\mu}\mathbb{A}^{\nu} \otimes [\gamma_{\nu}, \gamma_{\mu}]. \qquad (11.2.2.9)$$

At this point we may note that the quantity known as the *metric* now makes an appearance when the trace of $\delta \not{\mathbb{A}}$ is taken. Readers who have taken the exercise suggested in the paragraph following equation (9.4.37) will not be surprised to learn that

$$\mathrm{tr}(\gamma_m \gamma_n) = 4\eta_{mn}, \qquad (11.2.2.10)$$

where $\eta_{mn}$ is given in equation (11.1.19). This matrix is known for historical reasons as the *Minkowski metric*, and its well-known transformation properties now follow from equation (9.4.39). The matrix $\eta^{mn}$, which is the inverse of $\eta_{mn}$, may be used to raise and lower Latin indices in a way familiar to students of relativity, namely:

$$[\ ]^m \equiv \eta^{mk}[\ ]_k \qquad (11.2.2.11)$$

etc. (Of course, $\eta^{mn} = \eta_{mn}$.)

We define a Greek metric by

$$g_{\mu\nu} \equiv \frac{1}{4}\mathrm{tr}(\gamma_\mu \gamma_\nu)$$

$$= \frac{1}{4} e_\mu^k e_\nu^l \,\mathrm{tr}(\gamma_k \gamma_l)$$

$$= e_\mu^k e_\nu^l \eta_{kl}. \qquad (11.2.2.12)$$

Denoting by $g^{\mu\nu}$ the matrix inverse of the matrix $g_{\mu\nu}$, it is an elementary exercise to show that

$$g^{\mu\nu} = e_k^\mu e_l^\nu \eta^{kl}. \qquad (11.2.2.13)$$

These may be used to raise and lower Greek indices.

We note in passing how the last equation may be used to relate $\det E$ to the metric. First, recalling §9.4, write this equation in matrix form:

$$M = E^{-1}\eta(E^{-1})^T. \qquad (11.2.2.14)$$

Here, $M$ denotes the matrix whose entries are $g^{\mu\nu}$ and $\eta$ the matrix whose entries are $\eta^{mn} = \eta_{mn}$. Taking determinants we obtain

$$\det M = -(\det E^{-1})^2 \qquad (11.2.2.15)$$

so

$$(\det E)^{-1} = (-\det M)^{1/2}. \qquad (11.2.2.16)$$

In the literature one often finds $\det M$ denoted by $g$.

Thus, the measure that arises when a macroscopic experimenter integrates a Lagrangian density of the form equation (10.1.21) against $d^4x_\mu$ may be written

$$(-g)^{1/2} d^4 x_\mu . \qquad (11.2.2.17)$$

This measure is invariant with respect to transformations of the vierbein itself: cf. equation (9.4.1.9). For, $d^4 x_\mu$ acquires the factor $\det \Lambda$ while at the same time $(\det E)^{-1}$ acquires the factor $(\det \Lambda)^{-1}$.

Now we note that the well-known and easily checked anti-commutation relations

$$\gamma_k \gamma_l + \gamma_l \gamma_k = 2\eta_{kl} 1_V \qquad (11.2.2.18)$$

and the definition

$$\gamma_k \gamma_l - \gamma_l \gamma_k = -2i\sigma_{kl} \qquad (11.2.2.19)$$

imply

$$\gamma_k \gamma_l = \eta_{kl} 1_V - i\sigma_{kl} \qquad (11.2.2.20)$$

from which it follows that

$$\gamma_\mu \gamma_\nu = g_{\mu\nu} 1_V - i\sigma_{\mu\nu}, \qquad (11.2.2.21)$$

where

$$\sigma_{\mu\nu} \equiv e_\mu^k e_\nu^l \sigma_{kl}$$

$$= \frac{i}{2}[\gamma_\mu, \gamma_\nu]. \qquad (11.2.2.22)$$

These last two identities may be used to reexpress the right-

hand side of equation (11.2.2.9) as follows:

$$\delta \mathbb{A} = -\Delta^{\nu} \mathbb{A}^{\mu} \otimes (g_{\mu\nu} 1_V + i\sigma_{\mu\nu}) + \mathbb{A}^{\mu}\mathbb{A}^{\nu} \otimes 2i\sigma_{\mu\nu}$$

$$= -\Delta^{\nu} \mathbb{A}^{\mu} \otimes g_{\mu\nu} 1_V + i(-\Delta^{\nu} \mathbb{A}^{\mu} + 2\mathbb{A}^{\mu}\mathbb{A}^{\nu}) \otimes \sigma_{\mu\nu}$$

$$= -\Delta^{\nu} \mathbb{A}^{\mu} \otimes g_{\mu\nu} 1_V + \frac{i}{2}\left(\Delta^{\mu} \mathbb{A}^{\nu} - \Delta^{\nu} \mathbb{A}^{\mu} + 2[\mathbb{A}^{\mu}, \mathbb{A}^{\nu}]\right) \otimes \sigma_{\mu\nu} \tag{11.2.2.23}$$

using the skew-symmetry of the $\sigma_{\mu\nu}$ in the second term.

From equation (11.2.1.16) and the correspondence principle, we find (dropping the $V$ subscript from the 1)

$$\left[\!\left[ \delta \mathbb{A} \right]\!\right]_C = (i\tau\partial^{\nu})(-i\tau \mathbb{A}_C^{\mu}) \otimes g_{\mu\nu} 1 +$$

$$+ \frac{i}{2}\left((-i\tau\partial^{[\mu})(-i\tau \mathbb{A}_C^{\nu]}) + 2(-i\tau)^2 [\mathbb{A}_C^{\mu}, \mathbb{A}_C^{\nu}]\right) \otimes \sigma_{\mu\nu}$$

$$= \frac{\tau^2}{2}\left(2\partial^{\nu} \mathbb{A}_C^{\mu} \otimes g_{\mu\nu} 1 - i(\partial^{[\mu} \mathbb{A}_C^{\nu]} + 2[\mathbb{A}_C^{\mu}, \mathbb{A}_C^{\nu}]) \otimes \sigma_{\mu\nu}\right). \tag{11.2.2.24}$$

That is, defining

$$\mathbb{F}_C^{\mu\nu} \equiv \partial^{[\mu} \mathbb{A}_C^{\nu]} + 2[\mathbb{A}_C^{\mu}, \mathbb{A}_C^{\nu}] \tag{11.2.2.25}$$

we have

$$\left[\!\left[ \delta \mathbb{A} \right]\!\right]_C = \frac{\tau^2}{2}\left(2\partial^{\nu} \mathbb{A}_C^{\mu} \otimes g_{\mu\nu} 1 - i\mathbb{F}_C^{\mu\nu} \otimes \sigma_{\mu\nu}\right). \tag{11.2.2.26}$$

The "kinetic" quantity of interest (from the analog of equation (10.1.21), with the operator on the right-hand side of equation (11.2.8) replacing $\not{A}$) is

$$\mathrm{tr}\left[\!\left[ e^{\delta\not{\mathbb{A}}} - 1 \right]\!\right]_C^\dagger = \mathrm{tr}\left[\!\left[ \delta\not{\mathbb{A}} + \frac{1}{2!}(\delta\not{\mathbb{A}})^2 + \ldots \right]\!\right]_C^\dagger . \qquad (11.2.2.27)$$

Here we encounter a fork in the road, which, being quantum experimenters, we are compelled to take. Recall that the second quantized version of $V$, with Pauli–Dirac adjointness, etc., was originally invoked in §10.2 in order to amplify a macroscopic experimenter's registration of the dynamics associated with the bare defect, which would otherwise be invisible. That is to say, the use of these devices (namely second quantization, etc.) had the effect of increasing the experimenter's degree of resolution (or correspondingly lowering the degree of externality): indeed, we were left with operators, not amplitudes. In §11.2 we extended these methods to achieve a similar effect for the case of the clothed defect. This led to the interaction terms displayed there, which are similarly operators, or selective acts, representing amplifications of acts pertaining to a deeper level. If we proceed at this resolution (or non-maximal degree of externality) then equation (11.2.2.27) supplies us with a power series in $\tau$ of extra interaction terms starting with

$$-\tau^5 \overline{\Psi}_{\mathbb{W}} (\partial^\nu (\mathbb{A}_C)_\nu - i\mathbb{F}_C^{\mu\nu}\sigma_{\mu\nu})\Psi_{\mathbb{W}} . \qquad (11.2.2.28)$$

Thus we find spin couplings and their higher order correspondents. Some of these terms may supply ignorable total derivatives that would be eliminated by a suitable path integration, some may not. On the other hand, if we are prepared to believe that $\tau$ is very small indeed at the relatively macroscopic scales that we are expressly interested in, then we may ignore these terms from a certain point on in the series, and $O(\tau^5)$ seems a good place to perform this Procrustean process, since the Lagrangian density we obtain from the analog of

equation (10.1.21) would in any case be too full of dimensions if any of these terms were retained.

The other branch of the fork appears when we view equation (11.2.2.27) back at the maximally external resolution, which was too coarse to reveal any contributions from the lower order terms without amplification. This is the resolution at which we originally derived our Lagrangian and at which $\mathbb{W}$ and $V$ are considered as representing two non-interfacing (or non-interacting) quantum systems, except where a possible internal interface may be assumed to take precedence. This latter situation obtains for the right-most component of $\mathbb{W}$, as mentioned earlier. Except in this case, the amplitudes so produced will yield a macroscopic experimenter's Lagrangian describing the dynamics of the field $\mathbb{A}_C$ itself, decoupled from the associated fermionic selective acts which are elements of $\mathrm{End}V$. In the former case, namely that of the right-most component, where inner-to-outer full contraction is presumed to take precedence, the resulting Lagrangian must pertain to the transitions—or (self) interactions—between static and spinnable defect systems, and presumably describes locally possible flows of spin, as remarked earlier. On the other hand, our original models for the Dirac maps were cotangent representers, and we realized them as infinitesimal "parallel transporters," so this spin-flow must in a sense contain the sort of information macroscopic experimenters might interpret as geometrical. (In fact, there is a certain strong sense in which $\mathrm{End}V$, where $V$ now denotes the spinor bundle over a spin manifold, and the associated Dirac operator, contains *all* the geometrical information inherent in the manifold: cf. Connes 1994, VI.) Moreover, since we already have an interaction term in this case, namely equation (11.2.1.31), the additional interactive contribution resulting from this internal contraction is sure to lead to an extremely nonlinear field, which, we recall, couples to everything in sight. Macroscopic continuum-dwelling experimenters will recognize in it the hallmarks of the field known to them as *gravitational*.

Before we launch into the computations of the factors contributing to the right-hand side of equation (11.2.2.27), some further

remarks are in order. These concern the degree of resolution associated with the acts represented by the elements of $\mathbb{W}$. Since we have interpreted its spin component as $V$ itself, we may assume that the degree of resolution attributable to this component is as we have already interpreted it: namely, that high degree of resolution associated with the pristine reticular defect whose space of initial acts it represents. The degree of resolution embodied in our choices of $W_2$ and $W_3$ are, on the other hand, conditioned by macroscopic desiderata, since their symmetries derived from the dragged-back continuum-based Schrödinger vacuum. (These symmetries were employed, *a fortiori*, as coarsened macroscopic operators upon $\mathbb{W}$ in equation (11.1.23).) Consequently we should assume that acts represented by elements in $W_2$ and $W_3$ are associated with a coarser degree of resolution, relative to the resolution of the acts represented by elements of $V$. This resolutional mismatch does little harm when the associated interaction and "kinetic" terms (shortly to be derived) are considered separately, but presents a problem if these terms are to be combined, since the kinetic terms represent amplitudes associated with selective acts upon $W_2$ and $W_3$ which are decoupled from those associated with spinors, and therefore remain at a presumably coarser level of resolution. This issue will be addressed in §11.3. It is worth noting again that the problem does not arise for the spin component.

Once the correct acts and selective acts have been determined so as to redress this resolutional imbalance, an experimenter will still be faced with the problem of second quantizing the c-number expressions which emerge as kinetic terms. Since $\mathbb{W}$ was derived from a bosonic vacuum, one might be forgiven for believing that the $\mathbb{A}$ fields become boson creators. In fact, we have now met up with standard physics and may refer the reader to works already or to be cited, many of which start at, or near, this point.

Returning to the right-hand side of equation (11.2.2.27) it is clear that it decomposes into a sum of corresponding traces for each component. We shall consider each in turn, starting with the three-dimensional component.

From equations (11.2.1.20) and (11.2.2.25)

$$(F_3^{\mu\nu})_C \equiv \partial^{[\mu}(A_3^{\nu]})_C + 2[(A_3^{\mu})_C,(A_3^{\nu})_C]$$

$$= -\frac{i}{2}\partial^{[\mu}G^{\nu]} + 2\left(-\frac{i}{2}\right)^2[G^{\mu},G^{\nu}]$$

$$= -\frac{i}{2}\left(\partial^{[\mu}G^{\nu]} - i[G^{\mu},G^{\nu}]\right)$$

$$\equiv -\frac{i}{2}G^{\mu\nu}, \qquad (11.2.2.29)$$

so that from equation (11.2.2.26)

$$[\![\,\delta\not{A}_3\,]\!]_C = \frac{\tau^2}{2}\left(2\partial^{\nu}(A_3^{\mu})_C \otimes g_{\mu\nu}1 - i(F_3^{\mu\nu})_C \otimes \sigma_{\mu\nu}\right)$$

$$= -\frac{\tau^2}{2}\left[i\partial^{\nu}G_{\nu}\otimes 1 + \frac{1}{2}G^{\mu\nu}\otimes\sigma_{\mu\nu}\right]. \qquad (11.2.2.30)$$

Thus

$$\mathrm{tr}[\![\,\delta\not{A}_3\,]\!]_C = 0 \qquad (11.2.2.31)$$

and

$$[\![\,(\delta\not{A}_3)^2\,]\!]_C = [\![\,\delta\not{A}_3\,]\!]_C^2$$

$$= \frac{\tau^4}{4}\left[-(\partial^\nu G_\nu)^2 \otimes 1 + \frac{1}{4} G^{\mu\nu} G^{\rho\tau} \otimes \sigma_{\mu\nu}\sigma_{\rho\tau} + \text{trace free terms}\right]$$

(11.2.2.32)

so

$$\mathrm{tr}\left[\!\!\left[ \delta \mathcal{A}_3 + \frac{1}{2!}(\delta \mathcal{A}_3)^2 + \ldots \right]\!\!\right]_C^\dagger$$

$$= \frac{\tau^4}{2!\,4}\left[-4\,\mathrm{tr}(\partial^\nu G_\nu)^2 + \frac{1}{4}\mathrm{tr}(G^{\mu\nu}G^{\rho\tau})\,\mathrm{tr}(\sigma_{\mu\nu}\sigma_{\rho\tau})^\dagger\right] + O(\tau^6).$$

(11.2.2.33)

It remains to compute $\mathrm{tr}(\sigma_{\mu\nu}\sigma_{\rho\tau})^\dagger$. To this end we note (Weinberg 1995, p. 372) that

$$\mathrm{tr}(\gamma_k\gamma_l\gamma_m\gamma_n) = 4(\eta_{kl}\eta_{mn} + \eta_{kn}\eta_{lm} - \eta_{km}\eta_{ln}),$$

(11.2.2.34)

an identity that depends only upon the anticommutation relations equation (11.2.2.18), from which it follows immediately that

$$\mathrm{tr}(\gamma_\lambda\gamma_\mu\gamma_\nu\gamma_\rho) = 4(g_{\lambda\mu}g_{\nu\rho} + g_{\lambda\rho}g_{\mu\nu} - g_{\lambda\nu}g_{\mu\rho}).$$

(11.2.2.35)

A tedious calculation now yields

$$\mathrm{tr}(\sigma_{\mu\nu}\sigma_{\rho\tau}) = 4(g_{\mu\rho}g_{\nu\tau} - g_{\mu\tau}g_{\nu\rho}), \qquad (11.2.2.36)$$

which is real, so that

$$\mathrm{tr}(\sigma_{\mu\nu}\sigma_{\rho\tau})^{\dagger} = \overline{\mathrm{tr}(\sigma_{\mu\nu}\sigma_{\rho\tau})}$$

$$= \mathrm{tr}(\sigma_{\mu\nu}\sigma_{\rho\tau}). \qquad (11.2.2.37)$$

Thus

$$\frac{1}{4}\mathrm{tr}(G^{\mu\nu}G^{\rho\tau})\,\mathrm{tr}(\sigma_{\mu\nu}\sigma_{\rho\tau})^{\dagger} = \mathrm{tr}(G^{\mu\nu}G^{\rho\tau})(g_{\mu\rho}g_{\nu\tau} - g_{\mu\tau}g_{\nu\rho})$$

$$= \mathrm{tr}(G^{\mu\nu}G^{\rho}_{\nu}g_{\mu\rho}) - \mathrm{tr}(G^{\mu\nu}G^{\rho}_{\mu}g_{\nu\rho})$$

$$= \mathrm{tr}(G^{\mu\nu}G_{\nu\mu}) - \mathrm{tr}(G^{\mu\nu}G_{\mu\nu})$$

$$= -2\,\mathrm{tr}(G^{\mu\nu}G_{\mu\nu}). \qquad (11.2.2.38)$$

Putting all this together, and retaining only the terms up to $\tau^4$, we obtain as our Lagrangian density (cf. equation (10.1.21)) for the $\mathfrak{su}(3)$ contribution

$$\mathcal{L}_3 \equiv \tau^{-4}(-g)^{1/2}\left(\frac{\tau^4}{2!\,4}\left[-4\,\mathrm{tr}(\partial^{\nu}G_{\nu})^2 - 2\,\mathrm{tr}(G^{\mu\nu}G_{\mu\nu})\right]\right)$$

$$= (-g)^{1/2}\left[-\frac{1}{2}\mathrm{tr}(\partial^{\nu}G_{\nu})^2 - \frac{1}{4}\mathrm{tr}(G^{\mu\nu}G_{\mu\nu})\right]. \qquad (11.2.2.39)$$

The second term in this expression is exactly the famous Yang–Mills Lagrangian for the ($\mathfrak{su}(3)$-) *Yang–Mills* fields $G^{\mu\nu}$. The central rôle played by this particular form of Lagrangian in modern physics, its geometrical underpinnings (which we seem to have managed to avoid), physical and mathematical ramifications, etc., are well and oft-told tales. The reader may consult, among many others: Ryder 1996, Weinberg 1996, Sterman 1993, Nachtmann 1990,

Collins, Martin and Squires 1989, Shuryak 1988, Pokorski 1987, Commins and Bucksbaum 1983, Ynduráin 1983, Aitchison 1982, Aitchison and Hey 1982, Ramond 1981, and Taylor 1976.

The Yang–Mills Lagrangian's most significant attribute is its invariance with respect to certain transformations of the "potentials," proportional to $G^{\mu}$ in this case, called *gauge* transformations. (Please see §12.5 for a discussion of these.) Since path integrals must now be taken over possible histories of these fields, vast and divergent overcounting would result if the first term in $\mathscr{L}_3$ were absent, for then the gauge invariance of the Yang–Mills term would permit contributions to the path integral from histories of gauge equivalent fields, though these are physically indistinguishable. As it happens, the first term in $\mathscr{L}_3$, a *gauge-fixing* term, famously suppresses this overcounting, at the (affordable and necessary) cost of gauge-invariance of the whole expression. This particular gauge-fixing term—which is usually put in "by hand," but which we seem to have obtained free of charge—is named for Feynman: see the references cited above.

For the complexified $\mathfrak{su}(2)$ contribution, we have (cf. equation (11.2.1.32)) the following component of $\mathbb{A}^{\mu}_{C}$:

$$(A_1^{\mu})_C I_2 + (A_2^{\mu})_C = -\frac{i}{2}(B^{\mu} I_2 + W^{\mu}) \qquad (11.2.2.40)$$

so the appropriate component of equation (11.2.2.25) reads (cf. equation (11.2.2.29)):

$$-\partial^{[\mu}\left(\frac{i}{2}(B^{\nu]} I_2 + W^{\nu]})\right) + 2\left(-\frac{i}{2}\right)^2 [B^{\mu} I_2 + W^{\mu}, B^{\nu} I_2 + W^{\nu}]$$

$$= -\frac{i}{2}\partial^{[\mu} B^{\nu]} I_2 - \frac{i}{2}\left(\partial^{[\mu} W^{\nu]} - i[W^{\mu}, W^{\nu}]\right)$$

$$\equiv -\frac{i}{2}(B^{\mu\nu}I_2 + W^{\mu\nu}) \qquad (11.2.2.41)$$

so that the appropriate component of equation (11.2.2.26) is (cf. equation (11.2.2.30)):

$$\left[\!\left[\delta\mathbb{A}_{1,2}\right]\!\right]_C \equiv -\frac{\tau^2}{2}\left[i\partial^\nu(B_\nu I_2 + W_\nu)\otimes 1 + \frac{1}{2}(B^{\mu\nu}I_2 + W^{\mu\nu})\otimes\sigma_{\mu\nu}\right]. \qquad (11.2.2.42)$$

Thus

$$\mathrm{tr}\left[\!\left[\delta\mathbb{A}_{1,2}\right]\!\right]_C^\dagger = \frac{\tau^2}{2}[\,8i\partial^\nu B_\nu\,]$$

$$= 4i\tau^2\partial^\nu B_\nu, \qquad (11.2.2.43)$$

which, being a total derivative, may be discarded from the Lagrangian.

