Undergraduate Lecture Notes in Physics

Francesco Lacava

Classical Electrodynamics

From Image Charges to the Photon Mass and Magnetic Monopoles

Springer

Undergraduate Lecture Notes in Physics

Undergraduate Lecture Notes in Physics (ULNP) publishes authoritative texts covering topics throughout pure and applied physics. Each title in the series is suitable as a basis for undergraduate instruction, typically containing practice problems, worked examples, chapter summaries, and suggestions for further reading.

ULNP titles must provide at least one of the following:

- An exceptionally clear and concise treatment of a standard undergraduate subject.
- A solid undergraduate-level introduction to a graduate, advanced, or non-standard subject.
- A novel perspective or an unusual approach to teaching a subject.

ULNP especially encourages new, original, and idiosyncratic approaches to physics teaching at the undergraduate level.

The purpose of ULNP is to provide intriguing, absorbing books that will continue to be the reader's preferred reference throughout their academic career.

More information about this series at http://www.springer.com/series/8917

Francesco Lacava

Classical Electrodynamics

From Image Charges to the Photon Mass and Magnetic Monopoles

Springer

Francesco Lacava
Università degli studi di Roma
"La Sapienza"
Rome
Italy

ISSN 2192-4791 ISSN 2192-4805 (electronic)
Undergraduate Lecture Notes in Physics
ISBN 978-3-319-39473-2 ISBN 978-3-319-39474-9 (eBook)
DOI 10.1007/978-3-319-39474-9

Library of Congress Control Number: 2016943959

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

To the memory of
my parents Biagio and Sara
and my brother Walter

Preface

In the undergraduate program of Electricity and Magnetism emphasis is given to the introduction of fundamental laws and to their applications. Many interesting and intriguing subjects can be presented only shortly or are postponed to graduate courses on Electrodynamics. In the last years I examined some of these topics as supplementary material for the course on Electromagnetism for the M.Sc. students in Physics at the University *Sapienza* in Roma and for a series of special lectures. This small book collects the notes from these lectures.

The aim is to offer to the readers some interesting study cases useful for a deeper understanding of the Electrodynamics and also to present some classical methods to solve difficult problems. Furthermore, two chapters are devoted to the Electrodynamics in relativistic form needed to understand the link between the electric and magnetic fields. In the two final chapters two relevant experimental issues are examined. This introduces the readers to the experimental work to confirm a law or a theory. References of classical books on Electricity and Magnetism are provided so that the students get familiar with books that they will meet in further studies. In some chapters the worked out problems extend the text material.

Chapter 1 is a fast survey of the topics usually taught in the course of Electromagnetism. It can be useful as a reference while reading this book and it also gives the opportunity to focus on some concepts as the electromagnetic potential and the gauge transformations.

The expansion in terms of multipoles for the potential of a system of charges is examined in Chap. 2. Problems with solutions are proposed.

Chapter 3 introduces the elegant method of image charges in vacuum. In Chap. 4 the method is extended to problems with dielectrics. This last argument is rarely presented in textbooks. In both chapters examples are examined and many problems with solutions are proposed.

Analytic complex functions can be used to find the solutions for the electric field in two-dimensional problems. After a general introduction of the method, Chap. 5 discusses some examples. In the Appendix to the chapter the solutions for two-dimensional problems are derived by solving the Laplace equation with boundary conditions.

Chapter 6 aims at introducing the relativistic transformations of the electric and magnetic fields by analysing the force on a point charge moving parallel to an infinite wire carrying a current. The equations of motion are formally the same in the laboratory and in the rest frame of the charge but the forces acting on the charge are seen as different in the two frames. This example introduces the transformations of the fields in special relativity.

In Chap. 7 a short historic introduction mentions the difficulties of the classical physics at the end of the 19th century in explaining some phenomena observed in Electrodynamics. The problem of invariance in the Minkowsky spacetime is examined. The formulas of Electrodynamics are written in covariant form. The electromagnetic tensor is introduced and the Maxwell equations in covariant form are given.

Chapter 8 presents a lecture by Feynman on the capacitor at high frequency. The effects produced by iterative corrections due to the induction law and to the displacement current are considered. For very high frequency of the applied voltage, the capacitor becomes a resonant cavity. This is a very interesting example for the students. The students are encouraged to refer to the Feynman lectures for further comments and for other arguments.

The energy and momentum conservation in the presence of an electromagnetic field are considered in Chap. 9. The Poynting's vector is introduced and some simple applications to the resistor, to the capacitor and to the solenoid are presented. The transfer of energy in an electric circuit in terms of the flux of the Poynting's vector is also examined. Then the Maxwell stress tensor is introduced. Some problems with solutions complete the chapter.

The Feynman paradox or paradox of the angular momentum is very intriguing. It is very useful to understand the dynamics of the electromagnetic field. Chapter 10 presents the paradox with comments. An original example of a rotating charged system in a damped magnetic field is discussed.

The need to test the dependence on the inverse square of the distance for the Coulomb's law was evident when the law was stated. The story of these tests is presented in Chap. 11. The most sensitive method, based on the Faraday's cage, was introduced by Cavendish and was used until the half of last century. After that time the test was interpreted in terms of a test on a non-null mass of the photon. The theory is shortly presented and experiments and limits are reported.

Chapter 12 introduces the problem of the magnetic monopoles. In a paper Dirac showed that the electric charge is quantized if a magnetic monopole exists in the Universe. The Dirac's relation is derived. The properties of a magnetic monopole crossing the matter are presented. Experiments to search the magnetic monopoles and their results are mentioned.

In the Appendix the general formulas of the differential operators used in Electrodynamics are derived for orthogonal systems of coordinates and the expressions for spherical and cylindrical coordinates are given.

I wish to thank Professors L. Angelani, M. Calvetti, A. Ghigo, S. Petrarca, and F. Piacentini for useful suggestions. A special thank is due to Professor M. Testa for helpful discussions and encouragement. I am grateful to Dr. E. De Lucia for reviewing the English version of this book and to Dr. L. Lamagna for reading and commenting this work.

Rome, Italy
May 2016

Francesco Lacava

Contents

Chapter 1
Classical Electrodynamics: A Short Review

The aim of this chapter is to review shortly the main steps of Classical Electrodynamics and to serve as a fast reference while reading the other chapters. This book collects selected lectures on Electrodynamics for readers who are studying or have studied Electrodynamics at the level of elementary courses for the degrees in Physics, Mathematics or Engineering. Many text books are available for an introduction[1] to Classical Electrodynamics or for more detailed studies.[2]

1.1 Coulomb's Law and the First Maxwell Equation

Only two kinds of charges exist in Nature: positive and negative. Any charge[3] is a negative or positive integer multiple of the elementary charge $e = 1.602 \times 10^{-19}$ Coulomb, that is equal to the absolute value of the charge of the electron.

The law of the force between two point charges was stated[4] by Coulomb in 1785. In vacuum the force $\mathbf{F}_{21}$ on the point charge q_2, located at $\mathbf{r}_2$, due to the point charge q_1, located at $\mathbf{r}_1$, is:

$$\mathbf{F}_{21} = \frac{1}{4\pi\epsilon_0} q_1 q_2 \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \tag{1.1}$$

[1]For instance: D.J. Griffiths, *Introduction to Electrodynamics*, 4th Ed. (2013), Pearson Prentice Hall; R.P. Feynman, R.B. Leighton, M. Sands, *The Feynman Lectures on Physics*, Vol. II; E.M. Purcell, *Electricity and Magnetism, Berkeley Physics Course*, Vol II, McGraw-Hill

[2]For instance: J.D. Jackson, *Classical Electrodynamics*, 3rd Ed. (1999), John Wiley & Sons Inc., and the books cited in the following chapters.

[3]Quarks have charge $\frac{1}{3}e$ or $\frac{2}{3}e$ but are confined in the hadrons. The charge quantization is examined in Chap. 12.

[4]A short historic note on the discovery of the Coulomb's law is given in Chap. 11.2.

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9_1

with the permittivity constant $\epsilon_0 = 8.564 \times 10^{-12}$ F/m (Farad/meter). The force on the charge q_1 is $\mathbf{F}_{12} = -\mathbf{F}_{21}$ as required by Newton's third law.

It is experimentally proved that in a system of many charges the force exerted on any charge is equal to the vector sum of the forces acted by all the other charges (*superposition principle*).

If $\mathbf{F}$ is the net electrostatic force on the point charge q located at $\mathbf{r}$, the electric field $\mathbf{E}_0$ in that position is defined[5] by the relation:

$$\mathbf{E}_0 = \lim_{q \to 0} \frac{\mathbf{F}}{q} .$$

The electric field $\mathbf{E}(r)$, due to a point charge Q located in the frame origin, is:

$$\mathbf{E}_0(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r^2} \hat{\mathbf{r}} \tag{1.2}$$

and from the superposition principle the electric field at a point $\mathbf{r}$ due to a system of point charges q_i, located at $\mathbf{r}_i$, is equal to the vector sum of the fields produced at the position $\mathbf{r}$, by all the point charges:

$$\mathbf{E}_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{i=N} q_i \frac{(\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3} .$$

For a continuous distribution with charge density $\rho(\mathbf{r}') = dQ/d\tau$ over a volume τ, the electric field is:

$$\mathbf{E}_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_\tau \frac{(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}')\, d\tau' .$$

The electric field (1.2) of a charge Q is radial and then it is conservative, so the field can be written as the gradient[6] of a scalar electric potential V_0 that depends on the position:

$$\mathbf{E}_0 = -grad\ V_0 = -\nabla V_0 \tag{1.3}$$

For a point charge at the origin the potential is:

$$V_0(r) = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} + C$$

with an arbitrary additive constant C that becomes null if $V_0(\infty) = 0$ is assumed. From the superposition principle for the electric field, the electric field of a system

[5]The limit is needed to avoid the point charge q modifies the field due to charges induced in conductors or from the polarization of the media.

[6]In the book the nabla operator will be used for the differential operators: $grad\ f = \nabla f$, $div\ \mathbf{v} = \nabla \cdot \mathbf{v}$ and $curl\ \mathbf{v} = \nabla \times \mathbf{v}$.

of charges is also conservative, and the potential at a given point, is equal to the sum of the potentials at that point from all the charges in the system. Thus it follows:

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{i=N}\frac{q_i}{|\mathbf{r}-\mathbf{r}_i|} + C \qquad V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_\tau \frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d\tau' + C \tag{1.4}$$

where C is an arbitrary constant. For a confined distribution of charges the potential can be fixed null at infinity and thus $C = 0$.

The closed curve line integral of $\mathbf{E}_0$ is null and, for the Stokes's theorem, the field $\mathbf{E}_0$ is irrotational:

$$\nabla \times \mathbf{E}_0 = 0$$

that is the local form to state the electric field is conservative.

The Gauss's law[7] is very relevant in electrostatics. It states that the flux Φ_S of $\mathbf{E}_0$ through a closed surface S is equal to the total charge inside the surface, divided by ϵ_0:

$$\Phi_S(\mathbf{E}) = \int_S \mathbf{E}_0 \cdot \hat{\mathbf{n}}\,dS' = \frac{Q}{\epsilon_0} \tag{1.5}$$

with $\hat{\mathbf{n}}$ the outward-pointing unit normal at each point of the surface, and the charges outside the surface do not contribute to the flux Φ_S.

From this law the Coulomb's theorem: near the surface of a conductor with surface charge density $\sigma(x, y, z)$, the electric field is equal to:

$$\mathbf{E}_0 = \frac{\sigma}{\epsilon_0}\hat{\mathbf{n}}$$

with $\hat{\mathbf{n}}$ the versor with direction outside of the conductor at that point. If $\rho(\mathbf{r})$ is the charge density in the volume τ enclosed by S, the total charge is Q:

$$Q = \int_\tau \rho(\mathbf{r}')\,d\tau' \tag{1.6}$$

and substituting this expression in the relation (1.5) and using the divergence theorem, we find the first Maxwell equation in vacuum:

$$\nabla \cdot \mathbf{E}_0 = \frac{\rho}{\epsilon_0} \tag{1.7}$$

that is the differential (or local) expression of the Gauss's law.

[7] See also Chap. 11.1.

After the substitution of the relation (1.3) in the last equation, the Poisson's equation for the potential is found:

$$\nabla^2 V_0 = -\frac{\rho}{\epsilon_0}\,.$$

For a given distribution of charges and for fixed boundary conditions, due to the unicity of the solution, the solution of Poisson's equation is given by (1.4).

