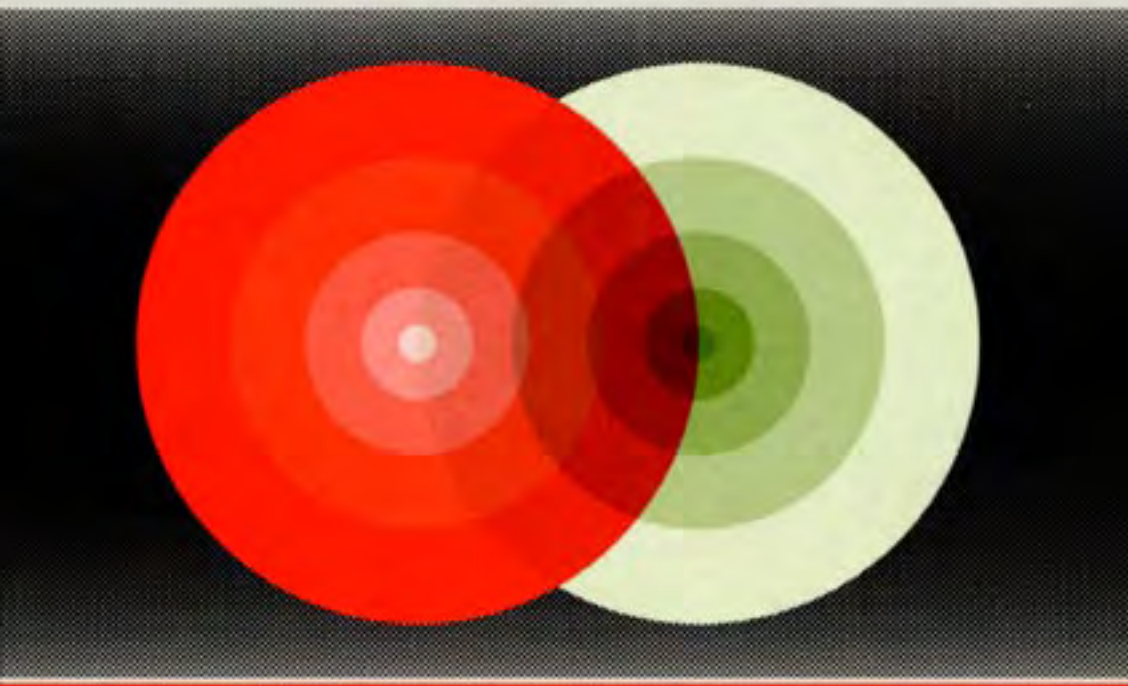
Theory of the Electron

A Theory of Matter from START

by

Jaime Keller

Kluwer Academic Publishers



Fundamental Theories of Physics

Theory of the Electron

Fundamental Theories of Physics

An International Book Series on The Fundamental Theories of Physics: Their Clarification, Development and Application

Editor:
ALWYN VAN DER MERWE, *University of Denver; U.S.A.*

Editorial Advisory Board:
JAMES T. CUSHING, *University of Notre Dame, U.S.A.*
GIANCARLO GHIRARDI, *University of Trieste, Italy*
LAWRENCE P. HORWITZ, *Tel-Aviv University, Israel*
BRIAN D. JOSEPHSON, *University of Cambridge, U.K.*
CLIVE KILMISTER, *University of London, U.K.*
PEKKA J. LAHTI, *University of Turku, Finland*
ASHER PERES, *Israel Institute of Technology, Israel*
EDUARD PRUGOVECKI, *University of Toronto, Canada*
TONY SUDBURY, *University of York, U.K.*
HANS-JÜRGEN TREDER, *Zentralinstitut für Astrophysik der Akademie der Wissenschaften, Germany*

Volume 115

Theory of the Electron

A Theory of Matter from START

by

Jaime Keller

CMS, Technical University of Vienna, Austria
& UNAM, Mexico

KLUWER ACADEMIC PUBLISHERS
NEW YORK / BOSTON / DORDRECHT / LONDON / MOSCOW

eBook ISBN: 0-306-47132-9
Print ISBN: 0-792-36819-3

©2002 Kluwer Academic Publishers
New York, Boston, Dordrecht, London, Moscow

Print ©2001 Kluwer Academic Publishers
Dordrecht

All rights reserved

No part of this eBook may be reproduced or transmitted in any form or by any means, electronic, mechanical, recording, or otherwise, without written consent from the Publisher

Created in the United States of America

Visit Kluwer Online at: http://kluweronline.com
and Kluwer's eBookstore at: http://ebooks.kluweronline.com

To my wife
Cristina

This page intentionally left blank.

Contents

This page intentionally left blank.

Preface

This book is devoted to the construction of a deductive theory of the electron, starting from first principles and using a single mathematical tool, geometric analysis. Its purpose is to present a comprehensive theory of the electron to that stage at which connection is made with the main approaches to the study of the electron in physics, in such a form that, once those have been derived anew, the reader can use them for particular purposes.

The electron has come, in the first century after its discovery, to be a fundamental element in the analysis of physical aspects of nature. It is at the basis of and gives its name to electrodynamics. Quantum theory is in practice modeled as a theory of the electron and its main examples are usually related to the behavior of the electron. Quantum electrodynamics is the starting point for the development of the standard model in elementary particle physics. Even in general relativity the electron is an acceptable test particle (although the universality of general relativity makes almost any physical object an acceptable test particle). Therefore a book on the theory of the electron belongs to the **melting pot** of physics (to use a concept of Professor P. O. Löwdin).

The book is therefore directed to two types of audiences. Primarily to theoretical physicists not only in the field of electron physics but also in the more general fields of quantum mechanics, of elementary particle physics, and of general relativity. Of course, theoretical chemists and quantum chemists are *de facto* involved in this branch of theoretical physics. The secondary audience is mathematicians in the field of geometric analysis, as they are always observing the applications of the field. The book is written, and in fact originated amongst, for an audience of graduate students in the subjects mentioned in the paragraph above. Nevertheless, it presents a new, comprehensive, fundamental approach to the theory of matter, and as such it is relevant to scientists

working in theoretical physics or analyzing theoretical physics. In this sense the book has kept the structure of a paper, suitable for this audience in mathematical physics, where physical reasoning is given more space than usual.

In the introduction the methodology is described. In Chapter 2 the concept of space–time–action relativity theory is presented in a form suitable for this book and in Chapter 3 the mathematical structures which describe action are analyzed. Chapters 4, 5 and 6 deal with the theory of the electron in a series of aspects in which the geometrical analysis is more relevant. Finally, in Chapter 7 we present the form of geometrical analysis which was used in the book, we felt this was needed because of the broad range of topics which were covered and because of the range of mathematical structures which are implicitly or explicitly included.

We close this preface with a philosophical and methodological remark. The theory of the electron developed in this book is based in two main theoretical considerations: on the nature of a scientific theory; and on the elements to describe nature. Additionally, in order to construct. a mathematical theory a series of postulates about the geometry relating the elements to describe nature and its relations are supplied. In the first chapter we state that the theory to be presented has as its basic purpose the description of what can be observed, inferred, related and predicted to be observed within the fundamental limitations of the experimental and theoretical purposes of science. We are not going beyond these limitations in any sense. This includes that we do not consider possibilities like trying to derive the fundamental concepts from other structures which can be supposed to be more fundamental. Also we are not assuming any particular model of the objects of nature based on hitherto unobserved entities.

About the basic elements of the physical objects and phenomena in nature we assume that we have three: time; space; and action density. The first element, **time,** corresponds to a primitive concept which is universally accepted. In a theory like the one presented in this book this primitive concept is defined by its mathematical properties and by accepting, without explaining, that for physical nature time is an ever growing variable for any observer. The concept of **space** is defined using the same considerations as that for time, through its mathematical description; this requires a set of three independent, variables to be in agreement with our anthropological apprehension of nature, the values the variables can take are considered to be as large as needed for the description of physical phenomena. We are not, considering boundaries or global topology for space, nor relations of the global boundaries or global

topology with the properties of the physical objects. The third element of physical nature which is considered here is given the, not very fortunate, name of density of **action,** assuming it is a one dimensional variable to be joined to the previous four in a geometrical unity. Man, as the scientific observer, considers that physical objects exist in space–time and, as time is always evolving, the observer naturally selects the evolution of the object and the evolution of the relation between the objects with respect to time as the dynamics of nature. One common observation is the permanence of these objects. A more scientific observation is the conservation of energy, in different 'forms' related to the description of those objects. Our actual knowledge of the physical objects is that, there is energy involved both in the existence of the object and in its relations. The concept of fields in physics has also defined the existence of an energy related to the field. Then, because fields and objects pervade all physical space–time, we have to accept that physics has come to the acceptance of an energy distribution, non-zero, at every point of space–time. This concept, hitherto not thoroughly explored, is, when joined with that of space–time, what creates the concept of action density. We have refrained from the idea of giving this concept a new name because we want to emphasize that we are not starting from new concepts or from new relations. We are systematically analyzing the geometrical relations amongst these three elements in order to construct a comprehensive theory of matter.

Also, when distributions of action in space–time are made to correspond to the physical objects, we have to conclude that the permanence of these objects as time evolves is related to a set of symmetry constraints on that action distribution. In our theory this is a set of self-generating symmetry constraints when the action distribution is described in a form suitable for dealing with the existence and interaction between the physical objects, and with the joint evolution of the objects and their interactions.

There is a number of papers in the scientific literature which propose mathematical models whose properties reproduce the properties of some specific physical object. This is an important point, in the scientific analysis of nature because these analogical procedures are, in general, a good guidance for the analysis of the physical phenomena. But when a general theory of matter is involved induction from specific models is difficult, and in all cases it has been only partially successful. Here we proceed in a deductive form: from general concepts to particular properties, using as the basic example the electron.

JK

Vienna, July 2000

This page intentionally left blank.

Acknowledgments

I have to thank first the colleagues in the mathematical physics community who have given me their points of view and described their mathematical tools, at least since the 1980 conference *The Mathematics of Physical Space–Time* which took place in México. Discussions with Professor Gaston Casanova and with Professor Zbigniew Oziewicz have been particularly important for me. The scientific exchange with a growing community, studying the use of geometric algebra and analysis in physics, has been specially relevant. Our common work searches both for the unification of the mathematical tools and of the mathematical formulation of the different theoretical approaches.

I am specially grateful for the periods of concentration which were given me by the invitations to work, asking me to lecture on this subject, from Professor Josep Manel Parra at Barcelona, to Professor Hans Christoph Siegmann at Zürich, and to Professor Peter Weinberger at Vienna where I enjoyed a longer stay, and there came the idea of writing this book providing me with the time to do so. Scientific discussions with the students were also always relevant; in particular, I want to thank my first student in this subject Suemi Rodríguez-Romo.

To my university, The National Autonomous University of México UNAM, which has given me great academic opportunities, to my adjoint Irma Vigil de Aragón who has made the academic life of my chair possible and productive.

The help in all senses, including academic and practical matters of writing, of my wife Cristina, is gratefully and deeply acknowledged.

I enjoyed the specific support of DGAPA-UNAM during my stay in Vienna. The author is a member of the Sistema Nacional de Investigadores, Conacyt, México.

This page intentionally left blank.

Chapter 1

INTRODUCTION

A comprehensive geometric theory of fields of carriers of energy is formulated from the basic principles of relativity theory extended to include action (START). Its development is compared with the geometric content of the electron theory. Our approach contains, being a deductive theory, results of general relativity, of density functional theory, wave function quantum mechanics, the classical theory of the electron, the description of the electron as a lepton in elementary particles theory, and of the fundamentals of both electrodynamics and electroweak interactions. The formalism is a geometric self-contained theory for the electron fields.

In a series of papers [89]–[127] we have reviewed the geometric content of both the classical and the quantum theory of the electron. There we analyzed some apparent duplicities or discrepancies, for example in the description of the energy content in the classical approach to the theory of the electron, or when considering the electromagnetic energy of the electron fields and the mass content of the electron as a particle, or the change in energies from separated particles to the bound case. In the analysis of the wave mechanics approach to the theory of the electron we also met some puzzling double descriptions both at the level of the quantum mechanics formulation and, for example, facts like the spin of the electron described as being contained both in the electron Dirac field and in the electromagnetic field of the electron. The multi-vector approach to the analysis of the electron's quantum mechanics was also included.

We mentioned that it was generally accepted that a more fundamental approach to the theory of the electron which could discriminate between various possible description seems to be required. We also presented

an intermediate analysis of several points. In the present book we shall develop such an approach from first principles and from a very basic geometric fomulation of physical nature, using as a starting point the concept of the physical world being described as an energy distribution over space, which from a relativistic point of view corresponds to an action distribution over space–time. We have already developed the basis of this approach for describing basic structures of physics, including an analysis of general relativity, of quantum mechanics, and of the theory of elementary particles. Here we shall use it as a basic frame of reference for constructing a comprehensive theory of the electron. We shall develop this theory to cover most aspects of the theory of the electron, including those related to the theory of elementary particles and to general relativity. By necessity we shall have to consider basic problems related to quantum electrodynamics, to gauge theory, and to the theory of induced mass and charges.

1. THE NATURE OF A PHYSICAL THEORY

The present approach to the theory of the electron contains an inherent consideration about the purpose of mathematical physical theories. First it should be considered that it is accepted that there is a physical reality, knoweable by the observer, which should be described in its contents, relations and predictions for future experiments by physical theory. A complete theory should then not pretend to describe more than can be observed, inferred, related and predicted to be observed, within the fundamental limtations of those experimental or theoretical processes. An additional limitation arises from the finiteness of the data that can be handled accurately in a quantitative theory; this is a limitation related both to the observer and to the mathematical structure of the theory and its computational techniques as far as the existence of fundamental limitations at this level have to be either avoided or clearly stated.

In the prediction of future observations the analysis of all possibilities is fundamental. A theory, such as the one presented here, based on distributions has an inherent, probabilistic structure which is mixed with both probabilistic considerations about the description of a given state and with actual probabilities of the future observations, the disentanglement, of which can only be carried out when the roles of description and of prediction are analyzed specifically.

Without this separation the entangled mathematical descriptions are to be considered as the carriers of information waiting to be used for specific purpose.

The mathematical description used has to be compatible with the experimental knowledge of the objects and their relations. An incomplete knowledge, either of fundamental or of practical origin, requires the consideration of all different possibilities compatible with our observations. This is to be both part of the fundamental structure of the theory and of its practical use. The theory and its mathematical structure can not ignore its basic purpose, the understanding of the physical observations in terms of basic concepts and prediction of possible observations.

From the considerations in this book Quantum Theory is basically the theoretical structure needed for the description of the distribution of energy amongst carriers. It is holistic by construction, in as far as all energy carriers and all forms of energy distribution per carrier have to be considered jointly, and also because the boundary conditions are a fundamental part of the description.

Within this approach we shall consider that the best. available fundamental description of matter is the one obtained by the use of relativistic quantum mechanics to study the electron, based on the Dirac equation and then our development of a theory, with deductive character, should recover at least all the results of this model of matter.

2. THE DEVELOPMENT OF THE THEORY

In Chapter 2 we present space–time–action relativity theory (START) in a form suitable for our present purposes, in Chapter 3 a formulation of action density functional theory, the concepts of carriers, of auxiliary action amplitude functions, as derived from START, and a general formulation of a theory of interacting elementary particles. Then in Chapter 4 we present a theory of the electron. In Chapter 5 and 6 we analyze concepts of our present knowledge of the theory of the electron from the point of view and results of the previous chapters. In Chapter 7 we provide one particular mathematical formulation, as used in the text, and some relevant historical and mathematical remarks about the theory of the electron. This last chapter presents an approach to geometric analysis which covers, in a unifying treatment, the study of the START geometry and the main topics related to spinors and twistors in the form needed for the analysis in the book.

For the development of the material we start from the following fundamental points of view.

- We adopt the traditional (in physics) starting point of considering space–time as the geometrical frame within which physical phenomena are to be described. That is, we consider Einstein's space–time (ST) geometry, with its physical motivations. Therefore, as our pos-

tulates assume, space and time are geometrically unified by considering a geometric space with a quadratic form using the velocity of light c as the universal and fundamental constant, to express the time intervals as distances. Also we start by considering that there is a Minkowski space–time (ST) frame with a coordinate parametrization for each observer.

- We shall consider our formulation to be relativistic, and then the Minkowskian ST as described by each observer will be related to that of a different observer in relative uniform motion by the corresponding Lorentz transformation (which are then both considered as basic: geometric notions). That is adopting space–time relativity theory (STRT) as the fundamental geometrical frame for the description of the physical phenomena.

- For the description of physical phenomena, with their large variety of forms, the only common property which can be used universally is that, there is an energy density distribution associated with them, to be described by each observer at a given moment of time, as a distribution of energy in space. This distribution evolves with time and we then have a distribution of action (A) in space–time.

- The freedom of the observer to select the description of the energy distribution usually with specific, collective, or generic names is given a firm status if two observers can agree on the description of the energy distribution even if using a different partition (gauge freedom). This extends STRT to a niore general framework of space–time–action geometric space (STA) with a quadratic form, using a universal constant k_0 to transform action density 'intervals' into equivalent distances and simultaneously using a generalized transformation group **KL** (including the Lorentz transformations **L** as a subgroup) to relate the description of different, observers in relative uniform motion. This formulates a space–time–action relativity theory (START).

- This description is generally based on the permanence of some properties which characterize these energy carriers. An energy carrier is then an acceptable selection for the description as far as the carrier has some permanence, malting it an acceptable description. Crudely speaking the classical carriers were objects which could be considered to be the same during the time that is needed for the interpretation of the physical phenomena: particles, droplets, rigid bodies, etc., which by definition had enough permanence to be considered carriers. In microscopic physics the carriers are molecules, atoms, nuclei, or elementary particles with the addition of elementary excitations and of

the photon and the weak interaction and color quanta. These **carriers of energy** have also the required permanence, or **carry** at least, **a set of conserved properties,** which, when the carrier ceases to be an useful description, are to be found in the set of conserved properties of the carriers which are assumed to have resulted from the previous ones.

- In general the carriers considered to be fundamental are defined from some asymptotic properties: the 'free' carriers. This is also the case in classical physics as far as, for example, the concept of a, planet. starts with the definition of what would be that planet 'free' in space and then its orbital and other motions or deformations are introduced as modifications of the concept of the 'free' planet. The same is true in microscopic physics, where we use the concept of an isolated electron, nucleus, molecule, etc., as fundamental names for energy carriers and we keep the same basic definition when we consider a collection of carriers. For example, when we describe a helium atom either as a unit or as a composite unit of two electrons and one nucleus, assuming that in the beginning the electrons were each the same physical entities as two isolated electrons. There is a hierarchy of descriptions when the nucleus is considered as consisting of interacting baryons, each baryon conceived as what would be an isolated proton or an isolated neutron, and only when refining the dcscription the changes of the protons or of the neutrons would be considered. This concept, of carriers inherits one fundamental mathematical complication, the notion that for some purposes the approximation of the carrier as a point-like carrier can be fruitfully used, even if strictly speaking this is not allowed when adopting the Planck action as the minimal possible physical action.

- But it has to be explicitly considered that there are no isolated carriers nor point-like carriers. Then a fundamental theory of matter can not be based on these concepts or it will be biased from the beginning and will carry an insurmountable construction defect. One way of saying this is to accept as a basic postulate that, action can not be given or taken except in integer multiples of the fundamental Planck action constant h and, moreover, that no carrier can be isolated from the rest of the universe in a fundamental form, only that, the corrections to its description can be made smaller than some acceptable (for computational purposes) value. This last property is describable as the association of an analytical function with both the action and with the carrier fields.

- The same properties which define the type of carriers which can be considered as asymptotically free can also be used for giving acceptable status to a new type of carriers which can not be isolated in the sense above but which are useful enough because of their permanence, in the sense of the permanence of their characterizing properties. The best known examples are phonons, second sound, or, at a more fundamental level, the quark component of hadrons. None of these carriers can exist in (even partial) isolation. Thus quarks or elementary excitations and quasi-particles, for example composite fermions, are acceptable carriers as far as a set of conserved properties are associated with them.

As, from the beginning, we have adopted the ST geometry of the space–time relativity (with the fundamental development and implications as in *Poincaré* (1905, 1906) and in Einstein, Lorentz, Minkowski and Weyl (1952)) we have both the geometry of space–time and, because in our construction energy–momentum by definition corresponds to the tangent space of space–time in STA, we also have the geometry of the energy–momentum tangent vectors. Our geometry then gives a direct foundation of the use of the tangent space of space–time for the description of energy and momentum, provided the operators ∂_m are applied to the action distribution function $a(x)$. We also give (using $\hbar x \wedge \nabla_x$ and $\hbar\gamma_\mu \nu$ operating on $a(x)$) a geometrical meaning to angular momentum, including, as we shall show, spin as a natural component of angular momentum. In fact, because of the STA geometric properties, **for a fundamental process**, which in physics is a process associated with fundamental particles and their interaction fields, no energy–momentum, that is action, can be given or taken without angular momentum being exchanged. Then geometrically a field or a process where no angular momentum is involved must correspond to a composite field or a process where this quantity cancels.

In the development of the theory we shall have to use a set of approaches. This is not related to our unique starting point but to the different postulates of the set of three theoretical approaches we want to derive from a single set of postulates. First Relativity Theory which considers a universal geometrical framework with test particles following geodesic trajectories. Second, Wave Function Quantum Theory which considers a (set of) particle(s) in the presence of an external potential, coupled through their charges (and in mutual interaction through the same set of charges). Third, Quantum Electrodynamics which considers a variable set of particles in interaction with quantized ‘interaction’ fields originated by the particles themselves or by an external collection of particle currents.

3. SOME METHODOLOGICAL CONSIDERATIONS

There has been a large series of efforts to extend and unify the basic theoretical structures of physics. Until recently, for several fundamental reasons, most approaches have studied massive (charged) particles with spin, where the electron is the main example. The purpose is twofold: first, to search for the basic underlying structure of the theory or model, within its different mathematical presentations in search of a common mathematical background; and second, the analysis of the underlying assumptions of each model or theory to see whether a unity of presentation and mathematical procedures could be found by collecting together similar starting points or starting assumptions.

This inductive approach has given first a classification of the main models for describing nature, and, second, a clear indication that there is only a reduced set of concepts which can be used in all cases. Surprisingly, they are already encountered in classical physics, modified by two main concepts: a) that of a geometrical unity of space and time through the use of a universal constant (the velocity of light c) and the definition of a space–time geometry; and b) the fundamental concept that all physical phenomena could be, in a fundamental form, described by the action associated with them, including the basic principle that action could only be given or taken away from a system as a multiple of a fundamental universal unit of action (the Planck constant h). In both cases the constants are not adimensional, that is they are not universal numbers.

We present here a different approach, that of creating a theory with a deductive character, from which the models and theories, which have shown their utility in their present form, should be derived, from more fundamental principles. The derived theories are also bounded by the common analysis presented here and then, by necessity, interrelated and enlarged. The fact that the theory here developed relates the basic structures of physics is in itself a major advantage; moreover, we give a firm theoretical structure to present theories and models. This is important even if we are referring to theories and models which hitherto have been given acceptable status both by their universality and by their empirical success.

We consider for the purpose of our analysis in this hook the following basic structures which use concepts of, but go beyond, classical physics.

- The Theory of Relativity which presents a geometrization, basically for the relation between observers, and a universal geometrical formulation of gravitation. Despite the great success of restricted and

general relativity, it has not usefully been extended to include phenomena like the electromagnetic, weak, and color forces, even if a large number of papers can be found in the literature where these cases are studied.

- Wave Function Quantum Mechanics, as applied to any collection of matter fields with an assumed interparticle interaction or external potential, accepting that it can be used to describe most physical phenomena in atomic, molecular and condensed matter physics.

- Quantum Electrodynamics has successfully described physical phenomena in which the exchange of quanta of action is explicitly involved. This approach has been successfully extended to the experimentally well tested standard model of elementary particles and their interactions. Particles and interactions fields, in this context, are characterized by a set of conserved quantities. Quantum Electrodynamics is related to Wave Function Quantum Mechanics but structurally different.

These three great structures of physics seem, each, to be possible starting points for describing most of the physical phenomena which can be observed in the laboratory, and yet they are not in fact interchangeable and each has a well defined domain of applications.

Then our present theory of the electron, because both the ubiquitous presence of the electron field and the almost unlimited possibility of using electrons directly or indirectly to measure, produce, or directly enter into a physical process, should be comprehensive and possess the basic characteristic of a scientific method based theory in physics:

.— It should be very clear in the definition of its basic principles, basic mathematical structures and the rules for proceeding;

.— It should allow the subdivision of the study of physical phenomena into tractable parts with a clear definition of the purpose and the limitations of these parts;

.— On the other hand, it should be simple enough for its application to obtain, first, the basic relations, and, afterwards, the detailed practical description of physical phenomena;

.— Finally, it should be comprehensive, and we should then be able to **derive** the principles and limitations of all three basic mathematical structures: general relativity, quantum mechanics, and the standard model, with a minimum of additional concepts, the ones which define the purpose of the derived theories.

As these three main structures will be derived, in our approach, from basic principles the relation between the structures should also be clearly

presentable. The presentation includes the possibility of exploring what are the limitations in one approach which make it difficult to describe phenomena normally explained within the other approaches.

We have selected as our **basic concepts** those of space (S), of time (T) and of action (A). This last selection is not the typical approach; even if in fact, almost any mathematical formulation in physics can be recast in terms of an action (or at least as an effective action called a Lagrangian) and then the requirement of the extreme value of this action in a given space–time domain takes the role of the guiding principle. Most of the presentations do not consider action to be the basic variable but an auxiliary variable which is constructed from other in a sense, more basic quantities. The selection of these three fundamental concepts (STA) contains an implicit, geometrical, set of additional concepts: the rate of change for a given physical system or subsystem $\mathcal{P}$ of each one of the variables with respect to the other. Then three rates of change are related to space and time and joined into a velocity vector or, through a parametrization, to a space–time four vector. Also four rates of change are related to the action of $\mathcal{P}$ and its dependence on space–time and joined into an energy momentum vector. One more vector is in fact implicit in the analysis, the current as a space–time vector, which is related to our concept, of the permanence of some physical entities (characterized by a set of parameters) in a space–time interval in such a way that the current describes the permanence of a set of parameters.

From the considerations of the previous paragraph we immediately arrive at the basic concept that physical phenomena are associated with structures in a space–time–action geometry such that physical entities are described by distributions of currents in this geometry. As simple as this starting point is, it is nevertheless not easy to be considered a basic and sufficient starting point for the purpose stated above. Besides action not being traditionally a basic concept, and then energy and momentum not being traditionally considered as the space–time description of an action distribution, the concept of currents is also not considered to be a basic quantity as in general it is thought to correspond to a way of describing the behavior of the, otherwise undefinable, assumed physical entities. These physical entities are theoretically describcd by the permanence of some set of parametrized properties in a space–time interval.

An additional fundamental principle gives its basic status to action: the principle that action can not be given or taken away from a system if it is not in multiples of the Planck constant *h*. Action is then not only the quantity that can be defined as an invariant for a given physical system, in a domain in space–time, independently of the state of motion

of the observer, but also the variable required by Planck's principle, which is the origin of quantum mechanics and quantum electrodynamics, to be unambiguously introduced into the theoretical framework. As action can only change in multiples of h the actual action related to a physical phenomena can not change continuously, therefore we can only describe continuous exchange of action between arbitrary selected parts of a system, with given arbitrary amounts of action allocated to each part. This continuous change of description will be accounted for within the theory and will be called 'gauge freedom'. Otherwise **actual changes of action** should always be an integer number of units of the action constant h. A physical system is by definition contained in a spacial volume and defined, with dynamic changes, for a characteristic length of time.

Action will be described in STA, considering derivatives $\partial A/\partial x^\mu = p_\mu$, as the product of an intensive factor (energy–momentum p_μ, $\mu = 0, 1, 2, 3$) multiplied by an extensive factor (characteristic time or characteristic distance) for each elementary physical entity which is chosen as a basis for description. As we have said above, the selection of the description should be open to the needs of the observer and then these quantities will depend on its selection. This will require that besides the principle of gauge freedom for the description we require a **Principle of Acceptable Choice of Description,** that is, the choice of descriptions in terms of carriers and energy–momentum contributions per carrier which allow the observer to define the action distribution among the fundamental entities. As the definition of these fundamental entities is a form of describing the partition and distribution in space–time of energy and momentum these fundamental entities will be called carriers, more precisely they should be called energy–momentum carriers. A carrier is otherwise characterized by a set of some conserved quantities. The intensive factors and the set of conserved properties are, in fact, the quantities guiding the more useful, acceptable, definition of carriers. This point will be discussed at some length. This distinction of intensive and extensive factors will also guide the structure of the equations. For example, when, for a given observer, a particular value of the time coordinate is considered, only intensive quantities can be related by the equations. Conversely, when the set of extensive quantities (space, time and action intervals) are directly considered, as in general relativity, the intensive quantities can only appear as the value they would have at given space–time–action points, and as a result the basic description is given in terms of a purely geometrical structure relating the extensive quantities. In both cases stationary state systems are privileged because

of their easier description and the permanence of the relations amongst the intensive quantities.

The development of the theory for the electron will follow this scheme in such a form to obtain a comprehensive theory:

SPACE + TIME + ACTION

+

FUNDAMENTAL PRINCIPLES

and

POSTULATES

⇓

START GEOMETRY

↙ ↓ ↘

General Relativity Quantum Mechanics Standard Model

In this program the principles refer to the existence of an unified geometry for space, time and action. The postulates, to the geometrical union of the manifold of those variables through the use of two universal constants c and K_0. The geometry, we have called START, to that of the quadratic space which is obtained. Two additional principles introduce the unit of action h in relation to the amount of action that can be given or taken from a system (usually described as energy and angular momentum being exchanged) and to a freedom of description of matter within START. The fact that the electron is taken as the fundamental concept is related to the particular conclusion that there should be a fundamental matter field for which the descriptions are more directly related and which can be described in the three structures we shall examine: General Relativity, Quantum Mechanics and Standard Model. A

constant with units of mass $m_0 = 1/(ck_0)$ can be defined in this geometry which is the natural mass constant for this special matter field. m_0 is then a relation of constants which defines this field to be fundamental, in our case it is immediately identified with the electron. Otherwise the title and the subtitle of the book could be interchanged and we could consider that we are proposing a fundamental theory of matter which by construction defines a basic carrier: the electron. The concept of carrier of action or of carrier of energy has been mentioned above and will be discussed several times. For a theory of matter, we have to define sets of carriers: independent carriers, interacting carriers, composite carriers, and elementary carriers. We shall find that the elementary carriers can correspond to irreducible representations of the solutions of the basic equations, or correspond to reducible representations. The electron will be presented as an elementary interacting carrier corresponding to an irreducible representation.

Chapter 2

SPACE–TIME–ACTION RELATIVITY THEORY

Our basic and more fundamental idea is that the physical world can be described as a distribution of action density in space–time. The properties of matter fields and their interaction are represented by the mathematical properties of this distribution. Action is considered as a fundamental variable, not as a quantity resulting from some calculation.

In [125, 127] we analyzed a classical theory of fields in (complex) space–time geometry and arrived to the conclusion that this geometry corresponds to a unified space–time–action geometry. We started from three basic assumptions: a) The use of space–time as a basic frame of reference; b) The introduction of physical phenomena as an action density function over space–time; and c) The geometrical (vectorial) union of space, time, and action in a quadratic space where a relativistic condition $(dS)^2 = 0$ defines both kinematics and dynamics. The basic principles of this Space–Time–Action Relativity Theory (START) were formulated and related to our present knowledge of the basic structures of physics. We present here these concepts in a form directly useful for our purpose of developing a geometric theory of the electron.

1. MOTIVATION FOR THE USE OF SPACE–TIME–ACTION GEOMETRY

For the construction of the vector representation of the space–time–action geometry we should start, by first considering some properties of the action density dimension. We have given in Chapter 1 a motivation for its use, and here we consider its geometric character.

One, fundamental, initial consideration is that the concept of phase space (PHS), where energy–momentum is joined to time–space, has

proved useful and even basic (for example, in classical and quantum statistical mechanics) for the study of matter. Its symplectic structure is basic in quantum mechanics and it provides the more direct form of introducing the quantum of action *h*. Our theory should contain PHS in a natural form. An extremely subtle property in particular, for the analysis of the theory of the electron is that we must be reminded that there exists a numerical and dimensional relationship between the fundamental constants:

$$c\left|\frac{\mu_e}{r_0}\right| = g_D c = \left(\frac{\hbar}{2}\right) c = \frac{hc}{4\pi}; \quad \mu_e = g_D r_0 \mathbf{e}_s = e r_{\text{Compton}} \mathbf{e}_s = \frac{e\mathbf{s}}{m_0}, \tag{2.1}$$

relating the electron's electrical charge *e*, the unit of action *h* and the numerical value of the velocity of light *c*, μ_e the magnetic moment of the electron, r_{Compton} the Compton wavelength, r_0 the classical electron radius, m_0 the electron mass, **s**, an axial vector, the electron spin $|\mathbf{s}| = \hbar/2$, with $\mathbf{s} = \mathbf{e}_s$ and defining the hypothetical Dirac monopole charge g_D, which we shall call only *m*-charge as a reminder of its auxiliary theoretical character. These basic constants and relations should be explicit, and fundamental in the geometrical structure developed here.

In space–time geometry, spanned by the vectors $\{e_\mu;\ \mu = 0,1,2,3\}$ generating a geometry G_{ST}, the constants *e* and *c* are (units system dependent but otherwise) invariant, scalar quantities. In particular they are invariant under proper or improper Lorentz transformations. Consider also that in the basic relations of electromagnetism a moving charge would appear as a four current

$$\mathbf{j} = j^\mu e_\mu, \tag{2.2}$$

where $j^\mu = ev^\mu$ with $\mathbf{v} = v^\mu e_\mu$ the four velocity. The Maxwell equations would then relate **j** to a vector four-potential **A**

$$\partial^\mu \partial_\mu \mathbf{A} = \mathbf{j}, \tag{2.3}$$

and to the (bi-vector) electromagnetic strengths

$$\mathbf{F} = h^{\mu\nu} e_\mu e_\nu = \tfrac{1}{2}\left(\partial^\mu A^\nu\right) e_\mu e_\nu. \tag{2.4}$$

On the other hand, if a quantity such as the *m*-charge g_D existed there should be a pseudo-scalar g_D that will, when moving, originate a **tri-vector** current (usually called an axial vector four-current) **k**

$$\mathbf{k} = \mathbf{g}_D \cdot \mathbf{v} = \mathbf{g}_D v^\mu e_\mu = h^{\lambda\nu\rho} e_\lambda e_\nu e_\rho; \tag{2.5}$$

which will, through the natural extensions of the Maxwell equations to magnetic currents, originate an axial vector potential

$$\partial^{\mu}\partial_{\mu}\mathbf{B} = \mathbf{k} \quad \text{where} \quad \mathbf{B} = b^{\nu\lambda\rho}e_{\nu}e_{\lambda}e_{\rho}, \tag{2.6}$$

here all indices ν, λ, ρ are different from each other. Then, geometrically, the 'constant' $\mathbf{g}_D$ must have a (real or imaginary) pseudo-scalar property, that, is, defining the geometric quantity $e_4 = ie_0e_1e_2e_3$, $\mathbf{g}_D = g_De_4$ which results in **k** and **B** being axial vectors and will induce bi-vector electromagnetic fields strengths $\mathbf{F}_m$ of *m*-like current origin

$$e_{\mu}\partial^{\mu}\mathbf{B} = \mathbf{F}_m = \tfrac{1}{3}\left(\partial_{\mu}B^{\mu\lambda\rho} - \partial_{\mu}B^{\lambda\rho\mu}\right)e_{\lambda}e_{\rho}. \tag{2.7}$$

Here, in (2.5), (2.6) and (2.7), we have used the multi-vector scalar product, property $\frac{1}{2}(e_{\mu}e_n + e_ne_{\mu}) = g_{\mu\nu}\mathbf{1}$ with **1** the scalar unit of the geometric algebra and $g_{\mu\nu}$ the metric tensor. This is the reason for having used the symbol $\mathbf{g}_D = ig_De_0e_1e_2e_3$ in equation (2.5) above, that is, the *m*-charge strength multiplied by the (imaginary to be linearly independent) unit pseudo-scalar $ie_0e_1e_2e_3$. It is to be recall that the electroweak interaction theory is constructed using the concept of an (imaginary) axial current for the electron, this being fundamental for understanding the weak interaction (in computing this axial current we do not use the tri-vector $\gamma_{\mu\nu\rho}$ in the Dirac theory but the imaginary tri-vector $i\gamma_5\gamma_{\mu}$, then we are not using the dual of the electron current, computed with γ_{μ}, we use a geometrically new type of current).

The electromagnetic field strengths arc considered as complex quantities whilst the currents are considered real functions.

Geometrically in relation to (2.1), to construct a geometric; theory of the electron there are then two possibilities, either there is some unit magnitude geometric factor missing or quantities related to action are given, in relation to space–time, (real or imaginary) pseudo-scalar properties. We should remember that spin and the Compton radius are axial 3-D vector quantities (that is bi-vectors in space–time). The analysis below will show that, if considered jointly with space–time, this second option is the more useful identification of the geometric properties of the action. Otherwise, when action is considered separately it is always properly represented as a scalar quantity. From the point of view of units the action constant is then, geometrically, both the constant relating energy–momentum to space–time and the key to the construction of an unified geometry of space, time and action. The new geometry below is derived from the introduction of an action coordinate $x^4 = a(\mathbf{x})\,d_0/h$ where *a* is (*i* times) the density of action at a given point **x** of space–time, *h* is Planck's constant and d_0 is an invariant basic length to be determined below, basically $x^4 = k_0a(\mathbf{x})$.

Remark. We use the traditional indices 0,1,2,3 for time and space, also, when convenient the isomorphism between the Dirac gamma symbols with the vectors in the geometry of space–time $e_\mu \Rightarrow \gamma_\mu = \gamma(e_\mu)$ and, when necessary, the representation of the γ_μ either by real $R(4)$ or by complex 4 x 4 matrices $C(4)$. In fact a special property of the pseudo-scalar in space–time $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ is that $\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5$ (from $\gamma_\mu\gamma_n = -\gamma_n\gamma_\mu$, $\mu \neq \nu$) and then it has the same commuting properties with the generating vectors as the generating vectors among themselves. The linearly independent combination $e_4 + i\gamma_5$, has the same commuting properties. The multi-vector $i\gamma_5 = \gamma(e_4)$ is then the immediate candidate to introduce an additional basis vector, adding one more dimension and, through its use, obtain the five dimensional carrier space spanned by the basic vectors e_ν, $u = 0,1,2,3,4$ (identified as $e_0 \Rightarrow \gamma_0$, $e_1 \Rightarrow \gamma_1$, $e_2 \Rightarrow \gamma_2$, $e_3 \Rightarrow \gamma_3$ and $e_4 \Rightarrow i\gamma_5$) with metric $guv = -\text{diag}(1, -1, -1, -1, -1)$. Its use allows the construction of a geometrical framework for the description of physical processes: a unified space–time–action geometry G_{STA}, mathematically a vector space with a quadratic form (see Chapter 7). Here we are implicitly assuming the Majorana representation of the gamma matrices.

The product of $e_4 \Rightarrow i\gamma 5$ with any element of the original space–time Geometry G_{ST} is in the basis vector set a purely imaginary quantity with the result that the $2(2^4) = 2^5 = 32$ elements of the new (space–time–action) geometry G_{STA} are equivalent to $G_{STA} = G_{ST} \otimes C$. In the GSTA geometry the coordinates are real numbers. This is discussed in detail in Chapter 7.

The value of d_0 we shall use below (taken from the theory of the electron) will be $d_0 = r_{Compton} = r_0/2\alpha$, the Compton radius, where $r_0 = e^2/m_0c^2$, the radius that relates the mass of the electron to an electromagnetic equivalent energy, and also $r_0 = \mu/\mathbf{g}_D c$ the ratio of the electron magnetic moment to the hypothetical Dirac *m*-charge. With this choice the presentation of the theory will immediately be suitable for the study of elementary particles. Nevertheless, we shall show that the same units are practical in the study of gravitational interactions. With this choice $k_0 = 1/m_0c$, a dimensional universal constant, provides a geometric character, fundamental in the theory, to the rest mass of the electron m_0.

2. SPACE–TIME TO SPACE–TIME–ACTION

As stated in Chapter 1 our previous studies have followed an inductive process, searching for and analyzing in a uniform mathematical form the different geometrical structures in fundarriental physics. Here we directly present a deductive type of theory.

2.1 FORMAL PRESENTATION

The ideas developed in START (Space–Time–Action Relativity Theory) can be derived from the systematic use of the following principles and postulates.

First Principle: Constancy of the **velocity** of light in vacuum. Independently of the state of movement of the source or of the observer (Poincaré–Einstein Relativity [48]).

First Postulate: There is a geometry, corresponding to space–time, where the First Principle is satisfied (Minkowski space–time with local pseudo-Euclidean structure).

Second Principle: Constancy of the **action** corresponding to an elementary physical phenomenon. Independently of the state of movement of the phenomenon or of the observer. Each observer considers the physical entity as an amount of action *A* contained in a given space-time volume $V_{ST}^{observer}$, *A* is a relativistic invariant.

Second Postulate: There is a geometry corresponding to the union of space–time and action where the First and Second Principles are satisfied (pseudo-Euclidean structure).

Main Theorem KT: Complex Structure Theorem . The geometry where the Second Postulate is satisfied is a five-dimensional basis geometry, mathematically corresponding to a particular complexification of space–time.

Third Principle: The changes in action always occur as integer multiples of *h.* Equivalent to the action per cycle being an integer multiple of $\hbar/2$. (This has to be a constitutive part of the units and practical use of **KT** theorem).

The relation between a 5-dimensional geometry and the complexification of the basis set has been briefly presented in the introduction and will be discussed below.

Fourth Principle: The distribution of action in space–time corresponding to a physical system is unique and any description of this distribution should be equally acceptable.

Third Postulate: The equivalent acceptable descriptions of a physical system are related by isometries and gauge transformations in the space–time–action geometric space corresponding to the Second Postulate.

Proof of **KT:** We have the kinematical concept of trajectory ($\mu, \nu = 0, 1, 2, 3$) with a quadratic form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu , \tag{2.8}$$

generated by the dx^μ and we want to include as a fifth coordinate the dynamical concept of action and its distribution

$$dA = p_\mu dx^\mu \quad \text{which defines} \quad p_\mu = \partial A/\partial x^\mu, \tag{2.9}$$

then write $p_\mu = \tan\Theta(\mu)$ and join formally, defining $-\mathbf{i}^2 = \mathbf{1}$, into

$$dS^\mu = dx^\mu(\mathbf{1} + \mathbf{i}\kappa_0 \tan\Theta(\mu)), \tag{2.10}$$

to obtain from the real quadratic form

$$dS^2 = g_{\mu\nu}\tfrac{1}{2}\{dx^\mu(\mathbf{1} + \mathbf{i}\kappa_0 \tan\Theta(\mu))\, dx^\nu(\mathbf{1} - \mathbf{i}\kappa_0 \tan\Theta(\nu)) + \mathbf{c.c.}\}, \tag{2.11}$$

$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu \left(\mathbf{1} + \kappa_0^2 \tan\Theta(\mu)\tan\Theta(\nu)\right), \tag{2.12}$$

or, in five-dimensional formulation

$$dS^2 = g_{uv}dx^u dx^v = ds^2 + \kappa_0^2 dA^2; \quad u, v = 0, 1, 2, 3, 4, \tag{2.13}$$

where $k_0^2 dA^2$ corresponds to the square of action. The basic dynamical equation is

$$\delta \int dS^2 = 0, \tag{2.14}$$

in a joint minimization of trajectory and action. Gravitation will require extremum (shortest) trajectories and the common procedure of Langrangian minimization, the minimization of action. The universal constant k_0 expresses the action as an equivalent distance and $(dx^4)^2 = (k_0 dA)^2$, with $g_{vv} = \text{diag}(-1, 1, \mathbf{1}, \mathbf{1}, \mathbf{1})$ defining the metric of the equivalent five dimensional geometry basis vectors. Here the diagonal time-like term $g_{00} = -1$ and the action term $g_{44} = 1$. ■

2.1.1 ACTION DENSITY TO ACTION COORDINATE

Consider the space–time points ($x^0 = ct$, c velocity of light, t time), the set of basis vectors $\{e_\mu;\ \mu = 0, 1, 2, 3;\ e_\mu e_n = -e_n e_\mu\}$, such that

$$\mathbf{X} = x^\mu e_\mu, \quad \mu = 0, 1, 2, 3,$$

where an action density function *ia* (**X**) is defined as an equivalent length

$$\kappa_0 ia\,(\mathbf{X}) = ik\,(\mathbf{X}), \tag{2.15}$$

otherwise the amount $A\,(x)$ of action in a unit space–time volume $e_0e_1e_2e_3$ (at **X**)

$$A\,(\mathbf{X}) = a\,(x)\, e_0e_1e_2e_3 = a\,(\mathbf{X})\, e_5, \tag{2.16}$$

representing the product of the space–time vectors by

$$e_5 = e_0 e_1 e_2 e_3, \quad e_5^2 = -\mathbf{1}, \quad e_5 e_\mu = -e_\mu e_5, \tag{2.17}$$

combining the imaginary unit in (2.16) and the unit volume in (2.17) define (here the notation $A := B$ is not an equation relating A to B, it means that the object (A) on the left is defined by the expression (B) appearing on the right) the additional basis vector

$$e_4 := ie_5, \quad e_4^2 = \mathbf{1}, \quad e_4 e_\mu = -e_\mu e_4, \tag{2.18}$$

is then the new basis element to go from STRT→START. This generates the map

$$\text{complex } \mathcal{C}\ell_{1,3} \to \mathcal{C}\ell_{4,1}. \tag{2.19}$$

In general, for a set of multi-vector (functions) $M = m^A e_A$, where e_A= scalar, vector, bi-vector, tri-vector, pseudo-scalar, we have the complex-ification

$$(b_1 + ib_2)\, M \to b_1 M + b_2 \mathbf{J} M,$$

$$\mathbf{J} e_\mu = e_\mu \mathbf{J}, \quad \mathbf{J}^2 = -1. \tag{2.20}$$

The basis multi-vectors are mapped

$$\{e_A\} \to \{\{e_A\} \oplus \{\mathbf{J} e_A\}\}, \tag{2.21}$$

and in fact the formal definition of e_4 should have been written

$$e_4 = \mathbf{J} e_5 = \mathbf{J} e_0 e_1 e_2 e_3, \tag{2.22}$$

with the, by construction, properties

$$e_4^2 = -\mathbf{1}, \quad e_4 e_\mu = -e_\mu e_4. \tag{2.23}$$

This is a relativistic approach including action we have called START.

For the study of the units to be used in this unified geometry consider the definition

$$m_0 c^2 = \frac{hc}{4\pi r_{\text{Compton}}}, \quad r_{\text{Compton}} = \frac{\hbar}{2m_0 c} = \frac{r_0}{2\alpha}, \quad \alpha = \frac{e^2}{\hbar c} \approx \frac{1}{137}, \tag{2.24}$$

and the relation between mass and $k_0 = d_0/h = 4\pi r_{\text{Compton}}/\text{h} = 1/m_0 c$. In general relativity, as analyzed below, this will imply that a unit curvature of space–time in the action direction would correspond to that,

generated by an energy density proportional to the energy of the electron mass. The classical limit of the unification of action to space–time is obtained when $K_0 \rightarrow \infty$ in a form similar to the classical limit, of the unification of space and time being obtained when $c \rightarrow \infty$. Note that $k_0 >> c$. The unification of minimal trajectories and minimal action could also be relevant in the field of high energy physics.

This set of *principia* and postulates, together with the geometric minimum action principle $\delta(dS^2) = 0$, defines the START theory. In this approach: the physical world corresponds to a 4-dimensional surface in the space–time–action geometry; elementary phenomena to exchange of multiples of the elementary action h between distributions of action (provided that those exchanges preserve the local symmetry constrains); a creation of a pair of elementary particle fields (distributions) requires a vanishing sum of symmetry properties in the new fields (except for pre-existing constraints in the action distribution). These will be the guiding principles below.

If in a specific system action is conserved, then in G_{STA} energy–momentum is conserved, but overall conservation rules with local fluctuations are not excluded by the formalism.

3. THE REPRESENTATION OF THE START GEOMETRY

From the considerations above the fifth coordinate corresponds to the use of x^4 to represent **action density,** not the accumulated action. Because action for a given amount of energy distribution in space would be an ever increasing function of time, even if changes local in time are meaningful, a natural representation could also be the use of a **rate of change of action density** over space–time. This will be distribution of energy in space moving in time. A representation closer to our common visualization of nature. In this representation large enough, stable in time, densities of energy at rest with respect to an observer will be seen as mass distributions. The definitions given, and that, we have a variable action density, allow the introduction of a series of operators for computing the dynamical quantities defined so far. First, the energy–momentum operator $\hat{\mathbf{p}}$, which would then be, with the action units included in its definition,

$$\hat{\mathbf{p}} = \mathbf{i}h\gamma^{\mu}\partial_{\mu} \tag{2.25}$$

A more mathematical, less intuitive, representation would be that of the use of an auxiliary circular hypercomplex function Φ and its factorization into auxiliary amplitudes to compute from it the action density

$a(\mathbf{X})$

$$a(\mathbf{X}) = \hat{a}(\Phi), \quad \Phi = \rho e^{-2\pi \mathbf{i} a(\mathbf{X})/h} \triangleq \Psi \gamma^0 \Psi^\dagger, \tag{2.26}$$

where $\hat{a}(\Phi)$ is the action density operator: $\rho = \rho(\mathbf{X})$ is a carrier density, γ^0 a time unit vector and $\mathbf{i}$ would be an appropriate multi-vector such that $\mathbf{i}^2 = -\mathbf{1}$.

The representation in (2.26) of the action density will be discussed in Chapter 3, it leads immediately to the concept of an auxiliary action amplitude function Ψ and to the possibility of selecting the multi-vector imaginary unit in relation to our third principle above. It is a fundamental indication that the generators of angular momentum are the (bi-vectors in STA) $i\gamma_{ij}$ and that angular momentum is expressed in the same units as action see (2.1) above. Also Ψ has to be a multi-vector valued function of the space–time coordinates $\mathbf{X}$.

Here we can consider a fundamental concept related to our third principle above. Because action has to be supplied or emitted in multiples of h, when a given energy ε_0 or momentum p_0 is considered a characteristic associated time τ or distance λ are automatically defined in START:

$$\mathcal{E}_0 = \frac{h}{\tau} \quad \text{or} \quad p_0 = \frac{h}{\lambda}; \tag{2.27}$$

from (2.25) ε_0 and p_0 are geometrically in the tangent space of ST and the associated action in the $e_4 e_\mu$ planes which, from (2.27), have symplectic structure. An area in a plane can be associated with a radius, a characteristic radius which should obey (2.27) in the form

$$\tau = 2\pi\tau_0 \quad \text{or} \quad \lambda = 2\pi\lambda_0. \tag{2.28}$$

Then defining, as usual, $\hbar = h/2\pi$ we can write (2.27) in the form

$$\mathcal{E}_0 = \hbar/\tau_0 = \hbar\omega \quad \text{or} \quad p_0 = \hbar/\lambda_0 = \hbar\varpi. \tag{2.29}$$

With the definitions in (2.26) the energy–momentum operator becomes the usual in Quantum Mechanics

$$\hat{\mathbf{p}} = \mathbf{i}\hbar\gamma^\mu\partial_\mu = \mathbf{i}\hbar\Box, \tag{2.30}$$

where $\Box = \gamma^\mu\partial_\mu$ is the space–time gradient operator where $\mu = 0, 1, 2, 3$. We should remind the reader that our definitions here are the direct consequence of the principles above, not additional postulates. The formulas (2.26) and (2.30) are a necessary starting point for characterizing the action distribution corresponding to our basic principles.

The inclusion of γ^0 in (2.26) is needed to specify the observer's frame of reference to which the energy distribution is related as at rest. For

that 'observer', as a reference point, time is the only evolving variable. With this presentation of our formalism we could recover the formulation of the quantum mechanics of the electron in terms of multi-vectors. See Chapters 3 and 5.

4. THE KINETIC ENERGY IN START

From our definitions we are considering two quantities: a given amount of action A distributed over a space–time volume V_{ST} corresponding to an action density $a(\mathbf{X})$ at point $\mathbf{X}$ in space–time; and second, the usual quantities energy $\int dV\, \partial_t\, a(\mathbf{X})$ and the corresponding momenta $\int dV\, \partial_{xi}\, a(\mathbf{X})$. One of the basic relations in relativistic dynamics is the transformation of the above quantities with respect to observers in relative motion with a relative velocity υ_{12}.

For observer 1 the energy can be written

$$\mathcal{E} = mc^2 = \int dV\, \partial_t\, a(\mathbf{X}), \tag{2.31}$$

if by definition the energy corresponds to a mass m when we can consider that for this observer that object is at rest and then no momenta are involved. In our formulation c^2 is a universal constant, and then (2.31) is just a change of variables useful to make contact with standard approaches. From the definition

$$p_i = \int dV\, \partial_{x_i}\, a(\mathbf{X}), \tag{2.32}$$

we see that there is a geometrical constraint that any space variation of the action distribution, in the volume of integration, should be symmetric, for the observer in relation to which the distribution is at rest in order that the integral to vanish. In the case $p_i = 0$ the whole object of mass m is considered at rest, independently of its internal description. Notice that it is not a point-like object, even if it can be considered small in relation to some characteristic distance.

For an, in relative motion with respect to the first, observer 2 the same relations hold but this observer will anyhow determine that because the distribution is moving in his frame of reference there is an effective momenta in the direction of the relative translation. Another change is that this observer will consider that the local values of the density function have increased in his frame of reference because the same amount of action is distributed in a smaller space–time volume. The energy for this observer will be $\varepsilon' = m'c^2 = \int dV' \partial_t'\, a'(\mathbf{X}')$, larger, as we shall see below, than ε, given that in his frame of reference the derivatives have

to be computed with respect of its own, shorter time intervals, *dt'*, where

$$\frac{dt'}{\sqrt{1-\frac{v_{12}^2}{c^2}}} = dt \text{ from } (cdt)^2 - (v_{12}dt)^2 = (cdt')^2. \tag{2.33}$$

For the observer 2 the measurement corresponding to the prime system

$$\mathcal{E}' = \frac{\partial}{\partial t'}A = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\frac{\partial A}{\partial t} = \frac{\mathcal{E}}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{2.34}$$

The action *A* related to that physical entity is invariant

$$A = \int_{V_{ST}} dV_{ST}a(\mathbf{X}) = \int_{V'_{ST}} dV'_{ST}a'(\mathbf{X}), \tag{2.35}$$

because the density changes as the inverse volume

$$\frac{A}{V} \circledast \frac{A}{V'}, \tag{2.36}$$

but the integral is performed as the volume

$$\frac{A}{V}V - \frac{A}{V'}V'. \tag{2.37}$$

Then for the observer 1 in relative motion the action *A* corresponds to a smaller time $\Delta t'$ and consequently the associated energy is $m'c^2 = mc^2 / \sqrt{1 - v^2/c_2}$. If the observer 1 calls the (for him at rest) energy ε in his system 'the mass',

$$mc^2 = \varepsilon,$$

observer 2 concludes that in the moving system

$$\mathcal{E}' = \frac{mc^2}{\sqrt{1-\frac{v^2}{c^2}}} \circledast \text{ with } \lim(v \ll c) \to mc^2 + \tfrac{1}{2}mv^2 - ..., \tag{2.38}$$

and also, because the prime system is considered in motion, he can call the energy ε' the sum of the rest (mass) energy ε and the kinetic energy ε_k which for the slow motion is approximated by the typical $\varepsilon_k = \frac{1}{2}mv^2$ term. Otherwise for the system in motion a characteristic length of the energy distribution has been changed from p_0/h to p'_0/h, together with a characteristic time $\varepsilon/h, \to \varepsilon'/h$, these quantities should be part of the gauge-free description used in the theory.

Considering (2.38) again, we can write

$$\mathcal{E} = h\nu_0, \quad \mathcal{E}' = h\nu = h(\nu_0 + \Delta\nu), \tag{2.39}$$

and then in the low velocity approximation we can write

$$\mathcal{E}_k = h\nu_k \quad \text{where} \quad \nu_k = \Delta\nu \geq 0. \tag{2.40}$$

Here ν will be the characteristic frequency of the Dirac equation, whereas $\lambda = p/h$ and ν_k will be the characteristic wavelengths and frequencies of the Schrödinger equation as derived from START.

5. DYNAMICAL PRINCIPLES

In space–time–action geometry the main dynamical principle is that all trajectories should be minimal. Defining the (square of the) differential $(dS^2) = (ds)^2 - (da)^2$, where now g_{AB} = diag (1, –1, –1, –1), $(ds)^2 = g_{\mu n}dx^\mu dx^n$ is the space–time differential and $(da)^2$ the action differential. We have, taking $k_0 \to \infty$, that in a first, non-united geometry, approximation the minimal principle

$$\delta(dS)^2 = 0, \tag{2.41}$$

could be separated into the kinematical principle of (general) relativity

$$\delta(ds')^2 = 0, \tag{2.42}$$

and the principle of minimum action

$$\delta(da')^2 = 0. \tag{2.43}$$

Here we have defined

$$(ds')^2 = (ds)^2 - \left[(da)^2 - (da')^2\right], \tag{2.44}$$

as a modified space–time interval square which, in fact, corresponds to considering a curved effective space–time as will be shown below. The action interval square $(da')^2$ corresponds to some 'inactive' part of the action in relation to a given description.

For some phenomena, light as the main example, (2.42) and (2.43) are separately obeyed given that $(cdt)^2 - (dx)^2 = 0$ and $g_{\mu n}p^\mu x^n = 0$ because $g_{\mu n}$ = diag(1, –1, –1, –1) and $\varepsilon = pc = h\nu = hc/\lambda$ Otherwise, the principle of minimal action is universally accepted in the formulation of physical principles. If (2.41) is accepted a geometrical model for mass appears in our theory. See Chapter 4.

5.1 STA TRAJECTORIES

Consider again the space–time–action (STA) space described by an observer's basic flat STA geometry with local orthogonal vectors e_A; $A = 0, 1, 2, 3, 4$; where e_4 denotes the action coordinate corresponding to g_{AB} = diag (1, –1, –1, –1, –1). For this observer there are two types of trajectories to study fields:

1) massive (m_0 parameter), that when at rest have STA interval

$$dS^2 = g_{AB}dx^A dx^B = 0 = (cdt)^2 - \left(\kappa_0 m_0 c^2 dt\right)^2, \tag{2.45}$$

which in fact defines $k_0 = 1/m_0 c$ in terms of m_0 and c (in our case the electron mass and the velocity of light); and

2) massless ($e = h\nu$, $p = h/\lambda$, $c = \lambda\nu$ parameters) for which $x^2 + y^2 + z^2 = (ct)^2$ and $a^2 = 0$ as a consequence

$$dS^2 = g_{AB}dx^A dx^B = 0 = (cdt)^2 - \left(dx^2 + dy^2 + dz^2\right) - (\kappa_0 da)^2 \tag{2.46}$$

Here the action of one unit of action ($\mu = 0,1,2,3$)

$$-i\hbar a(x) = \left[g_{\mu\nu}p^\mu x^\nu\right]_{\text{unit of action}} = e\tau - p\lambda = h\left(\tau\nu - \lambda/\lambda\right) = 0,$$

is a null valued function with the property ($i = 1,2,3$)

$$i\hbar\partial_t a(x) = E \quad \text{and} \quad i\hbar\partial_i a(x) = p_i.$$

In both cases the local action density is $A(x) = i\hbar c_d \rho a(x)$ in space–time where c_d is a numerical coefficient of geometrical origin (this will be relevant to the study of the families of elementary particles and their mass), ρ the normalized density of action of the carrier.

For the system we want to describe below the observer uses the START-Principle of Acceptable Choice formalism in a form in which the fields can be considered as a distribution corresponding to a 'test' carrier in the geometry induced by a 'source', for this purpose we use the solution of the equations for the STA geometry when each carrier in turn is described by applying the overall condition of null curvature $R_{AB} = 0$ with

$$\int_{ct_0}\int_{\Omega_x} a(x)\, d\mathbf{x}\, dx^0 = 2; \quad m_0 c^2 t_0 = h,$$

for the observers' frame of reference. We shall consider two approaches, the first one corresponding to General Relativity (next section) and the second corresponding to Density Functional Theory and Quantum Mechanics (Chapter 3).

5.2 GAUGE FREEDOM IN START

In the description of the basic postulates we have, as shown in the KT theorem, introduced action as a complexification of space–time with the kinematical concept of trajectory (μ, ν = 0, 1, 2, 3) with a quadratic form

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{2.47}$$

generated by the dx^μ and the dynamical concept of action distribution (which in fact is introduced in a form equivalent to a fifth coordinate) starting from the usual basic relation

$$dA = p_\mu dx^\mu \quad \text{which defines} \quad p_\mu = \frac{\partial A}{\partial x^\mu}, \tag{2.48}$$

writing $p\mu$ = tan Θ (μ) and formally joining into

$$dS^\mu = dx^\mu (1 + \mathbf{i}\kappa_0 \tan\Theta(\mu)), \tag{2.49}$$

to obtain from the real quadratic form

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu \left(1 + \kappa_0^2 \tan\Theta(\mu) \tan\Theta(\nu)\right), \tag{2.50}$$

a five-dimensional formulation ($g_{\mu\nu}$ = diag (–1, 1, 1, 1, 1) = $-g_{AB}$) written as

$$dS^2 = g_{uv} dx^u dx^v = ds^2 + \kappa_0^2 dA^2, \quad u, v = 0, 1, 2, 3, 4, \tag{2.51}$$

where $k_0^2 dA^2$ is some equivalent interval square which corresponds to the square of action. The basic equation (2.50) is then

$$dS^2 = g_{\mu\nu}^{(0)} \left[f^{(\mu)} \overline{f}^{(\nu)} \right]_{\mathrm{Re}} dx^\mu dx^\nu, \tag{2.52}$$

where the space–time differentials $dx^\mu dx^\nu$ become a factor and the f_λ are complex functions which should allow a gauge free description of the action, introducing a phase factor φ with $\varphi\bar{\varphi}$ = 1

$$f^{(\lambda)} dx^\lambda \to \left(1 + i \sum_a \frac{\partial A_a}{\partial x^\lambda} \varphi\right) dx^\lambda, \tag{2.53}$$

such that if the action is taken to consists of several contributions

$$A = \sum_a A_a, \tag{2.54}$$

we can also define

$$\varphi = e^{i\sum_a A_a} \quad , \tag{2.55}$$

for the second term in the parenthesis of (2.53) to be

$$\partial_\lambda \varphi = i \sum_a \frac{\partial \Lambda_a}{\partial x^\lambda} \varphi, \tag{2.56}$$

with (2.54) containing the gauge freedom to describe the action.

5.2.1 THE ELECTRON AS A GENERAL RELATIVITY TEST PARTICLE

In general relativity, meant to be a comprehensive theory, the best known solutions are developed for the so called matter-free space and a test particle. We use this concept to show that (2.44) corresponds to the description of the action distribution which agrees with the conceptual development of General Relativity.

In our present theory there are two fundamental carrier structures: the massless fields and the massive electron field with basic relation

$$\mathcal{E}^2 = (\mathcal{E}_0 + \Delta\mathcal{E})^2 , \tag{2.57}$$

where $\Delta\varepsilon$ is a gauge-free energy contribution and $\varepsilon_0 = m_0c^2$ the energy, at rest relative to an observer, considered to be the mass of the carrier.

The concept of test particle in general relativity in the Schwarzschild solution is compatible with the Newtonian limit for the interaction gravitational energy

$$\Delta\mathcal{E}(r) = -m_0 \frac{GM}{r}, \tag{2.58}$$

and, conceptually, with the use of the action square difference, writing $\varepsilon = \varepsilon_0 + \Delta\varepsilon$ for large (classical limit) values of r

$$\begin{aligned} \mathcal{E}^2 - \mathcal{E}_0^2 &= \mathcal{E}_0^2 + 2\mathcal{E}_0\Delta\mathcal{E} + (\Delta\mathcal{E})^2 - \mathcal{E}_0^2 \\ &= 2\mathcal{E}_0\Delta\mathcal{E} + (\Delta\mathcal{E})^2 \to -2m_0c^2 m_0\frac{GM}{r} + \left(m_0\frac{GM}{r}\right)^2 , \end{aligned} \tag{2.59}$$

this corresponds to $(da)^2 - (da')^2$ if $(da')2 = (m_0c^2dt)^2$ also

$$\mathcal{E}^2 - \mathcal{E}_0^2 = (pc)^2 \to (p_r c)^2 = -2m_0c^2 m_0\frac{GM}{r} + \left(m_0\frac{GM}{r}\right)^2 , \tag{2.60}$$

and substituting in (2.51) using $k_0 = 1/m_0 c$ and space spherically symmetric coordinates t, r, θ, ϕ we obtain

$$(dS)^2 = \left(1 - 2\frac{MG}{c^2 r} + \left(\frac{GM}{cr}\right)^2\right) c^2 (dt)^2 - \left(1 + \frac{2GM}{c^2 r} - \left(\frac{GM}{cr}\right)^2\right) (dr)^2 - r^2 \left[(d\theta)^2 + \sin^2\theta\,(d\phi)^2\right], \tag{2.61}$$

which is the Schwarzschild 1916 [168] metric in the limit of $r \gg MG/c^2$ (sometimes (2.61) is called the Eddington form [47], notice $(1 - x)^{-1} \cong (1 + x)$, $x \ll 1$, this relation is also used below.).

It is customary to write [169]

$$f(r) = \left(1 - 2\frac{MG}{c^2 r} + \left(\frac{GM}{cr}\right)^2\right) \text{ and } h(r) = \left(1 + \frac{2GM}{c^2 r} - \left(\frac{GM}{cr}\right)^2\right), \tag{2.62}$$

or $f(r) = 1 + b^2(r)$ and $h(r) = 1 - b^2(r)$, also for $cr \gg MG$ we obtain the Schwarzschild relation $f \cong h^{-1}$, which can afterwards be used for all r in matter free space following the Einstein's definition which requires the curvature to be identically zero.

The result (2.61) shows that our approach provides a conceptual understanding of the role of sources carriers and test particles in general relativity. It also shows the possibility of extending the analysis to circumstances more difficult to consider within the traditional approaches.

Once we have obtained the Schwarzschild metric we can now find the **curved hypersurface in START corresponding to the curved space–time** where the test particles are assumed to move. Formally we need to define a set of vectors e_μ, $\mu = 0,1,2,3$, $g_{\mu\nu} = \text{diag}\,(1, -1, -1, -1)$, and their reciprocal, in terms of a vierbein using the Minkowski space reference vectors

$$\left\{e_\mu^{(0)} = \widehat{e}_\mu,\ \mu = 0,1,2,3;\ e_\mu^{(0)} e_\nu^{(0)} + e_\nu^{(0)} e_\mu^{(0)} = 2g_{\mu\nu}^{(0)}\right\}$$

For this obtain first the Fock–Ivanenko bi-vectors $\Omega(a) = a^\nu \Omega_\nu^{\alpha\beta} e_{\alpha\beta}^{(0)}$, starting from the induced metric

$$g_{\alpha\beta} = \begin{bmatrix} c^2 f(r) & 0 & 0 & 0 \\ 0 & -h(r) & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2\theta \end{bmatrix}, \tag{2.63}$$

and the inverse metric (both are diagonal)

$$g^{\alpha\beta} = \begin{bmatrix} 1/c^2 f(r) & 0 & 0 & 0 \\ 0 & -1/h(r) & 0 & 0 \\ 0 & 0 & -1/r^2 & 0 \\ 0 & 0 & 0 & -1/(r^2 \sin^2\theta) \end{bmatrix}. \tag{2.64}$$

From these two equations, it is clear how to construct an orthonormal system of vectors and dual vectors (in the Eddington's representation)

$$e^t = \frac{1}{cf^{\frac{1}{2}}}\widehat{e}^0 = \frac{1}{c^2 f} e_t, \tag{2.65}$$

$$e^r = \frac{1}{h^{\frac{1}{2}}}\widehat{e}^1 = -\frac{1}{h} e_r, \tag{2.66}$$

$$e^\theta = \frac{1}{r}\widehat{e}^2 = -\frac{1}{r^2} e_\theta, \tag{2.67}$$

$$e^\phi = \frac{1}{r\sin\theta}\widehat{e}^3 = -\frac{1}{r^2\sin^2\theta} e_\phi, \tag{2.68}$$

with the local vectors being

$$e_t = c\left(f(r)\right)^{\frac{1}{2}}\widehat{e}_0, \tag{2.69}$$

$$e_r = \left(h(r)\right)^{\frac{1}{2}}\widehat{e}_1, \tag{2.70}$$

$$e_\theta = r\widehat{e}^2, \qquad e_\phi = r\sin\theta\widehat{e}^3. \tag{2.71}$$

Since our metric is diagonal, we can use the following relation (see Chapter 7 and [169]) to compute the Fock–Ivanenko 2-vectors

$$\Omega_\alpha = \tfrac{1}{4} e^{\beta\alpha}\frac{\partial}{\partial u^\beta} g_{\alpha\alpha}, \tag{2.72}$$

where the index β is summed but α is not. From this one obtains

$$\Omega_t = \frac{c^2 f'}{4} e^{rt} = \frac{cf'}{4(fh)^{\frac{1}{2}}}\widehat{e}^{10}, \tag{2.73}$$

$$\Omega_r = 0, \tag{2.74}$$

$$\Omega_\theta = -\frac{\gamma}{2} e^{r\theta} = -\frac{1}{2h^{\frac{1}{2}}}\widehat{e}^{12}, \tag{2.75}$$

$$\begin{aligned}\Omega_\phi &= -\frac{r\sin^2\theta}{2} e^{r\phi} - \frac{r^2\sin\theta\cos\theta}{2} e^{\theta\phi} \\ &= -\frac{\sin\theta}{2h^{\frac{1}{2}}}\widehat{e}^{13} - \frac{\cos\theta}{2}\widehat{e}^{23}\end{aligned} \tag{2.76}$$

To obtain the curvature 2-form we use its expression in terms of the Fock–Ivanenko 2-vectors

$$\tfrac{1}{2}\mathcal{R}_{\alpha\beta} = \frac{\partial}{\partial u^{\alpha}}\Omega_{\beta} - \frac{\partial}{\partial u^{\beta}}\Omega_{\alpha} + \Omega_{\beta}\Omega_{\alpha} - \Omega_{\alpha}\Omega_{\beta}. \tag{2.77}$$

From this formula, we obtain

$$\begin{aligned}\tfrac{1}{2}\mathcal{R}_{rt} = \frac{\partial}{\partial r}\Omega_{t} &= \left[\frac{cf''}{4(fh)^{\frac{1}{2}}} - \frac{cf'(fh)'}{8(fh)^{3/2}}\right]\hat{e}^{10} \\ &= \left[\frac{f''}{4(fh)} - \frac{f'(fh)'}{8(fh)^{2}}\right]e_{rt},\end{aligned} \tag{2.78}$$

In a similar fashion we obtain the Ricci curvature components

$$\tfrac{1}{2}\mathcal{R}_{\theta t} = \frac{-f'}{4(fh)\,r}e_{\theta t}, \tag{2.79}$$

$$\tfrac{1}{2}\mathcal{R}_{\phi t} = \frac{-f'}{4(fh)\,r}e_{\phi t}, \tag{2.80}$$

$$\tfrac{1}{2}\mathcal{R}_{r\theta} = \frac{h'}{4(h)^{2}\,r}e_{r\theta}, \tag{2.81}$$

$$\tfrac{1}{2}\mathcal{R}_{\phi r} = \frac{h'}{4(h)^{2}\,r}e_{\phi r}, \tag{2.82}$$

$$\tfrac{1}{2}\mathcal{R}_{\theta\phi} = \frac{h-1}{2hr^{2}}e_{\theta\phi}, \tag{2.83}$$

therefore what are considered to be particular solutions of the Einstein equations can be directly obtained from our basic definitions.

The extension of this analysis to the space–time regions where the sources are considered to be (as an action distribution) will be given in the next section.

Notice that in (2.61), because of the use of k_0, the test particle parameters have cancelled. This is the reason for this approach to be universal. As a result the solutions do not depend at all in the test particle considered, provided that its definition is the Einstein definition: a sufficiently small mass to be introduced into the system without noticeably disturbing it. The action related to the intrinsic energy–momentum of the test

particle is 'inactive' in the analysis. Otherwise we could have introduced in (2.59), (2.60) and (2.61) other contributions to the action, but this can not be done in a universal form if they are of electromagnetic type (a factor e/m_0 can not be removed). Anyhow, it is suggestive that it can be done, new terms appear for **that** test particle, including cross terms like (QM/r^2) (e/m_0) and others. Also in (2.61) there is a quadratic, repulsive, term which can have physical significance, for distances smaller than the ones where general relativity has been successfully tested.

The discussion above shows that one of the possible symmetries in START is the transformation of position vectors **y** in START to a new set $\{\mathbf{y} = x^u \mathbf{e}_u;\ u = 0, 1, 2, 3, 4\}$

$$\mathbf{y}' = \mathbf{f}(\mathbf{y}) = x'^u e'_u, \tag{2.84}$$

which describes the curvature of the space–time part necessary for representing physical interactions, at the expense of defining 'test' carriers which are now considered as non-interacting amongst themselves, but defining the new representation (2.84) of position vectors and coordinates in START.

5.2.2 DESCRIPTION OF A SYSTEM THROUGH THE USE OF A TEST CARRIER

Before passing to the next section we consider now the full system with local energy density

$$\varepsilon(\mathbf{X}) = m_0 c^2 + m_0 V(\mathbf{X}) - \mathcal{M}(\mathbf{X}) c^2,$$

the first term corresponds to the test particle reference mass, the second to the interaction energy and the third M (**X**) c^2 to the mass of the system we want to describe. For the purposes of the methodology of general relativity, from START, we use the term $k_0^2\ \Delta a^2$ action allocated to the test particle by subtraction, from the total action squared, of the system's basic action (and of, by definition of test carrier, the carrier's constitutive action): $(\varepsilon_s\ (\mathbf{X})\ dt)^2\ -\ (\rho\,(\mathbf{X})\ m_0 c^2 dt)^2$,

$$\begin{aligned}
\kappa_0^2 \Delta a^2 &= \kappa_0^2 \left(\varepsilon(\mathbf{X})\, dt\right)^2 - \left(\varepsilon_s(\mathbf{X})\, dt\right)^2 - \left(\rho(\mathbf{X})\, m_0 c^2 dt\right)^2 \\
&= \kappa_0^2 2 m_0^2 \left[V(\mathbf{X}) + \frac{V^2(\mathbf{X})}{2c^2} + \frac{\mathcal{M}(\mathbf{X})(c^2 + V(\mathbf{X}))}{m_0}\right] c^2 (dt)^2 \\
&= \left[\frac{2V(\mathbf{X})}{c^2} + \frac{V^2(\mathbf{X})}{c^4} + \frac{2\mathcal{M}(\mathbf{X})(c^2 + V(\mathbf{X}))}{m_0 c^2}\right] c^2 (dt)^2 \\
&= b^2 c^2 (dt)^2 = g_{00}^0 b^2 \left(dx^0\right)^2,
\end{aligned} \tag{2.85}$$

the last two equalities are obtained from the particular value $k_0 = 1/(m_0 c)$ we have found for this universal constant. This defines the electron as a fundamental field in the study of matter including the analysis of general relativity. In (2.85) the analysis goes beyond the original terms, in those the nature of the test particle is not relevant to the definition of the curved space–time obtained from the distribution of neutral matter. We have two main regions for this term

a) when $r > R_D$ with R_D the largest radius for which $M(\mathbf{X}) \neq 0$, also the mass of the system to be described $M = \int_D M(\mathbf{X})\, d\upsilon$;

b) when $r \leq R_D$ where the last two terms dominate.

Consider $g_{00}^{GR} = g_{00}^{0}(1 + b^2(r))$ as above. The function $b^2(r)$ describing the curvature is, for large r, dominated by the Schwarzschild term $-2GM/rc^2$. For intermediate values, larger than R_D, the second term will also contribute. But for values $r \leq R_D$ the third term (which in practice is considered as already corrected by the fourth term, even if it is not so) is the dominant contribution.

A formal definition

$$\frac{\partial\left(g_{00}^{0}\frac{2\mathcal{M}(\mathbf{X})}{m_0}\right)}{\partial g_{00}^{0}} = M_{00},$$

gives

$$M_{00} = \left(\frac{2}{m_0}\right)\mathcal{M}(\mathbf{X}) = KT_{00},$$

that is the derivative of this term with respect to the local flat, space metric tensor component coincides with the Einstein's postulate of the Energy–Momentum–Stress tensor $T_{\mu\nu}$ being the quantity that causes the space–time curvature given that

$$\frac{\partial\mathcal{R}}{\partial g_{\mu\nu}} = \mathcal{R}_{\mu\nu} - \mathcal{R}g_{\mu\nu} = G_{\mu\nu},$$

is the Einstein's curvature tensor, and as far as

$$G_{\mu\nu} - KT_{\mu\nu} = 0,$$

by construction. The use of this analysis for all terms $T_{\mu\nu}$ gives then a derivation of the basic relations of General Relativity from START and from the conceptual postulates of Einstein. This procedure being a deductive approach which allows possible changes and extensions. The

relations above are in fact already an extension. From this point of view general relativity is, in its present form, a macroscopic theory corresponding to a geometric transformation which allows an acceptable description of physical phenomena as geometrical properties of space–time. The transformations are made in agreement with the Riemann concept of a manifold being more basic than the metric used to endow it with a geometrical structure.

5.2.3 GENERAL RELATIVITY IN START

From our previous analysis, the structure equivalent to general relativity [49, 50, 51, 52] in START is the following:

- In the flat space–time–action geometry a distribution of action is given and analyzed as corresponding to the total matter and interaction fields (radiation) content. For a given observer at a given time this will appear as an energy, momentum, and stress distribution. The usual name for this is the energy–momentum–stress density tensor, which according to the different descriptions may appear either as an energy–momentum vector in space–time (which has been transformed to the coordinate frame of a given observer, that is, a vector function of a vector) or as the type of description which corresponds to a matter distribution where, besides the local energy–momentum density, the boundary conditions for each volume element introduce in the description the concept of pressures and shear stresses.

- Basically one obtains the structure corresponding to general relativity by the process of transforming this 1 + 3 + 1 geometrical description into the equivalent 1 + 3 description given by a curved space–time.

- Even if the projection of the surface in five dimensions as a four-dimensional space corresponds to the curved space–time of general relativity, the physical meaning of this curved space–time is given by defining the trajectories of ‘test’ particles as the geodesics in this 4-D space. The definition of the test particle is that of a sufficiently small and sufficiently small energy–momentum physical entity such that its trajectory can be considered as that generated by a moving point which will not noticeably change the assumed energy–momentum–stress distribution. An elementary particle in an otherwise macroscopic system will, in practice, fulfill these conditions. In our case the electron is to be considered the reference elementary particle.

- General relativity was constructed with gravitation as the basic interaction; this is very important because gravitation has proved this

far to be universal, that is, any two amounts of energy, within experimental conditions, attract each other through a gravitational interaction. This interaction is weak enough to be non-relevant in the study of elementary particles in themselves or for the study of atoms and molecules, that is, we do not know experimentally whether there arc gravitational effects in the structure of the elementary particles themselves. Even the basic question of whether gravitation, which was discovered as an interaction between masses, plays a significant role in the determination of the mass itself has found no definitive answer. This point will not be discussed in this book even if we are providing a theoretical structure which allows the consideration of this problem.

- Other interactions than gravitation could also be included in general relativity either as the arnount of energy–momentum they represent or as part, of the description of the energy–momentum which is attributed as belonging to a given 'test' particle. See below.

In the construction of the mathematical structure corresponding to quantum mechanics or to the standard model below, we shall not use the concept of a curved space–time as a frame of reference. Therefore we shall introduce gravitation only as a gauge field both in quantum mechanics and in the standard model of elementary particles as deduced from START.

The analysis we have presented here corresponds to changing the status of general relativity from a physical model to a part of a deductive theory where a specific form of description has been selected: that of not using action as a fifth variable but as a guide for constructing the curvature of space–time to obtain the characteristic equations of general relativity. Once, as done below, other descriptions of the interaction have been introduced the equations for the general relativity formulation are straightforward There is nevertheless a basic consideration: the explicit, introduction of electroweak and color interactions in the description of the action distribution requires the definition of the associated charges (electric, weak and color charge) of the test particle itself. Consider the following example.

