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Variational Inequalities and Frictional Contact Problems

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In the memory of my father, Dumitru Stoleru

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Contents

Chapter 1
Introduction

Nowadays, the expression *Variational Inequalities and Contact Problems* can be considered as a syntagm since the variational methods have provided one of the most powerful techniques in the study of contact problems and, on the other hand, the variational formulations of the contact problems are, in most cases, variational inequalities.

We therefore considered a book on this subject as necessary, a book where the reader will find many results on variational inequalities and, at the same time, a detailed study of certain contact problems with non local Coulomb friction.

In the last 50 years, variational inequalities became a strong tool in the mathematical study of many nonlinear problems of physics and mechanics, as the complexity of the boundary conditions and the diversity of the constitutive equations lead to variational formulations of inequality type.

The theory of variational inequalities find its roots in the works of Signorini [38] and Fichera [14] concerning unilateral problems and, also, in the work of Ting [44] for the elasto-plastic torsion problem. The mathematical foundation of the theory was widened by the invaluable contributions of Stampacchia [41] and Lions and Stampacchia [26] and then developed by the French and the Italian school: Brézis [3, 4], Stampacchia [42], Lions [25], Mosco [28], Kinderlehrer and Stampacchia [22]. Concerning the approximation of the variational inequalities, we refer to the important contributions brought by Mosco [27], Glowinski et al. [17], or Glowinski [16].

We do not claim that this book covers all the aspects in the study of the variational inequalities. However, we intent to give the reader an overview on this huge subject in a unified form, containing a detailed and justified description of the results on existence, uniqueness, regularity or approximation of solutions of variational and quasi-variational inequalities, in the linear and nonlinear cases, for the static and quasistatic cases.

We also deal in this book with the study of certain static and quasistatic problems with friction whose weak formulations are variational or quasi-variational

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inequalities. More precisely, we address here frictional contact problems for a linearly elastic body which, under the influence of volume and surface forces, is in contact with a rigid foundation. The contact is modeled by Signorini's law, except for the last section where bilateral contact is considered. We also use a nonlocal version of Coulomb's friction law. Most of the results presented here are obtained by applying abstract results on variational inequalities.

The first results concerning the mathematical study of this kind of problems, in the case of Tresca's friction (i.e., with given friction), are due to Duvaut and Lions [12]. In the static case, important results concerning the study of contact problems of Signorini type with local or nonlocal friction have been obtained by Duvaut [11], Nečas et al. [29], Oden and Pires [31,32], Demkowicz and Oden [10], and Cocu [7]. In the quasistatic case, the first existence results were given by Andersson [1], Han and Sofonea [18], and Klarbring et al. [23] for problems with normal compliance. Their approach is based on incremental formulations obtained from the quasi-variational inequality by an implicit time discretization scheme. The same technique was used by Cocu et al. [9], Rocca [36], Andersson [2], and Cocu and Rocca [8] in their existence proofs for quasistatic problems of Signorini type with local or nonlocal friction or with friction and adhesion. The works of Panagiotopoulos [33,34], Glowinski et al. [17], Glowinski [16], Campos et al. [5], Kikuchi and Oden [21], Haslinger et al. [19], Hlaváček et al. [20], Shillor et al. [37], Eck et al. [13], and Sofonea and Matei [39] enriched, theoretically and numerically, the study of contact problems. Among those who developed algorithms of resolution of the unilateral contact problems with friction, let us quote Raous et al. [35], Sofonea et al. [40], and Lebon and Raous [24].

The book is divided into III parts and 9 chapters.

Part I reviews, in a general way, the fundamental definitions, notation and theorems of the functional analysis which will be essential to understand the following parts. So, Chap. 2 is a potpourri of standard topics on functional spaces, while Chap. 3 refers to spaces of vector-valued functions. The material we present in these two chapters is a classical one and can be found in many monographs. Also, throughout this book, when necessary, further basic results on functional analysis will be recalled.

Part II is concerned with the study of variational inequalities.

Chapter 4 presents some generally known existence and uniqueness results. More precisely, in Sect. 4.1 one considers elliptic variational inequalities of the first and second kind involving linear and continuous operators in Hilbert spaces (Sect. 4.1.1) or monotone and hemicontinuous operators in Banach spaces (Sect. 4.1.2). The results are established using projection or proximity operators, Weierstrass or Lax–Milgram theorems, Schauder or Banach fixed point theorems.

Section 4.2 deals with elliptic quasi-variational inequalities. In Sect. 4.2.1, we refer to the case of monotonous and hemicontinuous operators: the existence is obtained by using Kakutani fixed point theorem, while the uniqueness, only for strongly monotone operators, is obtained using Banach fixed point theorem. In Sect. 4.2.2 we consider the case of potential operators and we introduce and justify the concept of generalized solution of a quasi-variational inequality. We then apply, in Sect. 4.2.3, these results to prove the existence and the uniqueness of the generalized solution of a contact problem with friction for the operator of Hencky–Nadai theory.

Section 4.3 presents a strategy, rather new, for the study of a class of abstract implicit evolutionary quasi-variational inequalities which covers the variational formulation of many quasistatic contact problems. The method used rests, as in the typical cases, on incremental formulations.

In Chap. 5 we give two remarkable properties satisfied by the solutions of certain variational inequalities. In Sect. 5.1 one highlights a maximum principle which is then applied to a problem which models the flow of fluids through a porous medium and also to an obstacle problem. In Sect. 5.2, using the method of the translations due to Nirenberg [30] (as Brezis [4] did in his thesis for a scalar second order elliptic operator), local and global regularity results of the solutions of a class of variational inequalities of the second kind are established.

In Chap. 6 we present first a brief background on convex analysis, and we then recall some classical results of the Mosco et al. [6] (M-CD-M) duality theory in its form adapted by Telega [43] for the so-called implicit variational inequalities.

In Chap. 7 one can find details results on the discrete approximation of two general classes of variational inequalities. For the quasi-variational inequalities considered in Sect. 4.2.1, the convergence of an internal approximation is obtained in Sect. 7.1 and an abstract error estimate is given in Sect. 7.2. A convergence result for an internal approximation in space and a back difference scheme in time of implicit evolutionary quasi-variational inequalities introduced in Sect. 4.3 is proved in Sect. 7.3.

In Part III we study, in an almost exhaustive way, the problem of Signorini with nonlocal Coulomb friction in elasticity.

Chapter 8 deals with the static problem. The mechanical problem is described in Sect. 8.1 and its variational formulation is obtained in Sect. 8.2. The existence and, under certain assumptions on the data, the uniqueness of the solution are obtained in Sect. 8.3 by applying the theorems established in Sect. 4.2.1. Using the regularity results given in Sect. 5.2.2 and an argument due to Fichera [15], we get, in Sect. 8.4, a local regularity result for the solutions of the static problem. In Sect. 8.5 we derive two dual formulations, dual and dual condensed, which involve as unknown the stress field instead of the displacement field like in the case of the primal problem, i.e. the variational formulation considered in Sect. 8.2. The first dual formulation is obtained, by using Green's formula, from the mechanical problem in the same way as for the primal formulation. The second dual formulation, i.e. the dual condensed one, is a problem posed on the surface of possible contact only, obtained by applying the M-CD-M duality theory developed in Sect. 6.2. This condensed dual formulation could be useful in numerical calculations since one computes directly the stresses on the contact boundary and usually these are the quantities of interest. In Sect. 8.6 we consider a finite element approximation of the primal problem. We first obtain an error estimate, either directly or by applying the estimate given in Sect. 7.3. We then prove that a higher order of the approximation can be obtained for a suitable choice of the regularization which describes the nonlocal character of Coulomb law.

In Sect. 8.7 we consider the discretization by the equilibrium finite element method of the two stress formulations, i.e. the dual formulation and the dual condensed one. We prove the convergence of our approximations and we derive error estimates of these discretized problems in different cases of the data. Section 8.8 is devoted to the study of an optimal control problem related to the Signorini problem with nonlocal Coulomb friction. More precisely, one characterizes the coefficient of friction which leads to a given profile of displacements on the contact surface.

Chapter 9 deals with the quasistatic problem. In Sect. 9.1, using an implicit time discretization scheme and applying the results of Sect. 4.3, an existence result is obtained. We then consider, in Sect. 9.2, a space finite element approximation and an implicit time discretization scheme of this problem and, by using the results of Sect. 8.3, we prove the convergence of the approximation. In the last section we consider a mathematical model describing the quasistatic process of bilateral contact with friction between an elastic body and a rigid foundation. Our goal is to study a related optimal control problem which allows us to obtain a given profile of displacements on the contact boundary, by acting with a control on another part of the boundary of the body. Using penalization and regularization techniques, we derive the necessary conditions of optimality.

This book was written in the framework of the author's research activity within the Institute of Mathematics of the Romanian Academy, and the results presented here are partially based on the author's own research.

The book is intended to be self-contained and it addresses mathematicians, applied mathematicians, graduate students in mathematical and physical sciences as well researchers in mechanics and engineering.

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Part I
Preliminaries

Chapter 2
Spaces of Real-Valued Functions

This chapter is a brief background on spaces of continuous functions and some Sobolev spaces including basic properties, embedding theorems and trace theorems. Hence, we recall some classical definitions and theorems of functional analysis which will be used throughout this book. These results are standard and so they are stated without proofs; for more details and proofs, we refer the readers to the monographs [1, 3–7, 10, 11, 14].

In this book we only deal with real-valued functions. We assume that the reader is familiar with the basic concepts of general topology and functional analysis.

For a point $\boldsymbol{x} = (x_1, \cdots, x_d) \in \mathbb{R}^d$, we denote by D_i the differential operator $\dfrac{\partial}{\partial x_i}$ $(1 \leq i \leq d)$.

If $\boldsymbol{\alpha} = (\alpha_1, \cdots, \alpha_d)$ is a multi-index, then $D^{\boldsymbol{\alpha}}$ denotes the differential operator of order $\boldsymbol{\alpha}$, with $|\boldsymbol{\alpha}| = \sum_{i=1}^{d} \alpha_i$, defined by

$$D^{\boldsymbol{\alpha}} = D_1^{\alpha_1} \cdots D_d^{\alpha_d} = \frac{\partial^{|\boldsymbol{\alpha}|}}{\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha_d}}.$$

Obviously, D_i^0 denotes the identity operator.

If $A \subset \mathbb{R}^d$, we denote by $C(A)$ the space of real continuous functions on A.

Let Ω be an open set in $\mathbb{R}^d$ with its boundary Γ. We denote by $\overline{\Omega} = \Omega \cup \Gamma$ the closure of Ω.

For any nonnegative integer m, let $C^m(\Omega)$, respectively $C^m(\overline{\Omega})$, be the space of real functions which, together with all their partial derivatives of orders $\boldsymbol{\alpha}$, with $|\boldsymbol{\alpha}| \leq m$, are continuous on Ω, respectively, on the closure $\overline{\Omega}$ of Ω in $\mathbb{R}^d$, i.e.

$$C^m(\Omega) = \{v \in C(\Omega)\,;\; D^{\boldsymbol{\alpha}} v \in C(\Omega) \text{ for } |\boldsymbol{\alpha}| \leq m\}\,. \tag{2.1}$$

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When $m = 0$, we abbreviate $C(\Omega) \equiv C^0(\Omega)$ and $C(\overline{\Omega}) \equiv C^0(\overline{\Omega})$. Any function in $C(\overline{\Omega})$ is bounded and uniformly continuous on Ω, thus it possesses a unique, bounded, and continuous extension to $\overline{\Omega}$.

Let

$$C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$$

be the space of infinitely differentiable functions on Ω.

If K is a subset of Ω, we shall write $K \subset\subset \Omega$ if $\overline{K} \subset \Omega$ and $\overline{K}$ is a compact (i.e., bounded and closed) subset of $\mathbb{R}^d$.

The support of a function $v : \Omega \to \mathbb{R}$ is defined as the closed subset

$$\operatorname{supp} v = \overline{\{\boldsymbol{x} \in \Omega\,;\; v(\boldsymbol{x}) \neq 0\}}\,. \tag{2.2}$$

We shall say that a function v has compact support in Ω if there exists a compact subset K of Ω such that $v(\boldsymbol{x}) = 0\ \forall \boldsymbol{x} \in \Omega \backslash K$ or, equivalently, $\operatorname{supp} v \subset\subset \Omega$.

We shall denote by $C_0^m(\Omega)$ the subspace of $C^m(\Omega)$ consisting of all those functions which have compact support in Ω.

If $m < +\infty$ and Ω is bounded, then $C^m(\overline{\Omega})$ is a Banach space with the norm given by

$$\|v\|_{C^m(\overline{\Omega})} = \sum_{|\alpha| \le m} \max_{x \in \overline{\Omega}} |D^{\alpha} v(\boldsymbol{x})|\,. \tag{2.3}$$

In the sequel, for $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ two normed spaces with $X \subset Y$, we shall write $X \hookrightarrow Y$ to designate the continuously embedding of X in Y provided the identity operator $I : X \to Y$ is continuous. This is equivalent, since I is linear, to the existence of a constant C such that

$$\|u\|_Y \le C\,\|u\|_X \qquad \forall u \in X\,.$$

We also say that the normed space X is compactly embedded in the normed space Y and write $X \hookrightarrow_c Y$ if the identity operator I is compact, i.e. every bounded sequence in X has a subsequence converging in Y, or, equivalently, if $\{u_k\}_k$ is a sequence which converges weakly to u in X, and we write $u_k \rightharpoonup u$, then $\{u_k\}_k$ converges strongly to u in Y, and we write $u_k \to u$.

We denote by $L^p(\Omega)$, for $1 \le p < +\infty$, the space of (equivalence classes of) real functions v defined on Ω with the p-power absolutely integrable, i.e.

$$\int_{\Omega} |v(\boldsymbol{x})|^p \, \mathrm{d}x < \infty\,,$$

where $\mathrm{d}x = \mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_d$ is the Lebesque measure. The elements of $L^p(\Omega)$, being equivalence classes of measurable functions, are identical if they are equal almost everywhere (a.e.) on Ω. Thus, we write $v = 0$ in $L^p(\Omega)$ if $v(x) = 0$ a.e. $x \in \Omega$.

We also denote by $L^\infty(\Omega)$ the space consisting of all (equivalence classes of) measurable real functions v that are essentially bounded on Ω, i.e. there exists a constant C such that $|v(\boldsymbol{x})| \le C$ a.e. on Ω.

The space $L^p(\Omega)$ endowed with the norm

$$\|v\|_{L^p(\Omega)} = \begin{cases} \left(\int\limits_\Omega |v(\boldsymbol{x})|^p \,\mathrm{d}x\right)^{1/p} & \text{if } 1 \le p < +\infty \\ \operatorname*{ess\ sup}_{\boldsymbol{x}\in\Omega} |v(\boldsymbol{x})| = \inf\{C;\, |v(\boldsymbol{x})| \le C \text{ a.e. } \boldsymbol{x} \in \Omega\} & \text{if } p = +\infty \end{cases} \tag{2.4}$$

is a Banach space. In addition, the space $L^p(\Omega)$ is separable if $1 \le p < +\infty$ and reflexive if $1 < p < +\infty$.

If $p \in [1,\infty]$, then the exponent conjugate to p is the number denoted by p' defined by the relation

$$\frac{1}{p} + \frac{1}{p'} = 1$$

where we used the convention

$$p' = \begin{cases} \infty & \text{if } p = 1\,, \\ 1 & \text{if } p = \infty\,. \end{cases}$$

From Riesz representation Theorem 4.1 for Hilbert spaces it follows that, for $p \in [1,+\infty)$, the dual space of $L^p(\Omega)$ is the space $(L^p(\Omega))' = L^{p'}(\Omega)$ where p' is the exponent conjugate to p. The dual space of $L^\infty(\Omega)$ is a space larger than $L^1(\Omega)$ (for more details, see [14, p. 118]).

In the case $p = 2$, the space $L^2(\Omega)$ is a Hilbert space with respect to the inner product

$$(u,v)_{L^2(\Omega)} = \int\limits_\Omega u(\boldsymbol{x})\, v(\boldsymbol{x})\,\mathrm{d}x\,. \tag{2.5}$$

Definition 2.1. We say that a measurable function v defined a.e. on Ω is locally p-integrable on Ω if $v \in L^p(A)$ for every measurable set $A \subset\subset \Omega$.

We shall denote by $L^p_{\mathrm{loc}}(\Omega)$ the space of all locally p-integrable functions on Ω.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ be an open set. The following assertions hold.*

1) Let $1 < p, q < \infty$.
If $u \in L^p(\Omega)$ and $v \in L^q(\Omega)$, then $uv \in L^{\frac{pq}{p+q}}(\Omega)$.

If $u_n \to u$ *in* $L^p(\Omega)$ *and* $v_n \to v$ *in* $L^q(\Omega)$, *then* $u_n v_n \to uv$ *in* $L^{\frac{pq}{p+q}}(\Omega)$.

If $u \in L^p(\Omega)$ *and* $v \in L^{p'}(\Omega)$ *where* p' *is the exponent conjugate to* p, *then* $uv \in L^1(\Omega)$ *and the Hölder's inequality holds:*

$$\int_\Omega u(\boldsymbol{x})v(\boldsymbol{x})\,\mathrm{d}x \le \|u\|_{L^p(\Omega)}\|v\|_{L^{p'}(\Omega)}\,. \tag{2.6}$$

When $p = p' = 2$, *we get the Cauchy–Schwartz inequality.*

2) *For* $1 \le p \le \infty$, *every Cauchy sequence in* $L^p(\Omega)$ *has a subsequence converging pointwise a.e. on* Ω.
3) $L^p(\Omega) \subset L^1_{\mathrm{loc}}(\Omega) \quad \forall p$ *with* $1 \le p \le \infty$.
4) *Let* $v \in L^1_{\mathrm{loc}}(\Omega)$ *be such that* $\int_\Omega v(\boldsymbol{x})\varphi(\boldsymbol{x})\,\mathrm{d}x = 0 \quad \forall\,\varphi \in \mathscr{D}(\Omega)$. *Then* $v(\boldsymbol{x}) = 0$ *a.e. on* Ω.
5) $C_0^\infty(\Omega)$ *is dense in* $L^p(\Omega) \quad \forall p$ *with* $1 \le p < \infty$.

The following theorem gives an embedding result for the spaces $L^p(\Omega)$ and some of its consequences.

Theorem 2.2. *Let* $\Omega \subset \mathbb{R}^d$ *be an open set with vol* $(\Omega) = \int_\Omega \mathrm{d}x < \infty$. *Then the following statements are valid.*

1) *For all* p, q *such that* $1 \le p \le q \le \infty$, *we have* $L^q(\Omega) \hookrightarrow L^p(\Omega)$ *and*

$$\|v\|_{L^p(\Omega)} \le (vol\,\Omega)^{\frac{1}{p}-\frac{1}{q}}\|v\|_{L^q(\Omega)} \quad \forall\, v \in L^q(\Omega)\,.$$

2) $\lim_{p\to\infty} \|v\|_{L^p(\Omega)} = \|v\|_{L^\infty(\Omega)} \quad \forall v \in L^\infty(\Omega)$.
3) *Suppose that* $v \in L^p(\Omega)$ *for any* $1 \le p < \infty$ *and that there exists a constant* C *such that* $\|v\|_{L^p(\Omega)} \le C$. *Then* $v \in L^\infty(\Omega)$.

To better understand what is the meaning of the differential operator $D^{\boldsymbol{\alpha}} v$ for functions v whose derivatives do not exist in the classical sense, we briefly remind the definition of distributions on Ω.

We denote by $\mathscr{D}(\Omega)$, called the space of test functions, the space $C_0^\infty(\Omega)$ equipped with the inductive limit topology as in the Schwartz theory of distributions [11].

Definition 2.2. A sequence $\{\varphi_k\}_k \subset C_0^\infty(\Omega)$ is said to converge to a function $\varphi \in C_0^\infty(\Omega)$ in (the sense of the space) $\mathscr{D}(\Omega)$, provided the following conditions are satisfied:

i) There exists a compact subset K of Ω such that supp $(\varphi_k - \varphi) \subset K\,, \ \forall k$
ii) $D^{\boldsymbol{\alpha}}\varphi_k \to D^{\boldsymbol{\alpha}}\varphi$ uniformly on $K\,, \ \forall \boldsymbol{\alpha}$ multi-index .

The dual space $\mathscr{D}'(\Omega)$ of $\mathscr{D}(\Omega)$ is called the space of (Schwartz) distributions (or, generalized functions). Hence, any distribution T is a linear and continuous functional on $\mathscr{D}(\Omega)$, i.e. $T(\varphi_k) \to T(\varphi)$ in $\mathbb{R}$ whenever $\varphi_k \to \varphi$ in $\mathscr{D}(\Omega)$. As dual of $\mathscr{D}(\Omega)$, the space $\mathscr{D}'(\Omega)$ is equipped with the weak-star topology: $T_k \to T$ in $\mathscr{D}'(\Omega)$ if and only if $T_k(\varphi) \to T(\varphi)$ in $\mathbb{R}$, for every $\varphi \in \mathscr{D}(\Omega)$.

Every distribution is infinitely differentiable in the following sense: if $T \in \mathscr{D}'(\Omega)$ then, for all multi-index $\boldsymbol{\alpha}$, the function $D^{\boldsymbol{\alpha}} T$ defined on $\mathscr{D}(\Omega)$ by

$$D^{\boldsymbol{\alpha}} T(\varphi) = (-1)^{|\boldsymbol{\alpha}|} T(D^{\boldsymbol{\alpha}} \varphi) \quad \forall \varphi \in \mathscr{D}(\Omega)\,. \tag{2.7}$$

is a distribution. In addition, the operator $D^{\boldsymbol{\alpha}}$ from $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$ is continuous.

Any function $u \in L^1_{\text{loc}}(\Omega)$ generates a distribution $T_u \in \mathscr{D}'(\Omega)$ defined by

$$T_u(\varphi) = \int_\Omega u(\boldsymbol{x}) \varphi(\boldsymbol{x})\, \mathrm{d}x \quad \forall \varphi \in \mathscr{D}(\Omega)\,. \tag{2.8}$$

Therefore, for any multi-index $\boldsymbol{\alpha}$, there exists the $\boldsymbol{\alpha}$-th derivative of T_u, namely the distribution $D^\alpha T_u \in \mathscr{D}'(\Omega)$ defined by (2.7), i.e.

$$D^{\boldsymbol{\alpha}} T_u(\varphi) = (-1)^{|\boldsymbol{\alpha}|} T_u(D^{\boldsymbol{\alpha}} \varphi) \quad \forall \varphi \in \mathscr{D}(\Omega)\,.$$

But not any distribution is generated by a locally integrable function.

Definition 2.3. We shall say that the function $u \in L^1_{\text{loc}}(\Omega)$ possesses the distributional (or generalized or weak) partial derivative of order $\boldsymbol{\alpha}$ on Ω, denoted by $D^{\boldsymbol{\alpha}} u$, if there exists a function $v_\alpha \in L^1_{\text{loc}}(\Omega)$ which generates the distribution $D^\alpha T_u \in \mathscr{D}'(\Omega)$, i.e.

$$D^{\boldsymbol{\alpha}} T_u = T_{v_\alpha}\,.$$

Thus, from the last three relations, it follows that $D^{\boldsymbol{\alpha}} u = v_\alpha$ is the distributional partial derivative of u if $v_\alpha \in L^1_{\text{loc}}(\Omega)$ satisfies

$$\int_\Omega u(\boldsymbol{x})\, D^{\boldsymbol{\alpha}} \varphi(\boldsymbol{x})\, \mathrm{d}x = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega v_\alpha\, \varphi(\boldsymbol{x})\, \mathrm{d}x \quad \forall \varphi \in \mathscr{D}(\Omega)\,. \tag{2.9}$$

Obviously, the distributional derivative is uniquely defined up to a set of measure zero.

In fact, this definition generalizes the classical partial derivative, obtained, for a function $u \in C^{|\boldsymbol{\alpha}|}(\Omega)$, by integrating by parts $|\boldsymbol{\alpha}|$ times

$$\int_\Omega D^{\boldsymbol{\alpha}} u(\boldsymbol{x})\, \varphi(\boldsymbol{x})\, \mathrm{d}x = (-1)^{|\boldsymbol{\alpha}|} \int_\Omega u(\boldsymbol{x})\, D^{\boldsymbol{\alpha}} \varphi(\boldsymbol{x})\, \mathrm{d}x \quad \forall \varphi \in \mathscr{D}(\Omega)\,. \tag{2.10}$$

Of course, in this case, $D^{\boldsymbol{\alpha}} u$ is also a distributional partial derivative of u. However, it should be noted that the derivative in the sense of distributions of a function, even sufficiently smooth, may exist, even if it does not exist in the classical sense.

In particular, the relation (2.8) brings out a linear and continuous mapping $u \mapsto T_u$ from $L^p(\Omega)$ into $\mathscr{D}'(\Omega)$ and so, we may identify the distribution T_u with the integrable function u. The same identification may be made for $\mathscr{D}(\Omega)$. Thus, we have

$$\mathscr{D}(\Omega) \hookrightarrow L^p(\Omega) \hookrightarrow \mathscr{D}'(\Omega)\,.$$

Using this result and the definition (2.9), Sobolev [12] expanded in a natural way the space $L^p(\Omega)$ by considering those functions which, for some nonnegative integer m, possess distributional partial derivatives of all orders $|\boldsymbol{\alpha}| \leq m$ in $L^p(\Omega)$. This is the definition of the Sobolev space

$$W^{m,p}(\Omega) = \{v\,;\; D^{\boldsymbol{\alpha}} v \in L^p(\Omega)\,,\;\; \text{for } |\boldsymbol{\alpha}| \leq m\}\,.$$

The space $W^{m,p}(\Omega)$ is a Banach space with the norm

$$\|v\|_{W^{m,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\boldsymbol{\alpha}| \leq m} \|D^{\boldsymbol{\alpha}} v\|^p_{L^p(\Omega)} \right)^{1/p} & \text{if } p \in [1, \infty)\,, \\ \displaystyle\max_{|\boldsymbol{\alpha}| \leq m} \|D^{\boldsymbol{\alpha}} v\|_{L^\infty(\Omega)} & \text{if } p = \infty\,. \end{cases} \tag{2.11}$$

Obviously, $W^{0,p}(\Omega) = L^p(\Omega)$ for $p \in [1, \infty)$. The seminorm over $W^{m,p}(\Omega)$ is defined by

$$|v|_{W^{m,p}(\Omega)} = \begin{cases} \left(\displaystyle\sum_{|\boldsymbol{\alpha}| = m} \|D^{\boldsymbol{\alpha}} v\|^p_{L^p(\Omega)} \right)^{1/p} & \text{if } p \in [1, \infty)\,, \\ \displaystyle\max_{|\boldsymbol{\alpha}| = m} \|D^{\boldsymbol{\alpha}} v\|_{L^\infty(\Omega)} & \text{if } p = \infty\,. \end{cases} \tag{2.12}$$

We denote by $W^{m,p}_0(\Omega)$ the closure of $C^\infty_0(\Omega)$ in the space $W^{m,p}(\Omega)$ for the norm $\|\cdot\|_{W^{m,p}(\Omega)}$. For $p \in [1, \infty)$, we have the following chain of embeddings

$$W^{m,p}_0(\Omega) \hookrightarrow W^{m,p}(\Omega) \hookrightarrow L^p(\Omega)$$

and, since $C^\infty_0(\Omega)$ is dense in $L^p(\Omega)$, it is clear that $W^{0,p}_0(\Omega) = L^p(\Omega)$.

It is easy to see that, if the open set Ω is bounded, the seminorm $|\cdot|_{W^{m,p}(\Omega)}$ is a norm over $W^{m,p}_0(\Omega)$ equivalent to the norm $\|\cdot\|_{W^{m,p}(\Omega)}$.

In the case $p = 2$, we use the notation

$$H^m(\Omega) = W^{m,2}(\Omega)\,.$$

Endowed with the scalar product

$$(u, v)_{H^m(\Omega)} = \sum_{\alpha \leq m} (D^{\alpha} u, D^{\alpha} v)_{L^2(\Omega)}, \tag{2.13}$$

the Sobolev space $H^m(\Omega)$ is a Hilbert space. Also we denote $H_0^m(\Omega) = W_0^{m,2}(\Omega)$.

If Ω is bounded, then, without any hypothesis on the regularity of Ω, we have

$$H_0^1(\Omega) \hookrightarrow_c L^2(\Omega).$$

Many different symbols are being used to denote these norms, when no confusion may occur: $\|\cdot\|_{m,p,\Omega}$ or $\|\cdot\|_{m,p}$ instead of $\|\cdot\|_{W^{m,p}(\Omega)}$, $\|\cdot\|_{m,\Omega}$ or $\|\cdot\|_m$ instead of $\|\cdot\|_{H^m(\Omega)}$ and $\|\cdot\|_{0,\Omega}$ or $\|\cdot\|_0$ instead of $\|\cdot\|_{L^2(\Omega)}$.

If $m \geq 1$ and $1 \leq p < \infty$, we denote by $W^{-m,p'}(\Omega)$ the dual space of $W_0^{m,p}(\Omega)$, p' being the exponent conjugate to p (in fact, $W^{-m,p'}(\Omega)$ is the notation for a space of some distributions on Ω which is isometrically isomorphic to the dual space $(W_0^{m,p}(\Omega))'$; for details, see [1]). Endowed with the norm

$$\|f\|_{W^{-m,p'}(\Omega)} = \sup_{\substack{u \in W_0^{m,p}(\Omega) \\ u \neq 0}} \frac{\langle f, u \rangle}{\|u\|_{W^{m,p}(\Omega)}},$$

the space $W^{-m,p'}(\Omega)$ is a Banach space which is separable and reflexive if $1 < p < \infty$. Here $\langle \cdot, \cdot \rangle$ is the duality pairing between $W^{-m,p'}(\Omega)$ and $W_0^{m,p}(\Omega)$.

We note that if X, Y are two Hilbert spaces such that $X \hookrightarrow Y$ dense, then (see, for instance, [2, p. 51]) $Y^* \hookrightarrow X^*$ dense, where Y^* and X^* denote their dual spaces.

If Ω is bounded, then $\mathscr{D}(\Omega)$ is dense in $H_0^m(\Omega)$, and so, we can identify the dual space $H^{-m}(\Omega)$ of $H_0^m(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$:

$$\mathscr{D}(\Omega) \subset H_0^m(\Omega) \subset L^2(\Omega) \subset H^{-m}(\Omega) \subset \mathscr{D}'(\Omega).$$

Now, we notice that most of the important results involving Sobolev spaces are first obtained for regular functions and then extended to Sobolev spaces. The density theorems and the embedding theorems show how and whether an element of a Sobolev space can be approximated by smooth functions. Since these theorems require additional regularity properties for the open set Ω, we recall some definitions of them. Later, in Chaps. 5 and 8, we will use some of these assumptions on Ω for getting regularity properties of the solutions of some concrete variational inequalities.

Definition 2.4. We say that the open subset Ω of $\mathbb{R}^d$ has the cone property if there exists a finite open bounded cover $\{O_j\}_{j \in J}$ of the boundary Γ of Ω and, for any j, there exists a cone C_j with the vertex at 0, such that, for all $\boldsymbol{x} \in O_j \cap \Omega$, $\boldsymbol{x} + C_j$ do not intersect $O_j \cap \Gamma$.

Definition 2.5. We say that the open set $\Omega \subset \mathbb{R}^d$ has the segment property if there exists a locally finite open cover $\{U_j\}_j$ of the boundary Γ of Ω and a corresponding sequence $\{\boldsymbol{y}_j\}_j$ of nonzero vectors such that if $\boldsymbol{x} \in \overline{\Omega} \cap U_j$ for some j, then $\boldsymbol{x} + t\boldsymbol{y}_j \in \Omega$ for $0 < t < 1$. In this case, Ω must have $(d-1)$-dimensional boundary and cannot simlutaneously lie on both sides of its boundary.

Definition 2.6. Let $r \geq 1$ an integer. An open bounded set $\Omega \subset \mathbb{R}^d$ is said to be $\mathscr{C}^r$-smooth (or, of class $\mathscr{C}^r$) if there exists a covering of the boundary Γ of Ω by a finite number of bounded open subsets $\{U_j\}_{j\in J} \subset \mathbb{R}^d$ and, for any $j \in J$, there exists a C^r-homeomorphisms θ_j such that:

(i) $\theta_j(U_j) = S = \{\boldsymbol{y} = (\boldsymbol{y}', y_d) \in \mathbb{R}^d;\ |\boldsymbol{y}'| < 1,\ |y_d| < 1\}$,
(ii) $\theta_j(U_j \cap \Omega) = S_+ = \{\boldsymbol{y} \in S;\ y_d > 0\}$,
(iii) $\theta_j(U_j \cap \Gamma) = S_0 = \{\boldsymbol{y} \in S;\ y_d = 0\}$.

Concerning the approximation by smooth functions, we have the following results (see, for instance, [13, p. 11], [9, p. 44], or [8, p. 40]).

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^d$ be an open bounded set. Then, the following approximation results are true.*

1) $C_0^\infty(\Omega)$ is dense in $W_0^{m,p}(\Omega)$.
2) If Ω has the cone property, then $C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$.
3) If Ω is $\mathscr{C}^\infty$-smooth, then $\mathscr{D}(\overline{\Omega})$ is dense in $H^m(\Omega)$.

We now recall the following Sobolev embedding theorem (see [1,9,13] for more details and proofs) which will be used frequently in this book.

Theorem 2.4 (Sobolev Embedding Theorem). *Suppose that the open bounded set Ω has the cone property and $1 \leq p < \infty$. Then, the following assertions hold.*

1) If $mp < d$, then

i) $W^{m,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ where $p^* = \dfrac{\mathrm{d}p}{\mathrm{d} - mp}$.
ii) $W^{m,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ for any q with $1 \leq q < p^*$.

2) If $mp = d$, then
$W^{m,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ for any $1 \leq q < \infty$.
3) If $mp > d$, then

i) $W^{m,p}(\Omega) \hookrightarrow_c L^q(\Omega)$ for any $1 \leq q < \infty$.
ii) $W^{m,p}(\Omega) \hookrightarrow C^k(\overline{\Omega})$ for any integer k with $\dfrac{mp-d}{p} - 1 \leq k < \dfrac{mp-d}{p}$.

As a consequence of this theorem we have the following particular cases that we shall often use:

$$
\begin{aligned}
&H^1(\Omega) \hookrightarrow_c C(\overline{\Omega}) \quad \text{if } d = 1\,,\\
&H^1(\Omega) \hookrightarrow_c L^q(\Omega) \quad \text{where} \quad \begin{cases} q \in [1, \infty) & \text{if } d = 2\,,\\ q = 6 & \text{if } d = 3\,,\end{cases}\\
&H^2(\Omega) \hookrightarrow_c C(\overline{\Omega}) \quad \text{if } d \in \{1, 2\}\,.
\end{aligned}
$$

We note that a function $v \in H^1(\Omega)$ is not necessary continuous on Ω, neither on $\overline{\Omega}$, and so, we may not define, in the classical sense, the values of v on the boundary Γ of Ω. The trace theorems show how one can define, in the trace sense, the restriction on the boundary Γ of a function which is not necessary continuous. Their purpose is to determine the space of functions defined on the boundary Γ of Ω containing the traces of functions in $W^{m,p}(\Omega)$.

The next theorem (see [8, p. 40] or [13, p. 9]) allows to define every function $v \in H^1(\Omega)$ almost everywhere on Γ.

Theorem 2.5 (Trace Theorem for $H^1(\Omega)$). *Let Ω be an open bounded set in $\mathbb{R}^d$ of class $\mathscr{C}^1$ with its boundary Γ. Then, one can uniquely define the trace $\gamma_0 v$ of $v \in H^1(\Omega)$ on Γ such that $\gamma_0 v$ coincides with the usual definition*

$$
\gamma_0 v(x) = v(x) \quad x \in \Gamma\,, \tag{2.14}
$$

if $v \in C^1(\overline{\Omega})$. Moreover, the mapping $\gamma_0 : H^1(\Omega) \to L^2(\Gamma)$ is linear continuous and the range of $\gamma_0(H^1(\Omega))$ is a space smaller than $L^2(\Gamma)$ denoted by $H^{1/2}(\Gamma)$.

Now, if $v \in C^m(\overline{\Omega})$, let $\boldsymbol{\gamma} v$ be the linear mapping defined by

$$
\boldsymbol{\gamma} v = (\gamma_0 v, \gamma_1 v, \ldots, \gamma_{m-1} v)
$$

where $\gamma_0 v$ is "the trace of v" on Γ and $\gamma_j v,\ j = 1, \ldots, m-1$ is "the trace of order j of v"' defined as the j-th order derivative in the direction of the outward unit normal $\boldsymbol{\nu}$ to Γ, i.e.

$$
\begin{aligned}
\gamma_0 v(x) &= v(x) \quad x \in \Gamma\,,\\
\gamma_j v(x) &= \frac{\partial^j v}{\partial \nu^j}(x) \quad x \in \Gamma\,.
\end{aligned} \tag{2.15}
$$

The problem of characterizing the image of the space $H^m(\Omega)$ under the trace operator involves Sobolev spaces of fractional order. These spaces can be defined in different ways but, for Ω sufficiently smooth, these definitions give the same space.

A compact definition (see [1]) of $H^s(\Omega)$, for s a real number, is the space obtained as the closure of $C^\infty(\Omega)$ in the norm

$$
\|v\|^2_{H^s(\Omega)} = \|v\|^2_{H^{[s]}(\Omega)} + \sum_{|\boldsymbol{\alpha}|=[s]} \int_{\Omega\times\Omega} \frac{|D^{\boldsymbol{\alpha}} v(\boldsymbol{y}) - D^{\boldsymbol{\alpha}} v(\boldsymbol{x})|^2}{|\boldsymbol{y} - \boldsymbol{x}|^{d+2\{s\}}}\, \mathrm{d}y\, \mathrm{d}x\,,
$$

where $s = [s] + \{s\}$ with $[s]$ an integer and $0 < \{s\} < 1$.

Another approach (see [2, 10]) for the definition of the space $H^s(\Omega)$ uses the Fourier transformation of a function.

Definition 2.7. The Fourier transformation of a function $v \in L^2(\mathbb{R}^d)$ is the function $\hat{v} \in L^2(\mathbb{R}^d)$ defined by

$$\hat{v}(\boldsymbol{y}) = \frac{1}{(2\pi)^{d/2}} \int\limits_{\mathbb{R}^d} v(\boldsymbol{x}) \exp(-i\, \boldsymbol{x}\, \boldsymbol{y})\, \mathrm{d}x$$

where $i = \sqrt{-1}$.

For any real number $s \geq 0$, the space $\mathscr{S}(\mathbb{R}^d)$, of test functions of rapid decay, is defined by:

$$\mathscr{S}(\mathbb{R}^d) = \{\varphi\, ;\; \boldsymbol{x}^{\boldsymbol{\alpha}} D^{\boldsymbol{\beta}} \varphi \in L^2(\mathbb{R}^d)\,,\; \forall \boldsymbol{\alpha},\, \boldsymbol{\beta} \text{ multi-indices}\}\,,$$

and $\mathscr{S}(\mathbb{R}^d)'$, called the space of temperated distributions, is the dual space of $\mathscr{S}(\mathbb{R}^d)$.

Then, the fractional order Sobolev space $H^s(\mathbb{R}^d)$ is defined by

$$H^s(\mathbb{R}^d) = \{v \in \mathscr{S}(\mathbb{R}^d)'\; ;\; (1 + |\boldsymbol{y}|^2)^{s/2} \hat{v} \in L^2(\mathbb{R}^d)\}$$

with the norm

$$\|v\|_{H^s(\mathbb{R}^d)} = \|(1 + |\boldsymbol{y}|^2)^{s/2} \hat{v}\|_{L^2(\mathbb{R}^d)}\,.$$

If $s < 0$, one denotes by $H^s(\mathbb{R}^d)$ the dual space of $H^{-s}(\mathbb{R}^d)$.

If Ω is sufficiently smooth, then we define $H^s(\Omega)$ to be the space of restrictions to Ω of functions of $H^s(\mathbb{R}^d)$. The boundary Γ of Ω can be identified, by means of local coordinates, to $\mathbb{R}^{d-1}$, and we can define $H^s(\Gamma)$ to be isomorphic to the Sobolev space $H^s(\mathbb{R}^{d-1})$.

If the open bounded set Ω of $\mathbb{R}^d$ is $\mathscr{C}^\infty$-smooth, then $\mathscr{D}(\overline{\Omega})$ is dense in $H^m(\Omega)$ and so, it is possible to extend by continuity the classical definition (2.15) to a generalized one $\boldsymbol{\gamma} v$ for $v \in H^m(\Omega)$ (see, for instance, [9, p. 44], [10, p. 142]).

Theorem 2.6 (Trace Theorem for $H^m(\Omega)$). *Suppose that the open bounded set Ω is C^∞-smooth. Then, for any $m > 0$ integer, the trace operator*

$$\boldsymbol{\gamma} : \mathscr{D}(\overline{\Omega}) \to (\mathscr{D}(\Gamma))^m$$

can be extended to the continuous linear and surjective operator

$$\boldsymbol{\gamma} : H^m(\Omega) \to \prod_{j=0}^{m-1} H^{m-j-1/2}(\Gamma)\,. \tag{2.16}$$

Moreover, there exists a continuous linear inverse operator

$$\boldsymbol{\gamma}^{-1} : \prod_{j=0}^{m-1} H^{m-j-1/2}(\Gamma) \to H^m(\Omega)$$

such that

$$\boldsymbol{\gamma}_j(\boldsymbol{\gamma}^{-1}\mathbf{g}) = g_j \quad 0 \leq j \leq m-1\,, \quad \forall \mathbf{g} \in \prod_{j=0}^{m-1} H^{m-j-1/2}(\Gamma)\,.$$

Therefore, the space $H^{m-j-1/2}(\Gamma)$ can be seen as the space of traces of order j of $H^m(\Omega)$. In addition, the kernel of the operator $\boldsymbol{\gamma}$ is the space $H_0^m(\Omega)$, the completion of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{H^m(\Omega)}$.

Finally, we recall the following result (see [1, p. 114]).

Theorem 2.7. *Suppose that* Ω *is sufficiently smooth. Then*

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Gamma)$$

where $q = \dfrac{\mathrm{d}p - p}{\mathrm{d} - mp}$ *if* $mp < d$, *and* $1 \leq q < \infty$ *if* $mp = d$.

In particular, we have the following frequently useful results.

$$H^1(\Omega) \hookrightarrow L^q(\Gamma) \quad \text{where} \quad \begin{cases} q \in [1,\infty) & \text{if } d = 2\,, \\ q = \dfrac{2(n-1)}{n-2} & \text{if } d \geq 3\,. \end{cases}$$

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Chapter 3
Spaces of Vector-Valued Functions

In this chapter we will introduce additional tools which are fundamentals for the study of evolutionary problems studied later in this book. We consider some spaces of functions defined on a time interval $I \subset \mathbb{R}$ with values into a Banach or Hilbert space X. The results are presented without proofs. For additional information about the results stated below, we refer the readers to [1–9].

We consider a Banach space $(X, \|\cdot\|_X)$ and an open interval $I = (0, T) \subset \mathbb{R}$ with $0 < T < \infty$ fixed. We will use the notation $\overline{I} = [0, T]$.

Definition 3.1. A function $v : I \to X$ is (strongly) continuous on I if it is strongly continuous in X at every $t \in I$, i.e. for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$, such that

$$\|v(t) - v(s)\|_X < \epsilon \quad \forall s \in I \text{ with } |t - s| < \delta\,.$$

We will denote by $C(I; X)$, respectively, $C(\overline{I}; X)$, the space of all (strongly) continuous functions from I, respectively, from the closed interval $\overline{I}$, to X.

Definition 3.2. A subset A in a normed space V is said to be compact (respectively, weakly compact) if every sequence $\{x_n\}_n \subset A$ contains a subsequence $\{x_{n_k}\}_k$ such that $x_{n_k} \to x$ strongly (respectively, $x_{n_k} \rightharpoonup x$ weakly) in V as $k \to \infty$, with $x \in A$.

The following theorem provides useful criteria for compactness of subsets of $C(\overline{I}; X)$.

Theorem 3.1. *Let $\mathcal{M} \subset C(\overline{I}; X)$ be a family of functions such that:*

(1) there exists a constant $C > 0$ such that $\|v(t)\|_X \leq C \quad \forall t \in \overline{I}\,,\ \forall v \in \mathcal{M}$,
(2) $\mathcal{M}$ is equi-uniformly continuous, i.e. $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon)$ such that

$$\|v(t) - v(s)\|_X < \epsilon \quad \forall t, s \in \overline{I} \text{ with } |t - s| < \delta\,,\ \forall v \in \mathcal{M}\,,$$

(3) For each $t \in \overline{I}$, the set $\{v(t)\,;\, v \in \mathcal{M}\}$ is compact in X.
Then, $\mathcal{M}$ is compact in $C(\overline{I}; X)$.

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Definition 3.3. A function $v : \overline{I} \to X$ is (strongly) uniformly continuous on $\overline{I}$ if for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$, such that

$$\|v(t) - v(s)\|_X < \epsilon \quad \forall t, s \in \overline{I} \text{ with } |t - s| < \delta\,.$$

Definition 3.4. A function $v : \overline{I} \to X$ is said to be Lipschitz continuous on $\overline{I}$ if there exists a positive constant C such that

$$\|v(t) - v(s)\|_X \le C|t - s| \quad \forall t,\ s \in \overline{I}\,.$$

The smallest positive constant C which satisfies the above relation is called the Lipschitz constant.

Proposition 3.1. *If $v \in C(\overline{I}; X)$, then v is uniformly continuous on $\overline{I}$.*

Definition 3.5. We say that a function $v \in C(\overline{I}; X)$ is (strongly) differentiable on $\overline{I}$ if it is differentiable at any $t \in \overline{I}$, i.e. there exists an element in X, denoted by $v'(t)$ and called the (strong) derivative of V at t, such that

$$\lim_{h\to 0} \left\| \frac{v(t+h) - v(t)}{h} - v'(t) \right\|_X = 0\,,$$

for h sufficiently small such that $t + h \in \overline{I}$. It is natural that the derivative at $t = 0$ is defined as a right-sided limit while at $t = T$ as a left-sided limit.

If the function v is differentiable a.e. on $\overline{I}$, then the function v' (denoted also $\frac{\mathrm{d}v}{\mathrm{d}t}$) is called the (strong) derivative of v.

The (strong) derivative of order j, for $j \ge 2$, denoted by $v^{(j)}$ or $\frac{\mathrm{d}^j v}{\mathrm{d}t^j}$, is defined by $v^{(j)} = (v^{(j-1)})'$. Most often, we will use the notation $\dot{v}$ instead of v' or $\frac{\mathrm{d}v}{\mathrm{d}t}$ and the convention $v^{(0)} = v$.

For $m \ge 0$ integer, let $C^m(I; X)$, respectively $C^m(\overline{I}; X)$ be the space of functions m times continuous differentiable defined on I, respectively $\overline{I}$, with values in X.

Theorem 3.2. *The space*

$$C^m(\overline{I}, X) = \{v : \overline{I} \to X\,;\ v^{(k)} \in C(\overline{I}; X) \textit{ for } 0 \le k \le m\}$$

is a Banach space equipped with the norm

$$\|v\|_{C^m(\overline{I};X)} = \sum_{k\le m} \sup_{x\in\overline{I}} \|v^{(k)}(\boldsymbol{x})\|_X = \sum_{k\le m} \max_{x\in\overline{I}} \|v^{(k)}(\boldsymbol{x})\|_X\,. \tag{3.1}$$

We also will denote by $C^\infty(I;X)$ the space of all infinitely differentiable functions defined on I with values in X, by $\mathscr{D}(I;X)$ the space of all functions of $C^\infty(I;X)$ with compact support, and by $\mathscr{D}'(I;X)$ the space of vectorial distributions defined on I with values in X, i.e. $\mathscr{D}'(I;X) = \mathscr{L}(\mathscr{D}(I;X);X)$ where $\mathscr{L}(U,V)$ denotes the space of all linear continuous functions from U to V.

Definition 3.6. A function $v : I \to X$ is said to be simple function if there exist Leabesgue measurable subsets $I_1, I_2, \cdots, I_n$ of I and the functions $\alpha_1, \alpha_2, \cdots, \alpha_n \in X$ such that

$$v(t) = \sum_{j=1}^{n} \alpha_j \chi_{I_j}(t) \quad \text{a.e. } t \in I\,,$$

where χ_{I_j} is the indicator function of I_j.

Definition 3.7. We say that a function $v : I \to X$ is (strongly) measurable if there exists a sequence $\{v_n\}_{n\geq 0}$ of simple functions $v_n : I \to X$ such that

$$\lim_{n\to\infty} \|v_n(t) - v(t)\|_X = 0 \quad \text{a.e. } t \in I\,.$$

It is easy to proven the following properties (see, e.g. [4]):

Proposition 3.2. *The following assertions hold.*

1) If $v : I \to X$ is a measurable function, then $\|v\|_X : I \to \mathbb{R}$ is measurable.
2) Let $\{v_n\}_{n\geq 0}$ be a sequence of measurable functions from I to X and $v : I \to X$ be a function such that

$$\lim_{n\to\infty} \|v_n(t) - v(t)\|_X = 0 \quad \textit{a.e. } t \in I\,.$$

Then v is measurable.
3) Let $v : I \to X$ be a weakly continuous function, i.e. $v(t_n) \rightharpoonup v(t)$ weakly in X as $t_n \to t$. Then v is measurable.

Definition 3.8. We say that a function $v : I \to X$ is (Bochner) integrable if there exists a sequence $\{v_n\}_{n\geq 0}$ of simple functions $v_n : I \to X$ such that

$$\lim_{n\to\infty} \|v_n(t) - v(t)\|_X = 0 \quad \text{a.e. } t \in I\,,$$

and

$$\lim_{n\to\infty} \int_I \|v_n(t) - v(t)\|_X \,\mathrm{d}t = 0\,.$$

(as $\|v_n(t) - v(t)\|_X$ is measurable and nonnegative, it follows that $\int_I \|v_n(t) - v(t)\|_X \, dt$ makes sense).

Proposition 3.3. *Let $v : I \to X$ be an integrable function and let be the sequence $\{v_n\}_{n\geq 0}$ of simple functions $v_n : I \to X$ such that*

$$\lim_{n\to\infty} \int_I \|v_n(t) - v(t)\|_X \, dt = 0 \, .$$

Then, there exists an element in X, denoted by $\int_I v(t) \, dt$ and called the (Bochner) integral of v over I, such that

$$\lim_{n\to\infty} \left\| \int_I v_n(t) \, dt - \int_I v(t) \, dt \right\|_X = 0 \, .$$

We note that the integral of v is independent of the sequence of simple functions considered above. The properties of the Bochner integral are similar to those of the Lebesgue integral of integrable real-valued functions.

Theorem 3.3 (Bochner's Theorem). *Let $v : I \to X$ be a measurable function. Then v is integrable if and only if $\|v\|_X$ is integrable. Moreover, we have*

$$\left\| \int_I v(t) \, dt \right\|_X \leq \int_I \|v(t)\|_X \, dt \, .$$

Theorem 3.4 (Lebesgue's Dominated Convergence Theorem). *Let $\{v_n\}_{n\geq 0}$ be a sequence of integrable functions from I to X such that*

1) There exists a integrable function $u : I \to X$ such that

$$\|v_n(t)\|_X \leq u(t) \quad a.e. \ t \in I \, , \ \forall n \in \mathbb{N},$$

2) There exists a function $v : I \to X$ such that

$$\lim_{n\to\infty} \|v_n(t) - v(t)\|_X = 0 \quad a.e. \ t \in I \, .$$

Then v is integrable and

$$\lim_{n\to\infty} \int_I \|v_n(t) - v(t)\|_X \, dt = 0 \, .$$

Let $p \in [1, \infty]$. We denote by $L^p(I; X)$ the space of (equivalence classes of) measurable functions $v : I \to X$ such that the mapping $t \to \|v(t)\|_X$ belongs to $L^p(I)$. Endowed with the norm

$$\|v\|_{L^p(I;X)} = \begin{cases} \left(\int\limits_I \|v(t)\|_X^p \, \mathrm{d}t \right)^{1/p} < \infty & \text{if } p \neq \infty \,, \\ \operatorname*{ess\ sup}_{t \in (0,T)} \|v(t)\|_X & \text{if } p = \infty \,, \end{cases} \tag{3.2}$$

$L^p(I; X)$ is a Banach space.

In particular, if $(X, (\cdot, \cdot)_X)$ is a Hilbert space, then $L^2(I; X)$ is also a Hilbert space with respect to the inner product

$$(u, v)_{L^2(I;X)} = \int\limits_I (u(t), v(t))_X \, \mathrm{d}t \,.$$

Definition 3.9. We say that a measurable function $v : I \to X$ is locally p-integrable, and we write $v \in L^p_{\mathrm{loc}}(I; X)$, if $v \in L^p(J; X)$ for any closed interval $J \subset I$.

The following theorem summarizes some basic properties of the space $L^p(I; X)$.

Theorem 3.5. *Let* $1 \leq p \leq \infty$. *Then*

1) $\mathscr{D}(I; X) \subset L^p(I; X) \subset \mathscr{D}'(I; X)$.
2) *If* $p < \infty$, *then* $\mathscr{D}(I; X)$ *is dense in* $L^p(I; X)$.
3) *If* $p < \infty$ *and* X *is reflexive or* X *is separable, then the dual space of* $L^p(I; X)$ *is* $L^{p'}(I; X^*)$, p' *being the exponent conjugate to* p, *and* X^* *denoting the dual space of* X.

   In particular, if $(X, (\cdot, \cdot)_X)$ *is a Hilbert space, then the duality pairing* $\langle \cdot, \cdot \rangle$ *between* $L^p(I; X)$ *and its dual* $L^{p'}(I; X)$ *is given by*

$$\langle u, v \rangle = \int\limits_I (u(t), v(t))_X \, \mathrm{d}t \quad \forall u \in L^{p'}(I; X) \,, \ \forall v \in L^p(I; X) \,.$$

4) *If* $u \in L^p(I; X)$ *and* $v \in L^{p'}(I; X^*)$, *then* $t \to \langle u(t), v(t) \rangle_{X^* \times X}$ *is integrable and*

$$\int\limits_I |\langle u(t), v(t) \rangle_{X^* \times X}| \, \mathrm{d}t \leq \|u\|_{L^p(I;X)} \|v\|_{L^{p'}(I;X^*)} \,,$$

   where $\langle \cdot, \cdot \rangle_{X^* \times X}$ *denotes the duality pairing between the space* X *and its dual* X^*.
5) $L^q(I; X) \hookrightarrow L^p(I; X)$ *for* $1 \leq p \leq q \leq \infty$.

6) *Let* $\{v_n\}_{n\in\mathbb{N}}$ *be a bounded sequence of* $L^p(I;X)$ *and* $v : I \to X$ *such that*

$$v_n(t) \rightharpoonup v(t) \quad \textit{weakly in } X \textit{ as } n \to \infty\,, \ \textit{a.e. } t \in I\,.$$

Then $v \in L^p(I;X)$ *and*

$$\|v\|_{L^p(I;X)} \le \liminf_{n\to\infty} \|v_n\|_{L^p(I;X)}\,.$$

7) *Let* $v \in L^1_{\mathrm{loc}}(I;X)$ *be such that* $v = 0$ *in* $\mathscr{D}'(I,X)$. *Then* $v = 0$ *a.e. on* I.
8) *Let* $v \in L^p(\mathbb{R};X)$. *If we put*

$$v_h(t) = \frac{1}{h}\int_t^{t+h} v(s)\,\mathrm{d}s\,, \quad \textit{for almost all } t \in \mathbb{R} \textit{ and } h \neq 0\,,$$

then $v_h \in L^p(\mathbb{R};X) \cap C_b(\mathbb{R};X)$ *and*

$$\lim_{h\to 0} \|v_h - v\|_{L^p(\mathbb{R};X)} = 0\,,$$
$$\lim_{h\to 0} \|v_h(t) - v(t)\|_X = 0 \quad \textit{a.e. } t \in \mathbb{R}\,,$$

$C_b(\mathbb{R};X)$ *denoting the space of all continuous bounded functions from* $\mathbb{R}$ *to* X.

We now introduce a weaker notion of the differentiability of a vector-valued function which is a natural generalization of the definition for real-valued functions.

Definition 3.10. Let $v \in L^1_{\mathrm{loc}}(I;X)$. We say that the function v is weakly differentiable if there exists $u \in L^1_{\mathrm{loc}}(I;X)$ such that

$$\int_I v(t)\varphi'(t)\,\mathrm{d}t = -\int_I u(t)\varphi(t)\,\mathrm{d}t \quad \forall \varphi \in C_0^\infty(I)\,,$$

where the integrals are understood to be Bochner integrals. The function u will be denoted by v' or $\frac{\mathrm{d}v}{\mathrm{d}t}$ and it is called the weak derivative of v. In a similar way, for $j \ge 2$, we say that the function v possesses a j-th weak derivative if there exists a function $v^{(j)} \in L^1_{\mathrm{loc}}(I;X)$ (denoted also $\frac{\mathrm{d}^j v}{\mathrm{d}t^j}$) such that

$$\int_I v(t)\varphi^{(j)}(t)\,\mathrm{d}t = (-1)^j \int_I v^{(j)}(t)\varphi(t)\,\mathrm{d}t \quad \forall \varphi \in C_0^\infty(I)\,.$$

As in the case of strong derivatives, v' and $v^{(0)}$ are usually denoted by $\dot{v}$ and, respectively, v.

As we will see below, the existence of the weak derivative of a vector-valued function is related to the absolute continuity.

Definition 3.11. A function $v : \overline{I} \to X$ is said to be absolutely continuous on $\overline{I}$ if for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that, for any finite set of disjoint intervals $\{(t_n, t'_n)\}_n \subset \overline{I}$ with $\sum_n |t_n - t'_n| < \delta$, one has $\sum_n \|v(t_n) - v(t'_n)\|_X < \epsilon$.

It is known that a real-valued function $v : \overline{I} \to \mathbb{R}$ is absolutely continuous iff v is a.e. differentiable on I, with the derivative $\dfrac{\mathrm{d}v}{\mathrm{d}t} \in L^1(I)$, such that

$$v(s) = v(0) + \int_0^s \frac{\mathrm{d}v}{\mathrm{d}t}(t)\,\mathrm{d}t \quad s \in \overline{I}\,.$$

This property does not carry over to Bochner integrals in arbitrary Banach space. A Banach space for which any absolutely continuous vector-valued function has an integrable weak derivative is said to have the Radon–Nikodym property. One of these spaces is given by the following result (see [7, p. 40]).

Theorem 3.6. *Let X be a reflexive Banach space. Suppose that $v : \overline{I} \to X$ is an absolutely continuous function. Then v is a.e. differentiable on I, with the derivative $\dot{v} \in L^1(I; X)$, such that*

$$v(s) = v(0) + \int_0^s \dot{v}(t)\,\mathrm{d}t \quad s \in \overline{I}\,.$$

For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, let $W^{m,p}(I; X)$ be the space of all (equivalence classes of) measurable functions $v : I \to X$ whose weak derivatives of order $0 \leq j \leq m$ belong to $L^p(I; X)$. Endowed with the norm

$$\|v\|_{W^{m,p}(I;X)} = \begin{cases} \left(\displaystyle\sum_{j=0}^{m} \|v^{(j)}\|^p_{L^p(I;X)} \right)^{1/p} & \text{if } p \in [1, \infty)\,, \\ \displaystyle\max_{0 \leq j \leq m} \|v^{(j)}\|_{L^\infty(I;X)} & \text{if } p = \infty\,, \end{cases}$$

$W^{m,p}(I; X)$ is a Banach space.

The following result gives a characterization of $W^{m,p}(I; X)$ spaces by means of absolutely continuous functions.

Theorem 3.7. *For $m \in \mathbb{N}$ and $1 \leq p \leq \infty$, let $v \in L^p(I; X)$. Then the following conditions are equivalent:*

(1) $v \in W^{m,p}(I; X)$,

(2) there exists $w \in C^{m-1}(\overline{I};X)$ *such that:*

$$\begin{cases} v(t) = w(t) \quad \text{a.e. } t \in I\,, \\ \text{the strong derivatives } w^{(j)} \text{ are absolutely continuous for } 1 \leq j \leq m-1\,, \\ \text{the } m\text{-th strong derivative } w^{(m)} \text{ exists a.e. and } w^{(m)} \in L^p(I;X)\,. \end{cases}$$

Finally, we review some useful results on the space $W^{1,p}(I;X)$.

Theorem 3.8. *Let* $p \geq 1$. *We have*

1) $W^{1,p}(I;X) \subset C(\overline{I};X)$.
2) $C^\infty(\overline{I};X)$ *is dense in* $W^{1,p}(I;X)$.

Theorem 3.9. *Let* X *be a reflexive Banach space and* $1 \leq p \leq \infty$.

1) Let $v \in L^p(I;X)$. *Then* $v \in W^{1,p}(I;X)$ *if and only if there exists* $u \in L^p(I;\mathbb{R})$ *such that*

$$\|v(\tau) - v(t)\|_X \leq \left| \int_t^\tau u(s)\,\mathrm{d}s \right| \quad \text{for almost all } t, \tau \in I\,.$$

In that case, one has $\|\dot{v}\|_{L^p(I;X)} \leq \|u\|_{L^p(I;\mathbb{R})}$.
2) Let $v : \overline{I} \to X$ *be a bounded and Lipschitz continuous function with* C *the Lipschitz constant. Then* $v \in W^{1,\infty}(I;X)$ *and* $\|\dot{v}\|_{L^\infty(I;X)} \leq C$.
3) Let $p > 1$, $\{v_n\}_{n\in\mathbb{N}} \subset W^{1,p}(I;X)$ *be a bounded sequence and* $v : I \to X$ *be such that* $v_n(t) \rightharpoonup v(t)$ *weakly in* X *as* $n \to \infty$, *for almost all* $t \in I$. *Then* $v \in W^{1,p}(I;X)$ *and*

$$\liminf_{n\to\infty} \|\dot{v}_n\|_{L^p(I;X)} \geq \|\dot{v}\|_{L^p(I;X)}\,.$$

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Part II
Variational Inequalities

Chapter 4
Existence and Uniqueness Results

This chapter deals with existence and uniqueness results for variational and quasi-variational inequalities. With the intention of focusing the differences among the proofs of results, we first consider elliptic variational inequalities of the first and second kind with linear and continuous operators in Hilbert space or monotone and hemicontinuous operators in Banach space. Next, we deal with elliptic quasi-variational inequalities involving monotone and hemicontinuous or potential operators. The last section concerns the study of a class of evolutionary quasi-variational inequalities. The results presented here will be applied, in the last part of the book, in the study of frictional contact problems.

4.1 Elliptic Variational Inequalities

In this section we recall some classical existence and uniqueness results for elliptic variational inequalities of the first and second kind (see, for instance, [6, 21, 21, 27, 30, 31, 37, 38, 40]).

4.1.1 Variational Inequalities with Linear Operators

Let V be a real Hilbert space with the inner product $(\cdot,\cdot)$ and the associated norm $\|\cdot\|$.

As follows from the following result, we may identify the Hilbert space V with its dual V^* (see, e.g. [1]).

Theorem 4.1 (Riesz's Representation Theorem). *Let V be a Hilbert space, and let $f \in V^*$ be a given element. Then there exists a unique element $u_f \in V$ such that*

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$$f(v) = (u_f, v) \quad \forall v \in V\,.$$

In addition, we have

$$\|f\|_{V^*} = \|u_f\|_V\,.$$

Let K be a nonempty closed (i.e., K contains the limits of all strongly convergent sequences $\{v_n\}_n \subset K$) convex subset of V.

Let us consider a bilinear form $a : V \times V \to \mathbb{R}$. We assume that a is continuous, that is, there exists a positive constant M such that

$$|a(u,v)| \leq M\,\|u\|\,\|v\| \quad \forall u, v \in V\,. \tag{4.1}$$

We denote by $j : K \to \overline{\mathbb{R}}$ a proper, convex, and lower semicontinuous (l.s.c.) function, i.e.

$$j \text{ is not identically } +\infty \text{ and } j(v) > -\infty \quad \forall v \in V\,, \tag{4.2}$$

$$j((1-t)u + tv) \leq (1-t)\,j(u) + t\,j(v) \quad \forall t \in [0,1]\,,\ \forall u, v \in K\,, \tag{4.3}$$

$$\text{if } \{v_n\}_n \subset K \text{ converges strongly to } v \in K \text{ then } \liminf_{n\to\infty} j(v_n) \geq j(v)\,. \tag{4.4}$$

Let $f \in V$ be given. With the above notation, we consider the problem
Problem (P_1^a): Find $u \in K$ such that

$$a(u, v-u) + j(v) - j(u) \geq (f, v-u) \quad \forall v \in K\,, \tag{4.5}$$

called elliptic variational inequality of the second kind.

As a particular case, for $j \equiv 0$, we have the variational inequality of the first kind defined by

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ a(u, v-u) \geq (f, v-u) \quad \forall v \in K\,. \end{cases} \tag{4.6}$$

First, we prove the following equivalence result, due essentially to Minty [27].

Lemma 4.1. *Let the above assumptions hold. We suppose, in addition, that the form a is positive (i.e., $a(v,v) \geq 0\,,\ \forall v \in V$). Then, the variational inequality (4.5) is equivalent to*

$$\begin{cases} \textit{Find } u \in K \textit{ such that} \\ a(v, v-u) + j(v) - j(u) \geq (f, v-u) \quad \forall v \in K\,. \end{cases} \tag{4.7}$$

Moreover, the set of all solutions of the variational inequality (4.5) is a closed convex (it could be empty) subset of K.

Proof. If u is a solution of (4.5), then, since a is positive, it follows

$$a(v, v-u) + j(v) - j(u) \geq a(u, v-u) + j(v) - j(u) \geq (f, v-u) \quad \forall v \in K\,,$$

i.e., u verifies (4.7).

Conversely, as the set K is convex, we may take $v = (1-t)u + tw \in K$ in (4.7) with $w \in K$ arbitrary, and $t \in (0,1)$. Then, by the convexity of j, one obtains

$$a((1-t)u + tw, w-u) + j(w) - j(u) \geq (f, w-u) \quad \forall w \in K\,,$$

and, by passing to the limit with $t \to 0$, we get (4.5).

This equivalence implies that the set of all solutions of the variational inequality (4.5) can be written as

$$\chi = \{u \in K\,;\; a(v, v-u) + j(v) - j(u) \geq (f, v-u) \quad \forall v \in K\}\,.$$

Then, it is easy to verify that the set χ is convex since the functional j is convex. In order to prove that it is closed, let $\{u_n\}_n \subset \chi$ be a sequence such that $u_n \to u$ strongly in V. As K is closed, we have $u \in K$, and

$$\begin{aligned}
&a(v, v-u) + j(v) - j(u) \geq \lim_{n\to\infty} a(v, v-u_n) + j(v) - \liminf_{n\to\infty} j(u_n)\\
&\geq \limsup_{n\to\infty}(a(v, v-u_n) + j(v) - j(u_n)) \geq \lim_{n\to\infty} (f, v-u_n) = (f, v-u) \quad \forall v \in K\,,
\end{aligned}$$

i.e., $u \in \chi$, which completes the proof. □

We also have the following equivalence result.

Lemma 4.2. *Let the above assumptions hold. We suppose, in addition, that the form a is symmetric (i.e., $a(u,v) = a(v,u)$, $\forall u, v \in V$). Then, the variational inequality (4.5) is equivalent to the following minimization problem*

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ J(u) \leq J(v) \quad \forall v \in K\,, \end{cases} \tag{4.8}$$

where the function $J : V \to \overline{\mathbb{R}}$ is defined by

$$J(v) = \frac{1}{2}a(v,v) + j(v) - (f, v) \quad \forall v \in V\,. \tag{4.9}$$

Proof. If $u \in K$ is a solution of the variational inequality (4.5), then we have

$$J(v) - J(u) = a(u, v-u) + j(v) - j(u) - (f, v-u) + \frac{1}{2}a(u-v, u-v) \geq 0 \quad \forall v \in K\,.$$

Conversely, if u satisfies (4.8), then

$$J(u) \leq J((1-t)u + tv) \quad \forall v \in K\,,\ \forall t \in (0,1)\,,$$

hence

$$\frac{1}{2}t^2 a(u-v, u-v) + t a(u, v-u) + j((1-t)u + tv) - j(u) \geq t(f, v-u)\,.$$

Therefore, by using the convexity of j, dividing by t and passing to the limit with $t \to 0$, we conclude that u is a solution of (4.5). □

Let us recall a Weierstrass type minimization theorem (see, for instance, [31, p. 1181], [10, p. 62], [14, p. 596]) which will be used frequently in this chapter.

Theorem 4.2 (Weierstrass's Minimization Theorem). *Let $(X, \|\cdot\|_X)$ be a reflexive Banach space and let K be a nonempty weakly closed subset of X. Let $J : K \to \overline{\mathbb{R}}$ be a proper function. Assume that J is weakly l.s.c., i.e.*

$$\textit{if } \{v_n\}_n \subset K \textit{ converges weakly to } v \in K \textit{ then } \liminf_{n\to\infty} J(v_n) \geq J(v)\,.$$

Suppose that one of the following three conditions holds:

1) K is bounded;
2) J is coercive, i.e. $\lim_{\|v\|\to\infty} J(v) = +\infty$;
3) every minimizing sequence $\{v_n\}_n \subset K$ of J is bounded in V.

Then J is bounded below on K and it attains its minimum value on K.

In addition, if J is convex, then the set of all minimizers of J is closed convex subset of K.

Moreover, if the functional J is strictly convex on K (i.e., $J((1-t)u + tv) < (1-t)J(u) + tJ(v)\,,\ \forall t \in (0,1)\,,\ \forall u, v \in K\,,\ u \neq v$), then the minimizer of J is unique.

We now introduce two important classes of nonlinear operators defined in a Hilbert space, namely the proximity operator and the projection operator (see, e.g., [28, 29]).

Definition 4.1. Let $\varphi : V \to \overline{\mathbb{R}}$ be a proper convex l.s.c. function. The proximity operator with respect to the function φ is the operator $Prox_\varphi : V \to V$ defined by $Prox_\varphi(w) = u$, $\forall w \in V$, u being the unique element in V, called the proximal element of w with respect to the function φ, such that

$$\Phi_w(u) = \min_{v \in V} \Phi_w(v)$$

where

$$\Phi_w(v) = \frac{1}{2}\|v\|^2 + \varphi(v) - (w, v) \quad \forall v \in V\,. \tag{4.10}$$

This definition is justified by the following existence and uniqueness theorem.

Theorem 4.3 (Proximity Theorem). *Let $\varphi : V \to \overline{\mathbb{R}}$ be a proper convex l.s.c. function. Then, for any $w \in V$, there is a unique element $u \in V$ such that $u = Prox_\varphi(w)$.*

Proof. It is easy to see that the function $\Phi_w : V \to \overline{\mathbb{R}}$, defined by (4.10), is proper. As φ is convex and the norm $\|\cdot\|_V$ is strictly convex, it follows that Φ_w is strictly convex. Moreover, since the norm and the function $v \mapsto (w, v)$ are continuous, and φ is l.s.c., we deduce that Φ_w is a l.s.c. function.

Next, the hypotheses on φ imply that there exist $\lambda \in V$ and $\mu \in \mathbb{R}$ such that

$$\varphi(v) \geq (\lambda, v) + \mu \quad \forall v \in V\,,$$

and hence

$$\Phi_w(v) \geq \frac{1}{2}\|v\|^2 + (\lambda - w, v) + \mu = \frac{1}{2}\|v + \lambda - w\|^2 - \frac{1}{2}\|\lambda - w\|^2 + \mu\,.$$

Therefore, the function Φ_w is coercive. Applying Weierstrass Theorem 4.2, we conclude the proof. □

The following results give a characterization of the proximity operator and some of its properties.

Proposition 4.1. *Let $\varphi : V \to \overline{\mathbb{R}}$ be a proper convex l.s.c. function and let $w \in V$. Then the following assertions are equivalent:*

(1) $u = Prox_\varphi(w)$,
(2) $(u, v - u) + \varphi(v) - \varphi(u) \geq (w, v - u) \quad \forall v \in V$.

Proof. We apply Definition 4.1 and Lemma 4.2 for $a(u, v) = (u, v)$, $\forall u, v \in V$. □

Proposition 4.2. *Let $\varphi : V \to \overline{\mathbb{R}}$ be a proper convex l.s.c. function. Then the proximity operator $Prox_\varphi : V \to V$ is monotone and non-expansive, i.e.*

$$(Prox_\varphi(w_1) - Prox_\varphi(w_1), w_1 - w_2) \geq 0 \quad \forall w_1\,,\, w_2 \in V$$
$$\|Prox_\varphi(w_1) - Prox_\varphi(w_1)\| \leq \|w_1 - w_2\| \quad \forall w_1\,,\, w_2 \in V\,.$$

Proof. Let $w_i \in V$ and $u_i = Prox_\varphi(w_i)$ for $i = 1, 2$. Therefore, by Proposition 4.1, we have

$$(u_i, u_{3-i} - u_i) + \varphi(u_{3-i}) - \varphi(u_i) \geq (w_i, u_{3-i} - u_i) \quad i = 1,\, 2\,.$$

By adding the two inequalities, we obtain

$$(w_1 - w_2, u_1 - u_2) \geq \|u_1 - u_2\|^2 ,$$

thus

$$0 \leq \|Prox_\varphi(w_1) - Prox_\varphi(w_2)\|^2 \leq (Prox_\varphi(w_1) - Prox_\varphi(w_2), w_1 - w_2)$$
$$\leq \|Prox_\varphi(w_1) - Prox_\varphi(w_2)\| \, \|w_1 - w_2\|$$

which completes the proof. □

The proximity operator, being non-expansive, is continuous. In addition, by Proposition 4.1, we deduce that

$$u = Prox_\varphi(u) \Longleftrightarrow \varphi(u) \leq \varphi(v) \quad \forall v \in V . \tag{4.11}$$

An important particular case of the proximity operator $Prox_\varphi$ is obtained by taking $\varphi = I_K$ where I_K denotes the indicator function of K defined by:

$$I_K(v) = \begin{cases} 0 & \text{if } v \in K , \\ +\infty & \text{otherwise} . \end{cases}$$

In this case, we will denote the proximity operator $Prox_{I_K}$ by P_K, called the projection operator.

Definition 4.2. The projection operator $P_K : V \to K$ is defined by $P_K(w) = u$ where $u \in K$, called the projection of w onto K, is the unique minimizer of the functional $\Psi_w(v) = \|v - w\|$ on K (i.e., the distance between the given element $w \in V$ and the closed convex set K).

As $P_K = Prox_{I_K}$, we have the following existence and uniqueness result.

Theorem 4.4 (Projection Theorem). *For any $w \in V$ there is a unique element $u \in K$ such that*

$$\|u - w\| = \min_{v \in K} \|v - w\| .$$

From Propositions 4.2 and 4.1, we deduce the following results.

Proposition 4.3. *The projection operator $P_K : V \to K$ is monotone and non-expansive.*

Proposition 4.4. *Let $w \in V$. Then, the following assertions are equivalent:*

(1) $u = P_K(w)$,
(2) $(u, v - u) \geq (w, v - u) \quad \forall v \in K$.

It is easy to see that

$$u = P_K u \Longleftrightarrow u \in K\,. \tag{4.12}$$

We note that, in general, the projection operator is not linear. However, if M is a closed subspace of V, then the projection operator P_M is linear and $\|P_M\|_{\mathscr{L}(V,V)} = 1$.

We remind that a subset A of a normed space X is said to be compact (respectively, weakly compact) if every sequence in A has a subsequence converging strongly (respectively, weakly) in X to an element of A.

We recall now the following fixed point theorems (see, e.g., [3], [9, p. 12], [23, p. 530], [36, p. 16]) that will be useful in the sequel.

Theorem 4.5 (Brouwer's Fixed Point Theorem). *Let K be a nonempty, convex, compact subset of a finite dimensional normed linear space V. If the operator $T : K \to K$ is continuous, then T has a fixed point, i.e. there exists $u \in K$ such that $T(u) = u$.*

Theorem 4.6 (Schauder's Fixed Point Theorem). *Let V be a Banach space, and let $K \subset V$ be a nonempty, convex, compact subset. If the operator $T : K \to K$ is continuous, then T has a fixed point.*

Theorem 4.7 (Banach's Fixed Point Theorem). *Let $(V, \|\cdot\|_V)$ be a Banach space, and let K be a nonempty closed subset of V. Suppose that the operator $T : K \to K$ is a contraction, i.e. there exists a constant $c \in [0, 1)$ such that*

$$\|T(u) - T(v)\|_V \leq c\|u - v\|_V\,.$$

Then T has a unique fixed point.

In the finite dimensional case, we have the following existence result for the variational inequality (4.5) (see [21]).

Theorem 4.8 (Hartman–Stampacchia's Theorem). *Let K be a nonempty compact convex subset of a finite dimensional space V. If we suppose that $A : K \longrightarrow V$ is a continuous mapping and $j : K \longrightarrow (-\infty, +\infty]$ is a proper l.s.c. convex function, then there exists at least one $u \in K$ such that*

$$(Au, v - u) + j(v) - j(u) \geq (f, v - u) \quad \forall v \in K\,. \tag{4.13}$$

Proof. We consider the proper l.s.c. convex function $\varphi : V \longrightarrow (-\infty, +\infty]$ defined by

$$\varphi(v) = \begin{cases} j(v) & \text{if } v \in K\,, \\ +\infty & \text{otherwise}\,. \end{cases} \tag{4.14}$$

Let the operator $T : K \to K$ be defined by $T(w) = Prox_\varphi(w - Aw + f)$, $\forall w \in K$, where $Prox_\varphi$ is the proximity operator with respect to φ. We first remark that, from the definition (4.14) of φ and the definition of the proximity operator, it follows that $T(w) \in K$, $\forall w \in K$. Then, by Proposition 4.1, it follows that the inequality (4.13) is equivalent to $u = T(u)$.

The operators A and $Prox_\varphi$ are continuous, hence T is itself on the compact convex set K. Hence, from Schauder fixed point Theorem 4.6 or from Brouwer Theorem 4.5, it follows that there exists at least one element $u \in K$ such that $u = T(u)$ which completes the proof. □

We have the following results (see, e.g., [16]) on the weak compactness in reflexive Banach spaces.

Theorem 4.9 (Eberlein–Smulyan Theorem). *Let V be a reflexive Banach space. Then any bounded sequence in V contains a weakly convergent subsequence.*

Corollary 4.1. *Any nonempty, bounded, and weakly closed subset in a reflexive Banach space is weakly compact.*

The following existence result for variational inequalities holds.

Theorem 4.10. *Suppose that the hypotheses (4.1)–(4.4) hold. In addition, we assume that the form a is positive and that the closed convex set K is bounded. Then, the set of all solutions of the variational inequality (4.5) is a nonempty, convex, and weakly compact subset of K.*

Proof. From Lemma 4.1, the set of all solutions of (4.5) is

$$\chi = \bigcap_{v \in K} S(v) \quad \text{where} \quad S(v) = \{u \in K\,;\ a(v, v-u) + j(v) - j(u) \geq (f, v-u)\}\,.$$

The set χ being closed convex, it is weakly closed (i.e., it contains the limits of all weakly convergent sequences $\{v_n\}_n \subset \chi$) in V. On the other hand, as the set K is bounded and weakly closed in V, by Corollary 4.1, it follows that it is weakly compact. Therefore, we will prove that $\chi \neq \emptyset$ by proving that the family $\{S(v)\}_{v\in K}$ has the finite intersection property, i.e. any finite subcollection $K_Q \subset K$ has nonempty intersection. Let $\{v_1, \cdots, v_q\}$ be a finite part of K and $K_Q = K \cap Q$ where Q is the finite dimensional space spanned by the family $\{v_1, \cdots, v_q\}$. Then, from Hartman–Stampacchia Theorem 4.8, it follows that there exists a solution $u \in K_Q \subset K$ of the inequality

$$a(u, v-u) + j(v) - j(u) \geq (f, v-u) \quad \forall v \in K_Q\,,$$

i.e., there exists $u \in S(v)\,,\ \forall v \in K_Q$, hence $\bigcap_{v \in K_Q} S(v) \neq \emptyset$. □

We shall see below that, in the case of compact sets, the existence result is a consequence of Schauder fixed point Theorem 4.6.

Proposition 4.5. *Under the hypotheses of Theorem 4.10, if the set K is compact, then the set of all solutions of the variational inequality (4.5) forms a nonempty compact convex subset of K.*

Proof. The set χ of all solutions of (4.5) is closed and convex.

In order to prove that it is nonempty, let $T : K \to K$ be the continuous operator defined by $T(w) = Prox_{\varphi}(w - Aw + f)$, $\forall w \in K$, where the functional φ is defined by (4.14), and $A \in \mathscr{L}(V, V)$ is the operator associated with the bilinear continuous form $a(\cdot, \cdot)$, i.e.

$$(Au, v) = a(u, v) \quad \forall u, v \in V\,. \tag{4.15}$$

Hence, by Schauder fixed point Theorem 4.6, it follows that there exists $u \in K$ such that $u = T(u)$, that is

$$((u - Au + f) - u, v - u) \leq j(v) - j(u) \quad \forall v \in K\,,$$

and thus, the set of the solutions of (4.5) is nonempty. Moreover, as χ is closed in the compact set K, it follows that χ is compact.

Next, if we refer to the variational inequality of the first kind (4.6), then we have to consider the continuous operator $T : K \to K$ defined by $T(v) = P_K(v - Av + f)$ where $P_K : V \to K$ is the projection operator on the nonempty closed convex subset K. Therefore, by taking into account the characterization of the projection given by Proposition 4.4, from $u = Tu$ it follows

$$(u - (u - Au + f), v - u) \geq 0 \quad \forall v \in K\,.$$

In fact, it is enough to remark that the projection operator is a particular case of the proximity operator, namely $P_K = Prox_{I_K}$, I_K being the indicator function of K. □

However, the most interesting cases involve unbounded sets K. Existence results are obtained by requiring that the form a is coercive on V (or, V-elliptic), that is, there exists a positive constant α such that

$$a(u, u) \geq \alpha \|u\|^2 \quad \forall u \in V\,. \tag{4.16}$$

We first recall an important surjectivity result (see, e.g., [10, p. 42], [9, p. 41]).

Theorem 4.11 (Lax–Milgram's Theorem). *Let V be a real Hilbert space, and let $a : V \times V \to \mathbb{R}$ be a bilinear, continuous, and coercive form. Then, for each $f \in V^*$, there exists a unique element $u \in V$ such that*

$$a(u, v) = \langle f, v \rangle \quad \forall v \in V\,,$$

with V^* the dual space of V and $\langle \cdot, \cdot \rangle$ the duality pairing between V^* and V.

The mapping $f \mapsto u$ is one-to-one, continuous, and linear from V^* onto V.

Corollary 4.2. *Let V be a real Hilbert space, and let $A : V \to V^*$ be a bilinear, continuous, and coercive operator. Then, the operator A is bijective from V onto V^*.*

We have the following existence and uniqueness result for unbounded sets.

Theorem 4.12. *Suppose that the hypotheses (4.1)–(4.4) hold. If the bilinear continuous form a is coercive, then there exists a unique solution $u \in K$ of the variational inequality (4.5).*

Proof. In order to emphasize the difference between techniques involved in the proof, we are going to assume following cases:

1) If $a(u, v) = (u, v)$, $\forall u, v \in V$, then, from Propositions 4.1 and 4.4, it follows that the variational inequalities (4.5) and (4.6) are equivalent to $u = Prox_{\varphi}(f)$ and, respectively, $u = P_K(f)$, the function φ being defined by (4.14). Therefore, Theorems 4.3 and 4.4 conclude the proof.
2) If the form $a(\cdot, \cdot)$ is symmetric, then, by Proposition 4.2, it follows that the problem (4.5) is equivalent to the minimization problem

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ J(u) = \inf\limits_{v \in K} J(v) \end{cases}$$

where the function $J : V \to \overline{\mathbb{R}}$ is defined by (4.9).

Therefore, Theorem 4.12 takes the form of Weierstrass Theorem 4.2.

It is easy to verify that the hypothesis (4.16) implies that the form a is strictly convex. Indeed, we have

$$\begin{aligned} &a((1-t)u+tv, (1-t)u+tv) = (1-t)a(u,u) + ta(v,v) - t(1-t)a(u-v, u-v) \\ &\leq (1-t)a(u,u) + ta(v,v) - \alpha t(1-t)\|u-v\|^2 \\ &< (1-t)a(u,u) + ta(v,v) \quad \forall t \in (0,1),\ \forall u, v \in V,\ u \neq v\,. \end{aligned}$$

Hence the function J is strictly convex.

From the hypotheses (4.2)–(4.4), it follows that there exist $\lambda \in V$ and $\mu \in \mathbb{R}$ such that

$$j(v) \geq (\lambda, v) + \mu\,.$$

Thus, by using (4.16), we get

$$J(v) \geq \frac{\alpha}{2}\|v\|^2 + (\lambda - f, v) + \mu \geq \frac{\alpha - \epsilon}{2}\|v\|^2 - \frac{1}{2\epsilon}\|\lambda - f\|^2 + \mu\,,\ \forall \epsilon \in (0, \alpha)$$

and so, the function J is coercive. In addition, J is proper and weakly l.s.c. (for convex functions, the lower semicontinuity is equivalent to the weakly lower semicontinuity). Therefore, by Theorem 4.2 the assertion follows.

3) If $K = V$ and $j \equiv 0$, then, the variational inequality (4.5), or (4.6), becomes

$$Au = f$$

where A is the operator associated with the form a by (4.15). So, Theorem 4.12 expresses Corollary 4.2 of Lax–Milgram Theorem 4.11, and so, the operator A is invertible (it satisfies the hypothesis $\|Av\| \geq \alpha\|v\|$, $v \in V$ and it is bijective), i.e. the equation $Au = f$ has a unique solution $u = A^{-1}f$ where $A^{-1} \in \mathscr{L}(V, V)$ is the inverse operator of A.

4) In the general case, for any $\rho > 0$, the inequality (4.5) can be written as

$$(u - (u - \rho(Au - f)), v - u) \geq \rho\varphi(v) - \rho\varphi(u) \quad \forall v \in V$$

that is $u = Prox_{\rho\varphi}(u - \rho(Au - f))$, and, respectively, $u = P_K(u - \rho(Au - f))$ in the case of the variational inequality of the first kind (4.6). We define the operator $T_\rho : K \to K$ by

$$T_\rho(v) = Prox_{\rho\varphi}(v - \rho(Av - f)), \tag{4.17}$$

and, respectively,

$$T_\rho(v) = P_K(v - \rho(Av - f)). \tag{4.18}$$

Now, the existence and uniqueness of u follow from Banach fixed point Theorem 4.7 provided that there exists ρ such that the mapping T_ρ is a contraction on the nonempty closed subset K of the Banach space V. Indeed, since the proximity and projection operators are non-expansive, then by the relations (4.1), (4.16), we have

$$\|T_\rho(v_1) - T_\rho(v_2)\| \leq \sqrt{1 + \rho^2 M^2 - 2\alpha\rho}\, \|v_1 - v_2\| \quad \forall v_1, v_2 \in K,$$

hence, by choosing $\rho \in \left(0, \dfrac{2\alpha}{M^2}\right)$, one obtains $1 + \rho^2 M^2 - 2\alpha\rho \in (0, 1)$, i.e. T_ρ is a contraction. □

4.1.2 Variational Inequalities with Nonlinear Operators

This section contains existence and uniqueness results for the solutions of variational inequalities involving a large class of nonlinear operators, namely monotone and hemicontinuous operators (see [6, 21, 30, 37, 38]). These operators are considered in most of the applications of the elliptic boundary problems.

Let $(V, \|\cdot\|)$ be a real reflexive Banach space with its dual $(V^*, \|\cdot\|_*)$, and let $K \subset V$ be a nonempty closed convex subset. We consider a function $j : K \to \overline{\mathbb{R}}$ satisfying the following conditions

$$j \text{ is proper l.s.c. convex}\,, \tag{4.19}$$

and a monotone hemicontinuous operator $A : V \to V^*$, i.e.

$$\langle Au - Av, u - v\rangle \geq 0 \quad \forall u, v \in V\,, \tag{4.20}$$

$$\forall u, v \in V, \text{the map } t \in [0,1] \longrightarrow \langle A((1-t)u + tv, u - v\rangle \text{ is continuous}, \tag{4.21}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between V^* and V.

We will establish conditions which ensure the existence of the solutions of the variational inequality

Problem (P_2^a): Find $u \in K$ such that

$$\langle Au, v - u\rangle + j(v) - j(u) \geq \langle f, v - u\rangle \quad \forall v \in K\,, \tag{4.22}$$

for $f \in V^*$ given.

First, by proceeding in a similar way as in the proof of Lemma 4.1, one obtains:

Lemma 4.3. *Under the above hypotheses, an element $u \in K$ satisfies the inequality (4.22) if and only if it satisfies the inequality*

$$\langle Av, v - u\rangle + j(v) - j(u) \geq \langle f, v - u\rangle \quad \forall v \in K\,. \tag{4.23}$$

Moreover, the set of all solutions of the variational inequality (4.22) is convex closed in V.

The main result of this section is the following existence and uniqueness result (see, e.g., [37]).

Theorem 4.13. *Suppose the hypotheses (4.19)–(4.21) hold. If one of the following conditions is satisfied*

$$K \textit{ is bounded}\,, \tag{4.24}$$

$$0 \in K\,,\; j(0) = 0 \textit{ and } \lim_{\substack{\|v\| \to +\infty \\ v \in K}} \frac{\langle Av, v\rangle + j(v)}{\|v\|} = +\infty\,, \tag{4.25}$$

$$\exists v_0 \in K \textit{ such that } \lim_{\substack{\|v\| \to +\infty \\ v \in K}} \frac{\langle Av, v - v_0\rangle + j(v) - j(v_0)}{\|v\|} = +\infty\,, \tag{4.26}$$

then, there exists at least one solution $u \in K$ of (4.22). Moreover, the set of all solutions of the variational inequality (4.22) is convex closed bounded in V, and so, it is weakly compact.

In addition, if j is strictly convex, i.e.

$$j(\lambda u + (1-\lambda)v) < \lambda j(u) + (1-\lambda)j(v) \quad \forall \lambda \in (0,1)\,,\ \forall u, v \in V\,,\ u \neq v\,,$$

or A is strictly monotone, i.e.

$$\langle Au - Av, u - v \rangle > 0 \quad \forall u, v \in V\,,\ u \neq v\,,$$

then the solution of the variational inequality (4.22) is unique.

Proof. By Lemma 4.3, the set of all solutions of (4.22) is the following closed convex set

$$\chi = \bigcap_{v \in K} S(v) \subset K \quad \text{where} \quad S(v) = \{u \in K\,;\ \langle Av, v-u \rangle + j(v) - j(u) \geq \langle f, v-u \rangle\}\,.$$

If the hypothesis (4.24) holds, then, obviously χ is also bounded. Proceeding in a similar way as in the proof of Theorem 4.10, we get $\chi \neq \emptyset$.

Now, we suppose that the hypothesis (4.25) or (4.26) is satisfied. We then consider the following closed bounded convex subset of K

$$K_R = K \cap B(0, R)$$

where $B(0, R) = \{v \in V\,;\ \|v\| \leq R\}$. We may assume that R is large enough such that the set K_R is not-empty. Therefore, from the first part of the proof, it follows that there exists $u_R \in K_R$ such that

$$\langle Au_R, v - u_R \rangle + j(v) - j(u_R) \geq \langle f, v - u_R \rangle \quad \forall v \in K_R\,. \tag{4.27}$$

We shall show that any of the two coerciveness conditions (4.25) or (4.26) implies $\|u_R\| < R$. We suppose by contradiction that $\|u_R\| = R$.

If (4.25) holds, then

$$\langle Au_R, u_R \rangle + j(u_R) > \langle f, u_R \rangle\,,$$

On the other hand, taking $v = 0 \in K_R$ in (4.27), we get

$$\langle Au_R, u_R \rangle + j(u_R) \leq \langle f, u_R \rangle\,,$$

which represents a contradiction.

If (4.26) is true, then

$$\langle Au_R, u_R - v_0\rangle + j(u_R) - j(v_0) > \langle f, u_R - v_0\rangle \,.$$

We may always suppose that R is large enough such that $R \geq \|v_0\|$. Therefore, from (4.27) with $v = v_0 \in K_R$, we obtain the contradiction

$$\langle Au_R, v_0 - u_R\rangle + j(v_0) - j(u_R) \geq \langle f, v_0 - u_R\rangle \,.$$

We conclude that $\|u_R\| < R$.

We note that, for every $w \in K$, there exists $\epsilon = \epsilon(w) \in (0, 1]$ such that $v = u_R + \epsilon(w - u_R) \in K_R$. Indeed, if $w \in K_R$, then we take $\epsilon = 1$, and if $w \notin K_R$, then, by taking $0 < \epsilon \leq \dfrac{R - \|u_R\|}{\|w\| - \|u_R\|} \in (0, 1)$, one obtains $v \in K_R$. Therefore, from (4.27) and the convexity of j, it follows

$$\langle Au_R, w - u_R\rangle + j(w) - j(u_R) \geq \langle f, w - u_R\rangle \quad \forall w \in K \,,$$

that is u_R is a solution of (4.22).

Therefore, in order to prove the first part of the theorem, it is enough to prove that the set χ is bounded. If we suppose that, for all $R > 0$, there exists $u_R \in \chi$ such that $\|u_R\| > R$, then, for R sufficiently large, the coerciveness relations (4.25) or (4.26), and the inequality (4.22) give, as we have seen above, a contradiction.

Finally, if j is strictly convex or A is strictly monotone, we wish to prove the uniqueness of the solution of (4.22). Suppose that two solutions u_1, $u_2 \in K$ exist. Taking $v = \dfrac{u_1 + u_2}{2}$ in the corresponding inequalities, by adding them, we deduce

$$0 \leq \frac{1}{2}\langle Au_1 - Au_2, u_1 - u_2\rangle + j(u_1) + j(u_2) - 2j\left(\frac{u_1 + u_2}{2}\right) \leq 0 \,,$$

which, in any of the two hypotheses, implies $u_1 = u_2$. □

In the following we shall see that the hypotheses (4.24)–(4.26) can be replaced by a strong assumption on the operator A.