The remaining contribution (up to $\tau^4$) is

$$\frac{1}{2!}\mathrm{tr}\left(\left[\!\left[\delta\mathbb{A}_{1,2}\right]\!\right]_C^2\right)^\dagger =$$

$$= \mathrm{tr}\left[\frac{\tau^4}{2!\,4}\left(-(\partial^\nu(B_\nu I_2 + W_\nu))^2\otimes 1 + \frac{1}{4}(B^{\mu\nu}I_2 + W^{\mu\nu})(B^{\rho\tau}I_2 + W^{\rho\tau})\otimes\right.\right.$$

$$\left.\left.\otimes\,\sigma_{\mu\nu}\sigma_{\rho\tau} + \text{ trace free terms}\right)\right]$$

$$= \mathrm{tr}\left[\frac{\tau^4}{2!4}\left(-((\partial^\nu B_\nu)^2 I_2 + 2(\partial^\nu B_\nu)(\partial^\mu W_\mu) + (\partial^\mu W_\mu)^2)\otimes 1 + \right.\right.$$

$$\left.\left. + \frac{1}{4}(B^{\mu\nu}B^{\rho\tau}I_2 + W^{\mu\nu}W^{\rho\tau})\otimes\sigma_{\mu\nu}\sigma_{\rho\tau} + \text{ trace free terms}\right)\right]$$

$$= \tau^4\left[-(\partial^\nu B_\nu)^2 - \frac{1}{2}\mathrm{tr}(\partial^\nu W_\nu)^2 - \frac{1}{2}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}\mathrm{tr}(W^{\mu\nu}W_{\mu\nu})\right].$$

(11.2.2.44)

Thus we obtain the Lagrangian density

$$\mathcal{L}_{1,2} \equiv (-g)^{1/2}\left[-(\partial^\nu B_\nu)^2 - \frac{1}{2}\mathrm{tr}(\partial^\nu W_\nu)^2 - \frac{1}{2}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}\mathrm{tr}(W^{\mu\nu}W_{\mu\nu})\right].$$

(11.2.2.45)

It is worth noting that the fields $B^\mu$ and $W^\mu$ contribute separately (with the $B^\mu$ contribution doubled). The $B^\mu$ part of this Lagrangian exhibits the gauge symmetry associated with the group U(1).

Finally, we turn to the remaining $\mathfrak{so}(3,1)$ component. Here, from equation (11.2.1.22), the appropriate component of equation (11.2.2.26) is

$$[\![\delta\mathbb{A}_S]\!]_C =$$

$$= \frac{\tau^2}{2}\left[2\partial^\nu\left(\frac{i}{4}\omega^\mu_{kl}\sigma^{kl}\right)\otimes g_{\mu\nu}1 -\right.$$

$$-i\left(\partial^{[\mu}\left(\frac{i}{4}\omega_{kl}^{\nu]}\sigma^{kl}\right)+2\left(\frac{i}{4}\right)^2\left[\omega_{kl}^{\mu}\sigma^{kl},\omega_{rs}^{\nu}\sigma^{rs}\right]\right)\otimes\sigma_{\mu\nu}\Bigg]$$

$$=\frac{\tau^2}{4}\Bigg[i\partial^{\nu}\omega_{kl}^{\mu}\sigma^{kl}\otimes g_{\mu\nu}1+$$

$$+\left(\frac{1}{2}\partial^{[\mu}\omega_{kl}^{\nu]}\sigma^{kl}+\frac{i}{4}\left[\omega_{kl}^{\mu}\sigma^{kl},\ \omega_{rs}^{\nu}\sigma^{rs}\right]\right)\otimes\sigma_{\mu\nu}\Bigg].$$

(11.2.2.46)

At this point, as we have noted a few times, an internal contraction (across the ⊗s) is possible and will be assumed to take precedence. Then we obtain

$$[\![\delta\mathbb{A}_S]\!]_C=$$

$$=\frac{\tau^2}{4}\Bigg[i\partial^{\nu}\omega_{kl}^{\mu}\sigma^{kl}g_{\mu\nu}1+\frac{1}{2}\partial^{[\mu}\omega_{kl}^{\nu]}\sigma^{kl}\sigma_{\mu\nu}+\frac{i}{4}\left[\omega_{kl}^{\mu}\sigma^{kl},\ \omega_{rs}^{\nu}\sigma^{rs}\right]\sigma_{\mu\nu}\Bigg].$$

(11.2.2.47)

The trace of only the first term vanishes: to compute the other traces, we note, from equation (11.2.2.36) that

$$\mathrm{tr}(\sigma^{kl}\sigma_{\mu\nu})=\mathrm{tr}(\sigma^{kl}e_{\mu}^{m}e_{\nu}^{n}\sigma_{mn})$$

$$=e_{\mu}^{m}e_{\nu}^{n}\,\mathrm{tr}(\sigma^{kl}\eta_{ms}\eta_{nt}\sigma^{st})$$

$$=e_{\mu}^{m}e_{\nu}^{n}\eta_{ms}\eta_{nt}\,\mathrm{tr}(\sigma^{kl}\sigma^{st})$$

$$= 4e^m_\mu e^n_\nu \eta_{ms}\eta_{nt}(\eta^{ks}\eta^{lt} - \eta^{kt}\eta^{ls})$$

$$= 4e^m_\mu e^n_\nu (\eta_m{}^k \eta_n{}^l - \eta_n{}^k \eta_m{}^l). \qquad (11.2.2.48)$$

Thus

$$\partial^{[\mu}\omega^{\nu]}_{kl}\,\mathrm{tr}(\sigma^{kl}\sigma_{\mu\nu}) = 4e^m_\mu e^n_\nu (\partial^{[\mu}\omega^{\nu]}_{kl}\eta_m{}^k\eta_n{}^l - \partial^{[\mu}\omega^{\nu]}_{kl}\eta_n{}^k\eta_m{}^l)$$

$$= 4e^m_\mu e^n_\nu (\partial^{[\mu}\omega^{\nu]}_{kn}\eta_m{}^k - \partial^{[\mu}\omega^{\nu]}_{km}\eta_n{}^k)$$

$$= 4e^m_\mu e^n_\nu (-\partial^{[\mu}\omega^{\nu]}_{nk}\eta_m{}^k + \partial^{[\mu}\omega^{\nu]}_{mk}\eta_n{}^k)$$

$$= 4e^m_\mu e^n_\nu (\partial^{[\mu}\omega^{\nu]}_{mn} - \partial^{[\mu}\omega^{\nu]}_{nm})$$

$$= 8e^m_\mu e^n_\nu \partial^{[\mu}\omega^{\nu]}_{mn}. \qquad (11.2.2.49)$$

For the trace of the third term we have, from equation (11.1.18),

$$\mathrm{tr}([\omega^\mu_{kl}\sigma^{kl}, \omega^\nu_{rs}\sigma^{rs}]\sigma_{\mu\nu}) = i\omega^\mu_{kl}\omega^\nu_{rs}\,\mathrm{tr}((-\eta^{kr}\sigma^{ls} + \eta^{lr}\sigma^{ks} - \eta^{ls}\sigma^{kr} + \eta^{ks}\sigma^{lr})\sigma_{\mu\nu})$$

$$(11.2.2.50)$$

In view of equation (11.2.2.48) the first term gives

$$-\omega^\mu_{kl}\omega^\nu_{rs}\eta^{kr}\,\mathrm{tr}(\sigma^{ls}\sigma_{\mu\nu}) = \omega^\mu_{kl}\omega^\nu_{sr}\eta^{kr}\,\mathrm{tr}(\sigma^{ls}\sigma_{\mu\nu})$$

$$= \omega^\mu_{kl}\omega_s{}^{\nu k}(4e^m_\mu e^n_\nu (\eta_m{}^l\eta_n{}^s - \eta_n{}^l\eta_m{}^s))$$

$$= 4e^m_\mu e^n_\nu (\omega^\mu_{km}\omega_n^{\nu k} - \omega^\mu_{kn}\omega_m^{\nu k}). \quad (11.2.2.51)$$

The second term may now be obtained by interchanging the $k$ and $l$ labels in the last equation to obtain

$$-\omega^\mu_{lk}\omega^\nu_{rs}\eta^{lr}\,\mathrm{tr}(\sigma^{ks}\sigma_{\mu\nu}) = 4e^m_\mu e^n_\nu (\omega^\mu_{lm}\omega_n^{\nu l} - \omega^\mu_{ln}\omega_m^{\nu l}) \quad (11.2.2.52)$$

or

$$\omega^\mu_{kl}\omega^\nu_{rs}\eta^{lr}\,\mathrm{tr}(\sigma^{ks}\sigma_{\mu\nu}) = 4e^m_\mu e^n_\nu (\omega^\mu_{lm}\omega_n^{\nu l} - \omega^\mu_{ln}\omega_m^{\nu l}). \quad (11.2.2.53)$$

It follows similarly that the other terms also have this value so that

$$\mathrm{tr}([\omega^\mu_{kl}\sigma^{kl}, \omega^\nu_{rs}\sigma^{rs}]\sigma_{\mu\nu}) = 16ie^m_\mu e^n_\nu (\omega^\mu_{\cdot m}\omega_n^{\nu\cdot} - \omega^\mu_{\cdot n}\omega_m^{\nu\cdot}).$$

(11.2.2.54)

Putting all this together we obtain, from equation (11.2.2.47)

$$\mathrm{tr}\left[\!\left[\,\delta\mathbb{A}_S\,\right]\!\right]_C = \frac{\tau^2}{4}\left(4\partial^{[\mu}\omega^{\nu]}_{mn} - 4(\omega^\mu_{\cdot m}\omega_n^{\nu\cdot} - \omega^\mu_{\cdot n}\omega_m^{\nu\cdot})\right)e^m_\mu e^n_\nu$$

$$= \tau^2\left(\partial^{[\mu}\omega^{\nu]}_{mn} - \omega^\mu_{\cdot m}\omega_n^{\nu\cdot} + \omega_m^{\nu\cdot}\omega^\mu_{\cdot n}\right)e^m_\mu e^n_\nu$$

$$= \tau^2\left(\partial^{[\mu}\omega^{\nu]}_{mn} - \omega^\mu_{\cdot m}\omega_n^{\nu\cdot} + \omega^\nu_{\cdot m}\omega_n^{\mu\cdot}\right)e^m_\mu e^n_\nu .$$

(11.2.2.55)

The quantity

$$R^{mn}_{pq} = (\partial^{[\mu}\omega^{\nu]}_{pq} - \omega^\mu_{\cdot p}\omega_q^{\nu\cdot} + \omega^\nu_{\cdot p}\omega_q^{\mu\cdot})e^m_\mu e^n_\nu \quad (11.2.2.56)$$

arises classically where it is known as the *Riemann curvature tensor.* Its contracted form

$$R \equiv R^{mn}_{mn}, \tag{11.2.2.57}$$

which appears in equation (11.2.2.55), is known as the *scalar curvature.* (See Ramond 1981, equation (9.4.62), or Collins, Martin and Squires 1989, §11.5. There seems to be an annoying sign ambiguity in the definition of $R$ in the literature. Here we are consistent with Ramond 1981; the other reference just quoted adopts the other sign.)

Our rather formal view of dynamics has given rise to recognizable expressions of geometrical curvature. (The fields $\mathbb{F}_C$ of equation (11.2.2.25) are also expressions of curvature—of bundles, in fact: cf. §12.3.) Although it must be admitted that our development has been beset by macroscopic archetypes in numerous places, it still seems striking that spatial curvature should have made so explicit an appearance. Our major macroscopic geometrical interpolations concerned the notion of parallel transport: in its infinitesimal form *via* the Dirac operator, and independently in a globalized form *via* Chen series—though only in their simplest line-segment form, which, in any case, arose elsewhere as an aspect of the ubiquitous coherence phenomenon. Nowhere did we explicitly transport any ideal instrument around a spatial loop, such holonomies being the usual physical method for probing curvature. (See Ryder 1996, §3.6, for a particularly illuminating account of the relevant geometry.) Thus, although it would be too much to claim that we have achieved geometry without geometry, we do seem to have obtained curvature as a purely dynamic effect without having built any curvature explicitly into the prior structure, i.e. the net. (This must mean that the Dirac operator contains geometrical information: cf. Connes 1994, VI.)

This having been said, there is still the next $\tau^4$ term in this component of equation (11.2.2.27) to contend with. Now we must square the right-hand side of equation (11.2.2.46), contract across the tensor signs, take the adjoint, and finally take a half of the trace.

This computation has turned out to be intractable as far as this writer is concerned, involving the expansion of traces of products of up to twelve Dirac matrices. The result is a gauge dependent second degree polynomial in the spin coefficients, their first derivatives, and the metric. It will be interpreted in the next chapter as an uncalculated correction to the first term, which is essentially the Einstein–Hilbert Lagrangian for gravity.

### *11.2.3 Ghosts*

(This subsection may be skipped.)

In this subsection we shall assume that elements of $\mathfrak{a}_c$ have been second quantized and are now realized as operators not upon $\mathbb{W}$ but upon a bosonic second quantized version of it. That is to say, $\mathbb{W}$ is replaced by the space $\mathfrak{W}$ of putative "physical" gauge bosons, undoubtedly an extremely large space, known to contain a multiplicity of complicated vacua: cf. the references already cited, particularly Shuryak 1988. The only formal difference from our previous considerations is that the $w$ appearing for example in the expression $w \otimes s$ should now be considered to represent the act of injection of a possible bosonic vacuum. Since the putative vacuum $w$ no longer inhabits $\mathbb{W}$, the possibility arises that an act $w$ may be inadvertently chosen (or *seized*) by the experimenter that kills, or fails to register, the quantum symmetry we have taken such pains to expose. That is to say, it might happen that

$$\mathbb{A}w = 0 \tag{11.2.3.1}$$

for all $\mathbb{A}$ in $\mathfrak{a}_c$. Then, since

$$e^{\mathbb{A}}(w \otimes s) = w \otimes s, \tag{11.2.3.2}$$

the experimenter would find that the fundamental dynamical selective act, namely the operator emerging from equation (11.2.2),

would, on the space of such vacua, collapse back essentially to the bare case, and no transitions involving $\mathbb{A}$-quanta could be registered.

Thus, if experimenters embark upon an attempt to quantize the fields $\mathbb{A}^\mu$ in the presence of the quantum symmetry of the clothed defect, it behooves them to take some action at the outset to avoid the initial selection of a putative vacuum-injecting act from among those *inessential* acts satisfying equation (11.2.3.1).

Consider now the general case of a representation of a finite dimensional Lie algebra $\mathfrak{g}$ upon a not necessarily finite dimensional space $\mathfrak{W}$. With

$$\mathfrak{W}^{\mathfrak{g}} \equiv \{w \in \mathfrak{W} : L.w = 0 \text{ for all } L \in \mathfrak{g}\} \tag{11.2.3.3}$$

and

$$\phi : \mathfrak{W} \longrightarrow \mathrm{Hom}(\mathfrak{g}, \mathfrak{W}) \tag{11.2.3.4}$$

defined by

$$\phi(w)(L) = L.w, \tag{11.2.3.5}$$

where $\mathrm{Hom}(\mathfrak{g}, \mathfrak{W})$ denotes the $\mathfrak{g}$-module of linear maps from $\mathfrak{g}$ into $\mathfrak{W}$, we have an exact sequence of $\mathfrak{g}$-modules

$$0 \longrightarrow \mathfrak{W}^{\mathfrak{g}} \longrightarrow \mathfrak{W} \xrightarrow{\phi} \mathrm{Hom}(\mathfrak{g}, \mathfrak{W}). \tag{11.2.3.6}$$

It may be helpful to recall at this point that the $\mathfrak{g}$-module structure on $\mathrm{Hom}(E,F)$, for $\mathfrak{g}$-modules $E$, $F$ is given, for $f \in \mathrm{Hom}(E,F)$, $L \in \mathfrak{g}$, and $x \in E$ by the derivation-based formula

$$(L.f)(x) = L.f(x) - f(L.x). \tag{11.2.3.7}$$

The tensor product $E \otimes F$ of spaces becomes a $\mathfrak{g}$-module

with respect to the action determined, for $L \in \mathfrak{g}$ and $x \otimes y \in E \otimes F$, by the derivation-like recipe:

$$L.(x \otimes y) = (L.x) \otimes y + x \otimes (L.y). \qquad (11.2.3.8)$$

Since $\mathfrak{g}$ is finite dimensional we have the usual vector space isomorphism (cf. Mac Lane 1963, Proposition 4.2, p.147):

$$\mathfrak{g}^* \otimes \mathfrak{W} \cong \mathrm{Hom}(\mathfrak{g}, \mathfrak{W}) \qquad (11.2.3.9)$$

in which $f \otimes w \in \mathfrak{g}^* \otimes \mathfrak{W}$ is sent to the function $L \mapsto f(L)w$, $L \in \mathfrak{g}$. It is easily checked that this is also an isomorphism of $\mathfrak{g}$-modules when each side of equation (11.2.3.9) is given its appropriate $\mathfrak{g}$-module structure as described above, and where the action on $\mathbb{C}$ is always the trivial one: $\mathfrak{g}.\mathbb{C} = 0$.

Thus, by slightly abusing the notation, the exact sequence (11.2.3.6) of $\mathfrak{g}$-modules may be written

$$0 \longrightarrow \mathfrak{W}^{\mathfrak{g}} \longrightarrow \mathfrak{W} \xrightarrow{\phi} \mathfrak{g}^* \otimes \mathfrak{W} \qquad (11.2.3.10)$$

so we have a splitting

$$\mathfrak{W} \cong \mathfrak{W}^{\mathfrak{g}} \oplus \phi(\mathfrak{W}), \qquad (11.2.3.11)$$

with $\phi(\mathfrak{W}) \subset \mathfrak{g}^* \otimes \mathfrak{W}$: the initial acts of injection of the *essential* vacua of interest may thus be realized as elements of $\mathfrak{g}^* \otimes \mathfrak{W}$. Projecting out the inessential vacua lands us squarely inside the latter module.

If an experimenter were to go through this process of projecting out the inessential vacua at the reticular level, the effect on his or her dynamical considerations would be that the chosen vacuum $w$ would have to be replaced by some element in $\phi(\mathfrak{W})$, which, since the latter module may be regarded as a subspace of $\mathfrak{g}^* \otimes \mathfrak{W}$, can be expressed as a finite sum of tensors of the form $\eta \otimes w$, with $\eta$ an ele-

ment of $\mathfrak{a}_c^*$. Thus, where previously we had $dw$, we now would have a sum of elements of the form

$$d(\eta \otimes w) = d\eta \otimes w + \eta \otimes dw. \qquad (11.2.3.12)$$

Now, for a fixed $\mathbb{L} \in \mathfrak{a}_c^*$, we have

$$(d\eta)(\mathbb{L}) = d(\eta(\mathbb{L})) - \eta(d\mathbb{L}) \qquad (11.2.3.13)$$

where

$$\begin{aligned} d(\eta(\mathbb{L})) &= \Delta^\mu(\eta(\mathbb{L})) \otimes \gamma_\mu \\ &= (\Delta^\mu \eta)(\mathbb{L}) \otimes \gamma_\mu \end{aligned} \qquad (11.2.3.14)$$

and where, for $v$ in $\mathfrak{W}$,

$$\begin{aligned} (d\mathbb{L})(v) &= d(\mathbb{L}(v)) - \mathbb{L}(dv) \\ &= \mathbb{A}^\mu \mathbb{L}(v) \otimes \gamma_\mu - \mathbb{L}\mathbb{A}^\mu(v) \otimes \gamma_\mu \\ &= ([\mathbb{A}^\mu, \mathbb{L}] \otimes \gamma_\mu)(v). \end{aligned} \qquad (11.2.3.15)$$

Thus (equation (11.2.3.13))

$$\begin{aligned} (d\eta)(\mathbb{L}) &= (\Delta^\mu \eta)(\mathbb{L}) \otimes \gamma_\mu - \eta([\mathbb{A}^\mu, \mathbb{L}]) \otimes \gamma_\mu \\ &= (\Delta^\mu \eta)(\mathbb{L}) \otimes \gamma_\mu + (\mathbb{A}^\mu . \eta)(\mathbb{L}) \otimes \gamma_\mu \end{aligned}$$

from equation (11.2.3.7) for the module action of $\mathfrak{a}_c$ upon $\mathfrak{a}_c^*$

$$= ((\Delta^{\mu} + \mathbb{A}^{\mu}.) \otimes \gamma_{\mu})(\eta)(\mathbb{L}), \qquad (11.2.3.16)$$

where we have left the dot in as a reminder of the $\mathfrak{a}_c$-module action of the $\mathbb{A}$s in this context.