5.2.4 A CHARGED PARTICLE IN GENERAL RELATIVITY

A charged particle at rest which is acted on by gravitational and electromagnetic interactions will have for the (attributed) total energy (at distances large enough such that the collection of masses with which the test carrier interacts are collectively represented by the volume integral

of a mass density $M(\mathbf{r})$ in the presence of the mass $M = \int_D M(\mathbf{r})d\upsilon$, the following description:

$$\varepsilon = m_0c^2 - m_0\frac{GM}{r} + e\frac{Q}{r}.$$

Substituting this in (2.57)–(2.62) will change the functions $f(r)$ and $h(r)$ into

$$f(r) = 1 - 2\frac{MG}{c^2r} + \left(\frac{GM}{cr}\right)^2 - \frac{e}{m_0}Q\frac{MG}{c^2r^2} + 2\frac{eQ}{m_0c^2r} + \left(\frac{eQ}{m_0c^2r}\right)^2,$$

$$h(r) = 1 + 2\frac{MG}{c^2r} - \left(\frac{GM}{cr}\right)^2 + \frac{e}{m_0}Q\frac{MG}{c^2r^2} - 2\frac{eQ}{m_0c^2r} - \left(\frac{eQ}{m_0c^2r}\right)^2,$$

The analysis of these functions would lead to the following conclusions:

1. Besides the attractive gravitational term there is a (quadratic) repulsive term which will dominate at intermediate distances. This repulsive term, of the form which has been called by some experimentalists the fifth force, will act as an equivalent 'cosmological' constant which will force a distribution (considered to be a universe) to expand and will avoid gravitational (full) collapses of large matter densities.

2. The electric part of the interaction depends explicitly in the e/m_0 ratio of the test particle, and it can then not, be a universal behavior of a test particle, even if the basic geometry has a universal form. From this it is clear that no universal or global consequences can be derived from the geometry, as is customary in the cosmological analysis of the geometries related to the general relativity solutions.

Otherwise, when the relations corresponding to general relativity are derived from START, those entering into the experimental proofs of the validity of general relativity (considered this far) are not changed and retain their validation status. Otherwise the extrapolations of general relativity to the analysis of cases that are not those included in these proofs are to be analyzed anew.

For other interactions a choice (compatible with our principle of acceptable description choice (PAD)) would be that of studying the electromagnetic, etc., field equations in a curved space background, which is the normal approach in GR. Then the field of local tetrads generate a dielectric tensor of 'geometric' origin for the electromagnetic fields.

An important case of mathematical analysis is the description of the trajectory of a test particle in the presence of a massive rotating reference

system. Besides the Schwarzschild part there are two sets of terms: first those related directly to the rotation of the coordinate frame of reference relative motion; second the crossed terms of the rotation (considered as kinetic energy) and the gravitation potential energy. They are studied, for example, in the Lense and Thirring solution [135], which in START corresponds to the double product

$$-2\kappa_0^2 \frac{m_0 GM}{r} \frac{m_0 c}{\sqrt{c^2 - v^2}} \frac{v^2}{2}$$

with $\upsilon = wr \sin\Theta$ the angular velocity w being related to the angular momentum $J = Iw$ of the massive rotating system with moment of inertia $I = M\bar{r}^2$, with equivalent rotation radius $\bar{r}$ and mass M. From $w = d\phi/dt$ the term has the Kerr-like form

$$\frac{GMb(r)}{r} \sin^2\Theta d\phi dt.$$

5.3 THE MATHEMATICALSTRUCTURE OF GENERAL RELATIVITY FROM START

Once we have seen that an electron used as a test particle in the START geometry allows us to obtain the Schwarzschild metric we can now proceed to a systematic derivation of the structure of general relativity from START.

The main considerations are the following.

a) General relativity is a geometric theory describing the trajectories of test particles as the natural trajectories, geodesics, in curved space–time geometry. This is a fundamental postulate in the original formulation of Einstein.

b) The curved space–time is obtained by incorporating, within STA, space from the action part into the ST part. At the level of the test particle the action that was incorporated into the initially flat, space–time was the total additional action which is the result of considering that the rest of the system will have an interaction, gravitational in our example, with the test particle. This corresponds to the amount of action which would be described as belonging to the particle in classical mechanics when the particle is considered as being ‘acted’ upon by the gravitational field, in addition to the action resulting from the rest mass of the particle which was taken as a reference. The mathematical description of the fact, that the particle is now a non-disturbing test object is introduced by subtracting from the square of the total action the square of the action corresponding

to the direct description of the particle. That is, general relativity is a theory where the geometry describes everything that is to be described, through the curved space–time, and the test particle is only an auxiliary in this description. It was important that the STA geometry was created taking the electron as the reference fields and that an electron trajectory was also taken to create the concept, of test particle.

c) The quadratic form obtained was afterwards analyzed using intrinsic geometrical techniques to have, in accordance to the basic geometric postulate of Einstein, a purely geometrical theory. When this is done it is understood that the basic equations, everywhere in space, are the transfer of the relevant action (squared) to the otherwise flat, quadratic form of space–time. Then in order to proceed in the regions of space–time where Einstein considered that a distribution of matter existed we have to construct the equivalent action from the corresponding energy–momentum distribution of that matter which will determine the trajectory of the test particle in that region. For a theory in the continuum, (mainly when there is no warranty that a global Lorentz transformation can put adjacent points in space–time distribution of energy–momentum in a diagonal form for a given observer) the quantity to consider is the energy–momentum–stress tensor ($\mathcal{T}$). In fact $\mathcal{T}$ is a vector-valued function of a vector, which can be, locally, brought into a diagonal form which is the equivalent energy–momentum of the distribution. This is the quantity that will be used in our equations below.

d) If we proceed as in paragraph c) we shall have to follow the analysis of the new quadratic form to obtain the characteristics of the geometry, a lengthy and in some instances non unique procedure. Instead we can directly consider that the quadratic form defines the metric tensor of the new geometry, and then use the definition of the curvature from the metric tensor in the generated curved space–time, to obtain a relation between the curvature and the energy–momentum–stress tensor. The result is a derivation of the starting equations of Einstein which in his formulation are a basic postulate that the presence of matter induces the curvature of space–time.

e) Once we have recovered the structure of general relativity from START we can go beyond, in the sense that we have shown that there were terms beyond the Schwarzschild solution in the basic equations. We can proceed either to go beyond general relativity in our description of matter or to define more complicated test particles (particles

with charges) or to use the enlarged equations to study very suggestive possibilities for general relativity, for elementary particle theory, and for cosmology.

We now proceed to write in some detail the equations related to the five considerations above.

5.3.1 THE METRIC IN GR

Once we have created the equivalent curved space–time the metric in GR is given through the use of the line element (here $g_{\mu\nu} = g^{GR}_{\mu\nu}$ from the choice of action allocation to geometry and $g^{0}_{\mu\nu}$ corresponds to flat space–time)

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu = g^0_{\mu\nu} (1 + \Delta g_{\mu\nu}) dx^\mu dx^\nu , \tag{2.86}$$

which in turn defines local vector frames (up to a gauge transformation)

$$e^{GR}_\mu = \mathbf{h} (e_\mu) ,$$

such that

$$g_{\mu\nu} = e^{\mathrm{GR}}_\mu \cdot e^{\mathrm{GR}}_\nu , \tag{2.87}$$

with **h** (x) a vector-valued function of vectors usually represented through a vierbein h^{ν}_{μ}. In general, from our complex type of structure,

$$\mathbf{h} (\mathbf{v}) = \mathbf{v} + f^{\mu\nu} (x) \mathbf{v} \cdot (e_\mu e_\nu) . \tag{2.88}$$

In (2.62)

$$\mathbf{h} (\mathbf{v}) = \mathbf{v} + ib (r) \mathbf{v} \cdot (e_0 e_r) , \quad b^2 = \kappa_0^2 (\Delta a)^2 ,$$

with Δa the action being allocated to be represented as space–time curvature.

Anyhow there are additional Lorentz boosts and rotations which can be applied, without changing dS^2 to a set of reference functions $\mathbf{h}^{(0)}$. They are generated by the bi-vectors functions $\Omega(x)$ to obtain $\mathbf{h} = \Omega \mathbf{h}^{(0)}$. The set of fields $\{\mathbf{h}^{(0)} (x) , \Omega (x)\}$ define the local geometry of the 4-D space–time ST (curved when action has been described as part of the carrier space ST). In this space the invariant volume is given by $d^4x \sqrt{-g}$, where g = det ($g_{\mu\nu}$).

In terms of these fields the components of the covariant derivative for multi-vectors M are

$$D_\mu M \equiv e_\mu \cdot \nabla M + \Omega (e_\mu) \times M , \tag{2.89}$$

using

$$2(A \times B) \equiv AB - BA, \tag{2.90}$$

or

$$DM \equiv \mathbf{h}(e^{\mu})\, D_{\mu} M. \tag{2.91}$$

For a spinor ψ

$$D\psi = h(e^{\mu})\,(e_{\mu} \cdot \Box\psi + \tfrac{1}{2}\Omega(e_{\mu})\,\psi), \tag{2.92}$$

and then the $\Omega\ (e_{\mu})$ appear as the Fock–Ivanenko gauge fields (see Chapters 6 and 7). In this book we shall consider two types of gauge fields. The type $\Omega(e_{\mu})$ which is a vector-valued function of a vector and the type

$$\mathcal{A}(x) = (A_R + iA_I) + ie_0e_1e_2e_3\,(B_R + iB_I)\,, \tag{2.93}$$

(complex) vector and (complex) axial vector fields, which can also be considered as resulting from proper and improper Lorentz transformations R (the inclusion of action has enlarged the set of isometries to that corresponding to complex Lorentz transformations), they all will obey

$$\Omega'(e_{\mu}) = \widetilde{R}\Omega(e_{\mu})\,R - 2\widetilde{R}e_{\mu} \cdot \nabla R, \tag{2.94}$$

using $\widetilde{R}R = 1$ and

$$M \to M' = \widetilde{R}MR, \tag{2.95}$$

as the Lorentz transformation of the multi-vectors. For spinors

$$\psi \to \psi = \widetilde{R}\psi, \tag{2.96}$$

and the basic gauged spinor field equation being

$$D\psi \to D'\psi' = \widetilde{R}(D\psi), \tag{2.97}$$

requiring

$$D'\psi' = e^{\mu}\left(\partial_{\mu} + \tfrac{1}{2}\Omega'(e_{\mu}; x)\right)\psi', \tag{2.98}$$

where again

$$\Omega'(e_{\mu}; x) = \widetilde{R}(x)\,\Omega(e_{\mu}; x)\,R(x) - 2\widetilde{R}(x)\,\partial_{\mu}R(x)\,, \tag{2.99}$$

as above. In (2.96) to (2.99) the Lorentz transformation $\widetilde{R}$ appears as a geometrical phase factor. In our theory below $\widetilde{\mathrm{R}}$ can be $\widetilde{\mathrm{R}} \in \{\widetilde{\mathrm{K}}\}$ a

more general transformation belonging to the group of complex Lorentz transformations $\tilde{K}$.

In practice the metric appears as an independent field in START which is defined according to the PAD, then once it is chosen the condition of flat STA is that the total curvature vanishes. Otherwise (from the integral of the selected contributions to action)

$$A = \int a(x)\sqrt{-g}\,dx_0 dx_1 dx_2 dx_3, \tag{2.100}$$

with

$$g \equiv -\det(g_{\mu\nu}), \tag{2.101}$$

we can derive the effective energy–momentum–stress density tensor, corresponding to the selected contributions to action, as

$$\kappa_0 \frac{\delta A}{\delta g^{\mu\nu}(x)} \equiv \frac{\kappa_0}{2}\mathcal{T}_{\mu\nu}(x), \tag{2.102}$$

(the factor $\frac{1}{2}$ is needed for convention reasons); also, from the Ricci scalar curvature $\mathcal{R}$ which results from the chosen line elements

$$\frac{\delta \mathcal{R}}{\delta g^{\mu\nu}(x)} = \mathcal{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\mathcal{R}, \tag{2.103}$$

we obtain the equivalent to the Einstein basic equation

$$\mathcal{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\mathcal{R} + \frac{\kappa_0}{2}\mathcal{T}_{\mu\nu} = 0. \tag{2.104}$$

Here we should stress that our formulation is not, neither for the gravitational part nor for the electromagnetic part, a Kaluza–Klein theory. Nevertheless, a change of variables in (2.104) would allow the formal writing of

$$\mathcal{R}_{ab} = 0, \quad a,b = 0,1,2,3,4. \tag{2.105}$$

Notice that there are no particular considerations about torsion or other symmetry considerations. We only require a set of coordinates x^μ covering the (curved or not) region of space–time around x, the existence of tangent vectors $e_\mu = \partial_\mu x$ allowing the definition of vectors in that space

$$\mathbf{V} = V^\mu e_\mu \to \mathbf{V}' = V'^\mu e'_\mu, \quad V'^\mu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right) V^\nu,$$

and $e'_\mu = \partial_{x'^\mu} x = (\partial x^\nu / \partial x'^\mu)\, e\nu$ and of its derivatives in terms of the coefficients of affine connection $\Gamma^\rho_{\nu\mu}$

$$\partial_\mu e_\nu \equiv \Gamma^\rho_{\nu\mu} e_\rho, \tag{2.106}$$

such that

$$\partial_\mu V = \left(\partial_\mu V^\mu + \Gamma^\nu_{\rho\mu} V^\rho\right) e_\nu \equiv \left(\nabla_\mu V^\nu\right) e_\nu, \tag{2.107}$$

where we define the covariant derivative of the (contravariant) component of a vector.

The connection coefficient may thus have symmetric part *S*,

$$2S^\rho_{\mu\nu} \equiv \Gamma^\rho_{\mu\nu} + \Gamma^\rho_{\nu\mu}, \tag{2.108}$$

and antisymmetric part (or torsion tensor *T* components),

$$2T^\rho_{\mu\nu} \equiv \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu}, \tag{2.109}$$

which, being tensor components, can not be used to construct a gauge field.

The set of definitions above are usually related to the Riemann curvature tensor

$$R^\mu_{\nu\rho\sigma} \equiv \partial_\rho \Gamma^\mu_{\nu\sigma} + \partial_\sigma \Gamma^\mu_{\nu\rho} - \Gamma^\mu_{\kappa\rho}\Gamma^\kappa_{\nu\sigma} + \Gamma^\mu_{\kappa\sigma}\Gamma^\kappa_{\nu\rho}, \tag{2.110}$$

and Ricci contractions

$$R_{\mu\nu} \equiv R^\rho_{\mu\rho\nu},\ R \equiv g^{\mu\nu} R_{\mu\nu} \text{ also } T_\mu \equiv T^\rho_{\rho\mu}, \tag{2.111}$$

(see Frankel 1999), and to the concept of parallel transport along a curve $x^\mu(\tau)$

$$\frac{dx^\rho}{d\tau}\left(\nabla_\rho V^\mu\right) = 0, \tag{2.112}$$

using the arbitrary parameters τ, defining the curve along which the vector **V** remains unchanged.

This is when

$$\frac{dV^\mu}{d\tau} = -\Gamma^\mu_{\nu\rho}\left(\frac{dx^\rho}{d\tau}\right) V^\nu. \tag{2.113}$$

To find the curve itself the vector **V** is considered the tangent vector $V^\mu \rightarrow dx^\mu/d\tau$ to the curve $x^\mu(\tau)$ to obtain from (2.113) the well known condition for an extremum trajectory

$$\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\rho}\left(\frac{dx^\rho}{d\tau}\right)\left(\frac{dx^\nu}{d\tau}\right). \tag{2.114}$$

The metric itself can be considered to be the symmetric functional $g(,)$ of the basis vector pairs

$$g_{\mu\nu} = g\left(e_\mu, e_\nu\right) = g\left(e_\nu, e_\mu\right), \tag{2.115}$$

in particular

$$g_{\mu\nu}^{GR} = g\left(\mathbf{h}(e_\mu^{(0)}), \mathbf{h}(e_\nu^{(0)})\right) \tag{2.116}$$

In terms of these quantities we can understand the formulation of GR in the direct application of the Principle of General Covariance using (2.116) and (2.113), as a geometrical guide, and the Principle of Equivalence stating that at any point in space–time we can always find a local frame in which the gravitational field vanishes and the trajectories of test particles obey (2.113). Einstein guided his mathematical formulation of the basic GR equations by the property that because the covariant divergence of the energy–momentum–stress density tensor vanishes,

$$\Box_\mu \mathcal{T}^{\mu\nu} = 0, \tag{2.117}$$

and then it had to geometrically correspond to the Einstein, vanishing covariant divergence, geometric tensor $\mathcal{G}$

$$\mathcal{G}_{\mu\nu} = \mathcal{R}_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu}\mathcal{R}. \tag{2.118}$$

Our theory is otherwise a purely geometric theory where the description of space–time surfaces in STA is changed to that corresponding to that of a particular action density distribution following our principle of acceptable choice of description PAD.

Otherwise, in our approach, where we have as basic equation $\delta\,(dS)^2 = 0$, the minimization procedure allows the derivation of other relations when other quantities are considered as independent variables. The $\mathbf{h}$, Ω, or $\Gamma^\mu_{\nu\rho}$ can be so considered. The use of $\Gamma^\mu_{\nu\rho}$ as an independent variable in particular establishes the fundamental relation between $\Gamma^\mu_{\nu\rho}$ and $g_{\mu\nu}$ which corresponds to the basic definitions of the Christoffel $\Gamma^\mu_{\nu\rho}$ symbols in terms of the derivatives of the metric tensor components

$$\Gamma^\mu_{\nu\lambda} = \tfrac{1}{2} g^{\mu\rho}\left(\partial_\nu g_{\lambda\rho} + \partial_\lambda g_{\nu\rho} - \partial_\rho g_{\nu\lambda}\right). \tag{2.119}$$

In relation to notation: we can use index notation, as in the last part of this section, or multi-vectorial notation above, or, when useful, a mixed notation like that at the beginning of this section. Geometric analysis allows a smooth transition between the different notations.

5.4 LAGRANGIANS

In this book we are analyzing the consequences of assuming an action density distribution to describe the physical world. There is a closely related function for the purpose of deriving the mathematical structures, that is, the Lagrangian of the system. The Lagrangian is obtained by

adding to the equation of the action density a series of vanishing functions multiplied by the set of factors known as Lagrange multipliers. They have the form of a choice of description. For example, to state that the particle density is factorized we write an additional term

$$\mathcal{L}_a = \lambda_a \left[\int \psi_a^+ \psi_a dx^3 - n_a \right]. \tag{2.120}$$

Here $\int \psi_a^+ \psi_a dx^3 = n_a$ and the term in the square parenthesis vanishes, λ_a is the Lagrange multiplier which, for consistency, is the average energy per carrier of type a, and when $(dS)^2$ is being optimized we perform the variation (for each carrier $\mathcal{L}_a$ can contain as many conditions as needed)

$$\delta \left\{ (dS)^2 + \sum_a \int c dt \mathcal{L}_a \right\} = 0. \tag{2.121}$$

The first term will give the condition that there is an energy for the system, the second an equation for the auxiliary functions ψ_a or for any other auxiliary variable. In short, a Lagrangian function contains a **description of the action distribution** amongst, carriers (which is added to the action itself) according to the principle of acceptable choice of description (PAD).

5.5 PHASE SPACE

In START, because of its equivalent complex structure (2.10)–(2.11) we have, besides the geometrical space–time volume

$$\mathcal{V} = V e_5 = V e_0 e_1 e_2 e_3, \quad V = \Delta x^1 \Delta x^2 \Delta x^3 \Delta x^0, \tag{2.122}$$

the geometrical phase space volume

$$\mathcal{V}_{PhS} = (\mathcal{E}\Delta t)\,(p_1 \Delta x^1)\,(p_2 \Delta x^2)\,(p_3 \Delta x^3)\, e_0 e_1 e_2 e_3, \tag{2.123}$$

corresponding to the 'imaginary' parts in (2.10) multiplied by $(h/k_0)^4$. At a given time the usual consideration of phase space for an ensemble of n particles includes a collection of n phase space cells with a volume $\hbar^3$ per particle and a volume V_{PhS} for the system which could be:

$$\mathcal{V}_{PhS} \geq n\hbar^3 \quad \text{for fermions,}$$
$$\mathcal{V}_{PhS} < n\hbar^3 \quad \text{for bosons.}$$

This can offer a direct connection with statistical (quantum) mechanics, which is not dealt with in this book.

5.6 THE GROUP OF SYMMETRIES IN START

In START, because of its equivalent complex structure (2.10)–(2.11) and its quadratic forms (2.45)–(2.46), we have, besides the geometrical space–time Poincaré group $\mathcal{P}$ of transformations leaving the finite difference $(dx^0)^2 - (d\mathbf{x})^2$ invariant, an additional set of transformations related to the more general quadratic form $(dx^0)^2 - (d\mathbf{x})^2 - (dx^4)^2$. The additional operations are: a translation in the e^4 direction, three rotations in the $e^i e^4$, $i = 1,2,3$ planes and one 'boost' in the $e^0 e^4$ plane. These operations correspond to changes of the energy–momentum density, that is to a change of the local or global reference values of energy–momentum (see [127] where the full group is denoted as $\mathcal{K} \approx \mathcal{P} + i\mathcal{P}$).

Chapter 3

ACTION MATHEMATICAL STRUCTURES

We should now consider the mathematical structures in the space–time–action geometry we shall use to describe matter and interaction fields.

1. ACTION

In this section we shall present the concept of Action Density Functional Theory and discuss the origin and consequences of the formulation of an Action Amplitude Function Theory.

In the space defined above we now introduce a dimensionless function $\mathbf{K} = \mathbf{K}(\mathbf{X})$ defined at each point $\mathbf{X}$ of space–time. The properties of $\mathbf{K}$ will be defined operationally.

Here we have to make two crucial considerations about what we know about action and about quantum mechanics that will be basic for the systematic procedure presented in this section.

As we mentioned in the previous section, and have discussed elsewhere [127], for each observer the concept of action distribution $a\ (x, t)$ is, for each given value t of the observer's time coordinate, the distribution $\varepsilon\ (x, t) = \partial_t a\ (x, t)$ of the energy attributed to physical phenomena. Relativity theory showed that an energy–momentum content in a given volume presents an inertia M, mass, through the basic relation $M = E/c^2$, this total energy content is independent of the form we have chosen to describe that matter. Let us, for example, start by considering that we have a free particle of a solid material. Which means that we have chosen a macroscopic point of view and a separation into the material itself and some external forces representing the mutual interaction between the material and the rest of the physical system. For some practical purposes this could be a sufficient degree of description

whcrc only shape and density will be required. We also know that if we consider that this piece of material is in motion relative to some measuring device, quantum mechanics can be applied to the material as a unit. Otherwise we may consider that a solid particle consists of molecules in interaction, and that there is an effective interaction potential between the molecules (this is a very common case in the study, for example, of rare gas solids). Quantum mechanics should be applied again for this system of interacting molecules. If our decision is to describe the material as electrons and nuclei we shall again apply quantum mechanics at this level of detail in the description. Other steps, the study in terms of more and more fundamental physical entities are again admissible. In every case we shall have a total energy which should be equal to the total energy of the previous steps and we shall have the practical choice of separation at any degree of description of: constitutional energy or mass, kinetic energy and, interaction energy.

That is, **quantum mechanics is a universal description of the phenomena,** valid for any degree of detail we might have chosen for the description and can not be a property of the components but a basic property for the description of nature. Action and space–time are also fundamental concepts in the description of nature and not concepts dependent on the system we are describing. The relation between these quantities should also be a fundamental mathematical structure. Here we describe the close connection between the two.

1.1 CARRIERS

For the study of the distribution of action we consider that:

a) In the space–time–action picture, where the basic mathematical properties of space–time are assumed to correspond to the physical space-time, the action density is inhomogeneously distributed, corresponding to the different material objects to which this action corresponds, with all the inhomogeneities (objects in space) moving at the speed of light c, basically in the time direction and, also in a possible relative motion in the spacial directions with speeds $0 \leq \upsilon \leq c$.

b) The matter-like energy distributions are to be considered as sources of (infinite extension, in principle) decaying deformations of action distribution of several types: first, a part A_0 uniformly decaying with distance, which observers will interpret as gravitation; second, of a collection *(A, B, C, ...)* of vorticex fields, superimposed on the A_0 part, which can be felt selectively by responses of given internal vortices of the other matter-like distributions. This second property is not given *a priori* but it is a consequence of the description of the objects as developed in the previous section.

c) We introduce now a third fundamental concept: energy carriers. At a macroscopic level the energy carrier is defined by a density distribution and by the integral properties of the distribution. We shall keep this concept without considering that the distribution could be reduced to a point singularity in space (line in space–time). We shall, anyhow, use the (not very fortunate) name **carrier density** for this quantity, the main reason being that its integral will be taken to be an integer. An extra reason is the definition of identical carriers as a density in a space volume V_s such that at time $t = t'$

$$\int_{V_s} \rho_b dx = n_b, \quad h \int_{\mathbf{V}_s} \partial_t a_b dx = \int_{V_s} \rho_b \varepsilon_b dx = n_b \varepsilon_b = E_b,$$

and $E_t' = [\Sigma_b\ E_b]_{t'}$ for a collection {b} of (by practical construction) independent types of carriers. Below, in discussing quantum electrodynamics, we shall allow the n_b not to be integers, provided Et' is not changed.

Carrier density and density of action should be gauge invariant physical quantities, thus we need to develop a procedure which can allow gauge freedom, that is, a procedure which allows for arbitrary, but correct and useful, descriptions. This is possible with the introduction of a probability amplitude Ψ, required to contain the necessary information in a form compatible to the basic concept that the energy–momentum components are obtained by using the operator $i\hbar\partial_\mu$ applied to the function which describes the splitting of the action density into a carrier density ρ and the action per carrier. The definition $\rho = |\Psi|^2$ allows gauge independence. A set of Lagrange conditions and multipliers has to be used to define the carriers and their desired properties. This procedure can be carried at any level of description, hence the universality of mathematical descriptions equivalent to the use of Wave Equations in Quantum Mechanics.

Once we have established that we are: 1) defining a density of action in space–time which corresponds to an energy density $\mathbf{E}t'(\mathbf{x})$ in space, where $\mathbf{E}_t(\mathbf{x}) = \{i\hbar\partial\mathbf{K}(\mathbf{X})/\partial t\}_{t=t'}$ for a fixed observer (for the reason to use **i** see below in this section); and 2) using a universal description, which allows a choice of the level of detail (for example: molecules→atoms→electron and nuclei to nucleons→quarks, provided that at each step the decision is formally made by selection of the type of 'particle', the type of interaction between carriers and what is to be considered the internal energy of the carriers) we can now proceed to the steps creating a practical (action) density functional theory.

The energy density, assuming **indistinguishable independent carriers** of a given type, at, a. given time t' is written as a product $\rho(\mathbf{x})\varepsilon$ of

a carrier density and a (global) energy per carrier

$$E = \int E(\mathbf{x})d\mathbf{v} = \int \rho(\mathbf{x})\varepsilon d\mathbf{v} \\ = \int \rho(\mathbf{x})(\mathbf{m}_c c^2 + \mathbf{kin}(\mathbf{x}) + \mathbf{V}(\mathbf{x}) + \mathbf{V}_{xc}(\mathbf{x}) + \varepsilon_0(\mathbf{x}))d\mathbf{v}, \tag{3.1}$$

where we have defined the energy density $E(\mathbf{x})$, the carrier density $\rho(\mathbf{x})$, the (by the definition of carriers) actual local kinetic energy per carrier $\mathbf{kin}(\mathbf{x})$, the external and average internal potential energy per carrier $\mathbf{V}(\mathbf{x})$, the correction to the average kinetic and potential energy per carrier arising from the statistics of the type of carriers under consideration $\mathbf{V}_{xc}(\mathbf{x})$, this term is needed to define independent carriers, and a local energy $\varepsilon_0(\mathbf{x})$, a basic term required to compensate for any difference in the sum of the previous terms with respect to the average energy per carrier ε, this term defines that the carriers are indistinguishable among themselves. **Density functional theory** (DFT as originally proposed by Thomas and Fermi) describes the self-organization of the system with density $\rho(\mathbf{x})$ in the presence of some external potential.

In (3.1) we are, in fact, defining the carriers. First, when we consider the energy density to be given by the product $\rho(\mathbf{x})\varepsilon$ of a density of carriers and an average energy per carrier (the same for all) in a form which makes them indistinguishable. In that integral the domain of integration defines the system of carriers, within this domain all are equivalent. In the last term, when we refer to the kinetic energy we consider these carriers to be independent carriers, that is the reason for the last function being required to compensate both for the definition of carriers and for considering them as independent carriers. When we enter into the theory of the electron we shall consider that within our START formalism there are some elementary carriers (they will be identified as either massless fields or the electron field), all other carriers will correspond to less fundamental descriptions and then will require either the last terms in (3.1) or a procedure, described below, of introducing gauge fields and the action associated with them.

1.2 THE DENSITY AS THE BASIC VARIABLE

It is convenient to define the action in a form which distinguishes the part corresponding to the self-organization of the distribution and the part which corresponds to the ‘external’ influences on the distribution.

The volume (in space) of integration is considered large enough for the ‘kinetic’ energy to be internal, that is there is no need to change the integration domain as a function of time. If the external influence is represented by the external potential $\mathbf{v}(\mathbf{X})$ we can write for the total

(invariant) action

$$A = \int dt\,(\mathcal{E}_I[\rho(\mathbf{X})] + \int d\mathbf{x}\,\mathbf{v}(\mathbf{X})\rho(\mathbf{X})), \tag{3.2}$$

where the functional $\varepsilon_I[\rho(\mathbf{X})]$ corresponds to the energy of the distribution of carriers $\rho(\mathbf{X})$. This functional ε_I has the interesting property that at any fixed time $t = t'$

$$\frac{\delta\mathcal{E}_I}{\delta\rho(\mathbf{X})} = -\mathbf{v}(\mathbf{X}). \tag{3.3}$$

This is a basic relation in Action–DFT as far as there is an intrinsic definition of the external potential. This shows the tautological nature of the concept of carriers, once they are defined the external potential is defined through the definition of the carriers themselves by $\varepsilon_I[\rho(\mathbf{X})]$. The tautological cycle is closed when given $\mathbf{v}(\mathbf{X})$ and $\rho(\mathbf{X})$ the kinetic energy and the interaction terms define $\varepsilon_I[\rho(\mathbf{X})]$.

From the definition in section 1.1 we can extend the description to consider a set {b} of types of carriers, each with density ρ_b.

1.3 INTRODUCING GAUGE FREEDOM FOR THE DESCRIPTION OF THE ENERGY

The fact that we are arbitrarily defining the terms above requires the possibility of changing the description of the energy partitioning without changing the description of the density. The density $\rho(\mathbf{x})$ is required to be gauge invariant] whereas the description of the energy (action) is gauge dependent. This is achieved by constructing the energy density as the product of an average energy per carrier ε with two conjugated quantities $\Psi(\mathbf{x})$ and $\Psi\dagger(\mathbf{x})$ such that $\rho(\mathbf{x}) = \Psi\dagger(\mathbf{x})\Psi(\mathbf{x})$ is gauge invariant. Here we have defined an auxiliary quantity which can be essentially written in terms of the basic action $a_0(\mathbf{x})$ and the action introduced by the gauge freedom $\phi(\mathbf{x})$ both in units of $\hbar$, as

$$\Psi(\mathbf{x}) = \sqrt{\rho(\mathbf{x})}\; e^{-i a_0(\mathbf{x}) + i\phi(\mathbf{x})}, \tag{3.4}$$

where we are restricted, by definition, to

$$\hbar\frac{\partial(a_0(\mathbf{x}) - \phi(\mathbf{x}))}{\partial t} = \varepsilon, \tag{3.5}$$

showing the gauge freedom of the description of the energy associated with the carrier. By definition, at all position points $\mathbf{x}$ we have the same energy per carrier ε which only in the simplest cases would be the sum of a constitutional, a kinetic and a potential energy part in the traditional

sense. A well known example is the case of electron density functional theory where the kinetic energy is assumed the kinetic energy of the free electron gas and then the additional term $\varepsilon_0(\mathbf{x})$ in (3.1) will contain, among other energy contributions the difference between the actual kinetic energy and the free electron gas term. The term $\varepsilon_0(\mathbf{x})$ in (3.1) exists either from the incomplete description of the other terms, the usual case, or from inaccuracies in the computational procedure. It acts locally to distribute the density in the form which minimizes the total energy and corresponds then to a variational procedure in the formulation of the theory. In the definition above, if $\mathbf{kin}(x) + \mathbf{V}(\mathbf{x}) + \mathbf{V}_{xc}(\mathbf{x}) + \varepsilon_0(\mathbf{X})$ are properly defined, then we should require that $\int \rho(\mathbf{x})\varepsilon_0(\mathbf{x})d\mathbf{v} = 0$. We have then recovered the equivalent to the Hohenberg–Kohn Theorems [140] and the Kohn–Sham minimization procedure [130] as the definition of this two terms

$$\delta\left(\mathcal{E}[\rho] - \varepsilon\left\{\int \rho(\mathbf{x})\, d\mathbf{v} - N\right\}\right) = 0, \tag{3.6}$$

allows the direct self-consistent determination of $\rho(\mathbf{x})$ and ε (see [58]). The constant ε is otherwise the rate of change of the energy in the process of removing one (average) carrier.

In the formalism, by formal inversion with a resolvant G of the operators in the equations $\hat{O}\Psi = 0$ which the action amplitudes Ψ obey, we can define a useful self-consistency relation

$$\Psi(x) = \int w(\mathbf{x})G(\mathbf{x},\mathbf{x}')\Psi(\mathbf{x}') - \Psi_0(\mathbf{x}), \tag{3.7}$$

where

$$w(\mathbf{x}) = \Delta\rho(\mathbf{x})\varepsilon + \rho(\mathbf{x})\Delta\varepsilon(\mathbf{x}),$$

this introduces both, the response function of the system $G(\mathbf{x}, \mathbf{x}')$, and the effective potential which would be caused either by fluctuations of the density or by differences in the local definition of energy per carrier. This reaction would propagate to all points of the distribution to achieve stability.

1.4 THE RELATION TO STANDARD DFT

The auxiliary function Ψ in (3.1) is identical to the standard quantum mechanical ψ in the case of a 'one' carrier system, otherwise we should consider it as the function which represents the gauge dependent, square root of the density of action, obeying the equations (here $\varepsilon \rightarrow \varepsilon - m_b c^2$ in a non-relativistic approximation)

$$\widehat{H}_{\text{eff}}\Psi(\mathbf{r}) = \varepsilon\Psi(\mathbf{r}), \tag{3.8}$$

where we have defined an effective operator, using $p_\mu = i\hbar\partial\mu$

$$\widehat{H}_{\text{eff}} = -\frac{\hbar^2}{2m_b}\nabla^2 + \upsilon(\mathbf{r}) - \upsilon_{\text{eff}}(\mathbf{r}), \tag{3.9}$$

such that the system's energy, non relativistic limit, is to be obtained from

$$T_s[\rho] = \int \Psi^\dagger(\mathbf{r})\left(-\frac{\hbar^2}{2m_b}\nabla^2\right)\Psi(\mathbf{r})d\mathbf{r} + T_\theta[\rho], \tag{3.10}$$

defining, variationally, the effective potential

$$\frac{\delta T_s[\rho]}{\delta\rho(\mathbf{r})} = -\frac{\hbar^2\Psi^\dagger\nabla^2\Psi}{2m_b\Psi^\dagger\Psi} + \upsilon_\theta([\rho];\mathbf{r}), \tag{3.11}$$

$$\upsilon_\theta([\rho];\mathbf{r}) = \frac{\delta T_\theta[\rho]}{\delta\rho(\mathbf{r})}, \quad \frac{\delta T_s[\rho]}{\delta\rho(\mathbf{r})} + \upsilon_{KS}([\rho];\mathbf{r}) = \varepsilon_M, \tag{3.12}$$

then (3.8) reads

$$-\frac{\hbar^2}{2m_b}\nabla^2\Psi + \upsilon_{KS}([\rho];\mathbf{r})\Psi + \upsilon_\theta([\rho];\mathbf{r})\Psi - \Delta\bar{\varepsilon}(\mathbf{r})\Psi = \varepsilon\Psi(\mathbf{r}). \tag{3.13}$$

Note. Now we quote something which should, in fact, be a result of some considerations of the next section. From the definition of a set of auxiliary functions below, we obtain for the last term:

$$\Delta\bar{\varepsilon}(\mathbf{r})\rho(\mathbf{r}) = \sum_i |\psi_i(\mathbf{r})|^2 - \bar{\varepsilon}\rho(\mathbf{r}), \tag{3.14}$$

which is equivalent to identifying ε with an average eigenvalue $\bar{\varepsilon}$ and to the optimization of

$$\Omega = E[\rho] - \mu\left[\int \rho(\mathbf{r})d\mathbf{r} - \mathrm{N}\right] - \lambda\left[\int\sum w_i\varepsilon_i|\phi_i(\mathbf{r})|^2 d\mathbf{r} - \int\bar{\varepsilon}\rho(\mathbf{r})d\mathbf{r}\right], \tag{3.15}$$

using a set of auxiliary functions ϕ_i, with the index i running through all possible forms of extracting one carrier from the system ($\nabla^2 = \partial_i\partial_i$, $i = 1, 2, 3$)

$$-\frac{\hbar^2}{2m_b}\nabla^2\phi_i + [\upsilon_{xc}(\mathbf{r}) + \upsilon_{\text{Coul}}(\mathbf{r}) + \upsilon_{\text{ext}}(\mathbf{r})]\phi_i = \varepsilon_i\phi_i. \tag{3.16}$$

which will define the elementary excitations of the system corresponding to the removal of one carrier with rate of change of the energy ε_i

$$\sum_i w_i h_i^{\mathrm{KS}}(\mathbf{r}) = \sum_i w_i \varepsilon_i |\phi_i|^2, \tag{3.17}$$

this is equivalent to the Kohn–Sham procedure and the use of the Kohn–Sham effective Hamiltonian $h_i^{KS}(\mathbf{r})$ in standard DFT (for a ground state calculation usually $w_i = 1$, $i = 1, ..., N$ and $w_i = 0$, $j > N$, $\varepsilon_j > \varepsilon_N$)

$$h_i(\mathbf{r})^{KS} = -\phi_i^* \frac{\hbar^2}{2m_b} \nabla^2 \phi_i + \phi_i^* [v_{xc}(\mathbf{r}) + v_{\mathrm{Coul}}(\mathbf{r}) + v_{\mathrm{ext}}(\mathbf{r})] \phi_i. \tag{3.18}$$

The use of auxiliary functions is further discussed in the next sections.■

Corresponding to (3.8) we have the final equation (see [102, 58])

$$-\frac{\hbar^2}{2m_b} \nabla^2 \Psi + v^{FK}(\mathbf{r}) \Psi = \bar{\varepsilon} \Psi, \tag{3.19}$$

where we have defined the effective potential [172]

$$v^{FK}(\mathbf{r}) = v_{xc}(\mathbf{r}) + v_{\mathrm{Coul}}(\mathbf{r}) + v_{\mathrm{ext}}(\mathbf{r}) + \Psi^\dagger \frac{\hbar^2}{2m_b} \nabla^2 \Psi + k(\mathbf{r}) - \Delta\bar{\varepsilon}(\mathbf{r}). \tag{3.20}$$

The first, two of the last three terms correspond to the correct additional kinetic energy density, and the last one, as above, to the symmetry constraint potential arising from the local in space comparison between the actual values of the energy necessary to remove one electron from the system and the average energy per carrier:

$$\Delta\bar{\varepsilon}(\mathbf{r}) = \sum n_i \varepsilon_i \frac{|\phi_i(\mathbf{r})|^2}{\rho(\mathbf{r})} - \bar{\varepsilon}, \tag{3.21}$$

if carriers of type *b* are indistinguisliable they all have the same energy per carricr $\bar{\varepsilon}$; here above the ε_i refer to the process of changing the system from that of N to that of $N-1$ carriers, the set $\{\varepsilon_i, i = 1, ..., N\}$ corresponds to the set (of rate of change of) energies for which the system will respond when a quanta of action is given to the original (N carriers) system.

1.5 RELATION OF ACTION DENSITY FUNCTIONAL TO ACTION AMPLITUDE MECHANICS

We now proceed formally to show that the procedure within START described here is equivalent to the one within the usual postulation of the

wave equation approach to quantum theory. In the rest of this section we follow lines which have been defined in the reports at several conferences [127]. We do not use the low kinetic energy limit, here.

(1) We have defined, for the representation of the physical system, an adimensional density of action **K**(**X**) at each space-time point

$$\mathbf{X} = \mathbf{e}_\mu X^\mu = e_0 x^0 + \mathbf{x}.$$

(2) The action is factorized, for its study, into a density **n**(**X**) and a local average action per carrier **k**(**X**) in units of $\hbar$, the use of this average is required when **n**(**X**) = Σ_b ρ_b(**X**),

$$\mathbf{K}(\mathbf{X}) = \mathbf{n}(\mathbf{X})\mathbf{k}(\mathbf{X}). \tag{3.22}$$

(3) The energy of the system $E(\mathbf{X}) = i\hbar\partial\mathbf{K}(\mathbf{X})/\partial t$ is obtained from this action density or, in general,

$$\Pi_\mu(\mathbf{X}) = [i\hbar\partial\mathbf{K}(\mathbf{X})/\partial X^\mu]_R = \mathbf{n}(\mathbf{X})\mathbf{p}_\mu(\mathbf{X}), \tag{3.23}$$

for the energy–momentum four-vector $\prod$. This equation is in fact. the defining equation for **K.** Notice that p_μ(**X**) contains two types of terms: one from the derivatives of k(**X**); and one from the derivatives of **n**(**X**); they are accounted for, systematically, below.

(4) To see the correspondence to the space–time–action geometry (STA) we remind the reader that the fifth axis of this geometry was labelled by $e_4 := ie_5$ and that $e5 := e_0e_1e_2e_3$ is the unit four-volume in space–time, then the density

$$\mathbf{K}(\mathbf{X}) = (\mathbf{i}/\hbar)\mathbf{A}(\mathbf{X}) = -\mathbf{a}(\mathbf{X})ie_5e^5 = -\mathbf{i}\mathbf{a}(\mathbf{X}), \tag{3.24}$$

per unit space–time volume. We see that this quantity is a pure imaginary complex number. Note also that action acquires a negative sign from $(ie_5)(-ie_5) = (e_5)^2 = -1$, this was important in the previous chapter, section 5.

(5) This action distribution representing the physical system has two sources of gauge dependence: (a) the dependence on the definition of the reference space–time hypersurfaces in the STA space, a dependence related to gravitation; and (b) the dependence on the arbitrary (either by incomplete knowledge or by practical decision) choice of the type(s) of field(s) whose density is represented by **n**(**X**) and, by definition of the fields, their energy contributions. Here we should

consider from very simple, one type of action carriers, to complicated cases like a system of fields representing an electron $e-$ and a W^+ (which could also be a neutrino ν, where the description of the system should include a complete range of possibilities. To solve this description problem we introduce again a description and gauge dependent auxiliary function

$$\Psi(\mathbf{X}; \{\mathbf{x}_i, t_i; i = 1, ..., N\}), \tag{3.25}$$

depending on a set of coordinates, which allows us to write for the action density

$$\mathbf{K}(\mathbf{X}) = \overset{\{i\}}{tr}\hat{A}(\Psi(\mathbf{X})\Psi^\dagger(\mathbf{X})), \tag{3.26}$$

which for a single carrier $N = 1$ is

$$\mathbf{K}(\mathbf{X}) = \hat{A}(\Psi(\mathbf{X})\Psi^\dagger(\mathbf{X})). \tag{3.27}$$

(6) The self-consistent properties of $\Psi(\mathbf{X};\{\mathbf{x}_i,\ t_i;\ i = 1,\ \ldots,\ N\})$ (3.22) to (3.27) are then (concepts like temperature and free energy are not introduced in this book):

a) From the simplest case, that of the homogeneous distribution of action with an assumed single carrier, where (defining $\mathbf{k} = \prod/$)

$$\Psi(\mathbf{X}) = \rho^{1/2}e^{-i\mathbf{k}\cdot\mathbf{X}}, \tag{3.28}$$

and therefore

$$\Pi_\mu = [\Psi^\dagger i\hbar\partial_\mu\Psi]_R = \hbar\rho k_\mu = \rho\mathbf{p}_\mu, \quad m = (\Pi^\mu\Pi_\mu)^{1/2}, \tag{3.29}$$

defining the auxiliary reference energy–momentum m. Then introducing the multi-vector operator $D_0 =$ the basic equation is

$$i\hbar e^\mu\partial_\mu\Psi = D_0\Psi = m\Psi. \tag{3.30}$$

Which shows that the auxiliary function Ψ has the same initial properties as the standard wavefunction in quantum mechanics. Note that $D\Psi = m\Psi$ requires that Ψ should be a particular type of multi-vector (Chapter 6). It is now immediate that the description freedom corresponds to the gauge theory approach, where the 'Lagrangian' density is related to $\mathbf{K}(\mathbf{X})$, the Ψ are gauged and the D_0 operator is enlarged to the covariant derivative D,

to keep $\mathbf{K}(\mathbf{X})$ gauge invariant. The equation $D_0 D_0 \Psi = m^2 \Psi$ is secondary at this level.

b) In the case of several, amongst themselves, identical carriers we can now construct the Ψ as a complete set of permutations of ϕ_i auxiliary functions $\Psi = \Sigma_p (-1)^p \prod^{(i)} \phi i(x_i, t_i)$ and define the potentials $\mathbf{V}$ and $\mathbf{V}_{xc}$ accordingly or, as defined below, use a more complicated expression for Ψ and a (simpler) expression for $\mathbf{V}$ and $\mathbf{V}_{xc}$ in terms of an interacting carriers definition of Ψ, here $\mathbf{V}$ should contain the sum of the inter-carrier interactions. See the next subsection. Here we have a fundamental consequence of the amplitude function Ψ being a multi-vector obeying (3.29): the product of functions has to be invariant under the Lorentz group and the sum over all possible permutations p includes the sign factors which forces the Ψ to be antisymmetric on the exchange of any pair (i, j) and as a result invariant. For the form corresponding to an electron a special form of the many carriers action amplitude function will be obtained (see [45] in this respect).

c) The case of several types of carriers corresponds to (sums and) products of descriptions of type b). Products describing independent descriptions and sums describing alternative descriptions.

d) The use of a description of an evolving systern with changing types of carriers defines interaction Lagrangians in which the sum of products of descriptions of type b) are used to represent, our uncertainty in the actual distribution of action, keeping nevertheless the $\mathbf{K}(\mathbf{X})$ invariant. (Note, however, that a Lagrangian defines the 'active' part of the action density and also explicitly includes the symmetry constrains).

e) A special case, a frequent situation, is that of a stationtry system. Such a system often corresponds to an unlimited number of carriers, but the carrier density is normalized either to $N = 1$ or to a current corresponding to one incoming carrier. Then an additional probabilistic consideration is incorporated when the weights of the different contributions is interpreted as probabilities of different configurations, or the outgoing currents of carriers, as probabilities P_a of different processes a. This is specially and numerically allowed by the normalization procedure itself, given that probabilities should obey $\Sigma_a P_a = 1$.

All these descriptions obeying the principles above show the intrinsic connection between the postulates of START and the structure and interpretation of wave quantum mechanics.

1.6 DESCRIPTION IN TERMS OF INTERACTING CARRIER FIELDS

The basic description, which could be used to construct a geometric theory applicable to the electron in terms of carriers (carrier fields in practice in the sense above where $Nm_i = \int_v \rho_i m_i dV$, etc.) is that of N equivalent interacting carriers:

1) At **each point** the density ρ_i (of the carrier field i) is endowed of a self energy expressed as its mass mi; a spin si; and a collection of charges $\{q_i^{(g)}\}$ (one for every gauge field g).

2) The definition of associated charges $q_i^{(g)}$ is given by considering that all carriers i are subjected to local external potentials $V_i(\mathbf{x})$ representing the 'rest of the universe' effects where

$$V_i(\mathbf{x}) = m_i V^{grav}(\mathbf{x}) + \sum_g q_i^{(g)} V^{(g)}(\mathbf{x}), \tag{3.31}$$

the theory of a particular type of carrier should create, not postulate, the values of the charges. Note that V_i (**x**) is local.

3) All the N equivalent fields i are supposed to have an independent carriers local kinetic energy contribution $\mathbf{kin}_i$ (**x**).

4) There is an intercarrier pairwise potential energy ($i \neq j$)

$$\sum_q \frac{1}{2} \sum_i \sum_j q_i^{(g)} q_j^{(g)} f(\mathbf{u}_i, \mathbf{u}_j; r_{ij}), \tag{3.32}$$

which is proportional to the products of the charges and to a function of the set of four velocities $\mathbf{u}_i$ and of the distances r_{ij} between points of the fields related to the carriers, the basic example being the electromagnetic case $q_i^{(e)} q_j^{(e)} (\mathbf{u}_i \cdot \mathbf{u}_j/c^2)/r_{ij}$, where for carriers at relative rest $(\mathbf{u}_i \cdot \mathbf{u}_j/c^2) = 1$, this being the fundamental **definition** of an interaction as a force which decays as the surface of a sphere with the source at the center, that is the capability of a source to do work is constant on the surface of, for an observer concentric, spheres and its surface integral a constant proportional to the source strength. We justify this approach below.

5) Other interaction terms, depending on the masses or spins of the fields. Until now there has been no reason nor practical use of more complicated terms, such as terms depending on products of charges

of different gauge fields $q_i^{(g)}$ $q_j^{(g')}$. In practice, for non-elementary carrier fields, three body terms and 'effective' charges have been used, this being perhaps a guide to establishing that a carrier is not elementary. When r_{ij} is large with respect to a measure of the extent of the distribution of both the i and the j field, an equivalent distance between the center of distribution (equidistant) r_{ij}^{o} can be used in practice, which allows us to consider the fields i and j as equivalent to point-like objects and the use of simplified forms of the definitions of the $V^{(g)}$ in (3.31).

The use of (at least) 1), 2), 3) and 4) induces either:

a) the use of a non local function of a set of coordinated $\{\mathbf{x}_i\}$, that is $\Psi_{\text{non-local}} = \Psi_{\text{non-local}}(\{\text{xi}; i = 1, ..., N\})$; or

b) the local formulation obtained by the introduction of this non-locality as a self consistent local potential (which strictly speaking would require for its calculation the knowledge of $\Psi_{\text{non-local}}$, considered now as an auxiliary calculation procedure).

Case a) is similar to standard quantum mechanics where the auxiliary function Ψ is constructed as sums of products of sums ($\Sigma\, C_{sr} \prod b_i\, \Sigma\, a_r$) of basic functions. The last sum corresponds to an assumed reference independent carrier. The products correspond to considering a set of those reference carrier distribution as an independent carrier field scheme and the first sum to account both for the statistics induced by the condition of relativistic invariance of the set of auxiliary fields (see Duck and Sudarshan [45]) and for all possible forms of response of the system to the possibility of removing one carrier from it with the condition $\Delta A = nh$.

Case b) corresponds to keeping the local $\Psi(x)$ and introducing the result of the non-local interaction as corrections to the kinetic energies obtained from $\psi(x)$ and as an equivalent, average, local, inter-carrier potential, where also the effect of the from relativistic invariance required statistics and of the full response of the system are included. This last term is equivalent to what is known as the local exchange–correlation potential in standard DFT. The remaining term is equivalent to the one which was introduced [102, 58] in the study of $\Psi(x)$ for a many electron system.

1.6.1 THE INDEPENDENT GAUGE FIELDS APPROACH

This is not the only set of possibilities, a third major line of approach can be developed to study processes in a form which is useful mainly

in connection with high energy physics and the study of elementary carriers. It consists in formulating an independent carrier approach using an action related to the **local** effect of the gauge fields into the carrier fields through terms

$$q_i^{(g)} \mathbf{A}_{(g)}(\mathbf{x}) \bullet d\mathbf{x}_i, \tag{3.33}$$

the scalar product of the vector $\mathbf{A}_{(g)}$ and the vector $d\mathbf{x}_i$, or $\mathbf{u}_i = d\mathbf{x}_i/dt$ if energy is computed. This is achieved at the expense of allowing **independent existence** of the **gauge interaction fields.** This quantum field theoretical approach has the advantage of allowing the possibility of describing the gauge fields independently of the source or target particle-like carrier fields, and introduces the quantization of this gauge interaction fields which carry energy–momentum, spin and geometrical information of the possible source or target fields. Their interactions induce geometric phases in the carrier fields [159].

The gauge interaction fields are assigned a gauge independent field strength

$$F_{\mu\nu}^{(g)} = \partial_\mu A_\mu^{(g)} - \partial_\nu A_\mu^{(g)}, \tag{3.34}$$

introducing geometrical conditions, and an **action** density of the gauge field itself

$$a_{(g)}(x) = -\frac{1}{4} F_{(g)}^{\mu\nu}(x) F_{\mu\nu}^{(g)}(x), \tag{3.35}$$

to be added to the particle-like carrier's field action density. The sum is a local action and a local energy–momentum by consequence. In the case where there are no carriers which can respond to the quanta of the gauge fields the pass from case 1) to case 3) is straightforward by partial integration using a source equation ($\Box^2 = \partial^\mu \partial_\mu$, $\mu = 0,1,2,3$)

$$\Box^2 A^{(g)}(x) = \sum_{i \neq \text{test}} j_i^{(g)}(x). \tag{3.36}$$

It must be stressed that the energy related to (3.35) will require in general the integration over volumes much larger than those of the integration of $\rho_i(x)$. In both cases 1) and 3) the energy related to external sources of gauge fields should be added, because only the action related to the system of carriers $\{i\}$ has been included. This defines external sources as carriers not included in the domain of integration in (3.1).

There are technical difficulties in approach 3), which is in principle equivalent to quantum field theory based on quantum electrodynamics as the simplest case and equivalent to Maxwell theory in the classical

formulation. Some of them would disappear if a hybrid approach were taken, using 1) and allowing (3.35) for the description of the external influences. The problems related to the non-abelian character of the gauge fields would require nevertheless the use of the special mathematical techniques now in use in the standard model of elementary carriers. Pairwise interactions and gauge fields (not including free 'real' carriers of the interaction fields) are equivalent dual formulations which should be explicitly followed, without mixing or repeating descriptions.

The description here developed requires, at least, the analysis of two particular cases:

1) the consideration of the low energies limit, without gauge field real carriers;

2) the consideration of large inter-carrier separations.

In the first case the current j_μ is given by ρu_μ with $\mathbf{u} = \mathbf{u}(\mathbf{x})$ and $u^2 = u^\mu u_\mu = c^2$, then for the description of the dynamics the energy density is given not by the sum of the kinetic energy and the mass energy $(m_o c^2)_i$ but by the energy–momentum $m_i u^2$ and the interaction energy

$$\sum_{i<j} \frac{q_i q_j}{|\mathbf{x}_j - \mathbf{x}_i|} \frac{\mathbf{u}_i \cdot \mathbf{u}_j}{c^2}, \tag{3.37}$$

That is, from (3.37), each particle-like carrier is a test paticle-like carrier with respect to the others. Then for each (test) particle-like carrier j we can define and 'external' electromagnetic field (usually known as the retarded potential)

$$A_\mu^{(j)}(\mathbf{x}) = \sum_{k \neq j} \int \frac{\rho_k(\mathbf{x}_k) q_k}{|\mathbf{x} - \mathbf{x}_k|} \frac{u_\mu(\mathbf{x}_k)}{c}, \tag{3.38}$$

completely defined by the set of distributions $\{\rho_k(\mathbf{x}_k),\ q_k,\ u_\mu(\mathbf{x}_k);\ \text{all } k\}$, that is: there is no need to consider new degrees of freedom for the $A_\mu^{(j)}(\mathbf{x})$,and these quantities depend on the sources, not on the test particle-like carrier (one at a time). Test particle-like carriers could have zero charge.

The second point is related to the interaction between two or more carriers in relative motion when the final action can be described by a similar system of carriers but with n units of angular momentum exchanged, $\Delta l = \pm nh$, and a corresponding energy change $\Delta\varepsilon = \Sigma_m\ n_m h\nu_m$ with $\Sigma_m\ n_m = n$. This **real** exchange of energy and angular momentum processes then correspond to quantized emission or absorption of energy;

there is nothing in the basic principles forbidding these processes and they obey the third principle. The puzzling fact is that the processes can be described by $A_\mu(\mathbf{x})$ fields also and then the (carriers in interaction) picture above is transformed into a dual description where the gauge fields can also acquire independent physical existence. The fields $A_\mu(x)$ corresponding to the pairwise interaction are $A_{\mu 4}$ terms in the STA geometry, not to action density (A_{44}) terms, although an energy momentum tensor can be defined from their space–time derivatives. The set $\{n_m\}$ is not conserved in the presence of carriers, in so far as all possibilities to describe action have to be included. We are then forced to systematically enlarge the formalism to include these possibilities. This accounts for the fact that experimentally we can produce light and handle its energy by interactions with the appropriate set of carriers. Also, for description purposes, it is common to consider these energy quanta $h\nu_m$ carrying angular momentum h, as a sort of independent light beam which is deflected, or reflected, or refracted, even if this is not what experiment supports, because the description of reflections and refractions is, at the fundamental level the description, absorption and re-emission of the quanta.

The existence of these quanta stems from the theory being analyzed from the special mathematical structure of the interaction field carrier (corresponding to the photon) which obeys (3.37) and (3.38) self-consistently by elimination of the q_k in a circuitous analysis inverting (3.38) and substituting in (3.37).

1.7 THE INDUCED PROBABILISTIC INTERPRETATION OF Ψ

Because the auxiliary functions describing the action contributions will either appear as products of functions $\varphi_1\varphi_2 \ldots$ or as sums of functions $\phi_1 + \phi_2 + \ldots$ the use of the derivatives in D as operators originate both a probabilistic interpretation and, in fact as a consequence, a systematic method of obtaining Ψ.

In fact for a product $\varphi_1\varphi_2$, because $D\,(a\varphi_1\varphi_2) = a[(D\varphi_1)\varphi_2 + \varphi_1(D\varphi_2)]$ the energy–momentum contributions will appear as sums of independent terms. Also, because $D(a\phi_1 + b\phi_2) = aD\phi_1 + bD\phi_2$, the energy contributions from a sum of functions appears as a weighted sum of dependent contributions. This is typical of probability theory and a probabilistic language will faithfully be useful to describe the total action.

The auxiliary function acquires a hierarchy of forms and uses:

a) The simple form $\Psi(\mathbf{X})$ for the density of action, this requires the definition of carriers as equivalent, the same $\bar{\varepsilon}$ for all, and independent, the same $V(x)$ for all.

b) The simple product $\Psi(\mathbf{X}) = \prod_{(a)} \psi_{i\,a}(\mathbf{X})$, the index '*a*' running over all types of carriers, for each type $\bar{\varepsilon}a$ and $V_a(x)$ as in the single type case. This corresponds to a set {a} of carrier fields in a stationary state which allows the determination of $\{V_a\ (x)\}$ using the definitions of the previous section.

c) The matrix form $\Psi(\mathbf{X};\ \{x_i, t_i;\ i = 1, ...,\})$ for equivalent, interacting carriers. This allows the inclusion of the inter-carrier interaction potential $V_{ij}(x_i,\ t_i;\ x_j,\ t_j)$ besides ε and $V(x)$.

d) The general form with auxiliary functions Φ_{nst} is given by

$$\Psi(\mathbf{X};\ \{x_i, t_i;\ i = 1, ..., \}_T) = \sum_c a_c \prod_t b^{tc} \sum_s c^s \prod_n \Phi_{nst}(x_i, t_i)$$

for each type *t* of carrier and state *s* of that carrier *n*.

The auxiliary function in the case d) is equivalent to the use of the so called Configuration Interaction method in atomic and molecular physics. Its restrictions reduce to c), b), and a). The sum of products $\Sigma_s\ c^s \prod_n$ are equivalent to the use of the Slater determinants for fermions, the c^5 accounting for the statistics in the sense of the Pauli exclusion principle [45].

In (3.39) the sum corresponds to considering **alternative** contributions to the action, whereas the products correspond to considering **independent** contributions to the action. This is originated in the need to consider either real alternatives or alternatives stemming from our fundamental or practical ignorance of the state of the system.

In all cases the coefficients and other parameters of the auxiliary action amplitude function can be determined by the dynamic principle of minimal action as considered in Chapter 2 above.

It is important to remark that the action distribution in space–time as a description of matter agrees, without actual limitations, with our present experimental and theoretical knowledge of matter and interaction fields.

From the set of considerations above we also have to include a symmetry between the space and time distributions' dependence, including a distribution form in time even if each observer only considers a cross section (sometimes called a time slice) of that distribution in his own frame of reference.

In relation to our construction of the auxiliary function Ψ we can then use a trial set of contributions to the action

$$\Psi_t = \sum_i a_i \left(\prod_{j \in \{j\}_i} \varphi_j \right), \tag{3.39}$$

with the φ_j also a composition of functions representing some contributions to the action, the chain derivative

$$\sum_i \frac{\partial \Psi^\dagger}{\partial a_i} \frac{\partial E}{\partial \Psi^\dagger} = 0, \tag{3.40}$$

will allow the optimization of the description, then a variational principle for energy exists which is, in Action Density Functional Theory, the equivalent to the Hohenberg–Kohn Theorems [83].

1.8 QUANTUM MECHANICS FROM START

An additional probabilistic concept, different from the one described above, arises from the algebra of the operators themselves, because the action being $x_\mu p_\mu$, and its operator $\hat{a} = x_\mu p_\mu = x^\mu (i\hbar \partial_{x^\mu})$, we then, from the chain rule for derivatives, obtain the operator

$$[x^\mu, \hat{p}_\mu] \equiv x^\mu \hat{p}_\mu - \hat{p}_\mu x^\mu = i\hbar, \tag{3.41}$$

with the well known Heisenberg limitation (and the basis of his formulation of Quantum Mechanics), introducing an uncertainty in our possibilities of knowing (not the action but) the factors of the action, separately, for a given action distribution, up to the small but highly significant value of $\hbar$. Because this is a fundamental restriction on the description of the action distribution, this uncertainty is presented as a basic property of matter, independent of our choice of the form for describing it. We remind the reader that we are within the gauge-free use of START, and that we have not postulated quantum mechanics, only Planck's rule $\Delta A = nh$.

The fundamental relation (3.41), which in our approach is given by construction of the START geometry and the introduction of the action density function, was found sufficient by Pauli in 1926 [145] for deriving the quantum mechanics of the hydrogen atom. This, and similar examples, show that our approach is both compatible with, and contains, the basic principles of Quantum Mechanics.

The study of the different probabilistic and statistical contents in quantum theory is an active subject, see for example Isham [84] or Görnitz [71]. Here we have only considered stationary states and not,

the two additional complementary problems: the first refers to the use of stationary states to analyze evolving dynamical systems (considering the different contributions to the stationary state as different situations in time, from an assumed past to a selected future state of the system, where the probabilities are considered the ratios of incoming channels to outgoing channels and special care has to be taken in the identification of the channels and their probabilities); the second refers to the use of the analysis of stationary states for the study of equivalent oscillating systems (where the probabilistic weights are identified with the ratios of the frequency in which the different configurations are to be found, if a series of experiments were performed). These complicated probabilistic interpretations, not discussed in this book, have often been the origin of considering that the predictions of the quantum mechanical approach contradict our classical experiences.

1.9 SOME REMARKS ABOUT THE FORMALISM

We have presented an action–density functional formalism, developed it, and shown that not only a standard density functional theory is recovered —and that in a sense it is more fundamental than wave function wave mechanics— but also that the analysis of the mapping of the density matrix into a density allowable for density functional theory [98, 58] requires the introduction of auxiliary terms which represent the internal symmetries of the system.

Several otherwise basic principles of quantum mechanics are then natural structures in the approach developed here for describing matter as a distribution of action in space–time (energy distribution over space).

In the separation of carriers discussed here the interaction of a carrier with itself in the form called self-exchange, is by construction to be ruled out. If a diagrammatic procedure were used, diagrams relating a carrier with itself are also to be ruled out. Nevertheless, in a system a 'real' **interaction carrier** can be created as a result of energy transfer in a pairwise process and the emitted carrier could be reabsorbed by any one of the other carriers present in the system; this real process interaction has to be included in the present theory.

2. CURVATURE

In START the geometrical union of energy–momentum space to a carrier space, above, has allowed a re-derivation of the proportionality between stress–energy–momentum and curvature.

There are at least two approaches to the Riemann curvature tensor. One is space intrinsic and the second considers the notion of an extrinsic normal to the space. The corresponding formulas are, for the intrinsic formulation,

$$R^{\delta}_{\alpha\beta\gamma} := \partial_\beta \Gamma^{\delta}_{\gamma\alpha} - \partial_\gamma \Gamma^{\delta}_{\beta\alpha} + \Gamma^{\delta}_{\beta\nu}\Gamma^{\nu}_{\gamma\alpha} - \Gamma^{\delta}_{\gamma\mu}\Gamma^{\mu}_{\beta\alpha}, \tag{3.42}$$

defining the Riemann (–Christoffel) curvature tensor in terms of the Christoffel symbols, and, for the second approach, the Gauss identity

$$R^{\delta}_{\alpha\beta\gamma} = P^{\delta}_{\beta} P_{\alpha\gamma} - P^{\delta}_{\gamma} P_{\alpha\beta}, \tag{3.43}$$

derived from the Leibnitz rule

$$\frac{\partial \mathbf{X}}{\partial x_\gamma \partial x_\beta \partial x_\alpha} = \frac{\partial \mathbf{X}}{\partial x_\beta \partial x_\gamma \partial x_\alpha}, \tag{3.44}$$

using the embedded curved spaces quadratic form, where a normal **n** is considered,

$$P_{\alpha\beta} = \left(\frac{\partial \mathbf{X}}{\partial x_\beta \partial x_\alpha}, \mathbf{n} \right), \tag{3.45}$$

$P_{\alpha\beta}$ being the scalar product which computes the **n** component of the variation of a vector $e_\alpha(\mathbf{X})$ *when translated by* dx_β *in the* $e_\beta(\mathbf{X})$ **direction.** This definition of $R^{\delta}_{\alpha\beta\gamma}$ is then based on the assumed existence of a normal **n** to a space $M \in \mathbf{X}$. The consequence of the introduction of the action coordinate (ST→STA) is: a normal **n** exists at every point.

As the normal direction to space–time has the dimension of action, then (3.45) corresponds to computing the rate of change of action, that is energy–momentum, in relation to the local curvature of space–time.

The STA geometry is assumed to have a null Ricci tensor: $a\cdot\mathbf{R}(a\wedge b) = R(b) = 0$ obtained by contraction of the usual curvature $\mathbf{R}(a \wedge b) = a\cdot\Box\Omega(b) - b\cdot\Box\Omega(a) + \Omega(a)\times\Omega(b)$ with the bi-vector functions $\Omega((b),x)$ being the local Lorentz group connections $D_a B = a \Box B + \Omega(a) \times B$ in the STA space. The corresponding 5-dimensional (flat space) Einstein tensor would be $G_{AB} = e_A \,.\, (R(e_B) - Rg_{AB}e_A) = 0$. From Campbell (1926) [23] any analytic $(N-1)$-dimensional manifold can be nested in an N-dimensional flat manifold $R(b) = 0$. We can use now the well known geometric procedure of dividing the 5-D metric tensor G_{AB} into a 4-D geometric part and an 1-D induced energy part. We obtain from $R(b) = 0$, by direct substitution, the components of $G(a)$ rewritten

as geometry and action, in the form $G(a) = kT(a)$ with $T(a)$ in the canonical form e_μ . $T(e_\nu) = (p + p)u_\mu u_\nu - pg_{\mu\nu}$, $\mu, \nu = 0,1,2,3$. The complex space–time line element becomes

$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu + G_{44}\left(\frac{G_{4\mu}}{G_{44}}dx^\mu\right)^2,$$

where the space–time standard line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ acquires the necessary values for representing different physical conditions of the system. We use now the Kaluza–Klein-like [88, 129] identification of the energy density and electro(weak) potential $A_\mu = G_{4\mu}/G_{445}$ to obtain $dS^2 = g_{\mu\nu}dx^\mu dx^\nu - m^2(dx^4 + A_\mu dx^\mu)^2$ rewritten as

$$dS^2 = g_{\mu\nu}dx^\mu dx^\nu - \frac{f^2}{m^2}dS^2, \quad \left(1 + \frac{f^2}{m^2}\right)dS^2 = ds^2.$$

The definition of m^2 above corresponds to the norm of a scalar product. In the five dimensional space we set the equivalent 4-D metric tensor

$$G_{\mu\nu} = g_{\mu\nu} - g^2\left(k_\mu k_\nu - \frac{e^2}{g^2}A_\mu A_\nu\right). \tag{3.46}$$

Here the square of the action density appears as the sum of mass terms and an interaction fields term $k^\mu k_\mu = 1$ and the A_μ are weighted STA bi-vectors.

This is a straightforward procedure to show the physical implications of using STA geometry:

1. We consider the formalism above for an equivalent simple solution of the Einstein equations in the presence of dust matter.

2. If we interpret STA as complex coordinates in the way described above, we show that the Einstein equations are a condition for the relationship between the real part and the imaginary part of the line element, once (3.46) is defined.

3. A set of constants appears as a basic property of matter, which in our approach is related to k_0 in (2.10)

4. The use of the space–time–action (STA) geometry allows the construction of an equivalent, if the concept of test particle is introduced as in the previous chapter, to a generalization of the Kaluza–Klein theory (see, for example [186, 187]) with both induced matter and interaction fields.

3. MATTER, GEOMETRY AND MINIMAL TRAJECTORY CONDITION

The procedure to analyze the model as a geometric theory of matter has gone through the following steps:

1) Assume space and time geometrically unified in Minkowski space–time manifold **M**.

2) Provide it with a full geometric structure.

3) Introduce action density and define action as a fifth geometrical coordinate. Obtain its full geometry. Enlarge the isometries of space–time to a full Complex Poincaré (With the complex Lorentz group as a subgroup).

4) Project out a physical (in general curved) local **M** manifold, as in Chapter 2. Define a test particle.

5) The projection has Induced Energy Density from the considerations of the previous section above. In fact an **action density** has been defined at every point of **space–time** when **M** is defined.

And will proceed now through:

6) Provide a stable structure to the energy density through a chiral constrain to the massless matter fields, this projection is described below in the section referring to matter and interaction fields. This implies that when a pair of carriers are created from a given amount of energy the symmetries of the carriers should add to zero: $\Delta q = q_1 + q_2 = 0$, $\Delta \mathbf{s} = 0$, etc., some of these symmetries are known in the literature as internal symmetries. Also when energy is transferred from one carrier to the other the intermediate carrier should contain the change in symmetries of the first and produce changes in symmetries of the second, the best known example is the spin of the photon, $\pm\hbar$, taken or given to the carriers.

7) The procedure then generates charges, and, through gauging, interaction fields

8) Every point of the matter fields is a source of a geometric wave, owing to the continuity of the distribution of action in space–time, required by assuming analyticity. This wave, described by the Kaluza–Klein-like extension of the Einstein equations, is here related to the description of action in space–time. In classical physics it corresponds to the infinite extension of the gravitational and the electromagnetic fields.

9) To transform this geometrical theory into a physical theory, useful for describing observed matter, the resulting structures should possess all the symmetries of the (enlarged, as we discuss in the next chapter) Standard Model. See next chapter.

The mathematical structures presented here have to be related to our perception of the physical world; the observer analyses a system through changes of action produced or induced in the observer system. This set of changes are associated with both the energy–momentum tensor of the distribution and with its possible responses.

A basic quantity is the conservation law implicit, in the equations defining the distribution. For a reference system we have, at every point

$$D \bullet (\rho\upsilon) = 0, \tag{3.47}$$

which endows the distribution with a set of trajectories along which a set of quantities have conserved values. For other quantities an average value can be computed along the trajectories; special examples are the average space–time position and the average velocity, allowing the definition of an equivalent, center of mass, trajectory and the velocity of the motion of this special point with respect to some frame of reference. This allows the definition of internal (center of mass as a reference) description of the distribution and an external (co-moving with the center of mass) description.

The set of world lines depends (causally in space–time, that is, as retarded interactions) by construction on the boundary conditions.

The lowest energy of a non-interacting distribution would correspond to the most extended one. As a consequence, following our principle of minimum action for a system of indistinguishable identical carriers, the most extended distribution should be found unless in (18) terms of the type $V(x)$ or $V_{xc}(x)$ favor a self-organization into a more compact distribution. An example in real nature is to be found in the atoms in which the larger the nuclear charge Z the smaller the ionic size for. a given number $n \leq Z$ of electrons; there $-Ze^2/r$ can be considered as the external potential.

Another consequence is that singularities and discontinuities in the distribution should not appear in the present theory as they would be sources of diverging contributions to the energy–momentum An interaction-free steady state distribution should be space-time symmetrical to allow coherent descriptions by different observers.

To make the connection with accepted concepts consider a distribution ρ with the lowest inhomogeneity, we shall take it to tend to zero $\partial\rho/\partial x \rightarrow 0$ or if $\rho(x) = \rho_0 = n/V_0$ in a large volume V_0 then $\rho_0 \rightarrow 0$ with vanishing

momentum terms. If as a result of some interactions $\rho(x)$, $\int_{V_0} \rho(x) = n$ but locally $\partial\rho/\partial x \neq 0$, an energy term related to momentum will appear, in the $u << c$ approximation (using again $\varepsilon = \varepsilon_0 + \Delta\varepsilon$)

$$\Delta\varepsilon = \int_{V_0} \frac{\vec{p}^2}{2m_0}\rho dx > 0, \quad \varepsilon_0 = m_0c^2, \tag{3.48}$$

for the energy increase. For a **symmetric** distribution of that type of carriers (even if $n = 1$)

$$(\varepsilon_0 + \Delta\varepsilon)^2 - \left[c\int_{V_0} p(x)\rho(x)dx\right]^2 = (\varepsilon_0 + \Delta\varepsilon)^2 > \varepsilon_0^2, \tag{3.49}$$

and the increase in energy has to be considered as internal energy. This process requires an external system providing the balance in change in action, say a second type of carriers, in respect to which the first type can exchange (at least for description purposes) energy. For a non-trivial theory this has to be allowed. A mathematical modeling of the interaction would be that of carrier 2, possessing an interaction capacity in the domain V_0. In our gauge freedom description we need to introduce a gauge field $B^{(2)}$ which will interact with carrier 1 with a strength $q_1B^{(2)}$. The description has to be gauge-free also for carrier 2 then we have to allow for the term $q_2B^{(1)}$ to be used for carrier 2. The simplest gauge field created by a source carrier distribution at rest in some frame of reference S would be spherically symmetrical to avoid spurious direction dependence with constant total force in concentric volume surfaces

$$4\pi r^2\frac{\partial}{\partial r}B^c(r) = b^c, \quad B^c = -\frac{b^c}{4\pi r}, \tag{3.50}$$

$b^c/4\pi$ will characterize the carrier in respect to the auxiliary gauge field. See next sections.

If $b_{(c)} = 4\pi q_{(c)}$ then the energy (per unit carrier density) between carriers is q_1q_2/r. Our knowledge of the real physical world tells us that this simple picture is a useful first approximation, at least, at the level of elementary carriers. Below we shall analyze this approach to construct a theory of the electron and we postpone to that point a further analysis of this possibility. Back to the problem of the extra energy in building up a concentrated carrier density, the first, conclusion is that the carrier density will not increase in a region unless there is some interaction which makes this possibility feasible. Here we should remember that Lorentz transformations (boosts) will result in the observer at rest considering that characteristic length of moving field will appear shorter, this is a purely geometric phenomena in STA.

For further analysis start with a neutral action distribution, with no symmetry constrains, and allow it to split into $n_+ + n_-$ carriers $n_+ = n_-$ with symmetrical opposite constraints (called + and –) with a distance inverse (linear) rate of energy expenditure needed

$$\Delta\varepsilon(r_{12}) = \frac{b_0}{r_{12}} + \frac{b_0'}{(r_{12})^2} + \ldots, \quad b_0' \ll b_o. \tag{3.51}$$

Every point of each distribution acquires a velocity with respect to the common center of mass $\upsilon^{(i)}(x)$ where i represent the type of carrier. The kinetic energy increases

$$\begin{aligned}\frac{\partial}{\partial t}k &= \frac{d}{dt}\int_V \sum_i \rho^{(i)}(x)\mathbf{v}^{(i)}(x)^2 m_0^{(i)} dx \\ &= \tfrac{1}{2}\int \sum_i b^{(i)}\mathbf{v}^{(i)}(x)\cdot\sum_{j\neq i}\mathbf{E}^{(ji)}(x)\rho^i(x),\end{aligned} \tag{3.52}$$

where $E^{(ji)}$ is the equivalent force of carrier j over carrier i and by partial integration and definition of the solenoidal and rotationless parts in (3.33) to (3.38), that is space–space and space–time bi-vectors in STA

$$b^{(i)}\mathbf{v}^{(i)}(x) = \nabla^2 B = \frac{\nabla\times\mathbf{B}}{\mu_0} - \varepsilon_0\frac{\partial\mathbf{E}}{\partial t}, \tag{3.53}$$

we obtain

$$\frac{\partial k}{\partial t} = -\frac{d}{dt}\int_V \left(\frac{\mathbf{B}^2}{2\mu_0} + \frac{\varepsilon_0\mathbf{E}^2}{2}\right) d\mathbf{x}, \tag{3.54}$$

provided we can put $(1/\mu_0)\int_f df(\mathbf{B}\times\mathbf{E}) = 0$, where this last surface integral represents the flux of auxiliary field energy on the surface f of the volume V. The correct units are included in μ_0 and ε_0 and we have used an electromagnetic like notation.

An special case arises when the auxiliary field is not constant in time (the simpler oscillatory case is best suited for our purpose) and then from the definitions an oscillatory **B** field induces a perpendicular oscillatory **E** field and *vice versa* and an effective self-current is created from

$$\mathrm{j}_{\text{self–current}} = \Box^2 B = \frac{1}{\mu_0}\nabla\times\mathbf{B} + \varepsilon\frac{\partial\mathbf{E}}{\partial t}, \tag{3.55}$$

the analysis can be carried out for this case and an energy density

$$w(x) = \frac{\varepsilon\mathbf{E}^2(x)}{2} + \frac{\mathbf{B}^2(x)}{2\mu}, \tag{3.56}$$

can be defined, together with an energy flux $\mathbf{P}(x)$

$$\mu_0 \mathbf{P}(x) = \mathbf{E} \times \mathbf{B}, \tag{3.57}$$

and an angular momentum flux

$$N = \hbar \frac{\mathbf{P}(x)}{w(x)}, \tag{3.58}$$

It is important to remark that even if the energy attributed to the rotational part **B** is small the quantities of interest depend basically in its contribution. The procedure (3.55) to (3.58) defines what we have called above an independent existence interaction carriers.

Before entering to the formal presentation of the geometric theory of the electron we must mention that we have presented elsewhere the construction of a theory of lepton and quark fields [89, 105, 117] using chiral geometry for the formulation of the multi-vector generalization of the Dirac factorization of the four dimensional d'Alembert operator (notation remainder: we use $\nabla^2 = \partial_i \partial_i$; $i = 1,2,3$) $\Box^2 = D^2 = \partial_\mu \partial_\mu$, written in the Lorentz invariant form

$$D_{(d,f)} = i\hbar \Gamma^{\mu}_{(f)} \partial^{(d)}_{\mu}, \tag{3.59}$$

in order to show the relation to the Dirac's original factorization in the simplest, possible form. In (3.59) the $\Gamma^{\mu}_{(f)}$ are either the fundamental or generalized (reducible representation) Dirac γ^μ matrices. We have shown [127] that these fields constitute a set with all the known properties of an elementary carrier's family, the fields they represent are: massless or massive after interactions are considered; and charged (integer or fractional). There the collection of the constructed fields have weak charge and color, and in general the characteristics usually postulated on phenomenological basis like composites being colorless, confinement, etc., here being immediate consequences of the defining equations.

The principal change from the standard model is that we are dealing there with a **theory** where the equations have, as constitutive parts, a series of conditions reproducing what the phenomenological approach showed to be necessary. The conditions are related to the basic properties of an action density distribution over space–time as a frame of reference to describe physics.

Because of the appearance or not of the $i\gamma^5$ factors, the fields have definite chiral properties. One of the most relevant consequences for our present purpose is that the electron field in the theory may have both chiralities simultaneously and therefore can be massive, charged (reference charge ± 1) and weak charged.

4. CENTER OF MASS COORDINATES AND HAMILTONIAN FORMALISM

We include this short discussion to show that relativistic invariance can be properly handed, otherwise we refer the reader to Barut 1991.