Corollary 4.3. *Let $j : K \to \overline{\mathbb{R}}$ be a proper l.s.c. convex function and $A : V \to V^*$ a hemicontinuous and strongly monotone operator, i.e. A satisfies (4.21) and*

$$\exists \alpha > 0 \text{ such that } \langle Au - Av, u - v\rangle \geq \alpha \|u - v\|^2 \quad \forall u, v \in V \,. \tag{4.28}$$

Then, there exists a unique solution $u \in K$ of (4.22).

Proof. We shall prove that the coerciveness hypothesis (4.26) is satisfied. From the strongly monotonicity of A, we get

$$\lim_{\substack{\|v\|\to+\infty \\ v\in K}} \frac{\langle Av, v-v_0\rangle}{\|v\|} = +\infty \quad \forall v_0 \in K\,. \tag{4.29}$$

On the other hand, the hypotheses on j imply that there exist $\lambda \in V^*$ and $\mu \in \mathbb{R}$ such that

$$j(v) \geq \langle \lambda, v\rangle + \mu \geq -\|\lambda\|_*\|v\| + \mu \quad \forall v \in K\,. \tag{4.30}$$

Now, by choosing $v_0 \in \operatorname{dom} j = \{v \in K;\ j(v) < +\infty\}$ (obviously, as the function j is proper, one has: $\operatorname{dom} j \neq \emptyset$), the relations (4.29) and (4.30) get (4.26). □

It is known that (see, for instance, [17], p. 42) if a monotone hemicontinuous operator A is bounded (i.e., A maps bounded sets into bounded sets), then it is pseudo-monotone, that is

$i)$ A is bounded

$ii)$ $\forall \{u_n\}_n \subset K\,,\ \forall u \in K$ s.t. $u_n \rightharpoonup u$ weakly in V and $\limsup_{n\to\infty}\langle Au_n, u_n - u\rangle \leq 0\,,$

then $\liminf_{n\to\infty}\langle Au_n, u_n - v\rangle \geq \langle Au, u-v\rangle \quad \forall v \in K\,.$

Finally, we note that Theorem 4.20 is still valid for a pseudo-monotone operator A with a slightly modified proof (see, e.g., [30, 31, 40]).

4.2 Elliptic Quasi-variational Inequalities

The object of this section is to study the so-called quasi-variational inequalities, initially introduced by Bensoussan and Lions [4] in connection with some stochastic impulse control problems. The mathematical literature on quasi-variational inequalities contains many notable contributions including a wide set of applications in mechanics, engineering, economics, or game theory. We do dot claim to cover here this huge subject, we only focus our attention on two classes of quasi-variational inequalities involving monotone hemicontinuous operators and, respectively, potential operators.

4.2.1 Quasi-variational Inequalities with Hemicontinuous Operators

Let $(V, \|\cdot\|)$ be a real reflexive Banach space, $(V^*, \|\cdot\|_*)$ its dual, and let K be a nonempty closed convex subset of V. We denote by $\langle\cdot,\cdot\rangle$ the duality product between V^* and V.

For $f \in V^*$ given, we consider the following quasi-variational inequality:
Problem (P_3^a): Find $u \in K$ such that

$$\langle Au, v-u\rangle + j(u,v) - j(u,u) \geq \langle f, v-u\rangle \quad \forall v \in K\,, \tag{4.31}$$

where $A : V \to V^*$ is an operator and $j : V \times V \to (-\infty, +\infty]$ a function.

In this paragraph we indicate a quite wide class of operators A and functionals j which guarantees the existence and, eventually, the uniqueness of the solution.

Remark 4.1. The problem (4.31) is called (see, for instance, [31]) quasi-variational inequality of the second kind.

Remark 4.2. Suppose that $K = V$ (or, sufficiently, $\text{dom}\, j = K \times K$ where $\text{dom}\, j = \{(u,v) \in V \times V\,;\ j(u,v) < +\infty\}$ is the effective domain of j) and

$$j(u,v) = I_{Q(u)}(v) = \begin{cases} 0 & \text{if } v \in Q(u)\,, \\ +\infty & \text{if } v \notin Q(u) \end{cases}$$

where $Q : V \to 2^V$ is a multivalued mapping such that for every $u \in V$, $Q(u)$ is a nonempty closed convex subset of V. Then the variational inequality (4.31) becomes

$$\begin{cases} u \in Q(u) \\ \langle Au, v-u\rangle \geq \langle f, v-u\rangle \quad \forall v \in Q(u)\,. \end{cases}$$

This inequality is called quasi-variational inequality of the first kind (see [4, 39]). We remark that this problem involves implicit constraints, i.e. constraints that depend on the solution itself.

We recall that a functional $\varphi : K \to \mathbb{R}$ is said to be weakly upper semicontinuous (weakly u.s.c.) on K if

$$\limsup_{n\to\infty} \varphi(v_n) \leq \varphi(v) \text{ for every sequence } \{v_n\}_n \subset K \text{ converging weakly in } V \text{ to } v \in K.$$

Theorem 4.14. *Suppose the following hypotheses hold.*

$$A \textit{ is a monotone hemicontinuous operator,} \tag{4.32}$$

$$\textit{the function } j \textit{ is weakly l.s.c. on } K \times K\,, \tag{4.33}$$

$$\forall v \in V\,,\ \textit{the function } j(\cdot, v) : K \to (-\infty, +\infty] \textit{ is weakly u.s.c. on } K\,, \tag{4.34}$$

$$\forall u \in K, \textit{ the function } j(u,\cdot) : K \to (-\infty, +\infty] \textit{ is proper convex.} \tag{4.35}$$

Then, the set of all solutions of the quasi-variational inequality (4.31) is a nonempty weakly compact subset of K if one of the two conditions is satisfied:

$$K \text{ is bounded}, \tag{4.36}$$

$$\exists v_0 \in K \text{ s.t. } \lim_{\substack{\|v\| \to +\infty \\ v \in K}} \frac{\langle Av, v - v_0 \rangle + j(v, v) - j(v, v_0)}{\|v\|} = +\infty . \tag{4.37}$$

In order to prove this theorem, we introduce some useful definitions and properties for multivalued mappings (for details and proofs, see [23], p. 541).

Let E and F be two topological spaces. We consider a multivalued mapping $S : E \to 2^F$.

Definition 4.3. We say that S is u.s.c. in a point $x_0 \in E$ if, for any open subset U in F such that $S(x_0) \subset U$, there exists a neighborhood V of x_0 in E such that $S(V) \subset U$ where $S(V) = \bigcup_{v \in V} S(v)$.

Definition 4.4. The multivalued mapping S is called u.s.c. on E if it is u.s.c. in any point $x \in E$.

Definition 4.5. We say that the mapping S is closed if its graph

$$\mathcal{G}_S = \{(x, y) \in E \times F;\ y \in S(x)\}$$

is a closed subset of $E \times F$.

Definition 4.6. Let E and F be two topological vector spaces. We say that the multivalued mapping $S : E \to 2^F$ is K-map (or, Kakutani map) if it satisfies the following conditions:

(i) S is u.s.c. on E;
(ii) $\forall x \in E,\ S(x)$ is a nonempty convex compact subset of F.

Proposition 4.6. *Let E be a locally convex topological space and C a compact subset of E. Let $S : C \to 2^C$ be a closed multivalued mapping. Then, S is u.s.c. on E and, for any $x \in E$, $S(x)$ is a compact subset of C.*

The following theorem represents a generalization of the classical theorem due to Ky Fan [24].

Theorem 4.15 (Kakutani Fixed Point Theorem). *Let E be a locally convex topological vector space and C a nonempty convex compact subset of E. Let $S : C \to 2^C$ be a K-map. Then S has at least a fixed point in C, i.e. there exists $x \in C$ such that $x \in S(x)$.*

Proof of Theorem 4.14. Suppose that the hypothesis (4.36) is satisfied. For every $u \in K$, we put

$$S(u) = \{w \in K;\ \langle Aw, v-w\rangle + j(u,v) - j(u,w) \geq \langle f, v-w\rangle \quad \forall v \in K\}. \tag{4.38}$$

From the hypotheses (4.32) and (4.35), proceeding as in Lemma 4.2, it follows that

$$S(u) = \{w \in K;\ \langle Av, v-w\rangle + j(u,v) - j(u,w) \geq \langle f, v-w\rangle \quad \forall v \in K\}.$$

By applying Theorem 4.13, it follows that $S(u)$ is a nonempty convex weakly compact subset of K.

We prove now that the multivalued mapping $S : K \to 2^K$, defined above, is weakly closed. Let $\{u_n, w_n\}_n \subset K \times K$ be such that $w_n \in S(u_n)$, $\forall n \in N$ and $u_n \rightharpoonup u$, $w_n \rightharpoonup w$ weakly in V when $n \to +\infty$. Therefore, we have

$$\langle Av, v-w_n\rangle + j(u_n,v) - \langle f, v-w_n\rangle \geq j(u_n,w_n) \quad \forall v \in K,$$

hence, by passing to the limit and using (4.34) and (4.33), one gets

$$\langle Av, v-w\rangle + j(u,v) - \langle f, v-w\rangle \geq \limsup_{n\to+\infty}[\langle Av, v-w_n\rangle + j(u_n,v) - \langle f, v-w_n\rangle]$$

$$\geq \liminf_{n\to+\infty} j(u_n,w_n) \geq j(u,w) \quad \forall v \in K,$$

i.e., $w \in S(u)$.

We now apply Proposition 4.6 and Theorem 4.15 for $E = V$ endowed with the weak topology, and $C = K$. Then, by taking into account (4.38), it follows that the quasi-variational inequality (4.31) has at least one solution $u \in K$.

Next we prove that the set of all solutions of (4.31) is a weakly closed subset of K. Let $\{u_n\}_n \subset K$ be such that $u_n \rightharpoonup u$ weakly in V, and

$$\langle Au_n, v-u_n\rangle + j(u_n,v) - j(u_n,u_n) \geq \langle f, v-u_n\rangle \quad \forall v \in K.$$

This means that $u_n \in S(u_n)$, $\forall n \in N$, and hence, since the multivalued mapping S is weakly closed, we deduce that $u \in S(u)$, i.e. u is a solution of the quasi-variational inequality (4.31).

Finally, taking into account that K is a weakly compact subset of V, we conclude the proof.

Next, we assume that the coerciveness condition (4.37) is satisfied. Let $R > \|v_0\|$ be sufficiently large such that $K_R = K \cap B(0,R) \neq \emptyset$ where $B(0,R) = \{v \in V;\ \|v\| \leq R\}$. By applying the first part of the proof for the nonempty convex bounded closed set K_R, it follows that there exists an element $u_R \in K_R$ such that

$$\langle Au_R, v-u_R\rangle + j(u_R,v) - j(u_R,u_R) \geq \langle f, v-u_R\rangle \quad \forall v \in K_R. \tag{4.39}$$

Proceeding as in the proof of Theorem 4.13, one shows that the coerciveness hypothesis (4.37) implies $\|u_R\| < R$.

We shall prove that u_R is a solution of (4.31). Let $w \in K\backslash K_R$. Taking $0 < \epsilon \leq \dfrac{R - \|u_R\|}{\|w\| - \|u_R\|}$ and $v = u_R + \epsilon(w - u_R)$, it follows that $v \in K_R$. Then, from (4.39) with this v, we obtain

$$\epsilon\langle Au_R, w - u_R\rangle + j(u_R, u_R + \epsilon(w - u_R)) - j(u_R, u_R) \geq \epsilon\langle f, w - u_R\rangle \quad \forall w \in K .$$

Using now the convexity of $j(u_R, \cdot)$ and dividing with $\epsilon > 0$, one deduces

$$\langle Au_R, w - u_R\rangle + j(u_R, w) - j(u_R, u_R) \geq \langle f, w - u_R\rangle \quad \forall w \in K ,$$

that is u_R is a solution of the quasi-variational inequality (4.31).

Proceeding again as in the first part of the proof, we obtain that the set of all solutions of the inequality (4.31) is weakly closed. On the other hand, every solution u of the inequality (4.31) verifies the inequality (4.39) for any $R > 0$. If we choose $R > \|v_0\|$ sufficiently large, then, from (4.39) and the coerciveness condition (4.37), it follows that $\|u\| < R$. Therefore, we conclude that there exists $R > 0$ such that the set of all solutions of the inequality (4.31) is weakly closed in the bounded set K_R, and so, it is weakly compact in V. □

Under more restrictive hypotheses on A, one obtains the following existence and uniqueness result.

Theorem 4.16. *Let $A : V \longrightarrow V^*$ be a hemicontinuous and strongly monotone operator, that is A satisfies (4.21) and*

$$\exists\alpha > 0 \text{ such that } \langle Au - Av, u - v\rangle \geq \alpha\|u - v\|^2 \quad \forall u, v \in V . \tag{4.40}$$

We suppose that the functional $j : V \times V \longrightarrow (-\infty, +\infty]$ satisfies the following conditions:

$$\forall u \in V,\ j(u, \cdot) : V \to (-\infty,\ +\infty] \text{ is a proper l.s.c. convex function}, \tag{4.41}$$

$$\begin{cases} \exists k < \alpha \text{ such that } |j(u_1, v_1) + j(u_2, v_2) - j(u_1, v_2) - j(u_2, v_1)| \\ \qquad \leq k\|u_1 - u_2\|\,\|v_1 - v_2\| \quad \forall u_1, u_2, v_1, v_2 \in K . \end{cases} \tag{4.42}$$

Then, the quasi-variational inequality (4.31) has a unique solution.

Proof. The proof is based on a fixed point theorem and on Theorem 4.13.

The operator A being strongly monotone, we obtain

$$\begin{aligned} \frac{\langle Aw, w - v_0\rangle}{\|w\|} &\geq \alpha\|w\| - 2\alpha\|v_0\| - \|Av_0\|_* \\ &\quad + \frac{\alpha\|v_0\|^2 - \|Av_0\|_*\|v_0\|}{\|w\|} \quad \forall w, v_0 \in K , \end{aligned} \tag{4.43}$$

hence

$$\lim_{\|w\|\to+\infty} \frac{\langle Aw, w - v_0\rangle}{\|w\|} = +\infty\,. \tag{4.44}$$

From (4.41), it results that, for any $v \in K$, there exists $\lambda \in V^*$, $\lambda = \lambda(v)$, and $\mu \in \mathbb{R}$ such that

$$j(v, w) \geq \langle \lambda, w\rangle + \mu \geq -\|\lambda\|_* \|w\| + \mu \quad \forall w \in K\,, \tag{4.45}$$

which, together with (4.44), implies

$$\lim_{\|w\|\to+\infty} \frac{\langle Aw, w - v_0\rangle + j(v, w) - j(v, v_0)}{\|w\|} = +\infty \; \forall v_0 \in \operatorname{dom} j(v, \cdot),\ \forall v \in K\,, \tag{4.46}$$

and so, the coerciveness condition (4.26) holds.

Now, we denote by S the mapping $S : K \to K$ which associates with every $w \in K$ the unique solution of the variational inequality of the second kind

$$\begin{cases} Sw \in K \\ \langle A(Sw), v - Sw\rangle + j(w, v) - j(w, Sw) \geq \langle f, v - Sw\rangle \quad \forall v \in K\,. \end{cases} \tag{4.47}$$

From (4.46) and (4.40)–(4.41), by applying Theorem 4.13, we conclude that the inequality (4.47) has a unique solution, hence the mapping S is well defined.

We remark that the set of all fixed points of S coincides with the set of all solutions of the quasi-variational inequality (4.31). Therefore, the question on the existence and uniqueness of the solutions of (4.31) is reduced to the existence and uniqueness of the fixed points of S.

We shall prove that the mapping S is a contraction. Indeed, for $w_1, w_2 \in K$ arbitrarily chosen, let Sw_1 and Sw_2 be the corresponding solutions of the inequality (4.47). By adding the two inequalities for $v = Sw_2$ and, respectively, $v = Sw_1$, by using (4.40) and (4.42), we obtain:

$$\|Sw_1 - Sw_2\| \leq q\|w_1 - w_2\| \tag{4.48}$$

with $q = \dfrac{k}{\alpha} < 1$.

Hence, by Banach fixed point Theorem 4.7, it follows that the mapping S has a unique fixed point. Therefore there exists a unique solution of the quasi-variational inequality (4.31). □

The above proof suggests and justifies the application of the following algorithm of Bensoussan–Lions type [4] for the approximation of the solution of the quasi-variational inequality (4.31) : for $u^0 \in K$ given, we define the sequence $u^n = Su^{n-1}$, that is u^n is the unique solution of the variational inequality:

$$\langle Au^n, v - u^n\rangle + j(u^{n-1}, v) - j(u^{n-1}, u^n) \geq \langle f, v - u^n\rangle \quad \forall v \in K\,. \tag{4.49}$$

From (4.48), we deduce

$$\|u^n - u\| \leq q^n \|u^0 - u\| \leq C q^n \tag{4.50}$$

where $u = Su$ is the unique solution of the quasi-variational inequality (4.31), C is a positive constant independent of n, and $q < 1$. Therefore, we have $u^n \to u$ strongly in V.

4.2.2 Quasi-variational Inequalities with Potential Operators

Variational methods have proven to be a powerful tool in the study of linear and nonlinear operator equations. The classical result of Friedrichs (see [18] or [14], p. 134) on the extension of any linear positive definite and symmetric operator A to an operator $\tilde{A}$, also positive definite but surjective, allowed to introduce the concept of generalized solution (in the Sobolev sense) of the equation $Au = f$ as the classical solution of the equation $\tilde{A}u = f$. The class of operators for which the generalized solution can be defined was enlarged with linear operators with positive definite derivative [26, 33] and nonlinear operators [15, 25]. Other generalizations were obtained for multivalued operators; so, the equation $Au + \partial j(u) \ni f$ is studied in [35] and [13] for a linear and, respectively, nonlinear operator A with positive definite and symmetric derivative, and in [22] are considered K-variational problems of the type $Pu \ni f$.

In the variational theory, the variational inequalities have an important place thanks to the characterization of the classical solution of the equation $Au + \partial j(u) \ni f$ as the solution of the variational inequality:

$$(Au, v - u) + j(v) - j(u) \geq (f, v - u)\,.$$

In this section, following [7], we introduce the concept of generalized solution of nonlinear quasi-variational inequalities. Our approach differs from the standard techniques since a quasi-variational inequality cannot be written as an operator equation of the type $Pu \ni f$. More precisely, for $u^0 \in V$ arbitrarily and supposing that u^{n-1} is known, we define u^n as the generalized solution of the variational inequality

$$(Au^n, v - u^n) + j_n(v) - j_n(u^n) \geq (f, v - u^n) \quad \forall v \in V\,,$$

where $j_n(\cdot) = j(u^{n-1}, \cdot)$.

We shall prove that the sequence $\{u^n\}_n$ is convergent and that its limit is independent of u^0. This limit is called the generalized solution of the quasi-variational inequality

$$(Aw, v - w) + j(w, v) - j(w, w) \geq (f, v - w)\,,\ \forall v \in V\,.$$

This definition is justified by the properties given by Theorem 4.17 below.

We start by recalling some definitions and results concerning generalized solutions.

Let $(V, (\cdot,\cdot), \|\cdot\|)$ be a real Hilbert space and $\mathscr{D}(P) \subset V$ be a linear dense subspace of V. For $f \in V$ given, we consider the problem

$$(P + \partial\varphi)(u) \ni f \tag{4.51}$$

where $P : \mathscr{D}(P) \to V$ is a nonlinear operator, $\varphi : \mathscr{D}(P) \to (-\infty, +\infty]$ is a function and $\partial\varphi$ represents the subdifferential of φ at u, i.e. the set

$$\partial\varphi(u) = \{w \in V\,;\, \varphi(v) - \varphi(u) \geq (w, v - u) \quad \forall v \in \mathscr{D}(P)\}\,.$$

We suppose that the following hypotheses hold.

$$\begin{cases} P \text{ is a potential, i.e. there exists a functional} \\ \beta : \mathscr{D}(P) \to \mathbb{R} \text{ such that } D\beta(u)v = (Pu, v) \quad \forall u \in \mathscr{D}(P)\,,\ \forall v \in V\,, \end{cases} \tag{4.52}$$

$$P \text{ is monotone}\,, \tag{4.53}$$

$$\varphi \text{ is proper l.s.c. convex,} \tag{4.54}$$

where $D\beta$ denotes the Gâteaux differential of β, i.e.

$$D\beta(u)v = \lim_{t\to 0} \frac{\beta(u + tv) - \beta(u)}{t} \quad \forall u, v \in \mathscr{D}(P)\,.$$

Remark 4.3. Under the hypotheses (4.52) and (4.53) one proves (see, for instance, [13]) that

$$\begin{aligned} \beta(v) &= \int_0^1 (P(sv), v)\,\mathrm{d}s + const. \quad \forall v \in \mathscr{D}(P)\,, \\ \beta(v + h) - \beta(v) &= \int_0^1 (P(v + sh), h)\,\mathrm{d}s \quad \forall v,\ h \in \mathscr{D}(P)\,. \end{aligned} \tag{4.55}$$

Definition 4.7. A classical solution for the Eq. (4.51) is an element $u \in \mathscr{D}(P)$ which verifies the variational inequality of the second kind

$$(Pu, v - u) + \varphi(v) - \varphi(u) \geq (f, v - u) \quad \forall v \in \mathscr{D}(P)\,.$$

We have the following characterization of the classical solution.

Proposition 4.7. *An element $u \in \mathscr{D}(P)$ is a classical solution for the equation (4.51) if and only if u minimizes on $\mathscr{D}(P)$ the functional*

$$F_f(v) = \beta(v) + \varphi(v) - (f, v)\,. \tag{4.56}$$

Proof. If $u \in \mathscr{D}(P)$ is a classical solution of (4.51), then, by using $(4.55)_2$ and the hypothesis (4.53), we get

$$F_f(v) - F_f(u) = \int_0^1 (P(u + s(v-u)), v-u)\,\mathrm{d}s + \varphi(v) - \varphi(u) - (f, v-u)$$

$$\geq \int_0^1 (P(u + s(v-u)), v-u)\,\mathrm{d}s - (Pu, v-u)$$

$$= \int_0^1 (P(u + s(v-u)) - Pu, v-u)\,\mathrm{d}s \geq 0 \quad \forall v \in \mathscr{D}(P)\,.$$

Conversely, if $u \in \mathscr{D}(P)$ is a minimizer of F_f on $\mathscr{D}(P)$, then

$$F_f(u) \leq F_f(u + t(v-u)) \quad \forall v \in \mathscr{D}(P)\,,\ \forall t \in (0,1)\,,$$

and so, by using the convexity of φ, we get

$$\frac{\beta(u + t(v-u)) - \beta(u)}{t} + \varphi(v) - \varphi(u) - (f, v-u) \geq 0 \quad \forall v \in \mathscr{D}(P)\,,\ \forall t \in (0,1)\,.$$

Thus, by passing to the limit with $t \to 0$, the assertion follows. □

In the sequel we consider the following more restrictive hypothesis on P:

$$\begin{cases} P \text{ is a strongly monotone operator: } \exists \alpha > 0 \text{ such that} \\ (Pu_1 - Pu_2, u_1 - u_2) \geq \alpha \|u_1 - u_2\|^2 \quad \forall u_1, u_2 \in \mathscr{D}(P)\,, \end{cases} \tag{4.57}$$

which, obviously, implies the uniqueness of the classical solution of (4.51).

Lemma 4.4. *Assume that (4.52) and (4.57) hold. Then β is strongly convex, namely*

$$(1-t)\beta(u) + t\beta(v) - \beta((1-t)u + tv) \geq \frac{\alpha}{2} t(1-t)\|u-v\|^2 \quad \forall t \in [0,1],\ \forall u, v \in \mathscr{D}(P).$$

Proof. From $(4.55)_2$ and (4.57), we obtain

$$
\begin{aligned}
&(1-t)\beta(u)+t\beta(v)-\beta((1-t)u+tv)\\
&=t(\beta(u+(v-u))-\beta(u))-(\beta((1-t)u+tv)-\beta(u))\\
&=\int_0^1 (P(u+s(v-u))-P(u+st(v-u)),(1-t)s(v-u))\frac{t}{(1-t)s}\,\mathrm{d}s\\
&\geq \alpha t(1-t)\|u-v\|^2\int_0^1 s\,\mathrm{d}s=\frac{\alpha}{2}t(1-t)\|u-v\|^2 .
\end{aligned}
$$

□

Lemma 4.5. *Suppose that the hypotheses* (4.52), (4.54), *and* (4.57) *are satisfied. Let F_f be the functional defined by* (4.56). *Then,*

(1) The functional F_f is lower bounded on $\mathscr{D}(P)$.
(2) Every minimizing sequence for F_f on $\mathscr{D}(P)$ is convergent in V.
(3) All the minimizing sequences for F_f on $\mathscr{D}(P)$ have the same limit in V.

Proof. (1) The hypotheses (4.54) on φ imply the existence of $\lambda \in V$ and $\mu \in \mathbb{R}$ such that $\varphi(v) \geq (\lambda, v)+\mu$, $\forall v \in \mathscr{D}(P)$. Therefore, by using $(4.55)_1$ and (4.57), we get

$$
\begin{aligned}
F_f(v)&=\int_0^1 (P(sv),v)\,\mathrm{d}s+\varphi(v)-(f,v)\\
&\geq \int_0^1 \big((P(0),v)+\alpha s\|v\|^2\big)\,\mathrm{d}s+(\lambda-f,v)+\mu\\
&\geq \frac{\alpha}{2}\|v\|^2-\|P(0)+\lambda-f\|\,\|v\|+\mu\\
&\geq -\frac{1}{2\alpha}\|P(0)+\lambda-f\|^2+\mu>-\infty \quad \forall v\in\mathscr{D}(P)
\end{aligned}
$$

and hence F_f is lower bounded on $\mathscr{D}(P)$.

(2) Let $\{v_n\}_n \subset \mathscr{D}(P)$ be a minimizing sequence for F_f, i.e.

$$\lim_{n\to\infty} F_f(v_n)=d=\inf_{v\in\mathscr{D}(P)} F_f(v) .$$

Using Lemma 4.4 and the convexity of φ, we get

$$
\begin{aligned}
&\frac{\alpha}{4}\|u_n-u_m\|^2 \leq \beta(u_n)+\beta(u_m)-2\beta\Big(\frac{u_n+u_m}{2}\Big)\\
&\leq F_f(u_n)+F_f(u_m)-2F_f\Big(\frac{u_n+u_m}{2}\Big)\leq (F_f(u_n)-d)+(F_f(u_m)-d),
\end{aligned}
$$

and so, by passing to the limit with $n, m\to\infty$, we find that $\{v_n\}_n$ is a Cauchy sequence.

(3) If $\{v_n\}_n$, $\{w_n\}_n \subset \mathscr{D}(P)$ are two minimizing sequences for F_f, then we have

$$\frac{\alpha}{4}\|u_n - w_n\|^2 \leq (F_f(u_n) - d) + (F_f(w_n) - d)$$

which obviously gives $\lim_{n\to\infty} v_n = \lim_{n\to\infty} w_n$ in V.

□

Lemma (4.5) justifies the following definition:

Definition 4.8. The limit in V of any minimizing sequence for the functional F_f on $\mathscr{D}(P)$ is called the generalized solution of the Eq. (4.51).

Proposition 4.8. *Under the hypotheses* (4.52), (4.54), *and* (4.57), *the following assertions hold.*

(1) The generalized solution of the equation (4.51) exists and it is unique.
(2) If the generalized solution of the equation (4.51) belongs to $\mathscr{D}(P)$, then it is the classical solution.
(3) The classical solution (if there exists) is the generalized solution.
(4) If $\mathscr{D}(P) = V$, then, the classical solution of the equation (4.51) exists and it is unique.

We now consider the following quasi-variational inequality:
Problem (P_4^a): Find $u \in \mathscr{D}(P)$ such that

$$(Pu, v-u) + j(u,v) - j(u,u) \geq (f, v-u) \quad \forall v \in \mathscr{D}(P) \tag{4.58}$$

where the operator $P : \mathscr{D}(P) \to V$ satisfies the hypotheses (4.52) and (4.57), and $j : V \times \mathscr{D}(P) \to (-\infty, +\infty]$ is a function such that

$$\forall u \in V,\ j(u,\cdot) : \mathscr{D}(P) \to (-\infty,\ +\infty] \text{ is proper l.s.c. convex} \tag{4.59}$$

$$\begin{cases} \exists k < \dfrac{\alpha}{2} \text{ such that } |j(u_1,v_1) + j(u_2,v_2) - j(u_1,v_2) - j(u_2,v_1)| \\ \qquad \leq k\|u_1 - u_2\|\,\|v_1 - v_2\|\,,\ \forall u_1,u_2,v_1,v_2 \in \mathscr{D}(P)\,. \end{cases} \tag{4.60}$$

Remark 4.4. The hypotheses (4.57) and (4.60) ensure the uniqueness of the classical solution of the inequality (4.58). Indeed, if we have two solutions $u_1, u_2 \in \mathscr{D}(P)$, then, by taking $v = u_2$, and, respectively, $v = u_1$ in the inequality (4.58) satisfied by u_1, and, respectively u_2, one obtains

$$\begin{aligned} &\alpha\|u_1 - u_2\|^2 \leq (Pu_1 - Pu_2, u_1 - u_2) \leq j(u_1,u_2) + j(u_2,u_1) \\ &-j(u_1,u_1) - j(u_2,u_2) \leq k\|u_1 - u_2\|^2 \end{aligned} \tag{4.61}$$

and so, as $k < \alpha$, one follows that $u_1 = u_2$.

In order to introduce the concept of generalized solution of the quasi-variational inequality (4.58), we denote by $S : V \to V$ the mapping which associates with every element $w \in V$ the generalized solution $u = Sw \in V$, which there exists and it is unique, of the following equation:

$$(P + \partial j_w)(u) \ni f\,, \tag{4.62}$$

where $j_w(v) = j(w, v)\,,\ \forall v \in \mathscr{D}(P)$.

Lemma 4.6. *The mapping S is a contraction.*

Proof. Let $w_1, w_2 \in V$ be arbitrarily chosen and let $Sw_1, Sw_2 \in V$ be the corresponding generalized solutions, i.e. Sw_i $(i = 1, 2)$ is the limit in V of any minimizing sequence of the function $F_f^i : \mathscr{D}(P) \to (-\infty, +\infty]$ defined by

$$F_f^i(v) = \beta(v) + j(w_i, v) - (f, v)\,,\ \forall v \in \mathscr{D}(P)\,.$$

If $\{w_n^1\}_n, \{w_n^2\}_n \subset \mathscr{D}(P)$ are two minimizing sequences for F_f^1, and, respectively, for F_f^2, then $w_n^i \to Sw^i$ $(i = 1, 2)$ strongly in V when $n \to \infty$.

Using Lemma 4.4 and the convexity of $j(w_i, \cdot)$, we easily deduce that the functional F_f^i is strongly convex with the same constant $\dfrac{\alpha}{2}$ as β, i.e.

$$tF_f^i(v) + (1-t)F_f^i(u) - F_f^i(tv + (1-t)u) \ge \frac{\alpha}{2}t(1-t)\|u - v\|^2$$

$$i = 1, 2\,,\ \forall t \in [0, 1]\,,\ \forall u, v \in \mathscr{D}(P)\,.$$

Taking $v = w_n^i$ and $u = w_n^{3-i}$ $(i = 1, 2)$ in the above relation, we get:

$$\begin{aligned} &\frac{\alpha}{2}t(1-t)\|w_n^1 - w_n^2\|^2 \le tF_f^i(w_n^i) + (1-t)F_f^i(w_n^{3-i}) \\ &-F_f^i(tw_n^i + (1-t)w_n^{3-i}) \le tF_f^i(w_n^i) + (1-t)F_f^i(w_n^{3-i}) - d_i \qquad i = 1, 2\,, \end{aligned} \tag{4.63}$$

where

$$d_i = \inf_{v \in \mathscr{D}(P)} F_f^i(v) \le F_f^i(tw_n^i + (1-t)w_n^{3-i})\,.$$

By adding the inequalities (4.63) for $i = 1, 2$, we obtain:

$$\begin{aligned} \alpha t(1-t)\|w_n^1 - w_n^2\|^2 &\le t\left((F_f^1(w_n^1) - d_1) + (F_f^2(w_n^2) - d_2)\right) \\ &+(1-t)\left(F_f^1(w_n^2) + F_f^2(w_n^1) - d_1 - d_2\right)\,. \end{aligned} \tag{4.64}$$

Now, from the definition of F_f^i, we have:

$$F_f^i(w_n^{3-i}) = F_f^{3-i}(w_n^{3-i}) + j(w_i, w_n^{3-i}) - j(w_{3-i}, w_n^{3-i}) \qquad i = 1, 2$$

and, by using (4.60), this gives:

$$\begin{aligned} & F_f^1(w_n^2) + F_f^2(w_n^1) - d_1 - d_2 \leq (F_f^1(w_n^1) - d_1) + (F_f^2(w_n^2) - d_2) \\ & \quad + k\|w_1 - w_2\| \, \|w_n^1 - w_n^2\| \, . \end{aligned} \tag{4.65}$$

On the other hand, from the definition of the generalized solution Sw_i, for $i = 1, 2$, we have:

$$\lim_{n\to\infty} \|w_n^1 - w_n^2\| = \|Sw_1 - Sw_2\| \tag{4.66}$$

and

$$\lim_{n\to\infty} F_f^i(w_n^i) = d_i \, . \tag{4.67}$$

By passing to the limit, with $n \to \infty$, in (4.64) and using (4.65)–(4.67), one deduces

$$\alpha t \|Sw_1 - Sw_2\| \leq k\|w_1 - w_2\| \, ,$$

hence, by taking $t = \dfrac{1}{2}$, one obtains:

$$\|Sw_1 - Sw_2\| \leq q\|w_1 - w_2\| \tag{4.68}$$

with $q = \dfrac{2k}{\alpha} < 1$. This concludes the proof. □

Therefore, by Banach fixed point Theorem 4.7, it results that there exists a unique fixed point, denoted by u, of the mapping S.

Lemma 4.6 suggests the definition of the following sequence: for $u^0 \in V$ given, we put $u^n = Su^{n-1}$, i.e. u^n is the generalized solution of the equation:

$$(P + \partial j_n)(v) \ni f \, , \tag{4.69}$$

where $j_n(v) = j(u^{n-1}, v)\, , \ \forall v \in \mathscr{D}(P)$.

Remark 4.5. The sequence $\{u^n\}_n$, defined above, is convergent. Moreover, for any u^0, every sequence $\{u^n\}_n$ has the same limit, namely the unique fixed point u of the mapping S. Indeed, from (4.68), one has

$$\|u^n - u\| = \|Su^{n-1} - Su\| \leq \frac{2k}{\alpha}\|u^{n-1} - u\| \leq \ldots \leq \left(\frac{2k}{\alpha}\right)^n \|u^0 - u\| \, .$$

Definition 4.9. The limit u in V of the sequence $\{u^n\}_n$ of the generalized solutions of the equations (4.69) is called the generalized solution of the quasi-variational inequality (4.58).

This definition is justified by the following result:

Theorem 4.17. *Suppose that the hypotheses* (4.52), (4.57), (4.59), *and* (4.60) *hold. Then, we have:*

(1) The generalized solution of the quasi-variational inequality (4.58) exists and it is unique.
(2) If the generalized solution of the quasi-variational inequality (4.58) belongs to $\mathscr{D}(P)$, then it is also the classical solution.
(3) If the quasi-variational inequality (4.58) has a classical solution, then it is the generalized solution.

Proof. (1) It follows from Definition 4.9 and Remark 4.5.

(2) If $u \in \mathscr{D}(P)$ is the generalized solution of the quasi-variational inequality (4.58), then, from Remark 4.5, we have $u = Su$ and hence, the generalized solution Su of the Eq. (4.51), for $\varphi(v) = j(u, v) \quad \forall v \in \mathscr{D}(P)$, belongs to $\mathscr{D}(P)$. By applying Proposition 4.8(2), it follows that Su is also the classical solution, i.e.

$$(P(Su), v - Su) + j(u, v) - j(u, Su) \geq (f, v - Su) \quad \forall v \in \mathscr{D}(P) .$$

and so, $u = Su$ is the classical solution of (4.58).

(3) If u is the classical solution of the quasi-variational inequality (4.58), then u is the classical solution of the equation

$$(P + \partial j_u)(v) \ni f \tag{4.70}$$

where $j_u(v) = j(u, v)$, $\forall v \in \mathscr{D}(P)$. By applying Proposition 4.8(3) one obtains that u is the generalized solution of the equation (4.70). On the other hand, from the definition of the mapping S, the unique generalized solution of the equation (4.70) is Su. We conclude that $u = Su$ which, again by Remark 4.5, implies that u is the generalized solution of the quasi-variational inequality (4.58).

□

Corollary 4.4. *If $\mathscr{D}(P) = V$, then, there exists a unique classical solution of the quasi-variational inequality (4.58).*

4.2.3 Example

We consider a frictional contact problem for the operator of Hencky–Nadai theory (i.e., the theory of small elastic-plastic deformations (see, e.g., [12])). By applying the variational method developed in the above section, we prove the existence and the uniqueness of the generalized solution for this problem.

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be an open bounded set occupied by an elastic-plastic body in reference configuration. Let us denote by Γ, supposed sufficiently smooth, the boundary of Ω which is decomposed into three open disjoint parts Γ_0, Γ_1, Γ_2 such that $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_2$ and $\text{meas}(\Gamma_0) > 0$. On Γ_2 the body is in contact with a rigid foundation. We suppose that the foundation does not allow a detachment of Ω on Γ_2, and so, the normal displacement is zero while the tangential displacement is a displacement with friction. On Γ_0 one supposes that the body is clamped. The body is subject to the action of volume forces $\boldsymbol{f}$, given in Ω, and surface forces $\boldsymbol{g}$, given on Γ_1.

The classical formulation of this boundary problem is:

Problem ($\mathscr{P}_{ep}$): Find a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ such that

$$\begin{cases} -\text{div}\, \boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega\,, \\ \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0\,, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_1\,, \\ u_\nu = 0 \quad \text{on } \Gamma_2\,, \\ |\boldsymbol{\sigma}_\tau| \le \mu|\sigma_\nu^*| \text{ and} \\ \qquad |\boldsymbol{\sigma}_\tau| < \mu|\sigma_\nu^*| \;\Rightarrow\; \boldsymbol{u}_\tau = 0 \qquad\qquad \text{on } \Gamma_2 \\ \qquad |\boldsymbol{\sigma}_\tau| = \mu|\sigma_\nu^*| \;\Rightarrow\; \exists \lambda \ge 0\,,\; \boldsymbol{u}_\tau = -\lambda \boldsymbol{\sigma}_\tau \end{cases} \tag{4.71}$$

where μ is the coefficient of friction and $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}) = (\sigma_{ij})$ is the stress tensor related by the strain tensor $\boldsymbol{\epsilon} = (\epsilon_{ij})$ by the nonlinear Hooke's law:

$$\sigma_{ij}(\boldsymbol{u}) = 2\varphi(\gamma(\boldsymbol{u}))\epsilon_{ij}(\boldsymbol{u}) + \left(k - \frac{2}{3}\varphi(\gamma(\boldsymbol{u}))\right)\epsilon_{ij}(\boldsymbol{u})\delta_{ij} \tag{4.72}$$

with $k = \lambda + \dfrac{2}{3}\theta$, λ and θ being the Lamé coefficients of the material. We denoted by φ a given function and by γ the following form:

$$\begin{aligned} &\gamma(\boldsymbol{u}) = \bar{\gamma}(\boldsymbol{u}, \boldsymbol{u})\,, \\ &\bar{\gamma}(\boldsymbol{u}, \boldsymbol{v}) = 2\epsilon_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v}) - \frac{2}{3}\epsilon_{ii}(\boldsymbol{u})\epsilon_{jj}(\boldsymbol{v})\,. \end{aligned} \tag{4.73}$$

In (4.71), σ_ν^* represents a regularization for σ_ν (for more details, see Sect. 8.1).

In order to obtain a variational formulation of the problem (4.71), we make the following hypotheses:

$$\begin{cases} \sigma_\nu^*(\boldsymbol{u}) \in C^1(\Gamma_2) \quad \forall \boldsymbol{u} \in (H^1(\Omega))^d \,, \\ \mu \in L^\infty(\Gamma_2) \text{ such that } \mu \geq 0 \text{ a.e. on } \Gamma_2 \,, \\ \varphi \in C^2[0, +\infty) \,, \\ \boldsymbol{f} \in (L^2(\Omega))^d \,, \; \boldsymbol{g} \in (L^2(\Gamma_1))^d \,. \end{cases} \tag{4.74}$$

By using a similar technique as in [32] or in Sect. 8.2, one obtains that the variational formulation of the problem (4.71) is the following quasi-variational inequality:

$$\begin{cases} \boldsymbol{u} \in \mathscr{D}(P) \,, \\ (P\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) \geq \mathbf{L}(\boldsymbol{v} - \boldsymbol{u}) \quad \forall \boldsymbol{v} \in \mathscr{D}(P) \end{cases} \tag{4.75}$$

where

$$\begin{aligned} &\boldsymbol{V} = \{\boldsymbol{v} \in (H^1(\Omega))^d \,;\; \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_0 \,,\; v_\nu = 0 \text{ a.e. on } \Gamma_2\} \,, \\ &\mathscr{D}(P) = \boldsymbol{V} \cap (C^2(\Omega))^d \cap (C(\overline{\Omega}))^d \,, \\ &(P\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \sigma_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})\,\mathrm{d}x \quad \forall \boldsymbol{u} \in \mathscr{D}(P) \,,\; \forall \boldsymbol{v} \in \boldsymbol{V} \,, \\ &j(\boldsymbol{u}, \boldsymbol{v}) = \begin{cases} \displaystyle\int_{\Gamma_2} \mu |\sigma_\nu^*(\boldsymbol{u})|\, |\boldsymbol{v}_\tau|\,\mathrm{d}s \quad \forall \boldsymbol{u} \in \boldsymbol{V} \,,\; \forall \boldsymbol{v} \in \mathscr{D}(P) \,, \\ +\infty \quad \forall \boldsymbol{u} \in \boldsymbol{V} \,,\; \forall \boldsymbol{v} \in V \setminus \mathscr{D}(P) \,, \end{cases} \\ &\mathbf{L}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g} \cdot \boldsymbol{v}\,\mathrm{d}s \quad \forall \boldsymbol{v} \in \boldsymbol{V} \,. \end{aligned}$$

Remark 4.6. From Korn's inequality (8.29), it follows that a norm on V, equivalent to the norm $\|\cdot\|_1$, is defined by

$$\|\boldsymbol{v}\| = \int_\Omega \epsilon_{ij}(\boldsymbol{v})\epsilon_{ij}(\boldsymbol{v})\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V} \,.$$

Theorem 4.18. *Suppose that the hypotheses (4.74) are satisfied. In addition, we assume that there exist the constants φ_0 and φ_1 such that:*

$$0 < \varphi_0 \leq \varphi(s) \leq \frac{3}{2}k \quad \forall s \geq 0 \,, \tag{4.76}$$

$$\varphi(s) + 2s\varphi'(s) \geq \varphi_1 > 0 \quad \forall s \geq 0 \,. \tag{4.77}$$

Then, there exists a constant $\mu_1 > 0$ such that for any $\mu \in L^\infty(\Gamma_2)$ with $\mu \geq 0$ a.e. on Γ_2 and $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$, the problem (4.75) has a unique generalized solution $\boldsymbol{u} \in \boldsymbol{V}$. Moreover, if the generalized solution $\boldsymbol{u}$ belongs to $\mathscr{D}(P)$, then $\boldsymbol{u}$ is the classical solution of the problem (4.71).

Proof. We shall show that the hypotheses of Theorem 4.17 are satisfied.

Let $\beta : \mathscr{D}(P) \to \mathbb{R}$ be the functional defined by:

$$\beta(\boldsymbol{v}) = \int_\Omega \frac{1}{2}\left(k\epsilon_{ii}^2(\boldsymbol{v}) + \int_0^{\gamma(\boldsymbol{v})} \varphi(s)\,\mathrm{d}s\right)\mathrm{d}x\,.$$

By using the relations (4.72) and (4.73), we obtain the Gâteaux derivative of β:

$$\begin{aligned}
D\beta(\boldsymbol{u})\cdot\boldsymbol{v} &= \lim_{t\to 0}\frac{\beta(\boldsymbol{u}+t\boldsymbol{v})-\beta(\boldsymbol{u})}{t} = \int_\Omega \left[k\epsilon_{ii}(\boldsymbol{u})\epsilon_{jj}(\boldsymbol{v}) + \varphi(\gamma(\boldsymbol{u}))\bar{\gamma}(\boldsymbol{u},\boldsymbol{v})\right]\mathrm{d}x\\
&= \int_\Omega \left[k\epsilon_{ii}(\boldsymbol{u})\epsilon_{jj}(\boldsymbol{v}) + \varphi(\gamma(\boldsymbol{u}))\left(-\frac{2}{3}\epsilon_{ii}(\boldsymbol{u})\epsilon_{jj}(\boldsymbol{v}) + 2\epsilon_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})\right)\right]\mathrm{d}x\\
&= \int_\Omega \left[\left(k - \frac{2}{3}\varphi(\gamma(\boldsymbol{u}))\right)\epsilon_{ii}(\boldsymbol{u})\epsilon_{jj}(\boldsymbol{v}) + 2\varphi(\gamma(\boldsymbol{u}))\epsilon_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})\right]\mathrm{d}x\\
&= (P\boldsymbol{u},\boldsymbol{v}) \quad \forall \boldsymbol{u}\,,\ \boldsymbol{v} \in \mathscr{D}(P)\,,
\end{aligned} \tag{4.78}$$

and hence, the operator P is potential.

In order to prove that the operator P is strongly monotone, by using (4.78), we get:

$$\begin{aligned}
D^2\beta(\boldsymbol{u});\boldsymbol{w}\cdot\boldsymbol{v} &= \lim_{t\to 0}\frac{D\beta(\boldsymbol{u}+t\boldsymbol{w}) - D\beta(\boldsymbol{u})}{t}\cdot\boldsymbol{v}\\
&= \int_\Omega k\lim_{t\to 0}\frac{\epsilon_{ii}(\boldsymbol{u}+t\boldsymbol{w}) - \epsilon_{ii}(\boldsymbol{u})}{t}\epsilon_{jj}(\boldsymbol{v})\,\mathrm{d}x\\
&\quad + \int_\Omega \lim_{t\to 0}\frac{\varphi(\gamma(\boldsymbol{u}+t\boldsymbol{w})) - \varphi(\gamma(\boldsymbol{u}))}{t}\left(2\epsilon_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v}) - \frac{2}{3}\epsilon_{ii}(\boldsymbol{u})\epsilon_{jj}(\boldsymbol{v})\right)\mathrm{d}x\\
&\quad + \int_\Omega \lim_{t\to 0}\varphi(\gamma(\boldsymbol{u}+t\boldsymbol{w}))\left(2\epsilon_{ij}(\boldsymbol{w})\epsilon_{ij}(\boldsymbol{v}) - \frac{2}{3}\epsilon_{ii}(\boldsymbol{w})\epsilon_{jj}(\boldsymbol{v})\right)\mathrm{d}x\\
&= \int_\Omega (k\epsilon_{ii}(\boldsymbol{w})\epsilon_{jj}(\boldsymbol{v}) + 2\varphi'(\gamma(\boldsymbol{u}))\gamma(\boldsymbol{u},\boldsymbol{w})\gamma(\boldsymbol{u},\boldsymbol{v}) + \varphi(\gamma(\boldsymbol{u}))\gamma(\boldsymbol{w},\boldsymbol{v}))\,\mathrm{d}x\,,
\end{aligned} \tag{4.79}$$

hence

$$
\begin{aligned}
(P(\boldsymbol{u}+\boldsymbol{v}) - P\boldsymbol{u}, \boldsymbol{v}) &= D\beta(\boldsymbol{u}+\boldsymbol{v})\cdot\boldsymbol{v} - D\beta(\boldsymbol{u})\cdot\boldsymbol{v} = \int_0^1 D^2\beta(\boldsymbol{u}+t\boldsymbol{v});\boldsymbol{v}\cdot\boldsymbol{v}\,\mathrm{d}t \\
&= \int_0^1 \left(\int_\Omega k\epsilon_{ii}^2(\boldsymbol{v}) + 2\varphi'(\gamma(\boldsymbol{u}+t\boldsymbol{v}))\gamma^2(\boldsymbol{u}+t\boldsymbol{v},\boldsymbol{v}) + \varphi(\gamma(\boldsymbol{u}+t\boldsymbol{v}))\gamma(\boldsymbol{v})\,dx \right) \mathrm{d}t \\
&= \int_0^1 \left(\int_{\Omega_+} E(\boldsymbol{x})\,dx + \int_{\Omega_-} E(\boldsymbol{x})\,\mathrm{d}x \right) \mathrm{d}t
\end{aligned}
\tag{4.80}
$$

where

$$
E(\boldsymbol{x}) = (k\epsilon_{ii}^2(\boldsymbol{v}) + 2\varphi'(\gamma(\boldsymbol{u}+t\boldsymbol{v}))\gamma^2(\boldsymbol{u}+t\boldsymbol{v},\boldsymbol{v}) + \varphi(\gamma(\boldsymbol{u}+t\boldsymbol{v}))\gamma(\boldsymbol{v}))(\boldsymbol{x}),
$$

$$
\Omega_+ = \{\boldsymbol{x}\in\Omega\,;\ \varphi'(\gamma(\boldsymbol{u}))(\boldsymbol{x}) \geq 0\},
$$

$$
\Omega_- = \{\boldsymbol{x}\in\Omega\,;\ \varphi'(\gamma(\boldsymbol{u}))(\boldsymbol{x}) < 0\}.
$$

For $\boldsymbol{x}\in\Omega_+$, from (4.76) and the definition (4.73) of γ, we have

$$
E(\boldsymbol{x}) \geq \frac{2}{3}\varphi_0\epsilon_{ii}(\boldsymbol{v}) + \varphi_0\gamma(\boldsymbol{v}) = 2\varphi_0\epsilon_{ij}(\boldsymbol{v})\epsilon_{ij}(\boldsymbol{v}),
$$

and so,

$$
\int_0^1\int_{\Omega_+} E(\boldsymbol{x})\,\mathrm{d}x\,\mathrm{d}t \geq 2\varphi_0\int_{\Omega_+}\epsilon_{ij}(\boldsymbol{v})\epsilon_{ij}(\boldsymbol{v})\,\mathrm{d}x\,. \tag{4.81}
$$

For $\boldsymbol{x}\in\Omega_-$ we have

$$
E(\boldsymbol{x}) \geq \frac{2}{3}\varphi_0\epsilon_{ii}^2(\boldsymbol{v}) - 2|\varphi'(\gamma(\boldsymbol{u}+t\boldsymbol{v}))|\gamma^2(\boldsymbol{u}+t\boldsymbol{v},\boldsymbol{v}) + \varphi(\gamma(\boldsymbol{u}+t\boldsymbol{v}))\gamma(\boldsymbol{v})
$$

which, by the Schwartz inequality:

$$
\gamma^2(\boldsymbol{u}+t\boldsymbol{v},\boldsymbol{v}) \leq \gamma(\boldsymbol{u}+t\boldsymbol{v})\gamma(\boldsymbol{v})
$$

and the condition (4.77) implies the relations

$$
\begin{aligned}
E(\boldsymbol{x}) &\geq \frac{2}{3}\varphi_0\epsilon_{ii}^2(\boldsymbol{v}) - 2|\varphi'(\gamma(\boldsymbol{u}+t\boldsymbol{v}))|\gamma(\boldsymbol{u}+t\boldsymbol{v})\gamma(\boldsymbol{v}) + \varphi(\gamma(\boldsymbol{u}+t\boldsymbol{v}))\gamma(\boldsymbol{v}) \\
&= \frac{2}{3}\varphi_0\epsilon_{ii}(\boldsymbol{v}) = \big(\varphi(\gamma(\boldsymbol{u}+t\boldsymbol{v})) + 2\varphi'(\gamma(\boldsymbol{u}+t\boldsymbol{v},\boldsymbol{v})\gamma(\boldsymbol{u}+t\boldsymbol{v})\big)\,\gamma(\boldsymbol{v})
\end{aligned}
$$

$$\geq \frac{2}{3}\varphi_0\epsilon_{ii}(\boldsymbol{v}) + \varphi_1\gamma(\boldsymbol{v}) = \frac{2}{3}\varphi_0\epsilon_{ii}(\boldsymbol{v}) + \varphi_1\left(2\epsilon_{ij}^2(\boldsymbol{v}) - \frac{2}{3}\epsilon_{ii}^2(\boldsymbol{v})\right)$$
$$= \frac{2}{3}(\varphi_0 - \varphi_1)\epsilon_{ii}^2(\boldsymbol{v}) + 2\varphi_1\epsilon_{ij}(\boldsymbol{v})\epsilon_{ij}(\boldsymbol{v})\,.$$

We may suppose that $\varphi_0 \geq \varphi_1$, and thus

$$\int_0^1\int_{\Omega_-} E(\boldsymbol{x})\,\mathrm{d}x\,\mathrm{d}t \geq 2\varphi_1\int_{\Omega_-}\epsilon_{ij}(\boldsymbol{v})\epsilon_{ij}(\boldsymbol{v})\,\mathrm{d}x\,. \tag{4.82}$$

Therefore, by using (4.81) and (4.82) in (4.80), it results

$$(P(\boldsymbol{u}+\boldsymbol{v}) - P\boldsymbol{u}, \boldsymbol{v}) \geq 2\varphi_1\int_{\Omega}\epsilon_{ij}(\boldsymbol{v})\epsilon_{ij}(\boldsymbol{v})\,\mathrm{d}x\,,$$

hence, by Remark 4.6, it follows that the operator P satisfies the relation (4.57), that is P is strongly monotone.

Finally, it is easy to see that the functional j satisfies the hypothesis (4.59).

Next, we remark that we can write

$$\begin{aligned}
&|j(\boldsymbol{u}_1,\boldsymbol{v}_2) + j(\boldsymbol{u}_2,\boldsymbol{v}_1) - j(\boldsymbol{u}_1,\boldsymbol{v}_1) - j(\boldsymbol{u}_2,\boldsymbol{v}_2)|\\
&\leq \int_{\Gamma_2}\mu\,\big|\,|\sigma_\nu^*(\boldsymbol{u}_1)| - |\sigma_\nu^*(\boldsymbol{u}_2)|\,\big|\;\big|\,|\boldsymbol{v}_{1\tau}| - |\boldsymbol{v}_{2\tau}|\,\big|\,\mathrm{d}s\\
&\leq \int_{\Gamma_2}\mu|\sigma_\nu^*(\boldsymbol{u}_1) - \sigma_\nu^*(\boldsymbol{u}_2)|\;|\boldsymbol{v}_{1\tau} - \boldsymbol{v}_{2\tau}|\,\mathrm{d}s\\
&\leq C\|\mu\|_{L^\infty(\Gamma_2)}\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|\;\|\boldsymbol{u}_1 - \boldsymbol{u}_2\| \quad \forall \boldsymbol{u}_1,\,\boldsymbol{u}_2\,\boldsymbol{v}_1,\,\boldsymbol{v}_2 \in \mathscr{D}(P)\,.
\end{aligned}$$

If we choose $\mu_1 < \dfrac{\varphi_1}{C}$, then, for any $\|\mu\|_{L^\infty(\Gamma_2)} < \mu_1$, we obtain the relation (4.60) satisfied with $k = C\|\mu\|_{L^\infty(\Gamma_2)} < \varphi_1$.

Therefore, by applying Theorem 4.17, we deduce the assertion. □

4.3 Implicit Evolutionary Quasi-variational Inequalities

This section is concerned with the mathematical analysis, following the work [8], of a class of abstract implicit evolutionary variational inequalities which constitutes a generalization of variational inequalities related to various quasistatic contact problems and of some parabolic variational inequalities of the second kind (see, for instance, [11, 19, 20, 34]).

As usual for quasistatic and dynamic problems, by an implicit time discretization scheme, the incremental formulation of the considered problem is obtained. We prove that the incremental formulation has a unique solution and some a priori estimates are obtained. Next, by using the incremental solution, we construct a sequence of piecewise constant functions which verify the variational formulation for all $t \in [0, T]$, T being the time interval considered. The existence of a solution of the implicit evolutionary problem is obtained by proving that some subsequences of the above sequence have a weak limit which verifies this problem.

The results obtained here will be applied, in Chap. 9, to quasistatic contact problems with nonlocal friction in linear elasticity. We mention that these results can be used in the study of a large variety of contact conditions, as, for instance, the unilateral or bilateral contact with nonlocal friction between two elastic bodies, the frictional contact with normal compliance or the corresponding frictionless cases.

We denote by $(V, (\cdot, \cdot))$ a real Hilbert space with the associated norm $\|\cdot\|$. Let K be a convex closed cone contained in V with its vertex at 0, i.e. $\rho v \in K$, $\forall v \in K$, $\forall \rho \geq 0$.

We mention that, if $C \subset V$ is a cone of vertex at 0, then C is a convex cone if and only if $u + v \in C$, $\forall u, v \in C$. Indeed, if C is a convex cone, then $\dfrac{u+v}{2} \in C$, $\forall u, v \in C$. Since the vertex of C is at 0, it results that $2\dfrac{u+v}{2} = u + v \in C$, $\forall u, v \in C$. Conversely, if $u, v \in C$, then, C being a cone with its vertex at 0, it follows that $-v \in C$, and thus, $u - v \in C$. Then $\rho(u - v) \in C$, $\forall \rho \geq 0$, and so, $\rho(u - v) + v \in C$, $\forall \rho \geq 0$.

We consider a bilinear symmetric form $a : V \times V \to \mathbb{R}$. We suppose that the form $a(\cdot, \cdot)$ is V-elliptic (or, coercive) and continuous, that is

$$\begin{cases} \text{there exists } \alpha > 0 \text{ such that } a(v, v) \geq \alpha \|v\|^2 & \forall v \in V, \\ \text{there exists } M > 0 \text{ such that } a(u, v) \leq M \|u\| \, \|v\| & \forall u, v \in V. \end{cases} \tag{4.83}$$

We also consider a family $\{K(g)\}_{g \in V}$ of nonempty convex subsets of K such that $0 \in K(0)$. We denote $D_K = \{(g, v)/g \in V,\ v \in K(g)\} \subset V \times K$.

In the sequel we assume that the set D_K is strongly–weakly closed in $V \times V$ in the following sense:

$$\left.\begin{array}{l} \forall (g_n, v_n) \in D_K \text{ s.t.} \\ g_n \to g \ \text{ strongly in } V \\ v_n \rightharpoonup v \text{ weakly in } V \end{array}\right\} \implies (g, v) \in D_K\,. \tag{4.84}$$

Remark 4.7. Under the above hypothesis, it follows that, for any $g \in V$, the set $K(g)$ is weakly closed in V.

We denote by $(H, (\cdot,\cdot)_H)$ a real Hilbert space with its norm $\|\cdot\|_H$ and we consider the operator $\beta : D_K \to H$ which satisfies the hypotheses:
β is strongly–weakly continuous, i.e.

$$\left.\begin{array}{l}\forall (g_n, v_n) \in D_K \text{ s.t.}\\ g_n \to g \text{ strongly in } V\\ v_n \rightharpoonup v \text{ weakly in } V\end{array}\right\} \Longrightarrow \beta(g_n, v_n) \to \beta(g, v) \text{ strongly in } H\,, \tag{4.85}$$

and

$$\|\beta(g_1, v_1) - \beta(g_2, v_2)\|_H \le k_1(\|g_1 - g_2\| + \|v_1 - v_2\|)\ \forall (g_1, v_1), (g_2, v_2) \in D_K\,, \tag{4.86}$$

with k_1 a positive constant.

Let $j : D_K \times V \to \mathbb{R}$ be a functional such that for all $(g, v) \in D_K$, the functional $j(g, v, \cdot) : V \to \mathbb{R}$ is sub-additive and positively homogeneous, i.e.

$$j(g, v, w_1 + w_2) \le j(g, v, w_1) + j(g, v, w_2) \qquad \forall\, w_1, w_2 \in V, \tag{4.87}$$

$$j(g, v, \lambda w) = \lambda j(g, v, w) \qquad \forall w \in V,\ \forall \lambda \ge 0\,. \tag{4.88}$$

In addition, we suppose that

$$j(0, 0, w) = 0 \qquad \forall w \in V, \tag{4.89}$$

and

$$\begin{aligned}&|j(g_1, v_1, w_2) + j(g_2, v_2, w_1) - j(g_1, v_1, w_1) - j(g_2, v_2, w_2)|\\ &\le k_2(\|g_1 - g_2\| + \|\beta(g_1, v_1) - \beta(g_2, v_2)\|_H)\|w_1 - w_2\|\\ &\qquad \forall\, (g_1, v_1), (g_2, v_2) \in D_K,\ \forall w_1, w_2 \in V,\end{aligned} \tag{4.90}$$

where k_2 is a positive constant.

Lemma 4.7. *The functional j has the following properties:*

(i)

$$j(g, v, 0) = 0 \quad \forall (g, v) \in D_K\,, \tag{4.91}$$

(ii)

$$\forall (g, v) \in D_K \ \textit{the functional}\ j(g, v, \cdot) : V \to \mathbb{R}\ \textit{is convex}, \tag{4.92}$$

(iii) $\forall(g,v) \in D_K$*, the functional* $j(g,v,\cdot) : V \to \mathbb{R}$ *is Lipschitz continuous. More precisely, we have*

$$\begin{aligned} |j(g,v,w_1) - j(g,v,w_2)| &\le |j(g,v,w_1-w_2)| \\ &\le ((1+k_1)k_2\|g\| + k_1k_2\|v\|)\,\|w_1-w_2\| \qquad \forall w_1,w_2 \in V, \end{aligned} \tag{4.93}$$

(iv) The functional j *has the following continuity properties:*

$$\left.\begin{array}{l} \forall (g_n,v_n) \in D_K \ s.t. \\ g_n \to g \ \text{strongly in } V, \\ v_n \rightharpoonup v \ \text{weakly in } V, \end{array}\right\} \Rightarrow \lim_{n\to\infty} j(g_n,v_n,w) = j(g,v,w) \quad \forall w \in V, \tag{4.94}$$

$$\left.\begin{array}{l} \forall (g_n,v_n) \in D_K,\ \forall w_n \in V \ s.t. \\ g_n \to g \ \text{strongly in } V, \\ v_n \rightharpoonup v \ \text{weakly in } V, \\ w_n \rightharpoonup w \ \text{weakly in } V, \end{array}\right\} \Rightarrow \liminf_{n\to\infty} j(g_n,v_n,w_n) \ge j(g,v,w). \tag{4.95}$$

Proof. It is easy to see that hypothesis (4.88) implies (i) and the hypotheses (4.87) and (4.88) imply (ii).

In order to prove (iii), we first note that the hypothesis (4.87) implies

$$j(g,v,w_1) - j(g,v,w_2) = j(g,v,w_1-w_2+w_2) - j(g,v,w_2) \le j(g,v,w_1-w_2),$$

and hence, by taking in (4.90), $g_1 = g, g_2 = 0, v_1 = v, v_2 = 0, w_1 = w_1 - w_2, w_2 = 0$ and using (4.86), we obtain (4.93).

Now, if we take $g_1 = g_n,\ g_2 = g,\ v_1 = v_n,\ v_2 = v,\ w_1 = 0,\ w_2 = w$ in (4.90), it follows

$$|j(g_n,v_n,w_n) - j(g,v,w)| \le k_2(\|g_n - g\| + \|\beta(g_n,v_n) - \beta(g,v)\|_H)\|w\|\,,$$

hence, by taking into account (4.85), one obtains (4.94).

Finally, from (ii) and (iii), it results that, for every $(g,v) \in D_K$, the functional $j(g,v,\cdot)$ is weakly l.s.c. On the other hand, from the hypothesis (4.90) written for $g_1 = g_n,\ g_2 = g,\ v_1 = v_n,\ v_2 = v,\ w_1 = 0,\ w_2 = w_n$, by using (4.85) and taking into account the boundedness of the sequence $\{w_n\}_n$, it follows

$$\lim_{n\to\infty} (j(g_n,v_n,w_n) - j(g,v,w_n)) = 0\,.$$

Concluding, we obtain

$$\begin{aligned} &\liminf_{n\to\infty} j(g_n,v_n,w_n) \ge \lim_{n\to\infty} (j(g_n,v_n,w_n) - j(g,v,w_n)) + \liminf_{n\to\infty} j(g,v,w_n) \\ &\ge j(g,v,w)\,. \end{aligned}$$

□

We also introduce a functional $b : D_K \times V \to \mathbb{R}$ which satisfies the following conditions

$$\forall (g, v) \in D_K,\ \ b(g, v, \cdot) \text{ is linear on } V, \tag{4.96}$$

$$\left.\begin{array}{l} \forall (g_n, v_n) \in D_K \quad \forall w_n \in V \text{ s.t.} \\ g_n \to g \text{ strongly in } V, \\ v_n \rightharpoonup v \text{ weakly in } V, \\ w_n \to w \text{ strongly in } V \end{array}\right\} \Rightarrow \lim_{n\to\infty} b(g_n, v_n, w_n) = b(g, v, w), \tag{4.97}$$

$$\begin{array}{l} |b(g_1, v_1, w) - b(g_2, v_2, w)| \le k_3(\|g_1 - g_2\| + \|v_1 - v_2\|)\|w\| \\ \qquad \forall\, (g_1, v_1), (g_2, v_2) \in D_K,\ \forall w \in V\,, \end{array} \tag{4.98}$$

where k_3 is a positive constant.

For any $g \in V$, $d \in K$, we define the mapping $S_{g,d} : K(g) \to K$ by $S_{g,d}(w) = u_w$, $\forall w \in K(g)$ where u_w is the unique solution, according to Theorem 4.12, of the variational inequality

$$\begin{cases} u_w \in K \\ a(u_w, v - u_w) + j(g, w, v - d) - j(g, w, u_w - d) \ge 0 \qquad \forall v \in K\,. \end{cases} \tag{4.99}$$

We shall suppose that, for all $g \in V$, $d \in K$, the set $K(g)$ is stable under the mapping $S_{g,d} : K(g) \to K$, that is

$$S_{g,d}(K(g)) \subset K(g). \tag{4.100}$$

In the following we suppose that the constants k_1, k_2 and α satisfy the relation

$$k_1 k_2 < \alpha. \tag{4.101}$$

Lemma 4.8. *For all $g \in V$, $d \in K$, there exists a unique $u \in K(g)$ such that $u = S_{g,d}(u)$, i.e.*

$$\begin{cases} u \in K(g) \\ a(u, v - u) + j(g, u, v - d) - j(g, u, u - d) \ge 0 \quad \forall v \in K\,. \end{cases} \tag{4.102}$$

Proof. Let $w_1, w_2 \in K$ and $u_1 = S_{g,d}(w_1)$, $u_2 = S_{g,d}(w_2)$. By adding the inequalities (4.99) corresponding to u_1 and u_2 for $v = u_2$, and, respectively, for $v = u_1$, from (4.83), (4.90), and (4.86), one obtains

$$\begin{array}{l} \alpha\|u_1 - u_2\|^2 \le a(u_1 - u_2, u_1 - u_2) \le j(g, w_1, u_2 - d) + j(g, w_2, u_1 - d) \\ -j(g, w_1, u_1 - d) - j(g, w_2, u_2 - d) \le k_2 k_1 \|w_1 - w_2\|\, \|u_1 - u_2\| \end{array}$$

hence

$$\|S_{g,d}(w_1) - S_{g,d}(w_2)\| \leq \frac{k_1\,k_2}{\alpha}\|w_1 - w_2\|.$$

Consequently, from the hypothesis (4.101), it follows that the mapping $S_{g,d}$ is a contraction, and hence, by Theorem 4.7, there exists a unique $u \in K(g)$ such that $u = S_{g,d}(u)$. □

For all $g \in V$, $d \in K$, we consider the following auxiliary problems
Problem ($\tilde{\mathbf{Q}}^a$): Find $u \in K(g)$ such that

$$\begin{cases} a(u, v-u) + j(g,u,v-d) - j(g,u,u-d) \geq b(g,u,v-u) \quad \forall v \in V, \\ b(g,u,z-u) \geq 0 \quad \forall z \in K \end{cases} \tag{4.103}$$

and
Problem ($\tilde{\mathbf{R}}^a$): Find $u \in K(g)$ such that

$$a(u, v-u) + j(g,u,v-d) - j(g,u,u-d) \geq 0 \quad \forall v \in K. \tag{4.104}$$

We make the hypothesis

$$\text{If } u \text{ is a solution of } (\tilde{\mathbf{R}}^a)\text{, then } u \text{ is a solution of } (\tilde{\mathbf{Q}}^a). \tag{4.105}$$

Remark 4.8. If u satisfies ($\tilde{\mathbf{Q}}^a$), then u obviously satisfies ($\tilde{\mathbf{R}}^a$).

Let $f \in W^{1,2}(0,T;V)$, with $T > 0$, be given and let $u_0 \in K(f(0))$ be the unique solution, according to Lemma 4.8, of the following elliptic variational inequality

$$a(u_0, w-u_0) + j(f(0),u_0,w) - j(f(0),u_0,u_0) \geq 0 \quad \forall\, w \in K. \tag{4.106}$$

Remark 4.9. Taking into account that K is a cone with the vertex at 0, we may take in (4.106) $w = 2u_0$ and $w = 0$. We conclude that the variational inequality (4.106) is equivalent to the system

$$\begin{cases} a(u_0,u_0) + j(f(0),u_0,u_0) = 0, \\ a(u_0,w) + j(f(0),u_0,w) \geq 0 \quad \forall\, w \in K. \end{cases}$$

We consider the following evolutionary system of coupled variational inequalities:
Problem ($\mathbf{Q}^a$): Find $u \in W^{1,2}(0,T;V)$ such that

$$\begin{cases} u(0) = u_0\,,\ u(t) \in K(f(t)) \quad \forall\, t \in [0,T], \\ a(u(t), v-\dot{u}(t)) + j(f(t),u(t),v) - j(f(t),u(t),\dot{u}(t)) \\ \geq b(f(t),u(t),v-\dot{u}(t)) \quad \forall\, v \in V \text{ a.e. in }]0,T[, \\ b(f(t),u(t),z-u(t)) \geq 0 \quad \forall\, z \in K,\ \forall\, t \in [0,T] \end{cases} \tag{4.107}$$

where the dot denotes the time derivative, that is $\dot{u} = \dfrac{\partial u}{\partial t}$.

Our goal is to prove the existence of a solution for the problem ($\mathbf{Q}^a$). In a first step, we shall consider an incremental formulation obtained by an implicit time discretization scheme.

For $n \in N^*$, we put $\Delta t = T/n$ and $t_i = i\,\Delta t$ for $i = 0, 1, \dots, n$. If θ is a continuous function of $t \in [0, T]$, we use the notation

$$\theta^i = \theta(t_i), \ \forall\, i \in \{0, 1, \dots, n\}, \quad \partial\theta^i = \frac{\theta^{i+1} - \theta^i}{\Delta t}, \ \forall\, i \in \{0, 1, \dots, n-1\}.$$

We also put $K^i = K(f(t_i)) \ \forall\, i \in \{0, 1, \dots, n\}$ and $u^0 = u_0$. We then approximate the problem ($\mathbf{Q}^a$) by the following sequence $\{\, (\mathbf{Q}^a)^{\mathbf{i}}_{\mathbf{n}} \,\}_{i=0,1,\dots,n-1}$ of incremental problems

Problem $(\mathbf{Q}^a)^{\mathbf{i}}_{\mathbf{n}}$: Find $u^{i+1} \in K^{i+1}$ such that

$$\begin{cases} a(u^{i+1}, v - \partial u^i) + j(f^{i+1}, u^{i+1}, v) - j(f^{i+1}, u^{i+1}, \partial u^i) \\ \geq b(f^{i+1}, u^{i+1}, v - \partial u^i) \quad \forall\, v \in V, \\ \\ b(f^{i+1}, u^{i+1}, z - u^{i+1}) \geq 0 \quad \forall\, z \in K. \end{cases} \tag{4.108}$$

Lemma 4.9. *If u is a solution of* ($\mathbf{Q}^a$) *and* u^{i+1} *is a solution of* $(\mathbf{Q}^a)^{\mathbf{i}}_{\mathbf{n}}$, *then*

$$b(f(t), u(t), u(t)) = 0 \quad on \ [0, T] \tag{4.109}$$

$$b(f^{i+1}, u^{i+1}, u^{i+1}) = 0 \quad \forall\, i \in \{0, 1, \dots, n-1\}. \tag{4.110}$$

$$b(f(t), u(t), \dot{u}(t)) = 0 \quad a.e.\ on \ [0, T], \tag{4.111}$$

Proof. Since K is a cone with the vertex at 0, we may take in the second inequalities of ($\mathbf{Q}^a$) and $(\mathbf{Q}^a)^{\mathbf{i}}_{\mathbf{n}}$, $z = 2u(t)$ and $z = 0$, respectively, $z = 2u^{i+1}$ and $z = 0$, thus we obtain the relations (4.109) and (4.110).

Also, from the second inequality of ($\mathbf{Q}^a$), we deduce that, for all $t \in]0, T[$ and for all $\Delta t > 0$ sufficiently small, we obtain

$$b\left(f(t), u(t), \frac{u(t + \Delta t) - u(t)}{\Delta t} \right) \geq 0$$

and

$$b\left(f(t), u(t), \frac{u(t - \Delta t) - u(t)}{-\Delta t} \right) \leq 0,$$

which imply the relation (4.111). □

Now, for any $i \in \{0, \ldots, n-1\}$, we introduce the following problem
Problem $(\mathbf{R}^a)^{\mathbf{i}}_{\mathbf{n}}$: Find $u^{i+1} \in K^{i+1}$ such that

$$\begin{aligned} & a(u^{i+1}, w - u^{i+1}) + j(f^{i+1}, u^{i+1}, w - u^i) \\ & - j(f^{i+1}, u^{i+1}, u^{i+1} - u^i) \geq 0 \quad \forall\, w \in K. \end{aligned} \tag{4.112}$$

Lemma 4.10. *The problem* $(\mathbf{Q}^a)^{\mathbf{i}}_{\mathbf{n}}$ *is equivalent to the problem* $(\mathbf{R}^a)^{\mathbf{i}}_{\mathbf{n}}$.

Proof. By taking $v = \dfrac{w - u^i}{\Delta t}$ with w arbitrarily chosen in V, it is easy to prove that the first inequality of $(\mathbf{Q}^a)^{\mathbf{i}}_{\mathbf{n}}$ is equivalent to

$$\begin{aligned} & a(u^{i+1}, w - u^{i+1}) + j(f^{i+1}, u^{i+1}, w - u^i) - j(f^{i+1}, u^{i+1}, u^{i+1} - u^i) \\ & \geq b(u^{i+1}, w - u^{i+1}) \quad \forall w \in V\,. \end{aligned}$$

Therefore, from Remark 4.8 and the hypothesis (4.105) for $g = f^{i+1}$ and $d = u^i$, the assertion follows. □

Proposition 4.9. *Under the above hypotheses, there exists a unique solution* $u^{i+1} \in K^{i+1}$ *of the problem* $(\mathbf{Q}^a)^{\mathbf{i}}_{\mathbf{n}}$.

Proof. By applying Lemma 4.8, for $g = f^{i+1}$ and $d = u^i$, it results that the problem $(\mathbf{R}^a)^{\mathbf{i}}_{\mathbf{n}}$ has a unique solution. Hence by Lemma 4.10, the problem $(\mathbf{Q}^a)^{\mathbf{i}}_{\mathbf{n}}$ has. □

By using the fact that the function f is absolutely continuous, we give the following estimates.

Lemma 4.11. *Let* $u^{i+1} \in K^{i+1}$ *be the solution of* $(\mathbf{Q}^a)^{\mathbf{i}}_{\mathbf{n}}$, $i \in \{0, 1, \ldots, n-1\}$. *Then*

$$\|u^0\| \leq C_0 \|f\|_{C([0,T];V)}, \quad \|u^{i+1}\| \leq C_0 \|f\|_{C([0,T];V)}\,, \tag{4.113}$$

$$\|u^{i+1} - u^i\| \leq C_0 \int_{t_i}^{t_{i+1}} \|\dot{f}(\tau)\| \, \mathrm{d}\tau \leq C_0 \sqrt{\Delta t}\, \|\dot{f}\|_{L^2(0,T;V)}\,, \tag{4.114}$$

$$\sum_{i=0}^{n-1} \|u^{i+1} - u^i\|^2 \leq C_0^2 \Delta t \|\dot{f}\|^2_{L^2(0,T;V)}\,, \tag{4.115}$$

where

$$C_0 = \frac{(k_1 + 1)k_2}{\alpha - k_1 k_2}\,. \tag{4.116}$$

Proof. From Remark 4.9, by using the relations (4.83), (4.91), and (4.93) for $w_1 = u^0$, $w_2 = 0$, we get

$$\alpha\|u^0\|^2 \leq |j(f^0, u^0, u^0)| \leq \big((1+k_1)k_2\|f^0\| + k_1k_2\|u^0\|\big)\,\|u^0\|.$$

Since $f \in W^{1,2}(0,T;V) \subset C([0,T];V)$, we have

$$\|f^i\| = \|f(t_i)\| \leq \max_{t\in[0,T]} \|f(t)\| = \|f\|_{C([0,T];V)}\,, \ \ \forall i \in \{0,1,\ldots,n\},$$

thus the first part of the estimate (4.113) is true.

In the following we take into account Lemma 4.10, that is u^{i+1} is the unique solution of $(\mathbf{R}^a)^{\mathbf{i}}_{\mathbf{n}}$. By taking $w = 0$ in $(\mathbf{R}^a)^{\mathbf{i}}_{\mathbf{n}}$, it follows

$$a(u^{i+1}, u^{i+1}) \leq |j(f^{i+1}, u^{i+1}, -u^i) - j(f^{i+1}, u^{i+1}, u^{i+1} - u^i)|$$

which, together with (4.83) and (4.93), gives the second part of the estimate (4.113).

Next, if we take $w = u^i$ in $(\mathbf{R}^a)^{\mathbf{i}}_{\mathbf{n}}$ and $w = u^{i+1}$ in $(\mathbf{R}^a)^{\mathbf{i-1}}_{\mathbf{n}}$, then, by using (4.87), (4.90), (4.91), and (4.86), we get

$$\begin{aligned}
\alpha\|u^{i+1} - u^i\|^2 &\leq j(f^i, u^i, u^{i+1} - u^{i-1}) - j(f^i, u^i, u^i - u^{i-1}) \\
&\quad - j(f^{i+1}, u^{i+1}, u^{i+1} - u^i) \\
&\leq j(f^i, u^i, u^{i+1} - u^i) - j(f^{i+1}, u^{i+1}, u^{i+1} - u^i) \\
&\leq ((1+k_1)k_2\|f^{i+1} - f^i\| + k_1k_2\|u^{i+1} - u^i\|)\|u^{i+1} - u^i\|.
\end{aligned}$$

This implies (4.114). Indeed, by taking into account the regularity of f, Bochner Theorem 3.3 and Cauchy–Schwartz inequality, we have

$$\begin{aligned}
\|f(t) - f(s)\| &= \Big\|\int_s^t \dot f(\tau)\,\mathrm{d}\tau\Big\| \leq \int_s^t \|\dot f(\tau)\|\,\mathrm{d}\tau \\
&\leq C_0\sqrt{t-s}\ \|\dot f\|_{L^2(0,T;V)} \quad \forall s,t \in [0,T],\ s < t.
\end{aligned} \tag{4.117}$$

The last estimate (4.115) is easily obtained by writing

$$\|u^{i+1} - u^i\|^2 \leq C_0^2\left(\int_{t_i}^{t_{i+1}} \|\dot f(\tau)\|^2\,\mathrm{d}\tau\right)\left(\int_{t_i}^{t_{i+1}} \mathrm{d}\tau\right) = C_0^2\Delta t \int_{t_i}^{t_{i+1}} \|\dot f(\tau)\|^2\,\mathrm{d}\tau\,.$$

□

Now, if we define the functions

$$\begin{cases} u_n(0) = \hat{u}_n(0) = u^0\,, \\ f_n(0) = f^0\,, \\ \left.\begin{aligned} & u_n(t) = u^{i+1} \\ & \hat{u}_n(t) = u^i + (t - t_i)\partial u^i \\ & f_n(t) = f^{i+1} \end{aligned}\right\} \forall\, i \in \{0, 1, \ldots, n-1\} \quad \forall\, t \in (t_i, t_{i+1}]\,, \end{cases} \tag{4.118}$$

then we obviously have u_n , $f_n \in L^2(0, T; V)$ and $\hat{u}_n \in W^{1,2}(0, T; V)$. From $(\mathbf{Q}^a)^{\mathbf{i}}_{\mathbf{n}}$ we deduce that these functions satisfy, for all $t \in [0, T]$, the following incremental formulation

Problem $(\mathbf{Q}^a)_{\mathbf{n}}$: Find $u_n(t) \in K(f_n(t))$ such that

$$\begin{cases} a\left(u_n(t), v - \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) + j(f_n(t), u_n(t), v) - j\left(f_n(t), u_n(t), \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \\ \geq b\left(f_n(t), u_n(t), v - \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \quad \forall\, v \in V, \\ b(f_n(t), u_n(t), z - u_n(t)) \geq 0 \quad \forall\, z \in K. \end{cases} \tag{4.119}$$

Remark 4.10. From (4.118), (4.96) and the second inequality of $(\mathbf{Q}^a)^{\mathbf{i}}_{\mathbf{n}}$, we get

$$\begin{aligned} & b\left(f_n(t), u_n(t), v - \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) = b(f^{i+1}, u^{i+1}, v - \partial u^i) = b(f^{i+1}, u^{i+1}, v) \\ & + \frac{1}{\Delta t} b(f^{i+1}, u^{i+1}, u^i - u^{i+1}) \geq b(f_n(t), u_n(t), v) \;\; \forall\, t \in (t_i, t_{i+1}]\,,\; \forall\, v \in V, \end{aligned} \tag{4.120}$$

and hence,

$$b\left(f_n(t), u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \leq 0 \;\; \forall\, t \in (0, T)\,,\; \forall\, v \in V. \tag{4.121}$$

Lemma 4.12. *We have the estimates*

$$\|u_n(t)\| \leq C_0 \|f\|_{C([0,T];V)}, \quad \forall t \in [0, T]\,, \tag{4.122}$$

$$\|u_n(s) - u_n(t)\| \leq C_0 \int_s^{\min\left\{t + \frac{T}{n}, T\right\}} \|\dot{f}(\tau)\|\, \mathrm{d}\tau\,, \quad \forall s, t \in [0, T],\; s < t\,, \tag{4.123}$$

$$\|\hat{u}_n\|_{L^2(0,T;V)} \leq C_0 \sqrt{T} \|f\|_{C([0,T];V)}\,, \tag{4.124}$$

$$\|u_n - \hat{u}_n\|_{L^2(0,T;V)} \leq C_0 \frac{\Delta t}{\sqrt{3}} \|\dot{f}\|_{L^2(0,T;V)}\,, \tag{4.125}$$

$$\left\| \frac{\mathrm{d}}{\mathrm{d}t} \hat{u}_n \right\|_{L^2(0,T;V)} \leq C_0 \|\dot{f}\|_{L^2(0,T;V)}\,, \tag{4.126}$$

where C_0 is the constant defined by (4.116).

Proof. The estimate (4.122) follows from the definition (4.118) and the estimate (4.113).

For proving (4.123), let $s,\ t \in [0, T]$ with $s < t$ and $0 \leq i \leq j \leq n-1$ such that $s \in (t_i, t_{i+1}]$, $t \in (t_j, t_{j+1}]$. Then, from (4.114), we have

$$\begin{aligned}
\|u_n(s) - u_n(t)\| &= \|u^{i+1} - u^{j+1}\| \\
&= \|(u^{j+1} - u^j) + (u^j - u^{j-1}) + \ldots + (u^{i+2} - u^{i+1})\| \\
&\leq \sum_{k=i+1}^{j} \|u^{k+1} - u^k\| \leq C_0 \sum_{k=i+1}^{j} \int_{t_k}^{t_{k+1}} \|\dot{f}(\tau)\|\,\mathrm{d}\tau \\
&= C_0 \int_{t_{i+1}}^{t_{j+1}} \|\dot{f}(\tau)\|\,\mathrm{d}\tau \\
&\leq C_0 \int_{s}^{\min\{t+\Delta t, T\}} \|\dot{f}(\tau)\|\,\mathrm{d}\tau.
\end{aligned}$$

The estimate (4.124) is easily obtained by the following computations

$$\|\hat{u}_n\|^2_{L^2(0,T;V)} = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left\| \left(1 - \frac{t-t_i}{\Delta t}\right) u^i + \frac{t-t_i}{\Delta t} u^{i+1} \right\|^2 \mathrm{d}t$$

$$\leq \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left(\left(1 - \frac{t-t_i}{\Delta t}\right) \|u^i\| + \frac{t-t_i}{\Delta t} \|u^{i+1}\| \right)^2 \mathrm{d}t \leq C_0^2 T \|f\|^2_{C([0,T];V)}\,,$$

where we have used (4.113).

Next, from (4.118) and (4.115), we get

$$\begin{aligned}
&\|u_n - \hat{u}_n\|^2_{L^2(0,T;V)} = \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \left\| u^{i+1} - u^i - \frac{t-t_i}{\Delta t}(u^{i+1} - u^i) \right\|^2 \mathrm{d}t \\
&= \frac{1}{\Delta t^2} \sum_{i=0}^{n-1} \|u^{i+1} - u^i\|^2 \int_{t_i}^{t_{i+1}} (t_{i+1} - t)^2\,\mathrm{d}t = \frac{\Delta t}{3} \sum_{i=0}^{n-1} \|u^{i+1} - u^i\|^2 \\
&\leq \frac{C_0^2 \Delta t^2}{3} \|\dot{f}\|^2_{L^2(0,T;V)},
\end{aligned}$$

that is (4.125).

Finally, by taking into account that

$$\left\| \frac{\mathrm{d}\hat{u}_n}{\mathrm{d}t} \right\|_{L^2(0,T;V)} = \frac{\sqrt{3}}{\Delta t} \|u_n - \hat{u}_n\|_{L^2(0,T;V)},$$

we also deduce the estimate (4.126). □

Lemma 4.13. *There exist a subsequence* $\{(u_{n_k}, \hat{u}_{n_k})\}_{k\in\mathbb{N}^*}$ *of* $\{(u_n, \hat{u}_n)\}_{n\in\mathbb{N}^*}$ *and an element* $u \in W^{1,2}(0,T;V)$ *such that*

$$u_{n_k}(t) \rightharpoonup u(t) \quad \textit{weakly in } V \quad \forall\, t \in [0,T], \tag{4.127}$$

$$\hat{u}_{n_k} \rightharpoonup u \quad \textit{weakly in } W^{1,2}(0,T;V). \tag{4.128}$$

In addition, for all $s \in [0,T]$, *we have*

$$\liminf_{k\to\infty} \int_0^s a\left(u_{n_k}(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{n_k}(t)\right) \mathrm{d}t \geq \int_0^s a(u(t), \dot{u}(t))\,\mathrm{d}t\,, \tag{4.129}$$

$$\liminf_{k\to\infty} \int_0^s j\left(f_{n_k}(t), u_{n_k}(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{n_k}(t)\right) \mathrm{d}t \geq \int_0^s j(f(t), u(t), \dot{u}(t))\,\mathrm{d}t. \tag{4.130}$$

Proof. Applying a process of diagonalization, from (4.122) and (4.123), we deduce that we can extract a subsequence $\{u_{n_k}\}_k \subset \{u_n\}_n$ such that $u_{n_k}(t) \rightharpoonup u(t)$ weakly in V, $\forall\, t \in [0,T]$ with $u \in L^2(0,T;V)$. Indeed, let $E = \{\tau_j\}_{j\in\mathbb{N}} \subset [0,T]$ be a countable dense subset. From (4.122) it follows that, for all $j \in \mathbb{N}$ and for all $n \in \mathbb{N}^*$, we have $\|u_n(\tau_j)\| \leq C_1$ where $C_1 = C\|f\|_{C([0,T];V)}$. Therefore, by a process of diagonalization, we can extract $\{u_{n_k}\}_k \subset \{u_n\}_n$ such that, for all $j \in \mathbb{N}$, the sequence $\{u_{n_k}(\tau_j)\}_k$ converges weakly towards an element of V denoted by $u(\tau_j)$. For the sake of simplicity we shall omit the subscript k from now.

We shall prove that, for all $t \in [0,T]$, the sequence $\{u_n(t)\}_n$ is weakly Cauchy. For $\varphi \in V$, $t \in [0,T)$, $q > 0$ and $\tau_j \in E$ arbitrarily chosen, we have

$$\begin{aligned} &|(u_{n+q}(t) - u_n(t), \varphi)| \leq |(u_{n+q}(t) - u_{n+q}(\tau_j), \varphi)| + |(u_{n+q}(\tau_j) - u_n(\tau_j), \varphi)| \\ &+|(u_n(\tau_j) - u_n(t), \varphi)| \leq \|\varphi\|(\|u_{n+q}(t) - u_{n+q}(\tau_j)\| + \|u_n(\tau_j) - u_n(t)\|) \\ &+|(u_{n+q}(\tau_j) - u_n(\tau_j), \varphi)|. \end{aligned}$$

Taking $\tau_j > t$ and using (4.123), we deduce

$$
\begin{aligned}
|(u_{n+q}(t) - u_n(t), \varphi)| &\le C_0\|\varphi\| \left(\int_t^{min\{\tau_j + \frac{T}{n+q}, T\}} \|\dot{f}(\tau)\|\,\mathrm{d}\tau \right.\\
&\left. + \int_t^{min\{\tau_j + \frac{T}{n}, T\}} \|\dot{f}(\tau)\|\,\mathrm{d}\tau \right) + |(u_{n+q}(\tau_j) - u_n(\tau_j), \varphi)| \\
&\le 2C_0\|\varphi\| \int_t^{min\{\tau_j + \frac{T}{n}, T\}} \|\dot{f}(\tau)\|\,\mathrm{d}\tau + |(u_{n+q}(\tau_j) - u_n(\tau_j), \varphi)| \\
&\le 2C_0\|\varphi\| \sqrt{\frac{T}{n} + \tau_j - t}\,\|\dot{f}\|_{L^2(0,T;V)} + |(u_{n+q}(\tau_j) - u_n(\tau_j), \varphi)|.
\end{aligned}
\tag{4.131}
$$

As $E = \{\tau_j\}_{j\in\mathbb{N}}$ is dense in $[0, T]$, we can choose $\tau_j > t$ such that $\tau_j - t$ is sufficiently small. On the other hand, the sequence $\{u_n(\tau_j)\}_n$ being weakly convergent it is also weakly Cauchy. Therefore, from (4.131), it follows that the sequence $\{u_n(t)\}_n$ is weakly Cauchy, and so $u_n(t) \rightharpoonup u(t)$ weakly in V. As K is weakly closed, it follows that $u(t) \in K$ and $u_n \rightharpoonup u$ weakly in $L^2(0, T; V)$.

Next, from (4.124) and (4.126), one obtains that there exists a subsequence $\{\hat{u}_n\}_n$, still denoted by $\{\hat{u}_n\}_n$, and an element $\hat{u} \in W^{1,2}(0, T; V)$ such that $\hat{u}_n \rightharpoonup \hat{u}$ weakly in $W^{1,2}(0, T; V)$ (in fact, one considers that $\|\hat{u}_{n_k}\|_{W^{1,2}(0,T;V)}$ is bounded for the same indices n_k for which the subsequence $u_{n_k}(t)$ is weakly convergent, and one extracts from this sequence a subsequence $\hat{u}_{n_{k_p}}$ which converges weakly in $W^{1,2}(0, T; V)$ towards $\hat{u}$). We show that $u = \hat{u}$. Indeed, we have

$$
\begin{aligned}
\left|(\hat{u}_n - u_n, \varphi)_{L^2(0,T;V)}\right| &= \left| \sum_{i=0}^{n-1} \int_{t_i}^{t_i+1} \left(u^i + \frac{t - t_i}{\Delta t}(u^{i+1} - u^i) - u^{i+1}, \varphi(t) \right) \mathrm{d}t \right| \\
&\le \sum_{i=0}^{n-1} \int_{t_i}^{t_i+1} \|u^{i+1} - u^i\|\,\|\varphi(t)\|\,\mathrm{d}t \\
&\le \left(\sum_{i=0}^{n-1} \int_{t_i}^{t_i+1} \|u^{i+1} - u^i\|^2\,\mathrm{d}t \right)^{1/2} \left(\sum_{i=0}^{n-1} \int_{t_i}^{t_i+1} \|\varphi(t)\|^2\,\mathrm{d}t \right)^{1/2} \\
&\le C_0 \frac{T}{n} \|\dot{f}\|_{L^2(0,T;V)} \|\varphi\|_{L^2(0,T;V)}
\end{aligned}
$$

that is, $\hat{u}_n$ and u_n have the same weak limit in $L^2(0, T; V)$.

In order to prove (4.129), let $s \in]\,0, T\,]$ and $i \in \{0, \ldots, n-1\}$ be such that $s \in (t_i, t_{i+1}]$. Using the definitions (4.118) and the properties of a, we obtain

$$
\begin{aligned}
&\int_0^s a\left(u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\,\mathrm{d}t = \int_0^{t_{i+1}} a\left(u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\,\mathrm{d}t - R_n \\
&= \sum_{j=0}^{i} \int_{t_j}^{t_{j+1}} a\left(u^{j+1}, \frac{u^{j+1}-u^j}{\Delta t}\right)\,\mathrm{d}t - R_n \geq \frac{1}{2}\sum_{j=0}^{i}\left(a(u^{j+1}, u^{j+1}) - a(u^j, u^j)\right) \\
&-R_n = \frac{a(u^{i+1}, u^{i+1}) - a(u^0, u^0)}{2} - R_n = \frac{a(u_n(s), u_n(s)) - a(u^0, u^0)}{2} - R_n
\end{aligned}
\tag{4.132}
$$

where $R_n = \int_s^{t_{i+1}} a\left(u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\,\mathrm{d}t$.

