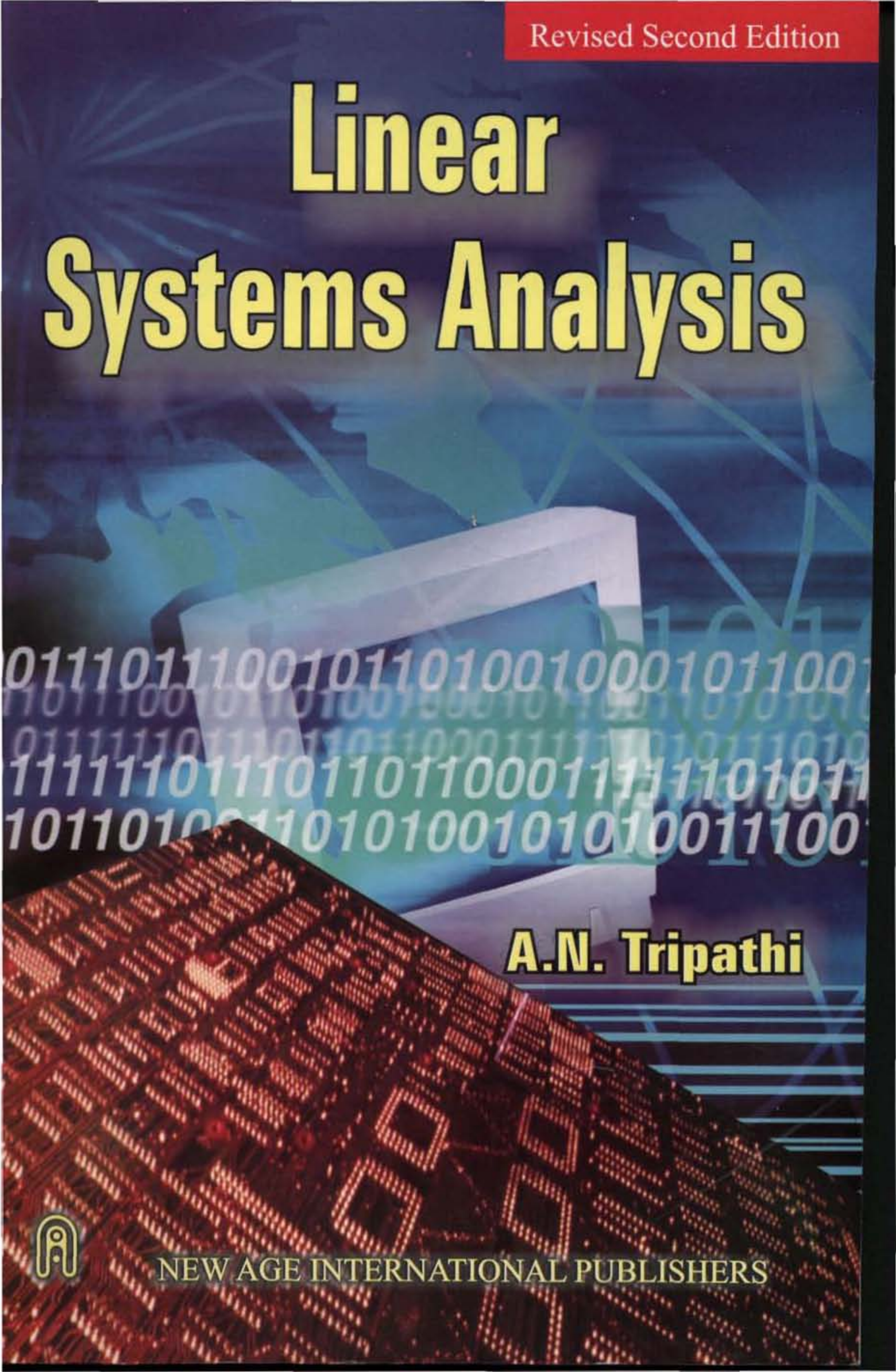

Revised Second Edition

Linear Systems Analysis

A.N. Tripathi

NEW AGE INTERNATIONAL PUBLISHERS

LINEAR SYSTEMS ANALYSIS

SECOND EDITION

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Preface to the Second Edition

The rate of obsolescence of knowledge in the field of technology is very fast these days. However, certain basic principles and methods of analysis remain the same even amidst rapid technological changes. The subject matter of linear systems is one such example. Therefore, most of the material in this second edition remains the same as in the first edition published about a decade ago. The text has been thoroughly edited to remove many errors which had crept into the first edition. Some of the explanations have been simplified to make them more understandable to the students. Additionally, there are a few significant changes also. These are:

- A new chapter on discrete-time systems has been added in view of the increasing importance of digital technology. The need for its inclusion was also pointed out by the reviewers and some users of the book.
- The chapter on analog computer simulation has been dropped as it has now become obsolete.
- The chapter on digital computer simulation has also been dropped, but because of an entirely different reason. Simulation studies have become so important that a number of packages with far greater degree of sophistication are now commercially available. Therefore the somewhat elementary treatment given to the topic in the first edition has now become redundant.

It is hoped that with these changes the book will prove more useful to the students and the teachers of the subject in different branches of engineering.

July, 1998 A.N. TRIPATHI

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Preface to the First Edition

The study of linear systems deals with the dynamic behaviour of physical systems. Such systems are usually represented by a set of linear mathematical relations, called the mathematical model of the system. Their analysis involves generating solutions to these mathematical equations under different working conditions. A number of mathematical techniques have been adopted, developed and refined to give the engineers different tools for the analysis of linear systems. It must, however, be emphasised that the engineer's interest in the subject of linear systems analysis is not merely to learn how to solve mathematical equations: he is interested in interpreting these solutions to find practically useful characteristics of engineering systems so that he can design better systems or operate the existing ones more efficiently. In representing the solution, and even in the process of arriving at them, he uses a combination of physical reasoning–based upon engineering common sense–and the usual mathematical methods. It is in this process of correlating the abstract mathematical arguments with the actual physical behaviour of systems that a student of engineering derives his twin levels of satisfaction: enjoyment of aesthetically pleasing mathematical methods of analysis and generation of satisfactory solutions for practical problems.

This book has been written for an undergraduate course in Linear Systems Analysis. Till about a decade back the topics of this course were taught partly in the Network Theory course and partly in the Control Systems course of Electrical and Electronics Engineering, usually in the final or the pre-final years. Realising the importance of these fundamental topics to several other courses, many Indian universities now offer a full course on the subject in the second year of the four-year degree programme. Some institutions have found it possible to give even two courses in this subject. However, in most of the older institutions, where it is not easy to reduce the burden of the traditional subjects, it does not appear feasible to have more than one such course. The selection of topics to be included in, or rather those which could be excluded from, such a single course becomes necessarily more difficult.

One approach, followed in some of the textbooks, is to reduce the system modelling aspects to a minimum and give full coverage only to the analysis techniques. This is because the modelling process of realistic practical systems requires a somewhat maturer understanding of the particular engineering discipline to which the system belongs. And since one cannot possibly master all the varied disciplines on whose problems the general systems approach is applicable, the task of formulating the mathematical model is not attempted in the introductory textbooks on systems analysis. This book adopts a different approach. It devotes its first two chapters to the mathematical modelling of systems from different engineering disciplines and their classification into different types from the systems viewpoint. Even though later on techniques for the analysis of only linear, time-invariant, continuous-time, lumped parameter systems are discussed, mathematical models are derived for non-linear, time-varying, discrete-time and distributed parameter systems also. It is felt that unless the student is exposed to realistic problems with different types of equations as their mathematical model, he may come out with the patently wrong view that the world consists of only linear systems; or that a particular model will represent the physical system under all working conditions. The experience of teaching these topics for over fifteen years, at different levels–starting from M.Tech. level to the second year B. Tech. level–indicates that usually a student does not appreciate the generality of the systems approach unless the details of the modelling process for problems from different engineering disciplines are explained to him. The selection of variables, the knowledge of appropriate physical laws applicable to the system, and most importantly, the simplifying assumptions to be made, require proper engineering judgement and engineering common sense. And this engineering attitude has to be cultivated right from the early undergraduate courses.

The author believes that a topic should be taught in the classroom and covered in the textbook in sufficient depth so that the student appreciates its origin, its development, its application and its relation to other topics. Pedagogically it is less than satisfactory to include a lot of topics and then given only a brief exposure to them in the hope that they will be mastered at a higher level. This approach has meant that certain important topics, like the discrete time systems, random variables, and distributed parameter systems had to be left out of the present book.

The above approach, however, does not mean that every detail and every derivation must be presented to the student so that he merely reads through them, leaving no new peaks to be conquered by the student himself. The most satisfying and exhilarating aspect of learning is the opportunity of deriving some new result or finding some new application for a given technique. A majority of the problems given at the end of chapters are meant for this purpose. They are not drill exercise where one substitutes the given values in the formulas already derived and follows predestined procedures to get the desired result. The problems usually call for sufficient intellectual exercise on the part of the student and complement the text by exploring new angles.

While the first two chapters are devoted to mathematical modelling and classification of different types of physical systems, the third chapter describes standard test signals and gives the differential equation approach, called the 'classical method', of analysing first and second order systems. Then follow chapters on special techniques of analysing linear systems: Fourier Series, Fourier Transform, Laplace Transform and State Variables. Chapter seven presents a concise treatment of the important aspects of feedback systems. Usually students of electrical and electronics engineering will have a full course on feedback control systems later on in their course and for them this chapter may be omitted. It may be found more useful for mechanical and chemical engineering students for whom it may not be possible to have another course in this area.

Chapter nine deals with analog computer simulation of linear as well as non-linear systems. Earlier this topic merited either a full course, or at least half a course as a part of Analog and Digital Computers. However, the march of digital technology has deprived this topic, a very interesting and useful one for engineers, of its independent status. It is now compressed into a single chapter and covered either as a part of linear systems course or as a part of the feedback control systems course. The last chapter, chapter ten, gives an introduction to the topic of Digital Computer Simulation of continuous time dynamic systems. It also develops a simple simulation package using FORTRAN language. It is hoped that students will have access to computer facility for using this package for solving their exercise problems in this as well as other courses.

The contents of this book have been taught for the past five years to the second year students of Electrical and Electronics Engineering of the Banaras Hindu University. The author is thankful to the students for providing feedbacks, which have gone into the improvement of the text, and also to his friends and colleagues for their helpful suggestions. He would also like to thank Sri Amit Tripathi for this help with drawings and the cover design.

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Contents

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CHAPTER 1

Systems and Their Models

LEARNING OBJECTIVES

After studying this chapter you should be able to:

(i) identify the variables and the parameters for electrical, mechanical, thermal and liquid level systems;

(ii) construct mathematical models for relatively simpler problems from different engineering disciplines;

(iii) obtain electrical analogies for many non-electrical systems;

(iv) appreciate the need and significance of simplifying assumptions and approximations usually made in mathematical modelling; and

(v) appreciate the generality of the systems approach.

One of the most important activities of an engineer is the application of his knowledge and creative skills in designing new systems, and redesigning existing ones to improve their performance. However, before he can start designing systems, the engineer must have a good understanding of the behaviour of the system under different working conditions. The process of determining how a system will behave under different conditions is called 'analysis'. Apart from being a prerequisite for design, a thorough understanding of the methods of analysis of engineering systems is essential for almost every other engineering activity, e.g. determining the optimum but safe working conditions, diagnosing and locating faults, preventive measures against failures, etc.

An alert reader would immediately ask several questions: (i) what is meant by 'behaviour' of a system? (ii) what are the different 'working conditions'? and (iii) what is a 'system' anyway? Instead of giving general definitions of these terms, it would be more instructive at this stage to take a few examples of engineering systems and to understand the concepts associated with these terms in the context of specific examples.

1.1 Automobile Ignition System

In the petrol-driven automobile engine, the power stroke is produced by igniting the compressed air-fuel mixture at the end of the compression stroke. The igniting spark is produced by flash-over across a small air-gap in the spark plug due to the high voltage applied across it. The function of the ignition system is to apply this high voltage at the appropriate time and in the appropriate cylinder of the engine.

A schematic diagram of the system is shown in Fig. 1.1. The main components of the system are; spark plug, ignition coil, spring-loaded contact points, engine-driven cam rotor, capacitor and the car battery. The closing and opening of the contact points by the cam causes sudden change in the primary current of the ignition coil, inducing very high voltage in the secondary winding. This high voltage appears across the spark plug producing the igniting spark. A capacitor is connected across the contact points to prevent sparking and consequent wear-out of the points. In the car engine there are four cylinders and four spark plugs. The high voltage of the secondary of the ignition coil passes through a 'distributor' which directs it to the appropriate cylinder. However, for our study it is sufficient to assume only one spark plug connected directly to the ignition coil, as shown in Fig. 1.1.

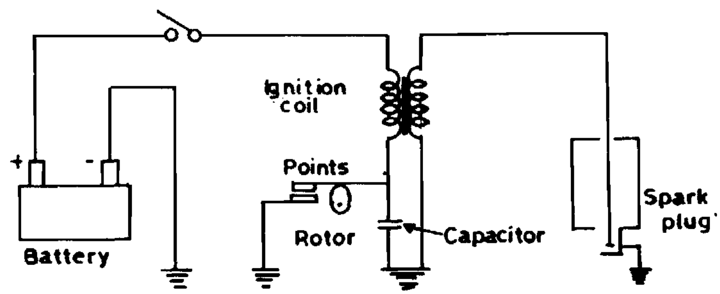


Fig. 1.1 Schematic Diagram of Automobile Ignition System

Let us now list some of the important questions whose answers will give us a proper understanding of the performance of this system:

1. How does the voltage across the spark plug change as a function of time? In other words, what is the waveform of the voltage across the spark plug?
2. What is the maximum value of this voltage and at what time does it occur?
3. How do the voltage waveform, its peak value and the time of peak voltage depend upon the rotor speed, battery voltage, number of primary and secondary turns and the value of capacitor?

The outcome of the analysis process will be the answer to these questions and will describe 'the behaviour of the system under different conditions'.

The system description given at the beginning of this subsection and the schematic diagram (Fig. 1.1) give us a broad qualitative understanding of how this system works. However, they are not sufficient to give quantitative answers to the questions raised here. To find such answers we must establish the 'structure' of the system in a more precise way. Further, we must establish quantitative relationships between the spark plug voltage and other quantities of the system. These two steps together will establish a mathematical model of the system.

Since the system is clearly an electrical system, its structure is best studied by drawing the equivalent circuit diagram. Electrical circuits are drawn in terms of idealised elements; resistance R, inductance L, capacitance C and ideal voltage and current sources. The equivalent electrical circuit of the automobile ignition system of Fig. 1.1 is shown in Fig. 1.2. In this circuit the physical battery is represented by an ideal d.c. voltage source E, the primary of the coil by the series combination of R and L, the coupling between the primary and the secondary windings by the mutual inductance M, the capacitor by C and the points by an ideal switch S. The spark plug is represented by the open terminals of the secondary. This representation is valid so long the flash-over has not taken place across the air-gap of the spark plug. The variable quantities of interest in this electrical system are the secondary voltage v_2, primary current i and the battery voltage E.

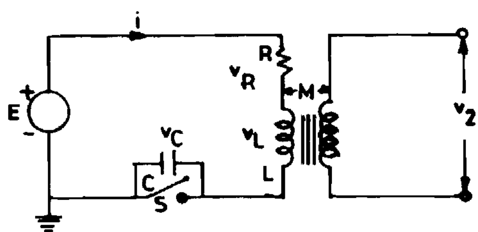


Fig. 1.2. Circuit Model of the Automobile Ignition System

1.1.1 *System Variables and Parameters*

The term 'variables' used in the previous paragraph has a special significance in systems analysis. The variables of interest in a system model can generally be classified as output variable(s), input variable(s) and internal variables. In the present problem the output variable is the secondary voltage v_2, the input variable is the battery voltage E and internal variables are primary current i and capacitor voltage v_c. These input, output and internal variables together are called the *system variables*. In general, they are functions of time. The values of these variables, at any instant of time, will obviously depend upon other conditions like the initial voltage across the capacitor or the initial current through the inductance at the instant when the input is applied. Such conditions are called the *initial conditions* of the system.

In the present problem there is only one input variable E and only one output variable v_2. Such systems are called *single-variable systems*. In many physical systems we have more than one input or more than one output variable. These are called *multi-variable systems*. In this book, we will be interested only in single-variable systems.

The designation of a variable as an input, output or internal variable depends upon the problem under study and also on the questions whose answers are to be given by the analysis. Consider, for example, another situation related to the automobiles ignition system. One may want to study the magnitude, waveform, peak value, etc. of the current in the primary of the ignition coil when the spark does take place. The circuit model will then be altered to have a short circuit across the secondary terminals. The output variable will then be the primary current i. The current flowing in the secondary winding will now be an internal variable.

Other pairs of words used to designate the input-output variables are: *stimulus-response, excitation-response, cause-effect,* etc.

In addition to the system variables, the behaviour of a system is dependent upon certain fixed quantities like, R, L and C in the present problem. Although the system variables are dependent on these quantities, the values of R, L and C are independent of the system variables. They are only dependent upon the elements of the system and their inter-connections. Such fixed quantities, not depending upon either the system variables or the initial conditions are called the *parameters* of a system. For electrical systems, the parameters are resistance, inductance and capacitance. Other types of systems will have different parameters, as we shall study later on. It is common to use small (lower case) letters to denote variables or functions of time and bold-face letters to denote parameters or constant quantities.

Let us summarise our study so far. The first step in the analysis of a physical system is to know the nature of study or the questions to be answered. Next, we must study the structure of the system and identify the parameters, the variables of interest and the initial conditions of the system. The change in the output and other internal variables, in response to different inputs and initial conditions, is called the 'behaviour' of the system and its study is the prime objective of systems analysis.

In order to study the behaviour of a system we must establish functional relationships between different variables. These relationships are given by a set of mathematical equations, called the *mathematical model* of the system.

1.1.2 *Mathematical Model*

Let us illustrate the method of obtaining mathematical models by modelling the automobile ignition system. As mentioned earlier, the mathematical model is a set of equations relating the different system variables. This relationship is governed by the physical laws applicable to the particular system. The system under study here is an electrical one and is represented by an electrical circuit. The appropriate physical laws applicable to such systems are the Kirchhoff's laws. Thus, the system model will be obtained by applying Kirchhoff's laws to the circuit of Fig. 1.2.

Let us assume that the current through the coil and the voltage across the capacitor are zero, prior to the closure of the switch. The instant at which the switch is closed is denoted by $t = 0$. Applying Kirchhoff's voltage law around the primary circuit we get,

$$v_L + v_R = E \quad \text{or} \quad L\frac{di}{dt} + iR = E. \tag{1.1}$$

This equation relates the input variable E with the internal variable i. The output variable v_2 is related to the internal variable i by the relationship,

$$v_2 = M\frac{di}{dt} \tag{1.2}$$

The set of eqns. (1.1) and (1.2), together with the initial condition $i(0) = 0$ and $v_c(0) = 0$ constitutes the mathematical model of the system for this operating condition.

There is another operating condition for which the above model does not provide the correct response. This condition occurs when the switch S is opened after remaining closed for some time. In this case the capacitor C will be in series with R and L. The circuit equation will be,

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,dt = E. \tag{1.3}$$

Differentiating once to remove the integral term and noting that E is a constant d.c. voltage we get,

$$L\frac{d^2 i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = 0. \tag{1.4}$$

Thus, for this operating conditions, eqn. (1.4) together with eqn (1.2) and appropriate initial conditions, constitutes the mathematical model of the system. Solution of these equations will give answers to the questions raised in the beginning. In later chapters, we shall study methods for solving these equations, the characteristics of their solutions and their engineering significance.

1.1.3 *Simplifying Assumptions*

The model-building process always involves approximations and simplifying assumptions, valid for a particular set of operating conditions. This is because the physical reality is always so complex that it is not possible to make a manageable model taking into account all the aspects with all their details. The ability to recognise those aspects which are essential and have to be retained, as against those which can be neglected, is extremely important in engineering analysis. In making and using a mathematical model, the approximations and the simplifying assumptions made, and the conditions under which they are valid must be clearly understood. A model valid for one set of assumptions for a given working condition may not be valid for another working condition.

In the ignition system model, the iron cored ignition coil has been modelled by a series combination of resistance R and inductance L. Let us examine the simplifying assumptions tacitly made in this modelling process. The value of inductance L is assumed to be independent of current i, that is, the inductance is assumed to be linear. Now, inductance of a coil is given by its flux linkages per ampere. Hence, a linear inductance L implies that the magnetising curve between current and flux of the coil is a straight line. This straight line approximation is usually satisfactory provided the core is not driven into the magnetic saturation region. If at any point of operation the primary current is large enough to saturate the core, the linear inductance model will not be valid and its use will give incorrect answers.

If the current through a coil does not change, then its inductance L has no effect and the coil can be modelled simply by a resistance R. For moderate rate of change of current, i.e. for low frequencies (up to a few kHz) the coil can be modelled by a series combination of R and L. At high frequencies (up to a few hundred kHz) the distributed inter-turn capacitances will also have to be taken into account. At still higher frequencies even the circuit theory approach will not be applicable and the mathematical model will have to be built using field theory and Maxwell's equations.

Another significant point concerns the resistance R of the coil. At low frequencies R will be just the ohmic resistance of the coil wire. At somewhat higher frequencies, R will have to include an additional component to account for the iron losses in the core. At still higher frequencies, we may have to add yet another component to account for the skin effect. All these considerations may not be applicable in the case of an ignition coil but are worth remembering whenever modelling an iron c re coil.

The above discussion shows that there is nothing like 'the' model for a given system. We can only have 'a' model valid for a set of assumptions and operating conditions. While the physical system remains the same its model will be different for different situations. Therefore, we should not equate the physical system with its mathematical model.

In this section we have considered the modelling process for an electrical system. Let us now consider a mechanical system and build its mathematical model using the techniques developed so far.

1.2 Automobile Suspension System

The object of this system is to damp out vibrations and jerks produced in a vehicle due to uneven road surface. The suspension system absorbs these jerks and prevents them from being transmitted to the passengers, thus improving the riding quality of the vehicle.

The weight of passengers and car body is transferred to the axle through a spring and a shock absorber. The shock absorber consists of an oil-filled cylinder having a piston with orifices in it. The piston-end is connected to the car body and the cylinder-end to the axle. Motion of the piston, and any mass connected to it, is opposed by the viscous friction of oil being forced through the orifices. The structure of the system is shown in Fig. 1.3. The mass of the car body and passengers is represented by the mass M, suspension springs by the ideal spring with spring constant K, and the shock absorber by an ideal dash-pot with coefficient of viscous friction D. Thus, the parameters of the system are: mass M, coefficient of viscous damping D and the spring constant K.

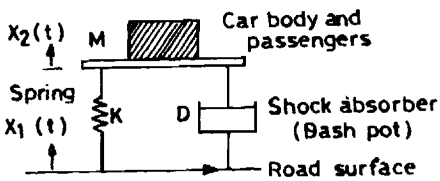


Fig. 1.3. Automobile Suspension System

The system variables are: displacement x, velocity u and acceleration a. In the present problem the input variable is the vertical displacement $x_1(t)$ of the axle due to undulations in the road surface. The output variable is the vertical motion $x_2(t)$ of the mass M. The mathematical model should relate these input and output variables.

The physical law governing the behaviour of this mechanical system is the D'Alembert's principle. It states that the algebraic sum of externally applied forces and the forces resisting motion in any given direction is zero. In the present case there is no explicit externally applied force. The net displacement of the mass is $(x_2 - x_1) = x$. The forces opposing the motion are:

by the mass M, $$Ma = M\frac{du}{dt} = M\frac{d^2x}{dt^2}$$

by the spring, $$Kx$$

by the dash-pot, $$Du = D\frac{dx}{dt}.$$

Applying D'Alembert's principle we get,

$$M \frac{d^2 x}{dt^2} + D \frac{dx}{dt} + K x = 0. \tag{1.5}$$

Equation (1.5), together with appropriate initial conditions, is the mathematical model of the system. Its solution will show how x varies as a function of time following an initial displacement $x(0)$. Whether the ride will be smooth or jerky can be assessed by the nature of this solution.

A number of simplifying assumptions have been implicitly made in the above modelling process. It is assumed that the shock absorber and the suspension spring can be represented by an ideal dash-pot and an ideal spring. Further, the spring effect of the inflated tube and the tyre and also of the seats has been neglected. The car body is supported at four places, near each wheel. In the present model, a single suspension has been considered independently, i.e. it has been assumed that the displacement of one suspension unit does not affect the others. In fact the coupled motion of the front and the rear suspensions gives rise to a 'pitching' motion of the vehicle. However, consideration of all these aspects would make the model more and more complicated. How complex a model one makes depends upon how precise answers are needed. It is usual in engineering practice to begin the analysis with a fairly simple model and then go on adding other complexities as required by the problem refinements.

1.3 Systems and Their Models

Let us now compare the previous two examples and generalise some of the systems concepts based on the experience of these two examples. The two problems are from two different areas of engineering: the ignition system from electrical engineering and the suspension system from mechanical engineering. The parameters, the variables, the laws governing their behaviour, the operating conditions and the simplifying assumptions made are all different for the two systems. And yet their mathematical models are very much similar. Both the models, given by eqns. (1.4) and (1.5), are ordinary, linear differential equations of order two. The methods for solving these two equations will be the same, the solutions will have the same characteristics and their physical interpretations would also be similar. On the basis of these similarities, both these systems are classified under one category—linear dynamic system of order two. The analysis technique, once developed for a second order system, would be applicable to *any* second order system, irrespective of whether the system is electrical, mechanical, thermal or hydraulic in nature. This general approach of developing an analysis technique, based on the type of mathematical model of the system, is the key factor which permits a common systems approach to be used for studying problems from different branches of engineering and even non-engineering disciplines, like biological systems, economic systems, management systems, etc.

We have repeatedly used the word system, without defining it. Defining such basic terms is always a somewhat difficult task. The definition given below should be interpreted as clarifying the concept associated with the term 'system' rather than a precise definition for it.

Definition of a system: A system is any structure of inter-connected components created to perform some desired function. It has distinct inputs and outputs and it produces an output signal in response to an input signal. The functional relationship between the inputs and the outputs is given by a set of mathematical equations and this set is called a *model of the system.*

The key factors to be examined in developing a mathematical model for a system are: (i) the structure, i.e. the components and their inter-connections; (ii) the parameters; (iii) the variables; (iv) the inputs and the outputs; and (v) the initial conditions. Once these factors are properly identified, the required model is obtained by making suitable simplifying assumptions and then applying the appropriate physical laws applicable to the system. Further analysis is carried out using this model.

The suitability of a model depends upon the working conditions and the questions one wants to answer. For example, the equivalent circuit and the mathematical model for determining the frequency response of a transistor is different for low, medium and high frequencies. The same transistor can also be modelled as a thermal system when we want to find its heat dissipation properties.

1.3.1 *Across and Through Variables*

Building mathematical models for an electrical system is very much simplified by first building its circuit model, i.e. its equivalent circuit. This circuit model is built using idealised two terminal elements R, L and C with well-defined terminal voltage (v)-current (i) relations:

$$v_R = R\,i,\ v_L = L\,\frac{di}{dt},\ \text{and}\ v_C = \frac{1}{C}\int i\,dt.$$

In addition to these passive elements, the circuit model also uses two terminal source elements—the ideal voltage source and the ideal current source.

The electrical properties of these two terminal elements are given in terms of the voltage 'across' the element and the current 'through' the element. An 'across' variable defines some state of one terminal with respect to the other and a 'through' variable, the flow or transmission of some quantity through the element. According to this general definition of 'across' and 'through' variables, the designation of voltage as an 'across' and current as a 'through' variable for electrical systems is self-explanatory. This concept is applicable for non-electrical systems also.

For translational mechanical systems, the three idealised elements are: the mass, the dash-pot and the spring. The parameters associated with these elements are mass M, coefficient of viscous friction D, and the spring constant K, respectively. The variables associated with these elements are force f and velocity u. The two terminal representations of these elements and their terminal force-velocity relations are given in Fig. 1.4. For the mass, one terminal is some stationary frame or the 'ground' with respect to which all the motion is described. Here force is the 'through' variable as it acts through the element, and velocity is the 'across' variable because it describes the motion of one terminal with respect to the other.

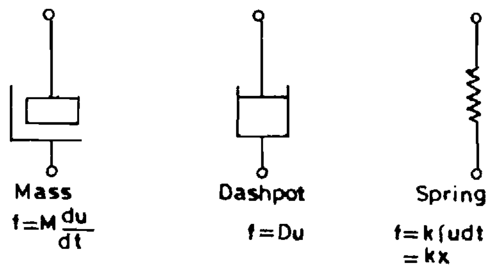


Fig. 1.4 Idealised Elements for Translational Mechanical Systems

In line with ideal voltage and ideal current sources for the electrical networks, ideal force source and ideal velocity source are used as the active elements in a mechanical system. An ideal force source produces a given force through its terminals, irrespective of the load connected between them. Similarly, an ideal velocity source produces the specified velocity across its terminals, irrespective of the load connected between them.

In rotational mechanical systems, the parameters are moment of inertia J, rotational damping coefficient D_θ and torsional constant K_θ. The variables are torque τ, angular velocity ω, angular displacement θ and angular acceleration α. By comparison with the translational system, the terminal relations are:

$$\tau = J\,\frac{d\omega}{dt}\,,\ \tau = D_\theta\,\omega \text{ and } \tau = K_\theta \int \omega\, dt = K_\theta\,\theta.$$

Angular velocity is the 'across' variables and torque the 'through' variable.

1.3.2 *Electrical Analogies*

Earlier in this section we noted that systems from different areas can be represented by the same mathematical model, e.g. the ignition system and the suspension system. Such systems, having the same type of defining equations, are called *analogous systems*. Because of the ease of experimental studies, it is useful to construct electrical analogy of non-electrical systems.

The mathematical model for a series RLC circuit shown in Fig. 1.5 (a) will be the same as eqn. (1.3) except for the different forcing function $e\ (t)$. Similarly the

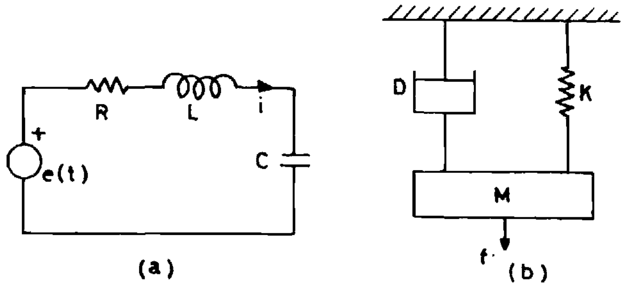


Fig. 1.5 Analogous Electrical and Mechanical Systems

mathematical model of the mechanical system of Fig. 1.5 (b) will be the same as eqn. (1.5) except for the forcing function $f(t)$. These are given by the following two equations:

$$L \frac{di}{dt} + Ri + \frac{1}{C} \int i\,d(t) = e(t). \tag{1.6}$$

$$M \frac{du}{dt} + Du + K \int u\,dt = f(t). \tag{1.7}$$

By comparing these two equations we can determine the analogous parameters and variables as in Table 1.1. In this method of developing an electrical analogy for the mechanical system, force is equated with voltage; hence it is called the *force-voltage* analogy.

Table 1.1 The Force-Voltage Analogy

Mechanical system	*Electrical system*
Force, $f(t)$	Voltage, $e(t)$
Velocity, $u(t)$	Current, $i(t)$
Mass, M	Inductance, L
Viscous friction, D	Resistance, R
Spring constant, K	Inverse capacitance, $1/C$

By the duality principle of electrical circuits, the series circuit of Fig. 1.5(a), driven by voltage source e, is equivalent to the parallel electric circuit of Fig. 1.6, driven by a current source i. The circuit equation for Fig. 1.6 is:

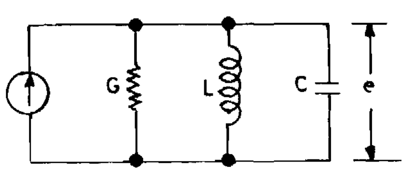


Fig. 1.6 Dual of Fig. 1.5(a)

$$C\frac{de}{dt} + Ge + \frac{1}{L}\int e\,dt = i. \tag{1.8}$$

Comparing elements of eqn. (1.7) with those of eqn. (1.8), we get the *force-current* analogy, as shown in Table 1.2.

Table 1.2 The Force-Current Analogy

Mechanical system	Electrical system
Force, $f(t)$	Current, $i(t)$
Velocity, $u(t)$	Voltage, $e(t)$
Mass, M	Capacitance, C
Viscous friction, D	Conductance, G
Spring constant, K	Inverse inductance, $1/L$

The force-current analogy is more direct because the across and through variables of both the systems are treated analogous to each other. In the force-voltage analogy, the across variable of one is treated as analogous to the through variable of the other.

For experimental studies, setting up a mechanical or a thermal system model is not so easy. However, its analogous electrical circuit can easily be set up with commonly available electrical components in the laboratory. Such an electrical model is quite flexible; changes in the parameters can be easily effected and different forcing functions can be easily simulated. Thus, the behaviour of the non-electrical system can be conveniently studied experimentally by studying the response of its analogous electrical model.

In the following sections of this chapter, we consider examples of systems from some other areas of engineering, as well as non-engineering disciplines.

1.4 An Electromechanical System: The Loudspeaker

A loudspeaker converts electrical signals into sound signals in the form of mechanical vibrations of the surrounding air; hence, it is an electromechanical system. It construction is shown in Fig. 1.7(a).

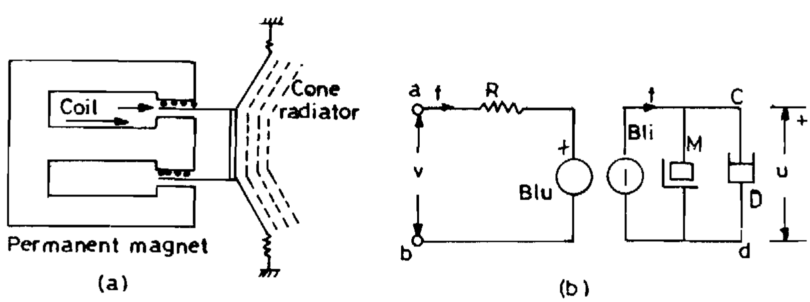


Fig. 1.7 (a) Loudspeaker Construction, (b) Equivalent Circuit

The permanent magnet produces a strong magnetic field in its air gap. A coil (called the *voice coil*) of a few turns of fine wire, wound on a light former, is able to move freely in the air-gap. The voice coil is attached to a light but stiff cone radiator made of paper or fibre. The cone radiator is clamped at its periphery through a flexible support so that it keeps the cone (and the attached voice coil) in proper position and yet permits oscillations of the cone as a whole in the axial direction.

The audio-frequency input signal voltage is applied across the coil. The resulting current in the voice coil interacts with the magnetic field to produce an axial force on it and the cone. The cone, and hence the air mass surrounding it, vibrates at the signal frequency to produce the output sound.

The input variable is the voltage signal $v(t)$ and the output variable is the axial velocity of the cone radiator $u(t)$. Let us now develop a mathematical model relating the input and the output variables.

The input voltage will produce a current in the voice coil which will be related to it by the Kirchhoff's voltage law. The voltages opposing the applied voltage are: (i) voltage drop in the resistance of the coil; (ii) voltage drop in the inductance of the coil; and (iii) the back e.m.f. induced in the coil due to its motion in a magnetic field. Because of the small number of turns, the coil inductance is small; hence, the voltage drop due to inductance may be neglected in comparison with the other two voltages. Equating the applied voltage to the sum of the opposing voltages we get,

$$Ri + Blu = v \tag{1.9}$$

where,

v = $v(t)$, the input voltage;

u = $u(t)$, the axial velocity of the coil and the cone;

i = $i(t)$, the current through the coil;

R = resistance of the coil;

B = flux density in the air gap; and

l = length of the coil.

The coil current i will react with the linking magnetic flux to produce an axial force f according to the relation,

$$f = Bli. \tag{1.10}$$

This force will produce motion of the cone and the surrounding air mass, which, according to the D'Alembert's principle will be given by the relation,

$$M \frac{du}{dt} + Du = f \tag{1.11}$$

where,

M = mass of the cone and the surrounding air mass in contact with it, and

D = coefficient of viscous friction due to the motion of the cone through the air.

In arriving at the relation (1.11), the spring effect of the cone suspension has been neglected.

The intermediate (or the internal) variables of the system are the coil current i and the force f. The parameters are R, B, l, M and D. Equations (1.9), (1.10) and (1.11) may be combined into one by eliminating the internal variables to give a single equation as the mathematical model of the system:

$$M \frac{du}{dt} + \left(\frac{B^2 l^2}{R} + D \right) u = \frac{Bl}{R} v. \tag{1.12}$$

Figure 1.7(b) shows a mixed electrical and mechanical network model of the system. Note the use of ideal force source with a symbol similar to a current source.

In addition to a number of simplifying assumptions made in the above derivation, the determination of parameters M and D is somewhat ambiguous. M includes the effective mass of the air being vibrated by the cone. Its determination will, at best, be an approximation. Further, both M and D will depend heavily on the shape, size and material of the enclosure in which the speaker is housed. The enclosure design is thus quite an important factor for determining the speaker performance.

1.4.1 *Frequency Response*

The main performance measure of interest in a speaker is its frequency response. An ideal speaker should reproduce all the input signals in the audio-frequency range, 20 Hz to 20 kHz, without any distortion. One way to measure this performance is to plot the ratio of the magnitudes of the output and the input signals for sinusoidal inputs, as a function of frequency. This is called *the magnitude response*. Ideally this curve should be a flat line, i.e., a line parallel to the frequency axis. This ideal is approximated to some extent in high fidelity speakers only. In ordinary speakers, the magnitude ratio decreases quite sharply with increasing frequency. For good reproduction at low frequencies the size of the cone should be large. However, a large cone has a high mass associated with it. Therefore, it will produce considerable attenuation at high frequencies, resulting in poor high frequency performance. In some high fidelity systems this difficulty is solved by having two speakers, a large one for low frequency signals (called the 'woofer') and a small one for high frequency signals (called the 'tweeter'). A frequency selective network in the amplifier separates the low and high frequency signals and directs them to their respective speakers.

In addition to the magnitude response curve, a plot of the phase difference between the sinusoidal input and the output versus frequency, called *the phase response*, is also of interest. The magnitude response and the phase response together are called the *frequency response* of a system.

1.4.2 *Transducers*

A loudspeaker may also be thought of as a device for converting electrical signals into sound signals. Such devices, which convert signals from one form of energy to another form, are called *transducers.* As an example of another electromechanical transducer we may mention the permanent magnet moving coil galvanometer. It converts electrical voltage or current signals into angular deflections of a needle. In electrical and electronic instrumentation systems, a large variety of transducers are used to convert non-electrical signals like temperature, pressure, flow, displacement, etc. into electrical form. Note that the term transducer is reserved for only signal level devices and not those which handle energy at power level. Power level energy conversion devices, like motors and generators, are called *energy converters* and not transducers.

1.5 A Thermal System

Efficient methods of heat dissipation from large power silicon devices like power transistors, rectifiers, thyristors, etc. is an important consideration in the design of power electronics equipment. The metal casing of such a device is embedded into a heat sink, an aluminium section with a number of cooling fins. The cooling fins increase the heat dissipation area and hence keep the device temperature within the required limits.

The power dissipation in the device, due to the flow of current through the silicon *p-n* junction, causes generation of heat at the junction. From the junction the heat flows to the outer metal casing, from the casing to the heat sink, and finally from the heat sink to the surrounding air mass. The junction temperature is a critical factor and it should not exceed the rated maximum value even under the worst loading condition. The junction temperature depends upon the rate at which the cooling system dissipates heat. The analysis of the heat-dissipating thermal system would aim at answering questions like:

(i) What is the steady state temperature of the junction for a steady current through the device?

(ii) How does the junction temperature change (as a function of time) for a specified variation in current (as a function of time)?

(iii) What is the peak value of temperature and after how much time is it reached, following a sudden increase in the load current?

Answers to these questions will enable the equipment designer to select suitable heat sink or cooling system to ensure safe operation of the device, even under adverse operating conditions.

Let us first identify the variables and the parameters of a thermal system. The variables of interest are the temperature τ and the rate of heat flow q. The temperature would be the 'across' variable and the heat flow rate the 'through' variable. The parameters for conduction of heat through a solid body are the thermal resistance and the thermal capacitance of the body. The two equations of heat flow are:

$$q = \frac{\tau}{R_T} \text{ and } q = C_T \frac{d\tau}{dt} = \frac{dw_T}{dt} \tag{1.13}$$

where,

q = rate of heat flow (cal/sec);

τ = temperature difference (degree Kelvin);

R_T = thermal resistance (sec–°K/cal);

C_T = thermal capacitance (cal/°K); and

w_T = thermal energy (cal).

The two heat flow relations (1.13) are very much similar to the current-voltage relations in electrical circuits. The similarity between the two systems becomes clearer if we note the analogies between the two systems, given in Table 1.3.

Note that in thermal systems there is no parameter analogous to inductance. (Why?)

Analogous to electrical systems, we try to model a thermal system in terms of idealised elements R_T and C_T. However, unlike the ordinary electrical circuits, the thermal resistance and capacitance are not located at one point of space but distributed over it. Accordingly the variables τ and q are also functions of space co-ordinates, in addition to being functions of time. Systems of this type are called *distributed parameter systems* and their modelling is done in terms of partial differential equations. However, a simplified model made by using lumped parameters is useful for approximate analysis and, in many cases, is adequate for practical purposes.

Table 1.3 Electrical Analogy of a Thermal System

Electrical system	*Thermal system*
Voltage, v	Temperature, τ
Current, i	Heat flow rate, q
Resistance, R	Thermal resistance, R_T
Capacitance, C	Thermal capacitance, C_T

Let us now construct a lumped parameter model using suitable approximations and assumptions. The heat flow path from the junction to the ambient air is divided into four parts: the junction, the casing, the heat sink and the atmosphere. Let us assume that the thermal capacities of the casing and the heat sink are C_{T1} aand C_{T2} repectively. The thermal capacity of the junction is small compared to those of the casing and the heat sink, and is therefore neglected. Let us assume that the thermal resistance between these parts is localised at their boundaries: R_{T1} between the junction and the casing, R_{T2} between the casing and the heat sink, and R_{T3} between the sink and the atmosphere. In line with the lumped parameter model, it is further assumed that the temperatures over each one of these parts is constant: T_J the junction temperature, T_C the casing temperature, T_S the heat sink temperature and T_A the ambient temperature.

Figure 1.8(a) shows the assumed temperature distribution by solid lines. The actual temperature distribution will be more like the dotted curve. Further, the heat flow is assumed to be in one direction only. Figure 1.8(b) shows the corresponding electrical network analogous to the thermal system. The object of the mathematical model is to relate the junction temperature with the rate of heat flow q_j. This heat rate q_j will be equal to the power dissipated at the junction in watts, multiplied by the Joule's constant.

The network model has been used as an intermediate stage in the development of the mathematical model. The derivation of the mathematical model for the system, using Kirchhoff's law on the analogous electrical system of Fig. 1.8(b), is left as an exercise for the reader.

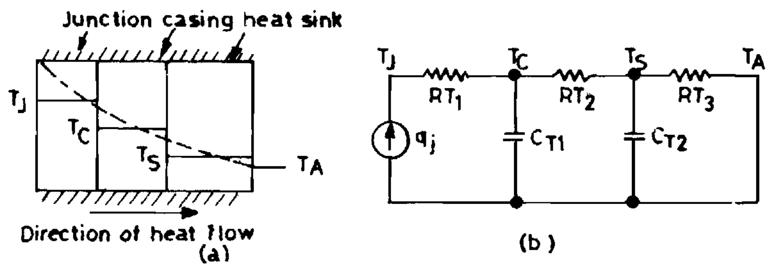


Fig. 1.8 Heat Dissipation from a Silicon Device: (a) Temperature Distribution and (b) Electrical Analogy

1.6 A Liquid Level System

Figure 1.9 shows a liquid level system. The inflow and the outflow from the tank is controlled by inlet and outlet valves. Under steady-state conditions the valve openings are such that the rate of inflow is equal to the rate of outflow. Under this condition, the liquid level in the tank will be constant. Now, assume that the inlet valve opening is suddenly increased, increasing the inflow rate. It is of interest to find how the liquid level in the tank will change as a function of time.

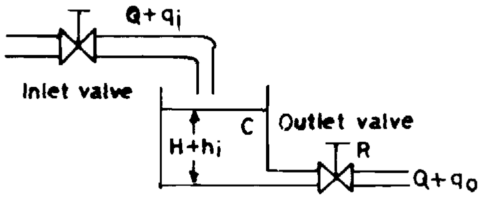


Fig. 1.9. A Liquid Level System

The variables of the system are the input and output flow rates and the liquid level in the tank. The parameters are the valve resistance and the area of cross-section of the tank. Let us first consider the valve characteristics.

If the flow through the valve is laminar, the flow rate and the difference in liquid levels across the valve, called the 'head', are related by an Ohm's law type of relation,

$$Q = \frac{H}{R} \tag{1.14}$$

where,

Q = liquid flow rate through the valve;

H = head across the valve; and

R = resistance of the valve.

However, more commonly the flow is turbulent. In that case, the relation between the flow rate and the head is non-linear and is given by:

$$Q = \sqrt{\frac{H}{R}}\ . \tag{1.15}$$

Use of such a non-linear relation will make the mathematical model also non-linear. Analysis with non-linear models is more complex and therefore the use of a linear model is preferable. To get a linear model, a technique called *linearisation around an operating point* is commonly used.

A plot of the non-linear relation (1.15) is shown in Fig. 1.10. The slope of the valve characteristic is different at different points. However, if we assume that the change in the head is small around an operating point P (Q_1, H_1), the 'incremental' resistance is constant around this point. That is

$$\frac{h}{q} = R \text{ or } q = \frac{h}{R}. \tag{1.16}$$

Equation (1.16) gives a liner relationship between small changes q and h around an operating point.

Let us now go back to the problem of modelling the liquid level system of Fig. 1.9. The question to be answered now is: what is the physical law governing the

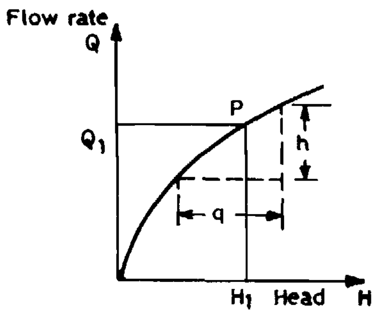


Fig. 1.10 Valve Characteristic

fluid flow in this situation? In other words, what is the relationship among the inflow rate, the outflow rate and the liquid level? Such a relationship can be derived from a very general principle which can be stated as,

$$\text{input} = \text{output} + \text{accumulation}. \tag{1.17}$$

The validity of eqn. (1.17) is self-evident.

Equation (1.17) is called the *continuity equation* and is useful for a number of physical systems like mass transfer, heat transfer, flow systems, etc. Even the Kirchhoff's current law at a node can be thought of as a particular form of the continuity equation. There can be no accumulation of charge at a node. Therefore the rate of charge inflow must be equal to the rate of charge outflow at a node. In other words, the current into a node must be equal to the current out of the node.

In the present problem, we are interested in the rates of input flow and output flow. Taking derivatives of terms in eqn. (1.17), we can rewrite the continuity equation as

$$\text{rate of inflow} = \text{rate of outflow} + \text{rate of accumulation}. \tag{1.18}$$

The accumulation of the liquid in the tank would be given by its area of cross-section multiplied by the change in liquid level, or the accumulation = $A \times h$. Therefore, the rate of accumulation = $A \dfrac{dh}{dt}$. Thus, eqn. (1.18) gives,

$$q_i = q_0 + A \frac{dh}{dt} \cdot \tag{1.19}$$

From eqn. (1.16) we have,

$$q_0 = \frac{h}{R}.$$

Substituting it in eqn. (1.19), and rearranging terms we get,

$$A R \frac{dh}{dt} + h = R q_i. \qquad (1.20)$$

This equation relates the input q_i with the output h and is the required mathematical model of the liquid level system.

Let us now compare this model with that of the electric circuit of Fig. 1.11. Writing the node equation we have,

$$C \frac{dv}{dt} + \frac{v}{R} = i$$

or

$$C R \frac{dv}{dt} + v = R i. \qquad (1.21)$$

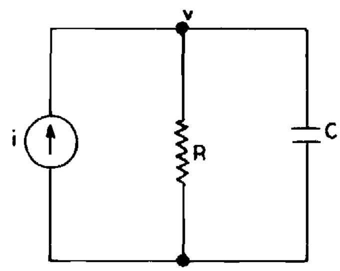


Fig. 1.11 Electric Circuit Model

Comparing eqns. (1.21) and (1.20), we can establish the analogy given in Table 1.4.

Table 1.4 Electrical Analogy of Liquid Level System

Electrical system	*Liquid level system*
Voltage, v	Head, h
Current, i	Flow rate, q
Resistance, R	Resistance, R
Capacitance, C	Area of cross-section of the tank, A

Liquid level and liquid flow systems of this type, and those of more complex types, are frequently encountered in process industries and water supply systems. The method of analysis given here can be generalised to include the flow of gases as well. Such analysis is also useful in pneumatic and hydraulic control systems.

1.7 A Biomedical System

In this section we take up the modelling of a non-engineering system. The developments in the succeeding paragraphs will show that the techniques of model building for such systems are not different from those for engineering systems. This will further illustrate the generality of the systems approach.

When a drug is injected into the body, it suddenly raises the concentration of that drug in the blood. In due course of time, some part of this drug is passed out from the blood stream (renal excretion) and the remaining part is converted into other chemicals (metabolism). As a result, the drug concentration in the body gradually reduces. We would like to build a mathematical model of the process from which the drug concentration at any time after the injection can be calculated. Note that the form of drug inlet may be a one-shot injection, injection at regular intervals or continuous infusion through a drip line.

The variables of interest in this problem are:

drug density or concentration = $c(t)$, (mg/litre)

drug inflow rate = $q_i(t)$, (mg/sec)

drug outflow rate = $q_0(t)$, (mg/sec)

The volume rate of outflow (renal excretion + metabolism) is generally a known constant, say K. Therefore,

$$q_0 = Kc. \tag{1.22}$$

Also, the total volume of blood in the body is a tant, say V. Thus, V and K are the two parameters of the system.

The physical law governing the system can be written in terms of the continuity equation, i.e., eqn. (1.18). Applying it to the present case we get,

$$q_i = q_0 + \frac{d(Vc)}{dt}$$

or

$$V\frac{dc}{dt} + q_0 = q_i. \tag{1.23}$$

Substituting for q_0 from eqn. (1.22) we have,

$$V\frac{dc}{dt} + Kc = q_i. \tag{1.24}$$

This is the required mathematical model. Note that this model is the same as that for the liquid level system, given by eqn. (1.20). Both are first order linear differential equations.

1.8 Concluding Comments

In this chapter we have studied the methodology for constructing mathematical models for common types of systems from different branches of engineering.

These models are important because the analytical study and the design of engineering systems is based on their mathematical models. The modelling process requires a knowledge of the structure of the system, its variables and parameters, and the physical laws governing the system. Generally, two types of relations between the system variables are needed: an Ohm's law types of relation and a continuity equation type of relation. A number of simplifying assumptions are usually required to obtain a simple enough model.

The analysis of systems involves solving the equations describing the mathematical model for different inputs and for different working conditions. The result of mathematical analysis is then interpreted in terms of the behaviour of the physical system under study. Commonly used methods of analysis will be developed in later chapters. The analysis procedure depends only on the type of mathematical equations of the model and not upon the branch of engineering to which the problem belongs. For example, the mathematical models for all the different systems discussed in this chapter are given by linear differential equations of order one or two. In systems terminology all these systems are classified as linear dynamic systems of order one or two. Their techniques of analysis and the characterisation of the solutions will be identical. This general method of studying systems, involving modelling, analysis, design, optimisation, etc., is the subject matter of a new branch of engineering, called the 'systems engineering'. The term 'engineering' used in this title is somewhat of a misnomer, as the same techniques are now being increasingly employed in studying and solving problems from a number of non-engineering disciplines like, biology, economics, management science, resource planning, etc. Thus, it is important for engineers to develop a 'systems approach' to the solution of real life problems, be they engineering or non-engineering problems.

GLOSSARY

System (definition): It is a structure of inter-connected components created to perform some desired function. It has distinct input(s) and output(s) and produces an output in response to an input.

Mathematical Model: A set of mathematical equations relating the output to the input is called a mathematical model (or simply a model) of the system.

System Variables: (Input, output and internal variables.) In general they are functions of time. The behaviour of the system is characterised by the system variables.

Across and Through Variables: An 'across' variable defines some state of one terminal with respect to the second terminal of a two-terminal device. The 'through' variable defines the flow or transmission of some quantity through the element.

Single and Multi-Variable Systems: When there is only one input and one output the system is called a *single-variable system*. If there are more than one inputs and/or outputs the system is called *a multi-variable system.*

Initial Conditions: The values of output and other internal variables of the system at the instant when the input is applied, i.e., at $t = 0$, are called *initial conditions*.

System Parameters: Those fixed quantities which characterise the properties of the system, and are not dependent upon system variables, are called *system parameters*.

Analogous Systems: Different types of systems (usually from different branches of engineering) having same type of defining mathematical model are called *analogous systems*.

Electrical Analogy: When a non-electrical system is equated to an electrical network with the corresponding variables and parameters of the two systems identified, the equivalent electrical network and its mathematical model is called *the electrical analogy of the non-electrical system*.

Transducers: Devices which convert signals from one form of energy to another are called *transducers*.

PROBLEMS

1.1 A π filter, shown in Fig. 1.12, is frequently connected between a rectifier and the load to smoothen out the ripples. Let the load be a pure resistance R_L. Let the input to the filter be a voltage source with open circuit voltage v_i and internal resistance R_i. Develop a mathematical model relating the input voltage v_i to the output load current i_L.

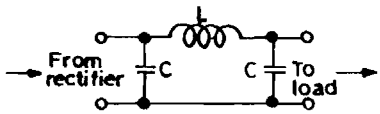


Fig. 1.12

1.2. A permanent magnet moving coil galvanometer may be considered a transducer, converting input electrical voltage $v(t)$ into angular deflection $\theta\,(t)$ of a pointer. Develop a mathematical model relating the input and the output.

1.3. Usually a d.c. motor is thought of as an energy converter. However, in many control applications, it is treated as an actuator for converting electrical voltage into angular motion of the rotor shaft. When the input voltage is applied to the armature, with field connected to a constant voltage supply, the motor is said to be armature controlled. Develop a mathematical model for the armature-controlled motor, treating angular velocity of the shaft as the output and the armature voltage as the input.

1.4. Develop both the f-v and the f-i analogies for the mechanical system shown in Fig. 1.13. Since this system has two dependent variables, x_1 and x_2, it is called a system with two degrees of freedom.

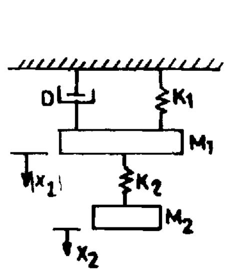


Fig. 1.13

1.5. In a rotational mechanical system, the gears perform the same function as a transformer in an electrical system. The shaft of a d.c. motor is coupled to a large flywheel through a pair of gears. Develop a mathematical model relating the angular position of the flywheel (output) $\theta(t)$, to the motor torque (input) $\tau(t)$. Also develop an analogous electrical circuit for this mechanical system.

1.6. A thermometer may be considered a thermal system. Develop its mathematical model relating the length of mercury column $x(t)$ as an output and the temperature $\tau(t)$ of the substance surrounding the bulb as the input. What assumptions are commonly made in using thermometers as temperature sensors?

1.7 An electric immersion heater has a rating of K kilowatts. The thermal conductance of the heater assembly is G_1 and its thermal capacitance C_1. The heater is immersed in a liquid tank with good thermal insulation on the tank walls. The thermal conductance and the thermal capacitance of the liquid are G_2 and C_2. Determine a mathematical model for finding the change in liquid temperature as a function of time when the heater is switched on.

Draw an analogous electrical circuit identifying the analogous variables and parameters of the thermal and the electrical systems.

1.8. Two water tanks of different areas of cross-section A_1 and A_2 and different liquid levels H_1 and H_2 are connected through a ground pipeline with a normally closed valve. Derive the equations for determining the liquid levels as functions of time, following a sudden opening of the connecting valve.

1.9. In the studies of population dynamism it has been empirically established that the rate of increase in the population of a species at any time is proportional to the total population at that time. Give a mathematical model for this situation. Now, assume that the food available to the species is limited. How would you modify the model to account for this fact?

1.10. The rate of dissolution of a chemical in water is proportional to a product of two factors; (i) amount of undissolved chemical, and (ii) the difference between the concentration in a saturated solution and the concentration in the actual liquid. Develop a mathematical model for calculating the variation of concentration as a function of time.

CHAPTER 2

Classification of Systems

LEARNING OBJECTIVES

After studying this chapter you should be able to:

(i) determine whether a system is linear or non-linear;

(ii) identify and characterise different types of commonly encountered non-linearities; and

(iii) classify a system according to the type of its mathematical model.

The definition of a system given in the previous chapter is very general and includes under it a large variety of different engineering and non-engineering systems. The methods of analysis and the characteristics of the solutions are also different for them. The object of systems classification is to bring under one category systems having certain common features. The basis of classification is the type of equation describing the mathematical model. Systems described by one type of equation are put under one category. In this chapter we shall classify systems into the following independent categories:

(1) Linear or non-linear system

(2) Dynamic or static system

(3) Time invariant or time-varying system

(4) Continuous time or discrete time system

(5) Lumped parameter or distributed parameter system.

(6) Deterministic or stochastic system

2.1 Linear and Non-linear Systems

The linearity principle: The word 'linear' intuitively suggests something pertaining to a straight line. If the input x and the output y of a system are related by a straight line, as shown in Fig. 2.1, we may say that it is a linear system. The mathematical equation relating the input to the output is then $y = mx$, a linear algebraic equation. A resistor obeying Ohm's law, or a spring obeying Hooke's law, are examples of such linear systems.

We will be dealing with different types of systems with different mathematical models. Most of them will not have simple algebraic equations as their models. Their models may be in terms of differential equations, difference equations, etc. We therefore need a more general criterion for deciding the linearity of a system. With this aim in view, let us re-examine the relationship shown in Fig. 2.1. One clearly noticeable property of such a relationship is that if the magnitude of the input is increased k times, from x_1 to kx_1, the output magnitude is also increased k times, from y_1 to ky_1. This property can be generalized for inputs and outputs which are functions of time. For a linear system if an input $x_1(t)$ gives an output $y_1(t)$, i.e., $x_1(t) \rightarrow y_1(t)$, then $kx_1(t) \rightarrow ky_1(t)$. This property is called *homogeneity* and is a property of all linear systems. In other words homogeneity is a necessary condition for a system to be linear.

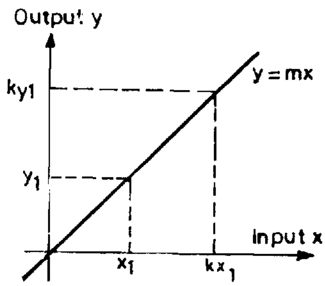


Fig. 2.1 Linear Relation

Another necessary condition for linearity is the property of *superposition*. Superposition implies that if an input $x_1(t)$ gives an output $y_1(t)$, i.e., $x_1(t) \rightarrow y_1(t)$, and another input $x_2(t) \rightarrow y_2(t)$ then, if the two inputs are applied together, the output will be the sum of the two individual outputs, i.e.,

$$\{x_1(t) + x_2(t)\} \rightarrow \{y_1(t) + y_2(t)\}.$$

These two conditions of homogeneity and superposition together constitute a set of necessary and sufficient conditions for a system to be linear. The following statement is a definition of linearity.

Definition: A system is called linear if and only if it possesses both homogeneity and superposition properties. That is, if $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$ and for *any* real numbers k_1 and k_2, the relationship

$$\{k_1x_1(t) + k_2x_2(t)\} \rightarrow \{k_1y_1(t) + k_2y_2(t)\}$$

is true, then the system is linear.

We may think of the physical system as a device for performing certain mathematical operations on the input, as given by the equations of its model, to produce the output. In this approach the physical system is conceived as a mathematical operator. In line with this view, the input-output relationship is written as,

$$y(t) = H[x(t)]$$

where H is a mathematical operator, representing the system. For linear systems the operator H will be a *linear operator*.

For actual physical systems superposition implies homogeneity. The inputs and outputs are real functions which exist physically. For such systems the property of homogeneity can be derived from the property of superposition. However, there are mathematical operators which satisfy superposition but not homogeneity. These are pathological cases and not of any interest in the analysis of physical systems. Therefore, for determining whether a physical system is linear or not we shall test only for superposition.

As will be discussed in the next section, a system whose mathematical model is an algebraic equation is called a *static system*. For such systems it is easy to determine whether the system is linear or not. Non-linear algebraic equations like, $y(t) = x^3(t)$ or $y(t) = \sqrt{x(t)}$, clearly fail to satisfy the superposition property and hence cannot represent linear systems.

Let us examine a system with model,

$$y(t) = m\,x(t) + c. \tag{2.1}$$

For input $x_1(t)$, output $y_1(t) = m\,x_1(t) + c$ and for input $x_2(t)$, output $y_2 = m\,x_2(t) + c$. For the combined input $\{x_1(t) + x_2(t)\}$, the output from eqn (2.1) is given by $y(t) = m\{x_1(t) + x_2(t)\} + c$, which is not equal to $y_1(t) + y_2(t)$. Hence, superposition does not apply for eqn. (2.1). Thus, we get the surprising result that the system represented by eqn. (2.1) is not linear. The input-output relationship of eqn. (2.1) is drawn in Fig. 2.2. If this straight line passed through the origin, the corresponding equation would obey superposition. The straight line passing through origin would mean that the output of the system

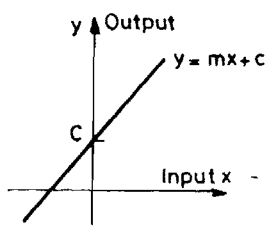


Fig. 2.2 Input-Output Relationship

is zero for zero input. Systems having this property are called *initially relaxed systems*. Thus, we arrive at the conclusion that an initially relaxed system, represented by a linear algebraic equation, will be linear. The techniques of linear analysis are still applicable to the system of eqn. (2.1) if c is treated as an initial condition.

In this book, we will be interested mainly in the techniques of analysing linear dynamic systems. Such systems have linear differential equations as their models. It can be easily verified that the operation of differentiation is a linear operation. Therefore, a differential equation containing linear terms in the dependent variable and its derivatives will be a linear differential equation. Presence of non-linear terms, like x^2 or $(dx)^2/dt$ or $x\,(d^2x)/dt^2$ will make the equation, and the system represented by it, non-linear.

Example 2.1:— An armature controlled d.c., motor is connected to a propeller air fan (Fig. 2.3). The speed-torque curve of the fan can be approximated by a square law relation, $\tau = k_1\,\omega^2$. Develop a mathematical model for determining the angular velocity ω of the motor as a function of time for a given input voltage $v_1(t)$.

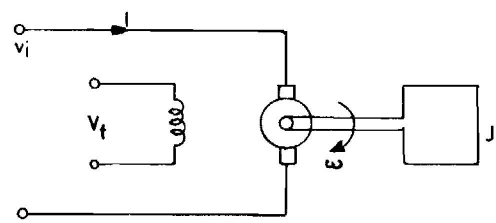


Fig. 2.3

The armature circuit can be modelled as a series combination of armature resistance, armature inductance and a source with a voltage equal to the back e.m.f. The armature inductance is small because of the small number of turns in the armature coil. Hence, the voltage drop across this inductance is neglected in comparison with the other voltages. The application of Kirchhoff's law around the armature circuit gives,

$$v_1 = i\,R + k_2\,\omega$$

where i is the armature current, R the armature resistance, ω the angular speed and $k_2\,\omega$ the back e.m.f.

As the field is connected to a constant voltage supply, the field current and hence the field flux, are constant. Therefore the torque developed by the armature, being proportional to the product of the armature current and the field flux, will be given by $\tau = k_3\,i$

Applying D'Alembert's principle to the mechanical load we get,

$$\tau = J \frac{d\omega}{dt} + k\,\omega^2$$

where J is the moment of inertia of the load and the armature. Combining the three equations above we get,

$$\frac{RJ}{k_3}\frac{d\omega}{dt} + k_2\,\omega + \frac{R\,k_1}{k_3}\,\omega^2 = v_r.$$

This is the required mathematical model. Note that the presence of ω^2 term makes it a non-linear, first order differential equation.

In engineering practice, it is usually an induction motor which is used for driving propeller fans. The speed-torque characteristics of the induction motor are highly non-linear. In such cases graphical, rather than analytical, techniques are used for determining system response.

Common types of non-linearities: Most of the components from which physical systems are constructed are essentially non-linear. However, their behaviour over a limited range of inputs can be approximated by linear relations. A resistor is linear for low values of current but becomes non-linear for higher values because of increase in temperature. An iron core coil has linear inductance for low values of current, but if the current is large enough to cause saturation of the core, its inductance becomes non-linear. All active devices like transistors, diodes, thyristors, etc., are intrinsically non-linear. Some of the non-linearities commonly encountered in engineering systems are discussed below.

1. Spring type non-linearity: The relationship between the applied force f and the resulting displacement x for a linear spring is given by $f = k_1 x$. However, for higher values of force, most springs deviate from this linear relationship. A better approximation of the spring characteristic is the non-linear equation

$$f = k_1 x + k_2 x^3. \tag{2.2}$$

   If k_2 is positive, we get the 'hard' spring characteristic and if it is negative we get the 'soft' spring characteristic, shown in Fig. 2.4. Any non-linearity of this type is commonly referred to as 'hard' spring or 'soft' spring non-linearity.

2. Saturation: An input-output characteristic of the type shown in Fig. 2.5 is called *saturation type non-linearity*. Transistors, operational amplifiers, magnetic and electromagnetic components all suffer from this type of non-linearity.

3. Friction: In mechanical systems we have two types of friction: (i) viscous friction and (ii) static or Coulomb's friction (see Fig. 2.6). For

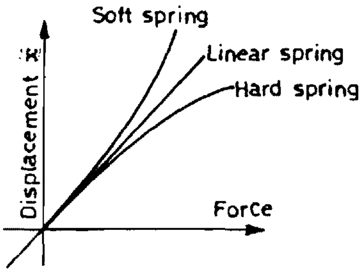


Fig. 2.4 Spring Characteristics

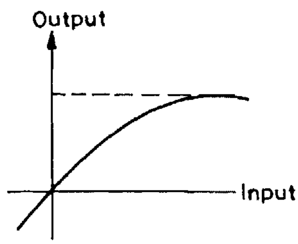


Fig. 2.5 Saturation Non-linearity

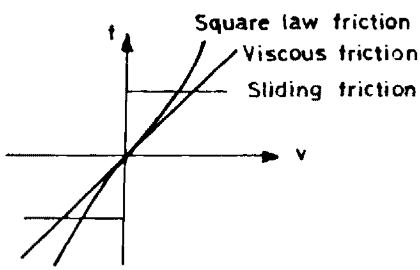


Fig. 2.6 Friction Characteristic

viscous friction, the force-velocity relation is $f = k\,v$, where k is the coefficient of viscous friction. This relationship is linear. Viscous friction arises in situations like motion of solid bodies through gases or liquids, relative motion of well-lubricated surfaces, etc. When there is relative motion between rough surfaces, the friction between them is called *sliding friction*. It produces an opposing force whose magnitude is nearly constant at all velocities. Just before the commencement of motion, the opposing friction force is higher than the sliding friction. It is called the *static friction*. The sliding and the static frictions are non-linear.

When a body moves at a very high speed through a fluid or when the fluid flow relative to the body is turbulent, the force of friction is nearly proportional to the square of velocity. The friction between air and the blades of a fan is of this type. This friction is again non-linear.

4. Hysteresis: When the increasing and decreasing portions of the input-output relationship follow different paths, as in the magnetisation curve of iron, we get hysteresis type of non-linearity (Fig. 2.7). Similar non-linearity also arises in case of backlash of gear trains, relays and electromechanical contactors.

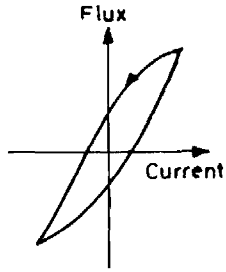


Fig. 2.7 Hysteresis

5. Dead zone: Figure 2.8 shows the dead zone type of non-linearity encountered in relays, motors and other types of actuators.

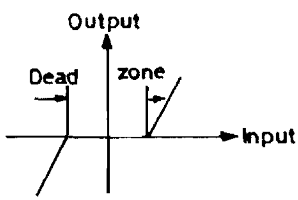


Fig. 2.8 Dead Zone

Non-linear systems: Presence of any one of the non-linearities listed above would make the system non-linear. The resulting mathematical model would also be non-linear. Analysis of such non-linear models is quite complex. Only some selected types of non-linear equations have been solved analytically, and that too with considerable approximations. Iterative numerical methods, suitable for computer analysis, are available for some more types of non-linear equations. However, there are no general methods for solving such equations. Therefore, it is a common engineering practice to make only linear models of physical system by either totally ignoring non-linearities or linearising them by the technique of section 1.6 or otherwise. This is because very elegant and extensive theories are available for the analysis of linear systems. The rest of this book will be devoted to a study of linear systems only.

If the linearisation process is too gross and the results predicted by the linear model do not match with experimental results, or where more accurate results are required, we will have to go in for a non-linear model. In such cases, computer simulation studies are more fruitful. Even in such situations, analysis is first done on an approximate linear model to get a feel for the nature of system behaviour. The non-linear effects are then introduced into this linear model as second order effects to obtain more accurate answers.

Non-linear systems have certain peculiar forms of behaviour which are not encountered in linear systems. Some of these peculiarities are noted here. In a linear system the form of output does not depend on the magnitude of the input. Increasing the input magnitude merely scales up the output, but the form remains the same. In non-linear systems the form of output may change, at times drastically, with changes in input magnitude. For example, in a non-linear system the output may settle down to finite values for small inputs, whereas for input magnitudes beyond a particular value, the output may go on increasing continuously, making the system unstable.

When a linear system is excited by a sinusoidal signal, its output will also be a sinusoid of the same frequency. In a non-linear system the output may be non-sinusoidal and may contain frequency components not present in the input. In some non-linear systems, on application of excitation, the output very quickly goes into modes of cyclic oscillations. These oscillations may persist even after removal of the excitation, provided the system has some energy source to replenish the losses inherent in any physical oscillation. Many electronic oscillators are built around such non-linearities. These are called 'relaxation' oscillators. Another peculiarity of non-linear systems is the 'jump' phenomenon. When the input signal frequency (or magnitude) is being continuously increased, the output magnitude also increases, but, at some point, it may suddenly jump from one value to another. Linear systems are better behaved and do not exhibit such peculiarities.

It should be noted that the classification of a system as linear or non-linear is really a classification of its selected model. In different situations the same system can be modelled as a linear or non-linear system. Thus, one should not take it that nature has created two types of systems, one linear and the other non-linear. In fact, this point is valid for all other categories of system classification.

2.2 Dynamic and Static Systems

Example 2.2:— Consider the resistive network shown in Fig. 2.9. The input-output relation, i.e., the mathematical model of this system is given by the algebraic equation,

$$i(t) = \frac{v(t)}{R_2 + R_3} = k\,v(t)\,.$$

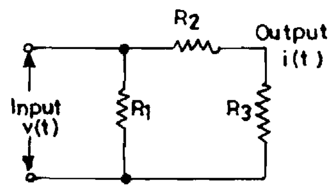


Fig. 2.9 A Resistive Network

In this case the output waveform will be a replica of the input waveform, its magnitude at any time being equal to the input magnitude at that time multiplied by constant k. Thus, the output at any instant depends upon the value of the input at that instant only. One can say that the system is *instantaneous or memoryless*. Such systems are called *static systems*. If the input to a static system does not change with time, then its output also will not change with time; i.e., there is no 'motion' and hence the designation static. When only the steady state is of interest, many systems are modelled by algebraic equations and treated as static systems.

As opposed to static systems, we have *dynamic systems* whose models are given by differential equations (or difference equations). In such differential equation models, time is the independent variable. The output is always a function of time, even when the input is a constant. Thus, there is 'motion' and hence the name dynamic. In such systems the output depends not only upon the input but upon the initial conditions also. Recall that the solution of an nth order differential equation requires a knowledge of the initial value of the depended variable and its first n–1 derivatives. These initial values of the system variables must be existing because of past inputs to the system. In fact the output at any instant t_1 is dependent upon the values of the input at *all* instants prior to and up to t_1, i.e., for time in the range $-\infty < t \leq t_1$. The system thus 'remembers' its past inputs or has 'memory'. All the models developed in Chapter 1 are dynamic models.

Let us examine the factors which make a system static or dynamic. Any electrical circuit made up of only resistive elements will be static, howsoever complex the circuit may be, while a circuit with even a single inductance are capacitance will always be dynamic. This is because a capacitor or inductor can store energy whereas a resistor can only dissipate it. Thus we conclude that the presence of an energy storage element makes a system dynamic. Mass and spring provide energy storage elements in a mechanical system. System variables associated with storage of energy cannot change instantaneously through or across that element; e.g., voltage across a capacitor, current through an inductor, velocity across a mass, force through a spring. A sudden change of current through inductor would mean infinite rate of change of energy, i.e, infinite power—a physical impossibility. Therefore, even for a sudden change in the input, the system variables in a dynamic system will not change instantaneously. Thus, dynamic systems are not 'instantaneous'.

There are many problems in the area of biological systems, economic systems, population dynamics, etc., where there is no well-defined concept of energy or energy storage elements. Even in these areas, we get systems modelled by differential equation. This happens whenever the rate of change (growth or decay rate) of a system variable is dependent upon the value of the variable at that instant. All systems having this characteristic are classified as dynamic systems.

Let us consider the experimental response of the static system of the resistive network of Fig. 2.9 to a sudden change in the value of input voltage from zero to V. Such signals are called *step signals* and the response to them, *step response*. This step response is plotted in Fig. 2.10(a) and (b). In Fig. 2.10(a) the time scale is in seconds and we get the output as an exact replica of the input, as predicted by its static model. In Fig. 2.10(b) the time scale is in nano-seconds. The experimentally determined output is no longer a replica of the input; more likely, it is changing as an exponential function, as in the case of a first order dynamic system. Thus, the static model is not valid. This is because stray capacitances between conductors and between the turns of the resistor will make the circuit an R-C network rather than a purely resistive one. In fact, probably every static system will behave as a dynamic system if the time scale is expanded into nano-seconds in the region where the input is changing.

Figure 2.10 illustrates once again that a mathematical model of a system is valid only for a set of assumptions and particular working conditions. Its classification into a particular category depends on the model selected. As far as the choice between static and dynamic models is concerned, we may generalise the previous example to conclude that if the transient response is much too fast and is not of much importance compared to the steady state response, the selected model will be static; if transients are of interest, the model for analysis should be dynamic.

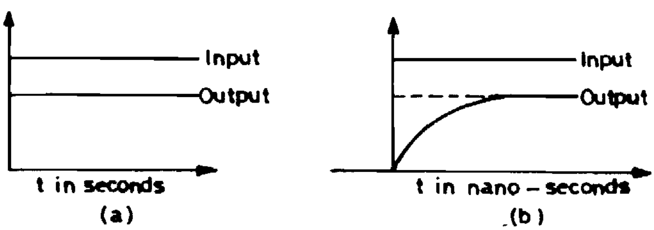


Fig. 2.10 Step Response of Static System

Static models are commonly encountered in socio-economic systems like resource allocation problems or optimisation problems. In such cases, they are usually functions of many variables. Their analysis is relatively straightforward requiring solution of linear algebraic equations only.

2.3 Time Invariant and Time-varying Systems

In section 1.1.1, we defined parameters of a system as the quantities which depend only on the properties of system elements and not on its variables. These parameters appear as coefficients of the dependent variable and its derivatives in the differential equation model of dynamic systems. For example, in eqn. (1.4) of the automobile ignition system,

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{i}{C} = 0$$

parameters R, L and C appear as coefficients. If these parameters become functions of the *dependent variable* [$i(t)$ in this case], then the system becomes non-linear. However, the coefficients can be functions of the *independent variable t* without affecting the property of linearity. When the parameters are fixed, i.e., do not change with time, the system to which they belong is called a *time invariant system*. When one or more parameters are functions of time, the system is called a *time-varying system*. The terms *stationary system* and *non-stationary system* are also used in place of time invariant and time-varying systems.

Example 2.3:— The equation of motion of a rocket in vertical flight is a good example of a time-varying system. As the rocket goes up, it burns fuel to develop the required thrust. Consequently its mass goes on reducing. Let its initial mass be M_o and k the rate at which the fuel is being burnt. Then its mass at any time t after the take-off would be given by $m(t) = M_o - kt$. The equation of motion, i.e., the mathematical model of the system will be,

$$m(t) \frac{d^2x}{dt^2} + D \frac{dx}{dt} = F \tag{2.3}$$

where,

$m(t)$ = time-varying mass of the rocket;

$x(t)$ = vertical displacement;

D = coefficient of friction between rocket body and atmosphere; and

F = thrust developed by the rocket.

Equation (2.3) is a time-varying differential equation. However, it is still a linear equation. The technique of solving time-varying differential equations is somewhat more involved than the method for time invariant equations. The main difference between these two types of systems is that the solution does not depend upon the instant of time at which an input is applied for the time invariant system, whereas it does change if the instant of application of the input is changed in the case of time-varying system. In this book we shall study only time invariant systems.

2.4 Continuous Time and Discrete Time Systems

In all the examples considered so far, the system variables like voltage $v(t)$, velocity $u(t)$, etc., were continuous functions of time, i.e., they are defined for every instant of time and their values can change at any instant. The systems to which such variables belong are called *continuous time systems*. As opposed to this, in the mathematical modelling of many processes, we encounter system variables which are either defined at only specified instants of time or can change their values only at these specified instants, as in a digital clock. Such signals are called *discrete time functions* and the systems to which they belong are called *discrete time systems*. A graphical plot of a discrete time function is shown in Fig. 2.11(b).

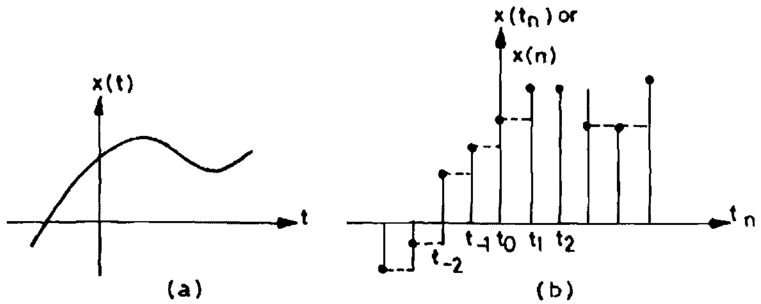


Fig.2.11(a) A Continuous Time Function and (b) A Discrete Time Function

It should be compared with the plot of a continuous time signal in Fig. 2.11(a). Here, t_0, t_1, t_2, ...,t_n are the specified instants of time, or the sequence of instants at which the function is defined or can change its magnitude. This second alternative is shown by the dotted lines in Fig. 2.11(b). The intervals between the specified instants may be fixed or variable.

For a continuous time function x, its functional dependence on the variable time is shown by writing it as $x(t)$. For discrete time functions, since the function is defined only at the sequence of instants t_n, it is written as $x(t_n)$ or simply $x(n)$. Just as the independent variable t_n is a sequence of time instants, variable $x(n)$ is a sequence of numbers, $x(1)$ corresponding to t_1, $x(2)$ corresponding to t_2, and so on.

Discrete time functions arise in the modelling of instrumentation and control systems employing multiplexing, computer control or digital signal processing. Before taking up an example of such a system let us clarify an area of possible confusion.

In the description 'continuous time function', the term continuous refers to the independent variable time and not to the function dependent on time. The function may possess discontinuities of different kinds, even when the time variable is continuous. Figure 2.12 shows some continuous time functions. In Fig. 2.12(a), the

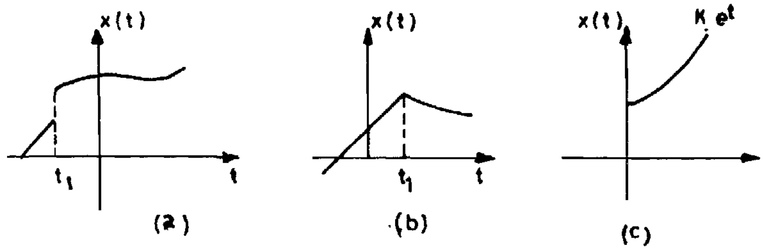


Fig. 2.12 Continuous Time Signals

function itself is discontinuous at $t = t_1$; in Fig. 2.12(b), the function is continuous but its first derivative is discontinuous at $t = t_1$; and in Fig. 2.12(c) we have a function which is continuous and possesses continuous derivatives of *all* orders (except at $t = 0$). Mathematicians have even devised a function which is discontinuous at every instant of time! However, in all these cases the time is still a continuous variable. The continuity, or otherwise, of the function has nothing to do with the concept of continuous time and discrete time functions.

Example 2.4:— We now take up the modelling of a computer-controlled furnace, shown in Fig. 2.13(a). The temperature of an electric furnace is to be varied in accordance with the curve given in Fig. 2.13(b). The slope and the duration of different segments of this curve may have to be varied depending upon the

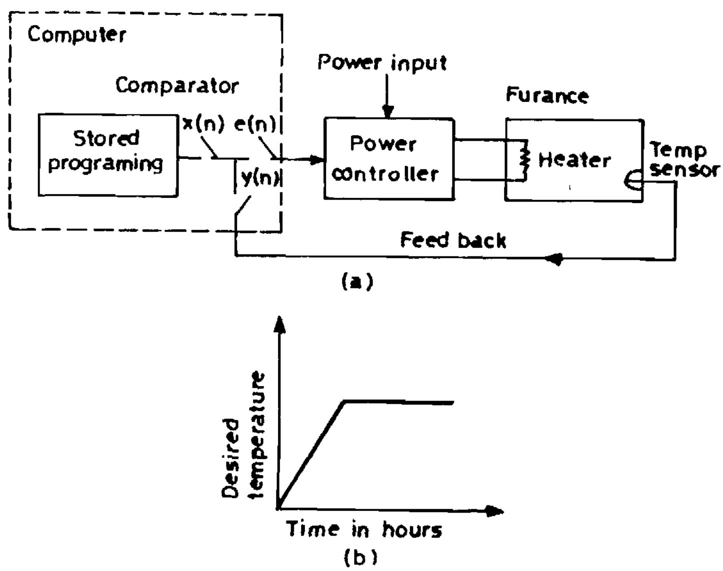


Fig. 2.13 Computer-controlled Furnace: (a) Schematic Diagram and (b) Desired Temperature Profile

material being heated and the nature of processing required. This type of 'programmable' furnace controller is required in processing of special ceramic materials. The required programme, i.e., the curve between time and the desired furnace temperature, is stored in the memory of a computer. For such a storage, both the time and temperature variables will have to be sampled and digitised. By sampling we mean that only specified values of time, say at one minute intervals, and the corresponding temperatures will be taken up for storage. Thus, the stored function of desired temperature will be a discrete function of time, represented by $x(n)$. In fact, storage of any time function in a computer can only be as a discrete function of time.

In practice a process computer for such an application will be used for performing many other tasks in addition to controlling the furnace temperature. That is, it will be used in a multiplexed or time-sharing mode. At specified instants, say, at 1-minute intervals, the computer will take up furnace control. At these instants it will compare the actual furnace temperature with the desired temperature at that instant stored in its memory. It will alter the setting of the power controller depending upon this comparison. For example, if the desired temperature and the actual temperature are equal it will make no change in the setting. If the actual temperature is lower than the desired temperature, it will step up the power input to the furnace. And so on.

For the purpose of performing such a temperature control, the information regarding the actual furnace temperature must be made available to the computer. Now the temperature is the output of the system and the computer is controlling the input to it. Therefore, we say a *feedback* must be provided to the computer regarding the value of the output. Control systems having feedback links are called *feedback control systems*. Most of the modern automatic control systems utilise this feedback principle.

The temperature transducer in the furnace will usually generate an 'analog' electrical signal proportional to the temperature. This analog signal will have to be digitised and time-sampled to make it intelligible to the computer. Thus, the feedback will also be a discrete time signal, represented by $y(n)$. The input signal to the power controller will also appear only at the specified instants of time. Hence this will also be a discrete time signal, represented by $e(n)$. To indicate the discrete time nature of these signals a switch has been included in series with the signal lines. It is assumed that all the switches are closed only at the specified instants of time. These specified instants are called 'sampling' instants and accordingly such a discrete time system is also called a *sampled data system*.

The discrete time system variables are thus identified as: actual furnace temperature, i.e., the output $y(n)$, desired temperature $x(n)$, and the input to the power controller $e(n)$. The input to the power controller will adjust the rate of heat flow $q(n)$ i.e., the power input to the furnace. This setting will remain fixed between the sampling intervals. The input heat rate $q(n)$ will be proportional to the setting $e(n)$. Let us assume that the 'control law' is such that $e(n)$ is directly proportional to the difference between $y(n)$ and $x(n)$. Thus

$$q(n) = K_1 e(n) = K_1 K_2 [x(n) - y(n)] \tag{2.4}$$

The mathematical model of the furnace, relating the input heat rate and the output temperature will be a first order differential equation. Therefore, for a fixed input heat rate between the sampling intervals, the temperature will increase exponentially. However, if the sampling interval is small compared to the thermal time constant of the furnace, we may assume that the temperature change is linear between two consecutive sampling instants. The rate of this change will be proportional to the input heat rate, or

$$\frac{y(n+1) - y(n)}{\Delta T} = K_3 q(n)$$

$$= K_1 K_2 K_3 [x(n) - y(n)] \tag{2.5}$$

where ΔT is the sampling interval, i.e. the time between t_n and t_{n+1}. Assuming ΔT to be constant, we may write eqn. (2.5) as

$$y(n+1) - y(n) = K[x(n) - y(n)]$$

where constant $K = \Delta T K_1 K_2 K_3$ Rearranging,

$$y(n+1) - (1 - K)\, y(n) = Kx(n)\,. \tag{2.6}$$

Equation (2.6) is the required mathematical model of the system.