Writing

$$\not{\Delta}_{\mathbb{A}} \equiv (\Delta^{\mu} + \mathbb{A}^{\mu}.) \otimes \gamma_{\mu} \qquad (11.2.3.17)$$

equation (11.2.3.12) now reads

$$d(\eta \otimes w) = \not{\Delta}_{\mathbb{A}}(\eta) \otimes w + \eta \otimes \not{\mathbb{A}} w$$

$$= (\not{\Delta}_{\mathbb{A}} \otimes 1 + 1 \otimes \not{\mathbb{A}})(\eta \otimes w). \qquad (11.2.3.18)$$

Thus, the introduction of the elements $\eta \in \mathfrak{a}_c^*$ would entail the replacement of $\mathbb{A}^{\mu}$ in our earlier dynamical considerations by the operator

$$\Delta^{\mu}_{\mathbb{A}} \otimes 1 + 1 \otimes \mathbb{A}^{\mu}. \qquad (11.2.3.19)$$

This operator is supposed to act in second quantized form upon some second quantized version of $\mathfrak{a}_c^* \otimes \mathfrak{W}$. Since $\mathfrak{W}$ is already assumed to be second quantized (as a space of bosonic quanta) we are confronted by the problem of second quantizing $\mathfrak{a}_c^*$. Now, if a Lie algebra $\mathfrak{g}$ is the Lie algebra of a compact Lie group, then $\mathfrak{g}^*$ may be identified with certain differential 1-forms on the group, namely those which remain invariant under a naturally defined ("left") action of the group upon forms. These 1-forms are named for *Maurer–Cartan* (cf. for instance Goldberg 1982). Moreover, the exterior algebra over $\mathfrak{g}^*$ comprises all left-invariant forms on the group (and is in fact a subcomplex of the de Rham complex). This observation seems to suggest that the appropriate second quantization of $\mathfrak{a}_c^*$ is the Fermi–Dirac one, namely this space of all left-invariant forms.

To redo the Lagrangian with (11.2.3.19) in place of $\mathbb{A}^\mu$ we note first that we now have an interaction term that requires an interpretation of

$$\mathrm{tr}[\![\mathbb{A}_{\mathbb{A}} \otimes 1_{\mathfrak{W}}]\!]_C^\dagger + \mathrm{tr}(1_{\mathfrak{a}_c^*})\,\mathrm{tr}[\![\mathbb{A}]\!]_C^\dagger . \qquad (11.2.3.20)$$

We may cope with the first term by regarding the operator in second quantization and then defining the trace as an operator as in §10.2. In this case, however, we may leave the Dirac matrices intact (since the $\Delta^\mu$ here act on $\mathfrak{a}_c^*$) and they may be used to kill the trace. The second term reproduces the earlier interaction term, but now carries the numerical factor

$$\mathrm{tr}(1_{\mathfrak{a}_c^*}) . \qquad (11.2.3.21)$$

Moving on to the kinetic term, we note that the $\delta^\nu$ acting on elements of $\mathfrak{a}_c^*$ should be *adjoint* to the $\delta^\nu$ used for operators upon $\mathfrak{a}_c$. To see in a formal way how an adjoint should be chosen, we note that the "coarsened" operator

$$e^{\delta^\nu} : \mathfrak{a}_c \longrightarrow \mathfrak{a}_c \qquad (11.2.3.22)$$

represents the macroscopic version of the selective act $\delta^\nu$, which is presumed to perform infinitesimal increments upon the elements of $\mathfrak{a}_c$. If a macroscopic experimenter were to demand that $\exp\delta^\nu$ preserve the *entire* quantum symmetry structure then $\exp\delta^\nu$ would be required to be *unitary* and it would follow that

$$(\delta^\nu)^\dagger = -\delta^\nu . \qquad (11.2.3.23)$$

(In a sense this requirement is tantamount to a requirement that $\delta$ preserve the algebra $\mathfrak{a}_c$ of quantum symmetries in a strong sense: namely, "at all stages.")

With this choice, instead of equation (11.2.2.6), we would

have

$$(\delta^{\nu})^{\dagger}(\Delta^{\mu}_{\mathbb{A}}) = \Delta^{\nu}(\Delta^{\mu}_{\mathbb{A}}). \qquad (11.2.3.24)$$

Then the analog of equation (11.2.2.23) is

$$\delta(\mathbb{A}_{\mathbb{A}}) = \Delta^{\nu}(\Delta^{\mu}_{\mathbb{A}}) \otimes g_{\mu\nu} 1 + \mathbb{G}^{\mu\nu} \otimes \sigma_{\mu\nu}, \qquad (11.2.3.25)$$

where $\mathbb{G}^{\mu\nu}$ is the appropriate "curvature" of $\Delta_{\mathbb{A}}$. Thus

$$\begin{aligned}\left[\!\left[ \delta(\mathbb{A}_{\mathbb{A}} \otimes 1 + 1 \otimes \mathbb{A}) \right]\!\right]^{\dagger}_{C} &= \left[\!\left[ \delta(\mathbb{A}_{\mathbb{A}}) \right]\!\right]^{\dagger}_{C} \otimes 1_{\mathfrak{W}} + 1_{\mathfrak{a}_c^*} \otimes \left[\!\left[ \delta\mathbb{A} \right]\!\right]^{\dagger}_{C} \\ &= \left( -\tau^2 (\partial^{\nu}(\partial^{\mu} + \mathbb{A}^{\mu}_{C\cdot}) \otimes g_{\mu\nu} 1 + \left[\!\left[ \mathbb{G}^{\mu\nu} \right]\!\right]_{C} \otimes \sigma_{\mu\nu} \right)^{\dagger} \otimes 1_{\mathfrak{W}} + \\ &\quad + 1_{\mathfrak{a}_c^*} \otimes \left[\!\left[ \delta\mathbb{A} \right]\!\right]^{\dagger}_{C}. \qquad (11.2.3.26)\end{aligned}$$

In addition to the new first term, the old one acquires a factor $1_{\mathfrak{a}_c^*}$. So, with

$$D^{\mu} \equiv \partial^{\mu} + \mathbb{A}^{\mu}_{C\cdot}, \qquad (11.2.3.27)$$

$$\mathrm{tr}\left[\!\left[ \delta(\mathbb{A}_{\mathbb{A}} \otimes 1 + 1 \otimes \mathbb{A}) \right]\!\right]^{\dagger}_{C} = -\tau^2 \mathrm{tr}(\partial^{\nu} D_{\nu})^{\dagger}.4 + (\dim \mathfrak{a}_c^*)\, \mathrm{tr}\left[\!\left[ \delta\mathbb{A} \right]\!\right]^{\dagger}_{C}.$$

(11.2.3.28)

Here we must interpret the first term as an operator, as in §10.2. Since the operators that will play the rôle analogous to the operators $\psi_{(k)}$ in equation (10.2.8) and that we shall denote by $\eta_a$, are actually second quantized versions of elements of $\mathfrak{a}_c^*$, we shall assume their macroscopic correspondents acquire factors similar to the

factors acquired by elements of $\mathfrak{a}_c$ (up to arbitrary extractable real constants): namely (cf. equation (11.2.1.16))

$$(\eta_a)_C = -\frac{i\tau}{2}(\dim \mathfrak{a}_c^*)^{1/2}\eta_a. \qquad (11.2.3.29)$$

Then

$$\begin{aligned}\mathrm{tr}(\partial^\nu D_\nu)^\dagger &= \mathrm{tr}((\partial^\nu)^\dagger (D_\nu)^\dagger)\\ &= \mathrm{tr}((-\partial^\nu)(-D_\nu))\\ &= \mathrm{tr}(\partial^\nu D_\nu)\\ &= (\dim \mathfrak{a}_c^*)(-\frac{i\tau}{2}\eta_a)^\dagger (\partial^\nu D_\nu)(-\frac{i\tau}{2}\eta_a)\\ &= (\dim \mathfrak{a}_c^*)\frac{\tau^2}{4}\eta_a^\dagger \partial^\nu D_\nu \eta_a, \qquad (11.2.3.30)\end{aligned}$$

where the $\eta_a$ constitute a basis of Maurer–Cartan forms regarded as Fermi–Dirac operators. Thus equation (11.2.3.28) becomes

$$\mathrm{tr}\left[\!\left[\delta(\not{\Delta}_{\mathbb{A}}\otimes 1 + 1\otimes \not{\mathbb{A}})\right]\!\right]_C^\dagger = (\dim \mathfrak{a}_c^*)\left[-\tau^4 \eta_a^\dagger \partial^\nu D_\nu \eta_a + \mathrm{tr}\left[\!\left[\delta \not{\mathbb{A}}\right]\!\right]_C^\dagger\right]. \qquad (11.2.3.31)$$

The next term contributing to the new Lagrangian is obtained by squaring equation (11.2.3.26) and taking half the trace. The first term on the right-hand side of this equation, when squared, yields a term in $\tau^4$. Since the act of taking the trace of the macroscopic operators acting on $\mathfrak{a}_c^*$ supplies an additional $\tau^2$ (equation (11.2.3.29)), the trace of this squared first term is of order $\tau^6$ and

may be discarded. The mixed terms similarly yield traces of order $\tau^6$: $\tau^4$ from the trace of the first term and $\tau^2$ from $\mathrm{tr}[\![\delta\mathbb{A}]\!]^\dagger_C$ (equation (11.2.2.26)). Thus the only term to keep here is the contribution from the trace of the square of the last term—which we computed earlier—multiplied by the dimension of the space $\mathfrak{a}_c^*$.

Summing the relevant terms yields (to order $\tau^4$) a new Lagrangian which may be obtained by multiplying the old one by $\dim \mathfrak{a}_c^*$ and adding one new term, namely

$$(-g)^{1/2}(\dim \mathfrak{a}_c^*)(-\eta_a^\dagger \partial^\nu D_\nu \eta_a). \qquad (11.2.3.32)$$

The overall constant is irrelevant to the field equations and may be discarded.

The fields $\eta_a, \eta_a^\dagger$ were, from the outset, merely bookkeeping devices concomitant with the joint activities of second quantization and the attempt to avoid inessential vacua. They did not emanate from the net itself and do not share any of the symmetries associated with the latter; moreover, they are *invariant* under the defect's symmetries. In particular, they ignore Lorentz transformations: that is, they are Lorentz scalars. On the other hand, they apparently obey Fermi–Dirac statistics, and therefore, as far as macroscopic experimenters who are convinced of the truth of the Spin–Statistics Theorem are concerned, they defy "physicality" and are called *ghosts*.

In fact, the expression appearing in (11.2.3.32) arises in various ways (different from ours) in standard treatments of the quantization problem for gauge theories where the ghosts are needed, roughly speaking, because there is a mismatch between the numbers of degrees of freedom available upon fixing the gauge while still allowing covariance in, for instance, the propagators of the theory. This has the effect (among others) of allowing an over-count of intermediate states—since gauge equivalences are included—in scattering calculations involving such propagators, thereby destroying unitarity. The inclusion of the unphysical ghost fields as in (11.2.3.32), which only appear as internal loops, has the apparently miraculous effect of exactly canceling this unphysical over-count. They were originally

cooked up in special cases expressly for this purpose by Feynman (and independently by de Witt) in the early 1960s, so, from this perspective, their appearance seems somewhat less than miraculous. However, what really does seem miraculous is their reappearance in a natural and elegant general form in the path integral formulation of Faddeev–Popov. (In addition to the references on gauge theories already cited, an extremely lucid account of these matters, and many others, may be found in Coleman 1985. The subject of gauge quantization has reached a high degree of maturity and technical complexity. For advanced accounts, see Henneaux and Teitelboim 1992 and Nakanishi and Ojima 1990.)

The ghosts have arisen in our context entirely differently, as a byproduct of an experimenter's attempt to avoid the use of inessential vacua: i.e. the use of non-injecting initial acts for gauge bosons. (Equation (11.2.3.1) involves an "incoming" quantum undergoing a null transition.) The spontaneous appearance of ghosts almost as soon as an experimenter acts to ensure the effective behavior of her or his "instrumentation" *vis à vis* the detection of gauge quanta is presumably a reward for the attempt to maintain "at all stages" at least some of the quantum symmetry of the defect. In a sense, by invoking the ghosts and thereby making canonical the implicit choice of an effective vacuum, an experimenter is "coordinating" his or her experimental arrangement with all "other" putative experimenters who do the same. How ghosts may be interpreted as parametrizing constraints pertaining to *other* experimenters (in the presence of gauge invariance) is beautifully explained by Finkelstein in an elementary fashion—and one quite different from ours—in QR, §4.11.9.

The full symmetry of the Lagrangian we have obtained (with the spin terms omitted), namely BRST symmetry, is named for Becchi, Rouet, Stora and, independently, Tyutin: see the references already cited.

## 11.3 Resolution and Rescaling

As noted, we have examined the individual terms that emerge from equation (11.2.8) at different resolutions: the interaction terms were considered at the degree of resolution that required the modularization and second quantization of $V$ to show them up, while the kinetic terms for $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$ were left at the presumably lower resolution determined by the macroscopically contaminated choice of spaces $W_2$ and $W_3$. The $\mathfrak{so}(3,1)$ component of $\mathbb{W}$ was chosen to be (the microscopic) $V$ itself, so the problem of a possible resolutional mismatch between its kinetic term(s) and its interaction term did not arise.

This mismatch must be resolved (for $\mathfrak{su}(2)$ and $\mathfrak{su}(3)$) at the algebraic level even before any presumed second quantization of the kinetic terms is contemplated. Let us attempt to address this in the $\mathfrak{su}(3)$ case. Recall that the triplet $\Psi_G$ appearing in the interaction term equation (11.2.1.30) is actually a condensed version of an act of type $W_3 \otimes V$, in which the $G^\mu$'s action upon $W_3$ has in a sense been lifted to an action upon the triplet of second quantized spinor operators. In this sense the "covariant" Dirac operator $\partial\!\!\!/ - i\mathcal{G}\!\!\!//2$ is to be regarded as operating at the higher degree of resolution embodied in the second quantized version of $V$, though $W_3$ has been left at its original macroscopically contaminated resolution.

This brings us to an issue we were able to evade earlier, namely the nature of the macroscopic interpretation of elements of $W_3$. Since this is a defect type we shall assume, as in the case of the bare defect type, that its macroscopic continuum interpretation differs from it only by a real scaling factor, which we take here to be dimensionless since the dimensional attributes of $\Psi_G$ are already accounted for in its bispinor factor. Such a rescaling of a vector does not of course change the act it represents but will affect amplitudes derived from it. Thus a change in "resolution" carries the implication that certain amplitudes may have a resolution dependency. (This observation will enable us to give a precise interpretation to the notion of "resolution": cf. §12.2.) Thus, a change in the resolution at which

the acts represented by elements of $W_3$ are used will register simply as a rescaling $w \mapsto \lambda w$ for all $w$ in $W_3$, for some real number $\lambda$.

Thus, we can rectify the resolutional mismatch within the interaction term by making a replacement $\Psi_G \mapsto \lambda\Psi_G$ while leaving intact the "microscopic" covariant operator within the interaction term. That is to say, we adopt the resolutional level determined by the operator $\not{G}/2$ as the *base* level, and change the resolutional level of $\Psi_G$ to meet it.

In contrast, the operators $G^\mu$ that enter into the kinetic term act on the macroscopically contaminated $W_3$ without any of the spinorial intimacy enjoyed by the covariant derivative in the interaction term: this difference is part and parcel of our assumption concerning the non-interfacing of $W_3$ and $V$ in this case (cf. the first paragraph following equation (11.2.2.28)). Consequently, to bring the level of resolution of this term up to that of the now resolutionally consistent interaction term, we should, as before, replace each $w$ by $\lambda w$, or, equivalently, apply the operator $\lambda I_3$ before applying $G^\mu$. This has the effect of replacing $G^\mu$ by $\lambda G^\mu$, an operation to be performed before the computation of the kinetic term is effected. If these two rescalings are performed, namely

$$\Psi_G \mapsto \lambda\Psi_G \text{ in the interaction term} \tag{11.3.1}$$

and

$$G^\mu \mapsto \lambda G^\mu \text{ in the kinetic term,} \tag{11.3.2}$$

then both resulting contributions to the Lagrangian pertain to the same level of resolution and may therefore be added together.

But if our interpretation of resolution-change as rescaling is correct, then, since $\not{G}/2$ *is* a rescaled version of $\not{G}$ which at the same time embodies by definition the correct base level of resolution of $\not{G}$, it must *be* $\lambda\not{G}$ for the $\lambda$ we seek. Thus $\lambda = \frac{1}{2}$.

Using (11.3.1), and temporarily ignoring the $(-g)^{1/2}$ factor, the interaction term (equation (11.2.1.30)) becomes

$$\frac{1}{4}i\overline{\Psi}_G\left(\not{\partial}-\frac{i}{2}\not{G}\right)\Psi_G, \tag{11.3.3}$$

while in view of (11.3.2), equation (11.2.2.29) now reads

$$(F_3^{\mu\nu})_C=-\frac{i}{2}\left(\frac{1}{2}\left(\partial^{[\mu}G^{\nu]}-\frac{i}{2}[G^{\mu},G^{\nu}]\right)\right). \tag{11.3.4}$$

In order to avoid further typographical proliferation, let us redefine $G^{\mu\nu}$ as the expression appearing in inner parentheses in the last equation. That is, we now write

$$G^{\mu\nu}\equiv\partial^{[\mu}G^{\nu]}-\frac{i}{2}[G^{\mu},G^{\nu}]. \tag{11.3.5}$$

Then the expression in equation (11.2.2.39) becomes

$$\frac{1}{4}(-g)^{1/2}\left[-\frac{1}{2}\mathrm{tr}(\partial^{\nu}G_{\nu})^2-\frac{1}{4}\mathrm{tr}(G^{\mu\nu}G_{\mu\nu})\right] \tag{11.3.6}$$

with $G^{\mu\nu}$ now given by equation (11.3.5). Consequently, the total $\mathfrak{su}(3)$ Lagrangian obtained by adding the interaction term to the kinetic term becomes

$$\frac{1}{4}(-g)^{1/2}\left[i\overline{\Psi}_G\left(\not{\partial}-\frac{i}{2}\not{G}\right)\Psi_G-\frac{1}{2}\mathrm{tr}(\partial^{\nu}G_{\nu})^2-\frac{1}{4}\mathrm{tr}(G^{\mu\nu}G_{\mu\nu})\right]. \tag{11.3.7}$$

(We shall ignore the ghost terms, which may be accommodated by rescaling the ghost fields as in (11.3.1).)

The complexified $\mathfrak{su}(2)$ component, which we will label as $\mathfrak{su}(2)_{\mathbb{C}}$, may be treated in an exactly similar fashion, and the chiral breaking is unaffected. The interaction term (11.2.1.3) similarly acquires the factor $1/4$, while the expression in equation (11.2.2.45) is changed by the replacement of $W^{\mu\nu}$ by the analog of the right-hand side of equation (11.3.5) and the acquisition of the factor $1/4$, a factor acquired also by the term $B^{\mu\nu}B_{\mu\nu}$ (inherited from the replacement $I_2 \mapsto \lambda I_2$ taking place in equation (11.2.2.41)), which is otherwise unchanged.

The spin term component remains unaffected by these activities, as noted.

Having equalized the degree of resolution of each component we may now add them up to obtain (from equations (11.2.1.30), (11.2.1.33), (11.2.2.39), (11.2.2.44), etc.)

$$\tau^{-4}(-g)^{1/2}(\tau^2 R) + ? + \frac{1}{4}(-g)^{1/2}\Big[i\overline{\Psi}_G\Big(\not{\partial} - \frac{i}{2}\not{G}\Big)\Psi_G - \frac{1}{4}\mathrm{tr}(G^{\mu\nu}G_{\mu\nu}) -$$

$$-\frac{1}{2}\mathrm{tr}(\partial^\nu G_\nu)^2\Big] + \frac{1}{4}(-g)^{1/2}\Big[i\overline{\Psi}_{W,R}\gamma_\mu\Big(\partial^\mu - \frac{i}{2}B^\mu I_2\Big)\Psi_{W,R} +$$

$$+ i\overline{\Psi}_{W,L}\gamma_\mu\Big(\partial^\mu - \frac{i}{2}B^\mu I_2 - \frac{i}{2}W^\mu\Big)\Psi_{W,L} - (\partial^\nu B_\nu)^2 - \frac{1}{2}B^{\mu\nu}B_{\mu\nu} -$$

$$-\frac{1}{2}\mathrm{tr}(\partial^\nu W_\nu)^2 - \frac{1}{4}\mathrm{tr}(W^{\mu\nu}W_{\mu\nu})\Big]$$

$$= \frac{1}{4}(-g)^{1/2}\Big[\Big(\frac{4}{\tau^2}\Big)R + ? + (\mathfrak{su}(3)\text{ term}) + (\mathfrak{su}(2)_{\mathbb{C}}\text{ term})\Big]$$

(11.3.8)

where the question mark denotes the uncalculated higher order spin term, where the indicated chiral doublets are defined by

$$\Psi_{W,R} \equiv \begin{pmatrix} 0 \\ \psi_{W,R} \end{pmatrix} \tag{11.3.9a}$$

$$\Psi_{W,L} \equiv \begin{pmatrix} \psi_{W,L} \\ 0 \end{pmatrix}, \tag{11.3.9b}$$

and where $W^{\mu\nu}$ and $G^{\mu\nu}$ are now given as in equation (11.3.5).