1.2 Charge Conservation and Continuity Equation

Charge conservation in isolated systems is experimentally proved. The charge in a volume τ, enclosed by the surface S, changes only if an electric current I, positive if outgoing, flows through S. So:

$$-\frac{dQ}{dt} = I \tag{1.8}$$

and I is the flux of the electric current density $\mathbf{J} = \rho\mathbf{v}$, with $\mathbf{v}$ the velocity of the charge, through the surface S:

$$I = \int_S \mathbf{J}\cdot\hat{\mathbf{n}}\,dS' \tag{1.9}$$

with $\hat{\mathbf{n}}$ the outward-pointing unit normal to the element of surface dS' of the closed surface. By substituting the relations (1.6) and (1.9) in Eq. (1.8) and applying the divergence theorem, the continuity equation is found:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{J} = 0 \tag{1.10}$$

that is the local expression of the charge conservation.

1.3 Absence of Magnetic Charges in Nature and the Second Maxwell Equation

The field lines of the magnetic induction B are always closed. This is easily seen by tracking with a small magnetic needle the field lines of $\mathbf{B}$ around a circuit carrying a current. Indeed in Nature no source (magnetic monopole) of magnetic field has ever been observed.[8] Thus the flux $\Phi_S(\mathbf{B})$ through a closed surface S is always null:

[8] Chapter 12 is devoted to the theory and the search of magnetic monopoles.

$$\Phi_S(\mathbf{B}) = \int_S \mathbf{B} \cdot \hat{\mathbf{n}}\, dS' = 0\,. \tag{1.11}$$

By applying the divergence theorem to this relation the second Maxwell equation can be found:

$$\nabla \cdot \mathbf{B} = 0$$

where the null second member corresponds to the absence of magnetic sources.

1.4 Laplace's Laws and the Steady Fourth Maxwell Equation

H.C. Oersted in 1820 discovered that an electric current produces a magnetic field. This observation led physicists to write the laws of magnetism in a short time.

The force acting on the element $\mathbf{dl}$ of a circuit carrying a current I, in the direction of $\mathbf{dl}$, located in a magnetic field $\mathbf{B}$, is:

$$\mathbf{dF} = I\ \mathbf{dl} \times \mathbf{B} \qquad \text{(Second Laplace's formula)}$$

from which the force on a point charge q moving with velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$:

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} \qquad \text{(Lorentz's force).}$$

The contribution to the field $\mathbf{B}_0(\mathbf{r})$ in vacuum at a point $P(\mathbf{r})$, given by $\mathbf{dl}'$, an element of a circuit, with a current I flowing in the direction of $\mathbf{dl}'$, located at $\mathbf{r}'$, is:

$$\mathbf{dB}_0(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{I\ \mathbf{dl}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \qquad \text{(First Laplace's formula)} \tag{1.12}$$

called Biot and Savart law, with μ_0 the permeability constant of free space ($\mu_0 = 4\pi \cdot 10^{-7}$ H/m (Henry/meter)).

The field $\mathbf{B}_0(\mathbf{r})$ from a circuit is:

$$\mathbf{B}_0(\mathbf{r}) = \frac{\mu_0}{4\pi} \oint \frac{I\ \mathbf{dl}' \times (\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}\,.$$

By integrating along a closed line the last formula the Ampère's law can be found:

$$\oint \mathbf{B}_0 \cdot \mathbf{dl} = \mu_0 \sum_{i=1}^{i=N} I_i$$

with the algebraic sum of all the currents I_i enclosed by the loop. Then applying the Stoke's theorem and the relation (1.9), the differential law is derived:

$$\nabla \times \mathbf{B}_0 = \mu_0 \mathbf{J} \tag{1.13}$$

that is the steady-state fourth Maxwell equation in vacuum.

1.5 Faraday's Law and the Third Maxwell Equation

Faraday's law of induction says that the induced electromotive force f in a circuit is equal to the negative of the rate at which the flux of magnetic induction $\Phi(B)$ through the circuit is changing:

$$f = -\frac{d\Phi}{dt} \qquad \text{(Faraday-Neumann-Lenz law)} \tag{1.14}$$

with f the closed line integral of an induced non conservative electric field $\mathbf{E}_i$:

$$f = \oint \mathbf{E}_i \cdot \mathbf{dl}\,. \tag{1.15}$$

The induction is observed in different situations. If the shape of the circuit is changed or some of its parts move in a steady magnetic field, the electromotive force and the induced electric field $\mathbf{E}_i$ can be related to the Lorentz's force acting on the free charges in the conductor. This is the case of a flux of magnetic field lines cut by parts of the circuit. Differently if the sources of the magnetic field (circuits carrying currents or permanent magnets) move while the circuit is at rest, the induced electric field is determined by the (relativistic) transformations of the fields $\mathbf{E}$ and $\mathbf{B}$ between different reference frames as discussed in Chaps. 6 and 7. But when the circuit and the sources of the field are at rest and the magnetic field changes (for instance due to the change of the current in one of the circuits used as sources) the induction effect implies a new physical phenomenon. By applying to the (1.15) the Stokes's theorem and substituting that in the first member of Eq. (1.14) while at the second member is written the flux of $\mathbf{B}$, as given in (1.11), the third Maxwell equations is found:

$$\nabla \times \mathbf{E}_i = -\frac{\partial \mathbf{B}}{\partial t}\,.$$

Thus in general the electric field is the superposition of the irrotational field $\mathbf{E}_e$ from the electric charges and a non irrotational electric field $\mathbf{E}_i$ induced by the rate of change of the magnetic field:

$$\mathbf{E} = \mathbf{E}_e + \mathbf{E}_i\,.$$

1.6 Displacement Current and the Fourth Maxwell Equation

Taking the divergence of the Eq. (1.13), while the first member is always null because $\nabla \cdot \nabla \times \mathbf{v} = 0$ is null for any vector $\mathbf{v}$, the second member $\nabla \cdot \mathbf{J}$ is null only in steady-state situations. To solve this fault Maxwell suggested to replace the charge density ρ in the continuity equation (1.10) with the expression for ρ from the first Maxwell equation (1.7). The result is the sum of two terms with always a null divergence:

$$\nabla \cdot \left(\epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \right) = 0\,.$$

Replacing $\mathbf{J}$ in the (1.13) with the sum of the two terms, the Eq. (1.13) becomes:

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right)$$

that is the correct fourth Maxwell equation with both members having a null divergence also for time varying fields and currents. The term added to the current density $\mathbf{J}$ at the second member, is the *displacement current density* $\mathbf{J}_S$:

$$\mathbf{J}_S = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$$

associated to the rate of change of the electric field $\mathbf{E}$.

1.7 Maxwell Equations in Vacuum

The four Maxwell equation in vacuum are:

$$\nabla \cdot \mathbf{E}_0 = \frac{\rho}{\epsilon_0} \qquad (I) \qquad\qquad \nabla \cdot \mathbf{B}_0 = 0 \qquad (II)$$

$$\nabla \times \mathbf{E}_0 = -\frac{\partial \mathbf{B}_0}{\partial t} \qquad (III) \qquad\qquad \nabla \times \mathbf{B}_0 = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}_0}{\partial t} \qquad (IV).$$

1.8 Maxwell Equations in Matter

In the presence of an external electric field the atomic and molecular dipoles in the media are polarized. The *electric polarization* $\mathbf{P}$ is the average dipole moment per unit volume. Charges due to the polarization are present on the surface of the dielectrics with charge density $\sigma_P = \mathbf{P} \cdot \hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ the outward-pointing unit normal vector to the surface, and in the volume with density $\rho_P = -\nabla \cdot \mathbf{P}$. So in the first Maxwell equation (1.7) the polarization charges have also to be taken into account:

$$\nabla \cdot \mathbf{E} = \frac{\rho + \rho_P}{\epsilon_0}.$$

By introducing the *displacement vector* $\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$ this equation becomes:

$$\nabla \cdot \mathbf{D} = \rho$$

where only the free charges are present.

The magnetization of the media can be described by the vector *magnetization* $\mathbf{M}$ that is the average magnetic moment per unit volume. The microscopic currents, associated to the magnetization, flow on the surface with current density $\mathbf{J}_{ms} = \mathbf{M} \times \hat{\mathbf{n}}$, with $\hat{\mathbf{n}}$ the outward-pointing unit normal vector to the surface, and in the volume with current density $\mathbf{J}_{mv} = \nabla \times \mathbf{M}$.

The current density $\mathbf{J}_{mv}$ has to be added to the free current density $\mathbf{J}$ in the fourth Maxwell equation (1.13):

$$\nabla \times \mathbf{B} = \mu_0 (\mathbf{J} + \mathbf{J}_{mv})$$

and defining the *magnetic field* $\mathbf{H}$:

$$\mathbf{H} = \frac{\mathbf{B} - \mu_0 \mathbf{M}}{\mu_0}$$

the steady fourth equation becomes:

$$\nabla \times \mathbf{H} = \mathbf{J}$$

with only the free current at the second member.

In matter the four Maxwell equations are:

$$\nabla \cdot \mathbf{D} = \rho \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}.$$

Of course to find the fields the constitutive relations $\mathbf{D} = \mathbf{D}(\mathbf{E})$ and $\mathbf{H} = \mathbf{H}(\mathbf{B})$ have to be known. In homogeneous and isotropic media these relations are:

$$\mathbf{D} = \epsilon \mathbf{E} \qquad \mathbf{P} = \epsilon_0 (\kappa - 1) \mathbf{E} \qquad \epsilon = \epsilon_0 \kappa$$

with ϵ the permittivity and κ the dielectric constant of the medium, and:

$$\mathbf{B} = \mu \mathbf{H} \qquad \mathbf{M} = (\kappa_m - 1)\mathbf{H} \qquad \mu = \mu_0 \kappa_m$$

with μ the permeability and κ_m the relative permeability of the medium.

1.9 Electrodynamic Potentials and Gauge Transformations

The Maxwell equations are four first-order equations that, with assigned boundaries conditions, can be solved in simple situations. It is often convenient to introduce potentials, that while are defined to satisfy directly the two homogenous equations, are determined by only two second-order equations.

In an isotropic and homogeneous medium with permittivity ϵ and permeability μ, the four Maxwell equations are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon} \qquad (I) \tag{1.16}$$

$$\nabla \cdot \mathbf{B} = 0 \qquad (II) \tag{1.17}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (III) \tag{1.18}$$

$$\nabla \times \mathbf{B} = \mu \mathbf{J} + \epsilon \mu \frac{\partial \mathbf{E}}{\partial t} \quad (IV)\,. \tag{1.19}$$

Electrodynamic Potentials

The divergence of a curl is always null ($\nabla \cdot \nabla \times \mathbf{v} = 0$), so the second equation is satisfied if $\mathbf{B}$ is the curl of a vector potential $\mathbf{A}$:

$$\mathbf{B} = \nabla \times \mathbf{A}\,. \tag{1.20}$$

With this definition the third equation becomes:

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0$$

and since the sum of the two terms has a null curl, it can be the gradient of a scalar potential V with a change of sign as in electrostatics:

$$\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$$

and thus, in terms of the potentials, the electric field is:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}\,. \tag{1.21}$$

Thus homogeneous equations are used to introduce a vector potential $\mathbf{A}$ and a scalar potential V that have to be determined.

Gauge Transformations

The vector potential $\mathbf{A}$ is determined up to the gradient of a scalar function φ. Indeed under the transformation:

$$\mathbf{A} \to \mathbf{A}' = \mathbf{A} + \nabla\varphi \tag{1.22}$$

the field $\mathbf{B}$, given by the relation (1.20), since $\nabla \times \nabla\varphi$ is always null, is left unchanged:

$$\mathbf{B}' = \nabla \times \mathbf{A}' = \nabla \times \mathbf{A} + \nabla \times \nabla\varphi = \nabla \times \mathbf{A} = \mathbf{B}\,.$$

It is easy to see that in order for the field $\mathbf{E}$ (1.21) to be also unchanged, the scalar potential has to transform as:

$$V \to V' = V - \frac{\partial \varphi}{\partial t}\,. \tag{1.23}$$

Indeed we find:

$$\mathbf{E}' = -\nabla V' - \frac{\partial \mathbf{A}'}{\partial t} = -\nabla V + \nabla\frac{\partial \varphi}{\partial t} - \frac{\partial \mathbf{A}}{\partial t} - \frac{\partial \nabla\varphi}{\partial t} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}\,.$$

The relations (1.22) and (1.23) are the *gauge transformations* of the electrodynamic potentials.

Equations of the Electrodynamic Potentials

To determine the potentials $\mathbf{A}$ and V we have to consider the two inhomogeneous Maxwell equations. Substituting the relations (1.20) and (1.21) in these equations we get[9] the two coupled equations:

$$\nabla^2\mathbf{A} - \mu\epsilon\frac{\partial^2 \mathbf{A}}{\partial t^2} - \nabla(\nabla \cdot \mathbf{A} + \mu\epsilon\frac{\partial V}{\partial t}) = -\mu\mathbf{J} \tag{1.25}$$

$$\nabla^2 V + \frac{\partial}{\partial t}\nabla \cdot \mathbf{A} = -\frac{\rho}{\epsilon}\,. \tag{1.26}$$

[9]Remind the relation:

$$\nabla \times \nabla \times \mathbf{v} = \nabla(\nabla \cdot \mathbf{v}) - \nabla^2\mathbf{v} \tag{1.24}$$

.