First, due to (4.83), (4.113), and (4.114), we have

$$
\begin{aligned}
|R_n| &= \frac{1}{\Delta t}\left|\int_s^{t_{i+1}} a(u^{i+1}, u^{i+1} - u^i)\,\mathrm{d}t\right| \leq M\,\frac{t_{i+1}-s}{\Delta t}\|u^{i+1}\|\|u^{i+1}-u^i\| \\
&\leq C_0^2 M\sqrt{\frac{T}{n}}\,\|f\|_{C([0,T];V)}\|\dot{f}\|_{L^2(0,T;V)}
\end{aligned}
$$

which implies

$$
\lim_{n\to\infty} R_n = 0\,.
$$

Therefore, as the form a is symmetric, from (4.132), we get

$$
\begin{aligned}
&\liminf_{n\to\infty}\int_0^s a\left(u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\,\mathrm{d}t \geq \frac{1}{2}\liminf_{n\to\infty} a(u_n(s), u_n(s)) - \frac{1}{2}a(u^0, u^0) \\
&\geq \frac{a(u(s), u(s)) - a(u(0), u(0))}{2} = \frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}a(u(t), u(t))\,\mathrm{d}t = \int_0^s a(u(t), \dot{u}(t))\,\mathrm{d}t\,,
\end{aligned}
$$

and thus, the relation (4.129) holds.

Next, since, for all $f, u \in V$, the functional $j(f, u, \cdot)$ is l.s.c. convex on V, it follows (see [5], p. 160) that the mapping $v \mapsto \int_0^s j(f(t), u(t), v(t))\,\mathrm{d}t$ is l.s.c. convex on $L^2(0, T; V)$. Thus

$$
\liminf_{n\to\infty}\int_0^s j\left(f(t), u(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\,\mathrm{d}t \geq \int_0^s j(f(t), u(t), \dot{u}(t))\,\mathrm{d}t. \tag{4.133}
$$

On the other hand, from (4.90), (4.91), and (4.126), one obtains

$$\left|\int_0^s \left(j\left(f_n(t), u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) - j\left(f(t), u(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\right) \mathrm{d}t\right|$$

$$\leq k_2 \left|\int_0^s (\|f_n(t) - f(t)\| + \|\beta(f_n(t), u_n(t)) - \beta(f(t), u(t))\|_H) \left\|\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right\| \mathrm{d}t\right|$$

$$\leq C_0 k_2 \|\dot{f}\|_{L^2(0,T;V)} \left[\left(\int_0^s \|f_n(t) - f(t)\|^2 \,\mathrm{d}t\right)^{\frac{1}{2}}\right.$$

$$\left. + \left(\int_0^s \|\beta(f_n(t), u_n(t)) - \beta(f(t), u(t))\|_H^2 \,\mathrm{d}t\right)^{\frac{1}{2}}\right]$$

from which, since $f, u \in W^{1,2}(0, T; V) \subset C([0, T]; V)$ and $f_n(t) \to f(t)$ in V $\forall\, t \in [0, T]$, by using the property (4.85) of β, one has

$$\lim_{n\to\infty} \int_0^s \left(j\left(f_n(t), u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) - j\left(f(t), u(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\right) \mathrm{d}t = 0. \tag{4.134}$$

Combining the relations (4.133) and (4.134), we deduce

$$\liminf_{n\to\infty} \int_0^s j\left(f_n(t), u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \mathrm{d}t$$

$$\geq \lim_{n\to\infty} \int_0^s \left(j\left(f_n(t), u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) - j\left(f(t), u(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\right) \mathrm{d}t$$

$$+ \liminf_{n\to\infty} \int_0^s j\left(f(t), u(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \mathrm{d}t \geq \int_0^s j(f(t), u(t), \dot{u}(t))\,\mathrm{d}t$$

which completes the proof. □

We now prove the following strong convergence result together with the main result of this section, namely the existence of a solution for the problem ($\mathbf{Q}^a$).

Theorem 4.19. *We suppose that the hypotheses (4.83)–(4.90), (4.96)–(4.98), (4.100), (4.101), and (4.105) are satisfied. Then the problem* ($\mathbf{Q}^a$) *has at least one solution. More precisely, there exists a subsequence* $\{(u_{n_k}, \hat{u}_{n_k})\}_{k\in\mathbb{N}^*}$ *of* $\{(u_n, \hat{u}_n)\}_{n\in\mathbb{N}^*}$ *such that*

$$u_{n_k}(t) \to u(t) \quad \textit{strongly in } V \quad \forall\, t \in [0, T], \tag{4.135}$$

$$\hat{u}_{n_k} \to u \quad \textit{strongly in } L^2(0,T;V), \tag{4.136}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{n_k} \rightharpoonup \dot{u} \quad \textit{weakly in } L^2(0,T;V), \tag{4.137}$$

as $k \to \infty$, where $u \in W^{1,2}(0,T;V)$ is a solution of the problem $(\mathbf{Q}^a)$.

Proof. Let $\{u_n\}_{n\in\mathbb{N}^*}$ be the subsequence given by Lemma 4.13. We first prove that its weak limit u is a solution of the problem $(\mathbf{Q}^a)$.

It is easy to show that $u(t) \in K(f(t))$, $\forall t \in [0,T]$. Indeed, since $(f_n(t), u_n(t)) \in D_K$, $\forall t \in [0,T]$, then, the convergences $f_n(t) \to f(t)$ strongly in V, $\forall t \in [0,T]$ and $u_n(t) \rightharpoonup u(t)$ weakly in V, $\forall t \in [0,T]$ imply, due to the hypothesis (4.84) on the set D_K, the assertion.

Let $s \in [0,T]$. Integrating the first inequality of $(\mathbf{Q}^a)_\mathbf{n}$ over $[0,s]$ and using (4.120), we obtain

$$\begin{aligned}
&\int_0^s a\left(u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\mathrm{d}t + \int_0^s j\left(f_n(t), u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\mathrm{d}t \\
&\le \int_0^s a(u_n(t), v(t))\,\mathrm{d}t + \int_0^s j(f_n(t), u_n(t), v(t))\,\mathrm{d}t \\
&- \int_0^s b(f_n(t), u_n(t), v(t))\,\mathrm{d}t \quad \forall\, v \in L^2(0,T;V)\,.
\end{aligned} \tag{4.138}$$

On the other hand, from (4.94) and (4.97), we have

$$\lim_{n\to\infty}\int_0^s j(f_n(t), u_n(t), v(t))\,\mathrm{d}t = \int_0^s j(f(t), u(t), v(t))\,\mathrm{d}t\,,$$

$$\lim_{n\to\infty}\int_0^s b(f_n(t), u_n(t), v(t))\,\mathrm{d}t = b(f(t), u(t), v(t))\,\mathrm{d}t.$$

Therefore, by passing to the limit in (4.138) and by using the convergences (4.129)–(4.130), we deduce that

$$\begin{aligned}
&\int_0^s a(u(t), v(t) - \dot{u}(t))\,\mathrm{d}t + \int_0^s j(f(t), u(t), v(t)\,\mathrm{d}t - \int_0^s j(f(t), u(t), \dot{u}(t))\,\mathrm{d}t \\
&\ge \int_0^s b(f(t), u(t), v(t))\,\mathrm{d}t \quad \forall\, v \in L^2(0,T;V)\,, \quad \forall s \in [0,T]\,.
\end{aligned} \tag{4.139}$$

Now, by taking $v \in L^2(0, T; V)$ defined by

$$v(t) = \begin{cases} w & \text{if } t \in [s, s+h] \\ \dot{u}(t) & \text{otherwise} \end{cases}$$

with $w \in V$ arbitrarily and $s \in [0, T]$, $h > 0$ such that $s + h \leq T$, we get

$$\int_s^{s+h} a(u(t), w - \dot{u}(t))\, \mathrm{d}t + \int_s^{s+h} j(f(t), u(t), w)\, \mathrm{d}t - \int_s^{s+h} j(f(t), u(t), \dot{u}(t))\, \mathrm{d}t$$
$$\geq \int_s^{s+h} b(f(t), u(t), w)\, \mathrm{d}t \quad \forall w \in V, \quad \forall s \in [0, T]$$

which gives, by passing to the limit with $h \to 0$, the inequality

$$\begin{aligned} &a(u(t), w - \dot{u}(t)) + j(f(t), u(t), w) - j(f(t), u(t), \dot{u}(t)) \\ &\quad \geq b(f(t), u(t), w) \quad \forall w \in V \text{ a.e. on }]0, T]. \end{aligned} \tag{4.140}$$

In order to show that u satisfies the second inequality of $(\mathbf{Q}^a)$, let us remark that, from the second inequality of $(\mathbf{Q}^a)_\mathbf{n}$, it follows

$$b(f_n(t), u_n(t), z) \geq b(f_n(t), u_n(t), u_n(t)) \quad \forall z \in K,$$

from which, by passing to the limit and taking into account the hypotheses (4.95) and (4.94), one obtains

$$b(f(t), u(t), z - u(t)) \geq 0 \quad \forall z \in K, \quad \forall t \in [0, T]. \tag{4.141}$$

Therefore, proceeding as in the proof of Lemma 4.9, one obtains

$$b(f(t), u(t), \dot{u}(t)) = 0. \tag{4.142}$$

From (4.140), (4.141), and (4.142) we conclude that u is a solution of $(\mathbf{Q}^a)$.

In order to prove the convergences (4.135)–(4.137), we shall use an argument due to Andersson [2]. We first prove that

$$\lim_{n \to \infty} \int_0^s a\left(u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \mathrm{d}t = \int_0^s a(u(t), \dot{u}(t))\, \mathrm{d}t \quad \forall s \in [0, T]. \tag{4.143}$$

Taking $v = 0$ in the first inequality of $(\mathbf{Q}^a)_\mathbf{n}$ and $w = 0$, $w = 2\dot{u}(t)$ in (4.140), and integrating these inequalities on $[0, s]$ for $s \in [0, T]$, one obtains

$$\begin{aligned}
0 &\geq \limsup_{n\to\infty} \int_0^s \left(a\left(u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) + j\left(f_n(t), u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\right) \mathrm{d}t \\
&\geq \liminf_{n\to\infty} \int_0^s \left(a\left(u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) + j\left(f_n(t), u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\right) \mathrm{d}t \\
&\geq \liminf_{n\to\infty} \int_0^s a\left(u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \mathrm{d}t + \liminf_{n\to\infty} \int_0^s j\left(f_n(t), u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \mathrm{d}t \\
&\geq \int_0^s a(u(t), \dot{u}(t))\,\mathrm{d}t + \int_0^s j(f(t), u(t), \dot{u}(t))\,\mathrm{d}t = 0
\end{aligned}$$

where we used (4.88), (4.96), (4.142), (4.129), (4.121), and (4.130). Therefore, we conclude

$$\liminf_{n\to\infty} \int_0^s a\left(u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \mathrm{d}t = \int_0^s a(u(t), \dot{u}(t))\,\mathrm{d}t\,, \tag{4.144}$$

$$\liminf_{n\to\infty} \int_0^s j\left(f_n(t), u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \mathrm{d}t = \int_0^s j(f(t), u(t), \dot{u}(t))\,\mathrm{d}t\,. \tag{4.145}$$

On the other hand, taking $v = \dot{u}(t)$ in the first inequality of $(\mathbf{Q}^a)_\mathbf{n}$ and integrating on $[0, s]$, from (4.145), (4.97), (4.142), (4.120), and (4.144), we have

$$\begin{aligned}
\limsup_{n\to\infty} \int_0^s a\left(u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \mathrm{d}t &\leq \lim_{n\to\infty} \int_0^s a(u_n(t), \dot{u}(t))\,\mathrm{d}t \\
&+ \lim_{n\to\infty} \int_0^s j(f_n(t), u_n(t), \dot{u}(t))\,\mathrm{d}t - \liminf_{n\to\infty} \int_0^s j\left(f_n(t), u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \mathrm{d}t \\
&- \lim_{n\to\infty} \int_0^s b(f_n(t), u_n(t), \dot{u}(t))\,\mathrm{d}t = \int_0^s a(u(t), \dot{u}(t))\,\mathrm{d}t \\
&= \liminf_{n\to\infty} \int_0^s a\left(u_n(t), \frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right) \mathrm{d}t\,,
\end{aligned} \tag{4.146}$$

and thus, (4.143). Obviously, from (4.143) and (4.132), we deduce

$$\frac{a(u(s),u(s))-a(u^0,u^0)}{2}=\int_0^s a(u(t),\dot{u}(t))\,\mathrm{d}t=\lim_{n\to\infty}\int_0^s a\left(u_n(t),\frac{\mathrm{d}}{\mathrm{d}t}\hat{u}_n(t)\right)\mathrm{d}t$$
$$=\frac{1}{2}\lim_{n\to\infty}a(u_n(s),u_n(s))-\frac{1}{2}a(u^0,u^0)\quad\forall s\in[0,T]$$

which gives

$$\lim_{n\to\infty}a(u_n(s),u_n(s))=a(u(s),u(s))\quad\forall\, s\in[\,0,T\,]\,.$$

Therefore, using the coerciveness of a, we deduce the strong convergence (4.135) which, obviously, implies

$$u_n\to u \text{ dans } L^2(0,T;V) \text{ fort}\,.$$

Thus, from (4.125), we obtain (4.136). Finally, the sequence $\{\hat{u}_n\}_n$ being bounded in $W^{1,2}(0,T;V)$, the convergence (4.136) implies the convergence (4.137) which completes the proof. □

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Chapter 5
Some Properties of Solutions

In this chapter one studies some properties of solutions of various variational inequalities of the first and second kind. We first consider a class of variational inequalities of the first kind and we emphasize, following the work [19], a property of solutions, namely a maximum principle. We illustrate it by a problem which models the flow of fluids through a porous medium and an obstacle problem. Next, following the work [10], we use the method of the translation to derive local and global regularity results of solutions of a class of variational inequalities of the second kind. In Sect. 8.4, these results will be applied to a frictional contact problem.

5.1 A Maximum Principle for a Class of Variational Inequalities

5.1.1 A General Result

The class of variational inequalities considered in this paragraph is characterized by the bilinear form

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v \, \mathrm{d}x \qquad \forall u, v \in H^1(\Omega) \tag{5.1}$$

where Ω is a bounded open subset of $\mathbb{R}^d$, with its boundary $\partial\Omega$ sufficiently smooth. We denote by K the set of constraints of our problem, defined by

$$K = \{v \in H^1(\Omega); v \geq 0 \text{ a.e. in } \Omega \text{ and } v = g \text{ a.e. on } \partial\Omega\}\,, \tag{5.2}$$

where g is a given function such that

$$g \in H^{1/2}(\partial\Omega)\,, \quad g = 0 \text{ a.e. on } \Gamma_0\,, \tag{5.3}$$

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Γ_0 being an open subset of $\partial\Omega$ such that meas $(\Gamma_0) > 0$. Thus, the set K can be written as

$$K = \{v \in V; v \geq 0 \text{ a.e. in } \Omega \text{ and } v = g \text{ a.e. on } \Gamma_1\} \tag{5.4}$$

where

$$V = \{v \in H^1(\Omega); v = 0 \text{ a.e. on } \Gamma_0\}, \tag{5.5}$$

Γ_0 and Γ_1 being open and disjoint sets such that $\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}_1$.

For $f \in L^2(\Omega)$ given, we consider the following variational inequality

$$\begin{cases} \text{Find } u \in K \text{ such that} \\ a(u, v-u) \geq (f, v-u) \quad \forall v \in K \end{cases} \tag{5.6}$$

where $a : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is the bilinear form defined by (5.1) and $(\cdot,\cdot)$ denotes the usual inner product on $L^2(\Omega)$.

Proposition 5.1. *Suppose that the hypothesis (5.3) is true. Then the variational inequality (5.6) has a unique solution.*

Proof. We show that the hypotheses of Theorem 4.12 are satisfied.

It is easy to see that the set K is convex subset of V.

For proving that it is closed, let us consider a sequence $\{v_n\}_{n\in\mathbb{N}} \subset K$ such that $v_n \to v$ strongly in $H^1(\Omega)$. It follows that $v_n \to v$ strongly in $L^2(\Omega)$, and hence, there exists a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ such that v_{n_k} converges pointwise a.e. on Ω towards v (see, for instance, [1], p. 27). As $v_{n_k}(x) \geq 0$ for almost everywhere $x \in \Omega$, it results that the pointwise a.e. limit v has the same property, i.e. $v \geq 0$ a.e. in Ω. As the trace operator $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$ is continuous (in fact, $\gamma = \gamma_0$ from Theorem 2.5, p. 17) and $H^{1/2}(\partial\Omega) \hookrightarrow L^2(\partial\Omega)$, it follows that the convergence $v_n \to v$ in $H^1(\Omega)$ implies the convergence $\gamma v_n \to \gamma v$ in $L^2(\partial\Omega)$, and so $\gamma v = g$ a.e. on Γ_1. We conclude that $u \in K$ and so, K is closed.

We shall show that the set K is nonempty. Let $\tilde{g} \in H^1(\Omega)$ be the extension of $g \in H^{1/2}(\partial\Omega)$ given by the surjectivity of the trace operator $\gamma : H^1(\Omega) \to H^{1/2}(\partial\Omega)$, i.e. $\gamma\tilde{g} = g$. Therefore, the positive part of $\tilde{g}$, denoted by $\tilde{g}^+ = \max\{\tilde{g}, 0\} \in H^1(\Omega)$, is such that $\tilde{g}^+ \geq 0$ a.e. on Ω and $\gamma\tilde{g}^+ = \max_{\partial\Omega}\{\gamma\tilde{g}, 0\} = \max_{\partial\Omega}\{g, 0\} = g$. Hence, $\tilde{g}^+ \in K$.

Next, by using the Schwartz inequality, we have

$$\begin{aligned} a(u,v) &= \sum_{i=1}^{n} \int_\Omega \frac{\partial u}{\partial x_i} \cdot \frac{\partial v}{\partial x_i}\,\mathrm{d}x \leq \left(\sum_{i=1}^{n}\int_\Omega \left(\frac{\partial u}{\partial x_i}\right)^2 \mathrm{d}x\right)^{1/2} \cdot \left(\sum_{i=1}^{n}\int_\Omega \left(\frac{\partial v}{\partial x_i}\right)^2 \mathrm{d}x\right)^{1/2} \\ &= |u|_1 \cdot |v|_1 \leq \|u\|_1 \cdot \|v\|_1 \qquad \forall u, v \in V\,. \end{aligned} \tag{5.7}$$

where $|\cdot|_1$ and $\|\cdot\|_1$ denote the seminorm and, respectively, the norm on $H^1(\Omega)$. Moreover, from the Poincaré–Friedrichs inequality

$$\|v\|^2_{L^2(\Omega)} \le C \sum_{i=1}^{n} \left\| \frac{\partial u}{\partial x_i} \right\|^2_{L^2(\Omega)} \qquad \forall v \in H^1(\Omega) \text{ such that } v = 0 \text{ on } \Gamma_0 ,$$

it follows that $\sqrt{a(u,u)}$ is a norm on V equivalent to the norm $\|\cdot\|_1$, i.e.

$$\sqrt{a(u,u)} = |u|_1 \le \|u\|_1 \le C\,|u|_1 \qquad \forall u \in V , \tag{5.8}$$

with C a positive constant depending only on Ω and on the dimension d. Thus, the bilinear continuous form $a(\cdot,\cdot)$ is coercive on V. By applying Theorem 4.12, we conclude the proof. □

Let $u \in K$ be the solution of the variational inequality (5.6). We denote by K_u the following set

$$K_u = \{w \in H^1(\Omega)\,;\, \exists v \in K \text{ and } \exists \epsilon \in \mathbb{R}_+ \text{ such that } w = \epsilon(v-u)\} . \tag{5.9}$$

It is easy to verify that the set K_u is nonempty closed in $H^1(\Omega)$ and $K_u \subset H^1_o(\Omega)$.

Lemma 5.1. *The solution $u \in K$ of the variational inequality (5.6) satisfies:*

$$a(u,w) \ge (f,w) \qquad \forall w \in K_u . \tag{5.10}$$

Proof. Let $w \in K_u$ and the corresponding $v \in K$, $\epsilon > 0$ such that $w = \epsilon(v-u)$. By writing (5.6) for this choice of v, the assertion is immediate. □

The following result shows that the solution $u \in K$ of (5.6) satisfies the same inequality on a set which is independent of g or u, namely on the cone of all non-negative elements of $H^1_0(\Omega)$.

Proposition 5.2. *The solution $u \in K$ of the variational inequality (5.6) verifies:*

$$a(u,w) \ge (f,w) \qquad \forall w \in H^1_0(\Omega) \textit{ with } w \ge 0 \textit{ a.e. in } \Omega . \tag{5.11}$$

Proof. It is easy to verify

$$\{w \in H^1_0(\Omega)\,;\, w \ge 0 \text{ a.e. in } \Omega\} \subset K_u . \tag{5.12}$$

Indeed, if $w \in H^1_0(\Omega)$ with $w \ge 0$ a.e. in Ω, then, it follows that $v = w + u \in K$, and hence $w \in K_u$. □

We now prove the main result of this section.

Theorem 5.1. *Under the hypothesis (5.3), let* $\overline{u} \in H^1(\Omega)$ *be a function which satisfies*

$$\begin{cases} a(\overline{u}, w) \geq (f, w) \quad \forall w \in H_0^1(\Omega) \quad w \geq 0 \text{ a.e. in } \Omega\,, \\ \overline{u} \geq 0 \quad \text{a.e. in } \Omega\,, \\ \overline{u} \geq g \quad \text{a.e. on } \partial\Omega\,. \end{cases} \tag{5.13}$$

Then

$$u \leq \overline{u} \quad \text{a.e. in } \Omega\,,$$

u being the unique solution of the variational inequality (5.6).

Proof. We first remark that

$$\min\{\overline{u}, v\} \in K \quad \forall v \in K\,.$$

Thus, by taking $v = \min\{\overline{u}, u\} = u - (u - \overline{u})^+$ in (5.6), we obtain

$$a(u, -(u - \overline{u})^+) \geq (f, -(u - \overline{u})^+)\,. \tag{5.14}$$

On the other hand, since $(u - \overline{u})^+ = u - \min\{u, \overline{u}\}$, it follows that $(u - \overline{u})^+ \in H_0^1(\Omega)$. Obviously, one has $(u - \overline{u})^+ \geq 0$ a.e. in Ω. By taking $w = (u - \overline{u})^+$ in $(5.13)_1$, we get

$$a(\overline{u}, (u - \overline{u})^+) \geq (f, (u - \overline{u})^+)\,. \tag{5.15}$$

By adding the relations (5.14) and (5.15), one obtains

$$a(u - \overline{u}, (u - \overline{u})^+) \leq 0\,. \tag{5.16}$$

We now remark that $a(v^+, v^-) = 0$, $\forall v \in H^1(\Omega)$, and thus $a(v, v^+) = a(v^+, v^+)$, $\forall v \in H^1(\Omega)$. This leads, thanks to the coerciveness of $a(\cdot, \cdot)$, to

$$\exists \alpha > 0 \text{ such that } \alpha \|v^+\|_1^2 \leq a(v, v^+) \quad \forall v \in H^1(\Omega) \text{ with } v^+ \in V\,. \tag{5.17}$$

Therefore, from (5.16) and (5.17), we have

$$\alpha \|(u - \overline{u})^+\|_1^2 \leq a(u - \overline{u}, (u - \overline{u})^+) \leq 0\,, \tag{5.18}$$

and thus $(u - \overline{u})^+ = 0$ a.e. in Ω, i.e. $u \leq \overline{u}$ a.e. in Ω which completes the proof. □

Remark 5.1. A function $\overline{u} \in H^1(\Omega)$ which satisfies (5.13) is called a supersolution of the variational inequality (5.6) (for general results and details, see [9]). Therefore, Theorem 5.1 asserts that the solution of the variational inequality (5.6) is the smallest supersolution of (5.6).

If the given function $f \in L^2(\Omega)$ is negative, we deduce the following maximum principle.

Corollary 5.1. *Let $u \in K$ be the solution of the variational inequality (5.6) for $f \le 0$ in Ω. Suppose that the function $g \in H^{1/2}(\partial\Omega)$ is bounded above by a positive constant C. Then $u \le C$ a.e. in Ω.*

Proof. The hypothesis on f implies $(f, w) \le 0 \quad \forall w \in H_0^1(\Omega)$ with $w \ge 0$ a.e. in Ω. This shows that $\overline{u} = C$ is a supersolution of (5.6). Hence, by Theorem 5.1, the assertion follows. □

In the sequel, we shall consider a slightly generalization of the above results. Let the set of constraints of the problem (5.6) be defined by

$$K(\psi, g) = \{v \in H^1(\Omega)\,; \quad v \ge \psi \text{ a.e. in } \Omega\,, \quad v = g \text{ a.e. on } \partial\Omega\} \tag{5.19}$$

with ψ a given function such that

$$\psi \in H^1(\Omega) \cap C^0(\overline{\Omega})\,, \quad \psi \le 0 \text{ on } \partial\Omega\,. \tag{5.20}$$

Proposition 5.3. *Suppose that the hypotheses (5.3) and (5.20) hold. Then, the variational inequality*

$$\begin{cases} \textit{Find } u \in K(\psi, g) \textit{ such that} \\ a(u, v-u) \ge (f, v-u) \quad \forall v \in K(\psi, g) \end{cases} \tag{5.21}$$

has a unique solution.

Proof. We first remark that the set $K(\psi, g)$ can be written as

$$K(\psi, g) = \{v \in V\,; \quad v \ge \psi \text{ a.e. in } \Omega\,,\, v = g \text{ on } \Gamma_1\}\,. \tag{5.22}$$

Proceeding as in the proof of Proposition 5.2, one proves that the set $K(\psi, g)$ is convex and closed in V. In order to prove that $K(\psi, g)$ is nonempty, let $\tilde{g} \in H^1(\Omega)$ be such that $\gamma\tilde{g} = g$. Therefore, $v = \tilde{g}^+ + \psi^+ \in K(\psi, g)$. By applying Theorem 4.12, the assertion follows. □

By similar proofs, we also obtain the analogous results to Theorem 5.1 and Corollary 5.1.

Theorem 5.2. *Suppose that the hypotheses (5.3) and (5.20) hold. Let $u \in K(\psi, g)$ be the unique solution of the variational inequality (5.21) and let $\overline{u} \in H^1(\Omega)$ be a function which satisfies the conditions*

$$\begin{cases} a(\overline{u}, w) \ge (f, w) \quad \forall w \in H_0^1(\Omega) \quad w \ge 0 \textit{ a.e. in } \Omega\,, \\ \overline{u} \ge \psi \quad \textit{a.e. in } \Omega\,, \\ \overline{u} \ge g \quad \textit{a.e. on } \partial\Omega\,. \end{cases} \tag{5.23}$$

Then $u \le \overline{u}$ a.e. in Ω.

Corollary 5.2. *Let $u \in K(\psi, g)$ be the solution of the variational inequality (5.21) for $f \in L^2(\Omega)$ such that $f \leq 0$ a.e. in Ω. We suppose that the functions ψ and g are bounded above in Ω, respectively, on $\partial\Omega$, by a constant C_1, respectively, by a constant C_2. Then $u \leq \max\{C_1, C_2\}$ a.e. in Ω.*

Remark 5.2. The conditions of the above corollary are satisfied if, for instance, $g \in C(\partial\Omega)$ and $\psi \in C(\overline{\Omega})$ or $\psi \in H^2(\Omega)$ and $d = 2$.

5.1.2 Examples

In the sequel, we shall consider applications of the above results to a dam problem and an obstacle problem.

1. A dam problem

We consider the problem of stationary flow of an incompressible fluid through a dam of an isotropic homogeneous porous media on a horizontal impervious base (see, for instance, [3, 5, 6, 18]). We suppose that the dam separates two tanks of different levels H and h with $H > h$ and that it is bounded by parallel vertical walls, so that the flow may be considered to be two-dimensional. The geometry of this problem, taking into account that the flow is the same for any normal section in the porous medium, is given in Fig. 5.1.

The dam cross-section is assumed to be the rectangle $D = (0, a) \times (0, H)$, i.e. a is the width of the dam. The flow region is a subset of D, with its boundary partially unknown, defined by

$$\Omega = \{(x, y) \in D\,; \quad 0 < x < a\,,\, 0 < y < \varphi(x)\}\,. \tag{5.24}$$

where $\varphi : [0, a] \to [0, H]$ is a continuous and strictly decreasing function such that $\varphi(0) = H$, $\varphi(a) \geq h$.

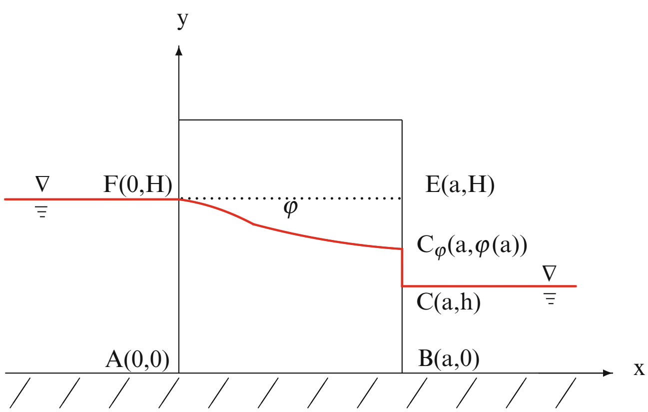

Fig. 5.1 A dam problem

If $\mathbf{v}$ denotes the velocity of the fluid, then by the continuity equation $\operatorname{div} \boldsymbol{v} = 0$ and the Darcy's law $\mathbf{v} = -\nabla u$ (the permeability coefficient of the porous medium is supposed equal to 1), we deduce

$$-\Delta u = 0 \qquad \text{in } \Omega\,, \tag{5.25}$$

where u is called a "potential velocity".

It is known (see [4]) that the above physical description leads to the following boundary conditions (according to Fig. 5.1):

$$\begin{cases} u = H & \text{on } [AF]\,, \\ u = h & \text{on } [BC]\,, \\ \dfrac{\partial u}{\partial \boldsymbol{v}} = 0 & \text{on } \widehat{FC}_\varphi \ \cup \ (AB)\,, \\ u = y & \text{on } \widehat{FC}_\varphi \cup [CC_\varphi]\,, \end{cases} \tag{5.26}$$

where the curve $\widehat{FC}_\varphi$ and the closed segment $[CC_\varphi]$ are, respectively, the free line and the seepage line, and $\dfrac{\partial u}{\partial \boldsymbol{v}}$ denotes the outward derivative normal.

The problem (5.25)–(5.26) has as unknown the triplet (φ, Ω, u). As in many free boundary problems, the solution cannot be directly written as the solution of a variational inequality. However, Baiocchi [3] introduced a change of unknown functions which allows to reduce this problem to a variational inequality of the type (5.6). Namely, supposing that the solution u, in the weak sense, of the problem (5.25)–(5.26) belongs to $H^1(\Omega) \cap C(\overline{\Omega})$, one defines the Baiocchi transformation:

$$w(x, y) = \int_y^H (\tilde{u}(x,t) - t)\,\mathrm{d}t \qquad \forall (x, y) \in D \tag{5.27}$$

where

$$\tilde{u}(x, y) = \begin{cases} u(x, y) & \text{if } (x, y) \in \overline{\Omega}\,, \\ y & \text{if } (x, y) \in \overline{D} \setminus \overline{\Omega}\,. \end{cases} \tag{5.28}$$

It is easy to show (for instance, see [6]) that w belongs to $H^1(D) \cap C(\overline{D})$ and it is the unique solution of the variational inequality

$$\int_D \nabla w \cdot \nabla (v - w)\,\mathrm{d}x\,\mathrm{d}y \geq -\int_D (v - w)\,\mathrm{d}x\,\mathrm{d}y \quad \forall v \in K \tag{5.29}$$

where

$$K = \{v \in H^1(D);\ v \geq 0 \text{ a.e. on } D\,,\ v = g \text{ on } \partial D\} \tag{5.30}$$

and

$$g(x, y) = w(x, y) \qquad \forall (x, y) \in \partial D\,. \tag{5.31}$$

From (5.26), (5.28), and (5.27), it is immediate that

$$g = \begin{cases} \dfrac{H^2}{2} - \dfrac{H^2 - h^2}{2a}x & \text{on } (AB)\,, \\ \dfrac{1}{2}(H-y)^2 & \text{on } [AF]\,, \\ \dfrac{1}{2}(h-y)^2 & \text{on } [BC]\,, \\ 0 & \text{on } \partial D \setminus ([AF] \cup (AB) \cup [BC])\,. \end{cases}$$

Now, if $w \in H^1(D) \cap C(\overline{D})$ is the unique solution of the variational inequality (5.29), then if we put

$$\Omega = \{(x, y);\ (x, y) \in D;\ w(x, y) > 0\}\,,$$

$$\varphi(x) = \sup\{y;\ (x, y) \in \Omega\} \qquad 0 < x < a\,,$$

$$\varphi(0) = \lim_{x \searrow 0} \varphi(x) \qquad \varphi(a) = \lim_{x \nearrow a} \varphi(x)\,,$$

$$u(x, y) = y - \frac{\partial w}{\partial y}(x, y) \quad \forall (x, y) \in \overline{\Omega}\,,$$

then, it follows that the triplet (φ, Ω, u) is the unique solution of the problem (5.25)–(5.26).

The function g has the properties: $g \geq 0$ on ∂D, $g = 0$ on $\Gamma_0 = (FE] \cup [EC)$, g is Lipschitz continuous and hence $g \in H^{1/2}(\partial\Omega)$ (see, e.g., [14]). Therefore, the problem (5.29) is a variational inequality of (5.6) type with $f = -1$.

Finally, by Corollary 5.1, we conclude that the unique solution of the variational inequality (5.29) satisfies:

$$w(x, y) \leq \frac{H^2}{2} \quad \text{in } D\,.$$

2. An obstacle problem.

We consider an elastic membrane, occupying an open bounded subset Ω of $\mathbb{R}^2$, which is fixed along its boundary Γ and must lie over an obstacle which is represented by a function $\psi : \overline{\Omega} \to \mathbb{R}$. The membrane is subject to the action of a vertical force of density f. We suppose that when $f = 0$, the membrane is in the plan of coordinates (x, y)) (Fig. 5.2).

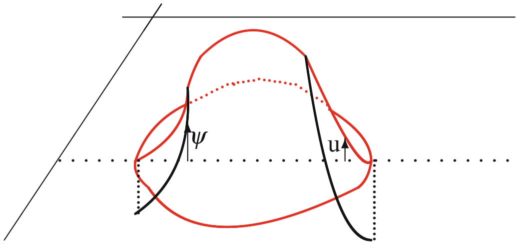

Fig. 5.2 An obstacle problem

The problem consists in finding the position of the membrane u and the free boundary γ such that:

$$\begin{cases} -\Delta u \geq f \quad \text{in } \Omega\,, \\ u \geq \psi \quad \text{in } \Omega\,, \\ (-\Delta u - f)(u - \psi) = 0 \quad \text{in } \Omega\,, \\ u = 0 \quad \text{on } \Gamma\,, \\ u^+ = u^0 \text{ on } \gamma\,. \end{cases} \tag{5.32}$$

We mention that γ is the part of the membrane which touches the obstacle, that is $\gamma = \partial\Omega^+ \cap \partial\Omega^0$ where $\Omega^+ = \{x \in \Omega\,;\, u(x) > \psi(x)\}$ and $\Omega^0 = \{x \in \Omega\,;\, u(x) = \psi(x)\}$. We also denoted $u^+ = u/\Omega^+$ and $u^0 = u/\Omega^0$.

Suppose that $f \in L^2(\Omega)$ and $\psi \in H^1(\Omega) \cap C(\overline{\Omega})$ with $\psi \leq 0$ on Γ. Then, it can be proved (see, for instance, [13], p. 26) that the variational formulation of this problem is a variational inequality of the form (5.21) with the notation

$$\begin{cases} V = H_0^1(\Omega)\,, \\ g = 0\,, \\ K(\psi, 0) = \{v \in V\,;\, v \geq \psi \text{ a.e. on } \Omega\}\,, \\ a(u, v) = \displaystyle\int_\Omega \nabla u \cdot \nabla v \,\mathrm{d}x\,\mathrm{d}y \quad \forall u, v \in V\,. \end{cases} \tag{5.33}$$

Applying Theorem 5.2 we find that the unique solution u of the variational inequality

$$a(u, v - u) \geq 0 \quad \forall v \in K(\psi, 0)$$

is the smallest supersolution of the above variational inequality. In addition, for $f \leq 0$ (we always may suppose $f \leq 0$), by applying Corollary 5.2, we deduce the expected physical result:

$$u \leq \max_{x \in \Omega} \psi(x) \quad \text{a.e. on } \Omega\,.$$

5.2 Regularity Properties

As the theory of variational inequalities represents a generalization of the theory of boundary value problems for partial differential equations, then it is of great interest to study the regularity of the solutions of variational inequalities.

Brézis and Stampacchia [8] studied this problem from an abstract point of view, considering a variational inequality of the first kind. The question is to find the conditions that ensure $Au \in \mathscr{W}$ if $f \in \mathscr{W}$, with $\mathscr{W}$ a subspace of V^*. Regularity results for the solutions of variational inequalities for a scalar second order elliptic operator have been obtained by many authors as, for instance, Lions [15], Necas [16], Duvaut [11].

In this section we study the regularity of the solutions of a class of variational inequalities of the second kind. We shall show, in Sect. 8.4, that these results allow us to obtain a local regularity for the solution of the quasi-variational inequality which models the Signorini problem with nonlocal Coulomb friction.

The section starts with some rappels on standard results which will be useful in what follows. Then, we state the variational inequality for which we obtain local and global regularity results. Our proof is based on the method of translation, due to Niremberg [17], as Brézis did in his thesis [7] for a scalar second order elliptic operator.

5.2.1 Notation and Preliminary Results

For a function v defined on $\mathbb{R}^d$, one introduces the notation

$$v_h^i(\mathbf{x}) = v(\mathbf{x} + h\mathbf{e}_i)\,, i \in \{1, ..., d\},$$

where $\mathbf{e}_i$ is the unit vector $(\delta_{1i}, \delta_{2i}, \ldots, \delta_{di})$, δ_{ji} being the Kronecker's symbol and h is a real number.

We first recall some standard results (for proofs, see, for instance, [2]).

Proposition 5.4. *Let Ω be an open set in $\mathbb{R}^d$. If $v \in H^1(\Omega)$ and $\varphi \in C^1(\overline{\Omega})$, then $v\varphi \in H^1(\Omega)$ and*

$$\frac{\partial}{\partial x_i}(v\varphi) = \frac{\partial v}{\partial x_i}\varphi + v\frac{\partial \varphi}{\partial x_i}\,, \quad i \in \{1, \ldots, d\}\,.$$

In the sequel we denote by C or C_j positive constants which we distinguish by subscripts if necessary.

Proposition 5.5. *Let Ω be an open set in $\mathbb{R}^d$ which has the segment property (cf. p. 16) and $v \in H^m(\Omega)$ with $m \geq 0$ an integer. If, for $i \in \{1, \ldots, d\}$, there exists a constant $C > 0$ such that*

$$\left\| \frac{v_h^i - v}{h} \right\|_{H^m(\Omega')} \leq C\,, \tag{5.34}$$

for every $\overline{\Omega}' \subset \Omega$ *and for all* $h \neq 0$ *with* $|h|$ *sufficiently small, then*

$$\left\| \frac{\partial v}{\partial x_i} \right\|_{H^m(\Omega)} \leq C\,.$$

If (5.34) holds for every $i \in \{1, \ldots, d\}$*, then* $v \in H^{m+1}(\Omega)$.

Proposition 5.6. *Let* Ω *be an open set in* $\mathbb{R}^d$. *Suppose that* $v \in H^m(\Omega)$, $m \geq 1$, *and let* $\overline{\Omega}' \subset \Omega$. *Then*

$$\left\| \frac{v_h^i - v}{h} \right\|_{H^{m-1}(\Omega')} \leq \|v\|_{H^m(\Omega)}\,,$$

for all $h \neq 0$ *such that dist* $(\overline{\Omega'}, \partial\Omega) > |h|$.

From the above proposition, we derive the following.

Corollary 5.3. *Let* $\eta \in C^\infty(\overline{S})$ *be such that supp* $\eta \subset S \cup \Sigma$*, where* $S = \{\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d\,;\ |\boldsymbol{\xi}| < 1\,,\ \xi_d > 0\}$, $d \geq 2$ *and* $\Sigma = \{\boldsymbol{\xi} \in \mathbb{R}^d\,;\ |\boldsymbol{\xi}| < 1\,,\ \xi_d = 0\}$. *Then, for every* $\boldsymbol{v} \in (H^1(S))^d$*, we have*

$$\left\| \frac{\eta(\boldsymbol{v}_i^h - \boldsymbol{v})}{h} \right\|_{(L^2(S))^d} \leq C\,\|\boldsymbol{v}\|_{(H^1(S))^d}\,,$$

for all $h \neq 0$ *with* $|h| <$ *dist* $(\partial S \backslash \Sigma$, *supp* $\eta)$ *and* $i \in \{1, 2, \ldots, d-1\}$.

Proof. Let $S_1 = \{\boldsymbol{\xi} \in S\,,\ \eta(\boldsymbol{\xi}) \neq 0\}$, $\tilde{S} = \{\boldsymbol{\xi} \in \mathbb{R}^d\,,\ |\boldsymbol{\xi}| < 1\}$ and $\tilde{S}_1 = S_1 \cup (\Sigma \cap \overline{S}_1) \cup \{\boldsymbol{\xi} = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d\,,\ (\xi_1, \ldots \xi_{d-1}, -\xi_d) \in S_1\}$.

For any function $\boldsymbol{w}$, we put

$$\tilde{\boldsymbol{w}}(\boldsymbol{\xi}) = \begin{cases} \boldsymbol{w}(\boldsymbol{\xi}) & \text{if } \xi_d \geq 0\,, \\ \boldsymbol{w}(\xi_1, \ldots, \xi_{d-1}, -\xi_d) & \text{if } \xi_d < 0\,. \end{cases} \tag{5.35}$$

It is easy to see that, if $\boldsymbol{w} \in (H^1(S))^d$, then $\tilde{\boldsymbol{w}} \in (H^1(\tilde{S}))^d$ and

$$\|\tilde{\boldsymbol{w}}\|^2_{(H^m(\tilde{S}))^d} = 2\|\boldsymbol{w}\|^2_{(H^m(S))^d} \quad \text{for } m \in \{0, 1\}\,. \tag{5.36}$$

Let $\tilde{\eta}$, $\tilde{\boldsymbol{v}}$ and $\tilde{\boldsymbol{v}}_h^i$ ($i \in \{1, \ldots, d-1\}$) be defined as in (5.35). Then supp $\tilde{\eta} = \tilde{\bar{S}}' \subset \tilde{S}$ and

$$\begin{aligned} &\left\| \frac{\eta(\boldsymbol{v}_h^i - \boldsymbol{v})}{h} \right\|_{(L^2(S))^d} = \left\| \frac{\eta(\boldsymbol{v}_h^i - \boldsymbol{v})}{h} \right\|_{(L^2(S'))^d} \\ &= \frac{1}{\sqrt{2}} \left\| \frac{\tilde{\eta}(\tilde{\boldsymbol{v}}_h^i - \tilde{\boldsymbol{v}})}{h} \right\|_{(L^2(\tilde{S}_1))^d} \leq \frac{1}{\sqrt{2}} C \left\| \frac{\tilde{\boldsymbol{v}}_h^i - \tilde{\boldsymbol{v}}}{h} \right\|_{(L^2(\tilde{S}_1))^d}\,. \end{aligned} \tag{5.37}$$

Applying Proposition 5.6 and using the relations (5.36) and (5.37), we conclude

$$\left\| \frac{\eta(\boldsymbol{v}_h^i - \boldsymbol{v})}{h} \right\|_{(L^2(S))^d} \le \frac{1}{\sqrt{2}} C \left\| \frac{\tilde{\boldsymbol{v}}_h^i - \tilde{\boldsymbol{v}}}{h} \right\|_{(L^2(\tilde{S}_1))^d} \le \frac{1}{\sqrt{2}} C \|\tilde{\boldsymbol{v}}\|_{(H^1(\tilde{S}))^d} = C \|\boldsymbol{v}\|_{(H^1(S))^d} .$$

□

5.2.2 Setting of the Problem and Regularity Results

Let Ω be an open bounded set in $\mathbb{R}^d$ with Γ an open subset of its boundary $\partial\Omega$. Let $x_0 \in \Gamma$. Suppose that Ω is C^3-smooth in x_0 (see Definition 2.6, 16), i.e. there exists a neighborhood I of x_0 such that the set $\overline{\Omega} \cap \overline{I}$ can be mapped C^3-homeomorphically onto $\overline{S}$ where $S = \{\boldsymbol{\xi} \in \mathbb{R}^d\ ;\ |\boldsymbol{\xi}| < 1\ ,\ \xi_d > 0\}$, such that the set $\partial\Omega \cap \overline{I}$ is mapped onto the set $\overline{\Sigma}$ where $\Sigma = \{\boldsymbol{\xi} = (\xi_1, \dots, \xi_d) \in \mathbb{R}^d\ ,\ |\boldsymbol{\xi}| < 1\ ,\ \xi_d = 0\}$. Without loss of generality, we may assume that $\partial\Omega \cap I \subset \Gamma$.

Let θ be the C^3-homeomorphism from $\overline{\Omega} \cap \overline{I}$ to $\overline{S}$. If $\boldsymbol{w}$ is a function defined on $\Omega \cap I$, we shall denote by $\tilde{\boldsymbol{w}}$ the function

$$\tilde{\boldsymbol{w}}(\boldsymbol{\xi}) = \boldsymbol{w}(\theta^{-1}(\boldsymbol{\xi})) \qquad \forall \boldsymbol{\xi} \in S .$$

Let $\boldsymbol{v} \in (H^1(\Omega))^d$. For $\eta \in \mathscr{D}(I)$ and h a real number, we set

$$\boldsymbol{v}_{h,\eta}^i(\boldsymbol{x}) = \begin{cases} \boldsymbol{v}(\boldsymbol{x}) + \eta(\boldsymbol{x}) \left(\tilde{\boldsymbol{v}}_h^i(\theta(\boldsymbol{x})) - \boldsymbol{v}(\boldsymbol{x})\right) & \text{if } \boldsymbol{x} \in \operatorname{supp} \eta \cap \Omega\,, \\ \boldsymbol{v}(\boldsymbol{x}) & \text{if } \boldsymbol{x} \in \Omega \setminus \operatorname{supp} \eta\,, \end{cases}$$

where $i \in \{1, \dots, d-1\}$ and $\tilde{\boldsymbol{v}}_h^i = (\tilde{\boldsymbol{v}})_h^i$. It is immediate that, for $|h|$ sufficiently small, $\boldsymbol{v}_{h,\eta}^i$ is well defined and $\boldsymbol{v}_{h,\eta}^i \in (H^1(\Omega))^d$.

In the sequel we use the summation convention.

We now define the bilinear form

$$b(\boldsymbol{u}, \boldsymbol{v}) = \int\limits_\Omega \Big(a_{ij}^{kl}(\boldsymbol{x}) \frac{\partial u_k}{\partial x_i} \frac{\partial v_l}{\partial x_j} + b_i^{kl}(\boldsymbol{x}) \frac{\partial u_k}{\partial x_i} v_l + c_i^{kl}(\boldsymbol{x}) \frac{\partial v_l}{\partial x_i} u_k$$
$$+ d^{kl}(\boldsymbol{x}) u_k v_l \Big)\, \mathrm{d}x \quad \forall \boldsymbol{u}, \boldsymbol{v} \in (H^1(\Omega))^d$$

where $a_{ij}^{kl},\ b_i^{kl},\ c_i^{kl},\ d^{kl} \in C^1(\overline{\Omega})$ and $a_{ij}^{kl} = a_{ji}^{kl},\ \forall i, j, k, l \in \{1, \dots, d\}$.

In the matrix form, we can write

$$b(\boldsymbol{u}, \boldsymbol{v}) = \int\limits_\Omega \left(\mathbf{A}_{ij}(\boldsymbol{x}) \boldsymbol{u}_{,i} \boldsymbol{v}_{,j} + \mathbf{B}_i(\boldsymbol{x}) \boldsymbol{u}_{,i} \boldsymbol{v} + \mathbf{C}_i(\boldsymbol{x}) \boldsymbol{u} \boldsymbol{v}_{,i} + \mathbf{D}(\boldsymbol{x}) \boldsymbol{u} \boldsymbol{v} \right) \mathrm{d}x \,,$$

where

$$\boldsymbol{w}_{,i} = \left(\frac{\partial w_1}{\partial x_i}, \dots, \frac{\partial w_d}{\partial x_i} \right),\ \mathbf{A}_{ij} = (a_{ij}^{kl})_{k,l}\,,\ \mathbf{B}_i = (b_i^{kl})_{k,l}\,,\ \mathbf{C}_i = (c_i^{kl})_{k,l}\,,\ \mathbf{D} = (d^{kl})_{k,l}\,.$$

For any open set ω in $\mathbb{R}^d$, we shall denote by $\|\cdot\|_{m,\omega}$ the norm on the product space $(H^m(\omega))^d$.

We suppose that there exists a constant $\alpha > 0$ such that

$$b(\boldsymbol{v},\boldsymbol{v}) \geq \alpha\|\boldsymbol{v}\|^2_{1,\Omega} \qquad \forall \boldsymbol{v} \in (H^1(\Omega))^d \text{ with } \operatorname{supp} \boldsymbol{v} \subset \overline{\Omega} \cap I\,. \tag{5.38}$$

We consider the functional $J : (H^1(\Omega))^d \to \mathbb{R}$ defined by

$$J(\boldsymbol{v}) = \int_\Gamma r(s)\psi(\boldsymbol{v}(s))\,\mathrm{d}s \quad \forall \boldsymbol{v} \in (H^1(\Omega))^d\,,$$

where $r \in H^1(\Gamma)$ with $r \geq 0$ a.e. on Γ and ψ is a seminorm on $\mathbb{R}^d$.

We denote by Q a nonempty closed convex subset of $(H^1(\Omega))^d$ such that

$$\begin{cases} \text{if } \boldsymbol{v} \in Q \text{ then } \boldsymbol{v}^l_{h,\eta} \in Q\,,\ \forall l \in \{1,\ldots,d-1\}\,,\ \forall \eta \in \mathscr{D}(I) \text{ with } 0 \leq \eta \leq 1\,,\\ \text{and } \forall h \neq 0 \text{ with } |h| < \operatorname{dist}(\partial S \backslash \Sigma,\ \operatorname{supp}\tilde{\eta})\,, \end{cases} \tag{5.39}$$

where $\tilde{\eta}(\boldsymbol{\xi}) = \eta(\theta^{-1}(\boldsymbol{\xi}))\,,\ \forall \boldsymbol{\xi} \in S$.

With the above notation, we consider the following variational inequality of second kind

$$\begin{cases} \boldsymbol{u} \in Q\,,\\ b(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + J(\boldsymbol{v}) - J(\boldsymbol{u}) \geq (\boldsymbol{L},\boldsymbol{v}-\boldsymbol{u}) \quad \forall \boldsymbol{v} \in Q \end{cases} \tag{5.40}$$

where $\boldsymbol{L} \in (L^2(\Omega))^d$ is defined by

$$(\boldsymbol{L},\boldsymbol{v}) = \int_\Omega L_i v_i\,\mathrm{d}x \quad \forall \boldsymbol{v} \in (H^1(\Omega))^d\,.$$

We can state our local regularity result.

Theorem 5.3. *Suppose that there exists a solution $\boldsymbol{u}$ of the variational inequality (5.40). Then, for any open set I' containing $\boldsymbol{x}_0$ such that $\overline{I}' \subset I$, we have $\boldsymbol{u} \in (H^2(\Omega \cap I'))^d$ and*

$$\|\boldsymbol{u}\|_{2,\Omega\cap I'} \leq C(\|\boldsymbol{u}\|_{1,\Omega\cap I} + \|r\|_{H^1(\Gamma\cap I)} + \|\boldsymbol{L}\|_{0,\Omega\cap I})\,. \tag{5.41}$$

Proof. Let $S' = \theta(\Omega \cap I')$. We shall prove that

$$\|\tilde{\boldsymbol{u}}\|_{2,S'} \leq C(\|\tilde{\boldsymbol{u}}\|_{1,S} + \|\tilde{r}\|_{H^1(\Sigma)} + \|\tilde{\boldsymbol{L}}\|_{0,S})\,.$$

First, let $\eta \in \mathscr{D}(\Omega)$ with $0 \leq \eta \leq 1$. It is easy to see that, if we take in (5.40) $\boldsymbol{v} = \boldsymbol{u}^l_{h,\eta}$ and, respectively, $\boldsymbol{v} = \boldsymbol{u}^l_{-h,\eta}$, $l \in \{1,\ldots,d-1\}$, we obtain by using local coordinates

$$\tilde{b}(\tilde{\boldsymbol{u}}, \tilde{\eta}(\tilde{\boldsymbol{u}}_h^l - \tilde{\boldsymbol{u}})) + \int_\Sigma \tilde{r}\tilde{\eta}\psi(\tilde{\boldsymbol{u}}_h^l)\,\mathrm{d}\sigma - \int_\Sigma \tilde{r}\tilde{\eta}\psi(\tilde{\boldsymbol{u}})\,\mathrm{d}\sigma \geq (\tilde{\boldsymbol{L}}, \tilde{\eta}(\tilde{\boldsymbol{u}}_h^l - \tilde{\boldsymbol{u}})) \tag{5.42}$$

$$\tilde{b}(\tilde{\boldsymbol{u}}, \tilde{\eta}(\tilde{\boldsymbol{u}}_{-h}^l - \tilde{\boldsymbol{u}})) + \int_\Sigma \tilde{r}\tilde{\eta}\psi(\tilde{\boldsymbol{u}}_{-h}^l)\,\mathrm{d}\sigma - \int_\Sigma \tilde{r}\tilde{\eta}\psi(\tilde{\boldsymbol{u}})\,\mathrm{d}\sigma \geq (\tilde{\boldsymbol{L}}, \tilde{\eta}(\tilde{\boldsymbol{u}}_{-h}^l - \tilde{\boldsymbol{u}})) \tag{5.43}$$

where

$$\tilde{b}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{v}}) = \int_S (\mathbf{A}_{ij}\tilde{\boldsymbol{u}}_{,i}\tilde{\boldsymbol{v}}_{,j} + \mathbf{B}_i\tilde{\boldsymbol{u}}_{,i}\tilde{\boldsymbol{v}} + \mathbf{C}_i\tilde{\boldsymbol{u}}\tilde{\boldsymbol{v}}_{,i} + \mathbf{D}\tilde{\boldsymbol{u}}\tilde{\boldsymbol{v}})\,\mathrm{d}\xi\,,$$

$$(\tilde{\boldsymbol{L}}, \tilde{\boldsymbol{v}}) = \int_S L_i\tilde{v}_i\,\mathrm{d}\xi\,.$$

Here, for the sake of simplicity, we do not change the notation for $\mathbf{A}_{ij}$, $\mathbf{B}_i$, $\mathbf{C}_i$, $\mathbf{D}$ and L_i.

Let $\varphi \in \mathscr{D}(I)$ be such that $0 \leq \varphi \leq 1$ and $\varphi \equiv 1$ in I'. Taking $\tilde{\eta} = \tilde{\varphi}^2$ in (5.42) and $\tilde{\eta} = \tilde{\varphi}_{-h}^2$ in (5.43), by adding the two inequalities, we get for $|h|$ sufficiently small

$$\begin{aligned} 0 &\leq b(\boldsymbol{u}, \varphi^2(\boldsymbol{u}_h - \boldsymbol{u})) - b(\boldsymbol{u}, \varphi_{-h}^2(\boldsymbol{u} - \boldsymbol{u}_{-h})) + \int_\Sigma r\varphi^2(\psi(\boldsymbol{u}_h) - \psi(\boldsymbol{u}))\,\mathrm{d}\sigma \\ &+ \int_\Sigma r\varphi_{-h}^2\,(\psi(\boldsymbol{u}_{-h}) - \psi(\boldsymbol{u}))\,\mathrm{d}\sigma + (\boldsymbol{L}, \varphi_{-h}^2(\boldsymbol{u} - \boldsymbol{u}_{-h})) - (\boldsymbol{L}, \varphi^2(\boldsymbol{u}_h - \boldsymbol{u})) \end{aligned} \tag{5.44}$$

where, for simplicity, we omitted the "~" and the index l. Therefore, from (5.44), we deduce that

$$\begin{aligned} & b(\varphi(\boldsymbol{u}_h - \boldsymbol{u}), \varphi(\boldsymbol{u}_h - \boldsymbol{u})) \leq b(\varphi(\boldsymbol{u}_h - \boldsymbol{u}), \varphi(\boldsymbol{u}_h - \boldsymbol{u})) + b(\boldsymbol{u}, \varphi^2(\boldsymbol{u}_h - \boldsymbol{u})) \\ & - b(\boldsymbol{u}, \varphi_{-h}^2(\boldsymbol{u} - \boldsymbol{u}_{-h})) + \int_\Sigma r\varphi^2(\psi(\boldsymbol{u}_h) - \psi(\boldsymbol{u}))\,\mathrm{d}\sigma \\ & + \int_\Sigma r\varphi_{-h}^2(\psi(\boldsymbol{u}_{-h}) - \psi(\boldsymbol{u}))\,\mathrm{d}\sigma + (\boldsymbol{L}, \varphi_{-h}^2(\boldsymbol{u} - \boldsymbol{u}_{-h})) - (\boldsymbol{L}, \varphi^2(\boldsymbol{u}_h - \boldsymbol{u}))\,. \end{aligned} \tag{5.45}$$

We now estimate the right-hand side of the inequality (5.45). First, from Proposition 5.4, we have

$$
\begin{aligned}
& b(\varphi(\boldsymbol{u}_h-\boldsymbol{u}),\varphi(\boldsymbol{u}_h-\boldsymbol{u}))+b(\boldsymbol{u},\varphi^2(\boldsymbol{u}_h-\boldsymbol{u}))-b(\boldsymbol{u},\varphi_{-h}^2(\boldsymbol{u}-\boldsymbol{u}_{-h}))\\
&=\int_S\{\mathbf{A}_{ij}[\varphi(\boldsymbol{u}_h-\boldsymbol{u})]_{,i}[\varphi(\boldsymbol{u}_h-\boldsymbol{u})]_{,j}+\mathbf{B}_i[\varphi(\boldsymbol{u}_h-\boldsymbol{u})]_{,i}[\varphi(\boldsymbol{u}_h-\boldsymbol{u})]\\
&+\mathbf{C}_i[\varphi(\boldsymbol{u}_h-\boldsymbol{u})][\varphi(\boldsymbol{u}_h-\boldsymbol{u})]_{,i}+\mathbf{D}\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})(\boldsymbol{u}_h-\boldsymbol{u})-\mathbf{A}_{ij}\boldsymbol{u}_{,i}[\varphi_{-h}^2(\boldsymbol{u}-\boldsymbol{u}_{-h})]_{,j}\\
&-\mathbf{B}_i\boldsymbol{u}_{,i}[\varphi_{-h}^2(\boldsymbol{u}-\boldsymbol{u}_{-h})]-\mathbf{C}_i\boldsymbol{u}[\varphi_{-h}^2(\boldsymbol{u}-\boldsymbol{u}_{-h})]_{,i}-\mathbf{D}\boldsymbol{u}\varphi_{-h}^2(\boldsymbol{u}-\boldsymbol{u}_{-h})\\
&+\mathbf{A}_{ij}\boldsymbol{u}_{,i}[\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})]_{,j}+\mathbf{B}_i\boldsymbol{u}_{,i}[\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})]+\mathbf{C}_i\boldsymbol{u}[\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})]_{,i}\\
&+\mathbf{D}\boldsymbol{u}\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})\}\,\mathrm{d}\xi=\int_S\{\mathbf{A}_{ij}\varphi_{,i}\varphi_{,j}(\boldsymbol{u}_h-\boldsymbol{u})(\boldsymbol{u}_h-\boldsymbol{u})\\
&+\mathbf{A}_{ij}\varphi_{,i}\varphi(\boldsymbol{u}_h-\boldsymbol{u})(\boldsymbol{u}_h-\boldsymbol{u})_{,j}+\mathbf{A}_{ij}\varphi\varphi_{,j}(\boldsymbol{u}_h-\boldsymbol{u})_{,i}(\boldsymbol{u}_h-\boldsymbol{u})\\
&+\mathbf{A}_{ij}\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})_{,i}(\boldsymbol{u}_h-\boldsymbol{u})_{,j}+\mathbf{B}_i[\varphi(\boldsymbol{u}_h-\boldsymbol{u})]_{,i}\varphi(\boldsymbol{u}_h-\boldsymbol{u})\\
&+\mathbf{C}_i\varphi(\boldsymbol{u}_h-\boldsymbol{u})[\varphi(\boldsymbol{u}_h-\boldsymbol{u})]_{,i}-[(\mathbf{A}_{ij})_h\boldsymbol{u}_{h,i}-\mathbf{A}_{ij}\boldsymbol{u}_{,i}][\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})]_{,j}\\
&-[(\mathbf{B}_i)_h\boldsymbol{u}_{h,i}-\mathbf{B}_i\boldsymbol{u}_{,i}]\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})-[(\mathbf{C}_i)_h\boldsymbol{u}_h-\mathbf{C}_i\boldsymbol{u}][\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})]_{,i}\\
&-(\mathbf{D}_h-\mathbf{D})\boldsymbol{u}_h\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})\}\,\mathrm{d}\xi=\int_S\{\mathbf{A}_{ij}\varphi_{,i}\varphi_{,j}(\boldsymbol{u}_h-\boldsymbol{u})(\boldsymbol{u}_h-\boldsymbol{u})\\
&-[(\mathbf{A}_{ij})_h-\mathbf{A}_{ij}]\boldsymbol{u}_{h,i}[\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})]_{,j}+\mathbf{B}_i[\varphi(\boldsymbol{u}_h-\boldsymbol{u})_{,i}\varphi(\boldsymbol{u}_h-\boldsymbol{u})\\
&+\mathbf{C}_i\varphi(\boldsymbol{u}_h-\boldsymbol{u})[\varphi(\boldsymbol{u}_h-\boldsymbol{u})]_{,i}-[(\mathbf{B}_i)_h\boldsymbol{u}_{h,i}-\mathbf{B}_i\boldsymbol{u}_{,i}]\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})\\
&-[(\mathbf{C}_i)_h\boldsymbol{u}_h-\mathbf{C}_i\boldsymbol{u}]\cdot[\varphi_{,i}\varphi(\boldsymbol{u}_h-\boldsymbol{u})+\varphi(\varphi(\boldsymbol{u}_h-\boldsymbol{u}))_{,i}]\\
&-(\mathbf{D}_h-\mathbf{D})\boldsymbol{u}_h\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})\}\,\mathrm{d}\xi\,.
\end{aligned}
$$

Now we can apply Corollary 5.3 by taking $\eta=\varphi$ and $\eta=\varphi_{,i}$. It follows that, for all $h\neq 0$ such that $|h|<\operatorname{dist}(\partial S\backslash\Sigma,\operatorname{supp}\varphi)$, we give

$$
\begin{aligned}
&\frac{1}{|h|}[b(\varphi(\boldsymbol{u}_h-\boldsymbol{u}),\varphi(\boldsymbol{u}_h-\boldsymbol{u}))+b(\boldsymbol{u},\varphi^2(\boldsymbol{u}_h-\boldsymbol{u}))-b(\boldsymbol{u},\varphi_{-h}^2(\boldsymbol{u}-\boldsymbol{u}_{-h}))]\\
&\leq C_1\|\boldsymbol{u}\|_{1,S}\|\varphi(\boldsymbol{u}_h-\boldsymbol{u})\|_{1,S}\,.
\end{aligned}
\tag{5.46}
$$

On the other hand, from Proposition 5.6, we have

$$
\begin{aligned}
&\int_\Sigma r\varphi^2[\psi(\boldsymbol{u}_h)-\psi(\boldsymbol{u})]\,\mathrm{d}\sigma+\int_\Sigma r\varphi_{-h}^2[\psi(\boldsymbol{u}_{-h})-\psi(\boldsymbol{u})]\,\mathrm{d}\sigma\\
&=\int_\Sigma(r-r_h)\varphi^2[\psi(\boldsymbol{u}_h-\psi(\boldsymbol{u})]\,\mathrm{d}\sigma\leq C_2|h|\,\|r\|_{H^1(\Sigma)}\|\varphi(\boldsymbol{u}_h-\boldsymbol{u})\|_{1,S}
\end{aligned}
\tag{5.47}
$$

and

$$
\begin{aligned}
&(\boldsymbol{L},\varphi_{-h}^2(\boldsymbol{u}-\boldsymbol{u}_{-h}))-(\boldsymbol{L},\varphi^2(\boldsymbol{u}_h-\boldsymbol{u}))=-(\boldsymbol{L},\varphi^2(\boldsymbol{u}_h-\boldsymbol{u})+\varphi_{-h}^2(\boldsymbol{u}_{-h}-\boldsymbol{u}))\\
&\leq C_3|h|\,\|\boldsymbol{L}\|_{0,S}\|\varphi(\boldsymbol{u}_h-\boldsymbol{u})\|_{1,S}\,.
\end{aligned}
\tag{5.48}
$$

Combining (5.45)–(5.48) and using the coerciveness condition (5.38), one deduces

$$\left\|\frac{\varphi(\boldsymbol{u}_h - \boldsymbol{u})}{h}\right\|_{1,S} \leq C_4(\|\boldsymbol{u}\|_{1,S} + \|r\|_{H^1(\Sigma)} + \|\boldsymbol{L}\|_{0,S}),$$

and hence

$$\left\|\frac{\boldsymbol{u}_h - \boldsymbol{u}}{h}\right\|_{1,S'} \leq C_4(\|\boldsymbol{u}\|_{1,S} + \|r\|_{H^1(\Sigma)} + \|\boldsymbol{L}\|_{0,S}). \tag{5.49}$$

We then conclude from Proposition 5.5 that

$$\frac{\partial^2 \boldsymbol{u}}{\partial \xi_i \partial \xi_j} \in (L^2(S'))^d \quad \text{for } i \in \{1, \ldots, d-1\} \text{ and } j \in \{1, \ldots, d\}.$$

Let us remark that $\boldsymbol{u}$ solves the system

$$-\frac{\partial}{\partial \xi_j}\left(\mathbf{A}_{ij}\frac{\partial \boldsymbol{u}}{\partial \xi_i}\right) + \mathbf{B}_i\frac{\partial \boldsymbol{u}}{\partial \xi_i} - \frac{\partial}{\partial \xi_i}(\mathbf{C}_i\boldsymbol{u}) + \mathbf{D}\cdot\boldsymbol{u} = \boldsymbol{L} \quad \text{in } S'. \tag{5.50}$$

The coerciveness condition (5.38) implies that $\det(\mathbf{A}_{dd}(\boldsymbol{\xi})) \neq 0, \ \forall \boldsymbol{\xi} \in S'$, and so, $\dfrac{\partial^2 \boldsymbol{u}}{\partial \xi_d^2}$ can be calculated from (5.50). Thus $\dfrac{\partial^2 \boldsymbol{u}}{\partial \xi_d^2} \in (L^2(S'))^d$, and, from (5.49), we obtain

$$\|\boldsymbol{u}\|_{2,S'} \leq C(\|\boldsymbol{u}\|_{1,S} + \|r\|_{H^1(\Sigma)} + \|\mathbf{F}\|_{0,S}).$$

Therefore, by transformation back to the $x_1, x_2, \cdots, x_d$ coordinates, we get the estimate (5.41). □

Remark 5.3. Using a similar technique as in the above theorem, we can obtain the known regularity result of the solution of the variational inequality (5.40) in a neighborhood of an interior point of Ω. Indeed, for any point $\boldsymbol{x} \in \Omega$, by taking $I = \{\mathbf{y} \in \Omega\,;\ |\mathbf{y} - \boldsymbol{x}| < R\}$ with R sufficiently small such that $\overline{I} \subset \Omega$, we can repeat the above proof with no need of using local coordinates, by requiring that

$$\begin{cases} \text{if } \boldsymbol{v} \in Q \text{ then } (1-\eta)\boldsymbol{v} + \eta\boldsymbol{v}_h^l \in Q\,,\ \forall l \in \{1, \ldots, d\}\,,\ \forall \eta \in \mathscr{D}(I) \\ \text{with } 0 \leq \eta \leq 1\,,\ \forall h \neq 0 \text{ such that } |h| < \text{dist}\,(\partial I,\ \text{supp}\,\eta)\,. \end{cases} \tag{5.51}$$

If Ω is C^3-smooth in $\boldsymbol{x} \in \partial\Omega$, one denotes by $I_{\boldsymbol{x}}$ the corresponding neighborhood of $\boldsymbol{x}$, and, if $\boldsymbol{x} \in \Omega$, one denotes by $I_{\boldsymbol{x}}$ the set $\{\mathbf{y} \in \mathbb{R}^d\,;\ |\mathbf{y} - \boldsymbol{x}| < R\}$ with R sufficiently small such that $I_{\boldsymbol{x}} \subset \Omega$.

Finally, from the local regularity result given by Theorem 5.3 and the above remark, we can easily obtain the following global regularity for the solution of the inequality (5.40).

Theorem 5.4. *Suppose that Ω is C^3-smooth in any point $\boldsymbol{x} \in \partial\Omega$ and $\Gamma = \partial\Omega$. Moreover, we suppose that the following assumptions hold*

$$\begin{cases} \text{if } \boldsymbol{v} \in Q \text{ then } \tilde{\boldsymbol{v}}_h^l \in Q\,,\ \forall l = 1,\ldots,d-1\,,\ \forall \eta \in \mathscr{D}(I_{\mathbf{x}}) \text{ with } 0 \le \eta \le 1\,, \\ \text{and } \forall h \neq 0 \text{ with } |h| < \text{ dist}\,(\partial S \backslash \Sigma,\ \text{supp}\ \tilde{\eta})\,,\ \forall \mathbf{x} \in \partial\Omega\,, \end{cases} \tag{5.52}$$

$$\begin{cases} \exists \alpha > 0 \text{ such that } b(\boldsymbol{v},\boldsymbol{v}) \ge \alpha \|\boldsymbol{v}\|^2_{1,\Omega}\,, \\ \quad \forall \boldsymbol{v} \in (H^1(\Omega))^d \text{ with supp } \boldsymbol{v} \subset I_{\boldsymbol{x}} \cap \overline{\Omega}\,,\ \forall \boldsymbol{x} \in \overline{\Omega}\,, \end{cases} \tag{5.53}$$

and

$$\begin{cases} \text{if } \boldsymbol{v} \in Q, \text{ then } (1-\eta)\boldsymbol{v} + \eta \boldsymbol{v}_h^l \in Q\,,\ \forall l = 1,\ldots,d\,,\ \forall \eta \in \mathscr{D}(I_{\boldsymbol{x}}) \\ \text{with } 0 \le \eta \le 1\,,\ \forall h \neq 0 \text{ such that } |h| < \text{ dist}\,(\partial I_{\boldsymbol{x}},\ \text{supp}\ \eta)\,,\ \forall \boldsymbol{x} \in \Omega\,. \end{cases} \tag{5.54}$$

If there exists a solution $\boldsymbol{u}$ of the variational inequality (5.40), then $\boldsymbol{u} \in (H^2(\Omega))^d$ and

$$\|\boldsymbol{u}\|_{2,\Omega} \le C\left(\|\boldsymbol{u}\|_{1,\Omega} + \|r\|_{H^1(\Gamma)} + \|\boldsymbol{L}\|_{0,\Omega}\right)\,. \tag{5.55}$$

Proof. For every $\boldsymbol{x} \in \overline{\Omega}$, let $I'_{\boldsymbol{x}}$ be an open set containing $\boldsymbol{x}$ such that $\overline{I}'_{\boldsymbol{x}} \subset I_{\boldsymbol{x}}$. As $\overline{\Omega}$ is compact, we can extract from $\{I'_{\boldsymbol{x}}\}_{\boldsymbol{x}\in\overline{\Omega}}$ a finite open covering $\{I'_{\boldsymbol{x}_i}\}_{i=1,\ldots,n}$. Therefore, from Theorem 5.3 and the above remark, we get

$$\boldsymbol{u} \in (H^2(\Omega \cap I'_{\boldsymbol{x}_i}))^d$$

and

$$\|\boldsymbol{u}\|_{2,\Omega\cap I'_{\boldsymbol{x}_i}} \le C_i\left(\|\boldsymbol{u}\|_{1,\Omega\cap I_{\boldsymbol{x}_i}} + \|r\|_{H^1(\Gamma\cap I_{\boldsymbol{x}_i})} + \|\boldsymbol{L}\|_{0,\Omega\cap I_{\boldsymbol{x}_i}}\right)\,, \tag{5.56}$$

for all $i \in \{1,\ldots,n\}$, C_i being a constant which depends on i. It follows that $\boldsymbol{u} \in (H^2(\Omega))^d$ and, by adding the relations (5.56) for all i, we obtain (5.55). □

Note that Theorems 5.3 and 5.4 still hold under a less restrictive assumption on Ω.

Remark 5.4. In more restrictive assumptions on b and Ω and for $J = 0$, analogous regularity results were obtained by Fichera [12].

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Chapter 6
Dual Formulations of Quasi-Variational Inequalities

The aim of this chapter is to derive dual formulations for quasi-variational inequalities. First, we present a brief background on convex analysis and, then, we recall the main ideas of the Mosco, Capuzzo-Dolcetta, and Matzeu (M–CD–M) duality theory [3] in its form adapted by Telega [14] for implicit variational inequalities.

As we saw in Lemma 4.2, for A symmetric (i.e., $\langle Au, v\rangle = \langle u, Av\rangle,\ \forall u, v \in V$), a variational inequality of the form (4.22) is equivalent to the minimization of the functional J defined by

$$J(v) = \frac{1}{2}\langle Av, v\rangle + j(v) - \langle f, v\rangle.$$

Generally speaking, the duality theory allows to associate with a minimization problem

$$\inf_{v\in K} J(v), \tag{6.1}$$

called primal problem, a maximization one, called dual problem, and to study the relationships between the two problems.

A large number of duality theories have been developed. The main idea in any duality theory is that a proper convex l.s.c. function is the upper envelope of its affine minorants, and so, we can write

$$J(v) = \sup_{\lambda\in\Lambda} \mathscr{L}(v, \lambda).$$

for various choices of $\mathscr{L}$, called the Lagrangian function, and of the set Λ of Lagrange multipliers λ. Hence, the primal problem (6.1) can be written as

$$\inf_{v\in K} \sup_{\lambda\in\Lambda} \mathscr{L}(v, \lambda). \tag{6.2}$$

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The dual problem is defined by

$$\sup_{\lambda \in \Lambda} \inf_{v \in K} \mathscr{L}(v, \lambda). \tag{6.3}$$

The oldest of the theories of duality is that based on the classical theorems of minimax of Fan [7] and Sion [13]. They studied the existence of saddle points for the Lagrangian function $\mathscr{L}$ (a saddle point for $\mathscr{L}$ is an element $(v^*, \lambda^*) \in K \times \Lambda$ such that $\mathscr{L}(v^*, \lambda) \leq \mathscr{L}(v^*, \lambda^*) \leq \mathscr{L}(v, \lambda^*)$, $\forall v \in K$, $\forall \lambda \in \Lambda$) and they give criteria (see also [5]) which ensure that $\sup_{\lambda \in \Lambda} \inf_{v \in K} \mathscr{L}(v, \lambda) = \inf_{v \in K} \sup_{\lambda \in \Lambda} \mathscr{L}(v, \lambda)$.

Another theory has been developed by Fenchel [6] and Rockafellar [11]. In their theory, the minimization problem is approached by a family of perturbed problems and the dual problem is defined by means of the conjugate functions. More details can be found in Rockafellar [12], Céa [4], Ekeland and Temam [5].

The duality theory has many applications in mechanics, numerical analysis, control theory, game theory, or economics. In addition, the so-called primal–dual algorithms are often used in solving the primal problem. Nevertheless, classical duality approaches do not apply to quasi-variational inequalities since they cannot be formulated as extremum problems. For this reason, within this chapter we do not want to develop classical duality methods, our intention is only to recall some results of the M–CD–M [3] duality theory for the so-called implicit variational problems. In Sect. 8.5, we will use this theory to derive the so-called condensed dual formulation for a frictional contact problem.

6.1 Convex Analysis Background

We recall some definitions and standard results which will be useful in the subsequent paragraph. Let V be a reflexive Banach space with its dual V^* (we note that almost all the results remain valid if V and V^* are two topological vector spaces which are in duality; see, for instance, [1, 5, 8, 10]). We denote by $\langle \cdot, \cdot \rangle_{V^* \times V}$ the duality pairing between V^* and V.

Let $f : V \longrightarrow \overline{\mathbb{R}}$ be a function.

Let us recall that the effective domain of f, the epigraph of f and, for any $a \in \overline{\mathbb{R}}$, the level sets are defined by

$$\text{dom } f = \{v \in V \, : \, f(v) < \infty\},$$
$$\text{epi } f = \{(v, a) \in V \times \overline{\mathbb{R}} \, : \, f(v) \leq a\},$$

and, respectively,

$$E_a(f) = \{v \in V \, : \, f(v) \leq a\}.$$

The function f is said to be proper if dom $f \neq \emptyset$ and $f(v) > -\infty, \forall v \in V$.

The convexity and the lower semicontinuity of functions can be characterized in the following way.

Proposition 6.1. *Let $f : V \longrightarrow \overline{\mathbb{R}}$ be a function. Then the following statements are equivalent:*

(i) the function f is convex and l.s.c. on V;
(ii) the set epi f is a convex and closed subset of $V \times \overline{\mathbb{R}}$.

Proof. For convexity, we only use its definition for functions and sets.

If f is l.s.c., then it is easy to show that epi f is closed in $V \times \overline{\mathbb{R}}$. Conversely, if the set epi f is closed in $V \times \overline{\mathbb{R}}$, then, for any $a \in \overline{\mathbb{R}}$, the level sets $E_a(f)$ are closed in V and so, the sets $\{v \in V \ : \ f(v) > a\}$ are open, i.e. the function f is l.s.c. on V. □

Definition 6.1. The function $f^* : V^* \longrightarrow \overline{\mathbb{R}}$ defined by

$$f^*(v^*) = \sup_{v \in V}\{\langle v^*, v\rangle_{V^* \times V} - f(v)\},$$

is called the Fenchel conjugate (sometimes also called convex conjugate, conjugate function, or polar function) to f.

In the particular case $V = \mathbb{R}$, f^* is the Young conjugate function to f.

An elementary property is the following Young inequality

$$f(v) + f^*(v^*) \geq \langle v^*, v\rangle_{V^* \times V} \quad \forall v \in V, \ \forall v^* \in V^*. \tag{6.4}$$

Remark 6.1. Let $C \subset V$ be a set such that $0 \in C$. Then

$$I_C^*(v^*) = \sup_{v \in C}\{\langle v^*, v\rangle_{V^* \times V}\} = I_{C^*}(v^*)$$

where $C^* = \{v^* \in V^* \ : \ \langle v^*, v\rangle_{V^* \times V} \leq 0, \ \forall v \in C\}$ is the polar cone of C and I_A is the indicator function of the set A.

We give below a separation theorem (see, e.g., [8]) which will be frequently used in the sequel.

Theorem 6.1. *Let M be a convex closed subset of V and let be $v_0 \in V$ such that $v_0 \notin M$. Then there exists $v^* \in V^*$, $v^* \neq 0$, strictly separating M and v_0, i.e. there exists $c \in \mathbb{R}$ such that*

$$\langle v^*, v_0\rangle_{V^* \times V} > c \geq \langle v^*, v\rangle_{V^* \times V} \quad \forall v \in M.$$

Proposition 6.2. *Let $f : V \longrightarrow \overline{\mathbb{R}}$ be a function. Then*

1) The conjugate function f^* is convex l.s.c. on V^*.
2) If f is proper convex l.s.c. on V, then f^* is proper.

Proof. 1) Let $u^*, v^* \in V$ and $t \in [0, 1]$. We have

$$f^*((1-t)u^*+tv^*) = \sup_{v\in V}\{(1-t)(\langle u^*, v\rangle_{V^*\times V}-f(v))+t(\langle v^*, v\rangle_{V^*\times V}-f(v))\}$$
$$\leq (1-t)f^*(u^*) + tf^*(v^*),$$

i.e. f^* is convex.

In order to prove that f^* is l.s.c., let be the sequence $\{u_n^*\}_n \subset V^*$ and let be $u^* \in V^*$ such that $u_n^* \to u^*$ strongly in V^*. Applying Young's inequality (6.4), we get

$$f^*(u_n^*) \geq \langle u_n^*, v\rangle_{V^*\times V} - f(v) \quad \forall v \in V,$$

and hence

$$\liminf_{n\to\infty} f^*(u_n^*) \geq \langle u^*, v\rangle_{V^*\times V} - f(v) \quad \forall v \in V.$$

This yields

$$\liminf_{n\to\infty} f^*(u_n^*) \geq f^*(u^*).$$

2) As f is proper, there exists $v_0 \in V$ such that $f(v_0) < \infty$. Hence, Young's inequality (6.4) yields

$$f^*(v^*) \geq \langle v^*, v_0\rangle_{V^*\times V} - f(v_0) > -\infty \quad \forall v^* \in V^*.$$

Let $d > 0$. Since $(v_0, f(v_0)-d) \notin \text{epi } f$ and epi f is convex closed in $V\times\mathbb{R}$, by the Separation Theorem 6.1, it follows that there exist $v_0^* \in V^*$, $v_0^* \neq 0$, and $\alpha \in \mathbb{R}$ such that

$$\langle v_0^*, v_0\rangle_{V^*\times V} + \alpha(f(v_0)-d) > \langle v_0^*, v\rangle_{V^*\times V} + \alpha a \quad \forall (v, a) \in \text{epi } f. \tag{6.5}$$

It is easy to prove that $\alpha < 0$. Indeed, if we suppose that $\alpha > 0$, then, for any $(v, a) \in \text{epi } f$, we can take $(v, a+n) \in \text{epi } f$ in (6.5), for any $n > 0$. Thus the right-hand side of (6.5) tends to $+\infty$ which is in contradiction with the relation (6.5). If $\alpha = 0$, then we obtain $\langle v_0^*, v_0\rangle_{V^*\times V} > \langle v_0^*, v\rangle_{V^*\times V}$, $\forall v \in V$ which contradicts $v_0 \in V$.

Therefore, if we put $v_1^* = -\dfrac{1}{\alpha}v_0^*$ in (6.5), in particular we deduce that

$$\langle v_1^*, v_0\rangle_{V^*\times V} - f(v_0) + d > \langle v_1^*, v\rangle_{V^*\times V} - f(v) \quad \forall v \in V,$$

and so, as $v_0 \in \text{dom}\,(f)$, we get

$$+\infty > \langle v_1^*, v_0\rangle_{V^*\times V} - f(v_0) + d > \sup_{v\in V}\{\langle v_1^*, v\rangle_{V^*\times V} - f(v)\} = f^*(v_1^*),$$

and hence, f^* is proper. □

Definition 6.2. Let $f^* : V^* \to \overline{\mathbb{R}}$ be the conjugate function to f. Then the function $f^{**} : V \to \overline{\mathbb{R}}$ defined by

$$f^{**}(v) = \sup_{v^*\in V^*} \{\langle v^*, v\rangle_{V^*\times V} - f^*(v^*)\},$$

is called the biconjugate function to f.

From Young's inequality (6.4), we always have $f^{**}(v) \leq \sup_{v^*\in V^*} \{\langle v^*, v\rangle_{V^*\times V} - \langle v^*, v\rangle_{V^*\times V} + f(v)\} = f(v)$, i.e.

$$f^{**}(v) \leq f(v) \quad \forall v \in V. \tag{6.6}$$

The following statement gives conditions which ensure the equality between a function and its biconjugate.

Theorem 6.2 (Fenchel–Moreau Duality Theorem). *Let $f : V \to \overline{\mathbb{R}}$ be a proper function. Then, f is l.s.c. and convex if and only if $f^{**} = f$.*

Proof. Suppose that f is l.s.c. and convex. By Proposition 6.2, it follows that f^* is a proper l.s.c. convex function, and so, f^{**} is a proper l.s.c. convex function.

As we always have $f^{**}(v) \leq f(v)$, suppose that there exists $v_0 \in V$ such that $f^{**}(v_0) < f(v_0)$. Thus $(v_0, f^{**}(v_0)) \notin \text{epi}\,(f)$. Applying the Separation Theorem 6.1, it follows that there exist $v_0^* \in V^*$, $v_0^* \neq 0$, and $\alpha \in \mathbb{R}$ such that

$$\langle v_0^*, v_0\rangle_{V^*\times V} + \alpha f^{**}(v_0) > \langle v_0^*), v\rangle_{V^*\times V} + \alpha a \quad \forall (v, a) \in \text{epi } f.$$

Proceeding as in the proof of Proposition 6.2 we conclude that $\alpha < 0$. If we put $v_1^* = -\dfrac{1}{\alpha} v_0^*$, then we deduce

$$\begin{aligned}\langle v_1^*, v_0\rangle_{V^*\times V} - f^{**}(v_0) &> \sup_{(v,a)\in \text{epi } f} \{\langle v_1^*), v\rangle_{V^*\times V} - a\} \\ &\geq \sup_{v\in V}\{\langle v_1^*), v\rangle_{V^*\times V} - f(v)\} = f^*(v_1^*)\end{aligned}$$

which contradicts the definition of $f^{**}(v_0)$.

Conversely, if $f = f^{**}$ then, by Proposition 6.2, it follows that f, as the conjugate to f^*, is l.s.c. convex on V. □

Remark 6.2. If $f : V \to \overline{\mathbb{R}}$ is a convex l.s.c. function which takes the value $-\infty$, then f is identically equal to $-\infty$. Therefore it is natural to consider convex l.s.c. functions $f : V \to (-\infty, +\infty]$.

Definition 6.3. Let $f : V \to (-\infty, +\infty]$ be a proper function and $u \in$dom (f). An element $u^* \in V^*$ is said to be subgradient of f at u (according to e.g., [9]) if

$$f(v) - f(u) \geq \langle u^*, v - u\rangle_{V^* \times V}, \quad \forall v \in V.$$

The set of all subgradients of f at u is called the subdifferential of f at u and is denoted by $\partial f(u)$,

$$\partial f(u) = \{u^* \in V^*;\ f(v) - f(u) \geq \langle u^*, v - u\rangle_{V^* \times V}, \quad \forall v \in V\}.$$

So, the subdifferential of f is the multivalued mapping $\partial f : V \to 2^{V^*}$ which associates with every $u \in V$ the subset $\partial f(u)$ of V^*.

The function f is said to be subdifferentiable at u, respectively, on V, if $\partial f(u) \neq \emptyset$, respectively, $\partial f(u) \neq \emptyset$, $\forall u \in V$.

The next result follows immediately from the definitions.

Theorem 6.3. *Let $f : V \to (-\infty, +\infty]$ be a proper function. Then, the following two conditions are equivalent:*

(1) $f(u) = \min_{v \in V} f(v)$,
(2) $0 \in \partial f(u)$

Theorem 6.4. *Let $f : V \to (-\infty, +\infty]$ be a function. Then the following two conditions are equivalent:*

(1) $f(u) + f^*(u^*) = \langle u^*, u\rangle$,
(2) $u^* \in \partial f(u)$.
Moreover, any of the above conditions implies
(3) $u \in \partial f^*(u^*)$.
In addition, if f is proper l.s.c. and convex, then the three above conditions are equivalent.

Proof. "(1) $\Rightarrow$ (2)" By using the hypothesis (1) and Young's inequality (6.4), we obtain

$$\langle u^*, u\rangle_{V^* \times V} - f(u) = f^*(u^*) \geq \langle u^*, v\rangle_{V^* \times V} - f(v) \quad \forall v \in V,$$

i.e. the condition (2).

"(2) $\Rightarrow$ (1)" If $u^* \in \partial f(u)$, then

$$\langle u^*, u\rangle_{V^* \times V} - f(u) \geq \langle u^*, v\rangle_{V^* \times V} - f(v) \quad \forall v \in V$$

and so,

$$\langle u^*, u\rangle_{V^*\times V} - f(u) \geq \sup_{\forall v\in V}\{\langle u^*, v\rangle_{V^*\times V} - f(v)\} = f^*(u^*).$$

Therefore, by Young's inequality (6.4), the assertion follows.

"(1) ⇒ (3)" By the definition of f^* and the hypothesis (1), we have

$$\begin{aligned} f^*(v^*) - f^*(u^*) &= \sup_{\forall v\in V}\{\langle v^*, v\rangle_{V^*\times V} - f(v)\} + f(u) - \langle u^*, u\rangle_{V^*\times V} \\ &\geq \langle v^*, u\rangle_{V^*\times V} - f(u) + f(u) - \langle u^*, u\rangle_{V^*\times V} \\ &= \langle v^* - u^*, u\rangle_{V^*\times V} \quad \forall v^* \in V^*, \end{aligned}$$

i.e. $u \in \partial f^*(u^*)$.

Suppose now that $f : V \to (-\infty, +\infty]$ is a proper l.s.c. convex function.

"(3) ⇒ (1)" If $u \in \partial f^*(u^*)$, then we have

$$\langle u^*, u\rangle_{V^*\times V} - f^*(u^*) \geq \langle v^*, u\rangle_{V^*\times V} - f^*(v^*) \quad \forall v^* \in V^*,$$

which implies

$$\langle u^*, u\rangle_{V^*\times V} - f^*(u^*) \geq \sup_{\forall v^*\in V^*}\{\langle v^*, u\rangle_{V^*\times V} - f^*(v^*)\} = f^{**}(u),$$

As Theorem 6.2 provides $f^{**}(u) = f(u)$, by Young's inequality (6.4), we conclude that $f^*(u^*) + f(u) = \langle u^*, u\rangle_{V^*\times V}$. □

Let $f_1, f_2 : V \to (-\infty, +\infty]$ be two proper functions.

Definition 6.4. The infimal convolution of functions f_1 and f_2, denoted by $f_1 \nabla f_2$, is the function defined by

$$(f_1 \nabla f_2)(u) = \inf_{v\in V}\{f_1(v) + f_2(u-v)\} = \inf_{\substack{v_1+v_2=u \\ v_1, v_2\in V}}\{f_1(v_1) + f_2(v_2)\} \quad \forall u \in V.$$

Definition 6.5. We say that the infimal convolution $f_1 \nabla f_2$ is exact at u if there exists $v \in V$ such that $(f_1 \nabla f_2)(u) = f_1(v) + f_2(u - v)$ or, equivalent, if there exist $v_1, v_2 \in V$ such that $v_1 + v_2 = u$ and $(f_1 \nabla f_2)(u) = f_1(v_1) + f_2(v_2)$.

Proposition 6.3. *Let $f_1, f_2 : V \to (-\infty, +\infty]$ be proper functions. Then*

(1) $(f_1 \nabla f_2)^* = f_1^* + f_2^*$,
(2) If $f_1 \nabla f_2$ is exact at u, i.e. there exists $u_1, u_2 \in V$ such that $u_1 + u_2 = u$ and $(f_1 \nabla f_2)(u) = f_1(u_1) + f_2(u_2)$, then $\partial(f_1 \nabla f_2)(u) = \partial f(u_1) \cap \partial f_2(u_2)$.
(3) If f_1, f_2 are convex, then $f_1 \nabla f_2$ is convex.

Proof. (1) By definitions, we have

$$
\begin{aligned}
(f_1 \nabla f_2)^*(u^*) &= \sup_{v \in V}\{\langle u^*, v\rangle_{V^*\times V} - \inf_{u\in V}\{f_1(u) + f_2(v-u)\}\} \\
&= \sup_{v \in V}\{\langle u^*, v\rangle_{V^*\times V} + \sup_{u\in V}\{-f_1(u) - f_2(v-u)\}\} \\
&= \sup_{u,v \in V}\{\langle u^*, v\rangle_{V^*\times V} - f_1(u) - f_2(v-u)\} \\
&= \sup_{u \in V}\{\langle u^*, u\rangle_{V^*\times V} - f_1(u) + \sup_{v\in V}\{\langle u^*, v-u\rangle_{V^*\times V} - f_2(v-u)\}\} \\
&= \sup_{u \in V}\{\langle u^*, u\rangle_{V^*\times V} - f_1(u)\} + f_2^*(u^*) = f_1^*(u^*) + f_2^*(u^*).
\end{aligned}
$$

(2) Theorem 6.4, the relation (1) and the hypothesis yield that we have the following sequence of equivalent assertions

$$
\begin{aligned}
& u^* \in \partial(f_1 \nabla f_2)(u) \\
\Longleftrightarrow\ & (f_1 \nabla f_2)^*(u^*) + (f_1 \nabla f_2)(u) = \langle u^*, u\rangle_{V^*\times V} \\
\Longleftrightarrow\ & f_1^*(u^*) + f_2^*(u^*) + f_1(u_1) + f_2(u_2) = \langle u^*, u_1\rangle_{V^*\times V} + \langle u^*, u_2\rangle_{V^*\times V}
\end{aligned}
$$

As from the Young inequality (6.4) we have

$$
\begin{aligned}
f_1^*(u^*) + f_1(u_1) &\geq \langle u^*, u_1\rangle_{V^*\times V}, \\
f_2^*(u^*) + f_2(u_2) &\geq \langle u^*, u_2\rangle_{V^*\times V},
\end{aligned}
$$

it follows that we must have $f_i^*(u^*) + f_i(u_i) = \langle u^*, u_i\rangle_{V^*\times V}$, for $i = 1, 2$. Again Theorem 6.4 provides $u^* \in \partial f_i(u_i)$, for $i = 1, 2$, i.e. $u^* \in \partial f_1(u_1) \cap \partial f_2(u_2)$.