For the study of the procedure described in this section consider the action distribution generated by the following, gauge dependent as shown below, total energy:

$$W = \int dx \left\{ \sum_i \overline{\psi}_i(x) \left(\gamma^\mu \partial_\mu - m_i\right) \psi_i(x) - j_\mu(x) A^\mu(x) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\}, \tag{3.60}$$

corresponding to a set of N similar carriers in interaction, and to an interaction field through A^μ and $F^{\mu\nu}$. Here similar carriers are defined as carriers with identical mass and differing only in the signs of their non contributing to the action symmetry constraints. The action related to the intrinsic angular momentum is the contained by definition in m_i. In order to see explicitly that (3.60) is a one-time coordinate (the same for all carrier fields) equation, we now separate center of mass and relative coordinates. We define (following the notation of Barut as closely as possible) the total center of mass momentum P_μ by the vector

$$P = \sum_{k=1}^{N} |p_k|, \tag{3.61}$$

and $N(N-1)/2$ relative momenta p_{ij}, not all independent, such that

$$p_k = \frac{1}{N} P + \frac{1}{N} \sum_{i<j} \alpha_k^{ij} p_{ij}, \tag{3.62}$$

where α_k^{ij} are constant mass ratios such that

$$\sum_k \alpha_k^{ij} = 0, \quad i, j = 1 \ldots N \tag{3.63}$$

If we insert (3.62) into (3.60), using, from $\gamma_\mu = \gamma(e_\mu)$, the dot product notation $\gamma \bullet n = \gamma_\mu n^\mu$, the kinetic energy then becomes

$$\sum_k \gamma \bullet n \cdots \overset{(k)}{\otimes} \gamma \left(\frac{P}{N} + \frac{1}{N} \sum_{i<j} \alpha_k^{ij} p_{ij} \right) \cdots \gamma \bullet n \tag{3.64}$$
$$- \gamma \bullet n \cdot \gamma \bullet n \cdots \overset{(k)}{I} \cdots \gamma \bullet n m_k$$
$$- \Gamma^\mu P_\mu + \sum_{i<j} \Lambda_\mu^{ij} P_{ij}^\mu - \underbrace{\sum_k \gamma \bullet n \cdots \overset{(k)}{I} \cdots \gamma \bullet n m_k}_{\Sigma = \Sigma^\lambda m_\lambda},$$

where

$$\Gamma^\mu = \frac{1}{N} \sum_{k=1}^{N} \gamma \bullet n \cdots \overset{(k)}{\gamma^\mu} \cdots \gamma \bullet n,$$
$$\Delta^{ij,\mu} = \frac{1}{N} \sum_{k=1}^{N} \alpha_k^{ij} \gamma \bullet n \cdots \overset{(k)}{\gamma^\mu} \cdots \gamma \bullet n, \tag{3.65}$$
$$\sum = \sum_{k=1}^{n} \gamma \bullet n \cdots \overset{(k)}{I} \cdots \gamma \bullet n m_k. \tag{3.66}$$

Now, because of the relation (3.63) we find that the component of Δ_μ^{ij} parallel to n^μ vanish:

$$\Delta_\mu^{ij} n^\mu = 0. \tag{3.67}$$

We can also split p_{ij}^μ into components parallel and perpendicular to n_μ:

$$p_{ij}^\mu = (p_{ij} \cdot n)\, n^\mu + P_{ij\perp}^\mu, \tag{3.68}$$

where only $P_{ij\perp}^\mu$ contribute. For $n^\mu = (1000)$, it means $\Lambda_0^{ij} = 0$ and only the three-dimensional vector part p_{ij} of the relative momenta enter the equation.

Hence we can write (3.60) as a wave equation

$$\left(\Gamma^\mu P_\mu + \Lambda_\mu^{ij} p_{ij}^\mu - \Sigma^\lambda m_\lambda - \sum_{i<j} e_i e_j \frac{G_{ij}}{r_{ij}} - V^{\text{self}} \right) \times \Phi(x_1 \cdots x_n) = 0, \tag{3.69}$$

with

$$G_{ij} = \gamma \bullet n \cdots \overset{(i)}{\gamma^{\mu}} \cdots \overset{(j)}{\gamma_{\mu}} \cdots \gamma \bullet n. \tag{3.70}$$

We shall now pass to the Hamiltonian form of our covariant N-body equation (3.69). The Hamiltonian can be identified with the component of P_μ parallel to n_μ, i.e., $P \cdot n$, which is just P_{ij} for $n = (1000)$. We therefore decompose the four vectors p_μ and Γ_μ also as in (3.68):

$$p_\mu = (p \cdot n)\, n_\mu + p_{\mu\perp}, \quad \text{and} \quad n^\mu p_{\mu\perp} = 0, \tag{3.71}$$

$$\Gamma_\mu = (\Gamma \cdot n) n_\mu + \Gamma_{\mu\perp}, \quad \text{and} \quad n^\mu \Gamma_{\mu\perp} = 0. \tag{3.72}$$

The normal component $P_{\mu\perp}$ is the kinetic momentum of the center of mass. Hence,

$$\left(\Gamma_\parallel P_\parallel + \Gamma_\perp^\mu \cdot P_{\perp\mu} + \Lambda_\perp^{ij} \cdot p_{ij\perp} - \Sigma^\lambda m_\lambda - \sum_{i<j} e_i e_j \frac{G_{ij}}{r_{ij1}} - V^{\text{self}} \right) \times \Phi = 0, \tag{3.73}$$

or, multiplying the equation from left with $\Gamma_\parallel^{-1}$, we can isolate $P_\parallel$

$$P_\parallel \equiv H = \alpha_\perp \cdot P_\perp - k_\perp^{ij} \cdot p_{ij\perp} + M^\lambda m_\lambda + \sum_{i<j} e_i e_j \frac{\Gamma_\parallel^{-1} G_{ij}}{r_{ij\perp}} + \Gamma_\parallel^{-1} V^{\text{self}}, \tag{3.74}$$

where, since

$$\Gamma_\parallel = \gamma \bullet n \gamma \bullet n \cdots \gamma \bullet n = \Gamma_\parallel^{-1}, \tag{3.75}$$

we have put

$$\Gamma_\parallel^{-1} \Gamma_\perp \equiv \alpha_\perp = \sum_k I \cdot I \cdots \overset{(\lambda)}{\alpha_\perp} \cdots I \equiv \sum_\lambda \alpha^\lambda, \tag{3.76}$$

$$\Gamma_\parallel^{-1} \Lambda^{ij} \equiv k_\perp^{ij} \equiv \frac{1}{N} \sum_k \alpha_\lambda^{ij} I \cdots \overset{(k)}{\alpha_\perp} \cdots I \equiv \frac{1}{N} \sum_\lambda \alpha_\lambda^{ij} \alpha^\lambda, \tag{3.77}$$

$$\Gamma_\parallel^{-1} \Sigma^k \equiv M^k = I \cdots \overset{(k)}{\gamma} \cdot n \cdots I, \tag{3.78}$$

$$\Gamma_\parallel^{-1} G_{ij} \equiv 1 - \alpha_\perp^i \cdot \alpha_\perp^j = I \cdots I - I \cdots \overset{(i)}{\alpha_\perp} \cdots \overset{(i)}{\alpha_\perp} \cdots I. \tag{3.79}$$

For n = (1000) we have the simpler equation

$$H = P_0 = \alpha \cdot \mathbf{P} - \mathbf{k}^{ij} \cdot \mathbf{p}_{ij} + \beta^k m_k + \sum_{i<j} e_i e_j \frac{1 - \alpha_i \cdot \alpha_j}{\mathbf{r}_{ij}} + \tilde{V}^{\text{self}}. \quad (3.80)$$

This is indeed a generalized Dirac Hamiltonian with a pair interactions of the Breit-type plus self-interaction terms.

For two-carriers it becomes from $H = P_0$

$$H = (\alpha_1 + \alpha_2) \cdot \mathbf{P} - (\alpha_1 - \alpha_2) \cdot \mathbf{p} + \beta_1 m_1 + \beta_2 m_2 + e_1 e_2 \frac{1 - \alpha_i \cdot \alpha_j}{\mathbf{r}} + \tilde{V}^{\text{self}}. \quad (3.81)$$

For three carriers, with relative momenta $P_{ij} = p_j - p_i$, we obtain from (3.62) and (3.63)

$$\begin{aligned} p_1 &= \tfrac{1}{3} P - \tfrac{1}{3} p_{12} - \tfrac{1}{3} p_{23}, \\ p_2 &= \tfrac{1}{3} P - \tfrac{1}{3} p_{12} - \tfrac{1}{3} p_{23}, \\ p_3 &= \tfrac{1}{3} P - \tfrac{1}{3} p_{12} - \tfrac{1}{3} p_{23}. \end{aligned} \quad (3.82)$$

Hence

$$\begin{aligned} k^{12} &= \tfrac{1}{3} (-2\alpha_1 + \alpha_2 + \alpha_3), \\ k^{23} &= \tfrac{1}{3} (-\alpha_1 - \alpha_2 + 2\alpha_3), \end{aligned} \quad (3.83)$$

and the three-body Hamiltonian becomes

$$\begin{aligned} H &= P_0 \\ &= \tfrac{1}{3} (\alpha_1 + \alpha_2 + \alpha_3) \cdot \mathbf{P} - \tfrac{1}{3} (2\alpha_1 - \alpha_2 - \alpha_3) \cdot \mathbf{p}_{12} \\ &\quad - \tfrac{1}{3} (\alpha_1 + \alpha_2 - 2\alpha_3) \cdot \mathbf{p}_{23} + \beta_1 m_1 + \beta_2 m_2 + \beta_3 m_3 \\ &\quad + \sum_{i<j} e_i e_j \frac{1 - \alpha_1 \cdot \alpha_2}{\mathbf{r}_{ij}} + V^{\text{self}}. \end{aligned} \quad (3.84)$$

Detailed properties of the two-body problem as well as some aspects of the three body problems have been discussed by Barut.

4.1 GAUGING OF THE TWO CARRIERS SYSTEM

As mentioned in the previous chapter, we consider the use of an auxiliary amplitude function Ψ approach for the study of the fundamental problem of the gauging of the two carriers system.

This is a very important analysis because it shows the role of the definition of a fundamental carrier as that physical object wich will be

observed if far from the rest of the objects. We transform the description of two carriers to that of two interacting carriers.

In the previous section, on the Hamiltonian Formalism, for two carriers, 1 and 2, we had to consider a multi-carrier amplitude function $\Psi = \psi_1 \otimes \psi_2$ which was written above as $\Psi = (\{x_c\};\ c = 1, ..., n)$ indicating that a set of carriers can be described by an amplitude function depending in a set of coordinates x^μ_c for each carrier. For the physical description a set of operators like $p^c = i\hbar\partial_\mu$ and of energy contributions V $(\{x^\mu_c\})$. This brought the additional concept of reducible representations of the geometrical quantities. The Γ^c_μ matrices of this representation are required to obey (upper index corresponds to carrier index, lower index to space–time coordinate components, Greek letters for space–time and Latin letters for space)

$$\begin{aligned} \Gamma^c_\mu\Gamma^j_\nu + \Gamma^c_\nu\Gamma^j_\mu &= 0, \qquad c \neq j, \\ \Gamma^c_\mu\Gamma^j_\nu + \Gamma^c_\nu\Gamma^j_\mu &= 2\eta_{\mu\nu}, \quad c = j, \\ \Gamma^c_\mu \cdot \Gamma^j_\nu &= \delta^{cj}\eta_{\mu\nu}, \end{aligned} \tag{3.85}$$

and the (per carrier) bi-vectors, etc., can be defined

$$\Gamma^1_{i0} = \Gamma^1_i\Gamma^1_0,\ \Gamma^2_{i0} = \Gamma^2_i\Gamma^2_0, \quad \text{etc..} \tag{3.86}$$

In the previous section we have used a collective symbol α_ϵ for all Γ^c_{io} of a given carrier field, as that discussion followed the work of Barut using an early type of notation. These objects commute

$$\Gamma^1_{i0}\Gamma^2_{j0} = \Gamma^2_{j0}\Gamma^1_{i0}. \tag{3.87}$$

We also define the pseudo-scalar as the product of the 4 + 4 = 8 vectors associated with the two carriers. We also define $\Gamma^i_5 = \Gamma^i_0\Gamma^i_1\Gamma^i_2\Gamma^i_3$ for each carrier and the global

$$\Gamma_5 = \Gamma^1_5\Gamma^2_5. \tag{3.88}$$

In this presentation the allowed set of algebraic solutions Ψ for the system are constructed, as in Chapter 6 below, from pairs of double projections, one double projector for each carrier, and have a matrix structure which can be represented by an equivalent column matrix, as the outer product of the representative column matrices for each carrier. For each carrier the column matrix has four entries as a result of the use for each carrier of the double projection $\frac{1}{2}(1 \pm \gamma_0)\frac{1}{2}(1 \pm i\gamma_{12})$ as in Chapter 6.

$$\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \text{etc..}$$

The first column describes carrier 1 and the second carrier 2, taken a given mass projector and a given spin projector of the system, in that case for the two carriers system with zero total spin

$$\psi\Gamma_5^1\Gamma_{30}^1 = -\psi\Gamma_5^2\Gamma_{30}^2,$$

and then we can, for the two particle auxiliary function ψ define

$$\psi = \psi\Gamma_5^1\Gamma_{30}^1\Gamma_5^2\Gamma_{30}^2,$$

a procedure equivalent to the use of the joint double projector

$$\psi = \psi\frac{1}{2}\left(1 + \Gamma_5^1\Gamma_{30}^1\Gamma_5^2\Gamma_{30}^2\right).$$

The extension allows the definition of a set of double projectors from $S_{\uparrow\uparrow}$

$$S_{\uparrow\uparrow} = \frac{1}{2}\left(1 - \Gamma_5^1\Gamma_{30}^1\Gamma_5^2\Gamma_{30}^2\right), \qquad S_{\uparrow\uparrow}^2 = S_{\uparrow\uparrow},$$

and its factorization

$$S_g = S_{\uparrow\uparrow}\,\Gamma_5^1\Gamma_{30}^1 = S_{\uparrow\uparrow}\,\Gamma_5^2\Gamma_{30}^2 = \frac{1}{2}\left(\Gamma_5^1\Gamma_{30}^1 + \Gamma_5^2\Gamma_{30}^2\right),$$

such that

$$S_g^2 = -S_{\uparrow\uparrow},$$

which allows us to obtain by projection $\frac{1}{2}(1 \pm i\Gamma_{12}^1)\frac{1}{2}(1 \pm i\Gamma_{12}^2)$ from a general solution the set of all four possible **spin** choices.

In the previous section we have used this technique to study the case of two and three carriers fields as special cases of the general formalism. Here we use the zero total spin case only.

Defining the space–time gradient operators

$$\Box^1 \equiv \Gamma_\mu^1\frac{\partial}{\partial r_\mu^1}, \qquad \Box^2 \equiv \Gamma_\mu^2\frac{\partial}{\partial r_\mu^2},$$

rewriting the free carrier equation

$$i\hbar\Box^c\psi = m_c\psi \quad \text{as} \quad \frac{i\hbar\Box^c}{m_c}\psi = \psi,$$

and using the properties

$$\psi\Gamma_0^c = \psi \quad \text{and} \quad \psi i\Gamma_{12}^c = \psi,$$

we can formally sum for the two 'free', non-interacting carriers, to obtain

$$\hbar\left(\frac{\Box^1}{m_1}+\frac{\Box^2}{m_2}\right)\psi(x)\left(\Gamma^1_{012}-\Gamma^2_{012}\right)=2\psi(x), \tag{3.89}$$

where we have used the joint coordinates

$$x=r^1+r^2=\Gamma^1_\mu r^{1\mu}+\Gamma^2_\mu r^{2\mu}.$$

The (projected by spin) amplitude function ψ corresponding to the description of two **independent** carriers can, by definition, be factorized

$$\psi=\psi^1\psi^2,$$

to obtain from (3.89), using the projector $S_{\uparrow\downarrow}$ to ensure the correct spin for each carrier, the amplitude function ψ^c

$$\hbar\left(\frac{\Box^1}{m_1}+\frac{\Box^2}{m_2}\right)\psi^1\psi^2 S_{\uparrow\downarrow}\left(\Gamma^1_5\Gamma^1_3-\Gamma^2_5\Gamma^2_3\right)=2\psi^1\psi^2 S_{\uparrow\downarrow}. \tag{3.90}$$

now following our procedure of redistributing the energy in a gauge-free form we transform (3.89) by a phase factor rotation. This requires us to understand the gauging properties of the equation by studying its behavior under Lorentz transformations.

For the transformation of the basic equations above into the gauged equations, where the action is shared through an interaction, consider a pair of Lorentz transformations R^1 and R^2 acting as

$$R^c\Gamma^c_\mu\widetilde{R^c}=\Gamma'^c_\mu,$$

and their action on the amplitude function

$$\psi(x)\longmapsto\psi'(x)\equiv R^1R^2\psi\left(\widetilde{R^2}\widetilde{R^1}xR^1R^2\right),$$

which is induced by the application of R^1R^2 to (3.89) or (3.90). For the analysis of the two carriers system consider the particular case

$$R^1\Gamma^1_{12}\widetilde{R^1}=\Gamma^1_{12},\quad R^2\Gamma^2_{12}\widetilde{R^2}=\Gamma^2_{12},$$

where by definition of space–time volume

$$R^1\Gamma_5\widetilde{R^1}=R^2\Gamma_5\widetilde{R^2}=\Gamma_5,$$

such that R^1 and R^2 are Lorentz boosts in the γ_3 direction, with opposite signs and rotations in the γ_{12} plane, also with opposite signs. With this choice $x\rightarrow x'=\tilde{R}^2\tilde{R}^1xR^1R^2$ corresponds to induced separation of the

carriers in the z direction with a synchronized rotation of the spin plane (perpendicular to z). This can be compensated if the new amplitude function ψ' obeys a gauged equation

$$\hbar\left(\frac{\Box^1}{m_1}+\frac{q_1}{m_1}A^1+\frac{\Box^2}{m_2}+\frac{q_2}{m_2}A^2\right)\psi'(x')\left(\Gamma^1_{012}-\Gamma^2_{012}\right)=2\psi'(x'), \tag{3.91}$$

For an observer the new situation is that of ψ' describing two interacting carriers in relative motion, with their spins having a relative additional rotation. The carriers have acquired kinetic energy and generated a reciprocal interaction proportional to q_c. If the total energy is required to be constant for all values of the joint position x, then the A^c obey the Maxwell equations. This is clarified in Chapter 6, section 5, where spin and electromagnetic interactions are analyzed. The final step is to define A^1 and A^2 from (3.38) to substitute this terms in (3.91) to obtain a pairwise interaction term as in (3.37).

Chapter 4

THE THEORY OF THE ELECTRON

We have already stated that there is no doubt that, the Dirac equation for the electron provides a sound starting point for the theory of the electron field both in the presence of electromagnetic and of weak interactions. The interaction fields are obtained as gauge fields of the matter field. Here we proceed to construct a theory of the electron from START and, because the Dirac equation is obtained from the first steps, the development of the theory will mostly appear as a theoretical derivation of the standard theory where the geometrical characterization of the different concepts (postulated in the Dirac theory) defines them and, most important, limits their usage. We are not otherwise proposing or justifying specific models or pictures of the electron, we only propose a geometric, fundamental, theory.

A fundamental reference for our present purposes in relation to the geometric content of the electron theory, is the 1929 work of Fock and Ivanenko [62] and the ones related to the discovery (1929–1935) of the use of a geometric approach based in recognizing the Dirac matrices algebra as a representation of abstract Clifford algebras to study the Maxwell field and the formulation of the Dirac theory [76]. This is important because the algebraic solution of the Dirac: equation gives a clue to the geometric meaning of many physical quantities. This is the subject of Chapters 5 and 6 below.

1. CARRIER FIELDS WITH ELEMENTARY TRAJECTORIES IN START

We shall now proceed to develop a theory for fields of carriers, embedded in the space–time–action geometry, with the simplest characteristic

allowed by START. At the end we shall show that the properties of these fields correspond to the experimental known properties of the electron then, within START, we are developing a theory of the electron.

There are two types of fundamental trajectories in START, the ones defined by

$$dS^2 = 0, \quad ds^2 = 0, \quad da^2 = 0, \tag{4.1}$$

corresponding to massless fields, and the ones defined by

$$dS^2 = 0, \quad ds^2 \neq 0, \quad da^2 \neq 0 \tag{4.2}$$
$$\text{with} \quad ds^2 - da^2 = 0.$$

The first one correspond to the cases of massless fields because the trajectories correspond to light-like paths and the energy–momentum relationship $E = pc$, combined with the light-like trajectories, ensure that $da^2 = 0$, that is

$$(cdt)2 - (dx)2 - (\mathrm{dy})^2 - (dz)^2 = 0 \tag{4.3}$$

$$(Edt)^2 - ((p_x dx)^2 + (p_y dy)^2 + (P_z dz)^2) = 0, \tag{4.4}$$

whereas the second type of trajectories correspond to the particular case where, using $m_0c^2 = E = h\nu$ and the particular choice of metric for the action coordinate $k_0 = d_0/h = 1/m_0c$ (where we have by definition a correspondence between d_0 and m_0), we have the relationship

$$(cdt)^2 - (K_0 m_0 c^2 dt)^2 = (cdt)^2 - (da)^2 = 0. \tag{4.5}$$

Then either we construct a theory with the first type of fields only (all starting fields are massless), or we construct directly a theory for a particular field where we have a rest mass m_0. The standard model of elementary carriers has a mathematical structure corresponding to the first type of approach, all fields are massless before interaction. The common theory of the electron and the practical use of the Dirac equation have shown that for all properties which are not related to weak interactions, the direct use of the consideration of the electron as a carrier with mass m_0 is a useful approach. Here we shall start with the second approach and in a second step show its extension to the first approach and then, by necessity, the inclusion of the electroweak interactions. We shall follow this procedure because we shall be analyzing different concepts in each case and because at the end there is no contradiction.

2. GAUGE FREE DESCRIPTION OF AN ENERGY DISTRIBUTION

According to our choice of description we consider an (considered at rest in some system) energy distribution $\varepsilon(\mathbf{X}) = \rho(\mathbf{x})\bar{\varepsilon}$ such that

$\partial\varepsilon/\partial t = 0$, through the density of n carriers

$$\int_V \rho(\mathbf{x})d\mathbf{x} = n, \tag{4.6}$$

and an average energy per carrier, for a free carrier $\bar{\varepsilon} = m_0c^2$. If the description corresponds to free carriers the relation (4.5) above is equivalent to $m_0c^2 = h\nu_0$ with a proper frequency ν_0. For $n = 1$ we require that the density is normalized within a domain V at instant t' for observer S

$$\int_V \rho(\mathbf{x})d\mathbf{x} = 1. \tag{4.7}$$

The normalization then corresponds to the definition of all the points of the distribution ρ as corresponding to the domain of the field of that type of carriers. We mentioned above that the requirement of an equal energy per carrier, of a given type, corresponds to defining a given domain as the space–time volume where we have a system of indistinguishable carriers. From the normalization (4.6) itself it also follows that the proper frequency is not a property of the distribution but a property of the carrier which is being described. Likewise, for an observer in relative motion a wave length will appear which is not a property of the distribution but a property of the carrier.

Of course, this distribution is at rest only with respect to one frame of reference S with space-time coordinates $\{x^\mu;\ \mu = 0,1,2,3$ and $x_0 = ct\}$. For any other frame of reference $S^{(0)}$ we must consider a Lorentz transformation R where $\mathbf{X} = x^\mu e_\mu \rightarrow \mathbf{X}' = R\mathbf{X}R^{-1}$.

Associated with each matter field there is an energy–momentum field $e_\mu p^\mu(x)$ (summation convention is used), denoting by $\mathbf{X} = e_\mu x^\mu$ points in the S observers frame of reference, such that from the covariance of the energy momentum vector

$$e_\mu p^\mu(x) = m_0 c e_0^{(0)}, \tag{4.8}$$

assuming that there exists a (local) frame $e_\mu^{(0)}$ where the energy–momentum is the one corresponding to that of a carrier's density at rest. The frame $e_\mu^{(0)}$ is related to the observers frame e_μ through the local Lorentz transformation

$$e_\mu^{(0)} = R(x)e_\mu R^{-1}(x), \quad R^{-1} = \bar{R} \ ; \quad R^{-1}R = RR^{-1} = 1, \tag{4.9}$$

then (4.8) becomes

$$e_\mu p^\mu(x) = m_0 c R(x) e_0 R^{-1}(x), \tag{4.10}$$

we multiply (4.10) by $R(x)$ on the right

$$e_\mu p^\mu(x) R(x) = m_0 c R(x) e_0, \tag{4.11}$$

As, a crucial step in our program, we now give a definition to the field distribution of the carriers, where: 1) the distribution is at rest in some frame of reference $\Sigma^{(0)}$ and for each point the space–time trajectory is given by the vector $e_0^{(0)}$ and we shall allow 2) for the field to have a possible spin **S** with plane of reference $e_1^{(0)}\, e_2^{(0)}$ (we should recall here that for any multi-vector M the local Lorentz transformations is $R(x)MR^{-1}(x)$). For this purpose we now use the multi-vector double projector $P+\uparrow$, with the properties

$$P_{+\uparrow} = e_0 P_{+\uparrow} = P_{+\uparrow} e_0 \quad \text{and} \quad P_{+\uparrow} = P_{+\uparrow} i e_1 e_2, \tag{4.12}$$

to obtain a projected transformation

$$e_\mu p^\mu R(x) P_{+\uparrow} = m_0 c R(x) P_{+\uparrow} i e_0 e_1 e_2. \tag{4.13}$$

Here the ie_1e_2 factor is to be kept for further reference to the fact that $P_{+\uparrow}$ was chosen as the appropriate projector, other choices could have been made. The up arrow refers to e_{12} as the direction of spin up and the plus sign to the choice of 'positive' mass m_0.

The representation of the e_μ defines the value of the spin, the standard representation corresponds to $|\mathbf{S}| = \frac{1}{2}\hbar$.

From our definitions of the gauge-free representation of $\varepsilon(\mathbf{X}) = \rho(\mathbf{x})\bar{\varepsilon}$, there is an action amplitude function (a multi-vector Dirac spinor in fact, as discussed in the next, chapter) in the space–time–action algebra, allowing us to obtain p^μ directly. We introduce the action amplitude function $\psi(x)$ from the definition in equations (3.23) to (3.30), a procedure which requires from (4.12)

$$\psi(x) = \mathcal{A}(x) R(x) P_{+\uparrow} \in \text{STA}, \tag{4.14}$$

with $A(x)$ containing the basic information about the action, such that (4.13) can finally be written, through the use of the operator $p_\mu = i\hbar\partial_\mu$,

$$\hbar e_\mu \partial^\mu \psi(x) = m_0 c \psi(x) e_0 e_1 e_2, \tag{4.15}$$

where the $i = \sqrt{-1}$ *has* been cancelled on both sides of (4.13). In the reference 'rest' frame of the field $R(x) = 1$ and free from gauge fields $A(x)$ should be such that

$$i\hbar e_\mu \partial^\mu \mathcal{A}(x) = m_0 c \mathcal{A}(x) e_0 e_1 e_2. \tag{4.16}$$

Our analysis here shows explicitly the multi-vector content of the Dirac spinor. The wavefunction (4.14) explicitly contains then 3 main contributions: the existence of the carriers' field in $A(x)$, the relative motion of the carriers' field in $R(x)$ and the reference time direction corresponding to a preferred sign of m_0 and spin in $P_{+\uparrow}$.

This is a derivation from first principles of the (until now only postulated) Dirac equation, being also an explanation of the geometric reason to consider a multi-vector equation which *goes* beyond the multi-vector analysis which was done (when wave function relativistic quantum mechanics was first developed) by solving the Dirac equation in terms of multi-vectors. The $\psi \in$ STA then contains a local Lorentz transformation and the information that a fixed time direction e_0 and a given plane e_1e_2 has been taken as an **overall** reference. But, yet another element of information should be contained in $A(x)$; from the normalization consideration $\int_D \rho_0 d\mathbf{x} = 1$ the quantity $|A(x)|^2$ should have the dimensions of a density, $A(x)$ then contains: a) a $\sqrt{\rho}$ factor; b) a gauge phase factor to allow for both interaction and freedom of description (discussed below) **;** and c) a basic gauge factor f with the effect of the rest mass in (4.15) of the carrier which should $f \to 1$ for a massless field.

Quantum mechanics is more general than (4.15), our analysis is in fact a starting point applied here to the election as elementary matter, spin $\frac{1}{2}$, fields, as a particular case, because the projector $P_{+\uparrow}$ and its eigenvector $ie_0e_1e_2$ was chosen as a reference. The general solution of (4.15), if $P_{+\uparrow}$ is not explicitly introduced in (4.15), will be an algebraic solution wave function ψ of the standard form, discussed in more detail below, it contains four minimal ideals into one single wave function, but the amount of information is redundant as discussed in Chapter 5. Each ideal contains a symmetry constrain, the sum has none. It is also important to note that the vector operators on the left and on the right of (4.13) are both geometrically odd, then the geometrically even and geometrically odd parts of ψ are linearly independent.

Because of the freedom to select, any useful description of carriers $A(x)$ can be given general multi-vector properties.

2.1 GAUGING OF THE DESCRIPTION OF THE DISTRIBUTION

We can now proceed to the gauging of the auxiliary function introduced in (4.14) and the corresponding gauging of the equation (4.15). Here a very special situation arises because we can either use a scalar phase or in general any multi-vector phase compatible with (4.15). Because the operators have on both sides an odd number of vectors (either e_μ or e_{012}) we can introduce a phase factor on both sides which has an

even number of vector factors without any internal contradiction. That is the allowed phase factors are

$$e^{i\phi(\mathbf{X})} \quad \text{with} \quad \phi(\mathbf{X}) = \phi_{\text{scalar}}(\mathbf{X})\mathbf{1} + \phi_{\text{pseudoscalar}}(\mathbf{X})ie^5 + \phi_{\alpha\beta}(\mathbf{X})e^{\alpha\beta},$$

the presence of the i in the pseudo-scalar phase is necessary to ensure that $e^{i\phi}$ is a phase factor.

In START the presence of the $ie^5 = e^4$, $\gamma(ie^5) = i\gamma^5$ phase is the reason for chirality of the currents of carriers being a basic property of nature as shown by the properties of the set of elementary carriers. This can be clearly seen with the induced gauging of the operators

$$D_{(d,f)} = \Gamma^{\mu}_{(f)}\left[\partial^{(d)}_{\mu} - i\frac{e}{\hbar}A^{(d)}_{\mu}(\mathbf{X})\right], \tag{4.17}$$

the gauging fields having the multi-vector composition (inducing chirality and local tetrads), using the representation $\gamma\,(e_{\mu}) = \gamma_{\mu}$

$$A^{(d)}_{\mu}(\mathbf{X}) = A^{d,\text{scalar(electromagnetic)}}_{\mu} + A^{d,\text{pseudoscalar(weak,color)}}_{\mu} i\gamma^5 + A^{\text{tensor(gravity)}}_{\alpha\beta,\mu}\gamma^{\alpha\beta}, \tag{4.18}$$

the labels are given because from the analysis of the Standard Model of elementary carriers, compared to our own, the gauging has electromagnetic, weak, color and gravity parts. The first two terms carry the index (d) because they are relative properties. Then the wave function becomes upon gauging (φ a reference spinor)

$$\psi_d(\mathbf{X}) = B\exp\left\{i(p^{\mu}_d x_{\mu} + \phi_d(\mathbf{X}))\right\}\varphi, \tag{4.19}$$

with the phase factor being a multi-vector

$$\phi_d(x) = \phi_{d,\text{scalar}}(\mathbf{X})\mathbf{1} + \phi_{d,\text{pseudoscalar}}(\mathbf{X})i\gamma^5 + \phi_{d,\alpha\beta}(\mathbf{X})\gamma^{\alpha\beta}, \tag{4.20}$$

the particular, relative, combinations for the phase, the $i\gamma^5$ terms, generate in our recent work, isospin and color and the $\gamma^{\alpha\beta}$ generate (as first shown by Fock and Ivanenko) the local Lorentz transformation which are a description of the effects of gravity.

In (4.17) a more general representation of the vectors e^{μ} is proposed $\Gamma_{(f)}\,(e^{\mu}) = \Gamma^{\mu}_{(f)}$ which for the case of the electron is $\Gamma^{\mu}_{1} = \gamma^{\mu}$ and for other series (families (f)) is a reducible representation $\Gamma^{\mu}_{(f)} = \gamma^{\mu} \otimes (1 \otimes 1 \otimes \ldots)_{2(f-1)\text{products}}$.

2.2 THE ELECTRON FIELD AS A SUM OF MASSLESS FIELDS

In the considerations of the previous sections the mass of the carrier appears from the particular selection of a field where the density corresponds to a density of paths in the space–time–action geometry. For comparison with the standard model of elementary carriers it seems preferable to start by considering massless fields, that is, fields which in our geometry obey the second postulate because they have space–time differential squared $(ds)^2 = 0$ and action differential square $(da)^2 = 0$.

For this purpose we start by considering two massless fields L_0 and R_0 for spin $\frac{1}{2}$, then by necessity of fixed chirality the first is left handed and the second is right handed. A sum of these two fields will also obey the massless wave equation. For the construction of the massive field, introducing a gauging of the action through a gauge factor, wc shall use now the following:

Theorem LK (Liu and Keller [136]). There exists a suitable complex vector k_μ such that if $\Psi_0 = L_0 + R_0$ satisfy the massless Dirac equation $i\gamma_\mu\partial_\mu\Psi_0 = 0$, then $\Psi = (\Psi_0)\exp(im\int k_\mu dx^\mu)$ will satisfy the massive Dirac equation $i\gamma^\mu\partial_\mu\Psi - m\Psi = 0$. Here

$$k_\mu = k_\mu^+ + k_\mu^- = \frac{\pi_\mu^+}{2\bar{R}_0 L_0} + \frac{\pi_\mu^-}{2\bar{L}_0 R_0}, \tag{4.21}$$

where $\pi_\mu = R_0\gamma_\mu R_0$, $\pi_\mu = L_0\gamma_\mu L_0$, $\pi_\mu = \pi_\mu + \pi_\mu = \Psi_0\gamma_\mu Y_0$.

The phase factor $\exp(i\,m\int k_\mu dx^\mu)$ corresponds exactly to our definition in (4.7) of a carrier with one given energy mc^2 at every point of the distribution. The unit vector $k_\mu k^\mu = 1$ is needed to preserve relativistic covariance. The definition (4.21) shows the dependence on the mixing of the different handedness fields when Lorentz transformations are performed. For a field at rest with respect to the observer the only component of k will be $k_0 = 1$. We see then that even the mechanism for symmetry breaking implied in (4.21) is of purely geometric nature. It can be cast anyhow in the form of a gauge field, as in section 5 below, as any other geometric contribution to the phase. Otherwise any interaction field which could be coupled to either L_0 or R_0 will have to be acted by the same phase factor and then this mechanism for creating rest mass should be universal. This will be used in section 5 in the description of the electroweak interactions, there when the weak interaction acts on the electron (positron) the corresponding carrier W^- (W^+) will carry away not only the charge of the field but also the coupling in the form of mass. See (4.76) in 6.4 below.

The fact that the carriers acquire mass from geometrical considerations should be reflected in any combinations of fields which can be at rest with respect to an observer. Only the total number of contributions can change, and then the masses of other elementary fields would be forced, by the present theoretical Considerations, to be expressible as some algebraic function of the mass of the electron or at least through a mechanism similar to our LK theorem (see [136] and [131, 132]).

2.3 ELECTRODYNAMICS

For the electrodynamics of the electron within our formulation it is important to emphasize that we only have to particularize the analysis given above (description in terms of interacting carriers Chapter 3) to the case where the charge of the carrier $q = e$, as far as the rest of the analysis is general and does not need any change for the case of the electron. See also Chapter 6, section 5. This is in agreement with our principle of acceptable description. The fact that the electron itself has been chosen as a carrier gives the basis for renormalization to be a guiding principle in quantum electrodynamics. There the variable number of quanta of the different carrier fields is an additional gauge freedom.

3. THEORETICAL DESCRIPTION OF THE ELECTRON IN START

It is convenient to summarize the resulting model for the electron when, in the description of a field, the basic properties of mass, spin and charge are included. The action distribution is given a set of geometrical symmetries by requiring that the field corresponds to the most, elementary field in START (see the discussion in Chapter 5, section 1).

The rest energy of the field, resulting from an integration where $\int_V \rho dx = 1$, is m_0c^2, and obeys a wave equation in the START geometry, the simplest representation corresponds to spin $\hbar/2$, its coupling to receive action from another field is given by e, the ratio e/m_0 corresponding to the rate of change of energy with action per unit energy of the original field. Correspondingly, when action is given to other fields the strength of this action is also proportional to e. When work can be clone on or by the field, by absorbing or by creating a quanta of the interaction field (redistributing the action), the emitted or absorbed energy per elementary action corresponds to a change of spin equal to $\hbar$, or to a corresponding change in angular momentum of orbital origin. We have to make a clear distinction between **actual** energies given to the field with **relative** energies which are described by a gauge field, even if in

both cases the description of energy demands the definition of a frame of reference with respect to which one carrier acquires more energy at the expense of the energy of the environment.

The elementary fields described by the model are required to be created or annihilated by at least pairs of fields with mutually canceling symmetry properties. Then a collection of fields, where no other fields are present which can cancel the symmetry properties of this collection, will correspond to stable matter. The distributions will exist in space but can only be created and annihilated in units of action, that is that, the change in spin has to be a multiple of $\hbar/2$ and the change in energy with respect to a reference observer must correspond to $h\nu$. When fields at rest in a frame of reference are created (annihilated) the energy of the distribution is $h\nu = m_0c^2$. These conditions originate the notion of carrier within the theory, a concept that will be even closer to the classical limit if the distribution domain is small compared to the distances involved in the global experimental observation. We can change the form of the action distribution in a continuous form, but we can not change the existence of the distribution except, in a quantized form. At the same time it is now clear that at the level of the elementary fields (first,) quantization corresponds to a type of description considering processes where the properties of the distributions can be changed, and second quantization to processes where distributions are created or annihilated.

3.1 THE LOCAL STRUCTURE OF THE ACTION DENSITY

The basic postulate for the theory of matter and interactions fields here presented is the existence of a distribution of action, with certain geometrical characteristics, in a region of space–time.

For massless fields a local system of vectors, at each point, defines the geometric characteristics: a vector a in direction e_4 for the intensity of action; a vector in direction e_0 for the (unavoidable) time direction. The rate of change of a with respect, to time being the density of energy of the massless field $\varepsilon = \rho(X)h\nu$. The remaining three directions are internally defined in this case because all massless fields are chiral, including the interaction fields. One of the space-like directions corresponds to the direction of propagation of the massless field, with momentum $p = \rho(X)h/\lambda$ parallel to velocity. The remaining two space-like directions define the plane of the spin, perpendicular by definition to the propagation direction. This is the local structure of the action density corresponding to massless matter and interaction fields, this structure is by no means trivial and is reflected in the gauging properties of the description of the field. We should stress here that every point of the

distribution is given the same set of local properties. The gauge freedom can only change the relative values of these sets of vectors at different space–time points with the condition that the local structure has to be respected. Chirality and the relation $\varepsilon'/p' = c$ are basic features of the description of the massless fields, the primed energy momentum components related to the unprimmed ones by a Lorentz transformation.

For **massive** fields the local structure is similar except that the set of three space-like vectors, even if their directions are related to momentum and spin, offers now the additional gauge freedom of the direction of the spin plane being orientable, independently of the momentum direction. In fact, for the electron there is at least one local frame of reference where $p = 0$ and, according to our principles above, $\partial a/\partial t = m_0c^2$ also $|(dx^4)^2| = (dx^0)^2$. For the massive fields, the electron being our example here, we then have that for a general observer there is a current **j** which is related to the local frame where the carrier field is at rest (frame vectors $\left\{\gamma_u^{(0)}\right\}$) through a Lorentz transformation

$$\mathbf{j} = j^\mu\gamma_\mu = \rho v^\mu\gamma_\mu = R\rho\gamma_0^{(0)}R^{-1}, \tag{4.22}$$

showing that to each point of the distribution corresponds a current **j,** this being one of the basic properties of the matter field. The direction of the current and the direction of the momentum are two different quantities, the current being related to the transformation of the local frame of reference and the momentum to the rate of change in space of the action distribution. In a similar form the spin s is related to a spin plane in the local frame of reference $\gamma_{12}^{(0)}$ by the same Lorentz transformation

$$\mathbf{s} = \rho s^{\mu\nu}\gamma_{\mu\nu} = R\rho\gamma_{12}^{(0)}R^{-1}. \tag{4.23}$$

Idealized currents can be considered for computational purposes. The best known example being perhaps that of a plane wave, even if in practice no actual currents of matter or radiation can be approximated by such a current, except for a very small region of space, and, moreover, only in the case where the actual current corresponds to a (steady) current of matter or radiation, consisting of a large number of matter or radiation units.

Because our description of matter and its interactions has as starting point a definition which corresponds to a definite picture of nature, there is a temptation to interpret several consequences of the theory as physical descriptions of nature, even if in many cases they are only one form of description among many, or if they correspond to an approximate description useful for calculations but not to a comprehensive description

of the different phenomena. In the other extreme we can analyze the different features of the theory in relation to well established mathematical models. Our theory reproduces the mathematical structure of density functional theory and the mathematical structure of quantum mechanics, then the calculational procedures can be carried on, having either of both presentations as a guide, given that the mathematical structure allows it. A basic difference is that we are using a continuous model for matter: a distribution of action where every point is endowed of a geometrical feature.

The local geometrical feature could also be modeled as a specific type of structure in space–time: there exists, at every point of the distribution (a mapping arising from the complex structure of space–time–action) an object corresponding to a spin $\frac{1}{2}$, Lense and Thirring or Kerr-like spinning particle. The Kerr geometry itself can be considered as a special type of string, as has been discussed in length by Burinskii [22]. The main difference is that while Burinskii and other authors, since the pioneering paper of Carter [27], consider the Kerr geometry (a geometric structure reduced to a point) as the particle itself and then do not use a continuous description of matter, our presentation starts with the distribution and only afterwards the local properties of the distribution could be recognized to correspond to a local spin $\frac{1}{2}$ Kerr-like geometry. Our theory is nevertheless an abstract geometrical description, with no intention to present a model. In our approach the distribution of action presents at every point a structure defined by the Dirac equation itself, providing geometrical constraints to the action, with a local weight $\rho(X)$. Below we discuss this local (abstract) structure in some detail.

4. DENSITIES AND CURRENTS OF THE ELECTRON

We have obtained the (electromagnetic field) gauged equation for the action amplitude auxiliary wave function ψ of an electron:

$$\gamma^{\eta}\left(\frac{\hbar}{i}\frac{\partial}{\partial x^{\eta}}+\frac{e}{c}A_{\eta}\right)\psi=-mc\psi, \tag{4.24}$$

in terms of the $A = \gamma^{\eta}A_{\eta}$ the electromagnetic potential vector.

In complex geometry the *complex conjugate* $\overline{M}$ of a multi-vector M is the complex conjugate of each coefficient of each multi-vector basis in the sum representing M. That is given ($i < j < k < m$)

$$\mathcal{M}=IM+\gamma_i M^i+\gamma_{ij}M^{ij}+\gamma_{ijk}M^{ijk}+\gamma_{ijkm}M^{ijkm}+\ldots, \tag{4.25}$$

$$\overline{\mathcal{M}}=I\overline{M}+\gamma_i\overline{M}^i+\gamma_{ij}\overline{M}^{ij}+\gamma_{ijk}\overline{M}^{ijk}+\gamma_{ijkm}\overline{M}^{ijkm}+\ldots, \tag{4.26}$$

where $\overline{M}^{i1i2...in}$ denotes the complex conjugate of $M^{i1i2...in}$. Similarly define the *complex reverse* M^{+} of a complex multi-vector M as the complex conjugate of M and the reverse of the order of all products of vectors generating the basis multi-vectors, for (4.25) above

$$\mathcal{M}^{+} = I\overline{M} + \gamma_i\overline{M}^{i} + \gamma_{ji}\overline{M}^{ij} + \gamma_{kji}\overline{M}^{ijk} + \gamma_{mkji}\overline{M}^{ijkm} + \ldots \quad (4.27)$$

If $M^{+} = M$ then M is Hermitian, and if $M^{+} = \overline{M}$ then M is unitary. The *complex scalar product* defined above becomes

$$\langle \mathcal{M}, \mathcal{B} \rangle = (\mathcal{M}^{\dagger}\mathcal{B})_0 , \quad (4.28)$$

again $(M+B)_0$ designates the (0-vector or) scalar part of $M+B$. See [169].

We use now the Fock and Ivanenko equation for curved space introducing the Fock-Ivanenko Ω (*a*) bi-vector fields coefficients. Locally the Dirac equation for curved space becomes the gauged with non-commuting fields

$$\gamma^{\eta}\left(\frac{\partial}{\partial x^{\eta}} - \Omega_{\eta} + \frac{ie}{\hbar c}A_{\eta}\right)\psi = -\frac{imc}{\hbar}\psi. \quad (4.29)$$

Rewriting the covariant derivative $D = g^{\mu}\Box_{\mu}$ of a multi-vector M in the form

$$\left(\frac{\partial}{\partial x^{\eta}} - \Omega_{\eta}\right)[\mathbf{M}] = \Box_{\eta}[\mathbf{M}] - [\mathbf{M}]\,\Omega_{\eta}. \quad (4.30)$$

We write (4.29) as

$$\gamma^{\eta}\Box_{\eta}\psi - \gamma^{\eta}\psi\mathcal{G}_{\eta} = -\frac{imc}{\hbar}\psi, \quad (4.31)$$

where the gauging fields (other terms could be added as discussed above, not included here to avoid the discussion of their chiral nature and their breaking of the chiral symmetry)

$$\mathcal{G}_{\eta} = \Omega_{\eta} - \frac{ie}{\hbar c}A_{\eta}I. \quad (4.32)$$

We now follow the usual steps to obtain the several currents. From the complex reverse of (4.29), one gets an equation for ψ^{+}

$$\left(\Box_{\eta}\psi^{+}\right)\gamma^{\eta} - \mathcal{G}_{\eta}^{+}\psi^{+}\gamma^{\eta} = \frac{imc}{\hbar}\psi^{+}. \quad (4.33)$$

The set of Ω_η are real bi-vectors (that is with real coefficients) and i appears in the vector gauge field A

$$\mathcal{G}_\eta^+ = -\mathcal{G}_\eta, \tag{4.34}$$

and

$$\left(\square_\eta \psi^+\right)\gamma^\eta + \mathcal{G}_\eta^+ \psi^+ \gamma^\eta = \frac{imc}{\hbar}\psi^+. \tag{4.35}$$

To obtain the vector current we multiply (4.35) from the right by ψ and multiply (4.31) from the left by ψ^+ and sum the resulting equations to obtain the η component of the current equations in curved space–time

$$\left(\square_\eta \psi^+\right)\gamma^\eta + \psi^+ \gamma^\eta \square_\eta \psi + \mathcal{G}_\eta\left(\psi^+ \gamma^\eta \psi\right) - \left(\psi^+ \gamma^\eta \psi\right)\mathcal{G}_\eta = 0. \tag{4.36}$$

as the scalar part of (4.36). As the scalar part is symmetric in the product of any two Clifford numbers the $()_0$ projection (scalar part) of the two last terms will cancel out to obtain

$$\left(\left(\square_\eta \psi^+\right)\gamma^\eta \psi + \psi^+ \gamma^\eta \square_\eta \psi\right)_0 = 0. \tag{4.37}$$

For the index free multi-vector ψ the covariant, derivative $\psi_{;\eta}$ is equal to $\square_\eta \psi$. Furthermore by construction $\gamma^\alpha_{;\eta} = 0$. Then (4.37) becomes

$$\left(\psi^+_{;\eta}\gamma^\eta \psi + \psi^+ \gamma^\eta_{;\eta}\psi + \psi^+ \gamma^\eta \psi_{;\eta}\right)_0 = 0,$$

or

$$\left(\left(\psi^+ \gamma^\eta \psi\right)_0\right)_{;\eta} = 0. \tag{4.38}$$

Allowing the definition of the current $\mathbf{J} = j^\mu \gamma_\mu$, where

$$J^\eta = e\left(\psi^+ \gamma^\eta \psi\right)_0, \tag{4.39}$$

can be interpreted as a conserved current satisfying

$$J^\eta_{;\eta} = 0. \tag{4.40}$$

A multi-vector interpretation of (4.39) is obtained from the use of the fact that one may exchange the order of the product, of two Clifford numbers before projecting out the scalar component without changing the result. Then

$$\left(\psi^+ \gamma^\eta \psi\right)_0 = \left(\gamma^\eta \psi^+ \psi\right)_0 = \left\langle \gamma^\eta, \psi^+ \psi \right\rangle,$$

allowing the definition of the current components

$$J^\eta = e\left\langle \gamma^\eta, \psi^+ \psi \right\rangle. \tag{4.41}$$

considering $\langle \gamma^\eta, \gamma_\alpha \rangle = \delta^\eta_\alpha$, as the vector part of the multi-vector product $\psi\psi^+$. That is

$$\psi\psi^+ = sI + \frac{J}{e} + \mathbf{M} + K + p\gamma_5, \tag{4.42}$$

with **M** an angular momentum field and *K* an axial current.

The Gordon decomposition is an analysis of the bi-vector components of $\psi\psi^+$ (Gordon 1928).

Following Gordon analysis

$$\psi^+\gamma^\eta\psi = \frac{1}{2}\left(\psi^+\gamma^\eta\psi\right) + \frac{1}{2}\left(\psi^+\gamma^\eta\psi\right). \tag{4.43}$$

is reinterpreted in terms of

$$\psi = \frac{i\hbar}{mc}\left(\gamma^\nu \Box_\nu \psi - \gamma^\nu \psi \mathcal{G}_\nu\right),$$

or considering again covariant derivatives

$$\psi = \frac{i\hbar}{e}\left(\gamma^\nu \psi_{;\nu} - \gamma^\nu \psi \mathcal{G}_\nu\right). \tag{4.44}$$

From the complex reverse of this last equation we obtain for ψ^+

$$\psi^+ = \frac{i\hbar}{mc}\left(\psi^+_{;\nu}\gamma^\nu + \mathcal{G}_\nu \psi^+ \gamma^\nu\right), \tag{4.45}$$

to obtain

$$\begin{aligned}\psi^+\gamma^\eta\psi = \frac{i\hbar}{2mc}\big[&\psi^+\gamma^\eta\gamma^\nu\psi_{;\nu} - \psi^+\gamma^\eta\gamma^\nu\psi\mathcal{G}_\nu \\ &-\psi^+_{;\nu}\gamma^\nu\gamma^\eta\psi - \mathcal{G}_\nu\psi^+\gamma^\nu\gamma^\eta\psi\big].\end{aligned}$$

The main point of the analysis arises from the basic geometrical definition $\gamma^\eta\gamma^\nu = \gamma^{\eta\nu} + \gamma^{\eta\nu}I$, to rewrite (4.43) in the form

$$\begin{aligned}\psi^+\gamma^\eta\psi = \frac{i\hbar}{2mc}\big[&\left(\psi^+\gamma^{\eta\nu}\psi_{;\nu} + \psi^+_{;\nu}\gamma^{\eta\nu}\psi\right) + g^{\eta\nu}\left(\psi^+\psi_{;\nu} - \psi^+_{;\nu}\psi\right) \\ &- g^{\eta\nu}\left(\psi^+\psi\mathcal{G}_\nu + \mathcal{G}_\nu\psi^+\psi\right) - \left(\psi^+\gamma^{\eta\nu}\psi\mathcal{G}_\nu - \mathcal{G}_\nu\psi^+\gamma^{\eta\nu}\psi\right)\big].\end{aligned} \tag{4.46}$$

Since $\gamma^{\eta\nu}_{;\nu} = 0$, the first pair of terms on the right-hand side of (4.46) may be combined into a single terms $(\psi^+\gamma^{\eta\nu}\psi)_{;\nu}$. In addition, if we project out the scalar component of both sides of (4.46), the last pair of terms will cancel out. We then have

$$\begin{aligned}J^\eta &= e\left(\psi^+\gamma^\eta\psi\right)_0 \\ &= \frac{ie\hbar}{2mc}\left(\left(\psi^+\gamma^{\eta\nu}\psi\right)_{;\nu}\right)_0 \\ &\quad + \frac{ie\hbar}{2mc}g^{\eta\nu}\left[\left(\psi^+\psi_{;\nu} - \psi^+_{;\nu}\psi\right)_0 - \left(\psi^+\psi\mathcal{G}_\nu + \mathcal{G}_\nu\psi^-\psi\right)_0\right].\end{aligned} \tag{4.47}$$

Following the Gordon interpretation the first term on the right hand side of (4.47) can be interpreted as the *proper current*:

$$J^{\eta}_{\mathrm{INT}} = \frac{ie\hbar}{2mc}((\psi^{+}\gamma^{\eta\nu}\psi)_{;\nu})_0 = \frac{ie\hbar}{2mc}\left(\left(\gamma^{\eta\nu}\psi\psi^{+}\right)_{;\nu}\right)_0 = \mathbf{M}^{\nu\eta}_{;\nu}. \tag{4.48}$$

It is fundamental that J^{η}_{INT} is covariant and satisfies a continuity equation.

Above the (antisymmetric) bi-vector **M**

$$\mathbf{M} = M^{\nu\eta}\gamma_{\nu\eta} = \frac{ie\hbar}{2mc}\left(\gamma^{\nu\eta}\psi\psi^{+}\right)_0\gamma_{\nu\eta}, \tag{4.49}$$

is defined. It obeys

$$J^{\eta}_{\mathrm{INT};\eta} = M^{\nu\eta}_{;\nu;\eta} = 0. \tag{4.50}$$

Gordon also defined the second term on the right hand side of (4.47) as a *convection current* J^{η}_{CONV}:

$$J^{\eta}_{\mathrm{CONV}} = \frac{ie\hbar}{2mc}g^{\eta\nu}\left[\left(\psi^{+}\psi_{;\nu} - \psi^{+}_{;\nu}\psi\right)_0 - 2\left(\mathcal{G}_{\nu}\psi^{+}\psi\right)_0\right], \tag{4.51}$$

which separately satisfies a continuity equation given that

$$J^{\eta}_{;\eta} = J^{\eta}_{\mathrm{INT};\eta} = 0. \tag{4.52}$$

The proper current J^{η}_{INT} is an internal current of the action distribution allowing the $M^{\nu\eta}$ to be interpreted as the electric and magnetic dipole moments of it. A reference system can be defined where the principal components are

$$P_z = -M^{30} = \frac{ie\hbar}{2mc}\left(\gamma^{30}\psi\psi^{+}\right)_0, \tag{4.53}$$

and

$$M_z = -M^{12} = \frac{ie\hbar}{2mc}(\gamma^{12}\psi\psi^{+})_0 = \frac{e}{mc}s_z, \tag{4.54}$$

with s_z the z-th component of the spin of the distribution.

The full set of currents is

$$e\psi\psi^{+} = IA + J^{\mu}\gamma_{\mu} - \frac{imc}{\hbar}M^{\mu\nu}\gamma_{\mu\nu} + i\gamma_5 K^{\mu}\gamma_{\mu} + p\gamma_5, \tag{4.55}$$

where the (imaginary) axial current is mentioned in Chapter 2. The auxiliary amplitude wave function ψ is not an eigenfunction of $i\gamma_5$ then the

last two terms do not correspond to gauge-free conserved currents, only the left handed part or the right handed part separately.

We have mentioned above the general gauge transformation for ψ. We have a non-abelian gauge group (this is more clearly seen with the analysis of Snygg 1997, chapter 8 [169]). If all physically measurable information stored in the wave function ψ is also stored in the product $\psi\psi^{+}$, then the wave function ψ must be considered equivalent to the wave function ψ*S* where *S* is any differentiable unitary Clifford number containing the information compatible with our Principle of Choice.

Then substituting

$$\psi' = \psi\mathcal{S}, \tag{4.56}$$

where ψ is a solution of (4.31). The ψ′ should be of

$$\gamma^{\eta}\Box_{\eta}\psi' - \gamma^{\eta}\psi'\mathcal{G}'_{\eta} = -\frac{imc}{\hbar}\psi', \tag{4.57}$$

where, by a gauge transformation

$$\mathcal{G}'_{\eta} = \mathcal{S}^{+}\mathcal{G}_{\eta}\mathcal{S} - (\Box_{\eta}\mathcal{S}^{+})\mathcal{S}. \tag{4.58}$$

then as $\Box vI = 0$

$$\left(\Box_{\nu}\mathcal{S}^{+}\right)\mathcal{S} = \Box_{\nu}\left(\mathcal{S}^{+}\mathcal{S}\right) - \mathcal{S}^{+}\Box_{\nu}\mathcal{S} = -\mathcal{S}\left(\Box_{\nu}\mathcal{S}\right), \tag{4.59}$$

and

$$\begin{aligned}\Box_{\nu}\mathcal{G}'_{\eta} - \mathcal{G}'_{\eta}\mathcal{G}'_{\nu} = {} & \mathcal{S}^{+}\left(\Box_{\nu}\mathcal{G}_{\eta} - \mathcal{G}_{\eta}\mathcal{G}_{\nu}\right)\mathcal{S} + \left(\Box_{\nu}\mathcal{S}^{-}\right)\mathcal{G}_{\eta}\mathcal{S} \\ & + \left(\Box_{\eta}\mathcal{S}^{+}\right)\mathcal{G}_{\nu}\mathcal{S} - \left(\Box_{\nu}\Box_{\eta}\mathcal{S}^{+}\right)\mathcal{S},\end{aligned} \tag{4.60}$$

from this relation it follows that

$$\begin{aligned}\Box_{\nu}\mathcal{G}'_{\eta} - \Box_{\nu}\mathcal{G}'_{\nu} - \mathcal{G}'_{\nu}\mathcal{G}'_{\eta} - \mathcal{G}'_{\eta}\mathcal{G}'_{\nu} = {} & \mathcal{S}^{+}\left(\Box_{\nu}\mathcal{G}_{\eta} - \Box_{\eta}\mathcal{G}_{\nu} - \mathcal{G}_{\nu}\mathcal{G}_{\eta} - \mathcal{G}_{\eta}\mathcal{G}_{\nu}\right)\mathcal{S} \\ & - \left(\left(\Box_{\nu}\Box_{\eta} - \Box_{\eta}\Box_{\nu}\right)\mathcal{S}^{+}\right)\mathcal{S},\end{aligned}$$

and from the definition of the Ricci curvature

$$\begin{aligned}\Box_{\nu}\mathcal{G}'_{\eta} - \Box_{\eta}\mathcal{G}'_{\nu} - \mathcal{G}'_{\nu}\mathcal{G}'_{\eta} - \mathcal{G}'_{\eta}\mathcal{G}'_{\nu} - \frac{1}{2}\mathcal{R}_{\nu\eta} & \\ = \mathcal{S}^{+}\left(\Box_{\nu}\mathcal{G}_{\eta} - \Box_{\eta}\mathcal{G}_{\nu} + \mathcal{G}_{\nu}\mathcal{G}_{\eta} - \mathcal{G}_{\eta}\mathcal{G}_{\nu} - \tfrac{1}{2}\mathcal{R}_{\nu\eta}\right)\mathcal{S}.\end{aligned} \tag{4.61}$$

In particular if $G_{\eta} = \Omega_{\eta} - (ie/\hbar c)\,A_{\eta}I$, then

$$\Box_{\nu}\mathcal{G}_{\eta} - \Box_{\eta}\mathcal{G}_{\nu} + \mathcal{G}_{\nu}\mathcal{G}_{\eta} - \mathcal{G}_{\eta}\mathcal{G}_{\nu} - \tfrac{1}{2}\mathcal{R}_{\nu\eta} = -\frac{ie}{\hbar c}F_{\nu\eta}I. \tag{4.62}$$

Since S commutes with I, it then follows that the left hand side of (4.61) is also equal to $-(ie/\hbar c)\, F_{\nu\eta} I$. But there are solutions of (4.62) more general than $\Omega_\eta - (ie/\hbar c)\, A_\eta I$, in particular the ones we have used to study the electroweak interaction, although as we mentioned above they break the chiral symmetry.

The curvature equation has been extended to include the geometric field intensities

$$\gamma^\eta \left(\Box_\nu \mathcal{G}_\eta - \Box_\eta \mathcal{G}_\nu + \mathcal{G}_\nu \mathcal{G}_\eta - \mathcal{G}_\eta \mathcal{G}_\nu\right) = \left(-\tfrac{1}{2} R_{\nu\eta} - \frac{ie}{\hbar c} F_{\nu\eta}\right) \gamma^\eta,$$

as already noted by Fock 1929.

This set of currents describe the symmetry constrains defining the electron as a stable carrier.

5. THE ELECTROWEAK INTERACTION OF THE ELECTRON FIELD

In order to understand the interactions of a field we have to consider first that the name 'interactive' itself expresses a relative property, the relation existing between two carrier fields which should have some gauge freedom of description allowing them to be considered together, as expressed in one of the sections above. We saw the case of the electromagnetic interaction. In the description of the gauging of the electron field we already mentioned the possibility of considering bi-vector valued phases or pseudo-scalar valued phases. The bi-vector-valued phases were found long time ago, by Fock and Ivanenko, to correspond to gravitational interactions. Here we shall describe the electrowcak interaction as an extension of the electromagnetic case. We shall show that it corresponds to a pseudo-scalar valued phase. For this purpose we first need to consider the partner of the electron in the weak interaction: the neutrino, next section, before analyzing the electroweak theory within our formalism.

5.1 THE THEORY OF THE NEUTRINO

We develop here a theory of the neutrino which is the natural complement of the formulation we have developed above for the electron.

We mentioned that we can define the electron field in an operational form: it is that field Ψ_e which obeys the Dirac equation (and its gauging)

$$(i\hbar D_0 - m)\Psi_e = 0, \quad -(i\hbar D_0 - m)(-i\hbar D_0 - m) = -\hbar^2 D_0^2 - m^2 \tag{4.63}$$

then the field has the correct mass, charge and spin density (of course magnetic and electric moment when gauged by the electromagnetic field).

The neutrino is considered to be massless, spin $\frac{1}{2}$, and uncharged, then without magnetic or electric moment and allows no gauging by the electromagnetic field. We propose then an operational definition: the neutrino field corresponds to that field which obeys the equation

$$i\hbar D_n \Psi_\nu = 0, \quad D_n D_n^\kappa = \partial^\mu \partial_\mu \quad , \tag{4.64}$$

with D_n such that the neutrino, besides being massless ($m = 0$ in (4.64)) is also neutral, no electromagnetic gauging (coupling) allowed and has the correct spin and chirality (left handed).

This properties are obtained from the definitions

$$\begin{aligned} D_n &= \gamma^0 \partial_0 + i\gamma^5 \gamma^j \partial_j, \quad j = 1,2,3, \\ D_n^\kappa &= \gamma^0 \partial_0 - i\gamma^5 \gamma^j \partial_j, \end{aligned} \tag{4.65}$$

where we should remark that from the metric and the anticommuting properties of the basis vectors

$$\gamma^0 \partial_0 \gamma^0 \partial_0 = \partial^0 \partial_0, \qquad -i\gamma^5 \gamma^i \partial_i i\gamma^5 \gamma^j \partial_j = -\partial^j \partial_j, \tag{4.66}$$

$$i\gamma^5 \gamma^0 \partial_0 \gamma^j \partial_j - \gamma^j \partial_j i\gamma^5 \gamma^0 \partial_0 = i\gamma^5 \gamma^0 \gamma^j (\partial_0 \partial_j - \partial_j \partial_0) = 0. \tag{4.67}$$

Also, if a mass term $m \neq 0$ were included in (4.64), the spurious term $D_n m - m D_n^k \neq 0$ would prevent the use of $m \neq 0$.

Additionally from (4.64) and (4.65) the requirement $i\gamma^5 \Psi_\nu = -\Psi_\nu$, for the neutrino imposes the condition for it to be a left handed carrier field. (D_n^k would be the operator for the right handed antineutrino field).

With respect to the gauging of the wave function Ψ_ν and the operator D_n, it is immediate that a term $\gamma^0 q_n A_0 g$ can not be cancelled by a gauge factor $g = e^{ia(t)/\hbar}$ acted upon by $i\hbar i\gamma^5 \gamma^0 \partial_0$ as far as $= -\hbar\gamma^5 \gamma^0 \partial g = -a'(t)\gamma^5 \gamma^0 g$.

On the other hand a term $\gamma^5 \gamma^0 \, A_0^{axial} \, g^{axial}$ will be cancelled by such a term. As in the case of the electron fields both possibilities are open, an axial electron current and an axial neutrino current can interact, this being the origin in our theory of the electron of the possible full electroweak interaction of the electron, but for the neutrino the interaction is restricted to the weak part (axial current only) and for the dual of the electron charge, that is g_D being the proper coupling constant.

5.2 THE ELECTROWEAK INTERACTION OF THE ELECTRON AND THE NEUTRINO

Now, from the description of the neutrino as above, and the possibility of both the electron left and right handed fields to be gauged by a pseudoscalar-valued phase, a Lagrangian can be written in which both fields

are together, the gauging of one corresponds to the opposite gauging of the other:

$$\nu_e + W^- \rightarrow e^- \quad \text{or} \quad e^- + W^+ \rightarrow \nu_e, \tag{4.68}$$

with the set of new gauge fields W^-, W^+, $[W^-, W^+] = Z^0$ carrying four physical properties: charge; angular momentum; vector (SU (2)) structure; and the possibility of interacting with carriers possessing a weak charge. Because this field interacts only with the left handed neutrino and the left handed part of the electron field, or the right handed antineutrino and the right handed part of the positron field, the weak field itself will have to interact with the mass producing phase factor of the LK theorem, then it will acquire mass from the same mechanism as the electron field (these matters are presented and analyzed in [105, 116]).

6. FORMULATION OF A THEORY OF ELEMENTARY PARTICLES FROM START

6.1 INTRODUCTION

As a generalization of the geometrical model of the electron we are developing a theory of elementary particles, reproducing the standard model (SM), which is derived from an analysis of wave equations in STA, using chirality as a basic symmetry. Besides a complete reproduction of SM and of its auxiliary conditions, our theory provides a systematic extension to SM and a geometrical theory of matter.

A fundamental concept here is that the current theory of elementary particles stresses the relations between fields. This is either by considering interactions or by considering (this is equivalent) decays. The most important properties are then relative (between fields).

For this purpose we start from the equivalent complex formulation of space–time and show that the natural structures in this geometry are associated with a set of equations for energy and for massless carriers fields by allowing a series of factorizations of the Laplacian operator and associated Dirac-like equations; this set of related equations generates three families of elementary particles with the experimentally observed lepton and quark content for each family and the experimentally observed electroweak color interactions and other related properties, when a gauge transformation of the different fields is performed and their relative gauging properties are compared. In this stage of the theory the factorizations $\Box^2 = (\Gamma^{\mu}_{(f)} \partial^{(d)}_{\mu}) \bullet (\Gamma^{\nu}_{(f)} \partial^{(d)}_{\nu})$ and the related Dirac-like equations $\Gamma^{\mu}_{(f)} \partial^{(d)}_{\mu} \psi_{(d,f)} = 0$ are studied and their symmetries are given. The $\Gamma^{\mu}_{(f)}$ generate the three families, the $\partial^{(d)}_{\mu}$ generate the observed lep-

ton and quark content of the families, this results from the inherent symmetry restrictions introduced by (d, f) in the equations and in the wavefunctions, and from the geometrical restrictions on the gauging, on the wave equations, and on the wavefunctions.

We have discussed elsewhere the use of multi-vectors as generators of Lie groups, as well as the generation of a multi-vector Dirac equation, see for example [101, 103, 104], where we analyze the construction within $C_{1,3}$ of frequently used groups as for example $SU(2,3)$, $SU(3)$ or $SU(2)$. Also the integration of spinors and multi-vectors in a geometric superalgebra [105, 107, 109].

In contrast to the usual approach to SM, the properties for the different, fields of the model are consequences of the relative properties of the equations, amongst themselves and in relation to space–time, and therefore they do not need to be postulates of the theory. Also we have shown that the formulation includes all possibilities open with higher-dimensional geometries, including the gauging of the geometry to generate (a gauge theory of) gravitation and, in fact, what has been called theories of induced (energy) matter and charges [5, 88, 129, 186, 187].

As mentioned above, all multi-vectors are operators on themselves and on their spinors. The best known examples are γ_0, generating the parity inversion *P;* the tri-vector γ_{123}, generating the time inversion *T*; the bi-vector γ_{0i} generating the Lorentz boosts $\mathcal{L}$; the bi-vector γ_{ij}, generating the space rotations $\mathcal{R}$; the tetravector γ_5, generating the duality transformation *D;* and the complex tetra-vector $i\gamma_5$, generating the chirality projection.

The pseudo-scalar unit in space–time is $\gamma_5 = \gamma_{\mu\nu\lambda\rho}\varepsilon^{\mu\nu\lambda\rho}/4!$. In complex space–time the equivalent, 5-dimensional pseudo-scalar is the pentavector $\mathbf{i}$ $(= \gamma_{01234} = \sqrt{-11})$.

Section 6.2 deals with some properties of the coniplexification of space–time. Section 6.3 with the study of chiral symmetry in complex space–time. Section 6.4 with the structural consequences of using complex space–time and their relation to phase space, induced energy concepts and to the process of creations of carriers.

The procedure we describe is equivalent to the following steps:

1) To assume a Minkowski space–time Manifold *M;*

2) To provide it with a Geometric Structure;

3) To obtain its Complexification (4-D 'transforms' into 5-D) ;

4) To project out a Physical (in general curved) local *M* Manifold;

5) To show that the Projection has Induced Energy Density;

6) To provide a Stable Structure for the Energy Density through a Chiral Constraint on the Massless Fields for Matter;

7) To show that the Procedure generates Charges, and, through Gauging, Interaction Fields;

8) To analyze the consequences derived from the fact that every Point of the Matter Fields is a Source of a Geometric Wave;

9) Show that the Resulting Structures reproduce the Enlarged Standard Model.

This equivalence is retaken in the concluding section.

Below we shall use a basis vector e_μ or its γ-representation $\gamma(e_\mu) = \gamma_\mu$ indistinctly.

6.2 COMPLEX SPACE–TIME GEOMETRY

In the auxiliary coordinate spanned by e_{n+1} in STA there are two types of contributions: the one, arising from a global reference action value $e = x^{n+1}_{(0)}$, affecting all space–time directions $x^{n+1}_{\{i\}} = x^{n+1}_{(0)} + \tan(a_i)x^i$; and the other related to the gauging in the complexification of the $e_i\ _{n+1}$ planes given by $x^i A_i$. In the correspondence between the geometry and physics l is related to curvature and the A_i to the gauge fields as in the electromagnetic case. All the degrees of freedom of $G_{p,q}$, $p + q = n$ are complexified, that is, we also have complex bi-vectors, tri-vectors, etc.. In particular, we have the mapping in space–time of a vector $p = p^\mu e_\mu$ such that $p^2 = p^\mu p_\mu = m^2$, with m a real number, to $p' = x^\mu i\gamma_5 e_\mu$, where $|p'|^2 = p^\mu p_\mu = m$ and $|A|^2 \equiv \frac{1}{2}(AA^* + A^*A)$, or to $p'' = x^\mu(\mathbf{1}\cos(n + t(\mu))\frac{1}{2}\pi + ig5\sin(n + t(\mu))\frac{1}{2}\pi)e_\mu$ with n and $t(\mu)$ integers and obtain again $|p''|^2 = m^2$, which in the study of the physical problems would correspond to the mixing of vector currents and axial vector currents, as in the theory of electroweak and color interactions. See (4.71) and (4.72) below.

6.3 CHIRAL SYMMETRY IN COMPLEX SPACE–TIME

We assume in accordance with the previous section that a local observer describes space–time by an orthonormal tetrad:

a) $(\gamma^0)^2 = -(\gamma^1)^2 = -(\gamma^2)^2 = -(\gamma^3)^2 = 1$; and $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ all $\mu \neq \nu$.

b) In this frame $\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3$ is both the duality transformation operator and the pseudo-scalar $(\gamma^5)^2 = -1$.

It is important that another observer can use a different coordinate system related by a, Lorentz transformation L, where the fundamental properties $(i\gamma^5)^2 = 1$ and $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ are also preserved, together with a) and b).

The handedness operator $H = i\gamma^5$ can be used to construct the chirality projectors P_R and P_L:

$$P_R + P_L = 1, \quad P_R P_R = P_R, \quad P_L P_L = P_L, \quad P_R P_L = P_L P_R = 0,$$

where $P_R = \frac{1}{2}(1+i\gamma^5)$, $P_L = +(1 - ig^5)$ or $P_{R,L} = \frac{1}{2}(1 + \pm H)$.

If a coordinate transformation $\psi^5 \rightarrow (\gamma_5)'$ is allowed where a), and consequently b), is not preserved (that is, if the determinant z of the transformation is not $z = +1$) then $H \neq i(\gamma_5)'$ showing that a chirality operator $H = i(g^5)'/z$, with $H^2 = 1$ in all frames has to be used. Here we shall assume $H = ig^5$, because of the restriction a) and the assumption that we have selected a 'right' handed frame of reference. The P_R and P_L can better be considered numbers of a new mathematical field, complex space–time, with basis 1 and H, in an hyper-complexification of the Clifford algebra. H is coordinate invariant.

6.4 CHIRAL GEOMETRY THEORY OF ELEMENTARY PARTICLES

Using spinors, vectors, and multi-vectors we shall now construct a theory of lepton and quark fields using the possible multi-vector generalization of the Dirac factorization of the Laplacian (the four-dimensional d'Alembert operator $\Box^2 = \partial_\mu \partial_\mu$). As in the particular study of the neutrino, we start by considering the Klein–Gordon equation and its factorization

$$(\partial^\mu \partial_\mu + m^2) = (D^\dagger + mi)(D - mi), \tag{4.69}$$

which requires that

$$-D^\dagger m + mD = 0 \quad \text{and} \quad D^\dagger D = \partial^\mu \partial_\mu = \Box^2, \tag{4.70}$$

we can have then a set of choices, either:

1) any value of m and $D^\dagger = D$ (the standard Dirac operator D_0);

2) or in the case where $m = 0$ the possibility $D^\dagger \neq D$ also become acceptable. Here we shall use the hyper-complexification of the field generated by 1 and H. The recent work using q-deformed Clifford algebras also falls in this category.

In multi-vector algebra the Dirac operator is the standard vector operator (using the vectors γ_μ) $D \rightarrow D_0 = \gamma_\mu \partial_\mu$ (sometimes $D \rightarrow \gamma^0 D_0 = g^{0\mu}\partial_\mu$ is used, which in fact was Dirac's original choice of 'alphas').

When gauging the D will become covariant derivatives, to include the effect of the interaction fields.

The basic requirement $D^{\dagger}D = DD^{\dagger} = \partial_{\mu}\partial_{\mu}$ limits the choices of D. Here they will be written in the Lorentz invariant form $D_{(d,f)} = \Gamma^{\mu}_{(f)}\partial^{(d)}_{\mu}$, in order to show the relation to Dirac's original factorization in the simplest possible form. The $\Gamma^{\mu}_{(f)}$ are generalized (reducible representation) Dirac γ^{μ} matrices, see below. The limitation is so strong that the only possible choice is where the multi-vector $i\gamma^5$, which has the same action on all g^{μ}, that is $i\gamma^5\gamma^{\mu} = -\gamma^{\mu}i\gamma^5$, is used (see Keller [105], page 158 and following). Using, in the following paragraphs, for simplicity of notation, a reference frame F in which a free field momentum is $p = p_0\gamma^0 + p_s\gamma^s$ with p_s being the spatial part of the momentum and $\gamma^s = (\gamma^1 + \gamma^2 + \gamma^3)/\sqrt{3}$, we define the hyper-complex differential

$$\partial^{(d)}_{\mu} = \left\{1\cos(n + t^d_{\mu})\frac{\pi}{2} + i\gamma^5 \sin(n + t^d_{\mu})\frac{\pi}{2}\right\}\partial_{\mu}, \tag{4.71}$$

with n and t^d_{μ} integers, a choice which results in the simplest multi-vector. Here, to take the electron as a reference, we use $n = 1$.

Then, in the particular frame F we have the 'diagonal' structure:

$$\partial^{(d)}_{\mu} = \begin{cases} \partial_{\mu} & \text{if } n + t^d_{\mu} \text{ are even,} \\ i\gamma^5\partial_{\mu} & \text{if } n + t^d_{\mu} \text{ are odd.} \end{cases} \tag{4.72}$$

The vectors, which are represented by the standard γ^{μ} matrices, correspond to an irreducible representation of $C_{1,3}$ and have been found to be useful for writing the wave equations of the fundamental family (e^-_R, e^-_L, ν^-_L, $\{u_L, d_L;$ color$\}$) of elementary particles. The electron requires a combination of two fields $e^- = (e_R, e_L)$ for the standard phenomenology of electroweak–color interactions. The study of the electron family suggests that for the second and third families a more general, reducible representation of $C_{1,3}$ could in fact be needed. They are collectively denoted by $\Gamma^{\mu}_{(f)}$.

From the considerations above we emphasize that Lorentz transformations change the multi-vectors $\Gamma^{\mu}_{(f)} \rightarrow \Gamma'^{\mu}_{(f)}$ but not the $\partial^{(d)}_{\mu}$.

The Γ^{μ} can all be written as exterior products of the $\gamma^{\mu}, \gamma^5, i\gamma^5$ and 1, for example a fundamental representation would be ($\otimes$ represents exterior product)

$$\Gamma^{\mu}_{(f)} = \gamma^{\mu} \otimes (1 \otimes 1 \otimes \ldots)_{2(f-1)\text{products}};$$

other different, but equivalent representations, being also possible. The corresponding spinors would then be the (totally antisymmetric) exterior products $\psi_{(f)} = \psi(x) \Lambda (\psi_1 \Lambda \psi_2 \Lambda \ldots)_2(f-1)_{\text{products}}$.

Where the ψ_i are $2(f-1)$ **constant** Dirac spinors which correspond to extra mathematical (that is internal) degrees of freedom of the diracon fields. Their spin should add to zero (f integer). The total antisymmetry of $\psi(f)$ limits the value of f to $f = 1, 2, 3$, otherwise the spinorial exterior product is null.

The degeneracy n_f of the representations of the $\Gamma_\mu$gives statistical weight to each family: $n_1 = 1$, $n_2 = 4$ and $n_3 = 24$. This will result in factors for the terms of the mass matrix.

The elementary fields thus described are mathematically composite, but still elementary in the sense that they can not be decomposed experimentally into some (not existing) components. No size of the particle is required by the theory, they are representations of the basic elementary fermion equations, no space–time structure is involved, there is only the mathematical complexity of the wave functions. Each family has an internal relationship to the fundamental family $f = 1$ and the same $SU(3)\text{color} \otimes SU(2) \otimes U(1)$ symmetry. No additional gauge interaction field is needed to relate the different families. They are algebraic families of otherwise structureless leptons and quarks. The algebra of the $\Gamma^\mu_{(f)}$ has been developed and studied by Królikowski [131, 132], as well as the consequences for the phenomenology of the elementary particle families.

Here we should remember that the idempotents $\frac{1}{2}(1 \pm i\gamma^5)$ correspond to the operators selecting handedness (or chirality) in space–time algebra. The set of t^d_μ are then restricted forms of handing the chiral symmetry of the different fields. The values given in table 1 below refer to the reference momentum above, they show the relation between the fields, the actual numbers will be those obtained after a rotation in relation to a common observer. The relative chiral symmetries of the fields are the relevant quantities. The properties are **relative** properties, only the relations are meaningful, not the actual components which are frame of reference dependent (or even coordinate dependent if general transformations are allowed). The group of these relations is the mathematical structure of physical interest. It is a $SU(2) \otimes SU(3)_c$ structure for each f. The $U(1)$ additional symmetry is related to the standard gauge freedom of the wave function.