Equations of this type are called *difference equations*. They are used to describe mathematical models of discrete time systems. Difference equations are analogous to differential equations for continuous time systems. Modelling and analysis techniques for discrete-time systems are discussed in chapter 9.

2.5 Lumped Parameter and Distributed Parameter Systems

Let us once again consider a thermal system. This time we would like to study the heating of an iron slab, like the ones encountered in steel mills. To simplify matters let us assume that it is being heated from only one end. Further, there is no dissipation of heat from the sides and that the heat flows in only one direction, as shown in Fig. 2.14. We will like to model it as a dynamic system so that we can answer

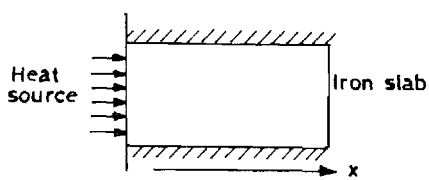


Fig. 2.14 One Dimensional Heat Flow

questions like; how does the temperature of the slab change as a function of time following a sudden change in the input heat rate? Thus the system variables, in this case the slab temperature, must be functions of time and the model a differential equation.

The next question which comes immediately to the mind is: temperature at which point in the slab? It is obvious that the temperature at any instant of time will be different at different points along the length of the slab. In fact there will be a temperature gradient along the length. Therefore the system variable temperature will have to be treated not only as a function of time but also as a function of space co-ordinate x. Thus, it becomes a function of two independent variables, time t and distance x along the length, i.e., $\tau = \tau(t, x)$. The differential equation relating the input heat rate to the output variable $\tau(t, x)$ will then be a *partial differential equation.*

In an earlier example of a thermal system (section 1.5, Fig. 1.8) we had divided the region of heat flow into three distinct regions and assumed that the temperature of each region (T_I, T_C *and* T_S) was the same over the whole region. Therefore, instead of having a single temperature variable as a function of length (and, of course, of time) we had three temperature variables as functions of only time. The thermal properties of the three regions were characterised by distinct two terminal elements. The system variables could change only at the nodes of the inter-connections of these two terminal elements. This is only an approximation because the fixed thermal characteristics of the system, i.e., the parameters of the system, thermal capacitance and thermal resistance, are actually *distributed* all over the space and not *lumped* between two terminals. Owing to this reason, the model of section 1.5 was called a lumped parameter system. In treating temperature $\tau = \tau(t, x)$ as a continuous function of space co-ordinate x (and a function of time), we are not making any lumping and treating the system parameters as continuously distributed in space. Such a system is called a *distributed parameter system* and its mathematical model is given by a partial differential equation.

Distributed parameter models are encountered in areas like heat flow, diffusion processes, torsion in long shafts, vibrating strings and air columns (e.g., in musical instruments like sitar and flute) and in long transmission lines. Because of its importance to electrical and electronics engineers, let us model a transmission line. The transmission line may be for power transmission or signal transmission.

Example 2.5:— A transmission line will have the parameters, series resistance, inductance, capacitance and parallel conductance associated with it. These parameters will be uniformly distributed over the length of the line. To simplify matters, let us assume that the line is *lossless*, i.e., the series line resistance and parallel line conductance are zero. In that case the circuit model of a small length Δx of the line will be as shown in Fig. 2.15. Here, L and C are the inductance and the capacitance per unit length of the line. The variables of the system are current $i(x, t)$ and voltage $v(x, t)$.

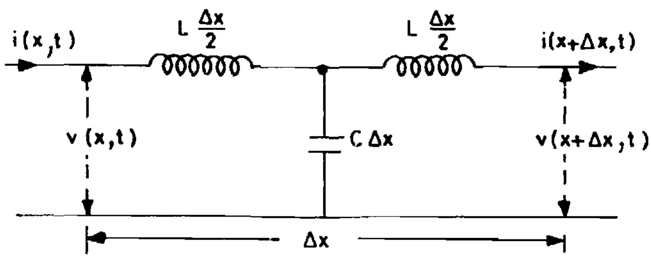


Fig. 2.15 A Small Section of Lossless Transmission Line

Applying Kirchhoff's voltage law around the loop Δx we get,

$$L\frac{\Delta x}{2}\frac{\partial i(x,t)}{\partial t} + L\frac{\partial x}{2}\frac{\partial i(x+\Delta x, t)}{\partial t} + v(x+\Delta x, t) = v(x, t)$$

or rearranging,

$$L\frac{\Delta x}{2}\frac{\partial}{\partial t}[i(x+\Delta x, t) + i(x, t)] = -[v(x+\Delta x, t) - v(x, t)].$$

Dividing by Δx we get,

$$\frac{L}{2}\frac{\partial}{\partial t}[i(x+\Delta x, t) + i(x, t)] = -\frac{[v(x+\Delta x, t) - v(x, t)]}{\Delta x}. \tag{2.7}$$

In the limit as $\Delta x \to 0$, the r.h.s. of eqn. (2.7) equals the derivative of $v(x, t)$ with respect to x. On the l.h.s. $i(x+\Delta x, t) \to i(x, t)$. That is, eqn. (2.7) becomes,

$$L\frac{\partial i(x,)}{\partial t} = -\frac{\partial v(x, t)}{\partial x}$$

or,

$$L\frac{\partial i}{\partial t} = -\frac{\partial v}{\partial x}. \tag{2.8}$$

Applying Kirchhoff's current law to the node in Fig. 2.15 and going through similar steps we get another equation,

$$C\frac{\partial v}{\partial t} = -\frac{\partial i}{\partial x}. \tag{2.9}$$

We now combine eqns. (2.8) and (2.9) into a single equation. With this in view we take the partial derivative of eqn. (2.8) with respect to x and that of eqn. (2.9) with respect to t. This gives,

$$L \frac{\partial^2 i}{\partial x \,.\, \partial t} = -\frac{\partial^2 v}{\partial x^2}. \tag{2.10}$$

and

$$C \frac{\partial^2 v}{\partial t^2} = -\frac{\partial^2 i}{\partial t \,.\, \partial x}. \tag{2.11}$$

Reversing the order of taking derivative, multiplying eqn. (2.11) by L and then substituting eqn. (2.10) in it we get,

$$LC \frac{\partial^2 v}{\partial t^2} = \frac{\partial^2 v}{\partial x^2}$$

or

$$\frac{\partial^2 v}{\partial t^2} = k^2 \frac{\partial^2 v}{\partial x^2} \tag{2.12}$$

where $k = 1/\sqrt{(LC)}$.

Equation (2.12) is the required mathematical model of the transmission line. This type of equation arises frequently in many other distributed parameter systems and is called the *wave equation*. The constant k used in eqn. (2.12) is called the *velocity of propagation*. An equation similar to eqn. (2.12) will relate time and space variations of current $i(x, t)$.

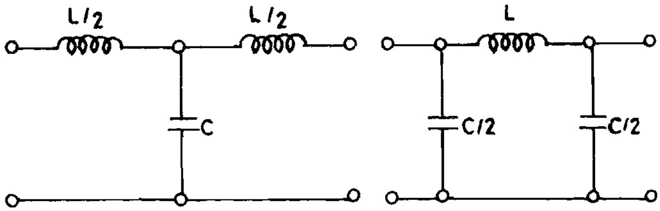
Equation (2.12) pertains to a lossless line where $R = G = 0$. Some other simplifying assumptions, leading to useful results are as follows:

(1) $R/L = G/C$. This case is called the *distortionless line*.

(2) $G = L = 0$. This is useful for a leakage-free, non-inductive cables.

In the above example of transmission line, the space distribution of the variables v and i were single dimensional. In other problems, we may encounter two dimensional distribution, e.g., vibration of plates, percussion type of musical instruments (say tabla), or three-dimensional distribution, e.g. vibration of air mass in a speaker enclosure, propagation of electromagnetic waves in space. These will give rise to more complex partial differential equations as their models.

In many cases the time and effort required to generate solutions for partial differential equation models are not justifiable. A simpler solution would do. In such cases the distributed parameter system is modelled by an equivalent lumped parameter model. For example, the lossless transmission line is frequently modelled by one or more sections of the equivalent T or π network shown in Fig. 2.16. The accuracy of representation improves as more number of such sections are connected in series.



(a) T-network (b) π-network

Fig. 2.16 Lumped Parameter Approximation of a Lossless Transmission Line

2.6 Deterministic and Stochastic Systems

Let us consider a ship-mounted naval gun, firing on a land target. For a specified gun angle (both in the horizontal and vertical planes) the firing of the gun should land the shells at the same spot. One would like to build a model relating the gun angle with the location of the point hit by the shell. Such a model will be useful if, for example, a computer were to control the gun. However the chances of all the shells hitting exactly the same spot are extremely low. At best, one can define a general area in which the shells will land. With somewhat greater insight, we can associate with each point in this area a probability of the shell landing there. A mathematical model, taking into account the uncertainty in the output and relating the probability of the output with the input is called a *stochastic model* and the system with which it is associated, a *stochastic system*. In a stochastic model the actual output for a given input is uncertain: only its probability of occurrence can be predicted. In contrast the systems modelled earlier were *deterministic systems* because their output could be exactly determined for a given input.

The uncertainty regarding the output in the naval gun problem is because of various factors like roll and pitch of the ship due to waves, slight differences in shell sizes, shape and fire power, effect of air currents and even our inability to set the gun angle to an exact value. These could be grouped into two categories: (i) uncertainties in the exact values of parameters and (ii) random noise signals picked up at the input and other points of the system. The effect of noise signals is particularly damaging in communication systems. A signal under transmission will always be corrupted by atmospheric and other noise signals. However, from a knowledge of the statistical properties of different types of noises, communication engineers are able to make stochastic models of the system and use these models to design filters to reduce the effect of noises.

2.7 Concluding Comments

In the preceding sections of this chapter we have had a bird's eyeview of the different classes of systems, nature of their mathematical models and the engineering and non-engineering areas in which they arise. In the recent past, engineers have been able to develop techniques for analysing all these systems. The fantastic growth of modern technology owes a great deal to these analytical techniques. However, in this introductory text we shall have time to explore the analytical techniques for only one class of systems. In terms of our classification scheme, this class can be correctly designated as linear, dynamic, time invariant, deterministic systems. We will include both the continuous time and discrete time systems in this class. For the sake of brevity, this string of adjectives is usually reduced to its first word, 'linear'. Henceforth, by the term linear system we will mean the class of systems defined above.

Linear system models are the ones most commonly used in engineering. This is because linear models are easier to build, several excellent techniques exist for their analysis and, finally, the answers given by linear models are satisfactory in majority of the situations.

GLOSSARY

Linearity: A system is called linear if, and only if, it possesses the properties of homogeneity and super-position, that is, if $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$, and for any real numbers k_1 and k_2 the following relationship is satisfied;

$$\{k_1 x_1(t) + k_2 x_2(t)\} \rightarrow \{k_1 y_1(t) + k_2 y_2(t)\}$$

Static versus Dynamic Systems: The mathematical models of static systems are given by algebraic equations whereas dynamic systems are modelled by differential equations. Static systems are instantaneous, i.e., their output at any instant of time depends upon the value of input at that instant only. Dynamic systems have memory, i.e., their output at any instant depends upon the whole past history of the input.

Time-varying Systems: When the parameters of a system are functions of time, the system is called a *time-varying system*. The mathematical model is given by time-varying (or non-stationary) differential equations.

Discrete Time Signals and Systems. If the value of a signal is defined only at a set of specified instants or if its value can change only at specified instants, the signal is called a *discrete time signal*. Systems in which such signals arise are called *discrete time systems*. Their mathematical model is given in terms of difference equations. Such systems are also called *sampled data systems*.

Distributed Parameter Systems: When the system parameters are distributed over space, the system variables become functions of space co-ordinates (in addition to being functions of time). Systems having such variables are called *distributed parameter systems* and their models are given by partial differential equations.

Stochastic Systems. When one or more variables or parameters of a system are not known precisely but can be described only in terms of its probability function, the system is called a *stochastic system*.

PROBLEMS

2.1 Classify the system models given by the following equations.

(i) $4\dfrac{d^2x}{dt^2} = x\dfrac{dx}{dt}$

(ii) $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 7x = \sin x.$

(iii) $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} = 5tx.$

(iv) $\dfrac{1}{x}\dfrac{d^2x}{dt^2} + \dfrac{1}{x}\dfrac{dx}{dt} + 1 = 0.$

(v) $\dfrac{d^2x}{dt^2} + 3\dfrac{dx}{dt} + 7x = \sin \omega t.$

2.2. What approximations/assumptions are generally made to obtain linear models for the following systems?

(i) A pendulum.

(ii) A transistor amplifier

(iii) A d.c. generator

(iv) A hydraulic press.

2.3. A power-dissipating resistor bank is made from a heating element whose temperature coefficient of resistance is α. The thermal resistance and capacitance of the bank are R_T and C_T Develop a mathematical model for determining the temperature of the heating element as a function of time when a constant voltage source is switched on across the bank. Comment on the type of model obtained.

2.4 The technique of linearisation around an operating point, mentioned in section 1.6, is based upon the expansion of a function by Taylor's series. Let the input-output be related by the equation,

$$y = f(x)$$

where $f(x)$ is a non-linear function of x. To linearise it around an operating point $x = x_1$, we first expand y around $x = x_1$ by Taylor's series as,

$$y = f(x_1) + (x - x_1)\left.\frac{d\,f(x)}{dx}\right|_{x=x_1} + \frac{(x - x_1)^2}{2!}\left.\frac{d^2 f(x)}{dx^2}\right|_{x = x_1} + $$

Since variations around x_1 are assumed to be small, terms containing $(x - x_1)^2$ and other higher powers may be neglected. Then,

$$y = f(x_1) + (x - x_1)\left.\frac{d\,f(x)}{dx}\right|_{x=x_1}$$

which is the required linear relation between y and x

Linearise the equation $y = 0.5\,x^3$ around point $x = 2$. Calculate the error involved in using this linear model for determining y at $x = 2$, 2.1, 2.2, 2.3 and 2.4.

2.5 A series RC circuit is connected to a d.c. source. The capacitor is a parallel plate capacitor, with one plate fixed and the other oscillated by a cam, such that the distance between the plates is a sinusoidal function of time. Determine the mathematical model for computing current in the circuit. To which class does this model belong?

2.6 An experimental study to determine the mathematical model of a continuous time system gave the following results.

Input = constant K

Output, measured at one-second intervals = 1, 3, 6, 10, 15, 21, 28, 36, 45, ... Determine the mathematical model of the system. To which class does it belong? Can you give example of a physical system corresponding to this model?

2.7 The network shown in the Fig. 2.17 is called a ladder network. Treating the voltage at the nodes as a discrete variable, determine a difference equation for the nth node voltage $v(n)$.

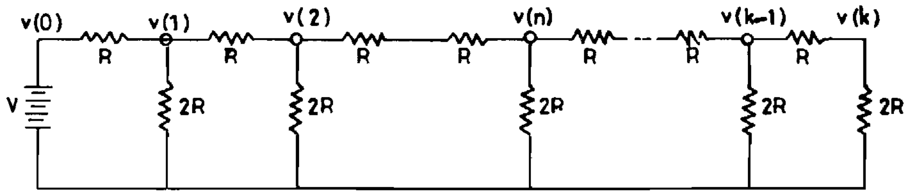


Fig. 2.17 Ladder Network

2.8. Suppose that a pair of rabbits can produce a new pair every month. Let us further suppose that the rabbits become fertile after one month. Develop an equation for determining the total number of rabbits in any month, starting with one pair of rabbits. To which class does this equation belong?

CHAPTER 3

Analysis of First and Second Order Systems

LEARNING OBJECTIVES

After going through this chapter you should be able to:

(i) know the mathematical representation and important characteristics of standard test signals;

(ii) obtain the transient and the steady state responses of linear systems as solutions of linear differential equations;

(iii) characterise the response of first order systems, and second order systems with underdamped, overdamped and critically damped characteristics;

(iv) represent second order systems in terms of a generalised equation using damping ratio ζ and frequency of natural oscillation ω_n as parameters; and

(v) obtain the numerical measures—rise time, peak time, percentage overshoot and settling time—for second order underdamped response in terms of ζ and ω_n.

The central object of analysis of physical systems is to generate solutions of the model equations for different types of inputs and initial conditions. Our interest here is limited to linear dynamic systems only. Their mathematical models are ordinary linear differential equations. The *order* of a system is defined as the order of its governing differential equation. In this chapter, we take up the analysis of only first and second order systems. It is presumed here that the reader is familiar with the methods of solving linear differential equations. Therefore, only a brief review of the classical method of solving such equations is given in the next section.

The engineer's interest in systems analysis does not end with solving linear differential equations. He or she wants to idenafy the specific characteristics of these solutions and to relate them to the actual modes of system behaviour. This understanding of system behaviour is essential to operate and maintain a given system in an efficient manner and to design better and more efficient systems. Therefore, mathematical analysis is merely a tool for the engineer. However, a mastery over this tool is essential for his successful performance. This book aims at helping the engineer learn these mathematical tools.

3.1 Review of the Classical Method of Solving Linear Differential Equation

Consider the general nth order, ordinary, linear, time-invariant differential equation,

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^n} + \ldots + a_1\frac{dy}{dt} + a_0 y = x \tag{3.1}$$

where $y = y(t)$ is the dependent variable and $x = x(t)$, the forcing function. The coefficients a_0, a_1. . ., a_{n-1} are constants. In the systems terminology x is the input, y the output and a's the parameters. The solution of eqn. (3.1) consists of two parts—the complementary function $y_c(t)$ and the particular integral $y_p(t)$. The complete solution is given by $y(t) = y_c(t) + y_p(t)$.

The complementary function: The complementary function $y_c(t)$ is the solution of the homogeneous equation corresponding to eqn. (3.1), obtained by equating its right-hand side to zero, i.e.,

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots + a_1\frac{dy}{dt} + a_0 y = 0 \tag{3.2}$$

Clearly, the form of the solution of eqn. (3.2) will not depend upon the forcing function x but only on the parameters of the system. Hence, in systems terminology $y_c(t)$ is called the *natural*, or the *unforced*, or the *source-free* response. The last term is more commonly used in network theory where the forcing function is in the form of voltage or current source.

The solution of the homogeneous eqn. (3.2) is obtained as,

$$y_c(t) = C_1 y_1(t) + C_2 y_2(t) + \ldots + C_n y_n(t) \tag{3.3}$$

where C_1, C_2...C_n are constants dependent upon the initial or the boundary conditions, and functions $y_1(t)$, ..., $y_n(t)$ depend on the roots of the characteristic algebraic equation,

$$r^n + a_{n-1} r^{n-1} + \ldots + a_1 r + a_0 = 0. \tag{3.4}$$

If the n roots, r_1, r_2, ..., r_n of eqn. (3.4) are all real and distinct then,

$$y_i(t) = \exp(r_i t), \quad i = 1, 2, \ldots n \text{ and}$$

$$y_c(t) = C_1 \exp(r_1 t) + C_2 \exp(r_2 t) + \ldots + C_n \exp(r_n t) \quad (3.5)$$

For other types of roots, the components $y_i(t)$ of $y_c(t)$ take the following forms:

(1) for each real distinct root r, the function e^{rt};

(2) for each real root of multiplicity k, the functions $e^{rt}, t\,e^{rt}, \ldots, t^{k-1} e^{rt}$;

(3) for each distinct complex pair of roots $a \pm jb$, the functions $e^{at} \cos bt$ and $e^{at} \sin bt$; and

(4) for each complex pair of roots $a \pm jb$ with multiplicity k the functions $e^{at} \cos bt$, $e^{at} \sin bt$, $t\,e^{at} \cos bt$, $t\,e^{at} \sin bt, \ldots, t^{k-1} e^{at} \cos bt$, $t^{k-1} e^{at} \sin bt$.

Example 3.1:— Solve the homogeneous differential equation,

$$\frac{d^3y}{dt^3} - 3\frac{dy}{dt} - 2y = 0.$$

The characteristic equation is,

$$r^3 - 3r - 2 = 0$$

or,

$$(r + 1)^2 (r - 2) = 0$$

and has three roots $r = -1, -1$ and $+2$. Thus, there is a root of multiplicity 2 at -1. Therefore, the complementary function is,

$$y_c(t) = (C_1 + C_2 t) e^{-t} + C_3 e^{2t}.$$

Example 3.2:— Solve the homogeneous differential equation,

$$\frac{d^3y}{dt^3} - \frac{d^2y}{dt^2} + \frac{dy}{dt} - y = 0.$$

The characteristic equation is,

$$r^3 - r^2 + r - 1 = 0$$

or

$$(r^2 + 1)(r - 1) = 0.$$

The three roots are $r = +j, -j$ and $+1$. There is a pair of imaginary roots $\pm j$. Therefore,

$$\begin{aligned} y_c(t) &= C_1 e^t + C_2 e^{jt} + C_3 e^{-jt} \\ &= C_1 e^t + C_2 \cos t + j\,C_2 \sin t + C_3 \cos t - j\,C_3 \sin t \\ &= C_1 e^t + (C_2 + C_3) \cos t + j(C_2 - C_3) \sin t \end{aligned}$$

The particular integral: The particular integral $y_p(t)$ is dependent upon the type of forcing function or input $x(t)$. Amongst the many methods for determining the particular integral, the method of undetermined coefficients is particularly useful in the analysis of physical systems. The forcing functions commonly encountered in engineering systems are: exponential (e^{at}), polynomial in t, sin ωt or cos ωt. Many other types of forcing functions can be equated or approximated by combinations of these functions. In the method of undetermined coefficients, we presume the form of solution $y_p(t)$ depending upon the form of the forcing function, as given in table 3.1.

Table 3.1: Assumed form of particular integral

Forcing function x(t)		*$y_p(t)$ to be assumed*
Exponential, Ke^{at}	(i)	Ae^{at} if a is not a root of the characteristic equation.
	(ii)	$At\,e^{at}$ if a is a simple root; and
	(iii)	$At^2\ e^{at}$ if a is a root of multiplicity 2, and so on.
Polynomial, Kt^k (k positive)	(i)	$A_0 + A_1t + \ldots + A_kt^k$ if the characteristic equation does not have zero as a solution;
	(ii)	Integral of polynomial (i) if zero is a solution of the characteristic equation.
K cos ωt or K sin ωt		A cos ωt + B sin ωt.

The values of constants in the assumed form of $y_p(t)$ are determined by substitution of the assumed $y_p(t)$ in the original equation.

Example 3.3:— Find the particular solution for the differential equation,

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = t^3 + t.$$

Assume,

$$y_p(t) = A_0 + A_1t + A_2t^2 + A_3t^3.$$

Substituting it in the given equation we have,

$$(A_0 + 2A_1 + 2A_2) + (A_1 + 4A_2 + 6A_3)\,t + (A_2 + 6A_3)\,t^2 + A_3t^3 = t + t^3.$$

Equating coefficients of like powers of t on both the sides we get,

$$A_0 + 2A_1 + 2A_2 = 0$$

$$A_1 + 4A_2 + 6A_3 = 1$$

$$A_2 + 6A_3 = 0$$

$$A_3 = 1$$

Solving these equations we get $A_0 = -26$, $A_1 = 19$, $A_2 = -6$ and $A_3 = 1$.

Therefore

$$y_p(t) = -26 + 19t - 6t^2 + t^3.$$

Example 3.4:— Find $y_p(t)$ for the differential equation,

$$\frac{d^3y}{dt^3} + y = 2\,e^{2t}\cos 3t.$$

Assume,

$$y_p(t) = A\,e^{2t}\cos 3t + B\,e^{2t}\sin 3t.$$

Substitution of this assumed $y_p(t)$ in the given equation we get,

$$(9B - 45A)\,e^{2t}\cos 3t - (9A + 45B)\,e^{2t}\sin 3t = e^{2t}\cos 3t.$$

Equating the coefficients of cos $3t$ and sin $3t$ on the two sides we get,

$$9B - 45A = 1 \text{ and } 9A + 45B = 0$$

or

$$A = \frac{-5}{234} \text{ and } B = \frac{1}{234}$$

Therefore,

$$y_pt = \frac{e^{2t}}{234}(\sin 3t - 5\cos 3t).$$

Initial conditions and the complete solution: The complete or the general solution is the sum of the complementary function $y_c(t)$ and the particular integral $y_p(t)$. That is,

$$y(t) = y_c(t) + y_p(t).$$

This solution, however, still contains unknown constants C_1, C_2, ..., C_n. The values of these constants should be so chosen that $y(t)$ equals at least one known solution point. For this, one must know the value of $y(t)$ and its first $(n-1)$ derivatives at some point of time. Usually, this point of time is the starting time t_o, i.e., $t = 0$. The value of $y(t)$ and its first $(n-1)$ derivatives at $t = 0$ are called the *initial conditions*. Sometimes this known solution point is given for some other value of time $t' \neq 0$. The values of y and its first n-1 derivatives at $t = t'$ are called *boundary conditions*. Thus, a knowledge of initial conditions or boundary conditions is necessary to determine the values of unknown constants in the complete response. Only then is the solution $y(t)$ completely specified.

Solution of problems with specified boundary values (these are called *boundary value problems*) is somewhat more difficult than that of problems with specified initial conditions (i.e., *initial value problems*). Fortunately, most of the engineering problems are initial value problems.

The initial conditions may also be considered as additional inputs similar to the forcing function. Thus, it can be assumed that the solution $y(t)$ or the output is because of two two factors—(i) due to the initial condition and (ii) due to the forcing function.

It should be noted that the values of C_1, C_2, ..., C_n depend not only on the initial conditions but also on the forcing function $x(t)$. Therefore, these constants should be determined only from the complete response $y(t)$ and not from $y_c(t)$ alone. These points are illustrated in the example that follows.

Example 3.5:— Determine the complete solution for the equation in example 3.3 with initial conditions,

$$y(0) = -25 \text{ and } \left.\frac{dy}{dt}\right|_{t=0} = 20.$$

The characteristic equation is $r^2 + 2r + 1 = 0$ or $(r+1)^2 = 0$. The roots are $r = -1, -1$. Thus, the complementary function is $y_c = C_1 e^{-t} + C_2 te^{-t}$.

The particular integral has already been determined in example 3.3 as $y_p(t) = -26 + 19t - 6t^2 + t^3$. The complete solution is,

$$\begin{aligned} y(t) &= y_c(t) + y_p(t) \\ &= C_1 e^{-t} + C_2 t e^{-t} - 26 + 19t - 6t^2 + t^3 \cdot \end{aligned}$$

Let us now determine the values of constants C_1 and C_2 to completely specify $y(t)$. At $t = 0$,

$$y(0) = C_1 - 26 = -25$$

Therefore, $C_1 = 1$. Now by differentiating the solution $y(t)$ we get,

$$\frac{dy}{dt} = -C_1 e^{-t} + C_2 e^{-t} - C_2 t e^{-t} + 19 - 12t + 3t^2 \cdot$$

At $t = 0$,

$$\left.\frac{dy}{dt}\right|_{t=0} = -C_1 + C_2 + 19 = 20$$

or $C_2 = 2$. Thus, the complete solution is,

$$y(t) = e^{-t} + 2te^{-t} - 26 + 19t - 6t^2 + t^3.$$

3.2 Transient and Steady-State Response

In the previous section we looked upon the problem of analysis as the problem of determining solutions of differential equations. Let us now interpret the form of this solution in terms of the general properties of linear dynamic systems.

The unforced or the natural term of the solution contains terms like $C_i \exp(r_i t)$. If even one of the r_i's is positive (or has positive real part for a complex r_i), then the response will contain an exponential term with a positive exponent. The magnitude of this positive exponential term will go on increasing as t increases, tending to ∞ as $t \to \infty$. Systems in which the response goes on continuously increasing, or becomes unbounded, are called *unstable systems*. Such systems do occur in engineering and other physical systems. However, it is clear that their usefulness is limited unless some means of control is applied to make the response bounded. Otherwise, the output and other system variables will reach such high magnitudes that either the system will be damaged or non-linearities would set in, limiting the magnitudes. The study of stability of systems is a very important area in the field of control theory and is discussed in Chapter 7.

For stable systems, the roots of the characteristic equation will be negative (or will have negative real part). In that case, the negative exponential terms in the complete response will go on reducing and finally tend to zero as $t \to \infty$. In most of the cases, the magnitudes of these components become zero in a very short time. Hence, this component of response, i.e., $y_c(t)$ is called the *transient response*.

As opposed to $y_c(t)$, the component $y_p(t)$ of the response does not decay to zero as $t \to \infty$ but persists as long as the input $x(t)$ remains applied. Its form is also similar to that of the applied input. Hence, this component of response is called the *steady-state response*. The complete response thus is the sum of the transient response and the steady-state response. In the period immediately following $t = 0$, the response is affected by both transient and steady-state components. However, after the transients have decayed only the steady-state component remains.

It should be noted that the form of the transient component is always a decaying exponential (real or complex), whereas its actual magnitude is dependent upon constants C_1, C_2 ,...,C_n which are affected both by the initial conditions and the applied input.

The choice of instant $t = 0$ is somewhat arbitrary. Generally, it is selected as the time at which the input is applied to the system. The values of initial conditions take care of the effect of all other past inputs. The value of response at any time $t_1 > 0$ is thus affected by both the initial conditions at $t = 0$ and the input from $t = 0$ up to $t = t_1$.

It should be noted that the phenomena of transient response is associated only with dynamic systems. A static system will have no transients. It can have no initial conditions or memory and its output at any instant will depend only on the value of the input at that instant of time. The output will not be affected by the past values of the input.

3.3 Standard Test Signals and Their Properties

As mentioned earlier, perhaps repeatedly, the essence of analysis is to generate solutions for different types of input signals (or excitations). In actual operation, physical systems may be subjected to almost all possible types of input signals. However, for the purpose of theoretical analysis, as well as for experimental studies, we use only some selected types of input signals. These signals are so chosen that the system response to them reveals significant properties of the system and gives an insight into its physical behaviour. We now study the forms, mathematical representations and the properties of these standard test signals.

Step signal: A unit step function is shown in Fig. 3.1. Its value prior to $t = 0$ is zero. It suddenly jumps to unit magnitude at $t = 0$ and remains at this magnitude for all times $t > 0$. It is represented by the symbol $u(t)$.

That is,

$$u(t) = 1 \text{ for } t \geq = 0$$
$$= 0 \text{ for } t < 0$$

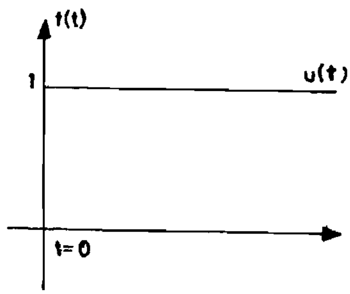


Fig. 3.1 A Unit Step Function

A step input of magnitude k will be represented as $ku(t)$. A step function at $t = t_1 \neq 0$ (Fig. 3.2) will be represented as a *delayed step* by $u(t - t_1)$. So long as the quotient in the brackets is negative, i.e., $t < t_1$, the function has zero value. It becomes unity for $t \geq t_1$.

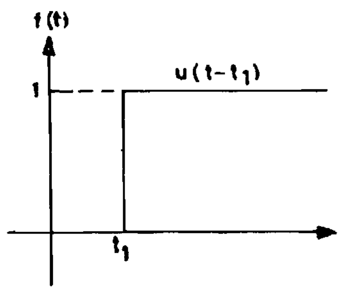


Fig. 3.2 Delayed Step

Step functions arise in practice when a switch is suddenly closed or opened in an electrical system or a valve is closed or opened in a fluid flow system or a mass is given an initial displacement and then allowed to oscillate in a mechanical system. The response to step input is called the *step response* of the system. It clearly reveals the dynamic and the transient behaviour of physical systems. It is also a very useful signal in the experimental study of physical systems. As we shall study later, the step response has an extremely useful property. If the response of a system to a step input is known, its response to *any* input can be determined.

Many other signals having discontinuities can be represented in terms of step signals. For example, a pulse can be represented as a sum of two steps, one positive step at $t = t_1$ and one negative step at $t = t_2$, i.e., the *pulse function* (Fig. 3.3) is $f(t) = Au(t-t_1) - Au(t-t_2)$.

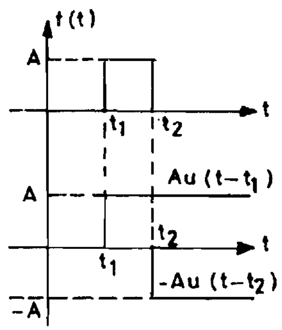


Fig. 3.3 Pulse Function

Another useful role of the step function is to define the range of time to lie between zero and infinity for a general signal $f(t)$. Normally the range of time is taken as from $-\infty$ to $+\infty$, i.e., $-\infty < t < +\infty$. However, if we want to define the function only for $0 \leq t < \infty$, we multiply it by a unit step function. Thus, the signal $f(t)\ u(t)$ means that its value is zero prior to $t = 0$ and equal to $f(t)$ for $t \geq 0$.

Ramp signal: A ramp signal is shown in Fig. 3.4 and is represented by $f(t) = k\,t\,u(t)$. The multiplication by u(t) is to show that the value of the function is zero prior to $t = 0$. The ramp signal may also be considered as a particular case of the more general polynomial function $f(t) = k_1 t + k_2 t^2 + \ldots$, when only k_1 is non-zero. Another way of looking at the ramp is as the integral of the step function.

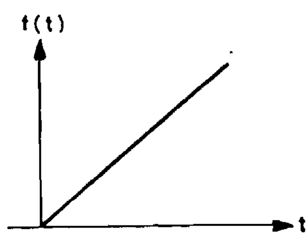


Fig. 3.4 A Ramp Signal

A ramp signal may also be used in representing signals of other forms. For example, the triangular pulse shown in Fig. 3.5 may be represented as

$$f(t) = tu(t) - 2(t-1)\,u(t-1) + (t-2)\,u(t-2) \qquad (3.6)$$

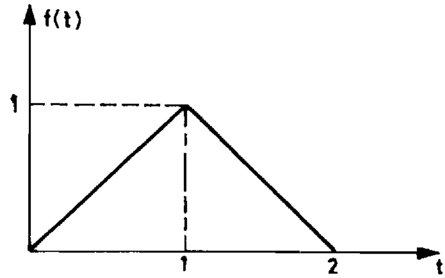


Fig. 3.5 A Triangular Pulse

Ramp signals arise in 'tracking' situations. For example, while tracking aircraft a radar antenna is rotated at a continuous angular velocity. The mechanism which controls the angular movement of the antenna normally responds to the input signals corresponding to the required angular position. Therefore, to make it rotate at a constant velocity, a ramp input signal must be given. Yet another example of the ramp signal is the input signal to the programmable temperature controller, shown in Fig. 2.13(b).

Exponential signal: Signals of the type $f(t) = k\,e^{at}$, with a positive or negative exponent are frequently encountered in linear dynamic systems. The natural response of systems consists of exponential terms only. If this output is an input to another physical system, we have to consider solutions of equations with this exponential function as the forcing function.

The most characteristic property of the exponential function can be expressed by its *time constant.* The time constant is the value of time for which the exponent becomes unity. Thus, time constant $\tau = 1/a$ for the exponential function e^{at}. The exponential signal has the interesting property that if a tangent to it is drawn at point p, it will meet the t-axis at a distance τ from the intercept of point p on the t-axis.

For the decaying exponential signal, the value of the signal at $t = 0$ is obviously k. At the instant $t = \tau$ the value becomes $f(t) = k\,e^{-1} = (k \times 1)/2.718 = 0.368\,k$. That is, in one time constant the exponential decays to 36.8% of its initial value (Fig. 3.6). The value of the time constant can therefore be defined as the time required for the decaying exponential to reduce to 36.8% of its initial value.

Although theoretically a decaying exponential never becomes zero, after a period of $t = 5\,\tau$, its value reduces to $k\,e^{-5} = 0.0067\,k$. This is small enough to be treated as zero for all practical purposes. Therefore, in engineering analysis it is common to assume that a decaying exponential becomes negligibly small after five time constants.

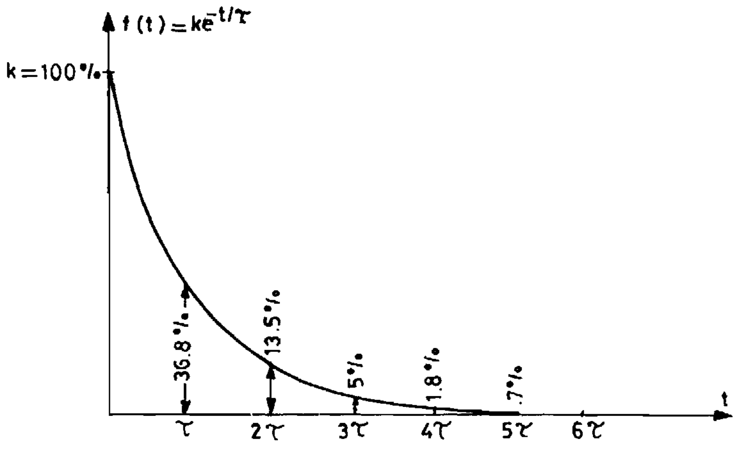


Fig. 3.6 Decaying Exponential Signal

Another interesting and useful property of an exponential function is that its derivatives of all orders are also exponential. That is why this function invariably appears in the solution of differential equations.

When the coefficient of t in the exponent becomes imaginary, the exponential function represents a sinusoidal function,

$$f(t) \;=\; k\; e^{\pm jbt} \;=\; k\,(\cos bt \,\pm\, j\;\sin bt) \tag{3.7}$$

as given by Euler's relation.

The sinusoidal signal represented by eqn. (3.7) has amplitude k and frequency b. If the exponent is complex (with negative real part), i.e.,

$$f(t) \;=\; k\, e^{(-a \pm jb)t} = k\, e^{-at}\; (\cos bt \,\pm\, j \sin bt\,) \tag{3.8}$$

the exponential function represents a decaying sinusoidal signal as shown in Fig. 3.7. We shall encounter such signals in the analysis of second order systems.

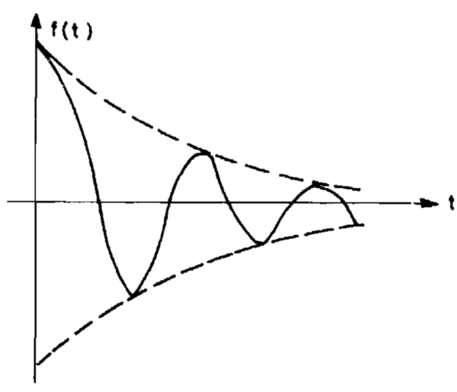


Fig. 3.7 Decaying Sinusoidal Signal

Sinusoidal Signal: The response of a system to sinusoidal signal reveals some of its most important properties. The analysis of dynamic systems relies on two main techniques—the time domain and the frequency domain techniques. The latter is dependent on the response of the system to sinusoidal inputs with frequency varying from zero to infinity. In practical testing also, response to sinusoidal signals or the frequency response of systems is very helpful.

A sinusoidal signal is shown in Fig. 3.8(a) and is represented by the equation,

$$f(t) = A \sin \omega t \tag{3.9}$$

Its peak amplitude (many a time referred to as just amplitude) is A and the frequency, ω radians/second. If at the instant $t = 0$, the value of the function is not zero, as in Fig. 3.8(b), the equation becomes,

$$f(t) = A \sin (\omega t + \theta) \tag{3.10}$$

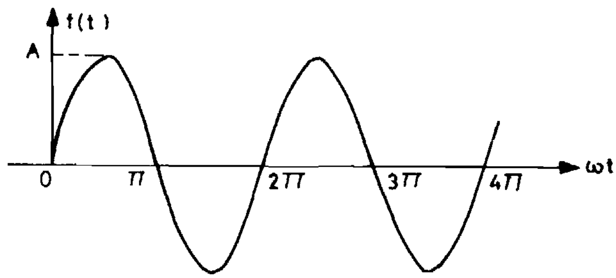


Fig. 3.8(a) A Sinusoidal Signal

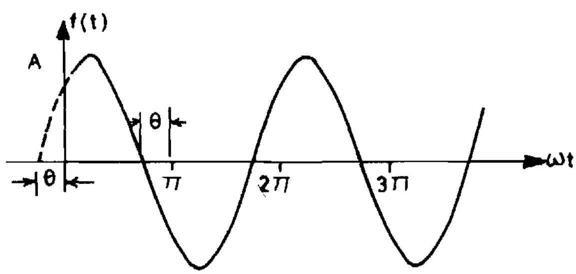


Fig. 3.8(b) Phase Shift

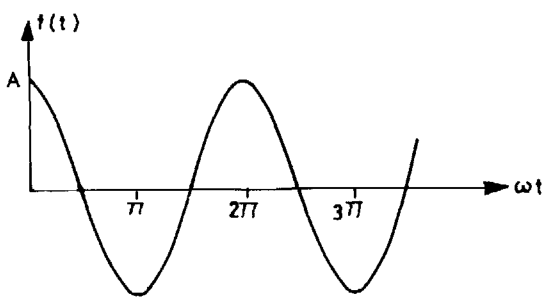


Fig. 3.8(c) Cosine Function

where angle θ is called the phase angle of the sinusoid. If the phase angle is $\pi/2$, the function $f(t) = A \sin(\omega t + \pi/2) = A \cos \omega t$, can also be called a *cosine function*.

The frequency can also be expressed as $\omega = 2\pi f$, where f is the frequency in hertz, abbreviated to Hz. In older times, f was expressed in cycles/second. The time period of the function is $T = 2\pi/\omega$ or $1/f$ seconds. The higher the frequency, the smaller will be its period. The power supply frequency is 50 Hz with a period of 20 milliseconds. The radio frequency range is 20 kHz to 1 GHz with a period from 0.05 milliseconds to 1 nano-second. The microwave range is 1 GHz to 100 GHz with periods from 1 nanosecond to 10 picoseconds.

As can be easily verified from trigonometric relations, the sum of two sinusoidal signals of the same frequency will also be sinusoidal, even if they have a phase shift between them. However, the sum of signals with different frequencies is not sinusoidal. In fact, non-sinusoidal signals can be shown to be equal to sum of different frequency sinusoidal signals. (This is the subject matter of the next chapter on *Fourier series*.)

Addition of sinusoidal functions of the same frequency is very much simplified by considering them as *rotating vectors*, as shown in Fig. 3.9.

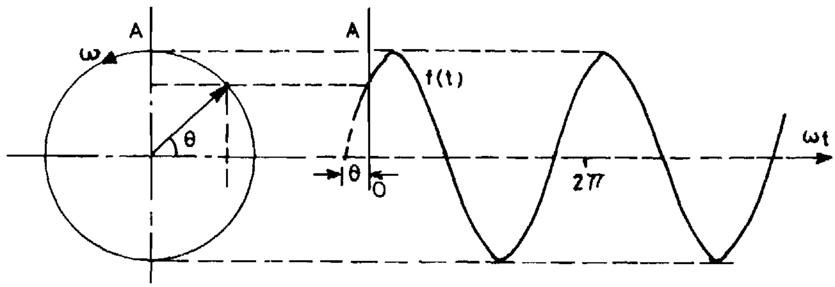


Fig. 3.9 Sinusoidal Signal as a Rotating Vector

The rotating vector with a magnitude A, angular velocity ω rad/sec and an initial displacement θ, will have intercepts on the vertical axis given by $f(t) = A \sin(\omega t + \theta)$, which is the same as the general form of the sinusoidal function given by eqn. (3.10). The function f can also be written as,

$$f = A \angle \theta \tag{3.11}$$

When written in this form, the function is called a *phasor* with magnitude A and phase angle θ. A *phasor diagram* like Fig. 3.10 is used as an aid in understanding and manipulation of phasors. For example, consider addition of two phasors,

$$f_1(t) = A_1 \sin(\omega t + \theta_1) = A_1 \angle \theta_1$$

and

$$f_2(t) = A_2 \sin(\omega t + \theta_2) = A_2 \angle \theta_2.$$

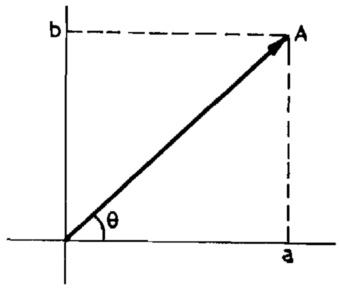


Fig. 3.10 Phasor Diagram

Their phasor representations are shown in the phasor diagram (Fig. 3.11). The sum of the two functions can be obtained by vector addition of A_1 and A_2 as shown in Fig. 3.11. Since A_1 and A_2 are special types of vectors (i.e., rotating vectors), called *phasors*, such addition is called *phasor addition*.

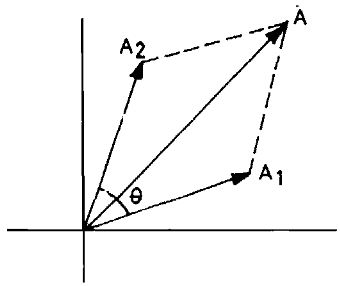


Fig. 3.11 Addition of Two Phasors

The horizontal and vertical axes in the phasor diagram may be considered as the real and the imaginary axes. Then the phasor A of Fig. 3.10 can also be expressed as a complex number, i.e.,

$$A \angle\theta = a + jb \tag{3.12}$$

Use of complex numbers simplifies the numerical work. For example, with this representation, $A_1 \angle\theta_1 = a_1 + j\,b_1$ and $A_2 \angle\theta_2 = a_2 + jb_2$. Their sum is given by,

$$A_1 \angle\theta_1 + A_2 \angle\theta = A \angle\theta = (a_1 + a_2) + j\,(b_1 + b_2)\ .$$

In the steady-state analysis of electrical networks, the phasor and complex number representations of sinusoids is used very frequently. Here the variables are voltages and currents. To make it easy for practical purposes, these variables are represented as phasors not with their peak amplitudes but with their r.m.s. values.

The complex exponential representation of the sinusoidal function is already given eqn. (3.7). This form is very useful for the dynamic analysis of systems and will be used quite frequently later in this text.

Impulse function: The impulse function is one of the most important functions encountered in the analysis of linear systems. Its characterisation and properties are dealt with in Section 5.3.

3.4 First Order Systems

As mentioned earlier, the order of a system is the same as that of its defining differential equation. A first order system is one whose mathematical model is a first order differential equation. As an example of a first order system let us analyse the response of a series *R-L* circuit.

Step response of an R-L circuit: Let us suppose that an electrical system is represented by a resistance R in series with an inductance L. Further, let the system be connected to a d.c. source of voltage E for a sufficiently long time. At time $t = 0$, the source is suddenly replaced by a short circuit. The object of the analysis is to determine current $i(t)$ for $t > 0$.

The mathematical model for the system, valid for t > 0, is given by,

$$L\frac{di}{dt} + Ri = 0 \tag{3.13}$$

The complementary function or the transient response is given by,

$$i_{tr} = C_1 \exp\left(-\frac{R}{L}t\right) = C_1\, e^{-t/\tau}\ .$$

The term $L/R = \tau$ is the time constant of the system. Since the right-hand side, i.e., the forcing function in eqn. (3.13) is zero, the particular integral or the steady-state response is also zero. Hence, the complete response is given by,

$$i = i_{tr} = C_1\, e^{-t/\tau} \tag{3.14}$$

The constant C_1 in eqn. (3.14) is determined by the initial condition, i.e., the value of current at $t = 0$. Prior to $t = 0$, the steady-state current in the circuit was E/R. This fact is represented mathematically by the statement $i(0^-) = E/R$. The minus sign over the zero indicates that it is the value at $t = 0$, when this point is approached from the negative side. However, the value of initial condition required is $i\,(0^+)$; the value of function $i\,(t)$ at $t = 0$, when this point is approached from the positive side. To determine $i(0^+)$ we take recourse to physical reasoning. We know that the current through an inductor cannot change in zero time. In other words, the current cannot be discontinuous. This is so because of the basic terminal $v - i$ relation of inductance, $v = Ldi/dt$. A discontinuous current will require an infinite voltage, a physical impossibility. Hence, we conclude that $i\,(0^+) = i(0^-) = E/R$. This type of physical reasoning is frequently required for the determination of initial conditions.

Substituting the initial condition at $t = 0$ in eqn. (3.14), we get $C_1 = E/R$. Therefore,

$$i = \frac{E}{R} e^{-t/\tau} \tag{3.15}$$

Equation (3.15) gives the solution of the problem. A plot of the response i as a function of time will be the same as that shown in Fig. 3.6 with $k = E/R$.

The transient response of a first order system is completely specified by only two factors: (i) the time constant τ and (ii) the initial magnitude.

Sinusoidal response: Let us now assume that the series R-L circuit is connected to a source with voltage $v = V_m \sin \omega t$, at $t = 0$. The mathematical model is given by,

$$L \frac{di}{dt} + Ri = V_m \sin \omega t. \tag{3.16}$$

The form of the transient response remains the same as given by eqn. (3.14),

$$i_{tr} = C_1 e^{-t/\tau}.$$

Let us assume that the form of the particular integral or the steady-state response is,

$$i_{ss} = A \cos \omega t + B \sin \omega t.$$

Substituting it in eqn. (3.16) we get,

$$(RA + B \omega L) \cos \omega t + (RB - A \omega L) \sin \omega t = V_m \sin \omega t.$$

Comparing the coefficients of sine and cosine terms on both sides of the equation,

$$RA + B\omega L = 0 \text{ and } RB - A\omega L = V_m$$

or,

$$A = \frac{-\omega L}{R^2 + (\omega L)^2} V_m \text{ and } B = \frac{R}{R^2 + (\omega L)^2} V_m$$

Therefore,

$$i_{ss} = \frac{RV_m}{R^2 + (\omega L)^2} \sin \omega t - \frac{\omega L V_m}{R^2 + (\omega L)^2} \cos \omega t .$$

Define $\sqrt{R^2 + (\omega L)^2} = Z$ and,

$$\frac{R}{Z} = \cos \theta \text{ and } \frac{\omega L}{Z} = \sin \theta$$

as per Fig. 3.12. Then,

$$i_{ss} = \frac{V_m}{Z} (\cos \theta \sin \omega t - \sin \theta \cos \omega t)$$

$$= \frac{V_m}{Z} \sin (\omega t - \theta)$$

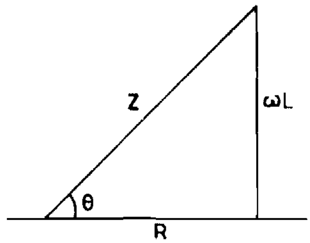


Fig. 3.12 Relation between R, ωL and Z

The symbol Z stands for the well-known concept of impedance of an a.c. circuit and θ is the phase difference between the current and the voltage.

The complete solution of the problem is,

$$i = i_{tr} + i_{ss}$$

$$= C_1 e^{-t/\tau} + \frac{V_m}{Z} \sin(\omega t - \theta)$$

If the circuit is assumed to be initially relaxed, then,

$$i(0) = 0 = C_1 - \frac{V_m}{Z} \sin\theta$$

or

$$C_1 = \frac{V_m}{Z} \sin\theta .$$

Then,

$$i = \frac{V_m}{Z} [e^{-t/\tau} \sin\theta + \sin(\omega t - \theta)] \tag{3.17}$$

A plot of the different components of the response is shown in Fig. 3.13.

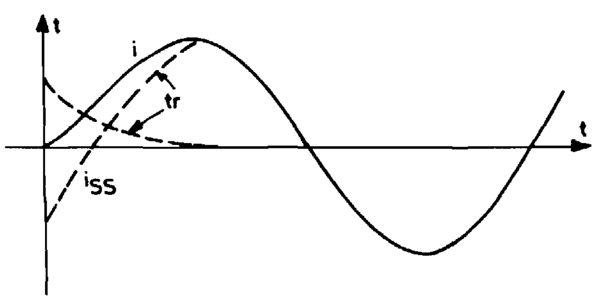


Fig. 3.13 Sinusoidal Response of a First Order System

An alternate problem would be the determination of $i(t)$ for $v = V_m \cos \omega t$. The reader is urged to work out this problem and compare the response with the previous problem.

3.5 Second Order Systems

Compared to the first order systems, the second order systems are by far more interesting and important in the analysis of physical systems—particularly those of interest to engineers. This is because a large number of important systems are intrinsically second order. Many a time systems of higher order can also be closely approximated by second order models. Even if this reduction in the order of the system is not possible, the properties of higher order systems can be understood in terms of the characteristics of the second order system.

In systems where the concept of energy applies, a system with two types of independent energy storage elements will give rise to a second order model. In electrical systems energy is stored in the magnetic field of an inductance and in the electric field of a capacitance. Hence, presence of both L and C in an electrical system will give rise to a second order model. In a mechanical system, kinetic energy is stored in the mass or inertia and potential energy in linear or torsional spring. Thus, a system having both mass and spring will have a second order model.

As a vehicle for understanding the properties of the second order systems let us consider the series *RLC* circuit of Fig. 3.14.

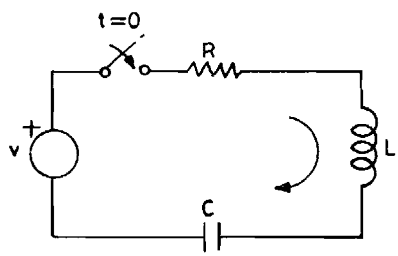


Fig. 3.14 Series RLC Circuit

Series RLC circuit: The switch is closed at time $t = 0$. Applying Kirchhoff's voltage law around the loop we get,

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i\,dt = v. \tag{3.18}$$

This is an integro-differential equation. To convert it into a differential equation, differentiate once to get,

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{i}{C} = \frac{dv}{dt}. \tag{3.19}$$

Transient response: The characteristic equation is

$$L\gamma^2 + R\gamma + \frac{1}{C} = 0\cdot$$

Its roots are,

$$\gamma_1, \gamma_2 = -\alpha \pm b = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Therefore,

$$i_n = C_1 e^{(-\alpha+b)t} + C_2 e^{(-\alpha-b)t}$$

where

$$\alpha = \frac{R}{2L} \text{ and } b = \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} .$$

While α will always be real, b may be real or imaginary depending on whether $R/2L$ is more than or less than $1/\sqrt{LC}$.

We thus have the following three cases:

(1) b is real, i.e., $\frac{R}{2L} > \frac{1}{\sqrt{LC}}$ or $R > 2\sqrt{\frac{L}{C}}\cdot$

Then α must be greater than b and we have

$$i_n = C_1 \exp(-a_1 t) + C_2 \exp(-a_2 t) \tag{3.20}$$

where $a_1 = \alpha - b$ and $a_2 = \alpha + b$.

The transient response will thus be the sum of two decaying exponentials which will tend to zero as $t \to \infty$ as in the case of a first order system. This is called the *overdamped* case.

(2) b is zero, i.e., $\frac{R}{2L} = \frac{1}{\sqrt{LC}}$ or $R = 2\sqrt{\frac{L}{C}}\cdot$

Then the two roots are equal and,

$$i_n = (C_1 + C_2 t)\, e^{-\alpha t}. \tag{3.21}$$

This is called the *critically damped case* and is the dividing line between the overdamped case and the underdamped case described next.

The value of α for critical damping is,

$$\alpha_c = \frac{R}{2L} = \frac{1}{\sqrt{LC}}\cdot$$

(3) b is imaginary: $\frac{R}{2L} < \frac{1}{\sqrt{LC}}$ or $R < 2\sqrt{\frac{L}{C}}$

Let $b = j\beta$ where β is a real number. Then,

$$i_n = e^{-\alpha t}(C_1 e^{j\beta t} + C_2 e^{-j\beta t})$$

$$= e^{-\alpha t}[(C_1 + C_2)\cos\beta t + j(C_1 - C_2)\sin\beta t]$$

$$= e^{-\alpha t}[B_1 \cos\beta t + B_2 \sin\beta t]$$

[where $B_1 = C_1 + C_2$ and $B_2 = j(C_1 - C_2)$]

$$= e^{-\alpha t} A \sin(\beta t + \theta), \tag{3.22}$$

where $A = \sqrt{B_1^2 + B_2^2}$ and

$$\theta = \tan^{-1}\frac{B_1}{B_2}.$$

Thus, the natural response in this case is a sinusoid with frequency β, phase angle θ, and the magnitude exponentially decaying by a factor α. This case is called the *underdamped* case, with a damping factor $\alpha = R/2L$ and frequency $\beta = \sqrt{(1/LC) - (R/2L)^2}$.

Thus, the amplitude and the duration of the transient response is very much dependent on the value of resistance R. If R is high, the system is overdamped. When R is progressively reduced, we get a critical value of resistance which gives critical damping. When R is reduced below this value, the response becomes oscillatory and the system is underdamped. The shape of the natural response of a second order system for the conditions of overdamping, critical damping, and underdamping is shown in Fig. 3.15.

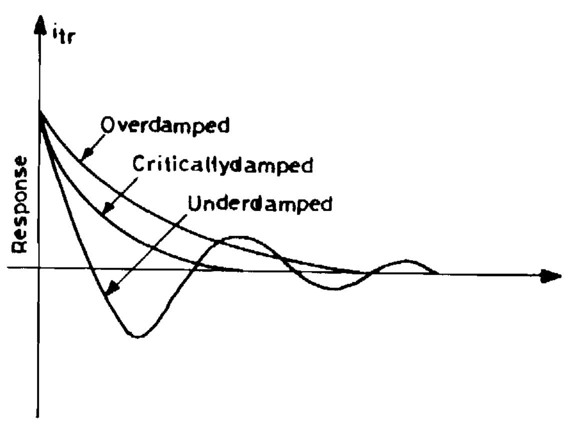


Fig. 3.15 Natural Response of Second Order System

Step input response with zero initial conditions: Let $v(t) = Ku(t)$. Then the forcing function dv/dt in eqn. (3.19) becomes zero for $t > 0$. Therefore, the steady-state response $i_{ss} = 0$. This is obvious because with a series capacitor steady-state current due to a constant voltage will be zero. Let us now determine the unknown constants in the complete response for the underdamped case:

$$i = i_{tr} + i_{ss} = i_{tr} = A\,e^{-\alpha t} \sin(\beta t + \theta)\,. \tag{3.23}$$

At $t = 0$, $i = 0$ as the system is assumed to be initially relaxed. Therefore, $i(0^+) = i(0^-) = 0 = A \sin \theta$. Since the magnitude A cannot be zero (otherwise the problem becomes trivial), θ must be zero. Thus, we get,

$$i(t) = A\,e^{-\alpha t} \sin \beta t.$$

Then,

$$\frac{di}{dt} = -\alpha A\,e^{-\alpha t} \sin \beta t + \beta A\,e^{-\alpha t} \cos \beta t \tag{3.24}$$

When the switch is closed at $t = 0$, initially the current is zero. Hence, there is no voltage across the resistor and $v_R(0) = 0$. Also, the integral of current would be zero. Hence the voltage across the capacitor, $v_c(0) = 0$. Thus, the whole of the applied voltage must appear across the inductance only. The voltage across the inductance is given by $L\,(di/dt)$. Therefore, from eqn. (3.24), we have,

$$\left.\frac{di}{dt}\right|_{t=0^+} = \beta A = \frac{K}{L}$$

or $A = K/(\beta L)$. Thus, the response of the system to a step input of magnitude K is given by,

$$i(t) = \frac{K}{\beta L}\,e^{-\alpha t} \sin \beta t. \tag{3.25}$$

An interesting case occurs when R is reduced to zero. For $R = 0$, the frequency of oscillation becomes $\beta = 1/\sqrt{LC}$ and the damping factor $\alpha = 0$. Since there is no damping in the system, it is called the *undamped system* and the frequency of oscillation is given a special name ω_n, the *frequency of undamped oscillation or natural oscillation*. Thus, $\omega_n = 1/\sqrt{LC}$. The response $i(t)$ given by eqn. (3.25) becomes,

$$i(t) = K\sqrt{\frac{C}{L}}\,\sin \omega_n t. \tag{3.26}$$

Equation (3.26) shows that the response of this undamped LC circuit, or the lossless LC circuit, to a step will be sustained sinusoidal oscillations. In fact, even after the step input is removed, the oscillations would continue indefinitely. Physically, what happens is that energy is continuously being exchanged between the magnetic field of the inductance and the electric field of the capacitance. As there is no resistance in the circuit to dissipate the initial energy, the oscillations would continue indefinitely with the constant initial amplitude. However, this situation is only an idealisation because any physical system will always possess some resistance and hence the magnitude of oscillations will decay according to eqn. (3.23).

With resistance $R = 0$, the mathematical model reduces to,

$$L\frac{d^2i}{dt^2} + \frac{i}{C} = \frac{dv}{dt}.$$

This is the equation of a simple harmonic motion, which, in general terms is written as,

$$\frac{d^2x}{dt^2} + \frac{x}{k} = f(t). \qquad (3.27)$$

The oscillatory solution has a frequency of $\sqrt{k}$.

3.6 The General Equation for Second Order Systems

In the previous section, the parameters of a second order system were redefined in terms of the undamped natural frequency ω_n, a damped frequency β and a damping factor α. It is of importance to know how much this damping factor deviates from the value α_c, required for critical damping. To focus attention on this property, a new parameter ζ is defined as the ratio of the actual damping factor α to the value of damping factor for critical damping α_c. That is $\zeta = \alpha/\alpha_c$. The factor ζ is called the *damping ratio*.

For studying the properties of second order systems, the system equation is written in the following general form:

$$\frac{d^2y}{dt^2} + 2\,\zeta\,\omega_n\frac{dy}{dt} + \omega_n^2\,y = \omega_n^2\,x(t) \qquad (3.28)$$

where ζ is the damping ratio and ω_n the undamped frequency of natural oscillations. For the electrical system of *RLC* circuit of Fig. 3.14 with eqn. (3.19) as the model, the new system parameters ζ and ω_n can be expressed in terms of *RLC* as,

$$\zeta = \frac{\alpha}{\alpha_c} = \frac{R/2L}{1/\sqrt{LC}} = \frac{R}{2}\sqrt{\frac{C}{L}} \quad \text{and} \quad \omega_n = \frac{1}{\sqrt{LC}}$$

For the second order mechanical system [Section 1.2, eqn. (1.5)] we have,

$$\zeta = \frac{D}{2\sqrt{KM}} \quad \text{and} \quad \omega_n = \sqrt{\frac{K}{M}}$$

Let us now obtain the solution for eqn. (3.28). The characteristic equation is

$$r^2 + 2\zeta\omega_n r + \omega_n^2 = 0$$

with roots: $r_1, r_2 = [-\zeta \pm \sqrt{\zeta^2 - 1}\,]\,\omega_n$.

If $\zeta > 1$, the roots are real and distinct and the system is overdamped. If $\zeta = 1$, the roots are equal and the system is critically damped. If $\zeta < 1$, the roots are complex leading to underdamped response. However, if ζ is negative, the roots will have a positive real part and the system will be unstable. Thus, a knowledge of the value of the damping ratio is sufficient to reveal the nature of the transient response.

Step response with zero initial conditions:

(i) *Overdamped case* ($\zeta > 1$) : The transient component of the solution of eqn. (3.28) with $x = u(t)$ is,

$$y_{tr} = C_1 \exp\,[(-\zeta - \sqrt{\zeta^2 - 1})\,\omega_n t] + C_2 \exp\,[(-\zeta + \sqrt{\zeta^2 - 1})\,\omega_n t].$$

The steady-state component is $y_{ss} = 1$. Therefore the complete response is,

$$y = 1 + C_1 \exp\,[(-\zeta - \sqrt{\zeta^2 - 1})\,\omega_n t] + C_2 \exp\,[(-\zeta + \sqrt{\zeta^2 - 1})\,\omega_n t].$$

The constants C_1 and C_2 are determined from the initial conditions,

$$y(0) = 0 \text{ and } \left.\frac{dy}{dt}\right|_{t=0} = 0.$$

Evaluating these constants and substituting them in the above expression we get,

$$y = 1 + \frac{\zeta - \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}} \exp\,[(-\zeta - \sqrt{\zeta^2 - 1}\,)\,\omega_n t]$$

$$- \frac{\zeta + \sqrt{\zeta^2 - 1}}{2\sqrt{\zeta^2 - 1}} \exp\,[(-\zeta + \sqrt{\zeta^2 - 1}\,)\,\omega_n t] \quad (3.29)$$

(ii) *Critically damped case* ($\zeta = 1$) : The roots of the characteristic equation will be equal, i.e., $r_1 = r_2 = -\omega_n$. Therefore, the solution is,

$$y = 1 + C_1 e^{-\omega_n t} + C_2\, t\, e^{-\omega_n t}$$

Evaluating the constants from the given initial conditions we get,

$$y = 1 - e^{-\omega_n t}\,(1 + t + \omega_n t) \quad (3.30)$$

(iii) *Underdamped case* ($\zeta < 1$) : In this case, the roots will be complex, i.e.,

$$r_1 = (-\zeta + j\sqrt{1 - \zeta^2}\,)\,\omega_n \text{ and } r_2 = (-\zeta - j\sqrt{1 - \zeta^2}\,)\,\omega_n.$$

The complete solution is,

$$y = 1 + C_1 \exp\,[(-\zeta + j\sqrt{1 - \zeta^2}\,)\,\omega_n t] + C_2 \exp\,[(-\zeta - j\sqrt{1 - \zeta^2}\,)\,\omega_n t].$$

Evaluating the constants C_1 and C_2 from the given initial conditions,

$$y(0) = 0 \text{ and } \left.\frac{dy}{dt}\right|_{t=0} = 0,$$

we get,

$$y = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\omega_n t} \sin [\omega_n (\sqrt{1-\zeta^2})\, t + \theta] \qquad (3.31)$$

where,

$$\theta = \tan^{-1} (\sqrt{1-\zeta^2}/\zeta).$$

Equation (3.31) reveals the oscillatory nature of the step response of an underdamped second order system. The frequency of oscillation is $\omega_n \sqrt{1-\zeta^2} = \omega_d$, the damped natural frequency, and varies with ζ. When $\zeta = 0$, the response becomes undamped and the oscillations continue indefinitely.

Transient response specifications: As indicated by eqn. (3.31), the transient response of a second order underdamped system is completely characterised by the system parameters, ζ and ω_n. A typical response of this type is shown in Fig. 3.16. From the practical application point of view the following numerical measures of this curve are of interest.

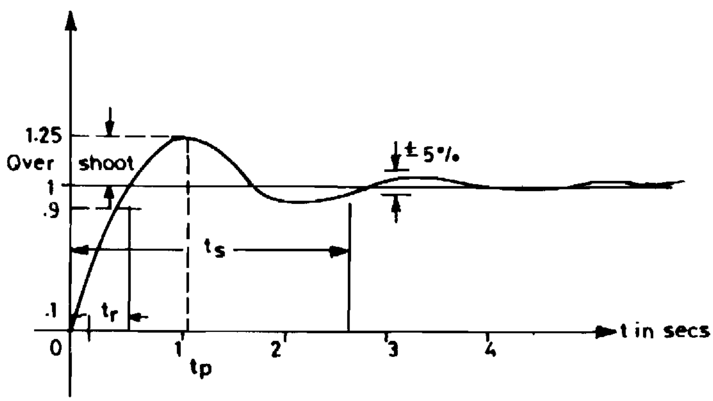


Fig. 3.16 Step Response of Second Order System for $\zeta = 0.4$, $w_n = 1.0$

(1) *Rise time* (t_r) : This is a measure of how fast the system is. It is normally taken as the time for the step response to rise from 10% to 90% of its final value. For underdamped systems, sometimes it is also taken as the time for 0-100% change.

(2) *Peak time* (t_p): This is the time required to reach the peak value of the step response.

(3) *Percentage overshoot* (*PO*): The percentage by which the maximum overshoot (which is the first overshoot and occurs at time $t = t_p$) exceeds the steady-

state value is called the *percentage overshoot.* It is a measure of the relative stability of the system.

(4) *Settling time* (t_s): The time at which the step response enters a ±5% band around the steady-state value and thereafter stays within this band is called the settling time.

These points are marked for the curve in Fig. 3.16.

The value of these performance measures, in terms of ζ and ω_n, can be derived from eqn. (3.31) as follows. Taking t_r as the time to reach 100% of its final value, $y(t) = 1$ at $t = t_r$. Substituting $y(t) = 1$ and $t = t_r$ in eqn. (3.31) we get,

$$\sin\left(\omega_n \sqrt{1 - \zeta^2}\, t_r + \theta\right) = 0$$

or,

$$t_r = \frac{\pi - \theta}{\omega_d} \qquad (3.32)$$

To obtain t_p we equate the derivative of $y(t)$ to zero. That is,

$$\frac{dy(t)}{dt} = \frac{\zeta\,\omega_n}{\sqrt{1 - \zeta^2}} \exp(-\zeta\,\omega_n t) \sin(\omega_d t_p + \theta)$$

$$- \omega_n \exp(-\zeta\,\omega_n t) \cos(\omega_d t_p + \theta) = 0$$

or,

$$\sin(\omega_d t_p + \theta) = \frac{\sqrt{1 - \zeta^2}}{\zeta} \cos(\omega_d t_p + \theta).$$

But, $\sqrt{1 - \zeta^2}/\zeta = \tan\theta = \sin\theta/\cos\theta$. Therefore at $t = t_p$,

$$\sin(\omega_d t_p + \theta) \cos\theta = \sin\theta \cos(\omega_d t_p + \theta).$$

The above relation is satisfied when $\omega_d t_p = \pi$. Therefore,

$$t_p = \frac{\pi}{\omega_d} \qquad (3.33)$$

Substituting expression (3.33) for t_p in eqn. (3.31), and noting that $\sin\theta = \sqrt{1 - \zeta^2}$ we get,

$$y(t)_{\max} = 1 + \exp\left(-\zeta\pi/\sqrt{1 - \zeta^2}\right).$$

Therefore, the percentage overshoot is given by,

$$PO = 100 \exp\left(-\zeta\pi/\sqrt{1 - \zeta^2}\right). \qquad (3.34)$$

The settling time t_s can be approximated by the expression,

$$t_s = \frac{3}{\zeta\,\omega_n}. \qquad (3.35)$$

Sometimes the ± 5% criterion is replaced by ± 2%. In that case,

$$t_s = \frac{4}{\zeta \omega_n}$$

It is normally assumed that during the period 0 to t_s the system gives its transient response and thereafter the steady-state response.

It is also important to be able to visualise the shape of the transient response of a second order system for different values of its parameters ζ and ω_n. Figure 3.17 shows the variation of the shape for different values of ζ for a fixed ω_n = 1 rad/sec. It shows that as ζ is reduced, the maximum overshoot increases and the response becomes more oscillatory. For ζ = 0, the system has no damping and it exhibits continuous oscillations. Too high an overshoot may cause damage to the system while a high value of ζ will make the system slow. As shown by the figure, values of ζ greater than unity make the system overdamped. Such a system does not have any oscillatory response.

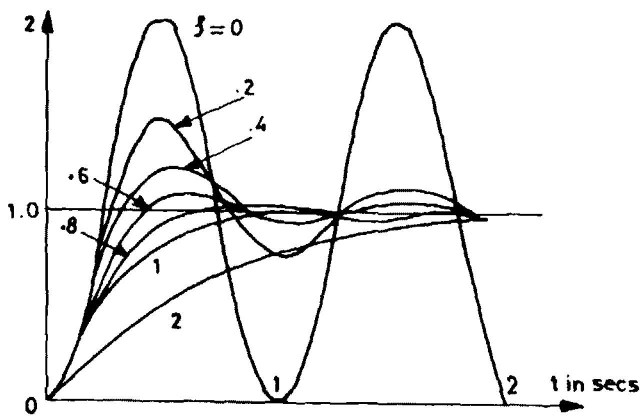


Fig. 3.17 Variation of the Transient Response of a Second Order System with Damping Ratio for a Fixed w_n = 1

Figure 3.18 shows the variation in the transient response with varying ω_n and a fixed ζ. The effect of decreasing ω_n is like expanding the response along the time axis. While the percentage overshoot remains fixed, both the rise time and the settling time increase with reduced ω_n.

The method of analysis using differential equations and their solutions is called the *classical method of analysis*. The algebra of the classical method becomes somewhat tedious even for second order systems with inputs other than the step input. Therefore, such problems are better handled by the use of more powerful transform techniques discussed in Chapter 6. However, the steady-state analysis for sinusoidal inputs is well established and straightforward. This method uses the impedance concept and phasor diagrams and is well known to electrical and

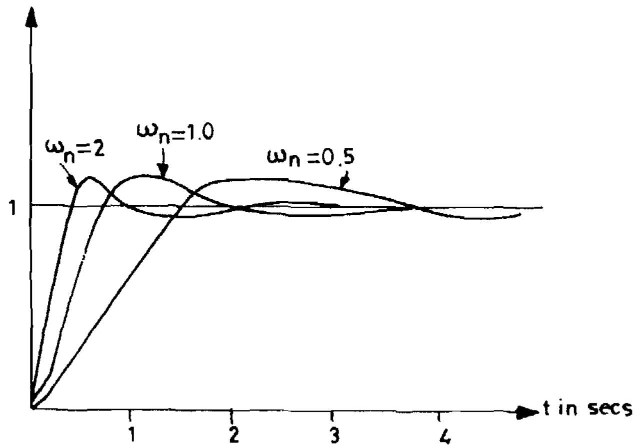


Fig. 3.18 Variation of the Transient Response of a Second Order System with w_n for a Fixed $\zeta = 0.5$

electronic engineers. The application of the impedance concept to non-electrical systems, however, is not very common. Later on, we shall define a generalised impedance concept, applicable to any type of system, in connection with the Laplace transform techniques.

The algebra involved in the solution of higher order equations becomes very much more tedious. It becomes still more difficult if the solution has to be generated for different types of working conditions. Numerical methods, implemented on computers, can take care of the tedium but they do not indicate the general properties of the system. Therefore, for the analysis of higher order systems, recourse is almost always taken to the more powerful Laplace transform techniques discussed in Chapter 6.

GLOSSARY

Order of a System: The order of the differential equation representing the model of the system is called *the order of the system.*

Natural Response (or unforced response or source free response): This is the solution of the homogeneous equation, obtained by replacing the r.h.s. or the forcing function of the differential equation equal to zero: i.e., it is the complementary function of the differential equation model.

Initial Conditions: The value of the dependent variable and its (n–1) derivatives for an *n*th order system, at the time at which the input is applied (usually $t = 0$), are called the *initial conditions of the system.*

Time Constant: The time required by a decaying exponential function to reduce to 36.8% of its initial value is called the time constant of the exponential function.

Overdamped, Underdamped and Critically Damped Systems: In a second order system, when the roots of the characteristic equation are real, the response decays to zero without changing sign, i.e., without oscillations. Such a system is called *overdamped system* When the roots are complex the response is oscillatory—a decaying sinusoid. Such a system is called *underdamped*. The borderline between these two cases occurs when the roots are real and equal. This system is called a *critically damped system*.

Damping Factor: In a second order system, the response decays exponentially. The coefficient α of time t in the exponent is called *the damping factor*.

Damping Ratio: The ratio of the actual damping factor to its value required for critical damping is called the *damping ratio* ζ. $\zeta > 1$ means an overdamped system, $\zeta < 1$ is an underdamped system and $\zeta = 1$ is a critically damped system. $\zeta = 0$ means an undamped system. $\zeta < 0$ means an unstable system.

PROBLEMS

3.1 Determine the complete solution of the equation,

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \frac{dx}{dt} + x$$

for (i) $x = t^2$ and (ii) $x = e^{-2t}$ with initial conditions,

$$y(0) = \left.\frac{dy}{dt}\right|_{t=0} = 0.$$

3.2. Describe a problem where it would be difficult to distinguish between the steady-state and the transient parts of the solution.

3.3. Write mathematical expressions for the functions of time illustrated in Fig. 3.19.

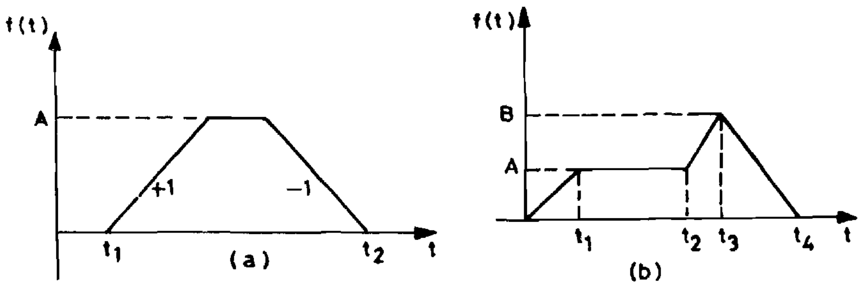


Fig. 3.19

3.4. Determine the mathematical model of the circuit shown in Fig. 3.20, treating e_i as the input and e_o as the output. This circuit is also called a *differentiating circuit*. Find the conditions under which the circuit can work as a differentiator. How would you experimentally demonstrate that the circuit acts as a differentiator?

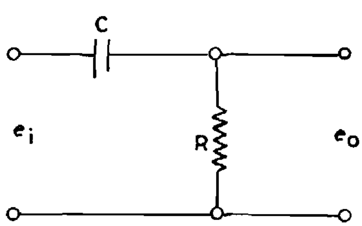


Fig. 3.20

3.5. Determine the mathematical model of the circuit shown in figure 3.21 and show that it can work as an integrating circuit.

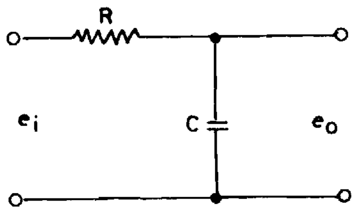


Fig. 3.21

3.6. If the current through an inductive circuit is suddenly interrupted, e.g., by opening a series switch, dangerously high voltages may be generated in the circuit (why?). What possible damages can this voltage cause? One of the methods of reducing this high induced voltage is to connect a resistance in parallel to the inductive circuit. The following problem indicates how the rating of this resistor can be calculated.

The field coil of a 200 V d.c. shunt generator has an inductance of 20 H and resistance of 200 ohms. Calculate the ratings of the resistor to be connected in parallel with it to reduce the maximum voltage across the field coil to 2000 V when its circuit is suddenly opened. (Note that the ratings of a resistor would include its ohmic value, power dissipation, maximum current and maximum voltage.) Can you suggest other possible methods for reducing the voltage?

3.7. A device represented by a series *RC* circuit is connected to a sinusoidal source through a switch. The switch is electronically controlled such that it can be closed at any phase angle θ of the sinusoidal wave. The peak amplitude of the source voltage is 10 V. There is an initial voltage of 5 V across the capacitor. Find the value of θ for which there will be no transient.

3.8. A quantity of 0.25 g of a drug was injected into a patient at $t = 0$. Thereafter, the concentration of the drug in the blood was determined at regular intervals. The experimental observations could be described satisfactorily by the equation,

$$c(t) = 0.0125\ e^{-0.084t}$$

where $c(t)$ is the drug concentration in *g/l* and t is in hours.

Determine and sketch $c(t)$ if the same drug is infused continuously for 24 hours at the rate of 0.25 *g*/hr starting at $t = 0$.

3.9. A thyristor used for power control is controlled such that it is on for one second, off for the next second, on again for one second and so on. (Such an arrangement is called *pulse width control with a 50% duty cycle.*) When on, the thyristor carries a constant current of 10A with a voltage drop of 1 V across it. A heat sink is connected to its casing. The parameters of the system are as follows: Thermal resistance; junction to case $RT_1 = 0.15\ ^\circ C/W$; case to heat sink $RT_2 = 0.1\ ^\circ C/W$; and heat sink to ambient air $RT_3 = 0.25\ ^\circ C/W$.

Thermal capacitance: case $c_{p1} = 0.1 \times 10^{-6}$ cal/°C and heat sink $C_{p2} = 0.2 \times 10^{-6}$ cal/°C.

Ambient temperature = 30°C.

(i) Determine and plot junction temperature as a function of time.

(ii) Study the effect of duty cycle on the junction temperature T_J and determine T_J for duty cycle of (a) 25%, (b)75%.

3.10. Determine the response of the general second order system with zero input and initial conditions is (0) = 0 and $y(0) = K$. Sketch the response as a function of time.

3.11. In the automobile ignition system of Section 1.1, assume the following values of parameters: $L = 50$ mH, $C = 0.5$ μF, $R = 20$ ohms, $M = 5$ H and $V = 12$ V. Determine the waveform and the maximum value of voltage across the spark plug. Does this maximum value occur on the 'make' of the switch or on the 'break' of the switch? Find out what is the normal peak firing voltage for car engines and compare the value obtained in this problem with the actual required value. What are the desirable features of the ignition system for improving the efficiency of the engine?

3.12. The natural response of a second order system is shown in Fig. 3.22. Determine ζ, ω_d and ω_n. Does this problem give you an idea of how to determine system parameters experimentalty?

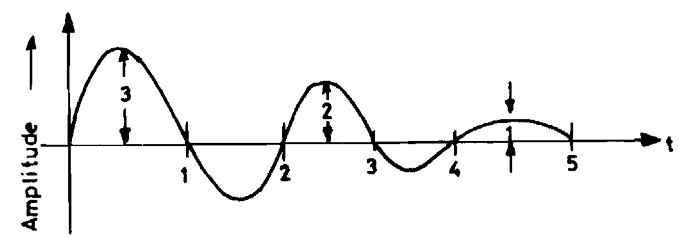


Fig. 3.22

3.13. A mass $M = 100$ kg is moving with a constant steady-state velocity $V_o = 10$ m/sec on frictionless rollers until it is engaged by a dashpot to decelerate it (Fig. 3.23). Determine the damping coefficient B of the dashpot such that the velocity of the mass is reduced to less than 0.05 V_0 in 5 seconds after the engagement.

3.14. A platform of weight 100 kg rests on a spring-dashpot unit. The spring coefficient is 3000 N/m and the coefficient of viscous friction of the dashpot is 300 N-sec/m. A man weighing 100 kg suddenly steps on the platform. Determine and sketch the subsequent motion of the platform.

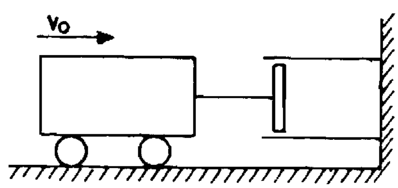


Fig. 3.23

3.15. For a series RLC circuit, connected to a constant voltage source V through a switch, determine the system equations, treating voltage across the capacitor as the output. Choose circuit parameters to obtain $\omega_n = 100$ rad/sec and $\zeta = 0.1$. Find the maximum voltage across the capacitor when the switch is closed.

CHAPTER 4

Fourier Series

LEARNING OBJECTIVES

After studying this chapter you should be able to:

(i) represent a periodic function in terms of the trigonometric or the exponential form of the Fourier series;

(ii) determine the magnitudes of the harmonic components of a non-sinusoidal waveform;

(iii) calculate the r.m.s. values of non-sinusoidal voltages and currents and the power in a circuit having such waveforms;

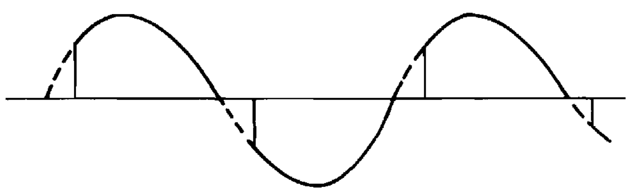
(iv) obtain the response of a linear system to non-sinusoidal periodic inputs, using the Fourier series; and

(v) calculate the value of the Fourier coefficients graphically or numerically.

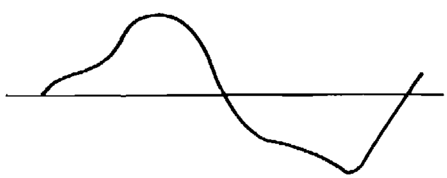
In the previous three chapters we have studied the mathematical modelling of linear systems and the classical method of solving differential equations to determine and characterise the response of these systems to different types of inputs. In this chapter and the remaining chapters, we will study some of the special tools developed to aid the analysis and to give a better appreciation of the properties of physical systems. The Fourier series is one such powerful tool. Originally developed by John Baptiste Joseph Fourier (1768-1830), a French mathematician, for the study of heat conduction in metallic rods, it is now widely used in electrical, electronic and other systems for analysing non-sinusoidal periodic signals and for finding the response of linear systems to such signals.

In areas like control, communication, network analysis, power control, etc., the steady-state response of systems to sinusoidal signals is of crucial importance. This is because sinusoidal signals occur naturally in most of these systems. The system response to these signals reveals a number of very significant properties of linear systems. This response can be very easily calculated using the impedence concept and phasor diagrams.

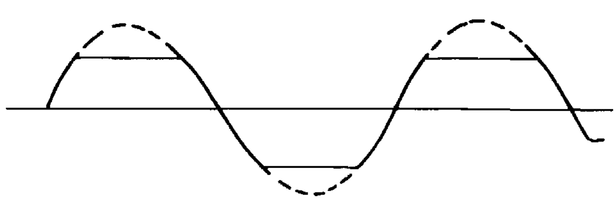
In addition to sinusoidal signals, one frequently encounters non-sinusoidal signals also in modern electrical and electronic systems. In the field of power electronics, where large amounts of power are controlled by solid-state devices, like thyristors, power transistors etc., the voltage and current waveforms are almost always non-sinusoidal. In electromechanical power converters, like transformers, alternators, motors, etc., magnetic saturation, electrical and mechanical imbalances, etc., make the current and voltage waveforms non-sinusoidal. In electronic systems, the inherent device non-linearities, saturation in amplifiers, effect of noise, etc., produce non-sinusoidal signals. Signal analysis techniques are now employed for studying diverse types of non-sinusoidal signals like those arising in speech analysis, seismic signals, atmospheric studies, bioelectric signals like ECG, EEG, EMG, etc. Increasing use of digital control and instrumentation systems has made response to rectangular pulses an important area of study. The techniques used for handling all such diverse types of signals are the Fourier series and the Fourier transforms. Some of the commonly encountered non-sinusoidal waveforms are shown in Fig. 4.1.