(3) As f_1, f_2 are convex, it follows that epi f_1 and epi f_2 are convex sets in $V \times \overline{\mathbb{R}}$. We prove that

$$
\text{epi } (f_1 \nabla f_2) = \text{epi } (f_1) + \text{epi } (f_2),
$$

from which the assertion follows. Indeed, we have

$$
\begin{aligned}
& (u, a) \in \text{epi } (f_1 \nabla f_2) \\
\Longleftrightarrow\ & \inf_{\substack{v_1 + v_2 = u \\ v_1, v_2 \in V}} \{f_1(v_1) + f_2(v_2)\} \leq a \\
\Longleftrightarrow\ & \exists u_1, u_2 \in V,\ u_1 + u_1 = u \text{ s.t. } f_1(u_1) + f_2(u_2) \leq a \\
\Longleftrightarrow\ & f_1(u_1) \leq a_1,\ f_2(u_2) \leq a_2,\ u_1 + u_2 = u,\ a_1 + a_2 = a \\
\Longleftrightarrow\ & (u_1, a_1) \in \text{epi } f_1,\ (u_2, a_2) \in \text{epi } f_2,\ u_1 + u_2 = u,\ a_1 + a_2 = a \\
\Longleftrightarrow\ & (u, a) = (u_1, a_1) + (u_2, a_2) \in \text{epi } f_1 + \text{epi } f_2.
\end{aligned}
$$

□

We now recall the Fenchel's duality theorem. The proof is available in [9], so we omit here.

Theorem 6.5 (Fenchel's Duality Theorem). *Let $f, -g : V \to (-\infty, +\infty]$ be two proper convex l.s.c. functions. Suppose that there exists $u_0 \in dom(f) \cap dom(-g)$ such that f or g is continuous at u_0. Then*

$$\inf_{v \in V}\{f(v) - g(v)\} = \max_{v^* \in V^*}\{g_*(v^*) - f^*(v^*)\}, \tag{6.7}$$

where g_* is the concave conjugate function to g, i.e.

$$g_*(v^*) = \inf_{v \in V}\{\langle v^*, v\rangle_{V^* \times V} - g(v)\}.$$

Proposition 6.4. *Let $f_1, f_2 : V \to (-\infty, +\infty]$ be two proper convex l.s.c. functions. If there exists $u_0 \in dom(f_1) \cap dom(f_2)$ such that f_1 or f_2 is continuous at u_0, then*

$$(f_1+f_2)^*(u^*)=(f_1^* \nabla f_2^*)(u^*)=f_1^*(u_1^*)+f_2^*(u_2^*) \quad \forall u^* \in V^* \text{ with } u_1^*+u_2^*=u^*$$

i.e. $f_1^* \nabla f_2^*$ is exact on V^*.

Proof. Let $u^* \in V^*$. We apply Fenchel's Duality Theorem 6.5 for

$$f(v) = f_2(v),\ g(v) = \langle u^*, v\rangle_{V^* \times V} - f_1(v) \quad \forall v \in V.$$

It is easy to verify that

$$\inf_{v \in V}\{f(v) - g(v)\} = -\sup_{v \in V}\{\langle u^*, v\rangle_{V^* \times V} - (f_1 + f_2)(v)\} = -(f_1^* + f_2^*)(u^*)$$

and

$$\begin{aligned}\max_{v^* \in V^*}\{g_*(v^*) - f^*(v^*)\} &= -\min_{v^* \in V^*}\{f_1^*(u^* - v^*) + f_2^*(v^*)\} \\ &= -f_1^*(u_1^*) - f_2^*(u_2^*), \quad u_1^* + u_2^* = u^*.\end{aligned}$$

On the other hand, from the definition of the infinimal convolution, we have

$$\min_{v^* \in V^*}\{f_1^*(u^* - v^*) + f_2^*(v^*)\} = (f_1^* \nabla f_2^*)(u^*).$$

Therefore, by (6.7), we get $(f_1^* + f_2^*)(u^*) = (f_1^* \nabla f_2^*)(u^*) = f_1^*(u_1^*) + f_2^*(u_2^*)$, $\forall u^* \in V^*$, and $u_1^* + u_2^* = u^*$, which completes the proof. □

Theorem 6.6. *Let $f_1, f_2 : V \to (-\infty, +\infty]$ be two proper convex l.s.c. functions. Suppose that there exists $u_0 \in dom(f_1) \cap dom(f_2)$ such that f_1 is continuous at u_0. Then*

$$\partial(f_1 + f_2)(u) = \partial f_1(u) + \partial f_2(u) \quad \forall u \in V.$$

Proof. Let $u \in V$.

We first prove that $\partial(f_1 + f_2)(u) \subset \partial f_1(u) + \partial f_2(u)$. Let $u^* \in \partial(f_1 + f_2)(u)$. Applying Theorem 6.4 and Proposition 6.4, we get

$$\begin{aligned} \langle u^*, u\rangle_{V^*\times V} &= (f_1 + f_2)(u) + (f_1 + f_2)^*(u^*) \\ &= f_1(u) + f_1^*(u_1^*) + f_2(u) + f_2^*(u_2^*) \quad \text{with } u_1^* + u_2^* = u^*. \end{aligned} \tag{6.8}$$

Since from the Young inequality we have

$$\begin{aligned} f_1(u) + f_1^*(u_1^*) &\geq \langle u_1^*, u\rangle_{V^*\times V}, \\ f_2(u) + f_2^*(u_2^*) &\geq \langle u_2^*, u\rangle_{V^*\times V}, \end{aligned}$$

the relation (6.8) implies

$$\begin{aligned} f_1(u) + f_1^*(u_1^*) &= \langle u_1^*, u\rangle_{V^*\times V}, \\ f_2(u) + f_2^*(u_2^*) &= \langle u_2^*, u\rangle_{V^*\times V}, \end{aligned}$$

and so, again by Theorem 6.4, $u_1^* \in \partial f_1(u)$ and $u_2^* \in \partial f_2(u)$ with $u_1^* + u_2^* = u^*$, i.e. $u^* \in \partial f_1(u) + \partial f_2(u)$.

The reverse $\partial f_1(u) + \partial f_2(u) \subset \partial(f_1 + f_2)(u)$ holds without any hypotheses on f_1 or f_2. Indeed, if $u^* \in \partial f_1(u) + \partial f_2(u)$, then there exist $u_1^*, u_2^* \in V^*$ such that $u^* = u_1^* + u_2^*$, $u_1^* \in \partial f_1(u)$ and $u_2^* \in \partial f_2(u)$, i.e.

$$\begin{aligned} f_1(v) - f_1(u) &\geq \langle u_1^*, v - u\rangle_{V^*\times V} \quad \forall v \in V, \\ f_2(v) - f_2(u) &\geq \langle u_2^*, v - u\rangle_{V^*\times V} \quad \forall v \in V. \end{aligned}$$

By adding them, we have

$$(f_1 + f_2)(v) - (f_1 + f_2)(u) \geq \langle u^*, v - u\rangle_{V^*\times V} \quad \forall v \in V,$$

which means $u^* \in \partial(f_1 + f_2)(u)$. □

6.2 M–CD–M Theory of Duality

We present here the main ideas for obtaining a dual formulation in the sense of M–CD–M (see [3, 14]) of an abstract problem.

Let $(V, V^*, \langle\cdot,\cdot\rangle_{V^*\times V})$ and $(Y, Y^*, \langle\cdot,\cdot\rangle_{Y^*\times Y})$ be two reflexive Banach spaces with their duals and their duality pairings. We consider the following primal problem:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ \varphi(Lu, u) + \psi(u, u) \le \varphi(Lu, v) + \psi(u, v) \quad \forall v \in V \end{cases} \tag{6.9}$$

where the operator $L : V \to Y$ and the functions $\varphi : Y \times V \to (-\infty, +\infty]$ and $\psi : V \times V \to \mathbb{R}$ satisfy the following hypotheses:

$$L \text{ is a linear continuous operator,} \tag{6.10}$$

$$\forall u \in V,\ \varphi(Lu, \cdot) \text{ is proper convex l.s.c.} \tag{6.11}$$

$$\forall u \in V,\ \psi(u, \cdot) \text{ is convex and } \psi(u, u) \text{ is continuous} \tag{6.12}$$

$$\begin{cases} \forall u \in V,\ \text{the mapping } v \mapsto \psi(u, v) \text{ has a Gâteaux derivative } D_2\psi(u, v) \\ \text{with respect to the second variable at } v = u \text{ such that, for any} \\ v^* \in V^*,\ \text{the set } \{u \in V\ ;\ D_2\psi(u, u) = v^*\} \text{ contains at most one} \\ \text{element denoted by } (D_2\psi)^{-1}(v^*). \end{cases} \tag{6.13}$$

We recall that the Gâteaux derivative with respect to the second variable of $\psi(u, \cdot)$ at v is defined by

$$\langle D_2\psi(u, v), w\rangle_{V^*\times V} = \lim_{t\to 0+} \frac{\psi(u, v + tw) - \psi(u, v)}{t}.$$

The dual problem of (6.9) is constructed by means of Fenchel conjugates of φ^* and ψ^* with respect to the second variable, defined by

$$\varphi^* : Y \times V^* \to (-\infty, +\infty], \quad \varphi^*(Lu, v^*) = \sup_{v\in V} \left(\langle v^*, v\rangle_{V^*\times V} - \varphi(Lu, v)\right),$$
$$\psi^* : V \times V^* \to (-\infty, +\infty], \quad \psi^*(u, v^*) = \sup_{v\in V} \left(\langle v^*, v\rangle_{V^*\times V} - \psi(u, v)\right).$$

We also denote, for all $u \in V$, the subdifferentials of $\psi(u, \cdot)$ and $\varphi^*(Lu, \cdot)$ with respect to the second variable by $\partial_2\psi(u, \cdot)$, and respectively, by $\partial_2\varphi^*(Lu, \cdot)$, where

$$\partial_2\psi(u, z) = \{v^* \in V\ ;\ \psi(u, v) - \psi(u, z) \ge \langle v^*, v - z\rangle_{V^*\times V},\ \forall v \in V\} \quad \forall z \in V,$$
$$\partial_2\varphi^*(Lu, z^*) = \{v \in V\ ;\ \varphi^*(Lu, v^*) - \varphi^*(Lu, z^*)$$
$$\ge \langle v^* - z^*, v\rangle_{V^*\times V},\ \forall v^* \in V^*\} \quad \forall z^* \in V^*.$$

With the above notation, the dual problem of (6.9) is

$$\begin{cases} \text{Find } (u, u^*) \in V \times V^* \text{ such that} \\ -u^* \in \partial_2 \psi(u, u) \\ u \in \partial_2 \varphi^*(Lu, u^*) \end{cases} \tag{6.14}$$

or, equivalently

$$\begin{cases} \text{Find } (u, u^*) \in V \times V^* \text{ such that} \\ \psi(u, v) - \psi(u, u) \geq \langle -u^*, v - u \rangle_{V^* \times V} \qquad \forall v \in V, \\ \varphi^*(Lu, v^*) - \varphi^*(Lu, u^*) \geq \langle v^* - u^*, u \rangle_{V^* \times V} \qquad \forall v^* \in V^*. \end{cases} \tag{6.15}$$

The relationship between the primal problem and the dual problem is given by the next result (see, e.g., [2, 15]).

Theorem 6.7. *Suppose the hypotheses (6.10)–(6.12) are satisfied.*

(i) If u is a solution of the primal problem (6.9), then there exists $u^* \in V^*$ such that (u, u^*) is a solution of the dual problem (6.14).
(ii) If (u, u^*) is a solution of the dual problem (6.14), then u is a solution of the primal problem (6.9).

In addition, the following extremality conditions hold:

$$\begin{cases} \varphi(Lu, u) + \varphi^*(Lu, u^*) = \langle u^*, u \rangle_{V^* \times V}, \\ \psi(u, u) + \psi^*(u, -u^*) = -\langle u^*, u \rangle_{V^* \times V}. \end{cases} \tag{6.16}$$

Proof. (i) Let u be a solution of (6.9) and $f(v) = \varphi(Lu, v) + \psi(u, v)$. It follows that

$$f(u) \leq f(v) \qquad \forall v \in V,$$

and so, by using Theorems 6.3, 6.4 and Proposition 6.4, we get

$$0 \in \partial f(u) \quad \Longleftrightarrow \quad u \in \partial f^*(0) = \partial(f_1^* \nabla f_2^*)(0), \tag{6.17}$$

where $f_1(v) = \varphi(Lu, v)$ and $f_2(v) = \psi(u, v)$.

On the other hand, from Proposition 6.4, the infimal convolution $f_1^* \nabla f_2^*$ is exact at 0. Hence, by Proposition 6.3$_2$, we deduce that $\partial(f_1^* \nabla f_2^*)$ is exact at 0, i.e. there exists $u^* \in V^*$ such that

$$\partial(f_1^* \nabla f_2^*)(0) = \partial f_1^*(u^*) \cap \partial f_2^*(-u^*). \tag{6.18}$$

Now, the relations (6.17) and (6.18) yield that there exists $u^* \in V^*$ such that $u \in \partial_2 \varphi^*(Lu, u^*) \cap \partial_2 \psi^*(u, -u^*)$. We conclude, by Theorem 6.4, that $u \in \partial_2 \varphi^*(Lu, u^*)$ and $-u^* \in \partial_2 \psi(u, u)$, i.e. (u, u^*) is a solution of (6.14).

(ii) If (u, u^*) is a solution of (6.14), then, from Theorem 6.4, we obtain

$$u \in \partial_2\varphi^*(Lu, u^*) \cap \partial_2\psi^*(u, -u^*)$$

and, proceeding as in the first part (i), the assertion follows.

Finally, as

$$-u^* \in \partial_2\psi(u, u) \quad \text{and} \quad u \in \partial_2\varphi^*(Lu, u^*),$$

Theorem 6.4 provides the extremality conditions (6.16).

□

The first variable u from the solution (u, u^*) of the dual problem (6.14) is eliminated by using the assumption (6.13).

Theorem 6.8 (M–CD–M Theorem). *Let the hypotheses (6.10)–(6.13) be satisfied. Then, u is a solution of the primal problem (6.9) if and only if $u^* = -D_2\psi(u, u)$ is a solution of the following dual problem*

$$\begin{cases} \text{Find } u^* \in V^* \text{ such that} \\ \varphi^*(L(D_2\psi)^{-1}(-u^*), v^*) - \varphi^*(L(D_2\psi)^{-1}(-u^*), u^*) \\ \qquad \geq \langle v^* - u^*, (D_2\psi)^{-1}(-u^*)\rangle_{V^*\times V} \quad \forall v^* \in V^*. \end{cases} \tag{6.19}$$

Moreover, the extremality conditions (6.16) hold.

Proof. We first remark that the hypothesis (6.13) implies

$$-u^* = D_2\psi(u, u) \iff u = (D_2\psi)^{-1}(-u^*). \tag{6.20}$$

Now, if u is a solution of the primal problem (6.9), then, by Theorem 6.7, one has $u \in \partial_2\varphi^*(Lu, u^*)$, and hence, by the characterization (6.20), one obtains

$$(D_2\psi)^{-1}(-u^*) \in \partial_2\varphi^*(L(D_2\psi)^{-1}(-u^*), u^*).$$

Therefore, from the definition of the subdifferential of ψ with respect to the second variable, we conclude that u^* solves (6.19).

Conversely, if $u^* = -D_2\psi(u, u)$ is a solution of the dual problem (6.19), then $(D_2\psi)^{-1}(-u^*) \in \partial_2\varphi^*(L(D_2\psi)^{-1}(-u^*), u^*)$ which, together with (6.20), gives

$$\begin{cases} u \in \partial_2\varphi^*(Lu, u^*), \\ -u^* = D_2\psi(u, u) = \partial_2\psi(u, u), \end{cases}$$

that is (u, u^*) is a solution of (6.14). Finally, from Theorem 6.7, we conclude the proof. □

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Chapter 7
Approximations of Variational Inequalities

This chapter is devoted to the discrete approximation of abstract elliptic and implicit evolutionary quasi-variational inequalities. We restrict ourselves to present convergence results for internal approximations in space of elliptic quasi-variational inequalities together with a backward difference scheme in time of implicit evolutionary quasi-variational inequalities. For more details we refer the reader to Glowinski, Lions and Trémolières [6], Glowinski [5], and the bibliography of these works. Here, following the works of Capatina and Cocu [7] and Capatina, Cocou and Raous [1], numerical analysis is carried out on general problems. Also, a general error estimate is derived. The results obtained in this chapter, representing generalizations of the approximations of variational inequalities of the first and second kinds, can be applied to a large variety of static and quasistatic contact problems, including unilateral and bilateral contact or normal compliance conditions with friction. In particular, static and quasistatic unilateral contact problems with nonlocal Coulomb friction in linear elasticity will be considered in Chaps. 8 and 9.

7.1 Internal Approximation of Elliptic Variational Inequalities

In this section one considers the internal approximation of the following abstract quasi-variational inequality.

Problem ($\mathbf{P}^a$): Find $u \in K$ such that

$$\langle Au, v-u\rangle + j(u,v) - j(u,u) \geq \langle f, v-u\rangle \quad \forall v \in K, \tag{7.1}$$

where $(V, \|\cdot\|)$ is a real reflexive Banach space with $(V^*, \|\cdot\|_*)$ its dual and $\langle\cdot,\cdot\rangle$ the duality product between V^* and V. We denote by K a nonempty closed convex

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subset of V and let $f \in V^*$ be given. One supposes that the operator $A : V \longrightarrow V^*$ is Lipschitz continuous and strongly monotone, i.e.

$$\exists M > 0 \text{ such that } \|Au - Av\|_* \leq M\|u - v\| \quad \forall u, v \in V\,, \tag{7.2}$$

$$\exists\, \alpha > 0 \text{ such that } \langle Au - Av, u - v\rangle \geq \alpha\|u - v\|^2 \quad \forall u, v \in V\,. \tag{7.3}$$

In addition, we assume that the function $j(\cdot,\cdot) : V \times V \longrightarrow (-\infty, +\infty]$ satisfies the conditions of Theorem 4.16, so

$$\forall u \in V,\ j(u,\cdot) : V \to (-\infty,\ +\infty] \text{ is a proper convex l.s.c. function,} \tag{7.4}$$

$$\begin{cases} \exists k < \alpha \text{ such that } |j(u_1, v_1) + j(u_2, v_2) - j(u_1, v_2) - j(u_2, v_1)| \\ \leq k\|u_1 - u_2\|\,\|v_1 - v_2\| \quad \forall u_1, u_2, v_1, v_2 \in K\,. \end{cases} \tag{7.5}$$

From the existence and uniqueness proof of Theorem 4.16, the following algorithm of Bensoussan–Lions type for the numerical approximation of Problem $(\mathbf{P}^a)$ follows: let $u^0 \in K$ be arbitrary and

$$u^n = Su^{n-1}\,, \quad n \geq 1 \tag{7.6}$$

where $S : K \to K$ is the mapping which associates with every $w \in K$ the unique solution $Sw \in K$ of the following variational inequality of the second kind:

$$\langle A(Sw), v - (Sw)\rangle + j(w, v) - j(w, (Sw)) \geq \langle f, v - (Sw)\rangle \quad \forall v \in K\,.$$

The hypothesis $k < \alpha$ implies (see p. 50) that the quasi-variational inequality (7.1) has a unique solution $u = Su$ and

$$u^n \to u \qquad \text{strongly in } V \text{ as } n \to \infty\,. \tag{7.7}$$

We shall consider an internal approximation of Problem $(\mathbf{P}^a)$.

Let h be a parameter which converges to zero. Let us consider a family $\{V_h\}_h$ of closed subspaces of V (in applications, we often take V_h to be finite dimensional), and a family $\{K_h\}_h$ of nonempty convex closed subsets of V_h which approximates K in the following sense (see, e.g., [6]):

$$\begin{cases} (i) \quad \forall v \in K,\ \exists r_h v \in K_h \text{ such that } r_h v \to v \text{ strongly in } V\,, \\ (ii) \quad \forall v_h \in K_h \text{ with } v_h \rightharpoonup v \text{ weakly in } V, \text{ then } v \in K\,. \end{cases} \tag{7.8}$$

Often one uses approximations A_h, f_h, and j_h for A, f and j, usually obtained by a process of numerical integration. Nevertheless, since the use of approximations A_h and f_h does not bring any major change comparatively with the use of A and f, here we only consider an approximate of the function $j(\cdot,\cdot)$ by a family $\{j_h\}_h$ of functions which, for every $u \in V$, satisfies the following conditions (see also [5]):

$$\forall h,\ j_h(u,\cdot) : V_h \to (-\infty,\ +\infty] \text{ is a convex l.s.c. function,} \tag{7.9}$$

$$\begin{cases} \text{the family } \{j_h(u,\cdot)\}_h \text{ is uniformly proper, i.e.} \\ \exists\lambda = \lambda(u) \in V^*,\ \exists\mu = \mu(u) \in \mathbb{R} \text{ such that} \\ j_h(u, v_h) \geq \langle \lambda, v_h\rangle + \mu \quad \forall v_h \in V_h,\ \forall h, \end{cases} \tag{7.10}$$

$$\liminf_{h\to 0} j_h(u, v_h) \geq j(u, v) \quad \forall v_h \in V_h \text{ such that } v_h \rightharpoonup v \text{ weakly in } V\,, \tag{7.11}$$

$$\lim_{h\to 0} j_h(u, r_h v) = j(u, v) \quad \forall v \in K\,. \tag{7.12}$$

In addition, we suppose that, for every h, j_h satisfies

$$\begin{aligned} &|j_h(u_h^1, v_h^2) + j_h(u_h^2, v_h^1) - j_h(u_h^1, v_h^1) - j_h(u_h^2, v_h^2)| \\ &\leq k\|u_h^1 - u_h^2\|\,\|v_h^1 - v_h^2\| \quad \forall u_h^1, u_h^2, v_h^1, v_h^2 \in K_h\,. \end{aligned} \tag{7.13}$$

Under the previous assumptions, one formulates the following discrete problem.
Problem $(\mathbf{P}^a)_\mathbf{h}$: Find $u_h \in K_h$ such that

$$\langle Au_h, v_h - u_h\rangle + j_h(u_h, v_h) - j_h(u_h, u_h) \geq \langle f, v_h - u_h\rangle \quad \forall v_h \in K_h\,. \tag{7.14}$$

Arguing as in the proof of Theorem 4.16, it follows that the mapping $S_h : K_h \to K_h$ defined, for every $w_h \in K_h$, as the unique element $S_h w_h \in K_h$ which verifies

$$\langle A(S_h w_h), v_h - S_h w_h\rangle + j_h(w_h, v_h) - j_h(w_h, S_h w_h) \geq \langle f, v_h - S_h w_h\rangle \quad \forall v_h \in K_h\,,$$

is a contraction:

$$\|S_h w_1 - S_h w_2\| \leq \frac{k}{\alpha}\|w_1 - w_2\| \quad \forall w_1, w_2 \in K_h\,. \tag{7.15}$$

Hence, the following existence and uniqueness result holds.

Proposition 7.1. *The discrete quasi-variational inequality (7.14) has a unique solution $u_h = S_h u_h \in K_h$.*

As in the continuous case, we approximate the discrete solution u_h by the sequence $\{u_h^n\}_{n\geq 1}$ defined by

$$u_h^n = S_h u_h^{n-1}\,, \quad n \geq 1$$

where $u_h^0 \in K_h$ is given such that the sequence $\{u_h^0\}_h$ is bounded. Obviously, we have

$$\|u_h^n - u_h\| = \|S_h u_h^{n-1} - S_h u_h\| \leq \left(\frac{k}{\alpha}\right)^n \|u_h^0 - u_h\|\,. \tag{7.16}$$

Thus, in order to prove that the sequence $\{u_h^n\}_n$ is uniformly bounded in h, it is enough to prove the following result.

Lemma 7.1. *The sequence* $\{u_h\}_h$ *of the solutions of the quasi-variational inequality (7.14) is bounded.*

Proof. Let $v \in K$ and $r_h v \in K_h$ such that $r_h v \to v$ strongly in V as $h \to 0$. Taking $v_h = r_h v$ in (7.14), we obtain

$$\begin{aligned}&\alpha\|u_h - r_h v\|^2 \leq \langle Au_h - A(r_h v), u_h - r_h v\rangle \leq \langle A(r_h v), r_h v - u_h\rangle\\&+(j_h(u_h, r_h v) - j_h(u_h, u_h) + j_h(u, u_h) - j_h(u, r_h v))\\&-j_h(u, u_h) + j_h(u, r_h v) - \langle f, r_h v - u_h\rangle\,.\end{aligned} \tag{7.17}$$

From (7.12) we have

$$|j_h(u, r_h v)| \leq C_1,$$

and, since the sequence $\{r_h v\}_h$ is bounded, from (7.2), we get

$$\|A(r_h v)\|_* \leq C_2$$

with C_1 and C_2 positive constants independent of h. Therefore, from (7.17), (7.10) and (7.13), we obtain

$$\begin{aligned}&\alpha\|u_h - r_h v\|^2 - \|\lambda\|_*\|u_h\| - |\mu| \leq \langle Au_h - A(r_h v), u_h - r_h v\rangle + j_h(u, u_h)\\&\leq C_2\|r_h v - u_h\| + k\|u_h - u\|\,\|r_h v - u_h\| + C_1 + \|f\|_*\|r_h v - u_h\|,\end{aligned} \tag{7.18}$$

hence

$$\begin{aligned}&\left(\alpha - k - \frac{k\epsilon_1 + \epsilon_2 + \epsilon_3}{2}\right)\|u_h - r_h v\|^2 \leq \frac{\|\lambda\|_*}{2\epsilon_3} + \|r_h v\|\,\|\lambda\|_*\\&+\frac{k}{2\epsilon_1}\|r_h v - u\|^2 + \frac{(C_2 + \|f\|_*)^2}{2\epsilon_2} + C_1 + |\mu| \leq C\end{aligned} \tag{7.19}$$

where $\epsilon_1, \epsilon_2, \epsilon_3 > 0$ are chosen such that $\alpha - k - \dfrac{k\epsilon_1 + \epsilon_2 + \epsilon_3}{2} > 0$ (for instance, $\epsilon_1 = \dfrac{\alpha - k}{3k}$, $\epsilon_2 = \dfrac{5}{6}(\alpha - k)$, $\epsilon_3 = \dfrac{5}{12}(\alpha - k)$) and C is a positive constant independent of h. Therefore, according to the choice of $\{r_h v\}_h$, we conclude that the sequence $\{u_h - r_h v\}_h$ is bounded, and so, the sequence $\{u_h\}_h$ is. □

Now, from (7.16), the above lemma and the boundedness of $\{u_h^0\}_h$, it follows that

$$\|u_h^n - u_h\| \leq C q^n \tag{7.20}$$

with $q = \dfrac{k}{\alpha} < 1$ and C a positive constant independent of n and h, i.e. $\{u_h^n\}_n$ is uniformly bounded in h. Hence, for all $\epsilon > 0$, there exists $N = N_\epsilon$ such that

$$\|u_h^n - u_h\| \leq \epsilon \quad \forall n \geq N_\epsilon\,,\ \forall h > 0\,. \tag{7.21}$$

We recall that, for any $n \geq 1$, u^n, respectively u_h^n, are defined as the unique solutions of the following problems:

Problem $(\mathbf{P}^a)_\mathbf{n}$: Find $u^n \in K$ such that

$$\langle Au^n, v - u^n\rangle + j(u^{n-1}, v) - j(u^{n-1}, u^n) \geq \langle f, v - u^n\rangle \quad \forall v \in K\,, \tag{7.22}$$

respectively,

Problem $(\mathbf{P}^a)_{\mathbf{h},\mathbf{n}}$: Find $u_h^n \in K_h$ such that

$$\langle Au_h^n, v_h - u_h^n\rangle + j_h(u_h^{n-1}, v_h) - j_h(u_h^{n-1}, u_h^n) \geq \langle f, v_h - u_h^n\rangle \quad \forall v_h \in K_h\,. \tag{7.23}$$

This means that Problem $(\mathbf{P}^a)_\mathbf{n}$ is an iterative approximation of Problem $(\mathbf{P}^a)$, while Problem $(\mathbf{P}^a)_{\mathbf{h},\mathbf{n}}$ is an iterative approximation of Problem $(\mathbf{P}^a)_\mathbf{h}$.

In order to obtain the convergence of the sequence $\{u_h\}_h$ to u, as $h \to 0$, we introduce an auxiliary sequence of problems. So, for $w_h^0 \in K_h$ given such that the sequence $\{w_h^0\}_h$ is bounded, we denote by $w_h^n \in K_h$ the solution, that there exists and is unique, of the following problem.

Problem $(\mathbf{P}^a)_{\mathbf{n},\mathbf{h}}$: Find $w_h^n \in K_h$ such that

$$\langle Aw_h^n, v_h - w_h^n\rangle + j_h(u^{n-1}, v_h) - j_h(u^{n-1}, w_h^n) \geq \langle f, v_h - w_h^n\rangle \quad \forall v_h \in K_h\,, \tag{7.24}$$

where the sequence $\{u^n\}_n \subset K$ is defined by (7.6). We note that Problem $(\mathbf{P}^a)_{\mathbf{n},\mathbf{h}}$ is an internal approximation of Problem $(\mathbf{P}^a)_\mathbf{n}$.

We have the following convergence result.

Proposition 7.2. *The sequence $\{w_h^n\}_h$, defined by (7.24), approximates the solution u^n of (7.22) in the sense*

$$w_h^n \to u^n \text{ strongly in } V \text{ as } h \to 0.$$

Moreover, we have

$$\lim_{h\to 0} j_h(u^{n-1}, w_h^n) = j(u^{n-1}, u^n).$$

Proof. Let $v \in K$ be arbitrarily chosen. Taking $v_h = r_h v$ in (7.24), it results

$$\langle Aw_h^n, w_h^n\rangle + j_h(u^{n-1}, w_h^n) \leq \langle Aw_h^n, r_h v\rangle + j_h(u^{n-1}, r_h v) - \langle f, r_h v - w_h^n\rangle\,. \tag{7.25}$$

By using the hypotheses (7.3), (7.2), (7.10), and (7.12), one gets

$$\alpha\|w_h^n\|^2 \leq \|\lambda\|_*\|w_h^n\| + |\mu| + M\|w_h^n\|\|r_h v\| + C + \|f\|_*(\|r_h v\| + \|w_h^n\|) \leq C_1\|w_h^n\| + C_2$$

with C, C_1, and C_2 positive constants independent of h. Hence, the sequence $\{w_h^n\}_h$ is bounded and we can extract a subsequence $\{w_{h_p}^n\}_p$ such that $w_{h_p}^n \rightharpoonup w_n$ weakly in V, with $w^n \in K$ (from $(7.8)_2$). Now, from (7.25), by using (7.3), (7.11), and (7.12), we obtain

$$\langle Aw^n, w^n\rangle + j(u^{n-1}, w^n) \le \liminf_{h_p\to 0}(\langle Aw^n_{h_p}, w^n_{h_p}\rangle + j_h(u^{n-1}, w^n_{h_p}))$$
$$\le \langle Aw^n, v\rangle + j(u^{n-1}, v) - \langle f, v - w^n\rangle \quad \forall v \in K .$$

This implies $w^n = u^n$, where u^n is the unique solution of the variational inequality (7.22). Therefore, $w^n_h \rightharpoonup u^n$ weakly in V as $h \to 0$.

Finally, from (7.25) and using the hypotheses (7.11) and (7.12), we have

$$j(u^{n-1}, u^n) \le \liminf_{h\to 0} j_h(u^{n-1}, w^n_h) \le \liminf_{h\to 0}(\alpha\|w^n_h - u^n\|^2 + j_h(u^{n-1}, w^n_h))$$
$$\le \limsup_{h\to 0}(\alpha\|w^n_h - u^n\|^2 + j_h(u^{n-1}, w^n_h))$$
$$\le \lim_{h\to 0}(\langle Aw^n_h, r_h v\rangle + j_h(u^{n-1}, r_h v) - \langle f, r_h v - w^n_h\rangle - \langle Aw^n_h, u^n\rangle - \langle Au^n, w^n_h\rangle$$
$$+\langle Au^n, u^n\rangle) = \langle Au^n, v - u^n\rangle + j(u^{n-1}, v) - \langle f, v - u^n\rangle \quad \forall v \in K .$$

The proof is completed by taking $v = u^n$. □

We are now prepared to prove the main result of this section.

Theorem 7.1. *We suppose that (7.2)–(7.13) hold. Let u and u_h be the unique solutions of (7.1) and, respectively, (7.14). Then, we have*

$$u_h \to u \text{ strongly in } V \text{ as } h \to 0 . \tag{7.26}$$

Proof. We observe that we have

$$\|u_h - u\| \le \|u_h - u^n_h\| + \|u^n_h - u^n\| + \|u^n - u\| \quad \forall n \ge 0 . \tag{7.27}$$

First, from (7.7) and (7.21), it results that, for $\epsilon > 0$ given, there exists $N_\epsilon > 0$ such that

$$\|u^n_h - u_h\| + \|u^n - u\| \le \frac{\epsilon}{2} \quad \forall n \ge N_\epsilon . \tag{7.28}$$

In order to estimate the second term in the right-hand side of (7.27), we deduce, form the definitions of u^n_h and w^n_h, that

$$\alpha\|u^n_h - w^n_h\|^2 \le \langle Aw^n_h - Au^n_h, w^n_h - u^n_h\rangle$$
$$\le j_h(u^{n-1}, u^n_h) + j_h(u^{n-1}_h, w^n_h) - j_h(u^{n-1}, w^n_h) - j_h(u^{n-1}_h, u^n_h) ,$$

from which, using (7.13), we deduce

$$\|u^n_h - w^n_h\| < \|u^{n-1}_h - u^{n-1}\| . \tag{7.29}$$

Now, by choosing $w^0_h = u^0_h$, we shall prove by recurrence, that

$$\|u^n_h - u^n\| \le \sum_{i=0}^{n} \|w^i_h - u^i\| \quad \forall n \ge 0 . \tag{7.30}$$

Indeed, for $n = 0$ the result is obvious. If we suppose that (7.30) holds for $n-1$, then, from (7.29), we get

$$\|u_h^n - u^n\| \le \|u_h^n - w_h^n\| + \|w_h^n - u^n\| \le \|u_h^{n-1} - u^{n-1}\| + \|w_h^n - u^n\| \le \sum_{i=0}^{n} \|w_h^i - u^i\| .$$

It follows that the relation (7.30) holds for every $n \ge 0$.

Choosing $n = N_\epsilon$ in (7.27) and taking into account (7.30) and (7.28), we obtain

$$\|u_h - u\| \le \frac{\epsilon}{2} + \sum_{i=0}^{N_\epsilon} \|w_h^i - u^i\| . \tag{7.31}$$

But, from Proposition 7.2, it follows that, for every i, there exists $H_\epsilon^i > 0$ such that

$$\|w_h^i - u^i\| \le \frac{\epsilon}{2(N_\epsilon + 1)} \quad \forall h \le H_\epsilon^i . \tag{7.32}$$

Concluding, from (7.31) and (7.32), for $\epsilon > 0$ given, there exists $H_\epsilon = \min_{i=0}^{N_\epsilon} H_\epsilon^i$ such that

$$\|u_h - u\| \le \epsilon \quad \forall h \le H_\epsilon ,$$

hence $u_h \to u$ strongly in V as $h \to 0$. □

7.2 Abstract Error Estimate

The purpose of this section is to obtain a priori error estimate for the approximation (7.14) of the quasi-variational inequality (7.1). This estimate generalizes the estimates obtained by Cea [2, 3] and Falk [4] for the approximation of variational equations and, respectively, variational inequalities of the first kind.

Theorem 7.2. *Let u and u_h be the unique solutions of the quasi-variational inequality (7.1) and, respectively, (7.14).*

We suppose that (7.2)–(7.13) hold. Moreover, we assume that there exists a Hilbert space $(H, \|\cdot\|_H)$ and a Banach space $(U, \|\cdot\|_U)$ such that $V \hookrightarrow H$ dense, $V \subset U$ and

$$Au - f \in H , \tag{7.33}$$

$$|j_h(u, v_h) - j(u, v)| \le C_1 \|v_h - v\|_U \quad \forall v_h \in K_h , \ \forall v \in K , \tag{7.34}$$

where C_1 is a positive constant independent of h. Then, there exists a positive constant C, independent of h, such that the estimate

$$\|u_h - u\| \le C \left\{ \inf_{v_h \in K_h} \left(\|u - v_h\|^2 + \|Au - f\|_H \|u - v_h\|_H + C_1 \|u - v_h\|_U \right) + \inf_{v \in K} \left(\|Au - f\|_H \|u_h - v\|_H + C_1 \|u_h - v\|_U \right) \right\}^{1/2} \tag{7.35}$$

holds.

Proof. From (7.1) and (7.14), we get

$$\langle Au_h - Au, u_h - u\rangle \le \langle Au - f, v - u_h + v_h - u\rangle + \langle Au_h - Au, v_h - u\rangle + j_h(u_h, v_h) - j_h(u_h, u_h) + j(u, v) - j(u, u) \quad \forall v \in K \quad \forall v_h \in K_h \,. \tag{7.36}$$

Evaluating each term in the right-hand side, we have

$$\langle Au - f, v - u_h + v_h - u\rangle \le \|Au - f\|_H (\|v - u_h\|_H + \|v_h - u\|_H)\,, \tag{7.37}$$

$$\langle Au_h - Au, v_h - u\rangle \le M\|u_h - u\|\, \|v_h - u\| \tag{7.38}$$

and

$$\begin{aligned} &j_h(u_h, v_h) - j_h(u_h, u_h) + j(u, v) - j(u, u) \\ &\le |j_h(u_h, v_h) - j_h(u_h, u_h) + j_h(u, u_h) - j_h(u, v_h)| + |j_h(u, v_h) - j(u, u)| \\ &+|j(u, v) - j_h(u, u_h)| \le k\|u_h - u\|\, \|v_h - u_h\| + C_1(\|v_h - u\|_U + \|v - u_h\|_U) \\ &\le k\|u_h - u\|^2 + k\|u_h - u\|\, \|v_h - u\| + C_1(\|v_h - u\|_U + \|v - u_h\|_U)\,. \end{aligned} \tag{7.39}$$

By using (7.37)–(7.39) in (7.36), with (7.3), it follows

$$(\alpha - k)\|u_h - u\|^2 \le (M + k)\|u_h - u\|\, \|v_h - u\| + \|Au - f\|_H(\|v - u_h\|_H + \|v_h - u\|_H) + C_1(\|v_h - u\|_U + \|v - u_h\|_U) \quad \forall v \in K\,,\ \forall v_h \in K_h\,, \tag{7.40}$$

which, by Young'inequality : $ab \le \dfrac{\epsilon a^2}{2} + \dfrac{b^2}{2\epsilon}$ for $\epsilon = \dfrac{\alpha - k}{M + k}$, $a = \|u_h - u\|$ and $b = \|v_h - u\|$, implies

$$\frac{\alpha - k}{2}\|u_h - u\|^2 \le \frac{M + k}{2(\alpha - k)}\|v_h - u\|^2 + \|Au - f\|_H(\|v - u_h\|_H + \|v_h - u\|_H) + C_1(\|v_h - u\|_U + \|v - u_h\|_U) \quad \forall v \in K\,,\ \forall v_h \in K_h\,, \tag{7.41}$$

i.e. (7.35). □

Remark 7.1. If $K_h \subset K$, then the term

$$\inf_{v \in K} \left(\|Au - f\|_H \|u_h - v\|_H + C_1 \|u_h - v\|_U\right) ,$$

which is expected to have the highest weight in (7.35), vanishes, thus one obtains

$$\|u_h - u\| \leq C \left\{ \inf_{v_h \in K_h} \left(\|u - v_h\|^2 + \|Au - f\|_H \|u - v_h\|_H + C_1 \|u - v_h\|_U\right) \right\}^{1/2} .$$

This means that an optimal error estimate $\|u_h - u\|$ depends of the distance between the exact solution u and the finite dimensional subspace V_h of V. Hence, the more suitable construction of the space V_h is, the better order of the error estimate will be. As we shall see on concrete examples in Sect. 8.6, the order of approximation essentially depends on the chosen type of finite element approximation for the space V.

Remark 7.2. If $j(\cdot,\cdot) \equiv 0$, therefore, by taking $C_1 = 0$, we deduce

$$\|u_h - u\| \leq C \left\{ \inf_{v_h \in K_h} \left(\|u - v_h\|^2 + \|Au - f\|_H \|u - v_h\|_H\right) + \|Au - f\|_H \inf_{v \in K} \|u_h - v\|_H \right\}^{1/2} ,$$

so, the estimate obtained by Falk [4] for the internal approximation of variational inequalities of first kind with A a linear and continuous operator.

Remark 7.3. If $j(\cdot,,\cdot) \equiv 0$ and $K = V$, then, by taking $K_h = V_h$, from (7.35), we get

$$\|u_h - u\| \leq C \inf_{v_h \in V_h} \|u - v_h\|$$

so, the result given by Céa [3] for the operator equation $Au = f$ with A a linear and continuous operator.

Finally, the following form of the error estimate is obvious.

Theorem 7.3. *We suppose that the hypotheses of Theorem 7.2 are satisfied but with the condition (7.33) replaced by*

$$\langle Au - f, v\rangle \leq C_2 \|v\|_U \quad \forall v \in V . \tag{7.42}$$

Therefore, we have the estimate

$$\|u_h - u\| \leq C \left\{ \inf_{v_h \in K_h} \left(\|u - v_h\|^2 + (C_1 + C_2)\|u - v_h\|_U\right) + (C_1 + C_2) \inf_{v \in K} \|u_h - v\|_U \right\}^{1/2} \tag{7.43}$$

with C a positive constant independent of h.

7.3 Discrete Approximation of Implicit Evolutionary Inequalities

This section is concerned with the numerical analysis of a class of abstract implicit evolutionary variational inequalities. Convergence results are proved using a method based on a semi-discrete internal approximation and an implicit time discretization scheme.

More precisely, for $f \in W^{1,2}(0,T;V)$ given, one considers the problem (4.107) (p. 68), i.e.

Problem ($\mathbf{Q}^a$): Find $u \in W^{1,2}(0,T;V)$ such that

$$\begin{cases} u(0) = u_0\,,\ u(t) \in K(f(t)) \quad \forall\, t \in [0,T\,], \\ a(u(t), v - \dot{u}(t)) + j(f(t), u(t), v) - j(f(t), u(t), \dot{u}(t)) \\ \geq b(f(t), u(t), v - \dot{u}(t)) \quad \forall\, v \in V \text{ a.e. in }]0,T[, \\ b(f(t), u(t), z - u(t)) \geq 0 \quad \forall\, z \in K,\ \forall\, t \in [0,T\,], \end{cases} \tag{7.44}$$

where $(V, (\cdot,\cdot))$ is a real Hilbert space with the associated norm $\|\cdot\|$ and $K \subset V$ is a closed convex cone with its vertex at 0.

We suppose that $a(\cdot,\cdot)$, $j(\cdot,\cdot,\cdot)$, $b(\cdot,\cdot,\cdot)$ and $K(g)$ satisfy the hypotheses (4.83)–(4.90), (4.96)–(4.98), (4.100), (4.101), and (4.105). We recall that $u_0 \in K(f(0))$ is the unique solution of the following elliptic variational inequality

$$a(u_0, w - u_0) + j(f(0), u_0, w) - j(f(0), u_0, u_0) \geq 0 \quad \forall\, w \in K. \tag{7.45}$$

In order to obtain the discretization of Problem ($\mathbf{Q}^a$), we first consider a semi-discrete approximation of it. For a positive parameter h converging to 0, let $\{V_h\}_h$ be a family of finite dimensional subspaces of V and let $\{K_h\}_h$ be a family of closed convex cones with their vertices at 0 such that $K_h \subset V_h$ and $(K_h)_h$ is an internal approximation of K in the sense specified in Sect. 7.1, i.e.

$$\begin{cases} (i) \quad \forall v \in K,\ \exists r_h v \in K_h \text{ such that } r_h v \to v \text{ strongly in } V\,, \\ (ii) \quad \forall v_h \in K_h \text{ avec } v_h \rightharpoonup v \text{ weakly in } V, \text{ then } v \in K\,. \end{cases} \tag{7.46}$$

For any $h > 0$, let $\{K_h(g)\}_{g \in V}$ be a family of nonempty convex subsets of K_h such that $0 \in K_h(0)$. We put $D_{K_h} = \{(g, v_h) \in V \times K_h\,;\ v_h \in K_h(g)\}$ and we assume the following conditions hold:

$$\left.\begin{array}{l} \forall (g_n, v_{hn}) \in D_{K_h} \text{ such that} \\ g_n \to g \text{ strongly in } V\,,\ v_{hn} \rightharpoonup v_h \text{ weakly in } V \end{array}\right\} \Longrightarrow (g, v_h) \in D_{K_h} \tag{7.47}$$

$$\forall (g, v_h) \in D_{K_h} \text{ such that } v_h \rightharpoonup v \text{ weakly in } V \Longrightarrow (g, v) \in D_K \tag{7.48}$$

We assume that the functional $j : D_K \times V \to \mathbb{R}$ is approximated by a family $\{j_h\}_h$ of functionals $j_h : D_{K_h} \times V_h \to \mathbb{R}$ satisfying

$$\left.\begin{array}{l}\forall g \in C([0,T];V)\,,\ \forall v_h \rightharpoonup v \text{ weakly in } W^{1,2}(0,T,V) \text{ such that}\\ (g(t), v_h(t)) \in D_{K_h}\,,\ \forall t \in [0,T]\end{array}\right\}$$
$$\Rightarrow \liminf_{h\to 0} \int_0^s j_h(g(t), v_h(t), \dot v_h(t))\,\mathrm{d}t \ge \int_0^s j(g(t), v(t), \dot v(t))\,\mathrm{d}t\,,\ \forall s \in [0,T]\,, \tag{7.49}$$

and

$$\left.\begin{array}{l}\forall (g, v_h) \in D_{K_h},\ \forall w_h \in V_h \text{ such that}\\ v_h \rightharpoonup v \text{ weakly in } V\,,\\ w_h \to w \text{ strongly in } V\end{array}\right\} \Rightarrow \lim_{h\to 0} j_h(g, v_h, w_h) = j(g, v, w). \tag{7.50}$$

Furthermore, we suppose that, for all h, the following conditions are fulfilled:

$$\begin{array}{l}\forall (g, v_h) \in D_{K_h},\ j_h(g, v_h, \cdot) : V_h \to \mathbb{R} \text{ is a sub-additive and}\\ \text{positively homogeneous functional,}\end{array} \tag{7.51}$$

$$j_h(0,0,w_h) = 0 \quad \forall w_h \in V_h \tag{7.52}$$

$$\begin{array}{l}|j_h(g_1, v_{1h}, w_{1h}) + j_h(g_2, v_{2h}, w_{2h}) - j_h(g_1, v_{1h}, w_{2h}) - j_h(g_2, v_{2h}, w_{1h})|\\ \le k_2(\|g_1 - g_2\| + \|\beta_h(g_1, v_{1h}) - \beta_h(g_2, v_{2h})\|_H)\|w_{1h} - w_{2h}\|\\ \forall (g_i, v_{ih}) \in D_{K_h}, \forall w_{ih} \in V_h,\ i = 1,2\end{array} \tag{7.53}$$

where the operator $\beta_h : D_{K_h} \to H$ is such that

$$\begin{array}{c}\|\beta_h(g_1, v_{1h}) - \beta_h(g_2, v_{2h})\|_H \le k_1(\|g_1 - g_2\| + \|v_{1h} - v_{2h}\|)\\ \forall (g_1, v_{1h}),\ (g_2, v_{2h}) \in D_{K_h},\end{array} \tag{7.54}$$

with k_1, k_2 the positive constants from (4.86), (4.90) such that $k_1 k_2 < \alpha$ (i.e., condition (4.101) from p. 65).

From the properties of a, j_h and K_h and proceeding as in the continuous case, it follows that, for any $g \in V$, $d_h \in K_h$, $w_h \in K_h(g)$, the elliptic variational inequality

$$\left\{\begin{array}{l}\text{Find } u_h \in K_h \text{ such that}\\ a(u_h, v_h - u_h) + j_h(g, w_h, v_h - d_h) - j_h(g, w_h, u_h - d_h) \ge 0 \quad \forall\, v_h \in K_h\end{array}\right. \tag{7.55}$$

has a unique solution $u_h = u_h(g, d_h, w_h)$. Hence, we can define the mapping

$$S^h_{g,d_h} : K_h(g) \to K_h \text{ by } S^h_{g,d_h}(w_h) = u_h \tag{7.56}$$

and, as in Remark 4.8, one obtains that it is a contraction.

We suppose that, for all $g \in V$ and $d_h \in K_h$

$$S^h_{g,d_h}(K_h(g)) \subset K_h(g). \tag{7.57}$$

Let u_{0h} be the unique fixed point of the mapping $S^h_{f(0),0}$, so

$$\begin{cases} u_{0h} \in K_h(f(0)), \\ a(u_{0h}, w_h - u_{0h}) + j_h(f(0), u_{0h}, w_h) - j_h(f(0), u_{0h}, u_{0h}) \geq 0 \; \forall \, w_h \in K_h. \end{cases} \tag{7.58}$$

From Theorem 7.1, it follows that

$$u_{0h} \to u_0 \text{ strongly in } V, \tag{7.59}$$

as $h \to 0$, u_0 being the unique solution of (7.45).

Now, for all $g \in V$ and $d_h \in K_h$, we introduce the following two auxiliary problems.

Problem ($\tilde{\mathbf{Q}}^a_\mathbf{h}$): Find $u_h \in K_h(g)$ such that

$$\begin{cases} a(u_h, v_h - u_h) + j_h(g, u_h, v_h - d_h) - j_h(g, u_h, u_h - d_h) \\ \qquad \geq b(g, u_h, v_h - u_h) \quad \forall \, v_h \in V_h, \\ b(g, u_h, z_h - u_h) \geq 0 \quad \forall \, z_h \in K_h, \end{cases} \tag{7.60}$$

and

Problem ($\tilde{\mathbf{R}}^a_\mathbf{h}$): Find $u_h \in K_h(g)$ such that

$$a(u_h, v_h - u_h) + j_h(g, u_h, v_h - d_h) - j_h(g, u_h, u_h - d_h) \geq 0 \quad \forall \, v_h \in K_h. \tag{7.61}$$

We will suppose that

$$\text{If } u_h \text{ is a solution of } (\tilde{\mathbf{R}}^a_\mathbf{h}), \text{ then } u_h \text{ is a solution of } (\tilde{\mathbf{Q}}^a_\mathbf{h}). \tag{7.62}$$

Remark 7.4. It is obvious that, if u_h satisfies ($\tilde{\mathbf{Q}}^a_\mathbf{h}$), then u_h satisfies also ($\tilde{\mathbf{R}}^a_\mathbf{h}$).

Let us consider the following semi-discrete problem.

Problem ($\mathbf{Q}^a_\mathbf{h}$): Find $u_h \in W^{1,2}(0, T; V_h)$ such that

$$\begin{cases} u_h(0) = u_{0h}, \; u_h(t) \in K_h(f(t)) \quad \forall \, t \in [0, T], \\ a(u_h(t), v_h - \dot{u}_h(t)) + j_h(f(t), u_h(t), v_h) - j_h(f(t), u_h(t), \dot{u}_h(t)) \\ \qquad \geq b(f(t), u_h(t), v_h - \dot{u}_h(t)) \quad \forall \, v_h \in V_h \text{ a.e. in }]0, T[, \\ b(f(t), u_h(t), z_h - u_h(t)) \geq 0 \quad \forall \, z_h \in K_h \quad \forall \, t \in [0, T]. \end{cases} \tag{7.63}$$

The full discretization of $(\mathbf{Q}_\mathbf{h}^a)$ is obtained by using a backward difference scheme as in Sect. 4.3 for $(\mathbf{Q}^a)$: for $u_h^0 = u_{0h}$ and $i \in \{0, 1, \ldots, n-1\}$, we define u_h^{i+1} as the unique solution of the following problem.

Problem $(\mathbf{Q}_\mathbf{h}^a)_\mathbf{n}^\mathbf{i}$: Find $u_h^{i+1} \in K_h^{i+1}$ such that

$$\begin{cases} a(u_h^{i+1}, v_h - \partial u_h^i) + j_h(f^{i+1}, u_h^{i+1}, v_h) - j_h(f^{i+1}, u_h^{i+1}, \partial u_h^i) \\ \qquad\qquad \geq b(f^{i+1}, u_h^{i+1}, v_h - \partial u_h^i) \quad \forall\, v_h \in V_h\,, \\ b(f^{i+1}, u_h^{i+1}, z_h - u_h^{i+1}) \geq 0 \quad \forall\, z_h \in K_h\,, \end{cases} \tag{7.64}$$

where $K_h^{i+1} = K_h(f^{i+1})$.

By (7.62) and Remark 7.4, it is easy to see that Problem $(\mathbf{Q}_\mathbf{h}^a)_\mathbf{n}^\mathbf{i}$ is equivalent to the following quasi-variational inequality.

Problem $(\mathbf{R}_\mathbf{h}^a)_\mathbf{n}^\mathbf{i}$: Find $u_h^{i+1} \in K_h^{i+1}$ such that

$$\begin{cases} a(u_h^{i+1}, w_h - u_h^{i+1}) + j_h(f^{i+1}, u_h^{i+1}, w_h - u_h^i) \\ \qquad\qquad -j_h(f^{i+1}, u_h^{i+1}, u_h^{i+1} - u_h^i) \geq 0 \quad \forall\, w_h \in K_h. \end{cases} \tag{7.65}$$

From (4.83), (4.86), (4.90), (4.101), and (7.57), it follows that the mapping

$$S^h_{f^{i+1}, u_h^i} : K_h^{i+1} \to K_h^{i+1},$$

defined by (7.56), is a contraction, so that $(\mathbf{R}_\mathbf{h}^a)_\mathbf{n}^\mathbf{i}$ has a unique solution.

We now define, as in the continuous case, the functions

$$\begin{cases} u_{hn}(0) = \hat{u}_{hn}(0) = u_{0h}\,, \\ \left.\begin{array}{l} u_{hn}(t) = u_h^{i+1} \\ \hat{u}_{hn}(t) = u_h^i + (t - t_i)\partial u_h^i \end{array}\right\} \forall\, i \in \{0, 1, \ldots, n-1\} \quad \forall\, t \in (t_i, t_{i+1}]\,. \end{cases} \tag{7.66}$$

Then, the functions $u_{hn} \in L^2(0, T; V_h)$ and $\hat{u}_{hn} \in W^{1,2}(0, T; V_h)$ satisfy the following problem.

Problem $(\mathbf{Q}_\mathbf{h}^a)_\mathbf{n}$: Find $u_{hn}(t) \in K(f_n(t))$ such that

$$\begin{cases} a\left(u_{hn}(t), v_h - \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{hn}(t)\right) + j_h(f_n(t), u_{hn}(t), v_h) \\ -j_h\left(f_n(t), u_{hn}(t), \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{hn}(t)\right) \geq b\left(f_n(t), u_{hn}(t), v_h - \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{hn}(t)\right) \\ \quad \forall\, v_h \in V_h, \\ b(f_n(t), u_{hn}(t), z_h - u_{hn}(t)) \geq 0 \quad \forall\, z_h \in K_h\,. \end{cases} \tag{7.67}$$

Moreover, we have the analogues of Lemmas 4.12 and 4.13. Hence, we conclude, as in Theorem 4.19, that the following convergence and existence result holds.

Theorem 7.4. *Assume that the hypotheses (4.83)–(4.90), (4.96)–(4.98),(4.100), (4.101), (4.105), (7.46), (7.57), and (7.62) hold. Then, the problem* $(\mathbf{Q}_\mathbf{h}^a)$ *has at least one solution. In addition, there exists a subsequence of* $\{(u_{hn}, \hat{u}_{hn})\}_{n\in N^*}$*, still denoted by* $\{(u_{hn}, \hat{u}_{hn})\}_{n\in N^*}$*, such that*

$$u_{hn}(t) \to u_h(t) \quad \text{in} \quad V \quad \forall\, t \in [0, T] \quad \text{as } n \to \infty\,, \tag{7.68}$$

$$\hat{u}_{hn} \to u_h \quad \text{in} \quad W^{1,2}(0, T; V) \quad \text{as } n \to \infty\,, \tag{7.69}$$

where $u_h \in W^{1,2}(0, T; V_h)$ *is a solution of* $(\mathbf{Q}_\mathbf{h}^a)$.

We now proceed to find a priori estimates for the solutions of u_h of $(\mathbf{Q}_\mathbf{h}^a)$ which are limits of subsequences of $\{u_{hn}\}_n$.

Lemma 7.2. *For* $h > 0$*, let* u_h *be the solution of* $(\mathbf{Q}_\mathbf{h}^a)$ *given by Lemma 7.4. Then,*

$$\|u_h(t)\| \le C_0 \|f\|_{C([0,T];V)} \quad \forall\, t \in [0, T]\,, \tag{7.70}$$

$$\|u_h(s) - u_h(t)\| \le C_0 \int_s^t \|\dot{f}(\tau)\|\, \mathrm{d}\tau \quad \forall\, s,\, t \in [0, T],\ s < t, \tag{7.71}$$

$$\|u_h\|_{W^{1,2}(0,T;V)} \le C_0 \sqrt{T \|f\|^2_{C([0,T];V)} + \|\dot{f}\|^2_{L^2(0,T;V)}}\,, \tag{7.72}$$

where C_0 *is the constant, independent of h, given by the relation (4.116).*

Proof. Using the same arguments as in the proof of Lemma 4.12, we obtain the estimates

$$\|u_{hn}(t)\| \le C_0 \|f\|_{C([0,T];V)} \quad \forall\, t \in [0, T],$$

$$\|u_{hn}(s) - u_{hn}(t)\| \le C_0 \int_s^{\min\{t+\Delta t,\, T\}} \|\dot{f}(\tau)\|\, \mathrm{d}\tau \quad \forall\, s,\, t \in [0, T],\ s < t,$$

$$\|\hat{u}_{hn}\|^2_{W^{1,2}(0,T;V)} \le C_0^2 (T \|f\|^2_{C([0,T];V)} + \|\dot{f}\|^2_{L^2(0,T;V)})\,.$$

Combining these results with (7.68), (7.69) and taking into account that the norm is weakly lower semicontinuous, the estimates (7.70)–(7.72) follow. □

Now, we have in position to prove the following convergence result.

Theorem 7.5. *Under the assumptions (4.83)–(4.90), (4.96)–(4.98), (4.100), (4.101), (4.105), (7.46), (7.57), and (7.62), there exists a subsequence of* $\{u_h\}_h$*, still denoted by* $\{u_h\}_h$*, such that*

$$u_h(t) \to u(t) \quad \textit{strongly in} \quad V \quad \forall\, t \in [\,0, T\,] \quad \textit{as } h \to 0\,, \tag{7.73}$$

$$\dot{u}_h \rightharpoonup \dot{u} \quad \textit{weakly in} \quad L^2(0, T; V) \quad \textit{as } h \to 0\,, \tag{7.74}$$

where $u \in W^{1,2}(0, T; V)$ *is a solution of* $(\mathbf{Q}^a)$.

Proof. From Lemma 7.2, it follows that there exists a subsequence of $\{u_h\}_h$ and an element $u \in W^{1,2}(0, T; V)$ such that

$$u_h(t) \to u(t) \quad \text{strongly in} \quad V \quad \forall\, t \in [\,0, T\,]\,, \tag{7.75}$$

$$u_h \rightharpoonup u \quad \text{weakly in} \quad W^{1,2}(0, T; V)\,.. \tag{7.76}$$

Moreover, from (7.75) and (7.59), we get

$$\liminf_{h\to 0} \int_0^s a(u_h(t), \dot{u}_h(t))\, \mathrm{d}t \ge \frac{1}{2}(\liminf_{h\to 0} a(u_h(s), u_h(s)) - \lim_{h\to 0} a(u_{0h}, u_{0h}))$$
$$\ge \frac{1}{2}(a(u(s), u(s)) - a(u_0, u_0)) = \int_0^s a(u(t), \dot{u}(t))\, \mathrm{d}t \quad \forall\, s \in [\,0, T\,]. \tag{7.77}$$

On the other hand, from the hypothesis (7.49), we have

$$\liminf_{h\to 0} \int_0^s j_h(f(t), u_h(t), \dot{u}_h(t))\, \mathrm{d}t \ge \int_0^s j(f(t), u(t), \dot{u}(t))\, \mathrm{d}t\,. \tag{7.78}$$

Next, we prove that u satisfies (7.44). In order to pass to the limit in $(\mathbf{Q}^a_\mathbf{h})$, we will make a convenable choice of v_h in V_h. Let $\pi_h : L^2(0, T; V) \to L^2(0, T; V_h)$ be the projection operator defined by $a(\pi_h v, w_h) = a(v, w_h)$ $\forall v \in L^2(0, T; V)$, $\forall w_h \in V_h$. Obviously, the operator π_h is well defined and $\pi_h v(t) \to v(t)$ in V a.e. on $[\,0, T\,]$, hence, by (7.49) and (4.97), it follows that, for all $s \in [0, T]$, we have

$$\lim_{h\to 0} \int_0^s j_h(f(t), u_h(t), \pi_h v(t))\, \mathrm{d}t = \int_0^s j(f(t), u(t), v(t))\, \mathrm{d}t \;\; \forall v \in L^2(0, T; V)$$

and

$$\lim_{h\to 0} \int_0^s b(f(t), u_h(t), \pi_h v(t))\, \mathrm{d}t = \int_0^s b(f(t), u(t), v(t))\, \mathrm{d}t \;\; \forall v \in L^2(0, T; V)\,.$$

Since $b(f(t), u_h(t), \dot{u}_h(t)) = 0$ a.e. on $[0, T]$, by integrating ($\mathbf{Q}_\mathbf{h}^a$) over $[0, s]$ for $v_h = \pi_h \dot{u}$ and passing to the limit, we obtain that u satisfies the first inequality of (7.44).

Now, we prove the strong convergence (7.73). Using the same argument as in the proof of Theorem 4.19, by taking $v = 0$, $v = 2\dot{u}$ in (7.44), $v_h = 0$, $v_h = 2\dot{u}_h(t)$ in ($\mathbf{Q}_\mathbf{h}^a$) and using (7.77), (7.78), for all $s \in [0, T]$, we have

$$\liminf_{h \to 0} \int_0^s a(u_h(t), \dot{u}_h(t))\, \mathrm{d}t = \int_0^s a(u(t), \dot{u}(t))\, \mathrm{d}t, \tag{7.79}$$

$$\liminf_{h \to 0} \int_0^s j_h(f(t), u_h(t), \dot{u}_h(t))\, \mathrm{d}t = \int_0^s j(f(t), u(t), \dot{u}(t))\, \mathrm{d}t. \tag{7.80}$$

and, by taking $v_h = \pi_h \dot{u}(t)$ in ($\mathbf{Q}_\mathbf{h}^a$), we obtain

$$\limsup_{h \to 0} \int_0^s a(u_h(t), \dot{u}_h(t))\, \mathrm{d}t \le \int_0^s a(u(t), \dot{u}(t))\, \mathrm{d}t \quad \forall\, s \in [0, T]. \tag{7.81}$$

From (7.79) and (7.81), it follows

$$\lim_{h \to 0} \int_0^s a(u_h(t), \dot{u}_h(t))\, \mathrm{d}t = \int_0^s a(u(t), \dot{u}(t))\, \mathrm{d}t,$$

or

$$\lim_{h \to 0} (a(u_h(s), u_h(s)) - a(u_h(0), u_h(0))) = a(u(s), u(s)) - a(u_0, u_0)\,.$$

We recall that $u_h(0) = u_{0h}$ and $u_{0h} \longrightarrow u_0$ strongly in V. Hence, we conclude

$$\lim_{h \to 0} a(u_h(s), u_h(s)) = a(u(s), u(s)) \quad \forall s \in [0, T]$$

which, with the ellipticity of a, implies the strong convergence (7.73).

Finally, we prove that u satisfies the second inequality of (7.44). From ($\mathbf{Q}_\mathbf{h}^a$), as $j_h(f(t), u_h(t), \cdot)$ is sub-additive, we deduce that, for all $t \in [0, T]$, we have

$$a(u_h(t), v_h - u_h(t)) + j_h(f(t), u_h(t), v_h - u_h(t)) \ge 0 \quad \forall\, v_h \in K_h. \tag{7.82}$$

Let $v \in K$ be arbitrarily chosen. Then, from (7.46), there exists $r_h v \in K_h$ such that $r_h v \to v$ strongly in V. By passing to the limit in (7.82) for $v_h = r_h v$ and using (7.73) and (7.50), we get that u satisfies

$$a(u(t), v - u(t)) + j(f(t), u(t), v - u(t)) \ge 0 \quad \forall v \in K$$

which, by the hypothesis (4.105), implies that u satisfies the second inequality of (7.44). From (7.73) and (7.48), it results that $u \in K(f)$ which completes the proof. □

Using Theorems 7.4 and 7.5, we conclude with the following main approximation result.

Theorem 7.6. *Under the assumptions of Theorem 7.5, the sequence* $\{u_{hn}\}_{hn}$ *of all solutions of complete discrete Problem* $(\mathbf{Q^a_h})_\mathbf{n}$ *has a subsequence, still denoted by* $\{u_{hn}\}_{hn}$*, such that*

$$u_{hn}(t) \to u(t) \quad \textit{strongly in} \quad V \quad \forall\, t \in [0, T] \quad \textit{as } h \to 0\,, n \to \infty\,, \tag{7.83}$$

$$\dot{u}_{hn} \rightharpoonup \dot{u} \quad \textit{weakly in} \quad L^2(0, T; V) \quad \textit{as } h \to 0\,, n \to \infty, \tag{7.84}$$

where $u \in W^{1,2}(0, T; V)$ *is a solution of Problem* $(\mathbf{Q}^a)$.

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Part III
Contact Problems with Friction in Elasticity

Chapter 8
Static Problems

In this chapter we study, in an almost exhaustive way, a contact problem with friction which models the contact between an elastic body and a rigid foundation. The contact is modeled upon the well-known Signorini conditions and the friction is described by a nonlocal Coulomb friction law. The classical formulation of the model is described, and a variational formulation of the problem is derived. Under appropriate assumptions on the data, existence, uniqueness and regularity results are provided. We also derive two dual formulations of this problem. Numerical analysis is carried out and convergence results are proved. Finally, a related optimal control problem is studied.

Most of these results are obtained by applying abstract results on variational inequalities presented in Part II.

8.1 Classical Formulation

We study here a static unilateral contact problem with nonlocal Coulomb friction in linear elasticity.

Let us consider a linearly elastic body occupying a bounded open set $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a sufficiently smooth boundary Γ which is decomposed into three open and disjoint parts Γ_0, Γ_1, Γ_2 such that $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_2$. The body is subjected to the action of volume forces of density $\boldsymbol{f}$ given in Ω and surface tractions of density $\boldsymbol{g}$ given on Γ_1. The displacements are prescribed on Γ_0. For the sake of simplicity we suppose that the body is clamped on Γ_0 and, so, the displacement vector vanishes here. On Γ_2 the body is in unilateral contact with a rigid foundation (Fig. 8.1).

We denote by $\boldsymbol{u}$, $\boldsymbol{\epsilon}$, and $\boldsymbol{\sigma}$ the displacement vector, the infinitesimal strain tensor and, respectively, the stress tensor related, in the framework of linear elasticity, by the constitutive law:

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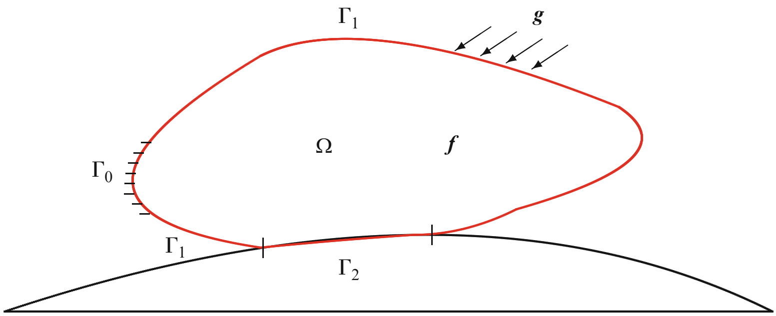

Fig. 8.1 The contact with a rigid support

$$\sigma_{ij} = a_{ijkh}\epsilon_{kh}(\boldsymbol{u}) .$$

This law expresses a linear relationship, the generalized Hooke's law, between the stress tensor and the small strain tensor (also, called linear, infinitesimal, or Cauchy's strain) defined by:

$$\epsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad 1 \le i, j \le d .$$

Here and below we adopt the usual summation convention. We suppose that the elasticity coefficients a_{ijkh} satisfy the usual symmetry conditions

$$a_{ijkh} = a_{jihk} = a_{khij} \quad 1 \le i, j, k, h \le d , \tag{8.1}$$

and the ellipticity condition

$$\exists\, \alpha > 0 \text{ such that } a_{ijkh}\xi_{ij}\xi_{kh} \ge \alpha|\boldsymbol{\xi}|^2 , \ \forall \boldsymbol{\xi} = (\xi_{ij}) \in \mathbb{R}^{d^2} . \tag{8.2}$$

We use a classical decomposition in the normal and the tangential components of the displacement vector and of the stress vector on Γ, i.e.

$$\begin{aligned} u_\nu &= u_i \nu_i \ , \qquad \boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu} \\ \sigma_\nu &= \sigma_{ij}\nu_i\nu_j \ , \ \sigma_{\tau_i} = \sigma_{ij}\nu_j - \sigma_\nu \nu_i \end{aligned}$$

where $\boldsymbol{\nu}$ is the exterior unit normal to Γ with the components $\boldsymbol{\nu} = (\nu_i)$.

The unilateral contact on Γ_2 is described by the Signorini's conditions (see, e.g., [51, 52]):

$$u_\nu \le 0 , \quad \sigma_\nu \le 0 , \quad u_\nu\sigma_\nu = 0 \quad \text{on } \Gamma_2 . \tag{8.3}$$

We see that, in the unilateral contact, the non-penetration condition ($u_\nu \leq 0$) of the body into the obstacle, which is assumed to be rigid and fixed, is taken into account. The Signorini's conditions express the fact that two cases are possible: the case when there is no contact between the body and the foundation (characterized by $u_\nu < 0$ and $\sigma_\nu = 0$) and the case when there is contact (characterized by $u_\nu = 0$ and $\sigma_\nu \leq 0$). Hence, the effective surface on which the body comes into contact with the obstacle is not known in advance and it is a part of Γ_2. But these conditions are not smooth since σ_ν is a multivalued application of u_ν. In fact, we have

$$\begin{cases} \sigma_\nu(u_\nu) = 0 \quad \text{if } u_\nu < 0\,, \\ \sigma_\nu(0) \in (-\infty, 0]\,. \end{cases}$$

We can regularize the Signorini's conditions by using the compliance model of [36, 45] in which σ_ν is considered to be a nonlinear function of u_ν. In fact, the condition of non-penetration, $u_\nu \leq 0$, is relaxed, i.e. the penetration is allowed but it is penalized with a normal compliance term. Though this model has the advantage of being simpler from mathematical point of view and having a mechanical meaning, it is not convenient for dealing with problems where the penetration into the obstacle is small: in this case σ_ν is a very stiff function of u_ν, i.e. σ_ν becomes almost a multivalued application of u_ν (see [49]).

We suppose that the contact on Γ_2 is with friction which is modeled by Coulomb's law. Outlined initially by Amontons [3], this law, which became famous, was presented to the Academy of Sciences of Paris in 1785, by the French engineer Charles-Augustin de Coulomb [19] under its form for static dry friction: "the relative sliding between two bodies in contact along plane surfaces will occur when the net share force parallel to the plane reaches a critical value proportional to the net normal force pressing the two bodies together". The constant of proportionality is called the coefficient of friction and it is dependent on the nature of the materials in contact. It should be mentioned that at the time when Coulomb has formulated his law, the concept of constraint and the general equations of the linear elasticity had not emerged. This law describes the effects of friction between two bodies and the raw slipping of the body relative to the foundation. The future developments of this law are written in terms of velocities: the frictional force required to initiate and to maintain the sliding is proportional with the magnitude of the normal force of contact and the tangential velocity is collinear to the tangential force. However, the static case, which is considered in this chapter, is a very convenient approach for problems describing monotonic loadings and, also, it can be considered as an intermediate problem in solving evolutionary problems by using incremental formulations (see Sects. 4.3 and 9.1). Therefore, the Coulomb's law is given by the following conditions:

$$|\boldsymbol{\sigma}_\tau| \leq \mu|\sigma_\nu| \text{ and } \begin{cases} |\boldsymbol{\sigma}_\tau| < \mu|\sigma_\nu| \;\Rightarrow\; \boldsymbol{u}_\tau = \boldsymbol{0} \\ |\boldsymbol{\sigma}_\tau| = \mu|\sigma_\nu| \;\Rightarrow\; \boldsymbol{u}_\tau = -\lambda\boldsymbol{\sigma}_\tau \end{cases} \text{ on } \Gamma_2\,, \tag{8.4}$$

where μ is the coefficient of friction and $|\cdot|$ denotes the absolute value if it is applied to a scalar or the Euclidean norm if it is applied to a vector.

This law shows that $\boldsymbol{\sigma}_\tau$ is a multivalued application of $\boldsymbol{u}_\tau$:

$$\begin{cases} \boldsymbol{\sigma}_\tau(\boldsymbol{u}_\tau) = -\mu\sigma_\nu \quad \text{if } \boldsymbol{u}_\tau < \mathbf{0}\,, \\ \boldsymbol{\sigma}_\tau(\mathbf{0}) \in (\mu\sigma_\nu\,,\, -\mu\sigma_\nu) \\ \boldsymbol{\sigma}_\tau(\boldsymbol{u}_\tau) = \mu\sigma_\nu \quad \text{if } \boldsymbol{u}_\tau > \mathbf{0}\,. \end{cases}$$

As in the case of unilateral contact, a regularization of this law may be obtained by considering a normal compliance penalization term (see [4, 32, 33]).

Finally, the classical formulation of this static unilateral contact problem with friction is given by the equations of equilibrium, the constitutive relation (linear elasticity), the kinematic equations (the linearized strain tensor), the boundary conditions on Γ_0 and Γ_1 and the unilateral contact conditions of Signorini and the friction law of Coulomb on Γ_2.

Find a displacement field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x})$ such that

$$\begin{cases} -\operatorname{div}\boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega\,, \\ \boldsymbol{\sigma} = \mathscr{A}\boldsymbol{\epsilon}\,, \qquad \boldsymbol{\epsilon} = \dfrac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T\right) \quad \text{in } \Omega\,, \\ \boldsymbol{u} = \mathbf{0} \quad \text{on } \Gamma_0\,, \\ \boldsymbol{\sigma}\cdot\boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_1\,, \\ u_\nu \le 0\,, \quad \sigma_\nu \le 0\,, \quad u_\nu\sigma_\nu = 0 \quad \text{on } \Gamma_2\,, \\ |\boldsymbol{\sigma}_\tau| \le \mu|\sigma_\nu| \text{ and } \begin{cases} |\boldsymbol{\sigma}_\tau| < \mu|\sigma_\nu| \;\Rightarrow\; \boldsymbol{u}_\tau = \mathbf{0} \\ |\boldsymbol{\sigma}_\tau| = \mu|\sigma_\nu| \;\Rightarrow\; \exists\lambda \ge 0\,,\; \boldsymbol{u}_\tau = -\lambda\boldsymbol{\sigma}_\tau \end{cases} \quad \text{on } \Gamma_2\,, \end{cases} \tag{8.5}$$

where $\mathscr{A} = (a_{ijkh})$ is the fourth order tensor of elasticity.

Remark 8.1. Obviously, the Coulomb law with unilateral contact should be written under the form

$$\begin{cases} u_\nu \le 0\,, \quad \sigma_\nu \le 0\,, \quad u_\nu\sigma_\nu = 0 \quad \text{on } \Gamma_2\,, \\ |\boldsymbol{\sigma}_\tau| \le -\mu\sigma_\nu \text{ and } \begin{cases} |\boldsymbol{\sigma}_\tau| < -\mu\sigma_\nu \;\Rightarrow\; \boldsymbol{u}_\tau = \mathbf{0} \\ |\boldsymbol{\sigma}_\tau| = -\mu\sigma_\nu \;\Rightarrow\; \exists\lambda \ge 0\,,\; \boldsymbol{u}_\tau = -\lambda\boldsymbol{\sigma}_\tau \end{cases} \quad \text{on } \Gamma_2\,. \end{cases}$$

However, we shall prefer to keep the formulation (8.5) unless a few special cases when the above formulation is more convenient (see Sects. 8.4 and 8.8). For the strengths and the weaknesses of the formulation (8.5), see also Remark 8.3 below.

Unfortunately, the writing at each point of the Coulomb law leads to great mathematical difficulties. Indeed, as we will see later, the variational formulation of this problem is a quasi-variational inequality which contains the term $\displaystyle\int_{\Gamma_2} \mu\,|\sigma_\nu(\boldsymbol{u})||\boldsymbol{v}_\tau|\,\mathrm{d}s$ or $\displaystyle\int_{\Gamma_2} \mu\,\sigma_\nu(\boldsymbol{u})|\boldsymbol{v}_\tau|\,\mathrm{d}s$. But, if

$$\boldsymbol{u} \in \mathscr{U}_{ad} = \{\boldsymbol{v} \in (H^1(\Omega))^d\,;\, \boldsymbol{v} = 0 \text{ a.e. on } \Gamma_0\}\,,$$

then $\sigma_\nu(\boldsymbol{u})$ is not defined on Γ_2. Moreover, even if $\boldsymbol{u} \in (H^1(\Omega))^d$ satisfies Eq. (8.5)$_1$ for $\boldsymbol{f} \in (L^2(\Omega))^d$, then $\sigma_\nu(\boldsymbol{u})$ is defined as a distribution only, and $|\sigma_\nu(\boldsymbol{u})|$ has no mathematical meaning. More precisely, it is known (see, for instance, [23]) that

$$\sigma_\nu(\boldsymbol{u}) \in H^{-1/2}(\Gamma), \quad \forall \boldsymbol{u} \in \boldsymbol{H}^1_{\text{div}}(\Omega) = \{\boldsymbol{v} \in (H^1(\Omega))^d \; ; \; \text{div}\, \boldsymbol{\sigma}(\boldsymbol{v}) \in (L^2(\Omega))^d\},$$

where $H^{-1/2}(\Gamma)$ denotes the dual space of $H^{1/2}(\Gamma)$. Also, the stress vector $\boldsymbol{\sigma} \cdot \boldsymbol{\nu}$ is defined, via an extension of the Green formula, by

$$\begin{aligned} \langle \sigma_{ij}(\boldsymbol{u})\nu_j, \gamma(v_i)\rangle_{1/2,\Gamma} &= \int_\Omega \sigma_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})\,\mathrm{d}x + \int_\Omega \sigma_{ij,j}(\boldsymbol{u})v_i\,\mathrm{d}x \\ \forall \boldsymbol{u} \in \boldsymbol{H}^1_{\text{div}}(\Omega) \quad &\forall \boldsymbol{v} \in (H^1(\Omega))^d, \end{aligned} \tag{8.6}$$

the symbol $\langle \cdot, \cdot \rangle_{1/2,\Gamma}$ denoting the duality pairing between $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$. Obviously, if $\boldsymbol{u}$ is a regular function, then we have the classical Green formula:

$$\int_\Gamma \sigma_{ij}(\boldsymbol{u})\nu_j\, v_i\,\mathrm{d}s = \int_\Omega \sigma_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})\,\mathrm{d}x + \int_\Omega \sigma_{ij,j}(\boldsymbol{u})v_i\,\mathrm{d}x \quad \forall \boldsymbol{v} \in (H^1(\Omega))^d \tag{8.7}$$

where $\mathrm{d}s$ is the surface measure element.

Therefore, if $\boldsymbol{u} \in (H^1(\Omega))^d$ satisfies Eq. (8.5)$_1$ for $\boldsymbol{f} \in (L^2(\Omega))^d$, then $\sigma_{ij,j} \in L^2(\Omega)$, which implies $\sigma_\nu(\boldsymbol{v}) \in H^{-1/2}(\Gamma)$ and $|\sigma_\nu(\boldsymbol{u})|$ has not a mathematical meaning on Γ. One can avoid this difficulty by using a nonlocal version of the Coulomb law. This law, introduced by Duvaut [22], stipulates that the motion at a point of contact between two deformable bodies may occur when the magnitude of the tangential stress vector at that point reaches a value proportional to an average of the normal stress vector in a neighborhood of the point. The character of the effective neighborhood and the manner in which the neighborhood stresses contribute to the slipping condition depend on the micro-structure of materials in contact. The nonlocal character of this law is given by the regularization of the normal stress vector σ_ν which is defined (see [20, 22, 46, 47]) at each point as a convolution on a small area surrounding the point:

$$\sigma^*_\nu(\boldsymbol{u})(x) = \int_{\Gamma_2} \omega_\rho(|x-y|)(-\sigma_\nu(y))\,\mathrm{d}y$$

where

$$\omega_\rho(x) = \begin{cases} C\, e^{\frac{\rho^2}{x^2-\rho^2}} & \text{if } 0 \le |x| \le \rho, \\ 0 & \text{if } |x| > \rho, \end{cases}$$

with C a constant such that $\displaystyle\int_{-e}^{e} \omega_\rho(x)\,\mathrm{d}x = 1$.

Hence, the local contact between the asperities is taken into account. So, in the sequel, we will consider the following problem.

Problem ($\mathscr{P}$): Find a displacement field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ such that

$$\begin{cases} -\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega\,, \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}) = \mathscr{A}\boldsymbol{\epsilon}\,, \qquad \boldsymbol{\epsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) \quad \text{in } \Omega\,, \\ \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0\,, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_1\,, \\ u_\nu \le 0\,, \quad \sigma_\nu \le 0\,, \quad u_\nu \sigma_\nu = 0 \quad \text{on } \Gamma_2\,, \\ |\boldsymbol{\sigma}_\tau| \le \mu |\mathscr{R}\sigma_\nu| \text{ and } \begin{cases} |\boldsymbol{\sigma}_\tau| < \mu|\mathscr{R}\sigma_\nu| \Rightarrow \boldsymbol{u}_\tau = 0 \\ |\boldsymbol{\sigma}_\tau| = \mu|\mathscr{R}\sigma_\nu| \Rightarrow \exists \lambda \ge 0\,,\ \boldsymbol{u}_\tau = -\lambda \boldsymbol{\sigma}_\tau \end{cases} \text{on } \Gamma_2\,, \end{cases} \tag{8.8}$$

where $\mathscr{R}\sigma_\nu$ denotes a regularization of σ_ν which will be specified later.

8.2 Displacement Variational Formulation

From now on, we will suppose that the elasticity coefficients a_{ijkh} satisfy the usual symmetry and ellipticity conditions (8.1) and (8.2).

In order to obtain the variational formulation of the mechanical problem **($\mathscr{P}$)**, we make the following regularity hypotheses on the data:

$$\begin{cases} \boldsymbol{f} \in (L^2(\Omega))^d\,,\ \boldsymbol{g} \in (L^2(\Gamma_1))^d\,, \\ a_{ijkl} \in L^\infty(\Omega),\ i,j,k,l = 1,\dots,d\,, \\ \mu \in L^\infty(\Gamma_2),\ \mu \ge 0 \text{ a.e. on } \Gamma_2 \\ \mathscr{R} : H^{-1/2}(\Gamma_2) \to L^2(\Gamma_2) \text{ is a linear continuous operator} \end{cases} \tag{8.9}$$

where $H^{-1/2}(\Gamma_2)$ is the dual space of $H^{1/2}(\Gamma_2) = \{v/\Gamma_2\,;\ v \in H^{1/2}(\Gamma)\,;\ v = 0 \text{ a.e. on } \Gamma\backslash\overline{\Gamma}_2\}$.

Let us introduce the following linear subspace

$$\boldsymbol{V} = \{\boldsymbol{v} \in (H^1(\Omega))^d\,;\ \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_0\} \tag{8.10}$$

of the Hilbert space $(H^1(\Omega))^d$ and the set of statically admissible displacement fields defined by

$$\boldsymbol{K} = \{\boldsymbol{v} \in \boldsymbol{V}\,;\ v_\nu \le 0 \text{ a.e. on } \Gamma_2\}\,. \tag{8.11}$$

We shall use the notation:

$$\begin{cases} a(\boldsymbol{w},\boldsymbol{v}) = \displaystyle\int_\Omega \boldsymbol{\sigma}(\boldsymbol{w})\boldsymbol{\epsilon}(\boldsymbol{v})\,\mathrm{d}x \quad \forall \boldsymbol{w},\ \boldsymbol{v} \in \boldsymbol{V}\,, \\ j_f(\boldsymbol{w},\boldsymbol{v}) = \displaystyle\int_{\Gamma_2} \mu|\mathscr{R}\sigma_\nu(P_f\boldsymbol{w})|\,|\boldsymbol{v}_t|\,\mathrm{d}s \quad \forall \boldsymbol{w}\,,\ \boldsymbol{v} \in \boldsymbol{V} \end{cases} \tag{8.12}$$

where $P_f : \boldsymbol{V} \to \boldsymbol{C}_f$ is the projection operator onto the closed convex set

$$\boldsymbol{C}_f = \{\boldsymbol{v} \in \boldsymbol{V} \,;\, a(\boldsymbol{v}, \boldsymbol{\varphi}) = (\boldsymbol{f}, \boldsymbol{\varphi}) \quad \forall \boldsymbol{\varphi} \in (\mathscr{D}(\Omega))^d\} \,.$$

Remark 8.2. Since $\sigma_\nu(\boldsymbol{w}) \in H^{-1/2}(\Gamma)$, for any $\boldsymbol{w} \in \boldsymbol{H}^1_{\mathrm{div}}(\Omega)$, we may define the functional

$$j(\boldsymbol{w}, \boldsymbol{v}) = \int_{\Gamma_2} \mu |\mathscr{R}\sigma_\nu(\boldsymbol{w})| \, |\boldsymbol{v}_t| \, \mathrm{d}s \quad \forall \boldsymbol{w} \in \boldsymbol{H}^1_{\mathrm{div}}(\Omega) \,, \quad \forall \boldsymbol{v} \in \boldsymbol{V} \,. \tag{8.13}$$

It is obvious that we have

$$j_f(\boldsymbol{w}, \boldsymbol{v}) = j(\boldsymbol{w}, \boldsymbol{v}) \quad \forall \boldsymbol{w} \in \boldsymbol{C}_f \,, \quad \forall \boldsymbol{v} \in \boldsymbol{V} \,.$$

Remark 8.3. As mentioned in Remark 8.1, the functional $j_f(\cdot, \cdot)$ can be defined by

$$j_f(\boldsymbol{w}, \boldsymbol{v}) = -\int_{\Gamma_2} \mu \mathscr{R}\sigma_\nu(P_f \boldsymbol{w}) \, |\boldsymbol{v}_t| \, \mathrm{d}s \quad \forall \boldsymbol{w}, \, \boldsymbol{v} \in \boldsymbol{V}$$

but, in this case, we will make the following additional assumption on the regularization operator $\mathscr{R}$:

$$\mathscr{R}(\tau) \le 0 \quad \forall \tau \in H^{-1/2}(\Gamma_2) \,,$$

and so, $\mathscr{R}$ must be more regular. However, the above assumption seems to be too strong taking into account that in problem $(\mathscr{P})$ one imposes $\sigma_\nu(\boldsymbol{u}) \le 0$ only for the solution $\boldsymbol{u}$.

We denote by $\boldsymbol{F}$ the element of $\boldsymbol{V}$ given by

$$(\boldsymbol{F}, \boldsymbol{v})_V = (\boldsymbol{f}, \boldsymbol{v})_0 + (\boldsymbol{g}, \boldsymbol{v})_{0,\Gamma_1} = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g} \cdot \boldsymbol{v} \, \mathrm{d}s \quad \forall \boldsymbol{v} \in \boldsymbol{V} \,, \tag{8.14}$$

where $(\cdot, \cdot)_V$ denotes the inner product over the space $\boldsymbol{V}$.

The variational formulation of $(\mathscr{P})$, in terms of displacements, is the following one.

Problem (P): Find $\boldsymbol{u} \in \boldsymbol{K}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j_f(\boldsymbol{u}, \boldsymbol{v}) - j_f(\boldsymbol{u}, \boldsymbol{u}) \ge (\boldsymbol{F}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall \boldsymbol{v} \in \boldsymbol{K} \,. \tag{8.15}$$

Remark 8.4. The condition of non-penetration is taken into account by the appurtenance of the displacement to the cone $\boldsymbol{K}$ of the Hilbert space $\boldsymbol{V}$.

Remark 8.5. If $\boldsymbol{u}$ is a solution of the problem (**P**), then, by taking $\boldsymbol{v} = \boldsymbol{u} \pm \boldsymbol{\varphi}$, with $\boldsymbol{\varphi} \in (\mathscr{D}(\Omega))^d$, we get $\boldsymbol{u} \in \boldsymbol{C}_{\boldsymbol{f}}$, and so,

$$j_{\boldsymbol{f}}(\boldsymbol{u}, \boldsymbol{v}) = j(\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V} .$$

Remark 8.6. For any $\boldsymbol{v} \in \boldsymbol{C}_{\boldsymbol{f}}$, we have

$$\|\sigma_\nu(\boldsymbol{v})\|_{-1/2,\Gamma} \leq C(\|\boldsymbol{v}\|_1^2 + \|\boldsymbol{f}\|_0^2)^{1/2} , \tag{8.16}$$

where C is a positive constant depending only on Ω and $\|\cdot\|_{-1/2,\Gamma}$, $\|\cdot\|_1$ and $\|\cdot\|_0$ denote the norms on $H^{-1/2}(\Gamma)$, $(H^1(\Omega))^d$ and, respectively, on $(L^2(\Omega))^d$.

Remark 8.7. For any $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$, we have

$$\|\sigma_\nu(P_{\boldsymbol{f}}\boldsymbol{u}) - \sigma_\nu(P_{\boldsymbol{f}}\boldsymbol{v})\|_{-1/2,\Gamma} \leq C\,\|\boldsymbol{u} - \boldsymbol{v}\|_1 . \tag{8.17}$$

Indeed, if $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$, then $P_{\boldsymbol{f}}\boldsymbol{u} - P_{\boldsymbol{f}}\boldsymbol{v} \in \boldsymbol{C}_{\boldsymbol{0}} = \{\boldsymbol{v} \in \boldsymbol{V}\,;\, a(\boldsymbol{v}, \boldsymbol{\varphi}) = 0\,,\ \forall \boldsymbol{\varphi} \in (\mathscr{D}(\Omega))^d\}$. From the relation (8.16), taking into consideration that $v \mapsto \sigma_\nu(v)$ is linear and the projection operator is non-expansive, we obtain

$$\|\sigma_\nu(P_{\boldsymbol{f}}\boldsymbol{u}) - \sigma_\nu(P_{\boldsymbol{f}}\boldsymbol{v})\|_{-1/2,\Gamma} \leq C\,\|P_{\boldsymbol{f}}\boldsymbol{u} - P_{\boldsymbol{f}}\boldsymbol{v}\|_1 \leq C\,\|\boldsymbol{u} - \boldsymbol{v}\|_1.$$

Theorem 8.1. *The problem* ($\mathscr{P}$) *is formally equivalent with the problem* (**P**) *in the following sense:*

(i) If $\boldsymbol{u}$ *is a sufficiently smooth function which verifies* ($\mathscr{P}$)*, then* $\boldsymbol{u}$ *is a solution of the quasi-variational inequality* (**P**).
(ii) If $\boldsymbol{u}$ *is a regular solution of the quasi-variational inequality* (**P**)*, then* $\boldsymbol{u}$ *satisfies* ($\mathscr{P}$) *in a generalized sense.*

Proof. (i) As usual, for obtaining a variational formulation, we suppose that all the functions are sufficiently smooth so that all the mathematical operations are justified. From $(\mathscr{P})_1$ we deduce directly that $\boldsymbol{u} \in \boldsymbol{C}_{\boldsymbol{f}}$, so $P_{\boldsymbol{f}}\boldsymbol{u} = \boldsymbol{u}$ and

$$j_{\boldsymbol{f}}(\boldsymbol{u}, \boldsymbol{v}) = j(\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V} . \tag{8.18}$$

Multiplying the equation $(\mathscr{P})_1$ with $\boldsymbol{v} - \boldsymbol{u}$ for $\boldsymbol{v} \in \boldsymbol{K}$ and integrating by parts over Ω, we get the Green formula:

$$(\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_0 = a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) - \int_\Gamma \sigma_{ij}(\boldsymbol{u})\nu_j(v_i - u_i)\,\mathrm{d}s . \tag{8.19}$$

Now, by using the conditions $(\mathscr{P})_3$–$(\mathscr{P})_5$ and taking into account that $\boldsymbol{u}$ and $\boldsymbol{v}$ belong to $\boldsymbol{K}$, we have

$$\int_{\Gamma} \sigma_{ij}(\boldsymbol{u})\nu_j(v_i - u_i)\,\mathrm{d}s = \int_{\Gamma_0} \sigma_{ij}(\boldsymbol{u})\nu_j(v_i - u_i)\,\mathrm{d}s + \int_{\Gamma_1} \sigma_{ij}(\boldsymbol{u})\nu_j(v_i - u_i)\,\mathrm{d}s$$
$$+ \int_{\Gamma_2} (\boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau) + \sigma_\nu(v_\nu - u_\nu))\,\mathrm{d}s = \int_{\Gamma_1} \boldsymbol{g} \cdot (\boldsymbol{v} - \boldsymbol{u})\,\mathrm{d}s$$
$$+ \int_{\Gamma_2} (\sigma_\nu v_\nu + \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau))\,\mathrm{d}s \geq \int_{\Gamma_1} \boldsymbol{g} \cdot (\boldsymbol{v} - \boldsymbol{u})\,\mathrm{d}s$$
$$+ \int_{\Gamma_2} \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\,\mathrm{d}s\,, \quad \forall \boldsymbol{v} \in \boldsymbol{K}\,. \tag{8.20}$$

Combining the relations (8.19) and (8.20), one obtains

$$\begin{aligned} &a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j(\boldsymbol{u}, \boldsymbol{v}) - j(\boldsymbol{u}, \boldsymbol{u}) - (\boldsymbol{F}, \boldsymbol{v} - \boldsymbol{u})_V \\ &\geq \int_{\Gamma_2} (\mu|\mathscr{R}\sigma_\nu(\boldsymbol{u})|(|\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|) + \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau))\ \mathrm{d}s \quad \forall \boldsymbol{v} \in \boldsymbol{K}\,. \end{aligned} \tag{8.21}$$

Next, we will prove that the conditions $(\mathscr{P})_6$ imply

$$\begin{aligned} &E = \mu|\mathscr{R}\sigma_\nu(\boldsymbol{u})|(|\boldsymbol{v}_\tau| - |\boldsymbol{u}_\tau|) + \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau) \geq 0 \quad \text{a.e. on } \Gamma_2, \\ &\quad \forall \boldsymbol{v} \text{ smooth function.} \end{aligned} \tag{8.22}$$

Indeed, if $|\boldsymbol{\sigma}_\tau| < \mu|\mathscr{R}\sigma_\nu(\boldsymbol{u})|$, then $\boldsymbol{u}_\tau = \boldsymbol{0}$ and, so

$$E = \mu|\mathscr{R}\sigma_\nu(\boldsymbol{u})|\,|\boldsymbol{v}_\tau| + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau \geq (\mu|\mathscr{R}\sigma_\nu(\boldsymbol{u})| - |\boldsymbol{\sigma}_\tau|)|\boldsymbol{v}_\tau| \geq 0\,.$$

If $|\boldsymbol{\sigma}_\tau| = \mu|\mathscr{R}\sigma_\nu(\boldsymbol{u})|$, then $\boldsymbol{u}_\tau = -\lambda\boldsymbol{\sigma}_\tau$, and, thus

$$\begin{aligned} E &= \mu|\mathscr{R}\sigma_\nu(\boldsymbol{u})|\,|\boldsymbol{v}_\tau| - \lambda\mu|\mathscr{R}\sigma_\nu(\boldsymbol{u})|\,|\boldsymbol{\sigma}_\tau| + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau + \lambda|\boldsymbol{\sigma}_\tau|^2 \\ &= |\boldsymbol{\sigma}_\tau|\,|\boldsymbol{v}_\tau| + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{v}_\tau \geq 0\,. \end{aligned}$$

Finally, from (8.21), (8.22), and (8.18), it follows

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + j_f(\boldsymbol{u}, \boldsymbol{v}) - j_f(\boldsymbol{u}, \boldsymbol{u}) - (\boldsymbol{F}, \boldsymbol{v} - \boldsymbol{u})_V \geq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}\,.$$

(ii) If $\boldsymbol{u}$ is a solution of the variational problem (**P**), then, by taking $\boldsymbol{v} = \boldsymbol{u} \pm \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in (\mathscr{D}(\Omega))^d$, one obtains

$$a(\boldsymbol{u}, \boldsymbol{\varphi}) = (\boldsymbol{f}, \boldsymbol{\varphi})_0\,, \quad \forall \boldsymbol{\varphi} \in (\mathscr{D}(\Omega))^d\,.$$

On the other hand, from the Green formula (8.7), one has

$$a(\boldsymbol{u}, \boldsymbol{\varphi}) = -\int_{\Omega} \operatorname{div} \boldsymbol{\sigma} \cdot \boldsymbol{\varphi}\,\mathrm{d}x\,, \quad \forall \boldsymbol{\varphi} \in (\mathscr{D}(\Omega))^d\,,$$

which gives $(\mathscr{P})_1$ in a generalized sense.