One more rescaling of $W_2$ and $W_3$ is required before we can begin a comparison with experiment. This rescaling will allow for the degree of resolution an experimenter might be expected to attain through the use of the macroscopically contaminated acts represented by these spaces.

In the preceding discussion we have taken pains to equalize the degree of resolution of the two species of terms in the contributions to the total Lagrangian associated with these acts. Having achieved this, we must now allow for a rescaling (applicable to all relevant terms) of these spaces in order to accommodate the absolute resolutional ambiguity in our choices of the fields $G^{\mu}$, $W^{\mu}$ and $B^{\mu}$. The (possibly experimenter-dependent) constant involved will reflect the (coarser) degree of resolution at which an experimenter may be attributed the status appropriate to the use of the correspondence principle and to which we have attached the name "macroscopic." This is actually something of a misnomer, however, since the degree of resolution will turn out to be very high indeed by the standards of terrestrial experimenters. At this resolution we will have passed beyond the Maxwell–Boltzmann phase into the realm of "objective" geometry, though we may not yet be confronted by a manifold. As usual, the bispinor space (assumed to emanate unscathed from the deepest recesses of the net) remains untouched at its base level of resolution.

We consider such an overall rescaling in the $\mathfrak{su}(3)$ case. As above we envisage a rescaling that effects, for some (real) constant g,

the following replacements:

$$\Psi_G \mapsto \mathrm{g}\Psi_G \tag{11.3.10}$$

$$G^\mu \mapsto \mathrm{g}G^\mu \tag{11.3.11}$$

where the latter replacement is now to be made in all relevant terms on the right-hand side of equation (11.3.8), not just in the kinetic term. The interaction term becomes

$$i\mathrm{g}^2\overline{\Psi}_G\left(\not{\partial} - \frac{i\mathrm{g}}{2}\not{G}\right)\Psi_G, \tag{11.3.12}$$

while $G^{\mu\nu}$ is replaced by

$$\mathrm{g}\left(\partial^{[\mu}G^{\nu]} - \frac{i\mathrm{g}}{2}[G^\mu, G^\nu]\right) \equiv \mathrm{g}G^{\mu\nu}, \tag{11.3.13}$$

where we have yet again redefined $G^{\mu\nu}$, this time as

$$G^{\mu\nu} \equiv \partial^{[\mu}G^{\nu]} - \frac{i\mathrm{g}}{2}[G^\mu, G^\nu]. \tag{11.3.14}$$

Thus, the kinetic term (on the right-hand side of equation (11.3.8)) yields

$$-\frac{\mathrm{g}^2}{2}\mathrm{tr}(\partial^\nu G_\nu)^2 - \frac{\mathrm{g}^2}{4}\mathrm{tr}(G^{\mu\nu}G_{\mu\nu}), \tag{11.3.15}$$

where $G^{\mu\nu}$ is now given by equation (11.3.14).

Since $W_2$ is presumably engaged by a macroscopic experimenter at the same degree of resolution as $W_3$, we should repeat this rescaling for it using the same constant g. Then an exactly similar

transformation of the $\mathfrak{su}(2)_{\mathbb{C}}$ terms occurs, with $gB^{\mu}$ and $gW^{\mu}$ appearing in the interaction term, which acquires a factor of $g^2$, with $W^{\mu\nu}$ being replaced as in equation (11.3.14), and with the kinetic terms also acquiring the $g^2$ factor.

Factoring out the $g^2$, the right-hand side of equation (11.3.8) now becomes

$$\frac{g^2}{4}(-g)^{1/2}\left[\left(\frac{4}{g^2\tau^2}\right)R + ? + \mathcal{L}(\mathfrak{su}(3)) + \mathcal{L}(\mathfrak{su}(2)_{\mathbb{C}})\right], \quad (11.3.16)$$

where

$$\mathcal{L}(\mathfrak{su}(3)) \equiv i\overline{\Psi}_G\left(\not{\partial} - \frac{ig}{2}\not{G}\right)\Psi_G - \frac{1}{2}\mathrm{tr}(\partial^{\nu}G_{\nu})^2 - \frac{1}{4}\mathrm{tr}(G^{\mu\nu}G_{\mu\nu})$$

(11.3.17)

with $G^{\mu\nu}$ given by equation (11.3.14), and where

$$\mathcal{L}(\mathfrak{su}(2)_{\mathbb{C}}) \equiv i\overline{\Psi}_{W,R}\gamma_{\mu}\left(\partial^{\mu} - \frac{ig}{2}B^{\mu}I_2\right)\Psi_{W,R} +$$

$$+ i\overline{\Psi}_{W,L}\gamma_{\mu}\left(\partial^{\mu} - \frac{ig}{2}B^{\mu}I_2 - \frac{ig}{2}W^{\mu}\right)\Psi_{W,L} - (\partial^{\nu}B_{\nu})^2 - \frac{1}{2}B^{\mu\nu}B_{\mu\nu} -$$

$$-\frac{1}{2}\mathrm{tr}(\partial^{\nu}W_{\nu})^2 - \frac{1}{4}\mathrm{tr}(W^{\mu\nu}W_{\mu\nu}), \quad (11.3.18)$$

with $W^{\mu\nu}$ being analogous to the right-hand side of equation (11.3.14).

The Lagrangian (11.3.16) is now ripe for comparison with the Standard Model, which is the subject of the next and final chapter.

# 12

# Comparisons, Interpretations and Speculations

In this final chapter we seek an accommodation with the Standard Model, a very brief outline of which is given in §12.1.

In §12.2 we compare our results with the Standard Model Lagrangian at the scale of "grand-unification" and give an estimate for $|\tau|$.

Section §12.3 contains a brief discussion of the spontaneous symmetry breaking mechanism that underlies the BCS theory of superconduction, an analog of which is believed to drive electroweak unification. This latter phenomenon is supposed to occur at a coarser resolution (or lower energy) than the one—it turns out—we have been primarily concerned with here: an account of it is included for the sake of completeness and also because we have appealed to the superconduction paradigm on earlier occasions.

In §12.4 we summarize certain implications for low-energy particle physics of the minimal kind of topological complexity one might expect to find in the manifold if there were pockets of non-manifold—such as Maxwell–Boltzmann surfaces—embedded within it.

In §12.5 we discuss very briefly a topic of central current interest, namely the so-called *loop state* approach to gauge quantization. This circle of ideas has culminated recently in a very significant ad-

vance in the search for a quantum theory of gravity. This theory starts from the traditional assumption of a manifold—albeit devoid of an explicit *a priori* background metric—and proceeds to the quantization through subtle methods. A basic relation between paths and connections lies at the heart of this approach: we indicate an entirely algebraic origin for this relation in the local or tangential picture afforded by the net algebra. It is interesting to note the appearance yet again of the "quantum duplication" coproduct operator.

In a final section we offer some comparisons with related current work and speculations about further developments.

## 12.1 An Abbreviated Sketch of the Standard Model

We very briefly review the main ingredients of the Standard Model, referring the reader to many excellent and complete treatments, some of which may be found in the references cited in §11.2.2.

In this model, three families, or generations, of fermion doublets are posited:

| **Family** | **Leptons** | **Quarks** |
|---|---|---|
| First | $(e^-, \nu_e)$ | $(d, u)$ |
| Second | $(\mu^-, \nu_\mu)$ | $(s, c)$ |
| Third | $(\tau^-, \nu_\tau)$ | $(b, t)$ |

(The classification into families, or generations, is determined by such attributes as similarities in masses or, equivalently, dominant decay modes.)

Each quark can exist in three forms distinguished by an attribute that has come to be called "color." This attribute is associated with an interaction among such quark color triplets, called the *strong* interaction, that is mediated by an SU(3) octet of massless gauge bosons called *gluons*. The leptons—reading down the first column:

electron, muon, tau particle, with the second entry in each case denoting the corresponding neutrino—are blind to this interaction.

The members within each quark pair in the above list of families is distinguished by an attribute called *flavor*. These flavors have come to be named *down, up, strange, charm, bottom, top*. The existence of all of them is now well established.

The *left-handed* component of each quark flavor pair constitutes the fermion doublet associated with an SU(2) gauge boson mediated interaction called the *weak* interaction, that is ignored by the right-handed components. The *left-handed* components of the lepton doublets similarly partake of this weak interaction, while the right-handed components ignore it entirely.

In addition to the "$SU(2)_L$" attributes of these doublets, the model also requires an additional U(1) attribute, called "weak hypercharge" so that the full gauge group for weakly interacting systems becomes $U(1) \times SU(2)_L$. One reason for thus supplementing the $SU(2)_L$ symmetry in the standard theory is so as to be in a position to implement the process known as *spontaneous symmetry breaking* at energies lower than about $10^2$ GeV (or length scales larger than about $10^{-16}$cm) that brings into being the electromagnetic interaction of macroscopic experience while simultaneously supplying masses to the fermions and the weak gauge bosons. We describe a toy geometrical model of this phenomenon in §12.3.

There is a large and complex system of other quantum numbers assigned to these particles and fields that we shall neglect here.

Comparing this abbreviated description of the Standard Model with our results in the previous chapter it seems that we have succeeded in deriving a reasonable analog of its basic architecture in Lagrangian form, prior to electroweak spontaneous symmetry breaking (with an additional gravitation-like term) even though we find no masses, no distinction between lepton and quark, no differentiation among possible coupling strengths, and no trace of fermion family/generational structure. With the exception of the last mentioned, these omissions may be expected to occur at a certain (high) energy scale (or (short) length scale) and their non-appearance will be

used shortly to infer an estimate for the length scale below which the Maxwell–Boltzmann phase may be presumed to set in, and an estimate for the value of $|\tau|$. This will be done by comparing our Lagrangian with the one given by the Standard Model at a certain energy (or length) scale.

Before such a comparison can be attempted, we must parametrize the Lie algebra representations used in the last chapter to consist with those used in the Standard Model. We start with SU(3). As a real Lie algebra it is $3^2-1=8$-dimensional and is parametrized in the Standard Model by a choice of basis, namely the one denoted by $\{i\lambda_a\}$, $a=1,\ldots,8$, where the (Hermitian, trace-free) $\lambda_a$ are called the *Gell–Mann* matrices, which satisfy

$$[\lambda_a,\lambda_b]=2if_{abc}\lambda_c \tag{12.1.1}$$

with certain real structure constants $f_{abc}$ that are totally antisymmetric in their indices, and

$$\mathrm{tr}(\lambda_a\lambda_b)=2\delta_{ab}. \tag{12.1.2}$$

Returning to the definition of $G^\mu$ given in equation (11.2.1.19) we note that it may be expanded in terms of the Gell–Mann basis as

$$G^\mu=G^\mu_a\lambda_a, \tag{12.1.3}$$

say, for real $G^\mu_a$. The "covariant derivative" in equation (11.3.12) then assumes the form

$$\partial^\mu-\frac{ig}{2}G^\mu_a\lambda_a \tag{12.1.4}$$

while the field $G^{\mu\nu}$ in its latter incarnation, namely equation (11.3.14), may be written

$$\partial^{[\mu}G^{\nu]} - \frac{ig}{2}[G^{\mu}, G^{\nu}] = \partial^{[\mu}G_a^{\nu]} - \frac{ig}{2}G_b^{\mu}G_c^{\nu}[\lambda_b, \lambda_c]$$

$$= (\partial^{[\mu}G_a^{\nu]} + gf_{bca}G_b^{\mu}G_c^{\nu})\lambda_a$$

from equation (12.1.1)

$$= (\partial^{[\mu}G_a^{\nu]} + gf_{abc}G_b^{\mu}G_c^{\nu})\lambda_a$$

$$\equiv G_a^{\mu\nu}\lambda_a. \qquad (12.1.5)$$

Thus

$$\mathrm{tr}(G^{\mu\nu}G_{\mu\nu}) = \mathrm{tr}(G_a^{\mu\nu}\lambda_a G_{b\mu\nu}\lambda_b)$$

$$= G_a^{\mu\nu}G_{b\mu\nu}\,\mathrm{tr}(\lambda_a\lambda_b)$$

$$= 2G_a^{\mu\nu}G_{a\mu\nu}, \qquad (12.1.6)$$

from equation (12.1.2), and

$$\mathrm{tr}(\partial^{\nu}G_{\nu})^2 = \mathrm{tr}[(\partial^{\mu}G_{a\mu}\lambda_a)(\partial^{\nu}G_{b\nu}\lambda_b)]$$

$$= (\partial^{\mu}G_{a\mu})(\partial^{\nu}G_{b\nu})\mathrm{tr}(\lambda_a\lambda_b)$$

$$= 2(\partial^{\nu}G_{a\nu})^2, \qquad (12.1.7)$$

the summation over $a$ being implicit in the last line.

Consequently, the $\mathfrak{su}(3)$ contribution assumes the form

(equation (11.3.17))

$$\mathcal{L}(\mathfrak{su}(3)) = 2\left[i\bar{f}_G\gamma_\mu\left(\partial^\mu - \frac{ig}{2}G_a^\mu\lambda_a\right)f_G - \frac{1}{4}G_a^{\mu\nu}G_{a\mu\nu} - \frac{1}{2}(\partial^\nu G_{a\nu})^2\right]$$

$$\equiv 2\mathcal{L}(G), \tag{12.1.8}$$

where the basic Standard Model bispinor variable $f$ is related to ours, $\Psi$, by

$$f \equiv \frac{\Psi}{\sqrt{2}} \tag{12.1.9}$$

(cf. proviso 3, below).

Similar considerations apply to the $\mathfrak{su}(2)_{\mathbb{C}}$ contribution, the analog of the Gell–Mann matrices being in the case of $\mathfrak{su}(2)$ just the $2^2 - 1 = 3$ Pauli matrices, that satisfy

$$[\sigma_a, \sigma_b] = 2i\varepsilon_{abc}\sigma_c \tag{12.1.10}$$

(the totally antisymmetric symbol $\varepsilon$ having its usual connotation, i.e. $\varepsilon_{123} = 1$, etc.). Thus, with

$$W^\mu = W_a^\mu\sigma_a \tag{12.1.11}$$

and

$$W_a^{\mu\nu} \equiv \partial^{[\mu}W_a^{\nu]} + g\varepsilon_{abc}W_b^\mu W_c^\nu \tag{12.1.12}$$

we have from equation (11.3.18)

$$\mathcal{L}(\mathfrak{su}(2)_{\mathbb{C}}) \equiv 2\left[i\bar{f}_{W,R}\gamma_\mu\left(\partial^\mu - \frac{ig}{2}B^\mu I_2\right)f_{W,R} +\right.$$

$$+ i\bar{f}_{W,L}\gamma_\mu\left(\partial^\mu - \frac{ig}{2}B^\mu I_2 - \frac{ig}{2}W^\mu\right)f_{W,L} - \frac{1}{4}W_a^{\mu\nu}W_{a\mu\nu} - \frac{1}{2}(\partial^\nu W_{a\nu})^2 -$$

$$-\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{2}(\partial^\nu B_\nu)^2\Big]$$

$$\equiv 2\mathcal{L}(W). \qquad (12.1.13)$$

It is worth noting that the $B$-kinetic terms inside the square brackets have now acquired a factor of $\frac{1}{2}$ giving these contributions the same form as the others.

Putting these contributions together with the spin term in equation (11.3.16) yields

$$\frac{g^2}{2}(-g)^{1/2}\left[\left(\frac{2}{g^2\tau^2}\right)R + ? + \mathcal{L}(G) + \mathcal{L}(W)\right]$$

$$\equiv \frac{g^2}{2}\mathcal{L}. \qquad (12.1.14)$$

We may now discard the overall constant $g^2/2$, whose absence will not affect the field equations, and adopt what remains as our final Lagrangian. Then we note that, with the following provisos, we have a perfect fit of the individual terms in the square brackets with the corresponding (un-ghosted) Lagrangians of the Standard Model:

1. All the coupling constants are the same, namely g.
2. There are no mass terms, no generational distinctions among fermion multiplets, and no distinctions among leptons and quarks: thus the couplings in $\mathcal{L}(W)$ are missing some of the quantum numbers

that reflect this distinction.

3. The basic bispinor field of the Standard Model has been identified with $\Psi/\sqrt{2}$. This seems harmless since any positive constant dimensionless factor could be used to adjust our definition of $\Psi_C$ in equation (10.2.15).
4. The gauge-fixing terms are supplied *ad hoc* in the Standard Model, and may be chosen differently to fulfill various technical demands without altering the physical content of the theory.
5. The Lagrangian for (classical) General Relativistic gravity is expressible in the Einstein–Hilbert form, namely

$$\mathcal{L}_{EH} = (-g)^{\frac{1}{2}} \frac{1}{16\pi G} R, \tag{12.1.15}$$

where $G$ is Newton's constant, that is related to the so-called *Planck* length $l_P$ *via* the equation

$$l_P = \left(\frac{\hbar G}{c^3}\right)^{\frac{1}{2}}$$

$$\approx 1.6 \times 10^{-33} \text{cm}. \tag{12.1.16}$$

In natural units $\hbar = c = 1$ and

$$G = l_P^2. \tag{12.1.17}$$

The problem of quantizing this theory is notoriously vexing and has repulsed all attacks until very recently: cf. §12.5, Isham 1989 and Butterfield and Isham 2001 for penetrating analyses of the major issues.

> Despite these difficulties, the Einstein–Hilbert Lagrangian seems to be generally acceptable as a low energy (or long distance) leading term even in those theories in which extra terms come into play at high energy (cf. Weinberg 1979). It may be noted in this connection that no short distance (high energy) gravitational experimental data is actually available.
>
> Thus it seems reasonable to assume that the constant factor accompanying $(-g)^{1/2}R$ in equation (12.1.15) is still there (or approximately there) at the short length scales we will soon find ourselves contemplating, whatever the nature of the unknown higher order term.

With these provisos in place we are led to institute a search for g, since, if it could be found, a comparison of our lowest order spin term with the Einstein–Hilbert Lagrangian would yield a value for $|\tau|$.

## 12.2 Asymptotic Freedom and Grand Unification

Physical couplings (of fields to fermions) arise in the action vector view espoused here as amplitudes associated with certain transitions, namely those represented by internal contractions among acts of the type exhibited in §11.2.1. Since the acts employed depend upon the degree of resolution available to the putative "external" experimenter, it would appear that the couplings themselves should depend upon the degree of resolution of the experimenter. Now the "physical" Yang–Mills couplings of the Standard Model—here we exclude the gravity term, which is not renormalizable—do indeed depend upon the energy (or length scale) at which they are measured, so we may interpret the "degree of resolution of the experimenter" as "the energy (or length scale) at which the experiments are performed." Consequently, our Lagrangian $\mathcal{L}$ (equation (12.1.14)), absenting the uncomputed spin term, conforms exactly with the La-

grangian of the Standard Model when the latter is considered at an energy sufficiently high that the Yang–Mills couplings become equal (to g) and relative to which all masses become negligible, assuming such an energy exists. If it does, we would have, at that energy, complete agreement between the Standard Model Lagrangian and the Yang–Mills portion of ours. The energy at which this occurs would then also establish a degree of resolution corresponding to a point at which the Maxwell–Boltzmann phase has given way to the "objective" geometry implicit in the interpretation and use of the correspondence principle by macroscopic external experimenters.

The precise delineation of the nature of the energy dependence of the coupling strength of a Yang–Mills gauge theory, due to Gross and Wilczek and, independently Politzer, in 1973, is one of the triumphs of the subject. Roughly speaking, they found that for gauge theories with a non-abelian gauge group, such as SU(2) or SU(3), the coupling strength increases with distance (or as the energy scale decreases) whereas for an abelian gauge group, such as the U(1) of the $B$-field, or electromagnetism, just the opposite happens. This difference is due to the presence of the commutator term (cf. equation (11.3.14)), which indicates self-interaction among the quanta of the field that is absent in the U(1) case. Since the effective coupling gets weaker as distance decreases in the non-abelian case, this phenomenon goes by the name "asymptotic freedom."

(Even more roughly speaking, the difference may be seen as follows. Consider a localized electric charge viewed from a distance *in vacuo*. The charge will polarize the vacuum in its vicinity by promoting the creation of electron-positron pairs. This polarized cloud will appear from a distance to screen the central charge and weaken the strength of the effective coupling to a charged probe. As the cloud is probed, there will be less screening and the effective coupling strength will increase. In the non-abelian case, since there is now available a multiplet of mutually interacting gauge bosons emanating from branching multilegged boson vertices, each boson line containing fermion loops, the vacuum is provoked into producing a cloud containing fermion-antifermion loops with the whole multiplicity of

different varieties (or colors, or flavors) of charge allowed to members of the multiplet. These do not cancel each other. Indeed, the further a probe is from the central charge the more charge it sees in the cloud surrounding the latter, and the stronger the effective coupling strength becomes: this is "antiscreening." As it moves closer the probe is surrounded by more cloud, and therefore the coupling strength becomes weaker, the interaction becoming "asymptotically free.")