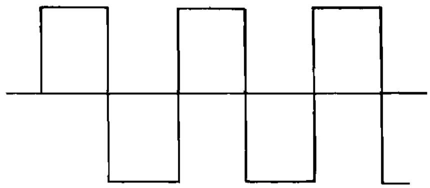
(a) Voltage waveform in a.c. power control with thyristor



(b) No load magnetising curve of a transformer



(c) Clipping action due to amplifier saturation



(d) Rectangular pulse train

Fig. 4.1 Some Non-sinusoidal Periodic Waveforms

4.1 Representation of a Periodic Function by Fourier Series

Let $f(t)$ be an arbitrary, though a well-behaved, periodic function of time with a period T. The significance of the term 'well-behaved' will be discussed later (Section 4.9). The requirement of this good behaviour on the part of the function is not very restrictive in the study of physical systems, because all physically occurring or physically realisable time signals are 'well-behaved'. Associated with the period T will be a frequency $\omega = 2\pi/T$. Fourier's theorem states that this arbitrary function can always be expressed as a sum of an infinite series as follows:

$$f(t) = \frac{a_0}{2} + a_1 \cos \omega t + a_2 \cos 2\omega t + \ldots + b_1 \sin \omega t + b_2 \sin 2\omega t + \ldots$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos n\omega t + b_n \sin n\omega t) . \quad (4.1)$$

Equation (4.1) is known as *the trigonometric form of the Fourier series.*

The frequency ω of the given function $f(t)$ is called the *fundamental frequency*. The Fourier series of eqn. (4.1) represents the given function $f(t)$ by the sum of sinusoidal components having frequencies which are integral multiples of the fundamental frequency. The component having frequency 2ω is called the *second harmonic*, 3ω the *third harmonic*, and so on. The component $a_0/2$ is the zero frequency, or the constant, or the d.c. component.

Like any sinusoid, the fundamental frequency component and each of the harmonics is completely specified by three factors: frequency, amplitude and phase angle. The frequencies of all the components are automatically fixed by the fundamental frequency of the given periodic function $f(t)$. Coefficients a_n and b_n, called the *Fourier coefficients*, determine the magnitude and the phase angle of the harmonic terms. This is more clearly brought out by writing the summation term in eqn. (4.1) as,

$$a_n \cos n\omega t + b_n \sin n\omega t = c_n \cos (n\omega t + \phi_n)$$

where $c_n = \sqrt{a_n^2 + b_n^2}$ is the amplitude of the nth component and $\phi_n = -\tan^{-1}(b_n / a_n)$ its phase angle. Also, $a_o = c_o$. Then eqn. (4.1) can be rewritten as,

$$f(t) = \frac{c_0}{2} + c_1 \cos(\omega t + \phi_1) + c_2 \cos(2\omega t + \phi_2) + \dots$$

$$= \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega t + \phi_n) \tag{4.2}$$

The next problem is to determine the values of the coefficients a_n and b_n. This problem is tackled as follows. Let us select the fundamental interval of the given function $f(t)$ as $-T/2$ to $+T/2$. Multiplying both sides of eqn. (4.1) by the factor $\cos m\omega t$, $m = 1, 2, 3, \dots$ and integrating from $-T/2$ to $+T/2$, we get,

$$\int_{-T/2}^{+T/2} f(t) \cos m\omega t \, dt = \frac{a_0}{2} \int_{-T/2}^{T/2} \cos m\omega t \, dt$$

$$+ \int_{-T/2}^{+T/2} \sum_{n=1}^{\infty} a_n \cos n\omega t \cos m\omega t \, dt.$$

$$+ \int_{-T/2}^{T/2} \sum_{n=1}^{\infty} b_n \sin n\omega t \cos m\omega t \, dt \tag{4.3}$$

The first and the third integrals on the right-hand side of eqn. (4.3) are zero for all values of m and n. The second integral is also zero for $m \neq n$. For $m = n$, its value is $(T/2)\, a_n$. Therefore,

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega t \, dt, \; n = 0, 1, 2, \dots \tag{4.4}$$

Note that with $n = 0$, eqn. (4.4) gives the value of the constant term a_0.

Similarly, multiplying both sides of eqn. (4.1) by $\sin m\omega t$, $m = 1, 2, 3, \dots$, and using procedures and arguments similar to those in the previous paragraph we get,

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega t \, dt, \; n = 1, 2, \dots \tag{4.5}$$

Once the coefficients a_n and b_n are determined from eqns. (4.4) and (4.5), the Fourier series representation of the given periodic function $f(t)$ is completely specified, either in the form (4.1) or in the alternative form (4.2). Let us now apply

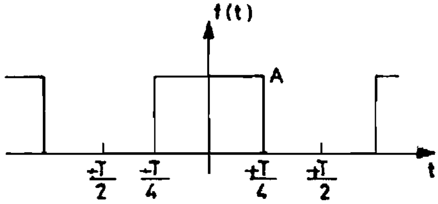


Fig. 4.2 Rectangular Waveform

these results for determining the Fourier series representation for a rectangular waveform shown in Fig. 4.2.

Example 4.1(a):— The waveform shown in Fig. 4.2 has an 'on' period = 'off' period = $T/2$ and is called a rectangular pulse train with 50% *duty cycle*. For this waveform, the Fourier coefficients a_n and b_n are determined as follows:

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos n\omega t\,dt$$

$$= \frac{2}{T}\int_{-T/4}^{T/4} A\cos n\omega t\,dt$$

$$= \frac{2A}{n\omega T}\Big[\sin n\omega t\Big]_{-T/4}^{T/4} = \frac{2A}{n\pi}\sin\frac{n\pi}{2},\quad n = 0, 1, 2, \ldots \tag{4.6}$$

Equation (4.6) gives,

$$a_1 = \frac{2A}{\pi},\; a_2 = 0,\; a_3 = -\frac{2A}{3\pi},\; a_4 = 0,\; a_5 = \frac{2A}{5\pi},\; \ldots$$

Substituting $n = 0$ directly in eqn. (4.6) makes $a_0 = 0/0$. Using L'Hospital's rule, we get,

$$a_0 = n \to 0 \frac{2A \cdot \frac{\pi}{2} \cdot \cos\frac{n\pi}{2}}{\pi} = A.$$

Therefore, the constant or the d.c. value $a_0/2 = A/2$. This average value $A/2$ could be determined from merely an inspection of the waveform of Fig. 4.2.

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin n\omega t\,dt$$

$$= \frac{2}{T}\int_{-T/4}^{T/4} A\sin n\omega t\,dt = 0.$$

Thus, the Fourier series for the given waveform becomes,

$$f(t) = \frac{A}{2} + \frac{2A}{\pi}\cos\omega t - \frac{2A}{3\pi}\cos 3\omega t + \frac{2A}{5\pi}\cos 5\omega t - \ldots \tag{4.7}$$

We note from eqn. (4.7) that in this series only cosine terms are present and all sine terms are zero. Further, it contains only odd harmonics: all even harmonics are zero.

Example 4.1(b):— Let us now shift the vertical axis of the waveform, as shown in Fig. 4.3. With this change,

$$a_n = \frac{2}{T} \int_0^{T/2} A \cos n\omega t \, dt$$

$$= \frac{2A}{n\omega t} \left[\sin n\,\omega t \right]_0^{T/2} = 0 \text{ for } n = 1, 2, 3, \ldots$$

For $n = 0$, application of L'Hospital's rule gives $a_o = A$.

$$b_n = \frac{2}{T} \int_0^{T/2} A \sin n\omega t \, dt = \frac{2A}{n\,\omega\,T} \left[-\cos n\,\omega\,t \right]_0^{T/2} = \frac{2A}{2\pi\,n} \left[1 - \cos n\,\pi \right].$$

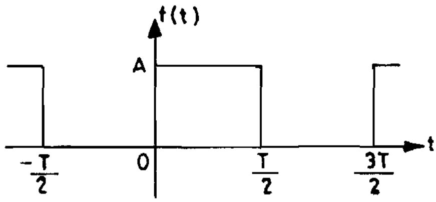


Fig. 4.3

The Fourier series representation becomes,

$$f(t) = \frac{A}{2} + \frac{2A}{\pi} \sin \omega t + \frac{2A}{3\,\pi} \sin 3\,\omega t + \frac{2A}{5\pi} \sin 5\omega t + \ldots \qquad (4.8)$$

Comparing eqns. (4.7) and (4.8), we note that the representation in (4.8) contains only sine terms while (4.7) contains only cosine terms. However, both (4.7) and (4.8) contain only odd harmonics. Thus, we may conclude that horizontal shifting of the vertical axis does not add any new frequencies: it merely alters the magnitudes of sine and cosine terms.

Example 4.1(c):— We now shift the horizontal axis as shown in Fig. 4.4. This shift will make the average or the d.c. value $a_0/2 = 0$. The Fourier series for Fig. 4.4 will be the same as eqn. (4.8), except that the constant term will be zero. We can thus conclude that shifting the horizontal axis only alters the average or the constant term; it has no effect on other components of the Fourier series.

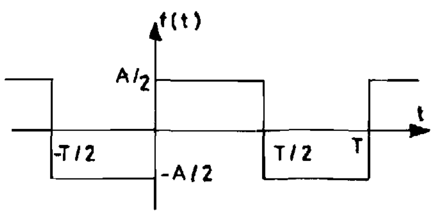


Fig. 4.4

As shown by eqns. (4.7) and (4.8), many of the Fourier coefficients may be zero in the Fourier series representation. In eqn. (4.7), $b_n = 0$ for all n and $a_n = 0$ for all even values of n. Similarly, in eqn. (4.8), $a_n = 0$ for all n and $b_n = 0$ for all even values of n. It would simplify matters if we could predict right at the beginning which of the Fourier coefficients would be zero. This aspect will be studied in the next section.

4.2 Symmetry Conditions

The waveform of Fig. 4.2 is symmetrical about the vertical axis, such that $f(t) = f(-t)$ for all values of t. Functions having this type of symmetry are called *even functions*. The Fourier series for such even functions should also contain only even functions. Now, cosine is an even function while sine is not. Hence, we conclude that the Fourier series expansion of an even function will contain only cosine terms. This is verified by example 4.1(a).

Functions which are anti-symmetrical about the vertical axis, such that $f(t) = -f(-t)$ for all t, are called *odd functions*. (The waveform shown in Fig. 4.4 is an odd function.) Since the sine function is an odd function while cosine is not, it is clear that the Fourier series for an odd function will contain only sine terms. This is verified by example 4.1(c).

We note that the Fourier series of example 4.1(b) (waveform of Fig. 4.3) also contains only sine terms and therefore it must be odd. However, this fact is not obvious from the waveform of Fig. 4.3. But, when the horizontal axis is shifted in Fig. 4.4 to remove the average or the d.c. value, the odd symmetry of the function is clearly brought out. Therefore, the property of evenness or oddness of a function should be explored only after removing the average value term by a suitable shift of the horizontal axis.

We further note that an even function, like that of Fig. 4.2, can be made odd by simply shifting the vertical axis by $T/4$. The choice of location of the vertical axis is arbitrary for periodic functions, which, in general, exist for all t, from $-\infty$ to $+\infty$. Therefore, the property of evenness or oddness of a function is not a fundamental property of the function but depends on where we choose to locate the $t = 0$ point.

The rectangular waveforms of Figs. 4.2, 4.3 and 4.4 have a more fundamental type of symmetry which is not affected by the shifts in horizontal and vertical axes. This is exemplified by the fact that the Fourier series expansions for all of them [eqns. (4.7) and (4.8)] contain only those frequency components which are odd multiples of ω. In other words, the expansions contain only *odd harmonics*: all even harmonic components have zero magnitude. To investigate this type of symmetry, let us start with Fig. 4.4, i.e. after the removal of the average value.

We note that the waveform is anti-symmetrical about the horizontal axis in every cycle, i.e., the positive and negative half cycles are equal in amplitude and opposite in sign. More precisely,

$$f\left(t \pm \frac{T}{2}\right) = -f(t). \tag{4.9}$$

Let us investigate the effect of this type of symmetry (also called *half-wave symmetry*) on the values of the Fourier coefficients.

The expression (4.4) for a_n can be broken into two parts as follows:

$$a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos n\omega t \; dt$$

$$= \frac{2}{T} \int_{-T/2}^{0} f(t) \cos n\omega t \; dt + \frac{2}{T} \int_{0}^{T/2} f(t) \cos n\omega t \; dt. \tag{4.10}$$

Let the variable in the first integral in eqn. (4.10) be changed from t to $(t - T/2)$. This does not cause any change in the value of the integral. With this change, the first integral in eqn. (4.10) becomes,

$$\int_{-T/2}^{0} f(t) \cos n\omega t \; dt = \int_{0}^{T/2} f\left(t - \frac{T}{2}\right) \cos n\omega \left(t - \frac{T}{2}\right) dt. \tag{4.11}$$

Now,

$$\cos n\omega \left(t - \frac{T}{2}\right) = \cos\left(n\omega t - n\omega \frac{T}{2}\right)$$

$$= \cos(n\omega t - n\pi)$$

$$= \cos n\omega t \cos n\pi.$$

Substituting this, and the symmetry condition (4.9), in eqn. (4.11) we get,

$$\int_{-T/2}^{0} f(t) \cos n\omega t \; dt = \int_{0}^{T/2} -f(t) \cos n\omega t \cos n\pi \; dt.$$

$$= -\cos n\pi \int_{0}^{T/2} f(t) \cos n\omega t \; dt. \tag{4.12}$$

Substituting (4.12) in (4.10) we get,

$$a_n = -\frac{2}{T} \cos n\pi \int_{0}^{T/2} f(t) \cos n\omega t \; dt + \frac{2}{T} \int_{0}^{T/2} f(t) \cos n\omega t \; dt$$

$$= \frac{2}{T}(1 - \cos n\pi) \int_{0}^{T/2} f(t) \cos n\omega t \; dt. \tag{4.13}$$

An entirely analogous derivation for b_n gives,

$$b_n = \frac{2}{T} (1 - \cos n\pi) \int_0^{T/2} f(t) \sin n\omega t \, dt. \tag{4.14}$$

When n = even, $\cos n\pi = 1$. Therefore, eqns. (4.13) and (4.14) give $a_n = b_n = 0$ for even values of n. Thus, we conclude that a function possessing the symmetry condition (4.9) will have only odd harmonics in its Fourier series expansion: coefficients of all even harmonics will be zero.

If, instead of the symmetry condition (4.9), a function satisfies the complementary symmetry condition,

$$f\left(t \pm \frac{T}{2} \right) = f(t) \tag{4.15}$$

then the multiplier term in eqns. (4.13) and (4.14) becomes $(1 + \cos n\pi)$. In that case a_n and b_n are zero for all odd values of n. We then conclude that the Fourier series expansion of a function satisfying the symmetry condition (4.15) will contain only even harmonics: the coefficients of all odd harmonics will be zero. However, this is not a very significant result because eqn. (4.15) merely means that the period of the function is half, i.e., $T/2$ instead of T. Naturally, all the frequency components then will be double of those corresponding to period T and hence will be even.

In all problems relating to Fourier series expansion of a given non-sinusoidal periodic waveform, the symmetries should first be noted (after removing the average term). Existence of symmetries considerably simplifies the evaluation of Fourier coefficients. After solving a few problems, it would be clear that if one symmetry exists the limit of integration in the expressions for a_n and b_n can be reduced to half the period with twice the function magnitude. With two symmetries, the limit becomes one-fourth of the period with four times the function magnitude.

If a given function has neither of the two symmetries, it can still be written as a sum of two functions, possessing complementary symmetries (see problem 4.3).

4.3 Convergence of Fourier Series

The amplitude of harmonic components of the rectangular waveform of example (4.1) was given by the expression $(2A)/n\pi$. Thus, the magnitude of a particular harmonic is inversely proportional to its order n. We also note that for the rectangular waveform the function is discontinuous, having two discontinuities in a period. In order to correlate the rapidity with which the magnitudes of harmonic components decrease as $n \to \infty$ and the smoothness property of a given function, we now take one more example—a triangular waveform.

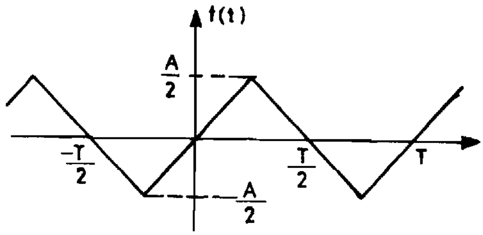


Fig. 4.5 Triangular Waveform

Example 4.2:— Let us determine the Fourier series expansion for the triangular waveform shown in Fig. 4.5. We note that here the function is continuous at every point, but its first derivative is discontinuous at two points in every cycle. Thus, the triangular waveform is more 'smooth' than the rectangular waveform.

We first note the symmetries of the waveform. Since $f(t) = -f(-t)$, it is an odd function. Therefore, its Fourier series will have only sine terms. Further $f[t \pm (T/2)] = -f(t)$. Therefore, the series will contain only odd harmonics. Also, because of the symmetry about the t axis, we conclude that the d.c. value will be zero. Hence, the form of the Fourier series will be,

$$f(t) = b_1 \sin \omega t + b_3 \sin 3\omega t + \ldots$$

The coefficients are given by,

$$b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin n\omega t \, dt = \frac{8}{T} \int_0^{T/4} f(t) \sin n\omega t \, dt.$$

In the range $0 \leq t \leq T/4$,

$$f(t) = \frac{A/2}{T/4} t = \frac{2A}{T} t.$$

Therefore,

$$b_n = \frac{8}{T} \int_0^{T/4} \frac{2A}{T} t \sin n\omega t \, dt$$

$$= \frac{16A}{T^2} \frac{1}{n\omega} \left[-t \cos n\omega t + \frac{\sin n\omega t}{n\omega} \right]_0^{T/4}$$

$$= \frac{16A}{T^2} \frac{1}{n^2 \left(\frac{2\pi}{T} \right)^2} \sin \frac{n\pi}{2} \left(\text{because } \omega = \frac{2\pi}{T} \right), \; n = 1, 3, 5, \ldots$$

$$= \frac{4A}{n^2 \pi^2} \sin \frac{n\pi}{2}$$

Thus,

$$f(t) = \frac{4A}{\pi^2}\left(\sin \omega t - \frac{\sin 3\,\omega t}{3^2} + \frac{\sin 5\,\omega t}{5^2} - \ldots\right) \tag{4.16}$$

Equation 4.16 reveals that the harmonic coefficients for the triangular wave are proportional to $1/n^2$. Thus, their magnitudes decay faster than the harmonic magnitudes for the square wave, as $n \to \infty$. For a waveform where both the function and its first derivative are continuous but the second derivative is discontinuous, the harmonic magnitudes decay still faster, being proportional to $1/n^3$. In general, if the function and its $(r-1)$ derivatives are continuous in the range $-T/2$ to $T/2$, but the rth derivative is discontinuous, then the Fourier coefficient for the nth harmonic is proportional to $1/n^{(r+1)}$. Thus, the 'smoother' the function is, the more rapidly the higher harmonic magnitudes decay. In fact, if the function and all its derivatives are continuous, its Fourier series converges most rapidly. Such a function will simply be a sinusoid and its Fourier series will contain just one term!

There is another type of convergence property which is of interest. The Fourier series expansion of a function contains an infinite number of terms. The sum of these terms, for any value of time, must equal the value of the function at that time, i.e., the Fourier series must converge at every point of $f(t)$. This is so for functions which are continuous. For functions having discontinuities, the Fourier series converges to the correct value of $f(t)$ at every continuous point; at the points of discontinuity it converges to a value equal to that of the midpoint of the discontinuity.

The problem of representation of any arbitrary function by trigonometric series had evoked considerable interest amongst mathematicians at the beginning of the nineteenth century. Other mathematicians of the time, like Euler, believed that such a representation was possible only for continuous functions. Fourier demonstrated in 1807 that it was possible for discontinuous functions as well. A problem then arose as to whether all functions could be represented by the Fourier series or whether there were some conditions to be satisfied by a function for having such a representation. This problem was solved by Dirichlet in 1837, who gave a set of sufficient conditions. These are called *Dirichlet conditions* and will be described in Section 4.9.

To get a feel of the idea of representing an arbitrary function by a trigonometric series, let us consider the representation of the rectangular waveform by eqn. (4.7). If we take only the first two terms of the series, i.e., the d.c. component and the fundamental component to represent the function, we get the result shown in Fig. 4.6(a). If we include the third harmonic also, we get the waveform shown in Fig. 4.6(b). Figure 4.6(c) show the effect of including terms up to the fifth harmonic. Thus, we see that as more and more terms are added to the series, its sum approaches closer and closer to the given function. In the limit, if we include an infinite number of terms, the sum of the Fourier series should exactly equal the

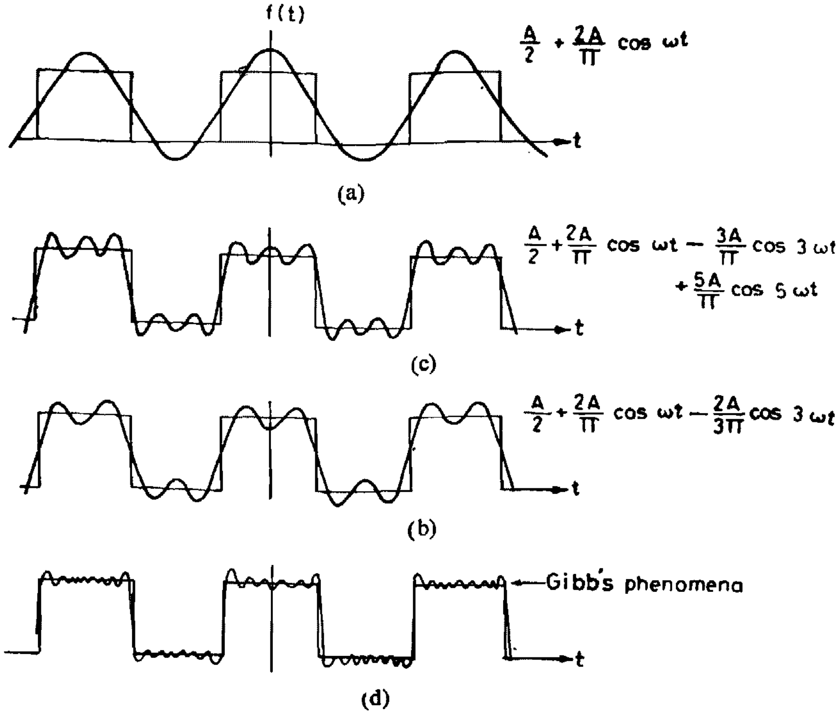


Fig. 4.6

given function at all instants of time. This is true for all instants of time except for those points where the function is discontinuous. In the case of a rectangular function, the sum of the Fourier series converges to $A/2$, the midpoint of discontinuity, or the average value of the function, as shown in Fig. 4.6(d).

Figure 4.6(d) points to another interesting fact regarding rectangular waveforms. The figure shows an overshoot on both the sides of the discontinuity. One would think that the magnitude of this overshoot would tend to zero as the number of terms in the Fourier series is increased to infinity. This is not so. In the limit, the overshoot converges to a constant amplitude of 18% of the average value in both the directions. This is called the *Gibb's phenomena*. It reveals that at a point of discontinuity, the given function cannot be approximated to a tolerance of better than ± 18%, even if very large number of terms are used in the Fourier series representation. Of course, a redeeming feature is that the time occupied by this overshoot does tend to zero and therefore in practice the overshoot may not have much effect. In situations where this error is not acceptable, other types of trigonometric series may be used to represent the function.

In approximating one function by another function, like in approximating the rectangular function by a partial sum of Fourier components in Fig. 4.6. a question arises as to what should be the criterion for judging the 'closeness' between the two functions? There are many possibilities: we may choose the maximum difference between the two functions or the average value of the difference over a period, or some similar function as this criterion. One commonly used criterion is the mean square error criterion. Suppose $f(t)$ is a given function, and $f'(t)$ an approximating function. Then $f(t)-f'(t) = e(t)$ is the difference or the error function. Its mean square value will be $\frac{1}{T}\int_{-t/2}^{+T/2} e^2(t)\,dt$. With this criterion for closeness, the approximating function which minimises this mean square value will be considered the closest to the given function. Now, the Fourier series approximation is only one of the several methods of approximating periodic functions. It has the important property of minimising the mean error square function. [It should be noted that the series given by eqn. (4.1) is a general trigonometric series: it becomes a Fourier series only when its coefficients a_n and b_n are calculated by eqns. (4.4) and (4.5).]

4.4 Exponential Form of Fourier Series

The harmonic components in the Fourier series can also be written in their exponential form using the relations,

$$\sin n\omega t = \frac{1}{2j}\left(\exp(jn\omega t) - \exp(-jn\omega t)\right)$$

$$\cos n\omega t = \frac{1}{2}\left(\exp(jn\omega t) + \exp(-jn\omega t)\right)$$

In terms of these exponential expressions, the trigonometric series (4.1) becomes,

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(\frac{a_n - jb_n}{2}\exp(jn\omega t) + \frac{a_n + jb_n}{2}\exp(-jn\omega t)\right) \tag{4.17}$$

The coefficients of the exponential terms are complex conjugates. Writing them as,

$$\alpha_n = \frac{a_n - jb_n}{2} \text{ and } \alpha_{-n} = \frac{a_n + jb_n}{2}$$

and for the sake of symmetry, $\alpha_0 = a_0/2$, eqn. (4.17) may be written compactly as,

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n \exp(jn\omega t) \tag{4.18}$$

The coefficient α_n in eqn. (4.18) is given by,

$$\alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t)\exp(-jn\omega t)\,dt,\ n = -\infty, \ldots -1, 0, 1, 2 \ldots +\infty \tag{4.19}$$

Equation (4.18), together with eqn. (4.19), gives the exponential *or the complex* form of the Fourier series. In comparison, the trigonometric form, i.e., eqn. (4.1), is called the *real form* of the Fourier series.

The Fourier coefficients α_n are complex numbers. They are related to the coefficients a_n and b_n of eqn. (4.1), or c_n of eqn. (4.2), by the relation,

$$|\alpha_n| = \frac{c_n}{2} = \frac{\sqrt{a_n^2 + b_n^2}}{2}$$

One of the advantages of this exponential form is that there is only one Fourier coefficient α_n to determine, as against two coefficients a_n and b_n in the real form. Further, this single complex coefficient contains both the magnitude and the phase angle information. For mathematical manipulations also the compact exponential form of eqn. (4.18) is often more useful than the real form of eqn. (4.1)

Example 4.3:—Find the complex Fourier coefficients for the half-wave rectified signal shown in Fig. 4.7.

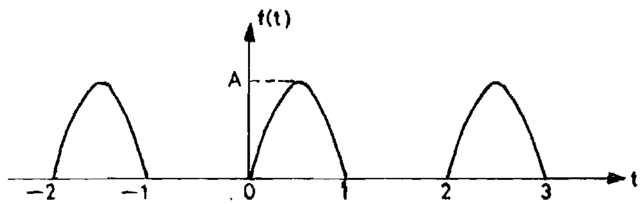


Fig. 4.7 Half-Wave Rectified Signal

Function $f(t)$ has a period $T = 2$ and frequency $\omega = 2\pi / T = \pi$. The mathematical description of the function over one period is given by,

$$f(t) = A \sin \omega t = A \sin \pi t = \frac{A}{2j}(\exp(j\pi t) - \exp(-j\pi t)) \text{ for } 0 \le t \le 1$$

$$= 0 \qquad \text{for} -1 \le t \le 0$$

Using eqn. (4.19) to determine α_n we get,

$$\alpha_n = \frac{1}{2}\int_0^1 A \sin \pi t \; \exp(-jn\pi t)\, dt$$

$$= \frac{A}{4j}\int_0^1 \{\exp(-j\pi(n-1)t) - \exp(-j\pi(n+1)t)\}\, dt$$

$$= \frac{A}{4j}\left[-\frac{\exp(-j\pi(n-1)t)}{j\pi(n-1)} + \frac{\exp(-j\pi(n+1)t)}{j\pi(n+1)}\right]_0^1$$

$$= \frac{A}{4j}\left[\frac{1-\exp(-j\pi(n-1))}{j\pi(n-1)} - \frac{1-\exp(-j\pi(n+1))}{j\pi(n+1)}\right]$$

$$= \frac{-A}{4\pi}\left[\frac{2-(n+1)\exp(-j\pi(n-1))+(n-1)\exp(-j\pi(n+1))}{n^2-1}\right]$$

$$= \begin{cases} \dfrac{A}{\pi} & \text{for } n = 0 \\ \mp\dfrac{jA}{4} & \text{for } n = \pm 1 \\ \dfrac{A}{\pi(1-n^2)} & \text{for } n = \text{even} \\ 0 & \text{for } n = \text{odd } (n \neq 1). \end{cases}$$

Thus, the exponential form of the Fourier series becomes,

$$f(t) = \frac{A}{\pi} - \frac{jA}{4}\exp(j\pi t) + \frac{jA}{4}\exp(-j\pi t) - \frac{A\exp(j2\pi t)}{3\pi} - \frac{A\exp(-j2\pi t)}{3\pi} - \ldots$$

4.5 Power and r.m.s. Values

In an electrical circuit if $v(t)$ and $i(t)$ are the voltage and current, then the average power consumed in the circuit is given by the relation,

$$W = \frac{1}{T}\int_0^T v(t)\, i(t)\, dt \tag{4.20}$$

When the voltage and current are sinusoidal and of the same frequency, with a phase angle ϕ between them, then elementary circuit theory derivation gives, $W = VI\cos\phi$, where V and I are the r.m.s. values of the voltage and the current. The ratio of the real power W to the apparent power VI is called the 'power factor' of the circuit and is equal to $\cos\phi$.

The question that arises next is: how to calculate the power and power factor in a circuit if either voltage or current or both are non-sinusoidal, periodic functions? In order to slightly generalise the problem, let us consider any two arbitrary periodic functions $f_1(t)$ and $f_2(t)$ and let us determine their average product:

$$W = \frac{1}{T}\int_0^T f_1(t) f_2(t)\, dt. \tag{4.21}$$

Expand both the functions into their Fourier series as,

$$f_1(t) = \sum_{n=-\infty}^{+\infty} \alpha_n e^{jn\omega t} \text{ and } f_2(t) = \sum_{m=-\infty}^{+\infty} \beta_m e^{jm\omega t}$$

Then, eqn. (4.21) becomes,

$$W = \frac{1}{T}\int_0^T \sum_{n,m=-\infty}^{+\infty} \alpha_n \beta_m e^{j(n+m)\omega t}\, dt\,.$$

Since the Fourier series for both f_1 and f_2 are convergent, their product will also be convergent. Hence, the integration of the product series can be carried out term by term, i.e.,

$$W = \frac{1}{T} \sum_{n, m = -\infty}^{+\infty} \alpha_n \beta_m \int_0^T e^{j(n+m)\omega t} dt$$

$$= \frac{1}{T} \sum_{n, m = -\infty}^{+\infty} \alpha_n \beta_m \left[\frac{e^{j(n+m)2\pi} - 1}{j(n+m)\omega} \right].$$

Since n and m are integers, the term $e^{j(n+m)2\pi}$ will have a value 1 for all n and m. Hence, the term inside the bracket will be zero for all n and m, except when $n = -m$, when it takes the form 0/0. Using L' Hospital's rule, this term becomes equal to $2\pi / \omega$ for $n = -m$. Therefore,

$$W = \frac{1}{T} \sum_{n = -\infty}^{+\infty} \alpha_n \beta_{-n} \frac{2\pi}{\omega}.$$

Replacing $(2\pi) / \omega$ by T we get,

$$W = \sum_{n = -\infty}^{+\infty} \alpha_n \beta_{-n} = \alpha_0 \beta_0 + \sum_{n=1}^{\infty} (\alpha_n \beta_{-n} + \alpha_{-n} \beta_n). \tag{4.22}$$

Both α_n and β_n are complex numbers. Therefore, they will have a magnitude and a phase angle. Let,

$$\alpha_n = |\alpha_n| \exp(j\theta_n) \text{ and } \beta_n = |\beta_n| \exp(j\theta'_n).$$

Then

$$\alpha_n \beta_{-n} + \alpha_{-n} \beta_n = |\alpha_n| |\beta_n| [\exp\{j(\theta_n - \theta'_n)\} + \exp\{-j(\theta_n - \theta'_n)\}].$$

Therefore,

$$W = \alpha_0 \beta_0 + 2 \sum_{n=1}^{\infty} |\alpha_n| |\beta_n| \cos(\theta_n - \theta'_n). \tag{4.23}$$

For electrical circuits. $f_1(t) = v(t)$ and $f_2(t) = i(t)$. Then $\alpha_0 = V_0$ and $\beta_0 = I_0$ are the d.c. values of voltage and current. For $n \neq 0$, $\alpha_n = (\sqrt{2}\, V_n) / 2$ and $\beta_n = (\sqrt{2}\, I_n) / 2$, where V_n is the r.m.s. value of the nth harmonic voltage and I_n the r.m.s value of the nth harmonic current. The nth harmonic component of voltage is $v_n(t) = V_{n\,max} \cos(n\omega t + \theta_n)$. In terms of α_n the nth harmonic voltage is,

$$\alpha_n \exp(jn\omega t) + \alpha_{-n} \exp(-jn\omega t) = |\alpha_n| \exp[j(n\omega t + \theta_n)] + |\alpha_n| \exp[-j(n\omega t + \theta_n)]$$

$$= 2|\alpha_n| \cos(n\omega t + \theta_n)$$

Hence, $V_{n\,max} = 2|\alpha_n|$.

The phase angle between the voltage and the current is $\phi_n = \theta_n - \theta'_n$. Therefore, the total power is given by,

$$W = V_0 I_0 + V_1 I_1 \cos\phi_1 + V_2 I_2 \cos\phi_2 + \ldots \tag{4.24}$$

Equation (4.24) gives the important result that the total power in the circuit is the sum of the d.c. power and the power due to each harmonic component. The power factor of each harmonic component will be different; being $\cos\phi_n$ for the nth harmonic.

Let us now consider the case $f_1(t) = f_2(t) = f(t)$. The average value of the product term can still be considered as a 'power' term, i.e.,

$$W = \frac{1}{T}\int_0^T f^2(t)\,dt. \tag{4.25}$$

From eqn. (4.23) we get,

$$W = \alpha_0^2 + 2\sum_{n=1}^{\infty} |\alpha_n|^2 \tag{4.26}$$

$$= W_0 + W_1 + W_2 + \ldots$$

Thus, W_n is the 'power' of the nth harmonic component. A plot of W_n versus $n\omega$ is called the *power spectrum* of a signal and is quite useful in the area of signal analysis.

If the function $f(t)$ in eqn. (4.25) is a voltage or a current, this equation defines the mean square value of the function. Designating the root mean square value (the r.m.s. value) of the voltage by V, and $f(t) = v(t)$, from eqns. (4.25) and (4.26) we get,

$$V^2 = \frac{1}{T}\int_0^T v^2(t)\,dt = \alpha_0^2 + 2\sum_{n=1}^{\infty} |\alpha_n|^2 = \alpha_0^2 + 2\sum_{n=1}^{\infty}\left(\frac{V_{n\,\max}}{2}\right)^2$$

$$= \alpha_0^2 + 2\sum_{n=1}^{\infty}\left(\frac{\sqrt{2}\,V_n}{2}\right)^2 = V_0^2 + (V_1^2 + V_2^2 + \ldots)$$

where V_0 is the d.c. component, V_1 the r.m.s. value of the fundamental, V_2 the r.m.s. value of the second harmonic, and so on.

4.6 Analysis with Fourier Series

In this section we consider some examples to show how Fourier series is used in the analysis of linear systems, particularly in the analysis of electrical circuits.

Example 4.4:—Figure 4.8(a) shows a waveform which is commonly encountered in thyristor a.c. voltage controllers. In every half cycle the 'firing angle' of the thyristor is delayed by an amount α. Thus, the conduction takes place for the interval $(\pi - \alpha)$. In the given waveform $\alpha = \pi/2$. Treating this waveform as the

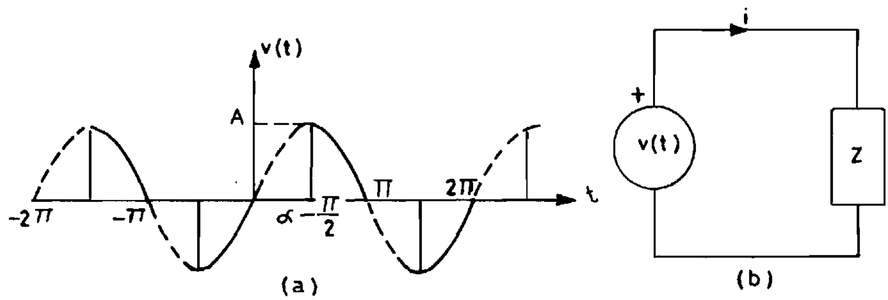


Fig. 4.8

input to an impedence Z, calculate the resulting output i, the current through the load.

Since the input is a non-sinusoidal periodic voltage, let us first expand it into its Fourier series. The waveform is symmetrical about the horizontal axis; hence, the average or the d.c. value will be zero. Further $f[t \pm (T/2)] = -f(t)$. Therefore, its Fourier series will contain only odd harmonics. Let us now determine the coefficients a_n and b_n. Before doing that, let us change the independent variable from t to ωt. The period then becomes $T = 2\pi$ and the integrals for a_n and b_n become,

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\omega t) \cos n\omega t \, d(\omega t)$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\omega t) \sin n \omega t \, d(\omega t) \,. \tag{4.27}$$

For this problem,

$$f(\omega t) = A \sin \omega t \text{ for } -\frac{\pi}{2} \le \omega t \le 0, \text{ and } +\frac{\pi}{2} \le \omega t \le \pi$$

and zero elsewhere in one period. Therefore,

$$a_n = \frac{1}{\pi} \int_{-\pi/2}^{0} A \sin \omega t \cos n\omega t \, d(\omega t) + \frac{1}{\pi} \int_{\pi/2}^{\pi} A \sin \omega t \cos n\omega t \, d(\omega t) \,. \tag{4.28}$$

For $n = 1$ the integral term becomes,

$$\int A \sin \omega t \cos n\omega t \, d(\omega t) = A \int \sin \omega t \cos \omega t \, d(\omega t)$$

$$= \frac{A}{2} \int \sin 2\omega t \, d(\omega t)$$

$$= -\frac{A}{4} \cos 2\omega t \,.$$

Putting the limits of integration as in eqn. (4.28) we get,

$$a_1 = -\frac{A}{4\pi}\Big[\cos 2\omega t\Big]_{-\pi/2}^{0} - \frac{A}{4\pi}\Big[\cos 2\omega t\Big]_{\pi/2}^{\pi}$$

$$= -\frac{A}{4\pi}\Big[(1+1)+(1+1)\Big] = -\frac{A}{\pi}.$$

For $n \neq 1$, the integral term becomes

$$\int A \sin \omega t \cos n\omega t \; d(\omega t) = \frac{A}{2}\int\Big[\sin(n+1)\,\omega t - \sin(n-1)\,\omega t\Big]\, d(\omega t)$$

$$= \frac{A}{2}\left[\frac{\cos(n-1)\,\omega t}{n-1} - \frac{\cos(n+1)\,\omega t}{n+1}\right].$$

Putting the limits of integration as in eqn. (4.28) we get,

$$\underset{(n\neq 1)}{a_n} = \frac{A}{2\pi}\left[\left\{\frac{\cos(n-1)\,\omega t}{n-1}\right\}_{-\pi/2}^{0} + \left\{\frac{\cos(n-1)\,\omega t}{n-1}\right\}_{\pi/2}^{\pi} - \left\{\frac{\cos(n+1)\,\omega t}{n+1}\right\}_{-\pi/2}^{0} - \left\{\frac{\cos(n+1)\,\omega t}{(n+1)}\right\}_{-\pi/2}^{\pi}\right].$$

Evaluating the above expression we get,

$$a_3 = \frac{A}{\pi};\; a_5 = -\frac{A}{3\pi};\; a_7 = +\frac{A}{3\pi};\; a_9 = -\frac{A}{5\pi};\; a_{11} = +\frac{A}{5\pi}$$

and so on.

Now, for the coefficients b_n we have,

$$b_n = \frac{1}{\pi}\int_{-\pi/2}^{0} A \sin \omega t \; \sin n\omega t \; d(\omega t) + \frac{1}{\pi}\int_{\pi/2}^{\pi} A \sin \omega t \; \sin n\omega t \; d(\omega t). \tag{4.29}$$

For $n = 1$, the integral term in eq. (4.29) becomes,

$$\int A \sin \omega t \; \sin n\omega t \; d(\omega t) = A\int \sin^2 \omega t \; d(\omega t)$$

$$= \frac{A}{2}\int (1 - \cos 2\omega t)\, d(\omega t)$$

$$= \frac{A}{2}\left(\omega t - \frac{\sin 2\omega t}{2}\right).$$

Putting the limits as in eqn. (4.29) we get,

$$b_1 = \frac{A}{2\pi}\left[\left\{\omega t - \frac{\sin 2\,\omega t}{2}\right\}_{-\pi/2}^{0} + \left\{\omega t - \frac{\sin 2\,\omega t}{2}\right\}_{\pi/2}^{\pi}\right] = \frac{A}{2}.$$

For $n \neq 1$, the integral term in eqn. (4.29) becomes,

$$\int A \sin \omega t \sin n \omega t \; d(\omega t) = \frac{A}{2} \int \left[\cos (n-1)\, \omega t - \cos (n+1)\, \omega t \right] d(\omega t)$$

$$= \frac{A}{2}\left[\frac{\sin (n-1)\, \omega t}{n-1} - \frac{\sin (n+1)\, \omega t}{n+1} \right].$$

Putting in the limits, as in eqn. (4.29) we get,

$$\underset{(n \neq 1)}{b_n} = \frac{A}{2\pi}\left[\left\{ \frac{\sin (n-1)\, \omega t}{n-1} \right\}_{-\pi/2}^{0} + \left\{ \frac{\sin (n-1)\, \omega t}{n-1} \right\}_{\pi/2}^{\pi} - \left\{ \frac{\sin (n+1)\, \omega t}{n+1} \right\}_{-\pi/2}^{0} - \left\{ \frac{\sin (n+1)\, \omega t}{(n+1)} \right\}_{\frac{\pi}{2}}^{\pi} \right].$$

From the above expression we have,

$$b_3 = \frac{A}{2\pi}\left[\left(\frac{\sin 2\, \omega t}{2} \right)_{-\pi/2}^{0} + \left(\frac{\sin 2\, \omega t}{2} \right)_{\pi/2}^{\pi} - \left(\frac{\sin 4\, \omega t}{4} \right)_{-\pi/2}^{0} - \left(\frac{\sin 4\, \omega t}{4} \right)_{\pi/2}^{\pi} \right] = 0.$$

In fact, for all higher odd values of n, all the terms will be identically zero. Hence, we get an interesting situation where only the fundamental frequency sine component is present.

Combining all the terms together we have,

$$v(t) = -\frac{A}{2} \cos \omega t + \frac{A}{2} \sin \omega t + \frac{A}{\pi} \cos 3\, \omega t - \frac{A}{3\pi} \cos 5\, \omega t$$

$$+ \frac{A}{3\pi} \cos 7\, \omega t - \frac{A}{5\pi} \cos 9\, \omega t + \frac{A}{5\pi} \cos 11\, \omega t - \ldots \tag{4.30}$$

Combining the first two terms we get,

$$-\frac{A}{\pi} \cos \omega t + \frac{A}{2} \sin \omega t = \sqrt{\left(\frac{A}{\pi}\right)^2 + \left(\frac{A}{2}\right)^2} \cos\left(\omega t - \frac{\pi}{2} - \phi \right)$$

where

$$\phi = \tan^{-1} \frac{A/\pi}{A/2} = \tan^{-1} \frac{2}{\pi}$$

To obtain numerical values, let us assume that $v(t)$ is 230 V, 50 Hz single phase supply voltage. Then $A = 230\sqrt{2}$ and eqn. (4.30) becomes,

$$v(t) = 192\cos(\omega t - 122.5^\circ) + \frac{230\sqrt{2}}{\pi}\left(\cos 3\omega t - \frac{\cos 5\omega t}{3} + \frac{\cos 7\omega t}{3} - \ldots\right)$$

$$= 192\cos(\omega t - 122.5^\circ) + 104\left(\cos 3\omega t - \frac{\cos 5\omega t}{3} + \frac{\cos 7\omega t}{3} - \ldots\right)$$

Let the load impedance Z be a series RLC circuit with $R = 10$ ohm, $L = 0.1$ H and $C = (10\pi^2)\ \mu F$. The value of impedance will be different for different frequency components. For the fundamental frequency $\omega = 2\pi f = 100\pi$ rad/sec,

$$Z(\omega) = R + j\omega L - \frac{j}{\omega C} = 10 + j\left(10\pi - \frac{1}{10 \times 100\pi \times \pi^2 \times 10^{-6}}\right)$$

$$= 10 + j(10\pi - 10\pi), \quad (\text{taking } \pi^2 = 10)$$

$$= 10 \text{ ohm.}$$

At the third harmonic frequency we have,

$$Z(3\omega) = 10 + j\left(30\pi - \frac{10\pi}{3}\right)$$

$$= 10 + j64 = 64.9\ \angle 81.1^\circ$$

$$Z(5\omega) = 10 + j\left(50\pi - \frac{10\pi}{5}\right)$$

$$= 10 + j151 = 151.1\ \angle 83.9^\circ$$

$$Z(7\omega) = 10 + j\left(70\pi - \frac{10\pi}{7}\right)$$

$$= 10 + j216 = 216\ \angle 84.1^\circ.$$

Since the system is linear, the principle of superposition applies. Therefore, the total current will be the sum of harmonic currents. We can calculate the harmonic currents individually by dividing the harmonic voltage by impedance at that frequency. Thus,

$$i_{\text{fundamental}} = \frac{192\angle -122.5^\circ}{10} = 19.2\ \angle -122.5^\circ$$

or

$$i_{\text{fundamental}} = 19.2\ (\cos\omega t - 122.5^\circ)$$

$$i_{\text{third harmonic}} = \frac{104}{64.9}\angle -81.1^\circ$$

$$= 1.6 \angle -81.1^\circ$$

$$i_{\text{fifth harmonic}} = \frac{104}{5 \times 151.1 \angle 83.9^\circ} = 0.33 \angle -83.9^\circ$$

$$i_{\text{seventh harmonic}} = \frac{104}{3 \times 216 \angle 84.1^\circ} = 0.16 \angle -84.1^\circ.$$

Thus, the expression for current is

$$i(t) = 19.2 \cos(\omega t - 122.5^\circ) + 1.6 \cos(3\omega t - 81.1^\circ)$$
$$- 0.33 \cos(5\,\omega t - 83.9^\circ) + 0.16 \cos(7\,\omega t - 84.1^\circ) + \ldots$$

It may be noted that the current harmonics decay much more rapidly than voltage harmonics because of the increased impedance at higher harmonic frequencies.

Example 4.5:— A 2 ohm resistive load is supplied from a full-wave rectifier connected to 230 V, 50 Hz single phase supply. Determine the average and the r.m.s. values of load current. Hence determine the proportion of d.c. power and a.c. power to the total power in the load. Investigate the effect of adding an inductance in series with the load.

Let us first determine the Fourier coefficients for the given waveform. For Fig. 4.9 we note that $v(t) = v(-t)$ and $v[t + (T/2)] = v(t)$. With these two symmetry conditions, the Fourier series will have only even, cosine terms. Also, the integration need be carried out only for the quarter period, i.e., from 0 to $\pi/2$, and the integral multiplied by 4. Thus, we have,

$$a_n = \frac{4A}{\pi} \int_0^{\pi/2} \sin \omega t \cos n\omega t \; d(\omega t), \quad n = 0, 2, 4, \ldots$$

$$= \frac{4A}{2\pi} \left[\frac{\cos(n-1)\,\omega t}{n-1} - \frac{\cos(n+1)\,\omega t}{n+1} \right]_0^{\pi/2}$$

$$= \frac{2A}{\pi} \left[\frac{\cos(n-1)(\pi/2)}{n-1} - \frac{\cos(n+1)(\pi/2)}{n+1} - \frac{1}{n-1} + \frac{1}{n+1} \right].$$

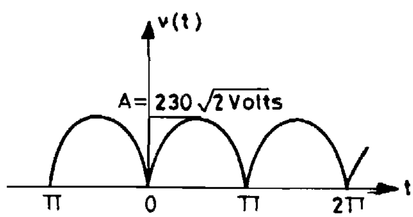


Fig. 4.9 Full Wave Rectifier Output

For all even values of n, the cosine terms inside the brackets will have a value 0. Therefore,

$$a_n = \frac{2A}{\pi}\left[\frac{1}{n+1} - \frac{1}{n-1}\right] = -\frac{4A}{\pi(n^2-1)} = -\frac{4\times 230\sqrt{2}}{\pi(n^2-1)} = \frac{-414}{(n^2-1)}.$$

The d.c. value $a_0/2 = +207$, $a_2 = -138$, $a_4 = -27.5$, $a_6 = -11.8$. Thus,

$$v(t) = 207 - 138\cos 2\omega t - 27.5\cos 4\omega t - 11.8\cos 6\omega t - \ldots$$

For the purely resistive load of 2 ohm, the load current is given by,

$$i(t) = 103.5 - 69\cos 2\omega t - 13.7\cos 4\omega t - 5.9\cos 6\omega t - \ldots \quad (4.31)$$

The average or the d.c. value of the load current is simply the first term, i.e., $I_{d.c.} = 103.5$ A.

The r.m.s. value of the load current is given by,

$$I_{r.m.s.} = [103.5^2 + \frac{1}{2}(69^2 + 13.7^2 + 5.9^2 + \ldots)]^{1/2} = 115 \text{ A}.$$

$$I_{a.c.} = [\frac{1}{2}(69^2 + 13.7^2 + 5.9^2 + \ldots)]^{1/2} = 50 \text{ A}.$$

$$\text{d.c. power} = R \times I_{d.c.}^2 = 2 \times 103.5^2 = 21.5 \text{ kW}.$$

$$\text{a.c. power} = R \times I_{a.c.}^2 = 2 \times 50^2 = 5 \text{ kW}.$$

$$\text{Total power} = 21.5 + 5 = 26.5 \text{ kW} = 2 \times 115^2.$$

$$\text{Ratio of d.c. power to total power} = \frac{21.5}{26.5} \times 100 = 81.1\ \%.$$

$$\text{Ratio of a.c. power to total power} = \frac{5}{26.5} \times 100 = 18.9\ \%.$$

Let us now investigate the effect of introducing an inductance $L = 3.18$ mH in series with the load. The load impedance for different harmonic components becomes,

$$Z_2 = \sqrt{R^2 + (2\pi f L)^2} \angle \cos^{-1}\frac{R}{Z_2} = 2.83 \angle 45°$$

$$Z_4 = 4.47 \angle 63.4°, Z_6 = 6.32 \angle 73.5°.$$

Thus,

$$i(t) = 103.5 - 48.8\cos(2\omega t - 45°) - 6.16\cos(4\omega t - 63.4°)$$

$$- 1.86\cos(6\omega t - 73.5°) - \ldots \quad (4.32)$$

Comparison of eqns. (4.31) and (4.32) shows that the effect of the series inductance is to reduce the magnitude of harmonics, specially those of higher harmonics. The d.c. value remains the same.

$$I_{\text{r.m.s.}} = [103.5^2 + \frac{1}{2}(48.8^2 + 6.16^2 + 1.86^2 + \ldots]^{1/2} = 108 \text{ A}.$$

$$I_{\text{a.c.}} = [\frac{1}{2}(48.8^2 + 6.16^2 + 1.86^2 + \ldots)]^{1/2} = \sqrt{1211} = 34.8 \text{ A}.$$

$$\text{d.c. power} = R \times I_{\text{d.c.}}^2 = 2 \times 103.5^2 = 21.5 \text{ kW}.$$

$$\text{a.c. power} = R \times I_{\text{a.c.}}^2 = 2 \times 34.8^2 = 2.42 \text{ kW}.$$

$$\text{Total power} = 23.92 \text{ kW}.$$

$$\text{Ratio of d.c. power to total power} = \frac{21.5}{23.9} \times 100 = 90\ \%.$$

$$\text{Ratio of a.c. power to total power} = \frac{2.42}{23.9} \times 100 = 10\ \%.$$

$$\text{Apparent power or volt-amperes} = 230 \times 108 = 24.8 \text{ kVA}$$

$$\text{Power factor of the load} = \frac{\text{real power (total power)}}{\text{apparent power}} = \frac{23.92}{24.8} = 0.96\,.$$

If the load contains an iron core, power will be dissipated as iron losses, in addition to the power consumed by the resistance of the load. These iron losses are dependent upon frequency. Hence higher harmonics, though small in magnitude, may still cause appreciable iron loss. The use of series inductance, called the 'filter choke', is helpful in reducing harmonics and hence these losses. However, this advantage must be compared with the cost of such chokes which must be capable of carrying full load current.

4.7 Graphical Method

The techniques developed so far can expand a given periodic waveform into its Fourier series, provided an analytical expression relating the function with time is known. The waveforms arising out of experimental work may have any odd shape and it may not be possible to express them analytically. Graphical methods may then have to be used to determine Fourier coefficients. In order to develop such a graphical procedure, let us have a second look at the determination of the Fourier coefficients for the rectangular waveform shown in Fig. 4.10. This is the same waveform as shown in Fig. 4.4.

The expression for coefficients b_n is given by,

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(\omega t) \sin n\omega t \, d(\omega t) = \frac{2}{\pi} \int_{0}^{\pi} \frac{A}{2} \sin n\omega t \, d(\omega t)\,.$$

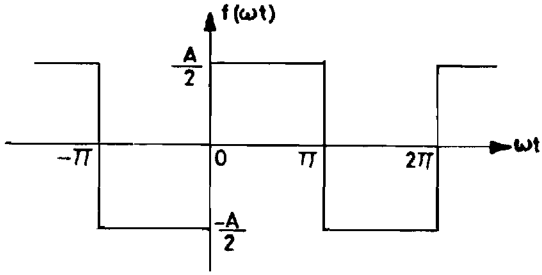


Fig. 4.10

Thus, the Fourier coefficient b_n is equal to the integral of the instantaneous product of the signal waveform and a unit sine wave of harmonic frequency. For the fundamental frequency the integral over 0 to π of the sine wave alone is given by,

$$\int_0^{\pi} \sin \omega t \, d(\omega t) = \Big[- \cos \omega t \Big]_0^{\pi} = 2.$$

For the second harmonic we have,

$$\int_0^{\pi} \sin 2\omega t \, d(\omega t) = \int_0^{\pi/2} \sin 2\omega t \, d(\omega t) + \int_{\pi/2}^{\pi} \sin 2\omega t \, d(\omega t)$$

$$= \frac{1}{2} \Big[- \cos 2\omega t \Big]_0^{\pi/2} + \frac{1}{2} \Big[- \cos 2\omega t \Big]_{\pi/2}^{\pi}$$

$$= \frac{1}{2} [+2-2] = 0.$$

For the third harmonic, we have

$$\int_0^{\pi} \sin 3\omega t \, d(\omega t) = \int_0^{\pi/3} \sin 3\omega t \, d(\omega t) + \int_{\pi/3}^{2\pi/3} \sin 3\omega t \, d(\omega t)$$

$$+ \int_{2\pi/3}^{\pi} \sin 3\omega t \, d(\omega t)$$

$$= - \frac{1}{3} \left[\Big(\cos 3\omega t \Big)_0^{\pi/3} + \Big(\cos 3\omega t \Big)_{\pi/3}^{2\pi/3} + \Big(\cos 3\omega t \Big)_{2\pi/3}^{\pi} \right]$$

$$= - \frac{1}{3} (-2+2-2) = 2/3.$$

Continuing in this way, the results up to fifth harmonic are shown in Fig. 4.11.

Since the magnitude of the signal waveform is constant $A/2$ over the period $T/2$, the coefficients b_n are given by,

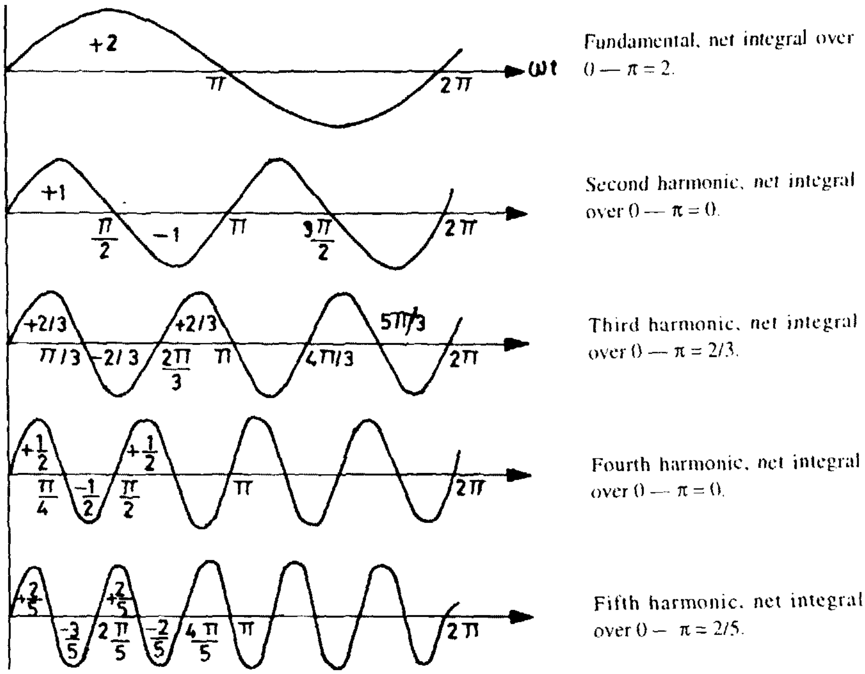


Fig. 4.11

$$b_1 = \frac{A}{\pi} \cdot 2 = \frac{2A}{\pi}$$

$$b_2 = \frac{A}{\pi} \cdot 0 = 0$$

$$b_3 = \frac{A}{\pi} \cdot \frac{2}{3} = \frac{2A}{3\pi}$$

$$b_4 = \frac{A}{\pi} \cdot 0 = 0$$

$$b_5 = \frac{A}{\pi} \cdot \frac{2}{5} = \frac{2A}{5\pi}.$$

These values are the same as those derived in eqn. (4.8).

Now consider a stepped waveform shown in Fig. 4.12. The values of the integrals of the unit sine wave over partial periods $(0 - \pi / 3)$, $(\pi / 3 - 2\pi / 3)$ and $(2\pi/3 - \pi)$, for the fundamental and other harmonics are shown in Fig. 4.12. Using these values, the integral of the product $f(t)\sin n\omega t$ over $0 - \pi$ will be as follows :

$$\text{Fundamental:} \quad \frac{A}{4} \cdot \frac{1}{2} + \frac{A}{2} \cdot 1 + \frac{A}{4} \cdot \frac{1}{2} = \frac{3A}{4}$$

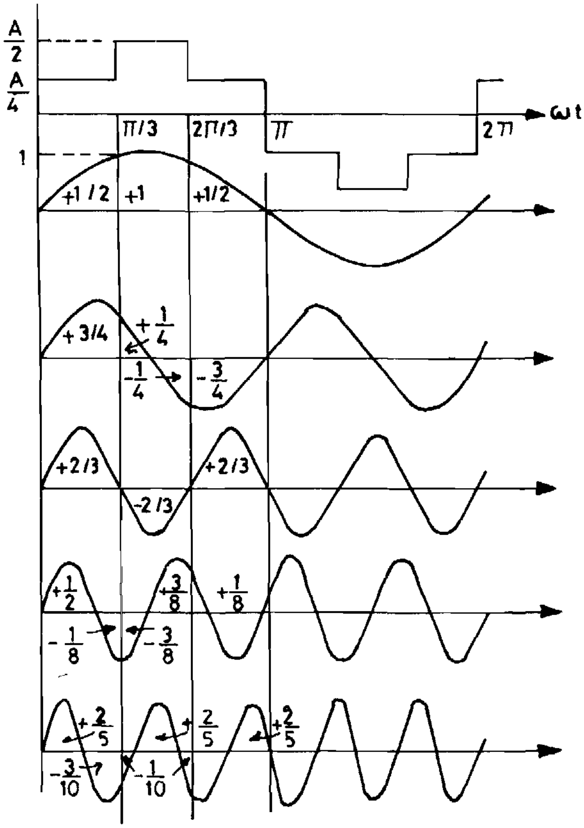


Fig. 4.12

Therefore, $b_1 = \frac{2}{\pi} \quad \frac{3A}{4} = \frac{3A}{2\pi}$

Second harmonic : $\frac{A}{4} \cdot \frac{3}{4} + \frac{A}{2} \cdot \frac{1}{4} - \frac{A}{2} \cdot \frac{1}{4} - \frac{A}{4} \cdot \frac{3}{4} = 0$

Therefore, $b_2 = 0.$

Third harmonic: $\frac{A}{4} \cdot \frac{2}{3} - \frac{A}{2} \cdot \frac{2}{3} + \frac{A}{4} \cdot \frac{2}{3} = 0$

Therefore, $b_3 = 0.$

Fourth harmonic: $\frac{A}{4} \cdot \frac{1}{2} - \frac{A}{4} \cdot \frac{1}{8} - \frac{A}{2} \quad \frac{3}{8} + \frac{A}{2} \quad \frac{3}{8}$

$$+ \frac{A}{4} \cdot \frac{1}{8} - \frac{A}{4} \cdot \frac{1}{2} = 0$$

Therefore, $b_4 = 0.$

Fifth harmonic: $\frac{A}{4} \cdot \frac{2}{5} - \frac{A}{4} \cdot \frac{3}{10} - \frac{A}{2} \cdot \frac{1}{10} + \frac{A}{2} \cdot \frac{2}{5} - \frac{A}{2} \cdot \frac{1}{10}$

$$- \frac{A}{4} \cdot \frac{3}{10} + \frac{A}{4} \cdot \frac{2}{5} = \frac{3A}{20}$$

Therefore, $b_5 = \frac{2}{\pi} \cdot \frac{3A}{20} = \frac{3A}{10\pi}$

and so on.

It is suggested that the reader should verify these results by calculating the values of these Fourier coefficients analytically, according to eqn. (4.5).

If the waveform has some arbitrary shape, it should first be approximated by a stepped waveform. This process, in effect, means that the interval $((0 - 2\pi)$ is divided into subintervals and in each subinterval, we approximate $f(t)$ by a constant equal to the value of $f(t)$ at the midpoint of the subinterval. The unit sine and cosine functions are also divided into the same number of subintervals. The values of their integrals over each subinterval are then determined. The integral of the product over the whole period 2π is the sum of integral products over the subintervals. This procedure thus gives a graphical or numerical method of determining the Fourier coefficients. A computer programme could be written to implement this procedure.

4.8 Frequency Spectrum

In sections 4.1 and 4.4, we developed two alternative but equivalent forms of Fourier series. The real form is given by eqn. (4.2) as,

$$f(t) = \frac{c_0}{2} + \sum_{n=1}^{\infty} c_n \cos(n\omega t + \phi_n)$$

and the complex form by eqn. (4.18) as,

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n \exp(jn\omega t) .$$

These representations show that the Fourier components have a magnitude $c_n = 2|\alpha_n|$ and a phase angle ϕ_n. A plot of c_n for different values of harmonic frequencies $n\omega$ is called the *amplitude spectrum* of the function $f(t)$. A plot of the phase angle ϕ_n versus $n\omega$ is called the *phase spectrum*. These two plots together are called the *Fourier spectrum* or the *frequency spectrum*. In many problems, the phase angle information is not so important, and only the amplitude spectrum is of interest.

Equation (4.2) or eqn. (4.18) gives the Fourier series for periodic functions, where the harmonic frequencies are integral multiples of the fundamental frequen-

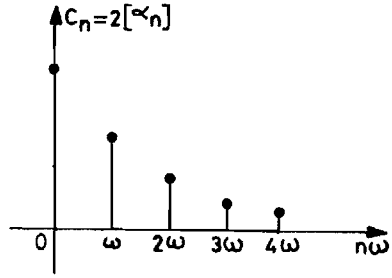


Fig. 4.13 A Line Spectrum

cy. Therefore, the independent variable $n\omega$ in the spectrum plot will be defined only for discrete values of $n = 0, 1, 2, 3$ ... That is, $n\omega$ will be a discrete variable. The amplitude spectrum will look like that given in Fig. 4.13 and will consist of a set of lines spaced at intervals of ω. Such a spectrum is called a *line spectrum*.

In the exponential form of eqn. (4.18), α_n is a complex number. We can also consider α_n as a function of discrete variable $n\omega$ and express it as $\alpha(n\omega)$. This function is defined by eqn. (4.19). Rewriting eqns. (4.18) and (4.19) together, we get the pair,

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n \exp(jn\omega t)$$

$$\alpha(n\omega) = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \exp(-jn\omega t)\, dt$$

Here, $n = -\infty, \ldots, -1, 0, 1, 2, \ldots, \infty$. Either of the two expressions in the above pair is a complete description of the given function. In eqn. (4.18) the independent variable is time t and, hence, $f(t)$ is the signal description in the *time domain*. In eqn. (4.19), the independent variable is the discrete variable $n\omega$ and, hence, $\alpha(n\omega)$ is the signal description in the *frequency domain*. The frequency spectrum plot contains all the information which $f(t)$ contains. In situations where the relative magnitude of frequency components is more important than the distribution of the signal magnitude as a function of time, e.g., in voice signals, frequency-modulated communication signals, EEG records, etc., the frequency domain description is more useful than the time domain description.

Example 4.6:—Determine and plot the Fourier spectrum for the square waveform shown in Fig. 4.4.

From eqn. (4.19) we get,

$$\alpha_n = \alpha(n\omega) = \frac{1}{T}\left[\int_{-T/2}^{0}\left(-\frac{A}{2}\right)e^{-jn\omega t}\,dt + \int_{0}^{T/2}\frac{A}{2}\,e^{-jn\omega t}\,dt\right].$$

For $n = 0$ we have,

$$\alpha_0 = \frac{1}{T}\left[-\frac{A}{2}\left(0+\frac{T}{2}\right)+\frac{A}{2}\left(\frac{T}{2}+0\right)\right] = 0.$$

For n = 1, 2, 3, ..., replacing T by $(2\pi)/\omega$ we get,

$$\alpha(n\omega) = \frac{\omega}{2\pi}\cdot\frac{A}{2}\left[-\int_{-\pi/\omega}^{0} e^{-jn\omega t}\,dt + \int_{0}^{\pi/\omega} e^{-jn\omega t}\,dt\right]$$

$$= \frac{\omega}{2\pi}\cdot\frac{A}{2}\left[\left(\frac{-e^{-jn\omega t}}{-jn\omega}\right)_{-\pi/\omega}^{0} + \left\{\frac{e^{-jn\omega t}}{-jn\omega}\right\}_{0}^{\pi/\omega}\right]$$

$$= \frac{A}{2}\,\frac{(1-\cos n\pi)}{jn\pi} \tag{4.33}$$

For all even values of n, $\cos n\pi = 1$ and, hence, $\alpha(n\omega) = 0$ for n = even. For all odd values of n, $\cos n\pi = -1$ and, hence, $\alpha(n\omega) = A/(jn\pi)$ for n = odd.

The amplitude and the phase angle are given by,

$$2|\alpha_n| = \frac{2A}{n\pi} \text{ and } \phi_n = -\frac{\pi}{2} \text{ for } n = \text{odd}.$$

Figure 4.14 shows the amplitude spectrum. The envelope of this line spectrum is simply a rectangular hyperbola, with an equation, $y = 1/x$. This figure places directly in evidence the convergence property discussed in Section 4.3.

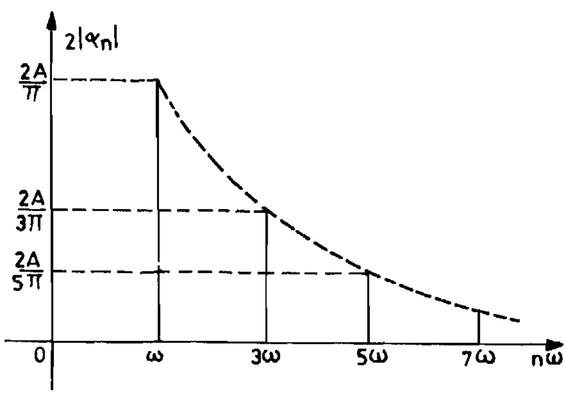


Fig. 4.14 Amplitude Spectrum of the Square Wave

Example 4.7:— Let us now determine the Fourier spectrum for the rectangular waveform with amplitude A, repetition period T, and a variable pulse period Tp, as shown in Fig. 4.15. It is also called the 'gate' function. This type of waveform is encountered in many signal-processing and power-processing applications. For example, in a battery-driven vehicle for controlling the speed of a d.c. motor, the battery voltage is 'chopped' into this rectangular waveform, using solid state choppers. The average value of voltage and current (and hence power) supplied to the motor can be varied by either varying T_p with fixed T, or varying T with fixed T_p. The former is called the *pulse width modulation* and the latter, *pulse frequency modulation.* It is apparent that this form of power control will produce harmonic voltages and currents in the motor which will contribute to the heating of the motor but will not contribute to its torque output. It is, therefore, necessary to know the magnitudes of these harmonics, i.e., the frequency spectrum, so that either arrangement may be made to reduce the harmonics or alternatively the motor specially designed to work satisfactorily even in the presence of these harmonics.

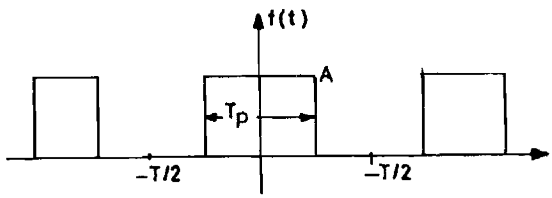


Fig. 4.15 A Rectangular Pulse Train with Variable Pulse-Period

From Fig. 4.15 we get,

$$\alpha_n = \frac{1}{T} \int_{-T_p/2}^{T_p/2} A\, e^{-jn\omega t}\, dt.$$

For $n = 0$,

$$\alpha_0 = \frac{1}{T} \Big[At \Big]_{-T_p/2}^{T_p/2} = \frac{AT_p}{T}$$

For $n = 1, 2, 3, \ldots$, we have,

$$\alpha_n = \frac{A}{T} \left[\frac{e^{-jn\omega t}}{-jn\omega} \right]_{-T_p/2}^{T_p/2}$$

$$= \frac{A}{jn\omega T} \left[e^{jn\omega T_p/2} - e^{-jn\omega T_p/2} \right]$$

$$= \frac{A}{n\pi} \sin \left(n\pi \frac{T_p}{T} \right)$$

Let us rewrite the final expression as,

$$\alpha_n = \frac{AT_p}{T} \frac{\sin \dfrac{n\pi T_p}{T}}{n\pi \dfrac{T_p}{T}} \tag{4.34}$$

The ratio T_p/T is called *duty ratio* of the waveform. A duty ratio of 0.5 gives a square waveform. Let us denote it by the symbol δ, i.e., $T_p/T = \delta$. Then eqn. (4.34) becomes,

$$\alpha_n = A\delta \frac{\sin n\delta\pi}{n\delta\pi}. \tag{4.35}$$

Equation (4.35) shows that the variation of α_n with n is given by a function of the form $(\sin m)/m$, $m = n\delta\pi$. Such a function is quite important in signal analysis and is therefore given a special symbol, 'Sinc m'. Thus, Sinc $m = (\sin m)/m$. It is illustrated in Fig. 4.16 and has the property that Sinc $m = 1$ for $m = 0$; for other integral values of $n\delta$, its value is zero.

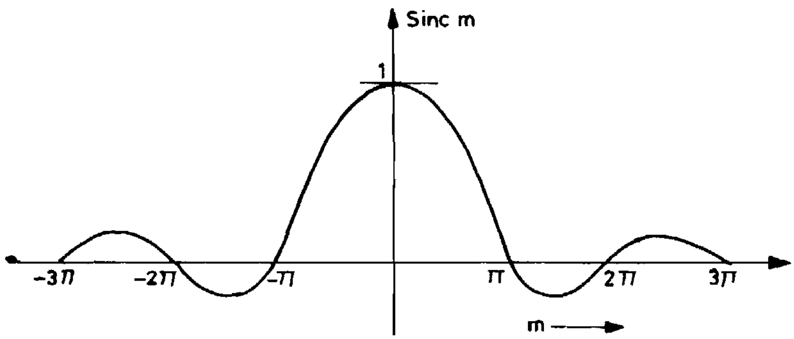


Fig. 4.16 Sinc Function

It oscillates with decreasing peak amplitudes as m increases; however, it is not a 'periodic' function. It is available as a tabulated mathematical function.

In terms of the Sinc function, eqn. (4.35) becomes $\alpha_n = A\ \delta$ Sinc $n\delta\pi$. The task of plotting the amplitude spectrum becomes considerably simplified by the use of Sinc function.

Let $T_p = 1/10$ and $T = 1/2$. Then $\delta = 1/5$ and $\alpha_n = A/5$ Sinc $(n\pi)/5$. The fundamental frequency of the signal is $\omega = (2\pi)/T = 4\pi$. Therefore, the lines in the line spectrum will be spaced 4π apart. The actual spectrum plot is shown in Fig. 4.17.

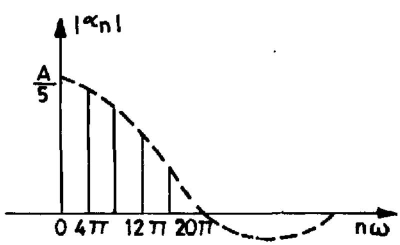


Fig. 4.17 Spectrum of a Rectangular Waveform

It shows that the d.c. value, corresponding to $n = 0$, has the highest amplitude. We also note that the values of the 5th, 10th and other 5th multiple frequencies have zero magnitude. This is a general result, i.e., all harmonics, whose frequencies are $1/\delta$ or its integral multiples, will have zero magnitude. This fact is made use of in suppressing selected harmonics, particularly 3rd, 5th etc., in chopper power converters.

Now suppose that the repetition period T is increased from $\frac{1}{2}$ to 1, keeping T_p fixed. Then $\delta = 1/10$: The fundamental frequency of the signal now reduces to $\omega = 2\pi$. Thus, the lines in the spectrum will be spaced only 2π apart. In other words, the spectrum becomes 'dense'. The maximum magnitude also reduces to $A/10$. If T is further increased, the spectrum will become denser still. In the limit as $T \rightarrow \infty$, the spacing between the lines tends to zero and the line spectrum becomes a continuous spectrum. The variable $\alpha_n = \alpha(n\omega)$ changes from a discrete variable to a continuous variable as $\omega \rightarrow 0$. In that case, the summation sign in eqn. (4.18) changes to an integration sign. Thus, the Fourier series becomes a *Fourier integral.* The pair of eqns. (4.18) and (4.19) then define a transform pair called the *Fourier transform.*

What is the physical significance of $T \rightarrow \infty$? It means that the repetition rate of the function is infinite. In other words, the function does not repeat itself, i.e., it is a *non-periodic* function. Such non-periodic functions in the time domain are represented by Fourier transform in the frequency domain. The study of Fourier transforms will be the subject matter of the next chapter.

4.9 Concluding Comments

Dirichlet conditions: In the beginning of section 4.1, it was implicitly stated that every periodic function may not necessarily have a Fourier series representation. In order to be representable by Fourier series, the function must be 'well behaved'. This requirement of good behaviour means that the periodic function $f(t)$ must satisfy a set of conditions, called the *Dirichlet conditions*, given as follows:

(1) $\int_{-T/2}^{T/2} |f(t)|\, dt < \infty.$

(2) $f(t)$ can have at the most only a finite number of discontinuities in one period.

(3) $f(t)$ can have at the most only a finite number of maxima and minima in one period.

These conditions are important from the point of view of mathematics. However, all physically realisable signals always satisfy these conditions and hence they do not impose any restrictions in the Fourier series analysis of physical systems.

Orthogonal functions: The determination of the Fourier coefficients a_n in eqn. (4.1) is possible by eqn. (4.4) only because the cosine function possesses the property,

$$\int_{-\pi}^{\pi} \cos n\omega t \cos m\omega t \; d(\omega t) = 0, \text{ for all } m \neq n.$$

Similarly b_n can be determined by eqn. (4.5) only because,

$$\int_{-\pi}^{\pi} \sin n\omega t \sin m\omega t \; d(\omega t) = 0, \text{ for all } m \neq n.$$

Functions possessing this property are called a set of *orthogonal functions*. Similarly, the exponential form of the Fourier series, eqn. (4.18), is possible because the set of functions $e^{jn\omega t}$, $n = 0, \pm 1, \pm 2, \ldots$ constitute an orthogonal set. An arbitrary function can be represented not only in terms of functions $e^{jn\omega t}$, but also in terms of any set of orthogonal functions. Some of the other useful orthogonal sets are as follows:

Legendre polynomials: $$P_n(t) = \frac{1}{2^n\, n!}\frac{d^n}{dt^n} \cdot (t^2 - 1)^n$$

Laguerre polynomials: $$L_n(t) = e^t \frac{d^n}{dt^n}\,(t^n e^{-t}) \,.$$

GLOSSARY

Harmonics: The frequency components with integral multiples of the fundamental frequency are called *harmonics*. If the fundamental frequency is ω, the component with frequency 2ω in its Fourier series representation is called the *second harmonic*, the component with frequency 3ω, the *third harmonic*, and so on.

Even Function. If $f(t) = f(-t)$, it is called an *even function*. An even function is symmetrical about the vertical axis and its Fourier series expansion contains only cosine terms.

Odd Function: If $f(t) = -f(-t)$, it is called an *odd function*. It is anti-symmetrical about the vertical axis and its Fourier series expansion contains only sine terms.

Fourier Spectrum or Frequency Spectrum: A plot of the amplitude of harmonic components versus frequency is called the *amplitude spectrum* and a plot of the phase angle against frequency, the *phase spectrum*. The two together are called the *Fourier or the frequency spectrum*.

Line Spectrum: When the frequency is a discrete variable, as in the case of periodic functions, the spectrum consists of a set of equally spaced lines. Such a spectrum is called a *line spectrum*.

Time Domain and Frequency Domain: When a function $f(t)$ is described as a function of time as the independent variable, it is called a description in *the time domain*. When the same function is described as a function of frequency (discrete variable $n\omega$ for periodic signals and continuous variable ω for non-periodic signal), it is called a *description in the frequency domain*.

Duty Ratio: For a periodic function, which has a non-zero value for a part of the period T_{on} and a zero value for the remaining part T_{off}, the ratio of the on-period to the total period $T = T_{on} + T_{off}$ is called the *duty ratio* δ. That is, $\delta = T_{on}/T$.

PROBLEMS

4.1. Sketch the waveform of a periodic function whose Fourier series has:

(i) Only odd harmonics but both sine and cosine terms are present.

(ii) Only even harmonics but both sine and cosine terms are present.

(iii) Only sine terms but all harmonics are present.

(iv) Only cosine terms but all harmonics are present.

4.2. Find the Fourier series for the waveform shown in Fig. 4.18, both in the real form and in the complex form.

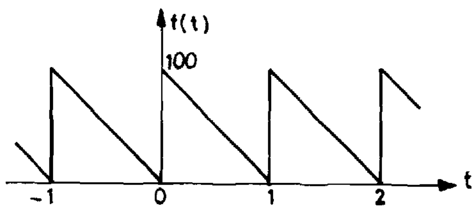


Fig. 4.18

4.3. Represent the waveform shown in Fig. 4.19 as a sum of an even function and an odd function. Determine its Fourier series using the even and the odd components.

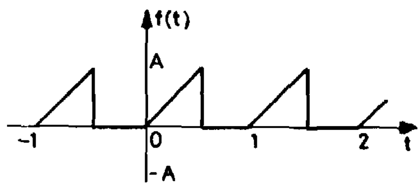


Fig. 4.19

4.4. The Fourier series of a voltage waveform is given by

$$v(t) = 10 + \frac{100}{\pi} \sin 100\,\pi t + \frac{100}{3\,\pi} \sin 300\,\pi\, t + \frac{100}{5\,\pi} \sin 500\,\pi\, t + \ldots$$

(i) Sketch the waveform, noting its important characteristics.

(ii) Give the reasoning followed in arriving at this waveform.

(iii) What would be the power developed by $v(t)$ in a 1-ohm resistor ?

(iv) What would be the power factor of the load when this voltage is applied across a series *RLC* circuit with R = 100 ohm, L = 1 H and $C = 100/\pi^2\ \mu F$?

4.5. Sketch a periodic function which is continuous with continuous first derivative but discontinuous second derivative. Determine its Fourier coefficients and show that their rate of convergence as $n \to \infty$ is proportional to $1/n^3$.

4.6. One method of a.c. power control using thyristors is called the *on-off method.* In this method, mains voltage is applied across the load for some time t_{on} and is then switched off for time t_{off}. Load power is controlled by varying the ratio of t_{on} and t_{off}. In a simplified situation assume that the 50 Hz mains supply is on for one cycle, off for the next cycle and on again. Determine the Fourier series for this voltage waveform.

4.7. Sketch the waveform of a periodic function from the following information:

(i) Its Fourier series contains only cosine terms.

(ii) All even harmonics are absent.

(iii) The period of the function is 4 seconds.

(iv) During the interval 0 to 1 second, the function is a ramp with slope +1.

4.8. Figure 4.20 shows the output voltage v_0 of a three-phase rectifier circuit. Determine the frequency spectrum of v_0. Compare it with the spectrum for full-wave single phase rectifier of Example 4.5.

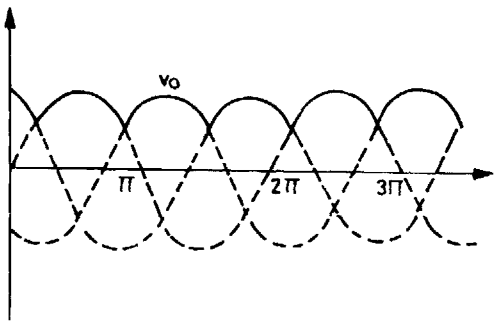


Fig. 4.20

4.9. A rectangular voltage waveform of magnitude 10 V, duty ratio 75% and frequency 50 Hz is applied across a resistance of 1 ohm in series with an inductance of 100 mH. Determine the steady-state current in the circuit. Also find the power and power factor of the load circuit.

4.10. The differential equation model of a first order system, with input $x(t)$ and output $y(t)$ is given by $dy/dt+2y = x$. Determine the steady-state response of the system if the input x is the triangular waveform of Fig. 4.5.

CHAPTER 5

Fourier Transform

LEARNING OBJECTIVES

After a study of this chapter you should be able to:

(i) determine the Fourier transform and the spectrum of non-periodic functions of time;

(ii) determine the impulse response from the differential equation model of a system;

(iii) determine the response of linear systems in the time domain, using the convolution integral; and

(iv) determine the response of linear systems in the frequency domain, using the Fourier transform techniques.

The previous chapter described how to represent a periodic signal in terms of its frequency components with the help of Fourier series. Such a representation permits determination of the response of a linear system to non-sinusoidal, periodic excitations in terms of its response to sinusoidal input. We know take up the next logical question of how to represent a non-periodic signal in terms of its frequency components. Such a representation is obtained with the help of the Fourier transform. It gives us a method of determining the response of a linear system to non-periodic excitations in terms of its response to the sinusoidal signal. Fourier transform techniques are particularly useful in the analysis of filters, communication systems, sampling processes and other areas of signal analysis.

5.1 From Fourier Series to Fourier Transform

In the development of Fourier series we used the symbol ω to denote the fundamental frequency of a periodic signal, its harmonics being $2\,\omega$, $3\,\omega$, $4\,\omega$, . . . In the study of Fourier transform, we will have to distinguish between the fundamental frequency of a given signal and the general frequency variable.

Therefore, we shall use the symbol ω_0 to denote the fundamental frequency of a particular periodic signal and reserve ω for the general frequency variable. The exponential form of Fourier series for a given periodic signal of fundamental frequency ω_0 can then be written as,

$$f(t) = \sum_{n=-\infty}^{\infty} \alpha_n \exp(jn\omega_0 t) \text{ and } \alpha_n = \frac{1}{T}\int_{-T/2}^{T/2} f(t) \exp(-jn\omega_0 t)\, dt \quad (5.1)$$

where $\omega_0 = 2\pi/T$, T being the fundamental period of the given waveform.

To extend the Fourier methods from periodic to non-periodic signals, let us start with a periodic rectangular pulse train of repetition period T and pulse period T_p. Its frequency spectrum will be a line spectrum with ω_0 as the spacing between the lines. The signal and its spectrum are shown in Figs. 5.1(a) and (b). Now, let the repetition period T be doubled, to $2T$, keeping T_p fixed. The fundamental frequency will be halved to $\omega_0/2$. The spectral lines will now be more closely spaced, with spacing of $\omega_0/2$, as shown in in Fig. 5.1(d).