The basic equations for the set of spinor fields being

$$D_{(d,f)}\psi_{(d,f)} = 0, \quad \psi^{(0)}_{(d,f)} = D^{(0)\dagger}_{(d,f)}\Phi \quad \text{and} \quad \partial^\mu\partial_\mu\Phi = 0, \tag{4.73}$$

where the sub-index d stands for (symmetry constrained **Dirac** fields) **Diracons.**

The actual wave functions can be written $\Psi^{(0)} = \tilde{\mathbf{R}}\Psi$, where $\tilde{\mathbf{R}}$ is a local boost and rotation from the frame of reference where the $D^{(0)\dagger}_{(d,f)}$ are defined to the frame of reference of the observer. Because of the Clifford algebra outermorphism

$$\tilde{\mathbf{R}}\gamma_\alpha\tilde{\mathbf{R}}^{-1} \to \tilde{\mathbf{R}}\gamma_{\alpha\beta\ldots}\tilde{\mathbf{R}}^{-1} = \tilde{\mathbf{R}}\gamma_\alpha\tilde{\mathbf{R}}^{-1}\tilde{\mathbf{R}}\gamma_\beta\tilde{\mathbf{R}}^{-1}\ldots . \tag{4.74}$$

We can write

$$D_{(d,f)}\psi_{(d,f)} = \tilde{\mathbf{R}}D^{(0)}_{(d,f)}\tilde{\mathbf{R}}^{-1}\tilde{\mathbf{R}}\psi_{(d,f)} = \tilde{\mathbf{R}}D^{(0)}_{(d,f)}\psi^{(0)}_{(d,f)} = 0, \tag{4.75}$$

and then a special frame of reference definition is consistent with a definition in any observers frame of reference

particle$^{\text{charge}}_{\text{handedness}}$		Weak Charge t_0	Electric Charge t_1	t_2	t_3	$q = \frac{(t_1+t_2+t_3)}{3}e$	
electron$^{-1}_{L}$		+1	+1	+1	+1	(*reference*)	
quark $u^{+2/3}_{L}$	r	−1	−1	0	−1	red	color
	b	−1	0	−1	−1	blue	
	g	−1	−1	−1	0	green	
quark $d^{-1/3}_{L}$	$\bar{r}$	−1	0	−1	0	$\overline{\text{red}}$	color
	$\bar{b}$	+1	+1	0	0	$\overline{\text{blue}}$	
	$\bar{g}$	+1	0	0	+1	$\overline{\text{green}}$	
neutrino$^{0}_{L}$		−1	0	0	0		

Table 1. The Symmetries Generated by the Set of t^d_μ Numbers

The weak interaction carries $W^{\pm}(\pm2, \pm1, \pm1, \pm1)$ information in relation to the t_μ numbers, with $SU(2)$ symmetry.

The color interaction carries $G_{cc'}$ (0, ±l, ±l, ±l) information in relation to the t_μ numbers, with $SU(3)$ symmetry.

An example of carrier interactions would be

$$\nu^0 + W^- \to e^- \quad \text{or} \quad (-1, 000) + (2,1,1,1) \to (1, 1, 1, 1),$$

for leptons and

$$u^{2/3}_r + G_{rb} \to u^{2/3}_b \quad \text{or} \quad (-1, -1, 0, -1) + (0, 1, -1, 1) \to (-1, 0, -1, 0),$$

for quarks.

We have shown [91, 96, 105] that they constitute a set with all the known properties of an elementary particle's family, the fields they represent can be:

— massless or massive after interactions are considered;
— charged (integer or fractional);

and (as is discussed in [105] pages 158 and following) the collection of the fields constructed with (4.71) and (4.73) have weak charge and color, and in general the characteristics, usually postulated on a phenomenological basis, such as composites being colorless, confinement, etc. being immediate consequences of the defining equations.

The principle change from the usual presentation of the Standard Model [35, 57, 67, 69] is that now the equations have, as constitutional parts, a series of conditions which the phenomenological approach showed to be necessary. The conditions are here related to the basic properties of space–time as a frame of reference to describe physics.

Because of the appearance, or not, of the $i\gamma^5$ factors in (4.72), the fields have definite chiral properties. Only one field in the theory may have both chiralities simultaneously, and therefore can be, as a free field, massive, charged (reference charge ±1), and weak charged: this is identified as the electron field. The different values of the index *f*, the family number, generate the families.

The resulting theory is a chiral geometry theory of charge, isospin and color. The theory has a Lagrangian formulation which reproduces all aspects of the standard theory. Even if the Higgs mechanism has, in its first approximation, the same motivation as in the standard theory, it has a purely geometric character in the present analysis.

The Lagrangian in the standard model for a fermion field with electroweak interactions and a symmetry breaking mass term is reproduced from the considerations above $\left(L_0 = \psi_{e_L}^{(0)},\ R_0 = \psi_{e_R}^{(0)}\right)$

$$
\begin{aligned}
\mathbf{L} = &+\tfrac{1}{2}\overline{L_0} i\gamma^\mu \left(\frac{1-i\gamma_5}{2}\right)\left(\partial_\mu L_0 + \frac{ig'}{2}B_\mu L_0 - \frac{ig}{2}A_\mu^i \tau_i L_0\right) \\
&- \tfrac{1}{2}\left(\partial_\mu \overline{L_0} - \frac{ig'}{2}B_\mu \overline{L_0} + \frac{ig}{2}\overline{L_0}\tau_i A_\mu^i\right) i\gamma^\mu \left(\frac{1-i\gamma_5}{2}\right) L_0 \\
&+ \tfrac{1}{2}\overline{R_0} i\gamma^\mu \left(\frac{1-i\gamma_5}{2}\right)\left(\partial_\mu R_0 + ig' B_\mu R_0\right) - \tfrac{1}{2}\left(\partial_\mu \overline{R_0}\right. \\
&\left. - ig' B_\mu \overline{R_0}\right) i\gamma^\mu \left(\frac{1-i\gamma_5}{2}\right) R_0 \\
&- g_e \left[\overline{R_0}\Phi^\dagger \left(\frac{1-i\gamma_5}{2}\right) L_0 + \overline{L_0}\left(\frac{1-i\gamma_5}{2}\right)\Phi R_0\right],
\end{aligned}
\tag{4.76}
$$

where the $\tau_i (i = 1,2,3)$ are Pauli matrices, Φ is a field corresponding to the LK Theorem and B_μ and A_μ^i are $U(1)$ and $SU(2)$ gauge fields. A further analysis of the fields in (4.76) shows that the coupling constants

g and g' correspond to the electromagnetic constant e and to its dual (axial) pair.

To (4.76) we should add the energy corresponding to the neutrino, the energy corresponding to the interactions fields and the possibility of the neutrino and the electron interacting via the axial current, which by definition is also the basic current of the neutrino from its chiral properties as a massless field. The neutrino, as mentioned above, can not interact with its polar current without violating space–time symmetry. The equations and Lagrangian for the different fields are given by the corresponding application on the analysis in Chapter 3, where we discuss how to introduce constrains and how to relate currents to the corresponding gauge fields created by those currents.

Confinement results, within the theory, from the requirement that the Lorentz symmetry should not be broken even at local level. The same requirement gives rise to the colorless condition for hadrons, the new feature is that hadrons should be both globally and locally colorless. Fractional charges are also a natural consequence of the gauging properties of the Lagrangian.

The theory shows the reason for chirality being a basic property of nature as shown by the set of elementary carriers. This can be clearly seen with the gauging of the Diracon equations, following the discussion for the electron,

$$D_{(d,f)} = \Gamma^{\mu}_{(f)} \left[\partial^{(d)}_{\mu} - i \frac{e}{\hbar} A^{(d)}_{\mu}(x) \right], \tag{4.77}$$

the gauging fields having, by selection in agreement with $\partial^{(d)}_{\mu}$ the multi-vector composition

$$A^{(d)}_{\mu}(x) = A^{d,\mathrm{scalar(electromagnetic)}}_{\mu} + A^{d,\mathrm{pseudoscalar(weak,color)}}_{\mu} i\gamma^5 + A^{\mathrm{tensor(gravity)}}_{\alpha\beta,\mu} \gamma^{\alpha\beta}, \tag{4.78}$$

that is, the combined gauging has 1 + 2 + 3 + 6 parts given, when interacting with other carrier fields, electromagnetic, weak, color, and gravity parts. The first two terms carry the index (d) because they are relative properties. Then the wave function becomes upon gauging (φ a reference spinor).

$$\psi_d(x) = B \exp\left\{ i(p^{\mu}_d x_{\mu} + \phi_d(x)) \right\} \varphi, \tag{4.79}$$

with the phase factor being a multi-vector

$$\phi_d(x) = \phi_{d,\mathrm{scalar}}(x)\mathbf{1} + \phi_{d,\mathrm{pseudoscalar}}(x) i\gamma^5 + \phi_{d,\alpha\beta}(x)\gamma^{\alpha\beta}, \tag{4.80}$$

the particular, relative, combinations for the phase, the $i\gamma^5$ terms, generate isospin and color and the g^{ab} generate the local Lorentz transformation which are a consequence of gravity. The symmetries of

$$\phi_{d,\text{scalar}}(x)+\phi_{d,\text{pseudoscalar}}(x)i\gamma^5,$$

generate the well known $SU(3)_C \otimes [SU(2) \otimes U(1)]_{ew}$ standard model.

The electroweak–color relative properties can be illustrated comparing the following complex currents vector for massless fields obeying (4.77). For a massless field $(j^0)^2 = (j^f)^2$ flowing, in the 8 cases below, in the $\mathbf{v}_f$ direction

$$\begin{array}{ll}
e & \mathbf{j}_1 = j^0 e_0 + \frac{1}{3} j^f (e_1 + e_2 + e_3), \\
d_r & \mathbf{j}_2 = j^0 e_0 + \frac{1}{3} j^f (e_4 e_1 + e_2 + e_3), \\
d_b & \mathbf{j}_3 = j^0 e_0 + \frac{1}{3} j^f (e_1 + e_4 e_2 + e_3), \\
d_g & \mathbf{j}_4 = j^0 e_0 + \frac{1}{3} j^f (e_1 + e_2 + e_4 e_3), \\
u_r & \mathbf{j}_5 = j^0 e_0 + \frac{1}{3} j^f (e_1 + e_4 e_2 + e_4 e_3), \\
u_b & \mathbf{j}_6 = j^0 e_0 + \frac{1}{3} j^f (e_4 e_1 + e_2 + e_4 e_3), \\
u_g & \mathbf{j}_7 = j^0 e_0 + \frac{1}{3} j^f (e_4 e_1 + e_4 e_2 + e_3), \\
\nu & \mathbf{j}_8 = j^0 e_0 + \frac{1}{3} j^f (e_4 e_1 + e_4 e_2 + e_4 e_3),
\end{array} \tag{4.81}$$

from the property $(e_4)^2 = 1$ we see that if we make the replacement $j^f \rightarrow j^f e_4$ the current $\mathbf{j}_8$ will be transformed into current $\mathbf{j}_1$ and the currents $\mathbf{j}_{5,6,7}$ into the currents $\mathbf{j}_{2,3,4}$ respectively, corresponding to the $SU(2)$ symmetry and an equivalent type of transformation will transform the sets $\{2,3,4\}$ and $\{5,6,7\}$ within themselves corresponding to the $SU(3)_c$ symmetry.

The semi-empirical mass matrix (based on Królikowski [131, 132]) for the families of elementary carriers has a very interesting form in its first approximation:

$$m_{(f,d)} = N_f m_d (5.75\,(1-(1-f)\,K_f/c^2_f) + \text{effect of nondiagonal terms}), \tag{4.82}$$

with $N_f = n_f c_f^2$ and $m_d = m_0 (n_c)^2_d Q^4_d$, where n_f is the degeneracy of the family's wave unction as mentioned above, $c_f = 2f - 1$ the number of spinors in the outer product of ψ, m_0 the electron mass, nc the number of color degrees of freedom: 1 for ν, e^- and 3 for the quarks; and Q_d the charge of the lepton or quark field. Then the masses are all, in a first approximation, proportional to the electron mass. That is, the action density is multiplied by this factor of geometrical origin. The factor Q_d^4 suggests that the additional interaction is directly related to the

electronic, gauge, field as of a self-interaction origin. The induced mass and charge discussions below are related to this point. The creation of a pair of elementary carriers at a given point, and its subsequent separation, involves the creation of their gauge fields, Q_d^2 is the factor for the energy required to create the carrier's electromagnetic field, an inseparable field from the concept of the existence of the carrier, whereas Q_d^4 should correspond to geometric self interaction.

In the theory we have presented here the physical properties are now a constitutive part of the wave equations. The relative properties are clearly shown [105] when super-matrices describe a collection of fields. Off diagonal terms couple them amongst themselves.

We have seen that space–time and its T_M (complex) allow enough degrees of freedom to construct a theory of elementary particles and their interactions. Specially important is that all known interactions are properly described. No additional isospin space is therefore needed, it is generated by the relative properties of the fields, the same applies to the color space.

Nucleons like proton or neutron and mesons are, within this theory, composite fields but elementary carriers. In fact these composite 'elementary' particles cannot, even if enough energy is available, be split into smaller components; the requirement of rotational invariance forces the 'colorless' combination of quarks, even to the smallest possible experimental probe size.

Otherwise the use of STA and its equivalent complex space–time, which results in a five-dimensional geometry, with 32 degrees of freedom, allows the construction of a theory with both induced matter and interaction fields and the new features consisting in a natural existence of the $SU(3) \otimes SU(2) \otimes U(1)$ theory for the elementary particles fields. The main difference is that here we do not have a model (corresponding to the Standard Model) but a theory of the elementary carriers and their interactions fields, where the SM is a natural structural part.

6.5 MASSES AND GEOMETRIC ANALYSIS

We consider here the problem of the masses of fundamental particles considering the electron as the reference field. We follow the ideas and analysis of Królikowski in a form suitable to our theory of carriers.

Introduction. The fundamental contribution of Królikowski [131, 132], independent but immediately relevant, to our theory, was the study of the puzzle in particle physics of the occurrence of three families of

leptons:

$$\begin{array}{llll} \nu_e & \nu_\mu & \nu_\tau & (\text{charge} \quad 0), \\ e^- & \mu^- & \tau^- & (\text{charge} \quad -1), \end{array} \tag{4.83}$$

and quarks:

$$\begin{array}{llll} u & c & t & (\text{charge} \quad 2/3) \\ d & s & b & (\text{charge} \quad -1/3) \end{array} \tag{4.84}$$

differing by (apparently nothing but) their masses. It has been important to consider the CERN measurements of total decay width for Z^0 gauge boson, manifesting itself as a resonance at *circa* 91 GeV of Center of Mass energy in the process

$$e^+e^- \rightarrow Z^0 \rightarrow \text{anything}, \tag{4.85}$$

showing that the number of different neutrino versions, lighter than $\frac{1}{2} m_z \simeq 46$ GeV, is just three. This result, followed by others, strongly suggests that the number of all lepton and quark families is equal to three if all neutrinos are light (or massless).

The basic consideration is that there should be several approaches to understand the different fields that can be described by the Dirac equation (using $\Gamma \cdot \mathrm{B} = \Gamma^\mu B_\mu$)

$$[\Gamma \cdot (p - gA) - M]\psi = 0, \tag{4.86}$$

where the geometric meaning of the gamma symbols is kept

$$\{\Gamma^\mu, \Gamma^\nu\} = 2g^{\mu\nu}\mathbf{1}. \tag{4.87}$$

The $g\Gamma \cdot A$ should represent the standard-model coupling, identical for all versions, while the mass operator M may depend on the version. Królikowski relates a set of versions with the three experimental families of leptons and quarks. In this set the argument expresses the idea of algebraic composedness of fundamental fermions, similar but different from the familiar notion of spatial composedness (so useful, for instance, in the case of pseudo-scalar and vector mesons built up of quark–antiquark pairs moving in the physical space).

Geometric composedness. Consider, in a systematic approach, the spatial composedness in the familiar Duffin-Kemmer-Petiau equation describing a particle with spin $0 \otimes 1$ (for instance, a pseudo-scalar or vector meson). In the free field case, it can be written in the form

$$\left[\tfrac{1}{2}(\gamma_1 + \gamma_2) \cdot P - M\right] \psi(X) = 0, \tag{4.88}$$

where γ_1^μ and γ_2^μ are two commuting sets of Dirac matrices,

$$\{\gamma_i^\mu, \gamma_i^\nu\} = 2g^{\mu\nu}, [\gamma_1^\mu, \gamma_2^\nu] = 0. \tag{4.89}$$

Here, the pairs $\frac{1}{2}(\gamma_1^\mu+\gamma_2^\mu)$ are represented by the 16 x 16 Duffin-Kemmer-Petiau matrices.

It can be readily seen that (4.88) may be considered as a point-like limiting form of the following two-body wave equation (Królikowski 1987, 1988):

$$\left[\gamma_1 \cdot \left(\tfrac{1}{2}P + p\right) + \gamma_2 \cdot \left(\tfrac{1}{2}P - p\right) - m_1 - m_2 - S(x)\right] \psi(X, x) = 0, \tag{4.90}$$

where (for simplicity) masses could be assumed equal: $m_1 = m_2$ (describing, for instance, a pair of a quark and an antiquark of the same sort). The internal interaction $S(x)$ in (4.90) can be related to the more familiar internal interaction $I(x)$ appearing in the Bethe-Salpeter equation through the formula

$$S(x) = \left[\frac{1}{\gamma_1 \cdot \left(\frac{1}{2}P + p\right) - m_1 + i\varepsilon} + \frac{1}{\gamma_2 \cdot \left(\frac{1}{2}P - p\right) - m_2 + i\varepsilon}\right] I(x). \tag{4.91}$$

Then, any of these two four-dimensional integral operators allows to calculate, step by step, a three-dimensional integral operator playing the role of internal interaction $V(x)$ (internal interaction energy) in the one-time two-body wave equation having the conventional form of the state equation. This equation (Królikowski 1955, 1956), reduces to the familiar Salpeter equation (Salpeter 1952) in the case of an instantaneous internal interaction.

Representations. He explores what would happen if in (4.88) the commuting γ_1^μ and γ_2^μ are replaced by anticommuting γ_1^μ and γ_2^μ (Królikowski 1986):

$$\{\gamma_i^\mu, \gamma_j^\nu\} = 2\delta_{ij}g^{\mu\nu}, \tag{4.92}$$

instead of (4.89). Note that the Clifford algebra (4.92) could be represented by

$$\gamma_1^\mu = \gamma^\mu \otimes 1, \gamma_2^\mu = \gamma^5 \otimes i\gamma^5\gamma^\mu, \tag{4.93}$$

with γ^μ, 1 and $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ being the usual Dirac 4 x 4 matrices.

In the case of (4.92), the counterpart of the Duffin-Kemmer-Petiau equation (4.88) (with the convenient coefficient $1/\sqrt{2}$ in place of 1/2),

$$\left[\frac{1}{\sqrt{2}}(\gamma_1^\mu + \gamma_2^\mu)\cdot P - M\right]\psi(X) = 0, \tag{4.94}$$

might be considered as a point-like limiting form of the two-body wave equation

$$\sqrt{2}\gamma_1\cdot\left(\tfrac{1}{2}P + p\right) + \sqrt{2}\gamma_2\cdot\left(\tfrac{1}{2}P - p\right) - m_1 - m_2 - S(x)\psi(X,x) = 0, \tag{4.95}$$

but the latter, in contrast to (4.90), could not be derived from the conventional quantum field theory. This is a consequence of the fact that the particle kinetic energy operators in the Fock space $\gamma_i^0(\gamma i\cdot Pi + m)$ all commute, if they are derived from the field kinetic-energy operator $\int d^3x\psi^+(x)\gamma^0(\gamma\cdot p + m)\psi(x)$, *so*, in such a case, all γ_i^μ must commute for different *i* (at least, when massive particles are considered; if an interaction with an external scalar field is introduced, also massless particles cannot escape from this conclusion).

Thus, while (4.94) (with (4.92)) may be investigated for some hypothetical particles, it cannot be considered as a point-like limiting form of a two-body wave equation following from the conventional field theory. So, $\psi = (\psi_{\alpha_1\alpha_2})$ displays an algebraic structure that, now, does not coexist with any spatial internal structure, at any rate, in the framework of the conventional quantum field theory (Królikowski 1991). This illustrates, therefore, the notion of *geometric composedness.*

The logical relationship between the notions of spatial composedness and geometric composedness reminds the, as discussed above, relationship between the notions of orbital angular momentum and spin. In fact, in these cases we have to do with similar acts of geometric structure related to some notions of spatial character.

It is important to note that due to the Clifford algebra (4.92) the matrices

$$\Gamma^\mu = \frac{1}{\sqrt{2}}(\gamma_1^\mu + \gamma_2^\mu), \tag{4.96}$$

appearing in (4.94) satisfy the Dirac algebra (4.87). This implies that (4.94) has the form of the Dirac equation (4.86) (in the free case). Thus, the hypothetical particles described by (4.94): when coupled to the magnetic field: should display (magnetically "visible") spin $\frac{1}{2}$ though any of them is a composite of two algebraic partons of spin $\frac{1}{2}$. There exists, therefore, another (magnetically "hidden") spin $\frac{1}{2}$ It is related to the

matrices $(1/\sqrt{2})(\gamma_1^\mu - \gamma_2^\mu)$ also fulfilling the Dirac algebra (4.87) and anticommuting with the matrices Γ^μ in (4.96).

Note further that the matrices (4.96) may be represented in the convenient form

$$\Gamma^\mu = \gamma^\mu \otimes 1, \tag{4.97}$$

if the representation (4.93) is changed into

$$\gamma_{1,2}^\mu = \frac{1}{\sqrt{2}} \left(\gamma^\mu \otimes 1 \pm \gamma^5 \otimes i\gamma^5\gamma^\mu \right). \tag{4.98}$$

So, (4.94) can be rewritten as

$$\left(\gamma_{\alpha_1\beta_1} \cdot P - \delta_{\alpha_1\beta_1} M\right) \psi_{\beta_1\alpha_2}(X) = 0, \tag{4.99}$$

where the second Dirac bispinor index α_2 is free. Such an equation is known as the Dirac form (Banks 1982) of the Kähler equation (Kähler 1962).

A sequence of equations for spin $\frac{1}{2}$ carriers. Therefore the Dirac algebra (4.87) admits the remarkable sequence $N = 1,2,3, \ldots$ of representations

$$\Gamma^\mu = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \gamma_i^\mu, \tag{4.100}$$

where the vectors γ_i^μ, $i = 1,2,3, \ldots, N$, satisfy the sequence $N = 1,2,3, \ldots$ of Clifford algebras

$$\left\{ \gamma_i^\mu, \gamma_j^\mu \right\} = 2\delta_{ij} g^{\mu\mu}. \tag{4.101}$$

With the matrices (4.100), (4.86) gives us a sequence $N = 1, 2,3, \ldots$ of Dirac-type equations (Królikowski 1990, 1992). Of course, for $N = 1$ (4.86) (with the vectors (4.100) inserted) is the usual Dirac equation, while for $N = 2$ it is equivalent to the Dirac form of the Kähler equation already discussed above (in the free case). For $N = 3, 4, 5, \ldots$ it provides us with new Dirac-type equations.

Except for $N = 1$, the representations (4.100) are reducible since they may be realized in the convenient form

$$\Gamma^\mu = \gamma^\mu \otimes \underbrace{1 \otimes \cdots \otimes 1}_{(N-1)\text{times}}, \tag{4.102}$$

with γ^μ and 1 standing for the usual Dirac vectors which can be rep resented by 4 x 4 matrices. In fact, for any $N > 1$ one can introduce, beside $\Gamma_1^\mu \equiv \Gamma^\mu$ given in (4.100) $N - 1$ other Jacobi-type independent combinations $\Gamma_2^\mu, \ldots, \Gamma_N^\mu$,

$$\Gamma_2^\mu = \frac{1}{\sqrt{2}}(\gamma_1^\mu - \gamma_2^\mu), \quad \Gamma_3^\mu = \frac{1}{\sqrt{6}}(\gamma_1^\mu + \gamma_2^\mu - 2\gamma_3^\mu)\ldots, \tag{4.103}$$

such that

$$\{\Gamma_i^\mu, \Gamma_j^\nu\} = 2\delta_{ij}g^{\mu\nu}, \tag{4.104}$$

(from (4.100) and (4.101)). In particular, for $N = 3$ one may use the representation

$$\Gamma_1^\mu = \gamma^\mu \otimes 1 \otimes 1, \ \Gamma_2^\mu = \gamma^5 \otimes i\gamma^5\gamma^\mu \otimes 1, \Gamma_3^\mu = \gamma^5 \otimes \gamma^5 \otimes \gamma^\mu. \tag{4.105}$$

In the representation (4.102), the Dirac-type equation (4.86) for any N can be rewritten in matrix index notation as

$$[\Gamma \cdot (p - gA) - M]_{\alpha_1\beta_1}\psi_{\beta_1\alpha_2\ldots\alpha_N} = 0, \tag{4.106}$$

where $M_{ij} = \delta_{ij}M$. Here, $\psi = (\psi_{a1a2\ldots aN})$ carries N Dirac bispinor indices α, $i = 1,2, \ldots, N$, of which only the first one is affected by the Dirac matrices Γ^μ and so is coupled to the particle's momentum and to the standard model gauge fields (among others, to the electromagnetic field). The rest of them are free. Thus, only a_1 is 'visible', say, in the magnetic field, while $a_2, \ldots, \alpha_N$ are 'hidden'. In consequence, a particle described by (4.86) or (4.106) can display, say, in the magnetic field only a 'visible' spin $\frac{1}{2}$, though it possesses also $N - 1$ 'hidden' spins $\frac{1}{2}$.

Because the Γ^μ are representations of the space–time multi-vectors the Lorentz group transformations can be applied to the field described by (4.86) or (4.106) for any N. The form $\psi^+\Gamma_1^0\Gamma_1^\mu\psi$ is not a relativistic covariant for $N > 1$, though (4.86) with $\Gamma^\mu \equiv \Gamma_1^\mu$ implies that always

$$\partial_\mu\psi^+\Gamma_1^0\Gamma_1^\mu\psi = 0. \tag{4.107}$$

In contrast, the form $\psi^+\Gamma_1^0\Gamma_2^0 \ldots \Gamma_N^0\Gamma_1^\mu\psi$ is a relativistic vector for any N, but (4.86) with $\Gamma^\mu \equiv \Gamma_1^\mu$ shows that the current density conservation of

$$\partial_\mu\psi^+\Gamma_1^0\Gamma_2^0 \ldots \Gamma_N^0\Gamma_1^\mu\psi = 0, \tag{4.108}$$

implies N to be odd.

Thus, the geometrical meaning of the quantities and the probability interpretation of the current of carriers requires that (i) only the odd terms

$$N = 1, 3, 5, \ldots, \tag{4.109}$$

should be present in the sequence of the Dirac-type equation (4.86) (if these are considered as wave equations), and (ii) the carrier current should have the form

$$j^{\mu} = \eta_N \psi^{+} \Gamma_1^0 \Gamma_2^0 \ldots \Gamma_N^0 \Gamma_1^{\mu} \psi. \tag{4.110}$$

Here, η_N is a phase factor making the parity P_h

$$P_h = \eta_N \Gamma_2^0 \ldots \Gamma_N^0 \tag{4.111}$$

Hermitian. Since due to (4.108) P_h is a constant of motion, Królikowski remarks that one can consistently impose on the amplitude function ψ in (4.86) the constraint

$$P_h \psi = \psi, \tag{4.112}$$

in order to guarantee the probability density to be positive:

$$j^0 = \eta_N \psi^{+} \Gamma_2^0 \ldots \Gamma_N^0 \psi > 0. \tag{4.113}$$

Exclusion principle. The Dirac-type equation (4.86) with $\Gamma^{\mu} \equiv \Gamma_1^{\mu}$ distinguishes the visible bispinor index α_2 from $N - 1$ hidden bispinor indices $\alpha_1, \ldots, \alpha_N$. About the latter indices, appearing in this scheme on the equal footing, Królikowski remarks that they represent physically non-distinguishable degrees of freedom obeying the Fermi statistics along with the Pauli exclusion principle. Then, the wave functions $\psi = (\psi_{\alpha_1 \alpha_2 \ldots \alpha_N})$ in the sequence (4.109) of the Dirac-type wave equations should be completely antisymmetric with respect to the indices $a_2, \ldots, \alpha_N$. This implies that the sequence (4.109) must terminate at $N = 5$,

$$N = 1, 3, 5, \tag{4.114}$$

leaving us with three and only three terms (4.114) in the sequence.

In the case of $N = 5$ our exclusion principle requires that

$$\psi_{abcde} \equiv \varepsilon_{bcde} \psi_a^{(5)}. \tag{4.115}$$

Thus, in this case there are 4!= 24 equivalent nonzero components (carrying the index *a*), all equal (up to the sign) to one Dirac function $\psi_a^{(5)}$. This reduces the Dirac-type equation to the usual Dirac equation. Here, of course, spin is $\frac{1}{2}$ and it is provided by the visible spin, while four 'hidden' spins sum up to zero.

The case of $N = 3$ is more complicated since then one should consider five candidates for relativistic covariants,

$$p_a = (C^{-1})_{bc}\psi_{abc}, \; s_a = (C^{-1}\gamma^5)_{bc}\psi_{abc}, \tag{4.116}$$

$$\alpha_a^\mu = (C^{-1}\gamma^\mu)_{bc}\psi_{abc}, v_a^\mu = (C^{-1}\gamma^5\gamma^\mu)_{bc}\psi_{abc}, \tag{4.117}$$

$$t_a^{\mu\nu} = \left(C^{-1}i\gamma^5\frac{i}{2}[\gamma^\mu,\gamma^\nu]\right)_{bc}\psi_{abc}, \tag{4.118}$$

Here, C denotes the usual charge conjugation matrix that in the chiral representation, where $\gamma^5 = \mathrm{diag}(1, 1, -1, -1)$, may be written as

$$C = \begin{pmatrix} 0 & -i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix} = C^{-1}. \tag{4.119}$$

Making use of (4.105), one can write the parity (4.111) in the form

$$P_{\mathrm{h}} = i\mathrm{I}_2^0\mathrm{I}_3^0 = 1 \otimes \gamma^0 \otimes \gamma^0, \tag{4.120}$$

where in the chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{4.121}$$

Then, the constraint (4.112) implies that,

$$\psi_{a11} = \psi_{a33}, \; \psi_{a22} = \psi_{a44}, \; \psi_{a12} = \psi_{a34}, \; \psi_{a21} = \psi_{a43}, \tag{4.122}$$

$$\psi_{a13} = \psi_{a31}, \; \psi_{a24} = \psi_{a42}, \; \psi_{a14} = \psi_{a32}, \; \psi_{a41} = \psi_{a23}. \tag{4.123}$$

Thus, the constraint (4.112) and our exclusion principle (requiring that $\psi_{abc} = -\psi_{acb}$) loads to the conclusion that from all components ψ_{abc} only

$$\psi_{a12} = -\psi_{a21} = \psi_{a34} = -\psi_{a43} \equiv \psi_a^{(3)}, \tag{4.124}$$

and

$$\psi_{a14} = -\psi_{a41} = \psi_{a32} = -\psi_{a23}, \tag{4.125}$$

may be non-zero. Then, after a simple calculation,

$$p_a = 0, \quad s_a = -4i\psi_{a12}, \tag{4.126}$$

$$a_a^\mu = 0, \quad v_a^\mu = \begin{cases} -4i\psi_{a14} & \text{for} \quad \mu = 0 \\ 0 & \text{for} \quad \mu = 1,2,3 \end{cases}, \tag{4.127}$$

$$t_a^{\mu\nu} = 0. \tag{4.128}$$

But, the Lorentz invariance of the vector v_a^μ given in (4.127) requires that $v_a^0 = 0$ since $v_a^\mu = 0$ for μ = 1,2,3. Hence $\psi_{a14} = 0$. In this way, we can see that all components ψ_{abc} must vanish except those in (4.124). So, in this case there are 4 equivalent non-zero components (carrying the index *a*), all equal (up to the sign) to the Dirac function $\psi_a^{(3)}$. This reduces again the Dirac-type equation to the usual Dirac equation. Here, spin is evidently $\frac{1}{2}$ and it is given by the visible spin, two 'hidden' spins being summed up to zero.

Concluding, in each of the three allowed cases $N = 1,3,5$ there exists one and only one Dirac particle (for any given color and up/down weak flavor described by the theory above). So, it is natural to connect these three versions of the Dirac particle with the three experimental families of leptons and quarks.

Amplitude functions. As for the wave functions with $N = 1,3,5$ the number of equivalent nonzero components (carrying the visible bispinor index) is 1, 4, 24, respectively, the following overall wave function comprising three sectors $N = 1,3,5$ (or three fundamental-fermion families) may be constructed:

$$\Psi = \frac{1}{\sqrt{29}} \begin{pmatrix} \psi_a^{(1)} \\ \sqrt{4}\psi_a^{(3)} \\ \sqrt{24}\psi_a^{(5)} \end{pmatrix} = \widehat{\omega} \begin{pmatrix} \psi_a^{(1)} \\ \psi_a^{(3)} \\ \psi_a^{(5)} \end{pmatrix}, \tag{4.129}$$

Here, the sector-weighting (or family-weighting) matrix

$$\widehat{\omega} = \frac{1}{\sqrt{29}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sqrt{4} & 0 \\ 0 & 0 & \sqrt{24} \end{pmatrix}, \tag{4.130}$$

appears.

Mass Spectral Formula. The three-family wave function (4.129) implies the following form of the mass matrix for any triple of fundamental fermions ordered in one line in (4.83) and (4.84):

$$\widehat{M} = \widehat{\omega} H \widehat{\omega}. \tag{4.131}$$

Here, H denotes a Higgs coupling strength matrix, while ω is given as in (4.130). So, there are four different matrices (4.131) corresponding to triples of neutrinos, charged leptons, up quarks and down quarks, respectively.

Among all 12 fundamental fermion masses, the masses m_e, m_μ, m_t of charged leptons e^-, μ^-, t^- are the best known. On the base of some numerical experience, Królikowski proposes the following phenomenological ansatz (in two options) for the matrix H in the case of charged leptons:

$$H = \begin{pmatrix} h^{(1)} & 0 & 0 \\ 0 & h^{(3)} & 0 \\ 0 & 0 & h^{(5)} \end{pmatrix}, \tag{4.132}$$

with

$$h^{(N)} = M_0\left(N^2 - \frac{1 \pm \varepsilon^2}{N^2}\right), \tag{4.133}$$

where $N = 1,3,5$. Here, $M_0 > 0$ and ε^2 denote two real constants independent of N. Then, the eigenvalues of the mass matrix (4.131) take the form

$$\begin{aligned} \mp m_e \equiv M^{(1)} &= \mp\frac{M_0}{29}\varepsilon^2, \\ m_\mu \equiv M^{(3)} &= \mp\frac{4}{9}\frac{M_0}{29}(80 \mp \varepsilon^2), \\ m_\tau \equiv M^{(5)} &= \mp\frac{24}{25}\frac{M_0}{29}(624 \mp \varepsilon^2), \end{aligned} \tag{4.134}$$

since the Dirac masses are defined as non negative (*a priori*, the second option seems to be more attractive). From the system of three equations (4.134) Królikowski obtains in terms of experimental m_e and m_μ the predictions (in two options) for the mass m_τ,

$$m_\tau = \frac{6}{125}(351 m_\mu \pm 136 m_e) = \begin{cases} 1783.47\ \text{MeV} \\ 1776.80\ \text{MeV} \end{cases}, \tag{4.135}$$

and for the parameters M_0 and ε^2,

$$M_0 = \frac{29}{320}(9 m_\mu \pm 4 m_e) = \begin{cases} 86.3629\ \text{MeV} \\ 85.9924\ \text{MeV} \end{cases}, \tag{4.136}$$

and

$$\varepsilon^2 = \frac{320 m_e}{9 m_\mu \pm 4 m_e} = \begin{cases} 0.171590 \\ 0.172329 \end{cases} \tag{4.137}$$

There is an excellent agreement between the predictions (4.135) for m_τ and its experimental value

$$m_\tau = 1784.1^{+2.7}_{-3.6}\ \text{MeV}, \tag{4.138}$$

cited for several years by Particle Data Group (1992) or

$$m_\tau = (1776.9 \pm 0.4 \pm 0.3)\ \text{MeV}, \quad m_\tau = (1776.3 \pm 2.4 \pm 1.4)\ \text{MeV}, \tag{4.139}$$

reported recently by Beijing Electron-Positron Collider Group (Qi 1992) and ARGUS Collaboration (Albrecht 1992), respectively.

In the picture which emerges from the argument, any fundamental fermion with $N = 1,3,5$ is composed of one 'visible' spin $\frac{1}{2}$ and $N - 1 = 0,2,4$ 'hidden' mutually cancelling spins $\frac{1}{2}$, the latter forming relativistic scalars. This is the analysis of Królikowski which in our theory represents, from the algebraic structure, both a magnetic self–energy and the **relation** of other elementary carriers to the fundamental carrier field of the electron.

This page intentionally left blank.

Chapter 5

GEOMETRY AND THE ELECTRON

1. INTRODUCTION

The experiments of Gauss and Weber in the 1830's on Ampère's law led to the concept of the electron and the atomic cores, before the confirmation by J. J. Thomson in 1897, as the carrier of negative charge and the first studies of these particles determined it to be a (point) charge with well defined trajectories and with a well defined charge to mass ratio. The charge $q = -1.602 \times 10^{-19}$ Coulomb and the mass $m_0 = 9.108 \times 10^{-31}$ kg. Then the electron was to be defined as a massive charged particle, point-like up to the experimental accuracy of that time. In fact, in the years 1846–1856 Weber developed a relative velocity dependent generalization of Ampère's law that led him in 1871 to the theoretical recognition of the existence of the charged atomic nucleus and oppositely charged orbiting electrons. In 1855 Weber determined the constant velocity, equal to $\sqrt{2}\,c$ to be known as Weber's constant. Bernard Riemann, observing the Weber experiment, noted that the value of $\sqrt{2}c$ is close to the velocity of light determined by Fizeau.

Weber (1871 [182]) arrived at. the charge to mass ratio and proton–electron mass ratio, derived the formula e^2/mc^2 known as the classical electron radius, and identified the nuclear binding force for which there was no empirical evidence until the 20th century. Weber also pointed out in 1871 that his constant velocity $\sqrt{2}\,c$ *must represent a limiting velocity for electrical particles.* This early history of atomic science *was* presented by Hecht (1996) [78]. The name *electron* was introduced in 1874 by G. Johnstone Stoney one year after the publication of the *Treatise on Electricity and Magnetism* by James Clerk Maxwell in 1873.

Let e and e' denote the charges and let m and m' be the masses. Weber denotes by f the acceleration not caused by mutual action of particles. Then the Weber law (1871) for the force between two charged particles in relative motion led him to the following comment:

From this its results that the law of electrical force is by no means as simple as we expect a fundamental law to be… . The particles do not by any means always repel each other; (relativity theory cleared up these relations 35 years afterwards).

These considerations were soon confronted with a basic problem: a charged particle generates an electrostatic field of intensity $E(r)$ which contains an electromagnetic static field energy

$$U_{em} = \int_a^{\infty} \frac{1}{2k}(\mathbf{E}^2 + c\mathbf{H}^2)4\pi r^2 dr \quad \text{with} \quad \mathbf{E}(r) = \frac{kq}{r^2}, \tag{5.1}$$

where a is the radius of an spherical distribution of the charge q, that is (without magnetic field contributions)

$$U_{\text{em}}(a) = \frac{kq^2}{2a}. \tag{5.2}$$

It is well known that, as for a point particle $a \to 0$, the energy of the electromagnetic field diverges. Moreover, the self-interaction of the field with the particle could also be considered $U_{self}(a) = +kq^2/a$ as an additional divergent quantity. When in the 20th century the mass–energy relation was discovered it was thought that $U_{em}(a)$ could be the origin of the electron's mass

$$m_a c^2 = U_{\text{em}}(a), \tag{5.3}$$

which defines a quantity $r_0 = 2a$, and if $m_a = m_0$, then

$$r_0 = \frac{q^2}{m_0 c^2 4\pi\epsilon_o}; \tag{5.4}$$

These models replace the electron's mass by the electron's radius parameter. It is also well known that further refinements give for the spherical distribution of charge moving with velocity υ

$$\epsilon_{\text{em}} = \gamma \frac{kq^2}{2a}\left(1 + \tfrac{1}{3}\frac{\upsilon^2}{c^2}\right), \tag{5.5}$$

$$P_{\text{em}} = \gamma \frac{2kq^2}{3ac^2}\upsilon, \tag{5.6}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$, for the electromagnetic energy $\in_{em}$ and the electromagnetic moment P_{em}. A discussion of these problems (Poincaré 1906 [149], Abraham and Lorentz 1909 [1], Cohen and Mustafa 1986 [36], Lozada 1989 [137], etc.) is still a live subject in the literature. No structure or size has been found for the electron this far, the **distribution** related to it can be made as small in volume as available energy allows.

The picture of the electron was more complete when it was discovered that besides its current j_μ and its associated mass m_0, another property, the electron's magnetic moment μ_e, had to be introduced (Stern and Gerlach 1921). This vectorial quantity in the presence of an external magnetic field $\mathbf{H} = h\vec{e}_3$ can have only two values $\mu_s = \pm e\hbar/2m_0\vec{e}_3$ (notice the change of units $q \to e$). The well known ideas of Pauli (1921) and of Goudsmit and Uhlenbeck (1925) conveyed to the introduction of the electron's intrinsic angular momenta, or spin, $\vec{s} = \hbar/2\vec{e}_3$.

Then already by 1925 the electron was to be considered as a massive charged particle with spin s and magnetic moment μ_e. A few years later quantum mechanics was introduced, and it was found that the theory of Dirac (1928) gives a correct description of the electron in the sense that it provides useful calculational procedures for the electron in the one particle approximation. Of course, it was a great success that the equation also described the positron. The latest definition of the electron could be then operational: *An electron is a particle (a carrier field in our approach) obeying Dirac's equation with charge –e and mass m_0. The same equation introduces both the spin and the correct magnetic moment of the electron as a structural consequence.*

Users of the Dirac equation can work with many-electron, fermion systems, if the self-Coulomb problem is avoided. Frequently this is done by solving at the same time the statistics problem through the systematic consideration of exchange and the Pauli exclusion principle, a common practice between atomic, molecular, and solid state physicists.

In this book we have, on the other hand, not separated the problem of the electron and its electromagnetic fields, this in order to search for new understanding of the electron's observed nature. Also we have shown that the natural mathematical formulation is given in terms of multi-vector algebra.

We could, otherwise, try to avoid the explicit reference to the electromagnetic fields, (Schwarzschild (1903) [167], Tetrode (1922) [176], Fokker (1929, 1932) [64], Wheeler and Feynman (1949) [188] or Sutherland (1989) [170]).

2. GEOMETRY AND THE ELECTRON

Here we analyze the different possibilities of studying the electron, in the single carrier (particle) approximation, mainly from the point of view of the space–time multi-vector geometry. Different points of view of the electron and their relation to geometric algebras are confronted.

We further assume that any physical quantity in the space–time should correspond either to a scalar (for example, a charge), a vector (the electron current or the energy–momentum) which, in principle, is a time component of a tensor, or the electromagnetic potentials A^μ (from which the electromagnetic magnetic field intensities can be derived by, outer differentiation), bi-vectors (as the electromagnetic field intensities), tri-vectors (the axial currents) or pseudo-scalars (the space–time hyper-volume itself or quantities related to chirality). We have reminded the reader of these examples to illustrate that the STA algebra is already the basis for the study of the electromagnetic field or of the properties usually associated with the electron matter field itself. It has been shown that specific combinations of multi-vector quantities are associated with spinors ψ_M, see, for example, Flügge (1947 [59]) or Hestenes (1975 [80]) or Boudet (1985 [18, 19, 20]). These multi-vectors are considered to belong specifically to the electron matter field, see [13, 21, 32, 29, 59, 74, 79, 141].

Those multi-vectors ψ_M correspond to ideals in the algebra [59]. By right multiplication by a constant (unit) **minimal ideal** η_i (a minimal ideal of the space–time geometric algebra corresponds to a spinor, the projection used here to the inverse Cartan map [26, 103]), the ideal ψ_M reduces to the well known Dirac spinors ψ. We have shown [105] that the particular choice $\eta_{M,i}$ of the constant multi-vector minimal ideals, where each i corresponds to a combination of signs in $(1 \pm g_0)(1 \pm i g_{12})$, in matrix representation:

$$\eta_{M,i} = \tfrac{1}{4}(1 \pm \gamma_0)(1 \pm i\gamma_{12}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ etc.,} \tag{5.7}$$

or the 'column' spinor equivalent η_i, for example for $i = 1$

$$\eta_i = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad \eta_i^T = (1000) \text{ such that } \eta_{M,i} \quad \eta_i \eta_i^T, \tag{5.8}$$

results in the reconstruction of the multi-vector ψ_M corresponding to a spinor set $\{\psi_i\}$

$$\psi_M = \sum_i \psi_i \eta_i^T , \tag{5.9}$$

and the standard Dirac theory can be faithfully mapped, Cartan map, to a multi-vector form of the Dirac theory ([85, 86, 139, 140, 59, 160, 161, 162], see also next chapter).

The multi-vector representing the electron obeys the Dirac wave equation (eigenvalue equation). The results obtained with the standard procedure of solving the Dirac equation with a column matrix representation of the spinor are not different from the results obtained with the multi-vector procedure. All results related to that formulation are kept in the new formulation: orthogonality, interference, dispersion, diffraction, etc.. Moreover, the statistical nature of the interpretation of the the–ory is enriched in the sense that we shall be working at two 'statistical' levels:

1) The wave nature of the fields of multi-vectors and their sum resulting in interference; and

2) The **distribution** nature of the different multi-vector fields which in the standard theory, and more widely used interpretation, are taken to be the 'probability density' of finding the particle at a given point, ρ, the probability density of finding the particle with a current *j* or a spin *s*, etc.. We should not forget that, as considered in Chapters 3 and 4, the theory for a collection of particles should be treated within many-body quantum theory with its particular set of premises and rules, densities being replaced by density matrices in which the fermion statistics are fully included. In our approach ρ is the carrier density.

A set of rules for the formalism in terms of multi-vectors and for the interpretation of the results computed with the theory allows the unambiguous calculation of the quantities obtained with the standard procedures. Nevertheless, a richer physical structure of the theory, expressed in terms of multi-vectors, has been shown to exist, in particular, in relation to our basic claim that the matter and interaction fields are both parts of one single physical reality.

Electromagnetism has been known for a long time to be a theory which can be constructed entirely from multi-vectors (Mercier 1932 [139, 140]). The now well known discussions of Hestenes (1966) [13, 77, 79, 80, 81] or Casanova (1976) [28, 29, 30, 31] illustrate its structure. The matter fields

enter into the theory as charge–current distribution densities $j_\mu \rightarrow (p, \mathbf{J})$. Nevertheless as the mapping multi-vectors $\rightarrow$ spinors is possible and the electromagnetic field needs a maximum of six quantities to be defined (spinors allow the use of six independent parameters), electromagnetism could also be considered a theory constructed through the use of Dirac spinors [24, 25, 39, 101, 103, 104]. Here, mainly to emphasize our point of view of matter fields as distribution of multi-vectors, we shall take the two possible formulations: the theory to be cast in terms of multi-vectors and in their multi-vector ideals. See below for the particular construction $\mathbf{F} = \bar{\psi}\gamma_{12}\psi *$.

In brief, for our analysis leading to the ideas below, we are either using the inverse Cartan map (Crawford 1985 [37], see also [103]) or we use the Cartan map to analyze the multi-vector theories for matter fields.

The currents **J** (for example those generated by the electron field) can, in general, be decomposed into their solenoidal j_{sol} and irrotational j parts (see section 4 of previous chapter). In Dirac's theory the solenoidal parts contain two components: one which is intrinsically solenoidal; and a second which is solenoidal only with reference to the boundary conditions and the observer's frame of reference. Then the electron sources of the electromagnetic fields are described in fact by a set of seven basic quantities:

$$\rho, j_i, j_j, j_k, j_{l,\mathrm{sol}}, j_{j,\mathrm{sol}} \quad \text{and} \quad j_{k,\mathrm{sol}}. \tag{5.10}$$

We have already reminded the reader that an electron cannot exist without its electromagnetic fields, that is, it exists with an electrostatic field generated by the electron's charge, an **intrinsic magnetic** field generated by its intrinsic **solenoidal current** and an additional magnetic field generated by the, extrinsic, electric current. We should remember that the intrinsic solenoidal current is explicit when the Dirac current is analyzed via the Gordon decomposition, (see Ohanian 1986 [143]). A satisfactory theory for the electron which is also a satisfactory functional definition of the electron should be obtained when the sources and the interaction fields are considered as a unit. Such a theory would be in accordance with a philosophical point of view that a physical entity is constituted by whatever is observable of it (an observation being understood as all that we can infer through experiment). The intrinsic solenoidal current, of the electron implicates (see the detailed discussion of Ohanian (1986)) not only a magnetic moment but also an angular momentum

$$S = \int S(x) = \frac{1}{2}\hbar, \tag{5.11}$$

then in (5.10) above j_{sol} could also be replaced by an angular moment field S(x). Dirac's theory shows that the magnitude of $\vec{S}(x)$ is

$$S(x) = S\rho(x), \tag{5.12}$$

then only the direction of $\vec{S}(x)$ is independent of $\rho(x)$ but not its magnitude, this is one of the most important features of the geometrical content of the electron theory. It says that, even if the analysis of an electron distribution shows some solenoidal current, there is a curl of the distribution at every point and, as is well known in vector analysis, the overall intrinsic solenoidal current is the result of the application of Gauss' theorem to the ensemble. Then an analysis of the spin as resulting from a macroscopic current is mathematically correct but physically misleading: *every point of the distribution contains the same amount of angular momentum per unit density.*

There is no indication whatsoever of a structure giving rise to spin and in fact a spin field $S = \psi g12 \psi^* = r\psi_{g12}\,\psi^{\dagger}$ is one of the most fundamental quantities of the theory

In all experiments performed up to date an electron appears as a distribution of charge, currents and electromagnetic (electroweak, in fact,) fields. Most problems arise from the attempt to rationalize the experimental facts starting from a point particle idea as the basis for the interpretation of experiment or for the interpretation of the results of the now standard quantum mechanical calculations. Experiment shows that there is no internal structure of the electron, but the experiment does not disagree with the existence of distribution. The 'interpretation' of the distribution is a fundamental question of quantum mechanics, not of the electron theory. That is, there is no experiment resolving the electron 'cloud' into instantaneous positions of a 'point' particle, nor, at the same time is there any evidence at all of a possible excitation of internal structures of an electron.

We could speak in terms of electromagnetic quantities alone. The densities, which we commonly refer to as the sources, can be substituted by electromagnetic quantities through the integral form of the Maxwell equations. For example to relate **E** and $\nabla \cdot \mathbf{E}$

$$\mathbf{E}(\vec{r}_2) = \frac{1}{4\pi\epsilon_o}\int \frac{\epsilon_o \nabla \cdot \mathbf{E}(\vec{r}_1)}{r_{12}^2}\,\vec{r}_{12}\,dV_1, \tag{5.13}$$

or to relate $\nabla \times \mathbf{H}$ and $\nabla \cdot \mathbf{E}$ for time independent **E**,

$$\nabla \times \mathbf{H} = (\nabla \cdot \mathbf{E})\mathbf{v}, \tag{5.14}$$

and we can even think of the electromagnetic potentials A^{μ} as quantities related to the sources in special forms

$$\nabla^2 A^o + \nabla \cdot \frac{\partial \vec{A}}{\partial t} = -\frac{\nabla \cdot \mathbf{D}}{\epsilon_o}. \tag{5.15}$$

We can then assume that besides the field intensities **E** and **H** we have a vector distribution

$$\nabla \cdot \mathbf{E} \longrightarrow \rho, \tag{5.16}$$

$$(\nabla \cdot \mathbf{E})\mathbf{v} \longrightarrow \mathbf{J}, \tag{5.17}$$

and the energy–momentum related to this vector being

$$\mathcal{E} = \gamma m_0 \epsilon_0 \nabla \cdot \mathbf{E}, \tag{5.18}$$

$$P = \gamma m_0 \epsilon_0 (\nabla \cdot \mathbf{E})\, \mathbf{v}'. \tag{5.19}$$

Here m_0 appears as a parameter providing the correct dimensions and **v'** corresponds to the relative velocity between the inertial system where $\nabla \cdot \mathbf{E}$ has been computed and that of the observer. In equation (5.6) the quantity γ is to be computed from **v'**. Remember that relativistically **E** and **H** can not be separated, nor have they a unique formulation, in fact they can always be expressed as Lorentz transformations and duality rotation of a reference bi-vector **H** = $\psi\gamma_{12}\psi^*$. See below.

The energy–momentum distribution of the electron will be given by (5.18) and (5.19) when **E, H,** and $\nabla \cdot \mathbf{E}$ are those (additive) quantities referred to the particular electron under consideration. **E** and **H** are to be used only in relation to other particles. Equations (5.16, 5.17, 5.18, 5.19) are nevertheless insufficient,. Also the electromagnetic field of the electron extends to infinity, making this approach more complicated, even if allowed by the Principle of Choice in our theory.

2.1 MAPPING: SPINOR TO / FROM MULTI-VECTOR

We briefly reproduce here what was presented elsewhere (Keller and Viniegra 1992). The direct and inverse Cartan maps between Dirac spinors and space–time multi-vectors, a procedure which incorporates Fierz identities and the Boudet(1985) relations, have allowed us to show the explicit relations. Here $\bar{\psi} = \psi^\dagger \gamma_o$, and $\rho_A = \bar{\psi}\gamma_A\psi$, where the γ_A are here any of the 16 **Hermitian** multi-vectors of the complex Dirac algebra, such that the corresponding real numbers are

$$\sigma = \bar{\psi}\psi, \quad \pi = \bar{\psi} i\gamma_5 \psi, \quad \Sigma_{\mu\nu} = \bar{\psi} i\gamma_{\mu\nu}\psi \tag{5.20}$$

$$j_\mu = \bar{\psi}\gamma_\mu\psi \quad \text{and} \quad k_\mu = \psi_1 i\gamma_\mu\gamma_5\psi.$$

related through the Fierz identities

$$j_\mu j^\mu = \sigma^2 + \pi^2, \tag{5.21}$$

$$k_\mu k^\mu = j_\mu j^\mu, \tag{5.22}$$

$$j_\mu k^\mu = 0, \tag{5.23}$$

$$\Sigma_{\mu\nu} = (\sigma^2 + \pi^2)^{-1}\{\sigma\varepsilon_{\mu\nu\rho\tau}j^\rho k^\tau - \pi(j_\mu k_\nu - j_\nu k_\mu)\}, \tag{5.24}$$

(here $M_\mu = M^\nu g_{\mu\nu}$). Through the use of Crawford's inversion theorem (Crawford 1985) the ψ_i mentioned above can be constructed

$$\psi_i = e^{-i\phi_i}\rho_A\gamma^A\eta_i, \tag{5.25}$$

where ϕ_i is an arbitrary phase and h_i is a reference constant, spinor, which projects the multi-vector $M = \rho_A\gamma^A$ into the space of the Dirac spinors. This multi-vector M is the one which corresponds to the additional vectors and other quantities in the extended matterinteraction field theory proposed above. The multi-vectors obey the Dirac like equation (remember the η_i are constant)

$$[(i\gamma^\mu\partial_\mu - e\gamma^\mu A_\mu)\psi_i]\,\eta_i^T = m_o c\psi_i\eta_i^T, \tag{5.26}$$

where $\psi_i\eta_i^T$ is a multi-vector and the operator $(i\gamma^\mu\partial_\mu - eg^\mu A_\mu)$ is a space–time vector operator. $\psi = \Sigma_i\ \psi_i\eta_i^T$ obeys the Dirac algebraic equation for a particular set of values of the ψ_i. The relative phases ϕ_i have to be given in such a form that $M =< e^{\beta\gamma^5}m >0= m\cos\beta$.

2.2 SPIN, DE BROGLIE WAVES, AND MASS

There is an analysis of the structure of the field associated with the electron which acquires special meaning in START, that of a local structure of the field's geometry in space–time. In fact, because the action coordinate allows the repeated use of space–time-like hyperplanes we can consider an auxiliary function $f(r; a)$ projected into one point, $(\mathbf{x}; 0)$ of space–time: $r = |(\mathbf{x}'; a) - (\mathbf{x}; a)|$; that is a projection of a function of r at a into a point at $a = 0$, the value of a could otherwise be $a = a(k)$ below.

Let us compute the spin of a de Broglie wave (de Broglie 1943 [43]) which we write here in the form proposed by Mackinnon (1981) [138]

$$\psi = f(r)g(\mathbf{x}, t)\eta_1, I^2_{+1}\eta_1 = \eta_1, \tag{5.27}$$

Mackinnon proposed a particular case with $f(r) = (\sin kr)/kr$ but, $f(r)$ can be in general a spherical Bessel (Newman) function. The momentum $\mathbb{P}$ of the field is

$$\mathbb{P} = \frac{\hbar}{4i}\left[\psi^\dagger \nabla\psi + \psi^\dagger \alpha(\alpha^\mu \nabla_\mu)\psi\right] + \text{h.c.}, \tag{5.28}$$

$$\mathbb{P} = \frac{\hbar}{2i}\left[\psi^\dagger \nabla\psi - (\nabla\psi^\dagger)\psi\right] + \frac{\hbar}{4}\nabla \times (\psi^\dagger \sigma\psi), \tag{5.29}$$

and, for $g(\mathbf{x}, t)$ corresponding to a particle at rest

$$\mathbb{P} = \frac{\hbar}{4}\left(\frac{k^3}{2\pi^2}\right)\frac{\partial}{\partial r}\left(\frac{\sin^2 kr}{k^2 r^2}\right)(-2y\hat{x} + 2x\hat{y}), \tag{5.30}$$

which represents a circular flow of the field in the plane $\hat{x} \wedge \hat{y}$. The circulation has a singularity (zero density weight nevertheless) at the origin. The angular momentum is given by

$$\mathbb{J} = \frac{\hbar}{2i}\int x \times [\psi^\dagger \nabla\psi - (\nabla\psi^\dagger)\psi]d^3x + \frac{\hbar}{2}\int \psi^\dagger \sigma\psi d^3x, \tag{5.31}$$

and again the second term, spin, will be the relevant quantity. If we assume f to be normalized then the integral of the spin part would be trivially of magnitude $\hbar/2$. As for a de Broglie wave packet $k = m_0 c/\hbar$ then the same prefactor $f(r)$ that generates the mass generates the spin of the total wave. This seems to be the real origin of the structural parts discussed above. Notice that $4\pi r^2 (f(r))^2 = 0$ as $r \to 0$.

A different, probabilistic, problem is related to the interpretation of the total ψ as a probability amplitude. As mentioned above each point in the space where the electron is considered is given the same factor m_0 and $s = \hbar/2$ for mass and spin magnitude. These properties are obtained by mapping a generating function $f(r; (\mathbf{x}, a))$ into the space–time point $(\mathbf{x}, a = a_0)$, the space–time of the observer describing nature. This is an allowed procedure in a geometrical theory as the one presented in this book.

The interpretation of $f(r)$ is consistent with the discovery that *Zitterbewegung* is mapped by a circular (helical) current for an electron at rest (in displacement). The classical pictures generated by space–time algebra act as 'localizer' of currents. This is seen both for bound states (see the discussion of the atomic calculation as plane solution in the book of Casanova (1976)) or for the traveling waves (Hestenes 1990 [82], Vaz and Rodrigues 1993 [181]) as *Zitterbewegung* .

The prefactor $f(r)$ provides, additionally, a connection with the standard model of elementary particles as far as

$$f(r) = \frac{\sin kr}{kr} = \frac{e^{ikr}}{ikr} - \frac{e^{-ikr}}{ikr}, \tag{5.32}$$

and it corresponds to a standing spherical wave. e^{ikr}/kr is an outgoing spherical wave and e^{-ikr}/kr an incoming spherical wave. Given a spin direction they will have opposite helicities and the standing spherical wave will be the realization of the well known sum of a left handed and a right handed wave. The $\pm i$ factor in the exponent would be the eigenvalue of the γ_5 operator. This is related to action as discuss in the previous chapter.

This procedure is not directly related to the mapping of other types of functions into the one describing the electron directly as a solution of the Dirac equation. An example described at some length below is the formation of non-dispersive waves for the relativistic wave equation (Maxwell) started at least with the work of Bateman (1915) [12]. But one should not be confused with the fact that the solution Φ of the $\Box^2\Phi = 0$ may be given with Φ belonging to any of the $D^{m,n}$ representations of the Lorentz group.

2.3 THE 1929 WORK OF FOCK AND IVANENKO

As we have mentioned above, the geometric, Clifford algebraic, content of the Dirac equation and the corresponding electron theory was clearly and explicitly discussed by V. Fock and D. Ivanenko in 1929. On the 20th of May 1929 a physics conference took place in Karkov (Ukranie) where Fock and Ivanenko presented the idea that the Dirac matrices had a pure geometrical meaning. The theory was developed in a series of 4 papers:

(A) V. Fock and D. Iwanenko, Über eine mögliche geometrische Deutung der relativistischen Quantcntheorie, Z. für Physik **54,** 798 (1929);

(B) V. Fock and D. Iwanenko, Géométrie Quantique Linéaire et Deplacement paralléle (presented by Maurice de Broglie) , C.R.. Acad. Sciences (Paris) **188,** 1470, (1929);

(C) V. Fock, Geometrisierung der Diracschen Theorie des Electrons, Z. für Physik **55,** 261 (1929) and;

(D) V. Fock, Sur les Equations de Dirac dans la Théorie de Relativité générale, C.R. Acad. Sciences (Paris), **189,** 25 (1929).

In the paper (C) the Dirac equations were given a general relativistic invariant form and the gauge invariance of general relativity and electro-

magnetism were recognized as having a geometrical meaning by themselves and a proper place in the geometric description of the physical world: gravitation through the Ricci coefficients and electromagnetism through, an independent geometrical quantity, the four potentials. In papers (A) and (B) they explicitly wrote

$$ds = \sum_{\nu} \gamma^{\nu} dx_{\nu}, \tag{5.33}$$

as a linear differential form whose square ds^2 should be equal to the Riemann ordinary quantity. They proposed calling this point of view of geometry **linear quantum geometry**. They also proposed that the Dirac spinor should be considered as a new geometric quantity (called semi-vector, following L.D. Landau). In short, they proposed that the purely geometric quantities should be the Dirac algebra set of element's and to consider them as operators. This allowed Fock and Ivanenko to consider all geometric transformations related to special and general relativity. Having found the geometric meaning of the Dirac γ_A matrices they had no problem in formulating the theory in general coordinate system.

In the first section of their *' Comptes Rendus Hebdomedaires des Seances de I'Academie des Sciences'* (1929B) paper these Rusian physicists state that in Riemannian geometry the fundamental quadratic form ds^2 explains gravitation, but for quantum and electromagnetic phenomena new geometric notions are needed ("... notions géometriques nouvelles et étrangéres á la géometrie de Riemann"). They propose:

"...Le caractére géometrique des operateurs α_k de Dirac a été signalé par les auteurs de cette Note [Paper (1029 A) Ueber eine mögliche geometrische Deutung der relativistischen Quantentheorie] qui ont proposé d'introduire les opérateurs analogues aux matrices de Dirac dans la géometrie et de considérer la forme différentielle linéaire

$$ds = \sum_{\nu} \gamma^{\nu} ds_{\nu}, \tag{5.34}$$

dont le carré donne le ds^2 ordinaire de Riemann" ... .

There is little doubt that this is the birth of the space–time geometric analysis within geometric Clifford algebra concepts.

In the second section of (1929 B) they state that from the *n-tuple* γ^{ν} of orthogonal directions at each point of space we can define the Dirac spinor ψ , which they call 'demi-vecteur'. Then, in analogy to the parallel displacement of a vector studied by Levi–Civita, the parallel displace-

ment of a semi-vector can be used to start linear geometry

$$\delta\Psi = \sum_{\nu} c_\nu ds_\nu \Psi, \tag{5.35}$$

where the c_ν are now operators (matrices) on the Φ and the ds_ν the infinitesimal displacements defined in (5.33)

$$\delta\Psi^\dagger = \Psi^\dagger \sum_{\nu} c_\nu^\dagger ds_\nu, \tag{5.36}$$

and if $\alpha_\nu \alpha_\mu + \alpha_\mu \alpha_\nu = 2\delta_{\mu\nu}$ and the $A_\nu = \Psi\dagger a_\nu \Psi$ behave as vector components, then

$$\delta A_\nu = \delta(\Psi^\dagger \alpha_\nu \Psi) = \Psi^\dagger \sum_{\mu} (c_\mu^\dagger \alpha_\nu - \alpha_\nu c_\mu) ds_\mu \Psi, \tag{5.37}$$

where δA_ν should otherwise be of the relativistic from

$$\delta A_\nu = \sum_{\lambda\mu} \Gamma^\lambda_{\mu\nu} A_\lambda ds_\mu, \tag{5.38}$$

if the $\Gamma^\lambda_{\mu\nu}$ are the Ricci coefficients, or

$$c_\mu^\dagger \alpha_\nu + \alpha_\nu c_\mu = \sum_{\lambda} \Gamma^\lambda_{\mu\nu} \alpha_\lambda, \tag{5.39}$$

then

$$c_\mu = g_\mu + i\Phi_\mu, \tag{5.40}$$

the g_μ commuting with the α_ν and

$$i(\alpha_\nu \Phi_\mu - \Phi_\mu \alpha_\nu) = \sum_{\lambda} \Gamma^\lambda_{\mu\nu} \alpha_\lambda, \tag{5.41}$$

a condition which, in modern Clifford algebra language defines the Φ_μ to be the bi-vectors associated with the Poincaré group. Think for example of the $\alpha_\nu = \gamma_0\gamma_\mu$ as bi-vectors in a space–time 'cut' of a vector set γ_ν, then to describe space–time general connections from point to point the Φ_μ would in general, be $\Phi_\mu = \gamma'_0\gamma'_\mu$ with $\gamma'_\nu = \gamma'_\nu(\underline{x})$ representing a field of space–time transformations for curved reference space–time relative to an arbitrary reference γ_ν. Fock and Ivanenko go on to consider the special case $g_\mu = 0$ and $\Phi = \phi^\mu \alpha_\mu e/\hbar c$ such that ϕ^μ will be considered to correspond to the electromagnetic vcctor potential and the covariant derivative being written

$$\nabla_\mu \Psi = \left(\frac{\partial}{\partial x_\mu} - \frac{e}{\hbar c} \phi^\mu \right) \Psi, \tag{5.42}$$

giving a geometric meaning to the electromagnetic field.

V. Fock, in his extensive paper 'Geometrisierung der Diracschen Theorie des Elektrons' (Zeitschrift für Physik, **55,** 261–277 (1929)), clearly states: 1) the geometric meaning of the Dirac γ_μ as vectors; 2) the formulation of the Dirac theory in general relativity; 3) the geometrical meaning of the electromagnetic field as a connection; and 4) the gauge theory formulation of gravitation and electromagnetism as geometric fields. Fock had just reviewed the algebra of bi-spinor quantities and, studying velocity in particular, found relations, which are now known in a general form as Fierz identities, and their transformation properties which he clearly related to the transformation properties of spinors.

The Clifford algebra behind the Fock line of analysis is, however, not clear and is by far not explicit. The reason for this is that he failed to discover that his 'quantum geometry' should in fact be the **standard** space–time geometry, or at least a realization of it. It was not until the work of Mercier in the early 1930's that this was clearly seen. An additional problem arose from his analysis being based in the Dirac α_μ matrices, and then space–time vectors were represented by quaternions, which actually correspond to bi-vectors. The concept of vector and bi-vector is then mixed throughout the paper. This is common in practice because both the 'normalization' $\psi^\dagger\gamma_0\psi$ and the velocity vector $v_\mu = \psi^\dagger\alpha_\mu\psi$, are based on considering a reference vector γ_0 (or $\alpha_\mu = \gamma_0\gamma_\mu$).

Two basic relations are nevertheless the key point in the analysis of the geometry being constructed; the construction of the local γ-matrices tetrad (here $e_0^2 = -e_1^2 = -e_2^2 = -e_3^2 = 1$)

$$\gamma_{(x)}^{\nu} = \sum_{\beta} e_\beta \alpha_\beta h_\beta^\nu(x), \tag{5.43}$$

(his equation (23)) and the treatment of those γ^μ matrices as vectors in curved space–time

$$\square_\alpha \gamma_{(x)}^{\sigma} = \frac{\partial \gamma_{(x)}^{\sigma}}{\partial \xi^\alpha} + \Gamma_{\alpha\rho}^{\sigma}\gamma_{(x)}^{\rho}, \tag{5.44}$$

(his note to his equation (43)) when he is constructing the Dirac field energy–momentum tensor.

2.4 THE DISCOVERY OF THE MULTI-VECTOR WAVE FUNCTION

Later in the 30's, came the pioneering papers of Sommerfeld (1939), Sauter (1930), Mercier (1934, 1935), etc.. The great difference is a clear distinction between what is geometry and what is a physical proposition.

Mercier discussed electrodynamics from the same geometric, Clifford algebraic, point of view . The thesis and two papers of Mercier (1934 and 1935) are crucial to formulating the equations of electromagnetism in Clifford algebra. He even went to consider thermodynamics from this point of view. Mercier had no doubt that he was dealing with space–time geometry and that the Clifford algebra approach was not to be circumscribed to relativistic quantum mechanics.

The work of Sauter is important because by recognizing that in the solution to the Dirac equation each of the columns of a $4 \times n$ matrix is a solution of the equation, a minimal left ideal, we could then think of 4×4 matrices as solutions, and then the solutions and the operators would then be members of the same algebra of square matrices, now known to represent the space–time geometric algebra.

Sommerfeld recognized that the Dirac equation can be solved without any representation of the Dirac algebra, and that then the solutions are explicitly members of the Dirac algebra.

Finally, in the work of Hestenes (1966), we find that the geometrical meaning of most quantities was clearly separated from other types of contributions.

2.5 ON THE ALGEBRAIC DIRAC EQUATION

We consider here aspects of the multi-vector form of the electron's wave equation. See, for example, [59] and Proca A. (1930a,b,c), Sauter F. (1930), Mercier A. (1934, 1935), Eddington A.S. (1936), Sommerfeld A. (1939), Riesz M. (1946, 1953, 1958), Quilichini P. (1957), Ravsevskii P.K. (1957), Teitler S. (1965a,b,c, 1966a,b), Hestenes D. (1966, 1975, 1979), Casanova G. (1970, 1976), Boudet R. (1971, 1974, 1985), Salingaros N. and Dresden M. (1979), Greider T.K. (1980), Keller J. (1982a,b, 1984, 1986a,b, 1991), Keller J. and Megy F. (1984), Crawford J.P. (1985), Keller J. and Rodríguez-Romo S. (1991), Snygg (1997). A recent analysis of some of these works can be found in [115, 125].

2.5.1 THE STRUCTURE OF THE WAVE FUNCTION

Here we want to show explicitly the multi-vector content of the Dirac spinor. We first rephrase some material from the previous chapter. To start, let us consider that associated with each matter field, corresponding to the field associated to a spin $\frac{1}{2}$ particle, there is an energy–momentum field $e_\mu p^\mu(x)$, such that, denoting by $x = e_\mu x^\mu$ points in the observers frame of reference,

$$e_\mu p^\mu(x) = m_0 c e_0', \tag{5.45}$$

assuming that there exists a (local) frame e'_μ where the energy momentum is tho one corresponding to that of a particle at rest. The frame e'_μ is related to the observers frame e_μ through the local Lorentz transformation

$$e'_\mu = R(x)e_\mu R^{-1}(x), \quad R^{-1} = \widetilde{R}, \tag{5.46}$$

then (5.45) becomes

$$e_\mu p^\mu(x) = m_0 c R(x) e_0 R^{-1}(x), \tag{5.47}$$

we multiply (5.47) by $R(x)$ on the right,

$$e_\mu p^\mu(x) R(x) = m_0 c R(x) e_0, \tag{5.48}$$

and use the multi-vector double projector $P_{+\uparrow}$, with the properties

$$P_{+\uparrow} = e_0 P_{+\uparrow} = P_{+\uparrow} e_0 \quad \text{and} \quad P_{+\uparrow} = P_{+\uparrow} i e_1 e_2, \tag{5.49}$$

to obtain

$$e_\mu p^\mu R(x) P_{+\uparrow} = m_0 c R(x) P_{+\uparrow} i e_0 e_1 e_2. \tag{5.50}$$

Here, the ie_1e_2 factor is to be kept for further reference to $P_{+\uparrow}$ having been chosen as the appropriate projector, other choices could have been made. The up arrow refers to γ_{12} as the direction of spin up and the plus sign to the choice of 'positive' mass m_0.

Now assume that there is a function

$$\psi(x) = A(x)R(x)P_{+\uparrow} \in \widehat{C}_{1,3}, \tag{5.51}$$

where $\widehat{C}_{1,3}$ is the Dirac space–time algebra and such that (5.50) can be written, allowing us to use the operator representation $p^\mu = i\hbar\partial^\mu$,

$$\hbar e_\mu \partial^\mu \psi(x) = m_0 c \psi(x) e_0 e_1 e_2, \tag{5.52}$$

where the i has been cancelled on both sides of (5.50). In the reference 'rest' frame of the field $R(x) = 1$ and $A(x)$ should be such that $i\hbar e_\mu \partial^\mu A(x) = m_0 c A(x)\, e_0 e_1 e_2$.

The wave function (5.51) then explicitly contains three main contributions: the existence of the particles' field in $A(x)$; the relative motion of the particles' field in $R(x)$; and the reference to a preferred sign of m_0 and spin in $P_{+\uparrow}$.

This derivation from first principles ([105]) is also an explanation of the geometric reason for considering a multi-vector equation (which goes beyond the multi-vector analysis which was done, at the beginning of

relativistic quantum mechanics, by solving the Dirac equation in terms of multi-vectors). The $\psi \in \hat{C}_{1,3}$ then contains a (local) Lorentz transformation and the information that a fixed time direction e_0, a given duality rotation (discussed below) and a given plane e_1e_2 have been taken as an overall reference. By definition of carrier, the normalization is $\int_V j_0 d\tau = 1$, V being some reference volume.

2.5.2 THE CARTAN MAPPING APPLIED TO THE DIRAC EQUATION

For the inverse procedure start now from the Dirac equation for massive spin $\frac{1}{2}$ fields

$$i\hbar e_\mu \partial^\mu \Psi(x) = m_0 c \Psi, \tag{5.53}$$

where $\Psi(x) \in \hat{\mathcal{L}}^{1,3}$ the space–time spinors and $e_\mu \partial^\mu \in \hat{C}_{1,3}$ is the gradient operator. Consider again the **constant** reference spinor $\eta^\dagger$ such that,

$$\eta^\dagger \eta = 1, \quad P_{+\uparrow}\eta = \eta \quad \text{and} \quad \eta^\dagger P_{+\uparrow} = \eta^\dagger \quad \text{then} \quad \eta^\dagger = \eta^\dagger P_{+\uparrow} i e_0 e_1 e_2, \tag{5.54}$$

and post-multiply (5.53) by $\eta^\dagger$ on the left hand side and by $\eta^\dagger P_{+\uparrow} i e_0 e_1 e_2$, on the right hand side (this is done to introduce into the equations the definitions (5.54)) to obtain

$$i\hbar e_\mu \partial^\mu \Psi(x)\eta^\dagger = m_0 c \Psi(x)\eta^\dagger i e_0 e_1 e_2, \tag{5.55}$$

the obvious definition

$$\psi(x) = \Psi(x)\eta^\dagger \text{ or } \Psi(x) = \psi(x)\eta, \tag{5.56}$$

brings (5.55) into (5.52). The relations (5.56) correspond to the Cartan map, and to the inverse Cartan map respectively (Crawford (1985) and [103]). For massless fields a different choice of the spinor defined in (5.54) should be made.

We now use the matrix representation of the geometric superalgebra $\hat{K}$ [109, 107], an example of Z_2 to illustrate the mathematical structure of (5.52), its equivalent standard form (5.53) and the new type of symmetries that are immediate in $\hat{K}$-algebra.

The basis vectors e_μ which are usually represented by the traceless γ_μ matrices, being elements of order 2 in Z_2 are now represented by the traceless supermatrices

$$e_\mu \to \tilde{\gamma}_\mu = \begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix}. \tag{5.57}$$

The Dirac spinor ψ corresponds in geometric superalgebra to elements of order

$$\Psi \rightarrow \widehat{\Psi} = \begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \widehat{\Psi}^{\dagger} = \begin{pmatrix} 0 & 0 \\ \Psi^{\dagger} & 0 \end{pmatrix} \tag{5.58}$$

The reference, spin up, positive mass spinor η obeying $\gamma_0\gamma_{12}\eta = -i\eta$ corresponds to

$$\eta \rightarrow \hat{\eta} = \begin{pmatrix} 0 & \eta \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{\eta}^{\dagger} = \begin{pmatrix} 0 & 0 \\ \eta^{\dagger} & 0 \end{pmatrix}, \tag{5.59}$$

with $\eta^{\dagger} = (1000)$ in the standard representation of the γ_μ matrices. It has, as mentioned above, the property

$$\hat{\eta}\hat{\eta}^{\dagger} = \hat{P}_{+\uparrow} = \frac{1}{2}(\hat{1} + \hat{\gamma}_0)\frac{1}{2}(\hat{1} + i\hat{\gamma}_{12}). \tag{5.60}$$

Here again (5.60) is an example of the Cartan map: spinors $\rightarrow$ multi-vectors expressed in $\widehat{K}$ algebra.

The standard Dirac equation (free space)

$$i\hbar\gamma_\mu\partial^\mu\Psi = m_0 c\Psi \rightarrow i\hbar\begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix}\partial^\mu\begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix} = m_0 c\begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix}, \tag{5.61}$$

is immediately mapped into the multi-vector Dirac equation, thus:

a) multiply (5.61) on the right by $\hat{\eta}^{\dagger}$ (remind $\hat{\eta}^{\dagger}$ is constant and $\hat{\eta}^{\dagger} = \eta^{\dagger} i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2 \qquad \eta^{\dagger} i\gamma_0\gamma_1\gamma_2$) v

$$i\hbar\begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix}\partial^\mu\begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \eta^{\dagger} & 0 \end{pmatrix} = m_0 c\begin{pmatrix} 0 & \Psi \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ \eta^{\dagger} & 0 \end{pmatrix}\begin{pmatrix} i\gamma_{012} & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.62}$$

and b) define $\phi = \Psi\eta^{\dagger}$ to obtain

$$i\hbar\begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix}\partial^\mu\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} = m_0 c\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} i\gamma_0\gamma_1\gamma_2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{5.63}$$

an equation where only multi-vectors occur, where again our choice for η of the reference positive mass and spin up, is explicitly shown, to obtain the projected Dirac (algebraic) equation in super-algebra form:

$$\hbar\begin{pmatrix} \gamma_\mu & 0 \\ 0 & 0 \end{pmatrix}\partial^\mu\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix} = m_0 c\begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \gamma_{012} & 0 \\ 0 & 0 \end{pmatrix} \tag{5.64}$$

It is clear that (5.64) is block diagonal and that the super-algebra here appears as redundant, that is in (5.64) we do not need, in practice, to

use the more general element $\ddot{\phi} = \begin{pmatrix} \phi & 0 \\ 0 & 0 \end{pmatrix}$, but the procedures (a) and (b) above are general and other options are open if super-algebra is used and not solely the upper left block.

As mentioned above, this mapping would not give back what is considered the more general standard solution of the Dirac equation in the multi-vector form. In fact, the multi-vector algebraic solution for the Dirac equation in free space is often written

$$\psi = \sqrt{\rho} e^{\beta \gamma_5 / 2} R', \tag{5.65}$$

where ρ is a scalar density, β a duality rotation angle, and R' a space–time rotation. As we mentioned, this ψ is, in fact, a multi-vector with four independent, left ideals related in a specific phase (here the bar denotes complex conjugation) which can be represented by a full 4 x 4 matrix, with the following structure

$$\psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix}, \tag{5.66}$$

with

$$\bar{\psi} = \gamma_0 \psi^+ \gamma_0 = \begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 & -\bar{\psi}_3 & -\bar{\psi}_4 \\ -\psi_2 & \psi_1 & -\psi_4 & \psi_3 \\ -\psi_3 & -\psi_4 & \psi_1 & \psi_2 \\ -\psi_4 & \psi_3 & -\psi_2 & \psi_1 \end{pmatrix}, \tag{5.67}$$

and

$$\psi\bar{\psi} = \Omega_1 + \Omega_2 \gamma_5 = \begin{pmatrix} \Omega_1 & 0 & i\Omega_2 & 0 \\ 0 & \Omega_1 & 0 & i\Omega_2 \\ i\Omega_2 & 0 & \Omega_1 & 0 \\ 0 & i\Omega_2 & 0 & \Omega_1 \end{pmatrix} \tag{5.68}$$

The Ω_1 and Ω_2 were denoted by δ and $i\pi$ in (5.20). The Cartan mapping should then be, as mentioned before,

$$\psi = \sum_{i=1}^{4} \Psi_i \eta_i^\dagger, \tag{5.69}$$

where each Ψ_i is each one of the columns of (5.67) and the $\eta_i^\dagger$ are row matrices which have zeros everywhere except at position *i*. Then a full

even multi-vector is reconstructed in (5.69). The phases of the Ψ_i are adjusted to give (5.69) the structure (5.65).

The direct solution of (5.64), given by (5.66) generates $\psi = \sqrt{\rho}e^{\gamma_5\beta/2}R'$ with $R' = Re^{\gamma_1\gamma_2 m_0 c^2 t/\hbar}$, where R is the rotation introduced in (5.46) above and the exponential in $\gamma_1\gamma_2$ has been the origin of most of the discussions presented below where the fundamental role of the spin bilinear form is considered.