This means that as we lower the energy scale (or degree of resolution) at which our Lagrangian $\mathcal{L}$ (equation (12.1.14)) is employed, relative to the value that corresponds to our unknown g, the $B$, $W$ and $G$ couplings will go their separate ways, those for $W$ and $G$ increasing, while that for $B$, decreasing. Indeed, at energies somewhat less than about $10^2$ GeV (or length scales somewhat greater than about $10^{-16}$ cm) the Standard Model is supposed to yield a $G$ coupling that is *strong*, a $W$ coupling that is *weak* and a $B$ coupling that is weaker still.

In fact, something peculiar is supposed to happen at approximately this energy ($10^2$ GeV) which we discuss briefly in the next section since it will involve a treatment of the superconduction paradigm that was invoked earlier.

Using the experimental values of the couplings obtained at the relatively low energies available to terrestrial experimenters, and the known energy dependence of the couplings, the point at which these three couplings coalesce may be estimated: see, for example, Weinberg 1996, Nachtmann 1990 or Collins, Martin and Squires 1989. The current estimate for the "grand-unified" energy at which this occurs seems to be about $10^{15}$ GeV (corresponding to a length scale of about $10^{-29}$ cm).

The so-called Grand Unified Theories (or GUTs: Georgi and Glashow, circa 1974) posit the existence of a single simple Lie group that dominates all symmetries at energies greater than the grand-unified scale, and that breaks down to $U(1) \times SU(2)_L \times SU(3)$ at lower energy scales. Remarkably, there are numerous candidate groups, the simplest being SU(5). These theories have a truly impres-

sive explanatory power, but unfortunately also suffer from fairly debilitating diseases, and predict a lifetime for the proton that seems too short. Nevertheless, their successes are extremely compelling.

Our approach seems to leave no room for a grand-unified *group*, since our quantization of $S_4$ gives rise to an already broken symmetry, with no intermediate group or algebra being apparent. This may be an artifact of the crudity of our quantization process and/or the macroscopic contamination introduced through our attempt to clothe the defect; or it may indicate that it is not a *group* that dominates the symmetries beyond the GUT scale but some, more subtle, quantum replacement for a group or Lie algebra: perhaps the correct quantum symmetry algebra of the true reticular defect, if there is one. In any event, our $\mathcal{L}$ coincides almost exactly with a snapshot of the Standard Model Lagrangian taken at the precise point of grand-unification: the ancestral symmetry has been broken down to the various gauge groups, yet the couplings are the same. On this assumption, we have

$$g^2 = 4\pi\alpha(M_X^2), \tag{12.2.1}$$

where $M_X^2$ is the grand-unified mass scale. Estimates based on SU(5) GUT considerations (see for instance Nachtmann 1990, p. 464) obtain

$$\alpha(M_X^2) \approx 0.022, \tag{12.2.2}$$

which yields the estimate

$$|g| \approx 0.53. \tag{12.2.3}$$

Then from equations (12.1.14), (12.1.15) and (12.1.17) we find

$$\frac{2}{g^2\tau^2} = \frac{1}{16\pi l_P^2}, \tag{12.2.4}$$

or

$$\begin{aligned} |\tau| &= \frac{\sqrt{32\pi}}{|g|} l_P \\ &\approx 19 l_P \\ &\approx 3\times 10^{-32}\text{cm}. \end{aligned} \tag{12.2.5}$$

Our model also seems to incorporate certain kinds of breakdown as we coarsen the resolution (or increase the length scale, or decrease the energy scale), namely the resolutional change from the net phase to the Maxwell–Boltzmann phase and then from the latter phase to some kind of objective geometry or manifold. Since the output from our correspondence principle is supposed to operate relatively macroscopically, in the presence at least of an objective geometry, we shall assume that an objective manifold-like structure is certainly present at and above the GUT scale (circa $10^{-29}$cm), and possibly well below this scale. However, in our picture, below the chrononic $\tau$-scale (circa $10^{-32}$cm) *length* becomes a meaningless concept: at these scales we must be impinging therefore upon some sort of non-objective geometrical phase. Consequently, the onset of the semi-classical objective phase must follow rapidly after the $\tau$-scale is reached, occurring presumably within the Maxwell–Boltzmann interregnum between these two scales. That is to say, the transition from "quantum" or non-objective geometry to objective (possibly manifold-like) geometry takes place at some scale between $19 l_P$ and $10^6 l_P$.

## 12.3 Superconduction and Electroweak Unification

In this section we give a brief account of the topics listed in its title, from a rather geometrical point of view. The phenomenon we are aiming to describe is believed to occur at length scales greater than or equal to $10^{-16}$cm, corresponding to energy scales less than or equal to $10^2$GeV. If our estimate in the last section of the scale at which we have emerged from the Maxwell–Boltzmann phase, namely $10^{-29}$cm, is even approximately correct, then we may expect an experimenter to be confronted by at least a rudimentary manifold structure at these length scales (i.e. greater than or equal to $10^{-16}$cm) and we may start using in earnest notions appropriate to such a geometry to describe such an experimenter's findings.

We suppose, then, that we are effectively confronted by a manifold, or a structure sufficiently close to being a manifold that we may discuss the notion of a *connection* on a bundle over it. (For details the reader is referred to one or more of the following, among many others: Darling 1994, Kobayashi 1987, Vaisman 1973 and Ward and Wells 1990. A rather abstract analytical treatment of the line bundle case may be found in Selesnick 1976, and a complete account of the geometry of arbitrary vector bundles from a similar but vastly more general point of view may be found in Mallios 1998.)

We briefly recall the notion of a *connection* on a vector bundle, a topic we will revisit in §12.5. One way they arise is when one tries to differentiate a section of such a bundle: it is found that the possible twisting of the fibre must be taken into account. Thus suppose $E$ denotes a complex vector bundle defined for simplicity over a manifold $M$. A general derivation-like operation defined on the bundle must take the form of an additive bundle map

$$D : E \longrightarrow E \otimes \Omega^1, \tag{12.3.1}$$

where $\Omega^1$ denotes the bundle of (complex) 1-forms, satisfying

$$D(sf) = sD(f) + f \otimes ds, \tag{12.3.2}$$

where $s$ is a complex differentiable function on $M$, $f$ a section of $E$, and $d$ denotes exterior differentiation. Such maps can always be found. In a coordinate patch $U_\alpha$ of $M$ over which $E$ is trivial $D$ must assume the form

$$D_\alpha \equiv d + \psi_\alpha, \tag{12.3.3}$$

where $\psi_\alpha$ is a linear bundle map $\psi_\alpha : E \longrightarrow E \otimes \Omega^1$ (restricted to $U_\alpha$) which, when a local basis for $E$ (restricted to $U_\alpha$) is chosen, is referred to collectively as a *connection matrix.* Since

$$\mathrm{Hom}(E, E\otimes\Omega^1) \cong \mathrm{Hom}(E,E)\otimes\Omega^1, \tag{12.3.4}$$

each $\psi_\alpha$ may be considered to be a matrix of local 1-forms. (In general there may be many candidates for such maps and consequently many choices of connection.)

Given a connection $D$ there is defined a bundle map

$$D^1 : E\otimes\Omega^1 \longrightarrow E\otimes\Omega^2 \tag{12.3.5}$$

(where $\Omega^p \equiv \bigwedge^p \Omega^1$ is the bundle of $p$-forms on $M$) satisfying

$$D^1(f\otimes\omega) = f\otimes d^1\omega + Df\wedge\omega. \tag{12.3.6}$$

It is easy and standard that

$$D^1D : E \longrightarrow E\otimes\Omega^2 \tag{12.3.7}$$

is in fact a linear map of bundles. That is, $D^1D$ is a global section of the bundle $\mathrm{Hom}(E, E\otimes\Omega^2) \cong \mathrm{Hom}(E,E)\otimes\Omega^2$ and hence may be thought of as a matrix of globally defined 2-forms whose coefficients may be chosen to lie in the Lie algebra of the structure group of the bundle. This is the *curvature matrix* of the connection $D$.

If $E$ is a *line* bundle, i.e. if $E$ is one-dimensional, we have

$$\mathrm{Hom}(E, E\otimes\Omega^2) \cong \mathrm{Hom}(E,E)\otimes\Omega^2 \cong \Omega^2, \qquad (12.3.8)$$

so that a curvature "matrix" for $E$ is represented simply by a global 2-form.

The last isomorphism in equation (12.3.8) follows from the fact that if $E$ is a line bundle then $\mathrm{Hom}(E,E)$ has a nowhere vanishing global section, namely the bundle map that is the identity on each fibre, and is therefore (isomorphic with) the trivial bundle, which is the identity for the $\otimes$ operation in this context. There are at least two ways to see that a line bundle that admits a nowhere vanishing global section must be trivial. One is to note that in this case the associated principal bundle has a section and is therefore trivial (since a principal bundle admits a section if and only if it is trivial), which implies the result. Another is to consider a specific trivialization of the bundle: that is, we consider an open covering $\{U_\alpha\}$ of the base manifold over each element of which the restriction of the bundle is trivial, as in the discussion leading to equation (12.3.3). Then a section is represented by a family $\{s_\alpha : U_\alpha \longrightarrow \mathbb{C}\}$ of differentiable functions satisfying (on each $U_\alpha \cap U_\beta$)

$$g_{\alpha\beta} s_\alpha = s_\beta, \qquad (12.3.9)$$

where $\{g_{\alpha\beta}\}$ is a Čech cocycle of *transition* functions for the bundle (defined on each $U_\alpha \cap U_\beta$). Then, if each $s_\alpha$ is nowhere zero,

$$g_{\alpha\beta} = s_\beta s_\alpha^{-1} \qquad (12.3.10)$$

and this shows that each $s_\alpha(x)^{-1}$, for $x$ in $U_\alpha$, regarded as an isomorphism $\mathbb{C} \longrightarrow \mathbb{C}$, is actually an isomorphism of the original bundle with the line bundle whose transition functions are constantly 1, i.e. the trivial line bundle. To see this, merely check that a section of $E$ when multiplied by $s_\alpha^{-1}$ yields a globally defined function, that is a

section of the trivial line bundle.

The curvature form of a line bundle is *closed*: that is to say, writing

$$D^1D(f) = f \otimes \Phi_E \,, \tag{12.3.11}$$

where $f$ is a section of $E$ and $\Phi_E$ denotes the curvature form of some connection, we have

$$d^2\Phi_E = 0. \tag{12.3.12}$$

Thus, a curvature form $\Phi_E$ of a line bundle determines a 2-dimensional de Rham cohomology class $[\Phi_E]$ that is easily shown to be independent of the connection that determines it. Moreover, the de Rham class $[-(2\pi i)^{-1}\Phi_E]$ is the image of an integral cohomology class, called the *Chern* class of the line bundle $E$. The Chern class is zero if and only if the bundle is trivial.

The reader will no doubt have noticed that we have already encountered connections in a form suitable to a single experimenter confined to a single trivializing patch: they have arisen as the operators within the terms we have called interaction terms (§11.2.1), where the differentials (or cotangents) $dx_\mu$ have been replaced by their "quantum versions" $\gamma_\mu$. If our single experimenter were to regard him or herself as one objective observer among a potentially infinite multiplicity of other equivalent observers, then he or she would expect objective consistency to obtain when appropriately transforming among connections belonging to different observers. When these consistency conditions are imposed we arrive at exactly the requirement that the manifold formed by collating all the putative experimenters' local patches has the structure of a Lorentz (spin) manifold, and that the local connections patch together on this manifold to form global connections (i.e. bundle maps, as in equation (12.3.1)) on the bundles involved. Moreover, the coefficient matrices of the curvature forms of these connections are proportional to the

tensor fields appearing in the Yang–Mills components of the kinetic terms for $\mathfrak{su}(3)$ and $\mathfrak{su}(2)_{\mathbb{C}}$.

We are now ready to follow historical precedent by considering the phenomenon of superconduction (*à la* BCS), albeit in a much simplified geometrical form, as a stepping-stone to the notion of spontaneous symmetry breaking. Thus, we wish to consider the assemblage of electrons within a (macroscopic!) superconductor. For simplicity, we shall consider such electrons to be adequately represented by just their pair of helicity states so that single electrons in the superconductor are represented by initial action vectors, (or wave-functions) belonging at each spacetime point to a two-dimensional space. Thus we obtain a two-dimensional vector bundle $E$, say, whose sections represent the wave-*fields* of single idealized electrons in the superconductor. We may think of this bundle as being supported on the world-tube of the interior of the physical superconductor considered as a spacelike three-dimensional subdomain of the spacetime manifold, from which we exclude a certain boundary layer in which we might expect to find diamagnetic screening or other surface effects that would perturb our idealized electron fields and destroy the bundle structure.

To obtain a bundle corresponding to an assemblage of such fermions we take the Fermi–Dirac Fock bundle, in other words the exterior algebra of the bundle, which is also a bundle, namely

$$\mathbf{1} \oplus E \oplus \Lambda^2 E, \tag{12.3.13}$$

where $\mathbf{1}$ denotes the trivial line bundle. The series terminates at this point with the dimension of the last component being one. (Although this is a finite dimensional bundle, its space of sections is not.) Let us denote the line bundle $\Lambda^2 E$ by $L$. The sections of this line bundle represent fields of pairs of idealized electrons within the superconductor. If the repulsive Coulombic interaction between the would-be electrons were switched on, these pairs would disappear into the Fermi sea of noninteracting states, in the absence of other effects. In a superconductor, however, thermodynamic and ionic ef-

fects conspire to counteract part of the interaction in such a way that the paired states near the Fermi surface are promoted out of the Fermi sea: the filled Fermi sea becomes unstable toward pair formation. These are the *Cooper pairs*, and the sections of $L$ represent possible Cooper pair condensates. Now, the structure group of any complex bundle over a manifold can always be reduced to the appropriate unitary group. That is, representative transitions functions for the bundle can always be chosen to have values in the appropriate unitary group: for line bundles this group is U(1). (As noted in §7.3 tensor powers of line bundles are necessarily symmetric so their spaces of sections describe systems satisfying Bose–Einstein statistics. Thus, in this idealization, the Cooper pair condensates have this character.)

Let the section $\varphi$ of $L$ represent a Cooper pair condensate. Then, as above, $\varphi$ can be represented on some trivialization of the bundle by a family $\{\varphi_\alpha\}$ of complex functions satisfying the transformation law

$$g_{\alpha\beta}\varphi_\alpha = \varphi_\beta, \tag{12.3.14}$$

where the scalar functions $g_{\alpha\beta}$ are transition functions for $L$ that may be taken to have modulus one. Thus, $\varphi$ has the appearance of an ordinary charged scalar field, and moreover

$$\varphi_\beta^\dagger \varphi_\beta = \varphi_\alpha^\dagger \varphi_\alpha . \tag{12.3.15}$$

In other words, $\varphi^\dagger\varphi$ is globally defined as a function on the support of the bundle, and we may write

$$\varphi^\dagger \varphi = \rho^2, \tag{12.3.16}$$

where $\rho$ represents the charge density of $\varphi$, and we may write locally

$$\varphi_\alpha = \rho \exp(-i\theta_\alpha) \tag{12.3.17}$$

for some real $\theta_\alpha$.

Which of these Cooper pair condensates, emerging from the Fermi sea, can represent a new ground state? In such a state the superconductor should be electrically neutral, and to counteract the constant uniform ambient ionic charge we must then have

$$\rho = \rho_0, \tag{12.3.18}$$

where $\rho_0$ is a nonzero constant. Thus, from equation (12.3.17) the possible ground states are U(1) degenerate in each neighborhood constituting a trivialization of the bundle, being parametrized by the fields $\theta_\alpha$. If the local U(1) symmetry of the bundle had instead been a global symmetry, we might have expected, in accordance with the Goldstone theorem, the "seizing" of the "vacuum" in equation (12.3.18) to be accompanied by the generation of a massless Goldstone boson tunneling between the possible ground states, all of which have the same energy. (This theorem asserts the existence of massless, spinless excitations in the presence of the spontaneous breakdown of a global gauge symmetry: cf. Taylor 1976, §5.4, for an account of a short, elegant proof in the relativistic case due to W. Gilbert.)

Indeed, the general Goldstone argument simplifies in this context as follows. To say that the symmetry is *global* is the same thing as saying that our trivialization has only one element, so we may drop the subscripted index. Then, noting that

$$\nabla\rho_0 = \frac{1}{2}\nabla(\varphi^\dagger\varphi) = 0, \tag{12.3.19}$$

the current produced in a ground state $\varphi$ of charge $2e$ is

$$\mathbf{j} = -2ie(\varphi\nabla\varphi^\dagger - \varphi^\dagger\nabla\varphi)$$

$$= -4ie\varphi\nabla\varphi^\dagger$$

$$= 4\rho_0^2 e \nabla\theta \ . \tag{12.3.20}$$

Thus

$$\begin{aligned} \nabla^2\theta &= (4\rho_0^2 e)^{-1} \nabla\cdot\mathbf{j} \\ &= -(4\rho_0^2 e)^{-1} \frac{\partial}{\partial t}\rho_0 \\ &= 0, \end{aligned} \tag{12.3.21}$$

which expresses the masslessness of the (static) field $\theta$ that implements the vacuum "tunneling" in this case.

However, this result is subverted in the case of the *local* symmetry of the bundle structure, if we are allowed to physically *identify* isomorphic bundles. For, as we have noted, the existence of a nowhere vanishing section of a line bundle implies that the bundle is trivializable. In particular, in the case at hand, any section of $L$ represented locally by the family $\{s_\alpha\}$, say, is mapped to a section of the trivial bundle by the isomorphism given locally by

$$s_\alpha \mapsto s_\alpha \exp(i\theta_\alpha) \tag{12.3.22}$$

since, from equations (12.3.14), (12.3.17) and (12.3.18),

$$g_{\alpha\beta} = \exp(-i\theta_\beta)\exp(i\theta_\alpha) \tag{12.3.23}$$

(cf. equation (12.3.10)).

A transformation of the type shown in equation (12.3.22) is called a *local gauge transformation* and to identify equivalent bundles implies that fields connected by such transformations should also be physically identified. In particular, from equation (12.3.17), it is ap-

parent that our ground state $\varphi$ is gauge equivalent to

$$\varphi_\alpha \exp(i\theta_\alpha) = \rho_0. \tag{12.3.24}$$

Thus the Goldstone-like $\theta$s vanish in this "gauge," and can have no physical significance. That is to say, the production of such bosons is suppressed if all local gauge transformations are allowed. By passing from the bundle $L$ to the isomorphic bundle $\mathbf{1}$, the section of $L$, whose local angular variation represented the Goldstone tunneling between ostensibly different states, has been frozen in direction, becoming the constant real section $\rho_0$ that has no angular variation, hence there are no "transverse" Goldstones.

This evasion of the conclusions of the Goldstone theorem in the presence of *local* gauge invariance was the circumstance that drew the attention of particle theorists to the BCS theory of superconduction as a mass-generation device, since the degree of freedom sacrificed by the $\theta$-field goes into contributing mass to impinging photons instead of generating massless Goldstones. To see this, we suppose our superconductor to be in the presence of an external electromagnetic field with four-potential $A^\mu$, which we shall assume to be sufficiently weak to not perturb the ground state, so that the bundle remains trivial. Then $\partial^\mu$ must be replaced by $\partial^\mu - ieA^\mu$ and produces a change in the current (equation (12.3.20)), which now becomes

$$\begin{aligned} \mathbf{j} &= 4\rho_0^2 e(\nabla\theta - e\mathbf{A}) \\ &= -4\rho_0^2 e^2 \mathbf{A}, \end{aligned} \tag{12.3.25}$$

since $\theta$ has been "gauged away." This last is a basic equation of superconductivity known as the *London* equation. Since the superconductor is assumed to be in equilibrium, it is immediate from the last equation that the electric field vanishes:

$$\mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} = 0. \tag{12.3.26}$$

But, by Ohm's law

$$\mathbf{E} = R\mathbf{j}, \tag{12.3.27}$$

$R$ here denoting the resistance, so

$$R = 0. \tag{12.3.28}$$

That is, the resistance in a superconductor is (eponymously) zero.

Moreover, on taking the curl of Ampère's equation, namely

$$\text{curl}\,\mathbf{B} = \mathbf{j}, \tag{12.3.29}$$

we get from equation (12.3.25), in transverse (electromagnetic) gauge,

$$\nabla^2\mathbf{B} = 4\rho_0^2 e^2\mathbf{B}, \tag{12.3.30}$$

which is a *screened* wave equation. Its exponentially decaying solution shows that the magnetic field dies out after penetrating a "characteristic" depth of $(4\rho_0^2 e^2)^{-1}$ into the superconductor. This expulsion of an impinging magnetic field is called the *Meißner* effect. In Lorentz covariant form the last equation reads

$$(\Box + 4\rho_0^2 e^2)A^\mu = 0, \tag{12.3.31}$$

which may be interpreted as asserting that the incoming photons acquire the *mass* $2\rho_0 e$ as they fail to penetrate into the superconductor.