Lorentz Gauge

The freedom in the definition of the potentials from the gauge transformations (1.22) and (1.23) gives the possibility to choose **A** and V in order they satisfy the *Lorentz condition*:

$$\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t} = 0 \tag{1.27}$$

useful to decouple the two Eqs. (1.25) and (1.26).

If not satisfied by **A** and V, this relation can be satisfied by two new potentials $\mathbf{A}'$ and V' that are gauge transformed of **A** and V by the (1.22) and (1.23). The (1.27) for the new potentials gives an equation for the scalar function φ used in the transformation:

$$\nabla^2\varphi - \mu\epsilon \frac{\partial^2\varphi}{\partial t^2} = -\left(\nabla \cdot \mathbf{A} + \mu\epsilon \frac{\partial V}{\partial t}\right) \tag{1.28}$$

that, with assigned boundaries conditions, at least in principle, can be solved. Since this gauge transformation is always possible, we can assume that the potentials satisfy the (1.27). We say we choose to work in the *Lorentz gauge*.

Gauge transformations of potentials which satisfy the Lorentz condition, give new potentials which observe the Lorentz condition if the function φ satisfies the equation:

$$\nabla^2\varphi - \mu\epsilon \frac{\partial^2\varphi}{\partial t^2} = 0\,.$$

Uncoupled Equations and Retarded Potentials

With the Lorentz conditions the two Eqs. (1.25) and (1.26) are decoupled and become:

$$\nabla^2\mathbf{A} - \mu\epsilon \frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J} \tag{1.29}$$

$$\nabla^2 V - \mu\epsilon \frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon} \tag{1.30}$$

or with the d'Alambertian operator:

$$\Box = \nabla^2 \; - \mu\epsilon \frac{\partial^2}{\partial t^2}$$

in a more compact form:

$$\Box\mathbf{A} = -\mu\mathbf{J} \qquad\qquad \Box V = -\frac{\rho}{\epsilon}\;.$$

The equations for **A** and V are four second-order scalar equations. Their particular solutions are the *retarded potentials*:

$$\mathbf{A}(\mathbf{r},t) = \frac{\mu}{4\pi}\int \frac{\mathbf{J}(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{v})}{|\mathbf{r}-\mathbf{r}'|}\, d\tau' + \mathbf{C} \tag{1.31}$$

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon}\int \frac{\rho(\mathbf{r}', t - \frac{|\mathbf{r}-\mathbf{r}'|}{v})}{|\mathbf{r}-\mathbf{r}'|}\, d\tau' + C' \tag{1.32}$$

with the constants $\mathbf{C}$ and C' null if the potentials are zero at infinity.

The potentials at the point $\mathbf{r}$ at time t depend on the values of the sources at $\mathbf{r}'$ at time $t' = t - (|\mathbf{r} - \mathbf{r}'|)/v$, where $\Delta t = (|\mathbf{r} - \mathbf{r}'|)/v$ is the time interval for the electromagnetic signal propagates from the source at $\mathbf{r}'$ to the point of observation at $\mathbf{r}$ with velocity $v = \frac{1}{\sqrt{\mu\epsilon}}$.

To get the solutions of the Eqs. (1.29) and (1.30), homogeneous solutions have to be added to the particular solutions (1.31) and (1.32). It is evident that the homogeneous solutions are waves that propagate. At large distance only the homogeneous solutions are present because the particular solutions vanish as $1/r$.

Coulomb Gauge

Another possible choice is the *Coulomb gauge* with the condition:

$$\nabla \cdot \mathbf{A} = 0$$

thus the Eq. (1.26) becomes the Poisson's equation:

$$\nabla^2 V = -\frac{\rho}{\epsilon}$$

with the instantaneous Coulomb potential as solution:

$$V(\mathbf{r},t) = \frac{1}{4\pi\epsilon}\int_\tau \frac{\rho(\mathbf{r},t)}{|\mathbf{r}-\mathbf{r}'|}\, d\tau' \tag{1.33}$$

while the Eq. (1.25) becomes:

$$\nabla^2\mathbf{A} - \mu\epsilon\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu\mathbf{J} + \mu\epsilon\nabla\left(\frac{\partial V}{\partial t}\right). \tag{1.34}$$

The instantaneous potential (1.33) does not take into account the propagation time of the electromagnetic signal and seems in contrast with the time interval required to propagate the information from the source to the position where the signal is observed. Actually the observable quantities are the fields $\mathbf{E}$ and $\mathbf{B}$ which depend also on the non instantaneous potential $\mathbf{A}$ given by the Eq. (1.34) thus their changes are also delayed with respect to the changes of the sources.

This gauge is also called *transverse gauge*. Indeed the density current $\mathbf{J}$ can be written as the sum[10] of a longitudinal or irrotational component $\mathbf{J}_l$, (such that $\nabla \times \mathbf{J}_l = 0$), and a transverse or solenoidal component $\mathbf{J}_t$, ($\nabla \cdot \mathbf{J}_t = 0$). From the continuity equation for the current we have that the term:

$$\epsilon \nabla \left(\frac{\partial V}{\partial t} \right)$$

in Eq. (1.34) is equal to the longitudinal component of the density current. Thus with only the transverse component $\mathbf{J}_t$ at the second member, the equation is:

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = -\mu \mathbf{J}_t$$

and the vector potential $\mathbf{A}$ depends only on the transverse component of the current density and is parallel to that.

The gauge $\nabla \cdot \mathbf{A} = 0$ is useful in the absence of sources ($\rho = 0$ and $\mathbf{J} = 0$). In this gauge, called *radiation gauge*, $V = 0$ or constant and the Eq. (1.25) becomes the wave equation:

$$\nabla^2 \mathbf{A} - \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} = 0$$

while the fields are given by the relations:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} \qquad\qquad \mathbf{B} = \nabla \times \mathbf{A} \,.$$

1.10 Electromagnetic Waves

In the absence of electric charges in a non conducting homogeneous and isotropic infinite medium, of permittivity ϵ and permittivity μ, the Maxwell equations are:

$$\nabla \cdot \mathbf{E} = 0 \qquad (I) \qquad\qquad \nabla \cdot \mathbf{B} = 0 \qquad (II)$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad (III) \qquad\qquad \nabla \times \mathbf{B} = \mu\epsilon \frac{\partial \mathbf{E}}{\partial t} \qquad (IV)$$

and non null solutions are possible for time-varying fields $\mathbf{E}$ and $\mathbf{B}$.

Taking the curl of the third equation, and substituting the fourth in the right hand side, by the first equation and the relation (1.24) the equation for the field $\mathbf{E}$ is:

[10] For this subject see J.D. Jackson, *Classical Electrodynamics*, cited, Sect. 6.5.

$$\nabla^2 \mathbf{E} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0\,.$$

From the curl of the fourth equation after the third is substituted at the right hand side, a similar equation is found for the field **B**:

$$\nabla^2 \mathbf{B} - \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0\,.$$

In the radiation gauge the same equation is found for the vector potential **A**. These are the equations for electromagnetic waves propagating in the medium with speed v:

$$v = \frac{1}{\sqrt{\epsilon\mu}} = \frac{1}{\sqrt{\epsilon_0\mu_0}}\frac{1}{\sqrt{\kappa\kappa_m}} \simeq \frac{c}{\sqrt{\kappa}} = \frac{c}{n} \qquad (\kappa_m \simeq 1)$$

where n is the refractive index of the crossed medium while:

$$c = \frac{1}{\sqrt{\epsilon_0\mu_0}}$$

is the speed of light in vacuum.

Plane Electromagnetic Waves

The plane electromagnetic waves have the fields **E** and **B** with same components over a plane surface. If this surface is parallel to the place yz the components depend only on the x component and on the time and the wave equations become:

$$\frac{\partial^2 \mathbf{E}}{\partial x^2} - \mu\epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \qquad \frac{\partial^2 \mathbf{B}}{\partial x^2} - \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2} = 0\,.$$

The most general solution is a function of the type:

$$f = f_1(x - vt) + f_2(x + vt) \qquad \text{with} \qquad v = \frac{1}{\sqrt{\mu\epsilon}}$$

that corresponds to the superposition of a wave $f_1(x - vt)$ travelling in the positive direction of the x axis and of a wave $f_2(x + vt)$ travelling in the opposite direction with *phase velocity* $\pm v$. For these two waves the fields **E** and **B** have the same components on the planes given by the relation:

$$x = \pm vt + const$$

and the fields are:

$$\mathbf{E} = \mathbf{E}_0\, f(x \mp vt) \qquad \mathbf{B} = \mathbf{B}_0\, f(x \mp vt).$$

Substituting these relations in the Maxwell equations it is easy to find:

$$\mathbf{E} = \mathbf{B} \times \mathbf{v}$$

so in a plane wave the fields **E** and **B** are orthogonal each other and also to the direction of propagation $\hat{\mathbf{v}}$. These waves are *transverse waves*.

For a monochromatic wave of frequency ω propagating in the $\hat{\mathbf{k}}$ direction, the most general solution for the fields is of the type:

$$f(\mathbf{r}, t) = Ae^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + Be^{i(\mathbf{k}\cdot\mathbf{r}+\omega t)}$$

where the *wave vector* **k** is:

$$\mathbf{k} = \frac{\omega}{v}\hat{\mathbf{v}} = \frac{2\pi}{\lambda}\hat{\mathbf{v}}$$

and λ is the wavelength.

Chapter 2
Multipole Expansion of the Electrostatic Potential

The potential of a localized charge distribution at large distance can be expanded as a series of multipole terms.[1] The terms of the series depend on the charge spatial distribution in the system and have different dependence from the distance. In this chapter we will first examine the electric dipole, the simplest system after the point charge. We will write the dipole potential and obtain the expressions of the electrostatic energy, the force and the torque acting on the dipole in an external field. Then we will derive the first terms of the multipole expansion for the potential from a charge distribution. Finally we will write the general expression for the multipole expansion together the formula for the expansion in terms of spherical harmonics.

2.1 The Potential of the Electric Dipole

The electric dipole is a rigid system of two point charges of opposite sign $+q$ and $-q$ separated by the distance δ. It is characterised by the *dipole moment* $\mathbf{p} = q\boldsymbol{\delta}$ with $\boldsymbol{\delta}$ oriented from the negative to the positive charge.

The potential generated by the electric dipole at point P at position $\mathbf{r}$ from its center, is the sum of the potentials of the two point charges:

$$V(P) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{r_- - r_+}{r_+ r_-}\right)$$

where r_+ and r_- are the distances of P from $+q$ and $-q$ respectively (see Fig. 2.1).

When $r \gg \delta$ first order approximation can be used:

[1] For this subject see for instance: D.J. Griffiths, *Introduction to Electrodynamics*, 4th Ed. (2013), Section 3.4, Pearson Prentice Hall; W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd Ed. (1962), Sections 1.7–8, Addison-Wesley.

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9_2

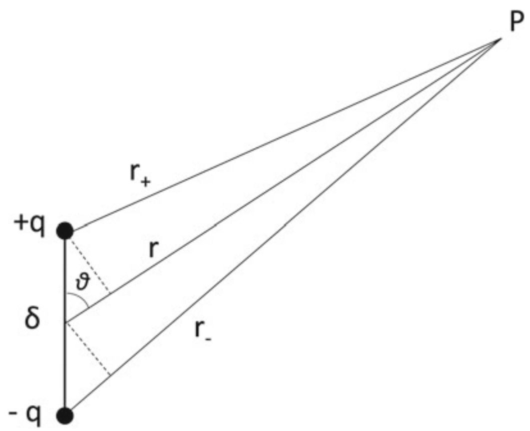


Fig. 2.1 The potential of the electric dipole at a point P at distance r, is the sum of the potentials of the two opposite point charges. The distances of the point P from the two charges in the approximation $r \gg \delta$ are $r_+ = r - \frac{\delta}{2}\cos\theta$ and $r_- = r + \frac{\delta}{2}\cos\theta$

$$r_- \simeq r + \frac{\delta}{2}\cos\theta \qquad r_+ \simeq r - \frac{\delta}{2}\cos\theta \qquad r_- - r_+ \simeq \delta\ \cos\theta$$

with θ the angle between $\mathbf{r}$ and $\boldsymbol{\delta}$, and:

$$r_+ r_- = r^2 - \frac{\delta^2}{4}\cos^2\theta \simeq r^2$$

and the potential at a large distance from the dipole becomes:

$$V(P) = \frac{1}{4\pi\epsilon_0}\frac{q\delta\cos\theta}{r^2} = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\mathbf{r}}{r^3}\,.$$

For increasing r this potential decreases as $1/r^2$ function, faster than the $1/r$ dependence of the point charge potential.