We multiply now $(\mathscr{P})_1$ with $\boldsymbol{v}-\boldsymbol{u}$, for $\boldsymbol{v}\in\boldsymbol{K}$, we integrate over Ω and we use the Gauss divergence theorem. This leads to

$$a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u})-(\boldsymbol{f},\boldsymbol{v}-\boldsymbol{u})_0=\int_{\Gamma}(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}s$$

and so, from the variational inequality **(P)**, we have

$$j(\boldsymbol{u},\boldsymbol{v})-j(\boldsymbol{u},\boldsymbol{u})-\int_{\Gamma_1}\boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}s+\int_{\Gamma}(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})\cdot(\boldsymbol{v}-\boldsymbol{u})\,\mathrm{d}s\ge 0\quad\forall\boldsymbol{v}\in\boldsymbol{K}\,,\tag{8.23}$$

where we took into consideration the Remark 8.5, i.e. $j_{\boldsymbol{f}}(\boldsymbol{u},\boldsymbol{v})=j(\boldsymbol{u},\boldsymbol{v})\,,\ \forall\boldsymbol{v}\in\boldsymbol{V}$, with j defined by (8.13).

Choosing $\boldsymbol{v}=\boldsymbol{u}\pm\boldsymbol{\varphi}$ with $\boldsymbol{\varphi}\in(H^{1/2}(\Gamma))^d$ and supp $\boldsymbol{\varphi}\subset\Gamma_1$, it follows

$$\int_{\Gamma_1}(\boldsymbol{\sigma}\cdot\boldsymbol{\nu}-\boldsymbol{g})\cdot\boldsymbol{\varphi}\,\mathrm{d}s=0$$

hence $(\mathscr{P})_4$ holds. Therefore, keeping in mind the inequality (8.23), one obtains

$$j(\boldsymbol{u},\boldsymbol{v})-j(\boldsymbol{u},\boldsymbol{u})+\int_{\Gamma_2}[\sigma_\nu(v_\nu-u_\nu)+\boldsymbol{\sigma}_\tau\cdot(\boldsymbol{v}_\tau-\boldsymbol{u}_\tau)]\,\mathrm{d}s\ge 0\quad\forall\boldsymbol{v}\in\boldsymbol{K}\,.\tag{8.24}$$

Now, we choose $\boldsymbol{v}=\boldsymbol{\varphi}_\tau+u_\nu\boldsymbol{\nu}$ with $\boldsymbol{\varphi}\in(H^{1/2}(\Gamma))^d$ such that supp $\boldsymbol{\varphi}\subset\Gamma_2$. Since $v_\nu=u_\nu$, $\boldsymbol{v}_\tau=\boldsymbol{\varphi}_\tau$ and $\boldsymbol{\sigma}_\tau\boldsymbol{\varphi}_\tau=\boldsymbol{\sigma}_\tau\boldsymbol{\varphi}$, we deduce

$$\int_{\Gamma_2}[\mu|\mathscr{R}(\sigma_\nu)|(|\boldsymbol{\varphi}_\tau|-|\boldsymbol{u}_\tau|)+\boldsymbol{\sigma}_\tau\cdot\boldsymbol{\varphi}]\,\mathrm{d}s\ge\int_{\Gamma_2}\boldsymbol{\sigma}_\tau\cdot\boldsymbol{u}_\tau\,\mathrm{d}s$$

thus, as $|\boldsymbol{\varphi}|\ge|\boldsymbol{\varphi}_\tau|$, we have

$$\int_{\Gamma_2}(\mu|\mathscr{R}(\sigma_\nu)|\,|\boldsymbol{\varphi}|+\boldsymbol{\sigma}_\tau\cdot\boldsymbol{\varphi})\,\mathrm{d}s-\int_{\Gamma_2}(\mu|\mathscr{R}(\sigma_\nu)|\,|\boldsymbol{u}_\tau|+\boldsymbol{\sigma}_\tau\cdot\boldsymbol{u}_\tau)\,\mathrm{d}s\ge 0\,.$$

Taking $\boldsymbol{\varphi}=\lambda\boldsymbol{\varphi}$ with $\lambda>0$, we get

$$\lambda T_1-T_2\ge 0\quad\forall\lambda>0$$

where

$$T_1=\int_{\Gamma_2}(\mu|\mathscr{R}(\sigma_\nu)|\,|\boldsymbol{\varphi}|+\boldsymbol{\sigma}_\tau\cdot\boldsymbol{\varphi})\,\mathrm{d}s\,,$$

$$T_2=\int_{\Gamma_2}(\mu|\mathscr{R}(\sigma_\nu)|\,|\boldsymbol{u}_\tau|+\boldsymbol{\sigma}_\tau\cdot\boldsymbol{u}_\tau)\,\mathrm{d}s\,.$$

Therefore, we have

$$T_1 \geq 0\,, \quad T_2 \leq 0 \quad \forall \boldsymbol{\varphi} \in (H^{1/2}(\Gamma))^d \text{ such that supp } \boldsymbol{\varphi} \subset \Gamma_2\,.$$

This implies, by taking $\boldsymbol{\varphi} = \pm\boldsymbol{\varphi}$,

$$\int_{\Gamma_2} |\boldsymbol{\sigma}_\tau|\,|\boldsymbol{\varphi}|\,\mathrm{d}s \leq \int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu)|\,|\boldsymbol{\varphi}|\,\mathrm{d}s \quad \forall \boldsymbol{\varphi} \in (H^{1/2}(\Gamma))^d \text{ with supp } \boldsymbol{\varphi} \subset \Gamma_2\,,$$

and so, $|\boldsymbol{\sigma}_\tau| \leq \mu|\mathscr{R}(\sigma_\nu)|$. Since $T_2 \leq 0$, it follows that $T_2 = 0$ and, thus

$$\mu|\mathscr{R}(\sigma_\nu)|\,|\boldsymbol{u}_\tau| + \boldsymbol{\sigma}_\tau \cdot \boldsymbol{u}_\tau = 0 \quad \text{a.e. on } \Gamma_2\,. \tag{8.25}$$

This gives $(\mathscr{P})_6$. Indeed, if $|\boldsymbol{\sigma}_\tau| < \mu|\mathscr{R}(\sigma_\nu)|$ then, assuming $\boldsymbol{u}_\tau \neq 0$, from (8.25), one obtains $\boldsymbol{\sigma}_\tau \cdot \boldsymbol{u}_\tau = -\mu|\mathscr{R}(\sigma_\nu)|\,|\boldsymbol{u}_\tau| < -|\boldsymbol{\sigma}_\tau|\,|\boldsymbol{u}_\tau|$ and, so, it is necessary to have $\boldsymbol{u}_\tau = \mathbf{0}$. If $|\boldsymbol{\sigma}_\tau| = \mu|\mathscr{R}(\sigma_\nu)|$, then, from (8.25), it follows $\boldsymbol{\sigma}_\tau \cdot \boldsymbol{u}_\tau = -|\boldsymbol{\sigma}_\tau|\,|\boldsymbol{u}_\tau|$, and so, there exists $\lambda \geq 0$ such that $\boldsymbol{u}_\tau = -\lambda\boldsymbol{\sigma}_\tau$.

In order to obtain the Signorini conditions $(\mathscr{P})_5$, we recall (8.24) and we take $\boldsymbol{v} = \varphi_\nu\boldsymbol{\nu} + \boldsymbol{u}_\tau$ where $\boldsymbol{\varphi} \in (H^1(\Gamma))^d$ with supp $\boldsymbol{\varphi} \subset \Gamma_2$ and $\varphi_\nu \leq 0$ a.e. on Γ_2. We obtain

$$\int_{\Gamma_2} \sigma_\nu\varphi_\nu\,\mathrm{d}s - \int_{\Gamma_2} \sigma_\nu u_\nu\,\mathrm{d}s \geq 0 \quad \forall \boldsymbol{\varphi} \in (H^1(\Gamma))^d \text{ with supp } \boldsymbol{\varphi} \subset \Gamma_2 \text{ and } \varphi_\nu \leq 0 \text{ a.e. on } \Gamma_2.$$

By taking, as we made in the cases of the conditions $(\mathscr{P})_6$, $\boldsymbol{\varphi} = \lambda\boldsymbol{\varphi}$ with $\lambda > 0$, we recover the conditions $(\mathscr{P})_5$. Obviously, as $\boldsymbol{u} \in \boldsymbol{K}$, the condition $u_\nu \leq 0$ a.e. on Γ_2 is satisfied. □

8.3 Existence and Uniqueness Results

In this section we give existence and, for a small enough coefficient of friction, uniqueness results for the solutions of the static unilateral contact problem with nonlocal Coulomb friction (**P**).

The beginning of the general theory of contact problems can be attributed to Duvaut and Lions [23] which gave a first mathematical formulation of the contact problem with friction in linear elasticity and proved an existence and uniqueness result for the solution of a bilateral contact problem (the contact is maintained independently of the direction of the efforts) with given friction (often called, by analogy with the corresponding plasticity law, Tresca friction). The first existence result for static unilateral contact problems with local Coulomb friction was obtained by Necas et al. [41] and then extended by Jarusek [30]. Concerning the static problem where the contact is described by a normal compliance law, we quote the results of Oden and Martins [45], Klarbring et al. [32, 33] and, for unilateral contact problems with nonlocal Coulomb friction, the results of Demkowicz and Oden [20], Oden and Pires [47], and Cocu [17].

We consider the case $d = 3$. We first prove the following existence result.

Theorem 8.2. *Let the hypotheses (8.9) hold. We assume that one of the following two conditions is satisfied:*

$$meas\ (\Gamma_0) > 0\,, \tag{8.26}$$

$$\Gamma_0 = \emptyset \text{ and } \mathscr{S} \cap \boldsymbol{K} = \{\mathbf{0}\}\,, \tag{8.27}$$

where $\mathscr{S} = \{\boldsymbol{v}\,;\ \boldsymbol{v}(x) = \mathbf{a} + \mathbf{b} \wedge \boldsymbol{x}$ *with* $\mathbf{a}, \mathbf{b} \in \mathbb{R}^d\}$ *is the set of rigid displacements.*

Then, the set of all solutions of the quasi-variational inequality **(P)** *is a nonempty weakly compact subset of* $\boldsymbol{K}$.

Proof. We shall show that the hypotheses (4.32)–(4.35) and (4.37) of Theorem 4.14 are satisfied. First, we remark that the operator $A : \boldsymbol{V} \to \boldsymbol{V}^*$ defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v}\rangle = a(\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}\,,$$

is linear and continuous. Here, $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\boldsymbol{V}^*$ and $\boldsymbol{V}$.

We will prove that any of the two conditions, (8.26) or (8.27), implies the existence of a constant $C > 0$ such that

$$\langle A\boldsymbol{v}, \boldsymbol{v}\rangle \geq C\,\|\boldsymbol{v}\|_1^2 \quad \forall \boldsymbol{v} \in \boldsymbol{K}\,, \tag{8.28}$$

i.e. the operator A is strongly monotone ($\|\cdot\|_1$ denotes the norm on $(H^1(\Omega))^d$).

If meas $(\Gamma_0) > 0$, then $\mathscr{S} \cap \boldsymbol{V} = \{\mathbf{0}\}$. On the other hand, if $\boldsymbol{v} \in \boldsymbol{V}$ such that $a(\boldsymbol{v}, \boldsymbol{v}) = 0$, then $\epsilon_{ij}(\boldsymbol{v}) = 0\,, \forall i, j \in \{1, 2, 3\}$. Hence there exist $\{a_{ij}\}_{i,j \in \{1,2,3\}} \subset \mathbb{R}$ such that

$$\begin{cases} v_i = a_{ii} + \sum\limits_{j \neq i} a_{ij} x_j \quad i \in \{1, 2, 3\} \\ a_{ij} + a_{ji} = 0 \quad \forall i \neq j\,, \end{cases}$$

and so, $\boldsymbol{v} = \mathbf{a} + \mathbf{b} \wedge \boldsymbol{x}$ with $\mathbf{a} = (a_{11}, a_{22}, a_{33})$ and $\mathbf{b} = (a_{32}, a_{13}, a_{21})$. Thus we have

$$a(\boldsymbol{v}, \boldsymbol{v}) = 0 \quad \boldsymbol{v} \in \boldsymbol{V} \Longleftrightarrow \boldsymbol{v} \in \mathscr{S}\,.$$

The above relation and the symmetry condition (8.1) imply that $\sqrt{a(\boldsymbol{v}, \boldsymbol{v})}$ is a norm on $\boldsymbol{V}$ which is equivalent to the norm $\|\boldsymbol{v}\|_1$. Indeed, as meas $(\Gamma_0) > 0$, Korn's inequality (see, e.g., [27]) holds, i.e.

$$\int\limits_\Omega \epsilon_{ij}(\boldsymbol{v})\epsilon_{ij}(\boldsymbol{v})\,\mathrm{d}x + \int\limits_\Omega v_i v_i\,\mathrm{d}x \geq C\,\|\boldsymbol{v}\|_1^2 \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,. \tag{8.29}$$

Hence, arguing by contradiction (see, for instance, [23, p. 116]), one proves that:

$$\int_{\Omega} \epsilon_{ij}(\boldsymbol{v})\epsilon_{ij}(\boldsymbol{v})\,\mathrm{d}x \geq C \int_{\Omega} v_i v_i\,\mathrm{d}x \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,,$$

with C a constant depending only on Ω and Γ_0.

Therefore, from the ellipticity condition (8.2), it follows that there exists a constant $C = C(\alpha, \Omega, \Gamma_0) > 0$ such that

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq C\,\|\boldsymbol{v}\|_1^2 \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,,$$

where α is the ellipticity constant from the relation (8.2).

If $\Gamma_0 = \emptyset$ and $\mathscr{S} \cap \boldsymbol{K} = \{\boldsymbol{0}\}$, then we decompose $\boldsymbol{V} = \mathscr{S} \oplus \mathscr{S}^{\perp}$ where $\mathscr{S}^{\perp}$ represents the orthogonal complement of $\mathscr{S}$ in $\boldsymbol{V}$ and, taking into account that Korn's inequality (8.29) still holds in $\mathscr{S}^{\perp}$, i.e.

$$a(\boldsymbol{w}, \boldsymbol{w}) \geq C\,\|\boldsymbol{w}\|_1^2 \quad \forall \boldsymbol{w} \in \mathscr{S}^{\perp}\,,$$

it follows that

$$a(\boldsymbol{v}, \boldsymbol{v}) \geq C\,\|\boldsymbol{v}\|_1^2 \quad \forall \boldsymbol{v} \in \boldsymbol{K}\,.$$

Concluding, the condition (8.26) or (8.27) is enough to ensure that the operator A, corresponding to the bilinear form $a(\cdot,\cdot)$, verifies (8.28). Therefore, by using the positivity of the function j_f, by taking $\boldsymbol{v}_0 = \boldsymbol{0}$, it follows that the condition (4.37) is satisfied.

It is easy to verify that, for any $\boldsymbol{u} \in \boldsymbol{V}$, the mapping $\boldsymbol{v} \mapsto j_f(\boldsymbol{u}, \boldsymbol{v})$ is proper convex continuous.

Next, we will show that, for any $\boldsymbol{v} \in \boldsymbol{V}$, the mappings $\boldsymbol{u} \mapsto j_f(\boldsymbol{u}, \boldsymbol{v})$ and $\boldsymbol{u} \mapsto j_f(\boldsymbol{u}, \boldsymbol{u})$ are weakly continuous on K. Let $\{\boldsymbol{u}_k\}_k \subset \boldsymbol{K}$ be a sequence which is weakly convergent to an element $\boldsymbol{u} \in \boldsymbol{K}$.

For any arbitrary element $\boldsymbol{u}^* \in \boldsymbol{C}_f$, we can write $\boldsymbol{C}_f = \boldsymbol{u}^* + \boldsymbol{C}_{\boldsymbol{0}}$. Hence, we have

$$P_f \boldsymbol{v} = \boldsymbol{u}^* + P_{\boldsymbol{0}}(\boldsymbol{v} - \boldsymbol{u}^*) \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,, \tag{8.30}$$

where $P_{\boldsymbol{0}} : \boldsymbol{V} \longrightarrow \boldsymbol{C}_{\boldsymbol{0}}$ is the projection operator on the linear subspace $\boldsymbol{C}_{\boldsymbol{0}}$ of $\boldsymbol{V}$. Indeed, for every $\boldsymbol{v} \in \boldsymbol{V}$, by denoting $\bar{\boldsymbol{v}} = \boldsymbol{u}^* + P_{\boldsymbol{0}}(\boldsymbol{v} - \boldsymbol{u}^*)$, one has $\bar{\boldsymbol{v}} \in \boldsymbol{C}_f$ and

$$(\bar{\boldsymbol{v}} - \boldsymbol{v}, \boldsymbol{w} - \bar{\boldsymbol{v}})_V = (P_{\boldsymbol{0}}(\boldsymbol{v} - \boldsymbol{u}^*) - (\boldsymbol{v} - \boldsymbol{u}^*), \mathbf{z} - P_{\boldsymbol{0}}(\boldsymbol{v} - \boldsymbol{u}^*))_V \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{C}_f$$

where $\mathbf{z} \in \boldsymbol{C}_{\boldsymbol{0}}$ such that $\boldsymbol{w} = \boldsymbol{u}^* + \mathbf{z}$. Therefore, the relation (8.30) is a consequence of the uniqueness of the projection of $\boldsymbol{v}$ by the operator P_f.

The relation (8.30) implies, P_0 being linear continuous, that $P_f \boldsymbol{u}_k \rightharpoonup P_f \boldsymbol{u}$ weakly in $\boldsymbol{V}$. In addition, the mapping $\boldsymbol{u} \mapsto \sigma_\nu(\boldsymbol{u})$ is weakly continuous, the trace operator from $\boldsymbol{V}$ to $H^{1/2}(\Gamma)$ is compact and, so it is $\mathscr{R}$. It follows that the mappings $\boldsymbol{u} \mapsto j_f(\boldsymbol{u}, \boldsymbol{v})$ and $\boldsymbol{u} \mapsto j_f(\boldsymbol{u}, \boldsymbol{u})$ are weakly continuous. Hence, by applying Theorem 4.14, the proof is complete. □

Remark 8.8. The existence of a solution for the problem **(P)** under the hypothesis (8.26) or (8.27) or

$$\Gamma_0 = \emptyset \ , \ (F, \boldsymbol{v}) < 0 \, , \ \forall \boldsymbol{v} \in \mathscr{S} \cap \boldsymbol{K} \backslash \{\boldsymbol{0}\} \tag{8.31}$$

is obtained in [17] by using the Ky-Fan inequality (see, for instance, [40, 44, 55]).

An uniqueness result for problem **(P)** is obtained in the case of a small enough coefficient of friction (see also [17]).

Theorem 8.3. *We assume that the assumptions (8.9) are satisfied and meas* $(\Gamma_0) > 0$. *Then there exists* $\mu_1 > 0$ *such that for any* $\mu \in L^\infty(\Gamma_2)$ *with* $\mu \geq 0$ *a.e. on* Γ_2 *and* $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$*, the problem* **(P)** *has a unique solution.*

Proof. This existence and uniqueness result can be obtained by applying Theorem 4.16. We will show that the functional $j_f(\cdot,\cdot)$ satisfies the relation (4.42). By using Schwartz's inequality (2.6) in $L^2(\Gamma_2)$ and taking into account that the operator $\mathscr{R}$ is linear continuous, we obtain

$$\begin{aligned}
&|j_f(\boldsymbol{u}_1, \boldsymbol{v}_2) + j_f(\boldsymbol{u}_2, \boldsymbol{v}_1) - j_f(\boldsymbol{u}_1, \boldsymbol{v}_1) - j_f(\boldsymbol{u}_2, \boldsymbol{v}_2)| \\
&= |\int_{\Gamma_2} \mu(|\mathscr{R}(\sigma_\nu(P_f \boldsymbol{u}_1))| - |\mathscr{R}(\sigma_\nu(P_f \boldsymbol{u}_2))|)(|\boldsymbol{v}_{2\tau}| - |\boldsymbol{v}_{1\tau}|)\, \mathrm{d}s| \\
&\leq |\int_{\Gamma_2} \mu |\mathscr{R}(\sigma_\nu(P_f \boldsymbol{u}_1 - P_f \boldsymbol{u}_2))|\, |\boldsymbol{v}_{2\tau} - \boldsymbol{v}_{1\tau}|\, \mathrm{d}s| \\
&\leq \|\mu\|_{L^\infty(\Gamma_2)} \|\mathscr{R}(\sigma_\nu(P_f \boldsymbol{u}_1 - P_f \boldsymbol{u}_2))\|_{L^2(\Gamma_2)} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{(L^2(\Gamma_2))^d} \\
&\leq C_1 \|\mu\|_{L^\infty(\Gamma_2)} \|\sigma_\nu(P_f \boldsymbol{u}_1 - P_f \boldsymbol{u}_2)\|_{-1/2,\Gamma_2} \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_{(L^2(\Gamma_2))^d} \ ,
\end{aligned} \tag{8.32}$$

with C_1 a positive constant.

So, since the trace operator from $\boldsymbol{V}$ into $(L^2(\Gamma_2))^d$ is continuous, with (8.17) we deduce that there exists a positive constant $C_2 = C_2(\Omega, \Gamma_2)$ such that

$$\begin{aligned}
&|j_f(\boldsymbol{u}_1, \boldsymbol{v}_2) + j_f(\boldsymbol{u}_2, \boldsymbol{v}_1) - j_f(\boldsymbol{u}_1, \boldsymbol{v}_1) - j_f(\boldsymbol{u}_2, \boldsymbol{v}_2)| \\
&\leq C_2 \|\mu\|_{L^\infty(\Gamma_2)} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1 \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_1 \quad \forall \boldsymbol{u}_1,\, \boldsymbol{u}_2,\, \boldsymbol{v}_1,\, \boldsymbol{v}_2 \in \boldsymbol{V} \ ,
\end{aligned} \tag{8.33}$$

i.e. $j_f(\cdot,\cdot)$ satisfied (4.42) with $k = C_2 \|\mu\|_{L^\infty(\Gamma_2)}$. Then, if we choose

$$0 < \mu_1 < \frac{\alpha}{C_2} \ , \tag{8.34}$$

it follows that, for any $\mu \in L^\infty(\Gamma_2)$ with $\mu \geq 0$ a.e. on Γ_2 and $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$, we have $k < \alpha$. Hence, the result follows from Theorem 4.16.

Let us also note that, using Theorem 8.2, this uniqueness result can be directly derived from the quasi-variational inequality (8.15). Indeed, supposing that there exist two solutions $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$ of Problem (**P**), we take $\boldsymbol{v} = \boldsymbol{u}_{3-i}$ in the inequality (8.15) corresponding to $\boldsymbol{u}_i$, $i = 1, 2$. Then, by adding the two inequalities and taking into account that $j_f(\boldsymbol{u}_i, \boldsymbol{v}) = j(\boldsymbol{u}_i, \boldsymbol{v})$, $\forall \boldsymbol{v} \in \boldsymbol{V}$, we obtain

$$\alpha \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1^2 \leq |j(\boldsymbol{u}_1, \boldsymbol{u}_2) + j(\boldsymbol{u}_2, \boldsymbol{u}_1) - j(\boldsymbol{u}_1, \boldsymbol{u}_1) - j(\boldsymbol{u}_2, \boldsymbol{u}_2)| \leq C \|\mu\|_{L^\infty(\Gamma_2)} \|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_1^2 ,$$

with C a positive constant. Hence, for $0 < \mu_1 < \dfrac{\alpha}{C}$ and $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$, we get $\boldsymbol{u}_1 = \boldsymbol{u}_2$. □

8.4 A Regularity Result

The solution of the quasi-variational inequality (**P**) is not, in general, enough smooth and, therefore, by Theorem 8.1, this solution does not satisfy the classical problem ($\mathscr{P}$). For this reason, it is called weak solution. Thus, it is very useful to find necessary conditions which ensure a good regularity for the solutions of the problem (**P**).

In this section, following the work [18], we give a local regularity result for the solutions of the problem (**P**). This result is obtained in more restrictive hypotheses on the data, namely

$$\Gamma_2 \neq \emptyset , \tag{8.35}$$

$$a_{ijkl} \in C^1(\overline{\Omega}) , \tag{8.36}$$

$$\mu \in C^1(\overline{\Gamma}_2) \ , \ \mu \geq 0 \text{ on } \Gamma_2 , \tag{8.37}$$

$$\begin{array}{l} \mathscr{R} : H^{-1/2}(\Gamma_2) \to C^1(\overline{\Gamma}_2) \text{ is a linear continuous operator such that} \\ \mathscr{R}(\tau) \leq 0 \, , \ \forall \tau \in H^{-1/2}(\Gamma_2) \end{array} \tag{8.38}$$

$$\boldsymbol{f} \in (L^2(\Omega))^d \ , \ \boldsymbol{g} \in (L^2(\Gamma_1))^d \ . \tag{8.39}$$

In what follows we also suppose the problem (**P**) has at least one solution (for instance, if one of the conditions (8.26), (8.27), or (8.31) is satisfied).

We remark that, if $\boldsymbol{u}$ is a solution of the problem (**P**), then the functional $j_f(\boldsymbol{u}, \cdot)$, defined by $(8.12)_3$, can be rewritten under the form

$$j_f(\boldsymbol{u}, \boldsymbol{v}) = -\int_{\Gamma_2} \mu \mathscr{R}(\sigma_\nu(\boldsymbol{u}))\, |\boldsymbol{v}_t| \, \mathrm{d}s \quad \forall \boldsymbol{v} \in \boldsymbol{V} .$$

Theorem 8.4. *Let* $\boldsymbol{u}$ *be a solution of the problem* (**P**). *Suppose that* Ω *is* C^3*-smooth in all* $\boldsymbol{x} \in \Gamma_2$. *Then, under the above hypotheses, for every open set* U *such that* $\overline{U} \subset \Omega \cup \Gamma_2$, *we have*

$$\boldsymbol{u} \in (H^2(U))^d .$$

Proof. For obtaining this result, we will apply Theorem 5.3.

Let $\boldsymbol{x} \in \Gamma_2$. From the definition of the C^3-smoothness of Ω at $\boldsymbol{x}$, it follows that there exists a neighborhood I of $\boldsymbol{x}$ such that $\overline{\Omega} \cap \overline{I}$ can be mapped by a C^3-homeomorphism onto $\overline{S}$ and the set $\partial\Omega \cap \overline{I}$ can be mapped onto the set $\overline{\Sigma}$ where the sets S and Σ are defined in Sect. 5.2.2 (see p. 94). We may assume, without loss of generality, that $\partial\Omega \cap I \subset \Gamma_2$.

Let $\boldsymbol{u}$ be a solution of the problem (**P**). Then, we remark that $\boldsymbol{u}$ verifies the following variational inequality

$$\begin{aligned}&\int\limits_{\Omega\cap I} a_{ijkl}\epsilon_{ij}(\boldsymbol{u})\epsilon_{kl}(\boldsymbol{v})\,\mathrm{d}x + \int\limits_{\Gamma_2\cap I} r(s)|\boldsymbol{v}_t|\,\mathrm{d}s - \int\limits_{\Gamma_2\cap I} r(s)|\boldsymbol{u}_t|\,\mathrm{d}s \\ &\geq \int\limits_{\Omega\cap I} f_i(v_i - u_i)\,\mathrm{d}x \qquad \forall \boldsymbol{v} \in \boldsymbol{K}_{\boldsymbol{u}} ,\end{aligned} \tag{8.40}$$

where $r(s) = -\mu(s)\mathscr{R}(\sigma_\nu(\boldsymbol{u}))(s)$, $\forall s \in \overline{\Gamma}_2$ and

$$\boldsymbol{K}_{\boldsymbol{u}} = \{\boldsymbol{w} \in (H^1(\Omega \cap I))^d \; ; \; \boldsymbol{w} = \boldsymbol{u} \text{ in } \Omega \cap \partial I, \; w_\nu \leq 0 \text{ on } \Gamma_2 \cap I\}.$$

Indeed, for any $\boldsymbol{w} \in \boldsymbol{K}_{\boldsymbol{u}}$, it follows that $\boldsymbol{w}' \in \boldsymbol{K}$ where

$$\boldsymbol{w}' = \begin{cases} \boldsymbol{w} & \text{on } \Omega \cap I , \\ \boldsymbol{u} & \text{on } \Omega \backslash I . \end{cases}$$

Thus, by taking $\boldsymbol{v} = \boldsymbol{w}'$ in (**P**), we get (8.40).

In order to apply Theorem 5.3, we use an argument due to Fichera [24]. The C^3 smoothness of Ω at $\boldsymbol{x}$ implies that, in every $\boldsymbol{y} \in \Omega \cap I$, there exists an orthogonal system of unit vectors $\boldsymbol{w}^1(\boldsymbol{y}), \ldots, \boldsymbol{w}^d(\boldsymbol{y})$ such that $\boldsymbol{w}^i \in (C^2(\overline{\Omega} \cap \overline{I}))^d$, $i = 1, \ldots, d$ and $\boldsymbol{w}^d(\boldsymbol{y}) = \boldsymbol{\nu}(\boldsymbol{y})$ for $\boldsymbol{y} \in \Gamma_2 \cap \overline{I}$. Hence, for any $\boldsymbol{v} \in (H^1(\Omega \cap I))^d$, we can write

$$\boldsymbol{v}(\boldsymbol{y}) = \bar{v}_i(\boldsymbol{y})\boldsymbol{w}^i(\boldsymbol{y}) \qquad \forall \boldsymbol{y} \in \Omega \cap I .$$

We put $\bar{\boldsymbol{v}} = (\bar{v}_1, \cdots, \bar{v}_d)$. Let $\mathbf{X}$ be the closed convex subset of $(H^1(\Omega \cap I))^d$ defined by

$$\mathbf{X} = \{\bar{\boldsymbol{v}} \in (H^1(\Omega \cap I))^d \; ; \; \bar{\boldsymbol{v}} = \bar{\boldsymbol{u}} \text{ in } \Omega \cap \partial I \text{ and } \bar{v}_d \leq 0 \text{ on } \Gamma_2 \cap I\} .$$

It is obvious that $\boldsymbol{v} \in \boldsymbol{K}_{\boldsymbol{u}}$ iff $\bar{\boldsymbol{v}} \in \mathbf{X}$.

Now, let us define the following forms:

$$b(\bar{\boldsymbol{v}},\bar{\boldsymbol{v}}') = \int\limits_{\Omega\cap I} \left(a_{ij}^{kl}(\boldsymbol{y})\frac{\partial \bar{v}_k}{\partial y_i}\frac{\partial \bar{v}_l'}{\partial y_j} + b_i^{kl}(\boldsymbol{y})\frac{\partial \bar{v}_k}{\partial y_i}v_l' + c_i^{kl}(\boldsymbol{y})\frac{\partial \bar{v}_l'}{\partial y_i}\bar{v}_k \right.$$
$$\left. + d^{kl}(\boldsymbol{y})\bar{v}_k\bar{v}_l' \right) \mathrm{d}y \quad \forall \bar{\boldsymbol{v}},\bar{\boldsymbol{v}}' \in \mathbf{X}\,,$$

$$J(\bar{\boldsymbol{v}}) = \int\limits_{\Gamma_2\cap I} r(s)\psi(\bar{\boldsymbol{v}})\,\mathrm{d}s \quad \forall \bar{\boldsymbol{v}} \in \mathbf{X}\,,$$

$$(\boldsymbol{L},\bar{\boldsymbol{v}}) = \int\limits_{\Omega\cap I} L_j\bar{v}_j\,\mathrm{d}y \quad \forall \bar{\boldsymbol{v}} \in \mathbf{X}\,,$$

where

$$\begin{aligned} a_{ij}^{kl} &= a_{ijqr}w_q^k w_r^l & b_i^{kl} &= a_{ijqr}w_q^k (w_r^l)_{,j} \\ c_i^{kl} &= a_{ijqr}(w_q^k)_{,j} w_r^l & d^{kl} &= a_{ijqr}(w_q^k)_{,i}(w_r^l)_{,j} \\ \psi(\bar{\boldsymbol{v}}) &= |(\bar{v}_1,\cdots,\bar{v}_{d-1},0)| & L_j &= f_i w_i^j\,. \end{aligned}$$

We directly obtain that

$$b(\bar{\boldsymbol{v}},\bar{\boldsymbol{v}}') = \int\limits_{\Omega\cap I} a_{ijkl}\epsilon_{ij}(\boldsymbol{v})\epsilon_{kl}(\boldsymbol{v}')\,\mathrm{d}x \quad \forall \bar{\boldsymbol{v}},\, \bar{\boldsymbol{v}}' \in \mathbf{X}\,,$$
$$J(\bar{\boldsymbol{v}}) = \int\limits_{\Gamma_2\cap I} r(s)|\boldsymbol{v}_t|\,\mathrm{d}s \quad \forall \bar{\boldsymbol{v}} \in \mathbf{X}\,,$$
$$(\boldsymbol{L},\bar{\boldsymbol{v}}) = \int\limits_{\Omega\cap I} f_i v_i\,\mathrm{d}x \quad \forall \bar{\boldsymbol{v}} \in \mathbf{X}\,,$$

hence, the variational inequality (8.40) becomes

$$b(\bar{\boldsymbol{u}},\bar{\boldsymbol{v}}-\bar{\boldsymbol{u}}) + J(\bar{\boldsymbol{v}}) - J(\bar{\boldsymbol{u}}) \geq (\boldsymbol{L},\bar{\boldsymbol{v}}-\bar{\boldsymbol{u}}) \quad \forall \bar{\boldsymbol{v}} \in \mathbf{X}\,. \tag{8.41}$$

It is easy to verify that J, $\boldsymbol{L}$, and $\mathbf{X}$, defined above, satisfy the hypotheses of Theorem 5.3 with Ω replaced by $\Omega \cap I$. Moreover, from Korn's inequality (8.29) it follows that b satisfies the relation (5.38) for any $\bar{\boldsymbol{v}} \in (H^1(\Omega \cap I))^d$ with supp $\bar{\boldsymbol{v}} \subset \overline{\Omega} \cap I$. Therefore, $\boldsymbol{u} \in (H^2(\Omega \cap I_{\boldsymbol{x}}'))^d$, $\forall I_{\boldsymbol{x}}' \ni \boldsymbol{x}$ with $\overline{I}_{\boldsymbol{x}}' \subset I$.

Arguing as in the proof of Theorem 5.3, we conclude that $\boldsymbol{u} \in (H^2(U))^d$. □

8.5 Dual Formulations for the Frictional Contact Problem ($\mathscr{P}$)

The variational formulation (**P**) (see p. 141) of the Signorini problem with nonlocal Coulomb friction is called the primal formulation of the mechanical problem **(**$\mathscr{P}$**)** (see p. 140), and has as unknown the field of displacements. In this formulation, the compatibility equations are verified in a strong manner while the equilibrium equations are verified in a weakly sense.

In what follows we will consider dual formulations of $(\mathscr{P})$ where the unknown will be the field of stresses defined on Ω or only on a part of Γ. The first dual formulation $(\mathbf{P}_1^*)$ is obtained from problem $(\mathscr{P})$ in a similar way as in the case of the formulation in terms of the displacement, and it is dual in the sense that the unknown, instead of the field of displacements, is the field of stresses. In contrast with the primal formulation $(\mathbf{P})$, the dual formulation $(\mathbf{P}_1^*)$ verifies the compatibility equations in a weakly sense and the equilibrium equations in a strong one.

The second dual formulation $(\mathbf{P}_2^*)$ is obtained from the primal formulation $(\mathbf{P})$ by applying the M–CD–M duality theory (see Sect. 6.2, p. 110), and it involves as unknown the stress field on the contact surface Γ_2 only. This is why it is called the dual condensed formulation.

We will suppose that the assumptions (8.9) are satisfied and that meas $(\Gamma_0) > 0$. Then, from Korn's inequality (8.29), it results that there exists a constant $C = C(\Omega, \Gamma_0)$ such that

$$\|\boldsymbol{\epsilon}(\boldsymbol{v})\|_{\boldsymbol{H}} \geq C\|\boldsymbol{v}\|_1 \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,, \tag{8.42}$$

where $\boldsymbol{H}$ is the Hilbert space

$$\boldsymbol{H} = \{\boldsymbol{\tau} = (\tau_{ij})\,;\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\,,\ 1 \leq i, j \leq d\}\,,$$

endowed with the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\boldsymbol{H}} = \int_{\Omega} \sigma_{ij}(\boldsymbol{x})\tau_{ij}(\boldsymbol{x})\,\mathrm{d}x \quad \forall \boldsymbol{\sigma}, \boldsymbol{\tau} \in \boldsymbol{H}\,,$$

and the corresponding norm $\|\cdot\|_{\boldsymbol{H}}$. Thus, the space $\boldsymbol{V}$ endowed with the inner product $(\cdot,\cdot)_{\boldsymbol{V}}$ defined by

$$(\boldsymbol{u}, \boldsymbol{v})_{\boldsymbol{V}} = (\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}\,,$$

is a Hilbert space, and

$$\|\boldsymbol{v}\|_1 \text{ is equivalent with } \|\boldsymbol{v}\|_{\boldsymbol{V}} \ \ \forall \boldsymbol{v} \in \boldsymbol{V}\,.$$

Also, we consider the following Hilbert space

$$\mathscr{H} = \{\boldsymbol{\tau} \in \boldsymbol{H}\,;\ \mathrm{div}\,\boldsymbol{\tau} \in (L^2(\Omega))^d\}\,,$$

equipped with the inner product

$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathscr{H}} = (\boldsymbol{\sigma}, \boldsymbol{\tau})_{\boldsymbol{H}} + (\mathrm{div}\,\boldsymbol{\sigma},\ \mathrm{div}\,\boldsymbol{\tau})_0\,,$$

and the corresponding norm $\|\cdot\|_{\mathscr{H}}$. Here, $(\cdot,\cdot)_0$ denotes the inner product on $(L^2(\Omega))^d$.

Now, we define the set of statically admissible stress fields by

$$\boldsymbol{\Sigma}(\boldsymbol{s}) = \{\boldsymbol{\tau} \in \boldsymbol{H}\,;\ (\boldsymbol{\tau}, \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} + \bar{j}(\boldsymbol{s}, \boldsymbol{v}) \geq (\boldsymbol{F}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{K}\},$$

where

$$\bar{j}(\boldsymbol{s}, \boldsymbol{v}) = \int_{\Gamma_2} \mu|\mathscr{R}(s_\nu)|\,|\boldsymbol{v}_\tau|\,\mathrm{d}s \quad \forall (\boldsymbol{s}, \boldsymbol{v}) \in \mathscr{H} \times \boldsymbol{V}.$$

This definition implies that, for every $\boldsymbol{s} \in \mathscr{H}$, the set $\boldsymbol{\Sigma}(\boldsymbol{s})$ is nonempty. Indeed, since

$$(\boldsymbol{\epsilon}(\boldsymbol{F}), \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} = (\boldsymbol{F}, \boldsymbol{v})_{\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{V},$$

and

$$\bar{j}(\boldsymbol{s}, \boldsymbol{v}) \geq 0 \quad \forall (\boldsymbol{s}, \boldsymbol{v}) \in \mathscr{H} \times \boldsymbol{V},$$

it follows that $\boldsymbol{\epsilon}(\boldsymbol{F}) \in \boldsymbol{\Sigma}(\boldsymbol{s})$, $\forall \boldsymbol{s} \in \mathscr{H}$.

We consider the following variational formulation in terms of the stress

Problem ($\mathbf{P_1^*}$): Find a stress field $\boldsymbol{\sigma} : \Omega \longrightarrow S_d$ such that

$$\begin{cases} \boldsymbol{\sigma} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}) \\ b(\boldsymbol{\sigma}, \boldsymbol{\tau} - \boldsymbol{\sigma}) \geq 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}) \end{cases} \tag{8.43}$$

where S_d is the space of second order symmetric tensors on $\mathbb{R}^d$ and

$$b(\boldsymbol{\sigma}, \boldsymbol{\tau}) = \int_\Omega \mathscr{G}\boldsymbol{\sigma} \cdot \boldsymbol{\tau}\,\mathrm{d}x \quad \forall \boldsymbol{\sigma}, \boldsymbol{\epsilon} \in \boldsymbol{W},$$

$\mathscr{G} = \mathscr{A}^{-1}$ being the compliance fourth order tensor.

Remark 8.9. If the condition $\boldsymbol{u} = \boldsymbol{0}$ on Γ_0 is replaced by $\boldsymbol{u} = \boldsymbol{u}_0$ with $\boldsymbol{u}_0$ given, then the dual formulation becomes

$$\begin{cases} \boldsymbol{\sigma} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}) \\ b(\boldsymbol{\sigma}, \boldsymbol{\tau} - \boldsymbol{\sigma}) \geq L(\boldsymbol{\tau} - \boldsymbol{\sigma}) \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}) \end{cases}$$

where

$$L(\boldsymbol{\tau}) = \int_{\Gamma_0} \boldsymbol{u}_0\,\boldsymbol{\tau} \cdot \boldsymbol{\nu}\,\mathrm{d}s.$$

Following a similar approach as in [21], one obtains the following relation between the primal formulation (**P**) and the dual formulation ($\mathbf{P_1^*}$).

Theorem 8.5. *(i) If $\boldsymbol{u}$ is a solution of the primal problem* $(\mathbf{P})$*, then $\boldsymbol{\sigma}$ defined by $\boldsymbol{\sigma} = \mathscr{A}\boldsymbol{\epsilon}(\boldsymbol{u})$ is a solution of the dual problem* $(\mathbf{P}_1^*)$.

(ii) Conversely, let $\boldsymbol{\sigma}^*$ be a solution of the problem $(\mathbf{P}_1^*)$*. Then, there exists a unique function $\boldsymbol{u} \in \boldsymbol{V}$ such that $\boldsymbol{\sigma}^* = \mathscr{A}\boldsymbol{\epsilon}(\boldsymbol{u}) = \boldsymbol{\sigma}(\boldsymbol{u})$. Moreover, $\boldsymbol{u}$ is a solution of the problem* $(\mathbf{P})$.

Proof. (i) If $\boldsymbol{u}$ is a solution of $(\mathbf{P})$, then, by taking $\boldsymbol{v} = \boldsymbol{u} \pm \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in (\mathscr{D}(\Omega))^d$, we deduce that $-\sigma_{ij,j} = f_i$ a.e. on Ω, i.e. $\boldsymbol{\sigma} \in \mathscr{H}$.

In addition, putting $\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v} = 2\boldsymbol{u}$ in $(\mathbf{P})$, we obtain the equivalence between the problem $(\mathbf{P})$ and the following one

$$\begin{cases} (\boldsymbol{\sigma}, \boldsymbol{\epsilon}(\boldsymbol{v}))_H + \bar{j}(\boldsymbol{\sigma}, \boldsymbol{v}) \geq (\boldsymbol{F}, \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \\ (\boldsymbol{\sigma}, \boldsymbol{\epsilon}(\boldsymbol{u}))_H + \bar{j}(\boldsymbol{\sigma}, \boldsymbol{u}) = (\boldsymbol{F}, \boldsymbol{u})_V, \end{cases} \tag{8.44}$$

thus, $\boldsymbol{\sigma} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma})$.

We now prove that $\boldsymbol{\sigma}$ verifies $(\mathbf{P}_1^*)$. From $(8.44)_2$ and the definition of the set $\boldsymbol{\Sigma}(\boldsymbol{\sigma})$, we have

$$\begin{aligned} b(\boldsymbol{\sigma}, \boldsymbol{\tau} - \boldsymbol{\sigma}) &= (\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\tau} - \boldsymbol{\sigma})_H = (\boldsymbol{\tau}, \boldsymbol{\epsilon}(\boldsymbol{u}))_H - (\boldsymbol{\sigma}, \boldsymbol{\epsilon}(\boldsymbol{u}))_H \\ &\geq (\boldsymbol{F}, \boldsymbol{u})_V - \bar{j}(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}, \boldsymbol{\epsilon}(\boldsymbol{u}))_H = 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}), \end{aligned}$$

which concludes the proof of the assertion.

(ii) Let $\boldsymbol{\sigma}^*$ be a solution of the dual problem $(\mathbf{P}_1^*)$ and $\mathbf{p} = \mathscr{G}\boldsymbol{\sigma}^*$. Then, one has

$$(\mathbf{p}, \boldsymbol{\tau} - \boldsymbol{\sigma}^*)_H \geq 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}^*). \tag{8.45}$$

We denote by $(\boldsymbol{\epsilon}(\boldsymbol{V}))^\perp$ the orthogonal complement in $\boldsymbol{H}$ of the closed subspace $\boldsymbol{\epsilon}(\boldsymbol{V}) = \{\boldsymbol{\epsilon}(\boldsymbol{v});\ \boldsymbol{v} \in \boldsymbol{V}\}$. Let $\bar{\mathbf{p}} \in (\boldsymbol{\epsilon}(\boldsymbol{V}))^\perp$, i.e.

$$(\bar{\mathbf{p}}, \boldsymbol{\epsilon}(\boldsymbol{v}))_H = 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V}.$$

Then, from the definition of $\boldsymbol{\Sigma}(\boldsymbol{\sigma}^*)$, we have

$$(\boldsymbol{\sigma}^* \pm \bar{\mathbf{p}}, \boldsymbol{\epsilon}(\boldsymbol{v}))_H + \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{v}) = (\boldsymbol{\sigma}^*, \boldsymbol{\epsilon}(\boldsymbol{v}))_H + \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{v}) \geq (\mathbf{F}, \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in \boldsymbol{K},$$

and, hence $\boldsymbol{\sigma}^* \pm \bar{\mathbf{p}} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}^*)$. By taking $\boldsymbol{\tau} = \boldsymbol{\sigma}^* \pm \bar{\mathbf{p}}$ in (8.45), we obtain $(\mathbf{p}, \bar{\mathbf{p}})_H = 0$, and thus $\mathbf{p} \in ((\boldsymbol{\epsilon}(\boldsymbol{V}))^\perp)^\perp = \boldsymbol{\epsilon}(\boldsymbol{V})$. This implies that there exists an element $\boldsymbol{u} \in \boldsymbol{V}$ such that $\mathbf{p} = \boldsymbol{\epsilon}(\boldsymbol{u})$, and, from the consequence (8.42) of the Korn inequality, we deduce the uniqueness of $\boldsymbol{u}$. Therefore,

$$\boldsymbol{\sigma}^* = \mathscr{A}\boldsymbol{\epsilon}(\boldsymbol{u}) = \boldsymbol{\sigma}(\boldsymbol{u}), \tag{8.46}$$

and the relation (8.45) can be written as

$$(\boldsymbol{\epsilon}(\boldsymbol{u}), \boldsymbol{\tau} - \boldsymbol{\sigma}^*)_H \geq 0, \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}^*). \tag{8.47}$$

Now, arguing by contradiction, we prove that $\boldsymbol{u} \in \boldsymbol{K}$. For this reason we consider the following inner product on $\boldsymbol{V}$

$$(\boldsymbol{w}, \boldsymbol{v})_A = a(\boldsymbol{w}, \boldsymbol{v}) = (\boldsymbol{\sigma}(\boldsymbol{w}), \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} = (\mathscr{A}\boldsymbol{\epsilon}(\boldsymbol{w}), \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} \quad \forall \boldsymbol{w}, \boldsymbol{v} \in \boldsymbol{V}, \tag{8.48}$$

and we denote by $\|\cdot\|_A$ the corresponding norm which is equivalent, thanks to the properties of $a(\cdot,\cdot)$, with the norm $\|\cdot\|_{\boldsymbol{V}}$. Hence, $(\boldsymbol{V}, \|\cdot\|_A)$ is a Hilbert space.

Let us suppose that $\boldsymbol{u} \notin \boldsymbol{K}$. If $P_{\boldsymbol{K}} : \boldsymbol{V} \longrightarrow \boldsymbol{K}$ denotes the projection operator of $(\boldsymbol{V}, \|\cdot\|_A)$ onto the nonempty closed convex subset $\boldsymbol{K}$ of $\boldsymbol{V}$, then we have

$$(P_{\boldsymbol{K}}\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{v})_A \geq (P_{\boldsymbol{K}}\boldsymbol{u} - \boldsymbol{u}, P_{\boldsymbol{K}}\boldsymbol{u})_A > (P_{\boldsymbol{K}}\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u})_A \quad \forall \boldsymbol{v} \in \boldsymbol{K},$$

and so, there exists $\alpha \in \mathbb{R}$ such that

$$(P_{\boldsymbol{K}}\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{v})_A > \alpha > (P_{\boldsymbol{K}}\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u})_A \quad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

Putting $\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma}(P_{\boldsymbol{K}}\boldsymbol{u} - \boldsymbol{u}) = \mathscr{A}\boldsymbol{\epsilon}(P_{\boldsymbol{K}}\boldsymbol{u} - \boldsymbol{u}) \in \boldsymbol{H}$, we get

$$\begin{aligned} &(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} = (\boldsymbol{\sigma}(P_{\boldsymbol{K}}\boldsymbol{u} - \boldsymbol{u}), \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} \\ &= (P_{\boldsymbol{K}}\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{v})_A > \alpha > (P_{\boldsymbol{K}}\boldsymbol{u} - \boldsymbol{u}, \boldsymbol{u})_A = (\tilde{\boldsymbol{\sigma}}, \boldsymbol{\epsilon}(\boldsymbol{u}))_{\boldsymbol{H}} \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \end{aligned} \tag{8.49}$$

and, by taking $\boldsymbol{v} = \boldsymbol{0}$, we deduce

$$(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\epsilon}(\boldsymbol{u}))_{\boldsymbol{H}} < \alpha < 0. \tag{8.50}$$

On the other hand, we will prove that

$$(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} \geq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}. \tag{8.51}$$

Indeed, let us suppose that there exists $\boldsymbol{v}_0 \in \boldsymbol{K}$ such that

$$(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\epsilon}(\boldsymbol{v}_0))_{\boldsymbol{H}} < 0. \tag{8.52}$$

Since $\lambda \boldsymbol{v}_0 \in \boldsymbol{K}$, $\forall \lambda \geq 0$, then, from (8.49), we obtain

$$\lambda(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\epsilon}(\boldsymbol{v}_0))_{\boldsymbol{H}} > \alpha, \quad \forall \lambda \geq 0.$$

By passing to the limit with $\lambda \longrightarrow +\infty$, together with (8.52), we deduce that $\alpha \leq -\infty$ which contradicts the hypothesis $\alpha \in \mathbb{R}$. Thus, the relation (8.51) is proved, and, since $\boldsymbol{\sigma}^* \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}^*)$, one has

$$(\boldsymbol{\sigma}^* + \tilde{\boldsymbol{\sigma}}, \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} + \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{v}) = (\boldsymbol{\sigma}^*, \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} + \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{v}) + (\tilde{\boldsymbol{\sigma}}, \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} \geq (\boldsymbol{F}, \boldsymbol{v})_{\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{K},$$

hence $\boldsymbol{\sigma}^* + \tilde{\boldsymbol{\sigma}} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}^*)$. Therefore, by choosing $\boldsymbol{\tau} = \boldsymbol{\sigma}^* + \tilde{\boldsymbol{\sigma}}$ in (8.47), we obtain

$$(\tilde{\boldsymbol{\sigma}}, \boldsymbol{\epsilon}(\boldsymbol{u}))_{\boldsymbol{H}} \geq 0, \tag{8.53}$$

which is in contradiction with the relation (8.50). We conclude that $\boldsymbol{u} \in \boldsymbol{K}$.

Next, we will see that $\boldsymbol{u}$ verifies $(\mathbf{P})$. First, from $\boldsymbol{\sigma}^* \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}^*)$, one has

$$(\boldsymbol{\sigma}^*, \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} + \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{v}) \geq (\boldsymbol{F}, \boldsymbol{v})_{\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \tag{8.54}$$

and hence, also

$$(\boldsymbol{\sigma}^*, \boldsymbol{\epsilon}(\boldsymbol{u}))_{\boldsymbol{H}} + \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{u}) \geq (\boldsymbol{F}, \boldsymbol{u})_{\boldsymbol{V}} \,. \tag{8.55}$$

Putting

$$J_{\sigma^*}(\boldsymbol{v}) = \begin{cases} \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{v}) & \text{if } \boldsymbol{v} \in \boldsymbol{K}, \\ +\infty & \text{otherwise,} \end{cases}$$

we deduce that the function $J_{\sigma^*} : \boldsymbol{V} \longrightarrow \bar{\mathbb{R}}$ is subdifferentiable on $\boldsymbol{V}$, and hence there exists $\boldsymbol{\sigma}_1 \in \boldsymbol{H}$ such that

$$(\boldsymbol{\sigma}_1, \boldsymbol{\epsilon}(\boldsymbol{v} - \boldsymbol{u}))_{\boldsymbol{H}} + \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{v}) - \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{u}) \geq (\boldsymbol{F}, \boldsymbol{v} - \boldsymbol{u})_{\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

By taking $\boldsymbol{v} = 2\boldsymbol{u}$ and $\boldsymbol{v} = \mathbf{0}$, we deduce

$$(\boldsymbol{\sigma}_1, \boldsymbol{\epsilon}(\boldsymbol{u}))_{\boldsymbol{H}} + \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{u}) = (\boldsymbol{F}, \boldsymbol{u})_{\boldsymbol{V}} \,, \tag{8.56}$$

$$(\boldsymbol{\sigma}_1, \boldsymbol{\epsilon}(\boldsymbol{v}))_{\boldsymbol{H}} + \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{v}) \geq (\boldsymbol{F}, \boldsymbol{v})_{\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{K}. \tag{8.57}$$

The second relation involves $\boldsymbol{\sigma}_1 \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}^*)$. Therefore, we can take $\boldsymbol{\tau} = \boldsymbol{\sigma}_1$ in (8.47), and so

$$(\boldsymbol{\sigma}_1, \boldsymbol{\epsilon}(\boldsymbol{u}))_{\boldsymbol{H}} \geq (\boldsymbol{\sigma}^*, \boldsymbol{\epsilon}(\boldsymbol{u}))_{\boldsymbol{H}}. \tag{8.58}$$

From (8.56) and (8.58), we have

$$(\boldsymbol{F}, \boldsymbol{u})_{\boldsymbol{V}} - \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{u}) \geq (\boldsymbol{\sigma}^*, \boldsymbol{\epsilon}(\boldsymbol{u}))_{\boldsymbol{H}} \,,$$

which, thanks to (8.55), gives

$$(\boldsymbol{\sigma}^*, \boldsymbol{\epsilon}(\boldsymbol{u}))_{\boldsymbol{H}} + \bar{j}(\boldsymbol{\sigma}^*, \boldsymbol{u}) = (\boldsymbol{F}, \boldsymbol{u})_{\boldsymbol{V}} \,. \tag{8.59}$$

Keeping in mind the relations (8.46), (8.54), (8.59), and (8.48), we conclude the proof. □

In the following we give an existence and uniqueness result for the solutions of the dual problem $(\mathbf{P}_1^*)$.

Theorem 8.6. *Suppose that meas* $(\Gamma_0) > 0$. *Then, under the hypothesis (8.9), there exists a constant* $\mu_1 > 0$*, depending only on* Ω*, such that, for every* μ *with* $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$*, the dual problem* $(\mathbf{P}_1^*)$ *has a unique solution* $\boldsymbol{\sigma}$. *Moreover, this solution is* $\boldsymbol{\sigma}(\boldsymbol{u})$ *where* $\boldsymbol{u}$ *is the unique solution of the primal problem* $(\mathbf{P})$.

Proof. Following Theorems 8.3 and 8.5, the result is immediate. Nevertheless, we give here a proof which puts into evidence two useful mappings for the study of the problems **(P)** and $(\mathbf{P_1^*})$.

Let the mappings $T : \mathscr{H} \longrightarrow \mathscr{H}$ and $U : \mathscr{H} \longrightarrow \mathscr{H}$ be defined, for any $\boldsymbol{s} \in \mathscr{H}$, by $T(\boldsymbol{s}) = \boldsymbol{\sigma}(\boldsymbol{u_s}) = \mathscr{A}\boldsymbol{\epsilon}(\boldsymbol{u_s})$ and $U(\boldsymbol{s}) = \boldsymbol{\sigma_s}$ where $\boldsymbol{u_s}$ is the unique solution of the problem $(\mathbf{P_1^{ax}})_s$ defined below by

Problem $(\mathbf{P_1^{ax}})_s$: Find $\boldsymbol{u_s} \in \boldsymbol{K}$ such that

$$a(\boldsymbol{u_s}, \boldsymbol{v} - \boldsymbol{u_s}) + \bar{j}(\boldsymbol{s}, \boldsymbol{v}) - \bar{j}(\boldsymbol{s}, \boldsymbol{u_s}) \geq (\boldsymbol{F}, \boldsymbol{v} - \boldsymbol{u_s})_V \quad \forall \boldsymbol{v} \in \boldsymbol{K} \tag{8.60}$$

and, respectively, $\boldsymbol{\sigma_s}$ is the unique solution of the following problem

Problem $(\mathbf{P_2^{ax}})_s$: Find $\boldsymbol{\sigma_s} \in \boldsymbol{\Sigma}(\boldsymbol{s})$ such that

$$b(\boldsymbol{\sigma_s}, \boldsymbol{\tau} - \boldsymbol{\sigma_s}) \geq 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{s}). \tag{8.61}$$

We note that the auxiliary problems $(\mathbf{P_1^{ax}})_s$ and $(\mathbf{P_2^{ax}})_s$ are the variational formulations of contact problems with given friction. The existence and the uniqueness of $\boldsymbol{u_s}$, and respectively $\boldsymbol{\sigma_s}$, follows from Theorem 4.12.

First, because $\boldsymbol{\sigma_s} \in \boldsymbol{\Sigma}(\boldsymbol{s})$, we have

$$(\boldsymbol{\sigma_s}, \boldsymbol{\epsilon}(\boldsymbol{v}))_H + \bar{j}(\boldsymbol{s}, \boldsymbol{v}) \geq (\boldsymbol{F}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{K}$$

hence, by taking $\boldsymbol{v} = \pm\varphi \in (\mathscr{D}(\Omega))^p$, we obtain that $-\text{div}\, \boldsymbol{\sigma_s} = \mathbf{f}$ a.e. on Ω, i.e. $\boldsymbol{\sigma_s} \in \mathscr{H}$, and thus the mappings U is well defined.

On the other hand, by taking $\boldsymbol{v} = 2\boldsymbol{u_s}$ and $\boldsymbol{v} = \mathbf{0}$ in $(\mathbf{P_1^{ax}})_s$ we get

$$\begin{cases} (\boldsymbol{\sigma}(\boldsymbol{u_s}), \boldsymbol{\epsilon}(\boldsymbol{u_s}))_H + \bar{j}(\boldsymbol{s}, \boldsymbol{u_s}) = (\boldsymbol{F}, \boldsymbol{u_s})_V, \\ (\boldsymbol{\sigma}(\boldsymbol{u_s}), \boldsymbol{\epsilon}(\boldsymbol{v}))_H + \bar{j}(\boldsymbol{s}, \boldsymbol{v}) \geq (\boldsymbol{F}, \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in \boldsymbol{K}, \end{cases} \tag{8.62}$$

hence $\boldsymbol{\sigma}(\boldsymbol{u_s}) \in \boldsymbol{\Sigma}(\boldsymbol{s})$, and so $\boldsymbol{\sigma}(\boldsymbol{u_s}) \in \mathscr{H}$, i.e. the mapping T is well defined.

We will prove that $U(\boldsymbol{s}) = T(\boldsymbol{s})$, $\forall \boldsymbol{s} \in \mathscr{H}$. Let $\boldsymbol{s} \in \mathscr{H}$ and $\boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{s})$, i.e.

$$(\boldsymbol{\tau}, \boldsymbol{\epsilon}(\boldsymbol{v}))_H + \bar{j}(\boldsymbol{s}, \boldsymbol{v}) \geq (\boldsymbol{F}, \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in \boldsymbol{K}.$$

Choosing $\boldsymbol{v} = \boldsymbol{u_s}$, we obtain

$$(\boldsymbol{\tau}, \boldsymbol{\epsilon}(\boldsymbol{u_s}))_H + \bar{j}(\boldsymbol{s}, \boldsymbol{u_s}) \geq (\boldsymbol{F}, \boldsymbol{u_s})_V,$$

which, together with the first relation of (8.62), gives

$$(\boldsymbol{\tau} - \boldsymbol{\sigma}(\boldsymbol{u_s}), \boldsymbol{\epsilon}(\boldsymbol{u_s}))_H \geq 0,$$

or

$$b(\boldsymbol{\sigma}(\boldsymbol{u_s}), \boldsymbol{\tau} - \boldsymbol{\sigma}(\boldsymbol{u_s})) \geq 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{s}). \tag{8.63}$$

Since $\boldsymbol{\sigma}(\boldsymbol{u}_s) \in \boldsymbol{\Sigma}(\boldsymbol{s})$, from (8.63) we deduce that $\boldsymbol{\sigma}(\boldsymbol{u}_s)$ is a solution of the problem $(\mathbf{P_2^{ax}})_s$. Hence, taking into account that the problem $(\mathbf{P_2^{ax}})_s$ has a unique solution, we conclude that $\boldsymbol{\sigma}(\boldsymbol{u}_s) = \boldsymbol{\sigma}_s$, i.e. $U(\boldsymbol{s}) = T(\boldsymbol{s})$.

The mapping T is a contraction on $\mathscr{H}$. Indeed, if $\boldsymbol{s}_1,\ \boldsymbol{s}_2 \in \mathscr{H}$, then, from $(\mathbf{P_1^{ax}})_{s_i}$, $i \in \{1, 2\}$, we have

$$-\mathrm{div}\,\boldsymbol{\sigma}(\boldsymbol{u}_{s_1}) = -\mathrm{div}\,\boldsymbol{\sigma}(\boldsymbol{u}_{s_2}) = \boldsymbol{f} \text{ a.e. on } \Omega\,,$$

and so,

$$\begin{aligned}&\|T(\mathbf{s}_1) - T(\mathbf{s}_2)\|_{\mathscr{H}} = \|\boldsymbol{\sigma}(\boldsymbol{u}_{\mathrm{s}_1}) - \boldsymbol{\sigma}(\boldsymbol{u}_{\mathrm{s}_2})\|_{\mathscr{H}} = \|\boldsymbol{\sigma}(\boldsymbol{u}_{\mathrm{s}_1} - \boldsymbol{u}_{\mathrm{s}_2})\|_{\boldsymbol{H}} \\ &\le C_1\|\boldsymbol{\epsilon}(\boldsymbol{u}_{\mathrm{s}_1} - \boldsymbol{u}_{\mathrm{s}_2})\|_{\boldsymbol{H}} = C_1\|\boldsymbol{u}_{\mathrm{s}_1} - \boldsymbol{u}_{\mathrm{s}_2}\|_{\boldsymbol{V}} \le C\,\|\boldsymbol{u}_{\mathrm{s}_1} - \boldsymbol{u}_{\mathrm{s}_2}\|_1\,.\end{aligned} \tag{8.64}$$

Now, by adding the inequalities $(\mathbf{P_1^{ax}})_{s_i}$, $i \in \{1, 2\}$, for $\boldsymbol{v} = \boldsymbol{u}_{s_{3-i}}$, we obtain

$$\begin{aligned}&\alpha\|\boldsymbol{u}_{s_1} - \boldsymbol{u}_{s_2}\|_1^2 \le a(\boldsymbol{u}_{s_1} - \boldsymbol{u}_{s_2}, \boldsymbol{u}_{s_1} - \boldsymbol{u}_{s_2}) \le |\bar{j}(\boldsymbol{s}_1, \boldsymbol{u}_{s_2}) - \bar{j}(\boldsymbol{s}_1, \boldsymbol{u}_{s_1}) + \bar{j}(\boldsymbol{s}_2, \boldsymbol{u}_{s_1}) \\ &-\bar{j}(\boldsymbol{s}_2, \boldsymbol{u}_{s_2})| \le \left|\int_{\Gamma_2} \mu|\mathscr{R}((\boldsymbol{s}_1)_\nu - (\boldsymbol{s}_2)_\nu)|\,|(\boldsymbol{u}_{s_1})_\tau - (\boldsymbol{u}_{s_2})_\tau|\,ds\right| \\ &\le C\,\|\mu\|_{L^\infty(\Gamma_2)}\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|_{\mathscr{H}}\|\boldsymbol{u}_{s_1} - \boldsymbol{u}_{s_2}\|_1\end{aligned}$$

thus

$$\|T(\boldsymbol{s}_1) - T(\boldsymbol{s}_2)\|_{\mathscr{H}} \le C\,\|\mu\|_{L^\infty(\Gamma_2)}\|\boldsymbol{s}_1 - \boldsymbol{s}_2\|_{\mathscr{H}}\,.$$

Therefore, for μ sufficiently small, it follows the mapping T is a contraction, which, together Banach's fixed point Theorem 4.7, implies the existence of a unique element $\boldsymbol{s}^*$ such that $T(\boldsymbol{s}^*) = \boldsymbol{s}^*$. Taking in mind that the set of all fixed points of the mappings T and U is the same with the set of all solutions of the problem (**P**), and, respectively, $(\mathbf{P_1^*})$, we proved the existence and the uniqueness of the solution $\boldsymbol{u}^*$ of (**P**), and respectively $\boldsymbol{\sigma}^*$ of $(\mathbf{P_1^*})$ with $\boldsymbol{\sigma}^* = \boldsymbol{\sigma}(\boldsymbol{u}^*)$. □

In the sequel, by applying the Mosco–Capuzzo-Dolcetta–Matzeu (M–CD–M) duality theory, described in Sect. 6.2, we obtain, following [14] (see, also [56, 57]), the so-called dual condensed formulation of the problem (**P**).

Let $\boldsymbol{u}$ be a solution of the problem (**P**). According to Remark 8.5 we have

$$j_f(\boldsymbol{u}, \boldsymbol{v}) = j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_2} \mu|\mathscr{R}(\sigma_\nu(\boldsymbol{u}))|\,|\boldsymbol{v}_\tau|\,\mathrm{d}s \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,,$$

and hence, $\boldsymbol{u}$ is a solution of the following problem

$$\begin{cases}\text{Find } \boldsymbol{u} \in \boldsymbol{V} \text{ such that} \\ \varphi(L\boldsymbol{u}, \boldsymbol{u}) + \psi(\boldsymbol{u}, \boldsymbol{u}) \le \varphi(L\boldsymbol{u}, \boldsymbol{v}) + \psi(\boldsymbol{u}, \boldsymbol{v})\ \forall \boldsymbol{v} \in \boldsymbol{V}\end{cases}$$

where

$$\begin{aligned}
&\psi(\boldsymbol{u},\boldsymbol{v}) = a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) - (\boldsymbol{F},\boldsymbol{v}-\boldsymbol{u})_V \quad \forall \boldsymbol{u}\,,\ \boldsymbol{v}\in \boldsymbol{V}\,,\\
&L:\boldsymbol{V}\longrightarrow Y = L^2(\Gamma_2), \quad L\boldsymbol{v} = \mathscr{R}(\sigma_\nu(\boldsymbol{v})) \quad \forall \boldsymbol{v}\in \boldsymbol{V}\,,\\
&\mathscr{J}(\theta,\boldsymbol{v}) = \int_{\Gamma_2} \mu|\theta|\,|\boldsymbol{v}_\tau|\,\mathrm{d}s \quad \forall\theta\in Y\,,\ \forall \boldsymbol{v}\in \boldsymbol{V}\,,\\
&\varphi(L\boldsymbol{u},\boldsymbol{v}) = \mathscr{J}(L\boldsymbol{u},\boldsymbol{v}) + I_K(\boldsymbol{v}) = j(\boldsymbol{u},\boldsymbol{v}) + I_K(\boldsymbol{v}) \quad \forall \boldsymbol{u}\,,\ \boldsymbol{v}\in \boldsymbol{V}\,.
\end{aligned}$$

It is easy to verify that ψ, L and φ, defined above, satisfy the hypotheses (6.10)–(6.12) from p. 111.

In order to derive the dual formulation, we take

$$\begin{aligned}
&C(\theta) = \{\boldsymbol{\tau}^*\in (H^{-1/2}(\Gamma_2))^d\,;\ \langle\boldsymbol{\tau}^*,\boldsymbol{v}_\tau\rangle_{1/2,\Gamma_2}\le \mathscr{J}(\theta,\boldsymbol{v})\,,\ \forall \boldsymbol{v}\in \boldsymbol{V}\} \quad \forall\theta\in Y\\
&K_\nu = \{z\in H^{1/2}(\Gamma_2)\,;\ z=\gamma\bar z/\Gamma_2 \text{ with } \bar z\in H^1(\Omega)\,,\ \bar z = 0 \text{ on } \Gamma_0\,,\ z\le 0\}\,,
\end{aligned} \tag{8.65}$$

where $\langle\cdot,\cdot\rangle_{1/2,\Gamma_2}$ denotes the inner product between $(H^{-1/2}(\Gamma_2))^d$ and $(H^{1/2}(\Gamma_2))^d$, or between $H^{-1/2}(\Gamma_2)$ and $H^{1/2}(\Gamma_2)$, and I_K denotes the indicator function of the set K.

An easy computation gives the following form for the Fenchel conjugate of φ with respect to the second variable

$$\begin{aligned}
\varphi^*(L\boldsymbol{u},\boldsymbol{v}^*) &= \sup_{\boldsymbol{v}\in(H^{1/2}(\Gamma_2))^d}\left\{\langle\boldsymbol{v}^*,\boldsymbol{v}\rangle_{1/2,\Gamma_2} - \mathscr{J}(L\boldsymbol{u},\boldsymbol{v}) - I_K(\boldsymbol{v})\right\}\\
&= \sup_{\boldsymbol{v}\in(H^{1/2}(\Gamma_2))^d}\left\{(\langle v_\nu^*,v_\nu\rangle_{1/2,\Gamma_2} - I_{K_\nu}(v_\nu)) + (\langle\boldsymbol{v}_\tau^*,\boldsymbol{v}_\tau\rangle_{1/2,\Gamma_2} - \mathscr{J}(L\boldsymbol{u},\boldsymbol{v}))\right\}\\
&= I^*_{K_\nu}(v_\nu^*) + I_{C(L\boldsymbol{u})}(\boldsymbol{v}_\tau^*) = I_{K_\nu^*}(v_\nu^*) + I_{C(\mathscr{R}(\sigma_\nu(\boldsymbol{u})))}(\boldsymbol{v}_\tau^*) \quad \forall \boldsymbol{v}^*\in (H^{-1/2}(\Gamma_2))^d\,,
\end{aligned}$$

where K_ν^* denotes the polar cone of the set K_ν in $H^{1/2}(\Gamma_2)$, i.e.

$$K_\nu^* = \{z^*\in H^{-1/2}(\Gamma_2)\,;\ \langle z^*,z\rangle_{1/2,\Gamma_2}\le 0\,,\ \forall z\in K_\nu\}.$$

From the definition of the set $C(\theta)$, and taking into consideration that $\mathscr{R}(\sigma_\nu(\boldsymbol{u}))\in L^2(\Gamma_2)$, we obtain

$$\varphi^*(L\boldsymbol{u},\boldsymbol{v}^*) = \begin{cases} 0 & \text{if } v_\nu^*\in K_\nu^*\,,\ \boldsymbol{v}_\tau^*\in C(L(\boldsymbol{u})),\\ +\infty & \text{otherwise.}\end{cases} \tag{8.66}$$

We now compute the Fenchel conjugate of ψ with respect to the second variable:

$$\begin{aligned}
\psi^*(\boldsymbol{u},\boldsymbol{v}^*) &= \sup_{\boldsymbol{v}\in V}\{\langle\boldsymbol{v}^*,\boldsymbol{v}\rangle - a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + (\boldsymbol{F},\boldsymbol{v}-\boldsymbol{u})_V\}\\
&= a(\boldsymbol{u},\boldsymbol{u}) - (\boldsymbol{F},\boldsymbol{u})_V + \sup_{\boldsymbol{v}\in V}\{\langle\boldsymbol{v}^*,\boldsymbol{v}\rangle - a(\boldsymbol{u},\boldsymbol{v}) + (\boldsymbol{F},\boldsymbol{v})_V\}\\
&= a(\boldsymbol{u},\boldsymbol{u}) - (\boldsymbol{F},\boldsymbol{u})_V + \sup_{\boldsymbol{v}\in V}\langle\boldsymbol{v}^* - A\boldsymbol{u} + \boldsymbol{F},\boldsymbol{v}\rangle\\
&= a(\boldsymbol{u},\boldsymbol{u}) - (\boldsymbol{F},\boldsymbol{u})_V + \begin{cases} 0 & \text{if } \boldsymbol{v}^* = A\boldsymbol{u}-\boldsymbol{F}\,,\\ +\infty & \text{other,}\end{cases} \qquad \forall \boldsymbol{v}^*\in \boldsymbol{V}^*
\end{aligned} \tag{8.67}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $\boldsymbol{V}^*$ and $\boldsymbol{V}$ and $A \in \mathscr{L}(\boldsymbol{V}, \boldsymbol{V}^*)$ is the operator defined by

$$\langle A\boldsymbol{u}, \boldsymbol{v} \rangle = a(\boldsymbol{u}, \boldsymbol{v}) \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}.$$

It is easy to verify that, for every $\boldsymbol{u} \in \boldsymbol{V}$, the function $\boldsymbol{v} \mapsto \psi(\boldsymbol{u}, \boldsymbol{v})$ is Gâteaux differentiable at $\boldsymbol{u}$ and

$$D_2\psi(\boldsymbol{u}, \boldsymbol{u}) = A\boldsymbol{u} - \boldsymbol{F}\,.$$

Since $A \in \mathscr{L}(\boldsymbol{V}, \boldsymbol{V}^*)$, we can define the Green's operator $G = A^{-1}$ for the boundary value problem in linear elasticity, and $G \in \mathscr{L}(\boldsymbol{V}^*, \boldsymbol{V})$. Thus, $\boldsymbol{u} = G(\boldsymbol{F}) + G(D_2\psi(\boldsymbol{u}, \boldsymbol{u}))$, and, if we put $\boldsymbol{u}^* = -D_2\psi(\boldsymbol{u}, \boldsymbol{u})$, it follows that $\boldsymbol{u} = (D_2\psi)^{-1}(-\boldsymbol{u}^*) = G(\boldsymbol{F}) + G(-\boldsymbol{u}^*)$.

If we suppose that the solution $\boldsymbol{u}$ of the problem **(P)** is smooth enough such that $\sigma_\nu(\boldsymbol{u}) \in L^2(\Gamma_2)$, then, by Theorem 6.8, the dual abstract formulation (6.19) applied to the problem **(P)** takes the form

$$\langle \boldsymbol{v}^* - \boldsymbol{u}^*, \boldsymbol{u} \rangle_{1/2,\Gamma_2} \leq 0 \quad \forall \boldsymbol{v}^* = (v_\nu^*, \boldsymbol{v}_\tau^*) \in K_\nu^* \times C(\sigma_\nu).$$

By putting $\boldsymbol{\sigma}^* = -\boldsymbol{u}^*$ and $\boldsymbol{\tau}^* = -\boldsymbol{v}^*$, it follows that the dual condensed formulation of the problem **(P)** can be written as the following quasi-variational inequality

Problem $(\mathbf{P_2^*})$: Find $\boldsymbol{\sigma}^* = (\sigma_\nu, \boldsymbol{\sigma}_\tau) \in (-K_\nu^*) \times C(\sigma_\nu)$ such that

$$\langle \boldsymbol{\tau}^* - \boldsymbol{\sigma}^*, G(\boldsymbol{\sigma}^*) + G(\boldsymbol{F}) \rangle_{1/2,\Gamma_2} \geq 0 \quad \forall \boldsymbol{\tau}^* = (\tau_\nu, \boldsymbol{\tau}_\tau) \in (-K_\nu^*) \times C(\sigma_\nu). \tag{8.68}$$

The extremality conditions (6.16) become

$$\begin{aligned} &a(\boldsymbol{u}, \boldsymbol{u}) - (\boldsymbol{F}, \boldsymbol{u})_V = \langle \sigma_\nu, u_\nu \rangle_{1/2,\Gamma_2} + \langle \boldsymbol{\sigma}_\tau, \boldsymbol{u}_\tau \rangle_{1/2,\Gamma_2}, \\ &j(\boldsymbol{u}, \boldsymbol{u}) = -\langle \boldsymbol{\sigma}_\tau, \boldsymbol{u}_\tau \rangle_{1/2,\Gamma_2}, \\ &\langle \sigma_\nu, u_\nu \rangle_{1/2,\Gamma_2} = 0. \end{aligned} \tag{8.69}$$

Concluding, by Theorem 6.8, we have the following result.

Theorem 8.7. *If meas* $(\Gamma_0) > 0$ *and the hypotheses (8.9) hold, then there exists a constant* $\mu_1 > 0$*, depending only on* Ω*, such that, for every* $\mu \in L^\infty(\Gamma_2)$ *with* $\mu \geq 0$ *a.e. on* Γ_2 *and* $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$*, the dual problem* $(\mathbf{P_2^*})$ *has a unique solution* $\boldsymbol{\sigma}^*$*. Moreover,* $\boldsymbol{u} = G(\boldsymbol{\sigma}^*) + G(\boldsymbol{F})$ *is the unique solution of the primal problem* **(P)**.

Since $(\mathbf{P_2^*})$ is a problem for the stresses on the contact surface Γ_2 only, it is called the condensed dual formulation of the problem **(P)**.

We remark that the problem $(\mathbf{P_2^*})$ cannot be separated into two inequalities in order to determine the contact normal stress σ_ν and the contact tangential stress $\boldsymbol{\sigma}_\tau$ on Γ_2 independently, their coupling being imposed by the appurtenance of $\boldsymbol{\sigma}_\tau$ to the set $C(\sigma_\nu)$. However, if we suppose that the Green's operator is known, the problem $(\mathbf{P_2^*})$ is very useful for the numerical determination of contact normal and tangential stresses (see [9, 10]). The same happens in particular cases, as we shall see in Sect. 8.7.

Finally, let us note that the M–CD–M duality theory can be applied to many other unilateral or bilateral contact problems as quasistatic or dynamic frictional problems for elastic materials or frictional problems involving contact normal compliance conditions for viscoelastic materials.

8.6 Approximation of the Problem in Displacements

In this section we consider (see, also [48]) the finite element approximation of the primal problem (**P**), p. 141. In order to derive some error estimates of this approximation, we limit our study to the case where the solution is unique. Therefore, we suppose that the regularity hypotheses (8.9) are satisfied and, in addition, we assume that

$$\begin{aligned} &\Omega \text{ is a bounded Lipschitzian domain in } \mathbb{R}^d\,,\\ &\text{meas}\,(\Gamma_0) > 0\,,\ \Gamma_2 \neq \emptyset\,,\\ &\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1\,, \end{aligned} \tag{8.70}$$

with μ_1 satisfying the relation (8.34) from the proof of Theorem 8.3.

Let $\boldsymbol{u}$ denote the unique solution of the problem (**P**). Then, by Remark 8.5, we have $j_f(\boldsymbol{u},\boldsymbol{v}) = j(\boldsymbol{u},\boldsymbol{v})$, $\forall \boldsymbol{v} \in \boldsymbol{V}$, with j defined by (8.13).

Let h be a given parameter converging to 0. We consider a family $\{\boldsymbol{V}_h\}_{h>0}$ of closed subspaces of $\boldsymbol{V}$ and a family $\{\boldsymbol{K}_h\}_{h>0}$ of nonempty convex closed subsets of $\boldsymbol{V}_h$ which approximates $\boldsymbol{K}$ in the sense of the internal approximation defined in Sect. 7.1, i.e. the conditions (7.8), from p. 116, are satisfied. Also we can approximate the functional j by a family $\{j^h\}_h$ which satisfies (7.9)–(7.13).

Proposition 7.1 and Theorem 7.1 lead to the following existence and uniqueness result.

Proposition 8.1. *The discrete quasi-variational inequality*

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + j^h(\boldsymbol{u}_h, \boldsymbol{v}_h) - j^h(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq (\boldsymbol{F}, \boldsymbol{v}_h - \boldsymbol{u}_h) \quad \forall \boldsymbol{v}_h \in \boldsymbol{K}_h$$

has a unique solution $\boldsymbol{u}_h \in \boldsymbol{K}_h$*. Moreover, we have*

$$\boldsymbol{u}_h \to \boldsymbol{u} \text{ strongly in } \boldsymbol{V},$$

$\boldsymbol{u}$ *being the unique solution of the problem* (**P**).

Now, we suppose that there exists an operator $\Pi_h : \boldsymbol{V} \longrightarrow \boldsymbol{V}_h$ such that

$$\|\Pi_h \boldsymbol{v} - \boldsymbol{v}\|_1 \leq C h \|\boldsymbol{v}\|_2 \quad \forall \boldsymbol{v} \in (H^2(\Omega))^d \cap \boldsymbol{V}\,, \tag{8.71}$$

$$\|\Pi_h \boldsymbol{v} - \boldsymbol{v}\|_{0,\Gamma_2} \leq C h^{3/2} \|\boldsymbol{v}\|_2 \quad \forall \boldsymbol{v} \in (H^2(\Omega))^d \cap \boldsymbol{V}\,, \tag{8.72}$$

where $\|\cdot\|_{0,\Gamma_2}$ and $\|\cdot\|_2$ denote the norms on $(L^2(\Omega))^d$, and, respectively, on $(H^2(\Omega))^d$.

Remark 8.10. Let Π_h be, as usual, the interpolation operator $\Pi_h : \boldsymbol{V} \to \boldsymbol{V}_h$. Then, it is known that the conditions (8.71) and (8.72) are satisfied (see, e.g., [5, 16]), if, for instance, Ω is a polygonal domain of $\mathbb{R}^d$, and $\boldsymbol{V}_h$ is defined by

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{V} \cap (C^0(\overline{\Omega}))^d \, ; \, \boldsymbol{v}/T \in (P_k)^d \, , \ \forall T \in \mathcal{T}_h\} \, ,$$

where P_k is the space of all polynomials of degree less than k in the variables $x_1, \cdots, x_d$ with $k \geq 1$. Here, we assume that $\mathcal{T}_h$ is a regular triangulation of the domain Ω such that

$$\overline{\Omega} = \bigcup_{T \in \mathcal{T}_h} T \, .$$

We say that a triangulation $\mathcal{T}_h$ is regular if all the angles of any element $T \in \mathcal{T}_h$ are bounded below from a positive constant, and there exists a positive constant δ such that the length of any side of any $T \in \mathcal{T}_h$ is at least δh.

We make the following additional assumptions about $\boldsymbol{K}_h$:

$$\boldsymbol{K}_h \subset \boldsymbol{K} \, , \tag{8.73}$$

$$\Pi_h \boldsymbol{u} \in \boldsymbol{K}_h \tag{8.74}$$

where $\boldsymbol{u}$ is the unique solution of the problem **(P)**.

Remark 8.11. The conditions (8.73), (8.74) are satisfied if, for instance, we take $d = 2, k = 1$, and

$$\begin{aligned} \boldsymbol{K}_h = \{ & \boldsymbol{v}_h \in \boldsymbol{V}_h \, ; \, v_{h\nu}(a_i) \leq 0 \text{ if } a_i \neq A_j \, , \ \forall j = 1, \cdots, N_\Omega \text{ and} \\ & (v_{h1}\nu_1^- + v_{h2}\nu_2^-)(a_i) \geq 0, (v_{h1}\nu_1^+ + v_{h2}\nu_2^+)(a_i) \geq 0 \text{ if there exists } j = 1, \cdots, N_\Omega \\ & \text{such that } a_i = A_j \, , \ i = 1, \cdots, N_2\} \end{aligned}$$

where $a_1, \cdots, a_{N_2}$ are the nodes of the triangulation $\mathcal{T}_h$ lying on Γ_2, $A_1, \cdots, A_{N_\Omega}$ are the vertices of Ω and $\boldsymbol{\nu}^- = (\nu_1^-, \nu_2^-)$, $\boldsymbol{\nu}^+ = (\nu_1^+, \nu_2^+)$ are the outward unit normal vectors on two adjacent edges (see [29]).

In the following, for simplicity, we shall consider that $j^h(\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) = j(\boldsymbol{v}_h^1, \boldsymbol{v}_h^2) \ \forall \boldsymbol{v}_h^1, \boldsymbol{v}_h^2 \in \boldsymbol{V}_h$. It is easy to verify that the conditions (7.9)–(7.13) are fulfilled in this case. Hence, we will consider the following finite element approximation of the problem **(P)**.

Problem $(\mathbf{P})_h$: Find $\boldsymbol{u}_h \in \boldsymbol{K}_h$ such that

$$a(\boldsymbol{u}_h, \boldsymbol{v}_h - \boldsymbol{u}_h) + j(\boldsymbol{u}_h, \boldsymbol{v}_h) - j(\boldsymbol{u}_h, \boldsymbol{u}_h) \geq (\boldsymbol{F}, \boldsymbol{v}_h - \boldsymbol{u}_h)_V \quad \forall \boldsymbol{v}_h \in \boldsymbol{K}_h \, . \tag{8.75}$$

Theorem 8.8. *Suppose that the conditions (8.9), (8.70)–(8.74), and (7.8) hold. Then, if the unique solution* $\boldsymbol{u}$ *of the problem* **(P)** *belongs to* $(H^2(\Omega))^d \cap \boldsymbol{K}$, *one has the error estimate*

$$\|\boldsymbol{u}_h - \boldsymbol{u}\|_1 \leq C h^{3/4} \|\boldsymbol{u}\|_2 \tag{8.76}$$

where $\boldsymbol{u}_h$ *is the unique solution of the problem* $(\mathbf{P})_h$, *and* C *is a positive constant independent of* h.

Proof. Taking $\boldsymbol{v} = \boldsymbol{u}_h$ in $(\mathbf{P})$ and $\boldsymbol{v}_h = \Pi_h \boldsymbol{u}$ in $(\mathbf{P})_h$, by addition we get

$$\begin{aligned}
&\alpha \|\boldsymbol{u}_u - \boldsymbol{u}\|_1^2 \le a(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{u} - \boldsymbol{u}_h) \le a(\boldsymbol{u}_h - \boldsymbol{u}, \Pi_h \boldsymbol{u} - \boldsymbol{u}) + (a(\boldsymbol{u}, \Pi_h \boldsymbol{u} - \boldsymbol{u}) \\
&- (\boldsymbol{F}, \Pi_h \boldsymbol{u} - \boldsymbol{u})_V) + (j(\boldsymbol{u}, \boldsymbol{u}_h) + j(\boldsymbol{u}_h, \Pi_h \boldsymbol{u}) - j(\boldsymbol{u}_h, \boldsymbol{u}_h) - j(\boldsymbol{u}, \Pi_h \boldsymbol{u})) \\
&+ (j(\boldsymbol{u}, \Pi_h \boldsymbol{u}) - j(\boldsymbol{u}, \boldsymbol{u}))\,.
\end{aligned} \tag{8.77}$$

Since $\boldsymbol{u} \in (H^2(\Omega))^d$, from the Green's formula, and the trace theorem, we deduce

$$\begin{aligned}
&a(\boldsymbol{u}, \Pi_h \boldsymbol{u} - \boldsymbol{u}) - (\boldsymbol{F}, \Pi_h \boldsymbol{u} - \boldsymbol{u})_V = \int_{\Gamma_2} \sigma_{ij}(\boldsymbol{u}) \nu_j (\Pi_h \boldsymbol{u} - \boldsymbol{u})_i \, \mathrm{d}s \\
&\le \|\boldsymbol{\sigma} \cdot \boldsymbol{\nu}\|_{0,\Gamma} \|\Pi_h \boldsymbol{u} - \boldsymbol{u}\|_{0,\Gamma_2} \le C_1 \|\boldsymbol{u}\|_2 \|\Pi_h \boldsymbol{u} - \boldsymbol{u}\|_{0,\Gamma_2}\,.
\end{aligned} \tag{8.78}$$

We also have

$$\begin{aligned}
&j(\boldsymbol{u}, \Pi_h \boldsymbol{u}) - j(\boldsymbol{u}, \boldsymbol{u}) \le C_2' \|\mathscr{R}(\sigma_\nu(\boldsymbol{u}))\|_{0,\Gamma_2} \|\Pi_h \boldsymbol{u} - \boldsymbol{u}\|_{0,\Gamma_2} \\
&\le C_2'' \|\sigma_\nu(\boldsymbol{u})\|_{H^{-1/2}(\Gamma)} \|\Pi_h \boldsymbol{u} - \boldsymbol{u}\|_{0,\Gamma_2} \le C_2 \|\boldsymbol{u}\|_2 \|\Pi_h \boldsymbol{u} - \boldsymbol{u}\|_{0,\Gamma_2}\,.
\end{aligned} \tag{8.79}$$

Now, combining (8.78)–(8.79), (8.33), (8.77), and using the continuity of the form $a(\cdot, \cdot)$, we get

$$(\alpha - k)\|\boldsymbol{u}_h - \boldsymbol{u}\|_1^2 \le (M + k)\|\boldsymbol{u}_h - \boldsymbol{u}\|_1 \, \|\Pi_h \boldsymbol{u} - \boldsymbol{u}\|_1 + C_3 \|\boldsymbol{u}\|_2 \|\Pi_h \boldsymbol{u} - \boldsymbol{u}\|_{0,\Gamma_2}\,. \tag{8.80}$$

Hence, by using the Young's inequality

$$ab \le \frac{\epsilon a^2}{2} + \frac{b^2}{2\epsilon} \qquad \forall \epsilon > 0\,,\ \forall a, b \in \mathbb{R}$$

for $\epsilon < \dfrac{2(\alpha - k)}{M + k}$, we deduce

$$\left(\alpha - k - \frac{M + k}{2}\epsilon\right) \|\boldsymbol{u}_h - \boldsymbol{u}\|_1^2 \le \frac{M + k}{2\epsilon} \|\Pi_h \boldsymbol{u} - \boldsymbol{u}\|_1^2 + C_3 \|\boldsymbol{u}\|_2 \|\Pi_h \boldsymbol{u} - \boldsymbol{u}\|_{0,\Gamma_2}\,. \tag{8.81}$$

Therefore, from (8.71), (8.72), and (8.81), the estimate (8.76) follows. □

Remark 8.12. The estimate (8.76) can be also obtained from Theorem 7.3. Indeed, by taking $U = (L^2(\Omega))^d$, and proceeding as in (8.78) and (8.79), we have

$$(A\boldsymbol{u} - \boldsymbol{F}, \boldsymbol{v}) \le C_1 \|\boldsymbol{u}\|_2 \|\boldsymbol{v}\|_{0,\Gamma_2} \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,,$$

$$|j^h(\boldsymbol{u}, \boldsymbol{v}_h) - j(\boldsymbol{u}, \boldsymbol{v})| = |j(\boldsymbol{u}, \boldsymbol{v}_h) - j(\boldsymbol{u}, \boldsymbol{v})| \le C_2 \|\boldsymbol{u}\|_2 \|\boldsymbol{v}_h - \boldsymbol{v}\|_{0,\Gamma_2} \ \forall \boldsymbol{v}_h \in \boldsymbol{K}_h\,,\ \forall \boldsymbol{v} \in \boldsymbol{V}\,,$$

and so, the hypotheses (7.42) and (7.34) are satisfied.