The Meißner effect has a simple geometrical interpretation. Since the external field must couple minimally to the condensate

fields,

$$D^{\mu} \equiv \partial^{\mu} - ieA^{\mu} \tag{12.3.32}$$

is required to be Lorentz covariant when applied to the fields $\varphi$ that are sections of the bundle $E$. That is, $D^{\mu}$ defines a connection on $E$ whose curvature is the applied field. Moreover, the correct gauge covariant condition for $\varphi$ to represent a ground state is

$$D^{\mu}\varphi = 0 \tag{12.3.33}$$

or

$$\partial^{\mu}\varphi = ieA^{\mu}\varphi \tag{12.3.34}$$

or

$$A^{\mu} = (ie)^{-1}\partial^{\mu} \ln \varphi \tag{12.3.35}$$

which is well defined since such a $\varphi$ can vanish nowhere. The impinging field, which is identical to a curvature of the bundle, is thus expelled from the support of the bundle, that is the interior of the superconductor excluding a certain boundary layer.

This in all essentials is the *Higgs mechanism* (in the scalar case) with $\varphi$ acting as *Higgs field.* In our formulation it has evolved from the single circumstance that the line bundle $L$ has a nowhere vanishing section that happens to represent a ground state. The choice, or *seizure*, of such a ground state constitutes a *spontaneous breaking* of the symmetry that in this case is a local gauge symmetry. Once seized, such a field contributes mass to those gauge bosons impinging upon the region in which the symmetry breaking takes place, and there are no massless Goldstones. The mass is acquired by the gauge bosons as they fail to tunnel into the symmetry breaking region that is the support of the bundle beyond the boundary layer: the force

mediated by such bosons becomes short ranged. The gauge bosons fail to tunnel into the support of the bundle because the structure of the bundle imposes *flatness* (i.e. vanishing curvature) upon the externally imposed connection. (This geometrical view of superconductivity as a spontaneous symmetry breaking of electromagnetic gauge invariance was put forward in Selesnick 1984. A masterful and properly physical treatment of this topic from a similar point of view may be found in Weinberg 1996.)

In the general case of a non-abelian gauge field associated with a bundle (over some subdomain of spacetime) of dimension greater than one, the Higgs field would be a section of the bundle and spontaneous symmetry breaking would occur if this field were to have a nowhere vanishing expectation value. Then the Higgs section would itself have to vanish nowhere and consequently its image in the bundle space would contain at least one flatness-imposing bundle that is split off from the original bundle. That is to say, the original bundle decomposes as the direct sum of at least one flatness-imposing trivial line bundle and another bundle of dimension one less than the original bundle. We conclude that spontaneous symmetry breaking of this type can occur only if the bundle involved admits a nowhere vanishing section.

To add to the peculiarities we have already found to be associated with the $\mathfrak{su}(2)_{\mathbb{C}}$ term, it is believed that a similar spontaneous symmetry breakdown phenomenon occurs in this sector at energies less than about $10^2$ GeV. (It would be interesting if the appearance of yet another quirk in this sector were not coincidence but could somehow be linked to the others.) Here we note that the bundle involved has as fibre the space we have written as $\mathbb{C}\otimes_{\mathbb{C}}W_2$, where an independent U(1) action upon the first factor has been introduced (by complexifying $\mathfrak{su}(2)$). The bundle may therefore be written as $Y\otimes W$, where $Y$ is a line bundle and $W$ is a two-dimensional bundle with structure group SU(2). We have already found the general connection matrix (or gauge potential) for this bundle at the grand-unification scale. It appears in the left-handed interaction term in equation (12.1.13). At the energy scale we are now considering the

couplings are different: we follow custom by denoting the $B$-coupling by $g'$ and the $W$-coupling by $g$. Then, written out explicitly in terms of the Pauli basis, the connection matrix on $Y \otimes W$ assumes the form

$$\mathbf{W}^{\mu} \equiv -\frac{i}{2}\begin{pmatrix} g'B^{\mu} + gW^{\mu}_{(3)} & gW^{\mu}_{+} \\ gW^{\mu}_{-} & g'B^{\mu} - gW^{\mu}_{(3)} \end{pmatrix}, \qquad (12.3.36)$$

where

$$W^{\mu}_{\pm} \equiv W^{\mu}_{(1)} \mp W^{\mu}_{(2)}. \qquad (12.3.37)$$

The theory of spontaneous symmetry breaking in this sector—called *electroweak unification* and named for Glashow, Salam and Weinberg—calls, in our language, for a Higgs section for this bundle that splits off one "flattening" trivial line bundle. In the associated symmetry breaking region—i.e. the support of the Higgs section—we then have a bundle isomorphism

$$Y \otimes W \cong L \oplus \mathbf{1} \qquad (12.3.38)$$

where $L$ is some line bundle and $\mathbf{1}$ is the trivial bundle that imposes flatness upon an appropriate impinging field.

The bundle on the right-hand side of equation (12.3.38) then has a connection matrix that may be written

$$\begin{pmatrix} ieA^{\mu} & 0 \\ 0 & 0 \end{pmatrix} \qquad (12.3.39)$$

for real $A^{\mu}$, which the theory identifies with the electromagnetic potential. (The zero entry in the lower right corner is a result of the flatness imposed by the bundle $\mathbf{1}$.) But from the isomorphism (12.3.38) the potentials $\mathbf{W}^{\mu}$ (equation (12.3.36)) and (12.3.39) are gauge

equivalent in the symmetry breaking region. Consequently the fields $W_+^\mu$, $W_-^\mu$ and $g'B^\mu - gW_{(3)}^\mu$ can be gauged away in this region and this signifies that the corresponding gauge bosons acquire mass as they fail to tunnel into the region and have the ranges of the forces they mediate curtailed. The other field survives in this region and is gauge equivalent there to electromagnetism so that we may write

$$g'B^\mu + gW_{(3)}^\mu = 2eA^\mu. \tag{12.3.40}$$

The fields $B^\mu$ and $W_{(3)}^\mu$ thus appear only in combination. They may be uncoupled, as Weinberg noticed, by choosing a special normalization for the massive neutral field. Thus, outside the symmetry breaking region we may write

$$kZ^\mu \equiv g'B^\mu - gW_{(3)}^\mu, \tag{12.3.41}$$

where $Z^\mu$ denotes a neutral field that acquires mass through the symmetry breaking mechanism, and

$$k^2 = g'^2 + g^2. \tag{12.3.42}$$

Then, with the Weinberg angle $\theta_W$ defined by

$$\tan\theta_W = \frac{g'}{g}, \tag{12.3.43}$$

and taking $k < 0$,

$$Z^\mu = W_{(3)}^\mu \cos\theta_W - B^\mu \sin\theta_W. \tag{12.3.44}$$

Let $X^\mu$ denote the linear combination of $B^\mu$ and $W_{(3)}^\mu$ that is orthogonal to $Z^\mu$. Then

$$B^{\mu} = X^{\mu}\cos\theta_W - Z^{\mu}\sin\theta_W \tag{12.3.45}$$

and

$$W^{\mu}_{(3)} = X^{\mu}\sin\theta_W + Z^{\mu}\cos\theta_W . \tag{12.3.46}$$

In the symmetry breaking region equation (12.3.40) gives

$$(g'\cos\theta_W + g\sin\theta_W)X^{\mu} = 2eA^{\mu} \tag{12.3.47}$$

up to a gauge transformation since $Z^{\mu}$ can be gauged away there. We may therefore make the identifications

$$A^{\mu} = X^{\mu} \equiv W^{\mu}_{(3)}\sin\theta_W + B^{\mu}\cos\theta_W \tag{12.3.48}$$

and

$$e = g'\cos\theta_W = g\sin\theta_W . \tag{12.3.49}$$

Thus, ordinary electromagnetism, with its massless photon, is the remnant of this spontaneous symmetry breaking mechanism that produces massive gauge bosons mediating the short range weak force. These bosons are supposed to acquire mass as they fail to penetrate into the symmetry breaking region: that is, as they meet and interact with the Higgs field. We can model this process by choosing an appropriate Higgs section for the broken bundle $L\oplus\mathbf{1}$, applying to it the unbroken "impinging" connection matrix $\mathsf{W}^{\mu}$ (equation (12.3.36)), and calculating the resulting mass term. An appropriate choice of Higgs is clearly

$$\varphi = \begin{pmatrix} 0 \\ \rho_0 \end{pmatrix}, \tag{12.3.50}$$

$\rho_0$ nonzero, giving a mass term proportional to

$$(\mathbf{W}^{\mu}\varphi)^{\dagger}(\mathbf{W}_{\mu}\varphi) = |\rho_0|^2 \begin{pmatrix} gW_{+}^{\mu} \\ kZ^{\mu} \end{pmatrix}^{\dagger} \begin{pmatrix} gW_{+\mu} \\ kZ_{\mu} \end{pmatrix}$$

$$= |\rho_0|^2 (g^2 \overline{W}_{+}^{\mu} W_{+\mu} + (g'^2 + g^2) Z^{\mu} Z_{\mu}).$$

(12.3.51)

The squared mass $M_W$ of the charged boson(s) is proportional to the coefficient of the $W$ term on the right-hand side of the last equation, whereas the squared mass $M_Z$ of the neutral boson is proportional to the coefficient of the $Z$ term. Thus,

$$\frac{M_W^2}{M_Z^2} = \cos^2\theta_W. \tag{12.3.52}$$

The conclusions of this amazing theory, that also furnishes the fermions with masses, have been verified experimentally, with the mass of the $W$s being found to be about 80 GeV, and that of the $Z$ to be about 91 GeV, the latter to great accuracy. Higgs parameters are uncertain and so far apparently undetected.

## 12.4 Long Distance Topological Implications

Might macroscopic experimenters be able to discern at lower energies (or longer distance scales) effects attributable to ultra-short distance singular behavior? If so, it would seem that such effects might be associated with purely topological properties of the classical manifold with which a macroscopic experimenter is confronted at long distance scales, since these would be independent of the distances and sizes involved. (Such effects are analogous to those induced in solutions to differential equations by their boundary condi-

tions: namely, the solutions are generally affected by these conditions at arbitrary distances from the boundary.) Here we make the simplifying assumption that these singularities—such as pockets of Maxwell–Boltzmann surface—may be smoothly, or at least continuously, excised so that we may still be left with a reasonable spacetime manifold when we avoid them.

Then we are faced with the topological problem of exploring the possible bundle structures a classical spacetime can support. Fortunately, this fundamental problem is completely solved in Avis and Isham 1978, that is at the same time a masterpiece of exposition, providing the reader with all necessary (and still current) topological background.

Low energy physics is dominated by the residual symmetries of the strong interaction, namely (SU(3)) QCD, with some weak remnants, such as fermion masses, electromagnetic effects, etc. There are several variations on the theme that low energy QCD may be describable by a phenomenological field theory of mesons, the idea being that at low energy mesonic degrees of freedom should dictate baryonic ones: when the color degrees of freedom are integrated out, one should be left with a non-linear theory of meson fields in terms of which baryons themselves should emerge as large scale, or topological, features in the mesonic landscape. (For a sampling of such theories the reader may consult Alkofer and Reinhardt 1995, Bhaduri 1988 and Shuryak 1988.)

We shall very briefly describe a slight variation of one such theory, namely the non-linear $\sigma$-model, and summarize some results that seem to implicate the topology in ways consistent with the kind of small scale topological distortions we have envisaged.

For $n_f$ flavors of massless quark the $\sigma$-model is rooted in the assumption that the $\mathrm{SU}(n_f)_L \times \mathrm{SU}(n_f)_R$ symmetry of the vacuum that would ordinarily obtain is, in nature, spontaneously broken down to the diagonal (axial) $\mathrm{SU}(n_f)$ action. Thus the diagonal axial group, denoted $\mathrm{SU}(n_f)_A$, acts principally upon the manifold of degenerate vacua to generate $n_f^2 - 1$ massless $0^-$-Goldstone bosons that tunnel between these degenerate vacua. These particles are identified

with the lightest physical $n_f^2 - 1$ $0^-$-mesons. There is an inherent localness in this scheme, since a vacuum at a point in spacetime is determined there only up to an $SU(n_f)$ phase that may differ at another point. This freedom to choose the vacuum is exploited in the non-linear $\sigma$-model to introduce parameters specifying a certain vacuum orientation at each point relative to a spontaneously seized global vacuum—namely that vacuum which sets the Goldstone fields locally to zero. Then a general axial rotation of the seized vacuum can be expressed in terms of (non-linear) functions of these parameters to furnish a non-linear realization of the original symmetry group out of which the basic dynamical field is constructed (cf. Pokorski 1987).

There is an equivalent formulation of this construction that exposes the possible involvement of the dynamics with an underlying topological structure. Namely, we regard the manifold of Goldstone vacua at each spacetime point, which is essentially a copy of $SU(n_f)$ since the group action is principal, as the fibre of a bundle over $M$, the spacetime manifold. (In this language, the "non-linear realization" is just the image of a global section of the principal $SU(n_f)$ bundle associated to the vacuum bundle: that is, the bundle obtained by replacing each fibre by the group $SU(n_f)$. This bundle is necessarily trivial for reasons to be explained, so such sections exist.)

How does this picture need to be adjusted in order to incorporate massive quarks? Let us denote the massless, or "gauge," quark eigenstates by $u'$, $d'$, $s'$, .... Then if the quark masses arise from Higgs interactions there is a natural way to incorporate the generational structure into the flavor bundle. Thus, the left-handed doublets (cf. the table in §12.1)

$$\begin{pmatrix} u' \\ d' \end{pmatrix}_L, \quad \begin{pmatrix} c' \\ s' \end{pmatrix}_L, \quad \begin{pmatrix} t' \\ b' \end{pmatrix}_L, \tag{12.4.1}$$

corresponding, respectively, to the lepton doublets, with charge superscripts dropped from the lower entries

$$\begin{pmatrix} \nu_e \\ e' \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\mu \\ \mu' \end{pmatrix}_L, \quad \begin{pmatrix} \nu_\tau \\ \tau' \end{pmatrix}_L, \tag{12.4.2}$$

can each be regarded as the doublet of a gauge symmetry whose spontaneous breaking gives mass to the quarks. That is, in the presence of the electroweak interaction the gauge symmetry of these doublets experiences a spontaneous breaking so that they can each be regarded as forming the fibres of a bundle of the form $\mathbf{1}\oplus L$ (where the summands carry the same connotations as they did in equation (12.3.38), though taken in a different order for conventional reasons) while at the same time the quarks acquire mass. Thus, the line bundles whose fibres are spanned by the *massive* eigenstates $u$, $c$, $t$ are in fact trivial and flattening. Then the flavor bundle that incorporates the *massive* quark eigenstates—i.e. in the presence of the spontaneous symmetry breaking—may be expressed, for fibres spanned by $u$, $d$, $s$, $c$, $b$, $t$, in the form

$$F \equiv F_{d,s,b} \oplus \mathbf{1} \oplus \mathbf{1} \oplus \mathbf{1}, \tag{12.4.3}$$

where $F_{d,s,b}$ denotes the direct sum of the line bundles $L$ corresponding to the indices shown. Now this latter three-dimensional bundle, being a sum of line bundles, need not admit an SU(3) structure group (or $\mathfrak{su}(3)$ connection matrix). However, any complex bundle admits a reduction of its structure group to the associated unitary group so one may certainly adopt U(3) as the structure group of $F_{d,s,b}$. Thus U(3) *may* act nontrivially on ($d$, $s$, $b$) in the presence of the mass-inducing spontaneous symmetry breaking, whereas ($u$, $c$, $t$), belonging as they do to the trivial part of the bundle, are left fixed. If this unitary action did not occur, the flavor bundle would remain unchanged in the presence of the symmetry breaking and no distinction could be made between massive quark eigenstates and massless ones. If the above rotation does occur, however, experimenters who seek to diagonalize their quark mass matrix relative to the massless gauge eigenstates or who otherwise effectively neglect to

distinguish between these two sets of eigenstates, would find, upon transforming to other Lorentz frames, that the actual mass eigenstates ($d$, $s$, $b$) had undergone a unitary transformation relative to their initial choice of massless gauge eigenstates ($d'$, $s'$, $b'$). Thus, the original quark eigenstates, that were presumed massless and should remain unmoved, would have to be reexpressed in terms of the rotated mass eigenstates in order to maintain their apparent masslessness or gauge properties in the presence of the weak interaction. That is, the massless eigenstates would appear to rotate relative to the massive ones, in the presence of the weak interaction. The fact that exactly such an effect, namely the *Cabibbo–Kobayashi–Maskawa* (or CKM) *rotation* is actually observed, would seem to support this view of flavor symmetry breaking. (Note that if the $u$, $c$, $t$ masses are attributable to the bundle structure being trivial, then the neutrinos—i.e. the top entries of the doublets shown in equation (12.4.2)—should also acquire mass through this mechanism, since they couple in exactly the same way to the same bundle.)

It is at this point that topological considerations enter the discussion. The mechanism described above fails to drive the CKM rotation if *all* the bundles are trivial: for then the structure group of $F_{d,s,b}$ may be reduced further to the identity, and *no* CKM rotation can be guaranteed. Indeed, this is exactly what happens in the massless SU(3) case, by a deep result of Avis and Isham 1978 (equation (4.28)). This result asserts that complex bundles over reasonable (non-compact) spacetimes are classified by their first Chern class: that is, such a bundle is trivial if and only if its first Chern class vanishes. The first Chern class of a complex bundle $E$, over $M$, denoted $c_1(E)$, is an element in the two-dimensional integral cohomology group $H^2(M,\mathbb{Z})$. We have briefly discussed one way of interpreting it if $E$ is a line bundle (§12.3). For a general bundle $E$ of dimension $n$, its first Chern class may in fact be realized as the (first) Chern class of an associated line bundle, namely

$$c_1(E) = c_1(\Lambda^n E). \tag{12.4.4}$$

Moreover, it is not hard to see that the structure group of $E$ may be reduced to SU($n$) if and only if $c_1(\Lambda^n E) = 0$. Putting this observation together with the result of Avis and Isham we conclude that an SU($n$) bundle must be trivial. Consequently, in order to be non-trivial, a bundle $E$ cannot be an SU($n$) bundle, and, moreover, its departure from triviality—which is simultaneously a departure from SU($n$) symmetry—is encapsulated in, and measured by, the curvature form of the bundle $\Lambda^n E$ (that represents its first Chern class).

Returning to the bundle $F_{d,s,b}$ it appears that a *small* CKM rotation could then be guaranteed if this bundle was in fact topologically *slightly* non-trivial, corresponding to a *slight* breaking of the SU(3) symmetry, a hypothesis for which we shall marshal some support shortly. The additional topological requirement on $M$ entailed by this assumption is simply that $H^2(M,\mathbb{Z}) \neq 0$. Under the reasonable assumption that $M$ is simply connected it follows that

$$H^2(M,\mathbb{Z}) \cong \mathrm{Hom}\,(H_2(M,\mathbb{Z}),\mathbb{Z}) \qquad (12.4.5)$$

and then a theorem of Hurewicz (Spanier 1966) asserting that $\pi_2(M) \cong H_2(M,\mathbb{Z})$ applies to produce at least one non-bounding 2-cycle in $M$, that is a homotopic image of the 2-sphere, supporting a nonzero cohomology class. Thus, for instance, just *one* such 2-cycle, surrounding perhaps an "initial" singularity, or pocket of Maxwell-Boltzmann surface, would suffice.

Certain consequences of this non-triviality assumption may be found in Selesnick 1984–1989. We briefly summarize some of these results.

Consider first the flavor bundle in the one and a half generation approximation: that is, we consider just $u$, $d$, and $s$. In this case a connection matrix for the bundle $\Lambda^3 F$ assumes the form $\mathrm{tr}(RA)$, where $A$ is the original unbroken $\mathfrak{su}(3)$ connection matrix and $R$ is the matrix

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_C & \sin\theta_C \\ 0 & 0 & 1 \end{pmatrix} \tag{12.4.6}$$

appropriate to this approximation: the angle $\theta_C$, that emerges here as the (small) symmetry breaking parameter, is identifiable as the *Cabibbo* angle. ($R$ contains the appropriate CKM matrix, or just the Cabibbo matrix in this case.) Then, since $\theta_C$ is assumed to be small, and $A$ is in $\mathfrak{su}(3)$

$$\begin{aligned} \mathrm{tr}(RA) &= A_{11} + A_{22}\cos\theta_C + A_{33} + A_{32}\sin\theta_C \\ &\approx A_{32}\sin\theta_C . \end{aligned} \tag{12.4.7}$$

The exterior derivative of the associated connection form is a curvature form of the line bundle $\Lambda^3 F$ and may be presumed to interfere with the Goldstone ($0^-$) boson generating mechanism that would otherwise be going on in the associated physical channels. Since $A_{32}$ corresponds to the neutral $\overline{K}^0$ Goldstone field, a PCAC-like analysis (Selesnick 1985) yields the following formula:

$$\sin^2\theta_C \approx \frac{m_{\pi^0}^2}{\sqrt{2}\, m_{K^0}^2}, \tag{12.4.8}$$

which gives

$$\sin\theta_C \approx 0.22804 \quad [0.2205 \pm 0.0018], \tag{12.4.9}$$

a recent experimental value appearing in square brackets.