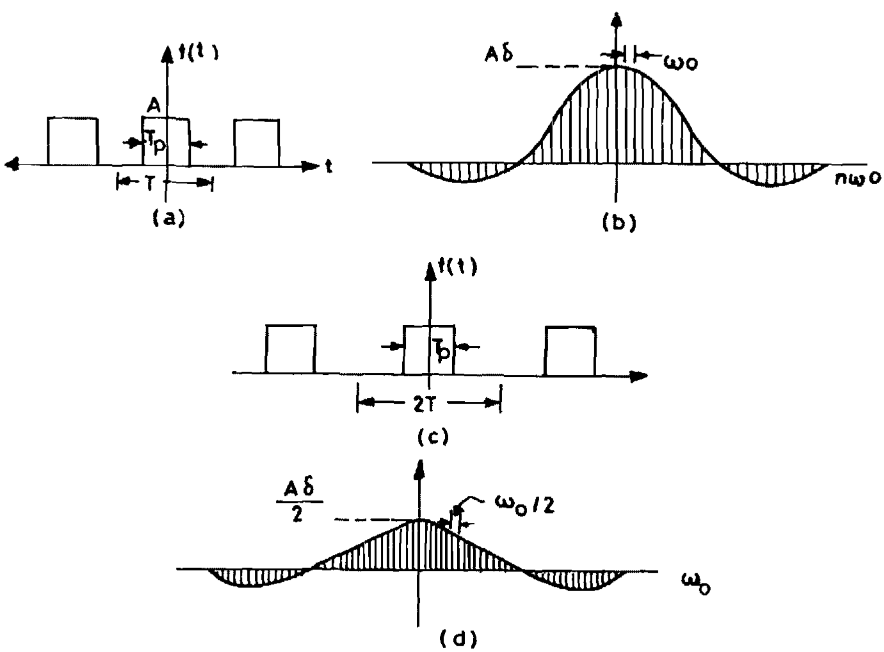


Fig. 5.1 The Effect of Increasing Repetition Period on the Line Spectrum of a Rectangular Pulse Train

Let this process of increasing T (keeping T_p fixed) continue still further till T tends to infinity. The pulse then repeats itself after infinite time, i.e., the pulse does not repeat itself. We have thus moved from a periodic signal to a non-periodic signal. In the process, the duty ratio $\delta = T_p/T \rightarrow 0$ and the fundamental frequency $\omega_0 = 2\pi/T$ becomes smaller and smaller, i.e., $\omega_0 \rightarrow d\omega$. Then, we can write $1/T = d\omega/2\pi$. The spectrum becomes more and more dense with the gap between the spectral lines reduced to $d\omega$. Thus, the frequency variable $n\omega_0$ of the line spectrum becomes a continuous variable ω. With these changes introduced in the Fourier series of eqn. (5.1), the coefficient α_n becomes a continuous function of ω. Or,

$$\alpha_n = \alpha(\omega) = \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt. \tag{5.2}$$

Substituting this value of α_n in the expression for $f(t)$ we get,

$$f(t) = \sum_{n=-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \right] e^{jn\omega_0 t}$$

With the variable $n\omega_0$ changing to a continuous variable ω, the summation sign in the above expression must change into an integration and we get,

$$f(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \right] e^{j\omega t}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \right] e^{j\omega t}\, d\omega. \tag{5.3}$$

The quantity inside the brackets in eqn. (5.3) will be a function of ω. Let us call it $F(\omega)$. Then, eqn. (5.3) becomes,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega. \tag{5.4}$$

Equation (5.4) is called the *Fourier integral*. Thus, the Fourier series of eqn. (5.1) turns into the Fourier integral of eqn. (5.4) as we go from a periodic signal to a non-periodic signal.

For a moment let us go back to the line spectrum of Fig. 5.1. As the lines come closer, the magnitude of the spectrum reduces. In the limit as $T \rightarrow \infty$, $\alpha_n \rightarrow 0$. Therefore, instead of α_n let us define $\alpha_n T$ as the spectrum. This quantity will be finite and, from eqn. (5.1) and definition of $F(\omega)$, will be equal to $F(\omega)$ in the limit. $F(\omega)$, defined as,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt \tag{5.5}$$

is a continuous function of ω and is called the *Fourier transform* of $f(t)$. It permits us to represent a function of time as a function of frequency. Correspondingly, relation (5.4) is called the *inverse Fourier transform* of $F(\omega)$ and gives a method of characterising a function of frequency as a function of time.

Relations (5.4) and (5.5) are called *the Fourier transform pair.* This pair is represented by $f(t) \leftrightarrow F(\omega)$, meaning that $F(\omega)$ is the Fourier transform of $f(t)$ and $f(t)$ is the inverse Fourier transform of $F(\omega)$. This relationship is also written as,

$$F(\omega) = \mathcal{F} f(t) \text{ and } f(t) = \mathcal{F}^{1} F(\omega).$$

Both the *time domain* description $f(t)$ and the *frequency domain* description $F(\omega)$ of the signal contain all the information about the signal and are equivalent, alternate methods of representing a given signal.

The quantity $F(\omega)$ is a complex function of ω. Graphically it can be plotted in two parts: (i) $|F(\omega)|$ vs. ω and (ii) $\angle F(\omega)$ vs. ω. The first one is the amplitude plot and the second, the phase plot. Together they constitute the *frequency spectrum* of the given signal. In this plot, it is necessary to consider both the positive and the negative frequencies, i.e., the range of ω is from $-\infty$ to $+\infty$. This is because a physically existing signal $f(t)$ can be realised from $e^{j\omega t}$ only if we consider both the positive and negative values of ω. For example, a physical signal

$$\cos \omega t = \frac{e^{j\omega t} + e^{-j\omega t}}{2}.$$

The conditions for the existence of the Fourier transform of a function $f(t)$ are similar to those for the existence of the Fourier series for periodic functions. In the present case, the conditions will be specified over the range $-\infty < t < +\infty$ instead of $-T/2 < t < +T/2$, which was the range for the periodic functions. These conditions may be stated as follows·

(1) The function must be absolutely integrable in the sense,

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty.$$

(2) It may have at the most a finite number of discontinuities in the range $-\infty < t < +\infty$.

(3) It may have at the most a finite number of maxima and minima in the range $-\infty < t < +\infty$.

The first condition rules out any periodic function or functions like step, ramp, etc., which exist for all the time. If the function has infinities, they must be integrable. A physically realisable function will always satisfy the other two conditions.

Let us re-examine the association between the line spectrum α_n of the Fourier series and the continuous spectrum $F(\omega)$ of the Fourier transform. The complex quantity α_n gives the magnitude and the phase angle of the nth harmonic component of a periodic signal and is defined as,

$$\alpha_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) \exp\,(-jn\omega_0 t)\; dt$$

where $\omega_0 = 2\pi / T$ is the frequency of the periodic signal $f(t)$. When we go over to the non-periodic case by letting $T \to \infty$, the discrete variable $\alpha_n = \alpha\,(n\,\omega_0)$ becomes a continuous variable $\alpha\,(\omega)$. But because of multiplication by $1/T$ in the above expression, the magnitude $|\alpha\,(\omega)| \to 0$. However the product $\alpha\,(\omega).\,T$ does approach a finite non-zero value and this value is $F(\omega)$. That is,

$$\lim_{T\to\infty} \alpha\,(\omega)\,.\,T = \int_{-\infty}^{\infty} f(t)\; e^{-j\omega t}\; dt = F\,(\omega),$$

where $f(t\,)$ satisfies Dirichlet's conditions given above. Although $F(\omega)$ is now called *the (continuous) spectrum of* $f\,(t)$, $|F(\omega)|$ does not give the amplitude of the frequency component ω.

To assign physical significance to $F(\omega)$ we now write it as,

$$F(\omega) = \lim_{T\to\infty} \alpha\,(\omega)\;.\;T = 2\pi \cdot \frac{\alpha\,(\omega)}{d\,\omega}$$

Since this expression involves division by $d\omega$, we may call $F\,(\omega)$ as the *frequency density* of the signal $f(t)$. Another descriptive term used for $F\,(\omega)$ is *relative frequency distribution*. This term arises because although the absolute amplitude of each frequency component is infinitely.small, their relative magnitudes are displayed by a plot of $|F\,(\omega)|$ vs. ω. We shall contend ourselves with the term *spectrum* for $F(\omega)$.

5.2 Fourier Transforms of Some Common Signals

A rectangular pulse: A rectangular pulse of amplitude A and duration $-T/2$ to $+T/2$ is shown in Fig. 5.2(a). Since this signal is encountered quite frequently it may be given a special symbol, $f_p(t)$. To derive its Fourier transform, we proceed as follows:

$$\begin{aligned} F\,(\omega) &= \int_{-\infty}^{\infty} f_p\,(t)\; e^{-j\omega t}\; dt \\ &= \int_{-T/2}^{T/2} A\; e^{-j\omega t}\; dt \\ &= \frac{A}{-j\,\omega} \left[\, e^{-j\omega T/2} - e^{+j\omega T/2} \,\right] \\ &= AT\; \frac{\sin\,(\,\omega\, T/2)}{(\omega\, T/2)} \end{aligned}$$

$$= AT \text{ Sinc } \frac{\omega T}{2} \cdot \tag{5.6}$$

Although the bandwidth of the spectrum shown in Fig. 5.2(b) is theoretically infinite, for most practical purposes it may be assumed that the significant frequency components lie in the range $-2\pi/T \leq \omega \leq +2\pi/T$. Thus, if a channel has a bandwidth $4\pi/T$, it will be able to transmit the rectangular pulse more or less faithfully. Now, if the pulse width T is reduced, the channel bandwidth, required for its faithful transmission will be more. For example, a pulse of 1 millisecond duration will require a channel of bandwidth 13 kHz while a 1 microsecond pulse will require 13 MHz. Realising a large bandwidth channel is more complex and expensive. Hence the statement, that it is more difficult to transmit (without distortion) a short duration pulse than a long duration pulse.

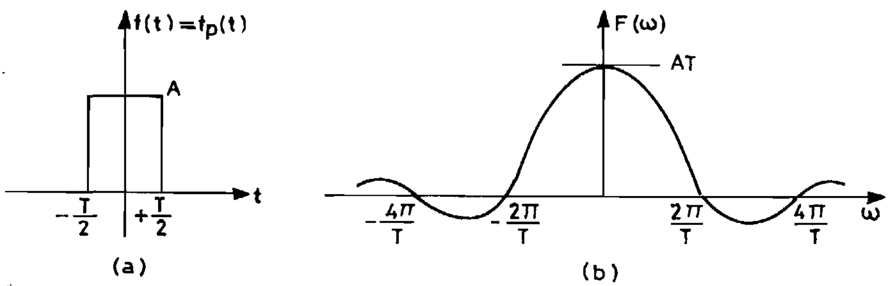


Fig. 5.2 Rectangular Pulse and Its Frequency Spectrum

If the rectangular pulse of Fig. 5.2(a) is shifted to the right by an amount $T/2$, so that its duration becomes 0 to T, its Fourier transform becomes,

$$F(\omega) = \int_0^T A\, e^{-j\omega t}\, dt.$$

Working it out we get,

$$F(\omega) = AT \frac{\sin(\omega T)/2}{(\omega T)/2} \exp(-j\omega T/2)$$

$$= AT \text{ Sinc } \frac{\omega T}{2}\, e^{-j\omega T/2}. \tag{5.7}$$

Equation (5.7) shows that the amplitude spectrum remains the same but a phase lag, equal to $\angle\, \omega T/2$, is added.

This is a general property of the Fourier transform, i.e., if the time function is shifted by an amount t_0, its Fourier transform is multiplied by $\exp(-j\omega t_0)$. That is, if $f(t) \leftrightarrow F(\omega)$, then $f(t - t_0) \leftrightarrow F(\omega)\exp(-j\omega t_0)$. This is called the *shifting property* of the Fourier transform.

A triangular pulse: Instead of determining the transform directly, we first note that the triangular pulse of Fig. 5.3(a) can be obtained by integrating a pair of square pulses shown in Fig. 5.3(b). We therefore enquire what happens to its Fourier transform when a time function is differentiated or integrated.

Let $f(t) \leftrightarrow F(\omega)$. Then,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega.$$

Differentiating both sides of this equation with respect to t, we get

$$\frac{df(t)}{dt} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[j\omega\, F(\omega) \right] e^{j\omega t}\, d\omega.$$

Therefore, we conclude that when a time function is differentiated, its Fourier transform is multiplied by $j\omega$. If the function is differentiated n times, the transform is multiplied by $(j\omega)^n$, i.e.,

$$\frac{d^n f(t)}{dt^n} \leftrightarrow (j\omega)^n\, F(\omega)\,. \tag{5.8}$$

Conversely, if the function is integrated its Fourier transform gets divided by $j\omega$, i.e.,

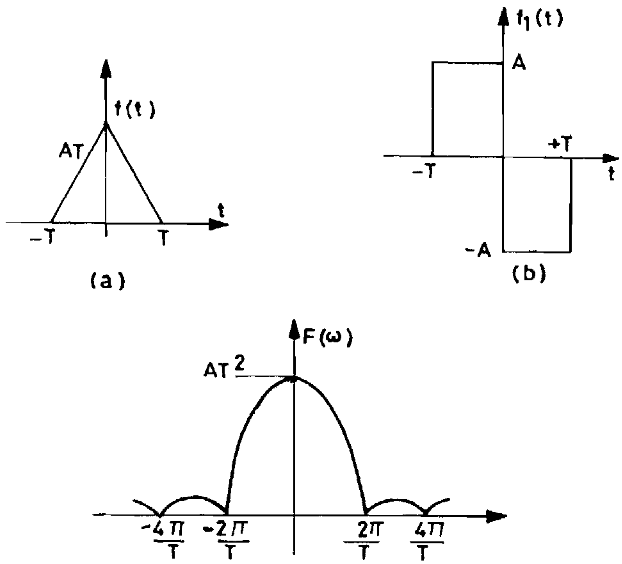


Fig. 5.3 (a) A triangular Pulse; (b) A Pair of Square Pulses; and (c) Spectrum of Triangular Pulse

$$\int f(t)\, dt \leftrightarrow \frac{F(\omega)}{j\omega} \tag{5.9}$$

Coming back to the triangular pulse, we note that the function $f_1(t)$ of Fig. 5.3(b) can be written as,

$$f_1(t) = f_p(t + T/2) - f_p(t - T/2).$$

Recalling that $f_p(t) \leftrightarrow AT$ Sinc $(\omega T/2)$ and using the shifting property, the Fourier transform of $f_1(t)$ is given by,

$$F_1(\omega) = AT\left[\text{Sinc}\ \frac{\omega T}{2}\ e^{j\omega T/2} - \text{Sinc}\ \frac{\omega T}{2}\ e^{-j\omega T/2}\right]$$

$$= AT\ \text{Sinc}\ \frac{\omega T}{2}\left[e^{j\omega T/2} - e^{-j\omega T/2}\right]$$

Now, since $f(t)$ of Fig. 5.3(a) is the integral of $f_1(t)$ of Fig. 5.3(b), using the integration property of eqn. (5.9), we get the Fourier transformation $F(\omega)$ of $f(t)$ as,

$$F(\omega) = \frac{F_1(\omega)}{j\omega}$$

$$= AT\ \text{Sinc}\ \frac{\omega T}{2}\ \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{j\omega}$$

$$= AT\ \text{Sinc}\ \frac{\omega T}{2}\ \frac{e^{j\omega T/2} - e^{-j\omega T/2}}{2j\,\frac{\omega T}{2}} \cdot T$$

$$= AT^2\ \text{Sinc}^2\ \frac{\omega T}{2} \tag{5.10}$$

The spectrum of the triangular wave is shown in Fig. 5.3(c).

The decaying exponential: The function $f(t) = A\, e^{-at}\, u(t)$ is shown in Fig. 5.4(a). Its Fourier transform $F(\omega)$ is given by,

$$F(\omega) = \int_0^{\infty} A\, e^{-at}\, u(t)\, \exp(-j\omega t)\, dt$$

$$= A\left[\frac{\exp(-(a+j\omega)t)}{-(a+j\omega)}\right]_0^{\infty}$$

$$= \frac{A}{a+j\omega}. \tag{5.11}$$

The magnitude and the phase angle of $F(\omega)$ are given by,

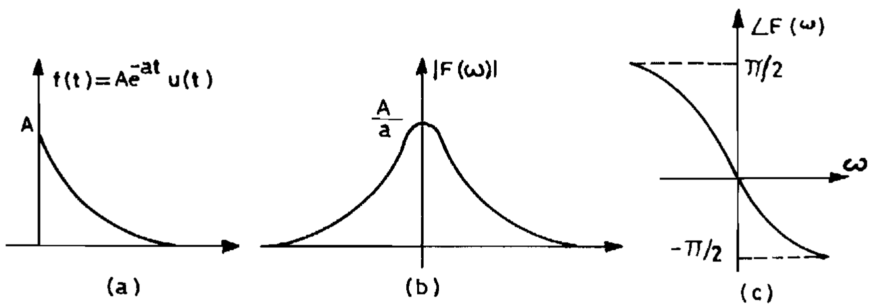


Fig. 5.4 The Decaying Exponential and Its Spectrum

$$|F(\omega)| = \frac{A}{\sqrt{a^2 + \omega^2}}$$

and $$\angle F(\omega) = \tan^{-1}(-\omega / a) .$$

The magnitude and phase spectra are shown in Figs. 5.4(b) and (c).

The Gausian function: The Gaussian function $f(t) = \exp(-\pi t^2)$ shown in Fig. 5.5(a) occurs frequently in the study of stochastic systems. The normalised form, with $f(0) = 1$, has the property that the area under the curve is unity, i.e.,

$$\int_{-\infty}^{\infty} f(t)\, dt = \int_{-\infty}^{\infty} \exp(-\pi t^2)\, dt = 1 .$$

The Fourier transform of the Gaussian function is given by,

$$F(\omega) = \int_{-\infty}^{\infty} \exp(-\pi t^2) \exp(-j\omega t)\, dt.$$

Replacing frequency ω (rad/sec) by h (Hz), $\omega = 2\pi h$, we have,

$$F(h) = \int_{-\infty}^{\infty} \exp(-\pi t^2) \exp(-j 2 \pi h t)\, dt$$

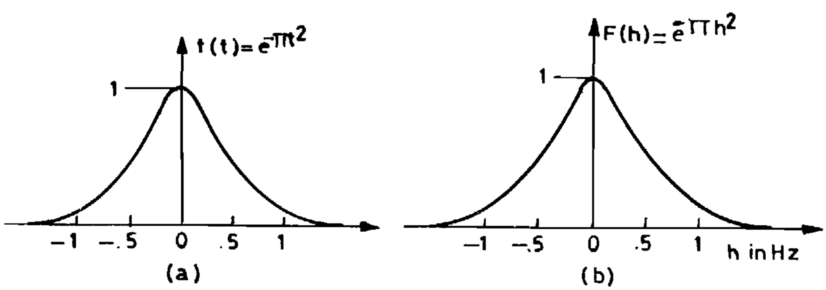


Fig. 5.5 The Gaussian Function and Its Spectrum

$$= \int_{-\infty}^{\infty} \exp\left[-\pi\,(t^2 + 2\,jht)\right]\,dt.$$

To complete the square in the exponent, multiply the function by $\exp[-\pi\,(jh)^{-2}] = \exp\,(\pi h^2)$, i.e.,

$$\exp\,(\pi\,h^2)\;F(h) = \int_{-\infty}^{\infty} \exp\,[-\pi\,\{t^2 + 2\,jht + (jh)^2\}]\,dt$$

or
$$F(h) = \exp\,(-\pi\,h^2) \int_{-\infty}^{\infty} \exp\,[\,-\pi\,(t + j\,h\,)^2\,]\,dt.$$

Let the variable of integration be changed from t to $\tau = t + jh$. Then $dt = d\tau$. However, the limits of integration remain the same for all finite h. Hence,

$$F(h) = \exp\,(-\pi\,h^2) \int_{-\infty}^{\infty} \exp\,(-\pi\,\tau^2)\,d\,\tau.$$

The integrand on the right-hand side is also a Gaussian function and hence the value of the integral is unity. Thus,

$$F(h) = \exp\,(-\pi\,h^2)\;. \tag{5.12}$$

Thus, the Gaussian function has the interesting property that its Fourier transform is also Gaussian as a function of frequency h in Hz.

Table of Fourier transform pairs: Starting with a few basic transforms and using the fundamental theorems and properties we can build up a fairly comprehensive table of Fourier transform pairs. Such a table (Table 5.1) is particularly useful for finding the inverse transforms.

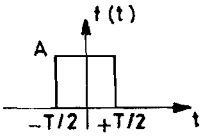
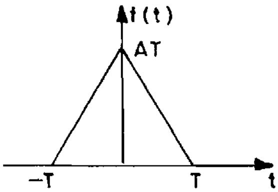
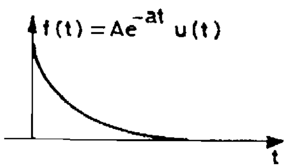
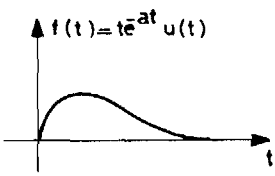
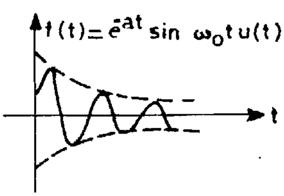
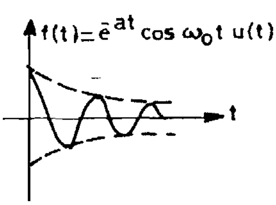
5.3 The Impulse Function

One of the key concepts in the analysis of linear systems is the impulse response of a system. An impulse is a singularity function and has properties which are somewhat different from those of ordinary functions of time. Ordinary functions are specified by defining the relation for obtaining the value of the function (a number) for every value of its argument in some specified field. But an impulse is defined by the effect it produces when interacting with an ordinary function of time. A unit impulse $\delta\,(t)$, occurring at $t = 0$, is defined by the property,

$$\int_{-\infty}^{\infty} \delta\,(t)\;f\,(t)\;dt = f\,(0) \tag{5.13}$$

where $f\,(t)$ is an ordinary function of time, continuous at the origin. A quantity defined in this special way, by the effect it produces, is called a *generalised function* or a *distribution*. Some of the properties of the generalised function $\delta(t)$, defined by eqn. (5.13), are now given.

Table 5.1 Fourier Transform Pairs

	Time function	Fourier transform
1	 Rectangular pulse or gate function	$AT\ \text{Sinc}\ \frac{\omega T}{2}$
2.	 Triangular pulse	$AT^2\ \text{Sinc}^2\ \frac{\omega T}{2}$
3.	 Decaying exponential	$\frac{A}{a+j\omega}$
4.		$\frac{1}{(a+j\omega)^2}$
5.		$\frac{\omega_0}{(a+j\omega)^2+\omega_0^2}$
6.		$\frac{a+j\omega}{(a+j\omega)^2+\omega_0^2}$

1. *Scaling*: This is given as,

$$\int_{-\infty}^{\infty} k\,\delta(t)\,f(t) = k f(0) \tag{5.14}$$

where k is a real constant and $k\,\delta(t)$ an impulse of strength k, occurring at $t = 0$.

2. *Shifting*: This is written as,

$$\int_{-\infty}^{\infty} \delta(t - t_1)\,f(t) = f(t_1) \tag{5.15}$$

provided $f(t)$ is continuous at $t = t_1$. Here, $\delta(t - t_1)$ is a unit impulse at $t = t_l$. From the above relations we can say that the effect of an impulse at $t = t_l$, interacting with a function $f(t)$, is to 'pluck out' the value of the function at $t = t_1$.

3. *Equivalence*: Let $\phi(t)$ be an ordinary function with the following properties: (i) it is infinitely smooth, i.e., it possesses derivatives of all orders, and (ii) the value of the function is zero outside some finite interval of time, i.e., for $t = -\infty$ and $t = +\infty$, $\phi(t) = 0$. Such a function is called a *test function*. Then, two generalised functions $g_1(t)$ and $g_2(t)$ are equivalent, i.e., $g_1(t) = g_2(t)$ if,

$$\int_{-\infty}^{\infty} g_1(t)\,\phi(t)\,dt = \int_{-\infty}^{\infty} g_2(t)\,\phi(t)\,dt. \tag{5.16}$$

Now let us define another generalised function $u(t)$ by the following effect it produces on any test function $\phi(t)$:

$$\int_{-\infty}^{\infty} u(t)\,\phi(t)\,dt = \int_{0}^{\infty} \phi(t)\,dt. \tag{5.17}$$

Thus, the effect of the generalised function $u(t)$, operating on the test function $\phi(t)$, is to find the area under the curve $\phi(t)$ in the interval 0 to ∞. Note that $u(t)$ is similar to the ordinary step function. However, as defined here, it has some special properties also, e.g., it possesses a derivative.

Let $u'(t)$ be the derivative of u(t). Let us determine the effect $u'(t)$ will produce on a test function $\phi(t)$. This effect is given by the defining integral,

$$\int_{-\infty}^{\infty} u'(t)\,\phi(t)\,dt.$$

Integrating by parts we get,

$$\int_{-\infty}^{\infty} u'(t)\,\phi(t)\,dt = u(t)\,\phi(t)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u(t)\,\phi'(t)\,dt.$$

The first term on the r.h.s. is equal to zero because of the property (ii) of the test signal $\phi(t)$. From the defining relation (5.17) the second term becomes,

$$\int_{-\infty}^{\infty} u(t)\,\phi'(t)\,dt = \int_{0}^{\infty} \phi'(t)\,dt$$

$$= \phi(t)\,\Big|_0^{\infty}$$

$$= -\phi(0)$$

Therefore we get,

$$\int_{-\infty}^{\infty} u'(t)\,\phi(t)\,dt = \phi(0)\,.$$

However, substituting $\phi(t)$ in place of $f(t)$ in eqn. (5.13) we also have,

$$\int_{-\infty}^{\infty} \delta(t)\;\phi\;(t)\,dt = \phi(0).$$

Therefore, from the equivalence property (5.16) we get,

$$u'(t) = \delta(t) \tag{5.18}$$

Thus, we conclude that the derivative of the generalised function $u(t)$ is the generalised function $\delta(t)$. Similarly, for higher derivatives, i.e.,

$u''(t) = \delta'(t)$, (called a 'doublet')

$u'''(t) = \delta''(t)$, (called a 'triplet'), and so on.

Defined as an ordinary function, the step function will not have derivatives. But defined as a generalised function, it can have derivatives of any order. It is one of the main advantages of the theory of generalised functions that we can define derivatives for discontinuous functions also. Such derivatives will be in terms of the impulse function and its derivatives.

As an ordinary function, the unit impulse may be thought of as the limiting case for a pulse of width Δ and amplitude $1/\Delta$ (i.e., with unit area) as Δ tends to zero. In that case, the unit impulse will be a function of infinite magnitude, zero duration, and finite area of unity, as shown by the sequence (a), (b) and (c) of Fig. 5.6.

Symbolically, the impulse is represented by Fig. 5.6(d), with its strength indicated by the number near the arrow. Although Fig. 5.6 shows impulse as the limiting case of a sequence of unit area rectangular pulses, the shape of the pulse is not material. We could get impulse as the limiting case of a unit area triangular pulse or a Gaussian pulse or any pulse with unit area.

Let us now determine the Fourier transform of an impulse. From the basic definitions of Fourier transform [eqn. (5.5)] and the unit impulse [eqn. (5.13)] we get,

$$\mathcal{F}\delta(t) = \int_{-\infty}^{\infty} \delta(t)\,\exp\,(-j\,\omega t)\,dt = \exp\,(-j\,\omega\,.\,0) = 1$$

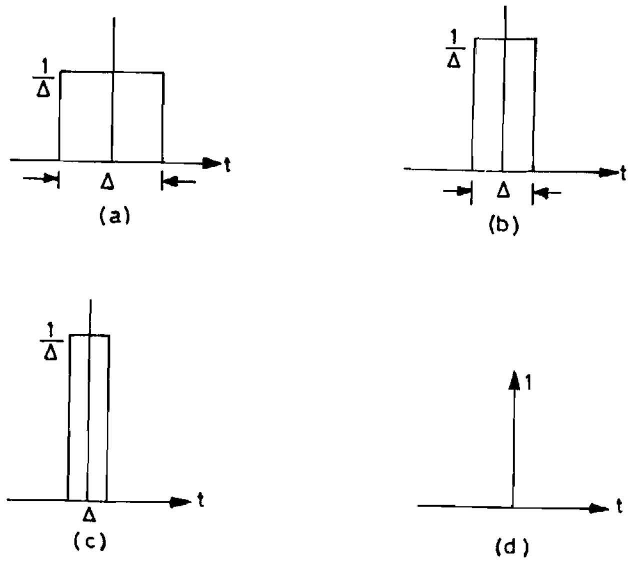


Fig. 5.6 Impulse Function

i.e.,

$$\delta(t) \leftrightarrow 1 \tag{5.19}$$

constitute a Fourier transform pair. This result can easily be extended to,

$$k\,\delta(t) \leftrightarrow k \text{ and } \delta(t-t_0) \leftrightarrow \exp(-j\,\omega\,t_0).$$

Equation (5.19) means that the spectrum of a unit impulse is a constant equal to unity for ω ranging from $-\infty$ to $+\infty$. This flat spectrum can be viewed as the limiting case for the spectrum of a pulse (Fig. 5.2) as the pulse tends to an impulse An impulse has zero duration: hence, pulse period T in Fig. 5.2(a) tends to zero. Thus, the curve in Fig. 5.2(b) will cut the ω axis at infinity. In other words, it will be a line parallel to the ω axis with a magnitude $AT = 1$. In practical terms, it means that to transmit a unit impulse faithfully we require a channel of infinite bandwidth.

5.4 Convolution

Chapter 3 described the classical method, using differential equations, for determining the response $y(t)$ of a linear system for an input $x(t)$. Since only functions of time, $x(t)$, $y(t)$ and their derivatives are involved in the solution process, we say that the classical method is a method in the *time domain*. There is another method for determining the system response in the time domain which uses the concepts of *impulse response* and *convolution.*

When an initially relaxed system is subjected to a unit impulse at its input, the resulting output of the system is called its *impulse response* (Fig. 5.7) and is represented by the symbol $h(t)$. Because the superposition principle is valid for linear

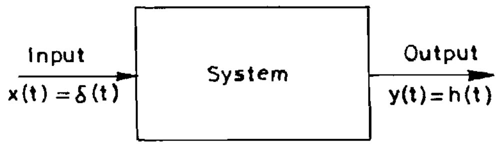


Fig. 5.7 Impulse Response

systems, the response to an impulse of strength k will be $kh(t)$. Since the system is time invariant, its response to a unit impulse occurring at $t = \tau$. i.e., to $\delta(t-\tau)$, will be $h(t-\tau)$. If two impulses $k_1\,\delta(t)$ and $k_2\,\delta(t-\tau)$ are applied together, the response will be $k_1\,h(t) + k_2\,h(t-\tau)$. If the system is subjected to an impulse train given by $\sum_{n=-\infty}^{\infty} k_n\,\delta(t-n\Delta)$, the response of the linear system will be,

$$\sum_{n=-\infty}^{\infty} k_n h(t-n\Delta).$$

Now, let the system be subjected to an arbitrary input $x(t)$. The function $x(t)$ can be approximated by a series of rectangular pulses as shown in Fig. 5.8. The width of each pulse is Δ and the height is equal to the value of $x(t)$ at the midpoint of the particular pulse, i.e., the pulse amplitude of the nth pulse (centered at $n\Delta$) will be $x(n\Delta)$. Thus, the area of the nth pulse is $\Delta x(n\Delta)$. The approximation to $x(t)$ becomes better and better as Δ is made smaller and smaller. Also, when Δ is sufficiently small, the response of the system to a pulse is the same as its response to an impulse of strength equal to the pulse area (as will be demonstrated later). Therefore, the response of the system to a very narrow pulse at $n\Delta$ will be $\Delta\,.\,x(n\Delta)\,.\,h(t-n\Delta)$, and the output $y(t)$ will be the sum of responses due to all the pulses, i.e.,

$$y(t) = \sum_{n=-\infty}^{\infty} \Delta \cdot x(n\Delta)\; h(t-n\Delta).$$

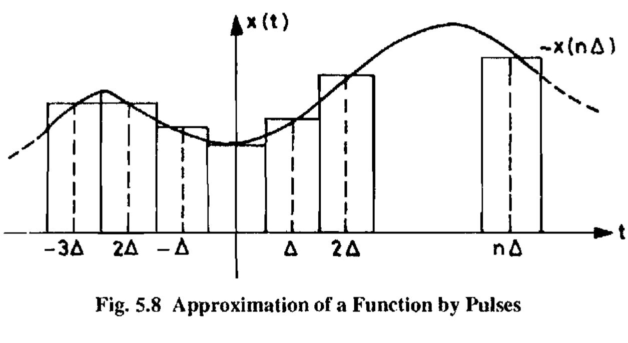


Fig. 5.8 Approximation of a Function by Pulses

In the limit, as $\Delta \to 0$, Δ can be replaced by $d\tau$, $n\Delta$ by the continuous variable τ, and the summation by integration. Thus,

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau. \qquad (5.20)$$

The integral in eqn. (5.20) is called the *convolution integral*. Equation (5.20) is symbolically written as,

$$y(t) = x(t) * h(t)$$

which means that the output of a linear, relaxed system to any arbitrary input can be obtained by *convolving* the input with the impulse response of the system. Thus, if the impulse response of a system is known, its response to any input can be obtained by relation (5.20). In other words, the impulse response completely characterises a system.

The convolution operation is a general mathematical operation between any two functions. That is,

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau.$$

It is easy to show that $f_1(t) * f_2(t) = f_2(t) * f_1(t)$, i.e., eqn. (5.20) can also be written as,

$$y(t) = \int_{-\infty}^{\infty} h(\tau)\, x(t-\tau)\, d\tau.$$

In the analysis of physical systems, $t = 0$ is chosen as the instant at which the input is applied to the system. Thus, $x(t) = 0$ for $t < 0$. Therefore, the lower limit of the convolution integral in eqn. (5.20) can be made zero instead of $-\infty$. Also, the output of a physical system at any time t_1 will depend upon the values of the input only up to t_1. This is obviously so because a physical system cannot respond to an input not yet applied to it. (This is called the property of *causality*.) Thus, the upper limit for τ in eqn. (5.20) will be t_1. That is, the output of a physical system at $t = t_1$ will be

$$y(t_1) = \int_0^{t_1} x(\tau)\, h(t-\tau)\, d\tau.$$

Since t_1 can have any value t, we have

$$y(t) = \int_0^{t} x(\tau)\, h(t-\tau)\, d\tau. \qquad (5.21)$$

Example 5.1:— The impulse response of a linear system is given by $h(t) = 2e^{-t}$ [Fig. 5.9(a)]. Determine its response to (i) a step input and (ii) a pulse of magnitude 10, duration 2, centred at $t = 3$.

Solution: The step response is determined directly from eqn. (5.21) as,

$$y(t) = \int_0^t u(\tau)\ 2\exp\ [-(t-\tau)]\, d\tau$$

$$= 2e^{-t}\int_0^t e^{\tau}\, d\tau = 2(1-e^{-t}).$$

(ii) The given pulse is shown in Fig. 5.9(b). It can be expressed mathematically as,

$$f(t) = 10\,[u(t-2) - u(t-4)]$$

Using eqn. (5.21) the response is,

$$y(t) = \int_0^t 10\,[u(\tau-2) - u(\tau-4)]\cdot 2\ \exp\,[-(t-\tau)]\, d\tau$$

$$= 20e^{-t}\left[\int_0^t u(\tau-2)\, e^{\tau}\, d\tau - \int_0^t u(\tau-4)\, e^{\tau}\, d\tau\right]\cdot$$

The first integral will be zero for τ up to 2 because of the multiplying factor $u(\tau-2)$. Therefore, the lower limit on the first integral should be 2. Similarly, the lower limit in the second integral should be 4. Thus,

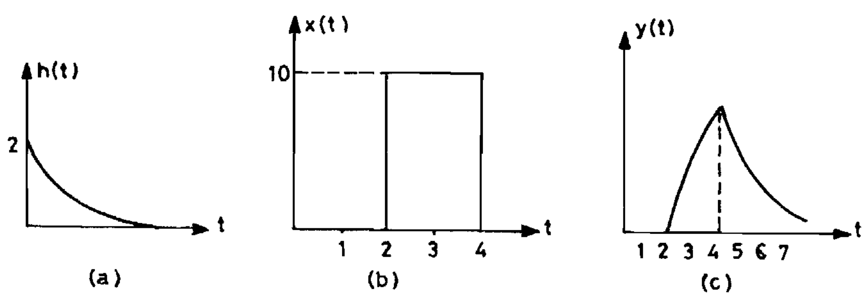
$$y(t) = 20e^{-t}\ [(e^t - e^2)\, u(t-2) - (e^t - e^4)\, u(t-4)]$$

$$= 20\,[\{1 - e^{-(t-2)}\}\, u(t-2) - \{1 - e^{-(t-4)}\}\ u(t-4)]$$

A plot of $y(t)$ is shown in Fig. 5.9(c).

Graphical convolution:

Impulse response testing is a very useful experimental technique. When a mathematical model cannot be built due to lack of information about the system, such experimental techniques are the only methods available for finding system characteristics. For practical test purposes, an impulse is approximated by a narrow pulse such that its width is less than the smallest time constant of the system. Of course,



(a) (b) (c)

Fig. 5.9(a) Impulse Response, (b)Input, (c) Output

the pulse should have sufficient energy to excite the system into giving a recordable output. In such a case, $h(t)$ will be known only graphically. Many a time even the system input may be known only graphically. In the absence of analytical expressions for either $x(t)$ or $h(t)$, the evaluation of the convolution integral cannot be done analytically. Recourse is then taken to either graphical or numerical techniques. Even when analytical expressions are available, it is conceptually helpful to visualise the convolution as a graphical process. We therefore consider a graphical interpretation of the convolution between two functions $f_1(t)$ and $f_2(t)$. To be specific, let $f_1(t)$ be the input pulse $x(t)$ of Fig. 5.9(b) and $f_2(t)$ the impulse response shown in Fig. 5.9(a). Then,

$$y(t) = f_1(t) * f_2(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau.$$

Now $x(\tau)$ is simply $x(t)$ with t replaced by τ. Let us find out $h(t-\tau)$ for a specific time $t = t_1$, i.e., $h(t_1 - \tau)$. It is easier first to find $h(\tau - t_1)$. It will be $h(\tau)$ shifted to the right by t_1. The effect of changing the sign of the argument, from $(\tau - t_1)$ to $(t_1 - \tau)$, is to take the mirror image of the function around the vertical line at t_1. The process is illustrated in Figs. 5.10 (b), (c) and (d).

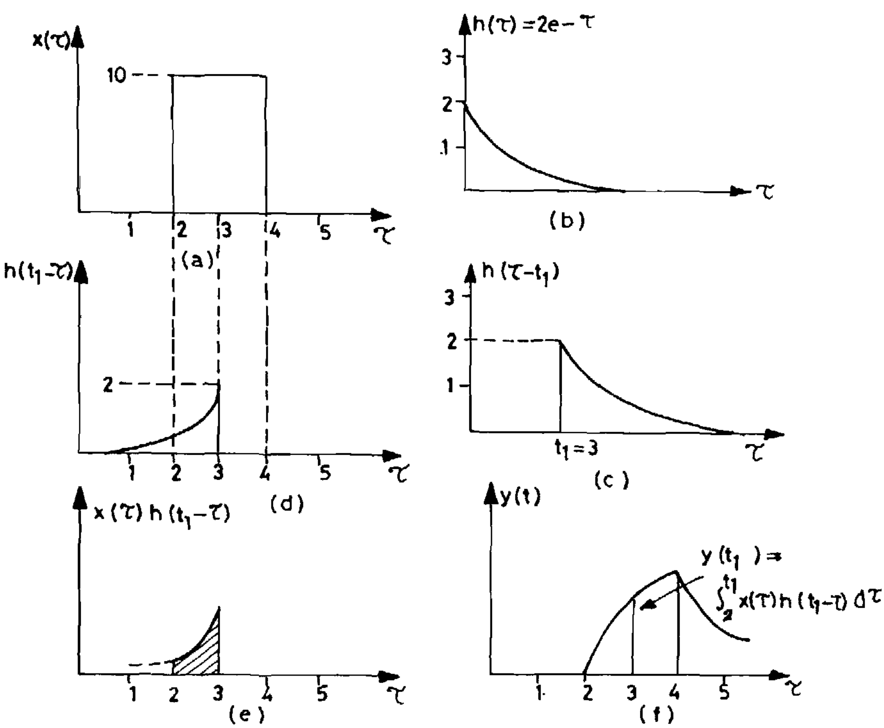


Fig. 5.10 Illustration of Graphical Convolution

Next, we multiply the functions in Figs. 5.10(a) and (d) to get the curve in Fig. 5.10(e). Its integral, i.e., the area under the curve in Fig. 5.10(e), will be the value of output $y(t_1)$ at the instant t_1. To obtain the output at another instant t_2, $h(t)$ is shifted to the right by t_2, folded along the vertical axis at t_2, the resulting figure multiplied by $x(t)$ and the product curve integrated to get $y(t_2)$. This process is repeated for all instants of time to get the complete $y(t)$.

In this particular problem for values of time less than 2 there will be no common overlapping area between $x(t)$ and $h(t-\tau)$. Hence, the product will be zero and the output will be zero up to $t = 2$.

If the values of $f_1(t)$ and $f_2(t)$ are stored as sequences of numbers in a digital computer, one could develop a computer programme for performing the convolution numerically.

Impulse response from differential equation model:

In linear systems analysis, the starting point is quite frequently the differential equation model of the system. In this subsection we develop a method for solving a given differential equation with impulse as the forcing function to determine the impulse response $h(t)$ of the system.

Let us start with a first order system. Assume the initial condition to be zero. That is, we want the solution of equation,

$$a\frac{dy}{dt} + by = \delta(t)\,,\, y\,(0) = 0. \tag{5.22}$$

Let us approximate the impulse response as the limiting case of the response for a rectangular pulse of magnitude $M = 1/\Delta$ and duration Δ (area equal to unity), as Δ tends to zero.

In Fig. 5.11(a) Δ_1 is many times larger than the system time constant a/b. So the output starts like the step response, reaches its steady-state value M_1/b before the pulse is removed, and then decays exponentially with the time constant a/b. That is, after the removal of the pulse the response is the same as the natural response with initial condition M_1/b. In Fig. 5.11(b) the pulse is made much narrower. The response start exponentially towards the steady-state value M_2/b. However, the pulse gets terminated much earlier, by which time the response has reached some value α_2. Thereafter, it behaves as the natural response with initial condition α_2.

In Fig. 5.11(c) the pulse width is still smaller. The output starts exponentially with an initial slope $(M_3/b)/(a/b) = M_3/a$. In the very short time Δ_3 the slope will not be altered much from its initial value. Hence, at the end of the pulse duration the output would have reached a value $(M_3/a)\Delta_3$. But $M_3 = 1/\Delta_3$. Therefore, $\alpha_3 = 1/a$. This means that if the pulse is sufficiently narrow, the output would reach a value $1/a$ at the end of the pulse period, irrespective of the actual pulse

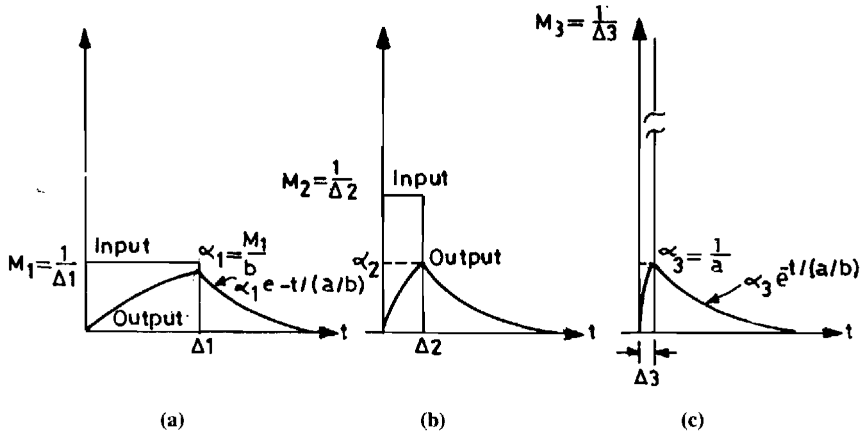


Fig. 5.11 Impulse Response of a First Order System as the Limiting Case of Its Pulse Response

width. In the limit as $\Delta \to 0$, the pulse becomes an impulse. In this case, the output would reach a value of $1/a$ in zero time. Thereafter the response would be the same as the natural response with initial condition $1/a$.

Thus, we reach the important conclusion that the response of a first order system, with its model given by eqn. (5.22), to a unit impulse is the same as its natural response with the initial condition $y(0) = 1/a$. In an electrical R-C circuit, one could think that the unit impulse source is able to transmit unit amount of charge to the capacitor, by a current of infinite amplitude and zero duration, and then the circuit behaves as if it had only an initial voltage across the capacitor.

The extension of this physical reasoning to a second order system,

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = \delta(t),$$

would show that its unit impulse response will be the same as its natural response with initial condition $\dot{y}(0) = 1/a_2$ and $y(0) = 0$. Similarly, the unit impulse response of an nth order system,

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \ldots a_1 \dot{y} + a_0 y = \delta(t),$$

will be the same as its natural response with initial conditions,

$$y^{(n-1)}(0) = 1/a_n \text{ and } y(0) = \dot{y}(0) = \ldots = y^{(n-2)}(0) = 0.$$

It has already been shown that an impulse may be thought of as the derivative of the step function. Thus, if we know the step response of a system, its impulse response will simply be the derivative of its step response.

Example 5.2:— Determine the output of the system shown in Fig. 5.12 for a single, half-wave sinusoidal pulse.

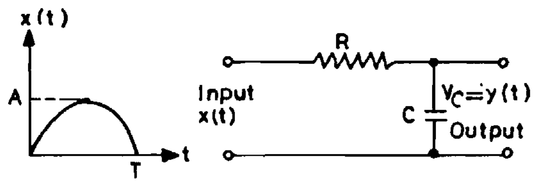


Fig. 5.12

Solution: Let us first determine the impulse response of the system. The system equation is,

$$C\,\frac{dy}{dt} = \frac{x-y}{R} \quad \text{or } RC\,\frac{dy}{dt} + y = x.$$

The impulse response will be the natural response with initial condition $y(0) = 1/RC$. The natural response is given by,

$$y_n = \frac{1}{RC}\,e^{-t/RC}, \quad \text{for } t > 0.$$

Therefore,

$$h(t) = y_n = \frac{1}{RC}e^{-t/RC}.$$

The input to the system, shown in Fig. 5.12, may be written as,

$$x(t) = A\sin\frac{\pi}{T}t, \quad 0 \le t \le T$$

$$= 0, \quad t > T.$$

The output $y(t)$ is given by,

$$y(t) = x(t) * h(t) = \frac{A}{RC}\int_0^t \sin\frac{\pi}{T}\tau\,\exp\{-(t-\tau)/RC\}\,d\tau, \quad 0 \le t \le T$$

$$= \frac{A}{RC}\int_0^T \sin\frac{\pi}{T}\tau\,\exp\{-(t-\tau)/RC\}\,d\tau, \qquad t > T.$$

Evaluating the integral on the r.h.s. by parts we get,

$$y(t) = \frac{AT^2}{(TRC)^2 + \pi^2}\left(\sin\frac{\pi}{T}t - \frac{\pi}{TRC}\cos\frac{\pi}{T}t + \frac{\pi}{TRC}e^{-t/RC}\right)$$

for $\quad 0 \le t \le T.$

For the particular case when $RC = 1$ and $T = \pi$ we get,

$$y(t) = \frac{A}{2}(\sin t - \cos t + e^{-t}), \quad \text{for } 0 \le t \le T.$$

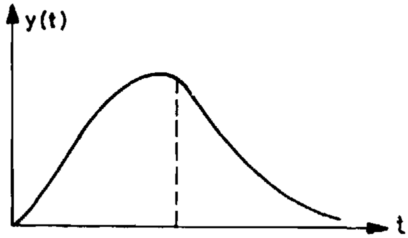


Fig. 5.13

For $t \geq T$, the upper limit of the integral is T. Substituting this value and solving we get,

$$y(t) = \frac{A\,e^{-t}}{2}(1 + e^{T}), \text{ for } t \geq T.$$

The solution is shown in Fig. 5.13.

Remarks: The use of convolution technique may look somewhat more complicated for pen-and-paper analysis. However, it is well suited for programming on a computer. It has many other useful applications: e.g., in filtering and smoothing of data, particularly in the digital handling of large observational data. More importantly, for the linear systems, it establishes the basis for the powerful transform methods of analysis in the frequency domain, as will be demonstrated in the next sub-section.

The convolution theorem :

Let $f_1(t)$ and $f_2(t)$ be two functions in the time domain with $F_1(\omega)$ and $F_2(\omega)$ as their Fourier transforms, i.e., $f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$. Let the convolution of $f_1(t)$ with $f_2(t)$ be equal to $y(t)$, i.e., $y(t) = f_1(t)*f_2(t)$. Let $y(t) \leftrightarrow Y(\omega)$. The questions we now pose is: what is the relationship between $Y(\omega)$, $F_1(\omega)$ and $F_2(\omega)$?

From the definition of convolution we have,

$$y(t) = \int_{-\infty}^{\infty} f_1(\tau)\, f_2(t-\tau)\, d\tau.$$

Therefore,

$$Y(\omega) = \int_{-\infty}^{\infty} \exp(-j\omega t)\, y(t)\, dt$$

$$= \int_{-\infty}^{\infty} \exp(-j\omega t) \int_{-\infty}^{\infty} [f_1(\tau)\, f_2(t-\tau)\, d\tau]\, dt.$$

Interchanging the order of integration we get,

$$Y(\omega) = \int_{-\infty}^{\infty} f_1(\tau) \left[\int_{-\infty}^{\infty} \exp(-j\omega t) f_2(t-\tau)\, dt \right] d\tau.$$

The quantity inside the brackets is, by definition, the Fourier transform of $f_2(t-\tau)$. From the shifting property of the Fourier transform we conclude,

$$f_2(t-\tau) \leftrightarrow F_2(\omega) \exp(-j\omega\tau).$$

Hence,

$$Y(\omega) = \int_{-\infty}^{\infty} f_1(\tau)\ F_2(\omega) \exp(-j\omega\tau)\ d\tau.$$

$$= F_2(\omega) \int_{-\infty}^{\infty} f_1(\tau) \exp(-j\omega\tau)\, d\tau.$$

The integral term above is equal to $F_1(\omega)$. Therefore,

$$Y(\omega) = F_1(\omega) \,.\, F_2(\omega) \,. \tag{5.23}$$

The result given in eqn. (5.23) is called the *convolution theorem*. Stated in words: the operation of convolution between two time functions results in the multiplication of their Fourier transforms. This is one of the most powerful results of the convolution theory.

To see the implication of the convolution theorem for linear systems analysis, let $f_1(t)$ be the input $x(t) \leftrightarrow X(\omega)$ and $f_2(t) = h(t) \leftrightarrow H(\omega)$, the impulse response of the system. Since $y(t) = x(t) * h(t)$, therefore,

$$Y(\omega) = X(\omega) \,.\, H(\omega) \,. \tag{5.24}$$

$H(\omega)$ is called the *system function*. Equation (5.24) means that in the frequency domain, the output is simply the product of the Fourier transform of the given input and the system function. Thus, the response of a system to any arbitrary input can be obtained if the system function is known. If the output is desired as a function of time, it can be obtained by taking the Fourier inverse of $Y(\omega)$. This is usually done with the help of a table of transform pairs. This means a great simplification of the analysis procedure. The differential and integral operations of the time domain methods are replaced by simple algebraic operations in the frequency domain. And this is why transform methods are so very popular.

A knowledge of the system function $H(\omega)$ permits us to find the output for any given input. Hence, $H(\omega)$ is yet another method of characterising the system and may be called its mathematical model in the frequency domain.

5.5 Analysis with Fourier Transforms

Example 5.3:— Determine the current i in the inductor of Fig. 5.14 for the applied voltage $v(t) = 10\, e^{-2t}$.

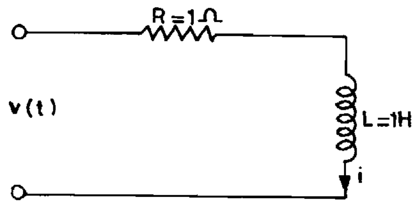


Fig. 5.14

Solution: First let us determine the system function $H(\omega)$. The mathematical model of the system is $di/dt + i = v(t)$. The impulse response of the system will be the natural response of the system with $i(0) = 1$. That is, $h(t) = e^{-t}$. Using table 5.1 the system function is,

$$H(\omega) = \mathcal{F}(e^{-t}) = \frac{1}{1 + j\omega}$$

The Fourier transform of the input is,

$$V(\omega) = \mathcal{F}[v(t)] = \mathcal{F}(10e^{-2t}) = 10/(2 + j\omega).$$

Output $\quad I(\omega) = V(\omega)\, H(\omega) = \dfrac{10}{(1 + j\omega)\,(2 + j\,\omega)}.$

The expression on the r.h.s. can be expanded into partial fractions as,

$$I(\omega) = 10\left[\frac{1}{1 + j\,\omega} - \frac{1}{2 + j\,\omega}\right].$$

We take the Fourier inverse of each term on the r.h.s. (using the table of Fourier transforms), to get ,

$$i(t) = 10\,(e^{-t} - e^{-2t}).$$

As illustrated by this example, the steps in this analysis procedure are as follows:

(1) Find the system function $H(\omega)$.

(2) Determine the Fourier transform of the given input function.

(3) Multiply the two together to obtain output as a Fourier transform.

(4) Determine output as a function of time by finding the inverse of its Fourier transform, using a table of Fourier transform pairs.

The first step will be explored further in the next subsection.

Determination of the system function $H(\omega)$:

In Example 5.3, the method for determining $H(\omega)$ was: write the differential equation model of the system; determine its impulse response $h(t)$; find the Fourier transform of $h(t)$ to get $H(\omega)$. This procedure is somewhat cumbersome and if followed for relatively more complex problems, it will reduce the advantage of simplicity of the Fourier transform analysis. We, therefore, explore the possibility of finding a more direct method of determining $H(\omega)$.

In the electrical circuit of Example 5.3, the input is a voltage and the output a current. If the input voltage were a periodic sinusoidal voltage of frequency ω_0, say $\hat{V}(\omega_0)$, then, the elementary circuit analysis tells us that the steady-state current $\hat{I}(\omega_0)$ will also be sinusoidal. The amplitude and the phase angle of the current $\hat{I}(\omega_0)$ may be found out from the basic relation,

$$\hat{V}(\omega_0) = Z(\omega_0)\,\hat{I}(\omega_0),$$

where $Z(\omega_0)$ is the impedance of the circuit at frequency ω_0. We are using a circumflex over the symbols for current and voltage variables to indicate that they are the phasor expressions for sinusoidal variables and not the Fourier transform of a transient, non-periodic time function. The above relation is valid for any frequency, i.e.,

$$\hat{V}(\omega) = Z(\omega)\,\hat{I}(\omega) \text{ or } \hat{I}(\omega) = G(\omega)\,\hat{V}(\omega)$$

where $G(\omega) = 1 / Z(\omega)$ is the admittance of the circuit.

The role of admittance $G(\omega)$ or impedance $Z(\omega)$ in the steady-state sinusoidal analysis can be viewed as follows. It modifies the magnitude and the phase angle of the sinusoidal input $\hat{V}(\omega)$ or $\hat{I}(\omega)$ to produce another sinusoidal signal of the same frequency but of different amplitude and phase angle. The modifications in the amplitude and the phase angle will be different at different frequencies because $G(\omega)$ and $Z(\omega)$ are frequency dependent. We now look again at the fundamental relation, i.e., eqn. (5.24):

$$Y(\omega) = H(\omega)\,X(\omega).$$

The Fourier transform $X(\omega)$ converts the function of time, input $x(t)$, into its sinusoidal components having different magnitudes $|X(\omega)|$ and phase angles $\angle X(\omega)$ for different frequencies. The system function $H(\omega)$ modifies these magnitudes and phase angles to produce output $Y(\omega)$. Thus, $H(\omega)$ performs the same function as $Z(\omega)$ or $G(\omega)$.

Therefore, we conclude that the system function $H(\omega)$ is the same as our familiar $Z(\omega)$ or $G(\omega)$ of the a.c. circuit theory, i.e., $H(\omega) = Z(\omega)$ [or$G(\omega)$]. The determination of the impedance or admittance of an electrical circuit is a straightforward matter. For Example 5.3, we can easily verify that its admittance is $1/(1+j\omega)$, the same as the expression for $H(\omega)$ determined from its impulse response.

In the elementary circuit theory, it is tacitly assumed that a sinusoidal voltage will always produce a sinusoidal current of the same frequency. This is a fundamental property of linear systems. That is, for a linear time invariant system, a sinusoidal input will produce a sinusoidal output of the same frequency. This result can be verified as follows.

Let the output of a linear, time invariant system be $y(t)$ for a sinusoidal input $x(t) = \exp(j\omega_0 t)$. Let us delay the input by t_1, i.e., let the input be $x(t-t_1) = \exp[j\omega_0(t-t_1)]$. Because the system is time invariant, the output to $x(t-t_1)$ will be $y(t-t_1)$. But the input function can be written as $x(t-t_1) = \exp(-j\omega_0 t_1)\exp(j\omega_0 t)$. Since the system is linear, the output will also be multiplied by the constant $\exp(-j\omega_0 t_1)$. Therefore,

$$y(t-t_1) = y(t)\exp(-j\omega_0 t_1) \text{ or } y(t) = y(t-t_1)\exp(j\omega_0 t_1).$$

The above result is true for any t_1, i.e., t_1 can have any arbitrary value. In particular, for $t_1 = t$, we get,

$$y(t) = y(0)\exp(j\omega_0 t)$$

Now, $y(0)$ is not a function of t. Therefore, we conclude that for a linear system, a sinusoidal input produces a sinusoidal output of the same frequency. The magnitude and the phase angle of the output will depend on the complex quantity $y(0)$.

In fact, we can generalise the result of the previous paragraph into the statement; a linear time invariant system will produce an exponential output, may be with different amplitude and time shift, for an exponential input e^{st}, whether the exponent s is real, imaginary or complex. This is because the model of a linear system will be a linear differential equation. Thus, the operations performed on the input to produce the output will be some combination of the operations of integration and differentiation. The exponential function has the interesting property that the result of integrating or differentiating it is also an exponential. And hence the statement that a linear system will give an exponential output for an exponential input. Note that when s in the exponent of e^{st} is imaginary, i.e., equal to $j\omega$, we get a steady sinusoidal variable. In such a case, the Fourier methods are useful for analysis. When s becomes complex, $s = \sigma + j\omega$ (with σ negative), e^{st} represents decaying sinusoids. The variable s is then called a *complex frequency*. Any given signal can be resolved into its components of complex frequency s by the use of Laplace transform, as will be studied in the next chapter.

Example 5.4:—For the network shown in Fig. 5.15(a), determine the system function $H(\omega)$ and hence its response to (i) a decaying exponential function $x(t) = e^{-t}$ shown in Fig. 5.15(c) and (ii) a decaying sinusoidal function $x(t) = e^{-t}\sin 2t$ shown in Fig. 5.15(d).

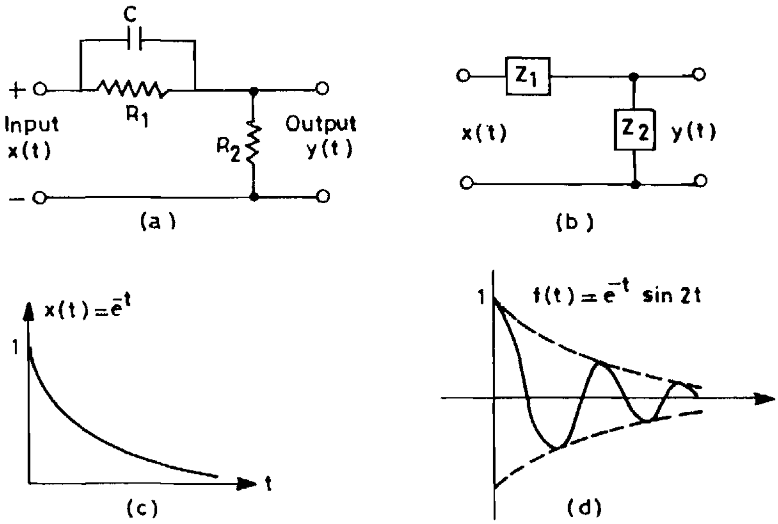


Fig. 5.15

Solution: From the equivalent circuit of Fig. 5.15(b) we have,

$$H(\omega) = \frac{Z_2}{Z_1 + Z_2} = \frac{R_2}{R_2 + \dfrac{R_1}{j\omega R_1 C + 1}} = \frac{j\omega + 1/R_1 C}{j\omega + (1/R_1 C + 1/R_2 C)}$$

For the particular case $R_1 C = R_2 C = 1$ we get,

$$H(\omega) = (1 + j\omega)/(2 + j\,\omega).$$

(i) For the input $x(t) = e^{-t}$,

$$X(\omega) = \mathcal{F}[x(t)] = \frac{1}{1 + j\,\omega}.$$

Output $\quad Y(\omega) = X(\omega)\; H(\omega)$

$$= \frac{1}{1 + j\,\omega} \cdot \frac{1 + j\,\omega}{2 + j\,\omega} = \frac{1}{2 + j\,\omega}.$$

Taking the Fourier inverse we get,

$$y(t) = \mathcal{F}^{-1}\,[Y(\omega)] = e^{-2t}.$$

(ii) For the input $x(t) = e^{-t}\sin 2t$,

$$X(\omega) = \mathcal{F}[x(t)] = \frac{2}{(1 + j\omega)^2 + 2^2} \text{ (from Table 5.1)}$$

$$Y(\omega) = X(\omega) . H(\omega) = \frac{2(1+j\omega)}{(2+j\omega)\,[(1+j\omega)^2+4]}.$$

The expression for $Y(\omega)$ can be expanded into partial fractions as,

$$Y(\omega) = \frac{A}{2+j\omega} + \frac{B\omega + C}{(1+j\omega)^2+4}.$$

Equating the numerators for these two expressions of $Y(\omega)$, we get $A = -2/5, B = 2j/5\,,\; C = 2$, or,

$$Y(\omega) = \frac{-2/5}{2+j\,\omega} + \frac{2+j\,2\,\omega/5}{(1+j\omega)^2+2^2}$$

$$= \frac{-2/5}{2+j\,\omega} + \frac{2}{5}\,\frac{1+j\omega}{(1+j\omega)^2+2^2} + \frac{4}{5}\,\frac{2}{(1+j\omega)^2+2^2}.$$

Taking the Fourier inverse of each term, with the help of Table 5.1, we get,

$$y(t) = \mathcal{F}^{-1}\,[(\,Y(\omega)] = -2/5\,.\,e^{-2t} + 2/5\,.\,e^{-t}\,\cos 2t + 4/5\;.\,e^{-t}\sin 2t.$$

Fourier methods enable us to resolve an arbitrary function of time into its sinusoidal frequency components. The Fourier series does it for periodic signals and the Fourier transform for non-periodic signals. Such a resolution of a signal in the time domain into a signal in the frequency domain is most useful in the analysis of those linear systems whose characteristics can be expressed more conveniently in terms of their frequency response. The analysis can then be carried out in the frequency domain as illustrated by the two problems above. Many problems, like filtering, sampling, modulation, etc., in control and communication systems, are analysed by the above techniques. In the following subsection, we will very briefly look at one of the problems in filtering to show the use of Fourier techniques in solving problems in that area.

Filters:

A filter is used either to suppress or to extract particular frequency components from a given signal. It may be realised using *RLC* components with, or without, active electronic components. It could also be any other signal-processing device, e.g., transducer, amplifier, transmission channel, etc. In fact, any linear system can be thought of as a filter, provided its frequency response is such that for some frequency range the output is zero. The characteristics of a filter are expressed by the amplitude and phase angle plots of its system function $H(\omega)$. The classification of filters into low pass, high pass and band pass filters is demonstrated in Fig. 5.16.

The magnitude plots shown in Fig. 5.16 are for ideal filters with sharp cut-off frequency ω_c. In actual filters, the cut off would be more gradual. For example, Fig. 5.17 shows the amplitude plot for a physical low pass filter.

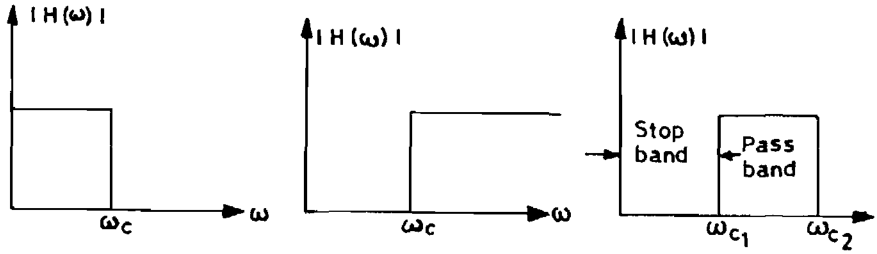


Fig. 5.16 Classification of Filters

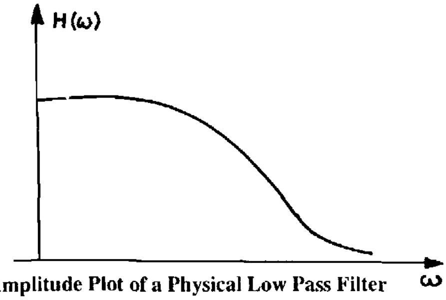


Fig. 5.17 Amplitude Plot of a Physical Low Pass Filter

Some of the problems studied in the theory of filters are: (i) What should be the system function $H(\omega)$ to extract a useful signal from a 'noisy' signal ? (ii) Given a particular filter characteristic $H(\omega)$, predict how it will alter the 'shape' of signals passing through it. (iii) What are the distortions produced by filters and on what factors do they depend? All these problems are studied with the help of Fourier transforms.

Example 5.5:—Consider the transmission of a pulse through a filter having the characteristics shown in Fig. 5.18.

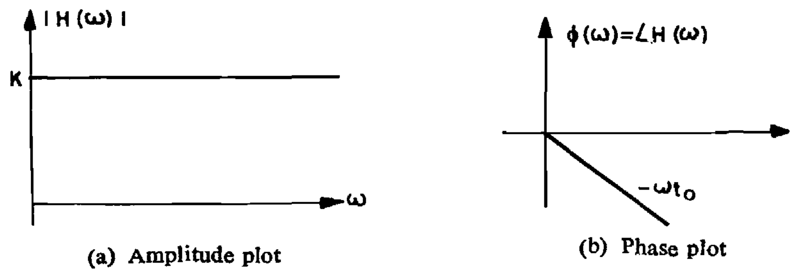


Fig. 5.18 Frequency Characteristics of an All Pass Filter

Because the amplitude spectrum is constant for all frequencies, such a filter is called an *all pass* filter. Further, its phase characteristic is a linear function of ω.

Let the Fourier transform of the input pulse shown in Fig. 5.19(a) be $X(\omega)$. Then the output of the filter will be,

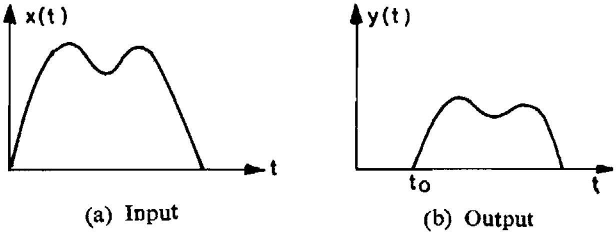


Fig. 5.19 Transmission of a Pulse through a Distortionless All Phase Filter

$$Y(\omega) = X(\omega)\, H(\omega)$$

$$= K\,|X(\omega)|\ \angle -\omega t_0$$

$$= K\,|X(\omega)|\ \exp\,(-j\,\omega\, t_0)$$

Now, from the shifting property of the Fourier transform, we know that a change in the phase angle in the frequency domain means a time shift in the time domain. Therefore, output,

$$y(t) = \mathcal{F}^{-1}\, Y(\omega) = \mathcal{F}^{-1}\, K\ |X(\omega)|\ \exp\,(-j\omega t_0) = Kx\,(t - t_0)\,.$$

The output pulse, shown in Fig. 5.18(b), has exactly the same shape as the input, except that its magnitude everywhere will be attenuated by a factor K and it will be delayed in time by t_0.

The information content of a signal is in its shape. The shape determines the relative magnitudes of its different frequency components. Multiplication by a constant and delay in time do not alter the relative magnitudes of the frequency components and, hence, the information content of the signal. In the present case, all the information coded into input pulse can be completely recovered from the output pulse. Filters having this property are called *distortionless* filters.

If the input signal is known to contain only limited frequency components (i.e., it is 'band limited'), then the filter will not produce any distortion if its pass band is reduced, so long as it is greater than the signal bandwidth. The phase relationship should of course be linear.

As mentioned earlier, a physical filter will always produce some distortion. These distortions are classified into two parts: (i) the amplitude distortion and (ii) the phase distortion. If the amplitude spectrum of the filter is not constant, it produces amplitude distortion and if its phase spectrum is not a linear function of ω, it produces phase distortion.

5.6 The DFT and the FFT

Fourier transforms are also used in analysing field data in areas like remote sensing, seismic signals, bioelectric signals (EEG, ECG, etc.). Because of the large number of data, recourse has to be taken to computer handling and analysis. The computer, however, will accept only digital values of signals at discrete instants of time. The Fourier transform determined from these discrete values of signal is called the *discrete Fourier transform* (DFT).

The DFT will correspond more closely to the Fourier transform of the original analog signal (or the continuous time signal) if the discrete instants of time, at which the signal is sampled, are close to each other. For this reason, the number of signal samples to be stored and handled by the computer will be quite large for the usual types of signals in the application areas mentioned above. Both the storage space and the computer time needed for computing the DFT may become prohibitively large. This is so because the number of multiplications needed, hence the computer time required, for calculating the DFT from N signal samples is proportional to N^2.

A technique called *the fast Fourier transform* (FFT), (developed in 1965) drastically reduces the computer time for computation of the discrete Fourier transform. In the FFT the number of multiplications required is proportional to $N \log_2 N$. If the number of samples is $N = 256$, for example, the ordinary DFT will require 256^2 multiplications, while the FFT will require only 256×8 multiplications. Thus, the computer time will be reduced by a factor of nearly 1/32.

GLOSSARY

Fourier Transform: The Fourier transform of a time function $f(t)$, satisfying the Dirichlet conditions, is given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-j\omega t}\, dt.$$

The time function $f(t)$ can be obtained from the frequency function $F(\omega)$ by the inverse Fourier transform,

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega.$$

This pair of transforms is symbolically represented by $f(t) \leftrightarrow F(\omega)$.

Shifting Property of Fourier Transform: If the time function $f(t)$ is shifted by an amount t_0, its Fourier transform is multiplied by exp $(j\omega t_0)$, i.e., if $f(t) \leftrightarrow F(\omega)$, then $f(t - t_0) \leftrightarrow F(\omega)$ exp $(-j\omega t_0)$.

Differentiation and Integration Property: If $f(t) \leftrightarrow F(\omega)$ then,

$$\frac{df(t)}{dt} \leftrightarrow j\omega\, F(\omega)$$

and

$$\int f(t)\, dt \leftrightarrow \frac{F(\omega)}{j\omega}.$$

Impulse Function: As a 'generalised function' the unit impulse $\delta(t)$ occurring at $t = 0$ is defined by,

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(0)$$

where function $f(t)$ is continuous at $t = 0$. As an ordinary function it may be thought of as the limiting case of a sequence of pulses of unit area, duration Δ and height $1/\Delta$ as $\Delta \to 0$.

Impulse Response: The output of a linear, initially relaxed system to a unit impulse input is called its *impulse response* and is represented by the symbol $h(t)$. The Fourier transform of unit impulse is 1.

Convolution Integral: The convolution between two functions $f_1(t)$ and $f_2(t)$ is given by,

$$f_1(t) * f_2(t) = \int_{-\infty}^{\infty} f_1(\tau) f_2(t-\tau)\, d\tau.$$

Convolution Theorem: Let $h(t) \leftrightarrow H(\omega)$, $x(t) \leftrightarrow X(\omega)$ and $y(t) \leftrightarrow Y(\omega)$. If $y(t) = x(t) * h(t)$, then $Y(\omega) = X(\omega) \cdot H(\omega)$.

System Function: The Fourier transform of the unit impulse response $h(t)$ is called the system function $H(\omega)$, i.e., $H(\omega) = \mathcal{F}[h(t)]$.

Filters: Any device used to either extract or suppress certain frequency components from a given signal is called a filter. A low pass filter suppresses high frequencies but allows low frequency components to pass through it. A high pass filter suppresses low frequencies and passes high frequencies. A band pass filter allows frequencies in a certain frequency range to pass through it and suppresses all frequency components above or below this pass band.

Distortions in a Filter: If the amplitude spectrum of the system function $H(\omega)$ of a filter is not constant w.r.t. ω in its pass band, it produces amplitude distortions. If its frequency spectrum is not a linear function of ω, it produces phase distortions.

PROBLEMS

5.1. Determine the Fourier transform for the following functions of time.

(i) $f(t) = t^2 e^{-2t} u(t)$.

(ii) $f(t) = 2a/(a^2 + t^2)$

(iii) $f(t) = e^{-at^2}$

(iv) $f(t) = \left[t^{n-1}/(n-1)!\right] e^{-at} u(t)$.

5.2. Determine the Fourier transform and sketch the spectrum for the functions of time described by Fig. 5.20(a-d).

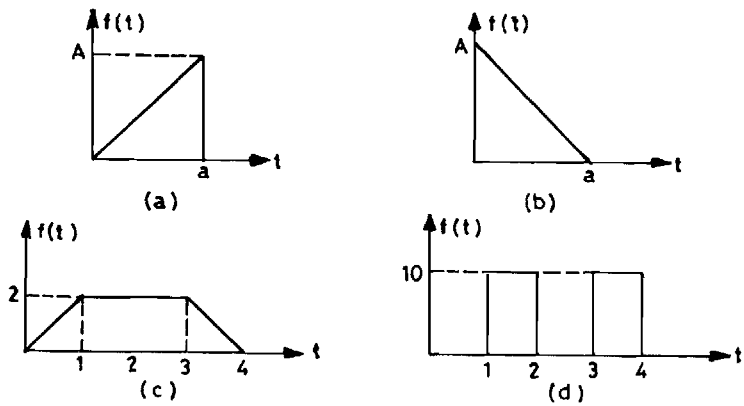


Fig. 5.20

5.3. Let $f(t) \leftrightarrow F(\omega)$. Prove the following properties of Fourier transforms:

(i) Scaling: $f(at) \leftrightarrow \frac{1}{|a|} F\left(\frac{\omega}{a}\right)$

(ii) Frequency modulation: $f(t) \exp(j\omega_0 t) \leftrightarrow F(\omega - \omega_0)$.

(iii) Frequency convolution: $\mathcal{F}[f_1(t) . f_2(t)] = 1/2\pi \, . \, F_1(\omega) * F_2(\omega)$

where $f_1(t) \leftrightarrow F_1(\omega)$ and $f_2(t) \leftrightarrow F_2(\omega)$.

5.4 Prove the statement that the convolution of a function with an impulse is the function itself, i.e.,

$$f(t) = \int_{-\infty}^{\infty} \delta(t-\tau) f(\tau) \, d\tau.$$

5.5. Determine and sketch the impulse response of a second order system with $\omega_n = 2$ rad/sec and $\zeta = 0.5$.

5.6. The impulse response of a system is given by $h(t) = t e^{-t}$, $t > 0$. Determine and sketch its response to a triangular pulse of magnitude 1, duration 2, centred at $t = 2$ using (i) the convolution integral and (ii) the Fourier transform.

5.7 Determine and sketch the voltage across the capacitor of a series RC circuit when its input is a voltage pulse of amplitude $1/\Delta$ and duration Δ, with: (i) $\Delta = 10\,RC$; (ii) $\Delta = RC$; and (iii) $\Delta = 0.1\,RC$ What conclusions do you derive from a comparison of these three cases?

5.8. The amplitude spectrum of an amplifier is constant over the frequency range 0-1 kHz. Its phase spectrum is given by $\angle H(\omega) = -2 \times 10^{-3} f$ radians where f is the frequency in Hz. Determine the minimum duration of a rectangular pulse which can be amplified by this amplifier without distortion.

5.9. A filter has the system function

$$H(\omega) = ja\omega / (1 + ja\omega).$$

(i) Plot the frequency response of the filter and, hence, determine the type of the filter

(ii) Find its output for a single sine wave input.

CHAPTER 6

Laplace Transform

LEARNING OBJECTIVES

After studying this chapter you should be able to:

(i) determine the Laplace transform $F(s)$ of any function of time $f(t)$;

(ii) determine the transfer function models of linear time invariant systems;

(iii) determine the Laplace inverse of a transform, using partial fraction expansion; and

(iv) determine the response to linear systems to any type of input, non-period or periodic, with or without initial conditions.

A non-periodic function of time $f(t)$ can be represented in the frequency domain by its Fourier transform $F(\omega)$, provided it is absolutely integrable in the sense,

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty .$$

However, many useful functions–like the step function, the ramp function, the exponentially increasing function–do not satisfy this condition and therefore do not possess a Fourier transform.* This difficulty is overcome in the Laplace transform by resolving the time function into decaying sinusoidal components. The frequency variable then becomes a complex frequency $s = \sigma + j\omega$. Thus, the Laplace transform is a generalisation of the Fourier transform and is perhaps the most powerful tool in the analysis of linear system.

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* Fourier transform for such functions can still be defined in terms of a limiting case of sequences. However, Fourier transforms, so defined, contain singularity functions of frequency and hence the analysis using such functions is not so straightforward.

6.1 From Fourier Transform to Laplace Transform

Let $f(t)$ be any function, including those which do not converge to zero as $t \to \infty$ (like a step function) and hence do not satisfy the integrability condition of the Fourier transform. To promote convergence of this function, let us multiply it by an exponentially decaying factor $e^{-\sigma t}$, where the value of the real constant σ is such that,

$$\int_{-\infty}^{\infty} |f(t)\, e^{-\sigma t}|\, dt < \infty .$$

Then, function $f(t)\, e^{-\sigma t}$ will have a Fourier transform given by,

$$F(\omega) = \int_{-\infty}^{\infty} f(t) \exp(-\sigma t) \exp(-j\omega t)\, dt$$

$$= \int_{-\infty}^{\infty} f(t) \exp\{-(\sigma + j\omega)\}t\, dt .$$

Giving $(\sigma + j\omega)$ a new symbol, s, i.e., $s = \sigma + jw$, we get,

$$F(s) = \int_{-\infty}^{\infty} f(t) \exp(-st)\, dt . \tag{6.1}$$

$F(s)$ is called the *Laplace transform* of $f(t)$ and is defined by eqn. (6.1). To get the inverse Laplace transform we start with eqn. (5.4) for the inverse Fourier transform which states that,

$$f(t) = \frac{1}{2\pi j} \int_{-\infty}^{\infty} F(\omega)\, e^{j\omega t}\, d\omega .$$

Change the variable of integration in this equation from ω to $s = \sigma + j\omega$. Then, $d\omega = ds/j$. The limits of integration $\omega = -\infty$ and $\omega = +\infty$ become $s = \sigma - j\infty$ and $s = \sigma + j\infty$. With these changes eqn. (5.4) becomes,

$$f(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) \exp(st)\, ds . \tag{6.2}$$

Equations (6.1) and (6.2) define the direct and the inverse Laplace transforms denoted by $F(s) = \mathcal{L} f(t)$ and $f(t) = \mathcal{L}^{-1} F(s)$. The pair is denoted by $f(t) \leftrightarrow F(s)$.

In the analysis of physical systems the values of input, output and other variables are usually counted after an instant $t = 0$, the time at which the input is applied to the system. That is, the values of functions prior to $t = 0$ are usually zero. Hence, the lower limit of the defining integral (6.1) may be made zero, i.e.,

$$F(s) = \int_{0}^{\infty} f(t) \exp(-st)\, dt . \tag{6.3}$$

The Laplace transform defined by eqn. (6.3) is called the *unilateral Laplace transform* as opposed to eqn. (6.1), which is called the *bilateral Laplace transform*. We are concerned only with the unilateral Laplace transforms.

The complex variable $s = \sigma + j\omega$ and can be represented graphically in a 'complex s-plane' or simply the s-plane. In the s-plane the real or the x-axis is the variable σ and the imaginary or the y-axis is $j\omega$. A particular value of $s = s_1$ is defined by its real and imaginary coordinates σ_1 and $j\omega_1$, as shown in Fig. 6.1.

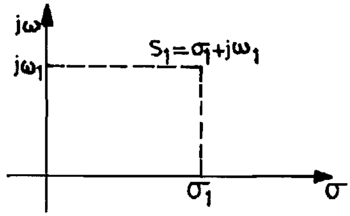


Fig. 6.1 The s-plane

The convergence factor σ, needed to force a function $f(t)$ to converge and hence have a Laplace transform will be different for different functions $f(t)$. In fact for every $f(t)$ there will be a range over which σ may take its values. This range of σ will define a region in the s-plane. This region is called the *region of convergence* of the Laplace transform of $f(t)$. The regions of convergence for some of the functions are shown in Fig. 6.2. For a function like $u(t)$, any positive value of σ, i.e.

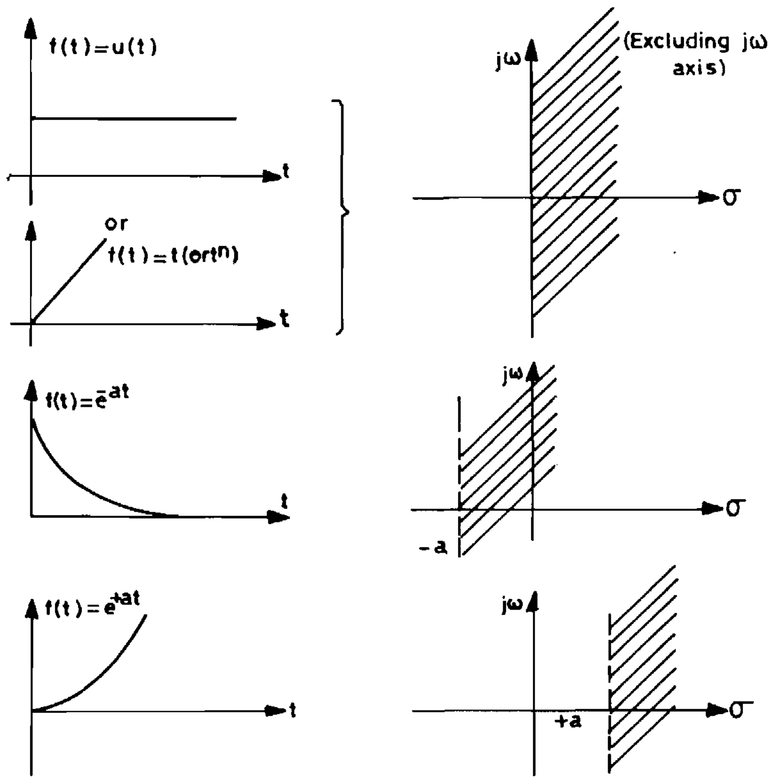


Fig. 6.2 Regions of Convergence for Some Functions

$\sigma > 0$, is sufficient to cause convergence. Hence, the entire right half of the s-plane, excluding the $j\omega$ axis, is the region of convergence for the Laplace transform of a step function. Ramp function $f(t) = t$, or any other power of t, will also have the entire right half of s-plane, except the $j\omega$ axis, as its region of convergence.

The above graphical considerations also give us a correlation between the Fourier transform and the Laplace transform. Since replacing the variable s by $j\omega$ in the definition of the Laplace transform gives the definition of the Fourier transform, it may be thought that replacing s by $j\omega$ in any Laplace transform expression will give the Fourier transform of the corresponding time function. However, all the functions which have Laplace transforms do not necessarily have Fourier transforms. In fact, only those functions will have a Fourier transform for which $\sigma = 0$ gives a valid Laplace transform. In other words, if the $j\omega$-axis is included in the region of convergence of the Laplace transform, only then replacing s by $j\omega$ converts the Laplace transform into the Fourier transform. If the $j\omega$-axis is not included in the convergence region, it means that the function is not integrable and, hence, has no Fourier transform.

From the above discussion, it should not be assumed that we can always find some σ which will make *any* given function convergent. For functions like e^{t^2} or t^t, there is no value of σ which will make the function convergent when multiplied by $\exp(-\sigma t)$. Therefore, such functions are not Laplace transformable. Functions for which there is some value of real constant σ_1 such that,

$$\int_0^\infty |f(t) \exp(\sigma_1 t)| \, dt < \infty ,$$

are called functions of *exponential order*. The Laplace transform is defined only for such functions. Fortunately in linear system analysis we seldom encounter functions which are not of exponential order.

6.2 Properties of Laplace Transform

Linearity: The Laplace transformation is a linear operation, i.e., if $f_1(t) \leftrightarrow F_1(s)$ and $f_2(t) \rightarrow F_2(s)$, then,

$$[A f_1(t) + B f_2(t)] \rightarrow [A F_1(s) + B F_2(s)] . \tag{6.4}$$

This property can be verified directly from the definition of the Laplace transform.

Shifting property: This is a direct extension of the shifting property of the Fourier transform, given in Section 5.2. If a function $f(t)$ is shifted to the right by an amount t_0, the Laplace transform of the shifted function is given by,

$$\mathcal{L}[f(t - t_0)\, u(t - t_0)] = \int_{t_0}^\infty f(t - t_0)\, e^{-st}\, dt .$$

Note that the lower limit of integration has been shifted to t_0, because the function is zero in the range 0 to t_0. Changing the variable of integration to $\tau = t - t_0$, we get $t = \tau + t_0$, and hence, $dt = d\tau$. The lower limit of integration changes back to zero.

With these changes, we have

$$\mathcal{L}[f(t - t_0)] = \int_{t_0}^{\infty} f(t - t_0)\, e^{-st}\, dt$$

$$= \int_0^{\infty} f(\tau) \exp[-s(\tau + t_0)]\, d\tau$$

$$= \exp(-st_0) \int_0^{\infty} f(\tau) \exp(-s\tau)\, d\tau = \exp(-st_0)\, F(s) .$$

That is, if $f(t) \leftrightarrow F(s)$ then,

$$f(t - t_0) \leftrightarrow \exp(-t_0 s)\, F(s) \tag{6.5}$$

This is called the *time shift* property.

Multiplication by e^{-at}: Let $f(t) \leftrightarrow F(s)$. Then,

$$\mathcal{L}[e^{-at} f(t)] = \int_0^{\infty} f(t)\, e^{-(s+a)}\, dt = F(s+a) . \tag{6.6}$$

This is called the *frequency shift* property.