2.2 Interaction of the Dipole with an Electric Field

We can consider the interaction of an electric dipole $\mathbf{p}$ with an external electric field $\mathbf{E}$ that in general can be non-uniform.

For the force $\mathbf{F}$, the F_x component of the total force on the dipole is the sum of the x-component of the forces on the two point charges:

$$F_x = -qE_x(x, y, z) + qE_x(x + \delta_x, y + \delta_y, z + \delta_z)$$

with $\mathbf{E}(x, y, z)$ the electric field at the position of the charge $-q$ and $\mathbf{E}(x + \delta_x, y + \delta_y, z + \delta_z)$ that at the position of $+q$ respectively.

Writing the second term as:

$$E_x(x+\delta_x, y+\delta_y, z+\delta_z) = E_x(x, y, z) + \nabla E_x \cdot \boldsymbol{\delta}$$

F_x becomes:

$$F_x = q\nabla E_x \cdot \boldsymbol{\delta} = (\mathbf{p}\cdot\nabla)E_x.$$

Similar expressions can be derived for F_y and F_z and therefore we have:

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\mathbf{E} = \nabla(\mathbf{p}\cdot\mathbf{E}) \tag{2.1}$$

where we have used relation[2] $(\mathbf{p}\cdot\nabla)\mathbf{E} = \nabla(\mathbf{p}\cdot\mathbf{E}) - \mathbf{p}\times(\nabla\times\mathbf{E}) = \nabla(\mathbf{p}\cdot\mathbf{E})$ with $\nabla\times\mathbf{E} = 0$ for the electrostatic field.

The potential energy of the dipole is equal to the sum of the potential energies of the two point charges:

$$U = -qV(x, y, z) + qV(x+\delta_x, y+\delta_y, z+\delta_z) = qdV = q\nabla V\cdot\boldsymbol{\delta} = -\mathbf{p}\cdot\mathbf{E}.$$

The work $\delta W = -dU$ done by the electric field when the dipole is displaced by $\mathbf{ds}$ by a force $\mathbf{F}$ and is rotated by $\boldsymbol{\delta\theta}$ around an axis $\hat{\boldsymbol{\theta}}$ by a torque $\mathbf{M}$, is:

$$\delta W = \mathbf{F}\cdot\mathbf{ds} + \mathbf{M}\cdot\mathbf{d\boldsymbol{\theta}} = -dU = -\nabla U\cdot d\mathbf{s} - \frac{\partial U}{\partial\theta}\delta\theta$$

and from the potential energy expression:

$$\mathbf{F} = \nabla(\mathbf{p}\cdot\mathbf{E}) \qquad\qquad \mathbf{M} = \mathbf{p}\times\mathbf{E} \tag{2.2}$$

where the expression for $\mathbf{F}$ is that already found in (2.1).

Supplemental problems are available at the end of this chapter as additional material on the interaction between two dipoles.

2.3 Multipole Expansion for the Potential of a Distribution of Point Charges

The potential $V_0(P)$ at the point $P(x, y, z)$ due to a distribution of N point charges q_i (see Fig. 2.2), is equal to the sum of the potentials at P from each charge of the system (principle of superposition of the electric potentials):

---

[2]Note that, since ∇ is a vector operator, we can get the relation used in the formula by substituting ∇ to $\mathbf{B}$ in the vector relation $\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = \mathbf{B}(\mathbf{A}\cdot\mathbf{C}) - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$.

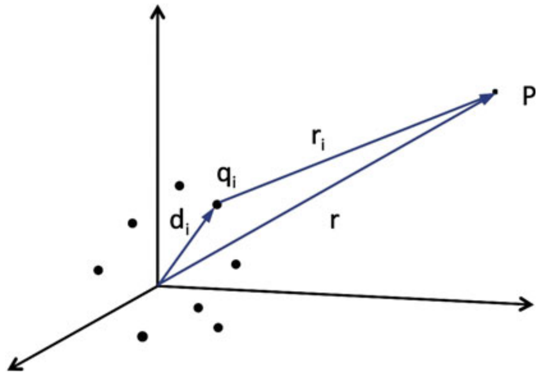


Fig. 2.2 The distribution of point charges. The position of the point P relative to the point charge q_i is $\mathbf{r}_i = \mathbf{r} - \mathbf{d}_i$

$$V_0(P) = \sum_{i=1}^{i=N} V_{0i}(P) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{i=N} \frac{q_i}{r_i} \tag{2.3}$$

where r_i is the distance between P and the ith charge.

If the distance between P and the system of the charges is much larger than the dimensions of the system, it is useful to approximate the potential at P as done for the electric dipole.

In the reference frame with origin in close proximity to the point charges system, we can write:

$$\mathbf{r} = \mathbf{r}_i + \mathbf{d}_i \qquad \text{and} \qquad \mathbf{r}_i = \mathbf{r} - \mathbf{d}_i$$

where $\mathbf{d}_i$ and $\mathbf{r}$ are the vector positions of the charge q_i and point P, respectively, and $\mathbf{r}_i$ the vector from the ith charge to P. Then:

$$\frac{1}{r_i} = \frac{1}{|\mathbf{r} - \mathbf{d}_i|} = \frac{1}{[(\mathbf{r} - \mathbf{d}_i) \cdot (\mathbf{r} - \mathbf{d}_i)]^{\frac{1}{2}}} = \frac{1}{[r^2 - 2\mathbf{r} \cdot \mathbf{d}_i + d_i^2]^{\frac{1}{2}}} = \frac{1}{r} \cdot \frac{1}{[1 + \frac{(d_i^2 - 2\mathbf{r} \cdot \mathbf{d}_i)}{r^2}]^{\frac{1}{2}}} .$$

If we define $\alpha = \frac{d_i^2 - 2\mathbf{r} \cdot \mathbf{d}_i}{r^2}$, since $r \gg d_i$, α is very small and the fraction in the last equation can be expanded in a power series:

$$\frac{1}{[1 + \alpha]^{\frac{1}{2}}} = 1 - \frac{1}{2}\alpha + \frac{3}{8}\alpha^2 - \frac{15}{48}\alpha^3 + \cdots$$

and keeping terms only to second order in α:

$$\frac{1}{r_i} = \frac{1}{r} \cdot \left(1 - \frac{1}{2}\frac{(d_i^2 - 2\mathbf{r} \cdot \mathbf{d}_i)}{r^2} + \frac{3}{8}\frac{(d_i^2 - 2\mathbf{r} \cdot \mathbf{d}_i)^2}{r^4}\right)$$

and neglecting terms with higher power than d_i/r squared we get:

$$\frac{1}{r_i} = \frac{1}{r} + \frac{(\mathbf{d}_i \cdot \hat{\mathbf{r}})}{r^2} + \frac{1}{r^3}\left[\frac{3}{2}(\mathbf{d}_i \cdot \hat{\mathbf{r}})^2 - \frac{1}{2}d_i^2\right]. \tag{2.4}$$

Using this relation in Eq. 2.3, the potential in the point P becomes:

$$V_0(P) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{i=N}\frac{q_i}{r} + \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{i=N}\frac{q_i\mathbf{d}_i \cdot \hat{\mathbf{r}}}{r^2} + \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{i=N}\frac{q_i}{r^3}\left[\frac{3}{2}(\mathbf{d}_i \cdot \hat{\mathbf{r}})^2 - \frac{1}{2}d_i^2\right]$$

that can be written as:

$$V_0(P) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\sum_{i=1}^{i=N}q_i + \frac{1}{4\pi\epsilon_0}\frac{1}{r^2}\sum_{i=1}^{i=N}q_i\mathbf{d}_i \cdot \hat{\mathbf{r}} + \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\sum_{i=1}^{i=N}q_i\left[\frac{3}{2}(\mathbf{d}_i \cdot \hat{\mathbf{r}})^2 - \frac{1}{2}d_i^2\right].$$

If we define the *total charge* of the system:

$$Q_{TOT} = \sum_{i=1}^{i=N} q_i \tag{2.5}$$

the *electric dipole moment*:

$$\mathbf{P} = \sum_{i=1}^{i=N} q_i\mathbf{d}_i \tag{2.6}$$

and the *electric quadrupole moment* relative to the direction $\hat{\mathbf{r}}$:

$$Q_{quadr} = \sum_{i=1}^{i=N} q_i\left[\frac{3}{2}(\mathbf{d}_i \cdot \hat{\mathbf{r}})^2 - \frac{1}{2}d_i^2\right] \tag{2.7}$$

the potential of the system of point charges at the point P can be written in the form:

$$V_0(P) = \frac{1}{4\pi\epsilon_0}\frac{Q_{TOT}}{r} + \frac{1}{4\pi\epsilon_0}\frac{\mathbf{P} \cdot \hat{\mathbf{r}}}{r^2} + \frac{1}{4\pi\epsilon_0}\frac{Q_{quadr}}{r^3}.$$

This relation represents the multipole expansion for the potential of the system of point charges, truncated to the second order. The first term depends on the total charge Q_{TOT} of the system and behaves as the $1/r$ potential of a point charge; the second term is related to the dipole moment $\mathbf{P}$ of the system and behaves as the $1/r^2$ potential of an electric dipole; the third term depends on the quadrupole moment and decreases as $1/r^3$. The contribution of these terms to the total potential decreases at higher terms.

If the total charge Q_{TOT} is equal to zero, the most relevant term is the dipole term, and if also this term is null, the potential is determined by the quadrupole moment. If also this term is zero the multipole expansion should be extended to include terms of higher order.

For a continuous distribution of charge limited to a volume τ, described by the density $\rho(\mathbf{r'})$, the summation in (2.5), (2.6) and (2.7) has to be replaced by an integral:

$$Q_{TOT} = \int_\tau \rho(\mathbf{r'})\, d\tau'$$

$$\mathbf{P} = \int_\tau \mathbf{r'}\, \rho(\mathbf{r'})\, d\tau'$$

$$Q_{quadr} = \int_\tau \rho(\mathbf{r'}) \left[\frac{3}{2}(\mathbf{r'} \cdot \hat{\mathbf{r}})^2 - \frac{1}{2}(r')^2\right] d\tau' .$$

The measurement of the terms in the potential expansion of a charged structure gives information on the distribution of the charge.

Molecules, in which the centers of the positive and negative charge do not coincide, have a permanent electric dipole moment. The dipole moment for molecules of H_2O in its vapour state is 6.1×10^{-30} C m, for HCl is 3.5×10^{-30} C m and for CO is 0.4×10^{-30} C m. In atoms and molecules, also with a null dipole moment, the action of an external field may separate the centers of positive and negative charges and produce an *induced electric dipole*. Uncharged atoms and molecules can therefore have *dipole-dipole* or *dipole-induced dipole* interactions.

2.4 Properties of the Electric Dipole Moment

When the total charge of a system is null, the dipole moment is independent from the point (or the reference frame) chosen for the calculation of the moment, and the dipole moment becomes an intrinsic feature of the system.[3] Indeed if the vector $\mathbf{a} = \overrightarrow{O'O}$ determines the position of the origin O of the first reference frame in a new frame with origin O' (see Fig. 2.3) we can write: $\mathbf{d'_i} = \mathbf{a} + \mathbf{d}_i$, and the dipole moment $\mathbf{P}'$ is:

$$\mathbf{P}' = \sum_{i=1}^{i=N} q_i \mathbf{d'_i} = \sum_{i=1}^{i=N} q_i (\mathbf{a} + \mathbf{d}_i) = \left(\sum_{i=1}^{i=N} q_i\right) \mathbf{a} + \sum_{i=1}^{i=N} q_i \mathbf{d}_i = Q_{TOT}\, \mathbf{a} + \mathbf{P} = \mathbf{P}$$

because $Q_{TOT} = 0$.

[3]This is a general property of the first non null term in the multipole expansion.

Fig. 2.3 Position of the point charge relative to two different frames

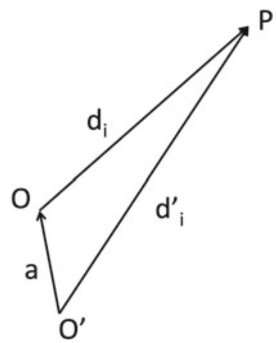
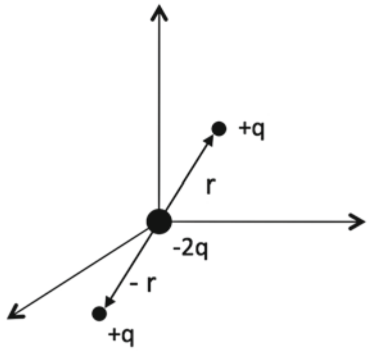


Fig. 2.4 Example of a charge distribution symmetric relative to a point



If a system of charges has a symmetry center, then the dipole moment is null. For instance for the system of three charges in Fig. 2.4:

$$\mathbf{P} = \sum_{i=1}^{i=3} q_i \mathbf{d}_i = q\mathbf{r} + q(-\mathbf{r}) + (-2q)\mathbf{0} = 0\,.$$

2.5 The Quadrupole Tensor

A rank two tensor can be associated to the quadrupole moment.