In our theory above we have presented the explanation: spin and action have similar axial properties and proportional to each other; also, for the electron within our geometry the spin density is immediately mapped through the Compton radius $\hbar/2 = |s| = m_0 c r_{\text{Compton}}$.

2.5.3 ON THE BILINEAR COVARIANTS IN THE DIRAC THEORY

In the Cartan and inverse Cartan mappings above we have used the well known bilinear covariants, given by (5.20), in the standard spinor formulation of the Dirac theory, they all correspond to densities of tensor quantities of the general form

$$T_A(x) = \widehat{\Psi}\gamma_A\Psi, \tag{5.70}$$

where the subindex A corresponds to the multi-vector characters and the γ_A should be Hermitian matrices, that is with a factor $i = \sqrt{-1}$ included whenever necessary. But in the multi-vector formulation there is a second set of bilinear covariants which correspond to quantities which are also relativistic covariants and can be (as are the $T_A(x)$ above) gauge invariants: they are constructed from the inverse of the wave function multi-vector

$$\psi^{-1}\psi = \psi\psi^{-1} = 1, \tag{5.71}$$

which can be unambiguously defined for the multi-vector form. In this case we will not be dealing with densities $T_A(x) = \psi\gamma_A\tilde{\psi}$ (corresponding to the T_A of the standard approach as discussed, for example, by Casanova (1976)) but with **structural numbers** of the theory, of the general form

$$\mathcal{T}_A(x) = \psi\gamma_A\psi^{-1}. \tag{5.72}$$

They have been discussed by Daviau (1989 [39]) where he found that, from the canonical form of the Dirac algebraic wave function, the gauge

invariant and covariant structural numbers of the theory are

$$f_0 = \psi\gamma_0\psi^{-1} = \sqrt{\rho}e^{\frac{-\beta}{2}\gamma_5}R\gamma_0\frac{1}{\sqrt{\rho}}e^{\frac{\beta}{2}\gamma_5}\tilde{R} \tag{5.73}$$

$$= e^{\frac{\beta}{2}\gamma_5}R\gamma_0\tilde{R}e^{\frac{-\beta}{2}\gamma_5} = e^{\beta\gamma_5}e_0 = e_0e^{-\beta\gamma_5},$$

$$\frac{2}{\hbar}S = \psi\gamma_1\gamma_2\psi^{-1} = \frac{1}{k_1}F = e_1e_2, \tag{5.74}$$

$$f_3 = \psi\gamma_3\psi^{-1} = e^{\beta\gamma_5}e_3 = e_3e^{-\beta\gamma_5}. \tag{5.75}$$

This $\mathcal{T}_4$ covariants should really be taken as intensive quantities, they do not depend on the electron density ρ. In terms of Daviau these quantities are remarkable because their intensity is rigorously the same both where the electron field has a high density or where it has a vanishing density. They can be studied in the multi-vector formalism because it is in this formalism where ψ is invertible.

A quantity like $\mathcal{T}_0 = f_0$ is the one that seems, in principle, associated with the mass term of the electron theory. In the analysis below a mapping of the Dirac equation will be presented, discussed and compared with the approach of Daviau and Lochak (1991 [41]) (see also Daviau 1993 [40]) where they use densities of bilinear forms and then have to speak in terms of 'a variable mass term'.

As in the case of the ordinary Dirac bilinear forms, so well known because of the Pauli–Fierz identities, we can use the inverse Cartan map on the intensive bilinears and start from the Dirac spinor and the corresponding mapping of the inverse Dirac multi-vector:

$$\psi^{-1}\eta_i = \Psi_{(\psi^{-1})} \quad \text{and then} \quad \mathcal{T}_A = \overline{\Psi}_{(\psi^{-1})}\gamma_A\Psi. \tag{5.76}$$

2.6 APPENDIX

This section is included to show the further possibilities of gauge freedom and of the freedom of choice of acceptable descriptions according to the PAD. It is otherwise not essential for the analysis of the basic theory of matter as presented in this book.

2.6.1 THE BI-VECTOR/SPINOR MAPPING

There were two series of papers, which were almost equivalent, developing the idea of a mapping of the Dirac equation into a Maxwell-like form, it is that the Dirac equation could be written in the form $\Box F = 0$. Simultaneously, the idea was introduced that the basic Maxwell equation

$$\Box F = \frac{4\pi}{c}J, \tag{5.77}$$

which in multi-vector algebra is a relation between the divergence of the bi-vector F and the vector J can be mapped into a Dirac like equation for a 'spinor' ψ which can generate the bi-vector F in the **multi-vector** bilinear map

$$\psi\gamma_1\gamma_2\psi^* = F. \tag{5.78}$$

The equation takes a special simple form in the case of $J = 0$.

The work of Campolattaro (1980) started with the analysis of the Maxwell equation by writing F in the equivalent bilinear form $\bar{\Psi}\gamma_\mu\gamma_\nu\Psi$ corresponding to (5.78) above, and by simple replacement into the Maxwell equation (5.77) together with the use of the constant $\bar{\Psi}\Psi = b$, he obtained the Dirac-like, non-linear, equation for Ψ.

In his two following papers Campolattaro (1990) pursued his mapping and showed that the inverse procedure can be used for the Dirac equation, arriving at the Maxwell-like equation obeyed by an F defined in exactly the same form as equation (5.77).

In what can be thought of as a completely independent work, Daviau proposed in 1988 that both the Dirac and the Maxwell equations could be generalized and mapped into each other in such a way that it is clear that they are mathematically equivalent (provided we fix the solutions by the use of subsidiary constrains) and interchangeable for the Dirac or for the Maxwell fields (Daviau (1989) [39]). This paper made full use of space–time Clifford algebra, whereas the work of Campolattaro considered the Dirac matrix ring algebra and spinors, F was represented by him in component tensor form. The related problem of presenting the Maxwell equations in Dirac form has also been considered (Keller 1993 [114]).

The Campolattaro reinterpretation of the Dirac equation for the free electron. Let us consider the Dirac equation for the free electron, i.e.,

$$(\gamma^\mu\partial_\mu + im)\Psi = 0. \tag{5.79}$$

By multiplying equation (5.79) on the left by $\Psi\gamma^\nu$ one has ($\gamma^{\nu\mu} = -\gamma^{\mu\nu}$)

$$\bar{\Psi}\gamma^\nu\gamma^\mu\partial_\mu\Psi + im\bar{\Psi}\gamma^\nu\Psi = 0, \tag{5.80}$$

Using $S^{\mu\nu} = \frac{i}{2}\gamma^{\mu\nu}$ and $\gamma^\mu\gamma^\nu = \eta^{\mu\nu} + \gamma^{\mu\nu}$ equation (5.80) reads

$$2i\bar{\Psi}S^{\mu\nu}\Psi_{,\mu} + \eta^{\mu\nu}\bar{\Psi}\partial_\mu\Psi + im\bar{\Psi}\gamma^\nu\Psi = 0. \tag{5.81}$$

By taking the Hermitian conjugate of equation (5.81), one has

$$2i\bar{\Psi}_{,\mu}S^{\mu\nu}\Psi - \eta^{\mu\nu}(\partial_\mu\bar{\Psi})\Psi + im\bar{\Psi}\gamma^\nu\Psi = 0, \tag{5.82}$$

and by adding equations (5.81) and (5.82) one obtains, for the antisymmetry of $S^{\mu\nu}$

$$(\bar{\Psi}S^{\nu\mu}\Psi)_{,\mu} = \eta^{\mu\nu}Im(\bar{\Psi}\Psi_{,\mu}) + m\bar{\Psi}\gamma^\nu\Psi. \tag{5.83}$$

Similarly, by multiplying equation (5.79) on the left by $\bar{\Psi}\gamma_5\gamma_\nu$ and by repeating the steps followed in the previous lines, one has

$$(\bar{\Psi}\gamma^5 S^{\nu\mu}\Psi)_{,\mu} = \eta^{\mu\nu}Im(\bar{\Psi}\gamma^5\Psi_{,\mu}). \tag{5.84}$$

Equations (5.83) and (5.84) are completely equivalent to the Dirac equation (5.79). Therefore one has, by using the results expressed by equations (5.79-5.84) above, that the Dirac equation is equivalent to the Maxwell equations for an 'electromagnetic field' $F^{\mu\nu}$ defined by

$$\bar{F}^{\mu\nu} = \bar{\Psi}S^{\mu\nu}\Psi, \tag{5.85}$$

and thence

$${}^*\bar{F}^{\mu\nu} = \bar{\Psi}\gamma^5 S^{\mu\nu}\Psi, \tag{5.86}$$

generated by the two currents

$$j^\mu = \eta^{\mu\nu}Im(\bar{\Psi}\Psi_{,\nu}) + m(\Psi\gamma^\mu\Psi), \tag{5.87}$$

and

$$g^\mu = \eta^{\mu\nu}Im(\bar{\Psi}\gamma^5\Psi_{,\nu}), \tag{5.88}$$

the first, electronic in nature and the second magnetic monopolar, or simply monopolar. The gauge conditions are automatically satisfied because each of the four components of the Dirac spinor satisfies the Klein–Gordon equation and the current $m\Psi\gamma_\mu\Psi$ is conserved.

The work of Daviau and the analysis of Rodrigues et al. As the use of STA allows a more comprehensive treatment, or at least a simple account, we shall follow the Daviau approach. He starts by showing that a matrix representation of the complex space–time Clifford algebra allows a straightforward analysis of several relations in (STA). He analyses scalars, vectors, bi-vectors, axial vectors, and pseudo-scalars in some detail in order to have a quick reference to the relationship between them. He also writes the gradient operator for space–time in matrix form

and describes the dot product of that operator with each of the multi-vector fields. He can then write the analysis (which was already known some 60 years before from the work of Mercier (1932)), to proceed to the (for him) obvious space–time generalization of the Maxwell equation for the case where:

a) The current is enlarged to have both a vector and tri-vector parts $J = J_v + J_t$. As we know the tri-vector part, the dual of the vector part, can be interpreted in Maxwell theory as a current of magnetic monopoles in the sense that the electric and magnetic fields are dual to each other.

b) In a second step Daviau generalizes the concept of the bi-vector F to study the possible inclusion of the full even part of the space–time algebra $F = F_s + F_b + F_p$, which, in the analysis of the equivalent to spinors in space–time algebra, corresponds to a spinor which can be written $\psi = \sqrt{\rho}e^{\gamma_5\beta/2}R$ and interpreted as a weight $\sqrt{\rho}$, a duality rotation phase $e^{g5b/2}$ and a Lorentz transformation R. Here no mass term $e^{\gamma_1\gamma_2 m_0 c^2 t/\hbar}$ is needed.

The two successive generalizations yield both a generalized Maxwell theory and the possibility of interpreting F as generated by the Ψ and then the possible mapping of the Maxwell equation into a Dirac-like equation for Ψ.

But before doing this mapping he first proceeds to map the algebraic form of the Dirac equation back into a Maxwell-like form by interpreting the solution of the Dirac equation as the sum of a scalar, an electric, a magnetic, and a pseudo-scalar part.

The algebraic manipulations behind these mappings are much better understood following the analysis presented by Vaz and Rodrigues (1992, 1993, 1994) because they start not by mapping the equations but, by examining the consequences of the free Maxwell equations $\Box F = 0$ when $F = b\psi\gamma\gamma^2\psi^*$.

We shall follow the analysis of Rodrigues and Vaz (RV) starting from the multi-vector

$$F = He^{\beta\gamma_5}R\gamma_1\gamma_2R^*, \tag{5.89}$$

and for the electromagnetic field writing $H \equiv b\rho$ with $\rho \leq 0$, we then have the formal

$$F = b\psi\gamma_1\gamma_2\psi^*, \tag{5.90}$$

the bi-vector which we consider to obey the free Maxwell equation:

$$\Box F = 0. \tag{5.91}$$

RV then look for solutions of (5.91) of the form (5.89); they consider that since (5.90) is valid when F is non-null ($F^2 \neq 0$) then planewave

solutions of (5.91) are excluded (since in this case $F^2 = 0$). This is already an important consideration about the difference between the Dirac and Maxwell fields even if both can be described by (5.91) for the source-free case. RV proceed to consider the trivial non-null solution of (5.91) given by F = constant, the case when b and ρ in (5.90) are constants or related in a special form. They suppose first b, ρ and β to **be** constants. From (5.90) in (5.91) RV obtain the non-linear Heisenberg-like spinor equation:

$$\Box\psi\gamma_1\gamma_2 + \mathcal{F}(\psi) = 0, \{b, \rho\} = \text{const.}, \tag{5.92}$$

with

$$\mathcal{F}(\psi) = \gamma^\mu\psi\gamma_1\gamma_2(\partial_\mu\psi^*)\psi(\psi\psi^*)^{-1}. \tag{5.93}$$

From $RR^* = R^*R = 1$ it follows that $(\partial_\mu R)R^* + R(\partial_\mu R^*) = 0$, and Hestenes (see Hestenes 1991) writes

$$\partial_\mu R = \tfrac{1}{2}\Omega_\mu R, \tag{5.94}$$

where $(\Omega_\mu + \Omega^*_\mu = 0 \text{ or } \Omega^*{}_\mu = -\Omega_\mu)$

$$\Omega_\mu = 2(\partial_\mu R)R^*. \tag{5.95}$$

Since we have supposed ρ and β constant, (5.94) can be written as

$$\partial_\mu\psi = \tfrac{1}{2}\Omega_\mu\psi, \tag{5.96}$$

using $\partial_\mu R^* = \frac{1}{2}R^*(R\partial_\mu R^*)$. If we introduce (5.96) into (5.93) and define the intensive, structural, multi-vector bilinear, 2-form S as

$$S \equiv \frac{\hbar}{2}R\gamma_1\gamma_2R^* = \frac{\hbar}{2}\psi\gamma^1\gamma^2\psi^{-1}, \tag{5.97}$$

where the constant $\hbar$ is identified by RV with the (reduced) Planck constant, then (5.92) acquires the relevant, (non-linear) form:

$$-\frac{1}{\hbar}S\Omega_\mu = R\gamma_1\gamma_2R^*R(\partial_\mu R^*),$$

an intensive or structural relation which, because $\{b, \rho\}$ = const., can be written

$$\Box\psi\gamma_1\gamma_2 - \frac{1}{\hbar}\gamma^\mu S\Omega_\mu\psi = 0. \tag{5.98}$$

RV point out that (5.98) is an interesting result: it is equivalent to the free Maxwell equations (5.91) under the above assumptions. The Daviau covariant S corresponds to spin.

Given that both S and Ω_μ are 2-forms, the product $S\Omega_\mu$ in (5.98) results in the sum of a scalar, a 2-form and a pseudo-scalar that is

$$S\Omega_\mu = -\rho_\mu + E_{\mu,\alpha\beta}(\gamma^\alpha \wedge \gamma^\beta) + \gamma^5 r_\mu, \tag{5.99}$$

where ρ_μ, $E_{\mu,\alpha\beta}$ and r_μ are scalars. Consider as a guide the counterpart of an electron at rest in the Dirac algebraic equation, then $R = e^{-\gamma_1\gamma_2 m_o c^2 i/\hbar}$ and ($R^* = R = e^{+\gamma_1\gamma_2 m_o c^2 t/\hbar}$ $S = (\hbar/2)\, R_{\gamma_1\gamma_2}\tilde{R} = (\hbar/2)\gamma_{12}$) also $\Omega_0 = 2(\partial_o R)R^* = 2m_o c\gamma_1\gamma_2/\hbar$; $\Omega_i = 0, i = 1,2,3$ and $S\Omega_o = m_o c$, $S\Omega_i = 0$ or $\hbar\partial\psi\gamma_{12} - m_o c\psi\gamma^o = 0$.

Then the product $S\Omega_\mu$ reduces to only one scalar constant and (5.98) becomes the Dirac equation, the same is true if a more general R is used than that of an electron at rest. The more general analysis of RV follows: Let us first suppose that $S\Omega_\mu$ possesses only a scalar part then

$$\gamma^\mu S\Omega_\mu = -\rho_\mu\gamma^\mu \equiv -\rho. \tag{5.100}$$

Now, given the velocity field u, defined as

$$v \equiv R\gamma^0 R^*, \tag{5.101}$$

so that $\rho v = \psi\gamma^0\psi^*$, let us define the mass m in such a way that

$$\rho\psi \equiv e^{\beta\gamma^5} mcv\psi, \tag{5.102}$$

to obtain

$$\rho\psi = mc\psi\gamma^0. \tag{5.103}$$

When we insert (5.100) into (5.98) and then use (5.103), we eventually end up with a *linear* equation:

$$\Box\psi\gamma_1\gamma_2 + \frac{mc}{\hbar}\psi\gamma^0 = 0, \tag{5.104}$$

for a free particle (electron).

At this point RV go beyond the 1980 Campolattaro analysis and introduce what is, in fact, a mathematical basis to consider interaction fields. In fact, the equations below could also be derived by gauging. Campolattaro considered this extension in Campolattaro (1990a and b). Now RV go on assuming b, ρ and β to be still constant, but instead of supposing that $S\Omega_\mu$ possesses only a scalar part they use its general expression, (5.99):

$$E_{\mu,\alpha\beta}\gamma^\mu(\gamma^\lambda \wedge \gamma^\beta) = -\frac{e}{c}A_\mu\gamma^\mu - \gamma^5\frac{g}{c}B_\mu\gamma^\mu = -\frac{e}{c}A - \gamma_5\frac{g}{c}B, \tag{5.105}$$

where A_μ and B_μ are

$$\frac{e}{c}A_\mu \equiv \eta_{\nu\sigma}E_{\nu,\mu\sigma}, \tag{5.106}$$

$$\frac{g}{c}A_\mu \equiv \eta_{\mu\nu}\epsilon_{\nu\sigma\rho\tau}E_{\sigma,\rho\tau}, \tag{5.107}$$

with $\in_{\nu\sigma\rho\tau}$ = +1(–1) for even (odd) permutations of [0, 1, 2, 3] while $\in_{\nu\sigma\rho\tau}$ = 0 when two indices are equal. Here A and $\gamma^5 B$ play the role of electromagnetic potentials when *external* (electromagnetic) fields are present: the potentials associated to an electric charge e and a magnetic monopole $\gamma^5 g$, respectively. Here we can recall, as we have mentioned above (see Sutherland (1989) [170]) that there is a gauge for the Dirac equation in which the mass term disappears explicitly. They now define, in analogy with (5.102),

$$r\psi \equiv e^{\beta\gamma^5}\mu c v\psi = \mu c\psi\gamma^0, \tag{5.108}$$

where $r = r_\mu\gamma^\mu$, then (5.98) assumes the form:

$$\Box\psi\gamma_1\gamma_2 + (m + \gamma_5\mu)\frac{c}{\hbar}\psi\gamma^0 + (eA + \gamma_5 gB)\frac{1}{\hbar c}\psi = 0. \tag{5.109}$$

A later analysis shows that ρ and β are constants only if A and B are also constants in (5.105).

Campolattaro, Daviau, Vaz and Rodrigues approaches. Since the early work of Fock and Ivanenko it is clear that spinor formulations and multi-vector spinor formulations can be mapped into each other, although they are not strictly equivalent, as we have discussed above in the previous chapters. Vaz and Rodrigues (1993) have recently shown the faithfulness of the mapping, and we present here the main steps they followed. Vaz and Rodrigues (1993) wrote in detail the proof of the equivalence between the non-linear (multi-vector) spinor equation for the Maxwell field and the equivalent equation derived by Campolattaro for the Dirac equation to transform it into a Maxwell-like equation, which are in fact different because Campolattaro uses Ψ, which is the Dirac column spinor, and Vaz and Rodrigues (Daviau) use an (algebraic) matrix spinor with four columns. The analysis of Vaz and Rodrigues goes as follows:

a. They write the Dirac spinor in the column matrix representation

$$\mathbf{Y} = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}, \tag{5.110}$$

$$\bar{\Psi} = (\bar{\psi}_1, \bar{\psi}_2, -\bar{\psi}_3, -\bar{\psi}_4,). \tag{5.111}$$

b. They substitute into the Campolattaro equation

$$\gamma^\mu \partial_\mu \Psi = -i\gamma^\mu \frac{e^{\gamma^5 \alpha}}{\rho} \{ Im(\partial_\mu \bar{\Psi}\Psi) - \gamma_5 Im(\partial_\mu \bar{\Psi}\gamma_5 \Psi) \} \Psi, \tag{5.112}$$

and then they transform that equation into an equation for column matrices

$$\begin{aligned} \frac{-i}{2}\xi \equiv \mathrm{Im}(\partial\bar{\Psi}\Psi) = \frac{-i}{2}[&(\partial\bar{\psi}_1\psi_1 - \bar{\psi}_1\partial\psi_1) + (\partial\bar{\psi}_2\psi_2 - \bar{\psi}_2\partial\psi_2) \\ &- (\partial\bar{\psi}_3\psi_3 - \bar{\psi}_3\partial\bar{\psi}_3) - (\partial\bar{\psi}_4\psi_4 - \bar{\psi}_4\partial\bar{\psi}_4)], \end{aligned} \tag{5.113}$$

$$\begin{aligned} \frac{-1}{2}\eta \equiv \mathrm{Im}(\partial\bar{\Psi}\gamma_5\Psi) = \frac{-1}{2}[&(\partial\bar{\psi}_1\psi_3 + \bar{\psi}_3\partial\psi_1) + (\partial\bar{\psi}_2\psi_4 + \bar{\psi}_4\partial\psi_2) \\ &- (\partial\bar{\psi}_3\psi_1 - \bar{\psi}_1\partial\psi_3) - (\partial\bar{\psi}_4\psi_2 + \bar{\psi}_2\partial\psi_4)], \end{aligned} \tag{5.114}$$

to obtain

$$\{\mathrm{Im}(\partial_\mu\bar{\Psi}\Psi) - \gamma_5\,\mathrm{Im}(\partial_\mu\bar{\Psi}\gamma_5\Psi)\}\Psi = \frac{-i}{2} \quad \xi \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} - \eta \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \tag{5.115}$$

Thus equation (5.112) is explicitly

$$\gamma^\mu\partial_\mu \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} = -\frac{1}{2}\gamma^\mu \frac{e^{\gamma^5\alpha}}{\rho} \begin{pmatrix} \xi\psi_1 + \eta\psi_3 \\ \xi\psi_2 + \eta\psi_4 \\ \xi\psi_3 + \eta\psi_1 \\ \xi\psi_4 + \eta\psi_2 \end{pmatrix} = -\frac{1}{2}\gamma^\mu \frac{e^{\gamma^5\alpha}}{\rho} \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix}. \tag{5.116}$$

c. For the non linear equation of Vaz and Rodrigues

$$\gamma^\mu\partial_\mu\psi\gamma_1\gamma_2 + \mathcal{F}(\psi) = 0, \tag{5.117}$$

with

$$\mathcal{F}(\psi) = \gamma^\mu\psi\gamma_1\gamma_2(\partial_\mu\psi^*)\psi(\psi^*\psi)^{-1}, \tag{5.118}$$

they write the ψ in full

$$\psi = \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix}, \quad \psi^* = \begin{pmatrix} \bar{\psi}_1 & \bar{\psi}_2 & -\bar{\psi}_3 & -\bar{\psi}_4 \\ -\psi_2 & \psi_1 & -\psi_4 & \psi_3 \\ -\bar{\psi}_3 & -\bar{\psi}_4 & \bar{\psi}_1 & \bar{\psi}_2 \\ -\psi_4 & \psi_3 & -\psi_2 & \psi_1 \end{pmatrix}, \tag{5.119}$$

and by direct substitution they obtain a 4 x 4 matrix equation

$$\gamma^\mu \partial_\mu \psi = -\tfrac{1}{2}\gamma^\mu \frac{e^{-\gamma^5\beta}}{\rho}\chi, \tag{5.120}$$

or, in terms of components,

$$\gamma^\mu \partial_\mu \begin{pmatrix} \psi_1 & -\bar{\psi}_2 & \psi_3 & \bar{\psi}_4 \\ \psi_2 & \bar{\psi}_1 & \psi_4 & -\bar{\psi}_3 \\ \psi_3 & \bar{\psi}_4 & \psi_1 & -\bar{\psi}_2 \\ \psi_4 & -\bar{\psi}_3 & \psi_2 & \bar{\psi}_1 \end{pmatrix} = -\frac{1}{2}\gamma^\mu \frac{e^{-\gamma^5\beta}}{\rho} \begin{pmatrix} \chi_1 & -\bar{\chi}_2 & \chi_3 & \bar{\chi}_4 \\ \chi_2 & \bar{\chi}_1 & \chi_4 & -\bar{\chi}_3 \\ \chi_3 & \bar{\chi}_4 & \chi_1 & -\bar{\chi}_2 \\ \chi_4 & -\bar{\chi}_3 & \chi_2 & \bar{\chi}_1 \end{pmatrix} \tag{5.121}$$

Comparison of (5.116) with (5.121) shows that every column of (5.121) obeys the Campolattaro equation. But it should be stressed that the ψ of Daviau is a multi-vector, and as such it has additional geometrical content.

The mapping of the Daviau–Vaz–Rodrigues F into the Campolattaro $F^{12} = \bar{\Psi}\gamma^{12}\Psi$ is not reduced to what was presented in (5.111-5.121) above, because $\psi = \Sigma_i \Psi_i \eta_i^\dagger$ as discussed in IV and then

$$F = \left(\sum_j \eta_j \bar{\Psi}_j\right) \gamma^{12} \left(\sum_i \Psi_i \eta_i^\dagger\right) I \tag{5.122}$$

and then, besides the $\bar{\Psi}_i \gamma^{12} \Psi_i$ terms another 12 terms $\bar{\Psi}_j \gamma^{12} \Psi_i$, $i \neq j$ are to be considered for the construction of the multi-vector F. We repeat, the multi-vector F contains more information than the minimum needed to satisfy the Campolattaro equations, equivalent to the Dirac equation by construction. As presented above the F is equivalent to the multi-vector solution of the Dirac equation.

In an interesting extension of their analysis Rodrigues, Vaz and Recami (1993) continue the study of $\Box F = 0$ when the canonical form $\psi = be^{\gamma^5\beta/2}R'$ is not restricted.

RVR pass to the most general case, by *eliminating* the restriction that b is constant, to obtain:

$$\Box\psi\gamma_1\gamma_2 + \mathcal{F}(\psi) = -\Box(\ln b)\psi\gamma_1\gamma_2, \tag{5.123}$$

which *generalizes* the non-linear equation (5.92), letting ρ and β be variables here, they

$$\partial_\mu \psi = [\partial_\mu \ln(\rho e^{\beta\gamma_5})^{1/2}]\psi + \tfrac{1}{2}\,\Omega_\mu \psi, \tag{5.124}$$

which, using (5.97), results in:

$$\Box\psi\gamma_1\gamma_2 - \frac{1}{\hbar}\gamma^\mu S\Omega_\mu\psi = -(\Box \ln b)\psi\gamma_1\gamma_2 - [\Box \ln(\rho e^{\beta\gamma_5})^{1/2}]\psi\gamma_1\gamma_2. \tag{5.125}$$

RVR rewrite the left hand side of (5.125) in order to have

$$\left[\Box R\gamma_1\gamma_2 - \frac{1}{\hbar}\gamma^\mu S\Omega_\mu R\right](\rho e^{\beta\gamma_5})^{1/2} = -(\Box \ln b)\psi\gamma_1\gamma_2 - 2[\Box \ln(\rho e^{\beta\gamma_5})^{1/2}]\psi\gamma_1\gamma_2. \tag{5.126}$$

The left hand side of (5.126) vanishes once R is required to satisfy (5.98) (which was written in terms of ψ because ρ, β were there supposed to be constants) we must then have [K being a constant]

$$\Box \ln b = -2\Box \ln(\rho e^{\beta\gamma_5})^{1/2} \Longleftrightarrow b = \frac{K}{\rho e^{\beta\gamma_5}}, \tag{5.127}$$

which implies that F is proportional to $R\gamma_1\gamma_2 R^*$. Equation (5.127), therefore, implies a (non-null) constant field F. Notice, incidentally, that in (5.127) it must be either β = 0 or β = π, since b is a *scalar*, and this is a consequence of supposing R to obey the Dirac algebraic equation.

Putting (5.127) into (5.125), RVR finally obtain the generalized spinor equation

$$\Box\psi\gamma_1\gamma_2 - \frac{1}{\hbar}\gamma^\mu S\Omega_\mu\psi = (\Box \ln\psi_0)\psi\gamma_1\gamma_2, \tag{5.128}$$

with $\psi_0 \equiv (\rho e^{\beta\gamma_5})^{1/2}$. And again, if $S\Omega_\mu$ has only a scalar part, (5.128) can be written —according to our previous discussion— as

$$\Box\psi\gamma_1\gamma_2 + \frac{mc}{\hbar}\psi\gamma_0 = (\Box \ln\psi_0)\psi\gamma_1\gamma_2, \tag{5.129}$$

which is a (non-linear) generalized Dirac equation. RVR show that if one applies the Dirac operator □ to the above equation one obtains [$\Box^2 \equiv \partial^\mu\partial_\mu$]

$$\Box^2\psi + \left(\frac{mc}{\hbar}\right)^2\psi = \frac{\Box^2\psi_0}{\psi_0}\psi + W\psi, \tag{5.130}$$

where

$$W\psi = \eta^{\mu\nu}(\partial_\nu \ln \psi_0)\Omega_\mu \psi. \tag{5.131}$$

The term W_ψ can be easily calculated in the rest frame. The result, is that it vanishes, i.e.,

$$W\psi = 0. \tag{5.132}$$

Then (5.130) assumes the interesting, simple form

$$\Box^2\psi + \left(\frac{mc}{\hbar}\right)^2 \psi = \frac{\Box^2\psi_0}{\psi_0}\psi. \tag{5.133}$$

This is a *non-linear* Klein–Gordon equation, which exactly coincides with the equation proposed by Gueret and Vigier (1982), and possesses *localized, non-dispersive* solutions. The term $\Box^2\phi_0/\phi_0$ is usually called the 'quantum potential'. Following Gueret and Vigier (1982) and also Mackinnon (1981) one can propose

$$\Box^2\phi_0 = \left(\frac{mc}{\hbar}\right)^2 \phi_0, \tag{5.134}$$

and then

$$\Box^2\phi = 0, \tag{5.135}$$

which is just the case discussed by Mackinnon (1981). In particular, (5.135) admits a non-trivial solution, representing a non-dispersive *soliton* (localized wave–packet) which moves *undeformed* with subluminal speed.

Let us recall here that in 1915 Bateman had looked for 'solitonic' solutions of Maxwell equations.

RVR further notice that when we replace $S\Omega_\mu$ in (5.128) by its full expression (5.99), containing a scalar, a 2-form, and a pseudo-scalar part, (5.128) then its most general form

$$\Box\psi\gamma_1\gamma_2 + (m + \gamma_5\mu)\frac{c}{\hbar}\psi\gamma^0 + (eA + \gamma_5 gB)\frac{1}{\hbar c}\psi = (\Box \ln \psi_0)\psi\gamma_1\gamma_2, \tag{5.136}$$

Actually, (139) is a particular case of (153), valid when the non-linear term $(\partial \ln \psi_0)\psi\gamma_1\gamma_2$ can be neglected.

Maxwell-like bi-vectors. Given an electromagnetic field F, from the Rainich (Misner–Wheeler) theorem, it is deduced that

$$F = b\psi\gamma_1\gamma_2\psi^*, \tag{5.137}$$

where ψ is a multi-vector spinor field (whose canonical form is $\psi = \rho^{1/2} e^{\gamma_5 \beta/2} R$). Rodrigues and Vaz showed that even supposing b, ρ and β to be *not* constant, correctly represents the electromagnetic field, the field solves the *free* Maxwell equations (without sources) $\boxed{F} = 0$.

On the other hand, F can be written

$$F = K \Phi \gamma_1 \gamma_2 \Phi^*, \tag{5.138}$$

where $b \equiv K/\rho e^{\beta\gamma_5}$ and where $\Phi \equiv R A_o(t)$ is a Dirac spinor field which satisfies the (linear) Dirac equation.

Those spinor fields are then related by the rest mass factor; the reader should be reminded here that for an electron there is at every point x a factor m_0 and a spin in $\hbar i \gamma_{12}/2$ as a structural part, then both are acceptable operators, also $\gamma_{12} \Phi = \pm i \Phi$ and $\pm i m_0 \Phi = \hbar \gamma_{12} \Phi$,

$$A_o(t) = e^{\gamma^{12} m_o c^2 t/\hbar}, \tag{5.139}$$

and in particular, for the 'electron solutions' (i.e.,for $\beta = 0$), by:

$$\psi = \rho^{1/2} \Phi, \tag{5.140}$$

which coincides, as shown by RV, with a well known expression in de Broglie's theory of double solution. (For the positon one would obtained $\psi = -\gamma_5 \rho^{1/2} \Phi$). Let us recall that in such as de Broglie's theory ψ was the electron (total) quantum-probabilistic wave-function, but the multi-vector Φ (which obeys the Dirac equation) was a physical wave.

An important remark of the RV approach, is that the multi-vector spinor field Φ is related to the *rotor* part of ψ and therefore has only 6 degrees of freedom, the same as the electromagnetic field F (while ψ possesses 8 degrees of freedom) and, in this case, the spinor field Φ could ultimately be analyzed having an electromagnetic nature, as suggested by equations (5.137) and (5.140). For instance, the basic equation (5.90) or (5.137) shows the strict relation existing between the non-null electromagnetic field F (present in the absence of sources) and the electron wave-function ψ.

They conclude that many of de Broglie's ideas concerning the interpretation of quantum mechanics should be seriously **revisited,** whilst the language of Clifford algebras appears to be particularly convenient for that purpose. In this and the previous chapter we have carried out this analysis.

Chapter 6

ALGEBRAIC ANALYSIS OF THE ELECTRON THEORY

1. THE ALGEBRAIC SOLUTION OF THE DIRAC EQUATION

We have mentioned that we are using the analysis of all our quantities within either the geometry of space–time, ST, or the geometry of the space–time–action space, STA. Both are defined as spaces with a quadratic form, this induces a multi-vector geometry and a particular form of geometric analysis (see next chapter and[128] and references therein). This is often overlooked, even if used in practice. The main reason is the immediate use of a representation of the algebra and the analysis through the use of the 32 Dirac matrices (or in terms of the Pauli matrices, or an indice notation arising in general from the Pauli algebra and its spinors).

We briefly review here the algebraic solution to the Dirac equation. We follow for example the textbook of Flügge [59].

We want to solve the Dirac equation ($\lambda = 0,1,2,3$)

$$i\hbar\gamma'_\lambda\partial^\lambda\bar{\psi}(z)\gamma'_0\psi = m_0c\psi, \tag{6.1}$$

gauged to represent the presence of an external potential $V(z)$. Define $\varkappa = m0c/\hbar$ to rewrite

$$\left(i\gamma_\mu\partial^\mu + \frac{V(z)\gamma_0}{\hbar c}\right)\psi = \varkappa\psi. \tag{6.2}$$

The solutions can be written

$$\psi(x,y,z,t) = e^{-iEt/\hbar}u(z)f(x,y), \tag{6.3}$$

obeying

$$i\gamma_3\frac{du}{dz}+\gamma_0 Q(z)u+\varkappa u=0, \qquad Q(z)=\frac{V(z)-E}{\hbar c}, \tag{6.4}$$

and $f(x, y)$ arbitrary. Because (6.4) only contains γ_3 and γ_0, we can use the multi-vector function

$$v(z)=A(z)+B(z)\gamma_3+C(z)\gamma_0+D(z)\gamma_3\gamma_0, \tag{6.5}$$

to find a solution in which

$$u=v\Gamma, \tag{6.6}$$

where Γ is any constant multi-vector. Defining the multi-vector σ_z commuting with γ_3 and γ_0

$$\sigma_z=-i\gamma_1\gamma_2, \tag{6.7}$$

the solution u can be an eigenspinor of σ_z

$$\sigma_z u=\sigma_z v\Gamma=v\sigma_z\Gamma, \tag{6.8}$$

where

$$\sigma_z\Gamma=\pm\Gamma, \quad \sigma_z u=\pm u. \tag{6.9}$$

The two possible solutions are

$$u(z)=v(z)(1\mp i\gamma_1\gamma_2), \tag{6.10}$$

and for a given V (z) we have to solve the coupled set of equations

$$\begin{aligned} -iu_3'+(Q+\varkappa)u_1&=0,\\ iu_4'+(Q+\varkappa)u_2&=0,\\ iu_1'+(\varkappa-Q)u_3&=0,\\ -iu_2'+(\varkappa-Q)u_4&=0, \end{aligned} \tag{6.11}$$

with the definitions

$$\begin{aligned} w_1&=\frac{1}{2}(B-D), & w_3&=\frac{1}{2}(A-C),\\ w_2&=\frac{1}{2}(A+C), & w_4&=\frac{1}{2}(B+D), \end{aligned} \tag{6.12}$$

and

$$u_1=w_1, \quad u_3=iw_3, \quad u_2=iw_2, \quad u_4=w_4, \tag{6.13}$$

then

$$v(z) = (w_2 + w_4\gamma_3)(1+\gamma_0) + (w_3 + w_1\gamma_3)(1-\gamma_0), \tag{6.14}$$

with current and density for the projection

$$a) \qquad u(z) = (w_3 + w_1\gamma_3)(1-\gamma_0)(1-i\gamma_1\gamma_2), \tag{6.15}$$

given by

$$\begin{aligned} s_3 &= ec\,(u_1^* u_3 + u_3^* u_1)\,\Gamma, \\ \rho &= e\left(|u_1|^2 + |u_3|^2\right)\Gamma, \\ \overline{u}u &= \left(|u_1|^2 - |u_3|^2\right)\Gamma, \end{aligned} \tag{6.16}$$

and three similar results for the linearly independent combinations

$$\begin{aligned} &b) \quad u(z) = (w_3 + w_1\gamma_3)(1+\gamma_0)(1+i\gamma_{12}), \\ &c) \quad u(z) = (w_3 + w_1\gamma_3)(1-\gamma_0)(1-i\gamma_{12}), \\ &d) \quad u(z) = (w_3 + w_1\gamma_3)(1+\gamma_0)(1-i\gamma_{12}). \end{aligned} \tag{6.17}$$

The solutions a) and b) have mutually cancelling constraints, the same as the pair of solutions c) and d). The set of solutions is complete, then any other solution can be written as a linear combination of this set of solutions.

For the purpose of our description in the text we can now consider a sum of two spinors solutions, say *a*) + *b*), but with $V^{(a)}(z) = -V^{(b)}(z)$ and $V^{(a)}(z) = V^{(a)}(z; \psi a, \overline{\psi} a)$ reciprocally $V^{(b)}(z) = V^{(b)}(z; \psi_a, \overline{\psi}_a)$, to describe now a pair of interacting carriers. The system not being totally defined until some distribution $\rho_{(i)}(x, y, z, t)$ is given and $V(z)$ can be properly defined. Externally, for each pair, the system will appear as a constrain free, but stable, distribution.

2. TWISTORS AND CARRIERS

We use a twistor approach starting frorn basic definitions to show both the twistor content of geometric analysis and the structure of gauged field solutions in the electromagnetic case.

See the next chapter for a formal presentation of twistors in terms of multi-vectors.

2.1 INTRODUCTION

One of the comprehensive developments of the idea of spinorization of the phase space has been achieved in twistor theory [147]. Using

the condition of masslessness the twistor approach established a deep relation between phase space variables of massless spinning particles and spinor wave functions. In this formalism the equations of motion of massless particles have been transformed into algebraic equations over oscillator ladder operators of the helicity.

The success of the twistorial description of massless particles has inspired several authors to apply this tool also to the construction of a twistorial representation of the phase space of massive spinning particles. Dirac's bispinors contain two independent Weyl spinors, and in that sense keeps twice as much information. This induces the idea [178, 179, 180, 15, 16, 17] of constructing a dynamics of massive spinning systems using pairs of twistors corresponding to (two) massless particles. It turns out that (a double) phase space of classical massless object may serve as a building block for the construction of the irreducible phase space of a massive spinning particle. In some sense the massive spinning particle obtained by the reduction procedure may be regarded as a bound (confined) system of two directly interacting massless spinning constituents.

The development of this approach in the quantum case requires the transformation of the Dirac equation into an algebraic system of equations over twistor variables. In the best approach to this program one has to exchange the mass parameter in the Dirac equation with two mutually complex conjugated values. This complex value arises as a simple consequence of the bispinor representation of momentum in the Dirac equation. The same result is obtained using the bispinor representation of the momentum in the basis of Dirac gamma matrices, in that case starting from Pauli–Fierz identities, we obtain a Dirac like equation in six dimensional momentum space, two components corresponding to the complex mass parameter. (Let us note that these components accept another interpretation if bispinors are substituted by twistor coordinates [119]).

In this section we examine both representations: the spinorial with well known van der Waerden symbols; and the vectorial expressed in terms of Dirac gamma matrices. In this way the bridge between the twistorial description of massive particle and the concept of screws (introduced in [118, 121]) is established.

In Section 2, for the convenience of the reader, we simply summarize the assumptions and definitions used in the twistor's theory of massless, spin $\frac{1}{2}$ carriers.

In Section 3 we build the twistor phase space for the massive spin $\frac{1}{2}$ carriers. We show that the twistor reprcscntation of momentum de-

mands a modification of the Dirac equation introducing a complex mass parameter.

In Section 4 the equation of motion of Lorentz–Bargmann–Michel–Telegdi in twistorial phase space deduced in full detail.

Several parts at the beginning of our presentation should be considered as an analysis and comparison of otherwise known formulations (see for example [122]).

2.2 MASSLESS CARRIERS

For the convenience of the reader we simply summarize the assumptions and definitions used in the twistor theory of massless fields. In the Minkowski space–time a particle M is described by the total four-momentum p^μ, total angular momentum $M^{\mu\nu}$ and its helicity s. Here, for practical purposes, the indices for space–time coordinates are Greek or lower Latin letters $\{\mu, \nu, c, d = 0, 1, 2, 3\}$, the indices for two component spinors are capital Latin letters $\{A, B, C = 1, 2\}$. Under the action of the Poincaré group the components of the pair $(p^\mu, M^{\mu\nu})$ transform covariantly. Regardless of whether a particle possesses mass or not, it will be called point-like if we have a splitting

$$M^{\mu\nu} = 2x^{[\mu}p^{\nu]} + S^{\mu\nu}, \tag{6.18}$$

with x^μ representing its position and with $S^{\mu\nu}$ representing intrinsic spin, such that

$$S^{\mu\nu}p_\nu = 0. \tag{6.19}$$

The masslessness of a system is expressed by the property

$$m^2 := p^\mu p_\mu = 0. \tag{6.20}$$

Using these relations we obtain that the Pauli–Lubanski four-vector

$$S_\mu = -\tfrac{1}{2}\epsilon_{\mu\nu cd}M^{\nu c}p^d. \tag{6.21}$$

is proportional to p^μ $S^\mu = sp_\mu$, with the factor of proportionality s being the helicity of the massless object. Equations (6.18–6.21) also imply that if $s \neq 0$ then x^μ may be taken as any point on a null hyperplane defined by $x^\mu p_\mu = d$, where d is a translationally dependent Lorentz scalar.

To obtain a twistor formulation of these relations corresponds to finding their (Weyl) spinor structure. The real four-momentum satisfying the zero rest mass condition (6.20) can be expressed as matrix factorization of the Hermitian matrix

$$P_{AA'} := p_\mu \sigma^\mu_{AA'} = \pi_A \pi_{A'}, \tag{6.22}$$

where π_A is the usual Weyl spinor. Thus the spinor π_A carries the information on the momentum of massless particles. To give a spinor representation of the six-angular momentum one needs an additional spinor ω^A carrying the information of space-time coordinates. It is defined as

$$\omega^A := -ix^\mu \sigma_\mu^{AA'} \pi_{A'} = -iX^{AA'} \pi_{A'}, \tag{6.23}$$

where ω^A is also a four-translation dependent Weyl spinor. Then the skew-symmetric angular momentum $M^{\mu\nu}$ is defined as ($\in^{AB}$ being the spinor metric)

$$M^{AA'BB'} = -i(\epsilon^{A'B'}(\omega^A \overline{\pi}^B + \omega^B \overline{\pi}^A) - \epsilon^{AB}(\overline{\omega}^{A'} \pi^{B'} + \overline{\omega}^{B'} \pi^{A'})). \tag{6.24}$$

Its dual, $M_{\mu\nu} = \frac{1}{2}\eta_{\mu\nu cd} M^{cd} = *M_{\mu\nu}$, can be written

$$M_{\mu\nu} \Rightarrow *M_{AA'BB'} = \epsilon_{A'B'}(\omega_A \overline{\pi}_B + \omega_B \overline{\pi}_A) + \epsilon_{AB}(\overline{\omega}_{A'} \pi_{B'} + \overline{\omega}_{B'} \pi_{A'}). \tag{6.25}$$

The quantities s and d also may be consistently expressed as functions of ω^A and π_A

$$s = \tfrac{1}{2}(\omega^A \overline{\pi}_A + \pi^{A'} \overline{\omega}_{A'}), \qquad d = \frac{i}{2}(\omega^A \overline{\pi}_A - \pi^{A'} \overline{\omega}_{A'}). \tag{6.26}$$

The twistor Z^α is directly defined as the mathematical object with components

$$Z^\alpha := (\omega^A, \pi_{A'}),$$

represented in the complexified Minkowski space–time *CM* picture as the locus of all complex points $X^{AA'}$. When the solution $X^{AA'}$ of (2.6) lies on a real null plane, Z_α is a null twistor. Twistor indices are raised and lowered by transposition $(\ )^T$ and complex conjugation: Z^α twistor has the simple conjugate Z^α of components

$$Z_\alpha = (\overline{\pi}_A, \overline{\omega}^{A'})^T,$$

so that contractions, of which the simplest representative is $Z^\alpha Z_\alpha$ are invariant under twistor transformations. This invariant has the form

$$Z^\alpha Z_\alpha = \overline{\pi}_A \omega^A + \pi_{A'} \overline{\omega}^{A'}. \tag{6.27}$$

Comparing (6.27) with (6.26) we find that the twistor norm expresses twice the value of helicity

$$s = \tfrac{1}{2} Z^\alpha Z_\alpha.$$

Then in twistor formalism the classical and quantum motions of a free massless and spinning relativistic particle may be described as a direct product of two Weyl spinors endowed with natural symplectic structure. This structure in twistor coordinates is given by

$$\Omega = idZ^{\alpha} \wedge dZ_{\alpha}.$$

The relations (6.22-6.25) express the transformation of momentum and angular momentum under the map $(x^{\mu},p_{\nu}) \rightarrow (\omega^{A},\pi_{A'}) = (Z^{\alpha})$ at the algebraic level. The Poincaré-covariance of the pair $(\omega^{A}, \pi_{A'})$ implies the Poincaré-covariance of the pair $(p_{\mu}, M^{\mu\nu})$. Let us now treat (ω^{A}, π_{A}') as the Poincaré-covariant coordinates of a point in four-dimensional complex vector space *T*. *T* is a symplectic vector space equipped with global Poincaré-covariant canonically conjugate coordinates: $-i\omega^{A}$ and $\overline{\pi}_{B}$ or, equivalently, $-i\pi_{A}'$ and $\overline{\omega}^{B'}$ fulfilling the following Poincaré-covariant canonical Poisson algebra

$$\{\omega^{A},\overline{\pi}_{B}\} = \delta^{A}_{B}, \qquad \{\pi^{A'},\overline{\omega}_{B'}\} = \delta^{A'}_{B'}. \tag{6.28}$$

Within the definitions (6.22–6.24) it is a straightforward tedious task to check that the canonical Poincaré-covariant Poisson bracket relations imply the Poincaré-covariant Poisson bracket realization of the algebra

In the twistor formalism the equations of motion for massless fields accept the form of algebraic equations, given in the language of creation and annihilation operators (represented by twistor variables). The Weyl equation for a free massless particle for spin $\frac{1}{2}\hbar$ is given by

$$P^{AA'}\pi_{A}\Phi = 0, P^{AA'}\pi_{A'}\Phi = 0,$$

where Φ is the holomorphic function of ω^{A}.

2.3 MASSIVE CARRIERS

We have seen in the previous section that the twistor description of the system is immediately available if the components of momentum satisfy the masslessness condition. One should define a similar fundamental object for the description of massive systems as well. When the spinor structure of the momentum is realized the condition of masslessness is transformed into Weyl algebraic identities. With the same aim one can obtain a bispinor representation of the momentum of massive particles, satisfying the relativistic relation between momentum and mass, if the Weyl algebraic identities may be extended for the massive Dirac equations.

Let us examine the momentum structure of the massive particle for the Dirac equation

$$(\gamma^{\mu}p_{\mu})\Psi = m\Psi.$$

Employing the chiral basis of Dirac γ-matrices where γ_5- diagonal in their van der Waerden description, we have considered as representations of basis vectors for space–time

$$\gamma_\mu = \begin{pmatrix} 0 & (\sigma_\mu)^{AA'} \\ (\sigma_\mu)_{BB'} & 0 \end{pmatrix},$$

with $(\sigma_\mu)_{AA'}$ the van der Waerden symbols, corresponding to Pauli matrices

$$(\sigma_0)_{AA'} = I, \qquad (\sigma_k)_{AA'} = \sigma_k, \qquad k = 1, 2, 3.$$

The Dirac bispinor Ψ over Weyl spinors is given by

$$\Psi = \begin{pmatrix} u_A \\ \overline{v}^{B'} \end{pmatrix}, \qquad \overline{\Psi} = \begin{pmatrix} v^B, & \overline{u}_{A'} \end{pmatrix}$$

Therefore in that description the Dirac equation becomes equivalent to the set of two equations

$$P_{AA'}u^{A'} = m\overline{v}_A, \tag{6.29}$$

$$P^{AA'}\overline{v}_A = mu^{A'}, \tag{6.30}$$

or

$$P_{AA'}v^{A'} = -m\overline{u}_A, \tag{6.31}$$

$$P^{AA'}\overline{u}_A = -mv^{A'}, \tag{6.32}$$

where

$$P_{AA'} = p_\mu \sigma^\mu_{AA'}. \tag{6.33}$$

From these equations we obtain

$$P^{AA'}(\overline{v}_A\overline{u}^B - \overline{u}_A\overline{v}^B) = m(u^{A'}\overline{u}^B + v^{A'}\overline{v}^B), \tag{6.34}$$

or

$$P^{AA'} = \frac{m}{(\overline{v}_C\overline{u}^C)}(u^{A'}\overline{u}^B + v^{A'}\overline{v}^B). \tag{6.35}$$

In (6.35) the momentum of the massive particle is given in the bispinor representation which has been directly deduced from the Dirac equation. Within conventional field theory the dynamical equations of motion for the massless particles (fermions) are understood as a consequence of the limit $m \to 0$ reducing the equations of the massive particles into the

equations of massless particles. One of the primary example of such transformation is the reduction of Dirac equation onto Weyl equations. In that case by putting $m = 0$ one obtains the splitting of the Dirac equation into two independent parts, where (only) one of the parts (namely left, handed) has been accepted as an equation for the neutrino. Since the other part (right handed) for the case $m = 0$ has to disappear, it is natural to suggest, that from $m = 0$ it follows $u^A = 0$, or *vice versa* Accordingly the expression for $P^{AA'}$ has to be reduced to the formula (6.22). It may be realized if in (6.35) we put

$$m = (\text{constant})(\overline{v}_C\overline{u}^C),$$

and thus we obtain the bispinor representation of the momentum for the massive particle

$$P^{AA'} = \pi^{A'}\overline{\pi}^A + \eta^{A'}\overline{\eta}^A. \tag{6.36}$$

This representation implies that the mass is given by

$$m^2 = p^\mu p_\mu = 2f\overline{f}, \tag{6.37}$$

with $f = (\pi^{A'} \quad \eta_{A'}), \overline{f} = (\overline{\pi}^A \quad \eta_A)$.

Using the spinor representation (6.36) one may build two sets of identities, which after appropriate quantization of twistor variables has to give back the Dirac equations. The first set of equations is given by

$$p_\mu\sigma^\mu_{AA'}\pi^{A'} = f\overline{\eta}_A; \tag{6.38}$$

$$p^\mu\sigma_\mu^{AA'}\overline{\eta}_A = \overline{f}\pi^{A'}.$$

The second has the form

$$p_\mu\sigma^\mu_{AA'}\eta^{A'} = -f\overline{\pi}_A, \tag{6.39}$$

$$p^\mu\sigma_\mu^{AA'}\overline{\pi}_A = -\overline{f}\eta^{A'}.$$

It is easy to see the difference between (6.38–6.39) and the value $m/\sqrt{2}$ in (6.30–6.32): in (6.38–6.39) we have instead the conjugations of the complex value f. One may use polar representation $f = m\,e^{i\phi}$. In this approach the problem of interpretation of the angle ϕ appears. Vaz and Rodrigues [181] had observed an interesting interpretation of the values

$m\cos(\phi)$ and $m\sin(\phi)$: these values would be that of longitudinal and transversal masses, respectively.

Let two massless carriers be described by two twistor fields

$$Z^{\alpha} = (\omega^{A}, \pi_{A'}), \qquad W^{\alpha} = (\lambda^{A}, \eta_{A'}), \tag{6.40}$$

respectively. The pair of twistors (Z^{α}, W^{α}) represents global Poincaré-covariant coordinates of a point in an eight-dimensional complex symplectic vector space $T \times T$ which defines a phase space for the two massless particles. Coordinates of (ω^{A}, $p_{B'}$) fulfill its own canonical Poincaré-covariant Poisson bracket algebra introduced in (6.28) and so do the coordinates of (λ_{A}, $\eta_{A'}$).

Assume now that the two massless particles form a massive and, in general, spinning point-like system. From now on we thus treat $T \times T$ as a reducible phase space of a massive and, in general, spinning physical system. However, it is necessary to remark that the two spinors contained in the Dirac's bispinor have different natures, one belongs to a right handed and the other to a left handed basis system. In accordance with this identification we identify the linear momentum four-vector of massive particle as

$$p^{\mu} = \sigma^{\mu}_{AA'}(P_1)^{AA'} + g^{\mu\nu}\sigma_{\nu}^{AA'}(P_2)_{AA'} = p_1^{i} + p_2^{i}, \tag{6.41}$$

where p_1 and p_2 are momenta belonging to the left and right handedness massless particles, and

$$(P_1)^{AA'} = \pi^{A'}\overline{\pi}^{A}, \qquad (P_2)_{AA'} = \eta_{A'}\overline{\eta}_{A}. \tag{6.42}$$

The complex coordinate of the 4-position $Z^{AA'}$ we define as solution of the system

$$\omega^{A} = iZ^{AA'}\pi_{A'}, \qquad \lambda^{A} = iZ^{AA'}\eta_{A'}, \tag{6.43}$$

Remembering $\pi^{A}\pi_{A} = 0$, and $\eta^{A}\eta_{A} = 0$, we look for $Z^{AA'}$ in the form

$$Z^{AA'} = \alpha^{A}\pi^{A'} + \beta^{A}\eta^{A'}. \tag{6.44}$$

Substitute (6.44) into (6.43) to obtain

$$\omega^{A} = iZ^{AA'}\pi_{A'} = i\beta^{A}(\eta^{A'}\pi_{A'}), \lambda^{A} = iZ^{AA'}\eta_{A'} = i\alpha^{A}(\pi^{A'}\eta_{A'}), \tag{6.45}$$

use the definition of f given in (6.37) to find

$$\beta^{A} = -\frac{i}{f}\omega^{A}, \qquad \alpha^{A} = -\frac{i}{f}\lambda^{A}. \tag{6.46}$$

This gives the solution of (6.43) as

$$Z^{AA'} = i\frac{1}{f}(\omega^A \eta^{A'} - \lambda^A \pi^{A'}). \tag{6.47}$$

We find the real and imaginary coordinates as real and imaginary parts of $Z^{AA'}$ correspondingly. The real part is given by

$$X^{AA'} = i\frac{1}{2f}(\omega^A \eta^{A'} - \lambda^A \pi^{A'}) - i\frac{1}{2\overline{f}}(\overline{\omega}^{A'} \overline{\eta}^A - \overline{\lambda}^{A'} \overline{\pi}^A). \tag{6.48}$$

For the imaginary part we obtain

$$Y^{AA'} = \frac{1}{2f}(\omega^A \eta^{A'} - \lambda^A \pi^{A'}) + \frac{1}{2\overline{f}}(\overline{\omega}^{A'} \overline{\eta}^A - \overline{\lambda}^{A'} \overline{\pi}^A). \tag{6.49}$$

Using the canonical and covariant twistor coordinates we define the Poincaré-invariant functions according to the following recipe

$$\begin{aligned} e &= Z^\alpha Z_\alpha + W^\alpha W_\alpha = (\omega^A \overline{\pi}_A + \overline{\omega}^{A'} \pi_{A'}) - (\lambda^A \overline{\eta}_A + \overline{\lambda}^{A'} \eta_{A'}), \\ k &= Z^\alpha Z_\alpha - W^\alpha W_\alpha = (\omega^A \overline{\pi}_A + \overline{\omega}^{A'} \pi_{A'}) - (\lambda^A \overline{\eta}_A + \overline{\lambda}^{A'} \eta_{A'}), \\ f &= I_{\alpha\beta} Z^\alpha W^\beta = \pi^{A'} \eta_{A'}; \qquad a = Z^\alpha W_\alpha = (\omega^A \overline{\eta}_A + \overline{\lambda}^{A'} \pi_{A'}). \end{aligned} \tag{6.50}$$

We then see the correspondence to the Dirac wave function as the bispinors defined by

$$\overline{\Psi}_1 = \begin{pmatrix} \eta_{A'} & \overline{\pi}^B \end{pmatrix}, \Psi_1 = \begin{pmatrix} \pi^{B'} \\ \overline{\eta}_A \end{pmatrix}, \tag{6.51}$$

$$\overline{\Psi}_2 = \begin{pmatrix} \pi_{A'} & \overline{\eta}^A \end{pmatrix}, \Psi_2 = \begin{pmatrix} \eta^{B'} \\ \overline{\pi}_A \end{pmatrix},$$

from which the MEXOR, corresponding to the massive spinning particle, is constructed in [121]. One may build the basis of four orthonormal vectors given by

$$\begin{aligned} L_1^{AA'} &= \frac{1}{m}(\overline{\pi}^A \pi^{A'} + \overline{\eta}^A \eta^{A'}), \\ L_2^{AA'} &= \frac{i}{m}(\overline{\pi}^A \pi^{A'} - \overline{\eta}^A \eta^{A'}), \\ L_3^{AA'} &= \frac{i}{m}(\overline{\pi}^A \eta^{A'} - \overline{\eta}^A \pi^{A'}), \\ L_4^{AA'} &= \frac{1}{m}(\overline{\pi}^A \eta^{A'} + \overline{\eta}^A \pi^{A'}). \end{aligned} \tag{6.52}$$

And one may expand the vector $Y^{AA'}$ using the basis $\{L_1^{AA'}, L_2^{AA'}, L_3^{AA'}, L_4^{AA'}\}$, obtaining

$$Y^{AA'} = l_1 L_1^{AA'} + l_2 L_2^{AA'} + l_3 L_3^{AA'} + l_4 L_4^{AA'}, \tag{6.53}$$

the coefficients of the expansion are defined by

$$\begin{aligned} l_1 &= (L_1)_{AA'} Y^{AA'}, & l_2 &= (L_2)_{AA'} Y^{AA'}, \\ l_3 &= (L_3)_{AA'} Y^{AA'}, & l_4 &= (L_4)_{AA'} Y^{AA'}. \end{aligned} \tag{6.54}$$

Substitute $Y^{AA'}$ from (6.49) and taking into account the formulae for $\{e, k, a, \bar{a}\}$ we obtain

$$l_1 = -\frac{e}{2m}, \quad l_2 = -\frac{k}{2m}, \quad l_3 = -\frac{1}{2m}(a - \bar{a}), \quad l_4 = \frac{1}{2m}(a + \bar{a}).$$

Introducing (6.52) and (6.54) into (6.53) we get

$$Y^{AA'} = \frac{1}{m^2}\left(a\bar{\pi}^A \eta^{A'} + \bar{a}\pi^{A'}\bar{\eta}^A - \frac{e+k}{2}\eta^{A'}\bar{\eta}^A - \frac{e-k}{2}\pi^{A'}\bar{\pi}^A\right).$$

Now let us redefine the basis $\{L_1^{AA'}, L_2^{AA'}, L_3^{AA'}, L_4^{AA'}\}$ in vectorial representation via Dirac matrices. For that purpose we introduce the following projective operators

$$\Pi^\mu = \frac{1}{2m}\gamma_\mu(1 - i\gamma_5), \quad K^\mu = \frac{1}{2m}\gamma_\mu(1 + i\gamma_5).$$

Remembering the correspondence given in (3.26) we obtain the mapping $\{L_1^{AA'}, L_2^{AA'}, L_3^{AA'}, L_4^{AA'}\} \to \{l_1^\mu, l_2^\mu, l_3^\mu, l_4^\mu\}$ from the spinorial representation to vectorial representation

$$l_1^\mu = \overline{\Psi_1}(\Pi^\mu + K^\mu)\Psi_1, \quad l_2^\mu = i\overline{\Psi_1}(K^\mu - \Pi^\mu)\Psi_1,$$

$$l_3^\mu = i\overline{\Psi_1}(\Pi^\mu - K^\mu)\Psi_2, \quad l_4^\mu = \overline{\Psi_1}(\Pi^\mu + K^\mu)\Psi_2. \tag{6.55}$$

In this basis the formulae for vectors p^μ, y^μ are given by

$$p^\mu = m l_1^\mu \quad \text{and} \quad y^\mu = \frac{1}{2m}\{(a + \bar{a})l_4^\mu + i(\bar{a} - a)l_3^\mu - e l_1^\mu - i k l_2^\mu\}.$$

The $T \times T$ is 16-dimensional and up to now we have identified 11 + 4 variables associated with the massive spinning point-like system. The sixteenth variable is provided by the angle of rotation of a space-like

two-plane in terms of the orthogonal unit four-vectors. This two-plane polarization may be chosen to be

$$E^{AA'} = \frac{i}{m\sqrt{a\bar{a}}}(a\bar{\pi}^A\eta^{A'} - \bar{a}\pi^{A'}\bar{\eta}^A), \tag{6.56}$$

for $\mu \neq 0$, and

$$E^{AA'} = \frac{i}{m}(\bar{\pi}^A\eta^{A'} - \pi^{A'}\bar{\eta}^A), \tag{6.57}$$

for $\mu = 0$. Also we define

$$F^{AA'} = \frac{1}{sm\sqrt{a\bar{a}}}\left[\frac{1}{2}k(a\bar{\pi}^A\eta^{A'} + \bar{a}\pi^{A'}\bar{\eta}^A) - a\bar{a}(\pi^{A'}\bar{\pi}^A - \eta^{A'}\bar{\eta}^A)\right], \tag{6.58}$$

for $\mu \neq 0$, and

$$F^{AA'} = \frac{i}{m}(\bar{\pi}^A\eta^{A'} + \pi^{A'}\bar{\eta}^A), \tag{6.59}$$

for $\mu = 0$.

In the vectorial basis these values are written as

$$\begin{aligned} e^\mu &= \frac{1}{2\sqrt{a\bar{a}}}\{(a+\bar{a})l_3^\mu + i(\bar{a}-a)l_4^\mu\}, \\ e^\mu &= l_3^\mu, \quad \text{for} \quad \mu = 0, \\ f^\mu &= \frac{1}{2\sqrt{a\bar{a}}}\{(a+\bar{a})l_4^\mu + i(\bar{a}-a)l_3^\mu - (a\bar{a})l_3^\mu\}. \\ f^\mu &= il_4^\mu \quad \text{for} \quad \mu = 0. \end{aligned}$$

Using the condition $m^2 = p_\mu p_\mu$ and the definition of the orbital angular momentum $M_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$ one may construct the relation

$$p_\nu M^{\mu\nu} = x^\mu m^2 - (p_\nu x^\nu)p^\mu, \tag{6.60}$$

from which one finds

$$x^\mu m^2 = p_\nu M^{\mu\nu} + (p_\nu x^\nu)p^\mu, \tag{6.61}$$

as the coordinates of the position four-vector. If we substitute the definition of orbital angular momentum into the Pauli–Lubanski four-vector formula (6.21) we obtain $S_\mu = 0$. This value is non-trivial if total angular momentum consists besides orbital part also spin part (see, (6.18)). Our goal is to define the spin part by extending the coordinate part. For that purpose let us before consider the massless case. In that case we expect that

$$S_\mu = sp_\mu. \tag{6.62}$$

To provide this relation we shall look for the structure of the spin part in the following form

$$S^{\mu\nu} = L^{\mu\nu c}p_c = \tfrac{1}{2}e^{\mu\nu cd}y_c p_d, \tag{6.63}$$

with

$$L^{\mu\nu c} = \tfrac{1}{2}e^{\mu\nu cd}y_d. \tag{6.64}$$

Substituting this into (6.21) we obtain

$$S_\mu = -\tfrac{1}{2}e_{\mu\nu cd}p^\nu S^{cd} = -\tfrac{1}{4}e_{\mu\nu cd}p^\nu e^{cdpq}y_p p_q = -p_\mu(y^\nu p_\nu) = sp_\mu, \tag{6.65}$$

with $(y^\nu p_\nu) = -s$.

We suggest that the spin part has the same structure in the massive case, to obtain

$$S_\mu = -\tfrac{1}{2}e_{\mu\nu cd}p^\nu S^{cd} = -\tfrac{1}{4}e_{\mu\nu cd}p^\nu e^{cdpq}y_p p_q = -p_\mu(y^\nu p_\nu) + m^2 y_\mu. \tag{6.66}$$

The definition of Pauli–Lubanski four-vector is also based on the suggestion that the total angular momentum is the sum of two operators constructed in analogy with the massless case

$$M_{AA'BB'} = \{i(\mu_{A'B'}\epsilon_{AB} - \overline{\mu}_{AB}\epsilon_{A'B'}) + i(\nu_{A'B'}\epsilon_{AB} - \overline{\nu}_{AB}\epsilon_{A'B'})\}, \tag{6.67}$$

where

$$\mu_{AB} = \omega_{(A}\overline{\pi}_{B)}, \quad \nu_{AB} = \lambda_{(A}\overline{\eta}_{B)}. \tag{6.68}$$

Let us substitute this formula into the Pauli–Lubanski formula. We write

$$S_\mu = \frac{1}{2}e_{\mu\nu cd}P^\nu M^{cd} = *M_{\mu\nu}p^\nu. \tag{6.69}$$

For the Pauli–Lubanski formula we then obtain

$$\begin{aligned} -S_\mu &= \tfrac{1}{2}e_{\mu\nu cd}\{(p^\nu_{(1)} - p^\nu_{(2)})(M^{cd}_{(2)} + M^{cd}_{(1)})\} \\ &= \tfrac{1}{2}(-s_1 p_{(1)_\mu} - s_2 p_{(2)_\mu}) + \tfrac{1}{2}e_{\mu\nu cd}\{p^\nu_{(1)}M^{cd}_{(2)} + p^\nu_{(2)}M^{cd}_{(1)}\}, \end{aligned} \tag{6.70}$$

where $s_1 = \frac{1}{2}(e + k)$, $s_2 = \frac{1}{2}(e - k)$. So, we need to calculate only the expression

$$\tfrac{1}{2}e_{\mu\nu cd}\{p^\nu_{(1)}M^{cd}_{(2)} + p^\nu_{(2)}M^{cd}_{(1)}\} \tag{6.71}$$

Using the spinor representations of p^ν and ($^*M_{cd}$) which are defined by

$$^*M_{AA'BB'} = \frac{1}{2}\{\epsilon_{A'B'}(\omega_A\overline{\pi}_B + \omega_B\overline{\pi}_A) + \epsilon_{AB}(\overline{\omega}_{A'}\pi_{B'} + \overline{\omega}_{B'}\pi_{A'}) + \epsilon_{A'B'}(\lambda_A\overline{\eta}_B + \lambda_B\overline{\eta}_A) + \epsilon_{AB}(\overline{\lambda}_{A'}\eta_{B'} + \overline{\lambda}_{B'}\eta_{A'})\}, \quad (6.72)$$

and

$$P^{AA'} = (\pi^{A'}\overline{\pi}^A + \eta^{A'}\overline{\eta}^A), \quad (6.73)$$

we find

$$^* M^{(1)}_{\mu\nu}p^\nu_{(2)} \quad (6.74)$$
$$= (\omega_A\overline{\pi}_B + \omega_B\overline{\pi}_A)\eta_{A'}\overline{\eta}^B + (\overline{\omega}_{A'}\pi_{B'} + \overline{\omega}_{B'}\pi_{A'})\eta^{B'}\overline{\eta}_A$$
$$= (\overline{\pi}_B\overline{\eta}^B)\omega_A\eta_{A'} + (\omega_B\overline{\eta}^B)\eta_{A'}\overline{\pi}_A + (\pi_{B'}\eta^{B'})\overline{\omega}_{A'}\overline{\eta}_A + (\overline{\omega}_{B'}\eta^{B'})\overline{\eta}_A\pi_{A'},$$

$$^* M^{(2)}_{\mu\nu}p^\nu_{(1)} \quad (6.75)$$
$$= (\lambda_A\overline{\eta}_B + \lambda_B\overline{\eta}_A)(\pi_{A'}\overline{\pi}^B + (\overline{\lambda}_{A'}\eta_{B'} + \overline{\lambda}_{B'}\eta_{A'})(\pi^{B'}\overline{\pi}_A)$$
$$= (\overline{\pi}^B\overline{\eta}_B)\lambda_A\pi_{A'} + (\lambda_B\overline{\pi}^B)\pi_{A'}\overline{\eta}_A + (\pi^{B'}\eta_{B'})\overline{\lambda}_{A'}\overline{\pi}_A + (\overline{\lambda}_{B'}\pi^{B'})\overline{\pi}_A\eta_{A'}.$$

Collecting all these terms we find

$$-m^2Y_{AA'} = \eta_{A'}\overline{\pi}_A(\omega_B\overline{\eta}^B + \overline{\lambda}_{B'}\pi^{B'}) + \overline{\eta}_A\pi_{A'}(\overline{\omega}_{B'}\eta^{B'} + \lambda_B\overline{\pi}^B), \quad (6.76)$$

where we have used

$$-m^2Y_{AA'} \quad (6.77)$$
$$= (im^2/2)(Z_{AA'} - \overline{Z}_{AA'})$$
$$= (\overline{\pi}^B\overline{\eta}_B)\lambda_A\pi_{A'} + (\pi^{B'}\eta_{B'})\overline{\lambda}_{A'}\overline{\pi}_A + (\overline{\pi}_B\overline{\eta}^B\omega_A\eta_{A'} + (\pi_{B'}\eta^{B'})\overline{\omega}_{A'}\overline{\eta}_A.$$

Substituting all these formulae into (6.75) we obtain

$$m^2Y^{AA'} - \frac{e}{2}P^{AA'} \quad (6.78)$$
$$= \tfrac{1}{2}\{-\tfrac{1}{2}(e+k)(\pi_{A'}\overline{\pi}_A) - \tfrac{1}{2}(e-k)(\eta_{A'}\overline{\eta}_A) - m^2Y_{AA'} - a\eta_{A'}\overline{\pi}_A - \overline{m\eta}_A\pi_{A'}\}$$
$$= \tfrac{1}{2}\{-\tfrac{1}{2}(e+k)(\pi_{A'}\overline{\pi}_A) - \tfrac{1}{2}(e-k)(\eta_{A'}\overline{\eta}_A) - m^2Y_{AA'}$$
$$+ \eta_{A'}\overline{\pi}_A(\omega_B\overline{\eta}^B - \overline{\lambda}_{B'}\pi^{B'}) + \overline{\eta}_A\pi_{A'}(\overline{\omega}_{B'}\eta^{B'} + \lambda_B\overline{\pi}^B)\}.$$

On the other hand according to (6.71) we can write

$$S^{AA'} = m^2Y^{AA'} + \frac{e}{2}P^{AA'}. \quad (6.79)$$

Indeed the formulae (6.69) and (6.79) are in agreement if we take into account, the representation of $Y^{AA'}$ given in (6.57). As a result we obtain following expression for the Pauli–Lubanski vector

$$S^{AA'} = \frac{k}{2}(-\eta^{A'}\overline{\eta}^{A} + \pi^{A'}\overline{\pi}^{A}) + a\overline{\pi}^{A}\eta^{A'} + \overline{a}\pi^{A'}\overline{\eta}^{A}). \tag{6.80}$$

In the vectorial representation this formula is given by

$$s^{\mu} = m\{-i\frac{k}{2}l_2^{\mu} + (a+\overline{a})l_4^{\mu} + i(\overline{a}-a)l_3^{\mu}\}, \tag{6.81}$$

in addition we have

$$s^{\mu}p_{\mu} = 0. \tag{6.82}$$

The vectors p^{μ}, s^{μ}, e^{μ}, f^{μ} form then the set of mutual orthogonal vectors. The spin vector squared is given by

$$s^2 = -\frac{1}{m^2}S^{\mu}S_{\mu} = \frac{1}{4}k^2 + a\overline{a}. \tag{6.83}$$

The canonical Poincaré-invariant twistor Poisson algebra on $T \times T$ implies the following physically meaningful Poincaré-covariant commutation relation

$$\{x^{\mu}, p^{\nu}\} = g^{\mu\nu}. \tag{6.84}$$

Thus x^{μ} and p^{μ} are conjugate variable but, as we shall see in a moment, they are canonically conjugate if and only if the system is spinless.

Defining the total angular four-momentum

$$L^{\mu\nu} := x^{\mu}p^{\nu} - x^{\nu}p^{\mu} + \frac{1}{2m^2}e^{\mu\nu cd}y_c p_d, \tag{6.85}$$

we obtain an important identity

$$\begin{aligned} L^{\mu\nu} &= M_1^{\mu\nu} + M_2^{\mu\nu} \\ &= i\sigma^{\mu}_{AA'}\sigma^{\nu}_{BB'}\{(\omega^{(A}\overline{\pi}^{B)}\epsilon^{A'B'} - \overline{\omega}^{A'}\overline{\pi}^{A')}\epsilon^{AB}) \\ &\quad + (\lambda^{(A}\overline{\eta}^{B}\epsilon^{A'B'} - \overline{\lambda}^{A'}\eta^{B')}\epsilon^{AB})\}. \end{aligned} \tag{6.86}$$

In the conclusion of this section let us demonstrate the connection the bispinor representation of momentum with the de Sitter surface equation. Define two vectors with following coordinates

$$\xi^{\mu} = \eta^{\mu} = (\overline{\pi}^0, \overline{\pi}^1, \pi^{0'}, \pi^{1'}), \qquad \eta_{\mu} = (\overline{\eta}_0, \overline{\eta}_1, \eta_{0'}, \eta_{1'}). \tag{6.87}$$

Construct the bilinears of η^μ, η_μ by using Dirac gamma matrices

$$P_k = \sum_{i,j=1}^{4} \xi^j \{\gamma_k\}_j^i \eta_i, \quad k = 0, 1, 2, 3, 4, 5. \tag{6.88}$$

For these values given the important following algebraic identity holds

$$P_0^2 - P_1^2 - P_2^2 - P_3^2 + P_4^2 = P_5^2. \tag{6.89}$$

That is the de Sitter surface equation. Furthermore in conformity with the definition (6.88) one may write also

$$\begin{pmatrix} P_4 & 0 & -P_1 + P_0 & -P_2 + iP_3 \\ 0 & P_4 & P_2 - iP_3 & P_1 + P_0 \\ P_1 + P_0 & P_2 - iP_3 & -P_4 & 0 \\ P_2 + iP_3 & -P_1 + P_0 & 0 & -P_4 \end{pmatrix} \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = P_5 \begin{pmatrix} \eta_0 \\ \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix}, \tag{6.90}$$

with

$$\begin{aligned} P_0 &= \eta^2\eta_0 + \eta^3\eta_1 + \eta^0\eta_2 + \eta^1\eta_3, \\ P_1 &= -\eta^2\eta_0 + \eta^3\eta_1 + \eta^0\eta_2 - \eta^1\eta_3, \\ P_2 &= -\eta^3\eta_0 - \eta^2\eta_1 + \eta^1\eta_2 + \eta^0\eta_3, \\ P_3 &= i(-\eta^3\eta_0 + \eta^2\eta_1 + \eta^1\eta_2 - \eta^0\eta_3), \\ P_4 &= \eta^0\eta_0 + \eta^1\eta_1 - \eta^2\eta_2 - \eta^3\eta_3, \\ P_5 &= \eta^0\eta_0 + \eta^1\eta_1 + \eta^2\eta_2 + \eta^3\eta_3. \end{aligned} \tag{6.91}$$

Comparing (6.91) with (6.38–6.39) we obtain $P_4 = -2if_2$, $P_5 = 2f_1$.

2.4 EQUATIONS OF MOTION

We shall now study the equations of motion known as the Lorentz–Bargmann–Michel–Telegdi equations.