Time differentiation and integration: In this case we have,

$$\mathcal{L}\left[\frac{df(t)}{dt}\right] = \int_0^{\infty} f'(t)\, e^{-st}\, dt .$$

Integrating by parts we get,

$$\int_0^{\infty} f'(t)\, e^{-st}\, dt = f(t)\, e^{-st}\Big|_0^{\infty} + s\int_0^{\infty} f(t)\, e^{-st}\, dt$$

$$= -f(0) + sF(s)$$

or $$\mathcal{L}[f'(t)] = sF(s) - f(0) . \tag{6.7a}$$

Extending the result to the nth derivative we get,

$$\mathcal{L}[f^n(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - f^{n-1}(0) . \tag{6.7b}$$

Similarly,

$$\mathcal{L}\left[\int_0^t f(t)\, dt\right] = \int_0^{\infty}\left[\int_0^t f(t)\, dt\right] e^{-st}\, dt .$$

Integrating by parts we get,

$$\mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{-e^{-st}}{s}\int_0^t f(t)\,dt\,\Bigg|_0^\infty + 1/s\int_0^\infty f(t)\,e^{-st}\,dt\,.$$

The first term on the r.h.s. is equal to zero. The second term is $F(s)/s$.

$$\therefore \qquad \mathcal{L}\left[\int_0^t f(t)\,dt\right] = \frac{F(s)}{s}\,. \tag{6.8}$$

Multiplication and division by t: Differentiating $F(s)$ w.r.t. s,

$$\frac{dF(s)}{ds} = \int_0^\infty f(t)\,\frac{de^{-st}}{ds}\,dt = -\int_0^\infty [\,t\cdot f(t)\,]\,e^{-st}\,dt\,.$$

Therefore,

$$\mathcal{L}\,[\,t f(t)\,] = -\frac{dF(s)}{ds}\,. \tag{6.9}$$

Further,

$$\int_s^\infty F(s)\,ds = \int_s^\infty\left[\int_0^\infty f(t)\,e^{-st}\,dt\right]ds\,.$$

Interchanging the order of integration we get,

$$\int_s^\infty F(s)\,ds = \int_0^\infty f(t)\left[\int_s^\infty e^{-st}\,ds\right]dt = \int_0^\infty\left[\frac{f(t)}{t}\right]e^{-st}\,dt\,.$$

Therefore,

$$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^\infty F(s)\,ds \tag{6.10}$$

Initial and final value theorems: From eqn. (6.7a) we have,

$$\mathcal{L}\,[f'(t)\,] = \int_0^\infty f'(t)\,e^{-st}\,dt = s\,F(s) - f(0)\,.$$

In the limit, as $s \to \infty$, the integral term becomes zero. Therefore,

$$\lim_{s\to\infty} s\,F(s) = f(0)\,. \tag{6.11}$$

Equation (6.11) is called the *initial value theorem* and permits the evaluation of the initial value of a function from its Laplace transform, without the need for finding its inverse. If $f(t)$ is discontinuous at $t = 0$, eqn. (6.11) gives the value of $f(0)$ when zero is approached from the positive side, i.e., $f(0^+)$.

Now, let s approach the lower limit, i.e., $s \to 0$. Then, from eqn. (6.7a) we have,

$$\lim_{s\to 0}\int_0^\infty f'(t)\,e^{-st}\,dt = \lim_{s\to 0}\,[\,s\,F(s) - f(0)\,]\,.$$

With $s \to 0$, the left-hand side integral becomes,

$$\int_0^\infty f'(t)\,dt = \lim_{t \to \infty} f(t) - f(0)\,.$$

Therefore,

$$\lim_{s \to 0} sF(s) = \lim_{t \to \infty} f(t)\,. \tag{6.12}$$

Equation (6.12) is called the *final value theorem.*

6.3 Laplace Transforms of Some Common Functions

The Laplace transforms of most of the commonly encountered functions can be derived from a knowledge of the transform for only a few elementary functions, and from the properties listed in the previous section.

In these derivations, it is assumed that $f(t) = 0$ for $t < 0$, i.e., all the functions are multiplied by $u(t)$.

1. The exponential function, $f(t) = e^{at}$:

$$\mathcal{L}\,[\,e^{at}\,] = \int_0^\infty e^{at}\,e^{-st}\,dt = \int_0^\infty e^{-(s-a)}\,dt = \frac{1}{(s-a)}\cdot \tag{6.13}$$

2. The unit step function, $f(t) = u(t)$:

If a is made equal to zero in the exponential function e^{at}, we get the step function. Therefore, making $a = 0$ in eqn. (6.13) we get,

$$\mathcal{L}\,u(t) = \frac{1}{s}\cdot \tag{6.14}$$

3. The sine function, $f(t) = A \sin \omega t$:

$$\mathcal{L}\,[\,A \sin \omega t\,] = \mathcal{L}\,\frac{A \exp\,(j\,\omega\,t)}{2j} - \mathcal{L}\,\frac{A\,\exp\,(-\,j\,\omega\,t)}{2j}$$

Using relation (6.13) for writing the Laplace transform of the exponential terms we get,

$$\mathcal{L}\,[\,A \sin \omega\,t] = \frac{A}{2j}\left[\frac{1}{s-j\omega} - \frac{1}{s+j\omega}\right].$$

Simplifying the r.h.s. we get,

$$\mathcal{L}\,[\,A \sin \omega\,t\,] = \frac{A\omega}{s^2+\omega^2}\cdot \tag{6.15}$$

4. The cosine function, $f(t) = A \cos \omega t$:

Following the same procedure as for the sine function we get,

$$\mathcal{L}\,[\,A \cos \omega t\,] = \frac{As}{s^2+\omega^2}\cdot \tag{6.16}$$

5. Decaying sine and cosine functions:

Using the frequency shift property [eqn. (6.6)], eqns. (6.15) and (6.16) directly give,

$$\mathcal{L}\,[\,e^{-at}\sin\omega t\,] = \frac{A\omega}{(s+a)^2+\omega^2} \tag{6.17}$$

$$\mathcal{L}\,[\,e^{-at}\cos\omega t\,] = \frac{A\,(s+a)}{(s+a)^2+\omega^2}\,. \tag{6.18}$$

6. The ramp function, $f(t) = t$:

Treating the ramp function as a multiplication of the unit step function by t, i.e., $f(t) = tu(t)$, and using eqn. (6.9) for multiplication by t we get,

$$\mathcal{L}\,[\,t\,u\,(t)\,] = -\,\frac{d\,(1/s)}{ds} = \frac{1}{s^2}\,. \tag{6.19}$$

7. Powers of $t, f(t) = t^n$:

A direct extension of the considerations for the ramp function gives,

$$\mathcal{L}\,[t^n] = \frac{n!}{s^{n+1}}\,. \tag{6.20}$$

8. $f(t) = t\,e^{-at}$:

Using eqn. (6.19) for the ramp function and the property given by [eqn. (6.6)] we get,

$$\mathcal{L}\,[\,t\,e^{-at}\,] = \frac{1}{(s+a)^2}\,. \tag{6.21}$$

9. $f(t) = (1 - e^{-at})\,u\,(t)$:

This frequently encountered function is the unit step response of a first order system. From the linearity property we have,

$$\mathcal{L}\,[f(t)\,] = \mathcal{L}\,[\,u\,(t) - e^{-at}\,u\,(t)\,]$$

$$= \frac{1}{s} - \frac{1}{(s+a)} = \frac{a}{s\,(s+a)}\,. \tag{6.22}$$

10. $f(t) = A\,e^{-at}\sin\,(\omega t + \theta)$:

Let us first find the Laplace transform for $A\sin\,(\omega t + \theta)$ and then use eqn. (6.6).

$$\mathcal{L}\,[\,A\sin\,(\omega t + \theta\,)\,] = A\;\mathcal{L}\,[\,\sin\omega t\cos\theta + \cos\omega t\sin\theta\,]$$

$$= A\left[\frac{\omega\cos\theta}{s^2+\omega^2} + \frac{s\sin\theta}{s^2+\omega^2}\right]$$

$$= A\left[\frac{\omega\cos\theta + s\sin\theta}{s^2+\omega^2}\right]$$

Therefore,

$$\mathcal{L}\ [\ A\ e^{-at} \sin(\omega t + \theta)\ ] = A\ \frac{\omega \cos\theta + (s+a)\sin\theta}{(s+a)^2 + \omega^2}\ . \tag{6.23}$$

11. *Rectangular pulse:*

Expressing the pulse as a sum of two step functions, $f(t) = u(t) - u(t-T)$, and using the shifting property [(eqn. 6.5)] for the second term we get,

$$\mathcal{L}\ [f(t)\ ] = \frac{1}{s} - \frac{e^{-Ts}}{s} = \frac{1 - e^{-Ts}}{s}\ . \tag{6.24}$$

12. *Unit impulse:*

In the definition [eqn. (5.13)] of a unit impulse,

$$\int_{-\infty}^{\infty} \delta(t) f(t)\, dt = f(o)\,,$$

we substitute $f(t) = e^{-st}$ to get the Laplace transform of the unit impulse. Since we are assuming all functions to be zero for $t < 0$, the lower limit of the above integral can be made zero. Thus, just like the Fourier transform, we have,

$$\mathcal{L}\ [\ \delta(t)\ ] = \int_0^{\infty} \delta(t)\, e^{-st}\, dt = (e^{-st})_{t=0} = 1\ . \tag{6.25}$$

The Laplace transform of an impulse of strength k will be simply k and that of $\delta(t-\tau)$ will be $e^{-s\tau}$.

For convenience, the common functions and their Laplace transforms are listed in Table 6.1.

Table 6.1 Laplace Transforms of Common Signals

	Time function $f(t)$	*Laplace transform* $F(s)$
1.	e^{at}	$1/(s-a)$
2.	$u(t)$	$1/s$
3.	$A \sin \omega t$	$A\omega / (s^2 + \omega^2)$
4.	$A \cos \omega t$	$As / (s^2 + \omega^2)$
5.	$A\, e^{at} \sin \omega t$	$A\omega / [(s+a)^2 + \omega^2]$
6.	$A\, e^{-at} \cos \omega t$	$A\,(s+a) / [(s+a)^2 + w^2]$
7.	t	$1/s^2$
8.	t^n	$n! / s^{n+1}$
9.	$t\, e^{-at}$	$1 / (s+a)^2$
10.	$t^n\, e^{-at}$	$n! / [\,(s+a)^{n+1}]$
11	$1 - e^{-at}$	$a / s\,(s+a)$
12.	$A\, e^{-at} \sin(\omega t + \theta)$	$A\,[\omega \cos\theta + (s+a)\sin\theta] / [\,(s+a)^2 + \omega^2)]$
13.	Rectangular pulse Unit Amplitude Duration $0 - T$	$(1 - e^{-Ts}) / s$
14.	Impulse of strength k, $k\,\delta(t)$	k

6.4 The Transfer Function

Let $x(t) \leftrightarrow X(s)$ be the input to a system and $h(t) \leftrightarrow H(s)$ the impulse response of the system. Then the output $y(t) \leftrightarrow Y(s)$ in the time domain is given by the convolution integral,

$$y(t) = \int_0^t x(\tau)\, h(t-\tau)\, d\tau . \tag{6.26}$$

Taking the Laplace transform of both sides of eqn. (6.26) we get,

$$Y(s) = \int_0^\infty e^{-st} \left[\int_0^t x(\tau)\, h(t-\tau)\, d\tau \right] dt .$$

The upper limit of the integral inside the brackets can be made ∞ without altering its value, because $h(t-\tau) = 0$ for $\tau > t$. Also, the order of integration can be interchanged. Thus,

$$Y(s) = \int_0^\infty x(\tau) \left[\int_0^\infty e^{-st}\, h(t-\tau)\, dt \right] d\tau .$$

From the shifting property [eqn. (6.5)], the term inside the brackets is equal to exp $(-s\tau)\, H(s)$. Therefore,

$$Y(s) = H(s) \int_0^\infty x(\tau) \exp(-s\tau)\, d\tau = H(s)\, X(s) . \tag{6.27}$$

Equation (6.27) is the result of application of the convolution theorem to the Laplace transform. A similar result is relation (5.24) for the Fourier transform. It means that the effect of convolution of two functions in the time domain is the multiplication of their Laplace transforms in the s-domain. The term $H(s) = \mathcal{L}[h(t)]$ is called the *transfer function*. Thus, if the transfer function $H(s)$ of a system is known, its response to any input can be found out, using relation (6.27). This fact is the cornerstone of the linear systems analysis.

Since relation (6.27) is an algebraic relation we could also write it as,

$$H(s) = \frac{Y(s)}{X(s)} . \tag{6.28}$$

Equation (6.28) gives another way of defining the transfer function of a linear system: it is the ratio of the Laplace transform of the output to the Laplace transform of the input for an initially relaxed system. (Note that the impulse response is, by definition, the response of an initially relaxed system to a unit impulse.) Thus, if we know the Laplace transforms of the input-output pair, we can find the transfer function of the system.

It can be seen that the system function $H(\omega)$, defined in connection with the Fourier transform, and the transfer function $H(s)$ defined here, are very much similar to each other. In fact, subject to certain conditions as mentioned in section

6.1, $H(\omega)$ can be obtained from $H(s)$ by replacing s by $j\omega$. However, $H(s)$ is more general in the same way as the Laplace transform is more general than the Fourier transform.

Since the knowledge of $H(s)$ completely characterises a system, it is also known as the mathematical model of the system in the s-domain.

The Laplace transform method of analysis makes it possible to obtain the output of a linear system to any type of input: periodic, non-periodic, converging or non-converging (provided it is of an exponential order). The basic steps in the analysis procedure may be summarised as follows:

(1) Obtain the Laplace transform of the input $X(s)$.

(2) Obtain the transfer function $H(s)$.

(3) Multiply the two together to get the Laplace transform of the output, i.e., $Y(s) = X(s)\,H(s)$.

(4) Find the Laplace inverse of $Y(s)$ to get $y(t)$.

The first step, i.e., the determination of the Laplace transform of the input has already been covered in the previous sections. We now discuss the second step.

Determination of transfer function:

If the analytical expression for the impulse response $h(t)$ of a linear system is known, the determination of its transfer function, which is the Laplace transform of $h(t)$, is quite straightforward. However, if we have to start from the given system itself, which is most frequently the case, we formulate the differential equation model first and determine the transfer function from this differential equation. This is illustrated by the following examples.

Example 6.1:—*Transfer function of a d.c. generator:* When considered as an electromechanical energy converter, the d.c. generator converts mechanical power into electrical power. In many control applications, a d.c. generator is used as a power amplifier, amplifying the small input power to its field winding into a very large armature power output delivered to the load. In the family of amplifiers, the d.c. generator is called a 'rotating amplifier'. Let us determine its transfer function, treating the signal voltage applied across the field winding as the input and the voltage across the armature terminals as the output, under the no-load condition.

Let us first write the system equations with reference to Fig. 6.3. The equation relating the input voltage to the field current is given by,

$$L\,\frac{di_f}{dt} + Ri_f = v_i\,. \qquad \text{(i)}$$

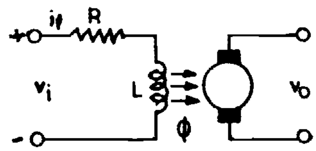


Fig. 6.3 A d.c. Generator

The field current produces magnetic flux ϕ, which in turn produces the output voltage v_o. In the portion prior to saturation, the magnetising curve of the d.c. generator may be assumed linear. In that case, the output voltage v_o is a linear function of the field current i, i.e.,

$$v_o = ki_f. \tag{ii}$$

Combining eqns. (i) and (ii) we get the differential equation of the system as,

$$\frac{L}{k}\frac{dv_o}{dt} + \frac{R}{k}v_o = v_i. \tag{iii}$$

Taking the Laplace transform of both the sides, with initial conditions assumed to be zero, we have,

$$\frac{L}{k}s\,V_o(s) + \frac{R}{K}V_o(s) = V_i(s). \tag{iv}$$

where $\quad v_o(t) \leftrightarrow V_o(s)$ and $v_i(t) \leftrightarrow V_i(s)$.

From eqn. (iv), the transfer function $H(s) = V_o(s) / V_i(s)$ is given as,

$$H(s) = \frac{1}{(L/k)s + R/k} = \frac{k}{Ls + R} = \frac{k/L}{s + R/L}. \tag{6.29}$$

The input may be viewed as the 'cause' and the output as the 'effect'. Thus, the transfer function relates the effect to the cause. This overall cause-effect relationship is made up of a number of links in the form of a cause-effect chain. In the previous problem, for example, the first cause v_i produces an immediate effect i_f. Treating i_f as the cause, we find that it produces a flux ϕ as the effect. Next, the cause ϕ produces the effect v_o. Each one of these cause-effect links gives rise to one system equation.

Unlike the previous example, which is quite simple, other problems may have a large number of cause-effect links and, hence a larger number of equations. One method of finding the transfer function is to combine all those differential/algebraic equations into a single equation, relating the input and the output, and then take its Laplace transform. A simpler method is to take the Laplace transform of each system equation and then combine these equations to get the overall transfer function. This second method is easier because the operation of taking the Laplace transform converts a differential equation into an algebraic equation, and combining these algebraic equations is easier than combining differential equations. The approach to be followed is the same as that described in Chapter 1 for making the differential equation model of a system.

Example 6.2:—*Vibration table:* High reliability electronic sub-assemblies for use in locations which are subjected to severe vibrations, like military mobile units, locomotives, aircraft, etc., have to be pretested for failures due to vibration. For this purpose, sample units are placed on a vibration table and vibrated continuously at controlled frequencies and amplitudes for a specified period, say, eight hours. A schematic diagram of the vibration table is shown in Fig. 6.4. The variable frequency alternating current flowing in the coil reacts with the flux of the permanent magnet yoke to produce an oscillatory force on the coil and the table attached to it. The test sample is suitably secured to this vibrating table. The object of analysis here is to determine the transfer function relating the output x(t), the vertical displacement of the plateform, to the input v(t), the voltage applied to the coil.

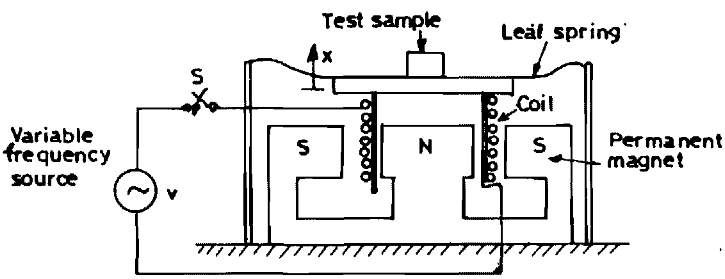


Fig. 6.4 Vibration Table

The input voltage $v(t) = V\cos\omega t$ produces a current i in the coil. The relationship between v and i can be obtained by applying Kirchhoff's voltage law around the electrical circuit. The applied voltage will be opposed by the voltage drop across the resistance R and inductance L of the coil. In addition, there will be an opposing back e.m.f. due to the motion of the coil in the magnetic field of the permanent magnet. Its magnitude will be proportional to the linear velocity of the coil. Thus, the electrical equation is,

$$Ri + L\frac{di}{dt} + K_1\dot{x} = v. \qquad \text{(i)}$$

Now, treating the current as the cause, the effect it produces is the force f. Since the magnetic flux is constant, f and i will be linearly related. i.e.,

$$K_2 i = f. \qquad \text{(ii)}$$

The force f, acting on the mechanical system, produces a displacement x. Neglecting damping due to air friction, the mechanical equation is,

$$M\ddot{x} + K_3 x = f. \tag{iii}$$

where M is the total moving mass and K_3 the spring constant of the leaf spring.

Eliminating the algebraic eqn. (ii) by writing $K_2 i$ *for* f in eqn. (iii) and taking this term to the left we get,

$$-K_2 i + M\ddot{x} + K_3 x = 0. \tag{iv}$$

Taking the Laplace transform of the two system eqns. (i) and (iv) (assuming all initial conditions to be zero) we get,

$$(R + Ls)\, I(s) + K_1 s\, X(s) = V(s). \tag{v}$$

$$-K_2 I(s) + (Ms^2 + K_3)\, X(s) = 0. \tag{vi}$$

Combining the algebraic eqns. (v) and (vi), we get the transfer function,

$$H(s) = \frac{X(s)}{V(s)} = \frac{K_2}{(R + Ls)(Ms^2 + K_3) + K_1 K_2 s}$$

$$= \frac{K_2}{LMs^3 + RMs^2 + (K_1 K_2 + K_3 L)\, s + K_3 R} \tag{6.30}$$

The differential equation model of the system shows that it is a third order system. Accordingly, the denominator of the transfer function, eqn. (6.30), is a third order polynomial. This is a general property of the transfer function, i.e., the order of the denominator polynomial is the order of the system.

Transfer function from the differential equation model (general case):

A general nth order system is represented by an nth order differential equation,

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y = x(t).$$

Taking the Laplace transform of both sides of this equation, with initial conditions assumed to be zero, gives,

$$(s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0)\, Y(s) = X(s).$$

Therefore, the transfer function of the system is,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

In general, the governing differential equation may contain derivatives of input also. The system equation will then be,

$$\frac{d^n y}{dt^n} + a_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 \frac{dy}{dt} + a_0 y$$

$$= \frac{d^m x}{dt^m} + b_{m-1} \frac{d^{m-1} x}{dt^{m-1}} \cdots + b_1 \frac{dx}{dt} + b_0 y$$

and its transfer function,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0} . \tag{6.31}$$

Equation (6.31) is the most general form of the transfer function of a linear system. It consists of a ratio of two polynomials in s i.e., it is a *rational function* of s. The powers of s are all integral powers. Another condition satisfied by physical systems is that the order of the numerator polynomial m is less than or at the most equal to the order of the denominator polynomial. To demonstrate the need for this condition, let us take an example which violates this requirement. Let,

$$H(s) = \frac{s^3 + 2s^2 + 2s + 1}{s^2 + 2s + 2} .$$

Performing long division, it can be rewritten as,

$$H(s) = s + \frac{1}{s^2 + 2s + 2} .$$

If the input to this system is $x(t) \leftrightarrow X(s)$, then the output $y(t) \leftrightarrow Y(t)$ will be given by,

$$Y(s) = s\,X(s) + \frac{X(s)}{s^2 + 2s + 2} .$$

The Laplace inverse of the first term would be the derivative of the input. That is, if the input is a step function, the output will have an impulse term. Such a behaviour is not encountered in physical systems. In dynamic systems, because of the differential relation, the input is always integrated to give an output, never differentiated. At worst, e.g., in a static system, the output may be a constant times the input. And hence the conclusion that in the transfer function of a physical system, the order of the numerator polynomial is either less than or, at the most, equal to the order of the denominator polynomial.

The form of response of a system is very much governed by the denominator of its transfer function. Hence, the denominator is called the *characteristic polynomial* of the system and the equation formed by equating it to zero, i.e.,

$$s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0 = 0$$

is called the *characteristic equation* of the system.

Poles and zeros:

Factorising the numerator and the denominator the general transfer function, eqn. (6.31), can also be written as,

$$H(s) = \frac{(s-z_1)(s-z_2)\cdots(s-z_m)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \tag{6.32}$$

There will be m factors of the mth order numerator polynomial and n factors of the nth order denominator polynomial. Constants $p_1, \ldots, p_n$ will be the roots of the characteristic equation. Similarly $z_1, \ldots, z_m$ will be roots of the equation $s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0 = 0$. These constants could be real or complex. Some of them could also be equal.

When the complex variable s assumes any one of the values $p_1, \ldots, p_n$ the value of the transfer function $H(s)$ becomes infinity. Hence, these values, i.e., $p_1, \ldots, p_n$ are called the *poles* of the system. Similarly, when s assumes any value equal to $z_1, \ldots, z_m$, $H(s)$ becomes zero. Hence, $z_1, \ldots, z_m$ are called the *zeros* of a system. An nth order system will always have n poles. The number of its zeros is not fixed and can have any value m, so long as $m \le n$. When the poles or zeros are complex, they will always occur in pairs as complex conjugates because the coefficients a's and b's in the numerator and denominator polynomials are all real numbers.

In Section 6.1, we defined a complex plane or the s-plane, with σ as the x-axis and $j\omega$ as the y-axis, over which the variable $s = \sigma + j\omega$ assumes values. Poles are special points in the s-plane where the function $H(s)$ and its derivatives do not exist. The magnitude of $H(s)$ tends to infinity at these points. The poles are therefore called *singularities* of $H(s)$.

Example 6.3:— Determine, and display graphically, the poles and zeros of the transfer function,

$$H(s) = \frac{s^2 - 1}{s^2 + 4s + 13}.$$

Solution: The roots of the numerator and denominator polynomials are $z_1, z_2 = \pm 1$ and $p_1, p_2 = -2 \pm j3$. Thus,

$$H(s) = \frac{(s+1)(s-1)}{(s+2+j3)\ (s+2-j3)} = \frac{(s+1)(s-1)}{[(s+2)^2 + 3^2]}.$$

The pole-zero diagram is shown in Fig. 6.5. This particular transfer function has two real zeros: one in the right half and one in the left half of the s-plane. It has two complex poles in the left half of the s-plane.

Pole-zero diagrams are very useful in the analysis of systems because all the dynamic properties of a system can be understood from a knowledge of the location of the poles and zeros of its transfer function.

6.5 Partial Fraction Expansion

The third step of the analysis procedure is to multiply the transform of the input $X(s)$ and the system transfer function $H(s)$ to obtain the transform of the output

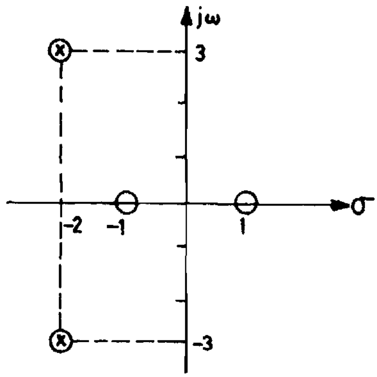


Fig. 6.5 A Pole-Zero Plot

$Y(s)$. This step requires no elaboration. We now come to the last step, i.e., finding the inverse transform of $Y(s)$ to get $y(t)$.

Application of the fundamental inversion formula (6.2) gives,

$$y(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} Y(s)\, e^{st}\, ds.$$

The integral above indicates contour integration in the complex s-plane and may be evaluated using the theory of complex variables. The evaluation of $y(t)$, however, can be done more simply in most of the cases using the method of partial fraction expansion.

$Y(s)$ is a product of two functions, $H(s)$ and $X(s)$. As we have already seen in the previous section, $H(s)$ for physical systems is a rational function of s, i.e., a ratio of two polynomials in s, with the order of the denominator polynomial higher than, or equal to, that of the numerator polynomial. A look at Table 6.1 of the Laplace transform pairs shows that the transforms for most of the common functions are also rational functions of s with the highest power in the denominator greater than that of the numerator. (Note that items 13 and 14 do not confirm to this pattern.) Therefore, the product function $Y(s)$ can also be expressed as a ratio of two polynomials in s, i.e.,

$$Y(s) = \frac{N(s)}{D(s)} = \frac{c_m s^m + c_{m-1} s^{m-1} + \cdots + c_1 s + c_0}{d_n s^n + d_{n-1} s^{n-1} + \cdots + d_1 s + d_0} \quad \text{with } m \le n$$

where c's and d's are real constants, and m and n are integers. Factorising the denominator we get,

$$Y(s) = \frac{N(s)}{d_n (s - p_1)\ (s - p_2)\ \cdots\ (s - p_n)} \tag{6.33}$$

vhere $p_1, \ldots, p_n$ are the roots of the denominator polynomial $D(s)$.

We now discuss the techniques of partial fraction expansion of expression (6.33) for two cases: (i) when all the roots are distinct and (ii) when some of the roots are repeated.

Partial fraction expansion for distinct roots:

The procedure is illustrated with the help of the following example.

Example 6.4:— Determine the Laplace inverse of the output transform,

$$Y(s) = \frac{s+1}{s^3 + 5s^2 + 6s}.$$

Solution: Factorising the denominator polynomial,

$$Y(s) = \frac{s+1}{s(s+2)(s+3)}.$$

Since all the roots of the denominator polynomial, $s = 0, -2, -3$, are distinct, the expression for $Y(s)$ can be expanded into partial fractions as

$$Y(s) = \frac{s+1}{s(s+2)(s+3)} = \frac{A_1}{s} + \frac{A_2}{s+2} + \frac{A_3}{s+3}. \qquad \text{(i)}$$

To obtain the value of the constant A_1, we multiply both sides of eqn. (i) by s and then equate s to 0. That is,

$$\left.\frac{s+1}{s(s+2)(s+3)} \cdot s \right|_{s=0} = \left[A_1 + \frac{A_2 s}{s+2} + \frac{A_3 s}{s+3}\right]_{s=0}$$

The effect of the above operation is that the r.h.s. will contain only A_1, Hence,

$$A_1 = \left.\frac{s+1}{s(s+2)(s+3)} \cdot s \right|_{s=0} = \frac{1}{6}.$$

Similarly, for A_2 we multiply both sides of eqn. (i) by $(s + 2)$ and then equate s to -2. Then,

$$A_2 = \left.\frac{s+1}{s(s+3)}\right|_{s=-2} = \frac{-1}{-2 \times 1} = \frac{1}{2}$$

and

$$A_3 = \left.\frac{s+1}{s(s+2)}\right|_{s=-3} = \frac{-2}{3}.$$

Therefore,

$$Y(s) = \frac{s+1}{s(s+2)(s+3)} = \frac{1/6}{s} + \frac{1/2}{s+2} - \frac{2/3}{s+3}. \qquad \text{(ii)}$$

The above result can be readily checked by evaluating both sides for some value of s, say $s = 1$. Then,

$$\text{l.h.s.} = \frac{2}{1 \times 3 \times 4} = \frac{1}{6}$$

$$\text{r.h.s.} = \frac{1}{6} + \frac{1}{2.3} - \frac{2}{3.4} = \frac{1}{6}\,.$$

The result of this partial fraction expansion is that the response transform $Y(s)$ has been expanded into a number of simpler terms. The inverse transform for each of these simple terms can be readily found by referring to the table of transforms. Each term of the form $A/(s+a)$ will have its inverse as $A\,e^{-at}$. Therefore, performing the inversion operation on eqn. (ii) term by term we get,

$$y(t) = \mathcal{L}^{-1}\,Y(s) = \mathcal{L}^{-1}\left[\frac{1/6}{s}\right] + \mathcal{L}^{-1}\left[\frac{1/2}{s+2}\right] - \mathcal{L}^{-1}\left[\frac{2/3}{s+3}\right]$$

$$= \frac{1}{6}\,u(t) + \frac{1}{2}\,e^{-2t} - \frac{2}{3}\,e^{-3t}\,.$$

Generalising on the basis of the above example, an nth order output transform can be expanded as,

$$Y(s) = \frac{A_1}{s+p_1} + \frac{A_2}{s+p_2} + \cdots + \frac{A_n}{s+p_n} \tag{6.34}$$

The value of the nth constant A_n in eqn. (6.34) is given by,

$$A_n = Y(s)\,(s-p_n)\,|_{s=p_n}\,. \tag{6.35}$$

n the previous example, all the roots of the denominator polynomial are real. Let us now take an example with complex roots.

Example 6.5:— Determine the Laplace inverse of,

$$Y(s) = 5/(s^3 + 5s^2 + 8s + 6)\,.$$

Solution: Factorising the denominator,

$$Y(s) = \frac{5}{(s+3)\,(s^2+2s+2)}\,.$$

The roots of the second factor in the denominator are $-1 \pm j1$. One way of finding the partial fraction expansion would be to treat the complex roots also in the same fashion as the real roots and proceed as in the previous problem. That is, write $Y(s)$ as,

$$Y(s) = \frac{5}{(s+3)\,(s^2+2s+2)}$$

$$= \frac{A_1}{s+3} + \frac{A_2}{(s+1+j1)} + \frac{A_3}{(s+1-j1)}$$

The constants A_2 and A_3 will then be complex conjugates. Evaluating them and combining their Laplace inverses, which will be in the form of exponentials with imaginary exponents, will be somewhat tedious. An easier approach is as follows.

The second order polynomial term may be written as $[(s+1)^2 + 1^2]$ after completing the square, and treated as a single factor in the partial fraction expansion. The numerator term assumed for this second order factor should be a first order polynomial. That is,

$$Y(s) = \frac{5}{(s+3)(s^2+2s+2)} = \frac{A_1}{s+3} + \frac{A_2 s + A_3}{[(s+1)^2 + 1^2]} \qquad \text{(i)}$$

$$A_1 = Y(s)(s+3)\Big|_{s=-3} = \frac{5}{9-6+2} = 1 .$$

To evaluate A_2 and A_3 we multiply both sides of the eqn. (i) by the entire denominator. Then the l.h.s. will be simply the numerator, i.e., 5, and

$$5 = A_1 (s^2 + 2s + 2) + (A_2 s + A_3)(s+3) .$$

Substituting the already calculated value of $A_1 = 1$, we get

$$5 = (1 + A_2) s^2 + (2 + 3A_2 + A_3) s + (2 + 3A_3) .$$

Comparing the coefficients of like powers of s on both sides of this equation, we get,

$$1 + A_2 = 0, \qquad \text{therefore } A_2 = -1$$

$$2 + 3A_3 = 5, \qquad \text{therefore } A_3 = 1.$$

Thus,

$$Y(s) = \frac{1}{s+3} + \frac{-s+1}{[(s+1)^2 + 1^2]}$$

$$= \frac{1}{s+3} - \frac{s+1}{[(s+1)^2 + 1^2]} + 2\,\frac{1}{[(s+1)^2 + 1^2]} .$$

A look at the Table 6.1 (items 5 and 6) shows that the second and the third factors are Laplace transforms of decaying cosine and sine functions, respectively. Therefore,

$$y(t) = \mathcal{L}^{-1} Y(s) = e^{-3t} - e^{-t}\cos t + 2e^{-t}\sin t .$$

Partial fraction expansion for repeated roots:

If some of the roots of a polynomial have equal values, we say more than one root exists at the same point in the s-plane or that the root has a multiplicity greater than one. The technique of partial fraction is somewhat different in this case, as illustrated by the following example.

Example 6.6:— Find the Laplace inverse of,

$$Y(s) = \frac{s^2 + 2s + 1}{(s + 1)^3}$$

Here, we have a root of multiplicity 3 at $s = -1$. In this case the partial fraction expansion takes the form,

$$Y(s) = \frac{s^2 + 2s + 2}{(s+1)^3} = \frac{A_1}{(s+1)^3} + \frac{A_2}{(s+1)^2} + \frac{A_3}{(s+1)} \quad \text{(i)}$$

To evaluate A_1, we multiply both sides by $(s + 1)^3$ to get,

$$s^2 + 2s + 2 = A_1 + A_2(s+1) + A_3(s + 1)^2 \quad \text{(ii)}$$

Now equating $s = -1$, we get $A_1 = 1$. That is,

$$A_1 = Y(s)\ (s + 1)^3 \big|_{s=-1} .$$

A_2 cannot be obtained by multiplying eqn. (i) by $(s + 1)^2$ and equating s to -1. Therefore, differentiate both sides of eqn. (ii) with respect to s. This gives,

$$\frac{d}{ds}(s^2 + 2s + 2) = \frac{d}{ds}(A_1) + \frac{d}{ds}[A_2(s+1)] + \frac{d}{ds}[A_3(s + 1)^2]$$

or,
$$2s + 2 = 0 + A_2 + \frac{d}{ds}[A_3(s+1)^2] . \quad \text{(iii)}$$

Now, equating s to -1 gives $A_2 = 0$. That is,

$$A_2 = \frac{d}{ds}[Y(s)(s+1)^3]_{s=-1} .$$

Similarly, for A_3 the formula is,

$$A_3 = \frac{1}{2!}\frac{d^2}{ds^2}\Big[Y(s)(s+1)^3 \Big]_{s=-1} .$$

Applying this formula, we get $A_3 = 1$. Thus,

$$Y(s) = \frac{1}{(s+1)^3} + \frac{1}{s+1} .$$

Therefore,

$$y(t) = \frac{t^2 e^{-t}}{2} + e^{-t} = \left(\frac{t^2}{2} + 1\right) e^{-t} .$$

Generalising on the experience of this example, if the jth root has multiplicity k, then corresponding to it the partial fraction expansion should have terms,

$$\frac{A_{j1}}{(s+p_j)^k} + \frac{A_{j2}}{(s+p_j)^{k-1}} + \cdots + \frac{A_{jk-1}}{(s+p_j)^2} + \frac{A_{jk}}{(s+p_j)}$$

The coefficients are determined from the formula

$$A_{jk} = \frac{1}{A_{k-1}\,!}\,\frac{d^{k-1}}{ds^{k-1}}\left[\,(s+p_j)^k\,Y(s)\,\right]_{s=-p_j} \tag{6.36}$$

Example 6.7:— Find the Laplace inverse of

$$Y(s) = \frac{s+1}{s^2\,(s^2+4)}\,.$$

Here we have a root of multiplicity 2 at the origin and a pair of complex roots at $s = \pm j2$. Expanding $Y(s)$ into partial fractions,

$$Y(s) = \frac{s+1}{s^2\,(s^2+4)} = \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{A_3\,s + A_4}{s^2+4}\,.$$

Multiply both sides by the denominator $s^2\,(s^2+4)$ we get,

$$\begin{aligned} s+1 &= A_1\,(s^2+4) + A_2\,s\,(s^2+4) + (A_3\,s + A_4)\,s^2 \\ &= (A_2 + A_3)\,s^3 + (A_1 + A_4)\,s^2 + 4\,A_2\,s + 4\,A_1\,. \end{aligned}$$

Equating coefficients of like powers of s on both sides of the equation we get,

$$A_2 + A_3 = 0,\; A_1 + A_4 = 0,\; 4\,A_2 = 1,\; 4\,A_1 = 1\,.$$

This gives,

$$A_1 = 1/4,\; A_2 = 1/4,\; A_3 = -1/4 \text{ and } A_4 = -1/4.$$

Therefore,

$$Y(s) = \frac{1}{4\,s^2} + \frac{1}{4s} - \frac{s}{4\,(s^2+4)} - \frac{1}{4\,(s^2+4)}$$

and

$$y(t) = \mathcal{L}^{-1}\;Y(s) = \frac{1}{4}\left[t + u(t) - \cos 2t - \frac{1}{2}\sin 2t\right]$$

Thus, we have the choice of either using formula (6.35) for the case of non-repeated roots or formula (6.36) for the case of repeated roots or the method of dividing by the denominator as shown in Examples 6.5 and 6.7 for determining the coefficients (A's). If the number of roots is small, say, 3 or 4, the second alternative is many a time easier. However, there is no clear-cut guideline as to which method to select for a particular problem.

6.6 Analysis with Laplace Transforms

Having learnt the mathematical techniques for performing all the four steps of the analysis procedure, we are now ready to solve some of the typical problems in linear systems analysis, using Laplace transform techniques.

Example 6.8:— The transfer function of an armature-controlled d.c. motor, relating the output speed to the input armature voltage is given by,

$$H(s) = \frac{0.03}{(s + .06)} .$$

Determine the output speed as a function of time when the armature is suddenly connected to a 240 V voltage source. Also determine the steady-state r.p.m. of the motor.

Solution: The input is a step voltage of magnitude 240 V. Therefore, its Laplace transform $X(s) = 240/s$. The output $Y(s)$ is given by,

$$Y(s) = X(s)\,H(s) = \frac{240 \times 0.03}{s\,(s + 0.06)} = \frac{7.2}{s\,(s + 0.06)} .$$

Expanding into partial fractions we get,

$$Y(s) = \frac{7.2}{s\,(s + 0.06)} = \frac{A_1}{s} + \frac{A_2}{s + 0.06} .$$

According to formula (6.35):

$$A_1 = \left.\frac{7.2}{(s + 0.06)}\right|_{s=0} = 120.$$

$$A_2 = \left.\frac{7.2}{s}\right|_{s=-0.06} = -120.$$

Therefore, $$Y(s) = 120/s - 120/(s + 0.06) .$$

Taking the Laplace inverse term by term we have,

$$y(t) = \mathcal{L}^{-1}\, Y(s) = 120\,u(t) - 120\,e^{-0.06t}$$

The output is the angular speed $\omega(t)$, i.e.,

$$\omega(t) = [\,120\,u(t) - 120\,e^{-0.06t}\,] \text{ rad/sec.}$$

The steady-state speed is achieved as $t \to \infty$. Thus,

$$\omega(t)\,|_{t\to\infty} = 120 \text{ rad/sec.} = 1150 \text{ r.p.m.}$$

This value of steady-state speed could have been obtained straightaway by using the final value theorem [eqn. (6.12)] as,

$$\omega(t)\,|_{t\to\infty} = s\,Y(s)\,|_{s\to 0} = \frac{7.2}{0.06} = 120 \text{ rad/sec.}$$

Examination of the response $y(t)$ indicates that it consists of two components, $120\,u(t)$ and $120\,e^{-.06t}$. The second component tends to zero as time tends to infinity. Hence, this is the transient component of the response. The first component persists as long as the input remains applied. Hence, it is the steady-state component of the response. The first component arises because of the term A_1/s associated with the input and the second term from $A_2/(s + 0.06)$, associated with the transfer function. Therefore, the first term is the forced response, while the second is the natural response of the system. This is a general feature of the response of any linear system. In general we have,

$$H(s) = \frac{N_1(s)}{D_1(s)} = \frac{N_1(s)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

and

$$X(s) = \frac{N_2(s)}{D_2(s)} = \frac{N_2(s)}{(s+q_1)(s+q_2)\cdots(s+q_r)} \, .$$

The partial fraction expansion of the output transform takes the form,

$$Y(s) = X(s) \, . \, H(s) = \frac{N_1(s)\, N_2(s)}{(s+p_1)\cdots(s+p_n)(s+q_1)\cdots(s+q_r)}$$

$$= \frac{A_1}{s+p_1} + \cdots + \frac{A_n}{s+p_n} + \frac{A_{n+1}}{s+q_1} + \cdots + \frac{A_{n+r}}{s+q_r} \, .$$

The output $y(t)$ is given by,

$$y(t) = L^{-1}\, Y(s) = [\, A_1 \exp(-p_1 t) + \cdots + A_n \exp(-p_n t) \,]$$

$$+ \, [\, A_{n+1} \exp(-q_1 t) + \cdots + A_{n+r} \exp(-p_1 t) \,] \, .$$

The first bracket gives the transient response, which is a combination of the *natural modes* of the system. The second bracket gives the steady-state or the forced component of response. The method of evaluation of the constants A_1, A_n and $A_{n+1}, \ldots, A_{n+r}$ indicates that their values are dependent upon both the p's and q's. Hence, we conclude that although the form of the transient response is dependent only on the system parameters, the actual value depends both on the system-parameters and the forcing function (or the input). This conclusion was also derived in Chapter 3 in the classical method of analysis using differential equations.

Example 6.9:— Now let us alter the previous problem as follows. Assume that the applied voltage is varied linearly from 0 to 240 V in 2 minutes (by a thyristor converter) and then kept constant at 240 V. Determine the output speed as a function of time.

The input signal is shown in Fig. 6.6. As a function of time, it can be written as,

$$x(t) = 2t - 2(t - 120)\, u(t - 120) \, .$$

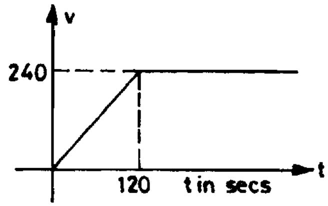


Fig. 6.6

That is, the input is a sum of two ramp functions: one starting at $t = 0$ with slope $+2$ and the other starting at $t = 120$ sec with slope -2. Using the time shift property [eqn. (6.5)] we get,

$$X(s) = \frac{2}{s^2} - \frac{2}{s^2} e^{-120s}.$$

$$Y(s) = X(s) \cdot H(s) = \frac{0.03}{s+0.06}\left[\frac{2}{s^2} - \frac{2}{s^2} e^{-120s}\right]$$

$$= \frac{0.06}{s^2(s+0.06)} - \frac{0.06}{s^2(s+0.06)} e^{-120s}$$

The Laplace inverse of the second term will be simply the first term shifted to the right by 120 seconds. Therefore, let us first determine,

$$y_1(t) = \mathcal{L}^{-1}\left[\frac{0.06}{s^2(s+0.06)}\right].$$

Now,

$$\frac{0.06}{s^2(s+0.06)} = \frac{A_1}{s^2} + \frac{A_2}{s} + \frac{A_3}{s+0.06}.$$

From eqn. (6.36) for repeated roots, we have,

$$A_1 = \frac{0.06}{s^2(s+0.06)} s^2 \Bigg|_{s=0} = \frac{0.06}{s+0.06}\Bigg|_{s=0} = 1$$

$$A_2 = \left[\frac{d}{ds}\frac{0.06}{s+0.06}\right]_{s=0} = -\frac{0.06}{(s+0.06)^2}\Bigg|_{s=0}$$

$$= -\frac{1}{0.06} = -16.37$$

$$A_3 = \frac{0.06}{s^2(s+0.06)}(s+0.06)\Bigg|_{s=-0.06} = \frac{0.06}{s^2}\Bigg|_{s=-0.06}$$

$$= \frac{1}{0.06} = 16.37.$$

Therefore,

$$y_1(t) = tu(t) - 16.37\, u(t) + 16.37\, e^{-0.06t}.$$

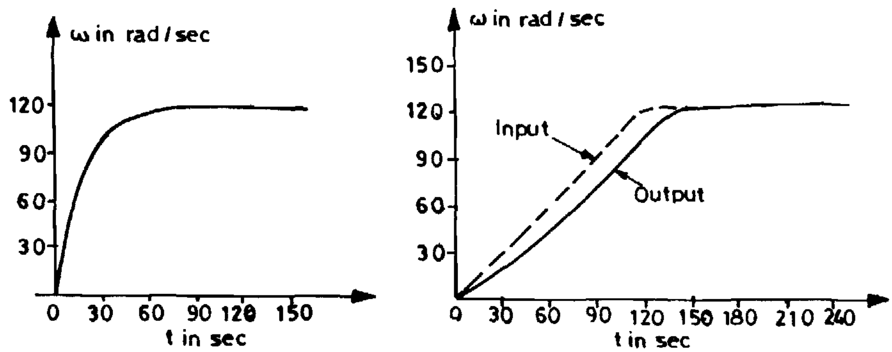
The complete response will be,

$$y(t) = y_1(t) - y_1(t-120)\, u(t-120)$$

$$= t\,[u(t) - (t-120)\, u(t-120)\,] - 16.37\,[u(t) - u(t-120)\,]$$

$$+\ 16.37\ [\,e^{-0.06t} - e^{-(.06-120)t}\ u(t-120)\,].$$

The plot of the responses for Examples 6.8 and 6.9 are shown in Fig. 6.7(a) and 6.7(b), respectively.



(a) (b)

Fig. 6.7 (a) Response of Example 6.8 and (b) Response of Example 6.9

Consideration of initial conditions:

The transfer function is defined for an initially relaxed system. That is, the very definition of transfer function assumes that all the initial conditions are zero. If the initial conditions are present, we must go back to the differential equation and include initial conditions while taking its Laplace transform. The technique is illustrated with the help of the following example.

Example 6.10:— Determine the response of the series *RLC* circuit of Fig. 6.8 with an initial current in the inductor and an initial voltage across the capacitor.

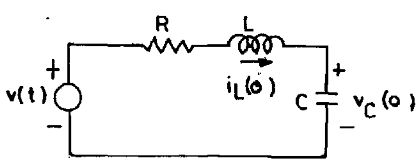


Fig. 6.8

Solution: Applying Kirchhoff's voltage law around the loop,

$$v_R + v_L + v_C = v$$

or

$$Ri + L \cdot \frac{di}{dt} + \frac{1}{C} \int_0^t i \, dt + v_c(0) = v \qquad \text{(i)}$$

Taking the Laplace transform of eqn. (i) we get,

$$RI(s) + L[sI(s) - i(0)] + \frac{I(s)}{Cs} + \frac{v_C(0)}{s} = V(s)$$

or

$$\left[R + LS + \frac{1}{Cs}\right] I(s) = V(s) + L\,i(0) - \frac{v_C(0)}{s}.$$

Therefore,

$$I(s) = \frac{V(s) + L\,i(0) - v_C(0)/s}{R + Ls + 1/(Cs)}. \qquad \text{(ii)}$$

The term $1/[R + Ls + 1/(Cs)]$ is the transfer function of the system. The effect of initial conditions is to alter the input excitation of the system. The numerator in eqn. (ii) may be considered as the 'total excitation'. Determination of the inverse transform of $I(s)$ to get $i(t)$ follows the usual partial fraction expansion techniques.

Solution of differential equations:

The Laplace transform method is commonly used for solving linear differential equations. The process of taking the Laplace transform converts a differential equation into an algebraic equation. Obtaining the solution (as a function of s) then involves only algebraic manipulations. The Laplace transform method also avoids the tedium of determining $(n - 1)$ unknown coefficients from the given initial conditions. The general procedure is illustrated by the following example.

Example 6.11:— Solve the second order differential equation,

$$\frac{d^2y}{dt^2} + a_1 \frac{dy}{dt} + a_0 y = x(t)$$

using Laplace transform.

Solution: Taking Laplace transform of both the sides,

$$[s^2 Y(s) - sy(0) - \dot{y}(0)] + a_1[sY(s) - y(0)] + a_0 Y(s) = X(s)$$

or

$$(s^2 + a_1 s + a_0) Y(s) = [\dot{y}(0) + sy(0) + a_1 y(0)] + X(s).$$

Therefore,

$$Y(s) = \frac{\dot{y}(0) + sy(0) + a_1 y(0)}{s^2 + a_1 s + a_0} + \frac{X(s)}{s^2 + a_1 s + a_0}.$$

The first term on the r.h.s. gives the response due to the initial conditions and the second term due to the forcing function. Combining the two terms we get,

$$Y(s) = \frac{[\dot{y}(0) + sy(0) + a_1 y(0)] + X(s)}{s^2 + a_1 s + a_0}.$$

Thus, the initial conditions are treated as additional inputs. The numerator may then be called the *total excitation*. The process of taking the Laplace inverse of $Y(s)$ to get the solution $y(t)$ follows the usual method of partial fraction expansion given in the previous section.

Response to sinusoidal signals: generalised impedance function

Example 6.12:— Determine the current in a series *RLC* circuit connected to a sinusoidal voltage source, switched-on at $t = 0$. Assume initial conditions to be zero.

Solution: For the sake of numerical simplicity, let us assume that $R = L = C = 1$ and the sinusoidal input voltage has a magnitude of 1 volt and frequency of 2 rad/sec. Then, from Example 6.10 we have,

$$H(s) = \frac{1}{R + Ls + 1/Cs} = \frac{Cs}{LCs^2 + RCs + 1} = \frac{s}{s^2 + s + 1} \qquad \text{(i)}$$

The Laplace transform of the input is,

$$X(s) = \mathcal{L}\ \sin 2t = 2/(s^2 + 2^2).$$

The output current $Y(s) = H(s) \times (s)$ or,

$$Y(s) = \frac{2s}{(s^2 + s + 1)(s^2 + 2^2)}.$$

Expanding into partial fractions we get,

$$Y(s) = \frac{2s}{(s^2 + s + 1)(s^2 + 2^2)} = \frac{As + B}{s^2 + s + 1} + \frac{Cs + D}{s^2 + 2^2} \qquad \text{(ii)}$$

To determine the constants A, B, C, D multiply eqn. (ii) by the entire denominator. Then we have,

$$2s = (As + B)(s^2 + 4) + (Cs + D)(s^2 + s + 1)$$

$$= (A + C)s^3 + (B + C + D)s^2 + (4A + C + D)s + (4B + D).$$

Equating coefficients of like powers on both sides of the equation we get,

$$A + C = 0,\ B + C + D = 0,\ 4A + C + D = 2,\ 4B + D = 0$$

Solving these we get,

$$A = 6/13,\ B = -2/13,\ C = -6/13,\ D = 8/13.$$

Thus,

$$Y(s) = \frac{2}{13}\left[\frac{3s-1}{(s^2+s+1)} - \frac{3s-4}{(s^2+2^2)}\right]$$

$$= \frac{2}{13}\left[3\,\frac{s+0.5}{(s+0.5)^2+(\sqrt{0.75})^2} - \frac{2.5}{(\sqrt{0.75})^2}\,\frac{\sqrt{0.75}}{(s+0.5)^2+(\sqrt{0.75})^2}\right]$$

$$- \frac{2}{13}\left[3\,\frac{s}{s^2+2^2} - 2\,\frac{2}{s^2+2^2}\right]$$

Therefore,

$$y(t) = \frac{6\,e^{-0.5t}}{13}\,(\cos\ 0.866\,t - \sin 0.866\,t) - \frac{2}{13}\,(3\cos 2t - 2\sin 2t)$$

$$= \frac{6\sqrt{2}}{13}\,e^{-0.5t}\sin(0.866\,t - \theta_1) + \frac{2}{\sqrt{13}}\,\sin\,(2\,t - \theta_2) \qquad \text{(iii)}$$

where $\sin\theta_1 = 1/\sqrt{2}$ or $\theta_1 = 45°$ and $\sin\theta_2 = 3/\sqrt{13}$ *or* $\theta_2 = 56.2°$.

The first term in the solution $y(t)$ is the transient response of the system arising from the natural mode of behaviour. This natural mode is an underdamped second order system with an undamped natural frequency of oscillation $\omega_n = 1$, damped natural frequency = 0.866 and damping ratio $\zeta = 0.5$. The second term in eqn. (iii) gives the steady-state response of the circuit. It shows that the peak amplitude of the current is $2/\sqrt{13}$ and it lags behind the applied voltage by an angle 56.2°. This steady-state value could have been obtained straightaway by using the elementary circuit theory technique of dividing the voltage phasor by the impedance phasor. In the present problem impedance Z is given by,

$$z = R + j\omega L + \frac{1}{j\,\omega\,C} = 1 + j\left(2 - \frac{1}{2}\right) = \frac{\sqrt{13}}{2}\ \angle - 56.2°.$$

The steady-state current is then given by

$$I_m = \frac{V_m}{Z} = \left|\frac{2}{\sqrt{13}}\right|\ \angle - 56.2°$$

or,

$$i_{ss} = \frac{2}{\sqrt{13}}\ \sin\ (2\,t - 56.2°).$$

This is the same result as that given by the steady-state term of eqn. (iii). We note that impedance $Z(\omega)$ is obtained by replacing $s = j\omega$ in the expression for $1/H(s)$. Thus, if only the steady-state response is desired we divide $V(\omega)$ by $Z(\omega)$. However, if the complete response, including the transient response, is desired we divide $V(s)$ by $H(s)$. Thus, $H(s)$ can be called the ***generalised impedance*** or ***transform impedance*** of a function of complex frequency s.

The generalised impedance $Z(s)$ is very useful in network analysis. [The reciprocal of $Z(s)$ is called *admittance.* When the reference is to either impedance or admittance, a common term 'immittance' is used.] The network immittance function completely characterises it and serves the same function as the transfer function. To simplify computation of the immittance function, the terminal voltage-current relations of the basic circuit elements are defined directly in terms of the transformed variables, as shown in Fig. 6.9.

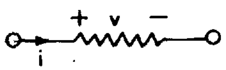
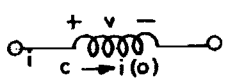
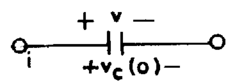
Circuit element	*v-i relations (in time domain)*	*V(s)-I(s) relations (in s-domain)*
	$v = Ri$	$V(s) = R\,I(s)$
	$v = L\,\frac{di}{dt}$	$V(s) = SL\,I(s) - Li(0)$
	$v = \frac{I}{C}\int_0^t i\,dt + v_C(0)$	$V(s) = \frac{I(s)}{Cs} + \frac{v_C(0)}{s}$

Fig. 6.9 Terminal *v–i* Relations of Basic Circuit Elements in Time and *s*-domains

The advantage of using generalised impedance function in network analysis is demonstrated by the following example.

Example 6.13:— Determine the step response of the thermal system for heat dissipation in a power transistor, described in Chapter 1, Section 1.5.

The electrical equivalent circuit of the given thermal system is reproduced in Fig. 6.10. The input is the rate of heat dissipation in the power semiconductor device i.e., power loss in watts. The output is the rise in the junction temperature T_J. R_T's are thermal resistances and C_T's thermal capacitances of various parts.

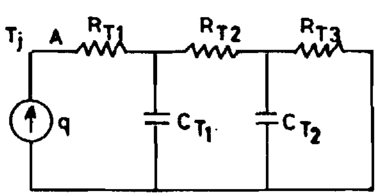


Fig. 6.10 Electrical Equivalent of a Thermal System

Electrical circuit theory tells us that the voltage across AB (i.e., T_J) will be the current through A (i.e., q) multiplied by the equivalent impedance Z of the network to the right of AB.

Since we are interested in the complete response, let us determine $Z(s)$ for the transformed network shown in Fig. 6.11. For numerical simplicity let us assume the values of all the parameters to be unity. The steps in the evaluation of $Z(s)$ are indicated in Fig. 6.12.

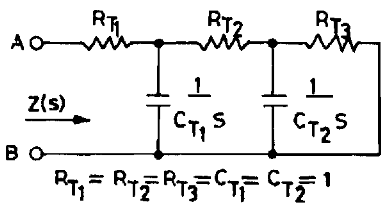
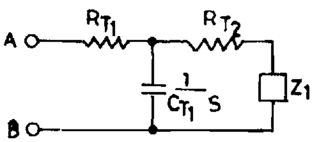
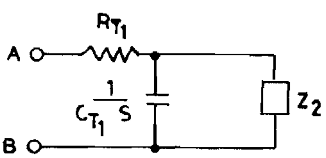


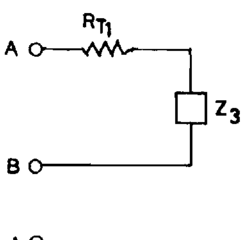
Fig. 6.11 Transformed Network of Fig. 6.10



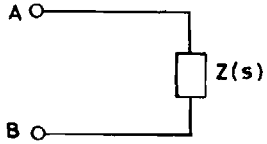
$$Z_1' = \frac{R_{T_3} \cdot 1/(C_{T_2} s)}{R_{T_3} + 1/(C_{T_2} s)} = \frac{1}{s+1}$$



$$Z_2 = R_{T_2} + Z_1 = \frac{s+2}{s+1}$$



$$Z_3 = \frac{1/(C_{T_1} s) \cdot Z_2}{1/(C_{T_1} s) + Z_2} = \frac{s+2}{s^2 + 3s + 1}$$



$$Z(s) = R_{T_1} + Z_3 = \frac{s^2 + 4s + 3}{s^2 + 3s + 1}$$

Fig. 6.12 Evaluation of Z(*s*) for Fig. 6.11

Now, temperature $T_J(s)$ will be given by,

$$T_J(s) = Q(s)\, Z(s)\,,$$

where $Q(s)$ = Laplace transform of the input heat rate $q(t)$. Since input is given as a unit step input, $Q(s) = 1/s$. Therefore,

$$T_J(s) = \frac{s^2 + 4s + 3}{s(s^2 + 3s + 1)} \cdot$$

Factorising the denominator and using partial fraction expansion we get,

$$T_J(s) = \frac{A}{s} + \frac{B}{s + 2.62} + \frac{C}{s + 0.38} \cdot$$

Evaluating A, B and C we get,

$$T_J(s) = \frac{3}{s} - \frac{0.1}{s + 2.62} - \frac{1.9}{s + 0.38} \cdot$$

Taking the Laplace inverse, we have,

$$T_J(t) = 3\,u(t) - 0.1\,e^{-2.62t} - 1.9\,e^{-0.38t}.$$

Analysis with periodic non-sinusoidal input:

A periodic signal with period T satisfies the relation,

$$f(t) = f(t + nT), \; n = 1, 2, 3, \ldots$$

Let $f_1(t)$ be the first cycle of the periodic function $f(t)$. Then the second cycle will be the first cycle shifted to the right by T, and so on. That is,

$$f(t) = f_1(t) + f_1(t - T)\,u(t - T) + f_2(t - 2T)\,u(t - 2T) + \ldots$$

Taking the Laplace transform, using the time shift property, we get,

$$F(s) = F_1(s) + F_1(s)\,e^{-sT} + F_1(s)\,e^{-2sT} + \ldots$$

$$= F_1(s)(1 + e^{-sT} + e^{-2sT} + \ldots) = \frac{F_1(s)}{1 - e^{-sT}} \cdot \qquad (6.37)$$

Example 6.14:— Determine the current in a series R-L circuit driven by a square wave voltage source of amplitude 1 and half period $T/2 = 1$, as shown in Fig. 6.13.

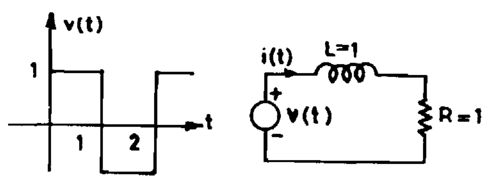


Fig. 6.13 *RL* Circuit Driven by Square Wave

Solution: Application of Kirchhoff's voltage law gives,

$$(Ls + R)\,I(s) = V(s)$$

or

$$I(s) = \frac{V(s)}{(Ls + R)} = \frac{V(s)}{s + 1}.$$

For determining the Laplace transform of the input square wave voltage, let us first determine the transform of the first cycle. As a time function, $v_1(t)$, the first cycle of $v(t)$ can be expressed in terms of step functions as,

$$v_1(t) = u(t) - 2u(t - 1) + u(t - 2)$$

Therefore,

$$V_1(s) = \frac{1}{s} - \frac{2e^{-s}}{s} + \frac{e^{-2s}}{s} = \frac{1}{s}(1 - 2e^{-s} + e^{-2s}).$$

Then, according to eqn. (6.37) for periodic functions,

$$V(s) = \frac{V_1(s)}{1 - e^{-sT}}.$$

In the present problem, $T = 2$. Therefore,

$$V(s) = \frac{1 - 2e^{-s} + e^{-2s}}{s(1 - e^{-2s})}.$$

Therefore,

$$I(s) = \frac{V(s)}{s + 1} = \frac{1 - 2e^{-s} + e^{-2s}}{s(s + 1)(1 - e^{-2s})}.$$

$$= \frac{1}{s(s + 1)}[(1 - 2e^{-s} + e^{-2s})(1 + e^{-2s} + e^{-4s} + e^{-6s} + \ldots)]$$

$$= \frac{1}{s(s + 1)}(1 - 2e^{-s} + 2e^{-2s} - 2e^{-3s} + 2e^{-4s} - \ldots)$$

Let $1/s(s + 1) = F(s)$ and Let $\mathcal{L}^{-1} F(s) = f(t)$. Then,

$$I(s) = F(s)(1 - 2e^{-s} + 2e^{-2s} - 2e^{-3s} + 2e^{-4s} - \ldots)$$

Each exponential term in the series on the r.h.s. shifts the function to the right by $T/2 = 1$. Therefore,

$$i(t) = f(t) - 2f(t - 1)u(t - 1) + 2f(t - 2)u(t - 2) - \ldots$$

Note that the r.h.s. of the above expression is not exactly an infinite series. In the first half cycle, only the first term is present. In the second half cycle, only the first and the second terms are present. And so on.

Let us now determine $f(t) = \mathcal{L}^{-1} F(s)$.

$$F(s) = \frac{1}{s(s + 1)} = \frac{1}{s} - \frac{1}{s + 1}$$

Therefore, $f(t) = (1 - e^{-t})\, u(t)$. Then in the nth half cycle the first n terms will be present. That is,

$$i(t) = [u(t) - 2u(t-1) + 2u(t-2) - \ldots + 2(-1)^{n-1} \cdot u\{t-(n-1)\}]$$
$$- [e^{-t} u(t) - 2e^{-(t-1)} u(t-1) + 2e^{-(t-2)} \cdot u(t-2) + \ldots$$
$$+ 2(-1)^{n-1} e^{-[t-(n-1)]} u\{t-(n-1)\}].$$

The term inside the first brackets on the r.h.s. is an oscillating term of the form [1 – 2 + 2 – 2 ...]. Therefore, in the nth half cycle its value will be simply $(-1)^{n-1}$. This gives, for $n - 1 \leq t \leq n$,

$$i(t) = (-1)^{n-1} - e^{-t}\, [1 - 2e + 2e^2 + \ldots + 2(-1)^{n-1} e^{n-1}].$$

The expression inside the brackets is a finite geometric series. For such a series, we have the basic mathematical result,

$$1 + x + x^2 + \ldots + x^{n-1} = \frac{1 - x^n}{1 - x}.$$

Writing the terms inside the square brackets as,

$$[2\{1 - e + e^2 - e^3 + \ldots + (-1)^{n-1} e^{n-1}\} - 1] = 2\, \frac{1-(-e)^n}{1-(-e)} - 1$$
$$= -\left[\frac{e-1}{e+1} + \frac{2(-e)^n}{e+1}\right]$$

we have,

$$i(t) = (-1)^{n-1} + e^{-t}\left[\frac{e-1}{e+1} + \frac{2(-e)^n}{e+1}\right]$$
$$= (-1)^{n-1} + \frac{2(-1)^n e^{n-t}}{e+1} + \frac{(e-1)\, e^{-t}}{e+1}.$$

This is the expression for current in the nth half cycle, where the value of t will be ranging from $(n-1)$ to n i.e. $(n-1) \leq t \leq n$. Hence the term e^{n-t} will always range from 0 to 1. Therefore, the first two terms give the steady-state response and the third term, which decays exponentially, is the transient response. That is,

$$i_{ss} = (-1)^{n-1} + \frac{2(-1)^n e^{n-t}}{e+1} \qquad \text{(i)}$$

$$i_{tr} = \frac{(e-1)}{(e+1)}\, e^{-t}.$$

The transient response, the steady-state response and the total response for the first four half cycles are shown in Fig. 6.14.

If only the steady-state response of the circuit is of interest, one need not resort to the complexity of first determining the complete response and then take out only

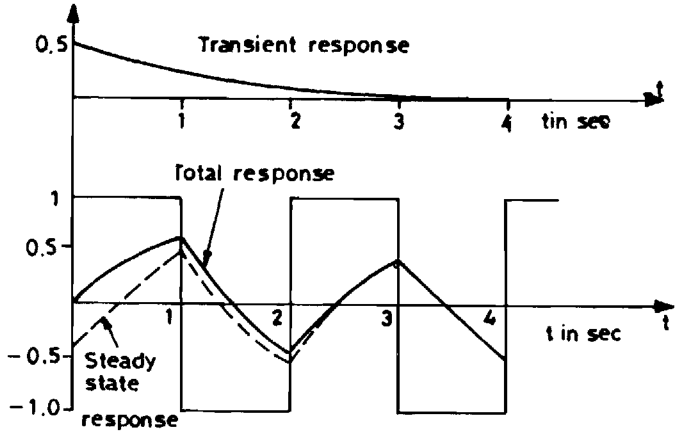


Fig. 6.14 Response of Circuit in Fig. 6.13

the steady-state term. For steady-state analysis with nonsinusoidal voltages, we have two alternatives. One method would be to use the Fourier series techniques of Chapter 4. When the object of analysis includes information like relative magnitudes of voltage and current harmonics, d.c. and a.c. power components, power factor, etc. Fourier series analysis is very useful. However, if the shape of the output waveform, that is, the output as a function of time, is of interest, adding harmonic components to get this waveform is not very convenient. In that case, the Laplace transform method now described is more useful.

To develop the technique for steady-state analysis, we note that the response of a linear system to a periodic input will also be periodic, i.e., the output wave shape will repeat itself after every period T. If the output function has no discontinuities, this means that its value at the beginning and at the end of any cycle will be the same. If $y_{ss}(t)$ is the steady-state output,

$$y_{ss}(nT) = y_{ss}[(n+1)T].$$

Since this is true for any n, it is also true for $n = 0$, and, hence,

$$y_{ss}(0) = y_{ss}(T).$$

This process of matching the values at the beginning and the end of a cycle of steady-state response simplifies its determination. This is demonstrated by determining only the steady-state response of Example 6.14.

In Example 6.14 the positive and negative half cycles of the input are equal and opposite to each other. Therefore, the half cycles of the output will also be equal and opposite. That is,

$$y_{ss}(0) = -y_{ss}(T/2).$$

We will use this condition for finding the steady-state response. Writing the system equation for the steady-state response for the first positive half cycle, we have,

$$L \frac{di_{ss}}{dt} + Ri_{ss} = 1, \ 0 \le t \le 1 \tag{ii}$$

Taking the Laplace transform we get,

$$Ls\, I_{ss}(s) - Li_{ss}(0) + RI_{ss}(s) = 1/s$$

or

$$I_{ss}(s) = \frac{1}{s(Ls + R)} + \frac{Li_{ss}(0)}{Ls + R} .$$

With numerical values $L = R = 1$,

$$I_{ss}(s) = \frac{1}{s(s+1)} + \frac{i_{ss}(0)}{s+1} .$$

Here, $i_{ss}(0)$ should not be confused with $i(0)$: $i_{ss}(0)$ is the value of the current at the beginning of a half cycle (any half cycle) when steady-state conditions have been reached, whereas $i(0)$ is the value of the initial current at $t = 0$ when the switch, connecting the source to the load, is closed. It should be noted that $i(0)$ will effect only the transient component of response; it will have no effect on the steady-state value. Also, it should be noted that so far $i_{ss}(0)$ is not known.

Taking the Laplace inverse of the expression for $I_{ss}(s)$, we get,

$$i_{ss}(t) = (1 - e^{-t}) + i_{ss}(0)\, e^{-t}, 0 \le t \le 1.$$

Now, applying the condition $i_{ss}(0) = -\, i_{ss}(T/2)$, we get,

$$i_{ss}(0) = -\,[(1 - e^{-T/2}) + i_{ss}(0)\, e^{-T/2})].$$

With numerical value $T/2 = 1$ we get,

$$i_{ss}(0)\,(1 + e^{-1}) = -\,(1 - e^{-1}).$$

Therefore,

$$i_{ss}(0) = -\frac{1 - e^{-1}}{1 + e^{-1}} = -\frac{e-1}{e+1} .$$

Substituting this in the expression for $i_{ss}(t)$ we get,

$$i_{ss}(t) = (1 - e^{-t}) - \frac{e-1}{e+1}\, e^{-t}$$

$$= 1 - \frac{2e^{(1-t)}}{1+e}, \ 0 \le t \le 1 \tag{iii}$$

The expression for i_{ss}, in eqn. (iii) is the same as in expression (i) obtained earlier for $n = 1$.

In the present problem, since the differential eqn. (ii) for i_{ss} (t) is a first order equation, it would have been equally easy to solve eqn. (ii) directly, with the condition i_{ss} $(0) = - i_{ss}(T/2)$, without taking the Laplace transform. However, for higher order systems, the solution of the differential equation by Laplace transform technique is easier.

GLOSSARY

Laplace Transform: The Laplace transform of a function $f(t)$ is defined by,

$$F(s) = \mathcal{L} f(t) = \int_0^\infty f(t) e^{-st} dt,$$

where s, the complex frequency variable, is given by $s = \sigma + j\omega$. The time function may be obtained from $F(s)$ by the inverse Laplace transform,

$$f(t) = \mathcal{L}^{-1} F(s) = \frac{1}{2\pi} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds.$$

s-plane: A plane with σ as its x-axis and $j\omega$ as y-axis is called the complex frequency plane or the s-plane. A point in this plane, with coordinates σ_1 and $j\omega_1$, defines a value of s, i.e., $s_1 = \sigma_1 + j\omega_1$.

Transfer Function: The ratio of the Laplace transform of the output to that of the input for an initially relaxed linear system is called its *transfer function* $H(s)$. $H(s)$ is also equal to the Laplace transform of the impulse response of the system, i.e., $H(s) = \mathcal{L} h(t)$.

Characteristic Equation: The denominator of the transfer function determines the form of its behaviour and, hence, is called the *characteristic polynomial* of the system. The denominator polynomial equated to zero is called the *characteristic equation* of the system.

Poles and Zeros: The transfer function can be written in the factorised form as,

$$H(s) = \frac{(s - z_1)\ (s - z_2)\ \ldots\ (s - z_m)}{(s - p_1)(s - p_2)\ldots(s - p_n)} \cdot m \le n.$$

The values of $s = z_1, z_2, \ldots, z_m$, for which $H(s)$ becomes zero, are called the zeros of the system. The values of $s = p_1, p_2, \ldots, p_n$ for which $H(s)$ becomes infinity are called the poles of the system. Zeros are the roots of the numerator polynomial and poles the roots of the denominator polynomial of the transfer function.

Generalised Impedance: In an electrical circuit, the ratio of the Laplace transform of the voltage to that of the current is called the *generalised impedance*, i.e., $Z(s) = V(s)/I(s)$. The ordinary impedance for sinusoidal inputs can be obtained from $Z(s)$ by replacing s by $j\omega$.

PROBLEMS

6.1. Determine the Laplace transforms of the following functions:

(a) $f(t) = t^2 \sin 2t$.

(b) $f(t) = e^{-at}/t$.

(c) $f(t) = 1/(1 + e^t)$.

(d) $f(t) = t e^{-at} \sin bt$.

(e) $f(t) = \sin \beta t / \beta t$.

(f) $f(t) = 1/t^2$.

(g) $f(t) = \sin(10t + 60°)$.

6.2. Determine the Laplace transform of the functions sketched in Figs. 6.15 (a-d).

6.3. Determine the Laplace inverse of the following functions and sketch them:

(a) $(s+2)/(s^2+1)^2$.

(b) $1/[s^2(s^2+2s+2)]$

(c) $1/[s(1-e^{-sT})]$

(d) $e^{-s}/(s-1)$.

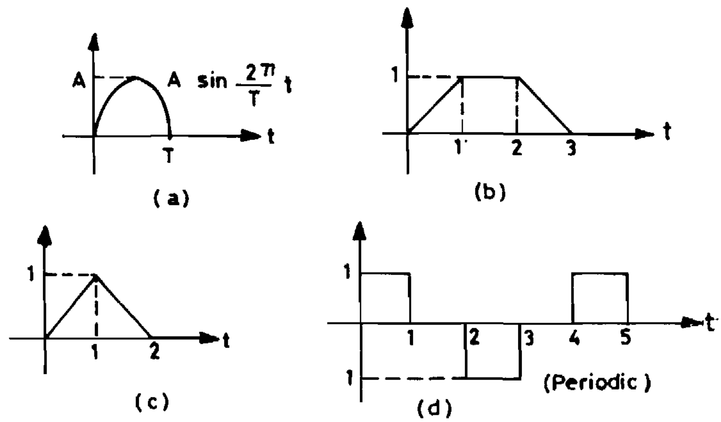


Fig. 6.15

6.4. Determine the initial value of $f(t)$, i.e., $f(0^+)$ and $f'(0^+)$ for $F(s) = 1/(s+1)^2$.

6.5. Determine the transfer functions for the following systems:

(i) The automobile ignition system of Chapter 1, Section 1.1.

(ii) π filter of problem 1.1.

(iii) The system whose response is shown in Fig. 3.22, problem 3.12.

(iv) The 'phase lead' and 'phase lag' networks shown in Fig. 6.16.

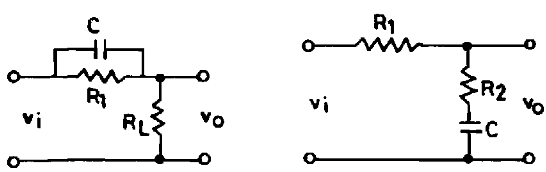


Fig. 6.16

6.6. Solve the following differential equations using Laplace transform:

(i) $2\ddot{x} + \dot{x} + 3x = u(t)$ with $\dot{x}(o) = 0$, $x(o) = -1$.

(ii) $\ddot{x} + 2\zeta\omega_n \dot{x} + \omega_n^2 x = 0$ with $\dot{x}(o) = a$ and $x(o) = b$.

(iii) $\dddot{x} + \ddot{x} + \dot{x} = e^{-t}$ with $\ddot{x}(o) = \dot{x}(o) = 0$ and $x(o) = 1$.

6.7. The impulse response of a system is given by $h(t) = (2e^{-t} + e^{-2t})\, u(t)$.

Determine its output when a triangular input of peak amplitude 10, duration 4, centred at 2 is applied as input to the system.

6.8. When connected to a 100 V d.c. source, an armature-controlled d.c. motor runs at a steady-state speed of 100 rad/sec. The armature resistance is 1 ohm, the torque constant 0.1 n-m/amp. and the time constant 4 sec. The armature circuit accidently gets opened at $t = 0$ and reclosed at $t = 1$. Determine and plot the motor speed as a function of time.

6.9. A sinusoidal voltage of magnitude 10V and frequency 10 rad/sec is switched on at $t = 0$ across a series *RLC* circuit with $R = L = C = 1$. Determine the transient and the steady-state values of the voltage across the resistor.

6.10. The input voltage v_i in Fig. 6.17 has a peak magnitude ± 1 V and half cycle duration of 1 sec. $R = L = C = 1$. Determine the steady-state output voltage v_o and sketch it. Determine the peak value of the voltage across *C* for $L = C = 1$ and $R = 0.1$, 1, and 10.

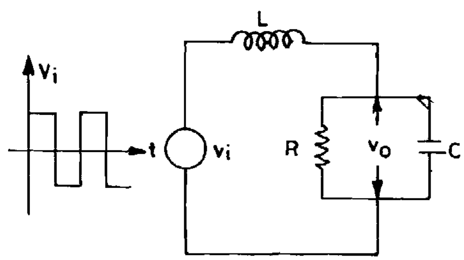


Fig. 6.17

6.11. When excited by a unit step input, the time response of a linear system can be approximated by $t - e^{-t}$ Find the transfer function of the system.

6.12. Figure 6.18 shows the schematic diagram of an accelerometer. The output casing is fixed on the body whose acceleration is to be measured, e.g., an aircraft. The deflection x of the mass and y of the case are with respect to the external inertial space. The deflection z of the pointer is $z = (x - y)$. Determine the transfer function with z as the output and acceleration of the case as the input. How should the parameters of the system be selected to make the pointer deflection z proportional to acceleration?

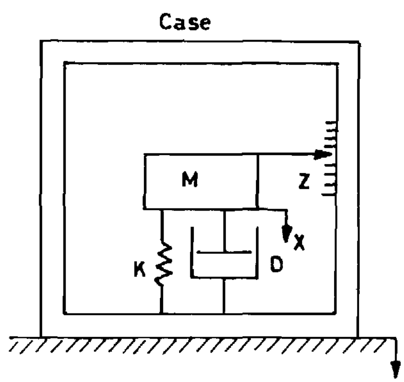


Fig. 6.18

CHAPTER 7

Feedback Systems

LEARNING OBJECTIVES

After studying this chapter you should be able to:

(i) determine the overall transfer function of an interconnected system, using block diagram reduction techniques;

(ii) represent a given system by its signal flow graph and use the techniques of signal flow graph reduction to determine its overall transfer function;

(iii) determine the response of feedback systems and to control the transient response by gain adjustment of the forward path;

(iv) determine whether a given system is stable or unstable, using the Routh-Hurwitz criterion;

(v) calculate the steady-state error of feedback systems due to step, ramp, and parabolic inputs; and

(vi) appreciate the advantages as well as the problems of feedback control systems.

7.1 Interconnection of Systems

Larger systems are usually made up of interconnected smaller systems, which are then called subsystems of the main system. The subsystems may themselves be interconnections of still smaller systems. Each one of these subsystems will be described by a transfer function relating its input and output variables. That is, for the ith subsystem, we have

$$Y_i(s) = X_i(s)\, H_i(s).$$

In this section we will study the methods for obtaining the transfer function of the overall system from a knowledge of the transfer function of its subsystems. We first start with only two subsystems. These two subsystems may be connected in the following three ways: (*i*) series or cascade connection, (*ii*) parallel connection and (*iii*) feedback connection.

Series connection :

The series connection of two subsystems is shown in Fig. 7.1. This connection arises when the output of one subsystem is the input to the next subsystem.

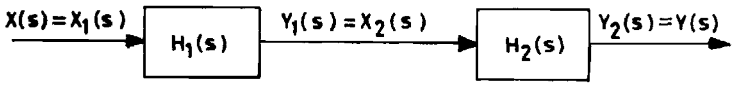


Fig. 1 Series Connection of Two Subsystems

The output $Y(s) = Y_2(s) = X_2(s)\,H_2(s)$. But $X_2(s) = Y_1(s) = X_1(s)\,H_1(s)$. Therefore, $Y(s) = X_1(s)\,H_1(s)\,H_2(s) = X(s)\,H_1(s)\,H_2(s)$. Thus the overall transfer function of the system relating the input $X(s)$ to the output $Y(s)$ is,

$$H(s) = H_1(s)\,H_2(s)\,. \tag{7.1}$$

In deriving eqn. (7.1) and in drawing Fig. 7.1, it is tacitly assumed that the process of connecting the second subsystem to the output of the first subsystem does not alter the relationships between the variables of the first subsystem. If this is not the case, we say that the second subsystem 'loads' the first subsystem. Equation (7.1) is then no longer valid. The overall transfer function of series-connected subsystems is equal to the product of the individual subsystem transfer functions only when a subsystem does not 'load' its preceding subsystem. This is illustrated in Fig. 7.2. The interconnection of the two tanks in Fig. 7.2(a) does not produce any 'loading effect'. Hence, the overall transfer function is the product of the transfer functions of the individual tanks. In Fig. 7.2(b), however, the outflow rate of the first tank depends not only on its inflow rate but also on the liquid level in the second tank. Hence, the second tank 'loads' the first one, and the transfer function is *not* equal to the product of the individual transfer functions. For finding the transfer function in this case, the equations of the whole system will have to be considered together. Similarly, if two low pass filters are cascaded directly the

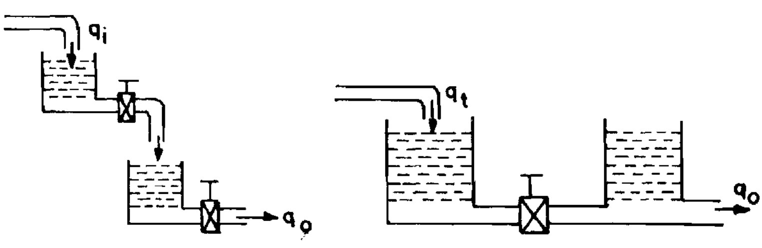


Fig. 7.2 Different Interconnections of Two Tanks

overall transfer function is not equal to the product of their individual transfer functions. However, if the two are connected through a buffer amplifier in between, no loading effect will be produced and $H(s) = H_1(s) H_2(s)$, as given by eqn. (7.1)

If two first order subsystems are cascaded, it is normal to assume that the overall system will be a second order system. However, in some cases it may not be so. For example, let,

$$H_1(s) = \frac{1}{s+1} \quad \text{and} \quad H_2(s) = \frac{s+1}{s+2}.$$

Then, $H(s) = H_1(s) H_2(s) = 1/(s+2)$.

The pole of the first subsystem at $s = -1$ gets cancelled by the zero of the second subsystem at the same location. Thus, in this case, the order of the overall system transfer function is less than the sum of the individual subsystem orders.

Parallel connection :

As shown in Fig. 7.3, the two subsystems receive the same input $X(s)$ in the case of a parallel connection. The two outputs, $Y_1(s)$ and $Y_2(s)$ are summed up at a summing junction to produce output $Y(s) = Y_1(s) + Y_2(s)$. The symbol ⊗ is used to indicate the summing operation.

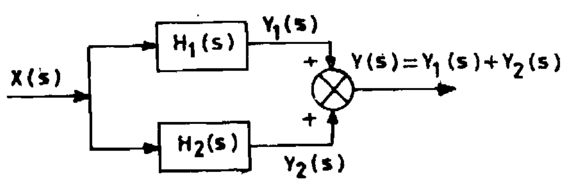


Fig. 7.3 Parallel Connection

Now,

$$Y(s) = Y_1(s) + Y_2(s) = X(s) H_1(s) + X(s) H_2(s)$$

$$= X(s) [H_1(s) + H_2(s)].$$

Thus, the overall transfer function for the parallel connection is,

$$H(s) = H_1(s) + H_2(s). \tag{7.2}$$

Feedback connection :

In the feedback connection, shown in Fig. 7.4, the summing junction adds the input $X(s)$ and the negative of the output $Y_2(s)$ of the second subsystem to produce the input to the first subsystem. In other words, $X_1(s) = X(s) - Y_2(s)$. The output $Y_1(s)$

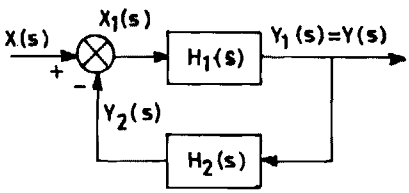


Fig. 7.4 Feedback Connection

of the first subsystem is the system output. That is $Y(s) = X_1(s)\,H_1(s)$. But $X_1(s) = X(s) - Y(s)\,H_2(s)$. Therefore,

$$Y(s) = [X(s) - Y(s)\,H_2(s)]\,H_1(s)$$

or

$$[1 + H_1(s)\,H_2(s)]\,Y(s) = X(s)\,H_1(s)$$

or

$$Y(s) = \frac{X(s)\,H_1(s)}{1 + H_1(s)\,H_2(s)}\,.$$

Thus, the overall transfer function is,

$$H(s) = \frac{Y(s)}{X(s)} = \frac{H_1(s)}{1 + H_1(s)\,H_2(s)}\,. \tag{7.3}$$

In the system of Fig. 7.4, the output signal is 'fed back' to the input through $H_2(s)$. At the summing junction, this feedback signal is subtracted from the input signal. Hence, this is called a *negative feedback* system. In case $Y_2(s)$ adds to the input $X(s)$, we have *positive feedback*. It is straight forward to show that for positive feedback the overall transfer function is.

$$H(s) = \frac{H_1(s)}{1 - H_1(s)\,H_2(s)} \tag{7.4}$$

Use of feedback is the key to almost all modern automatic control systems. Feedback is also used in a variety of signal-processing applications, e.g., feedback in amplifiers. In such applications, the feedback path is added by the designer to some existing system as an external element. However, feedback may be inherently present in the modelling process of many physical systems. This is illustrated by the following example.

Example 7.1 : — Develop a block diagram for a d.c. generator, used as a rotating amplifier, supplying current to a resistive load.