The Eq. (2.7):

$$Q_{quadr} = \frac{1}{r^2}\sum_{i=1}^{i=N} q_i \left[\frac{3}{2}(\mathbf{d}_i \cdot \mathbf{r})^2 - \frac{1}{2}d_i^2 r^2\right]$$

can be written as:

$$Q_{quadr} = \frac{1}{r^2}\sum_{i=1}^{i=N} q_i \left[\frac{3}{2}\left(\sum_{\mu=1}^{\mu=3} x_\mu d_{i\mu}\right)\cdot\left(\sum_{\nu=1}^{\nu=3} x_\nu d_{i\nu}\right) - \frac{1}{2} d_i^2 \sum_{\mu=1}^{\mu=3}\sum_{\nu=1}^{\nu=3} x_\mu x_\nu \delta_{\mu\nu}\right]$$

where $\delta_{\mu\nu}$ is the Kronecker delta ($\delta_{\mu\nu} = 1$ if $\mu = \nu$, and $\delta_{\mu\nu} = 0$ when $\mu \neq \nu$). Assuming the sum over any index that appears twice in:

$$Q_{quadr} = \frac{1}{r^2}\sum_{i=1}^{i=N} q_i \left[\frac{3}{2} d_{i\mu} x_\mu d_{i\nu} x_\nu - \frac{1}{2} d_i^2 \delta_{\mu\nu} x_\mu x_\nu\right] = \frac{1}{r^2}\sum_{i=1}^{i=N} q_i \left[\frac{3}{2} d_{i\mu} d_{i\nu} - \frac{1}{2} d_i^2 \delta_{\mu\nu}\right] x_\mu x_\nu$$

we can finally write:

$$Q_{quadr} = \frac{1}{r^2} Q_{\mu\nu} x_\mu x_\nu$$

where we have introduced a symmetric rank two tensor, the quadrupole tensor $Q_{\mu\nu}$ given by:

$$Q_{\mu\nu} = \sum_{i=1}^{i=N} q_i \left[\frac{3}{2} d_{i\mu} d_{i\nu} - \frac{1}{2} d_i^2 \delta_{\mu\nu}\right].$$

The quadrupole term in the multipole expansion can be then expressed in the form:

$$\frac{1}{4\pi\epsilon_0}\frac{1}{r^5} Q_{\mu\nu} x_\mu x_\nu \,. \tag{2.8}$$

2.6 A Bidimensional Quadrupole

As an example we can calculate the potential at large distance from the quadrupole[4] formed by four point charges as shown in Fig. 2.5.

The total charge is zero and the dipole moment is null because the point charges are placed symmetrically with respect to the origin. The first non null term in the multipole expansion is the quadrupole term.

[4] Note that the words *dipole*, *quadrupole*, etc. are used in two ways: to describe the charge distribution and secondly to designate the moment of an arbitrary charge distribution.

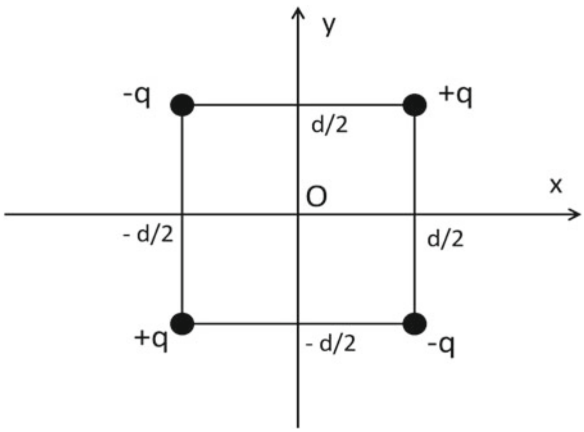


Fig. 2.5 The bidimensional electric quadrupole

For the tensor $Q_{\mu\nu}$ it is easy to find $Q_{xx} = Q_{yy} = Q_{zz} = Q_{xz} = Q_{zx} = Q_{yz} = Q_{zy} = 0$ and the only non null components are $Q_{xy} = Q_{yx} = \frac{3}{2}qd^2$.

Then from (2.8) the potential at a point $P(x, y, z)$ is:

$$V_0(x, y, z) = \frac{3qd^2}{4\pi\epsilon_0} \frac{xy}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}$$

that is zero in any point on the z axis.

Appendix

Higher Order Terms in the Multipole Expansion of the Potential

We have already seen the multipole expansion of the potential limited to the second order. To get the general expression[5] with all the terms of the expansion, we write the potential at a point $P(\mathbf{r}) = P(x, y, z)$ from a continuous charge distribution, limited in space, described by the density $\rho(\mathbf{r'}) = \rho(x', y', z')$. The distance of the point P from the elementary volume $d\tau'$ in the point $\mathbf{r'}$ is:

$$|\mathbf{r} - \mathbf{r'}| = \Delta r = \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}\,.$$

[5]For this expansion see for instance: W.K.H. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, 2nd Ed. (1962), Section 1.7, Addison-Wesley.

We set the origin of the frame inside the volume of the charge distribution or nearby. For a distance r large compared with the dimensions of the volume, we can expand the distance $|\mathbf{r} - \mathbf{r'}|$ as a Taylor series:

$$\frac{1}{|\mathbf{r}-\mathbf{r'}|} = \frac{1}{r} + x'_\alpha \left[\frac{\partial}{\partial\, x'_\alpha}\left(\frac{1}{\Delta r}\right)\right]_{x'_\alpha=0} + \frac{1}{2!}x'_\alpha x'_\beta \left[\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta}\left(\frac{1}{\Delta r}\right)\right]_{x'_\alpha=x'_\beta=0} + \cdots$$

$$\cdots + \frac{1}{n!}x'_\alpha x'_\beta x'_\gamma \ldots \left[\frac{\partial^n}{\partial x'_\alpha \partial x'_\beta \partial x'_\gamma \ldots}\left(\frac{1}{\Delta r}\right)\right]_{x'_\alpha=x'_\beta=x'_\gamma \ldots=0}$$

where we assume the sum over $\alpha, \beta, \gamma, \ldots = 1, 2, 3$, with $x_1 = x,\ x_2 = y,\ x_3 = z$.

The potential due to the charge distribution is:

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_\tau \frac{\rho(\mathbf{r'})}{|\mathbf{r}-\mathbf{r'}|}d\tau' \tag{2.9}$$

and by substituting the expression for $1/|\mathbf{r} - \mathbf{r'}|$ given before, we get:

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{1}{r}\int_\tau \rho(\mathbf{r}\,')d\tau' + \frac{1}{4\pi\epsilon_0}\left[\frac{\partial}{\partial\, x'_\alpha}\left(\frac{1}{\Delta r}\right)\right]_{x'_\alpha=0}\int_\tau x'_\alpha \rho(\mathbf{r}\,')d\tau'$$

$$+\frac{1}{4\pi\epsilon_0}\frac{1}{2!}\left[\frac{\partial^2}{\partial x'_\alpha \partial x'_\beta}\left(\frac{1}{\Delta r}\right)\right]_{x'_\alpha=x'_\beta=0}\int_\tau x'_\alpha x'_\beta\ \rho(\mathbf{r}\,')d\tau' + \cdots$$

$$+\frac{1}{4\pi\epsilon_0}\frac{1}{n!}\left[\frac{\partial^n}{\partial x'_\alpha \partial x'_\beta \partial x'_\gamma \ldots}\left(\frac{1}{\Delta r}\right)\right]_{x'_\alpha=x'_\beta=x'_\gamma \ldots=0}\int_\tau x'_\alpha x'_\beta x'_\gamma\ \ldots\ \rho(\mathbf{r}\,')d\tau'.$$

In this expression we can recognise the terms corresponding to the total charge, to the dipole and to the quadrupole moments that we have already seen and then the general form of the 2^n-pole.

Expansion in Terms of Spherical Harmonics

The multipole expansion of the potential from a charge distribution limited in space, can be also expressed in series of spherical harmonics.[6]

If $\mathbf{r'}$ gives the position of a point inside a sphere of radius R, and $\mathbf{r}$ that of a point outside, we can write for $1/|\mathbf{r} - \mathbf{r'}|$ the expansion in terms of the spherical harmonics $Y_{lm}(\theta, \varphi)$:

[6]For an exhaustive presentation see J.D. Jackson, *Classical Electrodynamics*, cited, Chapters 3 and 4.

$$\frac{1}{|\mathbf{r}-\mathbf{r}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{1}{2l+1} \frac{(r')^l}{r^{l+1}} Y_{lm}^{*}\left(\theta', \varphi'\right) \; Y_{lm}\left(\theta, \varphi\right) \; .$$

If the charge distribution $\rho(r')$ is confined inside the sphere of radius R we can substitute this expansion in (2.9) and we get:

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left\{ 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{1}{2l+1} \frac{1}{r^{l+1}} \left[\int Y_{lm}^{*}\left(\theta', \varphi'\right) (r')^l \rho(\mathbf{r}')\, d\tau' \right] Y_{lm}\left(\theta, \varphi\right) \right\}$$

and introducing the *multipole moments*:

$$q_{lm} = \int Y_{lm}^{*}\left(\theta', \varphi'\right) (r')^l \rho(\mathbf{r}')\, d\tau'$$

we get the expansion:

$$V_0(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \left\{ 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{m=l} \frac{1}{2l+1} \frac{1}{r^{l+1}}\, q_{lm}\, Y_{lm}\left(\theta, \varphi\right) \right\} .$$

Exercise With the formulas for the first spherical harmonics reported below, write the first three terms of the multipole expansion in spherical coordinates and compare with those expressed in cartesian coordinates.

$$l = 0 \qquad Y_{00} = \frac{1}{\sqrt{4\pi}}$$

$$l = 1 \qquad Y_{11} = -\sqrt{\frac{3}{8\pi}}\; \sin\,\theta\; e^{i\varphi}$$

$$Y_{10} = \sqrt{\frac{3}{4\pi}}\; \cos\,\theta$$

$$l = 2 \qquad Y_{22} = \frac{1}{4}\sqrt{\frac{15}{2\pi}}\; \sin^2\theta\; e^{2i\varphi}$$

$$Y_{21} = -\sqrt{\frac{15}{8\pi}}\; \sin\theta\; \cos\theta\; e^{i\varphi}$$

$$Y_{20} = \sqrt{\frac{5}{4\pi}}\; \left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)$$

with the relation:

$$Y_{l-m}(\theta, \varphi) = (-1)^m Y^*_{lm}(\theta, \varphi) \ .$$

Problems

2.1 Find the dipole moment of the system of four point charges q at $(a, 0, 0)$, q at $(0, a, 0)$, $-q$ at $(-a, 0, 0)$ and $-q$ at $(0, -a, 0)$.

2.2 Write the potential for the system of three point charges: two charges $+q$ in the points $(0, 0, a)$ and $(0, 0, -a)$, and a charge $-2q$ in the origin of the frame. Find the approximate form of this potential at distance much larger than a. Compare the result with the potential from the main term in the multipole expansion.

2.3 Two segments cross each other at the origin of the frame and their ends are at the points $(\pm a, 0, 0)$ and $(0, \pm a, 0)$. They have a uniform linear charge distribution of opposite sign. Write the quadrupole term for the potential at a distance $r \gg a$.

2.4 Calculate the quadrupole term of the expansion for the potential from two concentric coplanar rings charged with q and $-q$ and with radii a and b.

2.5 Write the interaction energy of two electric dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ with their centers at distance r.

2.6 Using the result of the previous problem write the force between the electric dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ at distance r. Then consider the force when the dipoles are coplanar oriented normal to their distance and they are parallel or antiparallel. Determine also the force when the dipoles are on the same line and oriented in the same or in the opposite direction.

2.7 Two coplanar electric dipoles have their centers a fixed distance r apart. Say θ and θ' the angles the dipoles make with the line joining their centers and show that if θ is fixed, they are at equilibrium when

$$\tan\theta' = -\frac{1}{2}\tan\theta \ .$$

Solutions

2.1 The dipole moment of the system has components:

$$p_x = \sum_1^4 q_i x_i = 2qa \qquad p_y = \sum_1^4 q_i y_i = 2qa \qquad p_z = \sum_1^4 q_i z_i = 0 \ .$$

The dipole moment is $\mathbf{p} = (2qa, 2qa, 0)$ with module $p = 2q\sqrt{2}a$. This is the moment of an elementary dipole with opposite charges $2q$ located in the centers of the positive and the negative charges which are distant $\sqrt{2}a$.