Another formula to emerge from this analysis is

$$m_\eta^2 \approx \sqrt{\frac{3}{2}}\, m_{K^0}^2, \tag{12.4.10}$$

which yields

$$m_\eta \approx 550.8 \text{ MeV} \quad [548.8 \text{ MeV}], \tag{12.4.11}$$

where $\eta$ denotes the physical meson of that name (a mixture of $\eta_1$ and $\eta_8$).

More striking effects are seen in the two generation approximation that includes $c$. The appropriate rotation matrix, acting on columns of the form $(u, d, s, c)$ is now

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_C & \sin\theta_C & 0 \\ 0 & -\sin\theta_C & \cos\theta_C & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{12.4.12}$$

and the analog of equation (12.4.7) is

$$\text{tr}(RA) \approx (A_{32} - A_{23}) \sin\theta_C \,. \tag{12.4.13}$$

The Goldstone field now appearing in the expression on the right-hand side of the last equation is the one associated with the CP-even neutral kaon eigenstate ($K_2$) and if this field is to be associated with a hard topological component that cannot be rotated away, then it must, in its own right, acquire a topological charge, as must the exterior square of the 2-form it represents, whose coefficient (an axial "anomaly") contributes to the axial charge in the appropriate Goldstone channel. The same remark holds for the entire expression, with the $\sin\theta_C$ included. The upshot of such considerations is the "rationality" formula

$$\sin^2\theta_C \approx \frac{1}{4\pi^2}\left(\frac{m}{n}\right), \tag{12.4.14}$$

where $m$ and $n$ are integers. It is argued in Selesnick 1989 that $m = 2n$, giving

$$\sin\theta_C \approx 0.2251 \quad [0.2205 \pm 0.0018]. \tag{12.4.15}$$

The CP-odd kaon eigenstate $K_1$ remains untouched by this topological effect and this dichotomy between $K_1$ and $K_2$ has interesting consequences when the "Goldstone" vacuum is identified with the QCD "$\theta$-vacuum." Namely, the topological $K_2$ acts on the $\theta$-vacuum so as to induce a CP violating tunneling amplitude between $K_S$ and $K_L$, a crude estimate of which may be found in Selesnick 1989. The result is

$$\langle K_S | K_L \rangle \approx 3.39 \times 10^{-3} \quad [3.27(\pm 0.12) \times 10^{-3}]. \tag{12.4.16}$$

This model may also be used to crudely estimate direct CP violation in kaon decays by taking into account the tunneling between inserted vacua.

The peculiar connections that may or may not obtain between spacetime singularities and T violation in physical processes are discussed in the fascinating Penrose 1978 and 1979. Note that in our picture there are only left-handers (associated with the spinor space $S$) at the deepest quantum level: cf. §7.1.

## 12.5 Quantization, Connections and Loop Quantum Gravity

As already noted, it has not been our purpose here to give an account of the full quantization of the field theories that arise from the Lagrangians. Such accounts may be found in the references already cited, at least for the particle spectrum excluding gravity. However, it appears that within even our minimalist scheme there is sufficient context to motivate a discussion of one important aspect of such a quantization program.

A fundamental step in the quantization of a gauge theory is

the specification of a viable measure on the "space" of gauge potentials (modulo gauge equivalence: see on), so that the path integrals of the theory may acquire a degree of credibility. For the purposes of canonical quantization an essentially equivalent problem is the specification of a viable Hilbert space to accommodate representations of the appropriate operator algebra. (Such a Hilbert space may be derived by suitably manipulating the $L^2$-space relative to the measure, assuming this to have been found.)

In the traditional context, in which one assumes an underlying manifold, the problem amounts to finding a measure on the space of connection forms (modulo gauge transformations) on a vector bundle defined over the manifold. In this form, the problem has proven to be immensely challenging, and its investigation has, within the last two decades, spawned vast tracts of noteworthy research including the opening up of new fields. We briefly review some of the relevant notions.

Let $E$, $F$ denote vector bundles defined over a manifold, sufficient smoothness being assumed for all concerned. Vector bundle isomorphisms $E \longrightarrow F$, where $E$, $F$ have the same structure group $G$, are called *gauge transformations* if a map of the associated principal bundles is induced by it. Equivalently, a gauge transformation $E \longrightarrow F$ is a vector bundle isomorphism that may be given locally upon sections of the respective bundles by actions lying locally in the structure group $G$. Two such bundles are equivalent as far as the physical fields they determine are concerned. (This generalizes the definition in the one-dimensional case treated in §12.3: namely, equation (12.3.22). We now drop the appellation "local.")

If $\phi : E \longrightarrow F$ is a gauge transformation and $D_E : E \longrightarrow E \otimes \Omega^1$ is a connection on $E$, then a connection $D_F$ on $F$ is physically equivalent to $D_E$ if, for any section $s$ of $E$,

$$D_F(\phi s) = (\phi \otimes 1) D_E s. \tag{12.5.1}$$

(Here $\phi$ is given locally by $G$-valued functions on the manifold.)

Writing locally, in an obvious notation,

$$D_E \equiv (\partial^\mu + A_E^\mu) \otimes dx^\mu , \tag{12.5.2}$$

$$D_F \equiv (\partial^\mu + A_F^\mu) \otimes dx^\mu , \tag{12.5.3}$$

where the $A^\mu$ may be taken to lie in the Lie algebra of $G$, equation (12.5.1) yields

$$A_E^\mu = \phi^{-1} \partial^\mu \phi + \phi^{-1} A_F^\mu \phi . \tag{12.5.4}$$

The right-hand side of this equation specifies the action of the "gauge group" upon the connection matrix $A_F^\mu$: it is this action that must be divided out of the set of connections on a given bundle, since elements in the same orbit are physically indistinguishable. Two such connections are said to be *gauge equivalent.*

The transformation shown on the right-hand side of equation (12.5.4) has, for obvious reasons, exactly the same form as that which obtains between local connection matrices on the overlap of two trivializing neighborhoods of a single bundle. The distinction between these local gauge equivalences within the same bundle, and those that arise in the case of a global isomorphism between ostensibly different bundles, is sometimes obscured in the physics literature. (For connections in general, see the references cited in the second paragraph of §12.3. In particular, Volume II of Mallios 1998 contains a treasure trove of detailed computations, conducted within a very general unified framework.)

The set of connections on a given bundle over a manifold may be parametrized in various ways, such a parametrization being the first step in the search for a measure and/or a Hilbert space. One such parametrization exploits a remarkable and deeply significant circumstance: namely, that connections on bundles upon manifolds are, in a sense, supported by one-dimensional polymer-like structures lying within the manifold. This rather surprising phenomenon is so fundamental that primitive traces of it can be found even among the

simple algebraic structures we have encountered here, and encountered, moreover, in the absence of an underlying manifold.

To motivate this, let us return to the context of §11.1. There we were concerned with actions upon elements in a representation space $\mathbb{W}$ of a Lie algebra $\mathfrak{a}_c$ induced by the unit truncations of net elements effected by the operators $\Delta^\mu$. Specifically, we associated an element of $\mathfrak{a}_c$ with each $\Delta^\mu$: cf. equation (11.1.22). This Lie algebra element represents, in a sense, the change in the "fibre" $\mathbb{W}$ as we "move" one chrononic unit in the $\mu$-direction. We now generalize this as follows. Let $G$ denote a compact Lie group with Lie algebra $\mathfrak{g}$. Then the data specifying a "connection matrix" with values in $\mathfrak{g}$ in the reticular context is simply an assignment of the form

$$\Delta^\mu \mapsto A^\mu, \tag{12.5.5}$$

where $A^\mu$ lies in $\mathfrak{g}$. This is a local version of the data specifying a connection matrix on vector bundle, $E$ say, with structure group $G$, over a manifold $M$, with the space spanned by the $\Delta^\mu$ playing the rôle of the tangent space at a generic point of $M$. In this case, a connection form is an element in the space of bundle maps

$$\begin{aligned} \mathrm{Hom}(E, E\otimes\Omega^1) &\cong \mathrm{Hom}(E,E)\otimes\Omega^1 \\ &\cong \mathrm{End}E\otimes\mathcal{T}^* \\ &\cong \mathrm{Hom}(\mathcal{T}, \mathrm{End}E), \end{aligned} \tag{12.5.6}$$

where $\mathcal{T}$ denotes the tangent bundle on $M$, which is dual to the bundle $\Omega^1$ of 1-forms on $M$. At each point $x$ of $M$ the image of the map corresponding to the connection in the right-hand side of the last equation lies in $\mathfrak{g}$, regarded as a subspace of $\mathrm{End}E_x$, where $E_x$ is the fibre of $E$ over $x$. This is in exact analogy with equation (12.5.5), at least locally at each point of $M$.

Suppose now that such a "reticular" connection, equation (12.5.5), is given. Let $i:\mathfrak{g} \longrightarrow U(\mathfrak{g})$ denote the canonical map of $\mathfrak{g}$ into its universal enveloping algebra (§3.1.9). Since $U(\mathfrak{g})$ is an associative algebra, the composition of $i$ with the linear map determined by equation (12.5.5) lifts to a map of associative algebras

$$\varphi_A: \mathbb{C}[\Delta^\mu] \longrightarrow U(\mathfrak{g}) \tag{12.5.7}$$

given by

$$\varphi_A(\Delta^{\mu_1} \ldots \Delta^{\mu_n}) = i(A^{\mu_1}) \ldots i(A^{\mu_n}), \tag{12.5.8}$$

where $\mathbb{C}[\Delta^\mu]$ is the algebra freely generated by the symbols $\Delta^\mu$: that is, it is the tensor algebra of the vector space generated by the symbols $\Delta^\mu$. To avoid confusion, since the $\Delta^\mu$ enter these considerations only formally (losing their original interpretation as operators), we shall replace them hereafter by (noncommuting) indeterminates denoted by $X^\mu$.

Now, it is easily verified that $\varphi_A: \mathbb{C}[X^\mu] \longrightarrow U(\mathfrak{g})$ is also a map of *coalgebras:* one merely checks that the maps traversing both circuits of the relevant diagram, which are maps of algebras, coincide on the generators $X^\mu$ of $\mathbb{C}[X^\mu]$, rendering the diagram commutative. Consequently, we obtain a map of the dual algebras:

$$\varphi_A^*: U(\mathfrak{g})^* \longrightarrow \mathbb{C}[X^\mu]^*. \tag{12.5.9}$$

The space $\mathbb{C}[X^\mu]^*$ may be identified with the space of formal power series $\mathbb{C}[[X^\mu]]$ in the noncommutative indeterminates $X^\mu$, and the algebra structure on the latter space that is dual to the coalgebra structure on $\mathbb{C}[X^\mu]$ coincides with that given by shuffle products of monomials. The proof of this assertion is identical to the proof of Proposition 8.2.1. We shall denote by $\mathbf{Sh}(X^\mu)$ this space equipped with the algebra structure given by the shuffle product.

Now we turn to the "gauge" group $G$. For a continuous

(hence analytic) representation $\sigma: G \longrightarrow \mathrm{GL}(V^{(\sigma)})$, where $V^{(\sigma)}$ is finite dimensional, denote by $d\sigma: \mathfrak{g} \longrightarrow \mathfrak{gl}(V^{(\sigma)})$ its differential (a Lie map). The Lie algebra $\mathfrak{gl}(V^{(\sigma)})$ of $\mathrm{GL}(V^{(\sigma)})$ is just the associative algebra $\mathrm{End}V^{(\sigma)}$ with Lie product given by the commutator. Thus, by the universal property for $U(\ )$, $d\sigma$ lifts uniquely to a map of algebras, which, for notational reasons, we shall write

$$L_\sigma: U(\mathfrak{g}) \longrightarrow \mathrm{End}V^{(\sigma)}. \tag{12.5.10}$$

If $\tau: G \longrightarrow \mathrm{GL}(V^{(\tau)})$ is another representation of $G$, then the differential of $\sigma \otimes \tau: G \longrightarrow \mathrm{GL}(V^{(\sigma)} \otimes V^{(\tau)})$ is the Lie map

$$d(\sigma \otimes \tau): \mathfrak{g} \longrightarrow \mathfrak{gl}(V^{(\sigma)} \otimes V^{(\tau)})$$

given by

$$d(\sigma \otimes \tau)(\ ) = d\sigma(\ ) \otimes 1 + 1 \otimes d\tau(\ ) \in \mathfrak{gl}(V^{(\sigma)} \otimes V^{(\tau)}). \tag{12.5.11}$$

Now note that, with $\psi$ denoting the coproduct in $U(\mathfrak{g})$, we have, for $A \in \mathfrak{g}$:

$$\begin{aligned}(L_\sigma \otimes L_\tau \circ \psi)(i(A)) &= (L_\sigma \otimes L_\tau)(1 \otimes i(A) + i(A) \otimes 1) \\ &= 1 \otimes L_\tau(i(A)) + L_\sigma(i(A)) \otimes 1 \\ &= (1 \otimes d\tau + d\sigma \otimes 1)(A). \end{aligned} \tag{12.5.12}$$

Thus, $L_\sigma \otimes L_\tau \circ \psi$ satisfies the commutation property uniquely satisfied by $L_{\sigma \otimes \tau}$ so we have proved

PROPOSITION 12.5.1

$$L_{\sigma\otimes\tau} = L_\sigma \otimes L_\tau \circ \psi. \tag{12.5.13}$$

For a representation $\sigma$ of $G$ we define a linear map

$$\theta_\sigma : V^{(\sigma)} \otimes V^{(\sigma)*} \longrightarrow U(\mathfrak{g})^* \tag{12.5.14}$$

determined, for $v \otimes f \in V^{(\sigma)} \otimes V^{(\sigma)*}$ and $w \in U(\mathfrak{g})$, by

$$\theta_\sigma (v \otimes f)(w) = f(L_\sigma (w) v). \tag{12.5.15}$$

Now consider $R(G)$, the Hopf algebra of representative functions on $G$ (§3.1.7). As noted, there is a coalgebra isomorphism (equation (3.1.7.4)) which may be reexpressed as

$$R(G) \cong \bigoplus\nolimits_{\sigma \in \hat{G}} V^{(\sigma)} \otimes V^{(\sigma)*}, \tag{12.5.16}$$

where $\hat{G}$ denotes the set of equivalence classes of irreducible (necessarily finite dimensional) unitary representations of $G$. This isomorphism assigns to the element $u_{ij}^{(\sigma)}$ of $R(G)$ (where

$$u_{ij}^{(\sigma)}(g) \equiv \langle \xi_i | \sigma(g) | \xi_j \rangle, \tag{12.5.17}$$

$g \in G$ and $\{\xi_i\}$ is an orthonormal basis of $V^{(\sigma)}$) the matrix $|\xi_j\rangle \otimes \langle \xi_i|$ in $V^{(\sigma)} \otimes V^{(\sigma)*}$.

Composing this isomorphism with $\oplus_\sigma \theta_\sigma$ (equation (12.5.15)) yields a map

$$\theta : R(G) \longrightarrow U(\mathfrak{g})^* \tag{12.5.18}$$

which, for $w \in U(\mathfrak{g})$, reads

$$\theta(u_{ij}^{(\sigma)})(w) = \theta_\sigma(|\xi_j\rangle\otimes\langle\xi_i|)(w)$$

$$= \langle\xi_i|L_\sigma(w)|\xi_j\rangle. \qquad (12.5.19)$$

PROPOSITION 12.5.2

*With the dual algebra structure on $U(\mathfrak{g})^*$ induced from the coproduct $\psi$ of $U(\mathfrak{g})$, $\theta$ is a map of commutative algebras.*

PROOF

Since $R(G)$ is finitely generated as an algebra by elements of the form given by equation (12.5.17) it suffices to show that $\theta$ acts multiplicatively on products of elements of this form. But, for example,

$$u_{ij}^{(\sigma)}u_{kl}^{(\tau)}(g) = \langle\xi_i^{(\sigma)}|\sigma(g)|\xi_j^{(\sigma)}\rangle\langle\xi_k^{(\tau)}|\tau(g)|\xi_l^{(\tau)}\rangle$$

$$= \langle\xi_i^{(\sigma)}\otimes\xi_k^{(\tau)}|(\sigma\otimes\tau)(g)|\xi_j^{(\sigma)}\otimes\xi_l^{(\tau)}\rangle. \qquad (12.5.20)$$

The result now follows immediately from equation (12.5.13).

(Hint: for $w\in U(\mathfrak{g})$ write $\psi(w) = \sum w_{(1)}\otimes w_{(2)}$ and compare $\theta(u_{ij}^{(\sigma)}u_{kl}^{(\tau)})(w)$ with $(\theta(u_{ij}^{(\sigma)})*\theta(u_{kl}^{(\tau)}))(w)$ *via* equation (12.5.13).) ∎

Composition of the algebra maps $\varphi_A^*$ (equation (12.5.9)) and $\theta$ (equation (12.5.7)) yields a map of commutative algebras

$$\Xi(A) \equiv \varphi_A^*\circ\theta : R(G) \longrightarrow \mathbf{Sh}(X^\mu). \qquad (12.5.21)$$

Now consider an algebra map

$$\gamma : \mathbf{Sh}(X^\mu) \longrightarrow \mathbb{R}. \qquad (12.5.22)$$

Composing it with $\Xi(A)$ yields an algebra map

$$h_\gamma^A \equiv \gamma \circ \Xi(A): R(G) \longrightarrow \mathbb{R} \tag{12.5.23}$$

trivially satisfying the property

$$h_\gamma^A(\bar{f}) = \overline{h_\gamma^A(f)}, \tag{12.5.24}$$

$f \in R(G)$, the bar denoting complex conjugation. Consequently, by the Tannaka–Krein Duality Theorem (§3.1.7), there exists a unique element $g \in G$ such that for all $f \in R(G)$,

$$h_\gamma^A(f) = f(g). \tag{12.5.25}$$

Thus, for a given "connection matrix"—that is, an assignment of the form (12.5.5)—each real-valued shuffle homomorphism $\gamma$ may be assigned an element of $G$. Now, given a curve (or path) $C$ in $\mathbb{R}^4$, the assignment

$$X^{\mu_1} \ldots X^{\mu_n} \mapsto \int_C dx_{\mu_1} \ldots dx_{\mu_n} \tag{12.5.26}$$

of iterated integrals, as noted in §8.2, determines such an algebra map (that will also be denoted $C$),

$$C: \mathbf{Sh}(X^\mu) \longrightarrow \mathbb{R}. \tag{12.5.27}$$

This map, which may also be described by the appropriate Chen series, essentially determines the curve completely up to translation, by the classical results of Chen (Chen 1958). In this case, the group element determined by $h_C^A$ according to the Tannaka–Krein theorem is exactly the standard holonomy along $C$ of the (constant)

connection $\partial^\mu \mapsto A^\mu$ on the tangent bundle of $\mathbb{R}^4$. (The *holonomy* of a connection along a path in the base manifold is that element of the structure group that effects parallel transport of fibres along the path: cf. Baez and Muniain 1994, Barrett 1991 among others.) To see this, merely unfold the above definitions and evaluate $h_C^A$ on elements of the form given by equation (12.5.17).

Thus, we obtain, through entirely algebraic means, a local version of the geometric notion of the holonomy of a connection along a path in a base manifold. In the case of an actual manifold $M$ we have, roughly speaking, for each connection (denoted $A$ for simplicity) on a given bundle on $M$ with structure group $G$, a holonomy map

$$\mathrm{hol}_A : \mathbf{P}M \longrightarrow G, \tag{12.5.28}$$

where $\mathbf{P}M$ denotes the space of paths in $M$, that takes path composition to group multiplication. ($G$ need not be compact here.) Part of the roughness in our speaking has to do with the proper definition of the word "path," which we will leave in abeyance in this brief overview. The assignment $A \mapsto \mathrm{hol}_A$ effects a parametrization of the set of connections, which, if care is taken, is in fact faithful: the bundle may be reconstructed from the set of data symbolized by equation (12.5.28). A very careful and comprehensive treatment of this topic may be found in Barrett 1991.

Thus, the set of connections may be, in a sense, embedded into the group $G^{\mathbf{P}M}$, which, as a sort of product of $G$s, carries a sort of product of the Haar measure on $G$, and hope is raised that a properly $G$-invariant measure may be found here. The implementation of such a plan has proven to be extremely challenging.