Solution: The circuit diagram of the system is shown in Fig. 7.5. The input signal is $V_i(s)$ and the output, $V_o(s)$. In addition, we have four intermediate variables— field current $I_f(s)$, armature current $I_a(s)$, air gap flux $\Phi(s)$ and induced voltage $E(s)$. Input $V_i(s)$ produces the field current $I_f(s)$, the two being related by the equation,

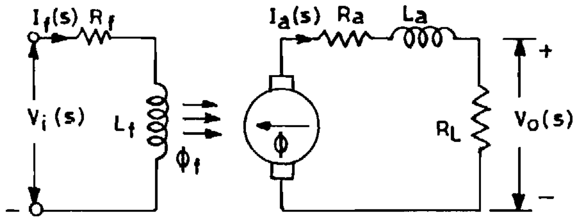


Fig. 7.5 d.c. Generator

$$(L_f s + R_f)\ I_f s = V_i(s)\,. \tag{i}$$

The field current produces the field flux,

$$\Phi_f(s) = K_1\ I_f(s) \tag{ii}$$

The air gap flux is the difference between the field flux and the armature reaction flux Φ_a, i.e.,

$$\Phi(s) = \Phi_f(s) - \Phi_a(s)\,. \tag{iii}$$

The air gap flux induces a voltages $E(s)$ in the armature, which is linearly proportional to it, i.e.,

$$E(s) = K_2\,\Phi(s). \tag{iv}$$

The induced voltage $E(s)$ causes the armature current I_a according to the relation,

$$(s\,L_a + R_a + R_L)\ I_a(s) = E(s) \tag{v}$$

The armature reaction flux is directly proportional to the armature current, i.e.,

$$\Phi_a(s) = K_3\,I_a(s) \tag{vi}$$

And finally the output,

$$V_0(s) = R_L\,I_a(s)\,. \tag{vii}$$

The cause-effect relationships given mathematically by eqns. (i) to (vii) can be displayed more effectively by the block diagram shown in Fig. 7.6. Figure 7.6 places in evidence the inherent feedback action of the armature reaction flux. In fact, whenever a subsequent variable affects preceding variables in the cause- effect chain, we have a feedback action.

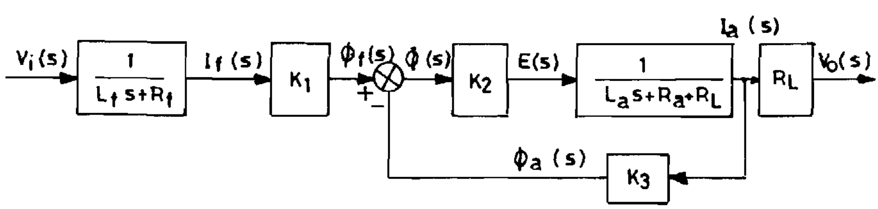


Fig. 7.6 Block Diagram of d.c. Generator

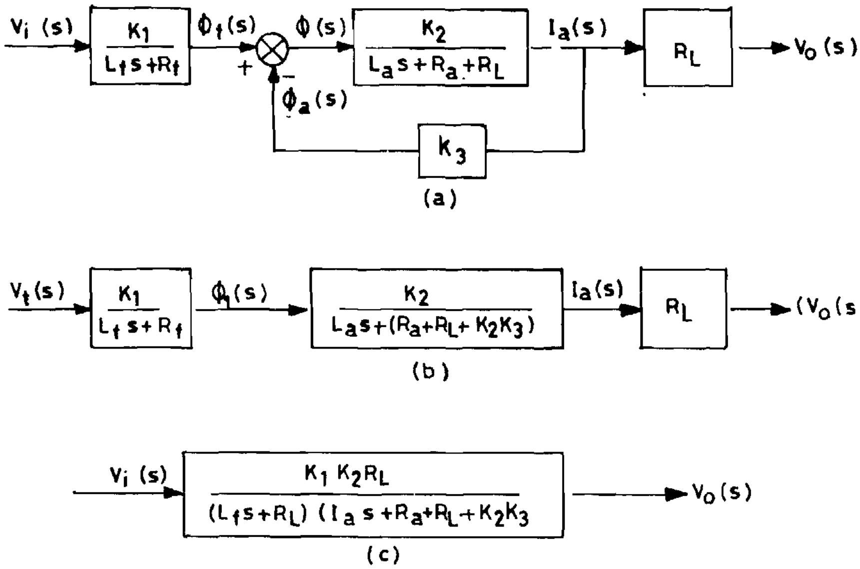


Fig. 7.7 Block Diagram Reduction of Fig. 7.6

7.2 Block Diagram Reduction

In situations like that of Example 7.1, the overall transfer function of the system may be obtained either by combining the system equations or by reducing the block diagram to a single block. The expression in this single block will be the system transfer function. In this section we study the techniques for reducing complex block diagrams into a single block.

Three types of reductions for combining blocks connected in series, parallel, or feedback configurations have already been described in the previous section. The steps involved in simplifying a block diagram, when these three reduction methods are sufficient, are shown by simplifying the block diagram of Fig. 7.6 in Figs. 7.7 (a), (b) and (c).

When the loops are intertwined in the case of more complex block diagrams, we need additional techniques for block diagram simplification. The techniques required for such cases are illustrated by the next example.

Example 7.2 : — Simplify the block diagram shown in Fig. 7.8.

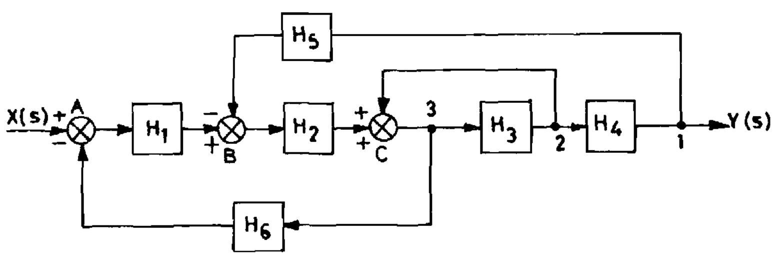


Fig. 7.8 A Complex Block Diagram

It is not possible to use any one of the three previous combinations because of the intertwining loops.

Movement of pick-off points: If the 'pick-off' points of the feedback loops could be altered by moving them forward (i.e., towards the input) or backward, some simplification could result. For example, if the pick-off point of H_6 could be moved back to location 2 from location 3, the H_3 block and the feedback around it could be combined into one block. However, this shift should not alter the feedback signal being received at the summing junction A. To ensure this, the transfer function H_6 should be altered to H_6/H_3. The shifting of pick-off point and the subsequent reductions are shown in Figs. 7.9 (a), (b), and (c).

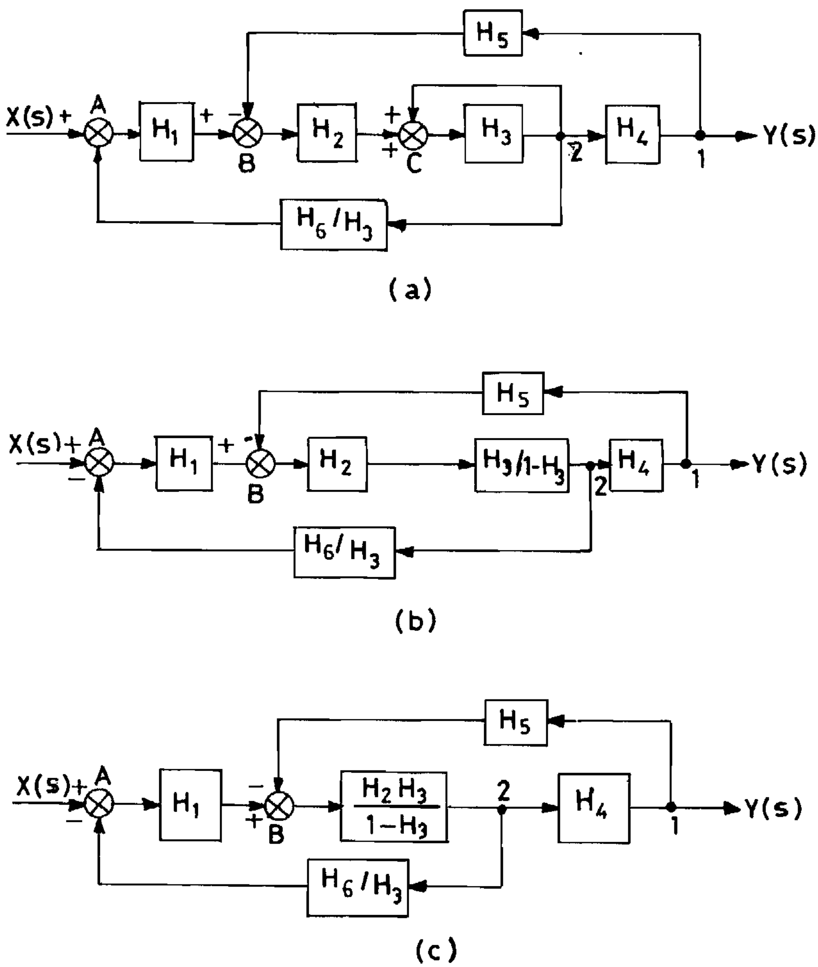


Fig. 7.9 Reduction of Block Diagram of Fig. 7.8

For further reduction, one more shift in the pick-off point is needed. We could shift either pick-off point of H_6/H_3 backwards to location 1 or shift pick-off point of H_5 forward to location 2. Let us follow the second alternative. Once again, the process of shifting the pick-off point should not alter the signal being received at the summing junction B. To ensure this, the transfer function H_5 should be multiplied by H_4. This shift and the resulting simplifications are shown in Fig. 7.10.

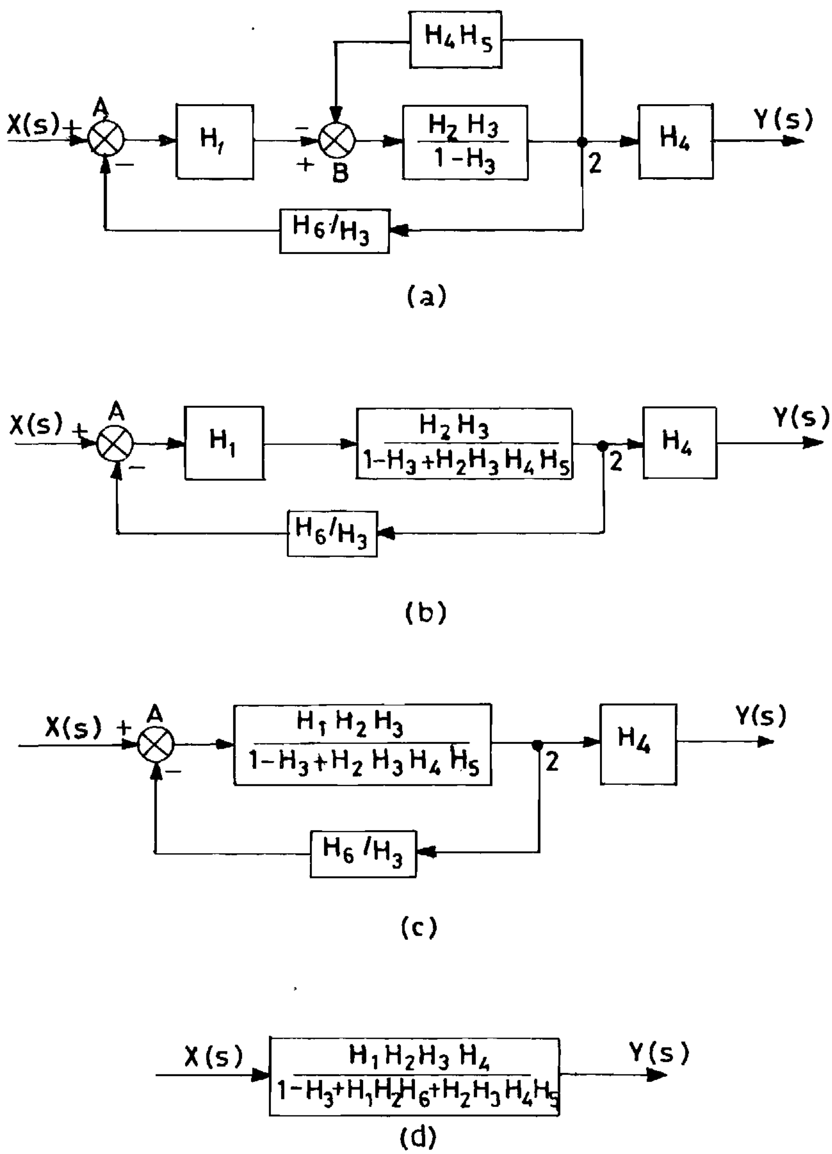


Fig. 7.10 Further Reduction of Fig. 7.8

Movement of summing junctions : The block diagram reduction of Fig. 7.8 has been achieved in Figs. 7.9 and 7.10 by the movement of pick-off points. In a similar fashion we could move summing junctions also. For example, let us move the summing junction into which H_1 feeds from B to A. The process of shifting should not alter the signal received at the input of H_2. To ensure this the feedback block H_5 should be divided by H_1. This shift and further reductions are shown in Fig. 7.11.

It should be noted that the locations of the two summing junctions at the input end of Fig. 7.11 (a) have been interchanged in Fig. 7.11 (b). This is quite permissible as it does not alter the signal being received at the input of the next block. The overall transfer function arrived at in Fig. 7.11 (d) is the same as the transfer function in Fig. 7.10 (d).

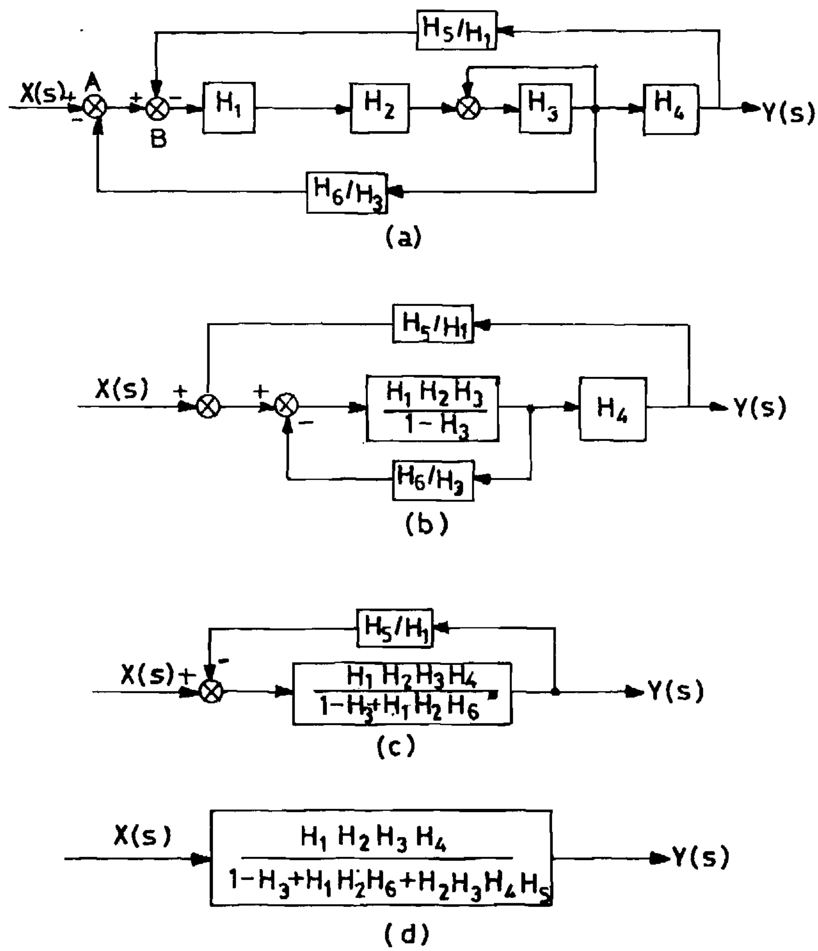


Fig. 7.11 Reduction of Fig. 7.9 by Movement of Summing Junction

7.3 Signal Flow Graph

A block diagram shows the interconnected parts through which the input signal moves towards the output. The characteristics of the different elements of the path are shown by the transfer function blocks. The same information can be displayed, somewhat more neatly, by a line diagram in which the summing junctions and pick-off points are represented simply as nodes: the paths are indicated by lines; the direction of flow of the signal by arrows; and the path characteristic by its transfer function, written along the line. Such a line diagram is called a *signal flow graph*. The signal flow graphs for Example 7.1 (Fig. 7.6) and Example 7.2 (Fig. 7.8) are shown in Figs. 7.12 (a) and (b).

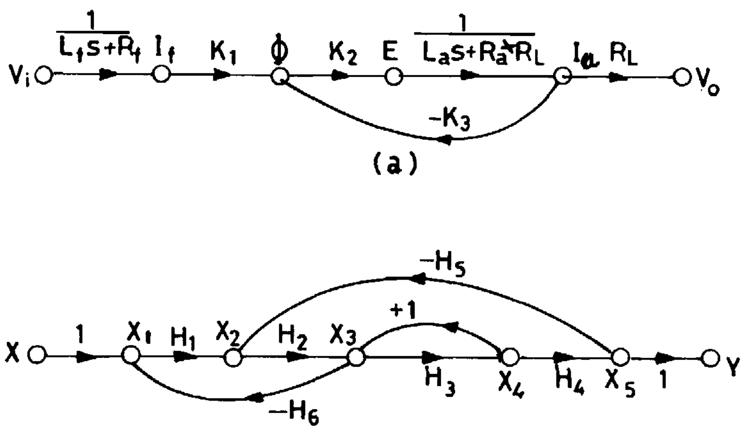


Fig. 7.12 Signal Flow Graph for (a) Fig. 7.6 and (b) Fig. 7.8

A system variable is associated with each node. The line joining any two nodes is called a *directed branch*. The transfer function relating the variable at the output end of the line to the variable at its input end is called the *branch transmittance*. The signal going to all the outward directed branches from a node is the sum of the signals coming on incoming branches. For example, for the node Φ in Fig. 7.12 (a), the incoming signal is $K_1 I_f - K_3 I_a = \Phi$. The outgoing signal is this sum. Therefore, $E = K_2 (K_1 I_f - K_3 I_a)$. If we write similar equations for each node, we get the original system eqns. (i) to (vii) of Example 7.1, written out in a slightly different form:

For node I_f : $\dfrac{V_i}{L_f s + R_f} = I_f$.

For node Φ : $K_1 I_f - K_3 I_a = \Phi$.

For node E : $K_2 \Phi = E$.

For node I_a : $\dfrac{E}{L_a s + R_a + R_L} = I_a$.

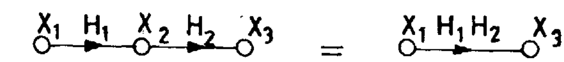
For node V_0 : $I_a R_L = V_0$.

The signal flow graph may, therefore, be thought of as a graphical representation for a set of algebraic equations. Instead of simplifying the algebraic equations step by step to obtain a single equation relating the input to the output, i.e., the transfer function of the system, we simplify the signal flow graph to obtain a single equivalent branch connecting the input node to the output node.

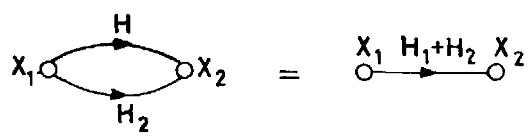
The rules for simplification of a signal flow graph are similar to those for block diagram manipulations. Four of these basic rules, which follow straightaway from the basic definitions, are summarised in Table 7.1 for quick reference.

Table 7.1 Rules for Simplification of Signal Flow Graph

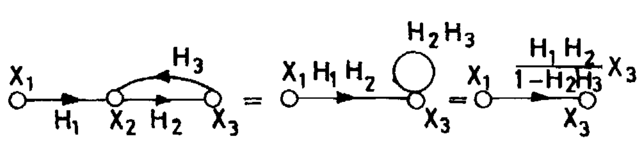
1 Series connection:



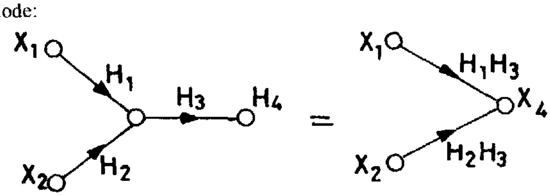
2. Parallel connection:



3. Feedback connection:



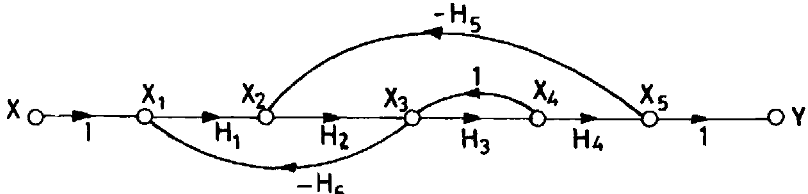
4. Removal of a node:



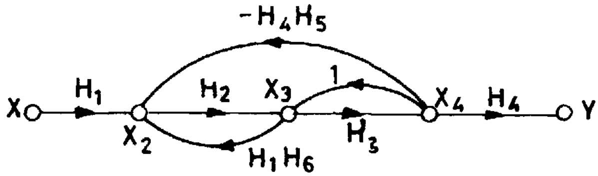
The following example illustrates the use of these rules for simplifying signal flow graphs.

Example 7.3 : — Simplify the signal flow graph of Fig. 7.12 (b) to obtain the transfer function between X and Y.

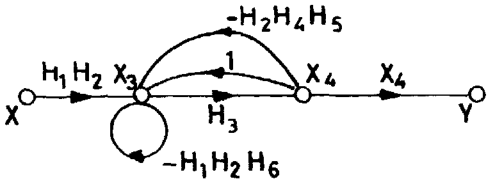
Solution: The signal flow graph is reproduced in Fig. 7.13 (a). The subsequent step-by-step reduction of the graph is shown in Fig. 7.13 (b) to (g).



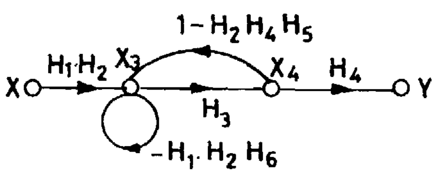
(a) Signal flow graph



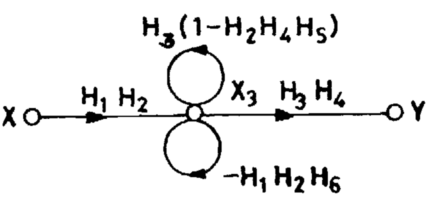
(b) Removal of nodes X_1 and X_5



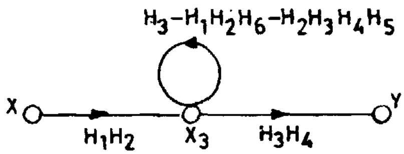
(c) Removal of node X_2



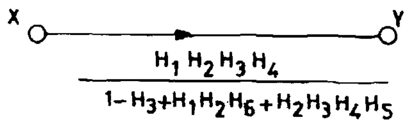
(d) Addition of parallel branches



(e) Removal of node X_4



(f) Addition of parallel loops



(g) Final graph

Fig. 7.13 Reduction of Signal Flow Graph

The transfer function derived from the signal flow graph reduction is the same as that obtained by the block diagram reduction in Example 7.2.

The sequence of steps for reduction of a signal flow graph to a single branch, as illustrated above, is not a unique one. It is usually not possible to ascertain beforehand which sequence will involve minimum computation. A major advantage of the signal flow graph technique is the availability of a formal procedure for reduction of a flow graph from mere inspection. This procedure is called *Mason's formula.* Certain terms need to be defined before this formula can be used. These terms are as follows.

Source node: The node at the input end which has only outward branches, e.g., node X in Fig. 7.13 (a).

Sink node: The node at the output end which has only inward branches, e.g., node Y in Fig. 7.13 (a).

Forward path (or simple path): A sequence of outward directed branches from source node to sink node such that no node is encountered more than once. Figure 7.13 (a) has only one forward path (X, X_1, X_2, X_3, X_4, Y_5, Y). Other graphs may have more than one forward path.

Forward path transmittance: The product of all the individual branch transmittances in a forward path, e.g., in Fig. 7.13(a) the forward path transmittance is H_1 H_2 H_3 H_4.

Loop: It is a path starting at a node and terminating at the same node. Figure 7.13 (a) has three loops: (i) $\{X_1, X_2, X_3, X_1\}$; (ii) $\{X_3, X_4, X_3,\}$; (iii) $\{X_2, X_3, X_4, X_5, X_2\}$.

Loop transmittance: The product of branch transmittances in a loop. The three loop transmittances of Fig. 7.13 (a) are: (i) $-H_1 H_2 H_6$; (ii) H_3; (iii) $-H_2 H_3 H_4 H_5$.

Non-touching or disjoint loops : Loops which do not have any common nodes. In Fig. 7.13 (a) none of the three loops is disjoint.

Determinant of a graph, Δ = 1 – (sum of all loop transmittances) + (sum of the products of all possible pairs of non-touching loop transmittances) – (sum of the products of all possible triplens of non-touching loop transmittances) + ...

Cofactor with respect to a particular forward path k, Δ_k = 1 – (sum of all loop transmittances of the loops that do not touch the k th forward path) + (sum of the products of all possible pairs of non-touching loop transmittances of loops which do not touch the kth forward path) — (sum of the products all possible triplens of non-touching loop transmittances of loops which do not touch the kth forward path) + ... Thus, Δ_k is the cofactor of the element corresponding to the kth forward path in the graph determinant Δ, with the transmittance of all the loops touching the kth path removed.

Mason's formula gives the net transmittance or the graph transmittance from a source node to a sink node. In our terminology, this graph transmittance is the transfer function relating the output to the input. The formula is,

$$H = \frac{1}{\Delta} \sum_k G_k \Delta_k \tag{7.5}$$

where

H = graph transmittance or the transfer function;

Δ = the determinant of the graph;

G_k = transmittance of the kth forward path; and

Δ_k = cofactor of the kth forward path, as defined avove.

The procedure for using this formula is illustrated by applying it to Example 7.3. By an inspection of Fig. 7.13 (a), we note that there is only one forward path with a transmittance $H_1 H_2 H_3 H_4$. There are three loops with transmittances (i) $-H_1 H_2 H_6$, (ii) H_3 and (iii) $-H_2 H_3 H_4 H_5$. All the loops touch each other, so there are no non-touching loops. Further, all the loops touch the forward path. Hence,

$$\Delta = 1 - H_3 + H_1 H_2 H_6 + H_2 H_3 H_4 H_6$$

$$\Delta_k = 1.$$

Therefore, $$H = \frac{H_1 H_2 H_3 H_4}{1 - H_3 + H_1 H_2 H_6 + H_2 H_3 H_4 H_6}.$$

Thus, Mason's formula permits reduction of the graph in almost a single step, without the need for a step-by-step graphical reduction procedure. This is a great help, especially in the case of more complex graphs.

Example 7.4 : — Find the overall transfer function of the system whose signal flow graph is shown in Fig. 7.14.

Solution: There are three forward paths:

$$G_1 = H_1 H_2 H_3 H_4 H_5 H_6; G_2 = H_1 H_7 H_4 H_5 H_6 \text{ and } G_3 = H_1 H_2 H_8 H_6 .$$

There are three loops:

$$L_1 = -H_4 H_{10} ; L_2 = -H_8 H_6 H_9 \text{ and } L_3 = -H_3 H_4 H_5 H_6 H_9 .$$

Out of these, L_1 and L_2 do not have any node in common and hence are non-touching loops. Therefore, the determinant of the graph is,

$$\Delta = 1 + H_4 H_{10} + H_6 H_8 H_9 + H_3 H_4 H_5 H_6 H_9 + H_4 H_{10} H_8 H_6 H_9.$$

For path G_1, all the loops touch it. Therefore its cofactor $\Delta_1 = 1$.

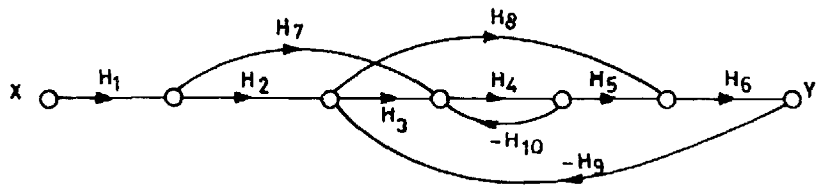


Fig. 7.14 A Signal Flow Graph

Similarly, $\Delta_2 = 1$. However, for the third path G_3, loop L_1 is non-touching. Hence $\Delta_3 = 1 + H_4 H_{10}$.

The overall transfer function, according to eqn. (7.5) is,

$$H = \frac{1}{\Delta} [G_1 \Delta_1 + G_2 \Delta_2 + G_3 \Delta_3]$$

$$= \frac{H_1 H_2 H_3 H_4 H_5 H_6 + H_1 H_7 H_4 H_5 H_6 + H_1 H_2 H_8 H_6 + H_1 H_2 H_8 H_6 H_4 H_{10}}{1 + H_4 H_{10} + H_6 H_8 H_9 + H_3 H_4 H_6 H_9 + H_4 H_{10} H_8 H_6 H_9}$$

It has been mentioned earlier that the signal flow graph may also be viewed as a graphical representation for simultaneous algebraic equations. The variables in these algebraic equations may be Laplace transforms of the system variables, or any other set of algebraic variables. Thus, the process of simplification of a graph is equivalent to solving a set of algebraic equations. Mason's formula for the signal flow graph corresponds to Crammer's rule for solving algebraic equations. This is demonstrated by the following example.

Example 7.5 : — Simplify the flow diagram of Fig. 7.15 to obtain the graph transmittance. Verify it by using Crammer's rule.

Solution: There are two forward paths:

$$G_1 = H_1 H_2 H_3 \text{ and } G_2 = H_4 H_3 .$$

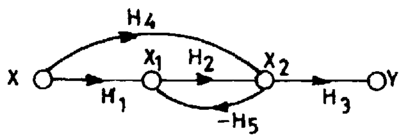


Fig. 7.15

There is only one loop; $L_1 = -H_2 H_3$. Therefore, $\Delta = 1 + H_2 H_5$ and $\Delta_1 = \Delta_2 = 1$. Therefore,

$$H = \frac{G_1 \Delta_1 + G_2 \Delta_2}{\Delta} = \frac{H_1 H_2 H_3 + H_3 H_4}{1 + H_2 H_5}$$

Now, let us write the algebraic equations for the given graph:

$$X_1 = H_1 X - H_5 X_2 \, ; X_2 = H_4 X + H_2 X_1 \, ; \text{ and } Y = H_3 X_2.$$

Rewriting these equations in the matrix form we get,

$$\begin{bmatrix} 1 & H_5 & 0 \\ -H_2 & 1 & 0 \\ 0 & -H_3 & +1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ Y \end{bmatrix} = X \begin{bmatrix} X_1 \\ H_4 \\ 0 \end{bmatrix}.$$

The determinant for Crammer's rule is,

$$\Delta = \begin{vmatrix} 1 & H_5 & 0 \\ -H_2 & 1 & 0 \\ 0 & -H_3 & +1 \end{vmatrix} = 1 + H_2 H_5$$

which is the same as the determinant of the signal flow graph. Cofactor for the output Y,

$$\Delta_Y = \begin{vmatrix} 1 & H_5 & H_1 \\ -H_2 & 1 & H_4 \\ 0 & -H_3 & 0 \end{vmatrix} = H_3 H_4 + H_1 H_2 H_5$$

and

$$\frac{Y}{X} = H = \frac{H_3 H_4 + H_1 H_2 H_5}{1 + H_2 H_5}$$

which is the same result as that obtained by the signal flow graph.

7.4 Feedback Control Systems

Automatic control action is vital to the modern technological achievements like space programmes, aviation, as well as to the established industries like power generation, chemical processing, paper mills, and steel mills. The basis of this automatic action is the feedback principle of control.

As an example of a feedback control system, let us consider the speed control of a d.c. motor for application in a paper mill. Paper from primary paper rolls (15- to 20-ft. wide paper) is passed through a number of rollers in a 'rewinder' mill to give it the required surface finish or to cut it into rolls of desired width. Paper is 'reeled out' by one d.c. motor and 'reeled in' at the other end of the roller bank by another d.c. motor. As the roll diameters change at these two ends, the paper tension also changes. The motor speeds have to be controlled continuously to keep the paper tension constant. If the tension is less, the quality of the surface finish gets altered. There is also a chance of warping and fouling of the paper line. If the tension is more, the paper may tear up leading to loss of paper and processing time.

Depending on the roll diameters and the required paper tension, the drive controller generates a signal for the desired motor speed. This signal, called the reference signal or the desired output, is the input to the speed controller. The actual motor speed is measured by a tachogenerator which gives a voltage proportional to the instantaneous speed. This is called the output signal. The input and the output signals are compared in a summing junction in the speed controller which generates a signal proportional to the difference between them. This signal is called the error signal. The voltage applied across the motor armature is proportional to this error signal. If the error is zero, i.e., the output is equal to the input, no control action is called for and the armature voltage remains unaltered at its previous value. If the error is positive, indicating that the output speed is less than the desired value, the armature voltage is increased to speed up the motor. When the input and output speeds match again, the voltage is not altered. In this fashion, by constantly comparing the input and the output signals, and then initiating appropriate control action, the motor speed automatically follows the set reference value. Since the output is fed back to the input, such a system is called a *feedback control system.*

Let us represent the speed control system in terms of the block diagram of Fig.7.16. The d.c. motor with its associated thyristor power converter and mechanical load connected to it (in this case, the paper roll) is called the 'plant' to be controlled. The input to this plant will be the thyristor firing angles (which, in turn, sets the armature voltage level) and the output will be the motor speed. The relationship between the input and the output of the plant will be given by its transfer function $G(s)$. The input to this plant will be given by the 'controller'. Actually, the summing junction is also a part of the controller only. The dynamics

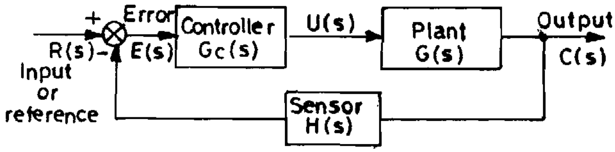


Fig. 7.16 Block Diagram of a Feedback Control System

of the controller are given by its transfer function $G_c(s)$. The output is sensed by some transducer which generates a signal proportional to the output. The characteristics of this sensor and other components in the feedback path are given by the transfer function $H(s)$. The block diagram of Fig. 7.16 is the representation for any general feedback control system. The symbols for the variables —$C(s)$ for the output; $R(s)$ for the reference input, $U(s)$ for the control signal; and $E(s)$ for the error signal—are also standard symbols in control theory.

Many a time the dynamics of the controller and/or feedback sensor can be neglected. Then the transfer functions $G_c(s)$ and $H(s)$ become constants. In many cases, the controller transfer function is simply the gain of some amplifier, i.e.,$G_c(s) = K$. Again, in many cases the feedback transfer function is simply unity. Such systems are called unity feedback systems and represented by the block diagram of Fig. 7.17. Note that Fig. 7.17 is a special case of Fig. 7.16 with $G_c(s) = K$ and $H(s) = 1$.

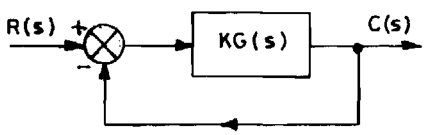


Fig. 7.17 Unity Feedback System

Let us now introduce some of the terms in control theory terminology. The 'open loop' transfer function of Fig. 7.16 is $G_c(s)\,G(s)\,H(s)$. The 'forward path' transfer function is $G_c(s)\,G(s)$, The 'closed loop' transfer function, according to eqn. (7.3), is given by,

$$\frac{C(s)}{R(s)} = \frac{G_c(s)\,G(s)}{1 + G_c(s)\,G(s)\,H(s)}. \tag{7.6}$$

The closed loop transfer function of the unity feedback system of Fig. 7.17 is given by,

$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)}. \tag{7.7}$$

As mentioned in Chapter 6, since $G(s)$ is a ratio of two polynomials in s, eqn. (7.6) or eqn. (7.7) will also be a ratio of two polynomials. That is,

$$\frac{C(s)}{R(s)} = \frac{N(s)}{D(s)} = \frac{c_m s^m + c_{m-1} s^{m-1} + \ldots + c_1 s + c_0}{s^n + d_{n-1} s^{n-1} + \ldots + d_1 s + d_0} \qquad (7.8)$$

with $m \leq n$. The denominator is called the *characteristic polynomial* and the equation,

$$s^n + d_{n-1} s^{n-1} + \ldots + ds + do = 0 \qquad (7.9)$$

is called the *characteristic equation* of the closed loop system. The roots of eqn. (7.9) are the closed loop poles and their locations determine the form of the system's natural response.

Some of the important problems in the analysis of feedback control systems are: (i) transient response; (ii) stability; (iii) accuracy; and (iv) sensitivity. In the following sections, we investigate the main points concerning these problems.

7.5 Transient Response

The unit step signal serves as a standard test signal for characterising the transient response of linear systems. The procedure for determining the complete response to a step input, using the Laplace transform method has already been described in Chapter 6.

Let us first investigate the effect of feedback on the transient response of a first order system. Consider the system shown in Fig. 7.18. The closed loop transfer function of the system is,

$$\frac{C(s)}{R(s)} = \frac{K}{s + 1 + Ka}.$$

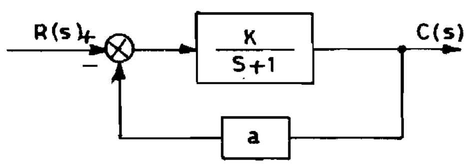


Fig. 7.18 A First Order System

The step response of the system is given by,

$$C(s) = \frac{K}{s(s + 1 + Ka)} = \frac{A_1}{s} + \frac{A_2}{s + 1 + Ka}$$

with $A_1 = K/(1 + Ka)$ and $A_2 = -K/(1 + Ka)$.

The Laplace inversion of $C(s)$ gives,

$$c(t) = A_1 u(t) + A_2 \exp[-(1 + Ka)t].$$

The first term gives the steady-state component and the second term the transient component of the response. The transient term is a decaying exponential term with a time constant equal to $1/(1+Ka)$, which can be controlled by a proper choice of K and a. K may be the gain of an amplifier and a the gain (rather the attenuation) of a potentiometer. By controlling these two constants, the time constant of the closed loop system can be made as small as desired. A smaller time constant means a faster acting system. Thus, even a slow acting plant can be made quick acting by the action of feedback.

The transient or the dynamic properties of a first order system are all summed up in a single factor; the time constant of the system. We now take up the more interesting case of a second order system.

Transient response of second order systems:

Second order systems are important because of two reasons. First, a number of physical systems give rise to second order models, e.g., R–L–C electrical circuit, mass-spring-dashpot mechanical system, etc. Second, the response of even higher order systems can usually be approximated by a second order system. In fact, the design specifications of most of the control systems are usually given in terms of performance criteria of second order systems.

As mentioned in Section 3.6, the general equation for a second order system with output $c(t)$ is given by,

$$\frac{d^2c}{dt^2} + 2\zeta\omega_n \frac{dc}{dt} + \omega_n^2 c = \omega_n^2 r(t) . \tag{7.10}$$

The input $r(t)$ has been multiplied by a constant ω_n^2 so that when $r(t)$ is constant, the steady-state output is equal to $r(t)$. In other words, the steady state gain of the system is unity. The transfer function of the system is,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} . \tag{7.11}$$

The system is overdamped when $\zeta > 1$, critically damped when $\zeta = 1$ and underdamped when $\zeta < 1$. The underdamped case is more important and henceforth, we assume the system to be underdamped.

The unit step response of eqn. (7.11) is,

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} .$$

Expanding by partial fractions we get,

$$C(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{A_1}{s} + \frac{A_2 s + A_3}{s^2 + 2\zeta\omega_n s + \omega_n^2} .$$

Multiplying both sides by the denominator we get,

$$A_1(s^2 + 2\zeta\omega_n s + \omega_n^2) + (A_2 s + A_3)s = \omega_n^2$$

or

$$(A_1 + A_2)s^2 + (2\zeta\omega_n A_1 + A_3)s + A_1\omega_n^2 = \omega_n^2 \cdot$$

Equating coefficients of like powers of s on both sides of the above equation we get,

$$A_1 = 1, \quad 2\zeta\omega_n A_1 + A_3 = 0 \text{ and } A_1 + A_2 = 0\,.$$

From the above we get,

$$A_1 = 1,\; A_2 = -1 \text{ and } A_3 = -2\zeta\omega_n\,.$$

Therefore,

$$C(s) = \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2}\,.$$

We now take the Laplace inverse of the above expression to get $c(t)$. For this purpose the second term on the r.h.s. is rewritten as,

$$\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\zeta\omega_n + (s + \zeta\omega_n)}{(s + \zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2}$$

$$= \frac{1}{\sqrt{1-\zeta^2}} \cdot \frac{(\omega_n\sqrt{1-\zeta^2})\,\zeta + (s + \zeta\omega_n)\sqrt{1-\zeta^2}}{(s + \zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2}$$

$$= \frac{1}{\sqrt{1-\zeta^2}} \frac{(\omega_n\sqrt{1-\zeta^2})\cos\theta + (s + \zeta\omega_n)\sin\theta}{(s + \zeta\omega_n)^2 + (\omega_n\sqrt{1-\zeta^2})^2}$$

where $\cos\theta = \zeta$ and $\sin\theta = \sqrt{1-\zeta^2}$.

The above expression now corresponds to item 12 of Table 6.1 and its Laplace inverse is given by,

$$\frac{1}{\sqrt{1-\zeta^2}} \exp(-\zeta\omega_n t)\sin\left\{\omega_n(\sqrt{1-\zeta^2})\,t + \theta\right\}$$

Therefore,

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} \exp(-\zeta\omega_n t)\,\sin\left\{\omega_n(\sqrt{1-\zeta^2})\,t + \theta\right\}$$

where $\theta = \tan^{-1}\sqrt{1-\zeta^2}/\zeta$.

Replacing $\omega_n\sqrt{1-\zeta^2}$ by ω_d we get,

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} \exp(1 - \zeta\omega_n t)\,\sin(\omega_d t + \theta) \tag{7.12}$$

Notice that relation (7.12) is the same as eqn. (3.31).

The first term on the r.h.s. of eqn. (7.12) is the steady-state response and the second term the transient response. The transient response is oscillatory, i.e., a decaying sinusoid with frequency $\omega_n \sqrt{1-\zeta^2}$. Here, ω_n is the natural frequency of undamped oscillations and $\omega_n \sqrt{1-\zeta^2} = \omega_d$, the frequency of damped oscillations. If the damping ratio is zero, the system will have oscillations of constant amplitude. However, such systems are called unstable systems and this condition is avoided. A plot of the transient response and its characteristics have already been described in Section 3.6.

In the design of control systems the transient behaviour is specified by the following measures: (i) rise time, t_r; (ii) percentage overshoot, PO; and (iii) settling time t_s. The expressions for these are (Section 3.6);

$$t_r = \frac{\pi - \theta}{\omega_d}; \ PO = 100 \ \exp(-\zeta\pi/\sqrt{1-\zeta^2}); \ t_s = \frac{3}{\zeta\,\omega_n}.$$

The parameters under control are ζ and ω_n. As the *PO* depends only on ζ, the damping ratio is selected first to satisfy the *PO* requirement. The value of ω_n is then chosen to satisfy either the t_s or the t_r requirement. Usually ζ is selected between 0.4 and 0.7; 0.4 gives an overshoot of 25% and 0.7 about 5%.

Correlation between transient response and location of poles:

As mentioned earlier, the shape of the transient response (i.e., the natural response) of a system is determined by the roots of its characteristic equation, i.e. the poles of the system. From eqn. 7.11, the characteristic equation of a second order system is $s^2 + 2\zeta\,\omega_n s + \omega_n^2 = 0$. The roots of this equation are,

$$s_1, s_2 = -\omega_n \pm j\,\omega_n \sqrt{1-\zeta^2}.$$

Thus, the location of the system poles in the s-domain is dependent on the parameters ζ and ω_n. Figure 7.19 shows this dependence. The distance of the poles from the origin is equal to ω_n and the intercept on the $j\omega$-axis equal to the damped

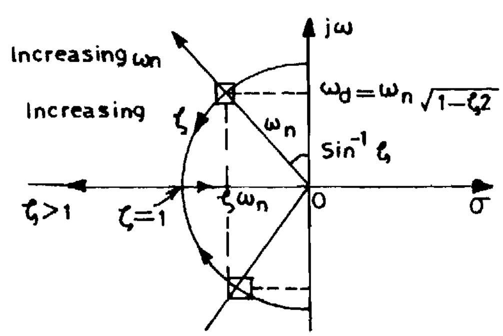


Fig. 7.19 Pole Locations in Terms of ζ and ω_n

frequency of oscillation ω_d. As ζ is varied keeping ω_n fixed, the poles move along the arc of a circle with radius ω_n and centre at origin. When $\zeta = 0$, both the poles lie on the $j\omega$-axis, giving steady-state oscillations. When $\zeta = 1$, both the poles move on to the same location on the σ-axis. Increasing ζ beyond 1 moves the poles apart but they remain on the real axis. Thus, poles on the real axis give an overdamped response and complex poles give an underdamped oscillatory response. Poles nearer the $j\omega$ axis cause more oscillations than those away from $j\omega$ axis. Of course, poles in the right half of the s-plane will make the system unstable.

We now take up an example of a feedback control system, study its basic components and their functions, obtain its mathematical model and study its transient behaviour.

Example 7.6 : — The schematic diagram of a position control system is shown in Fig. 7.20. The objective is to control the angular position θ_o of the output shaft. The input, i.e., the desired angular position θ_i is set on the potentiometer P_1. Potentiometer P_2 is coupled to the output shaft. The pair P_1 and P_2 act as the error detector, generating a voltage proportional to $\theta_i - \theta_o$. The amplifier amplifies this voltage to drive the armature of a d.c. motor.

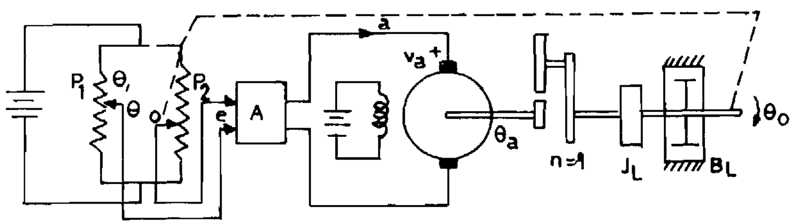


Fig. 7.20 Schematic Diagram of a Position Control System

The motor armature is coupled to the mechanical load through a gear train which reduces the speed in the ratio $n : 1$ ($n > 1$). Develop the transfer function of the system and find its step input response.

Solution: Let us first determine the transfer function relating the error signal e to the output θ_o. The cause-effect chain, relating e to θ_o, may be written as,

Armature voltage : $v_a = Ae.$

Armature current: $$i_a = \frac{v_a - K_1\dot{\theta}_a}{R_a} = \frac{v_a - k_1 n\dot{\theta}_o}{R_a}.$$

Torque: $\tau_a = K_2 i_a$.

These three equations can be combined into one as follows:

$$\tau_a(s) = \frac{K_2 A}{R_a} E(s) - \frac{K_1 K_2 n}{R_a} s\,\theta_o(s)$$

$$= K_3 E(s) - K_4 s\,\theta_o(s). \qquad \text{(i)}$$

The motor output torque τ_a will drive the mechanical load; the armature inertia J_a plus the load inertia and viscous friction, referred to the armature shaft. The load inertia J_L and viscous friction B_L will be multiplied by n^2 when referred to the armature shaft. Thus, the total inertia at the motor shaft is,

$$J' = J_a + n^2 J_L.$$

Similarly, the total viscous friction at the motor shaft is,

$$B' = B_a + n^2 B_L.$$

Thus, the equation of the mechanical motion is,

$$J'\ddot{\theta}_a + B'\dot{\theta}_a = \tau_a$$

or

$$J' n\,\ddot{\theta}_o + B' n\,\dot{\theta}_o = \tau_a.$$

Taking the Laplace transform, and replacing $J'n = J$ and $B'n = B$, we get,

$$Js^2\,\theta_o(s) + Bs\,\theta_o(s) = \tau_a(s). \qquad \text{(ii)}$$

Replacing $\tau_a(s)$ in eqn. (ii) by the expression (i) we have,

$$(Js^2 + Bs)\,\theta_o(s) = K_3 E(s) - K_4 s\theta_o(s) \qquad \text{(iii)}$$

Therefore, the plant transfer function is,

$$\frac{\theta_o(s)}{E(s)} = \frac{K_3}{s\,[Js + (B + K_4)]} \qquad \text{(iv)}$$

Now, the effect of feedback is to make

$$E(s) = K_5\,[\theta_i(s) - \theta_o(s)].$$

Substituting this expression in eqn. (iii) we get,

$$(Js^2 + Bs)\theta_o(s) = K_3K_5\,\theta_i(s) - (K_3K_5 + K_4 s)\,\theta_o(s).$$

Thus, the closed loop transfer function is,

$$\frac{\theta_o(s)}{\theta_i(s)} = \frac{K_3K_5}{Js^2 + (B + K_4)s + K_3K_5} \qquad \text{(v)}$$

Expression (v) can be rewritten in the standard form as,

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

where,

$$\omega_n = \sqrt{K_3K_5/J}; \quad \zeta = \frac{B+K_4}{2\sqrt{K_3K_5/J}},$$

Let the system parameters be so chosen that $\zeta = 0.5$ and $\omega_n = 4$ rad/sec, The performance measures of the step input response can then be calculated according to the eqns. (3.32) to (3.55) as follows:

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta} = \tan^{-1} 1.735 = 1.065 \text{ rad}$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2} = 3.47 \text{ rad/sec.}$$

Therefore,

$$t_r = \frac{\pi-\theta}{3.47} = 0.6 \text{ sec.}$$

$$t_p = \pi/\omega_d = 0.785 \text{ sec.}$$

$$PO = 100\exp(-\zeta\pi/\sqrt{1-\zeta^2}) = 16.4\%$$

$$t_s = 3/\zeta\omega_n = 1.5 \text{ sec.}$$

Control of transient response through feedback:

Figure 7.21 shows a second order overdamped plant. Feedback control is employed around it with a feedback gain a and forward path gain K. We now show that by controlling K and a, the system may be made to have any desired dynamic response.

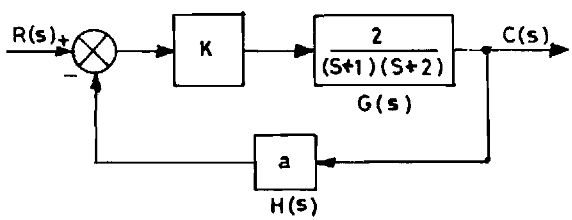


Fig. 7.21 Second Order Overdamped Plant

The closed loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1+KG(s)H(s)} = \frac{2K}{(s+1)(s+2)+2Ka}$$

or

$$\frac{C(s)}{R(s)} = \frac{2K}{s^2+3s+(2+2Ka)}.$$

Comparing with the standard form,

$$2\zeta\omega_n = 3 \text{ and } \omega_n^2 = 2 + 2Ka.$$

Therefore,

$$\omega_n = \sqrt{2 + 2Ka} \text{ and } \zeta = \frac{3}{2\sqrt{2 + 2Ka}}.$$

Thus, by suitably selecting the values of K and a, we can get any desired transient response.

The roots of the characteristic polynomial of the closed loop system are,

$$s_1, s_2 = -\frac{3}{2} \pm \frac{\sqrt{9 - 4(2 + 2Ka)}}{2}.$$

By suitably selecting K and a, the poles can be located anywhere on the real axis or on a vertical line passing through $\sigma = -3/2$. Thus, the feedback can be used to control the transient response of a feedback control system.

7.6 Stability

The transient response of linear systems consists of exponential terms of the form $c_i \exp(r_i t)$, where the coefficients r_i in the exponent are the roots of the characteristic equation of the system. In the transfer function representation of systems, the characteristic equation is the denominator polynomial equated to zero, and the roots r_i are the same as the poles of the system. If all the roots are negative (or have negative real parts in the case of a complex root), the transient response decays to zero as t increases to infinity. On the other hand, if even a single root is positive (or has a positive real part in the case of a complex root), the transient response will go on increasing without bounds. We then say that the system is unstable. Thus, the condition for stability of a linear system is that all the roots of its characteristic equation should be negative (or have negative real parts in the case of the complex roots) or all the system poles must lie in the left half of the s-plane.

When a pair of complex poles lie on the $j\omega$-axis, a step input response produces sinusoidal oscillations of constant magnitude. Such a system is on the borderline between stable and unstable conditions and is called a *marginally stable system*. However, even a slight change in the parameters may push the poles into the right half of the s-plane, making the system unstable. Hence, for practical purposes a marginally stable system is also classified as an unstable system.

The use of feedback usually has a tendency to destabilise a stable plant. The reason for this is as follows. A prime requirement of any control system is that it should have a low steady-state error. As we shall see later, this requires that the gain of the open loop transfer function, $|KG(s)H(s)|$, be high. This gain can easily be increased by increasing the forward path gain K, which is usually the gain of an amplifier. However, increasing K shifts the closed loop poles to the right, leading the system towards instability. This is illustrated by the following example.

Example 7.7 : — The open loop transfer function of a unity feedback system is,

$$G(s) = \frac{K}{(s+1)(s+2)(s+3)} = \frac{K}{s^3+6s^2+11s+6}.$$

Determine the location of its closed loop poles for increasing values of K. Also determine the limiting value of K beyond which the closed loop system with unity feedback becomes unstable.

Solution: All the three poles of the plant, $s_1 = -1$, $s_2 = -2$ and $s_3 = -3$, are negative. Hence, the plant, i.e., the uncontrolled system (without feedback) is stable. The transfer function of the closed loop system is,

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K}{s^3+6s^2+11s+6+K}.$$

The characteristic equation of the closed loop system is,

$$1 + G(s) = 0$$

or

$$s^3 + 6s^2 + 11s + 6 + K = 0.$$

In order to obtain the roots or the poles of the closed loop system, we must factorise the above closed loop characteristic equation for different values of K. This is done as follows.

(1) $K = 0$: The closed loop characteristic equation becomes the plant characteristic equation. Hence, the closed loop poles will be the same as the open loop poles. That is,

$$s_1 = -1, s_2 = -2, s_3 = -3.$$

(2) $K = 6$: The characteristic equation is

$$s^3 + 6s^2 + 11s + 12 = (s+4)(s^2+2s+3).$$

That is,

$$s_1, s_2 = -1 \pm j\sqrt{2} \text{ and } s_3 = -4.$$

(3) $K = 24$:

$$s^3 + 6s^2 + 11s + 30 = (s+5)(s^2+s+6).$$

That is,

$$s_1, s_2 = -0.5 \pm j\sqrt{23}/2 \text{ and } s_3 = -5.$$

(4) $K = 60$:

$$s^3 + 6s^2 + 11s + 66 = (s+6)(s^2+11).$$

That is,

$$s_1, s_2 = \pm j\sqrt{11} \text{ and } s_3 = -6.$$

The locus of the closed loop poles for increasing values of K is shown in Fig. 7.22 Such a diagram, showing the movement of the roots of the closed loop characteristic equation, is also called a *root locus diagram*. This figure shows that for values of gain K greater than 60, the closed loop poles will enter the r.h.s. of the s-plane. The system will then become unstable.

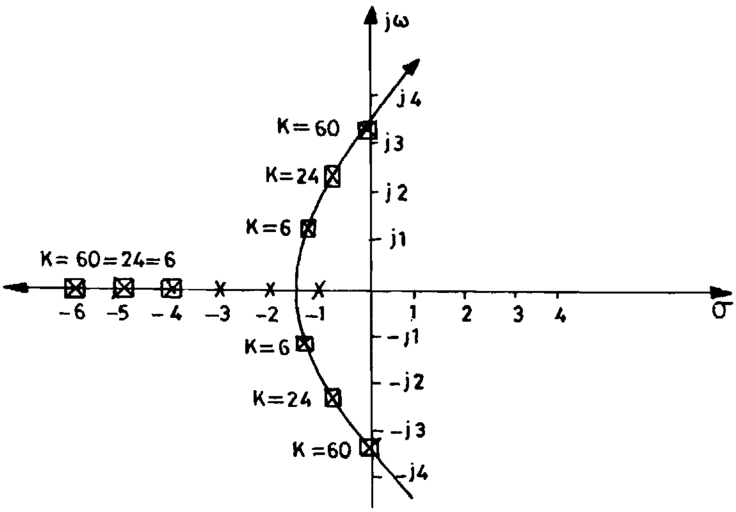


Fig. 7.22 Locus of Closed Loop Poles (Root Locus)

It is quite obvious that an unstable system cannot perform any useful function. Not only that, unstable operation is dangerous and may damage the plant. Therefore, it is absolutely necessary that the feedback control system be so designed that it does not become unstable under any condition of operation. The methods of determining whether a feedback system is stable or not form an important part of the study of control systems.

The most direct method for determining the stability is to factorise the closed loop characteristic polynomial $1 + G(s)\,H(s)$ in order to determine the roots of the characteristic equation $1 + G(s)\,H(s) = 0$. If all the roots have negative real parts, i.e., if all the closed loop poles lie in the l.h.s. of the s-plane, the system is stable. Otherwise it is unstable. However, factorising a polynomial is a tedious work and becomes almost impossible if the order of the polynomial is high. Therefore, different techniques have been developed for predicting the stability without the need for factorising the characteristic polynomial. In the following subsection we study one of these techniques.

The Routh-Hurwitz stability criterion:

For testing the stability of a system, we need not know the exact location of its poles. All that we need to test is whether all the system poles are in the l.h.s. of the s-plane or not. Routh Hurwitz criterion is a method for such a testing.

The characteristic equation of an nth order closed loop system is,

$$s^n + d_{n-1}s^{n-1} + d_{n-2}s^{n-2} + \ldots + d_1 s + d_0 = 0. \qquad (7.13)$$

Let the roots of this equation be $s_1, s_2, \ldots, s_n$. Then, from the properties of the roots of algebraic equations, we have,

$$d_{n-1} = -(s_1 + s_2 + \ldots + s_n)$$

$$d_{n-2} = +(s_1 s_2 + s_2 s_3 + s_1 s_3 + \ldots)$$

(i.e., the sum of the products of roots, taken two at a time)

$$d_{n-3} = -(s_1 s_2 s_3 + s_1 s_2 s_4 + \ldots)$$

(i.e., the sum of the products of roots, taken three at a time)

....

$$d_0 = (-1)^n s_1 s_2 \ldots . s_n. \qquad (7.14)$$

From the relations (7.14), we can derive the following necessary condition for all the roots $s_1, \ldots, s_n$ to be negative (or have negative real parts): all the coefficients $d_{n-1}, \ldots, d_0$ must be non-zero and positive. If this condition is violated, one or more of the roots will be positive and the system unstable. However, this is only a necessary condition and not a sufficient condition. If it is satisfied, we proceed to check for the Routh-Hurwitz criterion which is both a necessary and a sufficient condition.

We first form a Routh array as follows. The first two rows are made up of the coefficients of the characteristic eqn. (7.13) as,

Row 1, s^n	1	d_{n-2}	n_{n-4}	...
Row 2, s^{n-1}	d_{n-1}	d_{n-3}	d_{n-5}	...

Let us denote the elements of row 3 by $d_{31}, d_{32}, \ldots$. Then,

$$d_{31} = -\frac{1}{d_{n-1}} \begin{vmatrix} 1 & d_{n-2} \\ d_{n-1} & d_{n-3} \end{vmatrix}$$

$$d_{32} = -\frac{1}{d_{n-1}} \begin{vmatrix} 1 & d_{n-4} \\ d_{n-1} & d_{n-5} \end{vmatrix}$$

and so on. Thus, the array up to the third row is,

Row 1, s^n	1	d_{n-2}	d_{n-4}	...
Row 2, s^{n-1}	d_{n-1}	d_{n-3}	d_{n-5}	...
Row 3, s^{n-2}	d_{31}	d_{32}	d_{33}	...

The elements of the fourth row, $d_{41}, d_{42}, \ldots$ are formed as,

$$d_{41} = -\frac{1}{d_{31}}\begin{vmatrix} d_{n-1} & d_{n-3} \\ d_{31} & d_{32} \end{vmatrix}$$

$$d_{42} = -\frac{1}{d_{31}}\begin{vmatrix} d_{n-1} & d_{n-5} \\ d_{31} & d_{33} \end{vmatrix}$$

and so on. Thus a new row is formed out of the elements of the preceding two rows according to the formula given above. This procedure is continued till we reach the $(n + 1)$th row corresponding to s^0. The Routh-Hurwitz criterion is stated in terms of the first column of this array. It states: the characteristic polynomial (7.13) has no roots outside the l.h.s. of the s-plane if all the elements of the first column of the Routh array are non-zero, positive numbers. Further, the number of roots in the r.h.s. of the s-plane is equal to the number of sign reversals in the first column.

Example 7.8 : — Examine the stability of a system having the characteristic polynomial $s^4 + 10s^3 + 35s^2 + 50s + 24$.

Solution: Since all the coefficients of the given characteristic polynomial are positive we can proceed to perform the Routh-Hurwitz test. The first two rows of the Routh array are,

$$\begin{array}{c|ccc} s^4 & 1 & 35 & 24 \\ s^3 & 10 & 50 & 0 \end{array}$$

The missing elements are assumed to be zero. The elements of the third row are calculated according to the rule given above as,

$$d_{31} = -\frac{1}{10}\begin{vmatrix} 1 & 35 \\ 10 & 50 \end{vmatrix} = 30; \quad d_{32} = -\frac{1}{10}\begin{vmatrix} 1 & 24 \\ 10 & 0 \end{vmatrix} = 24.$$

The Routh array up to the third row is,

$$\begin{array}{c|ccc} s^4 & 1 & 35 & 24 \\ s^3 & 10 & 50 & 0 \\ s^2 & 30 & 24 & \end{array}$$

The elements of the fourth row are

$$d_{41} = -\frac{1}{30}\begin{vmatrix} 10 & 50 \\ 30 & 24 \end{vmatrix} = 42; \quad d_{42} = 0.$$

Thus the array up to the fourth row is,

$$\begin{array}{c|ccc} s^4 & 1 & 35 & 24 \\ s^3 & 10 & 50 & 0 \\ s^2 & 30 & 24 & \\ s & 42 & 0 & \end{array}$$

The element in the last row is,

$$d_{51} = -\frac{1}{42}\begin{vmatrix} 30 & 24 \\ 42 & 0 \end{vmatrix} = 24.$$

Thus, the complete array is,

s^4	1	35	24
s^3	10	50	0
s^2	30	24	
s	42		
s^0	24		

Examining the first column of the Routh array we find that there is no change in the sign of the elements. All the elements in the first row are non-zero positives. Hence, all the roots lie in the l.h.s. of the s-plane and the system is stable. We can verify this result by factorising the given polynomial as $(s+1)(s+2)(s+3)(s+4)$.

The results of the Routh-Hurwitz analysis are not altered if all the elements of a row are multiplied or divided by a positive constant. For example, in Example 7.8, the second row could be divided by 10 to give

s^4	1	35	24
s^3	1	5	0

This helps in simplifying the numerical work. The third row remains the same, i.e. (30, 24). Dividing each element by 6, we get (5, 4). The fourth row then becomes (21/5, 0), which can be written as (1, 0). The last row becomes 4. The complete array becomes

s^4	1	35	24
s^3	1	5	0
s^2	5	4	
s	1	0	
s^0	4		

The result about stability is the same since it is dependent only on the sign of the terms in the first column and not their numerical values.

Example 7.9 : — Let us now alter one of the poles from -1 to $+1$ in the previous example. The polynomial then becomes,

$$(s-1)(s+2)(s+3)(s+4) = s^4 + 8s^3 + 17s^2 - 2s - 24.$$

Since two of the terms have negative signs, it violates the necessary condition, and we can at once conclude that one or more roots are outside the l.h.s. s-plane and the system is unstable. Developing the Routh array we get,

s^4	1	17	−24
s^3	8	−2	
	(= 4	−1)	
s^2	18	−24	
	(= 3	−4)	
s	13/3		
s^0	−4		

There is one sign reversal in the first column, from + 13/3 to − 4. Hence, we correctly conclude that one of the roots is in the r.h.s. s-plane.

Example 7.10 : — Let us now shift one of the poles to the origin. The characteristic polynomial then becomes $s^4 + 9s^3 + 26\,s^2 + 24s + 0.s^0$.

Since one of the coefficients is zero, we may directly conclude that the system is unstable. However, let us continue with the development of the Routh array.

$$\begin{array}{c|ccc} s^4 & 1 & 26 & 0 \\ s^3 & 9 & 24 & \\ & (=3 & 8) & \\ s^2 & 70/3 & 0 & \\ s & 8 & & \\ s^0 & 0 & & \end{array}$$

Since one of the elements in the first column is zero, the system is unstable. Also the array tells us that one of the roots is at the $j\omega$-axis.

Example 7.11 : — Develop the Routh array for the polynomial,

$$(s^2 + 1)\,(s + 1) \;=\; s^3 + s^2 + s + 1.$$

Solution: The first two rows are,

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1 & 1 \end{array}$$

Then the first element of the third row,

$$d_{31} \;=\; -1 \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} \;=\; 0.$$

We cannot proceed any further if the first element of any row is 0, because the next term will have an impermissible multiplier − 1/0. To avoid this difficulty let us replace d_{31} by a small number ε. Then array up to the third row becomes,

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1 & 1 \\ s & \varepsilon & 0 \end{array}$$

Proceeding further,

$$d_{41} \;=\; -1/\varepsilon \begin{vmatrix} 1 & 1 \\ \varepsilon & 1 \end{vmatrix} \;=\; 1,$$

and the complete array is,

$$\begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 1 & 1 \\ s & \varepsilon = 0 & \\ s^0 & 1 & \end{array}$$

We conclude that the system is unstable because one of the elements in the first column is zero. We also derive another property. If the sign of the elements (in the first column) before and after a zero element is the same, then the system has a pair of poles on the $j\omega$-axis. In this case, the poles are located at $\pm j$. Another property of the Routh array is that if the last element in the first column is zero, (Example 7.10), the polynomial has a root at the origin.

Example 7.12 : — Develop the Routh array for the polynomial,

$$s^5 + 2s^4 + s^3 + 2s^2 - 2s - 4 = (s^2 - 1)(s^2 + 2)(s + 2).$$

Solution: The first two rows are,

$$\begin{array}{c|ccc} s^5 & 1 & 1 & -2 \\ s^4 & 2 & 2 & -4 \end{array}$$

Since the second row is equal to the first row multiplied by 2, all the elements in the next derived row, i.e., the third row, will be 0. Replacement of the zeros by ε will not help in this case. The procedure to be followed in such cases is as follows.

Form an auxiliary polynomial $P(s)$ with the coefficient of the second row: in this case, $P(s) = 2s^4 + 2s^2 - 4$. Then take the derivative of $P(s)$ to get $dP(s) / ds = 8s^3 + 4s$. The elements of the third row are the coefficients of this derivative. That is,

$$\begin{array}{c|ccc} s^5 & 1 & 1 & -2 \\ s^4 & 2 & 2 & -4 \\ s^3 & 8 & 4 & \\ & (=2 & 1) & \end{array}$$

Proceeding further we complete the array as,

$$\begin{array}{c|ccc} s^5 & 1 & 1 & -2 \\ s^4 & 2 & 2 & -4 \\ s^3 & 2 & 1 & \\ s^2 & 1 & -4 & \\ s & 5 & & \\ s^0 & -4 & & \end{array}$$

Since there is one sign reversal in the first column, the array correctly tells us that one of the roots is in the r.h.s. of the s-plane.

A situation like this, i.e., when all the elements of a derived row become equal to zero or when a row can be obtained by multiplying the preceding row by a constant, arises when the polynomial has pairs of roots located diagonally opposite to each other in the s-plane. In the present problem, we have two pairs of such roots: $s = \pm 1$ and $s = \pm j\sqrt{2}$. These roots are found by the auxiliary polynomial $P(s)$. In the present problem,

$$P(s) = 2s^4 + 2s^2 - 4 = 2(s^2 - 1)(s^2 + 2).$$

As far as stability is concerned, we conclude that if the elements of the second row of the Routh array are obtained by multiplying the elements of the first row by a constant, then the system is unstable.

7.7 Accuracy

The accuracy of a feedback system is measured by its steady-state error, i.e., by the difference between the input and the output under steady-state conditions. The smaller the error, the more accurate the system is. From the basic feedback configuration shown in Fig. 7.23, the expression for the error $E(s)$ is,

$$E(s) = R(s) - C(s)\,H(s).$$

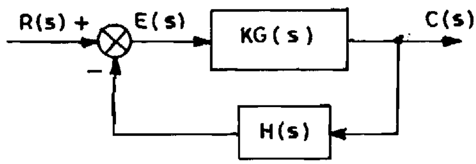


Fig. 7.23 Basic Feedback Configuration

The closed loop transfer function is,

$$\frac{C(s)}{R(s)} = \frac{KG(s)}{1 + KG(s)\,H(s)}.$$

Therefore,

$$C(s)\,H(s) = R(s) \cdot \frac{KG(s)\,H(s)}{1 + KG(s)\,H(s)}.$$

Hence,

$$E(s) = R(s)\left[1 - \frac{KG(s)\,H(s)}{1 + KG(s)\,H(s)}\right].$$

$$= R(s)\,\frac{1}{1 + KG(s)\,H(s)} \qquad (7.15)$$

The expression for steady-state error can be derived by using the final value theorem in eqn. (7.15). Then,

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + KG(s)\,H(s)}. \qquad (7.16)$$

For evaluation of e_{ss} in eqn. (7.16) the input function $R(s)$ must be known. However, the input can be *any* function of time. For the purpose of error analysis, i.e., for defining the steady-state accuracy of the system, we settle down to three basic input functions: the step, the ramp and the parabolic function. Analytically, these test functions are t^0, t^1 and t^2.

For the unit step input $r(t) = u(t)$, $R(s) = 1/s$ and eqn. (7.16) gives,

$$e_{ss} = \lim_{s \to 0} \frac{s/s}{1 + KG(s)\ H(s)} = \frac{1}{1 + KG(o)\ H(o)}$$

The constant term $KG(o)\ H(o)$ is given a special symbol K_p and the name *position error coefficient.* In terms of K_p,

$$e_{ss} = \frac{1}{1 + K_p} \cdot \tag{7.17}$$

To get a small steady-state error to a step input, K_p must be made high. This can be achieved by increasing the forward path gain K. Thus, the higher the gain, the smaller the error.

For a unit ramp input $r(t) = t$, $R(s) = 1/s^2$ and eqn. (7.16) gives,

$$e_{ss} = \lim_{s \to 0} \frac{s/s^2}{1 + KG(s)\ H(s)} = \lim_{s \to 0} \frac{1}{sKG(s)\ H(s)} \cdot$$

The constant term,

$$\lim_{s \to 0} sKG(s)\ H(s)$$

is called the *velocity error coefficient* K_v. In terms of K_v, the steady-state error due to a ramp input is,

$$e_{ss} = \frac{1}{K_v}. \tag{7.18}$$

Similarly, for a parabolic input $r(t) = t^2/2$, $R(s) = 1/s^3$ and,

$$e_{ss} = \lim_{s \to 0} \frac{s/s^3}{1 + KG(s)\ H(s)} = \lim_{s \to 0} \frac{1}{s^2\ KG(s)\ H(s)}.$$

Calling

$$\lim_{s \to 0} s^2\ KG(s)\ H(s)$$

the *acceleration error coefficient* K_a, the steady-state error due to a parabolic input is given by,

$$e_{ss} = \frac{1}{K_a} \tag{7.19}$$

In all the three cases, the steady-state error is inversely proportional to the error coefficient. This error coefficient can be increased, and consequently the error reduced, by increasing the gain K. However, increasing K may lead to instability. The main challenge of control system design is to reconcile these contradictory requirements of increasing K to reduce error and yet maintain a proper stability margin.

In most of the cases, it is required that the steady-state error of the closed loop system due to a step input be zero. For this to be so,

$$K_p = \lim_{s \to 0} KG(s)\, H(s)$$

must be infinite. Now, $KG(s)\, H(s)$ is the open loop transfer function, with a numerator polynomial and a denominator polynomial. In the factored form it can be written as,

$$KG(s)\, H(s) = \frac{K\,(s+a_1)\,(s+a_2)\,\ldots}{s^n\,(s+b_1)\,(s+b_2)\,\ldots}\;.$$

If the power n of the free s term, s^n, in the denominator is zero, then it is clear that

$$K_p = \lim_{s \to 0} KG(s)\, H(s)$$

will not be infinite. However, if $n = 1$ or more, K_p will always be infinite. Thus, the power n of the term s^n in the denominator of $KG(s)\, H(s)$ is important in deciding whether one or more of the error coefficients will be infinite (i.e., the corresponding steady-state error will be zero). Therefore, the open loop systems are classified according to the power n of the free s term, s^n, in the denominator of their transfer function. A system is called *type* 0 if $n = 0$, *type* 1, if $n = 1$, *type* 2, if $n = 2$ and so on. (Note that the *order* of a system is an entirely different entity from the *type* of a system.) For unity feedback systems, the type of the open loop transfer function is the same as the type of the plant transfer function. Let us now study the error coefficients and the steady-state errors for different *types* of systems.

Type 0 *system*: For ease of computation, let us assume unity feedback, i.e. $H(s) = 1$. Also, let $G(s) = 1/(s+1)$. The position error coefficient is $K_p = K$ and the steady-state error $e_{ss} = 1/(1+K)$. By making K large, e_{ss} can be reduced but it cannot be completely eliminated. Hence, a type 0 system will always have some steady-state error due to a step input.

The velocity error constant,

$$K_v = \lim_{s \to 0} s\, KG(s)\, H(s) = \lim_{s \to 0} Ks\,/\,(s+1) = 0.$$

Hence, steady-state error due to a ramp input will be infinite. That is, a type 0 system cannot follow a ramp input. Similarly, the steady-state error to a parabolic input will also be infinite. Thus, from the accuracy point of view, the type 0 system is not very effective.

Type 1 *system*: Let $H(s) = 1$ and $G(s) = 1/s\,(s+1)$. Then,

$$K_p = \lim_{s \to 0} K/s\,(s+1) = \infty.$$

Hence, the steady-state error due to a step input will be zero. The velocity error coefficient,

$$K_v = \lim_{s \to 0} K/(s+1) = K$$

will be finite. Therefore, the system will follow a ramp input with a finite steady-state error which can be made small by increasing K, but cannot be totally eliminated. The acceleration error coefficient K_a will be zero and, hence, the type 1 system cannot follow a parabolic input.

Type 2 system: It is straightforward to show that both K_p and K_v will be infinite for a type 2 system. Thus, the system will accurately follow step and ramp inputs without anysteady-state error. However, K_a will be finite and therefore there will be some steady state error to a parabolic input.

The free s term in the denominator represents integration of the input. A plant with a single s term in the denominator has one integration; with s^2 it has two integrations, and so on. Such integrations arise naturally in physical systems. For example, in an armature-controlled d.c. motor with the angular position of the shaft as the output, a constant voltage across the armature results in a continuously increasing output. Thus, the plant integrates the input once to produce the output and hence is of type 1 [eqn. (iv), Ex. 7.6]. However, if the shaft velocity is considered the output, there is no integration and the transfer function is type 0.

The terms 'position', 'velocity' and 'acceleration', used with error co-efficients, are derived from position control systems (or position 'servomechanism') where a step input produces a fixed output position; a ramp input gives a fixed output velocity; and a parabolic input gives a constant acceleration. However, the same terms are used for all control systems, whether the output is temperature or pressure or any other physical quantity.

Example 7.13 : — Determine the steady-state error due to step and ramp inputs for a unity feedback system with plant transfer function,

$$G(s) = \frac{K\,(s+1)}{s\,(s+2)\,(s+3)}$$

for $K = 1$, 6 and 60.

Solution: As the plant is type 1, $K_p = \infty$ and the steady-state error due to a step input will be zero. So, we consider only the ramp input (Fig. 7.24). From the definition of K_v

$$K_v = \lim_{s \to 0} s\,KG(s)\,H(s) = \lim_{s \to 0} \frac{K\,(s+1)}{(s+2)\,(s+3)} = \frac{K}{6}.$$

The steady-state error $e_{ss} = 1 / K_v = 6 / K$.

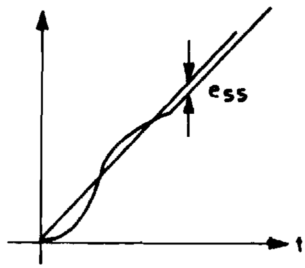


Fig. 7.24 Steady-state Error due to Ramp Input

For $K = 1,\ e_{ss} = 6.$

For $K = 6,\ e_{ss} = 1.$

For $K = 60,\ e_{ss} = 0.1.$

7.8 Sensitivity

Theoretically any desired dynamic response can be obtained from a plant by cascading a series controller with a proper transfer function $G_c(s)$, without using any feedback. Such a control is called 'open loop' control. However, the performance of an open loop control will change due to any change in its parameter values. The parameters may change because of environmental conditions or because of ageing; the resistance of coils change with temperature, the gain of transistors change with time, and the friction of a bearing is dependent upon the level of lubrication. In many cases, the parameter values may not even be known accurately or may be very much different from the values assumed at the design stage. The performance of the open loop system will also change because of unwanted 'noise' signals which will always be affecting any physical system. We then say that the performance of the system is 'sensitive' to parameter variations and to unwanted noise signals. One of the prime reasons for using feedback is that it reduces the sensitivity of the system to both parameter variations and external noises or disturbances.

To demonstrate the effect of feedback in reducing the sensitivity to parameter variations, consider a two-stage amplifier, each stage having a gain of 100, constant over the frequency range of interest. The transfer functions $G_1(s)$ and $G_2(s)$ of each stage are then just 100. The overall transfer function is 10,000. Now, suppose the gain of the second stage changes by 10%, to become 90. Then, the overall gain also changes by 10% to become 9000. Since a 10% change in the gain of one of the components causes a 10% change in the overall gain, we say that the sensitivity of the system is 1.

Now, suppose we put a feedback around the second unit with a resistive potential divider so that the feedback $H(s) = 0.5$. Then, the closed loop transfer function of the second unit becomes,

$$G_{2c}(s) = \frac{G_2(s)}{[1 + G_2(s)\,H(s)]} = \frac{100}{1 + 100 \times .5} = 1.965$$

The overall gain of the system becomes, $G(s) = G_1(s)\,G_{2c}(s) = 100 \times 1.965 = 1965$. With a change in the open loop gain of the second unit, the changed gain of the closed loop is $G'_{2c}(s) = 90/(1 + 90 \times 0.5) = 90/46 = 1.955$ and the overall gain $G'(s) = 1955$. The overall gain variation is now only 10, from 1965 to 1955, or $10 \times 100 / 1965 = 0.512\%$, as compared to 10% in the system without feedback. The sensitivity of the feedback system is only $0.512/10 = 0.0512$ as compared to in 1 the previous case. This demonstrates the effect of feedback in reducing the sensitivity of the system to parameter variations. Of course, the net gain of the amplifier has been reduced because of the feedback. To regain the original gain, one more stage (with appropriate gain) may be added to the amplifier.

For any feedback amplifier with a forward path gain A and feedback factor β, the closed loop gain is $A/(1 + \beta A)$. If the loop gain βA is much larger than 1, then the overall gain is simply $1/\beta$, which is independent of the forward path gain A. Hence, any variation in the value of A does not affect the overall gain. In other words the system becomes totally insensitive to changes in A.

Let us now consider any plant with a transfer function $G(s)$, input $R(s)$ and output $C(s)$. Then, for the open loop system without feedback, $C(s) = G(s)\,R(s)$. Now, suppose that the plant transfer function changes to $G(s) + \Delta\, G(s)$ because of parameter variations. Then, the open loop output becomes, $C'(s) = [G(s) + \Delta\, G(s)]\,R(s)$. The change in the output is,

$$[C'(s) - C(s)] = \Delta\, C(s) = \Delta\, G(s)\, R\,(s).$$

Let us now use a feedback around the plant with a transfer function $H(s)$. The closed loop transfer function (without parameter variation) is $G(s) / [1 + (G(s)\,H(s)]$ and the output,

$$C(s) = \frac{G(s)\,R(s)}{1 + G(s)\,H(s)}.$$

With $G(s)$ changed to $G(s) + \Delta\, G(s)$, the changed output becomes,

$$C'(s) = \frac{[G(s) + \Delta\, G(s)]\,R(s)}{1 + [G(s) + \Delta\, G(s)]\,H(s)}.$$

With $|\Delta\, G(s)| << |G(s)|$, the change in the output is,

$$C'(s) - C(s) = \Delta\, C(s) = \frac{\Delta\, G(s)}{1 + G(s)\,H(s)} R(s).$$

Thus the effect of feedback is to reduce the change in the output by a factor of $1/[1 + G(s)\,H(s)]$. If the loop gain $G(s)\,H(s)$ is kept large, which can easily be done by increasing the forward path gain K, the change in output due to changes in $G(s)$ can be kept as small as desired.

Let us now compare the effect of noise in the open loop and in the closed loop systems. In the open loop system of Fig. 7.25 (a), the transfer function from the noise to the output is,

$$\frac{C(s)}{N(s)} = G_2(s)\,.$$

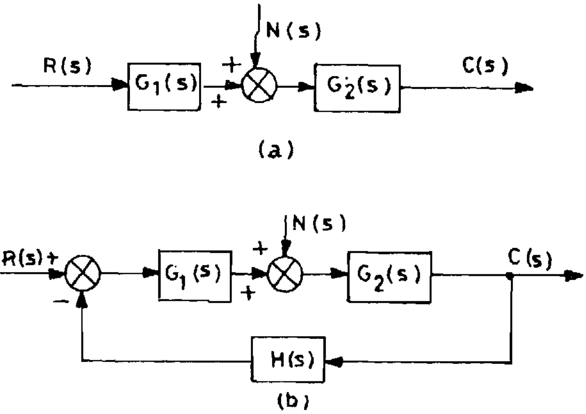


Fig. 7.25 Reduction of Noise by Feedback

For the closed loop system of Fig. 7.25 (b), input $R(s)$ is assumed to be zero for considering the transfer function from $N(s)$ to $C(s)$. Then,

$$C(s) = [N(s) - G_1(s)\,H(s)\,C(s)]\,G_2(s)$$

or,

$$[1 + G_1(s)\,G_2(s)\,H(s)]\,C(s) = N(s)\,G_2(s)$$

or,

$$\frac{C(s)}{N(s)} = \frac{G_2(s)}{1 + G_1(s)\,G_2(s)\,H(s)}.$$

Once again, by keeping the loop gain $G_1(s)\,G_2(s)\,H(s)$ large, the effect of noise on the output can be reduced.

Concluding comments: In this chapter, we have given a brief account of the important aspects of the feedback systems. These aspects are, transient response, stability, accuracy and sensitivity. All these aspects are affected by feedback. However, these requirements place conflicting demands on the feedback system.

Reconciling these conflicting demands is the interesting, though challenging, task of control system design.

GLOSSARY

Feedback: The process of comparing the output signal with the input signal to produce the control signal applied to a plant is called *feedback*. If the output signal opposes the input, it is called *negative feedback*; if it aids the input it is called *positive feedback*.

Block Diagram: Each subsystem in a large system is depicted graphically by a functional block, showing the input, the output and the transfer function relating the two. A diagram showing the interconnections of such functional blocks of a system is called a *block diagram*.

Signal Flow Graph: This is a line diagram, consisting of nodes and directed branches, which shows the flow of signals in a system. Each node represents a variable and each branch a subsystem with a specified transfer function. The overall transfer function of the system is obtained by simplifying its signal flow graph into a single branch between the input and the output nodes. Mason's formula gives a general method for signal flow graph reduction.

Stability: If the output of a system grows without bounds, it is called an *unstable system*. For the system to be stable all the roots of its characteristic equation must have only negative real parts. In other words, all the system poles must lie in the left half of the *s-plane*. The Routh-Hurwitz method is a technique for determining the stability of a system without finding the roots of its characteristic equation.

Error Coefficients: They determine the steady-state accuracy of a feedback system:

Position error coefficient: $K_p = \lim_{s \to 0} KG(s)\, H(s).$

Velocity error coefficient: $K_v = \lim_{s \to 0} s\, KG(s)\, H(s).$

Acceleration error coefficient: $K_a = \lim_{s \to 0} s^2\, KG(s)\, H(s).$

The steady-state error of a system due to a unit step input is $1/(1 + K_p)$, due to a ramp input, $1/K_v$, and due to a parabolic input ($\alpha\, t^2$), $1/K_a$.

Type of a System: In the factored form, if the denominator of $KG(s)\, H(s)$ has no free s term, it is called a *type* 0 system, if it has a free s terms it is called a *type* 1 system, and if it has a free s^2 term it is called a *type* 2 system. A type 0 system will have a finite steady-state error due to a step input. A type 1 system will have zero steady-state error due to a step input but a finite steady-state error due to a ramp input. A type 2 system will have zero steady-state error for both step and ramp inputs and a finite steady-state error due to a parabolic ($\alpha\, t^2$) input.

Sensitivity: The sensitivity of a performance measure p of a system to variations in the parameter a is defined as,

$$S_a^p = \frac{\%\ \text{change in}\ p}{\%\ \text{change in}\ a}.$$

PROBLEMS

7.1 Figure 7.26 shows an interconnection of two *RC* low pass filters. Treating each filter as a subsystem, develop a block diagram representation for this circuit. This problem illustrates that when loading effects are present, the overall transfer function of series-connected subsystems is not equal to the product of their individual transfer functions.

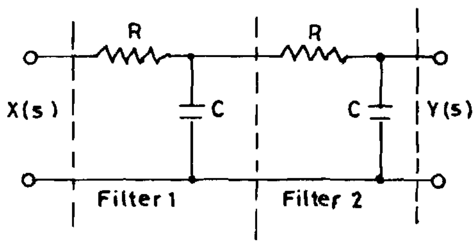


Fig. 7.26

7.2 Develop the block diagram representation for the two-tank system shown in Fig. 7.2 (b). Use block diagram reduction techniques to obtain its transfer function.