2.2 In spherical coordinates the potential depends only on the distance r from the origin and on the angle θ. Adding the potentials from the three charges we have:

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[-\frac{2q}{r} + \frac{q}{[r^2 - 2ra\cos\theta + a^2]^{\frac{1}{2}}} + \frac{q}{[r^2 + 2ra\cos\theta + a^2]^{\frac{1}{2}}}\right]$$

and expanding in power series as in (2.4) we find:

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\frac{qa^2}{r^3}\left(3\cos^2\theta - 1\right) . \tag{2.10}$$

In the multipole expansion for the system of the three charges the first non null term is the quadrupole moment. It is easy to find the components: $Q_{xz} = Q_{yz} = Q_{xy} = 0$, $Q_{xx} = Q_{yy} = -qa^2$ and $Q_{zz} = 2qa^2$ so that the potential is:

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{1}{r^5}\left(Q_{xx}x^2 + Q_{yy}y^2 + Q_{zz}z^2\right)$$

$$= \frac{1}{4\pi\epsilon_0}\frac{qa^2}{r^5}\left(2z^2 - x^2 - y^2\right)$$

that is the formula (2.10) written in cartesian coordinates.

2.3 For the given charge distribution it is easy to see that $Q_{xz} = Q_{yz} = Q_{xy} = 0$ and by simple calculations we find:

$$Q_{xx} = \frac{1}{3}qa^2 \qquad Q_{yy} = -\frac{1}{3}qa^2 \qquad Q_{zz} = 0$$

so that the quadrupole potential is:

$$V(x,y,z) = \frac{1}{4\pi\epsilon_0}\frac{qa^2}{3}\frac{(x^2 - y^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}} .$$

2.4 We consider the two rings on the plane $z = 0$ with their centers in the origin. The total charge of the system is zero and, for the symmetry of the charge distribution with respect to the origin, also the dipole moment is null. The quadrupole term is the first non null term. It is evident that $Q_{xz} = Q_{yz} = 0$ and by simple integrals we can get:

$$Q_{xx} = Q_{yy} = \frac{q}{4}(a^2 - b^2) \qquad Q_{zz} = \frac{q}{2}(b^2 - a^2) \qquad Q_{xy} = 0$$

so that the first term of the potential expansion is:

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\frac{q(a^2 - b^2)}{4}\frac{(x^2 + y^2 - 2z^2)}{(x^2 + y^2 + z^2)^{\frac{5}{2}}}.$$

2.5 The interaction energy of the two dipoles is equal to the potential energy of $\mathbf{p}_2$ in the field of $\mathbf{p}_1$. Saying $\mathbf{r}$ the vector from the center of $\mathbf{p}_1$ to that of $\mathbf{p}_2$, we can write:

$$U_{21} = -\mathbf{p}_2 \cdot \mathbf{E}_1(\mathbf{r}) = \mathbf{p}_2 \cdot \nabla V_1 = \mathbf{p}_2 \cdot \nabla\left(\frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}_1 \cdot \mathbf{r}}{r^3}\right) = \frac{1}{4\pi\epsilon_0}\mathbf{p}_2 \cdot \left[\frac{\nabla(\mathbf{p}_1 \cdot \mathbf{r})}{r^3} + (\mathbf{p}_1 \cdot \mathbf{r})\nabla\frac{1}{r^3}\right]$$

and since:

$$\nabla(\mathbf{p}_1 \cdot \mathbf{r}) = \mathbf{p}_1 \qquad \nabla\frac{1}{r^3} = -3\frac{\mathbf{r}}{r^5} \qquad (\text{note that } \nabla\frac{1}{r^n} = -n\frac{\mathbf{r}}{r^{n+2}})$$

we get:

$$U_{21} = U_{12} = \frac{1}{4\pi\epsilon_0}\left[\frac{\mathbf{p}_1 \cdot \mathbf{p}_2}{r^3} - 3\frac{(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})}{r^5}\right] \tag{2.11}$$

that is symmetric in $\mathbf{p}_1$ and $\mathbf{p}_2$.

2.6 From the solution of the previous problem and from the formula (2.2) for the force on a dipole in an electric field we get:

$$\mathbf{F}_2 = -\nabla U_{21} = \nabla(\mathbf{p}_2 \cdot \mathbf{E}_1(\mathbf{r}))$$

$$= -\frac{1}{4\pi\epsilon_0}\left[(\mathbf{p}_1 \cdot \mathbf{p}_2)\nabla\frac{1}{r^3} - 3(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})\nabla\frac{1}{r^5} - 3\frac{\nabla[(\mathbf{p}_1 \cdot \mathbf{r})](\mathbf{p}_2 \cdot \mathbf{r})]}{r^5}\right]$$

and then:

$$\mathbf{F}_2 = \frac{1}{4\pi\epsilon_0}\left[3\frac{(\mathbf{p}_1 \cdot \mathbf{p}_2)\mathbf{r} + \mathbf{p}_1(\mathbf{p}_2 \cdot \mathbf{r}) + (\mathbf{p}_1 \cdot \mathbf{r})\mathbf{p}_2}{r^5} - 15\frac{(\mathbf{p}_1 \cdot \mathbf{r})(\mathbf{p}_2 \cdot \mathbf{r})\mathbf{r}}{r^7}\right].$$

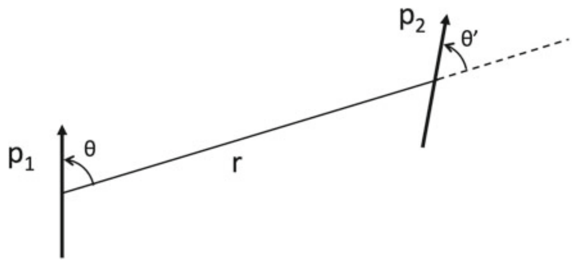
This formula is symmetric in $\mathbf{p}_1$ and $\mathbf{p}_2$ but $\mathbf{r}$ is directed from $\mathbf{p}_1$ to $\mathbf{p}_2$ so we get $\mathbf{F}_1$ changing $\mathbf{r}$ to $-\mathbf{r}$ and we have $\mathbf{F}_1 = -\mathbf{F}_2$ as expected.

For two coplanar parallel dipoles normal to the line joining their centres:

$$\text{with same direction}(\uparrow\ \uparrow): \mathbf{F}_2 = \frac{3}{4\pi\epsilon_0}\frac{p_1 p_2}{r^4}\hat{\mathbf{r}} \quad \text{a repulsive force}$$

$$\text{with opposite direction}(\uparrow\ \downarrow): \quad \mathbf{F}_2 = -\frac{3}{4\pi\epsilon_0}\frac{p_1 p_2}{r^4}\hat{\mathbf{r}} \quad \text{an attractive force}$$

Fig. 2.6 Coplanar dipoles $\mathbf{p}_1$ and $\mathbf{p}_2$ with their centers at fixed distance r and orientations θ and θ' with respect to $\mathbf{r}$



for the dipoles on the same line

$$\text{with same direction}(\rightarrow \quad \rightarrow) : \mathbf{F}_2 = -\frac{6}{4\pi\epsilon_0}\frac{p_1 p_2}{r^4}\hat{\mathbf{r}} \quad \text{an attractive force}$$

$$\text{with opposite direction}(\rightarrow \quad \leftarrow) : \mathbf{F}_2 = \frac{6}{4\pi\epsilon_0}\frac{p_1 p_2}{r^4}\hat{\mathbf{r}} \quad \text{a repulsive force.}$$

2.7 For the two coplanar dipoles shown in Fig. 2.6 the interaction energy (2.11) becomes:

$$U_{int} = \frac{1}{4\pi\epsilon_0}\frac{p_1 p_2}{r^3}\Big[\cos(\theta' - \theta) - 3\cos\theta\cos\theta'\Big].$$

At fixed θ we find the minimum of this energy solving the equation:

$$\frac{\partial U}{\partial \theta'} = \frac{1}{4\pi\epsilon_0}\frac{p_1 p_2}{r^3}\Big[-\sin(\theta' - \theta) + 3\cos\theta\sin\theta'\Big] = 0\,.$$

The solution is:

$$\tan\theta' = -\frac{1}{2}\tan\theta$$

$$\text{with the condition} \quad \frac{\partial^2 U}{\partial \theta'^2} = \frac{1}{4\pi\epsilon_0}\frac{p_1 p_2}{r^3}\Big[2\cos\theta\cos\theta' - \sin\theta\sin\theta'\Big] > 0\,.$$

For $\theta = \pi/2$ the minimum is at $\theta' = -\pi/2$, for $\theta = 0$ at $\theta' = 0$ and for $\theta = \pi$ at $\theta' = -\pi$.

Chapter 3
The Method of Image Charges

The potentials and the electric field in the presence of point charges and conductors are determined solving the Poisson equation and sometimes this may be very difficult. However in some particular configurations the equipotential surfaces of the conductors can be reproduced adding to the real point charges some *image* charges inside the conductors.[1] Using this technique the solution can be easily written as the sum of well known potentials of point charges. After the introduction of the *image* charges method, some relevant examples will be presented: (i) the point charge in front of a conductive plane, (ii) the point charge near a conductive sphere, (iii) a conductive sphere immersed in a uniform external electric field, (iv) a charged wire parallel to a cylindrical conductor. Additional examples will be available in the problems.

3.1 The Method of Image Charges

In a system of point charges and conductors at fixed potentials, Poisson equation has to be solved using as boundary conditions the potentials on the conductor surfaces.

Sometimes by adding to the system of real charges some point charges, called *image charges*, placed in suitable positions inside the conductors, the sum of the potentials due to both real and *image charges* can generate equipotential surfaces with the same shapes and potentials of the conductors.

Outside the space of the conductors the Poisson equation for the system of real and *image* charges depends on the real charges and has the same potentials of the real problem boundary conditions at the geometrical surfaces of the conductors. In the two cases therefore we have the same equation with the same conditions on the

[1]For the *image charges method* see also: J.D. Jackson, *Classical Electrodynamics*, cited, Sects. 2.1, 2; R.P.Feynman, R.B.Leighton, M.Sands, *The Feynman Lectures on Physics*, Vol. II, Sects. 6.7–9; L.D.Landau - E.M.Lifšits, *Electrodynamics of continuous media*, Chap. I, Sect. 3, with problems; D.J. Griffiths, *Introduction to Electrodinamics*, 4th Ed. (2013), Sect. 3.2, Pearson Prentice Hall. The method is examined also in J.C. Maxwell, *A treatise on Electricity & Magnetism*, Vol. I, Chap. XI.

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9_3

surfaces of the conductors, thus, given the uniqueness of the solution, the solution for the point charges (real and *image*) is also the solution of the problem with the real charges and the conductors. Outside the conductors the potential is then given by the sum of the potentials due to real and *image* charges. Inside the conductors voltages are constant and equal to the values on the surfaces.

It is worth noting that, outside the conductors, the sum of the potentials of the real charges represents the particular solution of the Poisson equation while the sum of the potentials from *image* charges is the solution of the associated Laplace equation.

3.2 Point Charge and Conductive Plane

A point charge q at a distance d from an infinite grounded conductive plane, shown in Fig. 3.1, represents the simplest example of the image charges method. The z axis can be defined as the line passing on the point charge and perpendicular to the plane located at $z = 0$. The Poisson equation for the point charge configuration has to be solved with boundary condition the null potential on the plane and as a consequence the electric field perpendicular to the plane.

Consider instead the electric field given by a charge q located at $(0, 0, d)$ and by the image charge $-q$ at $(0, 0, -d)$, mirror image of the charge q with respect to the xy plane.