Therefore, by applying Theorem 7.3 for $\boldsymbol{v}_h = \Pi_h \boldsymbol{u}$ and putting $\boldsymbol{v} = \boldsymbol{u}_h$ in (7.43), we obtain (8.76).

Now, we shall show that a higher order of approximation can be obtained for a suitable choice of the regularization operator $\mathscr{R}$, namely, we consider the mapping $\mathscr{R}$ given by the convolution

$$\mathscr{R}(\varphi) = \omega * \varphi \qquad \forall \varphi \in H^{-1/2}(\Gamma) \tag{8.82}$$

where $\omega \in \mathscr{D}(-\delta, \delta)$, with $\delta \in \mathbb{R}$, $\delta > 0$, is such that $\int_{-\delta}^{\delta} \omega(t)\,\mathrm{d}t = 1$ (see, e.g., [46]).

Theorem 8.9. *Let the assumptions (8.9), (8.70)–(8.74), and (7.8) hold. We suppose in addition that the mapping $\mathscr{R}$ is given by (8.82), and that the interpolation operator satisfies*

$$\|\,|\Pi_h \boldsymbol{v} - \boldsymbol{v}|\,\|_{H^{-1/2}(\Gamma)} \le C h^2 \|\boldsymbol{v}\|_2 \quad \forall \boldsymbol{v} \in (H^2(\Omega))^d \cap \boldsymbol{V}\,. \tag{8.83}$$

Then, if $\boldsymbol{u} \in (H^2(\Omega))^d \cap \boldsymbol{K}$, we have the estimate

$$\|\boldsymbol{u}_h - \boldsymbol{u}\|_1 \le C h \|\boldsymbol{u}\|_2\,. \tag{8.84}$$

Proof. If the solution $\boldsymbol{u}$ of (**P**) belongs to $H^2(\Omega))^d$, then it follows that $\mathscr{R}(\sigma_\nu(\boldsymbol{u})) \in H^{1/2}(\Gamma)$, and so

$$j(\boldsymbol{u}, \Pi_h \boldsymbol{u}) - j(\boldsymbol{u}, \boldsymbol{u}) \le C_4 \|\mathscr{R}(\sigma_\nu(\boldsymbol{u}))\|_{1/2,\Gamma} \|\,|\Pi_h \boldsymbol{u} - \boldsymbol{u}|\,\|_{-1/2,\Gamma}\,. \tag{8.85}$$

From the definition of the norm in $H^{1/2}(\Gamma)$:

$$\|\psi\|_{H^{1/2}(\Gamma)} = \inf\{\|v\|_1\,;\ v \in H^1(\Omega)\,,\ \psi = \gamma v\}\,,$$

where $\gamma : H^1(\Omega) \longrightarrow H^{1/2}(\Gamma)$ is the trace operator, we deduce

$$\|\mathscr{R}(\sigma_\nu(\boldsymbol{u}))\|^2_{H^{1/2}(\Gamma)} \le \|\omega * \sigma_\nu(\boldsymbol{u})\|_1^2 = \int_\Omega \left(\int_\Gamma \omega(|\boldsymbol{x} - \mathbf{y}|)(\sigma_\nu(\boldsymbol{u}))(\mathbf{y})\,\mathrm{d}y \right)^2 \mathrm{d}x$$
$$+ \sum_{i=1}^d \int_\Omega \left(\int_\Gamma \frac{\partial}{\partial x_i} \omega(|\boldsymbol{x} - \mathbf{y}|)(\sigma_\nu(\boldsymbol{u}))(\mathbf{y})\,\mathrm{d}y \right)^2 \mathrm{d}x \le C_5 \|\sigma_\nu(\boldsymbol{u})\|^2_{0,\Gamma}\,. \tag{8.86}$$

We also have

$$a(\boldsymbol{u}, \Pi_h \boldsymbol{u} - \boldsymbol{u}) - (\boldsymbol{F}, \Pi_h \boldsymbol{u} - \boldsymbol{u}) = \int_\Gamma \sigma_{ij}(\boldsymbol{u}) \nu_j (\Pi_h \boldsymbol{u} - \boldsymbol{u})_i \,\mathrm{d}s$$
$$- \int_{\Gamma_1} g_i (\Pi_h \boldsymbol{u} - \boldsymbol{u})_i \,\mathrm{d}s \le C_6' \left(\|\boldsymbol{\sigma} \cdot \boldsymbol{\nu}\|_{(H^{1/2}(\Gamma))^d} + \|\boldsymbol{g}\|_{(H^{1/2}(\Gamma))^d} \right) \|\Pi_h \boldsymbol{u} - \boldsymbol{u}\|_{(H^{-1/2}(\Gamma))^d}$$
$$\le C_6 \|\boldsymbol{u}\|_2 \|\Pi_h \boldsymbol{u} - \boldsymbol{u}\|_{(H^{-1/2}(\Gamma))^d} \tag{8.87}$$

where we have used the relation

$$\|\boldsymbol{g}\|_{(H^{1/2}(\Gamma))^d} \leq \|\boldsymbol{\sigma} \cdot \boldsymbol{\nu}\|_{(H^{1/2}(\Gamma))^d} .$$

It is easy to verify that

$$\|\,|z|\,\|_{H^{1/2}(\Gamma)} \leq \|z\|_{H^{1/2}(\Gamma)} \quad \forall z \in H^{1/2}(\Gamma)\,,$$

and thus

$$\|\,|\boldsymbol{v}|\,\|_{H^{-1/2}(\Gamma)} = \sup_{z \in H^{1/2}(\Gamma)} \frac{|(|\boldsymbol{v}|, z)_{0,\Gamma}|}{\|z\|_{H^{1/2}(\Gamma)}} \leq \sup_{z \in H^{1/2}(\Gamma)} \frac{(|\boldsymbol{v}|, |z|)_{0,\Gamma}}{\|\,|z|\,\|_{H^{1/2}(\Gamma)}} \;\forall \boldsymbol{v} \in (H^{1/2}(\Gamma))^d. \tag{8.88}$$

On the other hand, we have

$$\|\,|\boldsymbol{v}|\,\|_{H^{-1/2}(\Gamma)} \geq \sup_{\substack{z \in H^{1/2}(\Gamma)\\ z \geq 0}} \frac{|(|\boldsymbol{v}|, z)_{0,\Gamma}|}{\|z\|_{H^{1/2}(\Gamma)}} = \sup_{z \in H^{1/2}(\Gamma)} \frac{(|\boldsymbol{v}|, |z|)_{0,\Gamma}}{\|\,|z|\,\|_{H^{1/2}(\Gamma)}} \;\forall \boldsymbol{v} \in (H^{1/2}(\Gamma))^d. \tag{8.89}$$

From the last two relations, we get

$$\|\,|\boldsymbol{v}|\,\|_{H^{-1/2}(\Gamma)} = \sup_{z \in H^{1/2}(\Gamma)} \frac{(|\boldsymbol{v}|, |z|)_{0,\Gamma}}{\|\,|z|\,\|_{H^{1/2}(\Gamma)}} \quad \forall \boldsymbol{v} \in (H^{1/2}(\Gamma))^d\,, \tag{8.90}$$

and, thus

$$\|\boldsymbol{v}\|_{(H^{-1/2}(\Gamma))^d} \leq d\,\|\,|\boldsymbol{v}|\,\|_{H^{-1/2}(\Gamma)} \quad \forall \boldsymbol{v} \in (H^{1/2}(\Gamma))^d\,. \tag{8.91}$$

Finally, from (8.33), with (8.77), (8.85)–(8.87), and (8.91), the assertion follows. □

8.7 Approximation of Dual Problems by Equilibrium Finite Element Method

This section is concerned with the discrete approximations of the dual problem ($\mathbf{P_1^*}$), given by (8.43), and of the dual condensed problem ($\mathbf{P_2^*}$), given by (8.68). In our approach we use the equilibrium finite element method introduced in [25].

We suppose that hypotheses (8.9) and (8.70) hold.

We first recall the space

$$\mathscr{H} = \boldsymbol{H}(\mathrm{div}\,;\,\Omega) = \{\boldsymbol{\tau} \in \boldsymbol{H}\,;\; \mathrm{div}\,\boldsymbol{\tau} \in (L^2(\Omega))^d\}\,,$$

where

$$\boldsymbol{H} = \{\boldsymbol{\tau} = (\tau_{ij})\,;\ \tau_{ij} = \tau_{ji} \in L^2(\Omega)\,,\ 1 \le i, j \le d\}\,.$$

In the equilibrium methods, the finite dimensional spaces $\boldsymbol{V}_h$ and $\mathscr{H}_h$ are chosen such that the following condition is satisfied.

$$\left.\begin{array}{l}\forall \boldsymbol{\tau} \in \mathscr{H}_h \text{ such that} \\ \displaystyle\int_\Omega (\operatorname{div} \boldsymbol{\tau}) \boldsymbol{v}_h \,\mathrm{d}x = 0 \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h \end{array}\right\} \Longrightarrow \quad \operatorname{div} \boldsymbol{\tau} = 0 \text{ in } \Omega\,. \tag{8.92}$$

Moreover, we need to be able to construct a suitable interpolation operator $\Pi_h : \mathscr{H} \longrightarrow \mathscr{H}_h$ such that

$$\int_\Omega \operatorname{div}(\Pi_h \boldsymbol{\tau})\, \boldsymbol{v}_h \,\mathrm{d}x = \int_\Omega (\operatorname{div} \boldsymbol{\tau})\, \boldsymbol{v}_h \,\mathrm{d}x \quad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h\,,\ \forall \boldsymbol{\tau} \in \mathscr{H}. \tag{8.93}$$

For the sake of simplicity, we shall assume that the open bounded set $\Omega \subset \mathbb{R}^2$ is a convex polygonal. Let $\mathscr{T}_h$ be a regular family of triangulations of Ω, as in Sect. 8.6, such that

$$\overline{\Omega} = \bigcup_{T \in \mathscr{T}_h} T\,.$$

Johnson and Mercier [31] have proposed two different choices of the spaces $\mathscr{H}_h$ and $\boldsymbol{V}_h$ using low degree polynomials which meet the requirements (8.92) and (8.93). In both cases, they used composite piecewise linear finite elements, called macro elements (see Fig. 8.2), for the stresses, one triangular and one quadrilateral, together with piecewise linear discontinuous displacements.

We briefly recall one of these choices. We consider a composite triangle T, i.e. T is divided into three subtriangles T_1, T_2, T_3.

For each $T \in \mathscr{T}_h$, we define the finite dimensional space of piecewise linear stress tensors by

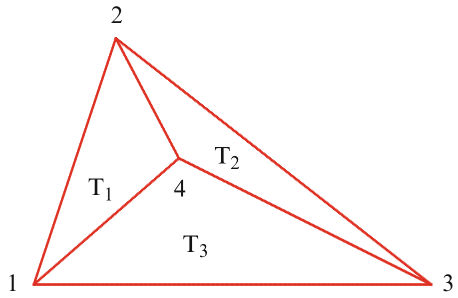

Fig. 8.2 Composite triangle T

$$\mathcal{H}_T = \{\boldsymbol{\tau} \in \hat{\mathcal{H}}_T \,;\ \boldsymbol{\tau}\cdot\boldsymbol{\nu} \text{ is continuous across the subtriangle boundaries } 1-4,\ 2-4,\ 3-4\} \subset \boldsymbol{H}(\mathrm{div}\,; T),$$

where

$$\hat{\mathcal{H}}_T = \{\boldsymbol{\tau} = (\tau_{ij})\,;\ \tau_{ij} = \tau_{ji} \in L^2(T),\ i,j = 1,2,\ \boldsymbol{\tau}/T_k \in (P_1(T_k))^4,\ k = 1,2,3\}\,,$$

P_1 denoting the space of all polynomials of degree less than 1 in the variables x_1, x_2.

The stress space $\mathcal{H}$ is then approximated by

$$\mathcal{H}_h = \{\boldsymbol{\tau} \in \hat{\mathcal{H}}_h\,;\ \mathrm{div}\,\boldsymbol{\tau} \in (L^2(\Omega))^2\} \subset \mathcal{H},$$

with

$$\hat{\mathcal{H}}_h = \{\boldsymbol{\tau} \in \boldsymbol{H}\,;\ \boldsymbol{\tau}/T \in \mathcal{H}_T,\ \forall T \in \mathcal{T}_h\},$$

and the displacement space $\boldsymbol{V}$ is approximated by

$$\boldsymbol{V}_h = \{\boldsymbol{v} \in \boldsymbol{V}\,;\ \boldsymbol{v}/T \in (P_1(T))^2\,,\ \forall T \in \mathcal{T}_h\}.$$

The interpolation operator $\Pi_h : \mathcal{H} \longrightarrow \mathcal{H}_h$ is uniquely defined by the following two requirements:

$$\int_S \boldsymbol{\nu}\cdot((\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau})\boldsymbol{\nu})\,\mathrm{d}s = 0 \quad \forall \boldsymbol{v} \in (P_1(S))^2\,,\ \forall \boldsymbol{\tau} \in (H^1(\Omega))^4, \tag{8.94}$$

for any side S of $\mathcal{T}_h$, $\boldsymbol{\nu}$ being the outward normal to S, and

$$\int_T (\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau})\,\mathrm{d}x = 0 \quad \forall T \in \mathcal{T}_h. \tag{8.95}$$

We shall use $\|\cdot\|_m$ and $|\cdot|_m$ to denote the usual norm and, respectively, seminorm on $(H^m(\Omega))^q$ with m and q integers.

By Green's formula, it follows that this interpolation operator satisfies the condition (8.93), and, in addition, we have the estimates

$$\|\mathrm{div}\,\Pi_h\boldsymbol{\tau}\|_0 \le C\|\mathrm{div}\,\boldsymbol{\tau}\|_0 \quad \forall \boldsymbol{\tau} \in \mathcal{H} \cap (H^1(\Omega))^4$$
$$\|\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}\|_0 \le Ch^2|\boldsymbol{\tau}|_2 \quad \boldsymbol{\tau} \in \mathcal{H} \cap (H^2(\Omega))^4.$$

Proceeding as in [31] or [16], we obtain

$$\|\boldsymbol{\tau} - \Pi_h\boldsymbol{\tau}\|_{\mathcal{H}} \le Ch|\boldsymbol{\tau}|_2 \quad \forall \boldsymbol{\tau} \in \mathcal{H} \cap (H^2(\Omega))^4. \tag{8.96}$$

In the sequel we shall give two error estimates for the equilibrium finite element approximations $\mathscr{H}_h$ and $\boldsymbol{V}_h$ of the dual problems $(\mathbf{P}_1^*)$ and $(\mathbf{P}_2^*)$. First, let us consider the following abstract variational inequality of the first kind:

$$\begin{cases} \boldsymbol{\sigma} \in \boldsymbol{\Sigma} \subset \mathscr{H} \\ b(\boldsymbol{\sigma}, \boldsymbol{\tau} - \boldsymbol{\sigma}) \geq L(\boldsymbol{\tau} - \boldsymbol{\sigma}), \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}, \end{cases} \tag{8.97}$$

where $\boldsymbol{\Sigma}$ is the set of the constraints of this problem. Then, for any internal approximation of this problem, we have the a priori error estimate (see Remark 7.2, p. 123 or [16, p. 292]):

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathscr{H}} \leq C \left(\inf_{\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h} (\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathscr{H}}^2 + \alpha \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathscr{H}}) + \alpha \inf_{\boldsymbol{\tau} \in \boldsymbol{\Sigma}} \|\boldsymbol{\sigma}_h - \boldsymbol{\tau}\|_{\mathscr{H}} \right)^{1/2} \tag{8.98}$$

where C is a constant independent of h, $\boldsymbol{\Sigma}_h$ is the discrete set corresponding to $\boldsymbol{\Sigma}$ and $\alpha = \|B\boldsymbol{\sigma} - L\|_{\mathscr{W}}$ with $\mathscr{W}$ a Banach space such that $\mathscr{H} \subset \mathscr{W}$ and $B\boldsymbol{\sigma} - L \in \mathscr{W}$, B being the continuous linear operator associated with the form b.

The difficulty in deriving an error estimate, obviously by using the properties of the interpolation operator Π_h, is to construct an approximation $\boldsymbol{\Sigma}_h$ of the set $\boldsymbol{\Sigma}$ such that $\Pi_h \boldsymbol{\sigma} \in \boldsymbol{\Sigma}_h$. We note also that the third term from the a priori estimate (8.98), which is presumed to have the greatest weight, vanishes if $\boldsymbol{\Sigma}_h \subset \boldsymbol{\Sigma}$ (a condition satisfied if, for instance, $\boldsymbol{\Sigma}_h = \boldsymbol{\Sigma} \cap \mathscr{H}_h$).

We shall consider two particular cases (see [14]):

(a) the problem without contact and without friction, i.e. the classic problem of linear elasticity;
(b) the problem with given friction.

For the general case, we only prove the convergence of the approximation.

As usual, for deriving error estimates, we will suppose that the solution $\boldsymbol{\sigma}$ of the problem $(\mathbf{P}_1^*)$ is sufficiently smooth.

(a) **Linear elasticity**

In this case, the problem $(\mathbf{P}_1^*)$ can be written under the form (8.97) with

$$\boldsymbol{\Sigma} = \{\boldsymbol{\tau} \in \mathscr{H}\,;\ -\mathrm{div}\,\boldsymbol{\tau} = \boldsymbol{f} \text{ a.e. in } \Omega\,,\ \boldsymbol{\tau} \cdot \boldsymbol{\nu} = \boldsymbol{g} \text{ a.e. on } \Gamma_1\}. \tag{8.99}$$

By applying the estimate (8.98), as $\alpha = 0$, one obtains

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathscr{H}} \leq C \inf_{\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathscr{H}}. \tag{8.100}$$

A natural choice of the discrete set $\boldsymbol{\Sigma}_h$ is to take the functions of $\mathscr{H}_h$ which satisfy the conditions of $\boldsymbol{\Sigma}$ only in the nodes of the triangulation, i.e.

$$\begin{aligned}\boldsymbol{\Sigma}_h = \{\boldsymbol{\tau}_h \in \mathscr{H}_h\,;\ -\mathrm{div}\,\boldsymbol{\tau}_h(x_i) = \boldsymbol{f}(x_i)\,,\ \forall x_i \in \mathscr{N}(\Omega) \text{ and}\\ (\boldsymbol{\tau}\cdot\mathbf{n})(y_i) = \boldsymbol{g}(y_i)\,,\ \forall y_i \in \mathscr{N}(\Gamma_1)\}\end{aligned} \tag{8.101}$$

where $\mathscr{N}(\Omega)$ and $\mathscr{N}(\Gamma_1)$ denote the set of the nodes of $\mathscr{T}_h$ lying on Ω, and, respectively, on Γ_1.

Then, we have the following result.

Theorem 8.10. *Under the above hypotheses, there exists a constant C, independent of h, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathscr{H}} \le Ch|\boldsymbol{\sigma}|_2. \tag{8.102}$$

Proof. From (8.100) and (8.96), it follows that we must only prove that $\Pi_h\boldsymbol{\sigma} \in \boldsymbol{\Sigma}_h$. In fact, we have even much more

$$\Pi_h\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \quad \forall\boldsymbol{\tau} \in \boldsymbol{\Sigma}.$$

Indeed, if $\boldsymbol{\tau} \in \boldsymbol{\Sigma}$, then, from the definition (8.99), using (8.93) and (8.94) for a suitable choice of $\boldsymbol{v}_h$, we obtain, from the definition (8.101), that $\Pi_h\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h$. □

Proposition 8.2. *If the volume force $\boldsymbol{f}$ is constant (weight) and the surface traction $\boldsymbol{g}$ is linear (linear distribution of forces), then $\boldsymbol{\Sigma}_h = \boldsymbol{\Sigma} \cap \mathscr{H}_h$.*

Proof. The assertion is immediate taking into account that the approximation $\mathscr{H}_h$ of the stress space $\mathscr{H}$ is constructed with piecewise linear finite elements, and so, if a such stress satisfies the condition of Σ in the nodes of the triangulation, then it satisfies these conditions in Ω, and, respectively, on Γ_1. □

(b) **Signorini problem with given friction**

As in the above case, the problem $(\mathbf{P}_1^*)$ can be written under the form (8.97) with

$$\boldsymbol{\Sigma} = \{\boldsymbol{\tau} \in \mathscr{H}\,;\ \langle\boldsymbol{\tau}, \boldsymbol{\epsilon}(\boldsymbol{v})\rangle_{\boldsymbol{H}} + J_\theta(\boldsymbol{v}) \ge (\boldsymbol{F}, \boldsymbol{v})_V\,,\ \forall \boldsymbol{v} \in \boldsymbol{K}\}\,, \tag{8.103}$$

where θ is a given friction (the problem of Tresca) and

$$J_\theta(\boldsymbol{v}) = -\int_{\Gamma_2} \mu\,\theta\,|\boldsymbol{v}_\tau|\,\mathrm{d}s\,.$$

We consider the approximation of $\boldsymbol{\Sigma}$ defined by

$$\boldsymbol{\Sigma}_h = \{\boldsymbol{\tau}_h \in \mathscr{H}_h\,;\ \langle\boldsymbol{\tau}_h, \boldsymbol{\epsilon}(\boldsymbol{v}_h)\rangle_{\boldsymbol{H}} + J_\theta(\boldsymbol{v}_h) \ge (\boldsymbol{F}, \boldsymbol{v}_h)_V\,,\ \forall \boldsymbol{v}_h \in \boldsymbol{K}_h\}\,. \tag{8.104}$$

Then, we have the following estimate results.

Theorem 8.11. *We assume that $\boldsymbol{f}$ is constant, $\boldsymbol{g}$ is linear, and θ is concave or piecewise linear. Then, there exist the constants C, which are independent of h, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathcal{H}} \le C h^{1/2} |\boldsymbol{\sigma}|_2 , \tag{8.105}$$

$$\|\sigma_\nu - (\sigma_\nu)_h\|_{H^{-1/2}(\Gamma)} \le C h^{1/2} |\sigma_\nu|_2 , \tag{8.106}$$

$$\|\boldsymbol{\sigma}_\tau - (\boldsymbol{\sigma}_\tau)_h\|_{(H^{-1/2}(\Gamma))^d} \le C h^{1/2} |\boldsymbol{\sigma}_\tau|_2 , \tag{8.107}$$

where $\boldsymbol{\sigma}$ *is the unique solution of the dual problem* $(\mathbf{P}_1^*)$ *and* $\boldsymbol{\sigma}^* = (\sigma_\nu, \boldsymbol{\sigma}_\tau)$ *is the unique solution of the dual condensed problem* $(\mathbf{P}_2^*)$.

Proof. By using the same arguments as in the proof of Proposition 8.2, we obtain $\boldsymbol{\Sigma}_h = \boldsymbol{\Sigma} \cap \mathcal{H}_h$. Taking now in (8.98), $\boldsymbol{\tau} = \boldsymbol{\sigma}_h$ and $\boldsymbol{\tau}_h = \Pi_h \boldsymbol{\sigma}$, we deduce the estimate (8.105).

Finally, in the dual condensed problem $(\mathbf{P}_2^*)$, the unknown $\boldsymbol{\sigma}^* = (\sigma_\nu, \boldsymbol{\sigma}_\tau)$ is searched in the set $(-K_\nu^*) \times C_\theta$ where

$$C_\theta = \{\boldsymbol{\tau}^* \in (H^{-1/2}(\Gamma_2))^d \,;\, \langle \boldsymbol{\tau}^*, \boldsymbol{v}_\tau \rangle_{\Gamma_2} \le J_\theta(\boldsymbol{v}) , \ \forall \boldsymbol{v} \in \boldsymbol{V}\} .$$

Hence, in this case with given friction, we can decompose the inequality $(\mathbf{P}_2^*)$ into two inequalities for obtaining independently σ_ν and $\boldsymbol{\sigma}_\tau$. Therefore, from (8.100), and the trace theorem, the assertion follows. □

(c) **Signorini problem with nonlocal Coulomb friction**

We remark that, in the general case, in order to derive an error estimate for the equilibrium finite element approximation of the problem $(\mathbf{P}_1^*)$, we must construct the approximation of $\boldsymbol{\Sigma}_h(\boldsymbol{\tau}_h)$ such that the following conditions are satisfied:

$$\left.\begin{aligned} &\Pi_h \boldsymbol{\sigma} \in \boldsymbol{\Sigma}_h(\boldsymbol{\sigma}_h) , \\ &\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}) . \end{aligned}\right\} \tag{8.108}$$

It is the case, for instance, if the solution $\boldsymbol{\sigma}$ is concave, but this condition cannot be imposed or controlled. The same conclusion we obtain for the dual condensed formulation $(\mathbf{P}_2^*)$. Hence, the obtaining of an optimal error estimate, for the general case, is an open problem as long as we are not able to construct an approximation which ensures the above conditions. However, in the general case, we shall obtain a convergence result of the approximation.

Before detailing the approaches we made for this convergence, it must be mentioned that we can approximate the dual condensed problem $(\mathbf{P}_2^*)$ in two different ways:

(i) We can consider an internal approximation $(\mathbf{P})_h$ of the primal problem $(\mathbf{P})$ (as, for instance, in [48]), and then we apply the M–CD–M duality theory to obtain the discretized dual formulation $(\mathbf{P})_h^*$;
(ii) We obtain, by the M–CD–M duality theory, the dual condensed formulation $(\mathbf{P}_2^*)$ of the primal problem $(\mathbf{P})$, and then, by using the equilibrium finite element method on plane surface, we deduce the discretized dual formulation $(\mathbf{P}_2^*)_h$.

We now consider the following approximation by the equilibrium finite elements of the problem $(\mathbf{P_1^*})$.

Problem $(\mathbf{P_1^*})_\mathbf{h}$: Find $\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h(\boldsymbol{\sigma}_h)$ such that

$$b(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h) \geq 0 \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_h(\boldsymbol{\sigma}_h)$$

where

$$\boldsymbol{\Sigma}_h(\boldsymbol{s}) = \{\boldsymbol{\tau}_h \in \mathscr{H}_h\,;\ \langle \boldsymbol{\tau}_h, \boldsymbol{\epsilon}(\boldsymbol{v}_h)\rangle_{\boldsymbol{H}} + \bar{j}(\boldsymbol{s}, \boldsymbol{v}_h) \geq (\boldsymbol{F}, \boldsymbol{v}_h)_{\boldsymbol{V}}\,,\ \forall \boldsymbol{v}_h \in \boldsymbol{K}_h\}. \tag{8.109}$$

Our goal is to study the behavior of the discrete solution $\boldsymbol{\sigma}_h$ when $h \longrightarrow 0$. The proof of Theorem 8.6 suggests the approach that we are going to do next.

For $\boldsymbol{u}^0 \in \boldsymbol{K}$ and $\boldsymbol{\sigma}^0 \in \mathscr{H}$ given, we consider the sequences $\{\boldsymbol{u}^n\}_n$ and $\{\boldsymbol{\sigma}^n\}_n$ defined by:

Problem $(\mathbf{P})_\mathbf{n}$: Find $\boldsymbol{u}^n \in \boldsymbol{K}$ such that

$$a(\boldsymbol{u}^n, \boldsymbol{v} - \boldsymbol{u}^n) + j_f(\boldsymbol{u}^{n-1}, \boldsymbol{v}) - j_f(\boldsymbol{u}^{n-1}, \boldsymbol{u}^n) \geq (\boldsymbol{F}, \boldsymbol{v} - \boldsymbol{u}^n)_{\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{K}\,,$$

Problem $(\mathbf{P_1^*})_\mathbf{n}$: Find $\boldsymbol{\sigma}^n \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}^{n-1})$ such that

$$b(\boldsymbol{\sigma}^n, \boldsymbol{\tau} - \boldsymbol{\sigma}^n) \geq 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}^{n-1})$$

i.e. $\boldsymbol{u}^n$ and, respectively, $\boldsymbol{\sigma}^n$ is the unique solution, no matter the coefficient of friction is, of the problem with given friction $(\mathbf{P})_\mathbf{n}$ and, respectively, $(\mathbf{P_1^*})_\mathbf{n}$.

We may suppose that $\boldsymbol{u}^0 \in \boldsymbol{C}_f$. Hence, we have $j_f(\boldsymbol{u}^n, \boldsymbol{v}) = j(\boldsymbol{u}^n, \boldsymbol{v})$, $\forall \boldsymbol{v} \in \boldsymbol{V}$, $\forall n \geq 0$. Also, as $\boldsymbol{u}$ is the unique solution of the problem $(\mathbf{P})$, we have $j_f(\boldsymbol{u}, \boldsymbol{v}) = j(\boldsymbol{u}, \boldsymbol{v})$, $\forall \boldsymbol{v} \in \boldsymbol{V}$.

Taking $\boldsymbol{v} = \boldsymbol{u}$ in $(\mathbf{P})_\mathbf{n}$ and $\boldsymbol{v} = \boldsymbol{u}^n$ in $(\mathbf{P})$, by adding the two inequalities and by using (8.33), we obtain:

$$\begin{aligned}\alpha\|\boldsymbol{u} - \boldsymbol{u}^n\|_1^2 &\leq a(\boldsymbol{u} - \boldsymbol{u}^n, \boldsymbol{u} - \boldsymbol{u}^n) \leq j(\boldsymbol{u}, \boldsymbol{u}^n) + j(\boldsymbol{u}^{n-1}, \boldsymbol{u}) - j(\boldsymbol{u}, \boldsymbol{u}) \\ -j(\boldsymbol{u}^{n-1}, \boldsymbol{u}^n) &\leq C_2\|\mu\|_{L^\infty(\Gamma_2)}\|\boldsymbol{u} - \boldsymbol{u}^{n-1}\|_1\|\boldsymbol{u} - \boldsymbol{u}^n\|_1\end{aligned}$$

hence

$$\|\boldsymbol{u} - \boldsymbol{u}^n\|_1 \leq k\|\boldsymbol{u} - \boldsymbol{u}^{n-1}\|_1$$

with $k < 1$ for μ chosen as in (8.70) and (8.34). It follows

$$\|\boldsymbol{u} - \boldsymbol{u}^n\|_1 \leq k^n\|\boldsymbol{u} - \boldsymbol{u}^0\|_1\,. \tag{8.110}$$

The relationship between the two problems $(\mathbf{P})_\mathbf{n}$ and $(\mathbf{P_1^*})_\mathbf{n}$ is given by the proposition below.

Proposition 8.3. *Let $\boldsymbol{u}^0 \in \boldsymbol{C}_f$ and $\boldsymbol{\sigma}^0 = \boldsymbol{\sigma}(\boldsymbol{u}^0) \in \mathscr{H}$ be given. Then, we have*

$$\boldsymbol{\sigma}^n = \boldsymbol{\sigma}(\boldsymbol{u}^n), \ \forall n \in \mathbb{N} \tag{8.111}$$

$$\lim_{n \longrightarrow \infty} \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}\|_{\mathscr{H}} = 0 \tag{8.112}$$

Proof. First, let us remark that the condition $\boldsymbol{\sigma}(\boldsymbol{u}^0) \in \mathscr{H}$ is not too restrictive. Indeed, a natural choice is to take $\boldsymbol{u}^0 \in \boldsymbol{K}$ as the unique solution of

$$a(\boldsymbol{u}^0, \boldsymbol{v} - \boldsymbol{u}^0) \geq (\boldsymbol{F}, \boldsymbol{v} - \boldsymbol{u}^0)_V \quad \forall \boldsymbol{v} \in \boldsymbol{K},$$

which implies that div $\boldsymbol{\sigma}(\boldsymbol{u}^0) = -\boldsymbol{f}$ a.e. in Ω, and so, $\boldsymbol{\sigma}(\boldsymbol{u}^0) \in \mathscr{H}$.

We shall prove by recurrence the relation (8.111). If we suppose that $\boldsymbol{\sigma}^{n-1} = \boldsymbol{\sigma}(\boldsymbol{u}^{n-1})$ holds, then, taking $\boldsymbol{v} = 2\boldsymbol{u}^n$ and $\boldsymbol{v} = \boldsymbol{0}$ in $(\mathbf{P})_{\mathbf{n}}$, we obtain

$$\langle \boldsymbol{\sigma}(\boldsymbol{u}^n), \boldsymbol{\epsilon}(\boldsymbol{u}^n)\rangle_H + \bar{j}(\boldsymbol{\sigma}^{n-1}, \boldsymbol{u}^n) = (\boldsymbol{F}, \boldsymbol{u}^n)_V, \tag{8.113}$$

$$\langle \boldsymbol{\sigma}(\boldsymbol{u}^n), \boldsymbol{\epsilon}(\boldsymbol{v})\rangle_H + \bar{j}(\boldsymbol{\sigma}^{n-1}, \boldsymbol{v}) \geq (\boldsymbol{F}, \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in \boldsymbol{K}. \tag{8.114}$$

The last relation gives

$$\boldsymbol{\sigma}(\boldsymbol{u}^n) \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}^{n-1}). \tag{8.115}$$

On the other hand, for every $\boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}^{n-1})$, one has

$$\langle \boldsymbol{\tau}, \boldsymbol{\epsilon}(\boldsymbol{u}^n)\rangle_H + \bar{j}(\boldsymbol{\sigma}^{n-1}, \boldsymbol{u}^n) \geq (\boldsymbol{F}, \boldsymbol{u}^n)_V. \tag{8.116}$$

From (8.113) and (8.116), we get

$$b(\boldsymbol{\sigma}(\boldsymbol{u}^n), \boldsymbol{\tau} - \boldsymbol{\sigma}(\boldsymbol{u}^n)) \geq 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}^{n-1}). \tag{8.117}$$

Since the solution of the problem $(\mathbf{P_1^*})_{\mathbf{n}}$ is unique, the relations (8.115) and (8.117) imply the assertion (8.111).

Finally, taking into account that div $\boldsymbol{\sigma}^n$ = div $\boldsymbol{\sigma} = -\mathbf{f}$ a.e. in Ω, and using the relations (8.64) and (8.110), we obtain

$$\|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}\|_{\mathscr{H}} = \|\boldsymbol{\sigma}(\boldsymbol{u}^n - \boldsymbol{u})\|_H \leq C\|\boldsymbol{u}^n - \boldsymbol{u}\|_1 \leq Ck^n, \tag{8.118}$$

with C a constant independent of n, and $k < 1$. Then, the relation (8.112) follows. □

In the same way, we approach the discrete problems $(\mathbf{P})_h$ and $(\mathbf{P_1^*})_{\mathbf{h}}$ by the following sequences of problems $\{(\mathbf{P})_{\mathbf{h,n}}\}_n$ and $\{(\mathbf{P_1^*})_{\mathbf{h,n}}\}_n$.

Problem $(\mathbf{P})_{\mathbf{h,n}}$: Find $\boldsymbol{u}_h^n \in \boldsymbol{K}_h$ such that

$$a(\boldsymbol{u}_h^n, \boldsymbol{v}_h - \boldsymbol{u}_h^n) + j(\boldsymbol{u}_h^{n-1}, \boldsymbol{v}_h) - j(\boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n) \geq (\boldsymbol{F}, \boldsymbol{v}_h - \boldsymbol{u}_h^n)_V \quad \forall \boldsymbol{v}_h \in \boldsymbol{K}_h,$$

Problem $(\mathbf{P}_1^*)_{\mathbf{h,n}}$: Find $\boldsymbol{\sigma}_h^n \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}_h^{n-1})$ such that

$$b(\boldsymbol{\sigma}_h^n, \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h^n) \geq 0 \quad \forall \boldsymbol{\tau}_h \in \boldsymbol{\Sigma}(\boldsymbol{\sigma}_h^{n-1}) .$$

with $\boldsymbol{u}_h^0 \in \boldsymbol{K}_h$ and $\boldsymbol{\sigma}_h^0 \in \mathscr{H}_h$ given.

We remark that, for any $n \in \mathbb{N}^*$, the problem $(\mathbf{P})_{\mathbf{h,n}}$, and respectively, $(\mathbf{P}_1^*)_{\mathbf{h,n}}$ has a unique solution. Arguing as in the continuous case, we find that the sequence $\{(\mathbf{P}_1^*)_{\mathbf{h,n}}\}_{n \in N^*}$ approximates the problem $(\mathbf{P}_1^*)_{\mathbf{h}}$ in the following sense.

Proposition 8.4. *We suppose that* $\boldsymbol{f}$ *is linear and* $\boldsymbol{\sigma}_h^0 = \boldsymbol{\sigma}(\boldsymbol{u}_h^0)$. *Then, we have*

$$\boldsymbol{\sigma}_h^n = \boldsymbol{\sigma}(\boldsymbol{u}_h^n) \quad \forall n \in \mathbb{N}, \tag{8.119}$$

and

$$\lim_{n \longrightarrow \infty} \|\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}_h\|_{\mathscr{H}} = 0. \tag{8.120}$$

Proof. The hypothesis on $\boldsymbol{f}$ and the definition of the spaces $\boldsymbol{V}_h$ and $\mathscr{H}_h$ imply that $\operatorname{div} \boldsymbol{\sigma}_h = \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_h^n) = -\boldsymbol{f}$ a.e. in Ω. Therefore, by a similar proof as for Proposition 8.3, the assertion follows. □

We are now prepared to prove the main convergence result of this section.

Theorem 8.12. *If we suppose that* $\boldsymbol{f}$ *is linear, then*

$$\boldsymbol{\sigma}_h \longrightarrow \boldsymbol{\sigma} \text{ strongly in } \mathscr{H} . \tag{8.121}$$

Proof. In order to obtain the convergence, we write

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{\mathscr{H}} \leq \|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^n\|_{\mathscr{H}} + \|\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}^n\|_{\mathscr{H}} + \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}\|_{\mathscr{H}} \quad \forall n \geq 0 . \tag{8.122}$$

From Propositions 8.3 and 8.4, it follows that, for any $\epsilon > 0$, there exists $N_\epsilon > 0$ such that

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^n\|_{\mathscr{H}} + \|\boldsymbol{\sigma}^n - \boldsymbol{\sigma}\|_{\mathscr{H}} \leq \frac{\epsilon}{2} \quad \forall n \geq N_\epsilon . \tag{8.123}$$

In order to estimate the second term in the right-hand side of (8.122), we recall that the construction of the spaces $\boldsymbol{V}_h$ and $\mathscr{H}_h$ was obtained with polynomials of degree one, $\boldsymbol{f}$ is linear, and so

$$\begin{aligned} \|\boldsymbol{\sigma}_h^n - \boldsymbol{\sigma}^n\|_{\mathscr{H}} &= \|\boldsymbol{\sigma}(\boldsymbol{u}_h^n) - \boldsymbol{\sigma}(\boldsymbol{u}^n)\|_{\mathscr{H}} = \|\boldsymbol{\sigma}(\boldsymbol{u}_h^n) - \boldsymbol{\sigma}(\boldsymbol{u}^n)\|_{\boldsymbol{H}} \\ &\leq C\,\|\boldsymbol{u}_h^n - \boldsymbol{u}^n\|_1 \leq C \sum_{i=0}^{n} \|\mathbf{w}_h^i - \boldsymbol{u}^i\|_1 \quad \forall n \geq 0 . \end{aligned} \tag{8.124}$$

In the last inequality we used the relation (7.30), $\mathbf{w}_h^i$ being the unique solution of the problem $(\mathbf{P}^a)_{\mathbf{i,h}}$ defined by (7.24).

Finally, from (8.123), (8.124) and Proposition 7.2, by taking $n = N_\epsilon$ in (8.122), we deduce that, for $\epsilon > 0$ given, there exists $H_\epsilon > 0$ such that

$$\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}\|_{\mathscr{H}} \leq \epsilon \quad \forall h \leq H_\epsilon \,, \tag{8.125}$$

and so, the convergence (8.121). □

8.8 An Optimal Control Problem

In the optimal control theory, the problems which are of interest concern the existence and, when possible, the uniqueness of an optimal control, and also the derivation of the necessary conditions of optimality or, better, the necessary and sufficient optimality conditions. This means to find an equation or an inequality that characterizes the optimal control. We remark that, if the relation control-state is linear, then the cost functional is convex differentiable, and so, the optimality conditions are easy to obtain.

For a better understanding of the problem we shall study in this section, we briefly recall the main ideas of the optimal control theory (see, for instance, [34,35]), in the simplest case of a system governed by a continuous linear operator.

We consider two Hilbert spaces: the space V of states and the space $\mathscr{U}$ of controls. Let $A \in \mathscr{L}(V, V^*)$ and $B \in \mathscr{L}(\mathscr{U}, V^*)$ be two operators, and $f \in V^*$ be a given element. Then, if $v \in \mathscr{U}$ is a given control, the state is defined as the solution $u \in V$, which, obviously, depends on v, of the following system (mechanical, physical, etc.) governed by the operator A:

$$Au = f + Bv \,.$$

We also suppose that there exists another Hilbert space W where one can obtain some observations of the state $u = u(v)$, i.e. one has $w(v) = C(u(v))$ where $C \in \mathscr{L}(V, W)$. It is natural to call W the space of observations.

Therefore, to any control v, we associate the cost functional defined by

$$J(v) = \|C(u(v)) - w_d\|_W^2 + J_0(v)$$

where $w_d \in W$ is a given observation. The additional term $J_0(v)$ is usually introduced to enrich the properties of the functional J. For instance, if the functional J_0 is coercive, then so is the functional J.

The control problem is the following constrained minimization problem

$$\inf_{v \in \mathscr{U}_{ad}} J(v)$$

where the set $\mathscr{U}_{ad}$, called the set of admissible controls, is convex closed in $\mathscr{U}$.

We say that v^* is an optimal control iff v^* achieves the minimum, i.e.

$$J(v^*) = \inf_{v \in \mathscr{U}_{ad}} J(v)$$

From the above example, it is obvious that an infinity of problems can be considered by taking into account the complexity of the involved systems, the diversity of the boundary conditions, the nature of the controls or observations (distributed or boundary).

The work of Lions [34] on the optimal control of systems governed by partial differential equations represents the foundation of the optimal control theory. The subject was developed by the contributions of Sprekels and Tiba [54] and Neittaanmaki and Tiba [43]. For the optimal control of variational inequalities, we refer to Friedman [26], Mignot [38], Barbu [6], Mignot and Puel [39], Barbu and Tiba [7], or Neittaanmaki et al. [42].

Despite the fact that there are many applications of the optimal control theory in mechanics (see, for instance, Abergel and Temam [2], Abergel and Casas [1], Capatina and Stavre [15]), the optimal control of contact problems is not very often addressed in the literature. We mention here the results obtained by Bermudez and Saguez [8], Capatina [13], Sofonea and Tiba [53], Matei and Micu [37] for static contact friction problems.

In this section we study, following the work [13], an optimal control problem governed by the problem **(P)**, p. 141. From the physical point of view, it is of great interest to determine the coefficient of friction, which depends on the nature of the materials in contact, such that one obtains certain displacements on the part of the contact boundary. The mathematical formulation of this mechanical model is an optimal control problem governed by the quasi-variational inequality (8.15). In such an approach one encounters considerable mathematical impediments, and the standard methods (see, for instance, [6, 35, 38]) cannot be applied for deriving the necessary optimality conditions. The difficulties in our problem are involved by: the state is the solution of a quasi-variational inequality, the control is a coefficient defined only on a part of the boundary, and the relationship between the control and the state is nonsmooth and nonconvex. In order to surpass these difficulties, we will approximate the given problem by a family of penalized control problems governed by a variational inequality, and then we will approximate each penalized problem by a family of regularized problems governed by an equation.

We assume that the following hypotheses hold:

$$\begin{aligned}
&\Gamma_2 \neq \emptyset\,, \\
&\text{meas}\,(\Gamma_0) > 0\,, \\
&\boldsymbol{f} \in (L^2(\Omega))^d\,,\ \boldsymbol{g} \in (L^2(\Gamma_1))^d\,, \\
&a_{ijkl} \in C^1(\overline{\Omega})\,, \\
&\mu \in L^2(\Gamma_2)\,,\ \mu \geq 0 \text{ a.e. on } \Gamma_2\,, \\
&\begin{cases} \mathscr{R} : H^{-1/2}(\Gamma_2) \to C(\overline{\Gamma}_2) \text{ is a linear continuous mapping such that} \\ \qquad\qquad \mathscr{R}(\sigma_\nu(\boldsymbol{w})) \leq 0\,,\ \forall \boldsymbol{w} \in \boldsymbol{W}\,, \end{cases}
\end{aligned} \tag{8.126}$$

where $\boldsymbol{W} = \{\boldsymbol{w} \in \boldsymbol{V}\,;\ \text{div}\,\boldsymbol{\sigma}(\boldsymbol{w}) \in (L^2(\Omega))^d\}$.

Then, it is easy to see that Theorem 8.2 is still valid. In addition, instead of relations (8.32) and (8.33), used in the proof of Theorem 8.3, we have

$$
\begin{aligned}
&|j_f(\boldsymbol{u}_1,\boldsymbol{v}_2)+j_f(\boldsymbol{u}_2,\boldsymbol{v}_1)-j_f(\boldsymbol{u}_1,\boldsymbol{v}_1)-j_f(\boldsymbol{u}_2,\boldsymbol{v}_2)| \\
&\le \|\mu\|_{L^2(\Gamma_2)}\|\mathscr{R}(\sigma_\nu(P_f\boldsymbol{u}_1-P_f\boldsymbol{u}_2))\|_{C(\overline{\Gamma}_2)}\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_{(L^2(\Gamma_2))^d} \\
&\le C\|\mu\|_{L^2(\Gamma_2)}\|\boldsymbol{u}_1-\boldsymbol{u}_2\|_1\|\boldsymbol{v}_1-\boldsymbol{v}_2\|_1 \quad \forall \boldsymbol{u}_1,\,\boldsymbol{u}_2,\,\boldsymbol{v}_1,\,\boldsymbol{v}_2\in \boldsymbol{V}\,,
\end{aligned}
$$

and so, by choosing $\mu_1 < \dfrac{C}{\alpha}$ and by applying Theorem 4.16, it follows that for any $\mu \in L^2(\Gamma_2)$ with $\mu \ge 0$ a.e. on Γ_2 and $\|\mu\|_{L^2(\Gamma_2)} \le \mu_1$, the problem **(P)** has a unique solution, denoted by $\boldsymbol{u}^\mu$.

The mathematical formulation of our optimal control problem is the following.

Problem (CP): Find $\mu^* \in M$ such that

$$J(\mu^*) = \min_{\mu\in M} J(\mu) \tag{8.127}$$

where

$$M = \{\mu \in L^2(\Gamma_2)\,;\ \|\mu\|_{L^2(\Gamma_2)} \le \mu_1\}, \tag{8.128}$$

and

$$J(\mu) = \frac{1}{2}\int_{\Gamma_2} (\boldsymbol{u}^\mu - \boldsymbol{u}_d)^2\,\mathrm{d}s\,, \tag{8.129}$$

with $u_d \in L^2(\Omega)$ given, representing a desired profile for displacements on Γ_2.

Also, to better highlight the dependence of the solution $\boldsymbol{u}^\mu$ of the coefficient of friction μ, we put

$$\varphi(\mu,\boldsymbol{w},\boldsymbol{v}) = -\int_{\Gamma_2} \mu\mathscr{R}(\sigma_\nu(\boldsymbol{w}))|\boldsymbol{v}_\tau|\,\mathrm{d}s \quad \forall \boldsymbol{w}\in \boldsymbol{W}\,,\ \forall \boldsymbol{v}\in\boldsymbol{V}\,, \tag{8.130}$$

and thus, the problem satisfied by $\boldsymbol{u}_\mu$ will be written under the following form.

Problem (P)$^\mu$: Find $\boldsymbol{u}^\mu \in \boldsymbol{K}$ such that

$$a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + \varphi(\mu,\boldsymbol{u},\boldsymbol{v}) - \varphi(\mu,\boldsymbol{u},\boldsymbol{u}) \ge (\boldsymbol{F},\boldsymbol{v}-\boldsymbol{u})_V \quad \forall \boldsymbol{v}\in\boldsymbol{K}. \tag{8.131}$$

The first result to prove is the existence of an optimal control.

Theorem 8.13. *The optimal control problem* **(CP)** *has at least one solution.*

Proof. The set M is bounded convex closed in the reflexive Banach space $L^2(\Gamma_2)$, and so it is weakly closed. Then, taking into account that the functional J is weakly lower semicontinuous on M, the assertion of the theorem follows by applying Weierstrass Theorem 4.2. □

Since the state $\boldsymbol{u}^{\mu}$ is the solution of a quasi-variational inequality and the control μ is a boundary coefficient which occurs only in the term defined on Γ_2, we have less informations about the relationship control-state. Hence, we cannot used the differentiability of J for deriving a characterization of an optimal control. For this reason, we will approximate the problem (8.127) by a family of penalized problems governed by a variational inequality.

More precisely, let us fix a solution μ^0 of the problem $(\mathbf{CP})$. Then, for any $\epsilon > 0$, we define the functional

$$\begin{aligned} J_\epsilon(\mu, \boldsymbol{w}) = \frac{1}{2}\int_{\Gamma_2} (\boldsymbol{u}^{\mu,\boldsymbol{w}} - \boldsymbol{u}_d)^2\,\mathrm{d}s + \frac{1}{2\epsilon}\|\boldsymbol{u}^{\mu,\boldsymbol{w}} - \boldsymbol{w}\|^2_{\boldsymbol{W}} \\ + \frac{1}{2}\|\mu - \mu^0\|^2_{L^2(\Gamma_2)} \quad \forall \mu \in M\,,\ \forall \boldsymbol{w} \in \boldsymbol{W} \end{aligned} \tag{8.132}$$

where the state $\boldsymbol{u}^{\mu,\boldsymbol{w}}$ is, this time, the unique solution of the following variational inequality of the second kind:

Problem $(\mathbf{P})^{\mu,\boldsymbol{w}}$: Find $\boldsymbol{u} \in \boldsymbol{K}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + \varphi(\mu, \boldsymbol{w}, \boldsymbol{v}) - \varphi(\mu, \boldsymbol{w}, \boldsymbol{u}) \geq (\boldsymbol{F}, \boldsymbol{v} - \boldsymbol{u})_V \quad \forall \boldsymbol{v} \in \boldsymbol{K}\,. \tag{8.133}$$

Now, we consider the following family of penalized problems

Problem $(\mathbf{CP})_\epsilon$: Find $(\mu^*_\epsilon, \boldsymbol{w}^*_\epsilon) \in M \times \boldsymbol{W}$ such that

$$J_\epsilon(\mu^*_\epsilon, \boldsymbol{w}^*_\epsilon) = \min_{(\mu,\boldsymbol{w})\in M\times \boldsymbol{W}} J_\epsilon(\mu, \boldsymbol{w})\,. \tag{8.134}$$

From the definition of the functional J_ϵ, it follows that, if $(\mu^*_\epsilon, \boldsymbol{w}^*_\epsilon)$ is an optimal control for the problem $(\mathbf{CP})_\epsilon$, then the corresponding solution $\boldsymbol{u}^*_\epsilon = \boldsymbol{u}^{\mu^*_\epsilon,\boldsymbol{w}^*_\epsilon}$ is forced to be very closed to $\boldsymbol{w}^*_\epsilon$, and also, the control μ^*_ϵ will be not far from the chosen optimal control μ^0 of the initial problem $(\mathbf{CP})$.

Remark 8.13. In our approach it is enough to consider only the constraint $\boldsymbol{w} \in \boldsymbol{W}$ instead of $\boldsymbol{w} \in \boldsymbol{K} \cap \boldsymbol{W}$. In addition, this constraint is convenient since it leads, as one see below, to an equation in the optimality system (obviously, the appurtenance to $\boldsymbol{K}$ would generate an inequality).

The first result for the family $(\mathbf{CP})_\epsilon$ is an existence one.

Proposition 8.5. *For any* $\epsilon > 0$ *there exists at least one solution* $(\mu_\epsilon, \boldsymbol{w}_\epsilon)$ *of the optimization problem* $(\mathbf{CP})_\epsilon$.

Proof. Every minimizing sequence for the functional J_ϵ is bounded. Indeed, let $\{(\mu^n_\epsilon, \boldsymbol{w}^n_\epsilon)\}_n$ be a minimizing sequence for J_ϵ on $M \times \boldsymbol{W}$, i.e.

$$\lim_{n\to\infty} J_\epsilon(\mu^n_\epsilon, \boldsymbol{w}^n_\epsilon) = \inf_{(\mu,\boldsymbol{w})\in M\times \boldsymbol{W}} J_\epsilon(\mu, \boldsymbol{w})\,.$$

If $\boldsymbol{u}_\epsilon^n = \boldsymbol{u}^{\mu_\epsilon^n, \boldsymbol{w}_\epsilon^n}$, then, by taking $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v} = 2\boldsymbol{u}$ in the problem $(\mathbf{P})^{\mu_\epsilon^n, \boldsymbol{w}_\epsilon^n}$, and using the positivity of φ, we get the estimate

$$a(\boldsymbol{u}_\epsilon^n, \boldsymbol{u}_\epsilon^n) \le a(\boldsymbol{u}_\epsilon^n, \boldsymbol{u}_\epsilon^n) + \varphi(\mu_\epsilon^n, \boldsymbol{w}_\epsilon^n, \boldsymbol{u}_\epsilon^n) = (\boldsymbol{F}, \boldsymbol{u}_\epsilon^n)_{\boldsymbol{V}} . \tag{8.135}$$

It follows, by using the coerciveness of a, that the sequence $\{\boldsymbol{u}_\epsilon^n\}_n$ is bounded in $\boldsymbol{V}$. Taking into account that div $\sigma(\boldsymbol{u}_\epsilon^n) = -\boldsymbol{f}$, a.e. on Ω, we deduce that $\{\boldsymbol{u}_\epsilon^n\}_n$ is also bounded in $\boldsymbol{W}$. As the sequence $\{\mu_\epsilon^n\}_n$ is bounded in $L^2(\Gamma_2)$, the boundedness of the sequence $\{\boldsymbol{w}_\epsilon^n\}_n$ in $\boldsymbol{W}$ is a consequence of the fact that the functional J_ϵ is proper.

Now we prove that the functional J_ϵ is weakly lower semicontinuous. Let $\{(\mu_\epsilon^n, \boldsymbol{w}_\epsilon^n)\}_n \subset M \times \boldsymbol{W}$ be a weakly convergent sequence to an element $(\mu_\epsilon, \boldsymbol{w}_\epsilon) \in M \times \boldsymbol{W}$. From (8.135), it follows that the sequence $\{\boldsymbol{u}_\epsilon^n\}_n$ is bounded in $\boldsymbol{V}$ where $\boldsymbol{u}_\epsilon^n = \boldsymbol{u}^{\mu_\epsilon^n, \boldsymbol{w}_\epsilon^n}$. Thus, we can extract a subsequence, denoted in the same way, such that $\boldsymbol{u}_\epsilon^n \rightharpoonup \boldsymbol{u}_\epsilon$ weakly in $\boldsymbol{V}$, with $\boldsymbol{u}_\epsilon \in \boldsymbol{K}$.

By passing to the limit in $(\mathbf{P})^{\mu_\epsilon^n, \boldsymbol{w}_\epsilon^n}$, from the uniqueness of the solution of $(\mathbf{P})^{\mu_\epsilon, \boldsymbol{w}_\epsilon}$, it follows that $\boldsymbol{u}_\epsilon = \boldsymbol{u}^{\mu_\epsilon, \boldsymbol{w}_\epsilon}$. This implies that $\boldsymbol{u}_\epsilon^n \rightharpoonup \boldsymbol{u}_\epsilon$ weakly in $\boldsymbol{W}$. Thus, from the lower weakly semicontinuity of the norm, we get

$$\liminf_{n\to\infty} \|\boldsymbol{u}_\epsilon^n - \boldsymbol{w}_\epsilon^n\|_{\boldsymbol{W}}^2 \ge \|\boldsymbol{u}_\epsilon - \boldsymbol{w}_\epsilon\|_{\boldsymbol{W}}^2 , \tag{8.136}$$

which, together the strong convergence $\boldsymbol{u}_\epsilon^n \to \boldsymbol{u}_\epsilon$ in $(L^2(\Gamma))^d$, gives

$$\liminf_{n\to\infty} J_\epsilon(\mu_\epsilon^n, \boldsymbol{w}_\epsilon^n) \ge J_\epsilon(\mu_\epsilon, \boldsymbol{w}_\epsilon) . \tag{8.137}$$

Finally, since the set $M \times \boldsymbol{K}$ is weakly closed, by applying Weierstrass Theorem 4.2, the assertion follows. □

The following result establishes the relationship between the family of penalized problems $(\mathbf{CP})_\epsilon$ and the control problem $(\mathbf{CP})$.

Proposition 8.6. *For any* $\epsilon > 0$, *let* $(\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*) \in M \times \boldsymbol{W}$ *be a solution of the problem* $(\mathbf{CP})_\epsilon$. *Then, we have*

$$\begin{cases} \mu_\epsilon^* \to \mu^0 & \textit{strongly in } L^2(\Gamma_2) , \\ \boldsymbol{w}_\epsilon^* \to \boldsymbol{u}^0 & \textit{strongly in } \boldsymbol{W} , \\ \boldsymbol{u}_\epsilon^* \to \boldsymbol{u}^0 & \textit{strongly in } \boldsymbol{W} , \end{cases} \tag{8.138}$$

when $\epsilon \to 0$, *where* $\boldsymbol{u}^0 = \boldsymbol{u}^{\mu^0}$ *and* $\boldsymbol{u}_\epsilon^* = \boldsymbol{u}^{\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*}$. *Moreover, we get*

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon} \|\boldsymbol{u}_\epsilon^* - \boldsymbol{w}_\epsilon^*\|_{\boldsymbol{W}} = 0 , \tag{8.139}$$

and

$$\lim_{\epsilon\to 0} J_\epsilon(\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*) = J(\mu^0) = \min_{\mu\in M} J(\mu) . \tag{8.140}$$

Proof. Let $\epsilon > 0$, and let $(\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*) \in M \times \boldsymbol{W}$ be a solution of the problem $(\mathbf{CP})_\epsilon$ and $\boldsymbol{u}_\epsilon^* = \boldsymbol{u}^{\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*}$. Since $\{\mu_\epsilon^*\}_\epsilon \subset M$, and $\boldsymbol{u}_\epsilon^*$ is a solution of $(\mathbf{P})^{\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*}$, it follows that the sequences $\{\mu_\epsilon^*\}_\epsilon$ and $\{\boldsymbol{u}_\epsilon^*\}_\epsilon$ are bounded in $L^2(\Gamma_2)$, and, respectively, in $\boldsymbol{V}$. Hence, there exist the subsequences $\{\mu_{\epsilon_p}^*\}_p$ and $\{\boldsymbol{u}_{\epsilon_p}^*\}_p$, and the elements $\mu^* \in M$ and $\boldsymbol{u} \in \boldsymbol{K}$ such that

$$\begin{cases} \mu_{\epsilon_p}^* \rightharpoonup \mu^* \quad \text{weakly in } L^2(\Gamma_2) , \\ \boldsymbol{u}_{\epsilon_p}^* \rightharpoonup \boldsymbol{u} \quad \text{weakly in } \boldsymbol{V} \end{cases} \tag{8.141}$$

when $p \to \infty$. Since $\boldsymbol{u}_\epsilon^*$ is a solution of $(\mathbf{P})^{\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*}$, one deduces that there exists $\boldsymbol{u}_1 \in \boldsymbol{W}$ such that

$$\boldsymbol{u}_{\epsilon_p}^* \rightharpoonup \boldsymbol{u}_1 \quad \text{weakly in } \boldsymbol{W} . \tag{8.142}$$

The sequence $\{\boldsymbol{w}_\epsilon^*\}_\epsilon$ is also bounded in $\boldsymbol{W}$. Indeed, since $\boldsymbol{u}^0$ is also a solution of $(\mathbf{P})^{\mu^0, \boldsymbol{u}^0}$, i.e. $\boldsymbol{u}^0 = \boldsymbol{u}^{\mu^0} = \boldsymbol{u}^{\mu^0, \boldsymbol{u}^0}$, we have

$$\|\boldsymbol{u}_\epsilon^* - \boldsymbol{w}_\epsilon^*\|_{\boldsymbol{W}}^2 \leq 2\epsilon J_\epsilon(\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*) \leq 2\epsilon J_\epsilon(\mu^0, \boldsymbol{u}^0) = 2\epsilon J(\mu^0) . \tag{8.143}$$

From (8.142) and (8.143), we conclude that

$$\boldsymbol{w}_{\epsilon_p}^* \rightharpoonup \boldsymbol{u}_1 \quad \text{weakly in } \boldsymbol{W} . \tag{8.144}$$

Now, by passing to the limit in $(\mathbf{P})^{\mu_{\epsilon_p}^*, \boldsymbol{w}_{\epsilon_p}^*}$, we obtain that $\boldsymbol{u}$ satisfies

$$a(\boldsymbol{u}, \boldsymbol{v} - \boldsymbol{u}) + \varphi(\mu^*, \boldsymbol{u}_1, \boldsymbol{v}) - \varphi(\mu^*, \boldsymbol{u}_1, \boldsymbol{u}) \geq (\boldsymbol{F}, \boldsymbol{v} - \boldsymbol{u})_{\boldsymbol{V}} \quad \forall \boldsymbol{v} \in \boldsymbol{K} ,$$

and hence, $\boldsymbol{u} \in \boldsymbol{W}$. Therefore, from (8.141) and (8.142), we get $\boldsymbol{u}_1 = \boldsymbol{u}$ and then, $\boldsymbol{u}$ is the unique solution of $(\mathbf{P})^{\mu^*}$, i.e. $\boldsymbol{u} = \boldsymbol{u}^{\mu^*}$.

In addition, since μ^0 is a solution of $(\mathbf{CP})$, we get

$$\begin{aligned} J(\mu^*) &= \frac{1}{2} \int_{\Gamma_2} (\boldsymbol{u} - \boldsymbol{u}_d)^2 \, \mathrm{d}s \leq \frac{1}{2} \int_{\Gamma_2} (\boldsymbol{u} - \boldsymbol{u}_d)^2 \, \mathrm{d}s + \frac{1}{2} \|\mu^* - \mu^0\|_{L^2(\Gamma_2)}^2 \\ &\leq \liminf_{\epsilon_p \to 0} J_{\epsilon_p}(\mu_{\epsilon_p}^*, \boldsymbol{w}_{\epsilon_p}^*) \leq \liminf_{\epsilon_p \to 0} J_{\epsilon_p}(\mu^0, \boldsymbol{u}^0) = J(\mu^0) \leq J(\mu^*) \end{aligned} \tag{8.145}$$

which implies $\mu^* = \mu^0$, $\boldsymbol{u} = \boldsymbol{u}^0$. Therefore, the whole sequence $\{(\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*, \boldsymbol{u}_\epsilon^*)\}_\epsilon$ converges to $(\mu^0, \boldsymbol{u}^0, \boldsymbol{u}^0)$ weakly in $L^2(\Gamma_2) \times \boldsymbol{W} \times \boldsymbol{W}$.

On the other hand, from (8.143), we have

$$\begin{aligned} &\frac{1}{2\epsilon} \|\boldsymbol{u}_\epsilon^* - \boldsymbol{w}_\epsilon^*\|_{\boldsymbol{W}}^2 + \frac{1}{2} \|\mu_\epsilon^* - \mu^0\|_{L^2(\Gamma_2)}^2 = J_\epsilon(\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*) \\ &- \frac{1}{2} \int_{\Gamma_2} (\boldsymbol{u}_\epsilon^* - \boldsymbol{u}_d)^2 \, \mathrm{d}s \leq J(\mu^0) - \frac{1}{2} \int_{\Gamma_2} (\boldsymbol{u}_\epsilon^* - \boldsymbol{u}_d)^2 \, \mathrm{d}s \end{aligned} \tag{8.146}$$

hence

$$\limsup_{\epsilon\to 0}\left(\frac{1}{2\epsilon}\|\boldsymbol{u}_\epsilon^* - \boldsymbol{w}_\epsilon^*\|_W^2 + \frac{1}{2}\|\mu_\epsilon^* - \mu^0\|_{L^2(\Gamma_2)}^2\right) \le J(\mu^0) - \frac{1}{2}\liminf_{\epsilon\to 0}\int_{\Gamma_2}(\boldsymbol{u}_\epsilon^* - \boldsymbol{u}_d)^2\,\mathrm{d}s \le 0\,,$$

and, so

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\|\boldsymbol{u}_\epsilon^* - \boldsymbol{w}_\epsilon^*\|_W = \lim_{\epsilon\to 0}\|\mu_\epsilon^* - \mu^0\|_{L^2(\Gamma_2)} = 0\,, \tag{8.147}$$

i.e. the relations $(8.138)_1$ and (8.139) hold.

Next, by choosing $\boldsymbol{v} = \boldsymbol{u}_\epsilon^*$ in $(\mathbf{P})^{\mu^0}$ and $\boldsymbol{v} = \boldsymbol{u}^0$ in $(\mathbf{P})^{\mu_\epsilon^*,\boldsymbol{w}_\epsilon^*}$, by adding them, we obtain

$$\begin{aligned}&\alpha\limsup_{\epsilon\to 0}\|\boldsymbol{u}_\epsilon^* - \boldsymbol{u}^0\|_V^2 \le \limsup_{\epsilon\to 0} a(\boldsymbol{u}_\epsilon^* - \boldsymbol{u}^0, \boldsymbol{u}_\epsilon^* - \boldsymbol{u}^0)\\ &\le \lim_{\epsilon\to 0}\left(\varphi(\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*, \boldsymbol{u}^0) - \varphi(\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*, \boldsymbol{u}_\epsilon^*) + \varphi(\mu^0, \boldsymbol{u}^0, \boldsymbol{u}_\epsilon^*) - \varphi(\mu^0, \boldsymbol{u}^0, \boldsymbol{u}^0)\right) = 0\end{aligned} \tag{8.148}$$

thus $(8.138)_3$, which, together with (8.143), leads to $(8.138)_2$. We remark that we have always used the fact that $\boldsymbol{u}_\epsilon^* = \boldsymbol{u}^{\mu_\epsilon^*,\boldsymbol{w}_\epsilon^*}$ and $\boldsymbol{u}^0 = \boldsymbol{u}^{\mu^0}$, hence $\|\boldsymbol{u}_\epsilon^* - \boldsymbol{u}^0\|_W = \|\boldsymbol{u}_\epsilon^* - \boldsymbol{u}^0\|_V$.

Finally, from the definitions (8.132) and (8.129) of J_ϵ, and respectively of J, by using $(8.138)_{1,3}$ and (8.147), we obtain (8.140). □

Until now we have reduced the constraints of our control problem to a variational inequality of the second kind. Unfortunately, even if the problem $(\mathbf{CP})_\epsilon$ is more simple than the original problem $(\mathbf{CP})$, it does not enable us to obtain the optimality conditions for an optimal control since the functional J_ϵ is not differentiable. In order to avoid this difficulty, for every $\epsilon > 0$, we consider a family $\{(\mathbf{CP})_{\epsilon\rho}\}_{\rho>0}$ of regularized problems.

Problem $(\mathbf{CP})_{\epsilon\rho}$: Find $(\mu_{\epsilon\rho}^*, \boldsymbol{w}_{\epsilon\rho}^*) \in M \times \boldsymbol{W}$ such that

$$J_{\epsilon\rho}(\mu_{\epsilon\rho}^*, \boldsymbol{w}_{\epsilon\rho}^*) = \min_{(\mu,\boldsymbol{w})\in M\times\boldsymbol{W}} J_{\epsilon\rho}(\mu, \boldsymbol{w}) \tag{8.149}$$

where $\{J_{\epsilon\rho}\}_\rho$ is the family of functionals defined by

$$\begin{aligned}J_{\epsilon\rho}(\mu, \boldsymbol{w}) &= \frac{1}{2}\int_{\Gamma_2}(\boldsymbol{u}_\rho^{\mu,\boldsymbol{w}} - \boldsymbol{u}_d)^2\,\mathrm{d}s + \frac{1}{2\epsilon}\|\boldsymbol{u}_\rho^{\mu,\boldsymbol{w}} - \boldsymbol{w}\|_W^2\\ &+\frac{1}{2}\|\mu - \mu^0\|_{L^2(\Gamma_2)}^2 \quad \forall\mu \in M\,,\ \forall\boldsymbol{w} \in \boldsymbol{W}\end{aligned} \tag{8.150}$$

$\boldsymbol{u}_\rho^{\mu,\boldsymbol{w}}$ being the unique solution of the variational equation defined below.

Problem $(\mathbf{P})_\rho^{\mu,\boldsymbol{w}}$: Find $\boldsymbol{u} \in \boldsymbol{V}$ such that

$$a(\boldsymbol{u}, \boldsymbol{v}) + \langle\nabla\varphi_{\mu,\boldsymbol{w}}^\rho(\boldsymbol{u}), \boldsymbol{v}\rangle + \langle\beta_\rho(\boldsymbol{u}), \boldsymbol{v}\rangle = (\boldsymbol{F}, \boldsymbol{v})_V \quad \forall\boldsymbol{v} \in \boldsymbol{V}\,. \tag{8.151}$$

We have denoted by $\beta_\rho : \boldsymbol{V} \to \boldsymbol{V}^*$ the Yosida approximation of the subdifferential $\partial I_K : \boldsymbol{V} \to 2^{\boldsymbol{V}^*}$ of the indicator function I_K of $\boldsymbol{K}$, i.e. $\beta_\rho(\boldsymbol{v}) = \dfrac{1}{\rho}(I - R_\rho)(\boldsymbol{v})$, $\forall \boldsymbol{v} \in \boldsymbol{V}$,‘ where I denotes the identity operator on V and $R_\rho : \boldsymbol{V} \to \boldsymbol{V}$ is the resolvent of the maximal monotone operator ∂I_K defined by $R_\rho(\boldsymbol{v}) = (I + \rho \partial I_K)^{-1}(\boldsymbol{v})$, $\forall \boldsymbol{v} \in \boldsymbol{V}$ (for more details, see [50, 58]). Also, we denoted by $\{\varphi^\rho_{\mu,\boldsymbol{w}}\}_\rho$ a family of convex functionals $\varphi^\rho_{\mu,\boldsymbol{w}} : \boldsymbol{V} \to \mathbb{R}$ which are of class C^2 weakly, i.e. $\nabla \varphi^\rho_{\mu,\boldsymbol{w}} : \boldsymbol{V} \to \boldsymbol{V}^*$ and $\nabla^2 \varphi^\rho_{\mu,\boldsymbol{w}} : \boldsymbol{V} \to \mathscr{L}(\boldsymbol{V}, \boldsymbol{V}^*)$ are weakly continuous, and, in addition, satisfy the following conditions

$$(\mu, \boldsymbol{w}) \longmapsto \varphi^\rho_{\mu,\boldsymbol{w}} \text{ is linear,} \tag{8.152}$$

$$\begin{cases} \lim\limits_{\rho \to 0} \varphi^\rho_{\mu_\rho, \boldsymbol{w}_\rho}(\boldsymbol{v}) = \varphi(\mu, \boldsymbol{w}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V} \\ \forall (\mu_\rho, \boldsymbol{w}_\rho) \rightharpoonup (\mu, \boldsymbol{w}) \ \text{ weakly in } \ L^2(\Gamma_2) \times \boldsymbol{W} \ , \end{cases} \tag{8.153}$$

$$\begin{cases} \liminf\limits_{\rho \to 0} \varphi^\rho_{\mu_\rho, \boldsymbol{w}_\rho}(\boldsymbol{v}_\rho) \geq \varphi(\mu, \boldsymbol{w}, \boldsymbol{v}) \\ \forall (\mu_\rho, \boldsymbol{w}_\rho, \boldsymbol{v}_\rho) \rightharpoonup (\mu, \boldsymbol{w}, \boldsymbol{v}) \ \text{ weakly in } \ L^2(\Gamma_2) \times \boldsymbol{W} \times \boldsymbol{V} \ , \end{cases} \tag{8.154}$$

$$\begin{cases} \lim\limits_{n \to \infty} \varphi^\rho_{\mu_n, \boldsymbol{w}_n}(\boldsymbol{v}_n) = \varphi^\rho_{\mu,\boldsymbol{w}}(\boldsymbol{v}) \\ \forall (\mu_n, \boldsymbol{w}_n, \boldsymbol{v}_n) \rightharpoonup (\mu, \boldsymbol{w}, \boldsymbol{v}) \ \text{ weakly in } \ L^2(\Gamma_2) \times \boldsymbol{W} \times \boldsymbol{V} \ , \end{cases} \tag{8.155}$$

Remark 8.14. We may choose

$$\varphi^\rho_{\mu,\boldsymbol{w}}(\boldsymbol{v}) = \varphi(\mu, \boldsymbol{w}, \theta_\rho(\boldsymbol{v}_\tau)) = -\int_{\Gamma_2} \mu \mathscr{R}(\sigma_\nu(\boldsymbol{w})) \theta_\rho(\boldsymbol{v}_\tau)\, \mathrm{d}s \ ,$$

where the function $\theta_\rho : (L^2(\Gamma_2))^d \to L^2(\Gamma_2)$ is an approximation (see, e.g., [45]) of the function $|\cdot| : (L^2(\Gamma_2))^d \to L^2(\Gamma_2)$ which is defined, for $\rho > 0$, $\boldsymbol{v} \in (L^2(\Gamma_2))^d$ and almost everywhere $x \in \Gamma_2$, according to

$$\theta_\rho(\boldsymbol{v}(x)) = \begin{cases} \dfrac{|\boldsymbol{v}(x)|^2}{\rho} \left(1 - \dfrac{|\boldsymbol{v}(x)|}{3\rho}\right) & \text{if } \ |\boldsymbol{v}(x)| \leq \rho \ , \\ \rho \left(\dfrac{|\boldsymbol{v}(x)|}{\rho} - \dfrac{1}{3}\right) & \text{if } \ |\boldsymbol{v}(x)| \geq \rho \ . \end{cases}$$

In this case, it is easy to verify that

$$\nabla \varphi^\rho_{\mu,\boldsymbol{w}}(\boldsymbol{v}) \cdot \boldsymbol{p} = -\int_{\Gamma_2} \mu \mathscr{R}(\sigma_\nu(\boldsymbol{w}))\ \theta'_\rho(\boldsymbol{v}_\tau) \cdot \boldsymbol{p}_\tau\, \mathrm{d}s \quad \forall \boldsymbol{v}, \boldsymbol{p} \in \boldsymbol{V} \ ,$$

where $\theta'_\rho : (L^2(\Gamma_2))^d \to (L^2(\Gamma_2))^d$, for a.e. $x \in \Gamma_2$, is given by

$$\theta'_\rho(\boldsymbol{v})(x) = \begin{cases} \dfrac{1}{\rho}\left(2 - \dfrac{|\boldsymbol{v}(x)|}{\rho}\right)\boldsymbol{v}(x) & \text{if } |\boldsymbol{v}(x)| \le \rho\,, \\ \dfrac{\boldsymbol{v}(x)}{|\boldsymbol{v}(x)|} & \text{if } |\boldsymbol{v}(x)| \ge \rho\,. \end{cases} \tag{8.156}$$

In addition, for every $\boldsymbol{v},\ \boldsymbol{p} \in (L^2(\Gamma_2))^d$ and for a.e. $x \in \Gamma_2$, one obtains

$$\theta''_\rho(\boldsymbol{v})\cdot\boldsymbol{p}(x) = \begin{cases} \dfrac{1}{\rho}\left(\left(2 - \dfrac{|\boldsymbol{v}(x)|}{\rho}\right)\boldsymbol{p} - \dfrac{1}{\rho}\dfrac{\boldsymbol{v}(x)\cdot\boldsymbol{p}}{|\boldsymbol{v}(x)|}\boldsymbol{v}(x)\right) & \text{if } |\boldsymbol{v}(x)| \le \rho\,, \\ \dfrac{1}{|\boldsymbol{v}(x)|}\left(\boldsymbol{p} - \dfrac{\boldsymbol{v}(x)\cdot\boldsymbol{p}}{|\boldsymbol{v}(x)|^2}\boldsymbol{v}(x)\right) & \text{if } |\boldsymbol{v}(x)| \ge \rho\,. \end{cases} \tag{8.157}$$

and

$$\langle \nabla^2\varphi^\rho_{\mu,\boldsymbol{w}}(\boldsymbol{v})\cdot\boldsymbol{p},\boldsymbol{q}\rangle = -\int_{\Gamma_2} \mu\mathscr{R}(\sigma_\nu(\boldsymbol{w}))\ \theta''_\rho(\boldsymbol{v}_\tau)(\boldsymbol{p}_\tau\cdot\boldsymbol{q}_\tau)\,\mathrm{d}s \quad \forall \boldsymbol{v},\boldsymbol{p},\boldsymbol{q} \in \boldsymbol{V}\,.$$

Now we prove that each problem $(\mathbf{CP})_{\epsilon\rho}$ has at least one solution and the family $(\mathbf{CP})_{\epsilon\rho}$ approximates $(\mathbf{CP})_\epsilon$ in a sense that we shall precise.

Proposition 8.7. *For every $\rho > 0$, there exists at least one solution $(\mu^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) \in M \times \boldsymbol{W}$ of the problem $(\mathbf{CP})_{\epsilon\rho}$. Moreover, there exist the elements $(\mu^*_\epsilon, \boldsymbol{w}^*_\epsilon, \boldsymbol{u}^*_\epsilon) \in M \times \boldsymbol{W} \times \boldsymbol{K}$ such that*

$$\begin{cases} \mu^*_{\epsilon\rho} \rightharpoonup \mu^*_\epsilon & \textit{weakly in } L^2(\Gamma_2)\,, \\ \boldsymbol{w}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{w}^*_\epsilon & \textit{weakly in } \boldsymbol{W}\,, \\ \boldsymbol{u}^*_{\epsilon\rho} \to \boldsymbol{u}^*_\epsilon & \textit{strongly in } \boldsymbol{V}\,, \end{cases} \tag{8.158}$$

when $\rho \to 0$, where $\boldsymbol{u}^*_{\epsilon\rho} = \boldsymbol{u}_\rho^{\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$. In addition, we have

$$\boldsymbol{u}^*_\epsilon = \boldsymbol{u}^{\mu^*_\epsilon,\boldsymbol{w}^*_\epsilon} \tag{8.159}$$

and

$$\lim_{\rho\to 0} J_{\epsilon\rho}(\mu^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) = J_\epsilon(\mu^*_\epsilon, \boldsymbol{w}^*_\epsilon) = \min_{(\mu,\boldsymbol{w})\in M\times \boldsymbol{W}} J_\epsilon(\mu,\boldsymbol{w})\,. \tag{8.160}$$

Proof. The functional $J_{\epsilon\rho}$ has the same form as J_ϵ with $\boldsymbol{u}$ satisfying a more simple and regularized constrain. So, arguing as in the proof of Proposition 8.5, one obtains the existence of an optimal control for $(\mathbf{CP})_{\epsilon\rho}$. We remark that the existence and the uniqueness of the solution $\boldsymbol{u}_\rho^{\mu,\boldsymbol{w}}$ of $(\mathbf{P})_\rho^{\mu,\boldsymbol{w}}$ is easy obtained by applying the Browder's surjectivity theorem (see, for instance, [11, 12, 28]) the operator $A + \nabla\varphi^\rho_{\mu,\boldsymbol{w}} + \beta^\rho : \boldsymbol{V} \to \boldsymbol{V}^*$ being strongly monotone, hemicontinuous, and coercive on $\boldsymbol{V}$, and so, it is bijective.

Let us prove the second part of the proposition. Let $(\mu^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) \in M \times \boldsymbol{W}$ be a solution of $(\mathbf{CP})_{\epsilon\rho}$. Since $\boldsymbol{u}^*_{\epsilon\rho} = \boldsymbol{u}_\rho^{\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}} \in \boldsymbol{W}$ and $\{\mu^*_{\epsilon\rho}\}_\rho \subset M$, it follows that the sequences $\{\boldsymbol{u}^*_{\epsilon\rho}\}_\rho$ and $\{\mu^*_{\epsilon\rho}\}_\rho$ are bounded, and so, there exist the elements $\mu^*_\epsilon \in M$ and $\boldsymbol{u}_\epsilon \in \boldsymbol{W}$ such that, on the subsequences, we have

$$\begin{cases} \mu^*_{\epsilon\rho_p} \rightharpoonup \mu^*_\epsilon & \text{weakly in } L^2(\Gamma_2)\,, \\ \boldsymbol{u}^*_{\epsilon\rho_p} \rightharpoonup \boldsymbol{u}_\epsilon & \text{weakly in } \boldsymbol{W} \end{cases} \tag{8.161}$$

when $p \to +\infty$.

For the sake of simplicity we shall omit the subscript p in what follows.

On the other hand, since $\varphi(\mu, \mathbf{0}, \boldsymbol{v}) = 0$, $\forall(\mu, \boldsymbol{v}) \in L^2(\Gamma_2) \times \boldsymbol{V}$, it follows $\boldsymbol{u}_\rho^{\mu^*_\epsilon,\mathbf{0}} = \boldsymbol{u}^{\mu^*_\epsilon,\mathbf{0}}$, and thus, we have

$$\|\boldsymbol{u}^*_{\epsilon\rho} - \boldsymbol{w}^*_{\epsilon\rho}\|^2_{\boldsymbol{W}} \le 2\epsilon J^\rho_\epsilon(\mu^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) \le 2\epsilon J^\rho_\epsilon(\mu^*_\epsilon, \mathbf{0}) = 2\epsilon J_\epsilon(\mu^*_\epsilon, \mathbf{0})$$

which implies that the sequence $\{\boldsymbol{w}^*_{\epsilon\rho}\}_\rho$ is bounded. Hence there exists $\boldsymbol{w}^*_\epsilon \in \boldsymbol{W}$ such that, on a subsequence, still denoted by $\boldsymbol{w}^*_{\epsilon\rho}$, one has

$$\boldsymbol{w}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{w}^*_\epsilon \quad \text{weakly in } \boldsymbol{W}\,,$$

when $\rho \to 0$.

Now, by passing to the limit in $(\mathbf{P})_\rho^{\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$ with $\rho \to 0$, we obtain that $\boldsymbol{u}_\epsilon = \boldsymbol{u}^*_\epsilon \in \boldsymbol{K}$ where $\boldsymbol{u}^*_\epsilon = \boldsymbol{u}^{\mu^*_\epsilon,\boldsymbol{w}^*_\epsilon}$.

Further, from $(\mathbf{P})^{\mu^*_\epsilon,\boldsymbol{w}^*_\epsilon}$, $(\mathbf{P})_\rho^{\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$, (8.153), (8.154), and taking into account that $\varphi(\mu, \boldsymbol{w}, \cdot)$ is weakly continuous, we get

$$\begin{aligned} &\alpha \limsup_{\rho\to 0} \|\boldsymbol{u}^*_{\epsilon\rho} - \boldsymbol{u}^*_\epsilon\|_1^2 \le \lim_{\rho\to 0} \varphi^\rho_{\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}(\boldsymbol{u}^*_\epsilon) - \varphi(\mu^*_\epsilon, \boldsymbol{w}^*_\epsilon, \boldsymbol{u}^*_\epsilon) \\ &- \liminf_{\rho\to 0} \varphi^\rho_{\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}(\boldsymbol{u}^*_{\epsilon\rho}) + \lim_{\rho\to 0} \varphi(\mu^*_\epsilon, \boldsymbol{w}^*_\epsilon, \boldsymbol{u}^*_{\epsilon\rho}) \le 0\,, \end{aligned}$$

i.e. the relation $(8.158)_3$.

In order to prove (8.160), let $(\mu^0_\epsilon, \boldsymbol{w}^0_\epsilon) \in M \times \boldsymbol{W}$ be a solution of $(\mathbf{CP})_\epsilon$, $\boldsymbol{u}^0_\epsilon = \boldsymbol{u}^{\mu^0_\epsilon,\boldsymbol{w}^0_\epsilon}$ and $\boldsymbol{u}^0_{\epsilon\rho} = \boldsymbol{u}_\rho^{\mu^0_\epsilon,\boldsymbol{w}^0_\epsilon}$. The sequence $\{\boldsymbol{u}^0_{\epsilon\rho}\}_\rho$ being bounded in $\boldsymbol{W}$, from the uniqueness of the solution of $(\mathbf{P})^{\mu^0_\epsilon,\boldsymbol{w}^0_\epsilon}$ and the properties (8.153)–(8.155) of $\varphi^\rho_{\mu,\boldsymbol{w}}$, we deduce that $\boldsymbol{u}^0_{\epsilon\rho} \to \boldsymbol{u}^0_\epsilon$ strongly in $\boldsymbol{W}$ when $\rho \to 0$. Hence

$$\lim_{\rho\to 0} \|\boldsymbol{u}^0_{\epsilon\rho} - \boldsymbol{w}^0_\epsilon\|_{\boldsymbol{W}} = \|\boldsymbol{u}^0_\epsilon - \boldsymbol{w}^0_\epsilon\|_{\boldsymbol{W}}\,.$$

Therefore

$$J_\epsilon(\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*) \leq \liminf_{\rho\to 0} J_\epsilon^\rho(\mu_{\epsilon\rho}^*, \boldsymbol{w}_{\epsilon\rho}^*) \leq \limsup_{\rho\to 0} J_\epsilon^\rho(\mu_{\epsilon\rho}^*, \boldsymbol{w}_{\epsilon\rho}^*)$$
$$\leq \limsup_{\rho\to 0} J_\epsilon^\rho(\mu_\epsilon^0, \boldsymbol{w}_\epsilon^0) \leq \frac{1}{2}\lim_{\rho\to 0}\int_{\Gamma_2} (\boldsymbol{u}_{\epsilon\rho}^0 - \boldsymbol{u}_d)^2\,\mathrm{d}s + \frac{1}{2\epsilon}\lim_{\rho\to 0}\|\boldsymbol{u}_{\epsilon\rho}^0 - \boldsymbol{w}_\epsilon^0\|_{\boldsymbol{W}}^2$$
$$+\frac{1}{2}\|\mu_\epsilon^0 - \mu^0\|_{L^2(\Gamma_2)}^2 = J_\epsilon(\mu_\epsilon^0, \boldsymbol{w}_\epsilon^0) \leq J_\epsilon(\mu_\epsilon^*, \boldsymbol{w}_\epsilon^*)$$

and, by (8.160), the proof is completed. □

Finally, by using all the previous convergences, we obtain that $\mu_{\epsilon\rho}^*$ is a suboptimal for the problem $(\mathbf{CP})$.

Proposition 8.8. *Let* $(\mu_{\epsilon\rho}^*, \boldsymbol{w}_{\epsilon\rho}^*) \in M \times \boldsymbol{W}$ *be a solution of the optimal control problem* $(\mathbf{CP})_{\epsilon\rho}$. *Then, we have*

$$\lim_{\epsilon,\rho\to 0} J(\mu_{\epsilon\rho}^*) = J(\mu^0)\,. \tag{8.162}$$

Proof. Let $\tilde{\boldsymbol{u}}_{\epsilon\rho} = \boldsymbol{u}^{\mu_{\epsilon\rho}^*}$ and $\boldsymbol{u}^0 = \boldsymbol{u}^{\mu^0}$. By taking $\boldsymbol{v} = \boldsymbol{u}^0$ in $(\mathbf{P})^{\mu_{\epsilon\rho}^*}$ and $\boldsymbol{v} = \tilde{\boldsymbol{u}}_{\epsilon\rho}$ in $(\mathbf{P})^{\mu^0}$, by adding them and by using the properties of a and φ, it follows

$$\begin{aligned}
&\alpha\|\tilde{\boldsymbol{u}}_{\epsilon\rho} - \boldsymbol{u}^0\|_{\boldsymbol{V}}^2 \leq a(\tilde{\boldsymbol{u}}_{\epsilon\rho} - \boldsymbol{u}^0, \tilde{\boldsymbol{u}}_{\epsilon\rho} - \boldsymbol{u}^0) \leq |\varphi(\mu_{\epsilon\rho}^*, \tilde{\boldsymbol{u}}_{\epsilon\rho}, \boldsymbol{u}^0) - \varphi(\mu_{\epsilon\rho}^*, \tilde{\boldsymbol{u}}_{\epsilon\rho}, \tilde{\boldsymbol{u}}_{\epsilon\rho})\\
&+\varphi(\mu^0, \boldsymbol{u}^0, \tilde{\boldsymbol{u}}_{\epsilon\rho}) - \varphi(\mu^0, \boldsymbol{u}^0, \boldsymbol{u}^0)| \leq |(\varphi(\mu^0, \boldsymbol{u}^0, \tilde{\boldsymbol{u}}_{\epsilon\rho}) - \varphi(\mu^0, \boldsymbol{u}^0, \boldsymbol{u}^0)\\
&+\varphi(\mu^0, \tilde{\boldsymbol{u}}_{\epsilon\rho}, \boldsymbol{u}^0) - \varphi(\mu^0, \tilde{\boldsymbol{u}}_{\epsilon\rho}, \tilde{\boldsymbol{u}}_{\epsilon\rho}))| + |(\varphi(\mu_{\epsilon\rho}^*, \tilde{\boldsymbol{u}}_{\epsilon\rho}, \boldsymbol{u}^0) - \varphi(\mu^0, \tilde{\boldsymbol{u}}_{\epsilon\rho}, \boldsymbol{u}^0))\\
&-(\varphi(\mu_{\epsilon\rho}^*, \tilde{\boldsymbol{u}}_{\epsilon\rho}, \tilde{\boldsymbol{u}}_{\epsilon\rho}) - \varphi(\mu^0, \tilde{\boldsymbol{u}}_{\epsilon\rho}, \tilde{\boldsymbol{u}}_{\epsilon\rho}))| \leq k\|\mu^0\|_{L^2(\Gamma_2)}\|\tilde{\boldsymbol{u}}_{\epsilon\rho} - \boldsymbol{u}^0\|_{\boldsymbol{V}}^2\\
&+k\|\mu_{\epsilon\rho}^* - \mu^0\|_{L^2(\Gamma_2)}\|\tilde{\boldsymbol{u}}_{\epsilon\rho}\|_{\boldsymbol{V}}\,\|\tilde{\boldsymbol{u}}_{\epsilon\rho} - \boldsymbol{u}^0\|_{\boldsymbol{V}}\,.
\end{aligned}$$

Now, by using the boundedness of the sequence $\{\tilde{\boldsymbol{u}}_{\epsilon\rho}\}_\rho$, and by taking $\mu_1 < \dfrac{\alpha}{k}$ (the same condition as in Sect. 8.3, Theorem 8.3, p. 148, for the uniqueness of the solution of the problem $(\mathbf{P})^\mu$), it follows

$$\|\tilde{\boldsymbol{u}}_{\epsilon\rho} - \boldsymbol{u}^0\|_{\boldsymbol{V}} \leq C\|\mu_{\epsilon\rho}^* - \mu^0\|_{L^2(\Gamma_2)}$$

with C a constant independent of ϵ and ρ.

Hence, the convergences (8.158) and (8.138) allow us to obtain (8.162). □

In order to obtain the necessary conditions of optimality for a solution of the problem $(\mathbf{CP})_{\epsilon\rho}$, we shall use the G-differentiability of the functional $J_{\epsilon\rho}$.

Lemma 8.1. *The functional* $J_{\epsilon\rho}$ *is G-differentiable and, for any* $(\mu^*, \boldsymbol{w}^*) \in M \times \boldsymbol{W}$, *we have*

$$\begin{aligned}
&\frac{\partial J_{\epsilon\rho}}{\partial\mu}(\mu^*, \boldsymbol{w}^*)\cdot(\mu - \mu^*) = \int_{\Gamma_2} \boldsymbol{z}_\mu^*(\boldsymbol{u}^* - \boldsymbol{u}_d)\,\mathrm{d}s\\
&+\frac{1}{\epsilon}(\boldsymbol{z}_\mu^*, \boldsymbol{u}^* - \boldsymbol{w}^*)_{\boldsymbol{W}} + (\mu - \mu^*, \mu^* - \mu^0)_{L^2(\Gamma_2)} \quad \forall\mu \in M
\end{aligned} \tag{8.163}$$

and

$$\frac{\partial J_{\epsilon\rho}}{\partial \boldsymbol{w}}(\mu^*, \boldsymbol{w}^*) \cdot \boldsymbol{w} = \int_{\Gamma_2} \boldsymbol{h}_{\boldsymbol{w}}^*(\boldsymbol{u}^* - \boldsymbol{u}_d)\,\mathrm{d}s + \frac{1}{\epsilon}(\boldsymbol{h}_{\boldsymbol{w}}^* - \boldsymbol{w}, \boldsymbol{u}^* - \boldsymbol{w}^*)_{\boldsymbol{W}} \quad \forall \boldsymbol{w} \in \boldsymbol{W} \tag{8.164}$$

where $\boldsymbol{u}^* = \boldsymbol{u}_\rho^{\mu^*,\boldsymbol{w}^*}$ *is the unique solution of* $(\mathbf{P})_\rho^{\mu^*,\boldsymbol{w}^*}$ *and* $\boldsymbol{z}_\mu^*$, $\boldsymbol{h}_{\boldsymbol{w}}^* \in \boldsymbol{W}$ *are the unique solutions of the problems*

$$\begin{aligned} &a(\boldsymbol{z}_\mu^*, \boldsymbol{v}) + \langle \nabla^2\varphi_{\mu^*,\boldsymbol{w}^*}^\rho(\boldsymbol{u}^*) \cdot \boldsymbol{z}_\mu^*, \boldsymbol{v}\rangle + \langle \nabla\beta_\rho(\boldsymbol{u}^*) \cdot \boldsymbol{z}_\mu^*, \boldsymbol{v}\rangle \\ &= \langle \nabla\varphi_{\mu^*-\mu,\boldsymbol{w}^*}^\rho(\boldsymbol{u}^*), \boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \end{aligned} \tag{8.165}$$

and, respectively,

$$\begin{aligned} &a(\boldsymbol{h}_{\boldsymbol{w}}^*, \boldsymbol{v}) + \langle \nabla^2\varphi_{\mu^*,\boldsymbol{w}^*}^\rho(\boldsymbol{u}^*) \cdot \boldsymbol{h}_{\boldsymbol{w}}^*, \boldsymbol{v}\rangle + \langle \nabla\beta_\rho(\boldsymbol{u}^*) \cdot \boldsymbol{h}_{\boldsymbol{w}}^*, \boldsymbol{v}\rangle \\ &= -\langle \nabla\varphi_{\mu^*,\boldsymbol{w}}^\rho(\boldsymbol{u}^*), \boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V} \end{aligned} \tag{8.166}$$

Proof. Let $(\mu^*, \boldsymbol{w}^*) \in M \times \boldsymbol{W}$.