7.3 Simplify the block diagram in Fig. 7.27 to determine the overall transfer function of the system.

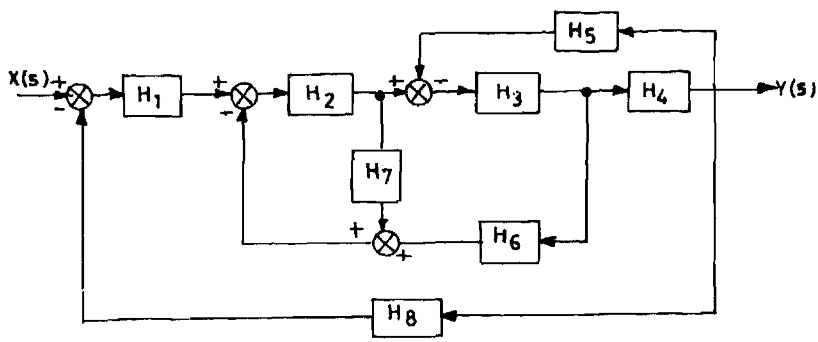


Fig. 7.27

7.4 Draw the signal flow graph for the network shown in Fig. 7.28. Determine the transfer function relating $V_o(s)$ and $V_i(s)$ by simplifying the flow graph.

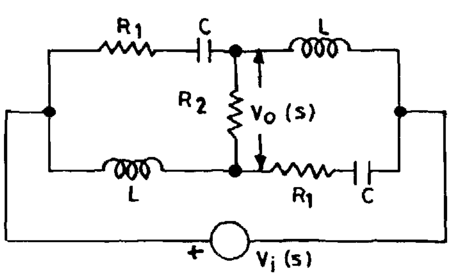


Fig. 7.28

7.5 Determine and sketch the unit impulse response of a second order system.

7.6 A first order plant has a time constant of 2 seconds. Discuss how its time constant can be halved using feedback.

7.7 Determine and sketch the unit ramp response of a second order system.

7.8 When supplied from a 200 V d.c. source, an armature-controlled d.c. motor develops a torque of 10 N-m and runs at a steady-state speed of 100 r.p.m. The supply is then switched off and the speed plotted as a function of time. This speed-time curve is a decaying exponential with a time constant of 2 seconds. From these experimental data, obtain the transfer function of the d.c. motor.

7.9 The forward path transfer function of a plant is,

$$G(s) = \frac{K}{s\,(s+1)}\,.$$

The constant feedback factor is 0.5. Show the location of closed loop poles for $K = 1$, 2 and 10. Sketch the step response of the closed loop system from a knowledge of pole locations.

7.10 A tachogenerator feedback is often used in position control systems to reduce the magnitude of transient oscillations. The effect of this feedback on the system is shown in Fig. 7.29. The forward path plant is such that it has a steady-state gain of 10, damping ratio 0.1 and $\omega_n = 4$ rad/sec. Determine the tachogenerator gain K_T to make the closed loop damping ratio = 0.6.

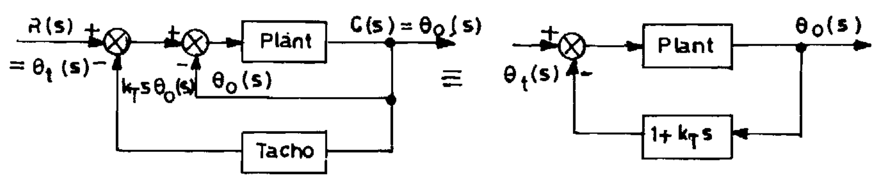


Fig. 7.29

7.11 Determine whether the closed loop systems, whose characteristic polynomials are given below, are stable or not. Also give information about the number of closed loop poles in the left half of the s-plane, on the $j\omega$-axis and in the right half of the s-plane:

(i) $s^3 + 2s + s + 2$.

(ii) $s^4 + 5s^3 + 13s^2 + 19s + 10$.

(iii) $s^4 + s^3 - s^2 + s - 2$.

7.12 The plant transfer function of a unity feedback system is,

$$G(s) = \frac{K}{s\,(s+1)\,(s+2)}\,.$$

Determine the range of K for stable operation of the closed loop system.

7.13 The characteristic polynomial of a third order system is given by $s^3 + d_2 s^2 + d_1 s + d_0$. Determine the relations to be satisfied by the coefficients d_o, d_1 and d_2 for stable operation of the system.

7.14 Feedback can also be used to stabilise an unstable plant. Determine the range of K for stable operation of the closed loop system having plant transfer function,

$$G(s) = K\,(s+2)/s\,(s-1)$$

and unity feedback.

7.15 A unity feedback system has the plant transfer function,

$$G(s) = \frac{K}{s\,(s+1)\,(s+10)}\,.$$

The input to the system is $r\,(t) = 1 + t$. Determine the value of K which will give a steady-state error of 0.1.

CHAPTER 8

State Variables

LEARNING OBJECTIVES

After studying this chapter you should be able to:

(i) formulate state variable equations for systems from their differential equation and transfer function models;

(ii) obtain the transfer function from a given state variable representation;

(iii) determine the state transition matrix and solve state variable equations; and,

(iv) diagonalise a matrix and derive the normal, or the standard, form of state variable representation.

As illustrated by many previous examples, the modelling process of linear systems involves setting up a chain of cause-effect relationships, beginning from the input variable and ending at the output variable. This case-effect chain includes a number of internal variables. These variables are eliminated, both in the differential equation model and in the transfer function model, to obtain the final relationship between the input and the output. Analysis of systems with this input-output relationship model will not give any information about the behaviour of the internal variables for different operating conditions. For example, the analysis of a vibration table (Ex. 6.2) with the transfer function model [eqn. (6.30)], gives no clue as to when the coil current will saturate the magnetic path or exceed its safe operating limit. Therefore, for a better understanding of the system behaviour its mathematical model should include the internal variables also. The state variable techniques of system representation and analysis make the internal variables an integral part of the system model and thus provide more complete information about the system behaviour. In order to appreciate how these internal variables are included in the system representation, let us once again examine the cause-effect chain, i.e., the mathematical modelling process, by means of a specific example.

Example 8.1 :— *Field-controlled d.c. motor.* Field-controlled d.c. motors are used as actuators in low power instrument servos, e.g., for controlling the position of a pointer over a dial. The control signal is applied across the field winding while the armature is supplied from a constant current source. This arrangement is more convenient because the field winding requires much less current than the armature. (However, for large power motors it becomes difficult to supply the required amount of constant armature current through a constant current source, and then armature control is used instead of field control.) The input signal is the voltage across the field winding, e, and the output is the angular position of the armature shaft, θ. The schematic diagram of such a system is shown in Fig. 8.1.

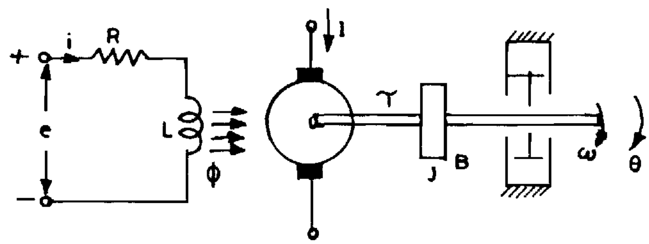


Fig. 8.1 Field-controlled d.c. motor

Treating input voltage e as the cause, the effect it produces is the field current i, the two being related by the equation,

$$L\frac{di}{dt} + Ri = e. \tag{i}$$

We now treat current i as the cause. The effect it produces is the field flux ϕ. Making the usual assumption of linear magnetic relation we get,

$$\phi = k_1 i \tag{ii}$$

where k_1 is a constant. The field flux interacts with the constant armature current I, to produce a torque τ. This relation is given by,

$$\tau = k_2\phi. \tag{iii}$$

The torque produces angular velocity ω. The relation between τ and ω is given by,

$$J\frac{d\omega}{dt} + B\omega = \tau \tag{iv}$$

where J and B are, respectively, the net moment of inertia and the coefficient of viscous friction on the armature shaft. And, finally the output is related to the angular velocity ω by the relation,

$$\frac{d\theta}{dt} = \omega. \tag{v}$$

The set of eqns. (i) to (v) gives the complete cause-effect chain, relating the input e to output θ, through the internal variables i, ϕ, τ and ω. The differential equation model, or the transfer function model, can be easily obtained by eliminating the internal variables and combining these five equations into one. But that is not our intention here. We would like to develop a model which includes the internal variables also.

8.1 State Variables

In Example 8.1, there are four internal variables and one output variable. As there is nothing to distinguish the internal variables from the output variables, we shall call the combined set of the internal variables and the output variables as *system variables*. Thus, for the field-controlled d.c. motor there are five system variables, $\{i, \phi, \tau, \omega, \theta\}$.

We also notice that in this set of five system variables, some are related to others through linear algebraic equations [eqns. (ii) and (iii)]. This means that their values (at all instants of time) can be obtained from the knowledge of other system variables, merely by linear combinations. In other words, even if the number of variables is reduced, this reduced set can still represent the system completely. For the purpose of finding a mathematical model to represent a system, we will naturally choose a set of variables with the minimum possible number of elements. Such a set would be obtained when none of the selected variables is related to the others through linear algebraic equations, i.e., when the variables are *linearly independent.**

A little consideration shows that the number of linearly independent variables in Example 8.1 is only three. However, the set of three linearly independent variables is not unique. The sets $\{i, \omega, \theta\}$, $\{\phi, \omega, \theta\}$ and $\{\tau, \omega, \theta\}$ are all linearly independent and any one of these could be used to represent the system. Choosing the first set $\{i, \omega, \theta\}$, we can write the system equations as,

$$L\frac{di}{dt} + Ri = e \text{ or } \frac{di}{dt} = -\frac{R}{L}i + \frac{e}{L} \qquad \text{(vi)}$$

$$J\frac{d\omega}{dt} + B\omega = k_1 k_2 i \text{ or } \frac{d\omega}{dt} = \frac{k_1 k_2}{J}i - \frac{B}{J}\omega \qquad \text{(vii)}$$

and

$$\frac{d\theta}{dt} = \omega. \qquad \text{(viii)}$$

---

* A set of variables $\{x_1, x_2, \ldots, x_n\}$ are called linearly independent if a linear algebraic equation, such as,

$$a_1 x_1 + a_2 x_2 + \ldots + a_n x_n = 0$$

, does not exist between them for any non-zero values of constants $a_1, a_2, \ldots, a_n$.

The set of eqns. (vi), (vii) and (viii) constitutes a mathematical model for the system. It is a set of three first order differential equations. Its complete solution for any given input $e(t)$ applied at $t = 0$, will require a knowledge of the value of selected variables $\{i, \omega, \theta\}$ at $t = 0$. To put it differently, we can say that if the values of $\{i, \omega, \theta\}$ at $t = 0$ are known, then the values of these variables at any time $t > 0$, in response to a given input $e(t)$, can be obtained by the solution of eqns. (vi), (vii) and (viii). A set of system variables having this property is called a set of *state variables*. The set of values of these variables at any time t_1 is called the *state* of the system at time t_1. The set of first order differential equations—like eqns. (vi), (vii) and (viii), relating the first derivative of the state variables with the variables themselves—is called a set of *state variable equations* and constitutes the state variable model of the system. It is also to be noted that the number of state variables needed to form a correct and complete model of the system is equal to the order of the system.

8.2 Standard Form of State Variable Equations

In the state variable method of analysis, standard symbols are used for state variables. The notations for writing state variable equations have also been standardised. To understand these, we start with eqns. (vi),(vii) and (viii) of the previous section.

Let the state variables be represented as, $\theta = x_1$, $\omega = x_2$ and $i = x_3$. Then, the state equations can be written as,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = -\frac{B}{J}x_2 + k_1 k_2 x_3$$

$$\dot{x}_3 = -\frac{R}{L}x_3 + \frac{1}{L}e\,.$$

These three equations can be written more systematically in the matrix form as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -B/J & k_1k_2 \\ 0 & 0 & -R/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L \end{bmatrix} e \qquad \text{(ix)}$$

The output variable is simply equal to one of the state variables in this problem. In general, the output can be any linear combination of the state variables. To provide for this generality, the output is also written in the matrix form as,

$$\theta = x_1 = [1 \;\; 0 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \qquad \text{(x)}$$

Equations (ix) and (x) constitute the state variable model of the field-controlled d.c. motor.

In the standard form the input variable is designated by the symbol u and the output by the symbol y. The state variable model for an nth order system is written as.

$$\dot{x} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{8.1}$$

$$y = \mathbf{c}\mathbf{x} \tag{8.2}$$

where $\mathbf{x}$ is $n \times 1$ vector, called the *state vector*, $\mathbf{A}$ is $n \times n$ matrix called the *state matrix*, $\mathbf{b}$ is $n \times 1$ *input matrix*, $\mathbf{c} = 1 \times n$ *output matrix*, and u and y are the scalar input and output. In the expanded form, we have,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}, \quad \mathbf{c} = [c_1 \; c_2 \; \ldots \; c_n] \tag{8.3}$$

The system variables $\mathbf{x}$, u and y are functions of time while matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$ consist of constant coefficients. Equation (8.1) is called a *vector differential equation* and is a compact way of writing a set of n first order differential equations.

In the standard form of eqns. (8.1) and (8.2), the variables are $\mathbf{x}$, u and y while the parameters are the matrices $\mathbf{A}$, $\mathbf{b}$ and $\mathbf{c}$. Assuming the standard form representation, a system is completely defined by these matrices. Therefore, many a time a system is referred to simply as the set $\{\mathbf{A}, \mathbf{b}, \mathbf{c}\}$.

Equations (8.1) and (8.2) represent a linear, time invariant, single-input-single-output system. However, the state variable technique provides an easy and general method of representing other classes of systems also. For example, if the system is not linear, the vector state equation is written as a general functional relationship between $\dot{\mathbf{x}}$ and $\mathbf{x}$ and u, i.e.,

$$\dot{\mathbf{x}} = \mathbf{f}\,(\mathbf{x}, u) \tag{8.4}$$

where $\mathbf{f}$ may be a set of non-linear relations.

State variable representation is also convenient for multivariable systems, where the number of inputs and outputs is more than one. Let there be m inputs and r outputs. Then the scalar u becomes a vector $\mathbf{u}$ with dimension m and output y becomes $\mathbf{y}$ with dimension r. The order of the system, that is, the number of state variables remains n. The representation for such a multivariable system is then,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \tag{8.5}$$

$$\mathbf{y} = \mathbf{Cx} \tag{8.6}$$

where the dimensions of parameter matrices **A, B, C** and variables **x, u** and **y** are:

$$\dim \mathbf{A} = n \times n \qquad \mathbf{x} = n \times 1$$

$$\dim \mathbf{B} = n \times m \qquad \mathbf{u} = m \times 1$$

$$\dim \mathbf{C} = r \times n \qquad \mathbf{y} = r \times 1$$

The output eqn (8.6) means that the output is a linear combination of state variables. The state variables, in turn, are effected by the input **u**. Thus, the input affects the output only through the state variables in these equations. However, it is possible that for some systems the input affects the output directly also. To allow the standard form of state variable representation to have sufficient generality to include this possibility also, the output equation is written as,

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}, \text{ with } \dim \mathbf{D} = r \times m.$$

Thus, the most general form of the state variable model for linear systems is the pair of equations,

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \tag{8.7}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du}. \tag{8.8}$$

As seen from Example (8.1), the set of linearly independent system variables which can be selected as state variables to represent a system is not unique. For field-controlled d.c. motor at least three different sets could be selected as state variables. Thus, the selection of state variables to represent any given system is not unique. Theoretically, the number of possible state variable sets for a system is infinite. For example, if $\{x_1, x_2, x_3\}$ is a set of state variables, then $\{(a_1x_1 + a_2x_2), x_2, x_3\}$ is also a set of state variables, for all different values of a_1 and a_2.

8.3 Phase Variables

In the state variable representation of Example 8.1, we had selected physically meaningful variables like current, velocity, angular displacement, as the state variables. From the engineering viewpoint, such a choice is obviously very desirable. However, from the mathematical point of view this is not necessary. The selected state variables may not correspond directly with any physical quantity. From a possible infinite number of state variable sets, we choose only those which give some desirable advantages. One such choice of state variables leads to mathematically convenient forms of matrices **A, b,** and **c**. In this choice, the output and its first $(n - 1)$ derivatives (the system is assumed to be of order n) are chosen as the state variables. Such a special set of state variables is called a set of *phase variables*. To illustrate the state variable representation of systems using phase variables, we again consider the field controlled d.c. motor.

Example 8.2:— Obtain the phase variable representation of the field-controlled d.c. motor.

Solution: Combining eqns. (i) to (v) of Example 8.1, the differential equation representation of the system is given by,

$$\frac{d^3\theta}{dt^3} + a_2 \frac{d^2\theta}{dt^2} + a_1 \frac{d\theta}{dt} + a_0 \theta = be$$

where,

$$a_2 = \left(\frac{B}{J} + \frac{R}{L}\right), \quad a_0 = 0$$

$$a_1 = \frac{RB}{JL}, \quad b = \frac{k_1 k_2}{JL}$$

Since it is a third order system, three state variables will be needed for its representation. Define them as,

$$x_1 = \theta,\ x_2 = \frac{d\theta}{dt} = \omega \text{ and } x_3 = \frac{d^2\theta}{dt} = \frac{d\omega}{dt}.$$

Then the state equations become,

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -a_0 x_1 - a_1 x_2 - a_2 x_3 + be.$$

In the matrix form the state equations become,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b \end{bmatrix} e$$

The output $y = \theta = x_1$. Therefore the output equation becomes,

$$y = [1 \ 0 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

This is the desired state variable representation.

Generalising on the basis of Example 8.2, for an nth order system with a differential equation model,

$$\frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \ldots + a_1 \frac{dx}{dt} + a_0 x = f(t),$$

the phase variables will be,

$$\begin{aligned}
x_1 &= x \\
x_2 &= \frac{dx}{dt} = \frac{dx_1}{dt}, \\
x_3 &= \frac{d^2x}{dt^2} = \frac{dx_2}{dt} \\
&\vdots \\
x_n &= \frac{d^{n-1}x}{dt^{n-1}} = \frac{dx_{n-1}}{dt}
\end{aligned}$$

For this choice of state variables, the system matrices will be,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & 0 & & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & & -a_{n-1} \end{bmatrix}$$

$$\mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad \text{and } \mathbf{c} = [1 \ 0 \ 0 \ \ldots \ 0] \tag{8.9}$$

Matrix **A** has a particular structure which is also called a *canonical form*. Given the system differential equation, the state variable representation in terms of phase variables can be written out directly by mere inspection of the differential equation. The particular forms of the matrices in eqn. (8.9) are helpful in many mathematical manipulations. The phase variable form of representation is also very convenient to derive if the initial system description is in the form of a transfer function.

Example 8.3:— The transfer function of a system is,

$$G(s) = \frac{2}{(s+1)(s+2)}$$

Obtain a state variable representation for the system.

Solution: From the given transfer function, $\frac{Y(s)}{U(s)} = \frac{2}{s^2+3s+2}$

or

$$(s^2 + 3s + 2)\, Y(s) = 2U(s)\,.$$

Taking the Laplace inverse of the above equation, the differential equation model of the system is,

$$\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 2u(t).$$

Choosing phase variables y and $\dot{y}$ as the state variables, the system matrices are,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}; \mathbf{b} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}; \mathbf{c} = [1 \quad 0].$$

In general, the transfer function may include zeros also. In that case, the numerator of the transfer function will also be a polynomial in s. The general form of the transfer function of an nth order system is,

$$G(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}, m < n \qquad (8.10)$$

To obtain a state variable representation for this transfer function we proceed as follows.

Let us break up the expression for $G(s)$ into two parts as,

$$G(s) = \frac{Y(s)}{U(s)} = \frac{X_1(s)}{U(s)} \cdot \frac{Y(s)}{X_1(s)}.$$

Let

$$\frac{X_1(s)}{U(s)} = \frac{1}{s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0} \qquad (8.11)$$

and

$$\frac{Y(s)}{X_1(s)} = b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0. \qquad (8.12)$$

Now, expression (8.11) does not contain any zeros. Therefore, its state variable description, using phase variables as state variables, can be obtained directly by the method described earlier. From expression (8.12) we get,

$$Y(s) = (b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0) X_1(s)$$

or

$$y(t) = b_m \frac{d^m x_1}{dt^m} + b_{m-1} \frac{d^{m-1} x_1}{dt^{m-1}} + \ldots + b_1 \frac{dx_1}{dt} + b_0 x_1$$

$$= b_m x_{m+1} + b_{m-1} x_m + \ldots + b_1 x_2 + b_0 x_1$$

(where $x_1, x_2, \ldots, x_m, x_{m+1}, \ldots, x_n$ are the phase variables) or,

$$y(t) = [b_0\, b_1 \ldots b_{m-1}\, b_m\; 0 \ldots 0] \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \vdots \\ x_n \end{bmatrix} \tag{8.13}$$

Hence, the effect of zeros is to alter the **c** matrix in the state variable representation as in expression (8.13). Matrices **A** and **b** remain the same as in eqn. 8.9.

As stated at the beginning of this section, though phase variable representation is convenient, the phase variables may not correspond to physically meaningful variables. For example, in a thermal system with temperature as the output, the phase variables will be first and higher derivatives of temperature. Now, the nth derivative of temperature is not a very meaningful physical quantity.

8.4 State Variables for Electrical Networks

The number of state variables needed for correct representation of a system is equal to the 'order' of the dynamic system. The order of a system, in turn, is equal to the number of independent energy storage elements, particularly in electromechanical systems where the concept of energy is valid. In electrical systems, energy is stored in the magnetic field of an inductor and in the electrical field of a capacitor. Therefore, it is natural to select the variables associated with inductors and capacitors as the state variables. These are the currents through the inductors and the voltages across the capacitors. The set consisting of currents in inductors and voltages across capacitors will be linearly independent. Also, if the values of these variables at time $t = 0$ are known, then their values for any instant $t > 0$ can be calculated for any given input. Thus, the set of currents in inductors and voltages across capacitors meets the conditions required to make it a set of state variables.

Example 8.4 :— Find a state variable model for the circuit in Fig.8.2. Voltage v_c across the capacitor is the output and $e(t)$ is the input.

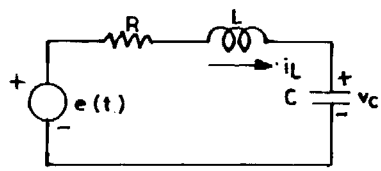


Fig. 8.2 *RLC* Series Circuit

Solution: Let the state variables be defined as $x_1 = v_c$ and $x_2 = i_L$. Then, the system equations are,

$$c\,\frac{dv_c}{dt} = i_c = i_L \quad \text{or} \quad \dot{x}_1 = \frac{1}{C}\,x_2\,. \tag{i}$$

Applying Kirchhoff's voltage law around the loop we get,

$$v_R + v_L + v_c = e$$

or

$$v_L = L \, di/dt = e - Ri_L - v_c$$

or

$$\dot{x}_2 = -\frac{1}{L} x_1 - \frac{R}{L} x_2 + \frac{1}{L} e . \tag{ii}$$

Writing eqns. (i) and (ii) in the standard matrix form, we get the state equations as,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} e .$$

The output equation is,

$$v_c = y = [1 \;\; 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example 8.5:— In the circuit shown in Fig. 8.3, the inputs are e_1 and e_2 and the outputs are v_1, v_2 and v_c. Obtain a state variable model for this system.

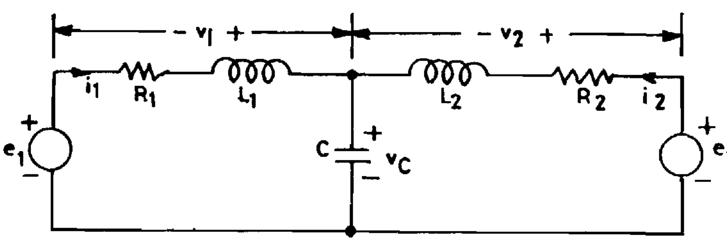


Fig. 8.3

Solution: We note that in this problem the number of input variables and the output variables is more than one. Hence, it is a multivariable system. The object of this example is to demonstrate that making a state variable model for a multivariable system is almost as easy as that for a single variable system.

As there are three independent energy storage elements, L_1, L_2 and C, three state variables will be required for a correct representation of the system. There are three outputs and two inputs.

Let the state variables be selected as,

$$i_{L1} = i_1 = x_1 \; ; \; i_{L2} = i_2 = x_2 \; ; \text{ and } v_c = x_3 .$$

The state equations will be the expressions for derivatives of selected state variables as functions of the state variables themselves. To obtain these expressions, we first apply Kirchhoff's voltage law to the two loops to get,

$$i_1 R_1 + L_1 \frac{di_1}{dt} + v_c = e_1$$

$$i_2 R_2 + L_2 \frac{di_2}{dt} + v_c = e_2 .$$

From these equations we get,

$$\dot{x}_1 = \frac{di_1}{dt} = -\frac{R_1}{L_1} i_1 - \frac{1}{L_1} v_c + \frac{1}{L_1} e_1 = -\frac{R_1}{L_1} x_1 - \frac{1}{L_1} x_3 + \frac{1}{L_1} e_1$$

$$\dot{x}_2 = \frac{di_2}{dt} = -\frac{R_2}{L_2} i_2 - \frac{1}{L_2} v_c + \frac{1}{L_2} e_2 = -\frac{R_2}{L_2} x_2 - \frac{1}{L_2} x_3 + \frac{1}{L_2} e_2 .$$

For the third state equation, we apply Kirchhoff's current law to the central node to get,

$$C \frac{dv_c}{dt} = (i_1 + i_2)$$

or

$$x_3 = \frac{1}{C} x_1 + \frac{1}{C} x_2 .$$

In the standard matrix form the state equations are,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & -\frac{1}{L_2} \\ \frac{1}{C} & \frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} & 0 \\ 0 & \frac{1}{L_2} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} .$$

The outputs, as linear combinations of state variables, are given by the relations,

$$y_1 = v_1 = e_1 - v_c ; \; y_2 = v_2 = e_2 - v_c ; \text{ and } y_3 = v_c .$$

or

$$y_1 = -x_3 + e_1 ; y_2 = -x_3 + e_2 ; \text{ and } y_3 = x_3 .$$

In the standard form, the output equations are,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix} .$$

Note that in this problem, the system representation uses the D matrix of eqn (8.8).

8.5 Transfer Function and State Variables

Both the transfer function and the state variable methods are alternate ways of representing the same physical system. Therefore, it may be required to obtain one type of representation from the other. Let us first consider how to obtain the transfer function representation from a given state variable representation.

Let us start with the standard from of state variable equations for single variable system [eqns. (8.1) and (8.2)],

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \quad \text{and} \quad y = \mathbf{c}\mathbf{x}\,.$$

Taking the Laplace transform of both sides of these two equations we get,

$$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{b}\,U(s)\,; \quad \text{and} \quad Y(s) = \mathbf{c}\mathbf{X}(s)\,.$$

In the transfer function representation, all initial conditions are assumed to be zeros. Therefore, $\mathbf{x}(0) = 0$. Then, the above equation becomes,

$$(s\mathbf{I} - \mathbf{A})\,\mathbf{X}(s) = \mathbf{b}U(s)$$

or

$$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}U(s)\,.$$

Substituting this expression for the Laplace transform of state variables into the output equation we get,

$$Y(s) = \mathbf{c}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}U(s)\,.$$

The transfer function is then given by,

$$G(s) = \frac{Y(s)}{U(s)} = \mathbf{c}\,(s\mathbf{I} - \mathbf{A})^{-1}\,\mathbf{b}$$

$$= \frac{\mathbf{c}\,[\,\text{adj}\,(s\mathbf{I} - \mathbf{A})\,]\,\mathbf{b}}{\det\,[\,s\mathbf{I} - \mathbf{A}\,]}\,. \tag{8.14}$$

Thus, for any given state variable representation **A, b, c,** eqn. (8.14) can be used to obtain its transfer function representation.

From the expression (8.14) we note that the poles of the system are the roots of the equation $\det\,[s\mathbf{I} - \mathbf{A}] = 0$. This expression is called the *characteristic equation* of the matrix A and its roots the *eigenvalues* of **A**. Thus, the poles of a system are the same as the eigenvalues of its state matrix.

Example 8.6 :— Obtain the transfer function representation for a system represented by matrices,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -10 & -2 & -3 \end{bmatrix}; \ \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \mathbf{c} = [2 \ \ 1 \ \ 0].$$

Solution: The solution is as follows:

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & -1 & 0 \\ 0 & s & -1 \\ 10 & 2 & s+3 \end{bmatrix}$$

$$\det [s\mathbf{I} - \mathbf{A}] = s(s^2 + 3s + 2) + 10 = s^3 + 3s^2 + 2s + 10$$

$$\text{adj } [s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s^2 + 3s + 2 & -10 & -10s \\ s+3 & s^2 + 3s & -2s - 10 \\ 1 & s & s^2 \end{bmatrix}^T$$

$$= \begin{bmatrix} s^2 + 3s + 2 & s+3 & 1 \\ -10 & s^2 + 3s & s \\ -10s & -2s - 10 & s^2 \end{bmatrix}$$

$$\mathbf{c} \, [\text{adj } (s\mathbf{I} - \mathbf{A})] \, \mathbf{b} = [2 \ \ 1 \ \ 0] \begin{bmatrix} 1 \\ s \\ s^2 \end{bmatrix} = s + 2$$

Therefore,

$$G(s) = \frac{s+2}{s^3 + 3s^2 + 2s + 10}.$$

As illustrated by example 8.6, it is easy to see that the transfer function corresponding to a given state variable representation will always be unique. However, since there is an element of choice in the selection of the state variables, starting from a given transfer function, we can arrive at different state variable representations. The method of deriving the state variable model from a given transfer function, with phase variables selected as state variables, is straightforward and has been described in Section 8.3. However, should we wish to choose physically meaningful state variables or any other set of state variables, a knowledge of the transfer function alone is not sufficient. We must know the details of the system in terms of all the equations in the cause-effect chain relating the input to the output or the corresponding block diagram representation.

8.6 Solution of State Equations

For the purpose of analysis, we have to solve the state equation (8.1), which is in the form of a vector differential equation. The solution is to be obtained for any arbitrary input function $u(t)$ for a given set of initial values of state variables, vector $\mathbf{x}|_{t=0} = \mathbf{x}(0)$.

The method of solution for the vector differential equation is similar to that for the scalar differential equation. Therefore, let us first review the steps in the solution of a scalar first order differential equation,

$$\dot{x} = ax + bu \text{ with } x\,|_{t=0} = x_0. \tag{8.15}$$

Transferring the ax term of eqn. (8.15) to l.h.s. and multiplying both sides by the integrating factor e^{-at} we get,

$$e^{-at}\dot{x} - ae^{-at}\,x = e^{-at}\,bu.$$

The l.h.s. of the above equation is the derivative of $e^{-at}x$. Therefore,

$$\frac{d\,(e^{-at}x)}{dt} = e^{-at}\,bu$$

or ,

$$e^{-at}x = \int_0^t e^{-a\tau}\,bu\,(\tau)\,d\tau + c_1$$

where c_1 is the constant of integration. Thus,

$$x = \int_0^t e^{a(t-\tau)}\,bu\,(\tau)\,d\tau + c_1\,e^{at}.$$

At $t = 0$, $x\,|_{t=0} = x_0$. Therefore, $c_1 = x_0$. Hence, the complete solution for eqn. (8.15) is,

$$x = x_0\,e^{at} + \int_0^t e^{a(t-\tau)}\,bu\,(\tau)\,d\tau. \tag{8.16}$$

The first term on the r.h.s. of eqn. (8.16) is the transient response and the second term the steady-state response, which is the convolution of the impulse response e^{at} and the forcing function $bu\,(t)$.

The integrating factor used above is, by definition,

$$e^{at} \triangleq 1 + at + \frac{a^2t^2}{2!} + \frac{a^3t^3}{3!} + \ldots \tag{8.17}$$

It could be used as an integrating factor to obtain the solution given by eqn. (8.16) because of the property,

$$\frac{d\,e^{at}}{dt} = a\,e^{at} \tag{8.18}$$

which can be verified directly from eqn. (8.17)

It appears reasonable to assume that a similar vector version of the integrating factor can be used to solve the vector differential equation,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{8.19}$$

for any given scalar input $u = u(t)$ and initial condition vector $\mathbf{x}(0)$. For this, we must first define the vector exponential function, analogous to the scalar e^{at} as follows:

$$[\exp(\mathbf{A}t)] = \left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \ldots\right]. \tag{8.20}$$

Differentiating the series in eqn. (8.20) term by term we get,

$$\begin{aligned}\frac{d\exp(\mathbf{A}t)}{dt} &= \left[\mathbf{O} + \mathbf{A} + \mathbf{A}^2 t + \frac{\mathbf{A}^3 t^2}{2!} + \ldots\right] \\ &= \left[\mathbf{A}\left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \ldots\right)\right] \\ &= \left[\left(\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \ldots\right)\mathbf{A}\right] \\ &= [\mathbf{A}\exp(\mathbf{A}t)] = [\{\exp(\mathbf{A}t)\}\mathbf{A}]\end{aligned} \tag{8.21}$$

Equation (8.21) is the vector version of result (8.18). Thus, we can use $\exp(-\mathbf{A}t)$ as the integrating factor for solving the vector differential equation (8.19).

Shifting the term $\mathbf{Ax}$ to the l.h.s. in eqn. (8.19) and then premultiplying both sides by the integrating factor $\exp(-\mathbf{A}t)$ we get,

$$[\exp(-\mathbf{A}t)]\,\dot{\mathbf{x}} - [\exp(-\mathbf{A}t)]\,\mathbf{Ax} = [\exp(-\mathbf{A}t)]\,\mathbf{b}u$$

or,

$$\frac{d}{dt}[\exp(-\mathbf{A}t)\,\mathbf{x}] = [\exp(-\mathbf{A}t)]\,\mathbf{b}u.$$

Integrating both sides of the above equation w.r.t. time and following the same steps as in the scalar case, we get the solution of eqn. (8.19) as,

$$\mathbf{x}(t) = \exp(\mathbf{A}t)\,\mathbf{x}(0) + \int_0^t \exp[\mathbf{A}(t-\tau)]\,\mathbf{b}u(\tau)\,d\tau \tag{8.22}$$

Equation (8.22) is the desired solution of the state equation. It is expressed in terms of the matrix function $\exp(\mathbf{A}t)$. Because of its importance in linear systems analysis, this function is given a special name, *the state transition matrix*, and assigned the symbol $\mathbf{\Phi}(t)$. Thus,

$$\mathbf{\Phi}(t) = \exp(\mathbf{A}t) = \left[\mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \ldots\right]. \tag{8.23}$$

In terms of $\mathbf{\Phi}(t)$, the solution of the state equation is written as,

$$\mathbf{x}(t) = \mathbf{\Phi}(t)\,\mathbf{x}(0) + \int_0^t \mathbf{\Phi}(t-\tau)\,\mathbf{b}u(\tau)\,d\tau. \tag{8.24}$$

The elements of matrix $\Phi(t)$, as defined by eqn. (8.23), will be in the form of infinite series. Therefore, the solution of the state equation $\mathbf{x}(t)$, given by eqn. (8.24), will also be in terms of infinite series. Presence of such terms will make it difficult to appreciate the characteristics of the solution or to use it for further analysis. Hence, it is imperative that we develop methods to compute the elements of $\Phi(t)$ in a closed from. There are a number of methods available for this purpose. In the following sections, we will study some of the more useful and commonly used methods. It may be mentioned that the function exp $(\mathbf{A}t)$ has many interesting properties and the study of these properties is an important area in the theory of functions of matrices (see problem 8.10).

8.7 Determination of $\Phi(t)$ Using Caley-Hamilton Theorem

Let us first recall some of the basic results from the theory of matrices. For an $n \times n$ matrix $\mathbf{A}$, and a scalar λ, the equation,

$$\det [\mathbf{A} - \lambda \mathbf{I}] = 0 \tag{8.25}$$

is called the *characteristic equation* of $\mathbf{A}$. Expanding the l.h.s. of eqn. (8.25), the characteristic equation can be written as,

$$\lambda^n + a_{n-1} \lambda^{n-1} + a_{n-2} \lambda^{n-2} + \ldots + a_1 \lambda + a_0 = 0 \tag{8.26}$$

The n roots of this nth order polynomial equation, $\lambda_1, \lambda_2, \ldots, \lambda_n$, are called the *eigenvalues* of matrix $\mathbf{A}$. The Caley-Hamilton theorem states that every matrix satisfies its own characteristic equation. In other words, eqn. (8.26) will be satisfied if λ is replaced by $\mathbf{A}$, i.e.

$$\mathbf{A}^n + a_{n-1} \mathbf{A}^{n-1} + \ldots + a_1 \mathbf{A} + a_0 \mathbf{I} = 0 \tag{8.27}$$

We will verify this result for a particular case, without proving the Caley-Hamilton theorem.

Example 8,7 :— Show that the Caley-Hamilton theorem is satisfied by the matrix,

$$\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 4 & 1 \end{bmatrix}.$$

Solution: The characteristic equation is given by,

$$\det \begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & -2-\lambda & 1 \\ 0 & 4 & 1-\lambda \end{bmatrix} = \lambda^3 - 2\lambda^2 - 9\lambda + 18 = 0.$$

The roots of the characteristic equation are $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = -3$. Now,

$$\mathbf{A}^2 = \begin{bmatrix} 9 & 0 & 0 \\ 0 & 8 & -1 \\ 0 & -4 & 5 \end{bmatrix} \text{ and } \mathbf{A}^3 = \begin{bmatrix} 27 & 0 & 0 \\ 0 & -20 & 7 \\ 0 & 28 & 1 \end{bmatrix}$$

Substituting these values in the characteristic equation, we get

$$\begin{bmatrix} 27 & 0 & 0 \\ 0 & -20 & 7 \\ 0 & 28 & 1 \end{bmatrix} - 2\begin{bmatrix} 9 & 0 & 0 \\ 0 & 8 & -1 \\ 0 & -4 & 5 \end{bmatrix} - 9\begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 4 & 1 \end{bmatrix} + 18\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 0$$

which verifies the theorem.

Our interest in the Caley-Hamilton theorem is because it provides a method for expressing the higher powers of a matrix **A** in terms of its lower powers. This can be better understood with reference to Example 8.7. Substituting **A** for λ in the characteristic equation we get,

$$\mathbf{A}^3 - 2\mathbf{A}^2 - 9\mathbf{A} + 18\mathbf{I} = 0$$

or

$$\mathbf{A}^3 = 2\mathbf{A}^2 + 9\mathbf{A} - 18\mathbf{I}.$$

Then,

$$\mathbf{A}^4 = \mathbf{A}\mathbf{A}^3 = \mathbf{A}(2\mathbf{A}^2 + 9\mathbf{A} - 18\mathbf{I})$$

$$= 2\mathbf{A}^3 + 9\mathbf{A}^2 - 18\mathbf{A}.$$

Substituting for $\mathbf{A}^3$ once again, we get,

$$\mathbf{A}^4 = 2(2\,\mathbf{A}^2 + 9\mathbf{A} - 18\mathbf{I}) + 9\mathbf{A}^2 - 18\mathbf{A} = 13\mathbf{A}^2 - 36\mathbf{I}$$

and so on for other higher powers of **A**. That is, $\mathbf{A}^3$, $\mathbf{A}^4$ and all other higher powers of **A** can be written in terms of a polynomial in **A** with the highest power term $\mathbf{A}^2$. Generalising this result for a matrix of dimension **n**, we can say that $\mathbf{A}^n$ and other higher powers of **A** can be expressed as a polynomial in **A** with highest power term $\mathbf{A}^{n-1}$ by the use of Caley-Hamilton theorem.

Now, in the infinite series expression (8.20) for $\Phi(t) = \exp(\mathbf{A}t)$, we have all powers of **A**, right up to infinity. The result of the previous paragraph means that all these higher powers of **A**, $\mathbf{A}^n$, $\mathbf{A}^{n+1}$, ... , can be expressed as polynomials in **A** with the highest power $(n-1)$. Therefore, it follows that the infinite series (8.20) can be written with only finite number of terms as,

$$\exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2 t^2}{2!} + \frac{\mathbf{A}^3 t^3}{3!} + \ldots$$

$$= f_0(t)\,\mathbf{I} + f_1(t)\,\mathbf{A} + f_2(t)\,\mathbf{A}^2 + \ldots + f_{n-1}(t)\,\mathbf{A}^{n-1} \qquad (8.28)$$

where $f_0(t), f_1(t), \ldots, f_{n-1}(t)$ are scalar functions of time. The highest power of **A** in eqn. (8.28) is only $n-1$ as opposed to infinity in eqn. (8.20).

Our next problem is the determination of $f_0(t), f_1(t), \ldots, f_n(t)$. The difficulty associated with a direct evaluation of these functions may be illustrated by considering the matrix **A** of Example 8.7. Using the expressions for $\mathbf{A}^3$, $\mathbf{A}^4$, ..., determined earlier, we can write,

$$\exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \frac{(2\mathbf{A}^2 + 9\mathbf{A} - 18\mathbf{I})t^3}{3!} + \frac{(13\mathbf{A}^2 - 36\mathbf{I})t^4}{4!} + \dots$$

$$= \left(1 - \frac{18t^3}{3!} - \frac{36t^4}{4!} - \dots\right)\mathbf{I} + \left(t + \frac{9t^3}{3!} + 0 + \dots\right)\mathbf{A}$$

$$+ \left(\frac{t^2}{2!} + \frac{2t^3}{3!} + \frac{13t^4}{4!} + \dots\right)\mathbf{A}^2$$

Thus, the coefficient terms $f_0(t), f_1(t)$ and $f_2(t)$ of eqn. (8.28) become,

$$f_0(t) = \left(1 - \frac{18t^3}{3!} - \frac{36t^4}{4!} - \dots\right)$$

$$f_1(t) = \left(t + \frac{9t^3}{3!} + 0 + \dots\right)$$

$$f_2(t) = \left(\frac{t^2}{2!} + \frac{2t^3}{3!} + \frac{13t^4}{4!} + \dots\right).$$

It is difficult to visualise the closed form expression for $f_0(t)$, $f_1(t)$ and $f_2(t)$ from the above expressions. Therefore, we will have to try some other approach for the determination of these functions.

Consider the expansion of function $e^{\lambda t}$ where λ is an eigenvalue of the state matrix **A**. This infinite series is given by,

$$e^{\lambda t} = 1 + \lambda t + \frac{\lambda^2 t^2}{2!} + \frac{\lambda^3 t^3}{3!} + \dots$$

Using eqn. (8.26), powers of λ equal to and greater than n can be written in terms of a polynomial of λ with the highest power equal to $n - 1$. Then, following the same procedure as that for $\exp(\mathbf{A}t)$, we can express $e^{\lambda t}$ in terms of a polynomial in λ with the highest power λ^{n-1}. A little observation will show that the coefficients of this polynomial will be the same functions $f_0(t), f_1(t), f_2(t), \dots, f_{n-1}(t)$ as in eqn. (8.28). That is,

$$e^{\lambda t} = f_0(t) + f_1(t)\lambda + f_2(t)\lambda^2 + \dots + f_{n-1}(t)\lambda^{n-1}. \quad (8.29)$$

Equation (8.29) is valid for every eigenvalue of matrix **A**. Hence,

$$\begin{aligned}
\exp(\lambda_1 t) &= f_0(t) + f_1(t)\lambda_1 + f_2(t)\lambda_1^2 + \dots + f_{n-1}(t)\lambda_1^{n-1} \\
\exp(\lambda_2 t) &= f_0(t) + f_1(t)\lambda_2 + f_2(t)\lambda_2^2 + \dots + f_{n-1}(t)\lambda_2^{n-1} \\
\vdots \quad & \quad \vdots \\
\exp(\lambda_n t) &= f_0(t) + f_1(t)\lambda_n + f_2(t)\lambda_n^2 + \dots + f_{n-1}(t)\lambda_n^{n-1}
\end{aligned} \quad (8.30)$$

In the set of n equations given by eqn. (8.30), the numerical values of $\lambda_1, \lambda_2, \ldots, \lambda_n$ are known. Hence, these are essentially n algebraic equations in n unknowns $f_0(t), f_1(t), \ldots, f_n(t)$, which can be straightaway obtained by solving equations (8.30).

Example 8.8 :— Determine the state transition matrix for the matrix **A** in Example 8.7.

Solution: Since **A** is a 3×3 matrix, $n = 3$. Therefore, the expansion of exp $(\mathbf{A}t)$ will contain $\mathbf{A}^2$ as the highest power of **A**. Thus,

$$\exp(\mathbf{A}t) = f_0(t)\,\mathbf{I} + f_1(t)\,\mathbf{A} + f_2(t)\,\mathbf{A}^2.$$

The eigenvalues of **A** have already been determined in Example 8.7 as, $\lambda_1 = 3$, $\lambda_2 = 2$ and $\lambda_3 = -3$. Therefore, eqn. (8.30) gives,

$$e^{3t} = f_0(t) + 3f_1(t) + 9f_2(t)$$

$$e^{2t} = f_0(t) + 2f_1(t) + 4f_2(t)$$

$$e^{-3t} = f_0(t) - 3f_1(t) + 9f_2(t).$$

Solving these three equations we get,

$$f_0(t) = -\,e^{3t} + \frac{9}{5}e^{2t} + \frac{\cdot}{5}\,e^{-3t}$$

$$f_1(t) = \frac{1}{6}\,e^{3t} - \frac{1}{6}e^{-3t}$$

$$f_2(t) = \frac{1}{6}e^{3t} - \frac{1}{5}\,e^{2t} + \frac{1}{30}\,e^{-3t}.$$

Substituting these relations, and already calculated values of **A** and $\mathbf{A}^2$, in the expression for exp $(\mathbf{A}t)$ we get,

$$\Phi(t) = \exp(\mathbf{A}\,t) = \begin{bmatrix} e^{3t} & 0 & 0 \\ 0 & \frac{1}{5}(e^{2t} + 4e^{-3t}) & \frac{1}{5}(e^{2t} - e^{-3t}) \\ 0 & \frac{4}{5}(e^{2t} - 3e^{-3t}) & \frac{1}{5}(4e^{2t} + e^{-3t}) \end{bmatrix}$$

Example 8.9 :— Determine the state transition matrix for the state matrix,

$$\mathbf{A} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

Solution: Let us first determine $\mathbf{A}^2$.

$$\mathbf{A}^2 = \mathbf{AA} = \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

Now,

$$\exp(\mathbf{A}t) = f_0(t)\,\mathbf{I} + f_1(t)\,\mathbf{A} + f_2(t)\,\mathbf{A}^2 \tag{i}$$

The characteristic equation is given by,

$$\det[\mathbf{A} - \lambda\mathbf{I}] = \begin{bmatrix} -1-\lambda & 1 & 0 \\ 0 & -1-\lambda & 1 \\ 0 & 0 & -2-\lambda \end{bmatrix} = (\lambda + 1)^2(\lambda + 2) = 0.$$

The eigenvalues are $\lambda_1 = \lambda_2 = -1$ and $\lambda_3 = -2$. Thus, unlike in Example 8.8, where all the roots were distinct, here we have a case of repeated roots. In this case, two of the equations in eqn. (8.30) will be indentical. The method for obtaining three independent equations is as follows.

The first equation in (8.30), corresponding to λ_1, is,

$$\exp(\lambda_1 t) = f_0(t) + f_1(t)\,\lambda_1 + f_2(t)\,\lambda_1^2 \tag{ii}$$

Take the derivative of this equation w.r.t. λ_1 to get,

$$t\,\exp(\lambda_1 t) = f_1(t) + 2f_2(t)\,\lambda_1 \tag{iii}$$

This is treated as the second equation in (8.30). The third equation corresponding to λ_3 is, of course,

$$\exp(\lambda_3 t) = f_0(t) + f_1(t)\,\lambda_3 + f_2(t)\,\lambda_3^2 \tag{iv}$$

Substituting the values of λ_1 and λ_3 in eqns. (ii), (iii) and (iv) we get,

$$e^{-t} = f_0(t) - f_1(t) + f_2(t)$$

$$te^{-t} = f_1(t) - 2f_2(t)$$

$$e^{-2t} = f_0(t) - 2f_1(t) + 4f_2(t)\,.$$

Solving these three equations we get,

$$f_0(t) = 2t\,e^{-t} + e^{-2t}$$

$$f_1(t) = (3t-2)\,e^{-t} + 2e^{-2t}$$

$$f_2(t) = (t - 1)\,e^{-t} + e^{-2t}.$$

Substituting the values of $\mathbf{A}$ and $\mathbf{A}^2$ in eqn. (i) we get,

$$\Phi(t) = \exp(\mathbf{A}t) = f_0(t)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + f_1(t)\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$+ f_2(t)\begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & -3 \\ 0 & 0 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} f_0 - f_1 + f_2 & f_1 - 2f_2 & f_2 \\ 0 & f_0 - f_1 + f_2 & f_1 - 3f_2 \\ 0 & 0 & f_0 - 2f_1 + 4f_2 \end{bmatrix} \qquad \text{(v)}$$

Substituting the expression for $f_0(t), f_1(t)$ and $f_2(t)$ in eqn. (v) we get,

$$\Phi(t) = \begin{bmatrix} e^{-t} & te^{-t} & (t-1)e^{-t} + e^{-2t} \\ 0 & e^{-t} & e^{-t} - e^{-2t} \\ 0 & 0 & e^{-2t} \end{bmatrix}$$

8.8 Determination of $\Phi(t)$ by Diagonalising A

Let us first assume that matrix **A** is given in a diagonal form to begin with. In the case of a third order matrix, for example, **A** may be,

$$\mathbf{A} = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

The characteristic equation will be,

$$\det[\mathbf{A} - \lambda \mathbf{I}] = (d_1 - \lambda)(d_2 - \lambda)(d_3 - \lambda) = 0.$$

Thus, the eigenvalues are $\lambda_1 = d_1$, $\lambda_2 = d_2$ and $\lambda_3 = d_3$, assuming distinct eigenvalues. In other words, the diagonal elements of **A** will be the eigenvalues themselves. That is,

$$\mathbf{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$$

The mathematical operations involved in determining the expression,

$$\exp(\mathbf{A}t) = \mathbf{I} + \mathbf{A}t + \frac{\mathbf{A}^2t^2}{2!} + \ldots$$

become quite straight forward when **A** is diagonal. For example,

$$\mathbf{A}^2 = \begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} \lambda_1^3 & 0 & 0 \\ 0 & \lambda_2^3 & 0 \\ 0 & 0 & \lambda_3^3 \end{bmatrix}$$

and so on. Thus,

$$\exp(\mathbf{A}t) = \begin{bmatrix} \sum_{i=0}^{\infty} \frac{(\lambda_1 t)^i}{i!} & 0 & 0 \\ 0 & \sum_{i=0}^{\infty} \frac{(\lambda_2 t)^i}{i!} & 0 \\ 0 & 0 & \sum_{i=0}^{\infty} \frac{(\lambda_3 t)^i}{i!} \end{bmatrix}$$

$$= \begin{bmatrix} \exp(\lambda_1 t) & 0 & 0 \\ 0 & \exp(\lambda_2 t) & 0 \\ 0 & 0 & \exp(\lambda_3 t) \end{bmatrix} \tag{8.31}$$

Thus, if **A** is given in the diagonal form, $\Phi(t)$ can be written by mere inspection of **A**. Substitution of eqn. (8.31) in eqn. (8.24) will give the solution of the sate equation as,

$$x_1 = \exp(\lambda_1 t)\, x_1(0) + \int_0^t \exp[\lambda_1 (t - \tau)]\, b_1\, u(\tau)\, d\tau$$

$$x_2 = \exp(\lambda_2 t)\, x_2(0) + \int_0^t \exp[\lambda_2 (t - \tau)]\, b_2\, u(\tau)\, d\tau \tag{8.32}$$

and so on.

When matrix **A** is diagonal, it means that the state equation are in a 'decoupled' from:

$$\begin{aligned} \dot{x}_1 &= \lambda_1 x_1 + b_1 u \\ \dot{x}_2 &= \lambda_2 x_2 + b_2 u \\ &\vdots \\ \dot{x}_n &= \lambda_n x_n + b_n u \end{aligned}$$

That is, the expression for $\dot{x}_1$ contains only x_1 and no other state variable. Each of the first order state equations can then be solved independently, resulting in the same expressions as those in eqn. (8.32). Determination of $\Phi(t)$, and expression of the solution in the form of eqn. (8.24) are then superfluous.

The assumption made at the beginning of this section that matrix **A** is given in the diagonal form is a bit too drastic. In system modelling, we seldom get the state matrix **A** in a diagonal form. Thus, in order to benefit from the simple procedure for obtaining $\Phi(t)$ described so far in this section, we must first establish a method for diagonalising a given non-diagonal matrix **A**. For this purpose we recall some more results of matrix theory.

The effect of multiplying a matrix by a vector (i.e., a column matrix) is to produce another vector whose elements are linear combinations of the original vector. That is, if **u** and **v** are two n-dimensional vectors such that,

$$\mathbf{Av} = \mathbf{u},$$

then the $n \times n$ matrix **A** is called a *linear transformation* which 'transforms' vector **v** to vector **u**. If **u** is looked upon as a vector in some n-dimensional space, the effect of the linear transformation is to produce a new vector in the same space. If the effect of this transformation on a vector is such that its direction does not get altered but only the magnitude changes by a factor equal to an eigenvalue of **A**, i.e.,

$$\mathbf{Av} = \lambda_1 \mathbf{v} \tag{8.33}$$

then vector **v** is called an *eigenvector* of **A**. There will be one eigenvector for each distinct root λ_i. Let us assume that all the roots of **A** are distinct. Then there will be n eigenvectors, $\mathbf{v}_1, \mathbf{v}_2, \ldots \mathbf{v}_n$. Let us call the matrix formed by these eigenvectors as P. That is,

$$\mathbf{P} = [\mathbf{v}_1 \; \mathbf{v}_2 \ldots \mathbf{v}_n].$$

Then it is clear that,

$$\mathbf{AP} = \Lambda\,\mathbf{P} = \mathbf{P}\Lambda \tag{8.34}$$

where,

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & & & \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}$$

Premultiplying eqn.(8.34) by $\mathbf{P}^{-1}$, we have,

$$\mathbf{P}^{-1}\,\mathbf{AP} = \Lambda. \tag{8.35}$$

Equation (8.35) shows how to transform a given matrix **A**, with distinct eigenvalues, into a diagonal matrix with eigenvalues as diagonal elements. The eigenvectors $\mathbf{v}_i$, $i = 1, 2, \ldots n$, can be obtained from eqn. (8.33) by first writing it as,

$$(\mathbf{A} - \lambda_i \mathbf{I})\; \mathbf{v}_i = 0\,, \quad i = 1, 2, \ldots, n$$

and then solving it for each λ_i to get the elements of $\mathbf{v}_i$. (8.36)

Example 8.10:— Find the linear transformation **P** for diagonalising the matrix,

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

Solution. The characteristic equation is,

$$\det\,[\mathbf{A} - \lambda\,\mathbf{I}] = \det \begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = 0$$

or

$$\lambda^2 + 3\lambda + 2 = (\lambda + 1)\,(\lambda + 2) = 0.$$

Therefore, the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$. Let the eigenvectors be $\mathbf{v}_1$ and $\mathbf{v}_2$. Then

$$[\mathbf{A} - \lambda_1 \mathbf{I}]\,\mathbf{v}_1 = 0 \text{ or} \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \mathbf{v}_1 = 0\,; \;\therefore\; \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$[\mathbf{A} - \lambda_2 \mathbf{I}]\,\mathbf{v}_2 = 0 \text{ or} \begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \mathbf{v}_2 = 0\,; \;\therefore\; \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Therefore,

$$\mathbf{P} = [\mathbf{v}_1 \, \mathbf{v}_2] = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

$$\mathbf{P}^{-1} = \frac{\text{adj}\,\mathbf{P}}{\det \mathbf{P}} = \frac{1}{-1}\begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$\mathbf{P}^{-1}\,\mathbf{A}\mathbf{P} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

This verifies the result of eqn. (8.35).

We now come back to the main job of determining the state transition matrix $\Phi(t) = \exp(\mathbf{A}t)$. Having diagonalised the matrix $\mathbf{A}$ by the linear transformation $\mathbf{P}$, we can write from eqn. (8.35),

$$\mathbf{A} = \mathbf{P}\Lambda\mathbf{P}^{-1} \tag{8.37}$$

Replacing expression (8.37) for $\mathbf{A}$ in the infinite series for $\exp(\mathbf{A}t)$, we get,

$$\exp(\mathbf{A}t) = \mathbf{I} + \mathbf{P}\Lambda\mathbf{P}^{-1}t + \frac{(\mathbf{P}\Lambda\mathbf{P}^{-1})^2 t^2}{2!} + \ldots$$

the kth term being $[(\mathbf{P}\Lambda\mathbf{P}^{-1})^k t^k]/k!$. However, we can write the kth matrix power as,

$$\mathbf{A}^k = (\mathbf{P}\Lambda\mathbf{P}^{-1})^k = (\mathbf{P}\Lambda\mathbf{P}^{-1})(\mathbf{P}\Lambda\mathbf{P}^{-1})\ldots(\mathbf{P}\Lambda\mathbf{P}^{-1}) = \mathbf{P}\Lambda^k\mathbf{P}^{-1}.$$

Therefore,

$$\exp(\mathbf{A}t) = \mathbf{P}\left[\mathbf{I}\,\Lambda t + \frac{\Lambda^2 t^2}{2!} + \frac{\Lambda^3 t^3}{3!} + \ldots\right]\mathbf{P}^{-1}$$

$$= \mathbf{P}\begin{bmatrix} \exp(\lambda_1 t) & 0 & \cdots & 0 \\ 0 & \exp(\lambda_2 t) & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & \exp(\lambda_n t) \end{bmatrix}\mathbf{P}^{-1}$$

Example 8.11:— Determine the state transition matrix for the state matrix $\mathbf{A}$ given in Example 8.10.

Solution: The linear transformation $\mathbf{P}$ for diagonalising $\mathbf{A}$ has already been developed in Example 8.10 as

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$$

The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -2$. Substituting these values in eqn. (8.38) we get,

$$\Phi(t) = \exp(\mathbf{A}t) = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} e^{-t} & 0 \\ 0 & e^{-2t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

Remark: The methods for determining the state transition matrix, described in Sections 8.7 and 8.8, are purely time domain methods and well suited for computer implementation. However, they are not very suitable for hand calculations. For this purpose, the technique using the Laplace transform of state equations is much better. This technique is described in the next section.

8.9 Determination of $\Phi(t)$ Using Laplace Transform

Taking the Laplace transform of the state equation,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\mathbf{u}$$

we get, $$s\mathbf{X}(s) - \mathbf{x}(0) = \mathbf{A}\mathbf{X}(s) + \mathbf{b}U(s)$$

or $$\mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\,U(s).$$

The matrix $(sI - \mathbf{A})^{-1}$ is called the *resolvent matrix* and given the symbol $\Phi(s)$, i.e.,

$$\Phi(s) = (s\mathbf{I} - \mathbf{A})^{-1}.$$

The solution for the state equation can then be written as,

$$\mathbf{X}(s) = \Phi(s)\,\mathbf{x}(0) + \Phi(s)\,\mathbf{b}U(s). \tag{8.39}$$

Taking the Laplace inverse of eqn. (8.39), we get,

$$\mathbf{x}(t) = \mathcal{L}^{-1}\,\mathbf{X}(s) = \mathcal{L}^{-1}\,[\Phi(s)]\,\mathbf{x}(0) + \mathcal{L}^{-1}\,[\Phi(s)\mathbf{b}\mathbf{U}(\mathbf{s})]$$

$$= \Phi(t)\,x(0) + \int_0^t \Phi(t-\tau)\,\mathbf{b}u(\tau)\,d\tau,$$

which is the same expression as eqn. (8.24). Thus we have,

$$\Phi(t) = \mathcal{L}^{-1}\,[\Phi(s)] = \mathcal{L}^{-1}\,[s\mathbf{I} - \mathbf{A}]^{-1}.$$

Example 8.12:— Determine $\Phi(t)$ for the state matrix $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Solution: This is the same matrix as used in Examples 8.10 and 8.11.

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\det\,[s\mathbf{I} - \mathbf{A}] = s(s+3) + 2 = s^2 + 3s + 2 = (s+1)(s+2)$$

$$\text{adj}\,[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+3 & -2 \\ 1 & s \end{bmatrix}^T = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

Therefore,

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\text{adj}\,[s\mathbf{I} - \mathbf{A}]}{\det\,[s\mathbf{I} - \mathbf{A}]}$$

$$= \begin{bmatrix} \dfrac{s+3}{(s+1)\,(s+2)} & \dfrac{1}{(s+1)\,(s+2)} \\ \dfrac{-2}{(s+1)\,(s+2)} & \dfrac{s}{(s+1)\,(s+2)} \end{bmatrix}$$

Taking the Laplace inverse of each term we get,

$$\Phi(t) = \mathcal{L}^{-1}\,\Phi(s) = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

which is the same expression as derived in Example 8.11.

8.10 Linear Transformation of State Variables

It has already been mentioned that the same physical system can be represented by different sets of state variables. For example, the field-controlled d.c. motor can be represented by three different sets of physical variables $\{i, \omega, \theta\}$, $\{\phi, \omega, \theta\}$, $\{\tau, \omega, \theta\}$ or phase variables $\{\theta, \dot{\theta}, \ddot{\theta}\}$. The system matrices **A**, **b** and **c** corresponding to these different sets of state variables will, of course, be different. However, since these different sets of state variables are the different representations of the same system, they must obviously be related to each other. A comparison of the four different sets of state variables for the field-controlled d.c. motor shows that one set can be obtained from the other by means of a linear combination. For example, in the sets $\{i, \omega, \theta\}$ and $\{\tau, \omega, \theta\}$, two of the state variables are the same while the third ones are related by the linear equation $\tau = k_1 k_2 i$. Thus,

$$\begin{bmatrix} \tau \\ \omega \\ \theta \end{bmatrix} = \begin{bmatrix} k_1 k_2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} i \\ \omega \\ \theta \end{bmatrix}$$

In general, if two sets of state variables, say, **x** and **z** represent the same system, they must be related to each other by a linear transformation,

$$\mathbf{x} = \mathbf{Q}\mathbf{z} \tag{8.40}$$

where **Q** is a non singular constant matrix.

Let the system matrices associated with set **x** be $\{\mathbf{A}, \mathbf{b}, \mathbf{c}\}$ and those with **z** be $\{\mathbf{A}^*, \mathbf{b}^*, \mathbf{c}^*\}$. The question we now ask is: how are these two sets of matrices related to each other when **x** and **z** are related by eqn. (8.40)?

For the choice **x**, the system representation is,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \tag{8.41}$$

$$y = \mathbf{c}\mathbf{x}$$

Differentiating both sides of eqn. (8.40) we get,

$$\dot{\mathbf{x}} = \mathbf{Q}\dot{\mathbf{z}} \tag{8.42}$$

Substituting eqns. (8.40) and (8.42) in eqn. (8.41) we get,

$$\mathbf{Q}\dot{\mathbf{z}} = \mathbf{AQz} + \mathbf{bu} \tag{8.43}$$

$$y = \mathbf{cQz}$$

Since **Q** is a non-singular matrix, premultiplying both sides of eqn. (8.43) by $\mathbf{Q}^{-1}$ we get,

$$\dot{\mathbf{z}} = \mathbf{Q}^{-1}\,\mathbf{AQz} + \mathbf{Q}^{-1}\,\mathbf{b}u$$

$$y = \mathbf{cQz} \tag{8.44}$$

Comparing eqn. (8.44) with the standard form of representation we have,

$$\mathbf{A}^{*} = \mathbf{Q}^{-1}\,\mathbf{AQ}\,;\ \mathbf{b}^{*} = \mathbf{Q}^{-1}\,\mathbf{b};\ \textbf{and}\ \mathbf{c}^{*} = \mathbf{cQ} \tag{8.45}$$

Equation (8.45) is the answer to the question of the previous paragraph. It shows the relationship between two sets of system matrices corresponding to two different choices of state variables.

We now derive an important property of the state variable representation: *the eigenvalues of a system remain unaltered under linear transformation of its state variables*. That is, for the linear transformation (8.40), the eigenvalues of **A** and $\mathbf{A}^{*} = \mathbf{Q}^{-1}\mathbf{AQ}$ are the same. To prove this result, we note that the eigenvalues of **A** are the roots of the characteristic eqn. (8.25):

$$\det\,[\mathbf{A} - \lambda\mathbf{I}] = \mathbf{0}.$$

Hence, if we can show that,

$$\det\,[\mathbf{A} - \lambda\mathbf{I}] = \det\,[\mathbf{A}^{*} - \lambda\mathbf{I}]$$

the above result will be proved. Now, $\mathbf{Q}^{-1}\,\mathbf{Q} = \mathbf{I}$. Therefore,

$$\begin{aligned}\det\,[\mathbf{A}^{*} - \lambda\mathbf{I}] &= \det\,[\mathbf{Q}^{-1}\,\mathbf{AQ} - \lambda\mathbf{Q}^{-1}\,\mathbf{Q}]\\ &= \det\,\mathbf{Q}^{-1}\,[\mathbf{A} - \lambda\mathbf{I}]\mathbf{Q}\\ &= \det\mathbf{Q}^{-1}\,.\,\det\,[\mathbf{A} - \lambda\mathbf{I}].\,\det\,\mathbf{Q}\end{aligned}$$

because the determinant of a product is equal to the product of the determinants. Since the determinant is a scalar quantity, we can rewrite the above expression as,

$$\det\,[\mathbf{A}^{*} - \lambda\mathbf{I}] = \det\,[\mathbf{A} - \lambda\mathbf{I}]\,\det\,\mathbf{Q}^{-1}\,.\,\det\mathbf{Q}.$$

But the product of determinants is equal to the determinant of the product. Therefore,

$$\det\mathbf{Q}^{-1}\,.\,\det\mathbf{Q} = \det\mathbf{Q}^{-1}\,\mathbf{Q} = \det\mathbf{I} = \mathbf{I}.$$

Therefore,

$$\det [\mathbf{A}^* - \lambda\mathbf{I}] = \det [\mathbf{A} - \lambda\mathbf{I}].$$

Hence the above result. This result shows that the eigenvalues of a system represent its truly invariant properties, being the same whatever be the state variable representation. These eigenvalues are therefore called the *natural modes* of the system.

In Section 8.8, we used a particular linear transformation **P** with the property of diagonalising matrix **A** (eqn.8.35), i.e.,

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \Lambda.$$

With this transformation, the new state variable set $\mathbf{z} = \mathbf{P}^{-1}\mathbf{x}$, with system matrices defined as,

$$\mathbf{A}^* = \mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \Lambda \; ; \; \mathbf{b}^* = \mathbf{P}^{-1}\mathbf{b} \text{ and } \mathbf{c}^* = \mathbf{c}\mathbf{P}$$

gives state equations which are 'decoupled', as explained in Section 8.8. This form of state variable representation is called the *normal form* representation. At times, it is also called the *canonical form*, though some authors call the phase variable form as the canonical form.

The state variable representation can also be given a geometrical interpretation. The state of a system at any time t_1 is an ordered set $[x_1(t_1), x_2(t_1), ..., x_n(t_1)]$, where $x_1\,(t_1)$ is the value of the state variable x_1 at time t_1. This set defines a point in an n-dimensional space, called the *state space*. As the time advances, the state of the system changes, either due to its stored energy alone, i.e, due to the initial conditions alone, or due to an external forcing function (the input) along with the initial conditions. The succession of these changing states generates a path in the state space, called the *state trajectory*. Since the matrix $\Phi\,(t)$ determines these successions, it is called the *state transition matrix*.

Every point in the state space when joined to the origin, gives rise to a state vector. The representation of any vector space is always with respect to a set of unit reference vectors. For example, in a two-dimensional space a common choice of reference vectors is a set of two unit vectors at right angles to each other, i.e., *orthogonal* to each other. The set of these reference vectors is called the *basis* of the vector space. There is no sanctity about a particular basis. We choose the basis according to some need and change it when the need changes. The same vector will be represented by different sets, e.g. $\{x_1(t_1),\ x_2(t_1),\ldots x_n(t_n)\}$ and $\{x_1^*\,(t_1),\, x_2^*\,(t_1),\,\ldots, x_n^*\,(t_1)\}$ when the basis is changed. The linear transformation discussed in this section can thus be seen to be a change in the basis of the state space.

8.11 Analysis with State Variables

Example 8.13:— For the *R-L-C* circuit shown in Fig. 8.4, the output is the voltage across the inductance. Determine the output for:

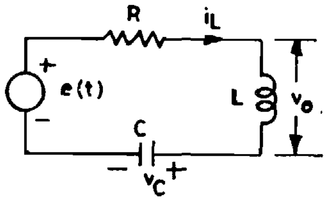


Fig. 8.4

(i) $i(0) = 0$, $v_c(0) = 10$ and $e(t) = 0$.

(ii) $i(0) = 1$, $v_c(0) = 0$ and $e(t)$ = unit step.

(iii) $i(0) = v_c(0) = 0$ and $e(t)$ = unit impulse.

Solution: This circuit has already been discussed in Example 8.4. As shown there, with state variables $x_1 = v_c$ and $x_2 = i_L$, the state equations are,

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1/C \\ -1/L & -R/L \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/L \end{bmatrix} e$$

However, the output equation in the present problem will be different. Application of Kirchhoff's voltage law gives,

$$v_0 = e - (iR + v_c) = -x_1 - Rx_2 + e.$$

In the standard matrix form,

$$v_0 = [-1 \quad -R] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + e$$

For the sake of numerical simplicity, let us choose the parameters as $C = 0.5$, L = 1 and $R = 2$. The system matrices then become,

$$\mathbf{A} = \begin{bmatrix} 0 & 2 \\ -1 & -2 \end{bmatrix}; \ \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \ \mathbf{c} = [-1 \quad -2].$$

In the present problem the expression for output v_0 contains not only the state variables x_1 and x_2 but also the input e. The direct effect of input on the output is expressed by the matrix **D** in a multivariable system [eqn. (8.8)]. In the case of single variable systems, matrix **D** reduces to a single constant d. Here $d = 1$.

Let us determine the state transition matrix $\Phi(t)$ by first determining the resolvent matrix $\Phi(s)$ and then taking its Laplace inverse:

$$[s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s & -2 \\ 1 & s+2 \end{bmatrix}$$

$$\det [s\mathbf{I} - \mathbf{A}] = s^2 + 2s + 2 = (s+1)^2 + 1^2.$$

The eigenvalues of **A** are $(-1 + j)$ and $(-1 - j)$.

$$adj\ [s\mathbf{I} - \mathbf{A}] = \begin{bmatrix} s+2 & 2 \\ -1 & s \end{bmatrix}$$

Therefore,

$$\Phi(s) = [s\mathbf{I} - \mathbf{A}]^{-1} = \frac{\text{adj}\,[s\mathbf{I} - \mathbf{A}]}{\det\,[s\mathbf{I} - \mathbf{A}]} = \begin{bmatrix} \dfrac{s+2}{(s+1)^2+1^2} & \dfrac{2}{(s+1)^2+1^2} \\ \dfrac{-1}{(s+1)^2+1^2} & \dfrac{s}{(s+1)^2+1^2} \end{bmatrix}$$

Taking the Laplace inverse term by term we get,

$$\Phi(t) = \begin{bmatrix} e^{-t}(\sin t + \cos t) & 2\,e^{-t}\sin t \\ -e^{-t}\sin t & e^{-t}(\cos t - \sin t) \end{bmatrix}$$

(i) $i(0) = 0$, $v_c(0) = 10$ and $e(t) = 0$: In this case the input or the forcing function is zero. Therefore, the state equations become homogeneous, i.e., $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. In control terminology, such systems are also called *autonomous systems*. The solution of state equations, given by eqn. (8.24), reduces to $\mathbf{x}(t) = \Phi(t)\,\mathbf{x}(0)$. In this case,

$$\mathbf{x}(0) = \begin{bmatrix} 10 \\ 0 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10\,e^{-t}(\sin t + \cos t) \\ -\,10\,e^{-t}\sin t \end{bmatrix}$$

From the output equation we have,

$$v_0 = -\,x_1 - 2\,x_2 = 10\,e^{-t}\,(\sin t - \cos t).$$

A plot of the state variables x_1, x_2 and the output v_o as functions of time is shown in Figs. 8.5(a), (b) and (c). Further, Fig. 8.6(a) shows the state space trajectory for this problem. Since it is a second order system, the state space reduces to a state plane which can be easily drawn on paper. This diagram shows the transition of the state from its initial value $x(0)$ to the final value, which is the origin in this case. This diagram is quite helpful in appreciating the dynamic properties of the system. The present system is only slightly underdamped. Therefore, the state plane trajectory goes only slightly into the negative x_1 region before reaching the origin. A system with much less damping may have a trajectory like Fig. 8.6(b). A system without damping will have a closed curve like Fig. 8.6(c) as its state trajectory, while an unstable system may have a trajectory like Fig. 8.6(d). When the state variables are phase variables, such figures are called the *phase-plane diagrams*. For non-linear systems, where the analytical methods are difficult to use, the *phase-plane analysis* is an important tool for their study.

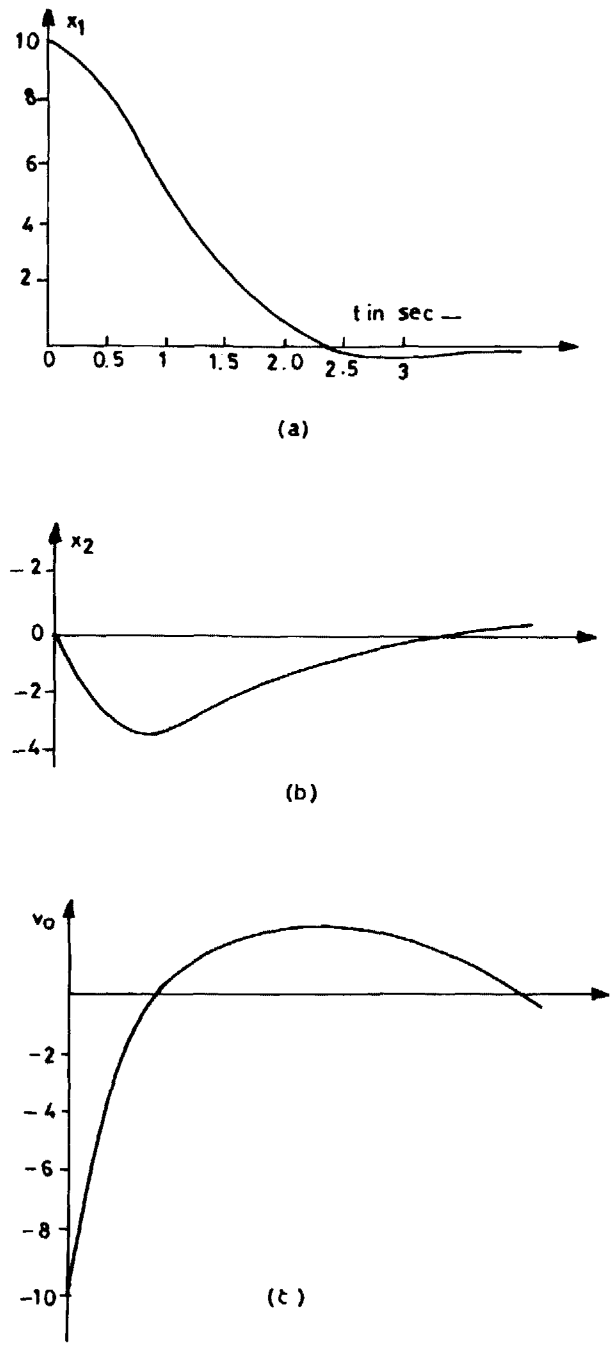


Fig. 8.5 Plot of State Variables and the Output

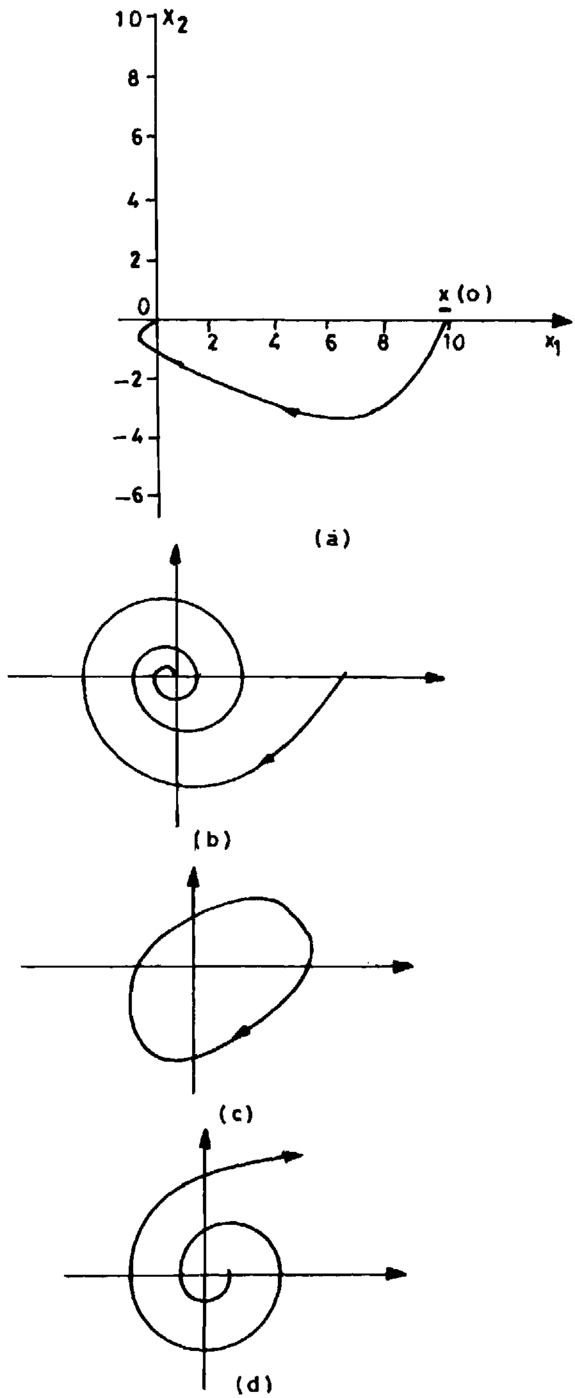


Fig. 8.6 State Plane Trajectories

(ii) $i(0) = 1$, $v_c(0) = 0$ and $e(t) = u(t)$: Here we have to determine the complete response. According to eqn. (8.24) we have,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \phi_{11}(t) & \phi_{12}(t) \\ \phi_{21}(t) & \phi_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$+ \int_0^t \begin{bmatrix} \phi_{11}(t-\tau) & \phi_{12}(t-\tau) \\ \phi_{21}(t-\tau) & \phi_{22}(t-\tau) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(\tau)\, d\tau .$$

In the present problem $x_1(0) = 0$ and $x_2(0) = 1$. Therefore, the response due to the initial conditions is,

$$\begin{bmatrix} \phi_{12}(t) \\ \phi_{22}(t) \end{bmatrix} = \begin{bmatrix} 2e^{-t} \sin t \\ e^{-t}(\cos t - \sin t) \end{bmatrix}$$

Since $b_1 = 0$ and $b_2 = 1$, the response due to the forcing function is,

$$\begin{bmatrix} \int_0^t 2e^{-(t-\tau)} \sin(t-\tau)\, u(\tau)\, d\tau \\ \int_0^t e^{-(t-\tau)} \{\cos(t-\tau) - \sin(t-\tau)\}\, u(\tau)\, d\tau \end{bmatrix}$$

Let $(t-\tau) = x$ in the above expression. Then $d\tau = -dx$, and the limits of integration become $\tau = 0 \rightarrow x = t$ and $\tau = t \rightarrow x = 0$. The expression for response due to the forcing function then becomes,

$$\begin{bmatrix} -2 \int_t^0 e^{-x} \sin x\, dx \\ -\int_t^0 e^{-x} \cos x\, dx + \int_t^0 e^{-x} \sin x\, dx \end{bmatrix}$$

The above integrals can be evaluated using the formulas:

$$\int e^{-x} \sin x\, dx = -\frac{e^{-x}(\sin x + \cos x)}{2}$$

$$\int e^{-x} \cos x\, dx = \frac{e^{-x}(\sin x - \cos x)}{2}$$

Thus, the response due to the forcing function is,

$$\begin{bmatrix} 1 - e^{-t}(\sin t + \cos t) \\ e^{-t} \sin t \end{bmatrix}$$

Summing the responses due to the initial conditions and the forcing function, we get the total response:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2e^{-t} \sin t + 1 - e^{-t}(\sin t + \cos t) \\ e^{-t}(\cos t - \sin t) + e^{-t} \sin t \end{bmatrix}$$

$$= \begin{bmatrix} 1 + e^{-t} \sin t - e^{-t} \cos t \\ e^{-t} \cos t \end{bmatrix}$$

The output v_0 is given by,

$$v_0 = -x_1 - 2x_2 + e(t)$$

Therefore,

$$v_0 = -\left\{1 + e^{-t}\sin t - e^{-t}\cos t\right\} - 2e^{-t}\cos t + 1$$

$$= -e^{-t}(\sin t + \cos t) = -\sqrt{2}\, e^{-t}\sin\left(t + \frac{\pi}{4}\right)$$

(iii) *Impulse response with zero initial conditions*: The output without initial conditions will be given by combining eqn. (8.24) with the output equation, i.e.,

$$v_0 = \int_0^t \mathbf{c}\,\Phi(t-\tau)\,\mathbf{b}\,\delta(\tau)\,d\tau + \delta(t).$$

Since the convolution of a function with an impulse is the function itself, we get,

$$v_0 = \mathbf{c}\,\Phi(t)\,\mathbf{b} + \delta(t)$$

Substituting the values of the matrices in the above expression we get,

$$v_0 = [-1 \;\; -2]\begin{bmatrix}\phi_{11}(t) & \phi_{12}(t)\\ \phi_{21}(t) & \phi_{22}(t)\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix} + \delta(t)$$

$$= -\phi_{12}(t) - 2\phi_{22}(t) + \delta(t)$$

$$= -2e^{-t}\sin t - 2e^{-t}(\cos t - \sin t) + \delta(t)$$

$$= -2e^{-t}\cos t + \delta(t)$$

8.12 Concluding Comments

An examination of Example 8.13 indicates that the system response due to initial conditions and due to forcing function can be calculated separately, once $\Phi(t)$ has been determined. In the transfer function approach, since the initial conditions are assumed to be zero, the presence of initial conditions necessitates additional steps of converting the transfer function back into the differential equation form. Thus, the state variable technique is more efficient in handling initial conditions.

The state variable technique gives not only the output response but the variation of all the state variables as functions of time. In Example 8.13, we know not only the output $v_0(t)$ but also $i(t)$ and $v_c(t)$. Thus, when the current will reach its maximum value, or what this maximum value will be, can easily be found from the solution. Such information is almost always necessary in practical problems.

State variable analysis is in the time domain, and hence, is well suited for numerical and computer techniques. Frequency domain techniques, like the

Fourier transform or the Laplace transform, are not very amenable to computer handling. The computer technique for solving a higher order differential equation is to first convert it into a set of first order differential equations, just like the state variable representation. Therefore, state variable techniques are very well suited for computer simulation and analysis of large interconnected systems. However, for pen and paper analysis, the transfer function methods are more appropriate.

The state variable formulation handles single variable as well as multivariable systems with about equal ease. Many of the important properties of such systems, as well as their design, are studied through state variable techniques. Similarly, the representation of discrete time, time varying as well as non-linear systems can easily be done in the state variable framework. And finally, the vector differential equation form of state variable technique provides a very compact form for writing a large number of equations for real life systems.

However, the above arguments do not mean that the state variable techniques are the best for all problems. For many problems, Fourier transform or Laplace transform methods are easier to use and provide a much better insight into the problem. We conclude by saying that an engineer should have mastery over all the different tools of analysis, so that he can select the one best suited for a particular problem or solve the same problem with two or more different techniques to get a better understanding of the problem.

GLOSSARY

State Variables: A set of minimum number of linearly independent system variables, such that their values for any time $t > 0$ can be calculated from a knowledge of their values at $t = 0$ for any given input, is called *a set of state variables*. The state of a system at any time t_1 is the set of values of its state variables at time t_1. The set of state variables representing a system is not unique; however, their number is always equal to the order of the system.

Phase Variables: When the set of state variables consists of the output and its first $(n-1)$ derivatives (for an nth order system), they are called *phase variables*.

State Variable Equations: The standard form of state variable equations for an nth order system, with m inputs and r outputs, is,

$$\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{Bu} \quad \text{(the state equation)}$$

$$\mathbf{y} = \mathbf{Cx} + \mathbf{Du} \quad \text{(the output equation)}$$

where the vectors **x**, **u** and **y** have dimensions, $n \times 1$, $m \times 1$ and $r \times 1$, respectively, and the dimensions of matrices are: dim $\mathbf{A} = n \times n$, dim $\mathbf{B} = n \times m$, dim $\mathbf{C} = r \times n$, dim $\mathbf{D} = r \times m$. For single variable system matrices, **B**, **C** and **D** become column matrix **b** with dimension $n \times 1$, row matrix **c** with dimension $1 \times n$ and constant d, respectively.

Canonical Form: The special structure of matrix **A**, given in eqn. (8.9), for phase variable representation is called the *canonical form*. Sometimes, this term is also used to designate the diagonal state matrix, with diagonal elements as the eigenvalues of the system.

State Transition Matrix: It is the name given to the matrix exp $(\mathbf{A}t)$ and is designated by $\Phi(t)$, *i.e.*, $\Phi(t) = \exp(\mathbf{A}t)$.

Linear Transformation of State Variables: Let **x** be a set of state variables for a system, represented by system equations $\dot{\mathbf{x}} = \mathbf{Ax} + \mathbf{b}u$, $y = \mathbf{cx}$. If a new set of state variables **z** is defined, then it must be related to the set **x** by a linear transformation $\mathbf{x} = \mathbf{Qz}$, where **Q** is a constant, non-singular, $n \times n$ matrix. If the system equations for representation with **z** are $\dot{\mathbf{z}} = \mathbf{A}^*\mathbf{z} + \mathbf{b}^* u$, then the system matrices of these two representations are related by,

$$\mathbf{A}^* = \mathbf{Q}^{-1}\mathbf{AQ}\,;\ \mathbf{b}^* = \mathbf{Q}^{-1}\mathbf{b};\ \mathbf{c}^* = \mathbf{cQ}.$$

The eigenvalues of a system remain unaltered under linear transformation of its state variables.