The potential by the two charges at the point $P(x, y, z)$ is:

$$V_0(x, y, z) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{r^2 + (z - d)^2}} - \frac{1}{\sqrt{r^2 + (z + d)^2}}\right\} \tag{3.1}$$

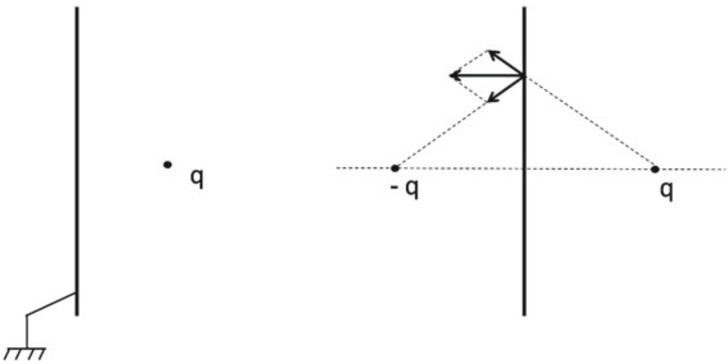


Fig. 3.1 Point charge near an infinite conductive plane at ground. Vector sum of the fields from the two point charges

where $r^2 = x^2 + y^2$ and the components of the electric fields are:

$$E_{0x}(x, y, z) = -\nabla_x V = \frac{q}{4\pi\epsilon_0}\left\{\frac{x}{(r^2 + (z-d)^2)^{\frac{3}{2}}} - \frac{x}{(r^2 + (z+d)^2)^{\frac{2}{3}}}\right\}$$

$$E_{0y}(x, y, z) = -\nabla_y V = \frac{q}{4\pi\epsilon_0}\left\{\frac{y}{(r^2 + (z-d)^2)^{\frac{3}{2}}} - \frac{y}{(r^2 + (z+d)^2)^{\frac{2}{3}}}\right\}$$

$$E_{0z}(x, y, z) = -\nabla_y V = \frac{q}{4\pi\epsilon_0}\left\{\frac{z-d}{(r^2 + (z-d)^2)^{\frac{3}{2}}} - \frac{z+d}{(r^2 + (z+d)^2)^{\frac{2}{3}}}\right\}.$$

The potential and the field components on the plane $z = 0$ are:

$$V_0(x, y, z=0) = 0 \qquad E_{0x}(x, y, z=0) = 0 \qquad E_{0y}(x, y, z=0) = 0$$

$$E_{0z}(x, y, z=0) = -\frac{q}{4\pi\epsilon_0}\frac{2d}{[r^2 + d^2]^{\frac{3}{2}}}.$$

As expected by symmetry the potential is null and the field is normal to the plane. The z-component of the field can be written in the form:

$$E_{0z}(x, y, z=0) = -2\frac{q}{4\pi\epsilon_0}\frac{1}{[r^2 + d^2]}\frac{d}{[r^2 + d^2]^{\frac{1}{2}}} = -2E^* \cos\theta$$

where the two charge contributions are evident (see Fig. 3.1) given the strength of the fields of the two charges in a point on the plane:

$$E^* = \frac{q}{4\pi\epsilon_0}\frac{1}{[r^2 + d^2]}.$$

The potential of the two charges for $z > 0$ is a solution of the Poisson equation and on the plane we have the same configuration required for the point charge and the conductive plane: null potential and normal field. Then the potential $V_0(x, y, z)$, given in (3.1), for $z > 0$ is also the solution for the problem of the point charge near the conductive plane.

From the Coulomb theorem the charge density induced on the conductive plane is:

$$\sigma(r) = \epsilon_0 E_{0z}(r, z=0) = -\frac{q}{2\pi}\frac{d}{[r^2 + d^2]^{\frac{3}{2}}} \tag{3.2}$$

and its integral:

$$\int_0^\infty \sigma(r)\, 2\pi r dr = -q\, d \left.\frac{1}{(d^2 + r^2)^{\frac{1}{2}}}\right|_0^\infty = -q$$

is equal to $-q$, the image charge located at $z = -d$, as expected from the Gauss theorem: the total flux out of the point charge has to enter the plane.

In the position of the charge q, the electrostatic field from the charge induced on the plane is equal to the field from the image charge, consequently the charge is attracted by the plane with a force:

$$F = \frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\,. \tag{3.3}$$

The electrostatic energy for the system of the point charge and the conductive plane is limited only at $z > 0$ and therefore is one half of the interaction energy of the two opposite charged points.

3.3 Point Charge Near a Conducting Sphere

An interesting case is that of a point charge q at a distance d from the center of a conductive sphere with radius $r < d$.

First we consider a grounded sphere as in Fig. 3.2. A charge is induced on the spherical surface and the field outside the sphere is determined by both the point charge and the induced charge distribution. To reproduce this field, we can introduce an image charge q' inside the sphere located, by symmetry, on the line from the point charge to the center of the sphere and at a distance x from the center of the sphere.[2]

The potential from the charges q and q' has to be zero on the spherical surface:

$$V_0(r) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{R_1} + \frac{q'}{R_2}\right\} = 0\,.$$

[2]For a more elegant solution we remind that the spherical surface is the locus of all points for which the distances from two given points are in a constant ratio.
Defining R_1 and R_2 the distances of the points on the sphere from the point charges q and q', the relation for a null potential on the spherical surface is:

$$V_0(r) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right) = 0$$

from which the condition:

$$\frac{R_2}{R_1} = -\frac{q'}{q} = const$$

equal, as pointed out, to that for a spherical surface. Using the ratios of the two distances from the points A and B in Fig. 3.2 and given the radius r of the sphere, we can find the distance x of the charge q' from the center of the sphere:

$$\frac{R_2(A)}{R_1(A)} = \frac{R_2(B)}{R_1(B)} \qquad \frac{r+x}{d+r} = \frac{r-x}{d-r} \qquad \frac{R_2}{R_1} = \frac{r}{d} \qquad x = \frac{r^2}{d}$$

and the value of q':

$$q' = -q\,\frac{r}{d}\,.$$

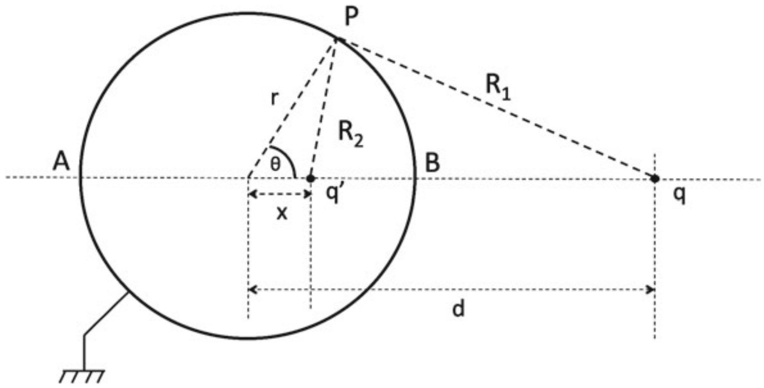


Fig. 3.2 Point charge near a grounded conductive sphere

From $R_1^2 = d^2 + r^2 - 2rd\cos\theta$ and $R_2^2 = r^2 + x^2 - 2xr\cos\theta$:

$$V_0(r) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q}{[d^2 + r^2 - 2rd\cos\theta]^{\frac{1}{2}}} + \frac{q'}{[r^2 + x^2 - 2xr\cos\theta]^{\frac{1}{2}}}\right\} = 0$$

and replacing $q' = -yq$ we have:

$$V_0(r) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{[d^2 + r^2 - 2rd\cos\theta]^{\frac{1}{2}}} - \frac{y}{[r^2 + x^2 - 2xr\cos\theta]^{\frac{1}{2}}}\right\} = 0$$

which gives:

$$(y^2d^2 + y^2r^2 - r^2 - d^2) = 2r(y^2d - x)\cos\theta\,.$$

This relation has to be verified for any $\cos\theta$ value, then the two expressions in parentheses have to be zero and therefore $y^2 = \frac{x}{d}$ and for x the two solutions: $x_1 = d$ and $x_2 = \frac{r^2}{d}$. The solution $x_1 = d$ adds a charge $q' = -q$ to the charge q with a null potential everywhere, which represents a trivial solution. For the second solution an image charge $q' = -q\frac{r}{d}$ is placed at a distance $x = \frac{r^2}{d}$ from the center of the sphere.

Outside the sphere the potential will be the sum of the potentials from both the charge q and the image charge q', with q' representing the total charge induced on the sphere. Inside the conductor the potential will be zero as on the surface.

If the sphere is insulated and initially uncharged, after the point charge q is brought near by, the total charge on the sphere has still to be zero. With the charge q', a charge $q'' = -q'$ has also to be present on the sphere. To have an equipotential surface this charge has to be uniformly distributed on the surface and the potential is that from the same charge located in the center of the sphere:

$$V_0 = \frac{1}{4\pi\epsilon_0}\,\frac{q''}{r}\,.$$

If on the isolated sphere is already present a charge Q and a charge q' is induced, a charge $q'' = Q - q'$ has to be uniformly distributed on the surface and determines the potential as in the previous configuration.

For a conductive sphere at the potential V_0, fixed by a generator, a second charge $q'' = 4\pi\epsilon_0 r V_0$, located in the center, has to be added to the image charge q'.

In all these configurations the force between the point charge and the conductive sphere is equal to the sum of the forces between the charge q and the charges q' and q''.

3.4 Conducting Sphere in a Uniform Electric Field

Two point charges of opposite sign produce a uniform electric field close to their midpoint. To solve the case of a sphere of radius R in a uniform electric field E_0, the conductive sphere is placed between two opposite point charges at large distance with respect to the radius of the sphere and then the point charges are moved to an infinite distance.

Consider the center of the sphere in the origin of the axis, a charge q on the z axis at the distance $-d$ and symmetrically a charge $-q$ at distance d as in Fig. 3.3. The two charges produce an induced charge on the spherical surface. From the result for a sphere in presence of a point charge, we know that the effect of the induced charge can be taken into account adding two image charges q_1 and q_2 in the points z_1 and z_2 inside the sphere, with:

$$q_1 = -q\,\frac{R}{d} \qquad z_1 = -\frac{R^2}{d} \qquad q_2 = q\,\frac{R}{d} \qquad z_2 = \frac{R^2}{d}\,.$$

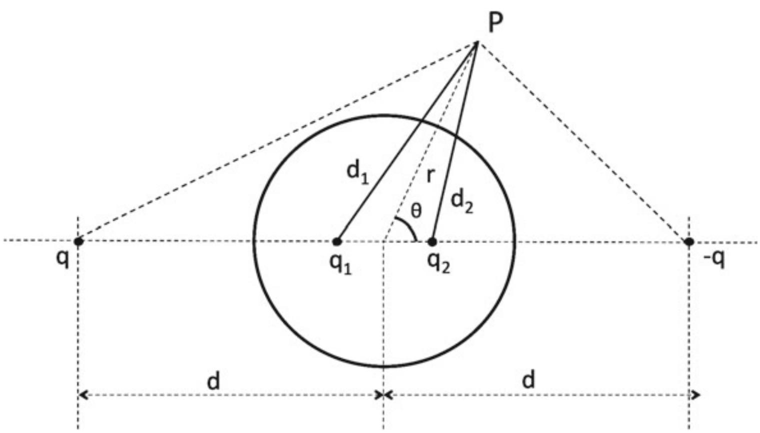


Fig. 3.3 Point charges to simulate a sphere in a uniform electric field

In the point P outside the sphere, at distance r from the origin and at an angle θ with the z axis (see the Fig. 3.3), the potential V is the sum of the potential V' from the charges at large distances and of the potential V'' from the two image charges.

In the limit $d \to \infty$ the two charges have to produce a uniform field E_0 so that their potential has to be:

$$V' = -E_0 z = -E_0\, r\, \cos\theta\,.$$

and the two image charges q_1 and q_2 become an electric dipole of infinitesimal dimension with dipole moment:

$$D = \left(-q\frac{R}{d}\right)\left(-\frac{R^2}{d}\right) + \left(q\frac{R}{d}\right)\left(\frac{R^2}{d}\right) = 2q\frac{R^3}{d^2}$$

where the charge q has to be determined.

The potential V'' from this dipole is:

$$V'' = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{D}\cdot\mathbf{r}}{r^3} = \frac{1}{4\pi\epsilon_0}\,2\,q\,\frac{R^3}{d^2}\,\frac{\cos\theta}{r^2}$$

so that the total potential is:

$$V(r,\theta) = -E_0\, r\, \cos\theta + \frac{1}{4\pi\epsilon_0}\,2\,q\,\frac{R^3}{d^2}\,\frac{\cos\theta}{r^2}.$$

Requiring a null potential on the surface of the sphere (at $r = R$), we get for q and D:

$$q = 2\pi\epsilon_0 d^2 E_0 \qquad D = 4\pi\epsilon_0 R^3 E_0$$

and the total potential[3] becomes:

---

[3]The potential can also be calculated directly as the sum of the four potentials of the point charges. The potential V' from the two charges at large distances is:

$$V_0'(r,\theta) = \frac{q}{4\pi\epsilon_0}\left\{\frac{1}{[d^2+r^2+2rd\cos\theta]^{\frac{1}{2}}} - \frac{1}{[d^2+r^2-2rd\cos\theta]^{\frac{1}{2}}}\right\}$$

$$V_0'(r,\theta) = \frac{q}{4\pi\epsilon_0 d}\left\{\frac{1}{\left[1+(\frac{r}{d})^2+\frac{2r}{d}\cos\theta\right]^{\frac{1}{2}}} - \frac{1}{\left[1+(\frac{r}{d})^2-\frac{2r}{d}\cos\theta\right]^{\frac{1}{2}}}\right\}$$

Moving the two charges to an infinite distance, in the limit $d \to \infty$, $(\frac{r}{d})$ is infinitesimal and the potential can be expanded in series:

$$V_0' = \frac{q}{4\pi\epsilon_0 d}\left\{1 - \frac{1}{2}\left[\left(\frac{r}{d}\right)^2 + \frac{2r}{d}\cos\theta\right] - 1 + \frac{1}{2}\left[\left(\frac{r}{d}\right)^2 - \frac{2r}{d}\cos\theta\right]\right\}$$

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

that is null on the surface of the sphere ($r = R$).