Consider the generating function of motion as a simple sum of two Poincaré invariants m^2 and s^2

$$H(m, s) = m^2 + s^2. \tag{6.92}$$

Taking into account the definitions (6.50) we can write

$$H(m, s) = P^i P_i - \frac{1}{m^2} S^i S_i = 2f\bar{f} + \frac{k^2}{4} + i\bar{\omega}. \tag{6.93}$$

The essential point is that the values *a, f, k, e* must be constants of motion, which implies the following form of the Hamiltonian function in the variables of twistors

$$\begin{aligned} &H(Z^\alpha, W^\beta, Z_\alpha, W_\beta) \\ &\quad = \tfrac{1}{4}k(Z^\alpha Z_\alpha + W^\beta W_\beta) + \tfrac{1}{2}aZ_\alpha W^\alpha + \tfrac{1}{2}\overline{a}Z^\alpha W_\alpha \\ &\quad + fZ_\alpha I^{\alpha\beta}W_\beta + \overline{f}Z^\alpha I_{\alpha\beta}W^\beta. \end{aligned} \tag{6.94}$$

The corresponding Hamilton equations for the twistor variables have the form

$$\begin{aligned} i\frac{d}{dt}Z^\alpha = \frac{\partial H}{\partial Z_\alpha}, \qquad -i\frac{d}{dt}Z_\alpha = c\frac{\partial H}{\partial Z^\alpha}, \\ i\frac{d}{dt}W^\alpha = \frac{\partial H}{\partial W_\alpha}, \qquad -i\frac{d}{dt}W_\alpha = \frac{\partial H}{\partial W^\alpha}, \end{aligned} \tag{6.95}$$

with that choice, the canonical flow is described by the following equations of motion

$$\begin{aligned} i\frac{d}{dt}Z^\alpha - \tfrac{1}{4}kZ^\alpha - \tfrac{1}{2}aW^\alpha - fI^{\alpha\beta}W_\beta = 0, \\ i\frac{d}{dt}W^\alpha + \tfrac{1}{4}kW^\alpha - \tfrac{1}{2}\overline{a}Z^\alpha + fI^{\alpha\beta}Z_\beta = 0, \end{aligned} \tag{6.96}$$

where $I^{\alpha\beta}$ is the antisymmetric tensor whose unique nonzero components are

$$I_{23} = -I_{32} = 1, \quad I^{01} = -I^{10} = 1.$$

Taking into account the expressions

$$fZ_\alpha I^{\alpha\beta}W_\beta = \overline{\pi}^A\overline{\eta}_A, \qquad fZ^\alpha I_{\alpha\beta}W^\beta = \pi^{A'}\eta_{A'}, \tag{6.97}$$

we obtain the equations of motion

$$\begin{aligned} i\frac{d}{dt}\omega^A - \tfrac{1}{4}k\omega^A - \tfrac{1}{2}a\lambda^A + f\overline{\eta}^A = 0, \\ i\frac{d}{dt}\pi_{A'} - \tfrac{1}{4}k\pi_{A'} - \tfrac{1}{2}a\eta_{A'} = 0, \\ i\frac{d}{dt}\lambda^A - \tfrac{1}{4}k\lambda^A - \tfrac{1}{2}\overline{a}\omega^A + f\overline{\pi}^A = 0, \\ i\frac{d}{dt}\eta_{A'} + \tfrac{1}{4}k\eta_{A'} - \tfrac{1}{2}\overline{a}\pi_{A'} = 0. \end{aligned} \tag{6.98}$$

The corresponding complex conjugated equations are

$$
\begin{aligned}
-i\frac{d}{dt}\overline{\omega}^{A'} &= \tfrac{1}{4}k\overline{\omega}^{A'} + \tfrac{1}{2}\overline{a}\overline{\lambda} - \overline{f}\eta^{A'} = 0, \\
-i\frac{d}{dt}\overline{\pi}_{A} &= \tfrac{1}{4}k\overline{\pi}_{A} + \tfrac{1}{2}\overline{a\eta}_{A}, \\
-i\frac{d}{dt}\overline{\lambda}^{A'} &= \tfrac{1}{4}k\overline{\lambda}^{A'} + \tfrac{1}{2}a\overline{\omega}^{A'} - \overline{f}\pi^{A'} = 0, \\
-i\frac{d}{dt}\overline{\eta}_{A} &= -\tfrac{1}{4}k\overline{\eta}_{A} + \tfrac{1}{2}a\overline{\pi}_{A}.
\end{aligned}
\tag{6.99}
$$

(6.100)

Now we shall modify these equations in such a way that. *a, f, k, e* remain constants of motion, choosing the four-velocity to remain parallel to the four-momentum, then the resulting equation for the four-momentum becomes the usual Lorentz force equation. In other words, we require

$$
\begin{aligned}
&\frac{d}{dt}(e, f, k, a) = 0, \\
&\frac{d}{dt}X^{i} = P^{i}, \\
&\frac{d}{dt}P^{i} = eF^{ij}P_{j}.
\end{aligned}
\tag{6.101}
$$

Following [15, 16, 17] we suggest the next twistor equations in the external electromagnetic field, from the results below the set is an algebraic form of the Dirac equation:

$$
\begin{aligned}
&i\frac{d}{dt}Z^{\alpha} - \tfrac{1}{4}kZ^{\alpha} - \tfrac{1}{2}aW^{\alpha} - fI^{\alpha\beta}W_{\beta} = g^{\alpha}, \\
&i\frac{d}{dt}W^{\alpha} - \tfrac{1}{4}kW^{\alpha} - \tfrac{1}{2}\overline{a}Z^{\alpha} + fI^{\alpha\beta}Z_{\beta} = h^{\alpha}.
\end{aligned}
\tag{6.102}
$$

In coordinate representation we obtain

$$
\begin{aligned}
&i\frac{d}{dt}\omega^{A} - \tfrac{1}{4}k\omega^{A} - \tfrac{1}{2}a\lambda^{A} + f\overline{\eta}^{A} = \gamma^{A} + iX^{AA'}\delta_{A'}, \\
&i\frac{d}{dt}\pi_{A'} - \tfrac{1}{4}k\pi_{A'} - \tfrac{1}{2}a\eta_{A'} = \delta_{A'}, \\
&i\frac{d}{dt}\lambda^{A} + \tfrac{1}{4}k\lambda^{A} - \tfrac{1}{2}\overline{a}\omega^{A} + f\overline{\pi}^{A} = \kappa^{A} + iX^{AA'}\theta_{A'}, \\
&i\frac{d}{dt}\eta_{A'} + \tfrac{1}{4}k\eta_{A'} - \tfrac{1}{2}\overline{a}\pi_{A'} = \theta_{A'}.
\end{aligned}
\tag{6.103}
$$

The corresponding complex conjugated equations with external electromagnetic potential have the form

$$i\frac{d}{dt}Z_\alpha = \tfrac{1}{4}kZ_\alpha + \tfrac{1}{2}\overline{a}W_\alpha + \overline{f}I_{\alpha\beta}W^\beta + \overline{g}_\alpha,$$
$$i\frac{d}{dt}W_\alpha = -\tfrac{1}{4}kW_\alpha + \tfrac{1}{2}aZ_\alpha - \overline{f}I^{\alpha\beta}Z^\beta + \overline{l}_\alpha, \tag{6.104}$$

where $Z_\alpha = (\overline{\pi}_A, \overline{\omega}^{A'}), W_\alpha = (\overline{\eta}_A, \overline{\lambda}^{A'})$. And in coordinate representation are given by

$$\begin{aligned}
-i\frac{d}{dt}\overline{\omega}^{A'} &= \tfrac{1}{4}k\overline{\omega}^{A'} + \tfrac{1}{2}\overline{a}\overline{\lambda}^{A'} - \overline{f}\eta^{A'} + \overline{\gamma}^A - iX^{AA'}\overline{\delta}_A,\\
-i\frac{d}{dt}\overline{\pi}_A &= \tfrac{1}{4}k\overline{\pi}_A + \tfrac{1}{2}\overline{a\eta_A} + \overline{\delta}_A,\\
-i\frac{d}{dt}\overline{\lambda}^{A'} &= \tfrac{1}{4}k\overline{\lambda}^{A'} + \tfrac{1}{2}a\overline{\omega}^{A'} - \overline{f}\pi^{A'} + \overline{\kappa}^A - iX^{AA'}\overline{\theta}_A,\\
-i\frac{d}{dt}\overline{\eta}_A &= -\tfrac{1}{4}k\overline{\eta}_A + \tfrac{1}{2}a\overline{\pi}_A + \overline{\theta}_A.
\end{aligned} \tag{6.105}$$

Let us put the external electromagnetic field in spinor notation as follows

$$F_{AA'BB'} = \frac{i}{e}\left(\overline{\mu}_{AB}\epsilon_{A'B'} - \mu_{A'B'}\epsilon_{AB}\right). \tag{6.106}$$

The functions g^α, h^α in spinorial components are chosen as

$$g^\alpha = \left(\gamma^A + iX^{AA'}\delta_{A'}, \delta_{A'}\right), \qquad h^\alpha = \left(\kappa^A + iX^{AA'}\theta_{A'}, \theta_{A'}\right), \tag{6.107}$$

where the spinors γ^A, $\delta_{A'}$, k^A, θ_A are given along particle's wordline location. Spinors γ^A, $\delta_{A'}$, K^A, θ_A may also be expressed as

$$\delta_{A'} = \mu_{A'B'}\xi^{B'}, \theta_{A'} = \mu_{A'B'}\psi^{B'}, \gamma^{A'} = \overline{\mu}^A_B\varsigma^B, \kappa^{A'} = \overline{\mu}^A_B\rho^B, \tag{6.108}$$

where $\xi^{B'}$ $\psi^{B'}$, ς^B, ρ^B are new variable spinors.

Now let us begin our explicit calculations. The condition

$$\frac{d}{dt}f = 0,$$

yields

$$\begin{aligned}
\frac{d}{dt}f &= \eta_{A'}\frac{d}{dt}\pi^{A'} + \pi^{A'}\frac{d}{dt}\eta_{A'}\\
&= \eta_{A'}\left(\tfrac{1}{4}k\pi^{A'} - \tfrac{1}{2}a\eta^{A'} + \delta^{A'}\right) + \pi^{A'}\left(-\tfrac{1}{4}k\eta_{A'} + \tfrac{1}{2}\overline{a}\pi_{A'} + \theta_{A'}\right)\\
&= \eta_{A'}\delta^{A'} + \pi^{A'}\theta_{A'}.
\end{aligned} \tag{6.109}$$

Here we have used the identity

$$i\overline{\pi}_A f\overline{\eta}^A - i\pi_{A'}\overline{f}\eta^{A'} = 0. \tag{6.110}$$

Inserting $\delta^{A'}, \theta_{A'}$ from (6.108) we obtain

$$\mu_{A'B'}\left(\pi^{A'}\psi^{B'} - \eta^{A'}\xi^{B'}\right) = 0. \tag{6.111}$$

As far as spinors $\psi^{B'}$, $\xi^{B'}$ are independent of the electromagnetic field this relation is to be fulfilled for arbitrary choice of $\mu_{A'B'}$. Then the following equation must be satisfied

$$\left(\pi^{A'}\psi^{B'} - \eta^{A'}\xi^{B'}\right) = \text{constant} \in_{A'B'}. \tag{6.112}$$

We find two kinds of solutions

(1): $\xi^{A'} = y\pi^{A'}, \quad \psi^{A'} = y\eta^{A'},$

(2): $\eta^{A'} = x\pi^{A'}, \quad \psi^{A'} = x\xi^{A'}.$

Now let us check the compatibility of the equations (6.102–6.105) with the Lorentz equation

$$\frac{d}{dt}P^a = eF^{a\nu}P_\nu. \tag{6.113}$$

Substituting on the left side P^a by its expression in twistor coordinates one obtains

$$\begin{aligned}
&\overline{\pi}^A\frac{d}{dt}\pi^{A'} + \pi^{A'}\frac{d}{dt}\overline{\pi}^A + \overline{\eta}^A\frac{d}{dt}\eta^{A'} + \eta^{A'}\frac{d}{dt}\overline{\eta}^A \\
&\quad = -i\overline{\pi}^A\left(\tfrac{1}{4}k\pi^{A'} + \tfrac{1}{2}a\eta^{A'} + \delta^{A'}\right) + i\pi^{A'}\left(\tfrac{1}{4}k\overline{\pi}^A + \tfrac{1}{2}\overline{a}\overline{\eta}^A - \overline{\delta}^A\right) \\
&\qquad - i\overline{\eta}^A\left(-\tfrac{1}{4}k\eta^{A'} + \tfrac{1}{2}\overline{a}\pi^{A'} + \theta^{A'}\right) + i\eta^{A'}\left(-\tfrac{1}{4}\overline{k}\overline{\eta}^A + \tfrac{1}{2}a\overline{\pi}^A + \overline{\theta}^A\right).
\end{aligned} \tag{6.114}$$

For the left hand side of (6.113) we then obtain

$$i\left(\pi^{A'}\overline{\delta}^A - \overline{\pi}^A\delta^{A'} + \eta^{A'}\overline{\theta}^A - \overline{\eta}^A\theta^{A'}\right) \tag{6.115}$$

For the right hand side of (6.113) write

$$\begin{aligned}
eF^{AA'BB'}\,P_{BB'} &= i\left(-\overline{\mu}_{AB}\epsilon^{A'B'} + \mu_{A'B'}\epsilon^{AB}\right)P_{BB'} \\
&= i\left(-\overline{\mu}_{AB}P_B^{A'} + \mu_{A'B'}P_{B'}^A\right) \\
&= i\left\{-\overline{\mu}_{AB}\left(p\overline{\pi}^{A'}\overline{\pi}_B + \eta^{A'}\overline{\eta}_B\right) + \mu_{A'B'}\left(\pi_A\overline{\pi}_B + \eta_{B'}\overline{\eta}^A\right)\right\}
\end{aligned} \tag{6.116}$$

Where there is a cancelation because

$$i\overline{\mu}_{AB}\pi_{A'}\left(\left(\overline{\pi}^{B}-\overline{\xi}^{B}\right)+\eta_{A'}\left(\overline{\eta}^{B}-\overline{\psi}^{B}\right)\right)$$
$$-i\mu_{A'B'}\left(\overline{\pi}_{A}\left(\pi^{B'}-\xi^{B'}\right)+\overline{\eta}_{A}\left(\eta^{B'}-\psi^{B'}\right)\right). \tag{6.117}$$

Let us check with the same procedure the independence of e and k from the evolution parameter

$$\frac{d}{dt}e=0,\quad \frac{d}{dt}k=0. \tag{6.118}$$

Write

$$\begin{aligned}\frac{d}{dt}\tfrac{1}{2}(e+k)&=\overline{\pi}_{A}\frac{d}{dt}\omega^{A}+\left(\frac{d}{dt}\overline{\pi}_{A}\right)\omega^{A}+\left(\frac{d}{dt}\overline{\omega}^{A'}\right)\pi_{A'}+\left(\frac{d}{dt}\pi_{A'}\right)\overline{\omega}^{A'}\\&=-i\overline{\pi}_{A}\left(\tfrac{1}{4}k\omega^{A}+\tfrac{1}{2}a\lambda^{A}-f\overline{\eta}^{A}+\gamma^{A}+iX^{AA'}\delta_{A'}\right)\\&\quad+i\omega^{A}\left(\tfrac{1}{4}k\overline{\pi}_{A}+\tfrac{1}{2}\overline{a\eta}_{A}+\overline{\delta}_{A}\right)\\&\quad+i\pi_{A'}\left(\tfrac{1}{4}k\overline{\omega}^{A'}+\tfrac{1}{2}\overline{a\lambda}^{A'}-\overline{f}\eta^{A'}+\overline{\gamma}^{A'}-iX^{AA'}\overline{\delta}_{A}\right)\\&\quad-i\overline{\omega}^{A'}(\tfrac{1}{4}k\pi_{A'}+\tfrac{1}{2}a\eta_{A'}+\delta_{A'})\\&=-ia\left(\overline{\omega}^{A'}\overline{\eta}_{A'}+\overline{\pi}_{A}\lambda^{A}\right)+i\overline{a}\left(\omega^{A}\overline{\eta}_{A}+\pi_{A'}\overline{\lambda}^{A'}\right)\\&\quad-i\overline{\pi}_{A}\gamma^{A}+\overline{\pi}_{A}X^{AA'}\delta_{A'}+i\omega^{A}\overline{\delta}_{A}\\&\quad+i\pi_{A'}\overline{\gamma}^{A'}+\pi_{A'}X^{AA'}\overline{\delta}_{A}-i\overline{\omega}^{A'}\delta_{A'}.\end{aligned} \tag{6.119}$$

Taking into account the definition $a=(\omega^{A}\overline{\eta}_{A}+\overline{\lambda}^{A'}\pi_{A'})$ for the first part we obtain

$$\begin{aligned}&-i\overline{\pi}_{A}\gamma^{A}+\pi_{A'}X^{AA'}\overline{\delta}_{A}+i\omega^{A}\overline{\delta}_{A}\\&\quad=-i\overline{\pi}_{A}\overline{\mu}^{A}_{C}\varsigma^{C}+\pi_{A'}X^{AA'}\overline{\mu}_{AB}\overline{\pi}^{B}+i\omega^{A}\overline{\mu}_{AB}\overline{\pi}^{B}\\&\quad=i\overline{\mu}_{AB}\overline{\pi}^{B}\left(\omega^{A}-i\pi_{A'}X^{AA'}+\varsigma^{A}\right).\end{aligned} \tag{6.120}$$

Remind $\omega^{A}=i\pi_{A}'\,(X^{AA'}+iY^{AA'})$ to find

$$\begin{aligned}&\overline{\mu}_{AB}\overline{\pi}^{B}\left(Y^{AB'}\pi_{B'}-\varsigma^{A}\right)-\text{c.c.}=0,\\&\overline{\mu}_{AB}\eta^{B}\left(Y^{AB'}\eta_{B'}-\rho^{A}\right)-\text{c.c.}=0.\end{aligned} \tag{6.121}$$

Thus to fulfill (6.117) and (6.121) the following relations should hold:

$$\xi_{B'}=\pi_{B'},\quad \psi_{B'}=\eta_{B'},\quad \varsigma^{A}=Y^{AA'}\pi_{A'},\quad \rho^{A}=Y^{AA'}\eta_{A'}. \tag{6.122}$$

In the Appendix we explicitly demonstrate that these relations, together with equations for twistors (6.102–6.105), imply the following equations of motion:

$$\frac{d}{dl}S^i = eF^{ij}S_j, \tag{6.123}$$

$$\frac{d}{dt}E^i = -sF^i - eF^{ij}E_j,$$

$$\frac{d}{dt}F^i = sE^i - eF^{ij}F_j. \tag{6.124}$$

From (6.123) it follows that the gyromagnetic ratio is necessarily equal to two. Equations (6.124) describe the evolution of the vectors tetrad in the polarization plane.

Appendix

Let us deduce the equations

$$\frac{d}{dt}E^i = -sF^i - eF^{ij}E_j,$$

$$\frac{d}{dt}F^i = sE^i - eF^{ij}F_j, \tag{6.125}$$

from twistor equations with the electromagnetic field interaction. Remember the definitions of $E^{AA'}$ and $F^{AA'}$ to compute $\frac{d}{dt}E^{AA'}$

$$\begin{aligned}\frac{d}{dt}E^{AA'} = {} & i\frac{i}{m\sqrt{a\overline{a}}}\left\{a\eta^{A'}\left(\tfrac{1}{4}k\overline{\pi}^A + \tfrac{1}{2}\overline{a\eta}^A + \overline{\delta}^A\right)\right. \\ & - a\overline{\pi}^A\left(-\tfrac{1}{4}k\eta^{A'} + \tfrac{1}{2}\overline{a}\pi^{A'} + \theta^{A'}\right) \\ & + \overline{a\eta}^A\left(\tfrac{1}{4}k\pi^{A'} - \tfrac{1}{2}a\eta^{A'} + \delta^{A'}\right) \\ & \left. - \overline{a}\pi^{A'}\left(-\tfrac{1}{4}k\overline{\eta}^A + \tfrac{1}{2}a\overline{\pi}^A + \overline{\theta}^A\right)\right\} \end{aligned} \tag{6.126}$$

$$\begin{aligned} = {} & -\frac{s}{sm\sqrt{a\overline{a}}}\left\{\tfrac{1}{4}k(a\eta^{A'}\overline{\pi}^A + \overline{a\eta}^A\pi^{A'}) - a\overline{a}\left(\eta^{A'}\overline{\eta}^A - \pi^{A'}\overline{\pi}^A\right)\right\} \\ & + i\frac{si}{sm\sqrt{a\overline{a}}}\left\{a\left(\overline{\mu}^{AB}\eta^{A'}\overline{\pi}_B\right) + a\left(-\mu^{A'B'}\eta_{B'}\overline{\pi}^A\right)\right. \\ & \left. + \overline{a}\left(\mu^{A'B'}\pi_{B'}\overline{\eta}^A\right) - \overline{a}\left(\overline{\mu}^{AB}\overline{\eta}_B\pi^{A'}\right)\right\}. \end{aligned} \tag{6.127}$$

The expression in the first brackets of (6.127) is $(-)sF^{AA'}$, whilst the expressions in the last brackets (6.127) are as follows

$$\begin{aligned} e\frac{i}{e}(\overline{\mu}^{AB}E_B^{A'} - \mu^{A'B'}E_{B'}^A) &= e\frac{i}{e}(\overline{\mu}^{AB}\epsilon^{A'B'} - \mu^{A'B'}\epsilon^{AB})E_{BB'} \\ &= eF^{AA'BB'}E_{BB'}. \end{aligned} \tag{6.128}$$

Equating the corresponding terms we obtain

$$\frac{d}{dt}E^i = -sF^i + eF^{ij}E_j.$$

Now let us calculate $\frac{d}{dt}F^{AA'}$

$$\begin{aligned}\frac{d}{dt}F^{AA'} &= \frac{1}{sm\sqrt{a\bar{a}}}\left\{\frac{k}{2}\frac{d}{dt}\left(a\eta^{A'}\overline{\pi}^{A} + \bar{a}\pi^{A'}\overline{\eta}^{A}\right) + a\bar{a}\frac{d}{dt}\left(\eta^{A'}\overline{\eta}^{A} - \pi^{A'}\overline{\pi}^{A}\right)\right\} \\ &= \frac{1}{sm\sqrt{a\bar{a}}}\left\{\frac{k}{4}(I) + a\bar{a}(II)\right\}. \qquad (6.129)\end{aligned}$$

In the bracket {..} for the part (I)

$$\begin{aligned}&\frac{d}{dt}\left(a\eta^{A'}\overline{\pi}^{A} + \bar{a}\pi^{A'}\overline{\eta}^{A}\right) \\ &\quad = ia\left\{\eta^{A'}\left(\frac{k}{4}k\overline{\pi}^{A} + \tfrac{1}{2}\overline{a}\overline{\eta}^{A} + \overline{\delta}^{A}\right) - \overline{\pi}^{A}\left(-\frac{k}{4}\eta^{A'} + \tfrac{1}{2}\overline{a}\pi^{A'} + \theta^{A'}\right)\right\} \qquad (6.130) \\ &\qquad + i\bar{a}\left\{-\overline{\eta}^{A}\left(\frac{k}{4}\pi^{A'} + \tfrac{1}{2}a\eta^{A'} + \delta^{A'}\right) + \pi^{A'}\left(-\frac{k}{4}\overline{\eta}^{A} + \tfrac{1}{2}a\overline{\pi}^{A} + \overline{\theta}^{A}\right)\right\} \\ &\quad = i\left\{\frac{k}{2}a\eta^{A'}\overline{\pi}^{A} - \tfrac{1}{2}\overline{a}a\left(\overline{\pi}^{A}\pi^{A'} - \overline{\eta}^{A}\eta^{A'}\right)\right\} \\ &\qquad + i\left\{-\tfrac{1}{2}k\bar{a}\pi^{A'}\overline{\eta}^{A} + \tfrac{1}{2}\overline{a}a\left(\overline{\pi}^{A}\pi^{A'} - \overline{\eta}^{A}\eta^{A'}\right)\right\} \\ &\qquad + ia\left(\overline{\delta}^{A}\eta^{A'} - \overline{\pi}^{A}\theta^{A'}\right) + i\bar{a}\left(\overline{\theta}^{A}\pi^{A'} - \overline{\eta}^{A}\delta^{A'}\right) \qquad (6.131)\end{aligned}$$

For the part (II) we obtain

$$\begin{aligned}&\frac{d}{dt}\left(\eta^{A'}\overline{\eta}^{A} - \pi^{A'}\overline{\pi}^{A}\right) \\ &\quad = i\left\{\eta^{A'}\left(-\frac{k}{4}\overline{\eta}^{A} - \tfrac{1}{2}a\overline{\pi}^{A} + \overline{\theta}^{A}\right)\right. \\ &\qquad - \overline{\eta}^{A}\left(-\frac{k}{4}\eta^{A'} + \tfrac{1}{2}\overline{a}\pi^{A'} + \theta^{A'}\right)\overline{\pi}^{A}\left(\tfrac{1}{4}k\pi^{A'} + \tfrac{1}{2}a\eta^{A'} + \delta^{A'}\right) \\ &\qquad \left. - \pi^{A'}\left(\frac{k}{4}k\overline{\pi}^{A} + \tfrac{1}{2}\overline{a}\overline{\eta}^{A} + \overline{\delta}^{A}\right)\right\} \\ &\quad = i\left\{\left(a\eta^{A'}\overline{\pi}^{A} - \bar{a}\pi^{A'}\overline{\eta}^{A}\right) + \left(\overline{\theta}^{A}\eta^{A'} - \overline{\eta}^{A}\theta^{A'}\right) + \left(\overline{\pi}^{A}\delta^{A'} - \pi^{A'}\overline{\delta}^{A}\right)\right\}. \\ &\qquad (6.132)\end{aligned}$$

Let us examine the parts (I) and (II) with interaction. We obtain for the part (I)

$$\begin{aligned} & ia\left(\overline{\delta}^{A}\eta^{A'}-\overline{\pi}^{A}\theta^{A'}\right)+i\overline{a}\left(\overline{\theta}^{A}\pi^{A'}-\overline{\eta}^{A}\delta^{A'}\right) \\ & \quad =i\left[a\left(\overline{\mu}^{AB}\overline{\pi}_{B}\eta^{A'}-\mu^{A'B'}\eta_{B'}\overline{\pi}^{A}\right)\right] \\ & \quad =i\left[\overline{\mu}^{AB}\left(a\overline{\pi}_{B}\eta^{A'}+\overline{a}\pi^{A'}\overline{\eta}_{B}\right)-\mu^{A'B'}\left(a\overline{\pi}^{A}\eta_{B'}+\overline{a}\pi_{B'}\overline{\eta}^{A}\right)\right] \\ & \quad =e\frac{i}{e}\left(\overline{\mu}^{AB}\epsilon^{A'B'}-\mu^{A'B'}\epsilon^{AB}\right)\left(a\overline{\pi}_{B}\eta_{B'}+\overline{a}\pi_{B'}\overline{\eta}_{B}\right), \end{aligned} \tag{6.133}$$

and for the part (II)

$$\begin{aligned} & i\left\{\left(\overline{\theta}^{A}\eta^{A'}-\overline{\eta}^{A}\theta^{A'}\right)+\overline{\pi}^{A}\delta^{A'}-\pi^{A'}\overline{\delta}^{A}\right\} \\ & \quad =i\left[\left(\overline{\mu}^{AB}\overline{\eta}_{B}\eta^{A'}-\overline{\mu}^{AB}\overline{\pi}_{B}\pi^{A'}+\mu^{A'B'}\pi_{B'}\overline{\pi}^{A}-\mu^{A'B'}\eta_{B'}\overline{\eta}^{A}\right)\right] \\ & \quad =\left[e\frac{i}{e}\left(\overline{\mu}^{AB}\epsilon^{A'B'}-\mu^{A'B'}\epsilon^{AB}\right)\left(\overline{\eta}_{B}\eta_{B'}-\pi_{B'}\overline{\pi}_{B}\right)\right]. \end{aligned} \tag{6.134}$$

Joining the results above and taking into account the factors $1/\left(sm\sqrt{a\overline{a}}\right)$, $\frac{1}{2}k$, $a\overline{a}$ we obtain

$$\begin{aligned} eF^{AA'BB'}F_{BB'} & =i\left(\overline{\mu}^{AB}\epsilon^{A'B'}-\mu^{A'B'}\epsilon^{AB}\right) \\ & \left[\frac{1}{sm\sqrt{a\overline{a}}}\left\{\frac{k}{2}\left(a\overline{\pi}_{B}\eta_{B'}+\overline{a}\pi_{B'}\overline{\eta}_{B}\right)+a\overline{a}\left(\overline{\eta}_{B}\eta_{B'}-\pi_{B'}\overline{\pi}_{B}\right)\right\}\right] \end{aligned} \tag{6.135}$$

For the part without interaction we have

$$i\left(\frac{k^{2}}{4}+a\overline{a}\right)\left(a\eta^{A'}\overline{\pi}^{A}-\overline{a}\pi^{A'}\overline{\eta}^{A}\right)=i\left(s^{2}\right)\left(a\eta^{A'}\overline{\pi}^{A}-\overline{a}\pi^{A'}\overline{\eta}^{A}\right) \tag{6.136}$$

Multiplying by the factor $1/\left(sm\sqrt{a\overline{a}}\right)$ we obtain: $sE^{AA'}$. Substitute the expressions (6.135-6.136) into the right side to get

$$\frac{d}{dt}F^{i}=sE^{i}+eF^{ij}F_{j}.$$

Finally let us check the equation for the spin

$$\frac{d}{dt}S^{i}=F^{ij}S_{j}.$$

Where

$$\frac{d}{dt}S^{AA'} = \frac{d}{dt}\left\{\frac{k}{2}\left(-\eta^{A'}\overline{\eta}^{A} + \pi^{A'}\overline{\pi}^{A}\right) + \left(a\overline{\pi}^{A}\eta^{A'} + \overline{a}\pi^{A'}\overline{\eta}^{A}\right)\right\} \quad (6.137)$$

Again let us calculate separately two terms in the bracket {..}. For the part (I)

$$\begin{aligned}
&\frac{d}{dt}\left(a\eta^{A'}\overline{\pi}^{A} + \overline{a}\pi^{A'}\overline{\eta}^{A}\right)\\
&= ia\left\{\eta^{A'}\left(\frac{k}{4}\overline{\pi}^{A} + \tfrac{1}{2}\overline{a\eta}^{A} + \overline{\delta}^{A}\right) - \overline{\pi}^{A}\left(\left(-\frac{k}{4}\eta^{A'} + \tfrac{1}{2}\overline{a}\pi^{A'} + \theta^{A'}\right)\right\}\right.\\
&\quad + i\overline{a}\left\{-\overline{\eta}^{A}\left(\frac{k}{4}\pi^{A'} + \tfrac{1}{2}a\eta^{A'} + \delta^{A'}\right) + \pi^{A'}\left(-\frac{k}{4}\overline{\eta}^{A} + \tfrac{1}{2}a\overline{\pi}^{A} + \overline{\theta}^{A}\right)\right\}\\
&= i\left\{\frac{ka}{2}\eta^{A'}\overline{\pi}^{A} - \tfrac{1}{2}\overline{a}a\left(\overline{\pi}^{A}\pi^{A'} - \overline{\eta}^{A}\eta^{A'}\right)\right\}\\
&\quad + i\left\{\frac{k\overline{a}}{2}a\pi^{A'}\overline{\eta}^{A} + \tfrac{1}{2}\overline{a}a\left(\overline{\pi}^{A}\pi^{A'} - \overline{\eta}^{A}\eta^{A'}\right)\right\}\\
&\quad + ia\left(\overline{\delta}^{A}\eta^{A'} - \overline{\pi}^{A}\theta^{A'}\right) + i\overline{a}\left(\overline{\theta}^{A}\pi^{A'} - \overline{\eta}^{A}\delta^{A'}\right).
\end{aligned} \quad (6.138)$$

For the part (II)

$$\begin{aligned}
&\frac{d}{dt}\left(\eta^{A'}\overline{\eta}^{A} - \pi^{A'}\overline{\pi}^{A}\right)\\
&= i\{\eta^{A'}\left(-\frac{k}{4}\overline{\eta}^{A} + \tfrac{1}{2}a\overline{\pi}^{A} + \overline{\theta}^{A}\right) - \overline{\eta}^{A}\left(-\frac{k}{4}\eta^{A'} + \tfrac{1}{2}\overline{a}\pi^{A'} - \theta^{A'}\right)\\
&\quad + \overline{\pi}^{A}\left(\frac{k}{4}\pi^{A'} + \tfrac{1}{2}a\eta^{A'} + \delta^{A'}\right) - \pi^{A'}\left(\frac{k}{4}\overline{\pi}^{A} + \tfrac{1}{2}\overline{a\eta}^{A} + \overline{\delta}^{A}\right)\}\\
&= i\left\{\left(a\eta^{A'}\overline{\pi}^{A} - \overline{a}\pi^{A'}\overline{\eta}^{A}\right) + \left(\overline{\theta}^{A}\eta^{A'} - \overline{\eta}^{A}\theta^{A'}\right) + \left(\overline{\pi}^{A}\delta^{A'} - \pi^{A'}\overline{\delta}^{A}\right)\right\}.
\end{aligned} \quad (6.139)$$

Now let us examine the parts (I) and (II) with interaction. We obtain for the part (I)

$$\begin{aligned}
&ia\left(\overline{\delta}^{A}\eta^{A'} - \overline{\pi}^{A}\theta^{A'}\right) + i\overline{a}\left(\overline{\theta}^{A}\pi^{A'} - \overline{\eta}^{A}\delta^{A'}\right)\\
&\quad = i\left[a\left(\overline{\mu}^{AB}\overline{\pi}_{B}\eta^{A'} - \mu^{A'B'}\eta_{B'}\overline{\pi}^{A}\right)\right]\\
&\quad = i\left[\overline{\mu}^{AB}\left(a\overline{\pi}_{B}\eta^{A'} + \overline{a}\pi^{A'}\overline{\eta}_{B}\right) - \mu^{A'B'}\left(a\overline{\pi}^{A}\eta_{B'} - \overline{a}\pi_{B'}\overline{\eta}^{A}\right)\right]\\
&\quad = i\left(\overline{\mu}^{AB}\epsilon^{A'B'} - \mu^{A'B'}\epsilon^{AB}\right)\left(a\overline{\pi}_{B}\eta_{B'} + \overline{a}\pi_{B'}\overline{\eta}_{B}\right).
\end{aligned} \quad (6.140)$$

and for the part (II)

$$
\begin{aligned}
& i\left\{\left(\overline{\theta}^{A}\eta^{A'}-\overline{\eta}^{A}\theta^{A'}\right)+\overline{\pi}^{A}\delta^{A'}-\pi^{A'}\overline{\delta}^{A}\right\} \\
& \quad = i\left[\left(\overline{\mu}^{AB}\overline{\eta}_{B}\eta^{A'}-\overline{\mu}^{AB}\overline{\pi}_{B}\pi^{A'}+\mu^{A'B'}\pi_{B'}\overline{\pi}^{A}-\mu^{A'B'}\eta_{B'}\overline{\eta}^{A}\right)\right| \\
& \quad = i\left(\overline{\mu}^{AB}\epsilon^{A'B'}-\mu^{A'B'}\epsilon^{AB}\right)\left(\overline{\eta}_{B}\eta_{B'}-\pi_{B'}\overline{\pi}_{B}\right), \qquad (6.141)
\end{aligned}
$$

Joining (6.139–6.141) and taking into account the factor *k/2*, we obtain

$$
\begin{aligned}
& i\left(\overline{\mu}^{AB}\epsilon^{A'B'}-\mu^{A'B'}\epsilon^{AB}\right)\left[\frac{1}{2}k(-\eta_{B'}\overline{\eta}_{B}+\pi_{B'}\overline{\pi}^{B})+(a\overline{\pi}_{B}\eta_{B'}+\overline{a}\pi_{B'}\overline{\eta}_{B})\right] \\
& = eF^{AA'BB'}S_{BB'}, \qquad (6.142)
\end{aligned}
$$

which gives the expected relation

$$\frac{d}{dt}S^{i}=eF^{ij}S_{j}.$$

This has to be seen in connection with the discussion of the two carriers system in Chapter 3. The boosts represented either the attraction or the repulsion of the carriers and the rotations were related to this formula which shows that the electromagnetic interaction carries an action over the spin. The two boosts were required to cancel in order to maintain the momentum of the original system and the two rotations were also required to cancel in order to maintain the original angular momentum of the system.

This page intentionally left blank.

Chapter *7*

GEOMETRICAL ANALYSIS

1. INTRODUCTION

This chapter is included for the sake of completeness. It describes the mathematical techniques which have been useful for studying the different approaches which are unified in the present theory, and the basic techniques used in the development of the theory. In particular, it is shown that geometry includes, in a natural form vectors, multi-vectors, spinors, and twistors, tools which some authors utilize as independent. The unity of the presentation should help to find the connections between theoretical approaches in the study of the basic structures of physics.

1.1 THE GEOMETRIC PROGRAM

Geometric analysis is that part of mathematics which is useful to solve geometrical problems. This definition is not void as there is a possibility of defining, beyond an intuitive approach, what is a geometrical problem.

This is more clearly seen if we write a hierarchy of problems and numbers such that a higher step includes the previous ones but contains a question which can not be solved within the domain of the previous steps.

The **natural** numbers are those required when questions such *as* 'if we *have n* object's of a given kind and we add *m* objects of the same kind then we want to know the total number of objects $n + m = p'$. This relation is closed by asking the reciprocal question: 'if we have n' objects and we want. to have a total of p' objects, how many more objects m' do we have to have to obtain p'?'. This enlarges the natural numbers *to* the **integers** and allows the definition of a negative integer instead of the conceptually more restricted question of defining the inverse operation

(+ ⇒ –) and requiring p > *m*

$$p' - m' = q'.$$

Proceeding further, if we start by saying that we have m'' objects a number n'' of times, then we have a total of p'' objects, the concept of multiplication including that of sum, has been created.

$$n''m'' = p'',$$

now we can ask again a reciprocal question: if we have p''' objects, and we want to consider them to be obtained from r sets of m''' object's each, we introduce the concept of **rational** numbers r such that

$$r \times m''' = p''',$$

the name arising froni the possibility of representing r by the definition of the inverse operation (x ⇒ /)

$$r = \frac{p'''}{m'''}.$$

Similarly the exponentiation includes the multiplication and the process of obtaining the root's generates the concept of irrational numbers. By requiring that any one of all these operations can be performed on a set of numbers we now have the concept of the field of **real** numbers which includes irrational, rational and integers. But as we also have to allow the operation of obtaining the square root of a negative number (and questions like **all** a and b such that $a^2 + b^2 = 1$) includes, for the purpose of our analysis, the **imaginary** numbers and the field has been enlarged to the complex field.

Standard algebra and calculus correspond to the study of the previous questions and numbers if the operations of integrals and derivatives are defined through the introduction of the concept of **function,** which to this level are complex-valued functions of complex numbers.

Now we can define what, is a geometric question. A geometric question is a question related to the process of considering the existence of a quadratic form (which can be a function) and the basic geometric question of factorizing the quadratic forms as the product of two linear forms which are called **vectors.** This can be traced back to the fundamental geometric relation (measuring in the Earth's surface) the Pythagoras Theorem (which is in fact a diagonal non degenerate quadratic form)

$$q = z^2 = x^2 + y^2,$$

when we write

$$q = x^2 + y^2 \quad \text{and} \quad \mathbf{r} = xe_1 + ye_2, \tag{7.1}$$

with

$$q = \langle \mathbf{r}, \mathbf{r} \rangle_S ; \tag{7.2}$$

where

$$\mathbf{r} \cdot \mathbf{r} = q\mathbf{1},$$

and

$$e_1^2 = e_2^2 = \mathbf{1},\ e_i\mathbf{1} = \mathbf{1}e_i = e_i,$$
$$e_1e_2 = -e_2e_1, \tag{7.3}$$

and its generalization to N-dimensional cases. Note that, $e_1e_2 \neq e_i$ and is defined by (7.5) below. This is what will be discussed below. The concept of function will be generalized to geometric objects valued functions of geometric objects and the discipline called **Geometric Analysis**. It turns out, as discussed below, that this problem has been considered from a series of starting points of view and that there is a particular one originated by Grassmann and analyzed by Clifford, known today as Clifford Algebra which is a backbone of this analysis because the Clifford algebra is by construction an algebra that allows the introduction of linear forms which factorize a quadratic form as in (7.2). Of course this is the highest step of the hierarchy we consider here and all the previous steps are to be included. The analysis of Riemann (1854) of a N manifold with a quadratic form is also included.

But there is a peculiarity in Geometric Analysis which is crucial to understanding its power as a mathematical tool. If M is a geometric number in an N-dimensional geometry G_0 we can define a mapping

$$M(a + ib) = (a + ib)M := aM + bJM \quad (a, b \in R). \tag{7.4}$$

of the product of a complex number $a + ib$ and the geometric number M through the introduction of two more geometric basis elements e_{N+1} and J together with the previous set of N elements $\{e_a;\ a = 1, ..., N\}$ and the centralizer unit element $\mathbf{1}$ where $\{ei;\ i = 1, 2, ..., N + 1\}$

$$\mathbf{1}e_i = e_i\mathbf{1} = e_i,$$
$$e_ie_j = 2g_{ij}\mathbf{1} - e_je_i, \tag{7.5}$$

and

$$e_iJ = Je_i,\ J^2 = -1 \quad \text{all } i, j,$$

which in fact, results in an $(N + 1)$-dimensional geometric algebra over the real field. For particular geometries there is an advantage in writing the element e_{N+1} as $e_{N+1} := Je_1e_2 \ldots e_N$, $(i^2 = -1)$, then $J = i\mathbf{1}$, as in Chapter 2. This possibility avoids the necessity of explicitly considering complex analysis because (in our approach) geometric real analysis is sufficient. Anyhow we shall discuss this point at some length, given that for practical purposes (and by the very definition of complex algebra as an algebra closed over a given set of operations) complex analysis can not be avoided.

The basic N-dimensional (carrier) space is by definition a Riemannian or a pseudo-Riemannian manifold. In the Grassmann program the elements $s\mathbf{1}$ geometrically represent scalar quantities, the $\Sigma_i\ v_ie_i$ vector quantities, the $\Sigma_{i,j}\ p^{ij}e_ie_j$ planes, the $\Sigma_{i,j,k}\ V^{ijk}e_ie_je_k$ volumes, etc.. We shall then be dealing in a restricted form with differentiable manifolds and their geometry. But even if restricted it is directly useful for the study of large class of problems in classical and quantum mechanics, either in non-relativistic or in relativistic approaches (restricted and general relativity) and as such in elementary particle physics.

From the mathematical point of view this approach is useful in algebraic geometry, Lie group theory, homogeneous spaces, probability theory, differential equations, and in that part of differential geometry which is associated with Riemannian geometry. The relations and results are equivalent and can be cast in the form of tensor calculus, suffix-free (when the indices refer to coordinates) calculus, calculus of exterior forms, moving frame methods or Clifford algebra methods. Spinors and twistors exist by definition (for $N \geq 2$) and have a natural and direct expression in the created multi-vector algebra or as (not necessarily included in the multi-vector algebra) minimal ideals.

The possibility of a straightforward procedure for the embedding (in a $(N + N')$-dimensional geometry) of the (carrier space) N-dimensional manifold, allows the simultaneous considerations of classical differential geometry with an embedding space or of the intrinsic approach to the study of the curvatures. In the same form the concept of fibre bundle and connection have an immediate use and representation. All this characteristic makes the approach described here specially useful for the purpose of the mathematical modelling of the physical phenomena. The formulation bypasses many aspects of the formal structure needed for the completely general study of differentiable manifolds, through the systematic use of the defining quadratic form and its factorization.

An important feature of the procedure followed here is that from the definition (7.4) the complex structure G of the multi-vector space G_0 with original multi-vector dimension $m = 2^N$ has dimension $2m = 2^{N+1}$,

and then, by Campbell's 1926 theorem, if the *N*-dimensional carrier vector space of G_0 is curved it can be locally embedded in its own complexified structure with an equivalent, could be assumed flat, $N + 1$ carrier vector space of *G*.

In the complex structure of the multi-vectors of the geometry *G* wc can alternatively define a complex structure vector space by considering a *c*-vector *X* and define (see, for example, Willmore 1998, pp. 152)

$$(a + ib)X = aX + bJX \quad \text{for} \quad X \in V_c \subset G \quad \text{and} \quad a, b \in R. \tag{7.6}$$

The *c*-vector has structure

$$X = \sum_j (a^j + ib^j)e_j, \tag{7.7}$$

associating a set of *N* complex numbers with the basis of generators of *G* given by (7.5), then (7.6) defines the linear endomorphism

$$J(X) = iX, \tag{7.8}$$

which allows the Proposition (5.1.1) of Willmore 1998 and then the direct use of his analysis of 'Complex and Almost Complex Manifolds'. Our presentation here will follow a different path which permits the extension of the previous presentations, we shall not consider complex coordinates and then the canonical complex structure induced by (7.8) is not used as such. Instead we shall consider, for the modelling of physical objects, the geometric interval

$$d\mathbf{S} = (1 + ip(\upsilon))d\mathbf{x}_\upsilon, \tag{7.9}$$

and the real quadratic interval

$$(dS)^2 = (g_{\mu\nu} + h^\lambda_\mu h^\rho_\nu p_\lambda p_\rho)dx^\mu dx^\nu, \tag{7.10}$$

which corresponds to (7.9) through the use of real quadratic form,

$$\sum_{\{\mu,\nu,\lambda,\rho\}} h^\lambda_\mu h^\rho_\nu p_\lambda p_\rho dx^\mu dx^\nu = \sum_{\{\mu,\nu\}} g_{\mu\nu} p(\mu) p(\nu) dx^\mu dx^\nu. \tag{7.11}$$

Another concept we want to introduce is that of a 'degenerate' geometry. This is the case when a basis set of vectors is defined which is not, linearly independent. In fact in that case the quadratic form (in two dimensions, higher dimensional cases follow by straightforward extension)

$$q = x^2 + 2xy\cos\theta + y^2, \tag{7.12}$$

corresponds not to cos $\theta = 0$, as in the orthonormal case, but to cos $\theta = 1$.

The linear form

$$\mathbf{r} = ze_1 + we_2, \tag{7.13}$$

which obeys the equivalent of (7.12) for $q = \langle \mathbf{r}, \mathbf{r} \rangle$, with $\mathbf{r}\cdot\mathbf{r} = q\mathbf{1}$ requires

$$\begin{aligned} &e_1^2 = 1, \quad e_2^2 = 0, \quad z = x - y, \\ &w = x - y, \qquad e_1 e_2 = -e_2 e_1. \end{aligned} \tag{7.14}$$

Then the quadratic form is diagonal in the new variables but one of the coefficient is zero (this is called a degenerate quadratic form)

$$\begin{aligned} &q_{ij} = \mathrm{diag}\,(1, 0), \\ &q = q_{11} z^2 + q_{22} w^2. \end{aligned}$$

An special case arises when $\{q_{ij} = 0$, all $i,j\}$ corresponding to the usually called Grassmann geometry. In all cases we are requiring the possibility of diagonalizing the otherwise symmetric quadratic form and in all our applications a non-degenerate geometry is considered.

Symplectic geometries are also not considered here. There is, in fact, a mapping of syrnplectic geometries into a complex geometry and, from our previous considerations into a 2*N*-dimensional real geometry. Our mapping is given by (7.10) where the symplectic related variables are p_μ and dx^μ, in pairs (or x^μ and dp^μ in its case). (7.9) is the mapping to a complex geometry and (7.5) the mapping to a real (2*N*-dimensional, non-symplectic) geometry.

1.2 ANALYSIS

A frequent approach in mathematical physics is the use of the geometric concept of one-dimensional, two-dimensional or three-dimensional spaces and of space–time as a frame of reference for the description of matter and interaction fields. In this paper we shall show that this corresponds to postulating a specific approach to geometry and to geometrical analysis. Space–time, in particular, not only describes our perception of physical nature, but is also a powerful analytical mathematical tool itself. Adopting space–time should imply that its structure and symmetries correspond to observed characteristics of the matter and interaction fields under consideration. If a contradictiori or insufficiency were found a wider reference frame should then be constructed and used, which, however, actually does not seem to be the case.

The purpose of this chapter is to present and develop an applied mathematics tool suitable for describing this type of geometries and the fields

representing matter and interaction in these geometries. A common feature which allows this program to be developed is the existence of a quadratic form $\{g_{ij};\ i,j = 1, \ldots, n\}$ for the n-dimensional 'carrier' space. As we shall show it is the very existence of this quadratic form that allows the definition of planes, volumes, hypervolumes, etc., in one multilinear algebra: the geometric algebra. A second, crucial, step is the introduction of dual coordinates and dual vectors allowing the introduction of multi-vector linear functions and derivative operators and then of an extension of real and complex analysis to the case considered here: geometric analysis.

The fact that linear functions and derivative operators are defined requires the study of their eigenfunctions in general. This part of the program brings spinors and twistors into geometric analysis; these geometric elements are defined in terms of their analytic and symmetry properties, md, once their structural relations are found, extended to the concepts of screws and mexors.

In section 3 polynomial algebras are embedded into the geometric algebra. The remaining sections treats fundamental aspects of the geometrical analysis.

Geometric analysis, GA, is then a specific applied mathematics tool which should be practical and sufficient for the purpose stated above. If differential geometry is comprehensive then GA should be contained in it. Otherwise we show here that GA contains, with an economy of postulates and principles, universal Clifford algebras (like the Pauli complex algebra of 3-D space or the Dirac real algebra of space–time), non-universal Clifford algebras (such as the algebra of quaternions), geometry generating algebras (such as spinors in the Cartan map or twistors in the twistor program) and algebras of analytical n-dimensional Riemannian manifolds which can then, by Campbell's theorem, be locally embedded in an $(n + 1)$-dimensional carrier space of null curvature (as in the Kaluza–Klein theories and their extensions, our physical approach above being not of this type). The minimum value of our program is its capacity of coherence and relation of the matheniatical structures mentioned in this paragraph, the systematic generation of their extension and of new structures.

1.3 THE MULTI-VECTOR ALGEBRA

As mentioned above, our formulation refers to and includes the algebras proposed by Grassmann and Clifford. We first recall that in textbooks related to algebras or to geometry (see e.g., Frankel T., 1997; Nakahara M., 1990; Porteous I.R., 1981, 1994, 1995) a standard approach to define a Clifford algebra is the following:

Let R_n be an associative algebra (over $\mathbb{R}$) with unit element I, generated by an n-dimensional vector subspace V^n. If (,) denotes any real symmetric quadratic form on V^n, and if V^n has a basis $e_1, \ldots, e_n$ satisfying

$$ejek + ekej = 2gjkI, \tag{7.15}$$

where $g_{jk} := \langle e_j, e_k, \rangle$ then $Rn = R(V^n)$ is called the Clifford algebra generated by V^n, induced by means of the quadratic form $\langle , \rangle$.

Some authors refer in a limiting form to Clifford Algebra as an algebra generated by a set of n mutually anti-commuting square matrices.

From here on we shall use the symbol $C\ell n$ for a real Clifford algebra generated by n elements $\langle e_i, i = 1, ..., n)$ without a specific definition of the quadratic form; the symbol $\mathcal{C}\ell_n^{(c)}$ will be used for a complex Clifford algebra generated by n elements $\langle e_i, i = 1, ..., n\rangle$ without a specific definition of the quadratic form. When a quadratic form is diagonal with respect to p positive and q negative elements g_{ii}, $p + q = n$, the notation will be $\mathcal{C}\ell_{p,q}$ or $\mathcal{C}\ell_{p,q}^{(c)}$. This far there are no requirements for the quadratic form $\langle , \rangle$: it can be Euclidean, pseudo-euclidean, symplectic, degenerate, etc.

As an example, consider a Clifford algebra generated by an n-dimensional vector space V^n with a quadratic form $\langle , \rangle$ identically 0. Then it is evident that we have, by necessity, generated a new set of elements e_{jk} corresponding to the anticommuting products (the symbol Λ is introduced to stress that independently of the chosen quadratic form, we consider here the totally associative (and antisymmetric in its basis vectors) part of the product (denoted in general by $A \Lambda B$ for any A and B in $C\ell_{p,q}$), that is for vectors a and b, using the symbol $\stackrel{e}{=}$ to denote a definition, $a \Lambda b \stackrel{e}{=} \frac{1}{2}(ab - ba)$),

$$e_{jk} \stackrel{e}{=} e_j \mathrm{L}\, e_k = -e_k \mathrm{L}\, e_j, \tag{7.16}$$

for all j, k, and of course $(e_j \Lambda e_j) = 0$. The resulting Clifford algebra, once all possible totally anticommuting products of 2, ..., n vectors are considered, is simply the exterior algebra, denoted by $\Lambda(V^n)$ in the following, based on the original vector space V^n, the total algebra being of dimension 2^n. The exterior product of a vector by itself is by definition not to be included.

In general $C\ell_n$ is generated by products of the form $e_ie_j \ldots e_k$. Each $(e_j)^2$ is a multiple of g_{jj} times the identity, and thus commutes with all elements $\in R^n$, i.e., there is no need to consider expressions containing a repeated basis vector e_j. From the above anticommutation properties (7.16) we need to consider only ordered products $e_{ij\ldots k} = e_ie_j \ldots e_k$, $i < j < \ldots < k$. It is then obvious that as a vector space (i.e., neglecting the product structure), the Clifford algebra $R(V^n)$ is isomorphic ($\approx$) to the exterior algebra $\Lambda(V^n)$ or $R(V^n) \approx \Lambda(u^n)$ and thus of dimension 2^n. This requires that **no** e_i can be represented by products of the remaining $e_j (j \neq i)$.

For example, the (complex) Pauli algebra, as a vector space, is isomorphic to the exterior algebra on $\mathbb{R}^3$ with an abstract basis given by σ_1,s_2, and σ_3, where $\sigma_1\sigma_2\sigma_3 = iI$ and the exterior product can be represented by the product of Pauli matrices, that is, by products of matrices defining a faithful representation of the basis vectors.

In an exterior algebra the real coefficients, that is the scalars, span a one-dimensional subspace. In a Clifford algebra the scalar multiples of the unit, element I form a one-dimensional subspace which can be identified with the coefficient field $\mathbb{R}$.

In order to form a Clifford algebra with generators $e_1, \ldots, e_n$ and a quadratic form $\langle , \rangle$, we simply consider all new elements, $e_{ij\ldots k} = e_i e_j \ldots e_k$ with $i < j < \ldots < k$, obeying relation (7.16). Some examples:

(1) $\mathcal{C}\ell_0 \cong \mathbb{R}$ is simply the algebra of real numbers, the unit element I being selfevident.

(2) Similarly let V^1 be a 1-dimensional vector space with the basis e_1, and consider the (negative-valued) quadratic form $\langle e_1,e_1 \rangle = -1$. Thus we have obtained a 2-dimensional vector space with a formal basis consisting of $\{I, e_1\}$, e_1 satisfying the a negative metric in (7.16), $(e_1)^2 = (-1)I$. From this definition the basis element. e_1 can then be called i, such that $(e_1)^2$ can be identified with the real number -1, with again the unit element I being selfevident, and the 2-dimensional vector over $\mathbb{R}$ is simply the algebra of complex numbers $a + b_i$, $\mathcal{C}\ell_{0,1} \cong \mathbb{C}$.

Since the early papers of Grassmann in the last century (Grassmann, 1844), the algebra described here was recognized as such, and in fact named a geometric algebra: the product of two non-collinear vectors corresponds to a unit plane; the product of three mutually non-collinear vectors to a volume, etc.. This program was given the name 'extensions program' by Grassmann himself.

In section 9 we briefly describe the historically more relevant developments of this program in the Nineteenth and Twentieth century, starting from the pioneering paper of Caspar Wessel, presented to *L 'Académie Royale des Sciences et des Lettres de Danmark* in 1797 (French translation 1897).

The vector algebra and analysis of Gibbs is an useful applied mathematics tool constructed for 3-D space where it is possible for a bi-vector to be faithfully represented by an axial vector and a volume by the triple (pseudo) scalar product. It can not be generalized to dimensions larger that 3 and it does not exist in 2-D.

In this century the main contributions to the use of geometric algebras in physics first of all came from the use of the Pauli algebra in the theory of spin, and second from the use of the Dirac algebra in the study of the electron, and from that of space–time. Fock and Ivanenko (Fock V., 1929; Fock V. and Iwanenko D., 1929) were the first ones to realize that

the Dirac algebra was a representation of geometry. Unfortunately they coined the name Quantum Geometry, only afterwards was it recognized as a space–time geometric algebra by a large series of authors, amongst whom probably the more relevant were: Proca A., 1930 a,b,c; Sauter F., 1930; Mercier A., 1934, 1935; Eddington A.S., 1936; Sommerfeld A., 1939; Riesz M., 1946, 1953, 1958; Schönberg M., 1956; Quilichini P., 1957; Ravsevskii P. K., 1957; Teitler S., 1965a,b,c, 1966a,b; Hestenes D., 1966, 1975, 1979; Casanova G., 1970, 1976; Boudet R., 1971, 1974, 1985; Salingaros N. and Dresden M., 1979; Greider T. K., 1980; Keller J., 1981–1999; Crawford J. P., 1985.

In the last decade the number of papers on this topic has increased considerably, and there is at least one journal (*Advances in Applied Clifford Algebras*) and a series of books and of conferences with proceedings devoted to geometric algebra (for example Altmann S. L., 1986; Artin E., 1957; Baylis W. E., 1996; Benn I. M. and Tucker R. W., 1987; Brackx F., Delanghe R. and Serras H., 1993; Chisholm J. S. R. and Common A. K., 1986; Habetha K., Dietrich V. and Jank G., 1998; Keller J. and Oziewicz Z., 1997; Micali A., Boudet R. and Helmstetter J., 1991; Snygg, 1997;), often using names like space–time Algebra (Hestenes D., 1966), Geometric Algebra, Clifford Algebras, Quaternionic Algebras, or less familiar names such as Manifolds with Grassmann Algebras or manifolds with Grassmann variables. Also very frequently a matrix representation of the algebras is considered rather than the abstract algebra itself, in particular if the names of Pauli or Dirac are associated with.

Below we shall give a special emphasis to the study of space–time in which (because of historical reasons) we use also the Dirac γ^μ as symbols for the basis vectors e^μ. The elements γ^A of the Dirac–Clifford algebra $C\ell_{1,3}$ are then the dimensionless totally antisymmetric Grassmann numbers: $\gamma^A = \gamma^i \wedge \gamma^j \wedge \ldots$. Here the basic Clifford numbers $(\gamma^\mu; \mu = 0,1,2,3)$ generate a Clifford algebra by means of a Clifford product such that the metric is defined by $(\gamma^0)^2 = -(\gamma^1)^2 = -(\gamma^2)^2 = -(\gamma^3)^2 = 1$. If an arbitrary element of the Clifford ring $\mathcal{C}\ell_{p,q}$ is denoted by γ^A; $A = 1, \ldots, 2^n$; $p + q = n$, then complexification of $\mathcal{C}\ell_{1,3} \to \mathcal{C}\ell_{1,3}^{(\mathcal{C})}$ refers to a complexification of the basis $\gamma^A + i\gamma^A$; $\gamma^A \subset \mathcal{C}\ell_{1,3}$. Remembering that, e.g., in the Dirac case $\gamma^0, i\gamma^{12}$ and $i\gamma^5$ are Hermitian, this clearly indicates that in physics the use of a complex version of a particular geometry is frequently needed and therefore will be introduced right, from the beginning of our present analysis.

1.4 SPINORS AND TWISTORS

The simplest mathematical objects which are eigenfunctions of the 3-D rotation group and the 4-D Lorentz group are the spinors η and η^+ (Cartan E., 1981) and of the 4-D Poincaré group are the twistors (Penrose R., 1967). This with basis elements $\left\{ \eta_a^0, \eta_a^{0+};\ a = 1, \ldots, 2^{\left[\frac{n+1}{2}\right]} \right\}$ where [*r*] denotes the integer part of *r*) are then fundaniental for the understanding of the symmetries of geometric spaces and for the construction of the mathematical structures modeling nature in these spaces. Their definition and properties is therefore a basic task in geometric analysis.

The Cartan map allows the inverse construction: the reconstruction of the multi-vector algebra *M* as a bilinear mapping of the spinor and twistor space: $M_A = M_A^{ab}\eta_a^0\eta_b^{0+}$, where $M_A \in M$, $M_A^{ab} \in \mathbb{C}$ $\eta_a \in \eta$ and $\eta_b^+ \in \eta^+$.

The projection of the minimal ideal and the bilinear reconstruction of multi-vectors is naturally contained, as a super algebra with its structure represented by the matrix relation, $Si \in \mathbb{C}$,

$$\begin{pmatrix} M_A & \eta_a \\ \eta_b^+ & S_1 \end{pmatrix} \begin{pmatrix} M_{A'} & \eta_{a'} \\ \eta_{b'}^+ & S_2 \end{pmatrix} = \begin{pmatrix} M_{A''} & \eta_{a''} \\ \eta_{b''}^+ & S_3 \end{pmatrix},$$

in geometric analysis. This part of the program is the subject of the last sections of this chapter.

1.5 GEOMETRIC ANALYSIS

Consider the following steps.

1.5.1 SELECTION OF A R^N MANIFOLD

Consider a set of *n* coordinates, collectively denoted by *X* (here the definition of patches and coordinates on patches is assumed, see also section 10 below):

(la) A manifold is transformed into a real quadratic space by introducing a quadratic form (usually associated with the names: scalar product, square, quadratic norm or quadratic polynomial), where for any arbitrary pair *xi,xj* ∈ *X:*

$$X^2 \to \mathbb{R};\ P(X,X) = \sum_{ij=1}^{n} a_{ij} x_i x_j, \tag{7.17}$$

such that $a_{ij}x_ix_j \in \mathbb{R}$ even if any one of the factors is not necessarily a real number The a_{ij} are defined as the elements of a (generalized) quadratic matrix of dimension *n* ($a_{ij} = a_ia_j \in \Xi^*$ and $x_ix_j \in$ X, such that $p'(\Xi^*, \Xi) \in \mathbb{R}$), where the x_ix_j are members of the (multi-vector) algebra Ξ defined in (lb) (below), although in general they and the a_{ij} are taken to be real numbers.

(1b) The geometrical meaning of X is introduced by means of a linear form

$$L(X) = \sum_{i=1}^{n} \alpha_i x_i, \tag{7.18}$$

such that

$$[(L(X))^2]_{\text{scalar}} = P(X, X) \in \mathbb{R}. \tag{7.19}$$

The geometry generated via (7.17) is fulfilled by squaring the linear form in (7.18). This procedure, as mentioned in the introduction, defines the mathematical elements corresponding to the antisymmetric product of the α_i as new algebraic elements belonging to a ring generated by the collection of all possible totally antisymmetric products of the α_i.

(1c) The above definitions allow the $\{x_i\}$ to be not only real variables, but also more general mathematical entities such as complex variables, quaternionic variables, Grassmann variables, other non-commuting variables, etc., such that $x_i x_j \in \Xi$, with $\Xi^* \Xi \in \mathbb{R}$. Simultaneously the set $\{\alpha_i\}$ can be represented by a standard non-commuting basis, called $e_i = e(\alpha_i)$ in (7.23) below, or by more general elements compensating the departure of the set $\{x_i\}$ from the standard choice as real numbers. This is the geometrical map of the algebras now named q-deformed or quantum algebras.

1.5.2 GEOMETRIC SPACE

Because the above paragraphs (1a–c) define an outer algebra with a quadratic form, we can now identify the a_i in terms of the e_i as generators from their antisymmetric products of the basis of a Clifford algebra $\mathcal{C}\ell_{p,q}$ with $p + q = n$, which automatically introduces a geometric space and its metric. The Clifford algebra is then generated by a set of (anticommuting) elements $\{e_i = \langle e_i \rangle_1\}$, $i = 1,2, \ldots, n$, with $\langle\rangle_r$ denoting the rank or number of factors in the geometric product defined below, considered as basis vectors with a metric $g_{ij}\mathbf{1} = \frac{1}{2}(e_i e_j + e_j e_i)$, defined also below, such that by definition $e_i e_j = -e_j e_i + 2g_{ij}\mathbf{1}$, where $\mathbf{1}$ denotes the unit element. The Clifford ring is then generated by the repeated geometric product of arbitrary pairs of elements e_i and e_j,

$$e_i e_j = \frac{1}{2}(e_i e_j + e_i e_j) + \frac{1}{2}(e_i e_j - e_i e_j). \tag{7.20}$$

The resulting product can be separated into a symmetric and an antisymmetric part; it is customary to denote these operations as clot and wedge parts of the product (it is unfortunate to call them products):

$$e_i e_j = e_i \bullet e_j + e_i \wedge e_j. \tag{7.21}$$

A set of basis vectors $e_i, e_j, e_k, \ldots \in \{e_\mu = \langle e_\mu \rangle_1\}$ is called orthonormal if the metric is diagonal with p assuming the value +1 and q the value

–1, or, equivalently,

$$g_{ij} = \text{diag}\ (1, \ldots, 1^{(p)}, -1, \ldots, -1^{(n)}). \tag{7.22}$$

A geometry, based now on a Clifford ring, is generated when the coordinates are introduced in the following manner, using the summation convention of upper and lower repeated indices

$$x \rightarrow \mathbf{x} = x^i e_i, \tag{7.23}$$

i.e., a geometric space with the properties of the corresponding Clifford algebra by multiplication of coordinates.

Other definitions of a metric tensor g_{ij} are, of course, also acceptable and correspond to a different set of n generators from those of the Clifford ring. This allows null vectors or symplectic scalar products to be used as basis vectors or as basic quadratic forms. They all can be mapped into a Clifford ring defined as above. Thus a full linear algebra with 2^n degrees of freedom can be generated. This in turn is equivalent to the following propositions:

> The real algebra of an n-dimensional quadratic space is a 2^n-dimensional linear algebra. In the following this algebra will be called a geometric space.

As can be seen from (7.23) the variable **x** carries geometrical information by either defining a set of operators which project relevant information from the manifold, or, because of the geometry provided by the e_i and their properties.

In a **quadratic space** the manifold R^n can be viewed in terms of a patchwise set of n coordinates $\{x_i;\ i = 1, \ldots, n\}$ and, additionally a dual set of n coordinates $\{x^i;\ i = 1, \ldots, n\}$. These two collections of possible coordinates induce, in fact, a metric for the geometry of the manifold. From these sets of coordinates we can define the basic operators $\{\hat{e}_i, \hat{e}^i, i = 1, \ldots, n\}$ creating thus the geometry $G(R^n, \mathcal{C}\ell_{p,q}, p + q = n)$ as

$$\begin{aligned} \hat{e}_i &\rightarrow \partial_{x^i}, \\ \hat{e}^i &\rightarrow \partial_{x_i}, \end{aligned} \tag{7.24}$$

with

$$\begin{aligned} \partial_{x^i}\mathbf{x} &= e_i, \\ \partial_{x_i}\mathbf{x} &= e^i, \end{aligned} \tag{7.25}$$

where the basic definition (see 5.1 below) is the operator relation

$$e^i e_j = \delta^i_j. \tag{7.26}$$

Relation (7.26) is similar to that amongst **forms** and **vectors** in differential geometry.

The properties which will be assigned to the vectors e_i, in our case a Clifford algebra, provide a geometry for the manifold. The basic manifold has no geometry, only a set of coordinate patches, unless a relation for the**x**, see (7.23), is defined from a given R^n. Different geometries can be created by assuming different properties of the basic operators creating the vectors, and from the properties of the vectors themselves.

1.5.3 PRODUCTS AND MULTIPLICATION TABLES

In the Clifford ring the multiplication table is usually given by grouping repeatedly the so called geometric multiplication into symmetric and antisymmetric parts and by introducing the, totally antisymmetric in their indices, symbols *e*...

$$e_i e_j = \frac{1}{2}(e_i e_j + e_j e_i) + \frac{1}{2}(e_i e_j - e_j e_i) = g_{ij}1 + e_{ij}, \qquad e_{ij} = -e_{ji}, \quad i,j = 1,...,n. \tag{7.27}$$

Our main example is the geometric algebra $C\ell_{1,3}$ of space–time $R^{1,3}$ generated by the Clifford algebra of the set of four vectors e_μ

$$\begin{aligned} &\mathbf{1},\ \mathbf{e}_\mu,\ e_\mu e_\nu = g_{\mu\nu} + e_{\mu\nu},\ g_{\mu\nu} = diag\,(1,-1,-1,-1) = g_{\nu\mu}, \\ &e_{\mu\nu} = -e_{\nu\mu},\ e_\rho e_{\mu\nu} = g_{\rho\mu}e_\nu - g_{\rho\nu}e_\nu - e_{\rho\mu\nu}, \\ &e_\lambda e_{\mu\nu\rho} = g_{\lambda\mu}e_{\nu\rho} - g_{\lambda\nu}e_{\mu\nu} + e_5 \quad \text{or} \quad e_5 = e_{0123}. \end{aligned} \tag{7.28}$$

and for all $\{\mu, \nu, \lambda, \rho\} = 0,1,2,3$.

This defines the basic multi-vectors $e_{\mu\nu...}$, totally antisymmetric with respect to the interchange of the vector indices $\mu, \nu, ...$ The number of indices defines the grade or blade of basis multi-vectors. In the same example, consider a Dirac spinor ψ (dual spinor $\psi^\dagger$) as a member of the minimum left (right) ideal of the geometric algebra $C\ell_{1,3}$ of space–time (see section 9). Multi-vectors have a representation in terms of the Dirac spinors $e_{\mu\nu...} \to (e_{\mu\nu...})^{\alpha\dot{\alpha}'}\psi_\alpha\psi^\dagger_{\dot{\alpha}'}$. The Dirac spinors themselves have four (complex) degrees of freedom $\Psi \to \Psi^a\psi_a$, where the ψ_a are basis spinors and $a = 1, 2,3,4$. Below we show that we can use the correspondence $(a = 1) \to (R, \uparrow)$, $(a = 2) \to (R, \downarrow)$, $(a = 3) \to (L, \Uparrow)$ and $(a = 4) \to (L, \Downarrow)$, that is, right ($R$) and left ($L$) handed spinors of spin up ($\uparrow$) and down ($\downarrow$), or chiral representation. If we write $e_ie_j = e_i.e_j + e_i\Lambda e_j$ then the symbol $\cdot$ and the symbol Λ clearly refer to operator relations.

For other applications we shall, additionally, allow an exterior Kronecker product $e_i \otimes e_j$ of the irreducible multi-vectors e_a.

Representations of x. Because of the definitions in (7.24) and (7.25) a representation of **x** can equivalently be given as

$$\mathbf{x} = x^i e_i = x_j e^j, \tag{7.29}$$

in the form 'induced' by the derivation operators. A possible representation of e^i, which explicitly includes its operator character, is $e^i = (g^{ij} e_j)\bullet$, the dot indicating the specific operation to perform. Clearly enough a different definition of the commuting properties of the x_i, or of the operators, modifies this representation, although some definitions could be equivalent.

1.5.4 MAPPINGS AND TRANSFORMATIONS

Consider a differentiable (divergence ∇x' can be defined) transformation of *n* chosen coordinates of the following kind

$$\mathbf{x} \to \mathbf{x}' = \mathbf{f}_c(\mathbf{x}), \qquad \mathbf{x} \in R^n, \tag{7.30}$$

where the coordinates **x**' contain now some particular characteristics. Clearly enough, whilst for the vector x, 'anchored' to some local origin of coordinates, we have (7.30), for an arbitrary 'free' vector $\mathbf{v} \in R^n$, a modified vector **v**' is obtained from

$$\mathbf{v} \to \mathbf{v}' = \mathbf{f}(\mathbf{v}), \tag{7.31}$$

considering the divergence of **x**' in the direction of **v** as **v**' = (**v** . ∇)**x**'.

Since not all transformations are related to coordinate transformations, therefore we have to study the general cases. Because there arc various forms of proceeding we can define the following set of four interrelated functions (explicitly presented in section 8 below)

$$\{\mathbf{f}, \mathbf{f}^{-1}, \mathbf{f}_D, \mathbf{f}_D^{-1}\}, \tag{7.32}$$

either to change from primed coordinates to unprimed coordinates and vice versa (passive transformation) or to change the vector fields themselves (active transformations). In this context the inverse operation corresponds to the definition

$$\mathbf{f}(\mathbf{f}^{-1}(\mathbf{v}) = \mathbf{f}^{-1}(\mathbf{f}(\mathbf{v})) = \mathbf{v}. \tag{7.33}$$

The dual operations $\mathbf{f}_D$ and $\mathbf{f}_D^{-1}$ allow such a change when both **f**(**v**) and **v** are expanded in relation to some particular vectors, the adjoint operation will be discussed in section 8 below.

This kind of procedure relates a particular geometry $G_{p,q}$ associated with a manifold R^n to the manifold itself. Given that $G_{p,q}$ has 2^n degrees

of freedom, which in turn corresponds to the 2^n elements of $\mathcal{C}\ell_{p,q}$, the geometry $G_{p,q}$ contains an implicit set of 2^n fields $X_M(\mathrm{x})$ (complex or real functions of the carrier space **x**), where M has 2^n values according to the construction $X_M(\mathbf{x})e_M$ and $\{e_M;\ M = 1, 2, \ldots, 2^n\}$ is a complete basis of the Clifford algebra.

1.5.5 ANALYSIS

Once a particular geometry $G_{p,q}$ is chosen one can use basic operators such as transformation operators, differential operators, integral operators, and other special operators in order to study the structure and the symmetry of both the geometry $G_{p,q}(R^n, \mathcal{C}\ell_{p,q})$ and the geometrical fields constructed over $G_{p,q}$. The definition (7.26) is basic here.

We could, of course, consider the use of the complexification of the basis of $G_{p,q}$. It should be noted, however, that, this is different from constructing a geometry from an n-dimensional complex space $\mathbb{C}^n$. In the former case one embeds the manifold into a Clifford algebra $\mathcal{C}\ell_{p',q'}$, $p' + q' = n + 1$, such that the number of degrees of freedom changes from $2^n \rightarrow 2 \times 2^n = 2^{n+1}$, then complexification of a geometry corresponds to a change from a $\mathbb{R}^n$ to a $\mathbb{R}^{n+1}$ carrier manifold whcrc the new degree of freedom can be a trivial coordinate with simplified topology (notation remark: we are using the symbol $\mathbb{R}^n$ for the carrier manifold and the symbol $\mathbb{R}_n$ for the resulting multi-vector algebra with 2^n degrees of freedom).

1.6 SUCCESSIVE COMPLEXIFICATION

As a complexification of an n-dimensional basic geometry G_n is equivalent to an $(n + 1)$-dimensional basis geometry G_{n+1} we can obtain successive, nested algebras and geometries by complexification of lower dimensional ones. Because every resulting geometry can itself be complexified, the algebras and geometries can be nested in even groups. This brings into the study of a n-dimensional space with complex structure the concepts of higher-dimensional spaces (Kaluza T., 1921, Klein O., 1926, Weyl H., 1952, Wesson P. S., 1999). Otherwise the geometries are in all cases multilinear over the field of the real numbers, there are no complex or imaginary coordinates being used and the symbol i such that $i^2 = -1$ refers to the definition of the elements of the ring $\mathcal{C}\ell_{n+1}$ generated by the e_u in terms of another ring $\mathcal{C}\ell_n$ generated by some given e_k with $\mathcal{C}\ell_n \in \mathcal{C}\ell_{n+1}$.

1.7 REPRESENTATIONS OF GEOMETRIES

Geometries can be represented by sets of square matrices $M(m)$ of dimension m such that the geometrical product is faithfully represented in terms of matrix multiplication. Since there are 2^n degrees of freedom in a chosen geometry, a representation in terms of real square matrices implies dimension of $m = 2^{[(n+1)/2)]}$. In many cases, however, the use of complex square matrices is more convenient. In going from $\mathbb{R}_n \rightarrow \mathbb{R}_{n+1}$ this is achieved by the step $\{M(m \times m)\} \rightarrow \{M(m \times m)(\mathbf{1} + \mathbf{i})\}$ with a suitable matrix representation of $\mathbf{1}$ and $\mathbf{i}$ (see Porteous I. R., 1981 and the review by Keller J., 1993). Spinors and twistors are faithfully represented by $M(1 \times m)$ and $M(m \times 1)$ for their duals.

2. THE EXAMPLE OF SPACE–TIME

The best known example is the standard reduction chain of even sub-algebras corresponding to the geometries of (in the geometrical sense) complex space–time, space–time, 3-D space, 2-D space and 1-D space, respectively:

$$\mathcal{D}_c \rightarrow \mathcal{D} \rightarrow \mathcal{P}_c \rightarrow \mathcal{P} \rightarrow C^1 \rightarrow R^1, \tag{7.34}$$

where $D_c = \mathcal{C}\ell_{0,5}$ is the complex Dirac algebra, $D = \mathcal{C}\ell_{1,3}$ is the Dirac algebra, P_c the complex Pauli algebra, P the Pauli algebra, $C' = \mathcal{C}\ell_{0,1}$ the complex algebra and $R = \mathcal{C}\ell_{0,0}$ is the real algebra, each with 2^n elements, $n = 5,4,3,2,1,0$, respectively.

Complexification is obtained by formally writing

$$\mathcal{D}_c = \mathcal{D}+i\mathcal{D}, \qquad \mathcal{D} = \mathcal{P}_c + i\mathcal{P}_c,$$
$$\mathcal{P}_c = \mathcal{P}+i\mathcal{P}, \qquad \mathcal{P} = C + iC, \qquad C = R + iR,$$

or, equivalently $D_c = D \otimes C$, etc.. Nesting then corresponds to formally considering the 'real' part only, namely $D = \mathrm{Re}(D_c)$, $P_c = Re(D)$, $P = \mathrm{Re}(P_c)$, $C = \mathrm{Re}(P)$ and $R = \mathrm{Re}(C)$. From here on an 'even' part refers to those basic geometric elements obtained by an even number of products of the basis vectors.

For a faithful representation with Dirac symbols $\gamma_A = \gamma(e_A)$ we first require the five basic vectors of D_c to be of the form $(\mathcal{C}\ell_{0,5})_1$ using u, υ, m, n, $w = 1, \ldots, 5$ and $\mathbf{i} = i\mathbf{1}$

$$\gamma_\nu = \{i\gamma_{123}, i\gamma_0, i\gamma_{01}, i\gamma_{02}, i\gamma_{03}\}, \quad \mathcal{D}_c = \{\mathbf{1}, \gamma_u, \gamma_{uv}, \gamma_{muv}, \gamma_{mnuv}, \mathbf{i}\},$$
$$\mathcal{D}_c = (\mathcal{D}_c)_{\mathrm{even}} + (\mathcal{D}_c)_{\mathrm{odd}} = \{\mathbf{1}, \gamma_{uv}, \gamma_{uvmn}\} + \{\gamma_\mu, \gamma_{muv}, \mathbf{i} = \gamma_{mnuvw}\},$$

(7.35)

such that its even part corresponds to $\mathcal{D} = \mathcal{C}\ell_{1,3}$.

Consider now spinors χ_α and the Cartan map $e_A = \epsilon_A^{ab}\chi_a\lambda_b^\dagger$. D_c and D share the same spinor basis, namely Dirac spinors. P_c and P have, both, two independent representations based on Pauli spinors (usually denoted by ($\chi_{\uparrow L}$, $\chi_{\downarrow L}$) and ($\chi_{\uparrow R}$, $\chi_{\downarrow R}$), called 'dotted' and 'undotted', respectively). The spinor basis of C^1 and R^1 is a trivial complex number or a real number, respectively. For the case of twistors see section 8 below.

In the possible use of the geometry for the study of mathematical structures in physics it is important to emphasize here that in quanturri mechanics all rotations in the representation of D_c are quantized (in the standard formulation of quantum mechanics, (Keller J., 1985)) and that. changes in the spinor basis $\chi_\alpha \rightarrow \chi_\alpha\ \epsilon_\alpha^\beta\chi_\beta$ correspond to supersymmetry changes, using the matrix operators (defined in the introduction), see, e.g., [105]. In general $\epsilon_\alpha^\beta = \epsilon_\alpha^\beta(\mathbf{x})$.

In this respect consider that not only the elements $i\gamma_{0i}i\gamma_{0j} = \gamma_{ij}$ represent rotation generators, but also terms like $i\gamma_{123}i\gamma 0$ are rotations in $\mathcal{C}\ell_{0,5}$ and have a physical meaning. In fact, the energy operator $\hat{H} = i\hbar\partial_t$, contains three factors: (1) i corresponds to the eigenvalue of an operator turning $\hat{H}$ into a Hermitian operator; (2) the 'scale' $\hbar$ determines the unit of 'action' (here the product of energy and time of one elementary physical action); and (3) with respect to the multi-vector aspect. the derivative ∂_t can geometrically be replaced by γ^0 if γ_0 is the unit time vector. Then from the definition of dual vectors $\gamma^0\gamma_0 = 1$ corresponds geometrically to the derivatives $\partial_t t$. Using the dual vector $\gamma^0 = \wedge\gamma_{123}\gamma_{0123}^{-1}$ (here we should recognize γ_{0123}^{-1} as $-\gamma_{0123} = -\gamma_5$, from $\gamma_5^2 = -1$), then from $\gamma^0 = -\Lambda\,\gamma_{123}\gamma_5$ it follows that $i\hbar\partial_t \rightarrow i\hbar\gamma_5\gamma_{123}\wedge$. We furthermore remind the reader that i can be viewed also as an eigenvalue α_5 of γ_5,

$$\gamma_5 F = a_5 F \quad ; \quad \gamma_5\gamma_5 F = a_5^2 F = -F \text{ or } a_5 = i, \tag{7.36}$$

where the F are suitable eigenfunctions.

As mentioned above, all (multi)-vectors can simultaneously be regarded as operators, the best known examples are: γ_0, generating the parity inversion P; the tri-vector γ_{123}, the time inversion T; the bi-vector γ_{0i}, the Lorentz boost's $\mathcal{L}$; the bi-vector γ_{ij}, the space rotations R: the tetravector γ_5, the duality transformation D; and the complex tetravector $i\gamma_5$, the chirality projection.

In space–time $\mathcal{C}\ell_{1,3}$ the pseudo-scalar unit is given by $\gamma_5 = \gamma_{\mu\nu\lambda\rho}\varepsilon^{\mu\nu\lambda\rho}/4!$, in complex space–time, however, it is simply the pentavector $\mathbf{i}$ $(= \mathbf{1}\sqrt{-1})$ in $\mathcal{C}\ell_{0,5}$. In fact, D_c can be regarded both as the complexification of the space–time geometric algebra or as a five-dimensional space whose even subalgebra corresponds to space–time.

2.1 CHIRAL SYMMETRY IN COMPLEX SPACE–TIME

It is assumed that, a local observer describes space–time by

(a) an orthonormal tetrad of $(\gamma^0)^2 = -(\gamma^1)^2 = -(\gamma^2)^2 = -(\gamma^3)^2 = 1$, and $\gamma^\mu\gamma^\nu = -\gamma^\nu\gamma^\mu$ all $\mu \neq \nu$. In this frame of reference we define:
(b) the handedness of $\gamma^5 = \gamma^0\gamma^{1\gamma 2}\gamma^3$, which is both the duality transformation operator and the pseudo-scalar $(\gamma^5)^2 = -1$.

It is important that another observer can use a different coordinate system related by a Lorentz transformation L, where together with (a) and (b) the fundamental properties, $(i\gamma^5)^2 = 1$ and $\gamma^5\gamma^\mu = -\gamma^\mu\gamma^5$ are also preserved.

The handedness operator $H = i\gamma^5$ can be used to construct the chirality projectors P_R and P_L such that

$$P_R + P_L = 1, \quad P_R P_R = P_R, \qquad P_L P_L = P_L, \quad P_R P_L = P_L P_R = 0, \tag{7.37}$$

where $P_R = \frac{1}{2}(1 + i\gamma^5)$, $P_L = \frac{1}{2}(1 - i\gamma^5)$,or $P_{R,L} = \frac{1}{2}(1 + \pm H)$.

If a coordinate transformation $\gamma^5 \rightarrow (\gamma^5)'$ is allowed where (a), and consequently (b), is not preserved (that is if the determinant ζ of the transformation is not $\zeta = +1$) then $H \neq i(\gamma^5)'$ shows that a chirality operator $H = i(\gamma^5)'/z$, with $H^2 = 1$ in all frames has to be used. Here we shall assume that $H = i\gamma^5$, because of restriction (a) and the assumption that, we have selected a 'right' handed frame of reference. The P_R and P_L can better be considered as numbers of a 'new' mathematical field, with the basis 1 and H, in an hyper-complexification of the Clifford algebra. Clearly, H is coordinate invariant.