A complementary approach (in the context of canonical quantization) is to try to specify directly a family of functions, or "states," that might form a basis for the relevant Hilbert space itself, thereby circumventing (or postponing) the problem of finding a measure. To motivate this, let us return to equation (12.5.21) and take linear duals:

$$\Xi(A)^*:\mathbf{Sh}(X^\mu)^* \longrightarrow R(G)^* \cong \prod_{\sigma\in\hat{G}} V^{(\sigma)}\otimes V^{(\sigma)*}, \tag{12.5.29}$$

so that we may regard the holonomy map as a map

$$\mathrm{hol}_A:\mathbf{P}\mathbb{R}^4 \longrightarrow \prod_{\sigma\in\hat{G}} V^{(\sigma)}\otimes V^{(\sigma)*}. \tag{12.5.30}$$

Equivalently, each connection gives rise to a map expressible in the form

$$\mathbf{P}\mathbb{R}^4 \times \hat{G} \longrightarrow \dot{\bigcup_\sigma} V^{(\sigma)}\otimes V^{(\sigma)*} \tag{12.5.31}$$

(the dot signifying disjoint union) given by

$$(C,\tau) \mapsto \tau(\mathrm{hol}_A(C)) \in V^{(\tau)}\otimes V^{(\tau)*}. \tag{12.5.32}$$

A numerical value may now be attached to $A$ by exploiting this map. For instance, one may fix a set of paths lying in the base manifold and assign an irreducible representation of $G$ to each path in the set in such a way that the assignments of operators to the resulting pairs (path, representation) *via* equation (12.5.32) may be composed and contracted to produce a number (for each $A$) in the appropriately $G$-invariant fashion. For example, a loop $L$ in the manifold (a set comprising one path), and choice of irreducible representation $\sigma$, produces a $G$-invariant function on connections *via* the assignment

$$A \mapsto \mathrm{tr}\,(\sigma(\mathrm{hol}_A(L))). \tag{12.5.33}$$

Such functions generate interesting states (the *Wilson loops*) but these are not the most general, and are in fact over-determined. More general graph-like patterns embedded in the manifold, embellished with irreducible representations attached to the paths constituting the edges, and with appropriate intertwiners for the representations involved provided at the vertices, may be used to produce functions, *via* equation (12.5.32), on the set of connections, which do indeed provide a basis for the relevant Hilbert space. These graph-like structures are called *spin nets,* a terminology stemming from the use of the group $SL(2,\mathbb{C})$ in their original construction by Penrose in the late 1960s (cf. Baez 1996). During the last decade and a half, dramatic progress has been made in this area, particularly as it applies to the effort to quantize (pure) gravity. (A list of names associated with this effort should include those of Ashtekar, Baez, Barrett, Bojowald, Isham, Jacobson, Rovelli, Smolin, Thiemann and many others. The reader may consult the superb overviews Rovelli and Upadhya 2002, Rovelli 1998, and Ashtekar 2002, 1999 and their references. The text Gambini and Pullin 1996 gives a comprehensive account of the loop representation in other gauge theoretic contexts in addition to gravity. For an encyclopedic account of the foundations of the gravitational theory, see the text Thiemann 2001.)

Since experimental data in the quantum regime of gravity would seem well beyond current technology, success in this context must be measured relative to criteria of consistency, either externally with respect to other parts of physics, or internally, in the sense that internal inconsistencies are convincingly removed. The infant quantum theory of gravity that has emerged through the application of these methods, dubbed *loop quantum gravity* (LQG), has recently scored on both counts. It gives rise spontaneously to a kind of "quantum geometry" in which operators corresponding to the measurement of certain geometrical quantities, namely area and volume, have discrete spectra. These sets of eigenvalues generally have a non-uniform structure, and tend to crowd together to yield back the continuum in semi-classical regimes. The smallest eigenvalues are on the scale of powers of the Planck length, rendering meaningless the at-

tempt to "observe" areas or volumes below this scale. Curiously, the operator corresponding to determinations of length itself does not seem to fit very comfortably into this picture: cf. Rovelli 1998.

Among the physical implications of the theory is a completely rigorous and satisfying account of the formerly heuristic and somewhat puzzling results dating from the 1970s concerning the extremely large entropy associated with black hole (and cosmological) horizons, which comes ultimately from quantum "microstates."

An even more impressive achievement of the theory is the miraculous resolution of the big bang singularity by M. Bojowald (Bojowald 2001, Ashtekar 2002). Big bang models emerge in classical general relativity when it is assumed, for instance, that the four-manifold representing spacetime is foliated by three-dimensional spatial leaves or slices, each of constant positive curvature. Each spatial leaf is therefore a three-sphere whose radius, $a$, characterizes the curvature at the corresponding instant of time. Solutions of the classical Einstein field equations with these symmetries predict a big bang origin: the scale factor $a$ goes to zero as time is reversed in the solution, and the curvature goes to infinity as $a^{-2}$. It is generally conceded that classical physics breaks down in some neighborhood of this singularity and the hope has been nursed by many over a number of decades that "quantum effects" might intercede to alleviate this difficulty. Bojowald's work seems to have fulfilled this hope, and has done so in a particularly beautiful manner. In his treatment the scale factor $a$ becomes an operator with discrete spectrum, whose values crowd together extremely rapidly beyond the Planck scale. Curvature becomes an observable that is bounded from above by the immense (but finite) value of about $10^{77}$ times the curvature of a solar mass black hole. The classical geometrical relation (curvature)$\,a^2 = 1$ breaks down in the quantum region—giving some inkling of the strangeness of actual "quantum geometry"—but is recovered in the semi-classical regime. Quantum evolution is well defined even at the big bang itself.

A particularly striking general feature of these calculations is the rapid onset of the semi-classical regime. This seems to occur at

hundreds of Planck lengths rather than many orders of magnitude greater, as has sometimes been proposed. Considering our use in §12.2 of a rather phenomenological GUT model, which includes other fields besides gravity, this conclusion is not inconsistent with our findings regarding the onset of the "objective" geometrical phase, such as they are.

## 12.6 Outlook

The hallmark of quantum experimentation is its non-objective nature: experimenters/observers *generically* disagree. For this reason, it would seem that accounts of the macroworld that start from quantum premises can at best only describe *local,* or per quantum experimenter, coordinatizations of the World, since that which may lie outside a quantum experimenter's purview—including knowledge of *other* experimenters—is intractable. (This circumstance is analogous to—indeed, the root cause of—the phenomenon of spacelike separation in classical relativity.) For such a quantum-based theory to encompass the full panoply of macroscopic physics in a semi-classical approximation it would seem necessary for it to effect, in some manner, a consistent piecing together of these local patches. How may this be done without introducing non-quantal macroscopically chauvinistic assumptions concerning other experimenters and their mutual relations? Ecumenical or democratic principles, such as general covariance *à la* classical general relativity, seem to fit the macroscopic evidence, but the assumption of such edicts would beg the question as to whether or not the World really is quantal at bottom.

Such theories as LQG (and the various string theories) start with global classical manifold assumptions emphatically and explicitly in place. Indeed, one of the critical assumptions of LQG is the presence of invariance with respect to diffeomorphisms of the base manifold—a rather strong form of the principle of democracy among local observers.

A somewhat less classically based approach is afforded by the

now highly developed methods of noncommutative geometry: cf. Gracia-Bondía *et al.* 2001, Loday 1998, Landi 1997, Madore 1995, Connes 1994, Karoubi and Leruste 1989, Karoubi 1987. However, the effective application of these methods to fundamental physics still seems to entail the assumption at the outset of certain global manifold structures. Indeed, a local version of the theory, together with a means of mediating between local and global—a noncommutative version of sheaf theory, for example—seems to remain underdeveloped at the time of this writing (but see for instance Mallios and Raptis 2001, 2002 and Raptis 2000b).

As noted, theories that start from the quantum end of the spectrum must address the issue of globalization if they are to viably reproduce the macrocosm of experience in their semi-classical regimes. Recent attempts to effect such a globalization center on the notion of a *topos*. This notion (due to A. Grothendieck), which we have already encountered, has wide and deep ramifications, amazingly drawing together both logic and geometry. Topos theory, which may be viewed from one perspective as a generalization of sheaf theory, provides a means of describing consistencies among "local" entities that are much more general than the kind of consistencies required among the local coordinate patches of a manifold. Very promising results have been obtained in a causal context close to that of our work here: cf. Mallios and Raptis 2001, 2002 and Raptis 1996. For other applications, see Butterfield and Isham 2000, Isham and Butterfield *et al.* 1998–2000, and the work of A. Guts, who maintains a valuable list of papers on this topic at the website: http://users.univer.omsk.su/~topoi/index.html.

In this work we have skirted the globalization issue, since our aim here—which seems to have been accomplished—was merely to compute Lagrangian densities, which are intrinsically local. The quantum net represents a generic local quantum structure whose localness is no impediment to such an endeavor.

The Lagrangian that has emerged from our macroscopically somewhat compromised non-objective considerations coincides almost exactly with a snapshot of the Standard Model taken at the

grand-unified scale. Compared to this scale the masses of the quanta involved are likely to be negligible, so their absence from our picture may not be a catastrophe. Of greater concern is the extra uncalculated gravitational term. This term, whose ramifications lie beyond the intellectual capacities of this writer to ascertain, would surely vitiate the standard gravitational equations (at least at small length scales), supplying a "cosmological" term near the big bang.

Our specification of the reticular vacuum is certainly a weak link in the chain, compounding macroscopic intrusions with an oversimplified group quantization scheme. One result seems to be that spaces of initial acts and their multiplicities are insufficiently restricted, and no generational structure is apparent.

On the other hand, we have found a rapid onset of the semiclassical regime, presumably mediated *via* a Maxwell–Boltzmann phase: a conclusion which seems to be consistent with recent results in LQG. Here we meet the symmetry- or group-quantization problem again in an acutely interesting form. At some point below the grand-unified (length) scale, we can no longer effectively speak of a manifold structure but must invoke instead some sort of nonobjective Maxwell–Boltzmann phase. What, then, is the appropriate quantized grand-unified symmetry algebra replacing the Lie algebra we have used? Whatever it is, it should preserve the virtues of the simple Lie group based GUTs and related inflationary scenarios.

As remarked, in the absence for the foreseeable future of experimental data pertaining to the quantum gravity regime, internal consistencies should be highly prized. It will be of great interest, therefore, to see whether or not these two classes of theory—local *vs.* global, each starting from diametrically opposing premises—might be engineered to meet upon some middle ground.

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# Citation Index

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# Notational Index and Glossary

The alphabetizable portion of this (partial) list of notations appears first.

$\mathfrak{a}_c$: equation (11.1.5).

bijective: injective and surjective, i.e. one-to-one and onto.

$\mathbb{C}[\Gamma^N]$: 7.2, see after equation (7.2.5).

$\mathbb{C}[\Gamma]$ : 7.2, see after equation (7.2.8).

$\mathbb{C}[[\Gamma]]$ : 8.1, see after equation (8.1.21).

$C_0(\ )$ : $C^*$-algebra of continuous complex functions vanishing at infinity on the locally compact Hausdorff argument, 1.2.

$\mathbb{C}$ : field of complex numbers.

$\Delta_N^\theta$ : diagram (8.1.3).

$\Delta_N^k$ : equation (9.3.8).

$\delta_{ij}$, $\delta_j^i$ : Kronecker delta, which has the value 1 when $i = j$ and 0 otherwise.

det( ) : determinant of the argument.

dim( ) : dimension of the vector space argument.

$E(\ )$ : exterior algebra of the argument, equation (2.2.3.4), 3.1.8.

End( ) : algebra of linear transformations of the vector space argument into itself, 1.3.

$\eta$ : Minkowski metric in diagonal matrix form, equation (11.1.19); entries in this matrix expressed as traces, equation (11.2.2.10).

$\hat{G}$ : set of equivalence classes of irreducible unitary representations of the compact group $G$, 3.1.7.

GL( ) : group of isomorphisms of the vector space argument, indentifiable with the group of invertible $n \times n$ matrices, where $n$ is the dimension of the argument.

g : unified coupling constant, equations (11.3.10), (11.3.11).

$g$ : $\det g^{\mu\nu}$, equations (11.2.2.14) – (11.2.2.16).

$g_{\mu\nu}$, $g^{\mu\nu}$ : equations (11.2.2.12), (11.2.2.13).

$\Gamma^{N}_{\Sigma\Sigma^{\sim}}$ : equation (7.2.5).

$\Gamma_{\theta}$, $\Gamma^{N}_{\theta}$ : 7.2.

$\gamma_{\Sigma_i\Sigma_j^{\sim}}$ : equation (9.2.18); in terms of the $\gamma_k$s, equation (9.4.31).

$\gamma_k$ : equation (9.4.30); in terms of the $\gamma_{\Sigma_i\Sigma_j^{\sim}}$s, equations (9.4.32a–d).

$\gamma_{\mu}$ : equation (9.4.1.12).

Hom( , ) : space of linear maps from first argument into second; similar notation for set of morphisms in other categories.

$\mathcal{H}_F$ : category of finite dimensional complex Hilbert spaces, 6.3.2.

iff : if and only if.

injective : one-to-one

Lorentzian transformations : an *extended* Lorentz transformation $\Lambda$ is a transformation on $\mathbb{R}^4$ leaving invariant the bilinear form $x_0^2 - x_1^2 - x_2^2 - x_3^2$, or equivalently satisfying the condition $\Lambda^T \eta \Lambda = \eta$,

: $\Lambda$ is *proper* if $\det \Lambda = 1$,

: $\Lambda$ is *orthochronous* if it is proper and $\Lambda^0_0 \geq 1$.

Note: The appellation *Lorentz transformation* is often reserved for the orthochronous ones.

$L^{\uparrow}_{+}$ : the group of orthochronous Lorentz transformations, called the *restricted* Lorentz group.

$L$( ) : Hilbert lattice: the lattice of closed subspaces of the Hilbert space argument, 1.2, 5.1.4; for operations on projections see 5.2.1.

$\lambda$(...) : $\lambda$-term, 4.2.

$\lambda_a$: Gell–Mann matrices, 12.1.

$l_P$ : Planck length ($\approx 1.6 \times 10^{-33}$cm), equation (12.1.16).

$p_k$ : equations (8.1.11), (9.3.1).

$q_k$ : equation (8.1.10).

$\mathfrak{q}_k$ : equation (8.2.9).

$\mathbb{R}$ : field of real numbers.

$R$ : scalar curvature, equation (11.2.2.57); also used briefly to denote electrical resistance, 12.3, and a certain rotation matrix, equation (12.4.6).

surjective : onto.

Ⓢ : symmetric product, 3.1.8.

$S(\ )$ : symmetric algebra of the argument, equation (2.2.4.3), 3.1.8.

$\mathcal{S}(g)$: bispinor representation, equation (9.2.25).

$\sigma_k$ : Pauli matrices, equations (9.4.16), (9.4.20)

$\sigma_{\Sigma_i\Sigma_j^{\sim}}$: equations (9.2.9), (9.2.10), (9.2.16a,b); in terms of Paulis, equations (9.4.24a–d).

$\sigma^{\Sigma_i\Sigma_j^{\sim}}$: equations (9.2.13), (9.2.14a,b); in terms of Paulis, equations (9.4.27a–d).

$\sigma_{mn}$ : equation (11.1.7).

$\sigma_{\mu\nu}$ : equation (11.2.2.22).

$\mathrm{SL}(n,\mathbb{C})$ : group of linear transformations of $\mathbb{C}^n$ having unit determinant.

$\mathfrak{sl}(n,\mathbb{C})$ : Lie algebra of $\mathrm{SL}(n,\mathbb{C})$, identifiable with the Lie algebra of linear transformations on $\mathbb{C}^n$ having zero trace, 3.2.1.

$\mathrm{SO}(m,n)$ : group of linear transformations of $\mathbb{R}^{m+n}$ with unit determinant preserving the form

$$x_1^2+\ldots+x_m^2-x_{m+1}^2-\ldots-x_{m+n}^2.$$

$\mathrm{SO}(n)$ : group of orthogonal transformations of $\mathbb{R}^n$, indentifiable as the group of real $n\times n$ matrices $A$ with unit determinant satisfying $A^TA=I$.

$\mathrm{SO}(n,\mathbb{C})$ : group of linear transformations of $\mathbb{C}^n$ with unit determinant preserving the quadratic form $z_1^2+\ldots+z_n^2$.

$\mathfrak{so}(n,\mathbb{C})$ : Lie algebra of $\mathrm{SO}(n,\mathbb{C})$.

$\mathfrak{so}(m,n)$ : Lie algebra of $\mathrm{SO}(m,n)$.

$\mathrm{SU}(n)$ : subgroup of $\mathrm{U}(n)$ whose elements have unit determinant.

$\mathfrak{su}(n)$ : Lie algebra of $\mathrm{SU}(n)$, identifiable with the set of $A$ in $\mathfrak{sl}(n,\mathbb{C})$ satisfying $A^\dagger+A=0$.

SU($n$) : subgroup of U($n$) whose elements have unit determinant.
$\tau$: Finkelstein's net constant, equation (8.1.10).
$T$: twist isomorphism, 3.1.
$T(\ )$ : tensor algebra of the argument, equation (2.2.2.3), 3.1.8.
tr( ) : trace of the argument, 1.3.
U( ) : universal enveloping of the Lie algebra argument, 3.1.9.
U($n$) : group of unitary linear transformations of $\mathbb{C}^n$, identifiable with the group of $n \times n$ complex matrices $U$ with $U^\dagger U = I$.
$V$: space of bispinors, equation (9.2.20).
$\mathbb{W}$ : representation space for $\mathfrak{a}_c$, equation (11.2.1.4).
$X^Y$: set of functions from set $Y$ into set $X$.
$x_\mu$ : equation (10.3.1.18).
$\mathbb{Z}_n$: cyclic group of order $n$, 3.2.1.
$\mathbb{Z}$ : ring of integers.

**2** : the two-element set.
∎ : end of proof.
$\propto$ : proportional to.
$\equiv$ : definition, usually of the left-hand side by the right-hand side; equal by definition.
$\exists$ : there exists.
$\forall$ : for all.
!( ) : *of course* operator, 6.2.
( )$^\dagger$: adjoint, 1.1; Hilbert space adjoint for operator argument, reducing to complex conjugate transpose on matrix arguments.
( )*: linear dual of argument reducing to transpose on matrix arguments.
( )$\hat{}$: natural linear embedding of a vector space into its second dual; an isomorphism in the finite dimensional case.
$\overline{(\ )}$ : complex conjugate for a complex argument; Pauli–Dirac adjoint for a bispinorial argument, 10.2.
( )°: modal translation of orthoformula argument, 6.3.1.
( )$^e$: translation into a **GQ** formula of the **IOL** formula argument,

6.3.1.

$( )^c$ : relative set complementation of the argument, 5.1.2.

$( )^\perp$ : set orthogonal to the argument, 5.1.2; subspace orthogonal to the argument, 5.1.3.

$(\not{\ })$: Dirac slashed vector argument *à la* Feynman: e. g. $\not{A}$, equation (10.0.2).

$( )_\mu$, $( )^\mu$: Greek, or curved, indexing, 9.4.1.

$| \rangle$ : Dirac ket, 1.1.

$\langle |$ : Dirac bra, 1.1.

$[\![ ]\!]$ : semantic function, 8.1.

$[\![ ]\!]_C$ : classical version of, 9.5.

$[a,b]$ : commutator $ab-ba$ of elements in an algebra (not to be confused with the closed interval this notation describes if $a,b \in \mathbb{R}$).

$*$ : convolution, or algebra product dual to a certain coproduct, equation (3.1.3.1).

$\ltimes$ : semi-direct product (of groups), equation (11.1.6).

$\int_C dx_{\mu_1} \ldots dx_{\mu_n}$ : Chen iterated integral, defined in equations (8.2.16) and (8.2.17).

$\wedge$ : (non-quantum ) logical conjunction, 4.2; meet or greatest lower bound in a distributive lattice; also E. Cartan's notation for exterior product, 2.3.

$\vee$ : (non-quantum) logical disjunction, 4.2; join or least upper bound in a distributive lattice; also Peano's notation for exterior product, 2.3.

$\neg$ : (non-quantum) negation.

$\sqcap$ : conjunction in **OL**, **IOL**, etc.; meet, or greatest lower bound, in ortholattices, 5.1.1, 5.2.1.

$\sqcup$ : disjunction in **OL**, **IOL**, etc.; join, or least upper bound, in ortholattices, 5.1.1, 5.2.1.

$\sim$ : negation in **OL**, **IOL**, etc., 5.1.1, 5.2.1.

$\Box$,$\Diamond$: modal operators, 5.1.2.

$\rightarrow$ : material implication, 5.1.2.
$\Rightarrow$ : intuitionistic implication, 4.1.
$\overrightarrow{q}$ : Sasaki hook, equation (5.1.4.8).
$aCb$: compatibility in an ortholattice, equation (5.1.4.5).
$a \leftrightarrow b$: commutativity in an orthomodular lattice, 5.1.4; double arrow sign also used to indicate informal correspondences at various places in the text.
$\models$ : validity in Kripke orthomodels, 5.1.2.
$\models$ : validity in Kripke B-models, 5.1.3.
$\vdash$ : turnstile, in sequent calculus, 4.1.
$\vdash_{B}$ : deducibility in the modal B-system, 5.1.3.
$\vdash_{GQ}$ : provability in **GQ**, 6.2.
$\vdash_{IO}$ : deducibility in intuitionistic orthologic (**IOL**), 6.3.
$\vdash_{O}$ : deducibility in orthologic (**OL**), 5.1.1.
$\vdash_{OM}$ : deducibility in orthomodular logic (**OML**), 5.1.4.
$\vdash_{Q}$ : deducibility in Quantum Logic (**QL**), 5.1.4.
$\varnothing$ : empty set.

# Subject Index