For every $t \in (0,1)$ and $\mu \in M$, let us denote $\mu_t = \mu^* + t(\mu - \mu^*)$ and $\boldsymbol{u}_t = \boldsymbol{u}_\rho^{\mu_t,\boldsymbol{w}^*}$. From $(\mathbf{P})_\rho^{\mu_t,\boldsymbol{w}^*}$, and using the positivity of $\nabla\varphi_{\mu_t,\boldsymbol{w}^*}^\rho$ and β_ρ, we get

$$\|\boldsymbol{u}_t\|_{\boldsymbol{V}} \leq C \tag{8.167}$$

with C a constant independent of t.

Putting $\boldsymbol{z}_t = \dfrac{\boldsymbol{u}_t - \boldsymbol{u}^*}{t}$ with $\boldsymbol{u}^* = \boldsymbol{u}_\rho^{\mu^*,\boldsymbol{w}^*}$, we obviously have $\boldsymbol{z}_t \in \boldsymbol{W}$, and, from $(\mathbf{P})_\rho^{\mu^*,\boldsymbol{w}^*}$ and $(\mathbf{P})_\rho^{\mu_t,\boldsymbol{w}^*}$, it follows that $\boldsymbol{z}_t$ satisfies

$$\begin{aligned} &a(\boldsymbol{z}_t, \boldsymbol{v}) + \frac{\langle \nabla\varphi_{\mu^*,\boldsymbol{w}^*}^\rho(\boldsymbol{u}^* + t\boldsymbol{z}_t) - \nabla\varphi_{\mu^*,\boldsymbol{w}^*}^\rho(\boldsymbol{u}^*), \boldsymbol{v}\rangle}{t} \\ &+ \frac{\langle \beta_\rho(\boldsymbol{u}^* + t\boldsymbol{z}_t) - \beta_\rho(\boldsymbol{u}^*), \boldsymbol{v}\rangle}{t} = \langle \nabla\varphi_{\mu^*-\mu,\boldsymbol{w}^*}^\rho(\boldsymbol{u}^* + t\boldsymbol{z}_t), \boldsymbol{v}\rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \end{aligned} \tag{8.168}$$

Taking $\boldsymbol{v} = \boldsymbol{u}_t - \boldsymbol{u}^*$ in (8.168), and thanks to the monotony of $\nabla\varphi_{\mu^*,\boldsymbol{w}^*}^\rho$ and β_ρ, we get

$$\begin{aligned} &\alpha\|\boldsymbol{u}_t - \boldsymbol{u}^*\|_{\boldsymbol{V}}^2 \leq a(\boldsymbol{u}_t - \boldsymbol{u}^*, \boldsymbol{u}_t - \boldsymbol{u}^*) \leq a(\boldsymbol{u}_t - \boldsymbol{u}^*, \boldsymbol{u}_t - \boldsymbol{u}^*) \\ &+\langle \nabla\varphi_{\mu^*,\boldsymbol{w}^*}^\rho(\boldsymbol{u}_t) - \nabla\varphi_{\mu^*,\boldsymbol{w}^*}^\rho(\boldsymbol{u}^*), \boldsymbol{u}_t - \boldsymbol{u}^*\rangle + \langle \beta_\rho(\boldsymbol{u}_t) - \beta_\rho(\boldsymbol{u}^*), \boldsymbol{u}_t - \boldsymbol{u}^*\rangle \\ &= t\langle \nabla\varphi_{\mu^*-\mu,\boldsymbol{w}^*}^\rho(\boldsymbol{u}_t), \boldsymbol{u}_t - \boldsymbol{u}^*\rangle \leq Ct\|\boldsymbol{u}_t - \boldsymbol{u}^*\|_{\boldsymbol{V}}\,\|\boldsymbol{u}_t\|_{\boldsymbol{V}} \end{aligned} \tag{8.169}$$

which, together with (8.167), gives

$$\|\boldsymbol{z}_t\|_{\boldsymbol{V}} \leq C. \tag{8.170}$$

Then, we can extract a subsequence $\{z_{t_k}\}_k$ such that $z_{t_k} \rightharpoonup z^*_\mu$ weakly in $\boldsymbol{V}$, and, thus, $\boldsymbol{u}_{t_k} \rightharpoonup \boldsymbol{u}^*$ weakly in $\boldsymbol{V}$. By passing to the limit in (8.168) with $t \to 0$, and taking into account that the operator $\nabla\varphi^\rho_{\mu^*-\mu,\boldsymbol{w}^*}$ is hemicontinuous, it follows that z^*_μ satisfies (8.165).

The uniqueness of the solution of (8.165) is immediate. Indeed, from the properties of $\nabla^2\varphi^\rho_{\mu,\boldsymbol{w}}$ and $\nabla\beta_\rho$, we have

$$\begin{aligned}&a(z_1-z_2,z_1-z_2) = -\langle\nabla^2\varphi^\rho_{\mu^*,\boldsymbol{w}^*}(\boldsymbol{u}^*)\cdot(z_1-z_2),z_1-z_2\rangle\\&-\langle\nabla\beta_\rho(\boldsymbol{u}^*)\cdot(z_1-z_2),z_1-z_2\rangle \le 0 \quad \forall z_1,\, z_2 \in \boldsymbol{W}\,.\end{aligned}$$

Now, we shall compute the Gâteaux differential of $J_{\epsilon\rho}$ at point $(\mu^*,\boldsymbol{w}^*)$ in the direction μ. We have

$$\begin{aligned}&\frac{\partial J_{\epsilon\rho}}{\partial\mu}(\mu^*,\boldsymbol{w}^*)\cdot(\mu-\mu^*) = \frac{1}{2}\lim_{t\to0}\left(\int_{\Gamma_2}\frac{(\boldsymbol{u}_t-\boldsymbol{u}_d)^2-(\boldsymbol{u}^*-\boldsymbol{u}_d)^2}{t}\,\mathrm{d}s\right.\\&\left.+\frac{1}{\epsilon}\frac{\|\boldsymbol{u}_t-\boldsymbol{w}^*\|^2_{\boldsymbol{W}}-\|\boldsymbol{u}^*-\boldsymbol{w}^*\|^2_{\boldsymbol{W}}}{t}+\frac{\|\mu_t-\mu^0\|^2_{L^2(\Gamma_2)}-\|\mu^*-\mu^0\|^2_{L^2(\Gamma_2)}}{t}\right)\\&=\lim_{t\to0}\left(\int_{\Gamma_2}(\boldsymbol{u}^*-\boldsymbol{u}_d)z_t\,\mathrm{d}s+\frac{t}{2}\int_{\Gamma_2}z_t^2\,\mathrm{d}s+\frac{1}{\epsilon}(\boldsymbol{u}^*-\boldsymbol{w}^*,z_t)_{\boldsymbol{W}}+\frac{t}{2\epsilon}\|z_t\|^2_{\boldsymbol{W}}\right.\\&\left.+(\mu^*-\mu^0,\mu-\mu^*)_{L^2(\Gamma_2)}+\frac{t}{2}\|\mu-\mu^*\|^2_{L^2(\Gamma_2)}\right)=\int_{\Gamma_2}z^*_\mu(\boldsymbol{u}^*-\boldsymbol{u}_d)\,\mathrm{d}s\\&+\frac{1}{\epsilon}(z^*_\mu,\boldsymbol{u}^*-\boldsymbol{w}^*)_{\boldsymbol{W}}+(\mu-\mu^*,\mu^*-\mu^0)_{L^2(\Gamma_2)}\quad\forall\mu\in M\,,\end{aligned}$$

i.e. (8.163).

Next, for every $t \in (0,1)$ and $\boldsymbol{w} \in \boldsymbol{W}$, we met $\boldsymbol{w}_t = \boldsymbol{w}^* + t\boldsymbol{w}$ and $\boldsymbol{h}_t = \dfrac{\tilde{\boldsymbol{u}}_t - \boldsymbol{u}^*}{t}$ with $\tilde{\boldsymbol{u}}_t = \boldsymbol{u}_\rho^{\mu^*,\boldsymbol{w}_t}$. It is easy to prove that $\boldsymbol{h}_t$ satisfies

$$\begin{aligned}&a(\boldsymbol{h}_t,\boldsymbol{v})+\frac{\langle\nabla\varphi^\rho_{\mu^*,\boldsymbol{w}^*}(\boldsymbol{u}^*+t\boldsymbol{h}_t)-\nabla\varphi^\rho_{\mu^*,\boldsymbol{w}^*}(\boldsymbol{u}^*),\boldsymbol{v}\rangle}{t}\\&+\frac{\langle\beta_\rho(\boldsymbol{u}^*+t\boldsymbol{h}_t)-\beta_\rho(\boldsymbol{u}^*),\boldsymbol{v}\rangle}{t}=-\langle\nabla\varphi^\rho_{\mu^*,\boldsymbol{w}}(\boldsymbol{u}^*+t\boldsymbol{h}_t),\boldsymbol{v}\rangle\quad\forall\boldsymbol{v}\in\boldsymbol{V}\,.\end{aligned}$$

Hence, by proceeding as in the first part of this proof, we deduce (8.164) where the weak limit $\boldsymbol{h}^*_w$ of the sequence $\{\boldsymbol{h}_t\}_t$ in $\boldsymbol{W}$ satisfies (8.166). □

The main result of this section gives the necessary optimality conditions for each problem $(\mathbf{CP})_{\epsilon\rho}$.

Theorem 8.14. *Let $(\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}) \in M \times \boldsymbol{W}$ be a solution of the control problem $(\mathbf{CP})_{\epsilon\rho}$. Then, there exist the unique elements $(\boldsymbol{u}^*_{\epsilon\rho},\boldsymbol{p}^*_{\epsilon\rho}) \in \boldsymbol{W}\times\boldsymbol{W}$ such that*

$$a(\boldsymbol{u}^*_{\epsilon\rho},\boldsymbol{v})+\langle\nabla\varphi^\rho_{\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}(\boldsymbol{u}^*_{\epsilon\rho}),\boldsymbol{v}\rangle+\langle\beta_\rho(\boldsymbol{u}^*_{\epsilon\rho}),\boldsymbol{v}\rangle=(\boldsymbol{F},\boldsymbol{v})\quad\forall\boldsymbol{v}\in\boldsymbol{V}\,,\tag{8.171}$$

$$\begin{aligned}&a(\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{v})+\langle\nabla^2\varphi^{\rho}_{\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}(\boldsymbol{u}^*_{\epsilon\rho})\cdot\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{v}\rangle\\&+\langle\nabla\varphi^{\rho}_{\mu^*_{\epsilon\rho},\boldsymbol{v}}(\boldsymbol{u}^*_{\epsilon\rho})\cdot\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{v}\rangle+\langle\nabla\beta_\rho(\boldsymbol{u}^*_{\epsilon\rho})\cdot\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{v}\rangle\\&=-\int_{\Gamma_2}(\boldsymbol{u}^*_{\epsilon\rho}-\boldsymbol{u}_d)\boldsymbol{v}\,\mathrm{d}s\quad\forall\boldsymbol{v}\in\boldsymbol{W}\,,\end{aligned}\tag{8.172}$$

$$\langle\nabla\varphi^{\rho}_{\mu-\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}(\boldsymbol{u}^*_{\epsilon\rho}),\boldsymbol{p}^*_{\epsilon\rho}\rangle+(\mu-\mu^*_{\epsilon\rho},\mu^*_{\epsilon\rho}-\mu^0)_{L^2(\Gamma_2)}\geq 0\quad\forall\mu\in M\,.\tag{8.173}$$

Proof. Let $\boldsymbol{u}^*_{\epsilon\rho}$ be the unique solution of (8.171), i.e. $\boldsymbol{u}^*_{\epsilon\rho}=\boldsymbol{u}_\rho^{\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$. We denote by $\boldsymbol{p}^*_{\epsilon\rho}$ the unique solution (we remark that $\nabla^2\varphi^\rho_{\mu,\boldsymbol{w}}(\boldsymbol{u})+\nabla\beta_\rho(\boldsymbol{u})\in\mathscr{L}(\boldsymbol{W},\boldsymbol{W}^*)$ is a positive operator) of the following problem

$$\begin{aligned}&a(\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{v})+\langle\nabla^2\varphi^{\rho}_{\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}(\boldsymbol{u}^*_{\epsilon\rho})\cdot\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{v}\rangle+\langle\nabla\beta_\rho(\boldsymbol{u}^*_{\epsilon\rho})\cdot\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{v}\rangle\\&=-\int_{\Gamma_2}(\boldsymbol{u}^*_{\epsilon\rho}-\boldsymbol{u}_d)\boldsymbol{v}\,\mathrm{d}s-\frac{1}{\epsilon}(\boldsymbol{u}^*_{\epsilon\rho}-\boldsymbol{w}^*_{\epsilon\rho},\boldsymbol{v})_{\boldsymbol{W}}\quad\forall\boldsymbol{v}\in\boldsymbol{W}\,.\end{aligned}\tag{8.174}$$

Since $(\mu^*_{\epsilon\rho},\boldsymbol{u}^*_{\epsilon\rho})$ is a solution of $(\mathbf{CP})_{\epsilon\rho}$, it follows

$$\begin{aligned}&\frac{\partial J_{\epsilon\rho}}{\partial\mu}(\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho})\cdot(\mu-\mu^*_{\epsilon\rho})\geq 0\quad\forall\mu\in M\,,\\&\frac{\partial J_{\epsilon\rho}}{\partial\boldsymbol{w}}(\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho})\cdot\boldsymbol{w}=0\quad\forall\boldsymbol{w}\in\boldsymbol{W}\,.\end{aligned}$$

Therefore, from (8.163), by using the relation (8.174) with $\boldsymbol{v}=\boldsymbol{z}_\mu$ and the relation (8.165) with $\boldsymbol{v}=\boldsymbol{p}^*_{\epsilon\rho}$, we get

$$\begin{aligned}&\frac{\partial J_{\epsilon\rho}}{\partial\mu}(\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho})\cdot(\mu-\mu^*_{\epsilon\rho})=\int_{\Gamma_2}\boldsymbol{z}^*_\mu(\boldsymbol{u}^*_{\epsilon\rho}-\boldsymbol{u}_d)\,\mathrm{d}s+\frac{1}{\epsilon}(\boldsymbol{z}^*_\mu,\boldsymbol{u}^*_{\epsilon\rho}-\boldsymbol{w}^*_{\epsilon\rho})_{\boldsymbol{W}}\\&+(\mu-\mu^*_{\epsilon\rho},\mu^*_{\epsilon\rho}-\mu^0)_{L^2(\Gamma_2)}=-a(\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{z}^*_\mu)\\&-\langle\nabla^2\varphi^{\rho}_{\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}(\boldsymbol{u}^*_{\epsilon\rho})\cdot\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{z}_\mu\rangle-\langle\nabla\beta_\rho(\boldsymbol{u}^*_{\epsilon\rho})\cdot\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{z}_\mu\rangle\\&+(\mu-\mu^*_{\epsilon\rho},\mu^*_{\epsilon\rho}-\mu^0)_{L^2(\Gamma_2)}=\langle\nabla\varphi^{\rho}_{\mu-\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}(\boldsymbol{u}^*_{\epsilon\rho}),\boldsymbol{p}^*_{\epsilon\rho}\rangle\\&+(\mu-\mu^*_{\epsilon\rho},\mu^*_{\epsilon\rho}-\mu^0)_{L^2(\Gamma_2)}\geq 0\quad\forall\mu\in M\,,\end{aligned}\tag{8.175}$$

i.e. (8.173).

Finally, taking $\boldsymbol{v}=\boldsymbol{h}_{\boldsymbol{w}}$ in (8.174) and $\boldsymbol{v}=\boldsymbol{p}^*_{\epsilon\rho}$ in (8.166), from (8.164), we obtain

$$\begin{aligned}&\frac{\partial J_{\epsilon\rho}}{\partial\boldsymbol{w}}(\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho})\cdot\boldsymbol{w}=\int_{\Gamma_2}\boldsymbol{h}^*_{\boldsymbol{w}}(\boldsymbol{u}^*_{\epsilon\rho}-\boldsymbol{u}_d)\,\mathrm{d}s+\frac{1}{\epsilon}(\boldsymbol{h}^*_{\boldsymbol{w}}-\boldsymbol{w},\boldsymbol{u}^*_{\epsilon\rho}-\boldsymbol{w}^*_{\epsilon\rho})_{\boldsymbol{W}}\\&=-a(\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{h}^*_{\boldsymbol{w}})-\langle\nabla^2\varphi^{\rho}_{\mu^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}(\boldsymbol{u}^*_{\epsilon\rho})\cdot\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{h}_{\boldsymbol{w}}\rangle\\&-\langle\nabla\beta_\rho(\boldsymbol{u}^*_{\epsilon\rho})\cdot\boldsymbol{p}^*_{\epsilon\rho},\boldsymbol{h}_{\boldsymbol{w}}\rangle-\frac{1}{\epsilon}(\boldsymbol{w},\boldsymbol{u}^*_{\epsilon\rho}-\boldsymbol{w}^*_{\epsilon\rho})_{\boldsymbol{W}}\\&=\langle\nabla\varphi^{\rho}_{\mu^*_{\epsilon\rho},\boldsymbol{w}}(\boldsymbol{u}^*_{\epsilon\rho}),\boldsymbol{p}^*_{\epsilon\rho}\rangle-\frac{1}{\epsilon}(\boldsymbol{w},\boldsymbol{u}^*_{\epsilon\rho}-\boldsymbol{w}^*_{\epsilon\rho})_{\boldsymbol{W}}=0\quad\forall\boldsymbol{w}\in\boldsymbol{W}\,,\end{aligned}\tag{8.176}$$

which, together with (8.174), leads to (8.172). □

We finally remark that, taking into account the convergences given by Propositions 8.6 and 8.7, Theorem 8.14 gives an approximating process for computing the optimal control μ^0 of the problem (**CP**).

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Chapter 9
Quasistatic Problems

This chapter deals with the study of quasistatic contact problems with a nonlocal Coulomb friction law. We first consider that the unilateral contact is modeled by the Signorini conditions. In this case, a variational formulation (see [7]) involves two inequalities with the simultaneous presence of the displacement field and of the velocity field. More precisely, the friction law generates an inequality with the velocity field as test function while the Signorini conditions lead to an inequality with the displacement field as test function. Applying Theorem 4.19 (p. 77), a known existence result (see [7]) is provided. We then prove, following the work [5], convergence results for a space finite element approximation and an implicit time discretization scheme of this problem. The last section is devoted, as in the work [6], to the study of a boundary control problem related to a quasistatic bilateral contact problem with nonlocal Coulomb friction.

Concerning the study of quasistatic contact problems in elasticity, we mention the existence and/or uniqueness results obtained, in the case of a normal compliance law, by Andersson [3] and Klarbring et al. [9], and, in the case of a local or nonlocal Coulomb law with unilateral contact, by Cocu et al. [7], Andersson [4], Cocou and Roca [8], Rocca [14]. For the study of quasistatic bilateral contact problems involving viscoelastic or viscoplastic materials, we refer to Shillor and Sofonea [15], Shillor et al. [16] and Amassad [1].

9.1 Classical and Variational Formulations

The quasistatic evolutionary of an elastic body in unilateral contact with a rigid foundation is considered. We suppose that the volume forces $\boldsymbol{f} = \boldsymbol{f}(\boldsymbol{x}, t)$ and the surface tractions $\boldsymbol{g} = \boldsymbol{g}(\boldsymbol{x}, t)$ are applied so slowly that the inertial forces may be neglected.

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With the notation adopted in Sect. 8.1, the classical formulation of the quasistatic problem is obtained, as in the static case, by considering the equilibrium equations, the constitutive equation, the kinematic relation, the boundary conditions, and the initial condition.

Problem ($\mathscr{Q}$): Find a displacement field $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t) : \Omega \times [0, T] \to \mathbb{R}^d$ such that

$$-\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega \times (0, T)\,, \tag{9.1}$$

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}) = \mathscr{A}\boldsymbol{\epsilon}\,, \qquad \boldsymbol{\epsilon} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T) \quad \text{in } \Omega \times (0, T)\,, \tag{9.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0 \times (0, T)\,, \tag{9.3}$$

$$\boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_1 \times (0, T)\,, \tag{9.4}$$

$$u_\nu \le 0\,, \quad \sigma_\nu \le 0\,, \quad u_\nu \sigma_\nu = 0 \quad \text{on } \Gamma_2 \times (0, T)\,, \tag{9.5}$$

$$|\boldsymbol{\sigma}_\tau| \le \mu |\sigma_\nu| \text{ and } \begin{cases} |\boldsymbol{\sigma}_\tau| < \mu |\mathscr{R}\sigma_\nu| \;\Rightarrow\; \dot{\boldsymbol{u}}_\tau = 0 \\ |\boldsymbol{\sigma}_\tau| = \mu |\mathscr{R}\sigma_\nu| \;\Rightarrow\; \exists \lambda \ge 0\,,\ \dot{\boldsymbol{u}}_\tau = -\lambda \boldsymbol{\sigma}_\tau \end{cases} \text{on } \Gamma_2 \times (0, T)\,, \tag{9.6}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega\,. \tag{9.7}$$

where $\mathscr{A} = (a_{ijkh})$ is the fourth order tensor of elasticity with the elasticity coefficients satisfying the symmetry and ellipticity conditions:

$$\begin{aligned} &a_{ijkh} = a_{jihk} = a_{khij}\,,\ \forall 1 \le i, j, k, h \le d\,, \\ &\exists \alpha > 0 \text{ tel que } a_{ijkh}\xi_{ij}\xi_{kh} \ge \alpha |\boldsymbol{\xi}|^2\,,\ \forall \boldsymbol{\xi} = (\xi_{ij}) \in \mathbb{R}^{d^2}\,. \end{aligned} \tag{9.8}$$

In order to derive a variational formulation of the problem (9.1)–(9.7), we suppose that

$$\begin{aligned} &\boldsymbol{f} \in W^{1,2}(0, T; (L^2(\Omega))^d)\,, \\ &\boldsymbol{g} \in W^{1,2}(0, T; (L^2(\Gamma_1))^d)\,, \\ &a_{ijkl} \in L^\infty(\Omega),\ i, j, k, l = 1, \dots, d, \\ &\mu \in L^\infty(\Gamma_2),\ \mu \ge 0 \text{ a.e. on } \Gamma_2 \\ &\mathscr{R} : H^{-1/2}(\Gamma_2) \to L^2(\Gamma_2) \text{ is a linear continuous operator.} \end{aligned} \tag{9.9}$$

We shall use the notation

$$\begin{aligned} \boldsymbol{V} &= \{\boldsymbol{v} \in (H^1(\Omega))^d\,;\ \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_0\}\,, \\ \boldsymbol{K} &= \{\boldsymbol{v} \in \boldsymbol{V}\,;\ v_\nu \le 0 \text{ a.e. on } \Gamma_2\}\,, \\ a(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega \boldsymbol{\sigma}(\boldsymbol{u}) \boldsymbol{\epsilon}(\boldsymbol{v})\, dx \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}\,. \end{aligned} \tag{9.10}$$

Let $\boldsymbol{F} \in W^{1,2}(0, T; \boldsymbol{V})$ where, for all $t \in [0, T]$, $\boldsymbol{F}(t)$ is the element of $\boldsymbol{V}$ defined by

$$(\boldsymbol{F}(t), \boldsymbol{v}) = \int_{\Omega} \boldsymbol{f}(t) \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}(t) \cdot \boldsymbol{v} \, \mathrm{d}s \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{9.11}$$

where we have denoted by $(\cdot, \cdot)$ the inner product over the space $\boldsymbol{V}$.

We also put

$$\boldsymbol{W} = \{\boldsymbol{w} \in \boldsymbol{V};\ \mathrm{div}\, \boldsymbol{\sigma}(\boldsymbol{w}) \in (L^2(\Omega))^d\}. \tag{9.12}$$

For simplicity, we denote by $\langle \cdot, \cdot \rangle$ the duality pairing between $(H^{-1/2}(\Gamma_2))^d$ and $(H^{1/2}(\Gamma_2))^d$ or between $H^{-1/2}(\Gamma_2)$ and $H^{1/2}(\Gamma_2)$. Then, as we have precise in Sect. 8.1, we have

$$\langle \boldsymbol{\sigma}(\boldsymbol{w}) \cdot \boldsymbol{\nu}, \boldsymbol{v} \rangle = \int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{w}) \boldsymbol{\epsilon}(\bar{\boldsymbol{v}}) \, \mathrm{d}x + \int_{\Omega} \mathrm{div}\, \boldsymbol{\sigma}(\boldsymbol{w}) \bar{\boldsymbol{v}} \, \mathrm{d}x \quad \forall \boldsymbol{w} \in \boldsymbol{W}, \ \forall \boldsymbol{v} \in (H^{1/2}(\Gamma_2))^d$$

where $\bar{\boldsymbol{v}} \in (H^1(\Omega))^d$ satisfies $\bar{\boldsymbol{v}} = \boldsymbol{v}$ almost everywhere on Γ_2.

Therefore, we define the normal component of the stress tensor $\sigma_\nu(\boldsymbol{w}) \in H^{-1/2}(\Gamma_2)$ by

$$\langle \sigma_\nu(\boldsymbol{w}), v \rangle = \int_{\Omega} \boldsymbol{\sigma}(\boldsymbol{w}) \boldsymbol{\epsilon}(\bar{\boldsymbol{v}}) \, \mathrm{d}x + \int_{\Omega} \mathrm{div}\, \boldsymbol{\sigma}(\boldsymbol{w}) \bar{\boldsymbol{v}} \, \mathrm{d}x \quad \forall \boldsymbol{w} \in \boldsymbol{W}, \quad \forall v \in H^{1/2}(\Gamma_2)$$

where $\bar{\boldsymbol{v}} \in (H^1(\Omega))^d$ satisfies $\bar{\boldsymbol{v}}_\tau = \boldsymbol{0}$ and $\bar{v}_\nu = v$ a.e. on Γ_2.

It is easy to verify that, for any $\boldsymbol{w} \in \boldsymbol{W}$, the above definitions of $\boldsymbol{\sigma}(\boldsymbol{w}) \cdot \boldsymbol{\nu}$ and $\sigma_\nu(\boldsymbol{w})$ are independent on the choice of $\bar{\boldsymbol{v}}$.

For all $\boldsymbol{\Theta} \in \boldsymbol{V}$, we introduce the functional $\tilde{j}_{\boldsymbol{\Theta}} : \boldsymbol{K}(\boldsymbol{\Theta}) \times \boldsymbol{V} \to \mathbb{R}$ defined by

$$\tilde{j}_{\boldsymbol{\Theta}}(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_2} \mu |\mathscr{R} \sigma_\nu(\boldsymbol{u})| \, |\boldsymbol{v}_t| \, \mathrm{d}s \quad \forall \boldsymbol{u} \in \boldsymbol{K}(\boldsymbol{\Theta}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{9.13}$$

where

$$\boldsymbol{K}(\boldsymbol{\Theta}) = \{\boldsymbol{w} \in \boldsymbol{K};\ a(\boldsymbol{w}, \boldsymbol{\psi}) = (\boldsymbol{\Theta}, \boldsymbol{\psi}),\ \forall \boldsymbol{\psi} \in \boldsymbol{V} \text{ such that } \boldsymbol{\psi} = 0 \text{ a.e. on } \Gamma_2\}.$$

A variational formulation of this problem (see [7]) involves two inequalities and the simultaneous presence of the displacement field and of the velocity field. More precisely, the friction law generates an inequality with the velocity field as test function while the Signorini conditions lead to an inequality with the displacement field as test function. So, we shall consider the following weak formulation of Problem $(\mathscr{Q})$.

Problem (Q): Find $\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V})$ such that

$$\begin{cases} \boldsymbol{u}(0) = \boldsymbol{u}_0\,, \quad \boldsymbol{u}(t) \in \boldsymbol{K} \quad \forall t \in [0,T] \\ a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + \tilde{j}_{\boldsymbol{F}(t)}(\boldsymbol{u}(t), \boldsymbol{v}) - \tilde{j}_{\boldsymbol{F}(t)}(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \\ \geq (\boldsymbol{F}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + \langle \sigma_\nu(\boldsymbol{u}(t)), v_\nu - \dot{u}_\nu(t) \rangle \quad \forall \boldsymbol{v} \in \boldsymbol{V} \quad \text{a.e. } t \in (0,T) \\ \langle \sigma_\nu(\boldsymbol{u}(t)), z_\nu - u_\nu(t) \rangle \geq 0 \quad \forall \boldsymbol{z} \in \boldsymbol{K}\,,\ \forall t \in (0,T)\,. \end{cases} \tag{9.14}$$

Remark 9.1. If $\boldsymbol{u}$ verifies the first inequality of Problem (**Q**), then $\boldsymbol{u}(t) \in \boldsymbol{K}(\boldsymbol{F}(t))$, $\forall t \in [0,T]$.

We suppose that the initial displacement $\boldsymbol{u}_0 \in \boldsymbol{K}$ satisfies the following compatibility condition

$$a(\boldsymbol{u}_0, \boldsymbol{v}) + \tilde{j}_{\boldsymbol{F}(0)}(\boldsymbol{u}_0, \boldsymbol{v}) \geq (\boldsymbol{F}(0), \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{K}\,. \tag{9.15}$$

In order to show that the classical formulation ($\mathscr{Q}$) and the variational formulation (**Q**) are equivalent, we first prove the following result.

Lemma 9.1. *Let $\tilde{\boldsymbol{u}} \in \boldsymbol{K} \cap \boldsymbol{W}$ be a regular function. Then, the following two conditions are equivalent:*

$$\tilde{u}_\nu \leq 0\,,\ \sigma_\nu(\tilde{\boldsymbol{u}}) \leq 0\,,\ \tilde{u}_\nu\, \sigma_\nu(\tilde{\boldsymbol{u}}) = 0 \quad \text{on } \Gamma_2 \tag{9.16}$$

$$\langle \sigma_\nu(\tilde{\boldsymbol{u}}), z_\nu - \tilde{u}_\nu \rangle \geq 0 \quad \forall \mathbf{z} \in \boldsymbol{K}\,. \tag{9.17}$$

Proof. If the unilateral contact conditions (9.16) hold, then we have

$$\langle \sigma_\nu(\tilde{\boldsymbol{u}}), z_\nu - \tilde{u}_\nu \rangle = \langle \sigma_\nu(\tilde{\boldsymbol{u}}), z_\nu \rangle - \langle \sigma_\nu(\tilde{\boldsymbol{u}}), \tilde{u}_\nu \rangle = \langle \sigma_\nu(\tilde{\boldsymbol{u}}), z_\nu \rangle \geq 0 \quad \forall \mathbf{z} \in \boldsymbol{K}\,.$$

Conversely, if (9.17) is satisfied, then, by taking $\mathbf{z} = \mathbf{0}$ and $\mathbf{z} = 2\boldsymbol{u}$, we obtain

$$\langle \sigma_\nu(\tilde{\boldsymbol{u}}), \tilde{u}_\nu \rangle = 0\,, \tag{9.18}$$

and hence, by the inequality (9.17), we get

$$\langle \sigma_\nu(\tilde{\boldsymbol{u}}), z_\nu \rangle \geq 0 \quad \forall \mathbf{z} \in \boldsymbol{K}\,. \tag{9.19}$$

Finally, from the relations (9.18), (9.19) and the definition of $\boldsymbol{K}$, we conclude the proof. □

Following the standard procedure, we derive the next result.

Theorem 9.1. *The mechanical problem ($\mathscr{Q}$) is formally equivalent to the weak formulation* (**Q**) *in the following sense:*

(i) If $\boldsymbol{u}$ is a sufficiently smooth function which verifies the mechanical problem (9.1)–(9.7), then $\boldsymbol{u}$ is a solution of the variational problem (9.14).

(ii) If $\boldsymbol{u}$ is a regular solution of the variational problem (9.14), then $\boldsymbol{u}$ verifies (9.1)–(9.7) in the distributional sense.

Proof. For simplicity, we shall omit the variable t.

(i) Multiplying Eq. (9.1) by $\boldsymbol{v}-\dot{\boldsymbol{u}}$ with $\boldsymbol{v}\in\boldsymbol{V}$ and integrating by parts over Ω, we obtain

$$a(\boldsymbol{u},\boldsymbol{v}-\dot{\boldsymbol{u}})-\int_{\Gamma}\boldsymbol{\sigma}\cdot\boldsymbol{\nu}(\boldsymbol{v}-\dot{\boldsymbol{u}})\,\mathrm{d}s=\int_{\Omega}\boldsymbol{f}(\boldsymbol{v}-\dot{\boldsymbol{u}})\,\mathrm{d}x\quad\forall\boldsymbol{v}\in\boldsymbol{V},$$

and so, by using (9.3) and (9.4), we get

$$a(\boldsymbol{u},\boldsymbol{v}-\dot{\boldsymbol{u}})-\int_{\Gamma_2}(\sigma_\nu(v_\nu-\dot{u}_\nu)+\boldsymbol{\sigma}_\tau(\boldsymbol{v}_\tau-\dot{\boldsymbol{u}}_\tau))\,\mathrm{d}s=(\boldsymbol{F},\boldsymbol{v}-\dot{\boldsymbol{u}})\quad\forall\boldsymbol{v}\in\boldsymbol{V}.\tag{9.20}$$

Hence, for $\boldsymbol{v}=\boldsymbol{\psi}+\dot{\boldsymbol{u}}$ with $\boldsymbol{\psi}\in\boldsymbol{V}$ such that $\boldsymbol{\psi}=0$ a.e. on Γ_2, we deduce that $\boldsymbol{u}\in\boldsymbol{K}(\boldsymbol{F})$.

On the other hand, the Coulomb friction law (9.6) implies

$$\tilde{j}_F(\boldsymbol{u},\boldsymbol{v})-\tilde{j}_F(\boldsymbol{u},\dot{\boldsymbol{u}})+\int_{\Gamma_2}\boldsymbol{\sigma}_\tau(\boldsymbol{v}_\tau-\dot{\boldsymbol{u}}_\tau)\,\mathrm{d}s\geq 0\quad\forall\boldsymbol{v}\text{ smooth function}.\tag{9.21}$$

Indeed, let us denote $E=\mu|\mathscr{R}\sigma_\nu|(|\boldsymbol{v}_\tau|-|\dot{\boldsymbol{u}}_\tau|)+\boldsymbol{\sigma}_\tau(\boldsymbol{v}_\tau-\dot{\boldsymbol{u}}_\tau)$.
If $|\boldsymbol{\sigma}_\tau|<\mu|\mathscr{R}\sigma_\nu|$, then $\dot{\boldsymbol{u}}_\tau=\boldsymbol{0}$, and hence

$$E\geq-|\boldsymbol{\sigma}_\tau|\,|\boldsymbol{v}_\tau|+\mu|\mathscr{R}\sigma_\nu|\,|\boldsymbol{v}_\tau|\geq 0.$$

If $|\boldsymbol{\sigma}_\tau|=\mu|\mathscr{R}\sigma_\nu|$, then we have $\dot{\boldsymbol{u}}_\tau=-\lambda\boldsymbol{\sigma}_\tau$, and so

$$E=\boldsymbol{\sigma}_\tau\boldsymbol{v}_\tau+|\boldsymbol{\sigma}_\tau|\,|\boldsymbol{v}_\tau|\geq 0.$$

Combining (9.20) and (9.21), we deduce that $\boldsymbol{u}$ verifies the first inequality of (9.14).

The second inequality of (9.14) is obtained from (9.5) and Lemma 9.1 for $\tilde{\boldsymbol{u}}=\boldsymbol{u}$.

(ii) If we take $\boldsymbol{v}=\dot{\boldsymbol{u}}\pm\boldsymbol{\varphi}$ in the first inequality of Problem (**Q**), with $\boldsymbol{\varphi}\in(\mathscr{D}(\Omega))^d$ and we apply Green's formula (8.7), then we obtain (9.1) in the distributional sense.

It is immediate, from Lemma 9.1 and the second inequality of (9.14), that the Signorini contact conditions (9.5) are satisfied.

In order to obtain (9.4), we multiply the relation (9.1) by $\boldsymbol{v}-\dot{\boldsymbol{u}}$ with $\boldsymbol{v}\in\boldsymbol{V}$, and so, by integrating by parts and using the first inequality of (9.14), we obtain

$$\tilde{j}_F(\boldsymbol{u},\boldsymbol{v})-\tilde{j}_F(\boldsymbol{u},\dot{\boldsymbol{u}})+\int_{\Gamma}(\boldsymbol{\sigma}\cdot\boldsymbol{\nu})(\boldsymbol{v}-\dot{\boldsymbol{u}})\,\mathrm{d}s-\int_{\Gamma_1}\boldsymbol{g}(\boldsymbol{v}-\dot{\boldsymbol{u}})\,\mathrm{d}s$$
$$\geq\langle\sigma_\nu(\boldsymbol{u}),v_\nu-\dot{u}_\nu\rangle\quad\forall\boldsymbol{v}\in\boldsymbol{V}.\tag{9.22}$$

By choosing $\boldsymbol{v} = \dot{\boldsymbol{u}} \pm \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in (C^\infty(\Omega))^d$ and supp $\boldsymbol{\varphi} \subset \Gamma_1$, we deduce

$$\int_{\Gamma_1} ((\boldsymbol{\sigma} \cdot \boldsymbol{\nu}) - \boldsymbol{g}) \cdot \boldsymbol{\varphi} \, \mathrm{d}s = 0,$$

that is the relation (9.4). Thus, the relation (9.22) becomes

$$\tilde{j}_{\boldsymbol{F}}(\boldsymbol{u}, \boldsymbol{v}) - \tilde{j}_{\boldsymbol{F}}(\boldsymbol{u}, \dot{\boldsymbol{u}}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau (\boldsymbol{v}_\tau - \dot{\boldsymbol{u}}_\tau) \, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{V} . \tag{9.23}$$

We now take $\boldsymbol{v} \in \boldsymbol{V}$ such that $\boldsymbol{v}_\tau = \pm\delta\boldsymbol{\varphi}$ with $\delta \in \mathbb{R}_+$, $\boldsymbol{\varphi} \in (C^\infty(\Omega))^d$ and supp $\boldsymbol{\varphi} \subset \Gamma_2$. As $\boldsymbol{\sigma}_\tau \boldsymbol{v}_\tau = \pm\delta\boldsymbol{\sigma}_\tau\boldsymbol{\varphi}_\tau = \pm\delta\boldsymbol{\sigma}_\tau\boldsymbol{\varphi}$, we obtain

$$\delta \int_{\Gamma_2} (\mu|\mathscr{R}\sigma_\nu|\,|\boldsymbol{\varphi}| \pm \boldsymbol{\sigma}_\tau\boldsymbol{\varphi}) \, \mathrm{d}s - \int_{\Gamma_2} (\mu|\mathscr{R}\sigma_\nu|\,|\dot{\boldsymbol{u}}_\tau| + \boldsymbol{\sigma}_\tau\dot{\boldsymbol{u}}_\tau) \, \mathrm{d}s \geq 0 \quad \forall \delta \geq 0$$

which gives

$$\begin{cases} \displaystyle\int_{\Gamma_2} (\pm\boldsymbol{\sigma}_\tau\boldsymbol{\varphi} + \mu|\mathscr{R}\sigma_\nu|\,|\boldsymbol{\varphi}|) \, \mathrm{d}s \geq 0 \\ \displaystyle\int_{\Gamma_2} (\boldsymbol{\sigma}_\tau\dot{\boldsymbol{u}}_\tau + \mu|\mathscr{R}\sigma_\nu|\,|\dot{\boldsymbol{u}}_\tau|) \, \mathrm{d}s \leq 0 \end{cases}$$

or, equivalently to

$$|\boldsymbol{\sigma}_\tau| \leq \mu|\mathscr{R}\sigma_\nu| \tag{9.24}$$

and

$$\boldsymbol{\sigma}_\tau\dot{\boldsymbol{u}}_\tau + \mu|\mathscr{R}\sigma_\nu|\,|\dot{\boldsymbol{u}}_\tau| \leq 0 . \tag{9.25}$$

It is easy to see that the relations (9.25) and (9.24) give

$$\boldsymbol{\sigma}_\tau\dot{\boldsymbol{u}}_\tau + \mu|\mathscr{R}\sigma_\nu|\,|\dot{\boldsymbol{u}}_\tau| = 0 . \tag{9.26}$$

Indeed, if $|\boldsymbol{\sigma}_\tau| < \mu|\mathscr{R}\sigma_\nu|$, then, supposing that $\dot{\boldsymbol{u}}_\tau \neq 0$, it follows that $0 > \boldsymbol{\sigma}_\tau\dot{\boldsymbol{u}}_\tau + |\boldsymbol{\sigma}_\tau|\,|\dot{\boldsymbol{u}}_\tau| \geq 0$, which is a contradiction. It follows that $\dot{\boldsymbol{u}}_\tau = \boldsymbol{0}$.

If $|\boldsymbol{\sigma}_\tau| = \mu|\mathscr{R}\sigma_\nu|$, then it follows that $0 = \boldsymbol{\sigma}_\tau\dot{\boldsymbol{u}}_\tau + |\boldsymbol{\sigma}_\tau|\,|\dot{\boldsymbol{u}}_\tau|$, and so, there exists $\lambda > 0$ such that $\dot{\boldsymbol{u}}_\tau = -\lambda\boldsymbol{\sigma}_\tau$. Therefore, the friction conditions (9.6) are satisfied and, by taking into account that $\boldsymbol{u}(0) = \boldsymbol{u}_0$ et $\boldsymbol{u}(t) \in \boldsymbol{K}$ for all $t \in [0, T]$, we conclude the proof. □

Using an implicit time discretization scheme (as in Sect. 4.3, p. 69), we obtain the following sequence $\{(\mathbf{Q})_{\mathbf{n}}^{\mathbf{i}}\}_{i=0,1,\ldots,n-1}$ of incremental formulations.

Problem $(\mathbf{Q})_\mathbf{n}^\mathbf{i}$: Find $\boldsymbol{u}^{i+1} \in \boldsymbol{K}^{i+1}$ such that

$$\begin{cases} a(\boldsymbol{u}^{i+1}, \boldsymbol{v} - \partial \boldsymbol{u}^i) + \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{v}) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \partial \boldsymbol{u}^i) \\ \quad \geq (\boldsymbol{F}^{i+1}, \boldsymbol{v} - \partial \boldsymbol{u}^i) + \langle \sigma_\nu(\boldsymbol{u}^{i+1}), v_\nu - \partial u_\nu^i \rangle \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}, \\ \langle \sigma_\nu(\boldsymbol{u}^{i+1}), z_\nu - u_\nu^{i+1} \rangle \geq 0 \quad \forall\, \boldsymbol{z} \in \boldsymbol{K} \end{cases} \tag{9.27}$$

where $\boldsymbol{K}^{i+1} = \boldsymbol{K}(\boldsymbol{F}^{i+1})$ and $\boldsymbol{u}^0 = \boldsymbol{u}_0$. By setting $\boldsymbol{w} = \boldsymbol{v}\Delta t + \boldsymbol{u}^i$, we deduce that the problem $(\mathbf{Q})_\mathbf{n}^\mathbf{i}$ is equivalent to the following problem $(\tilde{\mathbf{Q}})_\mathbf{n}^\mathbf{i}$.

Problem $(\tilde{\mathbf{Q}})_\mathbf{n}^\mathbf{i}$: Find $\boldsymbol{u}^{i+1} \in \boldsymbol{K}^{i+1}$ such that

$$\begin{cases} a(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) + \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) \\ \quad \geq (\boldsymbol{F}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) + \langle \sigma_\nu(\boldsymbol{u}^{i+1}), w_\nu - u_\nu^{i+1} \rangle \quad \forall\, \boldsymbol{w} \in \boldsymbol{V}, \\ \langle \sigma_\nu(\boldsymbol{u}^{i+1}), z_\nu - u_\nu^{i+1} \rangle \geq 0 \quad \forall\, \boldsymbol{z} \in \boldsymbol{K}. \end{cases} \tag{9.28}$$

In order to obtain an existence result for the problem **(Q)** (by applying Theorem 4.19), we first prove the following equivalence result which states that the hypothesis (4.105) of Theorem 4.19 is satisfied.

Theorem 9.2. *For all $i \in \{0, \ldots, n-1\}$, the problem $(\tilde{\mathbf{Q}})_\mathbf{n}^\mathbf{i}$ is equivalent to the problem $(\tilde{\mathbf{R}})_\mathbf{n}^\mathbf{i}$ defined below.*

Problem $(\tilde{\mathbf{R}})_\mathbf{n}^\mathbf{i}$: Find $\boldsymbol{u}^{i+1} \in \boldsymbol{K}^{i+1}$ such that

$$\begin{aligned} a(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) &+ \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) \\ &\geq (\boldsymbol{F}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) \quad \forall\, \boldsymbol{w} \in \boldsymbol{K}\,. \end{aligned} \tag{9.29}$$

To help the reader acquire a better understanding of the proof of Theorem 9.2, we divide it into two steps, Propositions 9.1 and 9.2 below. For this reason we introduce the following mechanical problem.

Problem $(\mathscr{Q})_n^i$: Find a displacement field $\boldsymbol{u}^{i+1} : \Omega \to \mathbb{R}^d$ such that

$$-\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}^{i+1}) = \boldsymbol{f}^{i+1} \quad \text{in } \Omega\,, \tag{9.30}$$

$$\boldsymbol{u}^{i+1} = \boldsymbol{0} \quad \text{on } \Gamma_0\,, \tag{9.31}$$

$$\boldsymbol{\sigma}(\boldsymbol{u}^{i+1}) \cdot \boldsymbol{\nu} = \boldsymbol{g}^{i+1} \quad \text{on } \Gamma_1\,, \tag{9.32}$$

$$u_\nu^{i+1} \leq 0\,, \quad \sigma_\nu(\boldsymbol{u}^{i+1}) \leq 0\,, \quad u_\nu \sigma_\nu(\boldsymbol{u}^{i+1}) = 0 \quad \text{on } \Gamma_2\,, \tag{9.33}$$

$$\begin{cases} |\boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})| \leq \mu |\mathscr{R}\sigma_\nu(\boldsymbol{u}^{i+1})| \text{ and} \\ |\boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})| < \mu |\mathscr{R}\sigma_\nu(\boldsymbol{u}^{i+1})| \;\Rightarrow\; \boldsymbol{u}_\tau^{i+1} = \boldsymbol{u}_\tau^i \\ |\boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})| = \mu |\mathscr{R}\sigma_\nu(\boldsymbol{u}^{i+1})| \;\Rightarrow\; \exists \lambda \geq 0\,,\; \boldsymbol{u}_\tau^{i+1} - \boldsymbol{u}_\tau^i = -\lambda \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1}) \end{cases} \text{on } \Gamma_2\,. \tag{9.34}$$

Lemma 9.2. *Let* $\boldsymbol{\Theta} \in \boldsymbol{V}$ *and* $\boldsymbol{d} \in \boldsymbol{K}$ *be given and let* $\tilde{\boldsymbol{u}} \in \boldsymbol{K}(\boldsymbol{\Theta})$ *be a regular function such that*

$$\tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \boldsymbol{w} - \mathbf{d}) - \tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} - \boldsymbol{d}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \tilde{\boldsymbol{u}}_\tau)\, \mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{K}\,. \tag{9.35}$$

Then $\tilde{\boldsymbol{u}}$ *verifies (in the distributional sense)*

$$\begin{cases} |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| \leq \mu|\mathscr{R}\sigma_\nu(\tilde{\boldsymbol{u}})| \text{ and} \\ |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| < \mu|\mathscr{R}\sigma_\nu(\tilde{\boldsymbol{u}})| \;\Rightarrow\; \tilde{\boldsymbol{u}}_\tau = \boldsymbol{d}_\tau \\ |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| = \mu|\mathscr{R}\sigma_\nu(\tilde{\boldsymbol{u}})| \;\Rightarrow\; \exists\lambda \geq 0\,,\ \tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau = -\lambda\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}}) \end{cases} \quad \text{on } \Gamma_2\,. \tag{9.36}$$

Proof. If we take $\boldsymbol{w} = \boldsymbol{d} + \delta\boldsymbol{\varphi}_\tau$ in (9.35), with $\boldsymbol{\varphi} \in (C^\infty(\Omega))^d$, $\operatorname{supp}\boldsymbol{\varphi} \subset \Gamma_2$ and $\delta > 0$, we obtain

$$\begin{aligned} &\int_{\Gamma_2} \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|(|\boldsymbol{w}_\tau - \boldsymbol{d}_\tau| - |\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau|) + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \tilde{\boldsymbol{u}}_\tau)\, \mathrm{d}s \\ &= \delta \int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,|\boldsymbol{\varphi}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})\boldsymbol{\varphi})\ \mathrm{d}s \\ &- \int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau))\ \mathrm{d}s \geq 0 \quad \forall\delta > 0\,, \end{aligned}$$

which gives, as $|\boldsymbol{\varphi}| \geq |\boldsymbol{\varphi}_\tau|$,

$$\int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,|\boldsymbol{\varphi}| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})\boldsymbol{\varphi})\, \mathrm{d}s \geq 0 \quad \forall\boldsymbol{\varphi} \in (C^\infty(\Omega))^d\,,\ \operatorname{supp}\boldsymbol{\varphi} \subset \Gamma_2\,, \tag{9.37}$$

and

$$\int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau))\, \mathrm{d}s \leq 0\,. \tag{9.38}$$

Putting $\boldsymbol{\varphi} = \pm\boldsymbol{\varphi}$ in (9.37), it results

$$\int_{\Gamma_2} |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})|\,|\boldsymbol{\varphi}|\, \mathrm{d}s \leq \int_{\Gamma_2} \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))||\boldsymbol{\varphi}|\, \mathrm{d}s \quad \forall\boldsymbol{\varphi} \in (C^\infty(\Omega))^d\,,\ \operatorname{supp}\boldsymbol{\varphi} \subset \Gamma_2\,,$$

i.e.

$$|\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| \leq \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,. \tag{9.39}$$

Therefore, (9.38) implies

$$0 \geq \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau) \geq (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))| - |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})|)\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| \geq 0$$

that is

$$\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau) = 0\,. \tag{9.40}$$

If $|\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| < \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|$, then, supposing $\tilde{\boldsymbol{u}}_\tau \neq \boldsymbol{d}_\tau$, (9.40) gives

$$0 = \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau) > |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})|\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau) \geq 0\,,$$

and so, we must have $\tilde{\boldsymbol{u}}_\tau = \boldsymbol{d}_\tau$

If $|\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| = \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|$, then (9.40) implies

$$|\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})|\,|\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau) = 0$$

and thus, there exists $\lambda \geq 0$ such that $\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau = -\lambda\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})$. □

Lemma 9.3. *Let* $\boldsymbol{\Theta} \in \boldsymbol{V}$ *and* $\boldsymbol{d} \in \boldsymbol{K}$ *be given. Let* $\tilde{\boldsymbol{u}} \in \boldsymbol{K}(\boldsymbol{\Theta})$ *be a sufficiently smooth function which verifies (9.36). Then*

$$\tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \boldsymbol{w} - \mathbf{d}) - \tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} - \boldsymbol{d}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \tilde{\boldsymbol{u}}_\tau)\,\mathrm{d}s \geq 0 \;\; \forall \boldsymbol{w} \textit{ smooth function}\,. \tag{9.41}$$

Proof. Let $\boldsymbol{w}$ be a smooth function.

If $|\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| < \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|$ and $\tilde{\boldsymbol{u}}_\tau = \boldsymbol{d}_\tau$, then one has

$$\begin{aligned}
&\tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \boldsymbol{w} - \mathbf{d}) - \tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} - \boldsymbol{d}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \boldsymbol{d}_\tau)\,\mathrm{d}s \\
&= \int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|\,|\boldsymbol{w}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \boldsymbol{d}_\tau))\,\mathrm{d}s \\
&\geq \int_{\Gamma_2} (\mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))| - |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})|)\,|\boldsymbol{w}_\tau - \boldsymbol{d}_\tau|\,\mathrm{d}s \geq 0
\end{aligned}$$

If $|\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})| = \mu|\mathscr{R}(\sigma_\nu(\tilde{\boldsymbol{u}}))|$ and $\tilde{\boldsymbol{u}}_\tau - \boldsymbol{d}_\tau = -\lambda\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})$, then one gets

$$\begin{aligned}
&\tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \boldsymbol{w} - \mathbf{d}) - \tilde{j}_{\boldsymbol{\Theta}}(\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{u}} - \boldsymbol{d}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \tilde{\boldsymbol{u}}_\tau)\,\mathrm{d}s \\
&= \int_{\Gamma_2} |\boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})|\,|\boldsymbol{w}_\tau - \boldsymbol{d}_\tau| + \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}})(\boldsymbol{w}_\tau - \boldsymbol{d}_\tau)\,\mathrm{d}s \geq 0\,,
\end{aligned}$$

which completes the proof of Lemma. □

Proposition 9.1. *The problem* $(\tilde{\mathbf{Q}})^{\mathbf{i}}_{\mathbf{n}}$ *is formally equivalent (in the sense considered in Theorem 9.1) to the mechanical problem* $(\mathcal{Q})^i_n$.

Proof. Let $\boldsymbol{u}^{i+1}$ be a regular solution of $(\tilde{\mathbf{Q}})^{\mathbf{i}}_{\mathbf{n}}$. If we chose, in the first inequality of (9.28), $\boldsymbol{w} = \boldsymbol{u}^{i+1} \pm \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in (\mathcal{D}(\Omega))^d$ and we apply the Green's formula, then we obtain (9.30).

From the second inequality of (9.28) and Lemma 9.1 for $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{i+1}$, we deduce (9.33).

Multiplying (9.30) by $\boldsymbol{w} - \boldsymbol{u}^{i+1}$ for $\boldsymbol{w} \in \boldsymbol{V}$, integrating by parts and using against the Green's formula and the first inequality of (9.28), we get

$$\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\tau - \boldsymbol{u}^{i+1}_\tau)\,\mathrm{d}s$$
$$+ \int_{\Gamma_1} (\boldsymbol{\sigma}(\boldsymbol{u}^{i+1}) \cdot \boldsymbol{\nu} - \boldsymbol{g}^{i+1})(\boldsymbol{w} - \boldsymbol{u}^{i+1})\,\mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{V}\,, \tag{9.42}$$

and thus, by taking $\boldsymbol{w} = \boldsymbol{u}^{i+1} \pm \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in (C^\infty(\Omega))^d$ and supp $\boldsymbol{\varphi} \subset \Gamma_1$, one obtains (9.32). Therefore, the relation (9.42) implies

$$\begin{aligned} &\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) \\ &+ \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\tau - \boldsymbol{u}^{i+1}_\tau)\,\mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{V}\,. \end{aligned} \tag{9.43}$$

Therefore, by Lemma 9.2 for $\boldsymbol{\Theta} = \boldsymbol{F}^{i+1}$, $\boldsymbol{d} = \boldsymbol{u}^i$ and $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{i+1} \in \boldsymbol{K}(\boldsymbol{F}^{i+1})$, it follows that the conditions (9.34) are satisfied. As $\boldsymbol{u}^{i+1} \in \boldsymbol{K} \subset \boldsymbol{V}$, it yields the condition (9.31) holds which completes the proof.

Conversely, let $\boldsymbol{u}^{i+1}$ be a sufficiently smooth solution of the mechanical problem $(\mathcal{Q})^i_n$. Then, by applying Lemma 9.1 for $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{i+1}$, it follows that $\boldsymbol{u}^{i+1}$ satisfies the second inequality of (9.28).

Next, from (9.34), by Lemma 9.3 for $\boldsymbol{\Theta} = \boldsymbol{F}^{i+1}$, $\boldsymbol{d} = \boldsymbol{u}^i$ and $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{i+1}$, we obtain

$$\begin{aligned} &\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i) \\ &+ \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\tau - \boldsymbol{u}^{i+1}_\tau)\,\mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{V}\,. \end{aligned} \tag{9.44}$$

On the other hand, multiplying (9.30) by $\boldsymbol{w} - \boldsymbol{u}^{i+1}$ with $\boldsymbol{w} \in \boldsymbol{V}$, integrating by parts and using (9.32), we deduce

$$a(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) = (\boldsymbol{F}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\tau - \boldsymbol{u}^{i+1}_\tau)\,\mathrm{d}s$$
$$+ \langle \sigma_\nu(\boldsymbol{u}^{i+1}), w_\nu - u^{i+1}_\nu \rangle \quad \forall\, \boldsymbol{w} \in \boldsymbol{V}. \tag{9.45}$$

Combining (9.44) and (9.45), we obtain the first inequality of (9.28) which completes the proof. □

Proposition 9.2. *The problem* $(\tilde{\mathbf{R}})^{\mathbf{i}}_{\mathbf{n}}$ *is formally equivalent to the mechanical problem* $(\mathscr{Q})^i_n$.

Proof. If $\boldsymbol{u}^{i+1}$ is a regular solution of $(\tilde{\mathbf{R}})^{\mathbf{i}}_{\mathbf{n}}$, then, with a similar proof as for Proposition 9.1, one obtains (9.30). Therefore, from (9.30) and (9.29), one gets

$$\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w}-\boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}-\boldsymbol{u}^i) + \int_{\Gamma_2} \boldsymbol{\sigma}(\boldsymbol{u}^{i+1})\cdot\boldsymbol{\nu}(\boldsymbol{w}-\boldsymbol{u}^{i+1})\,\mathrm{d}s$$
$$+\int_{\Gamma_1} (\boldsymbol{\sigma}(\boldsymbol{u}^{i+1})\cdot\boldsymbol{\nu}-\boldsymbol{g}^{i+1})(\boldsymbol{w}-\boldsymbol{u}^{i+1})\,\mathrm{d}s \geq 0 \quad \forall \boldsymbol{w}\in\boldsymbol{K}, \tag{9.46}$$

from which, by taking $\boldsymbol{w} = \boldsymbol{u}^{i+1} \pm \boldsymbol{\varphi}$ with $\boldsymbol{\varphi} \in (C^\infty(\Omega))^d$ and supp $\boldsymbol{\varphi} \subset \Gamma_1$, one deduces (9.32). Thus, the relation (9.46) becomes

$$\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w}-\boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}-\boldsymbol{u}^i)$$
$$+\int_{\Gamma_2} (\sigma_\nu(\boldsymbol{u}^{i+1})(w_\nu - u^{i+1}_\nu) + \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\tau - \boldsymbol{u}^{i+1}_\tau))\,\mathrm{d}s \geq 0 \quad \forall \boldsymbol{w}\in\boldsymbol{K}. \tag{9.47}$$

By choosing $\boldsymbol{w} = \delta\varphi_\nu\boldsymbol{\nu} + \boldsymbol{u}^{i+1}_\tau$ with $\boldsymbol{\varphi} \in (C^\infty(\Omega))^d$, $\varphi_\nu \leq 0$ on Γ_2 and $\delta > 0$, it follows

$$\delta\int_{\Gamma_2} \sigma_\nu(\boldsymbol{u}^{i+1})\,\varphi_\nu\,\mathrm{d}s \geq \int_{\Gamma_2} \sigma_\nu(\boldsymbol{u}^{i+1})\,\boldsymbol{u}^{i+1}_\nu\,\mathrm{d}s \quad \forall\delta > 0$$

which gives

$$\begin{cases} \displaystyle\int_{\Gamma_2} \sigma_\nu(\boldsymbol{u}^{i+1})\,\varphi_\nu\,\mathrm{d}s \geq 0 \quad \forall\boldsymbol{\varphi}\in\boldsymbol{V}\,,\ \varphi_\nu \leq 0 \text{ on } \Gamma_2\,, \\ \displaystyle\int_{\Gamma_2} \sigma_\nu(\boldsymbol{u}^{i+1})\,\boldsymbol{u}^{i+1}_\nu\,\mathrm{d}s \leq 0\,, \end{cases} \tag{9.48}$$

and, as $\boldsymbol{u}^{i+1} \in \boldsymbol{K}$, we obtain (9.33).

Now, if we choose in (9.47), $\boldsymbol{w} = u^{i+1}_n\boldsymbol{\nu} + \boldsymbol{v}$ with $\boldsymbol{v} \in \boldsymbol{K}$ arbitrary, we obtain

$$\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{v}-\boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}-\boldsymbol{u}^i) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{v}_\tau - \boldsymbol{u}^{i+1}_\tau)\,\mathrm{d}s \geq 0 \quad \forall\boldsymbol{v}\in\boldsymbol{K}$$

which gives, together with Lemma 9.2, for $\boldsymbol{\Theta} = \boldsymbol{F}^{i+1}$, $\boldsymbol{d} = \boldsymbol{u}^i$ and $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{i+1}$, the conditions (9.34).

Conversely, if $\boldsymbol{u}^{i+1}$ is a sufficiently smooth function which verifies $(\mathscr{Q})_n^i$, then, from Lemmas 9.3 and 9.1, we obtain

$$\tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{w}-\boldsymbol{u}^i) - \tilde{j}_{\boldsymbol{F}^{i+1}}(\boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1}-\boldsymbol{u}^i) + \int_{\Gamma_2} \boldsymbol{\sigma}_\tau(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\tau - \boldsymbol{u}_\tau^{i+1})\,\mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{K} \tag{9.49}$$

and

$$\int_{\Gamma_2} \boldsymbol{\sigma}_\nu(\boldsymbol{u}^{i+1})(\boldsymbol{w}_\nu - \boldsymbol{u}_\nu^{i+1})\,\mathrm{d}s \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{K}\,. \tag{9.50}$$

Next, by arguing as in the proof of Proposition 9.1, we conclude that $\boldsymbol{u}^{i+1}$ is a solution of $(\tilde{\mathbf{R}})_\mathbf{n}^\mathbf{i}$ which completes the proof. □

Proof of Theorem 9.2. Using Propositions 9.1 and 9.2, the assertion follows. However, we remark that if $\boldsymbol{u}^{i+1}$ is a solution of $(\tilde{\mathbf{Q}})_\mathbf{n}^\mathbf{i}$, then, obviously, $\boldsymbol{u}^{i+1}$ is a solution of $(\tilde{\mathbf{R}})_\mathbf{n}^\mathbf{i}$. Hence, in order to prove the condition (4.105), it would have been enough to prove that $(\tilde{\mathbf{R}})_\mathbf{n}^\mathbf{i} \Rightarrow (\mathscr{Q})_n^i \Rightarrow (\tilde{\mathbf{Q}})_\mathbf{n}^\mathbf{i}$. □

In the sequel we shall use the similar definitions to (4.118) (p. 72), i.e.

$$\begin{cases} \boldsymbol{u}_n(0) = \hat{\boldsymbol{u}}_n(0) = \boldsymbol{u}^0, \\ \boldsymbol{F}_n(0) = \boldsymbol{F}(0) = \boldsymbol{F}^0, \\ \left.\begin{array}{l} \boldsymbol{u}_n(t) = \boldsymbol{u}^{i+1} \\ \hat{\boldsymbol{u}}_n(t) = \boldsymbol{u}^i + (t-t_i)\partial \boldsymbol{u}^i \\ \boldsymbol{F}_n(t) = \boldsymbol{F}^{i+1} \end{array}\right\} \forall\, i \in \{0,1,\ldots,n-1\} \quad \forall\, t \in (t_i, t_{i+1}]\,, \end{cases} \tag{9.51}$$

Therefore, $\boldsymbol{u}_n \in L^2(0,T;\boldsymbol{V})$ and $\hat{\boldsymbol{u}}_n \in W^{1,2}(0,T;\boldsymbol{V})$ satisfy, for all $t \in [0,T]$, the following incremental problem.

Problem $(\mathbf{Q})_\mathbf{n}$: Find $\boldsymbol{u}_n \in K(\boldsymbol{F}_n(t))$ such that

$$\begin{cases} a\left(\boldsymbol{u}_n(t), \boldsymbol{v} - \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{u}}_n(t)\right) + \tilde{j}_{\boldsymbol{F}_n(t)}(\boldsymbol{u}_n(t), \boldsymbol{v}) \\ -\tilde{j}_{\boldsymbol{F}_n(t)}\left(\boldsymbol{u}_n(t), \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{u}}_n(t)\right) \geq \left(\boldsymbol{F}_n(t), \boldsymbol{v} - \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{u}}_n(t)\right) \\ +\left\langle \sigma_\nu(\boldsymbol{u}_n(t)), v_\nu - \dfrac{\mathrm{d}}{\mathrm{d}t}\hat{u}_{n\nu}(t)\right\rangle \quad \forall\, \boldsymbol{v} \in \mathbf{V}, \\ \langle \sigma_\nu(\boldsymbol{u}_n(t)), z_\nu - u_{n\nu}(t)\rangle \geq 0 \quad \forall\, \boldsymbol{z} \in \boldsymbol{K}\,. \end{cases} \tag{9.52}$$

We have the following convergence and existence result.

Theorem 9.3. *Suppose the hypotheses (9.8) and (9.9) hold and that meas $\Gamma_0 > 0$. Then, there exists a constant $\mu_1 > 0$ such that for any $\mu \in L^\infty(\Gamma_2)$ with $\mu \geq 0$ a.e. on Γ_2 and $\|\mu\|_{L^\infty(\Gamma_2)} < \mu_1$, the problem* **(Q)** *has at least one solution. More precisely, there exists a subsequence $\{(\boldsymbol{u}_{n_k}, \bar{\boldsymbol{u}}_{n_k}\}_k$ such that*

$$
\begin{aligned}
&\boldsymbol{u}_{n_k}(t) \to \boldsymbol{u}(t) \textit{ strongly in } \boldsymbol{V} \quad \forall\, t \in [0,T]\,,\\
&\hat{\boldsymbol{u}}_{n_k} \to \boldsymbol{u} \textit{ strongly in } L^2(0,T;\boldsymbol{V})\,,\\
&\frac{\mathrm{d}}{\mathrm{d}t}\hat{\boldsymbol{u}}_{n_k} \rightharpoonup \dot{\boldsymbol{u}} \textit{ weakly in } L^2(0,T;\boldsymbol{V})
\end{aligned}
$$

as $k \to \infty$*, where* $\boldsymbol{u}$ *is a solution of the problem* $(\mathbf{Q})$.

Proof. By putting

$$
j(\boldsymbol{\Theta},\boldsymbol{v},\boldsymbol{w}) = \tilde{j}_{\boldsymbol{\Theta}}(\boldsymbol{v},\boldsymbol{w}) - (\boldsymbol{\Theta},\boldsymbol{w}) \quad \forall \boldsymbol{\Theta} \in \boldsymbol{V}\,,\ \forall \boldsymbol{v} \in \boldsymbol{K}(\boldsymbol{\Theta})\,,\ \forall \boldsymbol{w} \in \boldsymbol{V}\,, \tag{9.53}
$$

it follows that

$$
\begin{aligned}
&|j(\boldsymbol{\Theta}_1,\boldsymbol{v}_1,\boldsymbol{w}_2) + j(\boldsymbol{\Theta}_2,\boldsymbol{v}_2,\boldsymbol{w}_1) - j(\boldsymbol{\Theta}_1,\boldsymbol{v}_1,\boldsymbol{w}_1) - j(\boldsymbol{\Theta}_2,\boldsymbol{v}_2,\boldsymbol{w}_2)|\\
&= \left| \int_{\Gamma_2} \mu(|\mathscr{R}\sigma_\nu(\boldsymbol{v}_1)| - |\mathscr{R}\sigma_\nu(\boldsymbol{v}_2)|)(|\boldsymbol{w}_{1\tau}| - |\boldsymbol{w}_{2\tau}|)\,\mathrm{d}s + (\boldsymbol{\Theta}_1 - \boldsymbol{\Theta}_2, \boldsymbol{w}_1 - \boldsymbol{w}_2) \right|\\
&\le C_1 \|\mu\|_{L^\infty(\Gamma_2)} \int_{\Gamma_2} |\mathscr{R}\sigma_\nu(\boldsymbol{v}_1) - \mathscr{R}\sigma_\nu(\boldsymbol{v}_2)|\,|\boldsymbol{w}_1 - \boldsymbol{w}_2|\,\mathrm{d}s + \|\boldsymbol{\Theta}_1 - \boldsymbol{\Theta}_2\|\,\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|\\
&\le C_2 \|\mu\|_{L^\infty(\Gamma_2)} (\|\boldsymbol{\Theta}_1 - \boldsymbol{\Theta}_2\| + \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|)\|\boldsymbol{w}_1 - \boldsymbol{w}_2\|\\
&\qquad\qquad \forall\, \boldsymbol{w}_i \in \boldsymbol{V}\,,\ \forall \boldsymbol{v}_i \in \boldsymbol{K}(\boldsymbol{\Theta}_i)\,,\ \forall \boldsymbol{w}_i \in \boldsymbol{V}\,,\ i = 1,2\,,
\end{aligned} \tag{9.54}
$$

where C_1, C_2 are positive constants and $\|\cdot\|$ denotes the norm over $\boldsymbol{V}$.

In order to apply Theorem 4.19, we put

$$
b(\boldsymbol{\Theta},\boldsymbol{v},\boldsymbol{w}) = \langle \sigma_\nu(\boldsymbol{v}), w_\nu \rangle \quad \forall \boldsymbol{\Theta} \in \boldsymbol{V}\,,\ \forall \boldsymbol{v} \in \boldsymbol{K}(\boldsymbol{\Theta})\,,\ \forall \boldsymbol{w} \in \boldsymbol{V}\,, \tag{9.55}
$$

$$
\begin{aligned}
&H = L^2(\Gamma_2)\,,\\
&\beta(\boldsymbol{\Theta},\boldsymbol{v}) = \mu|\mathscr{R}\sigma_\nu(\boldsymbol{v})| \quad \forall \boldsymbol{\Theta} \in \boldsymbol{V}\,,\ \forall \boldsymbol{v} \in \boldsymbol{K}(\boldsymbol{\Theta})\,.
\end{aligned}
$$

Therefore, the problem $(\mathbf{Q})$ can be written under the form (4.107) (p. 68) and the problem $(\tilde{\mathbf{Q}})^{\mathbf{i}}_{\mathbf{n}}$ can be written under the form (4.103) (p. 68), i.e.

$$
\begin{cases}
\boldsymbol{u}^{i+1} \in \boldsymbol{K}(\boldsymbol{F}^{i+1})\\
a(\boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) + j(\boldsymbol{F}^{i+1}, \boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^i) - j(\boldsymbol{F}^{i+1}, \boldsymbol{u}^{i+1}, \boldsymbol{u}^{i+1} - \boldsymbol{u}^i)\\
\ge b(\boldsymbol{F}^{i+1}, \boldsymbol{u}^{i+1}, \boldsymbol{w} - \boldsymbol{u}^{i+1}) \quad \forall\, \boldsymbol{w} \in \boldsymbol{V},\\
b(\boldsymbol{F}^{i+1}, \boldsymbol{u}^{i+1}, \boldsymbol{z} - \boldsymbol{u}^{i+1}) \ge 0 \quad \forall\, \boldsymbol{z} \in \boldsymbol{K}.
\end{cases} \tag{9.56}
$$

The hypothesis (4.105) is satisfied due to Theorem 9.2. The other hypotheses (4.83)–(4.90), (4.96)–(4.98), (4.100), and (4.101) of Theorem 4.19 are easy to prove, and so, the assertion follows. □

9.2 Discrete Approximation

This section deals with the discretization of the problem (**Q**) written under the form

$$\begin{cases} \boldsymbol{u}(0) = \boldsymbol{u}_0\,,\ \boldsymbol{u}(t) \in \boldsymbol{K}(\boldsymbol{F}(t)) \quad \forall\, t \in [0, T\,], \\ a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j(\boldsymbol{F}(t), \boldsymbol{u}(t), \boldsymbol{v}) - j(\boldsymbol{F}(t), \boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \\ \geq b(f(t), u(t), v - \dot{u}(t)) \quad \forall\, \boldsymbol{v} \in \boldsymbol{V} \text{ a.e. in } (0, T), \\ b(\boldsymbol{F}(t), \boldsymbol{u}(t), \boldsymbol{z} - \boldsymbol{u}(t)) \geq 0 \quad \forall \boldsymbol{z} \in \boldsymbol{K},\ \forall\, t \in [0, T\,], \end{cases} \tag{9.57}$$

with j and b defined by (9.53), respectively, (9.55).

We shall prove a convergence result for a method based on an internal approximation in space and a backward difference scheme in time.

Let $\mathscr{T}_h = (T_j)_{j \in \mathscr{J}_h}$ be a family of regular triangulations of Ω such that

$$\overline{\Omega} = \bigcup_{j \in \mathscr{J}_h} \overline{T}_j\,,$$
$$T_i \cap T_j = \emptyset \quad \forall i, j \in \mathscr{J}_h\,,\ i \neq j\,.$$

We define the following sets

$$\begin{aligned} &\boldsymbol{V}_h = \{\boldsymbol{v}_h \in (C^0(\overline{\Omega}))^d\,;\ \boldsymbol{v}_h/T_j \in (P_1(T_j))^d\,,\ \forall j \in \mathscr{J}_h\,,\ \boldsymbol{v}_h = \boldsymbol{0} \text{ on } \Gamma_0\}\,, \\ &\boldsymbol{K}_h = \{\boldsymbol{v}_h \in \boldsymbol{V}_h\,;\ \boldsymbol{v}_{h\nu} \leq 0 \text{ on } \Gamma_2\} \\ &S_h = \{\tau_h \in L^2(\Gamma_2)\,;\ \tau_h/\Gamma_{2,j} \in P_0(\Gamma_{2,j}) \quad \forall j \in \mathscr{J}_h \text{ such that } \Gamma_{2,j} \neq \emptyset\} \end{aligned}$$

where $P_k(\omega)$ denotes the space of polynomials of degree lower or equal to k on ω and $\Gamma_{2,j} = \Gamma_2 \cap \overline{T}_j$.

As in Sect. 7.3, p. 128, we consider the following semi-discrete problem.

Problem ($\mathbf{Q_h}$): Find $\boldsymbol{u}_h \in W^{1,2}(0, T; \boldsymbol{V}_h)$ such that

$$\begin{cases} \boldsymbol{u}_h(0) = \boldsymbol{u}_{0h},\ \boldsymbol{u}_h(t) \in \boldsymbol{K}_h(\boldsymbol{F}(t)) \quad \forall\, t \in [0, T\,], \\ a(\boldsymbol{u}_h(t), \boldsymbol{v}_h - \dot{\boldsymbol{u}}_h(t)) + j(\boldsymbol{F}(t), \boldsymbol{u}_h(t), \boldsymbol{v}_h) - j(\boldsymbol{F}(t), \boldsymbol{u}_h(t), \dot{\boldsymbol{u}}_h(t)) \\ \geq b(\boldsymbol{F}(t), \boldsymbol{u}_h(t), \boldsymbol{v}_h - \dot{\boldsymbol{u}}_h(t)) \quad \forall\, \boldsymbol{v}_h \in \boldsymbol{V}_h\,,\ \text{a.e. } t \in (0, T)\,, \\ b(\boldsymbol{F}(t), \boldsymbol{u}_h(t), \boldsymbol{z}_h - \boldsymbol{u}_h(t)) \geq 0 \quad \forall\, \boldsymbol{z}_h \in \boldsymbol{K}_h, \quad \forall\, t \in [0, T\,]\,. \end{cases} \tag{9.58}$$

and, for $i \in \{0, 1, \cdots, n-1\}$, the following full discretization of Problem (**Q**).

Problem $(\mathbf{R_h})^{\mathbf{i}}_{\mathbf{n}}$: Find $\boldsymbol{u}_h^{i+1} \in \boldsymbol{K}_h^{i+1}$ such that

$$\begin{cases} a(\boldsymbol{u}_h^{i+1}, \boldsymbol{w}_h - \boldsymbol{u}_h^{i+1}) + j(\boldsymbol{F}^{i+1}, \boldsymbol{u}_h^{i+1}, \boldsymbol{w}_h - \boldsymbol{u}_h^{i}) \\ -j(\boldsymbol{F}^{i+1}, \boldsymbol{u}_h^{i+1}, \boldsymbol{u}_h^{i+1} - \boldsymbol{u}_h^{i}) \geq 0 \quad \forall\, \boldsymbol{w}_h \in \boldsymbol{K}_h. \end{cases} \tag{9.59}$$

We also suppose that $\boldsymbol{u}_h^0 = \boldsymbol{u}_{0h}$ satisfies the compatibility condition

$$\begin{cases} \boldsymbol{u}_{0h} \in \boldsymbol{K}_h(\boldsymbol{F}(0))\,, \\ a(\boldsymbol{u}_{0h}, \boldsymbol{v}) + j(\boldsymbol{F}(0), \boldsymbol{u}_{0h}, \boldsymbol{v}) \geq 0 \quad \forall \boldsymbol{v} \in \boldsymbol{K}_h\,. \end{cases}$$

Theorems 7.5 (p. 128) and 7.6 (p. 131) give convergence and existence results for these problems.

In order to solve the problem $(\mathbf{R_h})^{\mathbf{i}}_{\mathbf{n}}$, we suppose that μ is constant and we choose as a regularization mapping $\mathscr{R}$, the projection on the finite dimensional space S_{h_0} for a given h_0 (see [10]). Thus within finite element approximation, the regularization can be considered as a natural consequence of the discretization.

In the sequel, for simplicity, we shall omit the index h. We shall denote the solution $\boldsymbol{u}^{i+1}$ of $(\mathbf{R_h})^{\mathbf{i}}_{\mathbf{n}}$ by $\boldsymbol{u}_n^{i+1}$, for $i \in \{0, 1, \cdot, n-1\}$. We also remark that, from the definition of the set $\boldsymbol{K}^{i+1}$ and Remark 9.1, it follows that for the solution $\boldsymbol{u}^{i+1}$ we have

$$j(\boldsymbol{F}^{i+1}, \boldsymbol{u}_n^{i+1}, \boldsymbol{v}) = -\int_{\Gamma_2} \mu \mathscr{R} \sigma_\nu(\boldsymbol{u}_n^{i+1})\, |\boldsymbol{v}_t|\, \mathrm{d}s \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,.$$

Let us denote

$$j(\boldsymbol{u}_n^{i+1}, \boldsymbol{v}) = -\int_{\Gamma_2} \mu \mathscr{R} \sigma_\nu(\boldsymbol{u}_n^{i+1})\, |\boldsymbol{v}_t|\, \mathrm{d}s \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,.$$

Therefore, the problem to solve can be written as

$$\begin{cases} \boldsymbol{u}_n^{i+1} \in \boldsymbol{K}^{i+1}, \\ a(\boldsymbol{u}_n^{i+1}, \boldsymbol{w} - \boldsymbol{u}_n^{i+1}) + j(\boldsymbol{u}_n^{i+1}, \boldsymbol{w} - \boldsymbol{u}_n^{i}) - j(\boldsymbol{u}_n^{i+1}, \boldsymbol{u}_n^{i+1} - \boldsymbol{u}_n^{i}) \\ \geq (\boldsymbol{F}^{i+1}, \boldsymbol{w} - \boldsymbol{u}_n^{i}) \ \forall\, \boldsymbol{w} \in \boldsymbol{K}\,. \end{cases} \tag{9.60}$$

It is easy to see that the solution $\boldsymbol{u}_n^{i+1} \in \boldsymbol{K}^{i+1}$ of (9.60) is the fixed point of the mapping $\mathscr{T} : S \to S$ defined by $\mathscr{T}(r) = \boldsymbol{u}_n^{i+1}(r)$, for all $r \in S$, where $\boldsymbol{u}_n^{i+1}(r)$ is the unique solution of the following variational inequality:

$$\begin{cases} \boldsymbol{u}_n^{i+1}(r) \in \boldsymbol{K}^{i+1}, \\ a(\boldsymbol{u}_n^{i+1}(r), \boldsymbol{w} - \boldsymbol{u}_n^{i+1}(r)) + \varphi(r, \boldsymbol{w} - \boldsymbol{u}_n^{i}(r)) - \varphi(r, \boldsymbol{u}_n^{i+1}(r) - \boldsymbol{u}_n^{i}) \\ \geq (\boldsymbol{F}^{i+1}, \boldsymbol{w} - \boldsymbol{u}_n^{i+1}(r)) \ \forall\, \boldsymbol{w} \in \boldsymbol{K}\,. \end{cases} \tag{9.61}$$

where

$$\varphi(r, \boldsymbol{w}) = -\int_{\Gamma_2} \mu \mathscr{R} \sigma_\nu(r) |\boldsymbol{w}_\tau|\, \mathrm{d}s \quad \forall \boldsymbol{w} \in \boldsymbol{V}.$$

This problem is equivalent, for $r \in S$ given, to the following minimization problem under constraints:

$$\mathscr{F}(\boldsymbol{u}_n^{i+1}(r)) = \min_{\boldsymbol{v} \in \boldsymbol{K}} \mathscr{F}(\boldsymbol{v})$$

where

$$\mathscr{F}(\boldsymbol{v}) = \frac{1}{2}a(\boldsymbol{v},\boldsymbol{v}) + \varphi(r, \boldsymbol{v} - \boldsymbol{u}_n^i(r)) - (\boldsymbol{F}^{i+1}, \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{V} .$$

This problem is very similar to a static problem except from the fact that the known solution $\boldsymbol{u}_n^i$ of the previous step appears in the friction term. The influence of the loading history, due to the velocity formulation of the friction, is characterized by this extra term. The convex K remains unchanged from one step to the next. This minimization problem can be solved by a Gauss–Seidel method with projection. This method is robust and very easy to implement on this kind of problem when dealing with the non-differentiable part relating to the friction term. Details on the convergence of the algorithm by using an Aitken acceleration procedure can be found in [5] or [13].

9.3 Optimal Control of a Frictional Bilateral Contact Problem

We consider a mathematical model describing the quasistatic process of bilateral contact with friction between an elastic body and a rigid foundation. Our goal is to study a related optimal control problem which allows us to obtain a given profile of displacements on the contact boundary, by acting with a control on another part of the boundary of the body. Using penalization and regularization techniques, we derive the necessary conditions of optimality.

As far as we know, there are few results concerning the optimal control of quasistatic frictional contact problems. We mention here the work of Amassad et al. [2] which treats a quasistatic bilateral contact problem with given friction, and so, an optimal control problem governed by a variational inequality which has, in addition, a unique solution.

9.3.1 Setting of the Problem

Let us consider a linearly elastic body occupying a bounded domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with a Lipschitz boundary $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, where $\Gamma_0,\ \Gamma_1,\ \Gamma_2$ are open and disjoint parts of Γ, with meas $(\Gamma_0) > 0$.

The body is subjected to the action of volume forces of density $\boldsymbol{f}$ given in $\Omega \times (0,T)$ and surface tractions of density $\boldsymbol{g}$ applied on $\Gamma_1 \times (0,T)$, where $(0,T)$ is the time interval of interest. The body is clamped on $\Gamma_0 \times (0,T)$ and, so, the displacement vector $\boldsymbol{u}$ vanishes here. On $\Gamma_2 \times (0,T)$, the body is in bilateral contact with a rigid foundation, i.e. there is no loss of contact between the body and the foundation. We suppose that the contact on Γ_2 is with friction modeled by a nonlocal variant of Coulomb's law. We suppose that $\boldsymbol{f}$ and $\boldsymbol{g}$ are acting slow enough to allow us to neglect the inertial terms.

The classical formulation of this mechanical problem, with the notation of Sect. 8.1, is:
Problem ($\mathscr{S}$): Find a displacement vector $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x}, t) : \Omega \times [0, T] \to \mathbb{R}^d$ such that

$$\begin{cases} -\operatorname{div} \boldsymbol{\sigma} = \boldsymbol{f} \quad \text{in } \Omega \times (0, T)\,, \\ \boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u}) = \mathscr{A}\boldsymbol{\epsilon}\,, \\ \boldsymbol{u} = \boldsymbol{0} \quad \text{on } \Gamma_0 \times (0, T)\,, \\ \boldsymbol{\sigma} \cdot \boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_1 \times (0, T)\,, \\ \left.\begin{cases} u_\nu = 0\,, \\ |\boldsymbol{\sigma}_\tau| \le \mu |\mathscr{R}\sigma_\nu| \\ |\boldsymbol{\sigma}_\tau| < \mu |\mathscr{R}\sigma_\nu| \;\Rightarrow\; \dot{\boldsymbol{u}}_\tau = 0 \\ |\boldsymbol{\sigma}_\tau| = \mu |\mathscr{R}\sigma_\nu| \;\Rightarrow\; \exists \lambda \ge 0\,,\ \dot{\boldsymbol{u}}_\tau = -\lambda \boldsymbol{\sigma}_\tau \end{cases}\right. \quad \text{on } \Gamma_2 \times (0, T)\,, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in } \Omega\,. \end{cases} \tag{9.62}$$

with $\mathscr{A} = (a_{ijkh})$ satisfying the conditions (9.8).

In order to write a variational formulation for the problem ($\mathscr{S}$), we define the following Hilbert spaces:

$$\begin{aligned} \boldsymbol{V} &= \{\boldsymbol{v} \in [H^1(\Omega)]^d\,;\ \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_0\,;\ v_\nu = 0 \text{ a.e. on } \Gamma_2\}\,, \\ \boldsymbol{W} &= \{\boldsymbol{v} \in \boldsymbol{V}\,;\ \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{v}) \in (L^2(\Omega))^d\}\,, \end{aligned}$$

endowed with the inner products

$$\begin{aligned} (\boldsymbol{u}, \boldsymbol{v})_V &= \int_\Omega \epsilon_{ij}(\boldsymbol{u})\epsilon_{ij}(\boldsymbol{v})\ \mathrm{d}x \quad \forall \boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{V}, \\ (\boldsymbol{u}, \boldsymbol{v})_W &= (\boldsymbol{u}, \boldsymbol{v})_V + (\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}), \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{v}))_{(L^2(\Omega))^d} \quad \forall \boldsymbol{u},\ \boldsymbol{v} \in \boldsymbol{W}\,. \end{aligned}$$

We make the following regularity assumptions on the data

$$\begin{cases} \boldsymbol{f} \in W^{1,2}(0, T; (L^2(\Omega))^d)\,, \\ \boldsymbol{g} \in W^{1,2}(0, T; (L^2(\Gamma_1))^d)\,, \\ a_{ijkl} \in L^\infty(\Omega),\ i, j, k, l = 1, \ldots, d\,, \\ \mu \in L^\infty(\Gamma_2),\ \mu \ge 0 \text{ a.e. on } \Gamma_2\,, \\ \mathscr{R} : H^{-1/2}(\Gamma_2) \to L^2(\Gamma_2) \text{ is a linear compact operator}\,, \\ \boldsymbol{u}_0 \in \boldsymbol{V}, \end{cases} \tag{9.63}$$

where $H^{-1/2}(\Gamma_2)$ is the dual space of $H^{1/2}(\Gamma_2) = \{v \in H^{1/2}(\Gamma)\,;\ v = 0 \text{ a.e. on } \Gamma \backslash \Gamma_2\}$.

Let $\boldsymbol{F} \in W^{1,2}(0, T; \boldsymbol{V})$, where, for all $t \in [0, T]$, $\boldsymbol{F}(t)$ is the element of $\boldsymbol{V}$ defined by (9.11) and let the symmetric, V-elliptic, continuous bilinear form $a : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ defined by (9.10)$_3$. We also denote by $j : \boldsymbol{W} \times \boldsymbol{V} \to \mathbb{R}$ the functional defined by

$$j(\boldsymbol{u},\boldsymbol{v}) = \int_{\Gamma_2} \mu |\mathscr{R}\sigma_\nu(\boldsymbol{u})|\,|\boldsymbol{v}_\tau|\,\mathrm{d}s \quad \forall \boldsymbol{u} \in \boldsymbol{W} \quad \forall \boldsymbol{v} \in \boldsymbol{V}. \tag{9.64}$$

The weak formulation of problem $(\mathscr{S})$, in terms of displacements, is the following quasi-variational inequality.

Problem (S): Find $\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V})$ such that

$$\begin{cases} a(\boldsymbol{u}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t)) + j(\boldsymbol{u}(t),\boldsymbol{v}) - j(\boldsymbol{u}(t),\dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{F}(t),\boldsymbol{v}-\dot{\boldsymbol{u}}(t))_{\boldsymbol{V}} \\ \qquad\qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,, \text{ a.e. } t \in (0,T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0. \end{cases}$$

We suppose that the initial displacement $\boldsymbol{u}_0 \in \boldsymbol{V}$ satisfies the following compatibility condition

$$a(\boldsymbol{u}_0,\boldsymbol{v}) + j(\boldsymbol{u}_0,\boldsymbol{v}) \geq (\boldsymbol{F}(0),\boldsymbol{v})_{\boldsymbol{V}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,. \tag{9.65}$$

We have the following existence result.

Theorem 9.4. *There exists $\mu_1 > 0$ such that for all $\mu \in L^\infty(\Gamma_2)$ with $\mu \geq 0$ a.e. on Γ_2 and $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$, the problem* **(S)** *has at least one solution $\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V})$.*

Proof. In order to apply Theorem 4.19, we put

$$\begin{aligned} &K = K(\boldsymbol{\Theta}) = \boldsymbol{W} \quad \forall \boldsymbol{\Theta} \in \boldsymbol{V}\,, \\ &D_K = \boldsymbol{W} \times \boldsymbol{V}\,, \\ &H = L^2(\Gamma_2)\,, \quad \beta(\boldsymbol{\Theta},\boldsymbol{v}) = \mu|\mathscr{R}\sigma_\nu(\boldsymbol{v})| \quad \forall \boldsymbol{\Theta} \in \boldsymbol{V}\,, \forall \boldsymbol{v} \in \boldsymbol{W}\,, \\ &j(\boldsymbol{\Theta},\boldsymbol{v},\boldsymbol{w}) = j(\boldsymbol{v},\boldsymbol{w}) - (\boldsymbol{\Theta},\boldsymbol{w})_{\boldsymbol{V}} \quad \forall \boldsymbol{\Theta},\, \boldsymbol{w} \in \boldsymbol{V}\,, \forall \boldsymbol{v} \in \boldsymbol{W}\,, \\ &b(\boldsymbol{\Theta},\boldsymbol{v},\boldsymbol{w}) = 0 \quad \forall \boldsymbol{\Theta},\, \boldsymbol{w} \in \boldsymbol{V}\,, \forall \boldsymbol{v} \in \boldsymbol{W}\,. \end{aligned}$$

It is easy to verify that the hypotheses (4.83)–(4.90), (4.96)–(4.98), and (4.100) are satisfied. In addition, both the problems $(\tilde{\mathbf{Q}}^a)$ and $(\tilde{\mathbf{R}}^a)$, p. 68, become the following problem

$$\begin{cases} \boldsymbol{u} \in \boldsymbol{W} \\ a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + j(\boldsymbol{u},\boldsymbol{v}-\mathbf{d}) - j(\boldsymbol{u},\boldsymbol{u}-\mathbf{d}) \geq (\boldsymbol{F},\boldsymbol{v}-\boldsymbol{u}) \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,, \end{cases}$$

and so, the hypothesis (4.105) is satisfied. As for μ_1 sufficiently small the hypothesis (4.101) is verified, the existence of a solution of the problem **(S)** follows from Theorem 4.19. □

In the sequel we shall suppose that $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_1$ with $\mu_1 > 0$ sufficiently small such that the problem **(S)** has at least one solution.

The following results will be frequently used.

Lemma 9.4. *The functional j, defined by (9.64), has the properties:*

$$j(\boldsymbol{w},\boldsymbol{v}) \geq 0 \quad \forall \boldsymbol{w} \in \boldsymbol{W},\ \forall \boldsymbol{v} \in \boldsymbol{V}, \tag{9.66}$$

$$j(\boldsymbol{w},\boldsymbol{v}_1) - j(\boldsymbol{w},\boldsymbol{v}_2) \leq j(\boldsymbol{w},\boldsymbol{v}_1 - \boldsymbol{v}_2) \quad \forall \boldsymbol{w} \in \boldsymbol{W},\ \forall \boldsymbol{v}_1,\boldsymbol{v}_2 \in \boldsymbol{V} \tag{9.67}$$

$$j(\boldsymbol{w},\boldsymbol{0}) = 0 \quad \forall \boldsymbol{w} \in \boldsymbol{W}. \tag{9.68}$$

Moreover, for all $s \in [0,T]$, we have

$$\begin{cases} \liminf\limits_{n\to\infty} \int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v}_n(t))\,\mathrm{d}t \geq \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v}(t))\,\mathrm{d}t, \\ \forall \boldsymbol{w}_n \rightharpoonup \boldsymbol{w} \text{ weakly in } L^2(0,T;\boldsymbol{W}),\ \forall \boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \text{ weakly in } L^2(0,T;\boldsymbol{V}), \end{cases} \tag{9.69}$$

and

$$\begin{cases} \lim\limits_{n\to\infty} \int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v}_n)\,\mathrm{d}t = \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v})\,\mathrm{d}t, \\ \forall \boldsymbol{w}_n \rightharpoonup \boldsymbol{w} \text{ weakly in } L^2(0,T;\boldsymbol{W}),\ \forall \boldsymbol{v}_n \rightharpoonup \boldsymbol{v} \text{ weakly in } \boldsymbol{V}. \end{cases} \tag{9.70}$$

Proof. The properties (9.66), (9.67), and (9.68) are obvious.