2.2 DEGENERATE REPRESENTATIONS

The basis vectors can be written in terms of degenerate representations Γ_μ which in turn can be written as exterior products of an irreducible basis. For (complex) space–time as an example, we can use as generators $\gamma^\mu, \gamma^5, i\gamma^5$ and 1, and a reducible representation (here $\otimes$ represents exterior product) can be formulated as (see Królikowski W., 1990, 1992):

$$\Gamma^\mu_{(f)} = \gamma^\mu \otimes (1 \otimes 1 \otimes ...)_{2(f-1)\text{products}} \tag{7.38}$$

The corresponding spinors would then be the totally antisymmetric, exterior $2(f - 1)$ products $\psi_{(f)} = \psi(x) \Lambda (\psi_1 \Lambda \psi_2 \Lambda ...)$, where the ψ_i are $2(f - 1)$ constant Dirac spinors which correspond to extra mathematical,

that is internal, degrees of freedom. Their spin should add to zero (f integer). The total antisymmetry of $\psi_{(f)}$ limits the value of f to f = 1, 2, 3, otherwise the spinorial exterior product is null.

2.3 MASSLESS WAVE EQUATIONS

The actual wave functions can be written as $\Psi^{(0)} = \tilde{\mathbf{R}}\Psi$ where $\tilde{\mathbf{R}}$ is a local boost and rotation from the frame of reference in which the transformations $D^{(0)\dagger}_{(d,f)}$, see below, are defined with respect to the frame of reference of the observer. Because of the following outermorphism

$$\widetilde{\mathbf{R}}\gamma_\alpha\widetilde{\mathbf{R}}^{-1} \rightarrow \widetilde{\mathbf{R}}\gamma_{\alpha\beta\ldots}\widetilde{\mathbf{R}}^{-1} = \widetilde{\mathbf{R}}\gamma_\alpha\widetilde{\mathbf{R}}^{-1}\widetilde{\mathbf{R}}\gamma_\beta\widetilde{\mathbf{R}}^{-1}\ldots, \tag{7.39}$$

we can write

$$D_{(d,f)}\psi_{(d,f)} = \widetilde{\mathbf{R}}D^{(0)}_{(d,f)}\widetilde{\mathbf{R}}^{-1}\widetilde{\mathbf{R}}\psi_{(d,f)} = \widetilde{\mathbf{R}}D^{(0)}_{(d,f)}\psi^{(0)}_{(d,f)} = 0. \tag{7.40}$$

In this case a definition of a special frame of reference is consistent with a definition in any observers frame of reference, i.e., the Dirac operator $D_0 = \gamma^\mu\partial_x\mu \stackrel{\circ}{=} \gamma^\mu\partial\mu$ can be generalized and gauged (Keller J., 1991, 1994. 1994b) to

$$D_{(d,f)} = \Gamma^\mu_{(f)}\left[\partial^{(d)}_\mu - i\frac{e}{\hbar}A^{(d)}_\mu(x)\right], \tag{7.41}$$

which in terms of the above frame of reference generates deductively the so called Standard Model of elementary particle physics.

2.4 STRUCTURAL CONSEQUENCES OF USING A COMPLEX SPACE–TIME ALGEBRA

Besides being a mathematical tool in many branches of physics , including the possibility of generating the (local) Standard Model, the use of a complex space–time algebra has relevant consequences for large scale or small scale physics. All applications refer to the fact that we have now both a five dimensional basic geometry and 2^5 = 32 degrees of freedom. We present here, also as an example, the geometric basis for generating mass and charges (Keller 1999).

Consider a space (R^5 structured with a $\mathcal{C}\ell_{2,3} \approx \mathcal{C}\ell_{0,5} \approx \mathcal{C}\ell_{4,1}$ Clifford algebra; in Chapter 2 the choice was induced by the complexification of $\mathcal{C}\ell_{1,3}$ to be $\mathcal{C}\ell_{0,5}$ which we have denoted by $G(R^5;\mathcal{C}\ell_{2,3})$ and called a geometry. The Clifford algebra imposes a metric g_{AB} = diag (1,1, –1, –1, –1) which we shall transpose for historical and mathematical reasons to g_{AB} = diag (1, –1, –1, –1,1). This geometry is con-

sidered as void of matter, and therefore assumed to have a Ricci tensor

$$a \cdot \mathbf{R}(a \wedge b) = R(b) = 0, \tag{7.42}$$

where the bi-vector function **R** of the bi-vector $a \wedge b$

$$\mathbf{R}(a \wedge b) = a \cdot \nabla\Omega(b) - b \cdot \nabla\Omega(a) + \Omega(a) \times \Omega(b), \tag{7.43}$$

is the curvature, the *a, b* being vectors of a basis set and using the cross product $AB - BA = A \times B$, if *A*, *B* are bi-vectors we can use $A \times B = -(\gamma_5 A) \cdot B$. The bi-vector functions $\Omega((b),x)$ being the local Lorentz group connections such that the covariant derivative of a bi-vector *B*

$$D_a B = a \cdot \nabla B + \Omega(a) \times B, \tag{7.44}$$

in the (curved or flat) space of $G(R^5; C\ell_{2,3})$.

The corresponding 5-D Einstein tensor is then given by the Ricci flat space condition

$$G_{AB} = e_A \cdot (R(e_B) - R g_{AB} e_A) = 0. \tag{7.45}$$

In the following we use the result of Campbell (1926), namely, that, any analytic $(N - 1)$-dimensional manifold can be locally nested in an *N*-dimensional flat manifold $R(b) = 0$.

Furthermore, let us define the complex space–time interval in the faithful representation

$$dS^2 = G_{AB} dx^A dx^B, \qquad A, B = 0, 1, 2, 3, 4, \tag{7.46}$$

where the fifth dimension has been introduced as a geometrical consequence of the complexification with a unit vector $\gamma_4 = -i\gamma_5$, and where $i = \sqrt{-1}$ was introduced, purposely, in order to show the complexification of the algebra. The element $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ is then a space–time pseudo-scalar and

$$\gamma_4^2 = G_{44}^0 = +1. \tag{7.47}$$

By projecting from (7.46) an ordinary space–time metric tensor,

$$g_{AB} = G_{AB} + P_A P_B, \tag{7.48}$$

using the projectors $P_A = \gamma_A \delta_4^A$ such that $g_{44} = g_{4A} = 0$, $\forall A$, the complex space–time line element becomes

$$dS^2 = g_{\mu\nu} dx^\mu dx^\nu + G_{44} \left(\frac{G_{4A}}{G_{44}} dx^A \right)^2,$$

which is the starting point of the induced mass and charges theories.

Examples for the mathematical structures in space–time can be found in the table below:

1) the generating vectors set γ_μ;
2) the 16, arbitrary, elements γ_m;
3) the rotation operators $\gamma_m^2 = -1$;
4) the translation operators $\gamma_m^2 = 0$;
5) the projection operators $\gamma_m^2 = +1$;
6) the group structures $G_{nm}\gamma_m = \gamma_n$.

Frequently (multi-)vectors are used also as generators of Lie groups (Porteous I. R., 1981), as well as to generate (multi-)vector equations and to construct groups from these (multi-) vectors (in particular the most frequently used groups $U(1)$, $SU(2,3)$, $SU(3)$ or SU(2). (See Keller J. and Rodriguez-Romo S., 1991). Another frequent procedure is the integration of spinors and multi-vectors in a geometric superalgebra (Keller J. and Rodriguez A., 1992).

This ends our presentation of space–time geometry as an example, we continue now with geometric analysis. The case of spinors and twistors is presented below.

3. POLYNOMIAL ALGEBRAS. GROUPS. MATRIX REPRESENTATIONS

The process of constructing a geometry over a R^m carrier space is related to some special cases of polynomial algebras (see Weinberger 1989). Otherwise polynomial algebras can be embedded into geometric algebras.

Let, $P_2(x)$ be a second order polynomial of the following form

$$P_2(x) = a_{21}\sum_{i\neq j} x_i x_j + a_{22}\sum_j x_j^2, \quad i,j = 1,2,\ldots,m, \tag{7.49}$$

where the a_{ij} are elements of a symmetric matrix. Consider further that the linear form

$$L(x) = \sum_{j=1}^{m} \alpha_j x_j\,, \tag{7.50}$$

satisfies the condition [158]

$$P_2(x) + L^2(x) = 0, \tag{7.51}$$

then the set of coefficients $\{\alpha_j\}$ has to satisfy the following properties:

$$i = j: \quad [\alpha_i, \alpha_j]_+ = -2a_{22}I, \tag{7.52}$$

$$i \neq j: \quad [\alpha_i, \alpha_j]_+ = -a_{21} I, \tag{7.53}$$

where I denotes the identity element in $\{\alpha_j\}$ and [,]+ anticommutators. The set of coefficients $\{\alpha_j\}$ is called an associative algebra. Two special cases carry famous names we have encountered above as our departing considerations, namely

$$a_{21} = a_{22} = 0 \quad \rightarrow \quad [\alpha_i, \alpha_j]_+ = 0, \tag{7.54}$$

as the so called Grassmann algebra and

$$a_{21} = 0\,, \quad a_{22} = -1 \quad \rightarrow \quad [\alpha_i, \alpha_j]_+ = 2\delta_{ij}, \tag{7.55}$$

as the so called Clifford algebra. It easily can be seen from the requirement, that even the simplest relativistic Hamiltonian $H = (\mathbf{p}^2 + m^2)^{\frac{1}{2}}$, $\mathbf{p}$ being the momentum operator and m the mass of the system, that exactly the case of the Clifford algebra is needed in tackling the problem of the linearization of the square root:

$$\underbrace{\sqrt{(\sum_{j=1}^{m} p_j^2)}}_{\Gamma_2(p)} = \underbrace{\sum_{j=1}^{m} \alpha_j p_j}_{L(p)}. \tag{7.56}$$

In the following first the case for $m = 2$ and 3 (Pauli spin theory) is discussed by considering the smallest groups with Clifford algebraic structure and only then in a similar way the Dirac problem ($m = 4$) is addressed.

3.1 THE PAULI GROUPS

For $m = 2$ the smallest set of elements α_i that shows group closure is given by

$$G_p^{(m=2)} = \{\pm I, \pm\alpha_1, \pm\alpha_2, \pm\alpha_1\alpha_2\}. \tag{7.57}$$

This group is of order 8 and has 5 classes (C_i), namely $C_1 = \{I\}$, $C_2 = \{-I\}$, $C_3 = \{\pm\alpha_i\}$, $C_4 = \{\pm\alpha_2\}$, $C_5 = \{\pm\alpha_1\alpha_2\}$ There are therefore 5 irreducible representations $(\Gamma_i^{(m=2)}, i = 1,5)$ of dimensions n_i such that

$$\sum_{i=1}^{5} n_i^2 = 8, \tag{7.58}$$

This implies that 4 irreducible representations $(\Gamma_i^{(m=2)}$, $i = 1, \ldots, 4)$ have to be one-dimensional and one two-dimensional. Since one dimensional representations are commutative, i.e., do not satisfy the conditions

of a Clifford algebra, only the two-dimensional representation $(\Gamma_5^{(m=2)})$ is of help. The matrices for this irreducible representation are listed below:

$$\Gamma_5^{(m=2)}(\pm I) = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \Gamma_5^{(m=2)}(\pm\alpha_1) = \pm \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{7.59}$$

$$\Gamma_5^{(m=2)}(\pm\alpha_2) = \pm \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \Gamma_5^{(m=2)}(\pm\alpha_1\alpha_2) = \pm \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Using this set of matrices it is easy to show that it indeed forms a representation of $G_P^{(m=2)}$ and that these matrices are Clifford algebraic. For the case of $m = 2$ the problem of the linearization of the square root, is therefore solved:

$$\sqrt{p_1^2 + p_2^2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = p_1 \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + p_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{7.60}$$

For $m = 3$ the smallest set of elements σ_i forming a group is given by

$$G_P^{(m=3)} = \{\pm I, \pm\alpha_1, \pm\alpha_2, \pm\alpha_3, \pm\alpha_1\alpha_2, \pm\alpha_1\alpha_3, \pm\alpha_2\alpha_3, \pm\alpha_1\alpha_2\alpha_3\}. \tag{7.61}$$

The order of this group is 16. It has 10 classes, namely

$$C_1 = \{I\}, \quad C_2 = \{-I\},$$

$$C_3 = \{\pm\alpha_1\}, \quad C_4 = \{\pm\alpha_2\}, \quad C_5 = \{\pm\alpha_3\},$$

$$C_6 = \{\pm\alpha_1\alpha_2\}, \quad C_7 = \{\pm\alpha_1\alpha_3\}, \quad C_8 = \{\pm\alpha_2\alpha_3\},$$

$$C_9 = \{\alpha_1\alpha_2\alpha_3\}, \quad C_{10} = \{-\alpha_1\alpha_2\alpha_3\}, \tag{7.62}$$

and therefore 10 irreducible representations,

$$\sum_{i=1}^{10} n_i^2 = 16, \tag{7.63}$$

of which 8 $(\Gamma_i^{(m=3)}, i = 1, 8)$ are one-dimensional and two $(\Gamma_i^{(m=3)}, i = 9,10)$ are two-dimensional. Again only the two-dimensional irreducible representations are Clifford algebraic.

For α_1 and α_2 one can use the same matrix representatives as in the $m = 2$ case,

$$\Gamma_9^{(m=3)}(\alpha_1) = \Gamma_5^{(m=2)}(\alpha_1), \qquad \Gamma_9^{(m=3)}(\alpha_2) = \Gamma_5^{(m=2)}(\alpha_2), \tag{7.64}$$

provided that the corresponding matrix for α_3 is defined by

$$\Gamma_9^{(m=3)}(\alpha_3) = -i\Gamma_9^{(m=3)}(\alpha_1)\Gamma_9^{(m=3)}(\alpha_2). \tag{7.65}$$

The second two-dimensional irreducible representation $(\Gamma_{10}^{(m=3)})$ is by the way the complex conjugate representation of $\Gamma_9^{(m=3)}$. It is rather easy to proof that these two irreducible representations are indeed non equivalent.

For the $m = 3$ case the problem of the linearization of the square root, reduces therefore to the following matrix equation:

$$\sqrt{p_1^2 + p_2^2 + p_3^2}\,\Gamma_9^{(m=3)}(I) = p_1\Gamma_9^{(m=3)}(\alpha_1) + p_2\Gamma_9^{(m=3)}(\alpha_2) + p_3\Gamma_9^{(m=3)}(\alpha_3). \tag{7.66}$$

The matrices

$$\Gamma_9^{(m=3)}(\alpha_1) \equiv \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_9^{(m=3)}(\alpha_2) \equiv \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},$$

$$\Gamma_9^{(m=3)}(\alpha_3) \equiv \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{7.67}$$

carry a famous name. They are the so-called Pauli spin matrices, usually (as indicated in the last equation) denoted simply by (σ_1, σ_2 and σ_3. For $m = 2, 3$ the corresponding groups are called Pauli group (as indicated by the indice *P*).

3.2 THE DIRAC GROUP

For $m = 4$ the following subset of the Clifford algebra forms the smallest group

$$G_{\mathrm{D}}^{(m=4)} = \{\pm I, \pm\alpha_i\ (i \le 4), \pm\alpha_i\alpha_j\ (i < j), \pm\alpha_i\alpha_j\alpha_k\ (i < j < k), \pm\alpha_5 \equiv \pm\alpha_1\alpha_2\alpha_3\alpha_4\}, \tag{7.68}$$

where 'traditionally' the elements α_i are usually also denoted by γ_μ. The order of this group is 32. It has 17 classes,

$$C_1 = \{I\}, \quad C_2 = \{-I\},$$

$$C_{3-6} = \{\pm\alpha_i \mid i \leq 4\}, \quad C_{7-12} = \{\pm\alpha_i\alpha_j \mid i < j \leq 4\},$$

$$C_{13-16} = \{\pm\alpha_i\alpha_j\alpha_k \mid i < j < k \leq 4\}, \quad C_{17} = \{\pm\alpha_1\alpha_2\alpha_3\alpha_4\}, \tag{7.69}$$

and therefore 17 irreducible representations. As can be checked in analogy to (7.58) 16 of these irreducible representations ($\Gamma_i^{(m=4)}$, $i = 1, \ldots, 16$) are one-dimensional and one is four-dimensional ($\Gamma_{17}^{(m=4)}$). Again only the matrices of the four-dimensional irreducible representation satisfy the conditions of the Clifford algebra. The following matrices

$$\Gamma_{17}^{(m=4)}(\alpha_i) \equiv \alpha_i \equiv \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad i = 1, 2, 3, \tag{7.70}$$

$$\Gamma_{17}^{(m=4)}(\alpha_4) \equiv \beta \equiv \gamma_4 = \begin{pmatrix} \underline{1}_2 & 0 \\ 0 & -\underline{1}_2 \end{pmatrix}, \tag{7.71}$$

where the σ_i are the Pauli spin matrices and $\underline{1}_2$ is a two-dimensional unit matrix, are irreducible representatives of the elements $\alpha_i \in G_D^{(m=4)}$. These particular representatives, usually denoted simply by α_i and β, are called Dirac matrices, $G_D^{(m=4)}$ the Dirac group.

3.3 RELATIONS BETWEEN THE DIRAC GROUP AND THE PAULI GROUP

3.3.1 THE SUBGROUP STRUCTURE

The Dirac group contains the Pauli groups as subgroups,

$$G_P^{(m=2)} \subset G_P^{(m=3)} \subset G_D^{(m=4)}, \tag{7.72}$$

whereby $G_P^{(m=2)}$ is a normal subgroup in $G_P^{(m=3)}$ and $G_D^{(m=4)}$. This implies that in a coset decomposition of $G_D^{(m=4)}$ in terms of $G_P^{(m=2)}$,

$$G_D^{(m=4)} = \{1G_P^{(m=2)}, \alpha_3 G_P^{(m=2)}, \alpha_4 G_P^{(m=2)}, \alpha_3\alpha_4 G_P^{(m=2)}\}, \tag{7.73}$$

the left, and right cosets are identical,

$$\begin{aligned} \alpha_3 G_P^{(m=2)} &= \{\pm\alpha_3, \pm\alpha_3\alpha_1, \pm\alpha_3\alpha_2, \pm\alpha_3\alpha_1\alpha_2\} \\ &= \{\pm\alpha_3, \pm\alpha_1\alpha_3, \pm\alpha_2\alpha_3, \pm\alpha_1\alpha_2\alpha_3\} = G_P^{(m=2)}\alpha_3, \end{aligned} \tag{7.74}$$

and that $G_P^{(m=2)}$ consists of complete classes of $G_D^{(m=4)}$, see (7.69), denoted for a moment as $C_i(G_D^{(m=4)})$,

$$\begin{aligned} G_P^{(m=2)} = \{&C_1(G_D^{(m=4)}), C_2(G_D^{(m=4)}), \\ &C_3(G_D^{(m=4)}), C_4(G_D^{(m=4)}), C_5(G_D^{(m=4)})\}. \end{aligned} \tag{7.75}$$

It should bc noted that $G_{\mathrm{P}}^{(m=3)}$ is not a normal subgroup in $G_{\mathrm{D}}^{(m=4)}$, since

$$C_{13}(G_{\mathrm{D}}^{(m=4)}) = C_9(G_{\mathrm{P}}^{(m=3)}) \cup C_{10}(G_{\mathrm{P}}^{(m=3)}). \tag{7.76}$$

3.3.2 SUBDUCED REPRESENTATIONS

The set of matrices

$$\Gamma_{17}^{(m=4)}(G_{\mathrm{P}}^{(m=2)}) \equiv \{\Gamma_{17}^{(m=4)}(\alpha), \forall \alpha \in G_{\mathrm{P}}^{(m=2)}\},$$

and

$$\Gamma_{17}^{(m=4)}(G_{\mathrm{P}}^{(m=3)}) \equiv \{\Gamma_{17}^{(m=4)}(\alpha), \forall \alpha \in G_{\mathrm{P}}^{(m=3)}\},$$

of course also forms a representation for $G_{\mathrm{P}}^{(m=2)}$ and $G_{\mathrm{P}}^{(m=3)}$, respectively, which, however, is reducible. Such representations are called subduced representations. Reducing these two representations (for example by means of the orthogonality relation for characters), one finds the following decompositions into irreducible representations:

$$\Gamma_{17}^{(m=4)}(G_{\mathrm{P}}^{(m=2)}) = 2\Gamma_5^{(m=2)}(G_{\mathrm{P}}^{(m=2)}), \tag{7.77}$$

and

$$\begin{aligned}\Gamma_{17}^{(m=4)}(G_{\mathrm{P}}^{(m=3)}) &= \Gamma_9^{(m=3)}(G_{\mathrm{P}}^{(m=3)}) + \Gamma_{10}^{(m=3)}(G_{\mathrm{P}}^{(m=3)}) \\ &= \Gamma_9^{(m=3)}(G_{\mathrm{P}}^{(m=3)}) + \left(\Gamma_9^{(m=3)}(G_{\mathrm{P}}^{(m=3)})\right)^*.\end{aligned} \tag{7.78}$$

Since the irreducible representation $\Gamma_{17}^{(m=4)}$ of $G_{\mathrm{D}}^{(m=4)}$ always subduces only the group of the Pauli spin matrices (and their complex conjugates), there is no way to deal with the problem posed by (7.50) in terms of 2 x 2 matrices only, i.e., to linearize properly the square root $\sqrt{\mathbf{p}^2 + m^2}$ for a three-component vector p! In other words: there is no other 'truly' relativistic description other than the one using the Dirac matrices.

One can summarize the properties of these three groups very compactly in the short table below:

m	Group-order	# of classes	# of one-dimensional irreps	# of two-dimensional irreps	# of four-dimensional irreps
2	8	5	4	1	0
3	16	10	8	2	0
4	32	17	16	0	1
	2^{m+1}	$m^2 + 1$	2^m		

(7.79)

The so called fundamental theorem of Dirac matrices, namely, that a necessary and sufficient condition for a set of 4 matrices γ_i' to be Dirac matrices, i.e., to be irreducible and Clifford algebraic, is that they have to be obtained via a similarity transformation W from the matrices in (7.70)–(7.71):

$$\gamma_i' = W^{-1}\gamma_i W, \quad i = 1, 4, \tag{7.80}$$

is in the context of the Dirac group nothing but Schur's lemma for irreducible representations.

4. DERIVATION OPERATORS

An important point of the presentation of geometric analysis in Keller and Weinberger (2000) is the clear distinction between the (at least,) four different types of derivation operators:

a) With respect to the carrier n-dimensional space coordinates;

b) With respect to the carrier space basis vectors;

c) The change of a function $F(M)$ of a multi-vector M with respect to a given multi-vector A;

d) The change of a function $F(M')$ of a variable multi-vector M' with respect to a variable multi-vector Z'.

The first type corresponds to the standard partial derivative ∂_{xi} and from a complete set a Laplacian operator can be defined which usually is now considered to be the Dirac operator.

The second is particular to geometric analysis given that the position vectors in the carrier space can be written as

$$\mathbf{x} = x^i e_i,$$

and then we can have either 'active' changes generated by ∂_{xi} or 'passive' changes generated by ∂_{ei}, that is, by a change in the description of the basis vectors themselves.

The third is also particular to geometric analysis given that we can explore the changes of the function $F(M)$ with respect to restricted changes of M of the form $M \to M + \tau A$.

The fourth type of change is the generalization of the complex derivative $\partial_z f(z)$ with $z = x + iy$, by defining a variable $Z' = \Sigma_a (Z')^a e_a$ where the different $(Z')^a$ replace x and y in complex analysis.

There is a big simplification, though, in the sense that all four cases can be considered as particular forms of a general derivation operator of the type introduced by Peano in 1888[146]. Then by discussing the case c) we cover the different situations. We shall anyhow discuss case b) in particular for its basic importance in the use of geometric analysis in quauntum mechanics.

4.1 THE DERIVATION IN GEOMETRIC ANALYSIS

In geometric analysis one very often deals with functions $F(M)$ of the variable (multi-)vectors M , considering a change when M changes in a given 'direction' A. The following quantity (Peano 1888),

$$\lim_{\tau\to 0}\frac{F(M+\tau A)-F(M)}{\tau},$$

is then a **constrained** rate of change when the (multi-)vector variable has a fixed form of change. This is the natural extension of the standard derivative (or $M = A = x$) or the complex derivative (using $M = Z = x\mathbf{1}+y\mathbf{I}$, $\mathbf{I}^2 = -1$) which are in fact written in the same form using $A = x$ or $A = Z$. Since any (multi-)vector of rank r can be decomposed in a basis of m elements $\gamma_{(m,r)}$,

$$M=\sum_r\sum_m M^m\gamma_{(m,r)}=\sum_{\{i,j,\ldots\}}M^{ij\ldots}\gamma_{ij\ldots}, \tag{7.81}$$

it is quite obvious to consider the $\gamma_{(m,r)}$ components of the derivative. First define the scalar part of the geometric product of two (multi-)vectors A and B as $A*B$ and demand that this operation has precedence over other, not explicitly defined by parenthesis, then we can define

$$A*\partial_M F(M)\equiv\lim_{\tau\to 0}\frac{F(M+\tau A)-F(M)}{\tau}, \tag{7.82}$$

as a scalar, then rank preserving, operator on the function F of the multi-vector M. The operator defined by (7.82) being rank preserving, A and the symbol ∂_M are to be understood as having the necessary multi-vector structure in order to allow $A * \partial_M$ to be a scalar operator.

Equation (7.82) allows the definition of a (multi-)vector-valued operator and the definition (Keller 1993) of a reciprocal (multi-)vector space (called dual also, as above, given their operator character):

$$\gamma^{r\ldots ji}\gamma_{ij\ldots r}=\delta^i_i\delta^j_j\ldots\delta^r_r\,, \tag{7.83}$$

in terms of the **operators**

$$\gamma^k=(-1)^{k-1}(\gamma_{12\ldots n})^{-1}\gamma_1\gamma_2\ldots\overset{\vee}{\gamma_k}\ldots\gamma_n\wedge, \tag{7.84}$$

where the symbol $\vee$ above γ_k denotes a factor that has to be removed from the product and we admit that the explicit outer product operates on either side:

$$\gamma^k\gamma_i=\gamma_i\gamma^k=\delta^k_i, \tag{7.85}$$

(here a more explicit notation would be $\gamma^k\gamma_i \Rightarrow (g^{kj}\gamma_j) * \gamma_i$, the matrix g^{kj} being the inverse of the matrix g_{jk}) with the reciprocal vectors generating also a Clifford algebra. In (7.84) we are assuming an orthogonal set $\{\gamma_\mu\}$ and the outer product is implied. The operator definition,

$$\partial_M \equiv \sum_{r=0}^{n} \sum_{i<j<\ldots<k} (\gamma^{ij\ldots k})^r (\gamma_{k\ldots ji})_r * \partial_M = \sum_{r=0}^{n} \sum_{i<j<\ldots<k} (\gamma^{ij\ldots k})^r \partial_{(k\ldots ji)}, \tag{7.86}$$

explicitly shows the multi-vector character of ∂_M as a sum of (multi-)vector valued combinations of scalar operators, from the definition

$$\gamma_{k\ldots ji} * \partial_M = \partial_{(k\ldots ji)}, \tag{7.87}$$

where the subindex $(k\ldots ji)$ is a tensor-like index necessary to construct the (multi-)vector-valued operator which otherwise is index free.

The Leibnitz differentiation rule. Consider now the derivative of the scalar part of the product of two (multi-)vectors M and B. The scalar part of the product will be denoted by $\left\langle M^m \gamma_m B^{m'} \gamma_{m'} \right\rangle_0$, $B = B^{m'}{}_{gm'}$ and $M = M^m\gamma_m$, where the $\langle\gamma_m\rangle_r = \gamma_m$ belong to the r-blade of the geometry. This derivative is then given by

$$\partial_M (MB) = \partial_M (B^{m'} M^m \gamma_{m'} * \gamma_m). \tag{7.88}$$

The scalar product $\gamma_m' * \gamma_m$ will be denoted below as $\eta_{mm'} = \eta_{m'm}$. Then, using $\partial_M \rightarrow \gamma^m \partial_{Mm}$, we obtain

$$\begin{aligned} \sum_m \gamma^m \gamma_m * \partial_M \langle MB \rangle &= \sum_m \gamma^m \lim_{\tau \to 0} \frac{\langle MB + \tau\gamma_m B - MB \rangle}{\tau} \\ &= \sum_m \gamma^m \gamma_m B = \sum_m \gamma^m B^{m'} \eta_{mm'}. \end{aligned} \tag{7.89}$$

In this context it must. be recalled that also $\langle\gamma^m\rangle r = \gamma^m$ by construction. This shows how the result of the derivation can be viewed as a projection of B in the space of the basis of M.

Clearly enough the Leibnitz rule for differentiation must be followed. Suppose from $M = M^m\gamma_m$ we define the scalar operator $\gamma_m * \partial_M \equiv \partial_{Mm}$. Consider now the typical example of the function $F(M) = (M^2 + AM + B)$ with A and B being constant (multi-)vectors and $\langle M\rangle_r = M$. Consider further the substitution $M \rightarrow M + \tau\gamma_m$ with $\langle\gamma_m\rangle r = \gamma_m$. The

limit of the difference

$$\lim_{\tau\to 0}\left[(M+\tau\gamma_m)(M+\tau\gamma_m)+A(M+\tau\gamma_m)+B-M^2-AM-B\right]$$
$$=\lim_{\tau\to 0}\left[\tau(\gamma_m M+M\gamma_m+\tau(\gamma_m)^2)+\tau A\gamma_m\right]+A\gamma_m, \tag{7.90}$$

is then to be compared with the algorithm

$$\gamma^m\partial_{M^m}F(M)=\gamma^n\partial_{M^m}\Big(\sum_{mm'}M^m M^{m'}\eta_{mm'}+\sum_n AM^m\gamma_m-B\Big)$$
$$=\gamma^m(M^m\eta_{mm'}+M^{m'}\eta_{mm'})+\gamma^m A\gamma_m. \tag{7.91}$$

The multi-vector Laplacian. In the same form we can define the reciprocal operator

$$\partial^M=\sum_m\gamma^m\partial^{M^m}, \tag{7.92}$$

and the (invariant) multi-vector Laplacian,

$$\partial_M\partial^M=\partial^M\partial_M=\sum_m^{\{m\}_r}\partial_{M^m}\partial^{M^m}\ldots m\in\{m\}_r, \tag{7.93}$$

which corresponds to the ordinary Laplacian if restricted to $\{m\}_r = \{\mu\}$, with $\{\mu\}$ corresponding to the set of generating vectors of the geometric algebra.

4.2 GEOMETRIC DERIVATIVE

In the geometry which has been discussed up to now we introduced the use of the dual frame of operators γ^α, which map vectors into scalars, defined as $\gamma^\alpha\gamma\beta = \gamma\beta\gamma^\alpha = \delta^\alpha_\beta$, and thus corresponds to a derivative ∂_{γ_a}, i.e., both $\gamma_\alpha\gamma_\beta = \delta^\alpha_\beta$ and therefore $\partial_{\gamma\alpha}\gamma_\beta = \delta^\alpha_\beta$, can be equivalently used. This operators follow the Leibnitz rule for differentiation when applied to a multi-vector.

Consider, for example, the generator $\gamma_{\alpha\beta}$ of the rotations and Lorentz boosts operator $\tilde{\mathbf{R}}$ and of the connections $\Omega(a)$ where a is a given arbitrary vector

$$\tilde{\mathbf{R}}=e^{\frac{1}{2}\theta(\alpha\beta)\gamma_\alpha\beta}\quad\text{and}\quad\Omega(a)=-2a\cdot\nabla\tilde{\mathbf{R}}\tilde{\mathbf{R}}^{-1}, \tag{7.94}$$

used to rotate a (multi-)vector $M = \{f^A(x)\gamma_A\}\gamma_A \in \{1, \gamma_\mu, \ldots .\}$,

$$M\to M'=\tilde{\mathbf{R}}M\tilde{\mathbf{R}}^{-1}, \tag{7.95}$$

in the plane spanned by γ_α and γ_β, with $\Omega(a)$ defining as above the transformation of the ordinary derivative of a multi-vector into a covariant derivative. In (7.95) $\tilde{\mathbf{R}}$ leaves $f^A(x)$ unchanged, but transforms the Clifford numbers $\gamma_A \to \gamma'_A = \tilde{\mathbf{R}}_{\gamma A}\tilde{\mathbf{R}}^{-1}$. Alternatively, we can, as very often is the case, write the complementary transformations as

$$M \to M^{(\tau)} = (Rf^A(x))\gamma_A, \tag{7.96}$$

where an equivalent rotation $f^A(x) \to (f^A(x))' = Rf^A(x)$ is used such that $\gamma_A \to \gamma_A$, and R is generated by

$$x^\beta \partial_{x^\alpha} - x^\alpha \partial_{x^\beta}. \tag{7.97}$$

The (multi-vector) algebra equivalent to (7.97) implies a change of the frame of reference, namely the passive transformation generated by

$$\gamma_\beta \partial_{\gamma_\alpha} - \gamma_\alpha \partial_{\gamma_\beta}, \tag{7.98}$$

where the operators $\partial_{\gamma_\mu} = \gamma^\mu$, and where, in an orthonormal frame of reference, the operators $\gamma^\mu = (\gamma^{\mu\nu}\gamma_\nu)^*$, with $\gamma_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$. Since for a proper space–time rotation we require $g^{\alpha\alpha} = g^{\beta\beta} = -1$, (7.98) yields

$$\gamma_\beta \partial_{\gamma_\alpha} - \gamma_\alpha \partial_{\gamma_\beta} = 2g^{\alpha\lambda}\gamma_\beta\gamma_\lambda = 2g^{\alpha\lambda}\gamma_{\beta\lambda} = 2\gamma_{\alpha\beta}, \tag{7.99}$$

which in turn shows the correspondence between (7.98) and the generators of (7.94). If for position vectors $\mathbf{x} = x^\alpha\gamma_\alpha$ eqs. (7.97) and (7.98) are applied simultaneously with angles θ and $-\theta$, they cancel each other.

A complete representation of the rotations of the Poincaré group is therefore given by

$$\begin{aligned} R(\alpha\beta) &= (x^\beta \partial_{x^\alpha} - x^\alpha \partial_{x^\beta}) + (\gamma_\beta \partial_{\gamma_\alpha} - \gamma_\alpha \partial_{\gamma_\beta}) \\ &= (x^\beta \partial_{x^\alpha} - x^\alpha \partial_{x^\beta}) + 2\gamma_{\alpha\beta}. \end{aligned} \tag{7.100}$$

Other examples for operators including derivatives are constructed in a similar form (Liu and Keller 1996).

5. STEPS TO BUILD A COMPLEX SPACE (–TIME)

In the following we first illustrate the procedure of building a complex space(–time) in one and two dimensions, where the basic ideas can be visualized, then we present the general case.

5.1 ONE- AND TWO-DIMENSIONAL CASES

Consider a one-dimensional geometry $G_{1,0}$ spanned by the vector e_1 and the unit scalar **1**, such that $e_1^2 = \mathbf{1}$ is the $\mathcal{C}\ell_{1,0}$ Clifford ring $(\pm\mathbf{1}, \pm\mathbf{e}_1)$ of that geometry. The usual complex plane is commonly represented as $z = x\mathbf{1} + \mathbf{i}y$, with the property $\mathbf{i}^2 = -1$, that is, in practice the ring has been enlarged to $(\pm\mathbf{1}, \pm e_1, \pm\mathbf{i})$ but, at the same time, is restricted by convention to $(\pm\mathbf{1}, \pm\ \mathbf{i})$. The complex plane is otherwise isomorphic; to

$$\mathbf{z} = xe_1 + ye_{n+1} = x^1 e_1 + x^{n+1}_{(1)} e_{n+1}, \tag{7.101}$$

or,

$$x^1 = x\ ,\ x^{n+1}_{(1)} = y, \tag{7.102}$$

where for a proper and faithful representation we require

$$e_1^2 = e_{n+1}^2 = \mathbf{1}, \tag{7.103}$$

and

$$e_1 e_{n+1} = e_{1\ n+1} = -e_{n+1} e_1 = -e_{n+1\ 1}, \tag{7.104}$$

to have the ring $\mathcal{C}\ell_{2,0}$ of $(\pm\mathbf{1}, \pm e_1, \pm e_{n+1}, \pm e_{1\ n+1})$, where again

$$(e_{1\ n+1})^2 = \mathbf{i}_1^2 = e_1 e_{n+1} e_1 e_{n+1} = -\mathbf{1}. \tag{7.105}$$

Hence the 'even' part (even number of products) of the ring $\mathcal{C}\ell_{2,0}$ is $(\pm\mathbf{1}, \pm\ e_{1\ n+1})$ and can be represented, using $\tan\theta_1 = x^{n+1}/x^1 = A_1$ by $\mathbf{z} \rightarrow z = e_1\mathbf{z}$ or the mappings:

$$\begin{aligned} \mathbf{p} &\Rightarrow z^1 = e_1\mathbf{p}, \\ \mathbf{p} &\Rightarrow z^1 = x^1\mathbf{1} + (x^1 \tan\theta_1)\mathbf{i}_1, \\ x^1 e_1 + x^{n+1} e_{n+1} &\Rightarrow x^1(\mathbf{1} + A_1\mathbf{i}_1). \end{aligned} \tag{7.106}$$

5.2 HIGHER-DIMENSIONAL CASES

A special situation arises for higher-dimensional spaces, $n \geq 2$ in which a full collection of 'complex' planes is generated by products of the form $e_i e_{n+1}$ (i = 1, ..., *n*). In particular, we have the collection of complex points $\{p(_i), i = 1, ..., n\}$ with coordinates $(x^i, \mathrm{x}^i\mathrm{A}_i)$, where $A_i =^{\mathrm{d}} \tan\theta_{i\ i}$.

As can be seen only one extra vector,

$$\{e_{n+1}; e_{n+1}e_i = -e_i e_{n+1}, i = 1, ..., n\},$$

is needed to complexify the algebra, generating in turn a collection of (bi-) vectors $\mathbf{i}_i =^{\mathrm{d}} e_i e_{n+1}$.

The possibility of mixed metrics $\mathcal{C}\ell_{p,q}$ is to be considered as a special *case,* because if for some k the ek_{n+1} are such that $(e_{k\ n+1})^2 = +\mathbf{1}$ then we have a hyperbolic complex plane k and $A_k = \tanh\theta_{i\,k.}$ The isomorphism $\mathcal{C}\ell_{p,q} \approx \mathcal{C}\ell_{p+2,q-2}$ can be exploited to obtain the largest possible number of ordinary complex planes where again $A_j = \tan\theta_j$. Here we have deliberately used the same notation as in the formula above. We see that in the auxiliary coordinate spanned by the e_{n+1} there are two types of contributions: the one, arising from a reference value $x^{n+1}_{(0)}$ to $x^{n+1}_{(i)} = x^{n+1}_{(0)} + \tan(a_i)x^i$, called l above; and the one related to the complexification of the ei_{n+1} planes given by $x^i A_i$. In the correspondence between the geometry and physics l is related to the curvature, and the A_i to the gauge fields such as the electromagnetic case (Kaluza–Klein or more general, string or superstring, theories). All degrees of freedom of $G_{p,q}$, $p + q = n$ are complexified, that is we also have complex (bi-, tri-, etc.) vectors. In particular we have the mapping in space–time of a vector

$$p = p^\mu e_\mu, \tag{7.107}$$

such that

$$p^2 = p^\mu p_\mu = m^2, \tag{7.108}$$

with m being a real number, to

$$p' = x^\mu i\gamma_5 e_\mu, \tag{7.109}$$

where

$$|p'|^2 = p'^\mu p_\mu = m^2 \tag{7.110}$$

and

$$|A|^2 \equiv \frac{1}{2}(AA^* + A^*A), \tag{7.111}$$

or to

$$p'' = x^\mu(\mathbf{1}\cos(n + t(\mu))\tfrac{1}{2}\pi + i\gamma_5 \sin(n + t(\mu))\tfrac{1}{2}\pi)e_\mu, \tag{7.112}$$

with n and $t(\mu)$ being integers. In this case we again obtain $|p''|^2 = m^2$, which in the study of physical problems would correspond to a mixing of the vector currents with the axial vector currents, as in the theory of electroweak and color interactions.

5.3 EMBEDDING

In space–time we have a R^4 carrier manifold which we want to study geometrically by endowing it with a $\mathcal{C}\ell_{1,3}$ or $\mathcal{C}\ell_{0,5}$ Clifford algebra as local geometric structure. There are, however, several possible embeddings of this manifold in the Clifford algebra, the best known examples are $R^4 \rightarrow \mathcal{C}\ell_{1,3} \rightarrow x^\mu\gamma_\mu$, $\mu = 0,1,2,3$, and its space–time 'cut' $x^\mu\gamma_0\gamma_\mu$, $\mu = 0,1,2,3$. The first is the general case, the second the choice of an observers reference frame with γ_0 being the time like vector of the observer. Another common example is $R \rightarrow \mathcal{C}\ell_{1,3} \Rightarrow x^\mu e_\mu$; $e_\mu = h^\nu_\mu \gamma_\nu$; $\mu = 0,1,2,3$, on which the vierbein e_μ is defined in terms of the (local) coefficients $h^\nu_\mu(x)$. Induced matter and charge is obtained by allowing $h^\nu_\mu(x) \rightarrow h^\nu_\mu(x, x^4)$ (Wesson, 1999, Keller, 1999). Other mappings, suitable for special purposes, are allowed, of course.

6. MAPPING COMPLEX INTO REAL GEOMETRIC SPACES

We have defined above a real geometry by means of a definition of its local geometrical properties corresponding to a Clifford Algebra $\mathcal{C}\ell_{p,q}$ of signature $g_{ab} = \text{diag}\,(1, \ldots, 1_p, -1, \ldots, -1_q)$, $p + q = n$, containing 2^n elements. We then showed that its complexification, containing 2^{n+1} elements, can be represented by a R^{n+1} carrier manifold which acquires a real geometry by the definition of its geometrical properties through a Clifford Algebra $\mathcal{C}\ell_{p+1,q}$. This is achieved by creating an n-dimensional metric tensor from the $(n + 1)$-dimensional equivalent geometry:

$$\begin{aligned} g^c_{\mu\nu} &= g_{\mu\nu} - \frac{g_{n+1,\mu} \cdot g_{n+1,\nu}}{g_{n+1,n+1}} \\ &= g_{\mu\nu} - m^2\left(k_\mu k_\nu - \frac{e^2}{m^2} A^a_\mu A^a_\nu\right), \end{aligned} \tag{7.113}$$

with $\mu,\nu = 1,2, \ldots, n + 1$, where the $g^c_{\mu\nu}$ are in practice n-dimensional because, by definition $g^c_{\mu,n+1} = 0$, and, in particular $g^c_{n+1,n+1} = 0$. The last equality shows the splitting of the complex term into a basic part $k_\mu k_\nu$ and a relative part $A^a_\mu A^a_\nu$.

The vector k_μ was introduced by Liu and Keller (1999) in their study of the gauge properties of the Dirac wave equation. We reproduce here their Theorem 1, to show the relation of this vector to the mass of a particle's field:

> There exists a suitable complex vector k_μ such that if $\Psi_0 \equiv L_0 + R_0$ satisfies the massless Dirac equation $i\gamma^\mu\partial_\mu\Psi_0 = 0$, then

$$\Psi = (\Psi_0)\exp(\text{im}\int k_\mu dx^\mu),$$

satisfies the massive Dirac equation $i\gamma^\mu\partial_\mu\Psi - m\Psi = 0$. Here

$$k_\mu = k_\mu^+ + k_\mu^- = \frac{\pi_\mu^+}{2\overline{R_0}L_0} + \frac{\pi_\mu^-}{2\overline{L_0}R_0} ,$$

where

$$\pi_\mu^+ = \overline{R_0}\gamma_\mu R_0, \quad \pi_\mu^- = \overline{L_0}\gamma_\mu L_0, \quad \pi_\mu = \overline{R_0}\gamma_\mu R_0 + \overline{L_0}\gamma_\mu L_0 = \overline{\Psi_0}\gamma_\mu\Psi_0.$$

and $\overline{R_0}$ and $\overline{L_0}$ refer to transposed complex conjugate functions.

Relation (7.113) is obtained from a local decomposition of the space into a part tangential to the original R^n manifold and a perpendicular part by means of the (multi-) vector projectors $P_\perp$ and $1 - P_\perp$. The geometry of G ($\mathcal{C}\ell_{p,q}$; R^n; $n = p + q$) is not restricted to a flat space but can be any space which is n-dimensional and analytical, and can using Campbell's theorem (Campbell J. E., 1926), be locally embedded in a R^{n+1} geometrical (flat) space. The coordinates are then given by $x_c^\mu = x_R^\mu + ix_I^\mu$ and the real scalar product is defined by $\frac{1}{2}\langle(AB^* + A^*B)$.

For many applications it is necessary to generate a real line element which, with the decomposition into a parallel and a tangential part to R^n is defined as $dS^2 = ds_0^2 + (g_{n+1},\mu^{dx\mu})^2$, with $ds_0^2 = g_{\mu\nu}^c dx_R^\mu dx_R^\nu$ being the line element of the n-dimensional real space (time), which, as should be noted, by definition has no $(n + 1)$th component.

7. LINEAR VECTOR TRANSFORMATIONS

A linear vector (or in general a multi-vector) transformation f maps a vector (multi-vector) **a** into another vector (multi-vector) **a′** whose components are linear functions of the original vector. In this subject in relation to the basic underlying Clifford algebra the books of Hestenes and Sobczyk (1984), Gilbert and Murray (1991) and of Baylis (1996) are basic references, the last one containing also a rich list of references. It is convenient to write a given transformation into a canonical form

$$f : \mathbf{a} \to \mathbf{a}' = \mathrm{f}(\mathbf{a}) \equiv \sum_{\mu\nu} f_\nu^\mu a^\nu \gamma_\mu = \sum_\mu (a')^\mu \gamma_\mu, \tag{7.114}$$

where

$$a^\nu = \gamma^\nu \mathbf{a}, \tag{7.115}$$

the reciprocal vectors γ^ν used above being defined by

$$\gamma^\nu \gamma_\mu = \delta_\mu^\nu. \tag{7.116}$$

Combining (7.114)–(7.116),

$$\mathbf{a}' = \sum_{\mu} \gamma_\mu (a')^\mu = \sum_{\mu} (\gamma^\mu \mathbf{a}') \gamma_\mu = \sum_{\mu} \gamma^\mu \mathbf{f}(\mathbf{a}) \gamma_\mu = \gamma^\mu \mathbf{f}(\sum_{\nu} (\gamma^\nu \mathbf{a}) \gamma_\nu, \tag{7.117}$$

implying linearity and using an implicit summation convention,

$$\gamma_\mu (\gamma^\mu \mathbf{f}(\gamma_\nu)) a^\nu = f_\nu^\mu a^\nu \gamma_\mu, \tag{7.118}$$

the usual tensorial components $f_{\mu\nu}$ are obtained from

$$g_{\mu\lambda} f_\nu^\lambda = f_{\mu\nu}. \tag{7.119}$$

It is then clear that linear transformations as defined here are vector valued functions of a vector and that the second rank tensor $f_{\mu\nu}$ is the result of obtaining the μ-th component of the transformation of the *ν*-th basis vector,

$$f_{\mu\nu} = \gamma_\mu \cdot \mathbf{f}(\gamma_\nu). \tag{7.120}$$

By using (7.115) and (7.118), we can write the canonical form of the operator **f** in geometric algebraic terms as

$$\mathbf{f} = \gamma_\mu f_\nu^\mu \gamma^\nu. \tag{7.121}$$

The inverse transformation defined as $\mathbf{f}\mathbf{f}^{-1} = \mathbf{f}^{-1}\mathbf{f} = \mathbf{1}$, namely

$$\mathbf{f}^{-1} = \gamma_\nu (f^{-1})_\mu^\nu \gamma^\mu \quad , \tag{7.122}$$

exists if **f** is not singular, see also below.

An important concept is that of the dual transformation,

$$\gamma_\mu \cdot \mathbf{f}(\gamma_\nu) = (\mathbf{f}^D(\gamma_\mu)) \cdot \gamma_\nu = (\gamma_\lambda f_\mu^{D\lambda}) \cdot \gamma_\nu, \tag{7.123}$$

or

$$g_{\mu\lambda} f_\nu^\lambda = g_{\lambda\nu} f_\mu^{D\lambda}. \tag{7.124}$$

It should be noted that for a symmetric transformation $\mathbf{f} \leftrightarrow \mathbf{f}^D$. The transformation is to be applied to every vector factor of a multi-vector) such that

$$\mathbf{f}(\gamma_A) = \mathbf{f}(\gamma_i) \wedge \mathbf{f}(\gamma_j) \wedge \ldots \mathbf{f}(\gamma_m), \tag{7.125}$$

is an equivalent transformation for a multi-vector.

Special cases are the generators $\gamma_{\mu\nu}$ for orthogonal transformations and the (rotation and boost) transformations they generate:

$$R\mathbf{v}\widetilde{R} = R\mathbf{v}R^{-1},$$

$$\mathbf{v} \to \mathbf{v}' = R\mathbf{v}R^{-1}, \tag{7.126}$$

For a multi-vector A,

$$A \to A' = RAR^{-1}, \tag{7.127}$$

one then obtains A' given in terms of $\gamma_i\gamma_j\gamma_k \ldots \gamma_m$. Furthermore, if $\gamma_i' = R_{\gamma i} R^{-1}$, we can write the product as

$$\gamma_{ijk\ldots m} \to \gamma'_{ijk\ldots m} = R\gamma_i R^{-1} R\gamma_j R^{-1} R\gamma_k R^{-1} \ldots R\gamma_m R^{-1}. \tag{7.128}$$

A very important case is the determinant. For a pseudo-scalar $\gamma_{12\ldots n}$ we can write:

$$\mathbf{f}(\gamma_{12\ldots n}) = \det(\mathbf{f})\gamma_{12\ldots n}, \tag{7.129}$$

provided that the pseudo-scalar is invariant under orthogonal transformations. In this case only scale factors enter. We should remind the reader that the determinant of a matrix is invariant under a similarity transformation, its value is given by its principal axis representation value where

$$f_{\mu\nu} = f_{(\mu)}\delta_{\mu\nu}. \tag{7.130}$$

When the transformations are generated by, or are chosen to be related or to induce a derivable transformation of coordinates.

$$\mathbf{x} \to \mathbf{x}' = \mathbf{f}_C(\mathbf{x}), \qquad \mathbf{x} \in R^n, \tag{7.131}$$

with the coordinates $\mathbf{x}'$ containing now some particular characteristics, 'anchored' to some local origin of coordinates, we induce from an arbitrary 'free' vector $\mathbf{v} \in R^n$, a vector $\mathbf{v}'$, formally obtained from the relation

$$\mathbf{v} \to \mathbf{v}' = \mathbf{f}(\mathbf{v}), \tag{7.132}$$

considering the divergence of $\mathbf{x}'$ in the direction of $\mathbf{v}$ as $\mathbf{v}' = (\mathbf{v} \cdot \nabla)\mathbf{x}'$, that is this time the mapping is generated by the coordinate transformation, otherwise we could invert the relations and search for the coordinate transformation which induces the transformation of $\mathbf{v}$ given by (7.132). This is in fact the geometrization process which corresponds to general relativity and to gauge principles considered as geometric transformations of a carrier space.

7.1 RESTRICTED TRANSFORMATIONS

By a restricted linear transformation one means as a restriction of the map generated by the transformation **f**(**v**) to some subspace $\mathbf{v}'_{(\mathbf{v}'\subset\mathbf{f}(\mathbf{v}))}$ of the vector valued functions **f** of the original vectors **v**.

The simplest formulation would be to consider a restriction of the type where one vector of the source space, say $\mathbf{a}_i$, is mapped into one and only one vector $\mathbf{b}_i$, of the same magnitude, of the target space

$$\mathbf{f}^{(n-1)}(\mathbf{a}_i) = \mathbf{b}_i, \|\mathbf{b}\| = \|\mathbf{a}\|,$$

where |||| denotes absolute magnitude and we have otherwise a free linear mapping of all other vectors $\mathbf{V}_R$. **a** = **0** into $\mathbf{v}'_R$. **b** = **0**.

If **v** = **VR** + **a** and **v**′ = $\mathbf{v}'_R$ + **b**, with the conditions given above, we define

$$\mathbf{f}^{(n-1,R(a,b))}(\mathbf{a}_i) = 0 \quad \text{and} \quad \mathbf{f}^{(n-1,R(a,b))}(\mathbf{v}_R) = \mathbf{v}'_R,$$

and the reciprocal vector $\mathbf{a}^+$ through

$$\mathbf{a}^+\mathbf{a} = \mathbf{1}.$$

The original linear transformation then becomes (b'_i is proportional to b_i or $b'_i = k(i)b_i$)

$$\mathbf{f}(\mathbf{v}) = \sum_{i=1}^{n} c_i(\mathbf{b}'_i\mathbf{a}_i^+ + \mathbf{f}^{(n-1,R(a_i,b_i))})(\mathbf{v}) = \mathbf{v}', \tag{7.133}$$

where the coefficient $c_i = (-1)^{i-1}$, and the number of linearly independent vectors $\mathbf{a}_i$ is the dimension of the basis *n*.

In tensor language the decomposition in (7.133) corresponds to the standard decomposition of a matrix into blocks. For the determinant of the transformation we have

$$\det(\mathbf{f}) = \sum_{\mu=1}^{n} f_{\mu\nu}(-1)^{\mu-1}\det(\mathbf{f}^{(\mu,\nu)}), \tag{7.134}$$

which, by successive application of (7.134), gives the well known result

$$\det(\mathbf{f}) = \sum_{\{\mu\}}\prod_{\{\nu\}} f_{\mu\nu}f_{\lambda\rho}\ldots\epsilon_{\mu\lambda\ldots}\epsilon^{\nu\rho\ldots}, \tag{7.135}$$

which in turn is sometimes considered to be the definition of the determinant, of a matrix. Here $\in_{\mu\nu}\ldots$ is the totally antisymmetric symbol with respect to the indices.

We can also define the derivative with respect to a linear function **f**(a). Using the linearity with respect to **a** the *j*-th component should be proportional to ($\mathbf{a}_{.ej}$) such that **f** can also be expanded into components

$$\partial_{\mathbf{f}(\mathbf{a})} \equiv \mathbf{a}\cdot e_j e_i \partial_{f_{ij}}. \tag{7.136}$$

The main relation is equivalent to a cancelation of the action of **f**,

$$\partial_{f_{ij}} \mathbf{f}(\mathbf{b})\cdot\mathbf{c} = \partial_{f_{ij}}(f_{lk}b^k c^l) = c^i b^j. \tag{7.137}$$

Multiplication by $\mathbf{a}.e_j e_i$, and in comparing with (7.136) one obtains,

$$\mathbf{a}\cdot e_j e_i \partial_{f_{ij}} \mathbf{f}(\mathbf{b})\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{b}\mathbf{c},$$

or,

$$\partial_{\mathbf{f}(\mathbf{a})}[\mathbf{f}(\mathbf{b})\cdot\mathbf{c}] = \mathbf{a}\cdot\mathbf{b}\mathbf{c}, \tag{7.138}$$

and, from the definition of $\mathbf{f}^{\mathrm{D}}$:

$$\partial_{\mathbf{f}(\mathbf{a})}\mathbf{f}^{\mathrm{D}}(\mathbf{b}) = \mathbf{b}\mathbf{a}. \tag{7.139}$$

8. IDEALS, SPINORS, TWISTORS, AND BEYOND

For a geometry $G(R^n,\ \mathcal{C}\ell_{p,q}, p+q=n)$ for which an in general complex matrix representation M can be constructed by using $M(m \times m)$ square matrices, where $m = 2^{(n/2)-1}$, and for which the geometric product is faithfully represented by matrix multiplication rules, we can consider left ideals ψ and right ideals $\psi^\dagger$ simply as one of the columns or one of the rows of those matrices acting from the left or from the right

$$M\psi = \psi' \quad \text{and} \quad \psi^\dagger M = \psi'^\dagger. \tag{7.140}$$

Furthermore, since by definition m is an even number, one can construct a matrix with only one non-zero column (or a matrix with only one non-zero row) in terms of direct application of u combinations of $m/2$ mutually commuting projectors $\{P_i;\ P_iP_j = P_jP_i\}$ to any matrix $M(a)$:

$$\begin{aligned}\psi_u(a) &= M(a)(P_1\ldots P_{m/2})_u = M(a)\mathbf{P}_u,\\ \psi_u^\dagger(a) &= (P_1\ldots P_{m/2})_u M(a) = \mathbf{P}_u M(a).\end{aligned} \tag{7.141}$$

There are $(m/2)$ combinations of projectors P_i and $\tilde{P}_i$, $i = 1, \ldots, m/2$

$$P_i + \tilde{P}_i = \mathbf{1}, \quad \text{or} \quad P_i = \tfrac{1}{2}(\mathbf{1}+\gamma_i), \quad \tilde{P}_i = \tfrac{1}{2}(\mathbf{1}-\gamma_i), \quad \gamma_i^2 = 1, \tag{7.142}$$

where $\gamma_i\gamma_j = \gamma_j\gamma_i$ by definition. This corresponds to every column (row) of the matrix $M(a)$ being itself a left (right) ideal:

$$M(a) = \sum_{u=1}^{m} M(a)\mathbf{P}_u. \tag{7.143}$$

That only one column (row) is non-zero in $\psi(\psi^\dagger)$ makes these ideals isomorphic and mathematically equivalent, to the spin (dual spinor) spaces $\varphi(\varphi^\dagger)$. The spinors can be used as a basis to construct a representation of the geometry,

$$M(a) = M^{uv}(a)\varphi_u\varphi_v^\dagger, \tag{7.144}$$

a procedure which should be called a Cartan map [46, 49].

As the relationships are independent of representations we now can write:

(a) A left ideal in a given geometry is generated by multiplication on the right with $m/2$ linearly independent mutually commuting projectors $\mathbf{P}_u = (P_1 \ldots P_{m/2})_u$. Equivalently, using a transposed definition a right ideal is obtained.

(b) The $\{P_u,\ u = 1, \ldots, m\}$ forms then a complete basis for the ideals space in the geometry.

(c) Every member of the ideal can in turn be analyzed by projecting its components, which in turn correspond to the customary spinor components

$$(\psi_u^{(a)})_v = \mathbf{P}_v M(a)\mathbf{P}_u \rightarrow \varphi_v(a, u). \tag{7.145}$$

(d) If a spinor basis set is defined by $\left\{\varphi_u^{(0)},\ u = 1, \ldots, m,\right\}$ (where m is even by definition) we can write

$$\mathbf{P}_v M(a)\mathbf{P}_u = \varphi_v^{(a)}\varphi_u^{\dagger(0)}, \tag{7.146}$$

such that

$$M(a) = M^{uv}(a)\varphi_v^{(0)}\varphi_v^{(0)\dagger}, \tag{7.147}$$

corresponds to the Cartan map. The symmetries of (7.147) are the geometrical basis of the methods called supersymmetries [49, 50] as mentioned in the introduction.

8.1 TWISTORS AS GEOMETRIC OBJECTS

A basic idea is that instead of considering a complex space (space–time) we should start from a more fundamental, complex formalism such that points and the space (space–time) continuum will be derived concepts. The basic objects were called twistors. Once they are defined, we show that twistors correspond to well defined geometrical objects. In

space–time, in particular, they can be completely described as members of the multi-vector algebra $\mathcal{C}\ell_{1,3}\otimes\mathbb{C}$ corresponding to the carrier space $R^{1,3}$.

The inverse logical derivation is that they can be used to generate the idea of space–time Penrose and collaborators showed that the fundamental mathematical objects should be conformally invariant This space is the projective twistor space constructed out of null geodesics and their ('half') complexification (see below) to obtain the three dimensional complex projective space CP^3. A point in this space is defined by the pair $(W^A, \Pi_{A'})$ which is required to obey (here A and A' are Pauli spinor indices, $Z^{AA'}$ and $X^{AA'}$ are (complex and real) four dimensional vectors in the Pauli algebra, that is $X = X^\mu\sigma'_\mu = X^{AA'}\xi_A\xi^\dagger_{A'}$, where $\{\sigma'_\mu\}$ are the Pauli matrices with $(\sigma'_i)^2 = -\mathbf{1}$, $\sigma_0 = \mathbf{1}$ and the $\xi_A(\xi^\dagger_A)$ are Pauli basis spinors (transpose spinors)),

$$W^A = iZ^{AA'}\Pi_{A'},$$

and

$$Z^{AA'} = Z_0^{AA'} + \lambda^A\Pi^{A'},$$

with the complex vector $Z_0^{AA'}$ and the Pauli spinor $\Pi^{A'}$ fixed and λ^A variable; $\Pi^{A'}$ is a spinor conjugated to $\Pi^{A'}$. If we now consider a real space–time point (in spinor notation) $X^{AA'}$ then the projective twistor $(iX^{AA'}\Pi_{A'}, \Pi_{A'})$ defines a real null geodesic through $X^{AA'}$ in the direction $\Pi^A\Pi^{A'}$ as described below. But the motivation of the twistor theory is to **derive** the concept of a space–time point. Twistor theory was afterwards used to describe zero rest mass fields and subsequently to construct space–time from some (deformed) twistor space.

Let us first show the geometric nature of the twistors, they are members of the (complex) Clifford algebra of space–time.

Consider a Dirac spinor ψ. We know that it is a member of the minimum left ideal of the geometric algebra $\mathcal{C}\ell_{1,3}$ of space–time $R^{1,3}$ generated by the Clifford algebra of the set of four vectors γ_μ. Multivectors have a representation in terms of the Dirac spinors $\gamma_{\mu\nu}... \rightarrow (\gamma_{\mu\nu...})^{aa'}\psi_a\psi^\dagger_{a'}$. The Dirac spinors themselves have four (complex) degrees of freedom $\Psi \rightarrow \Psi^a\psi_a$, where the ψ_a are basis spinors and a = 1,2,3,4. Below we show that we can use the correspondence: $(a = 1) \rightarrow (R,\uparrow)$, $(a = 2) \rightarrow (R, \downarrow)$, $(a = 3) \rightarrow (L, \uparrow)$ and $(a = 4) \rightarrow (L, \downarrow)$, that is right ($R$) and left ($L$) handed spinors of spin up ($\uparrow$) and down ($\downarrow$), or a chiral representation.

All elements $\mathbf{M}_p \in \mathbf{M} = \{\gamma_\mu, \gamma_{\mu\nu}, \gamma\lambda_{\mu\nu}, \gamma_5;\ \mathbf{1}, i\mathbf{1}\}$ of the (complex) Clifford algebra, called (complex) multi-vectors, such that $\mathbf{M}_p^2 = \mathbf{M}_p\mathbf{M}_p =$

1 can be used to construct projectors $\mathbb{P}_p = \frac{1}{2}(\mathbf{1} + \mathbf{M}_p)$, $\mathbb{P}_{-p} = \frac{1}{2}(\mathbf{1} - \mathbf{M}_p)$ with $\mathbb{P}_p\mathbb{P}_{-p} = 0$ and $\mathbb{P}_p + \mathbb{P}_{-p} = \mathbf{1}$. The combination of four commuting projectors $\{\mathbb{P}_p, \mathbb{P}_{-p}, \mathbb{P}_q, \mathbb{P}_{-q}\}$ suffices to classify the Dirac spinors

$$\psi = \psi_{pq} + \psi_{p(-q)} + \psi_{(-p)q} + \psi_{(-p)(-q)}, \tag{7.148}$$

where $\psi_{ab} = \mathbb{P}_a\mathbb{P}_b\psi = \mathbb{P}_{ab}\psi$, given that

$$\mathbb{P}_{pq} + \mathbb{P}_{p(-q)} + \mathbb{P}_{(-p)q} + \mathbb{P}_{(-p)(-q)} = 1. \tag{7.149}$$

There are two $\mathbf{M}_p$ (complex, or in fact imaginary) commuting unit multi-vectors $i\gamma_5$ and $i\gamma_{12}$ (that is $(i\gamma_5)^2 = (i\gamma_{12})^2 = 1$ and $\gamma_5\gamma_{12} = \gamma_{12}\gamma_5$) which are very convenient for an analysis of spinors; they generate the indice correspondence mentioned above for handedness and for spin.

Now the construction generated by the projectors $\mathbb{P}_R = \frac{1}{2}(1+i\gamma_5)$ and $T_x = 1 + \gamma_5\mathbf{x}$, with the position vector $\mathbf{x} = x^\mu\gamma_\mu$, $\mu = 0,1,2,3$ applied to a spinor ψ, is called a reference twistor η_x associated to $\mathbf{x}$ and ψ

$$\eta_{\mathbf{x}} = T_{\mathbf{x}}\mathbb{P}_R\psi \text{ or } \eta_{\mathbf{x}} = (\mathbf{1} + \gamma_5\mathbf{x})\mathbb{P}_R\psi = (\mathbf{1} + \gamma_5\mathbf{x})\Pi, \tag{7.150}$$

where $\Pi = \mathbb{P}_R\psi$.

Π is a right handed Dirac spinor which can be represented $\Pi \rightarrow \begin{pmatrix} 0 \\ \xi \end{pmatrix}$, as the couple of a Pauli ξ (usually called Weyl) spinor and zero. The adjoint spinor $\overline{\Pi} = (\xi^*, 0) = (\xi_1^*, \xi_2^*, 0, 0)$.

The transpose twistor, starting from $\psi = \psi^\dagger\gamma_0$ and considering $\mathbb{P}_R = \mathbb{P}_L^{tr}$ (where $M^k = (\Sigma A\ {}^{aA}\gamma A)^k = \Sigma a({}^{\alpha A})^*\gamma_A$, the star is used to indicate complex conjugation) is

$$\overline{\eta_{\mathbf{x}}} = \overline{\psi}\mathbb{P}_L(\mathbf{1} + \gamma_5\bar{\mathbf{x}}) = \overline{\Pi}(\mathbf{1} + \gamma_5\bar{\mathbf{x}}) \text{ product, for } \mathbf{x} \text{ real,} \tag{7.151}$$

is such that the scalar product

$$\overline{\eta}_{\mathbf{x}}\eta_{\mathbf{x}} = \overline{\Pi}\,\Pi + 2\overline{\Pi}\gamma_5\Pi + x^2\overline{\Pi}\,\Pi = 2\overline{\Pi}\gamma_5\ \mathbf{x}\Pi, \tag{7.152}$$

because

$$\overline{\Pi}\,\Pi = \overline{\psi}\frac{1}{2}(1 - i\gamma_5)\frac{1}{2}(1 + i\gamma_5)\psi = 0. \tag{7.153}$$

That is, it represents the expectation value of the (dual of the) position $\gamma_5\mathbf{x}$ with respect to spinor Π. The outer product

$$\eta_{\mathbf{x}}\overline{\Pi} = (\mathbf{1} + \gamma_5\mathbf{x})\Pi\,\overline{\Pi} = (\mathbf{1} + \gamma_5\mathbf{x})q, \tag{7.154}$$

here $q = \Pi\Pi$ is the right handed part of a null vector $Q = \psi\bar{\psi}$ (a single outer product of Dirac spinors or of Weyl spinors can only correspond to fixed handedness null vectors), that is the presence of

$$q = \mathbb{P}_R \psi\bar{\psi}\mathbb{P}_L, \tag{7.155}$$

gave origin to the Penrose interpretation of a twistor as a composite of a null vector q and a 'flag' $\gamma_5 xq$, as far as a bi-vector represents an oriented surface. The multi-vector corresponding to the twistor is

$$\underline{\eta}_{\mathbf{x}} = \eta_{\mathbf{x}}\bar{\Pi} = q + \gamma_5\,\mathbf{x}q = q + i\mathbf{x}q, \tag{7.156}$$

it contains the projection $Q \to q = \mathbb{P}_R Q \mathbb{P}_L$. In (7.154) we have $\gamma_5\mathbf{x} = -\mathbf{x}\gamma_5$ and $-\gamma_5 q = iq$ because q is a right handed projection.

A supermatrix representation of the above relations, although superfluous, is very helpful to visualize the different structures. The vectors

$$\gamma_\mu \to \left\{ \gamma_0 = \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \quad \gamma_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \quad i = 1,2,3 \right\}, \tag{7.157}$$

here the σ_i are the positive square Pauli matrices $(\sigma_i)^2 = \mathbf{1}$, $\sigma_1\sigma_2\sigma_3 = i\,1$ and $\sigma_i\sigma_j = i\sigma_k$; $i,j,k = 1,2,3$ cyclic.

The hypervolume $\gamma_5 = i\begin{pmatrix} \mathbf{1} & 0 \\ 0 & -\mathbf{1} \end{pmatrix} = \sigma_1\sigma_2\sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, then the twistor

$$\eta_{\mathbf{x}} = \begin{pmatrix} i\,\vec{x}\Pi \\ \Pi \end{pmatrix} \quad \text{remark} \quad \Pi = \mathbb{P}_R \eta_x, \tag{7.158}$$

where $\mathbf{x} = \begin{pmatrix} 0 & \vec{x} \\ \vec{x}^c & 0 \end{pmatrix}$, and $\vec{x}^c$ is the left-handed quaternion conjugate of $\vec{x}$. The use of the multi-vector $\underline{\eta}_x$ or of the twistor spinor ηx can be done indistinctly. ψ can also be replaced by a multi-vector as discussed in [127].

We then see that the admitted geometrical interpretation considers, as a result, not the vector $\mathbf{x}$ but its three-dimensional projection $\vec{x}$ and, moreover, the factor $i = \sigma_1\sigma_2\sigma_3$ is the three-dimensional volume element.

Let us now obtain the scalar products corresponding to what Penrose calls the twistor invariant, the product of two twistors when the transpose is taken considering that we can define $\gamma_5^\dagger = -\gamma_5$ (as is obvious from the representation (7.157) of γ_5 in terms of gamma matrices). Then instead of (7.151) we define the adjoint $\bar{\eta}$ as follows

$$\bar{\eta} = \bar{\psi}\mathbb{P}_R(1 + \gamma_5^\dagger \mathbf{x}) = \bar{\Pi}(1 - \gamma_5 \bar{\mathbf{x}}), \tag{7.159}$$

and consider the product of two twistors:

a) Corresponding to the same spinor ψ and to the same vector real **x**

$$J^{\psi\psi}_{\mathbf{xx}} = \hat{\eta}_{\mathbf{x}}\eta_{\mathbf{x}} = \bar{\Pi}\gamma_5(\mathbf{x}-\mathbf{x})\Pi = 0, \tag{7.160}$$

This is obviously invariant under multiplication of η_x by a complex factor, then from the eight degrees of freedom (four complex numbers) only six are geometrically significant and (7.160) reduces them to five real parameters. These parameters can be chosen to correspond too to the direction of the null ray (the light trajectory) and to its intersection with a base hyperplane $t = 0$.

b) Corresponding to the same spinor ψ, but to two different vectors **x** and **x**′

$$J^{\psi\psi}_{\mathbf{xx'}} = \bar{\Pi}\gamma_5(\mathbf{x}'-\bar{\mathbf{x}})\Pi, \tag{7.161}$$

which will be zero if it corresponds to a point where two (real x = x') light rays intersect. This is what could correspond to a definition of a point from twistors. For variable **x**′ and fixed **x** we define a congruence usually called the **Robinson congruence.**

Let us now consider the multi-vector twistors $\underline{\eta}_x$ and their generalization. Notation reminder: the multi-vector twistor

$$\underline{\eta}^{(\psi)}_{\mathbf{x}} = \eta_{\mathbf{x}}\bar{\Pi} = \mathbf{q} + \gamma_5\mathbf{x}\mathbf{q}, \tag{7.162}$$

where **q** is a null vector considered (by definition not included in (7.151)) supported at the position **x.** The obvious generalization is, both **q** and **x**, to be allowed to become arbitrary (complex) vectors.

8.2 REPRESENTATION OF THE POINCARÉ GROUP

The equations describing physical relations in space–time (or in ordinary three-dimensional space) should be covariant under both Lorentz transformations L and changes of origin d of the coordinate system. This set of operations, called the Poincaré group, of which rotations and changes of origin in ordinary three-dimensional space are subgroups, is characterized by the pair $\{L, d\}$.

The group product is $\{L_2, d_2\}\{L_1, d_1\} = \{L_3, d_3\}$. In the geometric algebra of space–time $R^{1,3}$ (the Clifford algebra denoted $R_{1,3}$ or $\mathcal{C}\ell_{1,3}$) a position vector $\chi_0 = \chi^\mu\gamma_\mu$ is transformed (d is a vector and L_n the exponential of a bi-vector)

$$\chi_0 \to \chi_1 = L_1\chi_0\tilde{L}_1 - d_1 \to \chi_2 = L_2\chi_1\tilde{L}_2 - d_2, \tag{7.163}$$

or

$$\chi_2 = L_2 L_1 \chi_0 \widetilde{L}_1 \widetilde{L} L_2 + L_2 d_1 \widetilde{L}_2 + d_2, \tag{7.164}$$

$$\chi_2 = L_3 \chi_0 \widetilde{L}_3 + d_3, \tag{7.165}$$

defining (the tilde operation reverses the product, of two multi-vectors)

$$L_3 = L_2 L_1 \quad \text{and} \quad d_3 = L_2 d_1 \widetilde{L}_2 + d_2. \tag{7.166}$$

The 'multiplication' of the Poincaré group is well defined but cumbersome. There are several representations, some of which are reasonable to handle. For example the use of the matrix form (here the L_i are square matrices and the χ_i and d_i column matrices, the 0^ℓ are row matrices)

$$\begin{pmatrix} L_1 & d_1 \\ 0^\ell & \mathbf{1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ 1 \end{pmatrix} = \begin{pmatrix} L_1\chi_0 + d_1 \\ 1 \end{pmatrix}, \tag{7.167}$$
$$\begin{pmatrix} L_2 & d_2 \\ 0^\ell & \mathbf{1} \end{pmatrix} \begin{pmatrix} \chi_1 \\ 1 \end{pmatrix} = \begin{pmatrix} L_2\chi_1 + d_2 \\ 1 \end{pmatrix},$$

and

$$\begin{pmatrix} L_3 & d_3 \\ 0^\ell & \mathbf{1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ 1 \end{pmatrix} = \begin{pmatrix} L_2 & d_2 \\ 0^\ell & \mathbf{1} \end{pmatrix} \begin{pmatrix} L_1 & d_1 \\ 0^\ell & \mathbf{1} \end{pmatrix} \begin{pmatrix} \chi_0 \\ 1 \end{pmatrix}, \tag{7.168}$$

then

$$\begin{pmatrix} \chi_0 \\ 1 \end{pmatrix} = \begin{pmatrix} L_2 L_1 & L_2 d_1 - d_2 \\ 0^\ell & \mathbf{1} \end{pmatrix}, \tag{7.169}$$

clearly shows that the 'product' of group elements are elements of the group. Group multiplication is matrix multiplication here.

In geometric algebra there is a representation of the elements of the group which allows **geometric multiplication** as the group (non abelian) multiplication

$$\{L_2, d_2\}\ \{L_1, d_1\} = \{L_3, d_3\}.$$

For this geometric algebra representation we use the product of the elements $(1 + \varepsilon d)$ and L which, separately have as group multiplication, the geometric product

$$L_3 = L_2 L_1 \quad \text{and} \quad (1 + \varepsilon d_3) = (1 + \varepsilon d_2)(1 - \varepsilon d_1), \tag{7.170}$$

where $d_3 = d_2 + d_1$ and $\varepsilon d_2 \varepsilon d_1 = 0$ requiring that either $\{\varepsilon^2 = 0,\ \varepsilon d = d\varepsilon\}$ or $\{\varepsilon + \varepsilon = 0,\ \varepsilon + d = d\varepsilon\}$. In the first case $\varepsilon^2 = 0$ is a nilpotent

operation commuting with the vectors *d.* In the second case $\varepsilon_+(\varepsilon_-)$ is a projector operator

$$\varepsilon_+\varepsilon_- = \varepsilon_-\varepsilon_+ = 0, \qquad \varepsilon_+ + \varepsilon_- = 1, \tag{7.171}$$

which can be written in terms of a unit multi-vector *e*, e^2 = **1** which, $ed = -de$, anticommutes with the vectors *d.* The $\varepsilon_+ = \frac{1}{2}(1 + e)$ and $\varepsilon = \frac{1}{2}(1 - e)$. In general a suitable $\{\varepsilon;\varepsilon^2 = 0,\ \varepsilon d = d\varepsilon\}$ or $\{e;\ e^2 = 1,\ ed = -de\}$ can only be found in an algebra of a dimension higher than the Clifford algebra $R_{p,q}$ corresponding to the space $R_{p,q}$. The formal definition of ε or *e* is enough for the purpose of studying the Poincaré group but the possibility of physical usefulness or insight would be lost.

In the Dirac algebra *D* corresponding to $R_{1,3}$ one usually admits its complexification, corresponding to the use of $R_{0,5} \simeq R_{2,3} \simeq R_{4,1}$, that is, $Dc = \{R_{1,3} \otimes iR_{1,3} \simeq R_{0,5}\}$. The commonly used operators $i\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$ and $i\gamma_{12} = i\gamma_1\gamma_2$ are good examples of this complexification of the algebra. It is $i\gamma_5$ which has the property $i\gamma_5\gamma_\mu = -\gamma\mu i\gamma_5$ and $(i\gamma_5)^2 = 1$ required for the use in (7.171) and (7.172). Then the group of translations Γ has two isomorphic representations $\Gamma : d \to d'$, which are $(1 + \mathbb{P}_R d)$ and $(1 + \mathbb{P}_L d)$ where

$$\mathbb{P}_R = \tfrac{1}{2}(1 + i\gamma_5) \quad \text{and} \quad \mathbb{P} = \tfrac{1}{2}(i\gamma_5),$$

$$\mathbb{P}_R + \mathbb{P}_L = 1, \quad \mathbb{P}_R\mathbb{P}_L = \mathbb{P}_L\mathbb{P}_R = 0, \tag{7.172}$$

and

$$\mathbb{P}_R\mathbb{P}_R = \mathbb{P}_R,\ \mathbb{P}_L\mathbb{P}_L = \mathbb{P}_L,\ \mathbb{P}_R d = d\mathbb{P}_L.$$

Here the $\mathbb{P}_R$ and $\mathbb{P}_L$ are the operators for right handed and left handed projection, respectively: $\psi = \mathbb{P}_R\psi + \mathbb{P}_L\psi = R_0 + L_0$.

The Lorentz transformations $L = \exp(\theta^{\mu\nu}\gamma_{\mu\nu}/2)$ by angles $\theta^{\mu\nu}$ in the planes $\gamma_{\mu\nu}$ act as

$$L : \chi \to \chi' = L\chi\tilde{L}, \quad L\tilde{L} = \tilde{L}L = 1, \tag{7.173}$$

do form a multiplication group

$$L_3 = L_2L_1, \qquad \tilde{L}_3 = \tilde{L}_1\tilde{L}_2, \tag{7.174}$$

which can be used to construct, together with the $(1 + \varepsilon d)$, a set of two **representations** of the Poincaré group

The representations of the Poincaré group are

$$\mathbb{P}_i^{(L)} = (1 + \mathbb{P}_L d_i)L_i, \qquad \mathbb{P}_i^{(R)} = (1 - \mathbb{P}_R d_i)L_i, \tag{7.175}$$

and ($A = L, R$, using $L_j \mathbb{P}_L = \mathbb{P}_L L_j$ and $\mathbb{P}_L d_n = d_n \mathbb{P}_R$)

$$\mathbb{P}_k^{(A)} = \mathbb{P}_j^{(A)} \mathbb{P}_i^{(A)} = (1 + \mathbb{P}_L d_j) L_j (1 + \mathbb{P}_L d_i) L_i,$$

$$\mathbb{P}_k^{(A)} = (1 + \mathbb{P}_L (d_j + L_j d_i L_j)) L_j L_i, \tag{7.176}$$

$$\mathbb{P}_k^{(A)} = (1 + \mathbb{P}_L d_k) L_k,$$

also

$$(1 + \mathbb{P}_L d_k)(1 - \mathbb{P}_L d_k) = 1.$$

The invertible operators $(1 + \mathbb{P}_A d)$ become then an important part of the study of the invariances and the symmetries related to relativistic quantum theory of particle and interaction fields. Their use to construct a new representation of the Poincaré Lie algebra is thus both straightforward and clarifies the reason for some features of the theory of elementary particles. The relations could have been the best starting point for the instruction of a theory of twistors. We should remember that any spinor can be written $\psi = L\psi_0$ and that $L\mathbb{P}_L = \mathbb{P}_L L$.

When we work in the realm of the Clifford algebras the complexification of a space corresponds to the increase of only one dimension in the space of basis vectors. Basis vectors should all anticommute among themselves. Other procedures, besides complexification, like duplexification or doubling, play a similar and equivalent role.