Normal Form Representation: When the state variables are so selected that the state matrix is diagonal, with diagonal elements equal to the eigenvalues (distinct eigenvalues), the representation is called the *normal form representation*.

State Space: The n-dimensional vector space, where the values of n state variables **x** at a time define a point, is called the *state space*.

Phase Plane: When the number of state variables is only two, the state space becomes a two-dimensional state plane, which can be easily drawn on paper. When the two state variables are phase variables, this plane is called the *phase plane*.

PROBLEMS

8.1. Develop a state variable model for the loudspeaker, described in Chapter 1, Section 1.4, treating displacement x of the cone as the output and voltage v to the voice coil as the input. Comment on the different possible choices of state variables.

8.2. Determine a state variable model for two water tanks, interconnected through a ground pipeline with a valve. Assume both tanks to have inflows from the top, with inflow rates q_1 and q_2, and outflow from the bottom, with outflow rates, q'_1 and q'_2, respectively.

8.3. Obtain a state variable model for a system described by the following differential equation:

$$10\frac{d^3y}{dt^3} + 5\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 3y = 20\frac{dx}{dt} + 10x.$$

8.4. Obtain the state equations for a system described by a pair of simultaneous differential equations,

$$\ddot{y}_1 + a_1(\dot{y}_1 - \dot{y}_2) + a_2(y_1 - y_2) = 0$$

$$\dot{y}_2 + b_1(\dot{y}_2 - \dot{y}_1) + b_2(y_2 - y_1) = k[f(t) - y_2].$$

8.5. Develop a state variable representation for the system shown by the block diagram of Fig. 8.7. Choose the state variables as the physical variables shown at the output of each block.

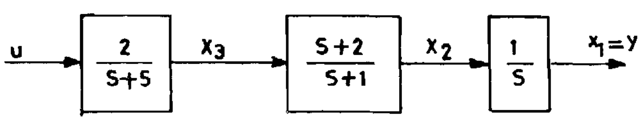


Fig. 8.7

8.6. Obtain a state variable model for the electrical circuit shown in Fig. 8.8. Can there be a combination of circuit parameters such that the output voltage is identically zero for any input $v_i(t)$?

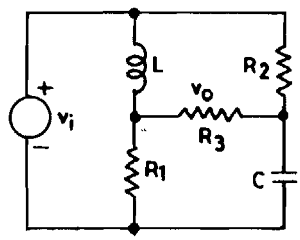


Fig. 8.8

8.7. A system is described by the matrices,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & -3 \end{bmatrix}; \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}; \quad \mathbf{c} = [1 \ 2 \ 0].$$

Determine its transfer function.

8.8 The system matrices for the state variable representation of a system are,

$$\mathbf{A} = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}; \mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}; \quad \mathbf{c} = [1, 0].$$

Determine the complete state response and the output response of the system for the initial state

$$\mathbf{x}(0) = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

and a ramp input, applied at $t = 1$. Sketch the time response and also the state space trajectory.

8.9 Find the linear transformation for diagonalising the state matrix **A** of Problem 8.8. Obtain the normal state variable representation for Problem 8.8, using this linear transformation.

8.10 Prove the following properties of the state transition matrix:

(i) $\Phi^{-1}(t) = \Phi(-t)$

(ii) $\Phi(t_1 + t_2) = \Phi(t_1)\,\Phi(t_2)$.

8.11 Obtain the unit impulse response of a system with system matrices

$$\mathbf{A} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}; \mathbf{B} = \begin{bmatrix} 2 & -2 \\ 0 & -3 \end{bmatrix}; \mathbf{C} = \begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}; \mathbf{D} = \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}.$$

8.12 Develop a phase variable representation for the system shown in figure 8.7 (problem 8.5). The physical variable representation for this system has already been derived in Problem 8.5. Determine the linear transformation **Q** which relates these two representations.

CHAPTER 9

Discrete-time Systems

LEARNING OBJECTIVES

After studying this chapter you should be able to:

(i) model a discrete-time system by difference equation, by z-transfer function and by state variables,

(ii) solve the system model to obtain response for different types of common test signals.

Discrete-time systems were introduced in Chapter 2. Section 2.4 gave a brief description of discrete-time signals. A mathematical model for a computer controlled furnace in terms of a difference equation was also presented. This chapter develops the modelling and analysis techniques for discrete-time systems.

9.1 Discrete-time Signals

As opposed to a continuous-time signal which is defined for all instants of time, a discrete-time signal is defined only at discrete instants of time $t_1, t_2, ..., t_n, ...$ Therefore it is represented by the symbol $x(t_n)$ or simply $x(n)$, x being the magnitude of the signal at the n^{th} instant. Such signals occur in many engineering and non-engineering systems. A common example is the signal given out by a computer or any other digital system used in control and instrumentation applications. It may also result because of sampling of a continuous-time signal at discrete instants of time. The time interval between two successive discrete instants.

$$\Delta T = t_n - t_{n-1}$$

may in general be variable, but in this study we will assume it to be fixed.

A discrete-time signal may be defined for both positive and negative integral values of n, as shown in figure 9.1. However in the study of physical systems it is generally assumed that a signal starts after some specified instant, which is taken

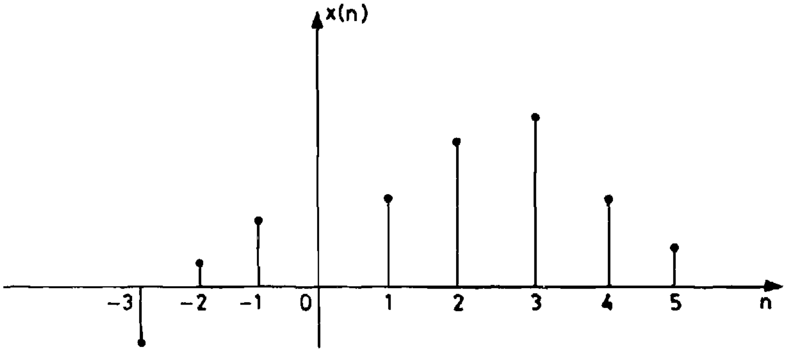


Fig. 9.1 Graphical representation of a discrete-time signal.

as instant $n = 0$. This fact is explicitly indicated by writing $x(n) = 0$ for $n < 0$. Unless stated otherwise this fact will be implicitly assumed in this study.

Figure 9.2 shows another kind of signal which can be characterised by discrete-time sequence $x(n)$. Actually it is a continuous-time signal whose values can change only at discrete instants of time. The value of $x(n)$ is the magnitude during the n^{th} time interval. Thus the discrete sequence $x(n)$ is $x(1)$, $x(2)$, $x(3)$, ... as shown in this figure.

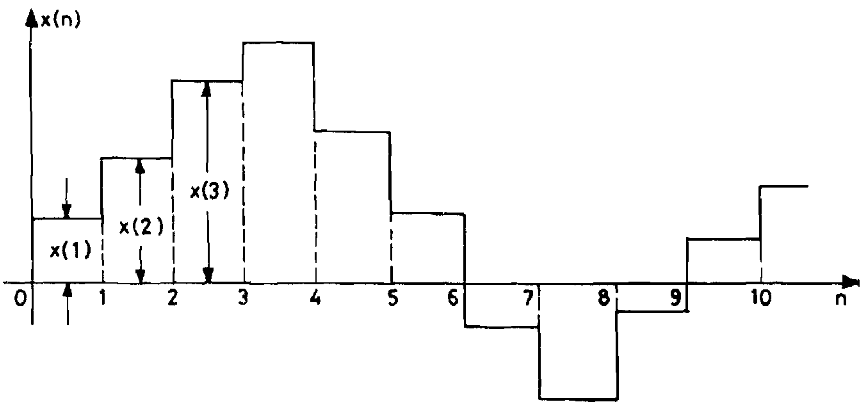


Fig. 9.2 A continuous-time signal modelled as discrete-time signal

Figure 9.3 illustrates graphically the effect of *shift* operations on a discrete time signal. A *left shift* by one step *advances* the signal by one step as shown by figure 9.2(b). Similarly a *right shift retards* the signal (figure 9.2(c)). Mathematically, if there is left shift by k steps, $x(n)$ is replaced by $x(n + k)$ and for a right shift, by $x(n - k)$.

The *transpose* of a discrete-time signal is obtained by replacing $-n$ for n in its mathematical description. The transpose of

$$x(\underline{n}) = e^{-5n}$$

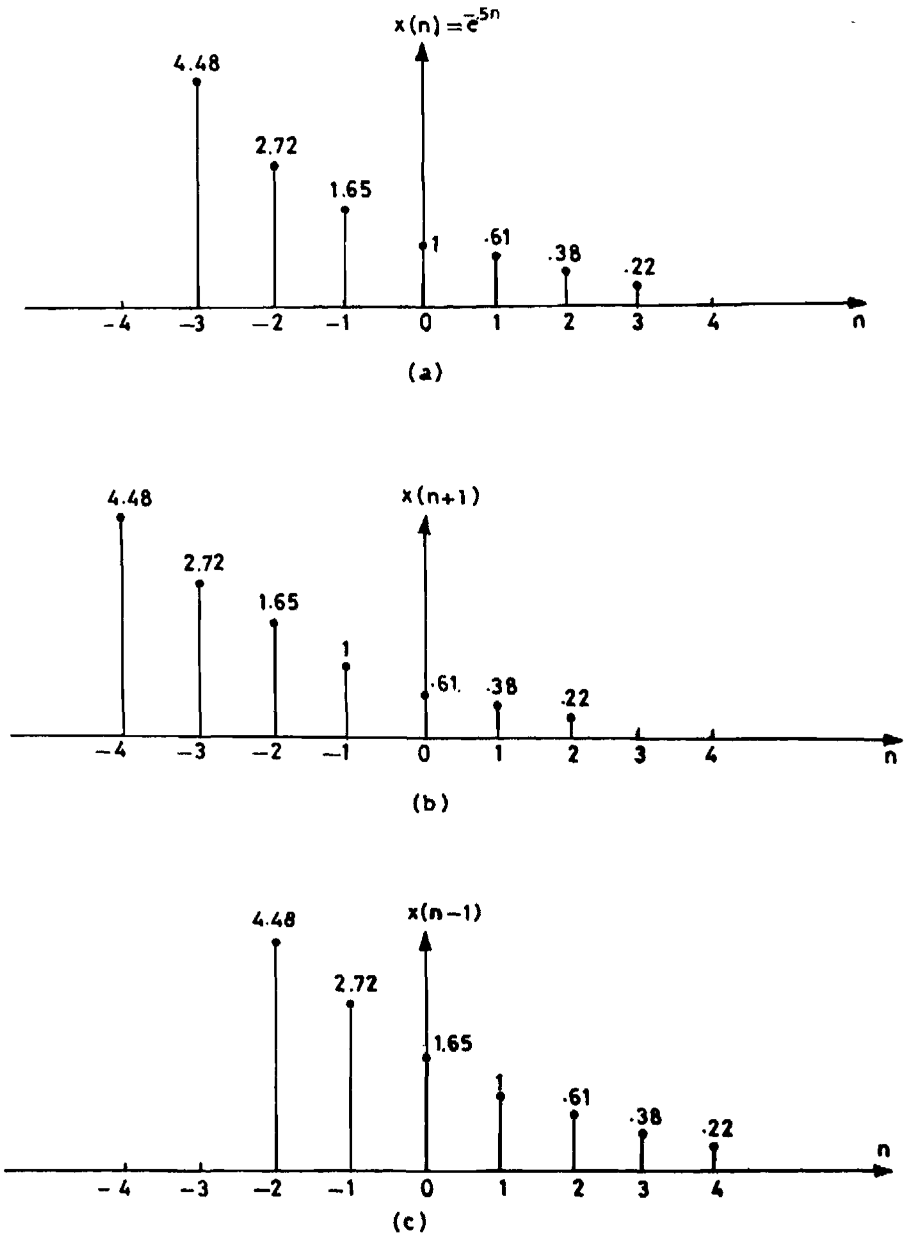


Fig. 9.3 Shift operation on a discrete-time signal;
(a) original signal $x(n) = e^{-.5n}$
(b) advance operation $x(n+1) = e^{-.5(n+1)}$
(c) retard operation $x(n-1) = e^{-.5(n-1)}$

is

$$x(-n) = e^{+.5n}$$

and is shown in figure (9.4).

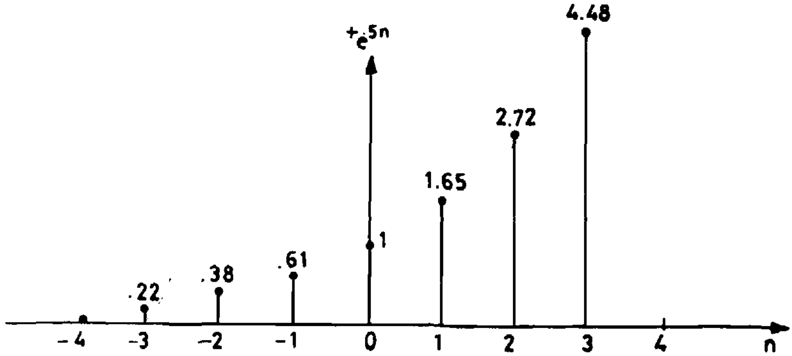


Fig. 9.4 Transpose of x (n) = $e^{-.5n}$

The integration operation of continuous-time signal $x(t)$ is equivalent to the summation operation for discrete-time signal $x(n)$. The result of summation is also a discrete-time signal and its value at any instant n is equal to,

$$\sum_{k=0}^{n} x(k) .$$

The difference operation corresponds to the differential operation on continuous-time signals. It is of two kinds,

forward difference $\Delta x(n) = x(n+1) - x(n)$

and backward difference $\nabla x(n) = x(n) - x(n-1)$.

9.2 Modelling of Discrete-time Systems

In a discrete-time system the input is a discrete-time signal $x(n)$ and the output another discrete-time signal $y(n)$, as shown in figure 9.5. In other words, the system receives a sequence $x(n) = \{x(1), x(2), \ldots, x(n), \ldots\}$ as input and processes it to produce another sequence $y(n) = \{y(1), y(2), \ldots, y(n), \ldots\}$ as the output.

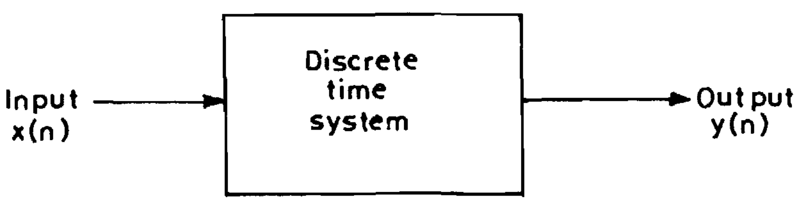


Fig. 9.5 Block diagram of a discrete-time system

A mathematical model of a discrete-time system in terms of difference equation was derived in section 2.4. We now consider some more examples of modelling engineering as well as non-engineering discrete-time systems using difference equations.

Example 9.1:— A bank pays an interest of a% per month on amounts standing for one full month. Part of a month and the date of deposit are not counted for interest calculation. A person makes regular deposits of Rs $x(n)$ in the nth month. What will be the standing amount to his credit at the beginning of a month?

Treating the bank account as a discrete-time system, its input is $x(n)$ and output $y(n)$, the standing amount at the beginning of a month. This amount will be a sum of three components: (i) standing amount at the beginning of the previous month, $y(n-1)$; (ii) interest for one month on $y(n-1)$; (iii) deposit during the $(n-1)^{th}$ month, $x(n-1)$. Writing it mathematically,

$$y(n) = y(n-1) + \frac{a}{100}\, y(n-1) + x(n-1)$$

Or, rearranging,

$$y(n) - \left(1 + \frac{a}{100}\right) y(n-1) = x(n-1) \tag{9.1}$$

This first order difference equation is the required mathematical model of the given system. We note that if we had started with a consideration of $y(n+1)$, the standing amount at the beginning of $(n+1)^{th}$ month, instead of $y(n)$, we would have got,

$$y(n+1) - \left(1 + \frac{a}{100}\right) y(n) = x(n) \tag{9.2}$$

as the mathematical model of the system. It is quite straight forward to see that equations (9.1) and (9.2) are equivalent.

Example 9.2:— A problem of pattern recognition.

A specified pattern, say 111, is said to occur at the nth digit in a binary sequence if the pattern is recognised when the nth digit is scanned, when scanning from left to right. For example, in the five digit sequence 10111, the pattern 111 is said to occur at the fifth digit. After one such successful recognition the search starts afresh. For example, in the eight digit sequence 11110111, the pattern 111 is recognised at the third and the eighth digits but not at the fourth. The problem is to find the number of n digit sequences $y(n)$ where the pattern 111 occurs at the nth place.

It is obvious that one and two digit binary sequences cannot contain the three digit pattern 111. Therefore, we have $y(1) = y(2) = 0$. There is only one three digit sequence in which 111 will be recognised at the third place. Therefore $y(3) = 1$. Similarly, 0111 is the only four digit sequence for which the given pattern will be recognised at the fourth place. Thus $y(4) = 1$. For $n = 5$ there are two sequences 00111 and 10111 where 111 will be recognised at the fifth place. Thus $y(5) = 2$. Proceeding on these line we can establish by direct enumeration that $y(6) = 5$, $y(7) = 9$, $y(8) = 18$. Thus the elements of sequence $y(n)$ are,

$$\begin{aligned} y(1) &= 0 \\ y(2) &= 0 \\ y(3) &= 1 \\ y(4) &= 1 \\ y(5) &= 2 \\ y(6) &= 5 \\ y(7) &= 9 \\ y(8) &= 18 \\ &\vdots \end{aligned}$$

The above pattern has the relationship,

$$\begin{aligned} y(3) + y(2) + y(1) &= 1 = 2^0 \\ y(4) + y(3) + y(2) &= 2 = 2^1 \\ y(5) + y(4) + y(3) &= 4 = 2^2 \\ &\vdots \end{aligned}$$

That is, for the n bit sequence we have the relation,

$$y(n) + y(n-1) + y(n-2) = 2^{n-3}, \; n = 3, 4, 5, \ldots \tag{9.3}$$

The solution of this second order difference equation will be the answer to our pattern recognition problem.

Example 9.3:— The ladder network shown in figure 9.6 is called on R-2R network because of its special topology. The network has $(r + 1)$ nodes, with node voltages ranging from v_0 to v_r. The problem is to find a general expression for determining the voltage v_n of the n^{th} node.

Applying Kirchhoff's current law at the node $v(n-1)$ we get,

$$\frac{v(n-2) - v(n-1)}{R} = \frac{v(n-1)}{2R} + \frac{v(n-1) - v(n)}{R}$$

Multiplying both sides by $2R$ and rearranging we get,

$$v(n) - \frac{5}{2} v(n-1) + v(n-2) = 0, \; n = 2, 3, \ldots, r \tag{9.4}$$

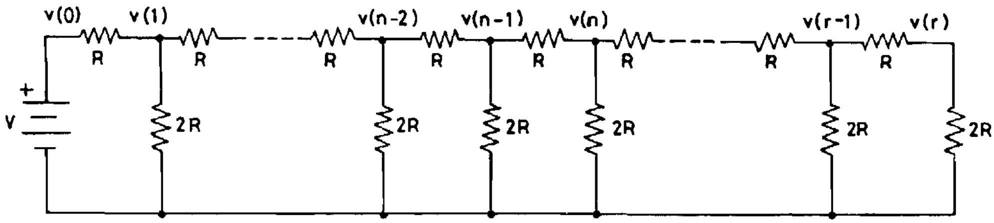


Fig. 9.6 A ladder network

For $n = 0$, $v(0) = V$, the given source voltage. To determine $v(1)$ we write the Kirchhoff's loop equation around the first loop to get,

$$3Ri = V \text{ or } i = \frac{V}{3R}$$

where i is the loop current. Now,

$$v(1) = 2Ri$$

Therefore,

$$v(1) = \frac{2}{3}V$$

The values of $v(0)$ and $v(1)$ are the initial conditions for the second order difference equation (9.4) whose solution will be the answer to the given problem.

9.3 Solution of Difference Equation

The method for solving difference equation is analogous to that for differential equation. Let us consider a first order difference equation without any forcing function,

$$y(n+1) - ay(n) = 0 \tag{9.5}$$

where a is a constant. This equation can also be written as,

$$\frac{y(n+1)}{y(n)} = a$$

which mean the ratio of two successive values of the sequence $y(n)$ is a constant. Such a sequence is a geometric sequence with ratio a. The value of an element in such a sequence is given by,

$$y(n) = Ka^n \tag{9.6}$$

where K is a constant. That (9.6) is the solution of (9.5) can be readily verified by direct substitution. To determine K, put $n = 0$ in (9.6). Then,

$$y(o) = K$$

Therefore the solution of the first order homogeneous difference equation (9.5) is,

$$y(n) = y(0)\ a^n. \tag{9.7}$$

Let us now include a forcing function of the form,

$$x(n) = c\ b^n$$

where c and b are constants. The difference equation then becomes,

$$y(n+1) = a\,y(n) + c\ b^n \tag{9.8}$$

The solution of equation (9.8) will be the sum of two components, the homogeneous solution or the unforced response given by (9.7) and a particular solution whose general form depends on the forcing function $x(n)$. For the given forcing function, the general form of the particular solution can be assumed as,

$$y_p(n) = P\ b^n \tag{9.9}$$

where P is another constant to be determined. Substituting (9.9) into (9.8) we get,

$$P\ b^{n+1} = a\ P\ b^n + c\ b^n$$

$\therefore$

$$P = \frac{c}{b-a}$$

and,

$$y_p(n) = \frac{c\ b^n}{b-a}\ . \tag{9.10}$$

The solution of (9.9) then is,

$$y(n) = K\ a^n + \frac{c\ b^n}{b-a}$$

For n = 0,

$$y(0) = K + \frac{c}{b-a}$$

$\therefore$

$$K = y(o) - \frac{c}{b-a}$$

Hence the complete solution is,

$$y(n) = \left\{ y(0) - \frac{c}{b-a} \right\} a^n + \frac{c\ b^n}{b-a} \tag{9.11}$$

Example 9.4:— Let us use the above result to solve the bank account problem of example 9.1. Let the interest rate be 1% per month and the monthly deposit Rs 100/-. With these values equation (9.1) becomes,

$$y(n) - 1.01\,y(n-1) = 100$$

which can also be written as,

$$y(n+1) = 1.01\,y(n) + 100.1^n \qquad (9.12)$$

Comparing (9.12) with (9.8) the parameters of the latter are,

$$a = 1.01,\ b = 1,\ c = 100$$

Further we see that $\quad y(o) = 0.$

Substituting these values in (9.11) we get the solution as,

$$y(n) = \frac{100}{0.01}(1.01)^n - \frac{100}{.01}$$

$$= 10{,}000\,[(1.01)^n - 1] \qquad (9.13)$$

Equation (9.12) can also be solved directly by calculating the values of $y(n)$ step-by-step.

For $\quad n = 0,\ y(1) = 1.01\ y(0) + 100 = 100$

$n = 1,\ y(2) = 1.01\ y(1) + 100$

$= 1.01 \times 100 + 100 = 201$

$n = 2,\ y(3) = 1.01\ y(2) + 100$

$= 1.01 \times 201 + 100 = 303.01$

and so on. Verify that the same values are given by the analytical solution (9.13) also.

We now study the method for solving difference equations of higher orders. The general form of a difference equation of order r can be written as,

$$y(n) + a_1\,y(n-1) + a_2\,y(n-2) + \ldots + a_{r-1}\,y(n-r+1) + a_r\,y(n-r)$$

$$= b_0\,x(n) + b_1\,x(n-1) + \ldots + b_{m-1}\,x(n-m+1) + b_m\,x(n-m) \qquad (9.14)$$

Since the input is applied at the instant $n = 0$, $x(n) = 0$ for $n < 0$. Also, in a causal system $m \leq r$.

Equation (9.14) means that the value of the output (or the response) at the nth instant of the system described by this equation, is dependent on the values of the output at $(n - r)$ previous instants, and the values of the input at the nth instant and at m previous instants.

Analogous to the differential equation the solution of the difference equation (9.14) also has two parts: the homogeneous or the complementary solution $y_c(n)$ and the particular solution $y_p(n)$. The complete solution is the sum of these two parts.

Homogeneous solution:— It is the solution of the homogeneous equation corresponding to (9.14), which is obtained by equating the right hand side to zero. That is,

$$y(n) + a_1 y(n-1) + \ldots + a_r y(n-r) = 0 \qquad (9.15)$$

Let us assume that the solution of equation (9.15) is of the form,

$$y(n) = A\alpha^n$$

where A is a constant and α some appropriately chosen complex number. Substituting this assumed solution in (9.15) we get,

$$A\alpha^n + a_1 A \alpha^{n-1} + \ldots + a_r A \alpha^{n-r} = 0$$

or

$$\alpha^n + a_1 \alpha^{(n-1)} + \ldots + a_r \alpha^{(n-r)} = 0$$

Taking out α^{n-r} as the common factor,

$$\alpha^{n-r} (\alpha^r + a_1 \alpha^{r-1} + \ldots + a_r) = 0$$

or

$$\alpha^r + a_1 \alpha^{r-1} + \ldots + a_r = 0 \qquad (9.16)$$

Equation (9.16) is called the *characteristic equation* of (9.15).

Let $\alpha_1, \alpha_2, \ldots, \alpha_r$ be the distinct roots of this characteristics equaton. Then the solution of (9.15) is given by,

$$y_c(n) = c_1 \alpha_1^n + c_2 \alpha_2^n + \ldots + c_r \alpha_r^n \qquad (9.17)$$

where $c_1, c_2, \ldots, c_r$ are constants which are determined by the given initial conditions.

Example 9.5:— Solve the homogeneous difference equation,

$$y(n) - y(n-1) - y(n-2) = 0$$

The corresponding characteristic equation is,

$$\alpha^2 - \alpha - 1 = 0.$$

Its roots are,

$$\alpha_1 = \frac{1+\sqrt{5}}{2} \text{ and } \alpha_2 = \frac{1-\sqrt{5}}{2}.$$

Therefore the solution is,

$$y(n) = c_1 \left[\frac{1+\sqrt{5}}{2}\right]^n + c_2 \left[\frac{1-\sqrt{5}}{2}\right]^n$$

where the unknown constants c_1 and c_2 can be found from the initial conditions.

The form of solution is different if the characteristic equation has repeated roots. Let α_1 be a root of multiplicity k. The term corresponding to α_1 in the solution will then have the form,

$$c_1 \alpha_1^n + c_2 n \alpha_1^n + c_3 n^2 \alpha_1^n + \ldots + C_k n^{k-1} \alpha_1^n.$$

Example 9.6:— Solve the homogeneous equation,

$$y(n) - 3y(n-2) - 2y(n-3) = 0.$$

The corresponding characteristic equation is,

$$\alpha^3 - 3\alpha - 2 = 0$$

Factorising,

$$(\alpha + 1)^2 (\alpha - 2) = 0.$$

The roots are -1, -1 and 2. Thus the root -1 is of multiplicity two. Therefore the solution is,

$$y(n) = c_1 (-1)^n + c_2 n(-1)^n + c_3 (2)^n$$

$$= (c_1 + c_2 n)(-1)^n + c_3 (2)^n.$$

Compare this example with example 3.1 of Chapter 3 to see the similarity in the solutions of differential and difference equations.

Particular Solution:— The form of particular solution $y_p(n)$ depends on the forcing function $x(n)$. The form of $y_p(n)$ to be assumed is given in table 9.1 for some simple cases.

Table 9.1

Forcing function $x(n)$	Form of particular solution $y_p(n)$
n^k	$A_0 + A_1 n + A_2 n^2 + \ldots + A_k n^k$
a^n	$A a^n$, if a is not a root of the characteristic equation.
	$A_1 n a^n + A_2 a^n$, if a is a distinct root of the characteristic equation.
	$A_1 n^2 a^2 + A_2 n a^n + A_3 a^n$, if a is a root of multiplicity 2, and so on.

The values of the unknown constants $A_0, A_1, \ldots$ etc. in the assumed form of $y_p(n)$ are determined by substituting the assumed expression in the given equation.

Complete Solution:— It is obtained by adding the complementary and the particular solutions and then finding the unknown constants $c_1, c_2, \ldots$ etc. from a knowledge of given initial or boundary conditions. The process is illustrated with the help of the following example.

Example 9.7:— Solve the difference equation,

$$y(n) + 3y(n-1) + 2y(n-2) = u(n)$$

with initial conditions $y(0) = 1$ and $y(-1) = 0$.

The character equation is,

$$\alpha^2 + 3\alpha + 2 = 0$$

or,

$$(\alpha + 1)(\alpha + 2) = 0$$

Therefore the roots of the characteristic equation are,

$$\alpha_1 = -1 \text{ and } \alpha_2 = -2.$$

Then the homogeneous solution is,

$$y_c(n) = c_1(-1)^n + c_2(-2)^n.$$

The forcing function $u(n)$, a discrete-time unit step function, can be written as,

$$u(n) = n^0.$$

Therefore the form of particular solution is,

$$y_p(n) = A_0$$

Substituting it in the given equation,

$$A_0 + 3A_0 + 2A_0 = 1$$

or,

$$A_0 = \frac{1}{6}\cdot$$

The form of complete solution is,

$$y(n) = y_c(n) + y_p(n)$$

$$= c_1(-1)^n + c_2(-2)^n + \frac{1}{6}$$

At $n = 0$,

$$y(0) = c_1 + c_2 + \frac{1}{6} = 1$$

At $n = -1$,

$$y(-1) = c_1(-1)^{-1} + c_2(-2)^{-1} + \frac{1}{6} = 0$$

Solving the above two simultaneous equations we get,

$$c_1 = -\frac{1}{2} \text{ and } c_2 = \frac{4}{3}$$

The complete solution is therefore,

$$y(n) = -\frac{1}{2}(-1)^n + \frac{4}{3}(-1)^n + \frac{1}{6} \cdot$$

9.4 Discrete-time Convolution

Convolution is a powerful technique for finding the response of a linear, discrete-time system to any arbitrary input. Application of this technique for continuous-time systems was given in section 5.4. On parallel lines we develop the convolution technique for discrete-time systems.

A *unit function* or *delta function* $\delta(n)$, analogous to the impulse function for continuous time systems, is defined as,

$$\begin{aligned} \delta(n) &= 1 \text{ for } n = 0 \\ &= 0 \text{ for } n \neq (0) \end{aligned} \tag{9.18}$$

A delta function occurring at instant j and having a magnitude a_j will be represented by,

$$\begin{aligned} a_j\ \delta(n-j) &= a_j \text{ for } n = j \\ &= 0 \text{ for } n \neq j \end{aligned}$$

Let the response of a discrete system to the unit delta function $\delta(n)$ be represented by the sequence $h(n)$. Then its response to $a_j\ \delta\ (n - j)$ will be $a_j\ h\ (n-j)$, i.e. $h(n)$ is shifted to the right by j instants and its magnitude multiplied by a_j.

Now, let $x(j)$ be a discrete-time sequence. It is easy to see that $x(j)$ is a train of delta functions. Since superposition is applicable for linear systems, the response $y(n)$ of the system to $x(j)$ will be the sum of responses due to the individual elements of this train. In other words,

$$y(n) = \sum_{j=0}^{\infty} x(j)\, h(n-j)\,. \tag{9.19}$$

Relation (9.19) is called the *discrete-time convolution* or the *convolution sum* between the input sequence $x(j)$ and the system's response $h(n)$, and is symbolically represented as,

$$y(n) = x(j) * h(n)\,. \tag{9.20}$$

Example 9.8:— The delta response of a system is given by,

$$h(n) = \frac{1}{3^n} \cdot$$

Determine its response to an input sequence $x(j)$ described by,

$$x(j) = 1 \text{ for } j = 0, 1, 2, 3, \text{ and}$$
$$= 0 \text{ for } j \geq 4.$$

Applying the convolution technique, the response is given by,

$$y(n) = \sum_{j=0}^{\infty} x(j)\, h(n-j)\,.$$

$$= x(0)\, h(n) + x(1)\, h(n-1) + x(2)\, h(n-2) + x(3)\, h(n-3)$$

$$= h(n) + h(n-1) + (n-2) + h(n-3)\,.$$

From this expression, the response can be calculated sequentially starting from $n = 0$. It should be noted that in this step-by-step calculation $h(n) = 0$ for $n < 0$. Proceeding in this manner we get,

$$y(0) = h(0) = \frac{1}{3^0} = 1$$

$$y(1) = h(1) + h(0) = \frac{1}{3} + 1 = \frac{4}{3}$$

$$y(2) = h(2) + h(1) + h(0) = \frac{1}{3^2} + \frac{1}{3} + 1 = \frac{13}{9}$$

$$y(3) = h(3) + h(2) + h(1) + h(0) = \frac{1}{3^3} + \frac{1}{3^2} + \frac{1}{3} + 1 = \frac{40}{27}$$

$$y(4) = h(4) + h(3) + h(2) + h(1) = \frac{1}{3^4} + \frac{1}{3^3} + \frac{1}{3^2} + \frac{1}{3} = \frac{40}{81}$$

$$y(5) = h(5) + h(4) + h(3) + h(2) = \frac{1}{3^5} + \frac{1}{3^4} + \frac{1}{3^3} + \frac{1}{3^2} = \frac{40}{243}$$

and so on. It is easy to develop an algorithm for computer solution. A plot of $h(n)$, $x(j)$ and $y(n)$ is shown in figure 9.7.

It should be noted that the delta response is defined for an initially relaxed system only, i.e. for a system with zero initial conditions. Therefore the response obtained by the convolution technique is the zero-state response. If the system has initial conditions, the response due to them alone is obtained by solving the systems homogeneous difference equation, as discussed earlier. This response is called the zero-input response. The complete response is the sum of zero-state and zero-input responses.

Application of the convolution technique requires a knowledge of the systems delta response. This response could be determined experimentally. The response to any arbitrary input can then be determined by either numerical or graphical

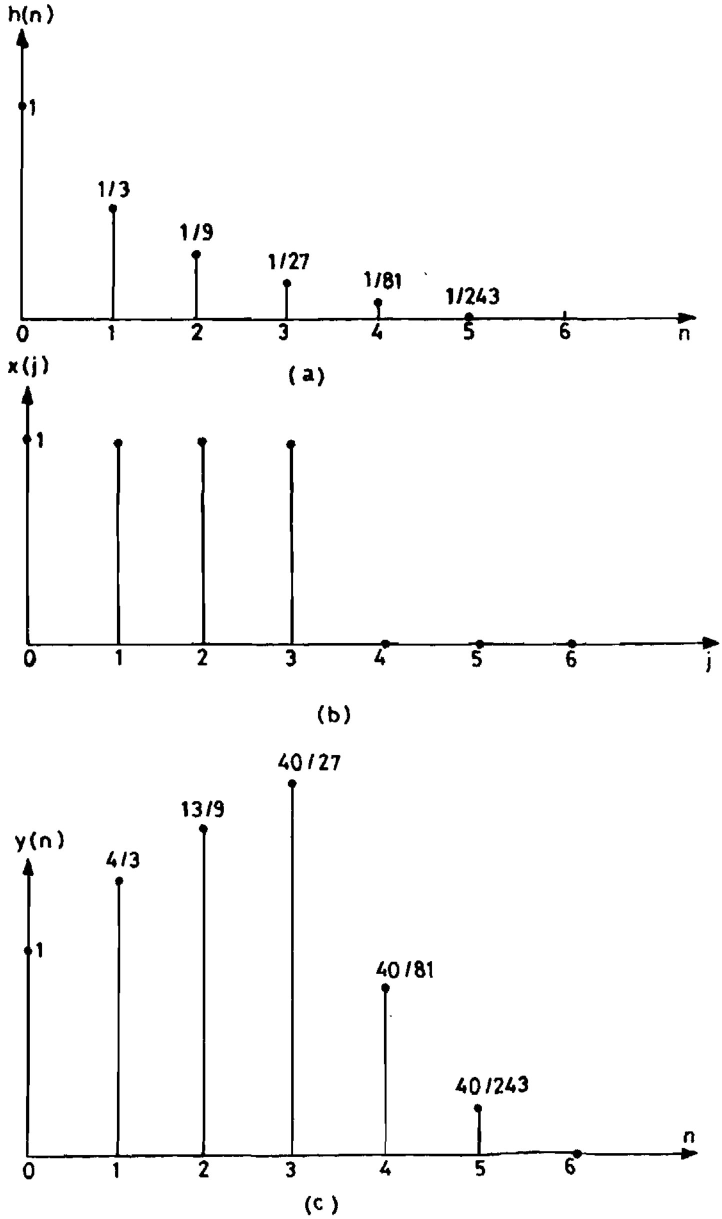


Fig. 9.7: (a) Delta response of the system, $h(n)$ (b) given input $x(j)$
(c) output $y(n) = x(j) * h(n)$

convolution. However if the systems model is given mathematically, its delta response needs to be derived analytically. For a first order system described by its difference equation model this derivation is straight-forward. Let the system model be,

$$y(n) = a\ y(n-1) + x(n),\ y(n) = 0 \text{ for } n \leq 0.$$

For obtaining its delta response, input *x(n)* is replaced by the unit delta function $\delta(n)$. The output then becomes the delta response $h(n)$. The system equation can then be written as,

$$h(n) = a\ h(n-1) + \delta(n).$$

Solving this equation sequential, starting with n = 0 we get,

$$h(0) = a\,h(-1) + \delta(0) = a.0 + 1 = 1$$

$$h(1) = a\,h(0) + \delta(1) = a.1 + 0 = a$$

$$h(2) = a\,h(1) + \delta(2) = a.a + 0 = a^2$$

Thus,

$$h(n) = a^n.$$

The derivation of delta response of higher order systems is easier if we use the z-transform technique. This technique is illustrated by example 9.11 in section 9.7.

9.5 The z-Transform

The z-transform is as powerful a tool of analysis for discrete-time systems as Laplace transform is for continuous-time systems. The z-transform of a discrete-time sequence *f(n)* is defined as,

$$Z\,f(n) = F(z) = f(0) + f(1)\,z^{-1} + f(2)\,z^{-2} + \ldots$$

where z is a complex variable. The above relation can also be written as,

$$F(z) = \sum_{n=0}^{\infty} f(n)\,z^{-n} \tag{9.21}$$

Compare this definition for z-transform with that of Laplace transform,

$$L\,f(t) = F(s) = \int_0^{\infty} f(t)\,e^{-st}\,dt,$$

as given by equation (6.3).

If the lower limit of summation in (9.21) is made $n = -\infty$ we get the double-sided z-transform. In physical systems the signal is applied at some specified instant which is treated as $n = 0$ and it is assumed that $f(n) = 0$ for $n < 0$. Therefore we will discuss here only the single-sided z-transform as defined by (9.21).

z-Transform of some common discrete sequences:—

1. *Delta function:*

It is defined as,

$$\delta(n) = 1 \text{ for } n = 0$$
$$= 0 \text{ for } n \neq 0$$

Using definition (9.21),

$$Z\,\delta(n) = \sum_{n=0}^{\infty} \delta(n)\, z^{-n}$$
$$= \delta(0)\, z^{0} + \delta(1)\, z^{-1} + \delta(2)\, z^{-2} + \ldots$$
$$= 1\,.$$

2. *Discrete unit step:*

It is defined as, $u(n) = 1$ for $n \geq 0$

$= 0$ for $n < 0$.

Using definition (9.21),

$$Z\, u(n) = U(z) = \sum_{n=0}^{\infty} u(n)\, z^{-n} = 1 + \frac{1}{z} + \frac{1}{z^2} + \ldots \qquad (9.22)$$

The infinite series (9.22) converges if $|z| > 1$, and in that case it can be written in the closed form as,

$$U(z) = \frac{1}{1 - z^{-1}} \qquad (9.23a)$$

$$= \frac{z}{z - 1} \qquad (9.23b)$$

Since z is a complex variable, the region of convergence of $U(z)$ given by (9.23) is represented graphically in the z-plane as the region outside the unit circle, as shown in figure 9.8.

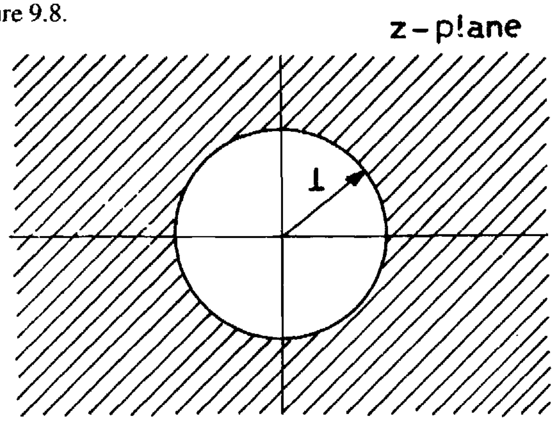


Fig. 9.8 Region of convergence of the z-transform for the discrete unit step function.

3. *Geometric sequence:*

Let $$f(n) = \left(\frac{1}{a}\right)^n \text{ for } n \geq 0$$

$$= 0 \text{ for } n < 0$$

Its z-transform is,

$$F(z) = \sum_{n=0}^{\infty} \left(\frac{1}{a}\right)^n z^{-n}$$

$$= 1 + \frac{1}{az} + \frac{1}{(az)^2} + \ldots$$

$$= \frac{1}{1 - \frac{1}{az}}$$

$$= \frac{az}{az - 1}$$

$$= \frac{z}{z - \frac{1}{a}} \text{ for } |z| > \frac{1}{a}$$

For $f(n) = a^n$, this gives,

$$F(z) = \frac{z}{z-a} = \frac{1}{1 - az^{-1}}, \ |z| > a.$$

4. *Function* $f(n) = \dfrac{a^n}{n!}$

Then, $$F(z) = \sum_{n=0}^{\infty} \frac{a^n}{n!} z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{a}{z}\right)^n$$

$$= 1 + \frac{a}{z} + \frac{1}{2!}\left(\frac{a}{z}\right)^2 + \ldots.$$

$$= \exp\left(\frac{a}{z}\right), \ |z| > 0$$

In this case the region of convergence is entire z-plane except the origin.

Properties of z-tranforms

1. *Linearity:*— It is straightforward to show that the z-transform is a linear operator. Thus if,

$$f_1(n) \leftrightarrow F_1(z) \text{ and } f_2(n) \leftrightarrow F_2(z)$$

are two z-transform pairs, then,

$$\left[A f_1(n) + B f_2(n)\right] \leftrightarrow \left[A F_1(z) + B F_2(z)\right] \tag{9.24}$$

2. *Right shift:*— Let a sequence $f(n)$ with z-transform $F(z)$ be shifted one step to the right. The shifted signal is represented by $f(n-1)$. The z-transform of the shifted signal is,

$$\begin{aligned} Z f(n-1) &= \sum_{n=0}^{\infty} f(n-1)\, z^{-n} \\ &= f(-1) + f(0) z^{-1} + f(1) z^{-2} + f(2) z^{-3} + \ldots \\ &= f(-1) + z^{-1}\left[f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + \ldots\right] \\ &= z^{-1} F(z) + f(-1). \end{aligned}$$

If $f(-1) = 0$, as will be the case if $f(n) = 0$ for < 0, then,

$$Zf(n-1) = z^{-1} F(z)$$

Generalising this result for right shift by k steps,

$$\begin{aligned} Zf(n-k) &= z^{-k} F(z) + z^{-k+1} f(-1) + z^{-k+2} f(-2) + \ldots . \\ &\quad + z^{-1} f(-k+1) + f(-k) \end{aligned} \tag{9.25}$$

and if $f(n) = 0$ for $n < 0$.

$$Z f(n-k) = z^{-k} F(z) \tag{9.26}$$

3. *Left shift:*—Let $F(z)$ be the z-transform of $f(n)$. Then the z transform of $f(n+1)$ is,

$$\begin{aligned} Z f(n+1) &= \sum_{n=0}^{\infty} f(n+1)\, z^{-n} \\ &= f(1) + f(2) z^{-1} + f(3)\, z^{-2} + \ldots . \\ &= z\left[f(1)\, z^{-1} + f(2) z^{-2} + f(3)\, z^{-3} + \ldots\right] \\ &= z[F(z) - f(0)] \end{aligned}$$

If the shift is k steps to the left,

$$Zf(n+k) = z^k\, [F(z) - f(0) - z^{-1} f(1) \ldots - z^{-(k-1)} f(k-1)] \tag{9.27}$$

4. *Initial value and final value theorems:—*

In the defining equation,

$$F(z) = \sum_{n=0}^{\infty} f(n)\, z^{-n}$$

$$= f(0) + f(1)\, z^{-1} + f(2)\, z^{-2} + \ldots$$

if $z \to \infty$, then all other terms of the r.h.s. except the first become zero.

Therefore,

$$f(0) = \lim_{z \to \infty} F(z) \tag{9.28}$$

This is the initial value theorem for the discrete-time function.

To derive the more useful final value theorem consider the expression,

$$\sum_{n=0}^{\infty} [f(n) - f(n-1)]\, z^{-n},\ f(n) = 0 \text{ for } n < 0$$

Using the right shift theorem we can write,

$$\sum_{n=0}^{\infty} [f(n) - f(n-1)]\, z^{-n} = F(z) - z^{-1} F(z) = (1 - z^{-1})\, F(z) \tag{9.29}$$

If we expand the l.h.s. of (9.29) and let $z \to 1$, all other terms will cancel out except $f(\infty)$. Therefore we have,

$$f(\infty) = \lim_{z \to 1} (1 - z^{-1})\, F(z) \tag{9.30}$$

which is the final value theorem. Equation (9.30) can also be written as,

$$f(\infty) = \lim_{z \to 1} (z - 1)\, F(z) \tag{9.31}$$

For (9.30) or (9.31) to be meaningful the indicated limits must exist. Otherwise the final value theorem is not applicable.

5. *Multiplication by n:—* Let $f(n) \leftrightarrow F(z)$ be a z-transform pair. If $f(n)$ is multiplied by n, the z-transform of the product is given by,

$$Z\, n f(n) = \sum_{n=0}^{\infty} n f(n)\, z^{-n}$$

$$= -z \sum_{n=0}^{\infty} f(n)\, (-n\, z^{(-n-1)})$$

The term $-n\, z^{(-n-1)}$ is the derivative of z^{-n}. Therefore,

$$Z\, n f(n) = -z \sum_{n=0}^{\infty} f(n) \frac{d}{dz} (z^{-n})$$

$$= -z \frac{d}{dz} \left[\sum_{n=0}^{\infty} f(n)\, z^{-n} \right]$$

$$= -z \frac{d}{dz} F(z)$$

Example 9.8:— Find the z-transform of a discrete ramp function with unit slope.

A discrete ramp is obtained by multiplying a discrete unit step function by n. That is,

$$f(n) = n\, u(n).$$

Then

$$Z f(n) = Z\, n\, u(n) = -z \frac{d}{dz} U(z)$$

Since,

$$U(z) = \frac{1}{1 - z^{-1}}$$

therefore,

$$Z\, f(n) = -z \frac{d}{dz} \left(\frac{1}{1 - z^{-1}} \right)$$

$$= \frac{z^{-1}}{(1 - z^{-1})^2}$$

$$= \frac{z}{(z - 1)^2}$$

Example 9.10:— Find the z-transform of a discrete sinusoidal signal.

Let the exponential signal,

$$f(t) = e^{j \omega t} = \cos \omega t + j \sin \omega t, \; t \geq 0$$

be sampled with a sampling interval T. The sampled sequence is,

$$f(n) = e^{j \omega n T} = [e^{j \omega T}]^n$$

Treating it as geometric sequence of form a^n,

$$Z\, [e^{j \omega T}]^n = \frac{z}{z - e^{j \omega T}}$$

$$= \frac{z}{z - (\cos \omega T + j \sin \omega T)}$$

$$= \frac{z}{(z - \cos \omega T) - j \sin \omega T}$$

$$= \frac{z^2 - z \cos \omega T}{z^2 - 2z \cos \omega T + 1} - j \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$$

Expanding the l.h.s. of the above equation,

$$Z [\cos \omega n T + j \sin \omega n T] = \frac{z^2 - z \cos \omega T}{z^2 - 2z \cos \omega T + 1} - j \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}.$$

Equating the real and the imaginary parts we get,

$$Z [\cos n \omega T] = \frac{z^2 - z \cos \omega T}{z^2 - 2z \cos \omega T + 1}$$

and

$$Z [\sin n \omega T] = \frac{z \sin \omega T}{z^2 - 2z \cos \omega T + 1}$$

A list of z-transforms for commonly used discrete-time functions is given in table 9.2

Table 9.2: Commonly used z-Transform Pairs

	Function $f(n)$, $n \geq 0$		z-Transform $F(z)$
1.	Unit delta	$\delta(n)$	1
2.	Unit step	$u(n)$	$\frac{z}{z-1}$ or $\frac{1}{1-z^{-1}}$
3.	Unit ramp	$n\,u(n)$	$\frac{z}{(z-1)^2}$ or $\frac{z^{-1}}{(1-z^{-1})^2}$
4.		a^n	$\frac{z}{z-a}$ or $\frac{1}{1-az^{-1}}$
5.		$n\,a^n$	$\frac{az}{(z-a)^2}$
6.		$(n+1)\,a^n$	$\frac{z^2}{z-a}$
7.		$\frac{a^n}{n!}$	$\exp\left(\frac{a}{z}\right)$
8.		$\sin \alpha n$	$\frac{z \sin \alpha}{z^2 - 2z \cos \alpha + 1}$
9.		$\cos \alpha n$	$\frac{z^2 - z \cos \alpha}{z^2 - 2z \cos \alpha + 1}$

9.6 The z-Transfer Function

As discussed in section 9.4, if $h(n)$ is the delta response of a discrete-time system, then its response to any input $x(n)$ is given by the convolution sum (9.19). That is,

$$y(n) = \sum_{j=0}^{\infty} x(j)\, h(n-j).$$

The z-transform of $y(n)$ is,

$$Y(z) = \sum_{n=0}^{\infty} y(n)\, z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[\sum_{j=0}^{\infty} x(j)\, h\,(n-j) \right] z^{-n}$$

$$= \sum_{n=0}^{\infty} z^{-n} \left[\sum_{j=0}^{\infty} x(j)\, h(n-j) \right]$$

Interchanging the order of summation,

$$Y(z) = \sum_{j=0}^{\infty} x(j) \sum_{n=0}^{\infty} z^{-n}\, h(n-j) \tag{9.32}$$

From the shifting property (9.26), the second summation in the above equation is,

$$\sum_{n=o}^{\infty} z^{-n}\, h(n-j) = Z h(n-j)$$

$$= z^{-j}\, Z h(n)$$

$$= z^{-j}\, H(z)$$

where $H(z)$ is the z-transform of $h(n)$.

Substituting this result in (9.32) we get,

$$Y(z) = \sum_{j=0}^{\infty} x(j)\, z^{-j}\, H(z)$$

$$= X(z) \cdot H(z) \tag{9.33}$$

Equation (9.33) means that the convolution of two discrete-time sequences results in the multiplication of their z-transforms in the z-domain. Similar result for the Laplace transform of continuous-time signals is given by equation (6.27).

From equation (9.33) we have,

$$H(z) = \frac{Y(z)}{X(z)} \tag{9.34}$$

$H(z)$ is called the *z-transfer function* of the discrete-time system. As seen from (9.34), it is the ratio of the z-transform of the output to the z-transform of the input. It has the same important role as the transfer function in the Laplace transform analysis of continuous-time linear systems. Note that $H(z)$ is the z-transform of the delta response $h(n)$ of an initially relaxed system, i.e., a system with zero initial conditions. Hence the z-transfer function is a mathematical model of an initially relaxed system. From a knowledge of this model the output of the system to any input can be obtained by using the relation (9.33) and then taking the z-inverse to obtain $y(n)$.

Determination of the z-transfer function:—

The initial analysis of a given discrete-time system usually results in its difference equation model. To use the more convenient z-transfer function method of analysis we need to derive the system's z-transfer function from this difference equation model.

Let the system be described by a general r^{th} order difference equation,

$$y(n) + a_1 y(n-1) + \dots + a_r y(n-r) = b_0 x(n)$$

$$+ b_1 x(n-1) + \dots + b_m x(n-m) \tag{9.35}$$

with $m \leq r$. Now we take the z-transform of both sides of equation (9.35). To do so we recall the shifting property (9.26),

$$Zf(n-k) = z^{-1} F(z)$$

where $F(z)$ is the z-transform of $f(n)$. Using this property we get,

$$[1 + a_1 z^{-1} + \dots + a_r z^{-r}]\ Y(z) = [b_0 + b_1 z^{-1} + \dots + b_m z^{-m}]\ X(z)$$

or

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \dots + b_m z^{-m}}{1 + a_1 z^{-1} + \dots + a_r z^{-r}} \tag{9.36}$$

Equation (9.36) gives s straightforward method of obtaining the z-transfer function from the given difference equation description of a system.

9.7 Analysis with z-Transform

The use of z-transform and the z-transfer function for finding the response of discrete-time systems is illustrated by the following example.

Example 9.11:— The z-transfer function of a system is given as,

$$H(z) = \frac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}}.$$

Find its (i) delta response and (ii) discrete unit-step response with zero initial conditions.

(i) Delta response:— The output is given by the relation,

$$Y(z) = X(z) \cdot H(z)$$

If the input is a delta function then $X(z) = 1$ from table 9.2, and

$$Y(z) = H(z) = \frac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

To obtain,

$$h(n) = Z^{-1}\ Y(z)$$

we expand *Y(z)* in partial fractions as,

$$H(z) = \frac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}} = \frac{1}{\left(z - \frac{1}{2}\right)\left(z - \frac{1}{4}\right)}$$

$$= \frac{4}{z - \frac{1}{2}} - \frac{4}{z - \frac{1}{4}}$$

To find the inverse z-transform of terms like these we note that,

$$\frac{1}{z-a} = z^{-1}\ \frac{z}{z-a}.$$

The inverse z-transform of the second factor on the r.h.s. is a^n and multiplication by z^{-1} means it is shifted one step to the right. Therefore,

$$Z^{-1}\ \frac{1}{z-a} = a^{n-1}.$$

Applying this result we get,

$$h(n) = Z^{-1}\ H(z) = 4\left(\frac{1}{2}\right)^{n-1} - 4\left(\frac{1}{4}\right)^{n-1}$$

(ii) For the discrete unit-step function, table 9.2 gives,

$$Z\,u(n) = \frac{z}{z-1}$$

Therefore,

$$Y(z) = \frac{z}{z-1} \cdot \frac{1}{\left(z-\frac{1}{2}\right)\left(z-\frac{1}{4}\right)}$$

$$= \frac{Az}{z-1} + \frac{Bz}{z-\frac{1}{2}} + \frac{Cz}{z-\frac{1}{4}}$$

Multipying both sides $(z\text{-}1)$ and then putting $z = 1$,

$$A = \frac{1}{\left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{4}\right)} = \frac{8}{4}$$

Similarly,

$$B = -8, \text{ and } C = \frac{16}{3}$$

Therefore,

$$Y(z) = \frac{8}{3}\frac{z}{z-1} - 8\frac{z}{z - \frac{1}{2}} + \frac{16}{3}\frac{z}{z - \frac{1}{4}}$$

Taking inverse z-transform, using table 9.2, we get the required solution,

$$y(n) = \frac{8}{3}\,u(n) - 8\left(\frac{1}{2}\right)^n + \frac{16}{3}\left(\frac{1}{4}\right)^n.$$

Example 9.12:— Find the unit-step response in the previous example with initial conditions $y(0) = 0,\ y(1) = 1$.

Since the z-transfer function is defined with zero initial conditions, we first convert the systems z-transfer function model into difference equation model to include the effect of initial conditions. The given model is,

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{z^2 - \frac{3}{4}z + \frac{1}{8}}$$

or

$$\left[z^2 - \frac{3}{4}z + \frac{1}{8}\right] Y(z) = X(z) = \frac{z}{z-1}$$

Taking inverse z-transform of both the sides,

$$y(n+2) - \frac{3}{4}\,y(n+1) + \frac{1}{8}\,y(n) = u(n)$$

Now, to find its z-transform with non-zero initial conditions we use equation (9.27) to get,

$$[z^2 Y(z) - z^2 y(0) - z\,y(1)] - \frac{3}{4}\,[z\,Y(z) - z\,y(0)] + \frac{1}{8}\,Y(z) = \frac{z}{z-1}\,.$$

Regrouping,

$$\left[z^2 - \frac{3}{4}z + \frac{1}{8}\right] Y(z) = \left[(z^2 - \frac{3}{4}z)\,y(0) + z\,y(1)\right] + \frac{z}{z-1}$$

or, $$Y(z) = \frac{(z^2 - \frac{3}{4}z)\,y(0) + z\,y(1)}{z^2 - \frac{3}{4}z + \frac{1}{8}} + \frac{z}{(z-1)\left(z^2 - \frac{3}{4}z + \frac{1}{8}\right)}$$

$$= Y_1(z) + Y_2(z)$$

$Y_1(z)$ is the zero-input response due to initial conditions alone and $Y_2(z)$ the zero-state response due to forcing function alone. Substituting the given values of initial conditions $y(0)$ and $y(1)$ we get,

$$Y_1(z) = \frac{z}{z^2 - \frac{3}{4}z + \frac{1}{8}} = \frac{4z}{\left(z-\frac{1}{2}\right)} - \frac{4z}{\left(z-\frac{1}{4}\right)}$$

Taking inverve z-transform,

$$y_1(n) = Z^{-1}\,Y_1(z) = 4\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{4}\right)^n.$$

The inverse z-transform of $y_2(z)$ is already found in the previous example as,

$$y_2(n) = \frac{8}{3}u(n) - 8\left(\frac{1}{2}\right)^n + \frac{16}{3}\left(\frac{1}{4}\right)^n.$$

Adding these two compcnents,

$$y(n) = y_1(n) + y_2(n)$$

$$= \left[4\left(\frac{1}{2}\right)^n - 4\left(\frac{1}{4}\right)^n\right] + \left[\frac{8}{3}u(n) - 8\left(\frac{1}{2}\right)^n + \frac{16}{3}\left(\frac{1}{4}\right)^n\right]$$

Regrouping,

$$y(n) = \frac{8}{3}u(n) - 4\left(\frac{1}{2}\right)^n + \frac{4}{3}\left(\frac{1}{4}\right)^n.$$

9.8 State-Variable Description

We have seen the advantages of state-variable technique for the analysis of continuous-time systems in chapter 8. This technique is even more advantageous for discrete-time systems. One reason for this advantage is that many discrete-time systems can be directly modelled in terms of state-variable description, as illustrated by the following example.

Example 9.13:— Let $x_1(n)$ be the number of small fish in an area during the nth time interval. In the absence of any predator fish, their rate of growth is dependent on their number and the amount of food available. The small fish population can then be described as,

$$x_1(n+1) = a\,x_1(n) + b\,F \tag{9.37}$$

where, F = amount of food available

and a, b = constants.

Let us now assume that the same area also has predator fish which eat the small fish. Let their number be $x_2(n)$ during the n[th] time interval. The biological law that the rate of growth of the species is dependent on their number and the amount of available food, is applicable to the predator fish also. Since the small fish is the food for them, the predator fish population can be described as,

$$x_2(n+1) = c\,x_2(n) + d\,x_1(n) \tag{9.38}$$

where c, d are constants.

The presence of predator fish will reduce the small fish population, the reduction being proportional to the number of predator fish. Therefore equation (9.37) is altered as,

$$x_1(x+1) = a\,x_1(n) - e\,x_2(n) + b\,F \tag{9.39}$$

Let us further assume that the economic worth of the total fish population $y(n)$ is given by,

$$y(n) = g\,x_1(n) + h\,x_2(n) \tag{9.40}$$

Treating $x_1(n)$ and $x_2(n)$ as the state-variables and $y(n)$ as the output, the state-variable description of the system is,

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} a & -e \\ d & c \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} b \\ o \end{bmatrix}^F$$

$$y(n) = [g \;\; h] \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}$$

In general, the output expression may contain a term directly proportional to the input. Including this generalisation, the standard form of state-variable representation for an r[th] order discrete-time system is written as,

$$\mathbf{x}(n+1) = \mathbf{A}\;\mathbf{x}(n) + \mathbf{b}\,u(n)$$

$$y(n) = \mathbf{c}\;\mathbf{x}(n) + d\,u(n) \tag{9.41}$$

where,

$\mathbf{A} = r \times r$ state matrix

$\mathbf{b} = r \times 1$ column matrix

$\mathbf{c} = 1 \times r$ row matrix

$u(n)$ = input sequence

$$y(n) = \text{output sequence}$$

[Note that in the context of state-variable analysis the symbol $u(n)$ represents **any** input sequence and not just the unit discrete-step sequence.]

State-variable model from difference equation

If the given description is in terms of a difference equation, the technique for obtaining the state-variable model is shown in the following example.

Example 9.14:— Obtain a state-variable model of the system represented by the difference equation,

$$3y(n) - 2y(n-1) + 2y(n-2) = 5f(n).$$

This second order difference equation will need two state-variables for its proper representation. Let us choose them as,

$$x_1(n) = y(n-2) \text{ and } x_2(n) = y(n-1).$$

From the given difference equation we get,

$$x_1(n+1) = y(n-1) = x_2(n)$$

$$x_2(n+1) = y(n) = \frac{2}{3}y(n-1) - \frac{2}{3}y(n-2) + \frac{5}{3}f(n)$$

$$= \frac{2}{3}x_2(n) - \frac{2}{3}x_1(n) + \frac{5}{3}f(n)$$

Writing the above equations in the standard matrix form of state-variable equations we get,

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{5}{3} \end{bmatrix} f(n)$$

$$y(n) = \begin{bmatrix} -\frac{2}{3} & \frac{2}{3} \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \frac{5}{3}f(n).$$

This procedure can be generalised for an rth order system. Let the given rth order difference equation be,

$$y(n) + a_1 y(n-1) + a_2 y(n-2) + \ldots$$

$$+ a_{r-1} y(n-r+1) + a_r y(n-r) = bf(n) \tag{9.42}$$

Define state-variables as,

$$x_1(n) = y(n-r)$$

$$x_2(n) = y(n-r+1)$$

$$\vdots$$

$$x_{r-1}(n) = y(n-2)$$

$$x_r(n) = y(n-1)$$

Then the state equations are,

$$x_1(n+1) = y(n-r+1) = x_2(n)$$

$$x_2(n+1) = y(n-r+2) = x_3(n)$$

$$\vdots \qquad \vdots$$

$$x_{r-1}(n+1) = y(n-2+1) = x_r(n)$$

$$x_r(n+1) = y(n)$$

$$= -a_r\, x_1(n) - a_{r-1}\, x_2(n) - \dots - a_1\, x_r(n) + b f(n)$$

In the standard matrix form the state-equation is,

$$\mathbf{x}(n+1) = \mathbf{A}\,\mathbf{x}(n) + \mathbf{b}\, f(n)$$

and the output equation,

$$y(n) = \mathbf{c}\,\mathbf{x}(n) + d f(n)$$

where,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & & & \\ 0 & 0 & 0 & \dots & 1 \\ -a_r & -a_{r-1} & -a_{r-2} & \dots & -a_1 \end{bmatrix} ; \ \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ b \end{bmatrix}$$

$$\mathbf{c} = [\,-a_r \ -a_{r-1} \ \dots \ -a_1\,] \ ; \ d = b$$

As with continuous-time systems, the state-variable model derived above is not unique. There are severale ways of choosing state-variables, and each choice will give a different state-variable model.

State-variable model from z-transfer function

Let us first consider the case where the given z-transfer function has no zeros, as in the following example.

Example 9.15:— Obtain the state-variable model of the system described by the z-transfer function,

$$H(z) = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}} \,.$$

We first convert the z-transfer function model to a difference equation model. For this let,

$$H(z) = \frac{Y(z)}{F(z)}$$

Then,

$$(1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3})\ Y(z) = F(z)$$

Taking its inverse z-transform we get,

$$y(n) + a_1 y(n-1) + a_2 y(n-2) + a_3\ y(n-3) = f(n)$$

Using the procedure of the previous example we get the state-variable model as,

$$\mathbf{x}(n) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_3 & -a_2 & -a_1 \end{bmatrix} \mathbf{x}(n) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} f(n)$$

$$y(n) = [-a_3\ -a_2\ -a_1]\ \mathbf{x}(n) + f(n)$$

where,

$$\mathbf{x}(n) = \begin{bmatrix} y(n-3) \\ y(n-2) \\ y(n-1) \end{bmatrix}$$

Let us now consider the general case where the given z-transfer function has both poles and zeros. Let,

$$H(z) = \frac{b_0 + b_1 z^{-1} + \ldots + b_m z^{-m}}{1 + a_1 z^{-1} + \ldots + a_r z^{-r}} \qquad m \le r \tag{9.43}$$

Let us express $H(z)$ as a product of two z-transfer functions. That is,

$$H(z) = \frac{Y(z)}{F(z)} = \frac{G(z)}{F(z)} \cdot \frac{Y(z)}{G(z)}$$

where,

$$\frac{G(z)}{F(z)} = \frac{1}{1 + a_1 z^{-1} + \ldots + a_r z^{-r}} \tag{9.44}$$

and,

$$\frac{Y(z)}{G(z)} = b_0 + b_1 z^{-1} + \ldots + b_m z^{-m} \tag{9.45}$$

The z-transfer function (9.44) is written as,

$$(1 + a_1 z^{-1} + \dots + a_r z^{-1})\, G(z) = F(z)$$

Taking the inverse z-transform of both the sides,

$$g(n) + a_1 g(n-1) + \dots + a_r g(n-r) = f(n) \tag{9.46}$$

which is the same as equation (9.42) with $b = 1$ and variable y replaced by g. Therefore the state-variables model of (9.46) will be the same as that for (9.42) with state-variables defined as,

$$\begin{aligned} x_1(n) &= g(n-r) \\ x_2(n) &= g(n-r+1) \\ &\vdots \\ x_r(n) &= g(n-1) \end{aligned}$$

Now consider the z-transfer function given by (9.45).

It can be written as,

$$Y(z) = \left[b_0 + b_1 z^{-1} + \dots + b_{m-1} z^{-(m-1)} + b_m z^{-m}\right] G(z)$$

Taking z-inverse of both the sides we get,

$$y(n) = b_0 g(n) + b_1 g(n-1) + \dots + b_m g(n-m).$$

Substituting for $g(n)$ from (9.46) we get,

$$\begin{aligned} y(n) &= -b_0 a_1 g(n-1) - \dots - b_o a_r g(n-r) + b_0 f(n) \\ &\quad + b_1 g(n-1) + \dots + b_m g(n-m) \\ &= (b_1 - b_0 a_1)\, x_r(n) + \dots + (b_m - b_0 a_m)\, x_{r-m+1}(n) \\ &\quad - b_0 a_{m+1}\, x_{r-m}(n) - \dots - b_0 a_r x_1(n) + b_0 f(n) \end{aligned}$$

In the matrix form the output equation above can be written as,

$$y(n) = [-b_0 a_r \dots - b_0 a_{m+1}\ (b_m - b_0 a_m) \dots (b_1 - b_0 a_1)] \begin{bmatrix} x_1(n) \\ \vdots \\ x_{r-m}(n) \\ x_{r-m+1}(n) \\ \vdots \\ x_r(n) \end{bmatrix} + b_0 f(n) \tag{9.47}$$

The presence of zeros thus alters the output equation as given by (9.47). The state equations remain the same as that for the case without zeros.

As pointed out earlier also, the state-variable model derived here is not unique. There are other procedures for choosing state-variables and they give different state-variable models.

9.10 Solution of State-variable Equations

We now look at the problem of solving state-variable equations to obtain the response of a discrete-time system to a given input. As for a continuous-time system, the response consists of two parts, the unforced or the zero-input response and the forced response.

The unforced response:— The state equation for an unforced system is,

$$\mathbf{x}(n+1) = \mathbf{A}\ \mathbf{x}\ (n) \tag{9.48}$$

This equation can be solved sequentially, starting with n=0.

For

$$n = 0, \quad \mathbf{x}(1) = \mathbf{A}\ \mathbf{x}(0)$$

$$n = 1, \quad \mathbf{x}(2) = \mathbf{A}\ \mathbf{x}(1) = \mathbf{A}^2\ \mathbf{x}\ (0)$$

$$n = 2, \quad \mathbf{x}(3) = \mathbf{A}\ \mathbf{x}(2) = \mathbf{A}^3\,\mathbf{x}(0)$$

$$\vdots$$

Thus if the initial state is $\mathbf{x}$ (0), the state at any instant n is,

$$\mathbf{x}\ (n) = \mathbf{A}^n\,\mathbf{x}\ (0)$$

This is the vector version of the unforced response of a first order difference equation given by (9.7). The $r \times r$ matrix $\mathbf{A}^n$ is called the *state transition matrix* for the discrete-time system and given the symbol $\mathbf{\Phi}$ (n). Thus,

$$\mathbf{x}(n) = \mathbf{A}^n\,\mathbf{x}(0) = \mathbf{\Phi}(n)\ \mathbf{x}(0) \tag{9.49}$$

The Forced response:— With a forcing function (i.e. input) present the system state equations are,

$$\mathbf{x}\ (n+1) = \mathbf{A}\ \mathbf{x}(n) + \mathbf{b}\ u(n) \tag{9.50}$$

where u (n) is the external forcing function. Following the sequential procedure once again we get,

$$\mathbf{x}(1) = \mathbf{A}\,x(0) + \mathbf{b}\,u(0)$$

$$\mathbf{x}(2) = \mathbf{A}\,x(1) + \mathbf{b}\,u(1) = \mathbf{A}^2\ \mathbf{x}(0) + \mathbf{A}\ \mathbf{b}\ u(0) + \mathbf{b}\ u(1)$$

$$\mathbf{x}(3) = \mathbf{A}\,x(2) + \mathbf{b}\,u(2) = \mathbf{A}^3\,\mathbf{x}(0) + \mathbf{A}^2\ \mathbf{b}\,u(0) + \mathbf{A}\ \mathbf{b}\ u(1) + \mathbf{b}\ u(0)$$

$$\vdots$$

$$\mathbf{x}(n) = \mathbf{A}^n\,\mathbf{x}\ (0) + \mathbf{A}^{n-1}\,\mathbf{b}\,u(0) + \mathbf{A}^{n-2}\,\mathbf{b}\,u(1) + \ldots + \mathbf{A}^0\,\mathbf{b}\,u(n-1)$$

$$= \mathbf{A}^n \mathbf{x}(0) + \sum_{j=0}^{n-1} \mathbf{A}^{n-(j+1)} \mathbf{b}\, u(j) \tag{9.51}$$

Using the notation of the state transition matrix this result can also be written as,

$$\mathbf{x}(n) = \Phi(n)\ \mathbf{x}(0) + \sum_{j=0}^{n-1} \Phi(n-j+1)\ \mathbf{b}\, u(j) \tag{9.52}$$

The first term on the r.h.s. of (9.52) is the *zero-input* or the unforced response which will be the system response with $u(n) = 0$. The second term is the response which will result if the initial state $\mathbf{x}(0) = 0$. Therefore it is called the *zero-state* response. The complete response is the sum of these two, zero-state and zero-input, responses.

Example 9.16:— The assets of an electric power distribution company can be divided into two parts; (i) low voltage (LV) equipment and (ii) high voltage (HV) equipment. The company earns a profit of 20% on its total current assets. All the profit is reinvested for purchasing additional equipment; 50% for LV and 50% for HV equipment. The company raises additional investment of Rs u(n) million in the nth year of its operation. 75% of this investment goes for purchasing LV equipment and 25% for HV equipment. The company starts from a scratch with an initial investment of Rs 800 million. The investment in succeeding years reduces as,

$$u(n) = 200\,(2^{-n+2}),\ n \geq 0$$

The rate of depreciation for LV equipment is 20% and 10% for HV equipment. Calculate the assets of the company for the first three years of its operation.

Let $x_1(n)$ and $x_2(n)$ be the worth of LV and HV equipments in the nth year. Then the profit earned in the nth year is $0.2\,\{x_1(n) + x_2(n)\}$. The worth of the LV equipment in the $(n+1)^{th}$ year will be equal to the sum of three components:

(i) The discounted value of worth of LV equipment in the nth year.

(ii) The worth of LV equipment purchased out of profit of nth year

(iii) The worth of LV equipment purchased out of fresh investment in the nth year.

That is,

$$x_1(n+1) = 0.8\,x_1(n) + 0.1\,\{x_1(n) + x_2(n)\} + 0.75\,u(n).$$

Similarly the worth of HV equipment is,

$$x_2(n+1) = 0.9\,x_2(n) + 0.1\,\{x_1(n) + x_2(n)\} + 0.25\,u(n).$$

Rearranging in matrix form,

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 1.0 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} u(n) \tag{9.53}$$

with initial state,

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ and } u(n) = 200(2^{-n+2})$$

and output (net worth) $y(n) = x_1(n) + x_2(n)$.

The state-variable equation (9.53) is solved sequentially, starting with n=0.

(i) For $n = 0$,
$$\begin{bmatrix} x(1) \\ x(2) \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 1.0 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} 200(2^{(-0+2)})$$
$$= \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 1.0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} 800$$
$$= \begin{bmatrix} 600 \\ 200 \end{bmatrix}$$

(ii) For $n = 1$,
$$\begin{bmatrix} x_1(1) \\ x_2(2) \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 1.0 \end{bmatrix} \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} + \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} 200(2^{(-1+2)})$$
$$= \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 1.0 \end{bmatrix} \begin{bmatrix} 600 \\ 200 \end{bmatrix} + \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} 400$$
$$= \begin{bmatrix} 560 \\ 260 \end{bmatrix} + \begin{bmatrix} 300 \\ 100 \end{bmatrix} = \begin{bmatrix} 860 \\ 360 \end{bmatrix}$$

(iii) For $n = 2$,
$$\begin{bmatrix} x_1(3) \\ x_2(3) \end{bmatrix} = \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 1.0 \end{bmatrix} \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} + \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} 200(2^{(-2+2)})$$
$$= \begin{bmatrix} 0.9 & 0.1 \\ 0.1 & 1.0 \end{bmatrix} \begin{bmatrix} 860 \\ 360 \end{bmatrix} + \begin{bmatrix} 0.75 \\ 0.25 \end{bmatrix} 200$$
$$= \begin{bmatrix} 810 \\ 446 \end{bmatrix} + \begin{bmatrix} 150 \\ 50 \end{bmatrix} = \begin{bmatrix} 960 \\ 515 \end{bmatrix}$$

Thus the net asset (in millions of rupees) of the company, $y(n) = \{x_1(n) + x_2(n)\}$ is,

$$y(0) = x_1(0) + x_2(0) = 0$$
$$y(1) = x_1(1) + x_2(1) = 800$$
$$y(2) = x_1(2) + x_2(2) = 1220$$
$$y(3) = x_1(3) + x_2(3) = 1475$$

We now look at the analytical solution (9.51) or (9.52) of the state-variable equation for calculating **x**(n) for any n without the necessity of first calculating the value of $x(n)$ of all the previous instants of n. To be able to do so we need a method to calculate the state-transition matrix $\Phi(n) = \mathbf{A}^n$. In chapter 8 we had studied three methods for computing the state-transition matrix $\Phi(t)$ for continuous-time systems. Note that the definition of this matrix is different for discrete-time systems. However the method of calculating $\Phi(n)$ is similar to the method for $\Phi(t)$ for continuous-time systems.

Calculation of state-transition matrix $\Phi(n)$ using Caley-Hamilton theorem:—

The objective is to determine $\mathbf{A}^n$ from the knowledge of its lower order characteristic polynomial.

Let $q(\lambda)$ be the r^{th} order characteristic polynomial of the $r \times r$ matrix $\mathbf{A}$. Then,

$$q(\lambda) = \det[\lambda \mathbf{I} - \mathbf{A}] = \lambda^r + a_1 \lambda^{r-1} + \ldots + a_r \tag{9.54}$$

Let $\lambda_1, \lambda_2, \ldots, \lambda_r$ be the roots of the characteristic equation,

$$q(\lambda) = \lambda^r + a_1 \lambda^{r-1} + \ldots + a_r = 0 \tag{9.55}$$

Now let us take the term λ^n, $n \geq r$, and divide it by $q(\lambda)$. This gives,

$$\frac{\lambda^n}{q(\lambda)} = M(\lambda) + \frac{N(\lambda)}{q(\lambda)} \tag{9.56}$$

where $M(\lambda)$ is the quotient polynomial and $N(\lambda)$ the remainder polynomial. Equation (9.56) can be written as,

$$\lambda^n = q(\lambda)\, M(\lambda) + N(\lambda) \tag{9.57}$$

At any of its roots λ_i ($= \lambda_1, \lambda_2, \ldots, \lambda_r$) the characteristic polynomial is zero. That is,

$$q(\lambda_i) = 0 \text{ for } \lambda_i = \lambda_1, \lambda_2 \ldots, \lambda_r.$$

Therefore substituting λ_i in (9.57) we get,

$$\lambda_i^n = N(\lambda_i) \tag{9.58}$$

The remainder polynomial $N(\lambda)$ has order $(r-1)$ and a form,

$$N(\lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2 + \ldots + \beta_{r-1} \lambda^{r-1}$$

Therefore,

$$\lambda_i^n = N(\lambda_i) = \beta_0 + \beta_1 \lambda_i + \beta_2 \lambda_i^2 + \ldots + \beta_{r-1} \lambda_i^{r-1} \tag{9.59}$$

$$i = 1, 2, \ldots, r$$

Now let us take recourse to the Caley-Hamilton theorem which states that every matrix satisfies its own characteristic equation. Therefore we can substitute A for λ in the characteristic equation (9.55), i.e.,

$$q(\mathbf{A}) = \mathbf{A}^r + a_1 \mathbf{A}^{r-1} + \ldots + a_{r-1} \mathbf{A} + a_r \mathbf{I} = 0$$

Similarly we can also substitute A for λ in equation (9.57) to get,

$$\mathbf{A}^n = q(\mathbf{A})\ M(\mathbf{A}) + N(\mathbf{A})$$

and since $q(\mathbf{A}) = 0$, we have

$$\mathbf{A}^n = N(\mathbf{A})$$

$$= \beta_0 \mathbf{I} + \beta_1 \mathbf{A} + \beta_2 \mathbf{A}^2 + \ldots + \beta_{r-1} \mathbf{A}^{r-1} \tag{9.60}$$

This is the desired result, which shows how to find $\Phi(n) = \mathbf{A}^n$ from a polynomial of lower order (r-1).

To be able to use (9.60) we need to know the values of coefficients $\beta_0, \beta_1, \ldots, \beta_{r-1}$. These are obtained from equation (9.59). For each value of characteristic root λ_i, equation (9.59) will give one algebraic equation. Thus for r roots $\lambda_1, \lambda_2, \ldots, \lambda_r$, their will be r equations. Knowing the values of the roots, these r equations are solved to get the values of coefficients $\beta_0, \beta_1, \ldots \beta_{r-1}$.

The use of this method is illustrated by the following example.

Example 9.17:— A nuclear reactor has two types of particles, α and β. The reaction is such that every second an α particle breaks-up into three β particles, and a β particle into one α and two β particles. How many particles of each type will be present in the reactor after n seconds if initially there is only one α particle in it?

Let $x_1(n)$ and $x_2(n)$ be the numbers of α and β particles after n seconds. From the given description of the system we can write the system's state equations as,

$$x_1(n+1) = x_2(n)$$

$$x_2(n+1) = 3x_1(n) + 2x_2(n)$$

with

$$x_1(0) = 1 \text{ and } x_2(0) = 0.$$

The solution of this homogeneous system is given by equation (9.49). However to use it we need to determine the state transition matrix $\Phi(n)$. In the present problem the state matrix $\mathbf{A}$ is,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$$

The characteristic equation for this $\mathbf{A}$ is,

$$\det \begin{bmatrix} \lambda & -1 \\ -3 & \lambda - 2 \end{bmatrix} = \lambda^2 - 2\lambda - 3 = 0$$

The roots of the characteristic equation are $\lambda_1 = -1$ and $\lambda_2 = 3$.

We now need to calculate the coefficients β_0 and β_1 of equation (9.60) for this second order system. Using equation (9.59) for each of the roots we get,

$$\beta_0 - \beta_1 = (-1)^n$$

and

$$\beta_0 + 3\,\beta_1 = 3^n$$

Solving these two simultaneous equations we get,

$$\beta_0 = \frac{3^n + 3(-1)^n}{4} \text{ and } \beta_1 = \frac{3^n - (-1)^n}{4}$$

The state-transition matrix given by (9.60) is,

$$\mathbf{\Phi}\,(n) = \mathbf{A}^n = \beta_0\,\mathbf{I} + \beta_1\,\mathbf{A}$$

with values of β_0 and β_1 as calculated above.

Let us now use the knowledge of $\mathbf{\Phi}\,(n)$ to calculate the number of particles, say after two seconds. For n = 2,

$$\beta_0\,(2) = \frac{3^2 + 3(-1)^2}{4} = 3 \text{ and } \beta_1\,(2) = \frac{3^2 - (-1)^2}{4} = 2$$

Therefore,

$$\mathbf{A}^2 = \beta_0\,(2)\;\mathbf{I} + \beta_1\,(2)\;\mathbf{A}$$

$$= \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 6 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 6 & 7 \end{bmatrix}$$

(Check this result by direct multiplication $\mathbf{A}^2 = \mathbf{A} \times \mathbf{A}$.)

The number of particles after two seconds is

$$\begin{bmatrix} x_1\,(2) \\ x_2\,(2) \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 6 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

Let us now determine the number of particles after 8 seconds. A direct calculation of $\mathbf{A}^8$ in this case would be tedious. Using the analytical method developed here,

$$\beta_0\,(8) = \frac{3^8 + 3(-1)^8}{4} = 1641$$

$$\beta_1\,(8) = \frac{3^8 - (-1)^8}{4} = 1640$$

Therefore,

$$\mathbf{A}^8 = \beta_0\,(8)\;\mathbf{I} + \beta_1\,(8)\;\mathbf{A}$$

$$= \begin{bmatrix} 1641 & 0 \\ 0 & 1640 \end{bmatrix} + 1640 \begin{bmatrix} 0 & 1 \\ 3 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1641 & 1640 \\ 4920 & 4921 \end{bmatrix}$$

Then,

$$\begin{bmatrix} x_1(8) \\ x_2(8) \end{bmatrix} = \mathbf{A}^n \mathbf{x}(0) = \begin{bmatrix} 1641 & 1640 \\ 4920 & 4921 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1641 \\ 4920 \end{bmatrix}$$

PROBLEMS

9.1 A batch process in a chemical plant has a mixing tank of 100 litre capacity with an inlet and an outlet valve. At the beginning of the n^{th} cycle of operation, 10 litres of solution with a fractional concentration of $y(n)$ is drawn out and an equal amount with fractional concentration $x(n)$ is added to the tank. The contents of the tank are then thoroughly mixed for a specified fixed time. The next cycle is started by drawing another 10 litres of solution, and the process is repeated.

(a) Derive the difference equation model of the process, relating output $y(n)$ with input $x(n)$.

(b) Solve the difference equation if the initial concentration $y(0) = 0.1$ and the input concentration is $x(n) = 0.8$.

9.2 Solve the following difference equations.

(a) $y(n) + 2y(n-1) = x(n) - x(n-1)$ with initial condition $y(0) = 1$ and $x(n) = n^2$.

(b) $y(n) + 3y(n-1) + 2y(n-2) = x(n) + x(n-1)$ with initial conditions $y(0) = y(1) = 0$ and $x(n) = (-2)^n$ for $n \geq 0$ and $= 0$ for $n < 0$.

9.3 The input sequence $x(n)$ to a discrete-time system is $x(0) = 1$, $x(1) = 2$, $x(2) = 3$ and $x(n) = 0$ for $n \geq 3$. The resulting output sequence is, $y(0) = 1$, $y(1) = -1$, $y(2) = 3$, $y(3) = -1$, $y(4) = 6$ and $y(n) = 0$ for $n \geq 5$.

Determine the impulse response $h(n)$ of the system.

9.4 Find the z-transform of the following functions.

(i) $x(n) = n(n+1)$

(ii) $x(n) = n^2$

(iii) $x(n) = \sin 2n$

(iv) $x(n) = n^2 a^{-n}$

9.5 Find the sum of the sequence,

$$\omega(n) = 2^0 \cdot 3^n + 2^1 \cdot 3^{n-1} + 2^2 \cdot 3^{n-2} + \ldots + 2^{n-1} \cdot 3^1 + 2^n \; 3^0$$

[**Hint**: Use convolution and z-transform.]

9.6 Solve the difference equations of problem 9.2 using the z-transform.

9.7 Two discrete-time systems, each halving a delta response of,

$$h(n) = 3^{-n}$$

are connected in cascade. Find the response of the composite system for inputs (i) unit delta function and (ii) unit discrete-step.

9.8 The z-transfer function of a system is given as,

$$H(z) = \frac{z(7z - 2)}{(z - 0.2)(z - 0.5)}$$

Find its response to a unit discrete-step input with zero initial conditions.

9.9 The mathematical model of a discrete-time system is described by the equation,

$$y(n) + 3y(n-1) + 2y(n-2) = f(n) - f(n-1).$$

Find a state-variable model of the system and solve it to obtain the response to a unit delta function with zero initial conditions.

9.10 Show that

$$\mathbf{A}^n = \begin{bmatrix} 2^n & n\,2^{(n-1)} \\ 0 & 2^n \end{bmatrix}$$

$$\text{if } \mathbf{A} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

9.11 Show how to use z-transform to find the state-transition matrix for a discrete-time system. Use this method to determine the state-transition matrix for the state-matrix,

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}.$$

9.12 Show how to obtain the z-transfer function $H(z)$ from a given state-variable model. Use this method to obtain $H(z)$ for a system described by,

$$\mathbf{A} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \mathbf{c} = [1 \;\; -4].$$

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