In order to prove (9.69), we write

$$\begin{aligned} &\left| \int_0^s \left(j(\boldsymbol{w}_n(t),\boldsymbol{v}_n(t)) - j(\boldsymbol{w}(t),\boldsymbol{v}_n(t)) \right) \mathrm{d}t \right| \\ &= \left| \int_0^s \int_{\Gamma_2} \mu(|\mathscr{R}\sigma_\nu(\boldsymbol{w}_n(t))| - |\mathscr{R}\sigma_\nu(\boldsymbol{w}(t))|)|(\boldsymbol{v}_n)_\tau(t)|\,\mathrm{d}s\,\mathrm{d}t \right| \\ &\leq \int_0^s \int_{\Gamma_2} \mu\,|\mathscr{R}\sigma_\nu(\boldsymbol{w}_n(t) - \boldsymbol{w}(t))|\ |(\boldsymbol{v}_n)_\tau(t)|\,\mathrm{d}s\,\mathrm{d}t \\ &\leq \int_0^s \|\mu\|_{L^\infty(\Gamma_2)} \|\mathscr{R}\sigma_\nu(\boldsymbol{w}_n(t) - \boldsymbol{w}(t))\|_{L^2(\Gamma_2)}\ \|(\boldsymbol{v}_n)_\tau(t)\|_{(L^2(\Gamma_2))^d}\,\mathrm{d}t \\ &\leq C\mu_1 \|\mathscr{R}\sigma_\nu(\boldsymbol{w}_n - \boldsymbol{w})\|_{L^2(0,T;L^2(\Gamma_2))}\ \|\boldsymbol{v}_n\|_{L^2(0,T,V)} \leq C_1 \|\mathscr{R}\sigma_\nu(\boldsymbol{w}_n - \boldsymbol{w})\|_{L^2(0,T;L^2(\Gamma_2))}, \end{aligned}$$

and hence, as the operator $\mathscr{R}$ is compact, it follows that

$$\lim_{n\to\infty} \int_0^s \left(j(\boldsymbol{w}_n(t),\boldsymbol{v}_n(t)) - j(\boldsymbol{w}(t),\boldsymbol{v}_n(t)) \right) \mathrm{d}t = 0 \quad \forall s \in [0.T]. \tag{9.71}$$

On the other hand, for any $\boldsymbol{w} \in L^2(0,T;\boldsymbol{W})$, the mapping $\boldsymbol{v} \longmapsto \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v}(t))\,\mathrm{d}t$ is convex l.s.c. on $L^2(0,T;\boldsymbol{V})$, thus

$$\liminf_{n\to\infty} \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v}_n(t))\,\mathrm{d}t \geq \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v}(t))\,\mathrm{d}t\,. \tag{9.72}$$

By combining the relations (9.71) and (9.72), we get:

$$\liminf_{n\to\infty} \int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v}_n(t))\,\mathrm{d}t \geq \lim_{n\to\infty} \int_0^s \left(j(\boldsymbol{w}_n(t),\boldsymbol{v}_n(t)) - j(\boldsymbol{w}(t),\boldsymbol{v}_n(t))\right)\,\mathrm{d}t$$
$$+ \liminf_{n\to\infty} \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v}_n(t))\,\mathrm{d}t \geq \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v}(t))\,\mathrm{d}t\,.$$

Next, we have

$$\left|\int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v}_n)\,\mathrm{d}t - \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v})\,\mathrm{d}t\right|$$
$$\leq \left|\int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v}_n)\,\mathrm{d}t - \int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v})\,\mathrm{d}t\right| + \left|\int_0^s j(\boldsymbol{w}_n(t),\boldsymbol{v})\,\mathrm{d}t - \int_0^s j(\boldsymbol{w}(t),\boldsymbol{v})\,\mathrm{d}t\right|$$
$$\leq C_1\|\boldsymbol{v}_n - \boldsymbol{v}\|_{(L^2(\Gamma_2))^d} + C_2\|\mathscr{R}\sigma\nu(\boldsymbol{w}_\nu - \boldsymbol{w})\|_{L^2(0,T;L^2(\Gamma_2))}$$

and hence, from the compactness of the trace map from $\boldsymbol{V}$ into $(L^2(\Gamma_2))^d$, the proof is completed. □

Now, we are interested in finding the surface tractions $\boldsymbol{g}$ acting on Γ_1 so that the resulting displacement on the contact boundary Γ_2 is as close as possible to a given profile $\boldsymbol{u}_d$, while the norm of these surface forces remains small enough. The mathematical formulation of this problem is a state-control boundary optimal control problem where the state is solution of the implicit evolutionary quasi-variational inequality (**S**).

We introduce the following control and, respectively, observation spaces:

$$\begin{aligned} \mathbf{H_g} &= W^{1,2}(0,T;(L^2(\Gamma_1))^d)\,, \\ \mathbf{H_u} &= L^2(0,T;(L^2(\Gamma_2))^d) \end{aligned} \tag{9.73}$$

and we define, for $\beta > 0$ and $\boldsymbol{u}_d \in \mathbf{H_u}$ given, the cost functional $J : \mathbf{H_g} \times W^{1,2}(0,T;\mathbf{V}) \to \mathbb{R}_+$ by:

$$J(\boldsymbol{g},\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u}-\boldsymbol{u}_d\|^2_{\mathbf{H_u}} + \frac{\beta}{2}\|\boldsymbol{g}\|^2_{\mathbf{H_g}}\,. \tag{9.74}$$

Due to the lack of uniqueness of solution for the quasi-variational inequality (**S**), the cost functional J, instead of depending, as usual, only on the "real" control $\boldsymbol{g}$, depends also on the state $\boldsymbol{u}$. For this reason, it is convenient to rewrite the variational problem (**S**), for $\boldsymbol{g} \in \mathbf{H_g}$, in the following form.

Problem $(\mathbf{S})^{\boldsymbol{g}}$: Find $\boldsymbol{u} \in W^{1,2}(0,T;\mathbf{V})$ such that

$$\begin{cases} a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j(\boldsymbol{u}(t), \boldsymbol{v}) - j(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{F}^{g}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \\ \qquad\qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,, \text{ a.e. } t \in (0,T) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0\,, \end{cases}$$

where

$$(\boldsymbol{F}^{g}(t), \boldsymbol{v})_V = \int_{\Omega} \boldsymbol{f}(t) \cdot \boldsymbol{v} \, \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}(t) \cdot \boldsymbol{v} \, \mathrm{d}s \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,.$$

We formulate now the control problem as follows:

Problem (CS): Find $(\boldsymbol{g}^*, \boldsymbol{u}^*) \in \mathscr{V}_{ad}$ such that

$$J(\boldsymbol{g}^*, \boldsymbol{u}^*) = \min_{(\boldsymbol{g},\boldsymbol{u}) \in \mathscr{V}_{ad}} J(\boldsymbol{g}, \boldsymbol{u})\,,$$

where

$$\mathscr{V}_{ad} = \{(\boldsymbol{g}, \boldsymbol{u}) \in \mathbf{H_g} \times W^{1,2}(0,T;\mathbf{V})\,; \boldsymbol{u} \text{ is a solution of } (\mathbf{S})^{\boldsymbol{g}}\ \}.$$

Remark 9.2. Let us assume that there exist $(\boldsymbol{g}^*, \boldsymbol{u}^*) \in \mathscr{V}_{ad}$ such that $J(\boldsymbol{g}^*, \boldsymbol{u}^*) = \min_{(\boldsymbol{g},\boldsymbol{u}) \in \mathscr{V}_{ad}} J(\boldsymbol{g}, \boldsymbol{u})$ and a function $\boldsymbol{g}_d \in \mathbf{H_g}$ such that $(\boldsymbol{g}_d, \boldsymbol{u}_d) \in \mathscr{V}_{ad}$. Then,

$$J(\boldsymbol{g}^*, \boldsymbol{u}^*) = \frac{1}{2}\|\boldsymbol{u}^* - \boldsymbol{u}_d\|^2_{\mathbf{H_u}} + \frac{\beta}{2}\|\boldsymbol{g}^*\|^2_{\mathbf{H_g}} \leq J(\boldsymbol{g}_d, \boldsymbol{u}_d) = \frac{\beta}{2}\|\boldsymbol{g}_d\|^2_{\mathbf{H_g}}$$

and, hence,

$$\|\boldsymbol{u}^* - \boldsymbol{u}_d\|^2_{\mathbf{H_u}} \leq \beta(\|\boldsymbol{g}_d\|^2_{\mathbf{H_g}} - \|\boldsymbol{g}^*\|^2_{\mathbf{H_g}})\,.$$

Therefore, for β arbitrarily small, we may hope to obtain, on the contact boundary, a displacement field $\boldsymbol{u}$ as closed as we want to the desired value $\boldsymbol{u}_d$.

As one can see, although the functional J has good properties on $\mathbf{H_g} \times W^{1,2}(0,T;\mathbf{V})$, the existence of a solution of the control problem (**CS**) cannot be obtained directly, since the correspondence control $\mapsto$ state is a multivalued mapping. In order to overcome this difficulty, we approximate the optimal control problem (**CS**) by a family of penalized optimal control problems, governed by a variational inequality.

We start by introducing a new control space:

$$\mathbf{H_w} = L^2(0, T; \boldsymbol{W}) .$$

Now, for $(\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$, we consider the variational inequality which models the problem $(\mathscr{S})$ in the case of Tresca friction.

Problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$: Find $\boldsymbol{u} \in W^{1,2}(0, T; \mathbf{V})$ such that

$$\begin{cases} a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j(\boldsymbol{w}(t), \boldsymbol{v}) - j(\boldsymbol{w}(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{F}^{\boldsymbol{g}}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \\ \qquad\qquad \forall\, \boldsymbol{v} \in \boldsymbol{V} \,, \text{ a.e. } t \in (0, T) \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 \,. \end{cases}$$

Using the same techniques as in [7] or Sect. 4.3 and taking into account the positivity of j, one can prove the following existence result.

Proposition 9.3. *For* $(\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$ *given, there exists a unique solution* $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}$ *of Problem* $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$. *Moreover, we have*

$$\|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}\|_{L^2(0,T;V)} \leq C(\|\dot{\boldsymbol{F}}^{\boldsymbol{g}}\|_{L^2(0,T;V)} + \|\boldsymbol{w}\|_{L^2(0,T;V)}) ,$$

with C *a positive constant.*

In the sequel, for $(\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$ given, we will denote by $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}$ the unique solution of Problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$.

Let us fix $\epsilon > 0$. We introduce the penalized functional $J_\epsilon : \mathbf{H_g} \times \mathbf{H_w} \to \mathbb{R}_+$ by

$$J_\epsilon(\boldsymbol{g}, \boldsymbol{w}) = J(\boldsymbol{g}, \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}) + \frac{1}{2\epsilon} \|\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}} - \boldsymbol{w}\|^2_{\mathbf{H_w}} \tag{9.75}$$

and we consider the control problem

Problem $(\mathbf{CS})_\epsilon$: Find $(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) \in \mathbf{H_g} \times \mathbf{H_w}$ such that

$$J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) = \min\{J_\epsilon(\boldsymbol{g}, \boldsymbol{w}) ;\ (\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}\} .$$

The following result establishes the existence of an optimal solution for this penalized control problem.

Proposition 9.4. *Let (9.63) and (9.65) hold. Then, for all* $\epsilon > 0$, *there exists a solution* $(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon)$ *of problem* $(\mathbf{CS})_\epsilon$.

Proof. Let $\{(\boldsymbol{g}^n_\epsilon, \boldsymbol{w}^n_\epsilon)\}_n \subset \mathbf{H_g} \times \mathbf{H_w}$ be a minimizing sequence for the functional J_ϵ. Then, from the definition (9.75) of J_ϵ, we deduce

$$\lim_{n\to\infty} J_\epsilon(\boldsymbol{g}^n_\epsilon, \boldsymbol{w}^n_\epsilon) = \inf\{J_\epsilon(\boldsymbol{g}, \boldsymbol{w}) ,\ (\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}\} \in [0, +\infty) , \tag{9.76}$$

which implies that the sequence $\{\boldsymbol{g}_\epsilon^n\}_n$ is bounded in $\mathbf{H_g}$. Obviously, the sequence $\{\boldsymbol{F}_\epsilon^n\}_n$ defined by

$$(\boldsymbol{F}_\epsilon^n(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t)\cdot \boldsymbol{v}\ \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}_\epsilon^n(t)\cdot \boldsymbol{v}\ \mathrm{d}s \tag{9.77}$$

is also bounded in $W^{1,2}(0,T;\mathbf{V})$.

Thus, there exists $(\boldsymbol{g}_\epsilon^*, \boldsymbol{F}_\epsilon^*) \in \mathbf{H_g} \times W^{1,2}(0,T;\mathbf{V})$ such that, passing to a subsequence still denoted in the same way, we have

$$\boldsymbol{g}_\epsilon^n \rightharpoonup \boldsymbol{g}_\epsilon^* \text{ weakly in } \mathbf{H_g}\,, \tag{9.78}$$

$$\boldsymbol{F}_\epsilon^n \rightharpoonup \boldsymbol{F}_\epsilon^* \text{ weakly in } W^{1,2}(0,T;\mathbf{V})\,, \tag{9.79}$$

where

$$(\boldsymbol{F}_\epsilon^*(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t)\cdot \boldsymbol{v}\ \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}_\epsilon^*(t)\cdot \boldsymbol{v}\ \mathrm{d}s\,.$$

Let $\boldsymbol{u}_\epsilon^n = \boldsymbol{u}^{\boldsymbol{g}_\epsilon^n, \boldsymbol{w}_\epsilon^n}$. Taking $\boldsymbol{v} = \mathbf{0}$ in $(\mathbf{S})^{\boldsymbol{g}_\epsilon^n, \boldsymbol{w}_\epsilon^n}$, integrating by parts on $[0,s]$ with $s \in [0,T]$ and taking into account the properties (9.66), (9.68) of the functional j, we have

$$\int_0^s a(\boldsymbol{u}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))\,\mathrm{d}t \le \int_0^s (\boldsymbol{F}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))_V\,\mathrm{d}t\,. \tag{9.80}$$

By using the V-ellipticity of $a(\cdot,\cdot)$, we obviously obtain

$$\begin{aligned}\int_0^s a(\boldsymbol{u}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))\,\mathrm{d}t &= \frac{1}{2}\int_0^s \frac{\mathrm{d}}{\mathrm{d}t} a(\boldsymbol{u}_\epsilon^n(t), \boldsymbol{u}_\epsilon^n(t))\,\mathrm{d}t\\ &= \frac{a(\boldsymbol{u}_\epsilon^n(s), \boldsymbol{u}_\epsilon^n(s)) - a(\boldsymbol{u}_0, \boldsymbol{u}_0)}{2} \ge \frac{\alpha\|\boldsymbol{u}_\epsilon^n(s)\|_V^2 - a(\boldsymbol{u}_0, \boldsymbol{u}_0)}{2}.\end{aligned} \tag{9.81}$$

On the other hand, we have

$$\begin{aligned}&\left|\int_0^s (\boldsymbol{F}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))_V\,\mathrm{d}t\right| = \left|\int_0^s \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{F}_\epsilon^n(t), \boldsymbol{u}_\epsilon^n(t))_V\,\mathrm{d}t - \int_0^s (\dot{\boldsymbol{F}}_\epsilon^n(t), \boldsymbol{u}_\epsilon^n(t))_V\,\mathrm{d}t\right|\\
&\le C\left(|(\boldsymbol{F}_\epsilon^n(s), \boldsymbol{u}_\epsilon^n(s))_V - (\boldsymbol{F}_\epsilon^n(0), \boldsymbol{u}_\epsilon^n(0))_V| + \int_0^s \|\dot{\boldsymbol{F}}_\epsilon^n(t)\|_V^2\,\mathrm{d}t + \int_0^s \|\boldsymbol{u}_\epsilon^n(t)\|_V^2\,\mathrm{d}t\right)\\
&\le C\left(\frac{\|\boldsymbol{F}_\epsilon^n(s)\|_V^2}{2\delta} + \frac{\delta\|\boldsymbol{u}_\epsilon^n(s)\|_V^2}{2} + \frac{\|\boldsymbol{F}_\epsilon^n(0)\|_V^2}{2} + \frac{\|\boldsymbol{u}_\epsilon^n(0)\|_V^2}{2}\right.\\
&\quad\left. + \int_0^s \|\dot{\boldsymbol{F}}_\epsilon^n(t)\|_V^2\,\mathrm{d}t + \int_0^s \|\boldsymbol{u}_\epsilon^n(t)\|_V^2\,\mathrm{d}t\right).\end{aligned}$$

By choosing $0 < \delta < \dfrac{\alpha}{C}$ in the last relation, from (9.80), (9.81) and Young's inequality, we get

$$\|\boldsymbol{u}_\epsilon^n(s)\|_V^2 \leq C \left(\|\boldsymbol{u}_0\|_V^2 + \|\boldsymbol{F}_\epsilon^n(s)\|_V^2 + \|\boldsymbol{F}_\epsilon^n(0)\|_V^2 + \int_0^s \|\dot{\boldsymbol{F}}_\epsilon^n(t)\|_V^2 \,\mathrm{d}t \right.$$
$$\left. + \int_0^s \|\boldsymbol{u}_\epsilon^n(t)\|_V^2 \,\mathrm{d}t \right),$$

and hence, by using Gronwall's inequality and the boundedness of $\{\boldsymbol{F}_\epsilon^n\}_n$, it follows that

$$\|\boldsymbol{u}_\epsilon^n(s)\|_V^2 \leq C \left(1 + \|\boldsymbol{F}_\epsilon^n(0)\|_V^2 + \|\dot{\boldsymbol{F}}_\epsilon^n\|_{L^2(0,T;V)}\right) \leq C \quad \forall s \in [0,T]. \tag{9.82}$$

Therefore, the sequence $\{\boldsymbol{u}_\epsilon^n\}_n$ is bounded in $L^\infty(0,T;\boldsymbol{V})$. In addition, from $(\mathbf{S})^{\boldsymbol{g}_\epsilon^n, \boldsymbol{w}_\epsilon^n}$, we have

$$\|\boldsymbol{u}_\epsilon^n\|_{\mathbf{H_w}}^2 = \|\boldsymbol{u}_\epsilon^n\|_{L^2(0,T;V)}^2 + \|\operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}_\epsilon^n)\|_{L^2(0,T;(L^2(\Omega))^d)}^2$$
$$= \|\boldsymbol{u}_\epsilon^n\|_{L^2(0,T;V)}^2 + \|\boldsymbol{f}\|_{L^2(0,T;(L^2(\Omega))^d)}^2 \leq C,$$

which, from the definition of J_ϵ and the boundedness (9.76) of J_ϵ, implies that the sequence $\{\boldsymbol{w}_\epsilon^n\}_n$ is bounded in $\mathbf{H_w}$.

Now, from Proposition 9.3, we obtain

$$\|\dot{\boldsymbol{u}}_\epsilon^n\|_{L^2(0,T;V)} \leq C. \tag{9.83}$$

Thus, we deduce that there exist the elements $\boldsymbol{u}_\epsilon^* \in W^{1,2}(0,T;\boldsymbol{V})$ and $\boldsymbol{w}_\epsilon^* \in \mathbf{H_w}$ and the subsequences, still denoted by $\{\boldsymbol{u}_\epsilon^n\}_n$ and $\{\boldsymbol{w}_\epsilon^n\}_n$, such that

$$\boldsymbol{w}_\epsilon^n \rightharpoonup \boldsymbol{w}_\epsilon^* \text{ weakly in } \mathbf{H_w}, \tag{9.84}$$

$$\begin{cases} \boldsymbol{u}_\epsilon^n \rightharpoonup \boldsymbol{u}_\epsilon^* \text{ weakly * in } L^\infty(0,T;\boldsymbol{V}), \\ \dot{\boldsymbol{u}}_\epsilon^n \rightharpoonup \dot{\boldsymbol{u}}_\epsilon^* \text{ weakly in } L^2(0,T;\boldsymbol{V}). \end{cases} \tag{9.85}$$

Using the embedding $W^{1,2}(0,T;\boldsymbol{V}) \hookrightarrow C([0,T];\boldsymbol{V})$, we also have

$$\boldsymbol{u}_\epsilon^n(t) \rightharpoonup \boldsymbol{u}_\epsilon^*(t) \text{ weakly in } \boldsymbol{V} \quad \forall t \in [0,T]. \tag{9.86}$$

Now, we shall prove the strong convergence of $\boldsymbol{u}_\epsilon^n$ to $\boldsymbol{u}$ in $L^2(0,T;\boldsymbol{V})$. Putting $\boldsymbol{v} = \boldsymbol{0}$ and $\boldsymbol{v} = 2\dot{\boldsymbol{u}}_\epsilon^n(t)$ in $(\mathbf{S})^{\boldsymbol{g}_\epsilon^n, \boldsymbol{w}_\epsilon^n}$, one obtains:

$$a(\boldsymbol{u}_\epsilon^n(t), \boldsymbol{v}) + j(\boldsymbol{w}_\epsilon^n(t), \boldsymbol{v}) \geq (\boldsymbol{F}_\epsilon^n(t), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \text{ a.e. } t \in (0,T),$$

Taking $\boldsymbol{v} = -\boldsymbol{v}$, it follows that

$$a(\boldsymbol{u}_\epsilon^n(t), \boldsymbol{v}) - j(\boldsymbol{w}_\epsilon^n(t), \boldsymbol{v}) \leq (\boldsymbol{F}_\epsilon^n(t), \boldsymbol{v})_V \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \text{ a.e. } t \in (0,T). \tag{9.87}$$

Passing to the limit with $n \to \infty$ in this inequality and taking into account the convergences (9.85), (9.84), and (9.79), we obtain

$$a(\boldsymbol{u}_\epsilon^*(t), \boldsymbol{v}) - j(\boldsymbol{w}_\epsilon^*(t), \boldsymbol{v}) \le (\boldsymbol{F}_\epsilon^*(t), \boldsymbol{v})_V \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,, \text{ a.e. } t \in (0, T)\,. \tag{9.88}$$

Setting $\boldsymbol{v} = \boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t)$ in (9.87) and $\boldsymbol{v} = \boldsymbol{u}_\epsilon^*(t) - \boldsymbol{u}_\epsilon^n(t)$ in (9.88), we get

$$\begin{aligned}
&\alpha \|\boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t)\|_V^2 \le a(\boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t), \boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t))\\
&\le C \|\mu\|_{L^\infty(\Gamma_2)} \left(\|\mathscr{R}\sigma_\nu(\boldsymbol{w}_\epsilon^n(t))\|_{L^2(\Gamma_2)} + \|\mathscr{R}\sigma_\nu(\boldsymbol{w}_\epsilon^*(t))\|_{L^2(\Gamma_2)} \right) \|\boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t)\|_{(L^2(\Gamma_2))^d}\\
&+ \|\boldsymbol{g}_\epsilon^n(t) - \boldsymbol{g}_\epsilon^*(t)\|_{(L^2(\Gamma_1))^d} \|\boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t)\|_{(L^2(\Gamma_1))^d} \le C \|\boldsymbol{u}_\epsilon^n(t) - \boldsymbol{u}_\epsilon^*(t)\|_{(L^2(\Gamma))^d}\,,
\end{aligned}$$

which, with (9.86) and the compactness of the trace map from $\boldsymbol{V}$ to $(L^2(\Gamma))^d$, implies

$$\boldsymbol{u}_\epsilon^n(t) \to \boldsymbol{u}_\epsilon^*(t) \text{ strongly in } \boldsymbol{V} \quad \forall t \in [0, T]\,. \tag{9.89}$$

Hence, by Lebesgue's Theorem 3.4, we obtain the strong convergence:

$$\boldsymbol{u}_\epsilon^n \to \boldsymbol{u}_\epsilon^* \text{ strongly in } L^2(0, T; \boldsymbol{V})\,. \tag{9.90}$$

We shall prove that $\boldsymbol{u}_\epsilon^* = \boldsymbol{u}^{\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*}$ and, from the uniqueness of the solution, we shall conclude that the convergences (9.78), (9.84), (9.85), and (9.89) hold true for the whole sequences.

For $s \in [0, T]$, from the convergences (9.85), (9.90), (9.84), (9.79) and the properties (9.69), (9.70), we have

$$\lim_{n\to\infty} \int_0^s a(\boldsymbol{u}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))\, \mathrm{d}t = \int_0^s a(\boldsymbol{u}_\epsilon^*(t), \dot{\boldsymbol{u}}_\epsilon^*(t))\, \mathrm{d}t\,, \tag{9.91}$$

$$\lim_{n\to\infty} \int_0^s a(\boldsymbol{u}_\epsilon^n(t), \boldsymbol{v}(t))\, \mathrm{d}t = \int_0^s a(\boldsymbol{u}_\epsilon^*(t), \boldsymbol{v}(t))\, \mathrm{d}t \quad \forall \boldsymbol{v} \in L^2(0, T; \boldsymbol{V})\,, \tag{9.92}$$

$$\lim_{n\to\infty} \int_0^s (\boldsymbol{F}_\epsilon^n(t), \boldsymbol{v}(t))_V\, \mathrm{d}t = \int_0^s (\boldsymbol{F}_\epsilon^*(t), \boldsymbol{v}(t))_V\, \mathrm{d}t \quad \forall \boldsymbol{v} \in L^2(0, T; \boldsymbol{V})\,, \tag{9.93}$$

$$\lim_{n\to\infty} \int_0^s j(\boldsymbol{w}_\epsilon^n(t), \boldsymbol{v}(t))\, \mathrm{d}t = \int_0^s j(\boldsymbol{w}_\epsilon^*(t), \boldsymbol{v}(t))\, \mathrm{d}t \quad \forall \boldsymbol{v} \in L^2(0, T; \boldsymbol{V})\,, \tag{9.94}$$

$$\liminf_{n\to\infty} \int_0^s j(\boldsymbol{w}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))\, \mathrm{d}t \ge \int_0^s j(\boldsymbol{w}_\epsilon^*(t), \dot{\boldsymbol{u}}_\epsilon^*(t))\, \mathrm{d}t\,. \tag{9.95}$$

Next, since we can write

$$\int_0^s (\boldsymbol{F}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))_V \, \mathrm{d}t = (\boldsymbol{F}_\epsilon^n(s), \boldsymbol{u}_\epsilon^n(s))_V - (\boldsymbol{F}_\epsilon^n(0), \boldsymbol{u}_0)_V \, \mathrm{d}t - \int_0^s (\dot{\boldsymbol{F}}_\epsilon^n(t), \boldsymbol{u}_\epsilon^n(t))_V \, \mathrm{d}t \,,$$

it follows that

$$\lim_{n\to\infty} \int_0^s (\boldsymbol{F}_\epsilon^n(t), \dot{\boldsymbol{u}}_\epsilon^n(t))_V \, \mathrm{d}t = \int_0^s (\boldsymbol{F}_\epsilon^*(t), \dot{\boldsymbol{u}}_\epsilon^*(t))_V \, \mathrm{d}t \,. \tag{9.96}$$

Now, by passing to the limit in $(\mathbf{S})^{g_\epsilon^n, w_\epsilon^n}$ with $n \to \infty$, one obtains

$$\begin{aligned} &\int_0^s a(\boldsymbol{u}_\epsilon^*(t), \boldsymbol{v}(t) - \dot{\boldsymbol{u}}_\epsilon^*(t)) \, \mathrm{d}t + \int_0^s j(\boldsymbol{w}_\epsilon^*(t), \boldsymbol{v}(t)) \, \mathrm{d}t - \int_0^s j(\boldsymbol{w}_\epsilon^*(t), \dot{\boldsymbol{u}}_\epsilon^*(t)) \, \mathrm{d}t \\ &\geq \int_0^s (\boldsymbol{F}_\epsilon^*(t), \boldsymbol{v}(t) - \dot{\boldsymbol{u}}_\epsilon^*(t))_V \, \mathrm{d}t \quad \forall\, \boldsymbol{v} \in L^2(0, T; \boldsymbol{V}) \,, \ \forall s \in [0, T] \,. \end{aligned} \tag{9.97}$$

Then, as usually, taking $\boldsymbol{v} \in L^2(0, T; \boldsymbol{V})$ defined by

$$\boldsymbol{v}(t) = \begin{cases} \boldsymbol{z} & \text{for } t \in [s, s+h] \,, \\ \dot{\boldsymbol{u}}_\epsilon^*(t) & \text{otherwise} \,, \end{cases}$$

with an arbitrary $\boldsymbol{z} \in \boldsymbol{V}$ and $h > 0$ such that $s + h \leq T$, one obtains

$$\begin{aligned} &\int_s^{s+h} a(\boldsymbol{u}_\epsilon^*(t), \boldsymbol{z} - \dot{\boldsymbol{u}}_\epsilon^*(t)) \, \mathrm{d}t + \int_s^{s+h} j(\boldsymbol{w}_\epsilon^*(t), \boldsymbol{z}) \, \mathrm{d}t - \int_s^{s+h} j(\boldsymbol{w}_\epsilon^*(t), \dot{\boldsymbol{u}}_\epsilon^*(t)) \, \mathrm{d}t \\ &\geq \int_s^{s+h} (\boldsymbol{F}_\epsilon^*(t), \boldsymbol{z} - \dot{\boldsymbol{u}}_\epsilon^*(t))_V \, \mathrm{d}t \quad \forall \boldsymbol{z} \in \boldsymbol{V} \,, \ \forall s \in [0, T) \,, \end{aligned} \tag{9.98}$$

which leads us, by passing to the limit with $h \to 0$, to the following inequality

$$\begin{aligned} &a(\boldsymbol{u}_\epsilon^*(t), \boldsymbol{z} - \dot{\boldsymbol{u}}_\epsilon^*(t)) + j(\boldsymbol{w}_\epsilon^*(t), \boldsymbol{z}) - j(\boldsymbol{w}_\epsilon^*(t), \dot{\boldsymbol{u}}_\epsilon^*(t)) \\ &\geq (\boldsymbol{F}_\epsilon^*(t), \boldsymbol{z} - \dot{\boldsymbol{u}}_\epsilon^*(t))_V \quad \forall \boldsymbol{z} \in \boldsymbol{V} \ \text{ a.e. } t \in (0, T) \,. \end{aligned} \tag{9.99}$$

Moreover, the pointwise convergence (9.89) and the initial condition $\boldsymbol{u}_\epsilon^n(0) = \boldsymbol{u}_0$ give us $\boldsymbol{u}_\epsilon^*(0) = \boldsymbol{u}_0$ and, so, $\boldsymbol{u}_\epsilon^* = \boldsymbol{u}^{g_\epsilon^*, w_\epsilon^*}$, i.e. $\boldsymbol{u}_\epsilon^*$ is the unique solution of problem $(\mathbf{S})^{g_\epsilon^*, w_\epsilon^n}$.

In order to end the proof of our existence result, let us notice that, from $(\mathbf{S})^{\boldsymbol{g}_\epsilon^n,\boldsymbol{w}_\epsilon^n}$ and (9.99), it follows that

$$\|\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^*\|_{\mathbf{H_w}} = \|\boldsymbol{u}_\epsilon^n - \boldsymbol{u}_\epsilon^*\|_{L^2(0,T;V)},$$

which obviously, from (9.90), gives

$$\boldsymbol{u}_\epsilon^n \to \boldsymbol{u}_\epsilon^* \text{ strongly in } \mathbf{H_w}.$$

Therefore, since the norm is weakly lower semicontinuous, from the convergence (9.84), we get

$$\liminf_{n\to\infty} \frac{1}{2\epsilon}\|\boldsymbol{w}_\epsilon^n - \boldsymbol{u}_\epsilon^n\|_{\mathbf{H_w}}^2 \geq \frac{1}{2\epsilon}\|\boldsymbol{w}_\epsilon^* - \boldsymbol{u}_\epsilon^*\|_{\mathbf{H_w}}^2. \tag{9.100}$$

Finally, by using the convergences (9.90), (9.78) and the relation (9.100), we have

$$\begin{aligned}&\inf\{J_\epsilon(\boldsymbol{g},\boldsymbol{w})\,;\ (\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H_g}\times\mathbf{H_w}\}\\ &= \lim_{n\to\infty} J_\epsilon(\boldsymbol{g}_\epsilon^n,\boldsymbol{w}_\epsilon^n) \geq \liminf_{n\to\infty} J_\epsilon(\boldsymbol{g}_\epsilon^n,\boldsymbol{w}_\epsilon^n) \geq J_\epsilon(\boldsymbol{g}_\epsilon^*,\boldsymbol{w}_\epsilon^*)\end{aligned}$$

and hence, we conclude

$$J_\epsilon(\boldsymbol{g}_\epsilon^*,\boldsymbol{w}_\epsilon^*) = \min\{J_\epsilon(\boldsymbol{g},\boldsymbol{w})\,;\ (\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H_g}\times\mathbf{H_w}\}.$$

□

Lemma 9.5. *If* $(\boldsymbol{g}_\epsilon^*,\boldsymbol{w}_\epsilon^*)$ *is an optimal control for* $(\mathbf{CS})_\epsilon$ *and* $\boldsymbol{u}_\epsilon^* = \boldsymbol{u}^{\boldsymbol{g}_\epsilon^*,\boldsymbol{w}_\epsilon^*}$*, then*

$$\lim_{\epsilon\to 0} \|\boldsymbol{w}_\epsilon^* - \boldsymbol{u}_\epsilon^*\|_{\mathbf{H_w}} = 0. \tag{9.101}$$

Proof. Indeed, if $(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}) \in \mathscr{V}_{ad}$, then $\tilde{\boldsymbol{u}} \in \mathbf{H_w}$, $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}}$ and, hence,

$$J_\epsilon(\boldsymbol{g}_\epsilon^*,\boldsymbol{w}_\epsilon^*) \leq J_\epsilon(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}) = J(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}). \tag{9.102}$$

Consequently, from the definition of J_ϵ, we get

$$\|\boldsymbol{w}_\epsilon^* - \boldsymbol{u}_\epsilon^*\|_{\mathbf{H_w}}^2 \leq 2\epsilon J_\epsilon(\boldsymbol{g}_\epsilon^*,\boldsymbol{w}_\epsilon^*) \leq 2\epsilon J(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}),$$

which implies (9.101). □

We are now in the position to prove the main result of this section, the existence of a solution to the optimal control problem **(CS)**.

Theorem 9.5. *For $\epsilon > 0$, let $(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) \in \mathbf{H_g} \times \mathbf{H_w}$ be an optimal control of* $(\mathbf{CS})_\epsilon$ *and $\boldsymbol{u}_\epsilon^* = \boldsymbol{u}^{\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*}$. Then, there exist the elements $\boldsymbol{u}^* \in W^{1,2}(0,T;\mathbf{V})$ and $\boldsymbol{g}^* \in \mathbf{H_g}$ such that*

$$\begin{aligned} &\boldsymbol{g}_\epsilon^* \to \boldsymbol{g}^* \ \textit{weakly in}\ \mathbf{H_g}\,,\\ &\boldsymbol{w}_\epsilon^* \rightharpoonup \boldsymbol{u}^* \ \textit{strongly in}\ \mathbf{H_w}\,,\\ &\boldsymbol{u}_\epsilon^* \rightharpoonup \boldsymbol{u}^* \ \textit{weakly in}\ W^{1,2}(0,T;\mathbf{V})\,,\\ &\boldsymbol{u}_\epsilon^* \to \boldsymbol{u}^* \ \textit{strongly in}\ L^2(0,T;V)\,. \end{aligned} \tag{9.103}$$

Moreover, $(\boldsymbol{g}^*, \boldsymbol{u}^*) \in \mathscr{V}_{ad}$ and

$$\lim_{\epsilon\to 0} J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) = J(\boldsymbol{g}^*, \boldsymbol{u}^*) = \min_{(\boldsymbol{g},\boldsymbol{u})\in\mathscr{V}_{ad}} J(\boldsymbol{g}, \boldsymbol{u})\,. \tag{9.104}$$

Proof. From the definition and the boundedness (9.102) of $J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*)$, it follows that the sequence $\{\boldsymbol{g}_\epsilon^*\}_\epsilon$ is bounded in $\mathbf{H_g}$. Therefore, there exists $\boldsymbol{g}^* \in \mathbf{H_g}$ such that, up to a subsequence, we have

$$\boldsymbol{g}_\epsilon^* \rightharpoonup \boldsymbol{g}^* \text{ weakly in } \mathbf{H_g}\,. \tag{9.105}$$

So,

$$\boldsymbol{F}_\epsilon^* \rightharpoonup \boldsymbol{F}^* \text{ weakly in } W^{1,2}(0,T,V)\,, \tag{9.106}$$

where

$$(\boldsymbol{F}_\epsilon^*(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\ \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}_\epsilon^*(t)\cdot\boldsymbol{v}\ \mathrm{d}s \tag{9.107}$$

and

$$(\boldsymbol{F}^*(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\ \mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}^*(t)\cdot\boldsymbol{v}\ \mathrm{d}s\,.$$

Using the same arguments as in the proof of Proposition 9.4, we deduce

$$\begin{cases} \boldsymbol{u}_\epsilon^* \rightharpoonup \boldsymbol{u}^* \text{ weakly } * \text{ in } L^\infty(0,T;V)\,,\\ \dot{\boldsymbol{u}}_\epsilon^* \rightharpoonup \dot{\boldsymbol{u}}^* \text{ weakly in } L^2(0,T;V)\,,\\ \boldsymbol{u}_\epsilon^* \to \boldsymbol{u}^* \text{ strongly in } L^2(0,T;V)\,,\\ \boldsymbol{w}_\epsilon^* \rightharpoonup \boldsymbol{w}^* \text{ weakly in } \mathbf{H_w}\,, \end{cases} \tag{9.108}$$

with $\boldsymbol{u}^* \in W^{1,2}(0,T;\mathbf{V})$ and $\boldsymbol{w}^* \in \mathbf{H_w}$.

Passing to the limit with $\epsilon \to 0$ in the integral form of $(\mathbf{S})^{g_\epsilon^*, w_\epsilon^n}$, we deduce that $\boldsymbol{u}^* = \boldsymbol{u}^{\boldsymbol{g}^*, \boldsymbol{w}^*}$. As

$$\|\boldsymbol{u}_\epsilon^* - \boldsymbol{u}^*\|_{\mathbf{H_w}} = \|\boldsymbol{u}_\epsilon^* - \boldsymbol{u}^*\|_{L^2(0,T;V)}\,,$$

we have

$$\boldsymbol{u}_\epsilon^* \to \boldsymbol{u}^* \text{ strongly in } \mathbf{H_w}\,,$$

and thus, from (9.101), we get $(9.103)_2$, $\boldsymbol{w}^* = \boldsymbol{u}^*$ and $(\boldsymbol{g}^*, \boldsymbol{u}^*) \in \mathscr{V}_{ad}$.

Next, from the definition of J_ϵ, we have

$$\begin{aligned}&\frac{1}{2\epsilon}\|\boldsymbol{w}_\epsilon^* - \boldsymbol{u}_\epsilon^*\|^2_{\mathbf{H_w}} = J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) - J(\boldsymbol{g}_\epsilon^*, \boldsymbol{u}_\epsilon^*)\\ &\le J_\epsilon(\boldsymbol{g}^*, \boldsymbol{u}^*) - J(\boldsymbol{g}_\epsilon^*, \boldsymbol{u}_\epsilon^*) = J(\boldsymbol{g}^*, \boldsymbol{u}^*) - J(\boldsymbol{g}_\epsilon^*, \boldsymbol{u}_\epsilon^*)\,,\end{aligned}$$

so,

$$0 \le \limsup_{\epsilon\to 0} \frac{1}{2\epsilon}\|\boldsymbol{w}_\epsilon^* - \boldsymbol{u}_\epsilon^*\|^2_{\mathbf{H_w}} \le J(\boldsymbol{g}^*, \boldsymbol{u}^*) - \liminf_{\epsilon\to 0} J(\boldsymbol{g}_\epsilon^*, \boldsymbol{u}_\epsilon^*) \le 0\,,$$

i.e.

$$\lim_{\epsilon\to 0} \frac{1}{\epsilon}\|\boldsymbol{w}_\epsilon^* - \boldsymbol{u}_\epsilon^*\|^2_{\mathbf{H_w}} = 0\,. \tag{9.109}$$

Finally, it is easy to see that

$$\begin{aligned}J(\boldsymbol{g}^*, \boldsymbol{u}^*) &\le \liminf_{\epsilon\to 0} J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) \le \limsup_{\epsilon\to 0} J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) \le \limsup_{\epsilon\to 0} J_\epsilon(\boldsymbol{g}^*, \boldsymbol{u}^*)\\ &= J(\boldsymbol{g}^*, \boldsymbol{u}^*)\end{aligned}$$

and

$$J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) \le J_\epsilon(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) = J(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) \quad \forall (\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) \in \mathscr{V}_{ad}\,,$$

which give us

$$J(\boldsymbol{g}^*, \boldsymbol{u}^*) = \lim_{\epsilon\to 0} J_\epsilon(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*) \le J(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) \quad \forall (\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) \in \mathscr{V}_{ad}\,.$$

So, $(\boldsymbol{g}^*, \boldsymbol{u}^*)$ is an optimal control for the cost functional J and the minimal value of J_ϵ converges to the minimal value of J. □

9.3.2 Regularized Problems and Optimality Conditions

Until now, we have reduced our optimal control problem to one governed by a variational inequality of the second kind. Unfortunately, the problem $(\mathbf{CS})_\epsilon$, despite the fact that it is simpler than the initial one, still involves a non-differentiable functional J_ϵ. Therefore, to attain our main goal, the obtaining of the optimality conditions, we shall consider a family of regularized problems associated with $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$, defined, for $\rho > 0$, by

Problem $(\mathbf{S})_{\rho}^{g,w}$: Find $\boldsymbol{u} \in W^{1,2}(0,T;\mathbf{V})$ such that

$$\begin{cases} \rho(\dot{\boldsymbol{u}}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V + a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j^{\rho}(\boldsymbol{w}(t), \boldsymbol{v}) - j^{\rho}(\boldsymbol{w}(t), \dot{\boldsymbol{u}}(t)) \\ \qquad \geq (\boldsymbol{F}^g(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_V \quad \forall \boldsymbol{v} \in \boldsymbol{V}, \text{ a.e. } t \in (0,T)\,, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0\,, \end{cases}$$

where, for $\boldsymbol{w} \in \boldsymbol{W}$, $\{j^{\rho}(\boldsymbol{w},\cdot)\}_{\rho}$ is a family of convex functionals $j^{\rho}(\boldsymbol{w},\cdot) : \boldsymbol{V} \to \mathbb{R}_+$, of class C^2, i.e. the gradients with respect to the second variable, $\nabla_2 j^{\rho}(\boldsymbol{w},\cdot) : \boldsymbol{V} \to \boldsymbol{V}^*$ and $\nabla_2^2 j^{\rho}(\boldsymbol{w},\cdot) : \boldsymbol{V} \to \mathscr{L}(\boldsymbol{V}, \boldsymbol{V}^*)$, are continuous. In addition, we suppose that the following conditions hold true:

$$j^{\rho}(\boldsymbol{w}, \mathbf{0}) = 0 \quad \forall \boldsymbol{w} \in \boldsymbol{W}\,, \tag{9.110}$$

$$|j^{\rho}(\boldsymbol{w}, \boldsymbol{v}) - j(\boldsymbol{w}, \boldsymbol{v})| \leq C\rho \|\boldsymbol{w}\|_V \quad \forall \boldsymbol{w} \in \boldsymbol{W},\ \forall \boldsymbol{v} \in \boldsymbol{V} \tag{9.111}$$

$$\begin{cases} \lim_{n\to\infty} \int_0^T \langle \nabla_2 j^{\rho}(\boldsymbol{w}_n(t), \boldsymbol{u}_n(t)), \boldsymbol{v}\rangle \,\mathrm{d}t = \int_0^T \langle \nabla_2 j^{\rho}(\boldsymbol{w}(t), \boldsymbol{u}(t)), \boldsymbol{v}\rangle \,\mathrm{d}t \\ \forall (\boldsymbol{w}_n, \boldsymbol{u}_n) \rightharpoonup (\boldsymbol{w}, \boldsymbol{u}) \text{ weakly in } \mathbf{H_w} \times L^2(0,T;\boldsymbol{V})\,,\ \forall \boldsymbol{v} \in \boldsymbol{V}\,, \end{cases} \tag{9.112}$$

where C is a constant independent of $\boldsymbol{v}$ and $\langle\cdot,\cdot\rangle$ denotes the duality pair between $\boldsymbol{V}^*$ and $\boldsymbol{V}$.

Remark 9.3. We can choose

$$j^{\rho}(\boldsymbol{w}, \boldsymbol{v}) = \int_{\Gamma_2} \mu |\mathscr{R}\sigma_{\nu}(\boldsymbol{w})|\, \theta_{\rho}(\boldsymbol{v}_{\tau})\,\mathrm{d}s \quad \forall (\boldsymbol{w}, \boldsymbol{v}) \in \boldsymbol{W} \times \boldsymbol{V}\,, \tag{9.113}$$

where the function $\theta_{\rho} : \mathbb{R}^p \to \mathbb{R}$ is an approximation (see [12] or [1]) of the function $|\cdot| : \mathbb{R}^p \to \mathbb{R}$, satisfying the following properties:

$$\begin{cases} \theta_{\rho} \text{ is a convex, nonnegative function of class } C^2\,, \\ \theta_{\rho}(\mathbf{0}) = 0\,, \\ \big|\theta_{\rho}(\boldsymbol{u}) - |\boldsymbol{u}|\big| \leq C_0\rho\,, \\ \big|\theta'_{\rho}(\boldsymbol{u}) \cdot \boldsymbol{v}\big| \leq C_1|\boldsymbol{v}|\,, \\ \big|\theta''_{\rho}(\boldsymbol{u})(\boldsymbol{v}\cdot\boldsymbol{q})\big| \leq C_2(\rho)|\boldsymbol{v}|\,|\boldsymbol{q}|\,, \end{cases} \tag{9.114}$$

with C_0, C_1, and $C_2(\rho)$ positive constants.

Then, after some computations, it follows that

$$\langle \nabla_2 j^{\rho}(\boldsymbol{w}, \boldsymbol{u}), \boldsymbol{v}\rangle = \int_{\Gamma_2} \mu |\mathscr{R}(\sigma_{\nu}(\boldsymbol{w}))| \theta'_{\rho}(\boldsymbol{u}_{\tau}) \cdot \boldsymbol{v}_{\tau}\,\mathrm{d}s\,,$$

$$\langle \nabla_2^2 j^{\rho}(\boldsymbol{w}, \boldsymbol{u})\boldsymbol{v}, \boldsymbol{q}\rangle = \int_{\Gamma_2} \mu |\mathscr{R}(\sigma_{\nu}(\boldsymbol{w}))| \theta''_{\rho}(\boldsymbol{u}_{\tau})(\boldsymbol{v}_{\tau} \cdot \boldsymbol{q}_{\tau})\,\mathrm{d}s\,.$$

For instance, if we take

$$\theta_\rho(\boldsymbol{v}) = \begin{cases} \dfrac{|\boldsymbol{v}|^2}{\rho}\left(1 - \dfrac{|\boldsymbol{v}|}{3\rho}\right) & \text{if } |\boldsymbol{v}| \le \rho\,, \\ \rho\left(\dfrac{|\boldsymbol{v}|}{\rho} - \dfrac{1}{3}\right) & \text{if } |\boldsymbol{v}| \ge \rho\,, \end{cases} \tag{9.115}$$

then θ'_ρ and θ''_ρ are defined by (8.156) and (8.157) (p. 182), and if we choose

$$\theta_\rho(\boldsymbol{v}) = \sqrt{\rho^2 + |\boldsymbol{v}|^2} - \rho\,, \tag{9.116}$$

then one has:

$$\theta'_\rho(\boldsymbol{u}) = \frac{\boldsymbol{u}}{\sqrt{\rho^2 + |\boldsymbol{u}(x)|^2}},$$

and

$$\theta''_\rho(\boldsymbol{u})(\boldsymbol{v}) = \frac{1}{\sqrt{\rho^2 + |\boldsymbol{u}(x)|^2}}\left(\boldsymbol{v} - \frac{(\boldsymbol{u}\cdot\boldsymbol{v})\boldsymbol{u}}{\rho^2 + |\boldsymbol{u}(x)|^2}\right).$$

It is easy to see that, in both cases, the functional j_ρ, defined by (9.113), satisfies the properties (9.110)–(9.112) and, in addition, we have

$$\begin{cases} |\nabla_2 j^\rho(\boldsymbol{w},\boldsymbol{u})\cdot\boldsymbol{v}| \le C_1\|\boldsymbol{v}\| \quad \forall \boldsymbol{u},\boldsymbol{v} \in \boldsymbol{V}\,, \\ |\langle \nabla_2^2 j^\rho(\boldsymbol{w},\boldsymbol{u})\cdot\boldsymbol{v},\boldsymbol{q}\rangle| \le C_2\|\boldsymbol{v}\|\,\|\boldsymbol{q}\| \quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{q} \in \boldsymbol{V}\,, \end{cases}$$

with $C_1 = C_1(\boldsymbol{w}) > 0$ and $C_2 = C_2(\boldsymbol{w},\rho) > 0$.

Obviously, the regularized problem $(\mathbf{S})_\rho^{\boldsymbol{g},\boldsymbol{w}}$ can be equivalently written as the following variational equality.

Problem $(\mathscr{S})_\rho^{\boldsymbol{g},\boldsymbol{w}}$: Find $\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V})$ such that

$$\begin{cases} \rho(\dot{\boldsymbol{u}}(t),\boldsymbol{v})_V + a(\boldsymbol{u}(t),\boldsymbol{v}) + \langle \nabla_2 j^\rho(\boldsymbol{w}(t),\dot{\boldsymbol{u}}(t)),\boldsymbol{v}\rangle \\ = (\boldsymbol{F}^g(t),\boldsymbol{v})_V\,,\ \forall\,\boldsymbol{v} \in \boldsymbol{V}\,,\ \text{a.e. } t \in (0,T)\,, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0\,. \end{cases}$$

We have the following existence and uniqueness result.

Proposition 9.5. *Let* $(\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$ *and* $\rho > 0$. *Then, there exists a unique solution* $\boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}} \in W^{1,2}(0,T;\mathbf{V})$ *of Problem* $(\mathscr{S})_\rho^{\boldsymbol{g},\boldsymbol{w}}$.

Proof. Arguing as in [2], one can prove the following main steps of the proof.

(1) For any $\boldsymbol{\alpha} \in W^{1,2}(0,T;\boldsymbol{V})$, the problem

$$\begin{cases} \boldsymbol{v}_{\rho\alpha}^{\boldsymbol{g},\boldsymbol{w}} \in W^{1,2}(0,T;\boldsymbol{V}) \\ \rho(\boldsymbol{v}_{\rho\alpha}^{\boldsymbol{g},\boldsymbol{w}}(t),\boldsymbol{v})_V + \langle \nabla_2 j^\rho(\boldsymbol{w}(t),\boldsymbol{v}_{\rho\alpha}^{\boldsymbol{g},\boldsymbol{w}}(t)),\boldsymbol{v}\rangle = (\boldsymbol{F}^{g(t)},\boldsymbol{v}) \\ -a(\boldsymbol{\alpha}(t),\boldsymbol{v})\ \forall\,\boldsymbol{v} \in \boldsymbol{V}\,,\ \ \forall t \in (0,T)\,, \end{cases} \tag{9.117}$$

has a unique solution $\boldsymbol{v}_{\rho\alpha}^{\boldsymbol{g},\boldsymbol{w}} \in W^{1,2}(0,T;\boldsymbol{V})$.

(2) Let $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}} : [0,T] \to \boldsymbol{V}$ be the function defined by

$$\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}}(t) = \int_0^t \boldsymbol{v}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}}(s)\,\mathrm{d}s + \boldsymbol{u}_0\,. \tag{9.118}$$

Then $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}} \in W^{2,2}(0,T;\boldsymbol{V})$ and $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}}(0) = \boldsymbol{u}_0$.

(3) We denote by $\Lambda_\rho : W^{1,2}(0,T;\boldsymbol{V}) \to W^{1,2}(0,T;\boldsymbol{V})$ the mapping defined by

$$\Lambda_\rho(\boldsymbol{\alpha})(t) = \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}}(t) \qquad \forall \boldsymbol{\alpha} \in W^{1,2}(0,T;\boldsymbol{V}),\ \forall t \in [0,T]\,. \tag{9.119}$$

One can prove that the map Λ_ρ has a unique fixed point $\boldsymbol{\alpha}^*$. Therefore, the function $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_{\rho\boldsymbol{\alpha}^*}$ defined by (9.118), is a solution of Problem $(\mathscr{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$. Finally, by using Gronwall' inequality and the properties (9.110)–(9.112) of the function j_ρ, from the formulation $(\mathscr{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$, the uniqueness follows. □

The regularized problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$ approximates the penalized problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$ in the following sense.

Proposition 9.6. *Let* $(\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$. *For* $\rho > 0$, *let* $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho$ *be the unique solution of problem* $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$. *Then*

$$\begin{aligned} \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho &\to \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}} \quad \textit{strongly in } L^\infty(0,T;\boldsymbol{V})\,,\\ \dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho &\rightharpoonup \dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}} \quad \textit{weakly in } L^2(0,T;\boldsymbol{V})\,, \end{aligned} \tag{9.120}$$

$\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}$ *being the unique solution of* $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$. *Moreover, there exists a constant* $C > 0$, *independent of* ρ, *such that*

$$\|\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho - \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}\|_{L^\infty(0,T;\boldsymbol{V})} \le C\sqrt{\rho}\left(1 + \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}\|^2_{L^2(0,T;\boldsymbol{V})}\right)\,. \tag{9.121}$$

Proof. Using the property (9.111) of j^ρ and taking $\boldsymbol{v} = \dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho$ in $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}$ and $\boldsymbol{v} = \dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}$ in $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$, we get

$$\begin{aligned}
&\rho\int_0^s \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho(t)\|^2_V\,\mathrm{d}t + \frac{\alpha}{2}\|\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}(s) - \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho(s)\|^2_V\\
&\le \int_0^s |j^\rho(\boldsymbol{w}(t),\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}(t)) - j(\boldsymbol{w}(t),\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}(t))|\,\mathrm{d}t\\
&\quad + \int_0^s |j(\boldsymbol{w}(t),\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho(t)) - j^\rho(\boldsymbol{w}(t),\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho(t))|\,\mathrm{d}t + \rho\int_0^s (\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho(t),\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}(t))_V\,\mathrm{d}t\\
&\le C\rho\int_0^s \|\boldsymbol{w}(t)\|_V\,\mathrm{d}t + \rho\int_0^s \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho(t)\|_V\,\|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}(t)\|_V\,\mathrm{d}t\\
&\le \rho\left(C_0 + \frac{\nu}{2}\int_0^s \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho(t)\|^2_V\,\mathrm{d}t + \frac{1}{2\nu}\int_0^s \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}(t)\|^2_V\,\mathrm{d}t\right)\,,\ \forall s \in [0,T]\,,
\end{aligned}$$

which implies, for $\nu > 0$ conveniently chosen, that

$$\|\dot{\boldsymbol{u}}_\rho^{\boldsymbol{g},\boldsymbol{w}}\|^2_{L^2(0,T;V)} \leq C(1 + \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}\|^2_{L^2(0,T;V)}) \tag{9.122}$$

and

$$\|\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}(s) - \boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}(s)\|^2_V \leq C\rho(1 + \|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}\|^2_{L^2(0,T;V)}) \quad \forall s \in [0,T]\,.$$

□

Now, we formulate an optimal control problem, governed by the regularized problem $(\mathbf{S})_\rho^{\boldsymbol{g},\boldsymbol{w}}$, in which the cost functional is defined similarly to J_ϵ, the only difference being that the state is, in this case, the solution of an equation. More precisely, we introduce the regularized functional:

$$\begin{aligned} J_{\epsilon\rho}(\boldsymbol{g},\boldsymbol{w}) =& J(\boldsymbol{g},\boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}) + \frac{1}{2\epsilon}\|\boldsymbol{w} - \boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}\|^2_{\mathbf{H_w}} \\ =& \frac{1}{2}\|\boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}} - \boldsymbol{u}_d\|^2_{\mathbf{H_u}} + \frac{\beta}{2}\|\boldsymbol{g}\|^2_{\mathbf{H_g}} + \frac{1}{2\epsilon}\|\boldsymbol{w} - \boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}\|^2_{\mathbf{H_w}}\,, \end{aligned} \tag{9.123}$$

$\boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}$ being the unique solution of the regularized problem $(\mathbf{S})_\rho^{\boldsymbol{g},\boldsymbol{w}}$ or, equivalently, of the variational equation $(\mathscr{S})_\rho^{\boldsymbol{g},\boldsymbol{w}}$.

For any $\rho > 0$, we consider the corresponding regularized optimal control problem.

Problem $(\mathbf{CS})_{\epsilon\rho}$: Find $(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) \in \mathbf{H}_{\boldsymbol{g}} \times \mathbf{H}_{\boldsymbol{w}}$ such that

$$J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) = \min\{J_{\epsilon\rho}(\boldsymbol{g},\boldsymbol{w})\,;\,(\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H}_{\boldsymbol{g}} \times \mathbf{H}_{\boldsymbol{w}}\}.$$

Theorem 9.6. *For $\rho > 0$, there exists a solution $(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho})$ of Problem $(\mathbf{CS})_{\epsilon\rho}$.*

Proof. Let $\{(\boldsymbol{g}^n_{\epsilon\rho}, \boldsymbol{w}^n_{\epsilon\rho})\}_n$ be a minimizing sequence for the functional $J_{\epsilon\rho}$. From the definition of $J_{\epsilon\rho}$, it follows that there exists $\boldsymbol{g}^*_{\epsilon\rho} \in \mathbf{H_g}$ such that, up to a subsequence, we have

$$\boldsymbol{g}^n_{\epsilon\rho} \rightharpoonup \boldsymbol{g}^*_{\epsilon\rho} \text{ weakly in } \mathbf{H_g}\,. \tag{9.124}$$

Let $\boldsymbol{u}^n_{\epsilon\rho} = u^{\boldsymbol{g}^n_{\epsilon\rho},\boldsymbol{w}^n_{\epsilon\rho}}$. Putting $\boldsymbol{v} = \dot{\boldsymbol{u}}^n_{\epsilon\rho}(t)$ in $(\mathscr{S})_\rho^{\boldsymbol{g}^n_{\epsilon\rho},\boldsymbol{w}^n_{\epsilon\rho}}$ and taking into account that (9.110) implies

$$\langle \nabla_2 j^\rho(\boldsymbol{w},\boldsymbol{u}),\boldsymbol{u}\rangle \geq 0\,,\ \forall (\boldsymbol{w},\boldsymbol{u}) \in \boldsymbol{W} \times \boldsymbol{V}\,, \tag{9.125}$$

we get

$$\begin{aligned} &\rho\int_0^s \|\dot{\boldsymbol{u}}^n_{\epsilon\rho}(t)\|^2_V\,\mathrm{d}t + \frac{\alpha}{2}\|\boldsymbol{u}^n_{\epsilon\rho}(s)\|^2_V \leq \frac{1}{2}a(\boldsymbol{u}_0,\boldsymbol{u}_0) + \int_0^s (\boldsymbol{F}^n_{\epsilon\rho}(t), \dot{\boldsymbol{u}}^n_{\epsilon\rho}(t))_V\,\mathrm{d}t \\ &\leq C + \frac{\nu}{2}\int_0^s \|\dot{\boldsymbol{u}}^n_{\epsilon\rho}(t)\|^2_V\,\mathrm{d}t + \frac{1}{2\nu}\int_0^s \|\boldsymbol{F}^n_{\epsilon\rho}(t)\|^2_V\,\mathrm{d}t\,, \end{aligned}$$

where

$$(\boldsymbol{F}^n_{\epsilon\rho}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\;\mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}^n_{\epsilon\rho}(t)\cdot\boldsymbol{v}\;\mathrm{d}s\,. \tag{9.126}$$

Thus, with (9.124), it follows that

$$\begin{aligned}
&\|\boldsymbol{u}^n_{\epsilon\rho}(s)\|^2_V \le C \quad \forall s\in[0,T]\,,\\
&\|\boldsymbol{u}^n_{\epsilon\rho}\|^2_{L^\infty(0,T;V)} \le C\,,\\
&\|\dot{\boldsymbol{u}}^n_{\epsilon\rho}\|^2_{L^2(0,T;V)} \le C_\rho\,,
\end{aligned} \tag{9.127}$$

with C and C_ρ positive constants. So, up to a subsequence, we have

$$\begin{cases}
\boldsymbol{u}^n_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^*_{\epsilon\rho} \text{ weakly * in } L^\infty(0,T;V)\,,\\
\boldsymbol{u}^n_{\epsilon\rho}(t) \rightharpoonup \boldsymbol{u}^*_{\epsilon\rho}(t) \text{ weakly in } V \quad \forall t\in[0,T]\,,\\
\dot{\boldsymbol{u}}^n_{\epsilon\rho} \rightharpoonup \dot{\boldsymbol{u}}^*_{\epsilon\rho} \text{ weakly in } L^2(0,T;V)\,.
\end{cases} \tag{9.128}$$

Therefore, since

$$\|\boldsymbol{u}^n_{\epsilon\rho}\|^2_{\mathbf{H_w}} = \|\boldsymbol{u}^n_{\epsilon\rho}\|^2_{L^2(0,T;V)} + \|\rho\dot{\boldsymbol{u}}^n_{\epsilon\rho} - \boldsymbol{f}\|^2_{L^2(0,T;(L^2(\Omega))^d)}\,,$$

we conclude that the sequence $\{\boldsymbol{u}^n_{\epsilon\rho}\}_n$ is also bounded in $\mathbf{H_w}$ and, from the definition and the boundedness of $\{J_{\epsilon\rho}(\boldsymbol{g}^n_{\epsilon\rho}, \boldsymbol{w}^n_{\epsilon\rho})\}_n$, it follows that the sequence $\{\boldsymbol{w}^n_{\epsilon\rho}\}_n$ is bounded in $\mathbf{H_w}$. So, up to a subsequence, we have

$$\boldsymbol{w}^n_{\epsilon\rho} \rightharpoonup \boldsymbol{w}^*_{\epsilon\rho} \text{ weakly in } \mathbf{H_w}\,, \tag{9.129}$$

with $\boldsymbol{w}^*_{\epsilon\rho}\in\mathbf{H_w}$.

Now, passing to the limit with $n\to\infty$ in $(\mathscr{S})_\rho^{\boldsymbol{g}^n_{\epsilon\rho},\boldsymbol{w}^n_{\epsilon\rho}}$ and using the convergences (9.124), (9.129), (9.128), and (9.112), we obtain that $\boldsymbol{u}^*_{\epsilon\rho} = \boldsymbol{u}^{\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$. From the uniqueness of the solution, we deduce that all the above convergences hold on the whole sequences.

Next, from $(\mathbf{S})^{\boldsymbol{g}^n_{\epsilon\rho},\boldsymbol{w}^n_{\epsilon\rho}}$ and $(\mathbf{S})^{\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$, we obtain

$$(\boldsymbol{u}^n_{\epsilon\rho}-\boldsymbol{u}^*_{\epsilon\rho},\varphi)_{\mathbf{H_w}} = (\boldsymbol{u}^n_{\epsilon\rho}-\boldsymbol{u}^*_{\epsilon\rho},\varphi)_{L^2(0,T;V)} + \rho(\dot{\boldsymbol{u}}^n_{\epsilon\rho}-\dot{\boldsymbol{u}}^*_{\epsilon\rho},\varphi)_{L^2(0,T;(L^2(\Omega))^d)}\ \forall\varphi\in\mathbf{H_w}\,,$$

which, together with $(9.128)_{1,3}$, implies

$$\boldsymbol{u}^n_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^*_{\epsilon\rho} \text{ weakly in } \mathbf{H_w}\,.$$

Therefore, by using the convergence (9.129), one gets

$$\liminf_{n\to\infty} \frac{1}{2\epsilon}\|\boldsymbol{w}^n_{\epsilon\rho} - \boldsymbol{u}^n_{\epsilon\rho}\|^2_{\mathbf{H_w}} \geq \frac{1}{2\epsilon}\|\boldsymbol{w}^*_{\epsilon\rho} - \boldsymbol{u}^*_{\epsilon\rho}\|^2_{\mathbf{H_w}}. \tag{9.130}$$

Finally, using the weakly lower semi-continuity of $J_{\epsilon\rho}$ and (9.130), one deduces

$$\begin{aligned}&\inf\{J_{\epsilon\rho}(\boldsymbol{g},\boldsymbol{w})\,;\ (\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H_g}\times\mathbf{H_w}\}\\ &= \lim_{n\to\infty} J_{\epsilon\rho}(\boldsymbol{g}^n_{\epsilon\rho},\boldsymbol{w}^n_{\epsilon\rho}) \geq \liminf_{n\to\infty} J_{\epsilon\rho}(\boldsymbol{g}^n_{\epsilon\rho},\boldsymbol{w}^n_{\epsilon\rho}) \geq J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho})\end{aligned}$$

and so, we conclude

$$J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}) = \min\{J_{\epsilon\rho}(\boldsymbol{g},\boldsymbol{w})\,;\ (\boldsymbol{g},\boldsymbol{w}) \in \boldsymbol{H}_g\times\boldsymbol{H}_w\}.$$

□

The following property of the solution of the regularized problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$ will allow us to prove an important result of this section, stated in Theorem 9.7, which gives the asymptotic behavior of the regularized optimal controls of problem $(\mathbf{CS})_{\epsilon\rho}$.

Proposition 9.7. *Let* $\{(\boldsymbol{g}_n,\boldsymbol{w}_n)\}_n \subset \mathbf{H_g}\times\mathbf{H_w}$ *be such that*

$$(\boldsymbol{g}_n,\boldsymbol{w}_n) \rightharpoonup (\boldsymbol{g},\boldsymbol{w}) \text{ weakly in } \mathbf{H_g}\times\mathbf{H_w}.$$

Then,

$$\boldsymbol{u}^{\boldsymbol{g}_n,\boldsymbol{w}_n}_\rho \to \boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho \quad \text{weakly in } W^{1,2}(0,T;\boldsymbol{V}),$$

$\boldsymbol{u}^{\boldsymbol{g}_n,\boldsymbol{w}_n}_\rho$ *being the unique solution of* $(\mathbf{S})^{\boldsymbol{g}_n,\boldsymbol{w}_n}_\rho$ *and* $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho$ *the unique solution of* $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$.

Proof. Let $\boldsymbol{u}_n = \boldsymbol{u}^{\boldsymbol{w}_n,\boldsymbol{g}_n}_\rho$. Taking $\boldsymbol{v} = \dot{\boldsymbol{u}}_n$ in $(\mathscr{S})^{\boldsymbol{g}_n,\boldsymbol{w}_n}_\rho$ and using the positivity (9.125), we deduce, for all $s \in [0,T]$, that

$$\rho\int_0^s \|\dot{\boldsymbol{u}}_n(t)\|^2_{\boldsymbol{V}}\,\mathrm{d}t + \frac{\alpha}{2}\|\boldsymbol{u}_n(s)\|^2_{\boldsymbol{V}} \leq \frac{1}{2\nu}\int_0^s \|\boldsymbol{F}^{\boldsymbol{g}_n}(t)\|^2_{\boldsymbol{V}} + \frac{\nu}{2}\int_0^s \|\dot{\boldsymbol{u}}_n(t)\|^2_{\boldsymbol{V}}\,\mathrm{d}t + C,$$

which, for $\nu > 0$ conveniently chosen, implies

$$\|\boldsymbol{u}_n(s)\|^2_{\boldsymbol{V}} \leq C(1 + \|\boldsymbol{F}^{\boldsymbol{g}_n}\|^2_{L^2(0,T;\boldsymbol{V})}) \quad \forall s \in [0,T],$$

$$\|\dot{\boldsymbol{u}}_n\|^2_{L^2(0,T;\boldsymbol{V})} \leq C_\rho(1 + \|\boldsymbol{F}^{\boldsymbol{g}_n}\|^2_{L^2(0,T;\boldsymbol{V})}).$$

Thus, there exists $\boldsymbol{u} \in W^{1,2}(0,T;\boldsymbol{V})$ such that, up to a subsequence, we have

$$\begin{aligned}&\boldsymbol{u}_n \to \boldsymbol{u} \text{ weakly * in } L^\infty(0,T;\boldsymbol{V}),\\ &\boldsymbol{u}_n \to \boldsymbol{u} \text{ weakly in } W^{1,2}(0,T;\boldsymbol{V}).\end{aligned} \tag{9.131}$$

Finally, by passing to the limit in $(\mathscr{S})_\rho^{\boldsymbol{g}_n,\boldsymbol{w}_n}$, with $n \to \infty$, and using (9.131), (9.112) and the hypotheses on $\{\boldsymbol{g}_n\}_n$ and $\{\boldsymbol{w}_n\}_n$, we get $\boldsymbol{u} = \boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}$. □

Now, we state the following convergence result.

Theorem 9.7. *Let* $(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho})$ *be a solution of problem* $(\mathbf{CS})_{\epsilon\rho}$ *and* $\boldsymbol{u}^*_{\epsilon\rho} = \boldsymbol{u}_\rho^{\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$. *Then,*

$$\begin{cases} \boldsymbol{g}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{g}^*_\epsilon \text{ weakly in } \mathbf{H_g}\,, \\ \boldsymbol{w}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{w}^*_\epsilon \text{ weakly in } \mathbf{H_w}\,, \\ \boldsymbol{u}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^*_\epsilon \text{ weakly in } W^{1,2}(0,T;V)\,, \end{cases} \tag{9.132}$$

where $\boldsymbol{u}^*_\epsilon = \boldsymbol{u}^{\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon}$. *Moreover,* $(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon)$ *is an optimal control for* J_ϵ *and*

$$\lim_{\rho\to 0} J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) = J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) = \min_{(\boldsymbol{g},\boldsymbol{w})\in\mathbf{H_g}\times\mathbf{H_w}} J_\epsilon(\boldsymbol{g},\boldsymbol{w})\,.$$

Proof. Let $(\tilde{\boldsymbol{g}}, \tilde{\boldsymbol{u}}) \in \mathscr{V}_{ad}$. Obviously, $\tilde{\boldsymbol{u}} = \boldsymbol{u}^{\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}}$ and, from Proposition 9.6, we have

$$\begin{aligned} \boldsymbol{u}_\rho^{\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}} &\to \tilde{\boldsymbol{u}} \quad \text{strongly in } L^\infty(0,T;V)\,, \\ \dot{\boldsymbol{u}}_\rho^{\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}} &\rightharpoonup \dot{\tilde{\boldsymbol{u}}} \quad \text{weakly in } L^2(0,T;V)\,. \end{aligned}$$

Therefore, we obtain

$$\lim_{\rho\to 0} J_{\epsilon\rho}(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}) = \lim_{\rho\to 0}\left(\frac{1}{2}\|\boldsymbol{u}_\rho^{\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}} - \boldsymbol{u}_d\|^2_{\mathbf{H_u}} + \frac{1}{2\epsilon}\|\boldsymbol{u}_\rho^{\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}} - \tilde{\boldsymbol{u}}\|^2_{\mathbf{H_w}} + \frac{\beta}{2}\|\tilde{\boldsymbol{g}}\|^2_{\mathbf{H_g}}\right) = J(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}})\,.$$

Since

$$J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) \le J_{\epsilon\rho}(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}})\,,$$

it follows that the sequence $\{J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho})\}_\rho$ is bounded. Hence, the sequence $\{\boldsymbol{g}^*_{\epsilon\rho}\}_\rho$ is bounded in $\boldsymbol{H}_g$.

Next, putting $\boldsymbol{v} = \mathbf{0}$ in $(\mathbf{S})^{\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$, integrating by parts on $[0,s]$ with $s \in [0,T]$ and taking into account the positivity and the property (9.110) of j_ρ, we get

$$\rho\int_0^s \|\dot{\boldsymbol{u}}^*_{\epsilon\rho}(t)\|^2\,\mathrm{d}t + \int_0^s a(\boldsymbol{u}^*_{\epsilon\rho}(t), \dot{\boldsymbol{u}}^*_{\epsilon\rho}(t))\,\mathrm{d}t \le \int_0^s (\boldsymbol{F}^*_{\epsilon\rho}(t), \dot{\boldsymbol{u}}^*_{\epsilon\rho}(t))_V\,\mathrm{d}t\,, \tag{9.133}$$

where

$$(\boldsymbol{F}^*_{\epsilon\rho}(t), \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\,\mathrm{d}x + \int_{\Gamma_1} \boldsymbol{g}^*_{\epsilon\rho}(t)\cdot\boldsymbol{v}\,\mathrm{d}s\,.$$

Proceeding like in the proof of Proposition 9.4, we deduce that the sequence $\{(\boldsymbol{u}^*_{\epsilon\rho}, \rho\dot{\boldsymbol{u}}^*_{\epsilon\rho})\}_\rho$ is bounded in $L^\infty(0,T;\boldsymbol{V}) \times L^2(0,T;\boldsymbol{V})$.

Thus, since

$$\|\boldsymbol{u}^*_{\epsilon\rho}\|^2_{\mathbf{H_w}} = \|\boldsymbol{u}^*_{\epsilon\rho}\|^2_{L^2(0,T;V)} + \|\rho\dot{\boldsymbol{u}}^*_{\epsilon\rho} - \boldsymbol{f}\|^2_{L^2(0,T;(L^2(\Omega))^d)},$$

it follows that the sequence $\{\boldsymbol{u}^*_{\epsilon\rho}\}_\rho$ is also bounded in $\mathbf{H_w}$. From the definition of $J_{\epsilon\rho}$ and the boundedness of the sequence $\{J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho})\}_\rho$, it follows that the sequence $\{\boldsymbol{w}^*_{\epsilon\rho}\}_\rho$ is bounded in $\mathbf{H_w}$. Thus, there exist the elements $\boldsymbol{g}^*_\epsilon \in \mathbf{H_g}$ and $\boldsymbol{w}^*_\epsilon \in \mathbf{H_w}$ and the subsequences, still denoted by $\{\boldsymbol{g}^*_{\epsilon\rho}\}_\rho$ and $\{\boldsymbol{w}^*_{\epsilon\rho}\}_\rho$, such that

$$\begin{aligned} \boldsymbol{g}^*_{\epsilon\rho} &\rightharpoonup \boldsymbol{g}^*_\epsilon \text{ weakly in } \mathbf{H_g}, \\ \boldsymbol{w}^*_{\epsilon\rho} &\rightharpoonup \boldsymbol{w}^*_\epsilon \text{ weakly in } \mathbf{H_w}. \end{aligned} \tag{9.134}$$

Applying Propositions 9.6 and 9.7, we deduce

$$\boldsymbol{u}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^*_\epsilon \quad \text{weakly in } W^{1,2}(0,T;\mathbf{V}), \tag{9.135}$$

where $\boldsymbol{u}^*_\epsilon = \boldsymbol{u}^{\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon}$. An easy computation gives

$$\boldsymbol{u}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^*_\epsilon \quad \text{weakly in } \mathbf{H_w}. \tag{9.136}$$

Let $(\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon)$ be a solution of problem $(\mathbf{CS})_\epsilon$, $\bar{\boldsymbol{u}}_\epsilon = \boldsymbol{u}^{\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon}$ and $\bar{\boldsymbol{u}}_{\epsilon\rho} = \boldsymbol{u}_\rho^{\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon}$. From Proposition 9.6, we get

$$\begin{aligned} \bar{\boldsymbol{u}}_{\epsilon\rho} &\to \bar{\boldsymbol{u}}_\epsilon \text{ strongly in } L^\infty(0,T;\boldsymbol{V}), \\ \dot{\bar{\boldsymbol{u}}}_{\epsilon\rho} &\to \dot{\bar{\boldsymbol{u}}}_\epsilon \text{ weakly in } L^2(0,T;\boldsymbol{V}), \end{aligned} \tag{9.137}$$

which, using $(\mathbf{S})_\rho^{\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon}$ and $(\mathbf{S})^{\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon}$, give

$$\bar{\boldsymbol{u}}_{\epsilon\rho} \rightharpoonup \bar{\boldsymbol{u}}_\epsilon \quad \text{strongly in } \mathbf{H_w}. \tag{9.138}$$

Therefore, the convergences (9.134)–(9.138) lead us

$$\begin{aligned} J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) &\le \liminf_{\rho\to 0} J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) \le \limsup_{\rho\to 0} J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) \\ &\le \limsup_{\rho\to 0} J_{\epsilon\rho}(\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon) = \lim_{\rho\to 0} J_{\epsilon\rho}(\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon) = J_\epsilon(\bar{\boldsymbol{g}}_\epsilon, \bar{\boldsymbol{w}}_\epsilon) \le J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon), \end{aligned} \tag{9.139}$$

i.e.

$$\lim_{\rho\to 0} J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) = J_\epsilon(\boldsymbol{g}^*_\epsilon, \boldsymbol{w}^*_\epsilon) = \min\{J_\epsilon(\boldsymbol{g}, \boldsymbol{w})\,;\ (\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}\}.$$

□

Finally, coupling the results proven in Theorems 9.7 and 9.5, we conclude that the regularized optimal problems represent a good approximation for the initial control problem.

Corollary 9.1. *Let* ϵ , $\rho > 0$ *and* $\{\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}\}_{\epsilon\rho}$ *be the sequence of solutions for problems* $(\mathbf{CS})_{\epsilon\rho}$*. Then, there exists* $(\boldsymbol{g}^*, \boldsymbol{u}^*) \in \mathscr{V}_{ad}$*, such that, up to a subsequence, for* ϵ , $\rho \to 0$*, we have*

$$\begin{cases} \boldsymbol{g}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{g}^* \text{ weakly in } \mathbf{H_g}\,, \\ \boldsymbol{w}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^* \text{ weakly in } \mathbf{H_w}\,, \\ \boldsymbol{u}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^* \text{ weakly in } W^{1,2}(0,T;\boldsymbol{V})\,, \end{cases} \tag{9.140}$$

where $\boldsymbol{u}^*_{\epsilon\rho} = \boldsymbol{u}^{\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}}$*. Moreover,*

$$\lim_{\epsilon,\rho\to 0} J(\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{w}^*_{\epsilon\rho}) = J(\boldsymbol{g}^*, \boldsymbol{u}^*) = \min_{(\boldsymbol{g},\boldsymbol{u})\in\mathscr{V}_{ad}} J(\boldsymbol{g}, \boldsymbol{u})\,. \tag{9.141}$$

In the sequel, we are concerned with the obtaining of the optimality conditions for the problem $(\mathbf{CS})_{\epsilon\rho}$, which means to derive the equations characterizing an optimal control from the fact that the differential of $J_{\epsilon\rho}$ vanishes at an extremum. We shall use the following result due to Lions [11].

Theorem 9.8. *Let* $\mathscr{B}$ *be a Banach space and* $\mathbf{X}$*,* $\mathbf{Y}$ *two reflexive Banach spaces. We consider two functions of class* C^1*,* $\mathscr{F} : \mathscr{B} \times \mathbf{X} \longrightarrow \mathbf{Y}$*, and* $\mathscr{J} : \mathscr{B} \times \mathbf{X} \longrightarrow \mathbb{R}$*.*

We suppose that, for all $\mathbf{h} \in \mathscr{B}$*,*

(i) there exists a unique solution $\boldsymbol{u}^h \in \mathbf{X}$ *of equation* $\mathscr{F}(\mathbf{h}, \boldsymbol{u}^h) = 0$*;*
(ii) the operator $\dfrac{\partial\mathscr{F}}{\partial\boldsymbol{u}}(\mathbf{h}, \boldsymbol{u}^h) : \mathbf{X} \longrightarrow \mathbf{Y}$ *is an isomorphism.*

Then, the function $J : \mathscr{B} \longrightarrow \mathbb{R}$*, defined by* $J(\mathbf{h}) = \mathscr{J}(\mathbf{h}, \boldsymbol{u}^h)$*, is differentiable and*

$$\frac{dJ}{d\mathbf{h}}(\mathbf{h})(\delta\mathbf{h}) = \frac{\partial\mathscr{J}}{\partial\mathbf{h}}(\mathbf{h}, \boldsymbol{u}^h)(\delta\mathbf{h}) - \left\langle \boldsymbol{q}^h, \frac{\partial\mathscr{F}}{\partial\mathbf{h}}(\mathbf{h}, \boldsymbol{u}^h)(\delta\mathbf{h}) \right\rangle_{Y^*,Y} \quad \forall\ \delta\mathbf{h} \in \mathscr{B}, \tag{9.142}$$

where the adjoint state $\boldsymbol{q}^h \in \mathbf{Y}^*$ *is the unique solution of*

$$\left\langle \left[\frac{\partial\mathscr{F}}{\partial\boldsymbol{u}}(\mathbf{h}, \boldsymbol{u}^h)\right]^* \cdot \boldsymbol{q}^h, \boldsymbol{v} \right\rangle_{X^*,X} = \frac{\partial\mathscr{J}}{\partial\boldsymbol{u}}(\mathbf{h}, \boldsymbol{u}^h)(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{X}. \tag{9.143}$$

First, let us remark that, for $(\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$, the regularized problem $(\mathbf{S})^{\boldsymbol{g},\boldsymbol{w}}_\rho$ has a unique solution $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho \in W^{1,2}(0,T;\boldsymbol{V})$ satisfying $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho(0) = \boldsymbol{u}_0$. Then, $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}_\rho = \boldsymbol{u}_0 + \tilde{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho$, where $\tilde{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}_\rho \in W^{1,2}(0,T;\boldsymbol{V})$ satisfies

$$\begin{cases} \rho(\dot{\tilde{\boldsymbol{u}}}_\rho^{\boldsymbol{g},\boldsymbol{w}}(t),\boldsymbol{v})_V + a(\tilde{\boldsymbol{u}}_\rho^{\boldsymbol{g},\boldsymbol{w}}(t)+\boldsymbol{u}_0,\boldsymbol{v}) + \langle\nabla_2 j^\rho(\boldsymbol{w}(t),\dot{\tilde{\boldsymbol{u}}}_\rho^{\boldsymbol{g},\boldsymbol{w}}(t)),\boldsymbol{v}\rangle \\ = (\boldsymbol{F}(t),\boldsymbol{v})_V \quad \forall\,\boldsymbol{v}\in\boldsymbol{V}\,,\quad \text{a.e. } t\in(0,T)\,, \\ \tilde{\boldsymbol{u}}_\rho^{\boldsymbol{g},\boldsymbol{w}}(0)=\boldsymbol{0}\,. \end{cases} \tag{9.144}$$

In order to apply Theorem 9.8, we take

$$\mathscr{B} = \mathbf{H_g}\times\mathbf{H_w}\,,$$
$$\mathbf{X} = \{\boldsymbol{v}\in W^{1,2}(0,T;\boldsymbol{V})\cap L^2(0,T;\boldsymbol{W})\,;\ \boldsymbol{v}(0)=\boldsymbol{0}\}\,,$$
$$Y = L^2(0,T;\boldsymbol{V}^*)\,,$$
$$\mathscr{F}:\mathscr{B}\times X\to Y\,,$$
$$\langle\mathscr{F}(\boldsymbol{g},\boldsymbol{w},\boldsymbol{u}),\boldsymbol{v}\rangle = \int_0^T\rho(\dot{\boldsymbol{u}}(t),\boldsymbol{v}(t))_V\,\mathrm{d}t + \int_0^T a(\boldsymbol{u}(t)+\boldsymbol{u}_0,\boldsymbol{v}(t))\,\mathrm{d}t$$
$$+\langle\nabla_2 j^\rho(\boldsymbol{w}(t),\dot{\boldsymbol{u}}(t)),\boldsymbol{v}(t)\rangle\,\mathrm{d}t - \int_0^T(\boldsymbol{f}(t),\boldsymbol{v}(t))_{(L^2(\Omega))^d}\,\mathrm{d}t$$
$$-\int_0^T(\boldsymbol{g}(t),\boldsymbol{v}(t))_{(L^2(\Gamma_1))^d}\,\mathrm{d}t\ \ \forall\,\boldsymbol{v}\in L^2(0,T;\boldsymbol{V})\,,$$
$$\mathscr{J}:\mathscr{B}\times X\to\mathbb{R}\,,$$
$$\mathscr{J}(\boldsymbol{g},\boldsymbol{w},\boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u}+\boldsymbol{u}_0-\boldsymbol{u}_d\|^2_{\mathbf{H_u}} + \frac{\beta}{2}\|\boldsymbol{g}\|^2_{\mathbf{H_g}} + \frac{1}{2\epsilon}\|\boldsymbol{u}+\boldsymbol{u}_0-\boldsymbol{w}\|^2_{\mathbf{H_w}}\,.$$

We remark that

$$\mathscr{J}(\boldsymbol{g},\boldsymbol{w},\tilde{\boldsymbol{u}}_\rho^{g,w}) = J_{\epsilon\rho}(\boldsymbol{g},\boldsymbol{w}) \quad \forall(\boldsymbol{g},\boldsymbol{w})\in\mathbf{H_g}\times\mathbf{H_w}\,.$$

In the sequel, to simplify the notation, we shall omit to write explicitly the indices ϵ, ρ, $\boldsymbol{g}$, and $\boldsymbol{w}$.

We state now the main result of this section.

Theorem 9.9. *Let $(\boldsymbol{g}^*,\boldsymbol{w}^*)\in\mathbf{H_g}\times\mathbf{H_w}$ be a solution of the optimal control problem* $(\mathbf{CS})_{\epsilon\rho}$. *Then, there exist the unique elements $\boldsymbol{u}^*\in\mathbf{X}$ and $\boldsymbol{q}^*\in\mathbf{Y}^*$ such that*

$$\begin{cases} \rho\displaystyle\int_0^T(\dot{\boldsymbol{u}}^*(t),\boldsymbol{v}(t))_V\,\mathrm{d}t + \int_0^T a(\boldsymbol{u}^*(t)+\boldsymbol{u}_0,\boldsymbol{v}(t))\,\mathrm{d}t \\ +\displaystyle\int_0^T\langle\nabla_2 j(\boldsymbol{w}^*(t),\dot{\boldsymbol{u}}^*(t)),\boldsymbol{v}(t)\rangle\,\mathrm{d}t = \int_0^T(\boldsymbol{f}(t),\boldsymbol{v}(t))_{(L^2(\Omega))^d}\,\mathrm{d}t \\ +\displaystyle\int_0^T(\boldsymbol{g}^*(t),\boldsymbol{v}(t))_{(L^2(\Gamma_1))^d}\,\mathrm{d}t \quad \forall\,\boldsymbol{v}\in L^2(0,T;\boldsymbol{V})\,, \end{cases} \tag{9.145}$$

$$\begin{cases} \int_0^T \rho(\dot{\boldsymbol{v}}(t), \boldsymbol{q}^*(t))_V \, \mathrm{d}t + \int_0^T a(\boldsymbol{v}(t), \boldsymbol{q}^*(t)) \, \mathrm{d}t \\ + \int_0^T \langle \nabla_2^2 j(\boldsymbol{w}^*(t), \dot{\boldsymbol{u}}^*(t)) \dot{\boldsymbol{v}}(t) - \nabla_2 j(\boldsymbol{v}(t), \dot{\boldsymbol{u}}^*(t)), \boldsymbol{q}^*(t) \rangle \, \mathrm{d}t \\ = \int_0^T (\boldsymbol{u}^*(t) + \boldsymbol{u}_0 - \boldsymbol{u}_d, \boldsymbol{v}(t))_{(L^2(\Gamma_2))^d} \, \mathrm{d}t \quad \forall \boldsymbol{v} \in \mathbf{X} \end{cases} \tag{9.146}$$

and

$$\beta(\boldsymbol{g}^*, \boldsymbol{g})_{\mathbf{H_g}} = (\boldsymbol{q}^*, \boldsymbol{g})_{L^2(0,T;(L^2(\Gamma_1))^p)} \quad \forall \boldsymbol{g} \in \mathbf{H_g} \, . \tag{9.147}$$

Proof. Let $\boldsymbol{u}^*$ be the unique solution of (9.144) corresponding to $(\boldsymbol{g}^*, \boldsymbol{w}^*)$. Some easy computations give:

$$\frac{\partial \mathscr{J}}{\partial \boldsymbol{w}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{w}) = \frac{1}{\epsilon}(\boldsymbol{u}^* + \boldsymbol{u}_0 - \boldsymbol{w}^*, \boldsymbol{w})_{\mathbf{H_w}} \quad \forall \boldsymbol{w} \in \mathbf{H_w} \, ,$$
$$\frac{\partial \mathscr{J}}{\partial \boldsymbol{g}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{g}) = \beta(\boldsymbol{g}^*, \boldsymbol{g})_{\mathbf{H_g}} \; \forall \boldsymbol{g} \in \mathbf{H_g} \, ,$$
$$\frac{\partial \mathscr{J}}{\partial \boldsymbol{u}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{u}) = (\boldsymbol{u}^* + \boldsymbol{u}_0 - \boldsymbol{u}_d, \boldsymbol{u})_{\mathbf{H_u}} + \frac{1}{\epsilon}(\boldsymbol{u}^* + \boldsymbol{u}_0 - \boldsymbol{w}^*, \boldsymbol{u})_{\mathbf{H_w}} \quad \boldsymbol{u} \in \mathbf{X} \, ,$$
$$\left\langle \frac{\partial \mathscr{F}}{\partial \boldsymbol{w}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{w}), \boldsymbol{v} \right\rangle = \int_0^T \langle \nabla_2 j(\boldsymbol{w}(t), \dot{\boldsymbol{u}}^*(t)), \boldsymbol{v}(t) \rangle \, \mathrm{d}t \; \forall (\boldsymbol{w}, \boldsymbol{v}) \in \mathbf{H_w} \times L^2(0,T;\boldsymbol{V}) \, ,$$
$$\left\langle \frac{\partial \mathscr{F}}{\partial \boldsymbol{g}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{g}), \boldsymbol{v} \right\rangle = -\int_0^T (\boldsymbol{g}(t), \boldsymbol{v}(t))_{(L^2(\Gamma_1)^d} \, \mathrm{d}t \; \forall \boldsymbol{g} \in \mathbf{H_g} \; \forall \boldsymbol{v} \in L^2(0,T;\boldsymbol{V}) \, ,$$
$$\left\langle \frac{\partial \mathscr{F}}{\partial \boldsymbol{u}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{u}), \boldsymbol{v} \right\rangle = \rho \int_0^T (\dot{\boldsymbol{u}}(t), \boldsymbol{v}(t))_V \, \mathrm{d}t + \int_0^T a(\boldsymbol{u}(t), \boldsymbol{v}(t)) \, \mathrm{d}t$$
$$+ \int_0^T \langle \nabla_2^2 j(\boldsymbol{w}(t), \dot{\boldsymbol{u}}^*(t)) \dot{\boldsymbol{u}}(t), \boldsymbol{v}(t) \rangle \, \mathrm{d}t \quad \forall \boldsymbol{u} \in \mathbf{X}, \; \forall \boldsymbol{v} \in L^2(0,T;\boldsymbol{V}) \, .$$

Thus, the operator $\frac{\partial \mathscr{F}}{\partial \boldsymbol{u}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*) : \mathbf{X} \to \mathbf{Y}$ is an isomorphism.

Using Theorem 9.8, the adjoint state $\boldsymbol{q}^* \in \mathbf{Y}^*$ is defined as being the unique solution of the following equation:

$$\left\langle \left[\frac{\partial \mathscr{F}}{\partial \boldsymbol{u}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*) \right]^* \cdot \boldsymbol{q}^*, \boldsymbol{v} \right\rangle = \frac{\partial \mathscr{J}}{\partial \boldsymbol{u}}(\boldsymbol{g}^*, \boldsymbol{w}^*, \boldsymbol{u}^*)(\boldsymbol{v}) \quad \forall \boldsymbol{v} \in \mathbf{X} \, .$$

Therefore, we have

$$\int_0^T \left[\rho(\dot{\boldsymbol{v}}(t), \boldsymbol{q}^*(t)) + a(\boldsymbol{v}(t), \boldsymbol{q}^*(t)) + \langle \nabla_2^2 j(\boldsymbol{w}^*(t), \dot{\boldsymbol{u}}^*(t))(\dot{\boldsymbol{v}}(t), \boldsymbol{q}^*(t)\rangle\right] \mathrm{d}t$$
$$= \int_0^T \left[(\boldsymbol{u}^*(t) + \boldsymbol{u}_0 - \boldsymbol{u}_d, \boldsymbol{v}(t))_{(L^2(\Gamma_2))^d} + \frac{1}{\epsilon}(\boldsymbol{u}_\rho^h + \boldsymbol{u}_0 - \boldsymbol{w}(t), \boldsymbol{v}(t))_W\right] \mathrm{d}t \;\; \forall \boldsymbol{v} \in \mathbf{X}.$$

Next, since $\mathbf{h}^* = (\boldsymbol{g}^*, \boldsymbol{w}^*)$ is a solution of the optimal control problem $(\mathbf{CS})_{\epsilon\rho}$, using Theorem 9.8, we obtain

$$\frac{dJ}{d\mathbf{h}}(\mathbf{h}^*)(\mathbf{h}) = \frac{\partial \mathscr{J}}{\partial \mathbf{h}}(\mathbf{h}^*, \boldsymbol{u}^*)(\mathbf{h}) - \left\langle \boldsymbol{q}^*, \frac{\partial \mathscr{F}}{\partial \mathbf{h}}(\mathbf{h}^*, \boldsymbol{u}^*)(\mathbf{h}) \right\rangle = 0 \;\; \forall \mathbf{h} = (\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w},$$

which gives

$$\int_0^T \frac{1}{\epsilon}(\boldsymbol{u}^*(t) + \boldsymbol{u}_0 - \boldsymbol{w}^*(t), \boldsymbol{w}(t))_W \,\mathrm{d}t + \beta(\boldsymbol{g}^*, \boldsymbol{g})_{\mathbf{H_g}} = \int_0^T \langle \boldsymbol{q}^*(t), \nabla_2 j(\boldsymbol{w}(t), \dot{\boldsymbol{u}}^*(t))\rangle \,\mathrm{d}t$$
$$-(\boldsymbol{q}^*, \boldsymbol{g})_{L^2(0,T;(L^2(\Gamma_1))^d)} \quad \forall (\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}.$$

Taking $\boldsymbol{g} = \mathbf{0}$, we deduce

$$\int_0^T \frac{1}{\epsilon}(\boldsymbol{u}^*(t) + \boldsymbol{u}_0 - \boldsymbol{w}^*(t), \boldsymbol{v}(t))_W \,\mathrm{d}t = \int_0^T \langle \boldsymbol{q}^*(t), \nabla_2 j(\boldsymbol{v}(t)(\dot{\boldsymbol{u}}^*(t))\rangle \,\mathrm{d}t \;\; \forall \boldsymbol{v} \in L^2(0,T;\boldsymbol{W})$$

and, so, we obtain (9.146) and (9.147). □

The asymptotic analysis (Corollary 9.1) of smoother problems $(\mathbf{CS})_{\epsilon\rho}$ provides that the sequence of optimal regularized controls $\{\boldsymbol{g}^*_{\epsilon\rho}, \boldsymbol{u}^*_{\epsilon\rho}\}_{\epsilon\rho}$ converges to an optimal control $(\boldsymbol{g}^*, \boldsymbol{u}^*)$ of the initial problem $(\mathbf{CS})$. Therefore, the system (9.145)–(9.147) can be useful in the numerical analysis of an optimal control.

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