8.3 SCREWS AND MULTI-VECTOR SCREWS

Let us analyze the possibilities open. The need for **q** to be a null vector stemmed from its introduction as a single outer product of a spinor Π and its conjugate Π. On the other hand **x** could have been null or general

$$\underline{\eta}_{\mathbf{x}}^{(\psi)} = (1 + \gamma_5 \mathbf{x})\mathbf{q} \to \underline{\eta}_{\mathbf{x},y} = (1 + \gamma_5 \mathbf{x} \mathbb{P}_A)\mathbb{P}_A Y, \tag{7.177}$$

where $\mathbb{P}_A = \{\mathbb{P}_L, \mathbb{P}_R\}$ is either one of the idempotent left or right handedness projectors, in particular we could have defined $\mathbf{q} = \mathbb{P}_L Y \gamma_0$ for Y such that **q** is a null vector. But in (7.177) we want to allow Y to be a general complex vector)

$$Y = Y^{ab} \xi_a \bar{\xi}_b \qquad a, b = 1, 2, 3, 4, \tag{7.178}$$

where ξ_a (and $\bar{\xi}_b$) are a general basis set for Dirac spinors (and adjoint Dirac spinors) $Y^{ab} \in \mathbb{C}$. If $\mathbf{q} = \mathbb{P}_L Y \gamma_0 = \tilde{\Pi}\Pi$ then $Y = \mathbb{P}_\uparrow Y$, (remember that up and down are only relative, otherwise free, directions).

The presence of g_0 in the definition of **q** from ***Y*** is necessary to connect, with (7.151) and (7.162) because **q** corresponds to a space–time 'cut' of a vector q_v (a space–time 'cut' is the multiplication of a multi-vector by a time vector γ_0)

$$\mathbf{q} = q_V \gamma_0 = q^\mu \gamma_\mu \gamma_0 = q^\mu \Sigma_\mu. \tag{7.179}$$

Where the space–time quaternions $\Sigma_\mu = \gamma_\mu\gamma_0 = \gamma_{\mu 0}$ can be represented by block diagonal matrices, the same as **q,** with σ_μ in the main diagonal, while the γ_μ and the particular time vector γ_0 were represented in (7.157) by block diagonal matrices with σ_μ in the second diagonal. $\Sigma_\mu = \gamma_\mu\gamma_0 \rightarrow \begin{pmatrix} \sigma_\mu & 0 \\ 0 & -\sigma_\mu \end{pmatrix}$; μ = 1,2,3 while Σ_0 = **1**. In (7.177) we find all elements of the Dirac (complex space–time) algebra. The *Y* are odd (so is $\mathbb{P}_L Y$) and the **X***Y* are even (the same as $\gamma_5 X \mathbb{P}_L Y$), the presence of γ_5 makes all elements, upon which it acts, become their dual: scalars to pseudo-scalars; vectors to tri-vectors; and space–space bi-vectors to space–time bi-vectors. $\underline{\eta}_{\mathbf{x},\mathbf{y}}$ is then a full (complex) multi-vector.

Here is where the discussion of part C, about the full Poincaré group is now directly relevant. The factors $(1 + \gamma_5 \mathbb{X}_A)$ are representations of the translations group

$$(1 + \gamma_5 \mathbf{X}\mathbb{P}_A)(1 + \gamma_5 \mathbf{X}'\mathbb{P}_A) = (1 + \gamma_5(\mathbf{X} + \mathbf{X}')\mathbb{P}_A). \tag{7.180}$$

Reminder: $\gamma_5 \mathbf{X}\mathbb{P}_A \gamma_5 \mathbf{X}'\mathbb{P}_A = \gamma_5 \mathbf{X} \gamma_5 \mathbf{X}' \mathbb{P}_B \mathbb{P}_A = 0$, where $B \neq$ A. These factors also commute with the rotations

$$\begin{aligned} R(1 + \gamma_5 \mathbf{X}\mathbb{P}_A)\mathbb{P}_A Y R^{-1} &= (1 + \gamma_5 R\mathbf{X}R^{-1}\mathbb{P}_A)\mathbb{P}_A R Y R^{-1} \\ &= (1 + \gamma_5 \mathbf{X}''\mathbb{P}_A)\mathbb{P}_A Y'', \end{aligned} \tag{7.181}$$

where X″ = R(X) and *Y*″ = *R*(*Y*) are the rotated vectors.

That is, the vector **X** is a **position** (fixed frame) vector, whereas *Y* is a free frame (sometimes called just **'free'**) vector *Y,* or, in the original, the twistor $\prod$, is to be acted on by rotations but not by translations. Then it should represent a physical phenomenon and *Y* is not a position vector. In geometry *Y* is a vector which should be associated with magnitude or direction, not with position.

If a twistor or our new objects, in fact a special combination of a vector and its product with a position vector, should become a field over **x**, it is the chiral part $\mathbb{P}_A Y = Y_A$ that should become $Y_A(\mathbf{X})$, even in the case where both *Y,* x $\in \mathbb{C}^4$. That is, the free vector (representing a vectorial magnitude field) is the part that carries the position dependence and **x** is the part that anchors the field.

In the use of the (multi-vector) twistors an integration over **x** averages the field and an integration over Y averages over the (auxiliary) field.

We can, moreover, take the imaginary phase out of the definition of the multi-vector pair, represented by γ_5 in (7.150) and (7.177); and keep the pair we shall call screw $S_{x,y}$

$$S^{(A)}_{\mathbf{x},\mathbf{y}} = (1 + \mathbf{X}\mathbb{P}_A)Y. \tag{7.182}$$

The screws are also faithful, nontrivial, representations of the Poincaré group. The factor $(1 + \mathbf{X}\mathbb{P}_A)$ is invertible (the factor $(1 + \gamma_5\mathbf{X}\mathbb{P}_A)$ was also invertible) and the factor Y can also be invertible. Consider

$$(1 - \mathbf{X}\mathbb{P}_A)(1 - \mathbf{X}\mathbb{P}_A) = 1, \tag{7.183}$$

because $\mathbf{X}\mathbb{P}_A\mathbf{X}\mathbb{P}_A = \mathbf{X}^2\mathbb{P}_B\mathbb{P}_A = 0$, B is $B \neq A$, also

$$YY^{-1} = Y\left(\frac{Y}{Y^2}\right) = 1 \qquad \text{if} \quad Y^2 \neq 0. \tag{7.184}$$

A uniform screw would be $S^{(A)}_{\mathbf{x},\mathbf{y}}$ where $Y = Y(\mathrm{X}) = \mathbb{R}(\mathbf{X})\ Y_0\mathbb{R}^{-1}\ (\mathbf{X})$ with $\mathbb{R}(\mathbf{X})$ a rotation generator linear in **X**. In Clifford algebra rotations are generated by the bi-vectors $\gamma_{ij}(i,j = 1,2,3)$, their general form is $\mathbb{R}(\mathbf{X}) = \exp(\frac{1}{2}\sigma_{ij}(\mathbf{X})\gamma_{ij})$. The uniform rotation would be a linear dependence on some $\mathbf{X}_1 = a\mathbf{X}_0$, such that $\sigma_{ij} = a\sigma^{ij}_1 + \sigma^{ij}_0$. Then the screw will correspond to rotations in the plane γ_{ij} proportional to the displacement, of the vector **X** according to the scalar product $(\mathbf{X}\mathbf{X}_0)_{\text{scalar}}$. We could also consider the screw to describe a helical path. The new objects are suitable candidates for robotics, vision analysis, or models of angular momentum carrying objects. Here we shall consider an electron field to be represented by combinations of screws. In (7.178) we can, of course, consider that Y is a multi-vector, all multi-vectors having the same decomposition, given by (7.178). The considerations of this paragraph applying equally to all multi-vectors given that rotations are multi-vector grade concerning functions. A multi-vector spinor will be used to represent an electron in the next section.

A final, geometrical, consideration. If **X** is a position vector and **q** a null vector, only the line where **q** is embedded can be known because we do not really know **X** but only the product **Xq** and this product is unchanged is we replace $\mathbf{X} \to \mathbf{X} + \alpha\mathbf{q}$, the scalar part is increased in $\alpha(q)^2_s = 0$ and the bi-vector part in, $\alpha q \wedge \mathbf{q} = 0$. Only if **X** itself is null then $\mathbf{X} \wedge \mathbf{q}/(\mathbf{X} \cdot \mathbf{q}) = \tan^2_2\theta$ and we can know the point, of 'support' of the null vector **q** by the null vector **X.**

This is the geometrical model of the original twistors, null rays supported by null rays.

If in the generalization we consider non-null vectors supported by non-null vectors, the product has a scalar part $X^{\mu}Y_{\mu}$ which changes if we charge $\mathbf{X} \rightarrow \mathbf{X}+\alpha Y$ because $\alpha(Y)^2$ will no longer be zero, then the product XY and the knowledge of Y determines $\mathbf{X}$ completely. We can, from the analysis of the generalized twistor, find the support position vcctor $\mathbf{X}$ and the supported free vector Y. The twistor η (see (7.150)) has a dual η^{D}. We have seen that η has 4 complex components, they may be labeled η^{α}, $\alpha = 1,2,3,4$, η^{0} also has 4 operator components η^{α}, $\alpha = 1,2,3,4$ and, being a function and a conjugated operator pair, they obey the commutation relations

$$[\eta^{\alpha}, \eta_{\beta}] = \delta^{\alpha}_{\beta}, \tag{7.185}$$

useful to construct field theories.

8.4 MULTI-VECTOR SCREW FOR THE ELECTRON FIELD

Let us now present as a final example the form in which a multi-vector screw can contain all the information we know to be necessary to describe an electron at the single particle level.

First, let us recall the physical content and the geometrical content of the (multi-vector) wavefunction ψ. A free electron wavefunction ψ_0 can be written (see Keller 1997a) in the well known form (see Casanova 1976 for a discussion)

$$\psi_0 = \sqrt{\rho}\, e^{\beta\gamma_5} R_0, \tag{7.186}$$

where ρ is the statistical weight ($\sqrt{\rho}$ is the amplitude) of the wave function at a point X, then the probabilistic nature of the wavefunction is contained in ρ. The factor $e^{\beta\gamma 5}$ with β the Takabayashi angle is the way to determine if we are describing a particle or its antiparticle and the rotor R_0 describes a rotation in the spin plane γ_{12} corresponding to intrinsic angular momentum of the particle's field at point X

$$R_0 = R(x)e^{ip_0 X/\hbar}. \tag{7.187}$$

Then this factor, the rotor R_0, contains the first quantization properties of the electron. An arbitrary wave function would be $\psi = R\psi_0$ with R a Lorentz transformation, which can be taken to be $R(X)$, to describe the gauge interactions of the electron. We have described elsewhere how R can also describe the full electroweak interaction, and not only the electromagnetic and gravitational parts.

Now the position vector of the electron X and the normalized spin vector Y of the electron can be incorporated in the full complex multi-

vector screw M

$$M = (1 + i(X + iX_I))\psi = (1 + iZ)\psi. \tag{7.188}$$

We propose to call this complex multi-vector a MEXOR or the **M**ulti-vector to represent the **E**lectron and its position **X** space–time vect**OR**. (We have also chosen this name for obvious geographycal reasons!) and it is the geometric object which completely represents a massive, interacting, spinning electron as will be analyzed in the next section.

But first let us connect with the already known concepts of Dirac spinor and twistor.

The Dirac spinor basis φ_i, $i = 1,2,3,4$, and the transpose $(\varphi^\dagger)^i$ obey

$$(\varphi^\dagger)^i\varphi_j = \delta^i_j, \tag{7.189}$$

they can generate the multi-vectors as

$$M = M^j_i\varphi_j(\varphi^\dagger)^i, \tag{7.190}$$

in particular, $\psi = \psi_k(\varphi^\dagger)^k$ with $k = \{a,b\}$ in (7.190) above and we can represent $\psi_k = \begin{pmatrix} \xi \\ \pi \end{pmatrix}$, where ξ and π are Weyl spinors and conjugated spinors respectively. And conversely any multi-vector can be projected into a Dirac spinor φ_k

$$\begin{aligned} M \to M_k &= M\varphi_k = M^j_i\varphi_j(\varphi^\dagger)^i\varphi_k \\ &= M^j_i\varphi_j\delta^i_k = M^j_k\varphi_j, \end{aligned} \tag{7.191}$$

(see Keller and Rodríguez 1991).

Then the multi-vector ψ contains, in fact, four components ψ_k which actually correspond, if we take the chiral basis representation, to the right handed spin up or down and to the left handed spin up or down components. In a 4 x 4 matrix representation of ψ each column ψ_k corresponds to each one of these four possibilities as a reference. And, from the use of the chirality projectors (7.150)

$$\mathbb{P}_L + \mathbb{P}_R = 1, \quad \mathbb{P}_R\mathbb{P}_L = \mathbb{P}_L\mathbb{P}_R = 0, \quad \mathbb{P}^2_R = \mathbb{P}_R \quad \text{and} \quad \mathbb{P}^2_L = \mathbb{P}_L, \tag{7.192}$$

we can write the MEXOR M in (7.188) as the multi-vector twistor pair

$$\begin{aligned} M &= (1 + iZ)\psi = (\mathbb{P}_L + \mathbb{P}_R + iZ(\mathbb{P}_L + \mathbb{P}_R))(\mathbb{P}_L + \mathbb{P}_R)\psi \\ &= M_L + M_R, \end{aligned} \tag{7.193}$$

and then for each basis spinor φ_k; $k = 1,2,3,4$ to project $\psi \to \psi_k = \psi\varphi_k$ as the four sets

$$M_k = (1 + iZ\mathbb{P}_L)\mathbb{P}_L\psi_k + (1 + iZ\mathbb{P}_R)\mathbb{P}_R\psi_k,$$

or

$$M_k = (M_k)_L + (M_k)_R, \tag{7.194}$$

with

$$M = M_k(\varphi^\dagger)^k,$$

and each M_k can describe an electron as a sum of two twistors in reference to each of the four Dirac spinors ψ_k for the electron being given.

8.4.1 A ONE PARTICLE FORMALISM FOR THE ELECTRON BASED ON MEXORS

Once the MEXOR corresponding to an electron has been defined we can use the analysis of twistors above to show that the formulation in terms of mexors contains some key new features for the determination of the theory of the electron.

We shall show that the Dirac equation with mass $m \neq 0$ for ψ_k is a consequence of having introduced the MEXOR. This is related to (7.193) not being Poincaré-covariant directly, but only its separated left and right handed parts M_L and M_R are faithful representations of the Poincaré group.

The twistor product of the M_k with itself and with its adjoint then has several terms which have different meanings:

$$(M_k)_L^\dagger\gamma_A(M_k)_L, \quad (M_k)_R^\dagger\gamma_A(M_k)_R, \quad (M_k)_L\gamma_A(M_k)_R,$$

$$(\widetilde{M}M_k)_L\gamma_A(M_k)_L, \quad (\widetilde{M_k})_R\gamma_A(M_k)_R \quad \text{and} \quad (\widetilde{M_k})_R\gamma_A(M_k)_L.$$

All of them related, however, to the analysis of section 3 above. We form combinations of the type of bilinear covariants and invariants with the form $\widetilde{M}_k\gamma_A M_k$, related to the Dirac theory with the γ_A the Hermitian operator obtained from the Clifford algebra (7.28).

The γ_A will contain three types of information. First that information related to geometric quantities: γ_μ for vector quantities like momentum or $\gamma_{\mu\nu}$ for bi-vector quantities like angular momentum or electromagnetic field strengths; second the twistor or spinor 'metrics': ε_{AB} is given by δ_1 and the twistor metric by γ_1, the twistor conjugation is given by $i\gamma_5\gamma_0$

represented by $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$; and third the projection which selects from the spinor or from the twistor the desired components, for example given

$$\eta_x = \begin{pmatrix} ix\Pi \\ \Pi \end{pmatrix} \rightarrow \mathbb{P}_R \eta_x = \Pi = \mathbb{P}_R \psi = \begin{pmatrix} 0 \\ \Pi \end{pmatrix}.$$

and then $\mathbb{P}_R$ selects the basic spinor from the twistor (in the usual twistor notation $Z^\alpha = (W^A, \pi_{A'}) \rightarrow \mathbb{P}_R Z^\alpha = (0, \pi_{A'})$ and $\mathbb{P}_L Z^\alpha = (W^A, 0)$.

First, remember that because in twistors we have, by definition,

$$V^A = iZ^{AA'}\xi_{A'}, \quad \text{and} \quad W^A = iZ^{AA'}\pi_{A'}, \tag{7.195}$$

from which we can form the invariant, using the twistor metric $m_{\alpha\beta}$, $\omega_0 = m_{\alpha\beta}V^\alpha W^\beta = \bar{\pi}\xi$, and we can also extract the complex vector Z from the twistors and the representation of the γ^μ basis vectors of the Clifford algebra

$$Z^\mu = i\omega_0^{-1}(\bar{V}\delta^\mu\pi - \bar{W}\delta^\mu\xi) = M_k^\dagger \gamma^\mu M_k, \tag{7.196}$$

with

$$X^\mu = \mathrm{Re}\,(Z^\mu) \quad \text{and} \quad X_I^\mu = \mathrm{Im}\,(Z^\mu), \tag{7.197}$$

we also obtain Y called the Pauli–Lubanski, or spin space–time, vector. This is more clearly shown if we form the momentum vector

$$P^\mu = \widetilde{M}_k \gamma^\mu M_k, \tag{7.198}$$

resulting in

$$P^\mu = \bar{\xi}(\sigma^\mu)\xi + \bar{\pi}\sigma^\mu\pi, \tag{7.199}$$

which is the momentum part, of the Gordon decomposition of the currents. The normalization of the ψ is given by (7.201) below; and then check that Y obeys

$$Y^\mu Y_\mu = 1, \quad Y^\mu P_\mu = 0. \tag{7.200}$$

Furthermore, from the existence of the cross products in (7.198) we find that

$$P^\mu P_\mu = 2\omega_0^* \omega_0 = m^2 > 0, \tag{7.201}$$

then for the determination of the wavefunction ψ the massive Dirac equation has to be used with the value of m resulting from (7.201). This

is in agreement with the Higgs postulate that mass is to be obtained from the coupling of the left handed and the right handed parts of the electron's field.

The cross products of the terms with $(\psi_k)_L$ and $(\psi_k)_R$ define a couple of vectors e_1 and e_2 which together with the normalized Y and P form an orthonormal tetrad with P and Y gauge invariant but e_1 and e_2 only defined up to a rotation angle in the plane they generate.

As we can recover the spinors from the twistors $\mathbb{P}_R\eta_R = \mathbb{P}_R\psi$ and $\mathbb{P}_L\eta_L = \mathbb{P}_L\psi$, then $\psi = \mathbb{P}_R\eta_R + \mathbb{P}_L\eta_L$ and all bilinear quantities and analysis of the standard theory are recovered. It is also clear that the fundamental differential equation becomes the Dirac equation applied to ψ. The bispinor ψ can also be mapped into a spinor pair

We can also map one of the two twistors, say the left handed, into a right handed twistor, by straightforward conjugation and then the MEXOR will result equivalent to a twistor pair as in the analysis of the massive particle with spin made by Tod (1975, 1977); Tod and Perjes (1976) and Bette (1984, 1988, 1989) with all their results recovered by the MEXOR approach.

Finally from the gauge freedom of the spinor $\psi \rightarrow R(x)\psi$ and of the twistor $Z^{AA'} \rightarrow Z^{AA'} + \lambda^A\pi^{A'}$ we introduce through $R(x)$ the gauge fields, and through the results (changes in Z given by the terms $\lambda^A\pi^{A'}$) describe the changes the gauge fields produce in Y and in the e_1 and e_2 vectors, besides the well known changes the gauge fields produce in the momentum vector P. This can also be described as a sum of spinors

$$\psi' = \psi + (R(x) - 1)\psi, \tag{7.202}$$

and therefore as the coupling with new MEXORS

$$(1 + iZ)[(R(x) - 1)\psi], \tag{7.203}$$

or their corresponding decomposition into a sum of (two additional) twistors.

We have described above how if the mass of the electron is given the masses of the other elementary particles are then describable in terms of numbers associated to the representations on the Dirac Clifford algebra.

The gauge approach to interactions is expressed in Clifford algebra with the use of the representations of the Lorentz transformations L given by (7.163) above. They are the R multi-vectors generated by the exponentiations of the bi-vectors $(\lambda \neq \{\mu \neq \nu\})\gamma_{mv}$

$$\gamma_{\mu\nu}\begin{cases} \gamma_\mu = -\gamma_\mu\gamma_{\mu\nu} = -g_{\mu\mu}\gamma_\nu, \\ \gamma_\nu = +\gamma_{\mu\nu}\gamma_\nu = +g_{\nu\nu}\gamma_\mu, \\ \gamma_\lambda = +\gamma_\lambda\gamma_{\mu\nu}, \end{cases}$$

(generators of the Lorentz transformations) which rotate vectors in the plane $\mu = \nu$ they represent and leave vectors orthogonal to that plane unchanged.

$$R = e^{\gamma_{\mu\nu}\phi^{\mu\nu}}$$

The action on multi-vectors is

$$M \to RM\tilde{R} \quad \text{with} \quad R\tilde{R} = \tilde{R}R = \mathbf{1}.$$

For example, applying $e^{\gamma_{\mu\nu}\varphi}$ with $(\gamma_{\mu\nu})^2 = -\mathbf{1}$, we obtain

$$\begin{aligned}(\cos\varphi + \gamma_{\mu\nu}\sin\varphi)\,\gamma_\lambda(\cos\varphi - \gamma_{\mu\nu}\sin\varphi)\\ = \gamma_\lambda\cos^2\varphi + \cos\varphi\sin\varphi(\gamma_{\mu\nu}\gamma_\lambda - \gamma_\lambda\gamma_{\mu\nu}) - \sin^2\varphi\gamma_{\mu\nu}\gamma_\lambda\gamma_{\mu\nu},\end{aligned}$$

corresponding to a rotation by an angle of 2φ of the vectors in the plane $\mu = \nu$. Whereas the action on spinors is

$$\psi \to R\psi,$$

corresponding to a rotation by an angle φ in the plane $\mu - \nu$.

These rotations and boosts change the energy–momentum of the particle represented by the field. Moreover, they change the components of the basic geometric quantities of the theory, in particular the local tetrad: X, Y, e_1 and e_2. From this point of view the action of the gauge fields is equivalent to local deformations of the reference space–time. This type of analysis constitutes, in fact, a total geometrization of the gauge fields and of the interactions they represent.

The changes in the spinor are

$$\psi' - \psi = (R - 1)\psi = R'\psi,$$

and then the changes in the MEXOR, assuming that the vector Y is also acted upon by the gauge rotor R, as $Y \to Y' = RY\tilde{R}$, is

$$(1 + iZ)\psi \to (1 + iZ')\psi',$$

with $Z' = X + iY'$ and then writing $Z' = Z + Y' - Y$

$$(1 + iZ)\psi \to (1 + iZ)\psi + (1 + iZ)R'\psi + (Y - Y')\psi',$$

but in practice Y is not used and the last term drops out. This shows that in principle the gauge fields correspond to the addition of a new MEXOR resulting from the change in the spinor part. Therefore the physics of the new model is in fact a manifestation of the spinor transformation

(gauge phase factors). The interaction field strengths $F^{\mu\nu}$ change the energy momentum

$$\dot{P}^{\mu} = eF^{\mu\nu}P_{\nu},$$

and induce a precession of the spin vector and an additional rotation of the spin plane

$$\dot{S}^{\mu} = eF^{\mu\nu}S_{\nu},$$

$$\dot{e}_1^{\mu} = -\omega e_2^{\mu} - eF^{\mu\nu}e_{1\nu},$$

$$\dot{e}_2^{\mu} = \omega e_1^{\mu} + eF^{\mu\nu}e_{2\nu},$$

where ω is the spin angular velocity $\hbar/2mc$.

9. HISTORICAL NOTES ABOUT GEOMETRIC ALGEBRA AND CALCULUS

The history of vector algebra and then of vector geometry is usually known (see the book of Crowe 1992) but often misunderstood, and characterized by, numerous 'rediscovery'.

9.1 NINETEENTH CENTURY

The idea of constructing a geometric algebra was started by

9.1.1 WESSEL

In a presentation to *L'Academie Royale des Sciences et des Lettres de Danemark*, 1797 (French translation 1897)

With the main concepts:

- '..... the direction of all lines in the same plane can be expressed analytically as their lengths ....',
- and achievements:
- to define the rules for such operations;
- to demonstrate their applications in 2D;
- to define the directions of lines in 3D;
- solve plane and spherical polygons;
- formulae for spherical trigonometry (3D).

9.1.2 GRASSMANN

In his 'Ausdehnungslehre', Leipzig 1844. He introduced:

- the geometric product: $\overline{AB} \wedge \overline{AC}$ = parallelogram;
- "… as my father has conceived it in his RAUMLEHRE…";
- non-commutativity: $\overline{AB} \wedge \overline{AD} = -\overline{AD} \wedge \overline{AB}$;
- extensions, volumes and hyper-volumes: $\overline{AB} \wedge \overline{AC} \wedge \overline{AD} \wedge \ldots$;
- the geometric product in correspondence to two of previous products, one being the symmetric part, the other one the antisymmetric part: $(\overline{AB})(\overline{AC}) = \overline{AB} \bullet \overline{AC} + \overline{AB} \wedge \overline{AC}$;
- the symmetric part usually being called dot product $\overline{AB} \bullet \overline{AC} = \overline{AC} \bullet \overline{AB}$;
- bi-vectors generate rotations;
- magnitudes: $(\overline{AB})(\overline{AB}) = (\overline{AB})^2$ = (length)^2; length is a (positive) magnitude.

9.1.3 HAMILTON

In his 'Lectures on Quaternions': 'It is proper to state here that a species of *non-commutative multiplication* for inclined lines (äussere Multiplikation) occurs in the very original and remarkable work of Prof. H. Grassmann … after years had elapsed from the invention and communication of quaternions... too late to acknowledge it ... perfectly distinct and independent …' (Dublin 1853).

There is an algebra of quaternions (a subalgebra of the 3-D space geometric algebra) where:

- i, j, k, and **1** are considered together, hence the word 'quaternions';
- $i^2 = j^2 = k^2 = ijk = \mathbf{-1}$ (ring);
- $ij = -ji$; $jk = -kj$; $ki = -ik$;
- $i\mathbf{1} = \mathbf{1}i$, etc..;
- Applications to geometry and mechanics:
- where, for example, the quaternion equation $q_1 q_2 = q_3$, with q_1 being an **operator** let us say for rotation, is a member of the algebra (but he fails to discover bi-vectors or to understand Grassniann), etc..;
- Definition of the equivalent Dirac Operator.

9.1.4 CLIFFORD

One algebra with variable number *n* of generator elements:

- $e_1, e_2, \ldots, e_n$,
- $e_1^2 = e_2^2 = \ldots = e_p^2 = +1$,
- $e_{p+1}^2 = e_{p+2}^2 = \ldots = e_n^2 = -1$;
- The acceptance of the Grassmann product with clear distinction of the symmetric and antisymmetric parts;
- Joins in with *Grassmann* and *Hamilton* in one formal algebra, extending Hamilton's multi-vector analysis.

9.1.5 PEANO

In his 1888 book extends the work of Grassmann, defines the different mathematical objects formally and, by creating the concept of multi-vector derivative, enlarges the geometric algebra to geometric analysis.

9.2 TWENTIETH CENTURY

- Recognition that there are 2^n geometrical elements in metric spaces with *n* dimensions.
- Use of mixed metrics. Description of curved spaces.
- Distinction from general manifolds (differential geometry was a great, step forward).
- Use of extensive and intensive variables (in electromagnetism **D**, **H** are extensive odd 2,1 forms; *E*, *B* intensive even 1,2 forms, etc.).
- Vectors and Forms.
- Recognition of Gibbs vector analysis as a mapping hidden in the concepts of polar vector, axial vector and pseudo-scalar triple scalar product.
- Reciprocal Spaces. Phase Space.
- Geometric Operators. Point Groups.
- Generators of Continuous Groups.
- Thorough analysis of Representation.
- Spinor and twistor spaces as minimal ideals in the algebra.
- Use of concepts as Grassmann variables and q-deformed algebraic structures.

9.3 EARLY TWENTY FIRST CENTURY

We estimate that initially it will consist of

- Unification and understanding of different approaches as selective points of view of basic geometries.
- Clear distinction of the mathematization level: geometry, statistics, etc.. Use of Functions, Functionals, Operators, Super-operators (in the form proposed by P. O. Lōwdin), etc., to analyze the use and the structure of the mathematical models.

10. BASIC DEFINITIONS

As we have required some well known mathematical concepts their definitions are presented here for purposes of completeness (see, for example, Frankel 1997).

Manifolds

An *n*-dimensional (differentiable) manifold M^n (briefly, an *n* manifold) is a topological space which is locally $\mathbf{R}^n$ in the following sense. It is covered by a family of local (curvilinear) coordinate systems

$$\{U; x_U^1, ..., x_U^n\},$$

consisting of open sets, or 'patches', U and coordinates x_U in U, such that a point $p \in U \cap V$ which lies in two coordinate patches will have its two sets of coordinates related differentiably

$$x_V^i(p) = f_{VU}^i(x_U^1, ..., x_U^n), \quad i = 1, 2, ..., n.$$

(If the function f_{UV} are C^∞, that is, infinitely differentiable, or real analytic, we say that M is C^∞, or real analytic).

Definitions related to a Manifold

Let M be any set (without a topology) that has a covering by subsets $M = U \cup V \cup ...$, where each subset U is in one-to-one correspondence $\phi_U : U \to \mathbf{R}^n$ with an open subset $\phi_U(U)$ of $\mathbf{R}^n$. By requiring that each $\phi_U(U \cap V)$ is an open subset of $\mathbf{R}^n$, we require that f_{VU},

$$f_{VU} = \phi_V \circ \phi_U^{-1} : \phi_U(U \cap V) \to \mathbf{R}^n,$$

i.e.,

$$\phi_U(U \cap V) \overset{\phi_U^{-1}}{\to} M \overset{\phi_V}{\to} \mathbf{R}^n,$$

is differentiable (we assume it is known what it means that a map $\phi_V \circ \phi_U^{-1}$ from an open set of $\mathbf{R}^n$ to $\mathbf{R}^n$ is differentiable). Each pair U, ϕ_U defines

a coordinate patch on M; to each $p \in U \subset M$ we may assign the n coordinates of the point $\phi_U(p)$ in $\mathbf{R}^n$. For this reason we shall call ϕ_U a coordinate map.

Complex Manifolds

A complex manifold is a set m together with a covering $m = U \cup V \cup ...$, where each subset U is in one-to-one correspondence $\phi_U : U \to \mathbb{C}^n$ with an open subset $\phi_U(U)$ of the complex n-space $\mathbb{C}^n$. We then require that F_{VU} mapping sets in $\mathbb{C}^n$ into sets in $\mathbb{C}^n$ is complex analytic, Thus if we write f_{UV} in the form of $w^k = w^k(z^1, ..., z^n)$, where $z^k = x^k + iy^k$ and $w_k = u^k + iu^k$, then u^k and v^k satisfy the Riemann–Cauchy equations with respect to each pair (x^r, y^r). Briefly speaking, each w^k can be expressed entirely in terms of $z^1, ..., z^n$, with no complex conjugates z^r appearing. The resulting manifold is called an n-dimensional complex manifold, although its topological dimension is $2n$.

Definitions related to a Ring

A ring $(R, +, .)$ is a set of elements $R = \{a, b, c, ..., ..\}$ closed under two binary operations , namely addition (+) and multiplication (.), satisfying the following properties (the identity element for addition is denoted by 0):

(a) $(R, +)$ is an abelian group;

(b) the multiplication is associative: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$;

(c) $a \cdot (b+c) = a \cdot b + a \cdot c$ and $(b+c) \cdot a = b \cdot a + c \cdot a$.

As in the case of groups, it is conventional to write ab instead of $a \cdot b$. A ring will be denoted simply by writing the set R, with the multiplication and addition being implicit in most cases. If $R \neq \{0\}$ and multiplication on R has an identity element, i.e., there is an element $1 \in R$ with $a1 = 1a = a$ for all $a \in R$ then R is said to be a ring an with identity. In this case $1 \neq 0$ (see, see the proposition below).

Proposition.

Let R be a ring. Then for arbitrary elements $a,b,c \in R$ the following rules apply:

(1) $a0 = 0a = 0$;

(2) $(-a)b = a(-b) = -(ab)$;

(3) $ab = (-a)(-b)$;

(4) $a(b - c) = ab - ac$ and $(a - b)c = ac - bc$;

(5) If R has an identity then $(-1)a = -a$;

(6) $1 \neq 0$.

If $a \neq 0$ and $b \neq 0$ are elements of R such that $ab = 0$ then a and b are called zero divisors of the ring R. If R has an identity, an element

$a \in R$ is a unit (unit element) if a has a multiplicative inverse, that, is there is a $b \in R$ with $ab = 1 = ba$. R^* denote the set of all units of R. R^* is a group, called the group of units of R.

Field

A ring R with identity is a division ring if $R^* = R/\{0\}$, i.e., every non-zero element of R has a multiplicative inverse. A field is a commutative division ring.

References

[1] Abraham, M. and Lorentz, H.A. 1909. *Theorie der Electrizität.* Leipzig: Teubner.

[2] Ahlfors, L. and Lounesto, P. 1989. Some remarks on Clifford algebras. *Complex Variables.* **12:** 201–209.

[3] Albrecht, H., et al. 1992. (ARGUS Collaborations), DESY report 92-086.

[4] Altmann, S.L. 1986. *Rotations, Quaternions and Double Groups.* Oxford: Clarendon Press.

[5] Appelquist, T., Chodos, A. and Freund, P.G.O. 1987. *Kaluza–Klein Theories.* Menlo Park: Addison-Wesley.

[6] Artin, E. 1957. *Geometric Algebra.* New York: Interscience.

[7] Atiyah, M.F., Bott, R. and Shapiro, A. 1964. Clifford modules. *Topology.* **3** (Supp. 1): 3-38.

[8] Baobour, J.B., Pfister, H. (eds.). 1995. *Mach's Principle: From Newton's Bucket to Quantum Gravity.* Boston: Birkhäuser.

[9] Banks, T., Dothan, Y. and Horn, D. 1982. *Phys. Lett.* **B 117:** 413. Cf. e.g. H.A. Bethe and E.E. Salpeter 1957, in *Encyclopedia of Physics,* Vol. **35,** Springer, Berlin-Göttingen-Heidelberg.

[10] Barut, A.O. 1991. The covariant many-body problem in quantumelectrodynamics. *J. Math. Phys.* **32**(4): 1091-1095.

[11] Barut, A.O. 1994. *Foundations of Physics.* **24**(4): 477-485.

[12] Bateman, H. 1915. *Electrical and Optical Wave Motions.* Cambridge: Cambridge University Press.

[13] Baylis, W.E. (Editor) 1996. *Clifford (Geometric) Algebras.* Boston: Birkhäuser.

[14] Benn, I.M. and Tucker, R.W. 1987. *An Introduction to Spinors and Geometry in Applications to Physics.* Bristol: Hilger.

[15] Bete, A. 1984. *J. Math. Phys.* **25** (8): 2456.

[16] Bete, A. 1988. In Proceedings of the Conference "Microphysical reality and quantum formalism" Urbino [1988] "Fundamental theories of Physics" , Reidel's Monograph Series.

[17] Bete, A. 1989. *Reports on Math. Phys.* **28:** 133.

[18] Boudet, R. 1971. C. *R. H. S. Acad. Sci. (Paris)* Serie A **272:** 767.

[19] Boudet, R. 1974. *C. R. H. S. Acad. Sci. (Paris)* Serie A **278:** 1063.

[20] Boudet, R. 1985. *C. R. H. S. Acad. Sci. (Paris)* Serie II **300:** 157.

[21] Brackx, F. Delanghe, R. and Serras, H. (Editors) (1993). *Clifford Algebras and their Applications in Mathematical Physics.* Dordrecht: Kluwer AP.

[22] Burinskii, A. 1998. *Phys. Rev.* **57** (4): 2392-2396 and references therein.

[23] Campbell, J.E. 1926. *A course on, Differential Geometry.* Oxford: Clarendon.

[24] Campolattaro, A.A. 1980. *Int. J. Theor.* Phys. **19:** 99; *ibid* **19:** 127.

[25] Campollataro, A.A. 1990. *Int. J. Theor. Phys.* **29**(2): 141; *ibid* **29**(5): 477.

[26] Cartan, E. 1981. *The Theory of Spinors.* New York: Dover (corresponding to the french edition Herniann ed. 1937).

[27] Carter B. 1968. *Phys. Rev.* **174:** 579.

[28] Casanova, G. 1970. C. *R.* H. *S. Acad. Sci. (Paris)* Serie A **270:** 1202

[29] Casanova, G. 1976. *L'algebre vectorielle.* Paris: Presses Universitaires de France.

[30] Casanova, G. 1993. *Advances in Applied Clifford Algebras.* **3** (2): 127-132.

[31] Casanova, G. 1997. *Advances in Applied Clifford Algebras* **7**(S): 163–166.

[32] Chisholm, J.S.R. and Common, A.K. (Editors) 1986. *Clifford Algebras and their Applications in Mathematical Physics.* Dordrecht : Kluwer AP.

[33] Clifford, W.K. 1876. Preliminary sketch of biquaternions. *Proc. London Math. Soc.* **4:** 381-395.

[34] Clifford, W.K. 1876. On The Classification of Geometric Algebras, published as Paper XLIII in *Mathematical Papers.* (Tucker, R., ed) . London: Macmillan (1882).

[35] Close, F.E. 1979. *An introduction to Quark and Partons.* New York: Academic Press.

[36] Cohen, J.M. and Mustafa, E. 1986. *International Journal of Theoretical Physics.* **25:** 717-726.

[37] Crawford, J.P. 1985. On the algebra of Dirac bispinor densities: factorization and inversion theorems. *J. Math. Phys.* **26:** 1439.

[38] Crowe, F. 1992. *A History* of *Vector Algebra.* New York: Dover.

[39] Daviau, C. 1989. *Ann. Fondation Louis de Broglie.* **14:** 373.

[40] Daviau, C. 1993. *Ann. Fondation Louis de Broglie.* **23:** 1431–1443.

[41] Daviau, C. and Lochak, G. 1991. *Ann. Fondation Louis de Broglie.* **16:** 43.

[42] Davison, A., and Owe, D. 1985. *Phys. Lett. B* **155:** 247.

[43] de Broglie, L. 1943. *Théorie Générale des Particules á Spin.* Paris: Gauthier–Villars; 1968. Ondes Electromagnétiques et Photons. Paris: Gauthier–Villars.

[44] Dirac, P.A.M. 1928. *Proc. Roy. Soc.* **A117:** 610; **A118:** 35.

[45] Duck, I. and Sudarshan, E.C.G. 1997. *Pauli and the Spin-Statistics Theorem.* London: World Scientific.

[46] Duff, M.J. 1996. *Int. J. Mod. Phys.* **A11:** 5623.

[47] Eddington, A.S. 1928. A symmetrical treatment of the wave equation. *Proceedings* of *the Royal Society.* Series A. **121:** 523–542.

[48] Einstein, A. 1905. Zur electrodynamik bewegter Körper. *Annalen der Physik*, Vierte folge, **17:** 891–921.

[49] Einstein, A. 1915a. Zur allgemeine Relativitätstheorie. *Sitz. Preuss. Akad. Wiss.:* 778–786 and 799–801.

[50] Einstein, A. 1915b. Die Feldgleichungen der Gravitation. *Sitz. Preuss. Akad. Wiss.:* 844–847.

[51] Einstein, A. 1915c. Erklärung der Perihelbewegung des Merkur aus der allgemeinen Relativitätstheorie. *Sitz. Preuss. Akad. Wiss. :* 831–839.

[52] Einstein, A. 1916. Die Grundlage der allgemeinen Relativitätstheorie. *Annalen der Physik*. Vierte folge, 17: 769**–**822.

[53] Einstein, A., Lorentz, H.A., Minkowski, H. and Weyl, H. 1952. *Principle of Relativity, collected Papers.* New York: Dover.

[54] Fermi, E. 1927. Applications of statistical gas methods to electronic system. *Atti. Accad. Naz. Lincei* (ser. 6). **6:** 602–606.

[55] Fermi, E. 1928. Statistical deduction of atomic properties. *Atti. Accad. Naz. Lincei* (ser. 6). **7:** 342-346.

[56] Fermi, E. 1928. Statistical methods of investigating electrons in atoms. *Z. Phys.* **48:** 73-79.

[57] Field, R.D. 1979. *Quantum Flavordynamics, Quantum chromodynamics and Unified Theories.* Nato Advanced Study Series B, Physics. **54** New York: Plenum Press.

[58] Flores, J.A. and Keller, J. 1992. Phys. Rev. A **45** (9): 6259–6262.

[59] Flügge, S. 1947. *Rechenmethoden der Quantentheorie.* (Springer, Berlin) exercises 197–202.

[60] Fock, V. 1929. Geometrisierung der Diracschen Theorie del Electrons. *Zeitschrift für Physik.* **55:** 261.

[61] Fock, V. 1929. Sur les étquation de Dirac dans la Théorie de Relativité Générale. *C. R. Acad. Sciences (Paris).* **189:** 25.

[62] Fock, V., and Iwanenko, D. 1929. Über eine Mögliche Geometrische Deutung der Relativistischen Quantentheorie. *Zeitschrift für physik.* **54:** 798.

[63] Fock, V., and Iwanenko, D. 1929. Géométrie Quantique Linéaire et, Deplacement Paralléle. *C. R. Acad. Sciences* (Paris). **188:** 1470.

[64] Fokker, A.D. 1929. *Z.Phys.* **58:** 386; 1929. *Physica* **9:** 33.

[65] Frankel, T. 1997. *The Geometry of Physics: An Introduction.* Cambridge: Cambridge U.P.

[66] Frappati, L., Sciarrino, A. and Sorba, P. 2000. *Dictionary on Lie Algebras and Superalgebras.* London: Academic Press.

[67] Fritzch, H., and Minkowski, P. 1974. *Ann. Phys. New York* **93:** 193.

[68] Gareev, F.A., Keller, J. and Yamaleev, R.M. 1998. On quantization of the decay products momenta in the elementary particle resonances. *Advances in Applied Clifford Algebras.* **8(2):** 255–270.

[69] Georgi, H., and Glashow, S.L. 1974. *Phys Rev. Lett.* **32:** 438.

[70] Gilbert, J.E. and Murray, M.A.M. 1991. *Clifford Algebras and Dirac Operators in Harmonic Analysis.* Cambridge: Cambridge University Press.

[71] Gornitz, T. 1999. *Quanten sind anders.* Berlin: Spectrum Acad. Verl.

[72] Grassmann, H. 1844. *Die Wissenschaft der extension Grösse oder die Ausdehnunglehre, eine neue mathematischen Disciplin.* Leipzig.

[73] Grassmann, H. 2000. *Extension theory (*1866*).* Translated by Kannenberg, L. History of Mathematics **19** The American Mathematical Society.

[74] Habetha, K., Dietrich, V. and Jank, G. (Editors) 1998. *Clifford Algebras and their Applications in Mathematical Physics.* Dordrecht: Kluwer Academic Publishers.

[75] Halsen, F., and Martin, A.D. 1984. *Quarks and Leptons.* New York: Wiley.

[76] Hamilton, W.R. 1844. On quaternions or on a new system of imaginaries in algebra. *Phil. Mag.* 25: 489–495.

[77] Hamilton, W.R. 1853. *Lectures on Quaternions.* Dublin: Hodges & Smith.

[78] Hecht, L. 1996. The significance of the 1845 Gauss-Weber correspondance, *21st Century Science & Technology.* **9**(3): 22–34.

[79] Hestenes, D. 1966. *Spacetime Algebra.* New York: Gordon and Breach.

[80] Hestenes, D. 1975. *J. Math. Phys.* **16:** 556 and references therein.

[81] Hestenes, D. and Sobczyk, G. 1984. *Clifford Algebra to Geometric Calculus.* Dordrecht: Reidel.

[82] Hestenes, D. 1990. The Zitterbewebung interpretation of Quantum Mechanics. *Found. Phys.* **20:** 1213–1232.

[83] Hohenberg, P., and Kohn, W. 1964. *Phys. Rev. B* **136:** 864–867.

[84] Isham, C. J. 1995. *Lectures on Quantum Theory. Mathematical and Structural Foundations.* London: Imperial College Press.

[85] Juvet, G. 1930. *Commentarii Mathematici Helvetici,* **2:** 225.

[86] Juvet, G. 1932. *Bulletin de la Societé Neuchateloise des Sciences Naturelles,* **57:** 127.

[87] Kahler, E. 1962. *Rendiconti di Matematica.* **21:** 425; cf. also Ivanenko, D. and Landau, L. 1928. *Z. Phys.* **48:** 341 published soon after the discovery of the Dirac equation.

[88] Kaluza, T. 1921. *Sitz. Preuss. Akad. Wiss.* **33:** 966–972.

[89] Keller, J. 1981. Masa y Carga como un problema de eigenvalores de variables estructurales. *Rev. Soc. Quim. Mex.* **25:** 28–31.

[90] Keller, J. 1982. Spacetime symmetries corresponding to elementary particle symmetries. In *Mathematics of the Physical Spacetime.* México: UNAM, pp. 117–32.

[91] Keller, J. 1982. Wave equation of symmetry constrained Dirac particles. *Int. J. of Theor. Phys.* **21**(10/11): 829–36.

[92] Keller, J. 1983. The geometry of spacetime and the Dirac equation. In *Proceedings* of *the NATO Institute Relativistic Effect in Atoms, Molecules and Solids.* (Malli, G., ed.). Canada: Plenum.

[93] Keller, J. 1984. Spacetime dual geometry theory of elementary particles and their interaction fields. *Int. J. of Theoretical Physics.* **23**(9): 817–837.

[94] Keller, J. and Megy, F. 1984. Geometría y la teoría del espin en mecánica cuántica no relativista. *Contactos.* **1**(1): 51–54.

[95] Keller, J. 1986. A system of vectors and spinors in complex spacetime and their application to elementary particle physics. In *Clifford Algebras and Their Applications in Mathematical Physics.* (Chisholm, J.S.R. and Common, A.K., eds.) Dordrecht: Reidel.

[96] Keller, J. 1986. A generalization of the Dirac equation admitting isospin and color symmetries. *Int. J.* of *Theoretical Physics.* **25** (8) : 779–806.

[97] Keller, J. 1986. *Int. J. Quantum Chem. Symp.* **20:** 767.

[98] Keller, J. and Ludeña, E. 1987. *Int. J. Quantum Chem. Symp.* **21:** 171.

[99] Keller, J. 1989. ?que es un electrón? *Acta Academica Potosina.* **XI**(2): 115–131.

[100] Keller, J. 1989. Clifford algebras and the electron. In *Clifford Algebras and Their Applications in Mathematical Physics.* (Helmstetter, J., Micali, A. and Boudet, R., eds.). Dordrecht: Kluwer Academic Pub., pp. 437–445.

[101] Keller, J. and Rodríguez-Romo, S. 1990. A multi-vectorial Dirac equation. *J. Math. Phys.* **31**(10): 2501–2510.

[102] Keller, J., Keller, A. and Flores, J.A. 1990. *Acta Chimica Theoretica Latina* **XVIII** (4): 175–186.

[103] Keller, J. and Rodriguez-Romo, S. 1991. Multivectorial generalization of the Cartan map. *J. Math. Phys.* **32**(6): 1591–1598; *Int. J. of Theoretical Physics.* **25**(8): 779–806.

[104] Keller, J. and Rodríguez-Romo, S. 1991. Multivectorial representation of lie groups. *Int. J. of Theoretical Physics.* **30**(2): 185–196.

[105] Keller, J. 1991. Spinors and multivectors as a unified tool for spacetime geometry and for elementary particle physics. *Int. Journal of Theoretical Physics.* **30**(2): 137–184.

[106] Keller, J. 1991. Dirac equations with electroweak and color symmetry. In *Differential Geometric methods in Theoretical Physics.* (Catto, S. and Rocha, A., eds.). **1** Singapore: World Scientific, pp. 355–361.

[107] Keller, J. 1991. Spinors as a basis of a geometric superalgebra. *Advances in Applied Clifford Algebras.* **1**(1): 31–50.

[108] Keller, J. and Viniegra, F. 1992. The multivector structure of the matter and interaction field theories. In *Clifford Algebras and Their Applications in Mathematical Physics.* (Helmsteller, J., Micali, A. and Boudet, R., eds.). Dordrecht: Kluwer Academic Publishers, pp. 437–445.

[109] Keller, J. and Rodriguez, A. 1992. Geometric superalgebra and the Dirac equation. *J. Math. Phy.* **33**(1): 161–170.

[110] Keller, J. 1992. The geometric content of the electron theory. Fock and Ivanenko 1929. *Advances in Applied Clifford Algebras.* **2**(2): 195–196.

[111] Keller, J. 1993. Tautology of quantum mechanics and spacetime. In *Vistas in Astronomy. 37 of Proceedings of the Symposium on Quantum Physics and the Universe, Tokyo, Japan, August 19-22, 1992.* pp. 283–286.

[112] Keller, J. 1993. Unified mathematical approach to spinors and multivectors, geometric superalgebra. *Int. J. of Modern Physics A.* **3A:** 511–514.

[113] Keller, J. 1993. Dual space and Clifford algebras. *Advances in Applied Clifford Algebras* **3**(1): 1–6.

[114] Keller, J. 1993. Dirac form of Maxwell equations; $\mathbb{Z}_n$-graded algebras. In *Spinors, Twistors, Clifford Algebras and Quantum Deformations.* (Oziewicz Z., Jancewicz B. and Borowiec A., eds.) Dordrecht: Kluwer Academic Publishers, pp. 189–196.

[115] Keller, J. 1993. The geometric content of the electron theory (part I). *Advances in Applied Clifford Algebras.* **3**(2): 147–200.

[116] Keller, J. 1994. Clifford algebra and the construction of a theory of elementary particle fields. *Advances in Applied Clifford Algebras (Proc. Suppl.).* **4**(S1): 379–393.

[117] Keller, J. 1994. Factorization of the Laplacian and families of elementary particles. In *Symmetry Methods in Physics.* (Sissakian, A.N., Pogosyan, G.S.and Vinitsky, S.I., eds.). **I.** Dubna: JIRN, pp. 236–241.

[118] Keller, J. 1995. Twistors as geometric objects in spacetime. In *Clifford Algebras and Spinor Structures.* (Lounesto, P. and Ablamowicz, R., eds.) Dordrecht: Kluwer Academic Publishers, pp. 133–135.

[119] Keller, J., Rodrigues, A. and Yamaleev, R. 1996. *Advances in Applied Clifford Algebras.* **6**(2): 275.

[120] Keller, J. 1997. On the electron theory "The Theory of the Electron". *Advances in Applied Clifford Algebras.* **7**(S): 3–26.

[121] Keller, J. 1997. Spinors, Twistors, Screws, Mexors and the massive spinning electron "The Theory of the Electron" (Keller, J. and Oziewicz, Z., eds.). *Advances in Applied Clifford Algebras.* **7**(S): 439–455.

[122] Keller, J. and Yamaleev, R.M. 1997. Equations of motion for massive particles over twistor fields. *Advances in Applied Clifford Algebras.* **7:** 141.

[123] Keller, J. and Ozciewicz, Z. (Editors) 1997. *The Theory of the Electron.* México: UNAM- Cuautitlán. *Advances in Applied Clifford Algebras.* **7**(S).

[124] Keller, J. 1998. Twistors and Clifford algebras. In *Clifford Algebras and their Applications in Mathematical Physics.* (Habetha, K., Dietrich, V. and Jank, G., eds.). Dordrecht: Kluwer Academic Publishers, pp. 161–173.

[125] Keller, J. 1999. The Geometric Content of the Electron Theory II. *Advances in Applied Clifford Algebras.* **9** (2): 309–395.

[126] Keller, J. 2000. The Foundations of Density Functional Theory and Wave Quantum Mechanics. *Rev. Soc. Quim. Mex.* **44**(1): 22–28.

[127] Keller, J. 1999. presented in the 1997 *Conference Symmetry in Physics*, Dubna, to be published. Also in *Proc. Int. Conf. "Geometrization of Physics IV"*, Kazan State University, Kazan, October 4–8, 1999; also in Proceedings of the Wigner Conference, Istambul 1999, to be published in the *Turkish Journal of Physics;* and in the Proceedings of the Clifford Analysis Conference, Beijing 2000, to be published in *Advances in Applied Clifford Algebras.* **10**(S).

[128] Keller, J. and Weinberger, P. 2000. to be published.

[129] Klein. O. 1926, Z. Phys. **37:** 895.

[130] Kohn, W. and Sham, L.J. 1965. *Phys. Rev. A.* **140:** 1133–1138.

[131] Królikowski, W. 1990. *Acta Physica Polonica.* **B21:** 871–879; 1992. *Phys. Rev.* D **45:** 3222.

[132] Królikowski, W. 1992. *Phys. Rev.* **D45:** 3222–3227.

[133] Królikowski, W. 1993. Clifford Algebras and Algebraic Structure of Fundamental Fermions. In *Spinors, Twistors, Clifford Algebras and Quantum Deformations.* (Oziewicz Z., Jancewicz B. and Borowiec A., eds.) Dordrecht: Kluwer Academic Publishers, pp. 171–181; Królikowski, W. and Rzewuski J. 1956. *Nuovo Cim.* **2:** 203; **4:** 974; Królikowski, W. 1986. *Acta Phys. Pol.* **B 17:** 813; Królikowski, W. 1988. *Acta Phys. Pol.* **B 18:** 111; *Nuovo Cim.* 100 **A:** 701.

[134] Królikowski, W. 1991. *Acta Phys. Pol.* **B 22:** 613; 1992. *Acta Phys. Pol.* **B 23:** 83.

[135] Lense, Von J. and Thirring, H. 1918. "Über den Einfluss der Eigenrotation Zentralkörper auf die Bewegung der Planeten und Monde nach der Einsteinsch Gravitationstheorie". *Physikalische Zeitschrift* (Leipzig). **19:** 156–163.

[136] Liu, Y.F. and Keller, J. 1996. A symmetry of massless fields. *J. Math. Phys.* **37**(9): 4320–4332.

[137] Lozada, A. 1989. *Journal of Mathematical Physics.* **30**(8): 1713–1720.

[138] Mackinnon, L. 1981. *Lett. Nuovo Cimento.* **32:** 311.

[139] Mercier, A. 1934. *Actes de la Societé Helvetique des Sciences Naturelles Zürich.* **1934:** 278.

[140] Mercier, A. 1935. These, Université de Génève. *Archives des Sciences Physiques et Naturelles (Suisse).* **17:** 278.

[141] Micali, A., Boudet, R. and Helmstetter, J. (Editors) 1991. *Clifford Algebras and their Applications in Mathematical Physics.* Dordrecht: Kluwer AP.

[142] Nakahara, M. 1990. *Geometry, Topology and Physics.* Bristol: IOP.

[143] Ohanian, H.C. 1986. *American journal of Physics* **54:** 504.

[144] Particle Data Group. Review of Particle Properties. 1992. *Phys. Rev.* **D 45,** Part 2.

[145] Pauli, W. 1926. *Z. Phys.* **36:** 336–363.

[146] Peano, G. 2000. Geometric Calculus (1888). Translated by Kannenberg, L. Boston: Birkhäuser.

[147] Penrose, R. 1967. *J. of Math. Phys.* **8:** 345.

[148] Poincaré, H. 1905. *Compt. Rend. Acad. Sci. Paris.* **140:** 1504.

[149] Poincaré, H. 1906. *Rend. Circ. Mat. Palermo.* **21:** 129–176.

[150] Porteous, I.R. 1981. *Topological Geometry.* Cambridge: Cambridge U.P.

[151] Porteous, I.R. 1994. *Geometric Differentiation.* Cambridge: Cambridge U.P.

[152] Porteous, I.R. 1995. *Clifford Algebras and the Classical Groups.* Cambridge:-Cambridge U.P.

[153] Proca, A. 1930a. C. *R. H. S.Acad. Sci. (Paris).* **190:** 1377.

[154] Proca, A. 1930b. *C. R. H. S.Acad. Sci. (Paris).* **191:** 26.

[155] Proca, A. 1930c. *Journal of Physics* **VII**(1): 236.

[156] Qi, N. 1992. *Bull. Am. Phys. Soc.* **37,** 957, caption only; information in: CERN *Courier* **32,** 13.

[157] Quilichini, P. 1971. *C. R. H. S.Acad. Sci. (Paris).* Serie B **273:** 829.

[158] Raghavacharyulu, I.V.V. and Menon, N.B. 1970. *Journal of Math. Physics.* **11:** 3055.

[159] Ravsevskii, P.K. 1957. *Transactions of the American Mathematical Society.* **6:** 1.

[160] Riesz, M. (1946). *Comptes Rendus du Dixieme Congres des Mathematiques des Pays Scandinaues.* **123,** Copenhagen.

[161] Riesz, M. 1953. *Comptes Rendus du Douzieme Congres des Mathematiques des Pays Scandinaves.* **241,** Copenhagen.

[162] Riesz, M. 1958. *Clifford Numbers and Spinors. Lecture series 38* (University of Maryland, Reprinted as facsimile, Bolinder, E.F. and Lounesto, P. eds. Kluwer 1993).

[163] Rodríguez-Romo, S., Viniegra, F. and Keller, J. 1992. Geometrical content of the Fierz identities. In *Clifford Algebras and Their Applications in Mathematical Physics.* Vol. **47** of *Fundamental Theories of Physics.* (Helmsteller, J., Micali, A. and Boudet, R., eds.). Dordrecht: Kluwer Academic Publishers, pp. 479–497.

[164] Salpeter, E.E. 1952 *Phys. Rev.* **87,** 328 .

[165] Sauter, F. 1930. *Z. Phys.* **63:** 803; **64:** 295.

[166] Schönberg, M. 1956. On the Grassman and Clifford Algebras I. *Anais da Academia Brasileira de Ciencias.* **28:** 11–19.

[167] Schwarzschild, K. 1903. *Nachr.* Ges. *Wiss Gottingen Math. Phys. K1. Fachgruppe.* **2:** 128–132.

[168] Schwarzschild, K. 1916. "Uber das Gravitationsfeld eines Massenpunktes Nach der Einsteinschen Theorie". *Sitzber. Deut. Akad. Wiss.* Berlin, Kl. Math.-Phys. Tech.: 189–196.

[169] Snygg, J. 1997. *Clifford Algebra - A Computational Tool for Physicists,* New York: Oxford University Press.

[170] Sutherland, R.I. 1989. *J. Math, Phys.* **30**(8): 1721–1726.

[171] Teitler, S. 1965a. *Nuovo Cimento Supplemento.* **3:** 1.

[172] Teitler, S. 1965b. *Nuovo Cimento Supplemento.* **3:** 15.

[173] Teitler, S. 1965c. *J. Math. Phys.* **6:** 1976.

[174] Teitler, S. 1966a. *J. Mnth. Phys.* **7:** 1730.

[175] Teitler, S. 1966b. *J. Math. Phys.* **7:** 1739.

[176] Tetrode, R.W. 1922. *Z.* Phys. **10:** 317.

[177] Thomas, L.H. 1927. Calculation of atomic fields. *Proc. Cambridge Philos. Soc.* **33:** 542–548.

[178] Tod, K.P. 1975 "Massive spinning particles and twistor theory", Doctoral Dissertation, Mathematical Institute, University of Oxford.

[179] Tod, K.P. 1977 *Rep. Math. Phys.* **11** (3): 339.

[180] Tod, K.P. and Perjes, Z. 1976. *GeneraI Relativity and Gravitation.* **7** (11): 903–913.

[181] Vaz, J.Jr. and Rodrigues W.A.Jr. 1993. Equivalence of Dirac and Maxwell Equations and Quantum Mechanics. *Int. J. of Theor. Phys.* **32**(6): 55.

[182] Weber, W.E. 1871. Electrodynamische Massbestimmungen, insbesoridere über das Princip der Erhaltung der Energie. *Abhandlungen der Mathematische Physische Klasse der Koenigles Sachsischen Gesellschaft der Wissenschaften* **X:** January; 1872. *Phil. Mag.* **43**(283): 1-20, 119–149.

[183] Weinberger, P. 1989. *Lectures Notes in Quantum Mechanics* Vienna: TU-Wien.

[184] Weintraub, S.H. 1997. *Differential Forms: A Complement to Vector Calculus.* San Diego: Academic Press.

[185] Wessel; C. 1797. *L'Academie Royale* des *Sciences et des* Lettres *de Danmark.* (French translation 1897).

[186] Wesson, P.S. 1999, *Space–Time–Matter: Modern, Kaluza–Klein Theory.* London: World Scientific Publishing Co.

[187] Weyl, H. 1952. *Space–Time–Matter (*translated from the 4th german edition by Brose, H.L). New York: Dover.

[188] Wheeler, J.A. and Feymann, R.P. 1949. *Rev. Mod. Phys.* **21:** 425.

[189] Willmore, T.J. 1998. *Riemannian Geometry.* New York: Oxford University Press.

This page intentionally left blank.

Index

This page intentionally left blank.

Fundamental Theories of Physics

Series Editor: Alwyn van der Merwe, University of Denver, USA

1. M. Sachs: *General Relativity and Matter.* A Spinor Field Theory from Fermis to Light-years. With a Foreword by C. Kilmister. 1982 ISBN 90-277-1381-2
2. G.H. Duffey: *A Development of Quantum Mechanics.* Based on Symmetry Considerations. 1985 ISBN 90-277-1587-4
3. S. Diner, D. Fargue, G. Lochak and F. Selleri (eds.): *The Wave-Particle Dualism.* A Tribute to Louis de Broglie on his 90th Birthday. 1984 ISBN 90-277-1664-1
4. E. Prugovecki: *Stochastic Quantum Mechanics and Quantum Spacetime.* A Consistent Unification of Relativity and Quantum Theory based on Stochastic Spaces. 1984; 2nd printing 1986 ISBN 90-277-1617-X
5. D. Hestenes and G. Sobczyk *Clifford Algebra to Geometric Calculus.* A Unified Language for Mathematics and Physics. 1984 ISBN 90-277-1673-0; Pb (1987) 90-277-2561-6
6. P. Exner: *Open Quantum Systems and Feynman Integrals.* 1985 ISBN 90-277-1678-1
7. L. Mayants: *The Enigma of Probability and Physics.* 1984 ISBN 90-277-1674-9
8. E. Tocaci: *Relativistic Mechanics, Time and Inertia.* Translated from Romanian. Edited and with a Foreword by C.W. Kilmister. 1985 ISBN 90-277-1769-9
9. B. Bertotti, F. de Felice and A. Pascolini (eds.): *General Relativity and Gravitation.* Proceedings of the 10th International Conference (Padova, Italy, 1983). 1984 ISBN 90-277-1819-9
10. G. Tarozzi and A. van der Merwe (eds.): *Open Questions in Quantum Physics.* 1985 ISBN 90-277-1853-9
11. J.V. Narlikar and T. Padmanabhan: *Gravity, Gauge Theories and Quantum Cosmology.* 1986 ISBN 90-277-1948-9
12. G.S. Asanov: *Finsler Geometry, Relativity and Gauge Theories.* 1985 ISBN 90-277-1960-8
13. K. Namsrai: *Nonlocal Quantum Field Theory and Stochastic Quantum Mechanics.* 1986 ISBN 90-277-2001-0
14. C. Ray Smith and W.T. Grandy, Jr. (eds.): *Maximum-Entropy and Bayesian Methods in Inverse Problems.* Proceedings of the 1st and 2nd International Workshop (Laramie, Wyoming, USA). 1985 ISBN 90-277-2074-6
15. D. Hestenes: *New Foundations for Classical Mechanics.* 1986 ISBN 90-277-2090-8; Pb (1987) 90-277-2526-8
16. S.J. Prokhovnik: *Light in Einstein's Universe.* The Role of Energy in Cosmology and Relativity. 1985 ISBN 90-277-2093-2
17. Y.S. Kim and M.E. Noz: *Theory and Applications of the Poincaré Group.* 1986 ISBN 90-277-2141-6
18. M. Sachs: *Quantum Mechanics from General Relativity.* An Approximation for a Theory of Inertia. 1986 ISBN 90-277-2247-1
19. W.T. Grandy, Jr.: *Foundations of Statistical Mechanics.* Vol. I: *Equilibrium Theory.* 1987 ISBN 90-277-2489-X
20. H.-H von Borzeszkowski and H.-J. Treder: *The Meaning of Quantum Gravity.* 1988 ISBN 90-277-2518-7
21. C. Ray Smith and G.J. Erickson (eds.): *Maximum-Entropy and Bayesian Spectral Analysis and Estimation Problems.* Proceedings of the 3rd International Workshop (Laramie, Wyoming, USA, 1983). 1987 ISBN 90-277-2579-9
22. A.O. Barut and A. van der Merwe (eds.): *Selected Scientific Papers of Alfred Landé.* [*1888-1975*]. 1988 ISBN 90-277-2594-2

Fundamental Theories of Physics

23. W.T. Grandy, Jr.: *Foundations of Statistical Mechanics.* Vol. II: *Nonequilibrium Phenomena.* 1988 ISBN 90-277-2649-3
24. E.I. Bitsakis and C.A. Nicolaides (eds.): *The Concept of Probability.* Proceedings of the Delphi Conference (Delphi, Greece, 1987). 1989 ISBN 90-277-2679-5
25. A. van der Merwe, F. Selleri and G. Tarozzi (eds.): *Microphysical Reality and Quantum Formalism, Vol. I.* Proceedings of the International Conference (Urbino, Italy, 1985). 1988 ISBN 90-277-2683-3
26. A. van der Merwe, F. Selleri and G. Tarozzi (eds.): *Microphysical Reality and Quantum Formalism, Vol. 2.* Proceedings of the International Conference (Urbino, Italy, 1985). 1988 ISBN 90-277-2684-1
27. I.D. Novikov and V.P. Frolov: *Physics of Black Holes.* 1989 ISBN 90-277-2685-X
28. G. Tarozzi and A. van der Merwe (eds.): *The Nature of Quantum Paradoxes.* Italian Studies in the Foundations and Philosophy of Modem Physics. 1988 ISBN 90-277-2703-1
29. B.R. Iyer, N. Mukunda and C.V. Vishveshwara (eds.): *Gravitation, Gauge Theories and the Early Universe.* 1989 ISBN 90-277-2710-4
30. H. Mark and L. Wood (eds.): *Energy in Physics, War and Peace.* A Festschrift celebrating Edward Teller's 80th Birthday. 1988 ISBN 90-277-2775-9
31. G.J. Erickson and C.R. Smith (eds.): *Maximum-Entropy and Bayesian Methods in Science and Engineering.* Vol. I: *Foundations.* 1988 ISBN 90-277-2793-7
32. G.J. Erickson and C.R. Smith (eds.): *Maximum-Entropy and Bayesian Methods in Science and Engineering.* Vol. II: *Applications.* 1988 ISBN 90-277-2794-5
33. M.E. Noz and Y.S. Kim (eds.): *Special Relativity and Quantum Theory.* A Collection of Papers on the Poincaré Group. 1988 ISBN 90-277-2799-6
34. I.Yu. Kobzarev and Yu.I. Manin: *Elementary Particles. Mathematics, Physics and Philosophy.* 1989 ISBN 0-7923-0098-X
35. F. Selleri: *Quantum Paradoxes and Physical Reality.* 1990 ISBN 0-7923-0253-2
36. J. Skilling (ed.): *Maximum-Entropy and Bayesian Methods.* Proceedings of the 8th International Workshop (Cambridge, UK, 1988). 1989 ISBN 0-7923-0224-9
37. M. Kafatos (ed.): *Bell's Theorem, Quantum Theory and Conceptions of the Universe.* 1989 ISBN 0-7923-0496-9
38. Yu.A. Izyumov and V.N. Syromyatnikov: *Phase Transitions and Crystal Symmetry.* 1990 ISBN 0-7923-0542-6
39. P.F. Fougere (ed.): *Maximum-Entropy and Bayesian Methods.* Proceedings of the 9th International Workshop (Dartmouth, Massachusetts, USA, 1989). 1990 ISBN 0-7923-0928-6
40. L. de Broglie: *Heisenberg's Uncertainties and the Probabilistic Interpretation of Wave Mechanics.* With Critical Notes of the Author. 1990 ISBN 0-7923-0929-4
41. W.T. Grandy, Jr.: *Relativistic Quantum Mechanics of Leptons and Fields.* 1991 ISBN 0-7923-1049-7
42. Yu.L. Klimontovich: *Turbulent Motion and the Structure of Chaos.* A New Approach to the Statistical Theory of Open Systems. 1991 ISBN 0-7923-1114-0
43. W.T. Grandy, Jr. and L.H. Schick (eds.): *Maximum-Entropy and Bayesian Methods.* Proceedings of the 10th International Workshop (Laramie, Wyoming, USA, 1990). 1991 ISBN 0-7923-1140-X
44. P. Pták and S. Pulmannova: *Orthomodular Structures as Quantum Logics.* Intrinsic Properties, State Space and Probabilistic Topics. 1991 ISBN 0-7923-1207-4
45. D. Hestenes and A. Weingartshofer (eds.): *The Electron.* New Theory and Experiment. 1991 ISBN 0-7923-1356-9

Fundamental Theories of Physics

46. P.P.J.M. Schram: *Kinetic Theory of Gases and Plasmas.* 1991 ISBN 0-7923-1392-5
47. A. Micali, R. Boudet and J. Helmstetter (eds.): *Clifford Algebras and their Applications in Mathematical Physics.* 1992 ISBN 0-7923-1623-1
48. E. Prugovecki: *Quantum Geometry.* A Framework for Quantum General Relativity. 1992 ISBN 0-7923-1640-1
49. M.H. Mac Gregor: *The Enigmatic Electron.* 1992 ISBN 0-7923-1982-6
50. C.R. Smith, G.J. Erickson and P.O. Neudorfer (eds.): *Maximum Entropy and Bayesian Methods.* Proceedings of the 11th International Workshop (Seattle, 1991). 1993 ISBN 0-7923-2031-X
51. D.J. Hoekzema: *The Quantum Labyrinth.* 1993 ISBN 0-7923-2066-2
52. Z. Oziewicz, B. Jancewicz and A. Borowiec (eds.): *Spinors, Twistors, Clifford Algebras and Quantum Deformations.* Proceedings of the Second Max Born Symposium (Wroclaw, Poland, 1992). 1993 ISBN 0-7923-2251-7
53. A. Mohammad-Djafari and G. Demoment (eds.): *Maximum Entropy and Bayesian Methods.* Proceedings of the 12th International Workshop (Paris, France, 1992). 1993 ISBN 0-7923-2280-0
54. M. Riesz: *CliffordNumbers and Spinors* with Riesz' Private Lectures to E. Folke Bolinder and a Historical Review by Pertti Lounesto. E.F. Bolinder and P. Lounesto (eds.). 1993 ISBN 0-7923-2299-1
55. F. Brackx, R. Delanghe and H. Serras (eds.): *Clifford Algebras and their Applications in Mathematical Physics.* Proceedings of the Third Conference (Deinze, 1993) 1993 ISBN 0-7923-2347-5
56. J.R. Fanchi: *Parametrized Relativistic Quantum Theory.* 1993 ISBN 0-7923-2376-9
57. A. Peres: *Quantum Theory: Concepts and Methods.* 1993 ISBN 0-7923-2549-4
58. P.L. Antonelli, R.S. Ingarden and M. Matsumoto: *The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology.* 1993 ISBN 0-7923-2577-X
59. R. Miron and M. Anastasiei: *The Geometry of Lagrange Spaces: Theory and Applications.* 1994 ISBN 0-7923-2591-5
60. G. Adomian: *Solving Frontier Problems of Physics: The Decomposition Method.* 1994 ISBN 0-7923-2644–X
61. B.S. Kerner and V.V. Osipov: *Autosolitons.* A New Approach to Problems of Self-Organization and Turbulence. 1994 ISBN 0-7923-2816-7
62. G.R. Heidbreder (ed.): *Maximum Entropy and Bayesian Methods.* Proceedings of the 13th International Workshop (Santa Barbara, USA, 1993) 1996 ISBN 0-7923-2851-5
63. J. Pe?ina, Z. Hradil and B. Jur?o: *Quantum Optics and Fundamentals of Physics.* 1994 ISBN 0-7923-3000-5
64. M. Evans and J.-P. Vigier: *The Enigmatic Photon.* Volume 1: The Field $\boldsymbol{B}^{(3)}$. 1994 ISBN 0-7923-3049-8
65. C.K. Raju: *Time: Towards a Constistent Theory.* 1994 ISBN 0-7923-3103-6
66. A.K.T. Assis: *Weber's Electrodynamics.* 1994 ISBN 0-7923-3137-0
67. Yu. L. Klimontovich: *Statistical Theory of Open Systems.* Volume 1: A Unified Approach to Kinetic Description of Processes in Active Systems. 1995 ISBN 0-7923-3199-0; Pb: ISBN 0-7923-3242-3
68. M. Evans and J.-P. Vigier: *The Enigmatic Photon.* Volume 2: Non-Abelian Electrodynamics. 1995 ISBN 0-7923-3288-1
69. G. Esposito: *Complex General Relativity.* 1995 ISBN 0-7923-3340-3

Fundamental Theories of Physics

70. J. Skilling and S. Sibisi (eds.): *Maximum Entropy and Bayesian Methods.* Proceedings of the Fourteenth International Workshop on Maximum Entropy and Bayesian Methods. 1996 ISBN 0-7923-3452-3
71. C. Garola and A. Rossi (eds.): *The Foundations of Quantum Mechanics Historical Analysis and Open Questions.* 1995 ISBN 0-7923-3480-9
72. A. Peres: *Quantum Theory: Concepts and Methods.* 1995 (see for hardback edition, Vol. 57) ISBN Pb 0-7923-3632-1
73. M. Ferrero and A. van der Merwe (eds.): *Fundamental Problems in Quantum Physics.* 1995 ISBN 0-7923-3670-4
74. F.E. Schroeck, Jr.: *Quantum Mechanics on Phase Space.* 1996 ISBN 0-7923-3794-8
75. L. de la Peña and A.M. Cetto: *The Quantum Dice.* An Introduction to Stochastic Electrodynamics. 1996 ISBN 0-7923-3818-9
76. P.L. Antonelli and R. Miron (eds.): *Lagrange and Finsler Geometry.* Applications to Physics and Biology. 1996 ISBN 0-7923-3873-1
77. M.W. Evans, J.-P. Vigier, S. Roy and S. Jeffers: *The Enigmatic Photon.* Volume 3: Theory and Practice of the $\boldsymbol{B}^{(3)}$ Field. 1996 ISBN 0-7923-4044–2
78. W.G.V. Rosser: *Interpretation of Classical Electromagnetism.* 1996 ISBN 0-7923-4187-2
79. K.M. Hanson and R.N. Silver (eds.): *Maximum Entropy and Bayesian Methods.* 1996 ISBN 0-7923-4311-5
80. S. Jeffers, S. Roy, J.-P. Vigier and G. Hunter (eds.): *The Present Status of the Quantum Theory of Light.* Proceedings of a Symposium in Honour of Jean-Pierre Vigier. 1997 ISBN 0-7923-4337-9
81. M. Ferrero and A. van der Merwe (eds.): *New Developments on Fundamental Problems in Quantum Physics.* 1997 ISBN 0-7923-4374-3
82. R. Miron: *The Geometry of Higher-Order Lagrange Spaces.* Applications to Mechanics and Physics. 1997 ISBN 0-7923-4393-X
83. T. Hakioglu and A.S. Shumovsky (eds.): *Quantum Optics and the Spectroscopy of Solids.* Concepts and Advances. 1997 ISBN 0-7923-4414-6
84. A. Sitenko and V. Tartakovskii: *Theory of Nucleus.* Nuclear Structure and Nuclear Interaction. 1997 ISBN 0-7923-4423-5
85. G. Esposito, A.Yu. Kamenshchik and G. Pollifrone: *Euclidean Quantum Gravity on Manifolds with Boundary.* 1997 ISBN 0-7923-4472-3
86. R.S. Ingarden, A. Kossakowski and M. Ohya: *Information Dynamics and Open Systems.* Classical and Quantum Approach. 1997 ISBN 0-7923-4473-1
87. K. Nakamura: *Quantum versus Chaos.* Questions Emerging from Mesoscopic Cosmos. 1997 ISBN 0-7923-4557-6
88. B.R. Iyer and C.V. Vishveshwara (eds.): *Geometry, Fields and Cosmology.* Techniques and Applications. 1997 ISBN 0-7923-4725-0
89. G.A. Martynov: *Classical Statistical Mechanics.* 1997 ISBN 0-7923-4774-9
90. M.W. Evans, J.-P. Vigier, S. Roy and G. Hunter (eds.): *The Enigmatic Photon.* Volume 4: New Directions. 1998 ISBN 0-7923-4826-5
91. M. Rédei: *Quantum Logic in Algebraic Approach.* 1998 ISBN 0-7923-4903-2
92. S. Roy: *Statistical Geometry and Applications to Microphysics and Cosmology.* 1998 ISBN 0-7923-4907-5
93. B.C. Eu: *Nonequilibrium Statistical Mechanics.* Ensembled Method. 1998 ISBN 0-7923-4980-6

Fundamental Theories of Physics

94. V. Dietrich, K. Habetha and G. Jank (eds.): *Clifford Algebras and Their Application in Mathematical Physics.* Aachen 1996. 1998 ISBN 0-7923-5037-5
95. J.P. Blaizot, X. Campi and M. Ploszajczak (eds.): *Nuclear Matter in Different Phases and Transitions.* 1999 ISBN 0-7923-5660-8
96. V.P. Frolov and I.D. Novikov: *Black Hole Physics.* Basic Concepts and New Developments. 1998 ISBN 0-7923-5145-2; PB 0-7923-5146
97. G. Hunter, S. Jeffers and J-P. Vigier (eds.): *Causality and Locality in Modern Physics.* 1998 ISBN 0-7923-5227-0
98. G.J. Erickson, J.T. Rychert and C.R. Smith (eds.): *Maximum Entropy and Bayesian Methods.* 1998 ISBN 0-7923-5047-2
99. D. Hestenes: *New Foundations for Classical Mechanics (Second Edition).* 1999 ISBN 0-7923-5302-1; PB ISBN 0-7923-5514-8
100. B.R. Iyer and B. Bhawal (eds.): *Black Holes, Gravitational Radiation and the Universe.* Essays in Honor of C. V. Vishveshwara. 1999 ISBN 0-7923-5308-0
101. P.L. Antonelli and T.J. Zastawniak: *Fundamentals of Finslerian Diffusion with Applications.* 1998 ISBN 0-7923-5511-3
102. H. Atmanspacher, A. Amann and U. Muller-Herold: *On Quanta, Mind and Matter Hans Primas in Context.* 1999 ISBN 0-7923-5696-9
103. M.A. Trump and W.C. Schieve: *Classical Relativistic Many-Body Dynamics.* 1999 ISBN 0-7923-5737-X
104. A.I. Maimistov and A.M. Basharov: *Nonlinear Optical Waves.* 1999 ISBN 0-7923-5752-3
105. W. von der Linden, V. Dose, R. Fischer and R. Preuss (eds.): *Maximum Entropy and Bayesian Methods Garching, Germany 1998.* 1999 ISBN 0-7923-5766-3
106. M.W. Evans: *The Enigmatic Photon Volume 5: O(3) Electrodynamics.* 1999 ISBN 0-7923-5792-2
107. G.N. Afanasiev: *Topological Effects in Quantum Mecvhanics.* 1999 ISBN 0-7923-5800-7
108. V. Devanathan: *Angular Momentum Techniques in Quantum Mechanics.* 1999 ISBN 0-7923-5866-X
109. P.L. Antonelli (ed.): *Finslerian Geometries A Meeting of Minds.* 1999 ISBN 0-7923-6115-6
110. M.B. Mensky: *Quantum Measurements and Decoherence Models and Phenomenology.* 2000 ISBN 0-7923-6227-6
111. B. Coecke, D. Moore and A. Wilce (eds.): *Current Research in Operation Quantum Logic.* Algebras, Categories, Languages. 2000 ISBN 0-7923-6258-6
112. G. Jumarie: *Maximum Entropy, Information Without Probability and Complex Fractals.* Classical and Quantum Approach. 2000 ISBN 0-7923-6330-2
113. B. Fain: *Irreversibilities in Quantum Mechanics.* 2000 ISBN 0-7923-6581-X
114. T. Borne, G. Lochak and H. Stumpf *Nonperturbative Quantum Field Theory and the Structure of Matter.* 2001 ISBN 0-7923-6803-7
115. J. Keller: *Theory of the Electron.* A Theory of Matter from START. 2001 ISBN 0-7923-6819-3