In spherical coordinates the components of the electric field are:

$$E_{0r} = -\frac{\partial V}{\partial r} = E_0\left(1 + 2\frac{R^3}{r^3}\right)\cos\theta$$

(Footnote 3 continued)
and becomes:

$$V_0' = -\frac{q}{2\pi\epsilon_0}\left\{\frac{r}{d^2}\cos\theta\right\} = -\frac{q}{2\pi\epsilon_0}\frac{z}{d^2}$$

where $z = r\cos\theta$.

From the electric field E_0:

$$E_0 = -\frac{\partial V'}{\partial z} = \frac{q}{2\pi\epsilon_0}\frac{1}{d^2}$$

we get the value of the charge q to get a field E_0: $q = 2\pi\epsilon_0 d^2 E_0$
and the potential is

$$V_0' = -E_0 z = -E_0 r\cos\theta.$$

We consider now the potential V'' from the two image charges (see Fig. 3.3). The distances d_1 and d_2 of the point P from the two charges, are:

$$d_1 = \left[\frac{1}{d^2}\left(r^2d^2 + R^4 + 2rR^2d\ \cos\theta\right)\right]^{\frac{1}{2}} \quad d_2 = \left[\frac{1}{d^2}\left(r^2d^2 + R^4 - 2rR^2d\ \cos\theta\right)\right]^{\frac{1}{2}}.$$

So that the potential is:

$$V_0''(r,\theta) = -\frac{qR}{4\pi\epsilon_0}\left\{\frac{1}{[r^2d^2 + R^4 + 2rR^2d\ \cos\theta]^{\frac{1}{2}}} - \frac{1}{[r^2d^2 + R^4 - 2rR^2d\ \cos\theta]^{\frac{1}{2}}}\right\}$$

$$V_0''(r,\theta) = -\frac{qR}{4\pi\epsilon_0 dr}\left\{\frac{1}{\left[1 + \frac{R^4}{r^2d^2} + \frac{2R^2}{rd}\cos\theta\right]^{\frac{1}{2}}} - \frac{1}{\left[1 + \frac{R^4}{r^2d^2} - \frac{2R^2}{rd}\cos\theta\right]^{\frac{1}{2}}}\right\}.$$

In the limit $d \to \infty$, $(\frac{R}{d})$ becomes infinitesimal and the potential expanded in power series is:

$$V_0''(r,\theta) = -\frac{qR}{4\pi\epsilon_0 dr}\left\{1 - \frac{1}{2}\left[\frac{R^4}{r^2d^2} + \frac{2R^2}{rd}\cos\theta\right] - 1 + \frac{1}{2}\left[\frac{R^4}{r^2d^2} - \frac{2R^2}{rd}\cos\theta\right]\right\}$$

that can be simplified in the form:

$$V_0''(r,\theta) = \frac{q}{2\pi\epsilon_0}\left\{\frac{R^3}{d^2r^2}\cos\theta\right\} = E_0\frac{R^3}{r^2}\cos\theta$$

where we have used the q found before.

The total potential at the point P is:

$$V(r,\theta) = V' + V'' = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta\,.$$

$$E_{0\theta} = -\frac{1}{r}\frac{\partial V}{\partial \theta} = -E_0 \left(1 - \frac{R^3}{r^3}\right) \sin\theta$$

$$E_{0\varphi} = -\frac{1}{r\sin\theta}\frac{\partial V}{\partial \varphi} = 0\,.$$

For $r \to R$ $E_{0\theta} = 0$ and only the radial component $E_{0r} = 3E_0 \cos\theta$ is non null, as expected for an electric field normal to the conductive spherical surface. From this component and the Coulomb theorem, we get the density of the induced charge on the spherical surface:

$$\sigma(\theta) = 3\,\epsilon_0\,E_0 \cos\theta\,.$$

3.5 A Charged Wire Near a Cylindrical Conductor

An infinite wire, with linear charge density λ is placed, at a distance d, parallel to the axis of an infinite charged cylindrical conductor of radius $R < d$ with a linear density $-\lambda$.

We can imagine to write the potential outside the cylinder as the superposition of the potentials from the real wire and from an *image wire*, with charge $-\lambda$, located at a distance $x < R$ from the axis of the cylinder, as in Fig. 3.4. The potentials $V(r_1)$ and $V(r_2)$ due to the two wires with respect to the potential, that we can assume zero, on a line[4] at the same distance r_0 from both wires, are:

$$V(r_1) = -\frac{\lambda}{2\pi\epsilon_0}\log\frac{r_1}{r_0} \qquad V(r_2) = \frac{\lambda}{2\pi\epsilon_0}\log\frac{r_2}{r_0}$$

$$\text{where } r_1^2 = r^2 + d^2 - 2rd\cos\theta \text{ and } r_2^2 = r^2 + x^2 - 2rx\cos\theta\,.$$

At the point $\mathbf{r}(r, \theta)$ the potential is:

$$V(r,\theta) = V(r_1) + V(r_2) = -\frac{\lambda}{4\pi\epsilon_0}\log\frac{r_1^2}{r_2^2} = -\frac{\lambda}{2\pi\epsilon_0}\log\frac{r_1}{r_2}\,. \tag{3.4}$$

The lateral surface of the cylinder has to be equipotential and the electric field has to be normal to this surface, that is with a radial direction with respect to the axis of the cylinder. This means that the component $E_\theta(R,\theta)$ has to be zero for any value of θ.

[4] This line can be any line, parallel to the two wires, that lies on the plane with respect to which the two wires are symmetrical.

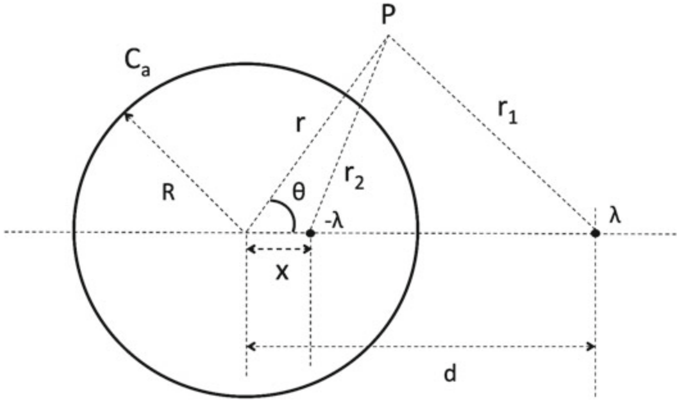


Fig. 3.4 Charged wire near a cylindrical conductor

The component $E_\theta(r, \theta)$ is:

$$E_\theta(r,\theta) = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{d\ \sin\theta}{r^2 + d^2 - 2rd\ \cos\theta} - \frac{x\ \sin\theta}{r^2 + x^2 - 2rx\ \cos\theta}\right]$$

and in the limit $r \to R$, from $E_\theta(R, \theta) = 0$ we get the equation:

$$[R^2 + x^2]d - [R^2 + d^2]x = 0$$

with the solutions:

$$x = d \quad \text{(a solution of no interest) and}\, x = \frac{R^2}{d}\,.$$

By substituting in (3.4) this last value for x, we get the potential of the cylinder:

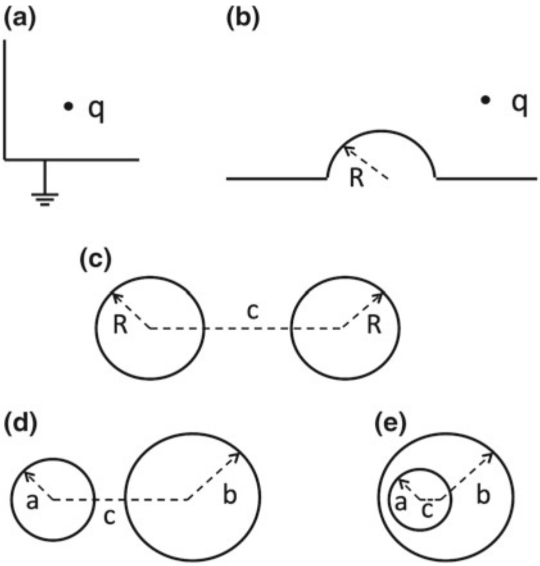
$$V(R) = -\frac{\lambda}{4\pi\epsilon_0}\log\frac{r_1^2}{r_2^2} = -\frac{\lambda}{2\pi\epsilon_0}\log\frac{d}{R}\,.$$

We have to underline that as a general result the two wires with opposite charge density produce cylindrical equipotential surfaces.

Problems

3.1 For a point charge q at distance d from the infinite conductive plane at ground, find: (a) the force acting on the charge by integrating the force from the distribution $\sigma(r)$ given by (3.2) and compare the result with (3.3); (b) the work to move the charge q to infinity and compare this work with the potential energy between the charge q and its image charge.

Fig. 3.5 Figure **a** for the Problem 2, **b** for 3, **c** for 9, **d** for 10, **e** for 11



3.2 Determine the image charges useful to describe the electrostatic field for a point charge q near the corner between two perpendicular conductive planes at ground (see Fig. 3.5a). Write the potential, the field and the force acting on the charge.

3.3 A hemispherical bump, of radius R, is present on an infinite conductive plane at ground. A charge q is located outside of the hemisphere as in Fig. 3.5b. Find the charge images to describe the field.

3.4 Consider a small dipole at distance d from an infinite conductive plane at ground. Find the force acting on the dipole. Determine the force when the dipole is parallel or normal oriented relative to the plane. Give also the torque acting on the dipole at a fixed distance from the plane.

3.5 Consider a point charge q at distance a from the center of thin conductive spherical surface, of inner radius $R > a$, at ground. Calculate the potential and the induced charge density on the inner surface of the conductor.

3.6 A charged wire, with linear density λ, is parallel to a conductive grounded plane. Consider how to write the potential in the half space with the wire.

3.7 An infinite wire with linear charge density λ, is parallel to the axis of a conductive cylindrical surface of infinite length, with linear charge $-\lambda$. The radius of the conductor is R and the distance of the wire from its axis is $a < R$. Say how to find the electric field inside and outside of the cylindrical surface.

3.8 An infinite uncharged conductive cylinder is located in a uniform electric field E_0 perpendicular to its axis. Write the electric field and the charge density induced on the cylinder.

3.9 Find the capacitance per unit length of a capacitor formed by two infinite cylindrical conductors of radius R with parallel axes at distance c ($c > 2R$) as in Fig. 3.5c. The configuration studied here is a particular case of the next problem.
Result:

$$C = \frac{\pi\epsilon_0}{\log \frac{c+\sqrt{c^2-4R^2}}{2R}} = \frac{2\pi\epsilon_0}{\log\left(\xi + \sqrt{\xi^2-1}\right)} = \frac{2\pi\epsilon_0}{\operatorname{acosh}\xi}$$

$$\text{where:}\quad \xi = \left(\frac{c^2}{2R^2} - 1\right). \quad \text{Note that:}\quad \operatorname{acosh}\xi = \ln\left(\xi + \sqrt{\xi^2-1}\right).$$

3.10 Determine the capacitance of two conductive cylinders of radii a and b with their parallel axes at a distance $c > a + b$ (see Fig. 3.5d).
Result:

$$C = \frac{2\pi\epsilon_0}{\operatorname{acosh}\xi} \quad \text{where} \quad \xi = \frac{c^2 - a^2 - b^2}{2ab}.$$

3.11 Calculate the capacitance of a condenser with the two cylindrical electrodes of radii a and b with parallel axes at a distance $c < |a - b|$ (see Fig. 3.5e). Note that $c = 0$ for the usual coaxial cylindrical condenser.
Result:

$$C = \frac{2\pi\epsilon_0}{\operatorname{acosh}\xi} \quad \text{where} \quad \xi = \frac{a^2 + b^2 - c^2}{2ab}.$$

Solutions

3.1 **(a)** We have to consider only the component of the force along the direction normal to the plane. An elementary ring of radius between r and $r + dr$ attracts the charge with a force:

$$dF = \frac{1}{4\pi\epsilon_0}\frac{q\sigma(r)2\pi r dr}{(d^2+r^2)}\frac{d}{(d^2+r^2)^{\frac{1}{2}}}.$$

By using the expression (3.2) for $\sigma(r)$ and integrating on the whole plane we find:

$$F = -\frac{q^2}{4\pi\epsilon_0}\frac{d^2}{2}\int_0^\infty \frac{dr^2}{(r^2+d^2)^3} = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}$$

that is the force (3.3) between the charge q and its image charge.
(b) to move the charge to an infinite distance we have to balance with an external force F_{ext} the force F applied on the charge by the conductive plane. The work is:

$$W = \int_d^\infty F_{ext} dr = \frac{q^2}{4\pi\epsilon_0}\int_d^\infty \frac{1}{(2x)^2}dx = \frac{1}{2}\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)}.$$

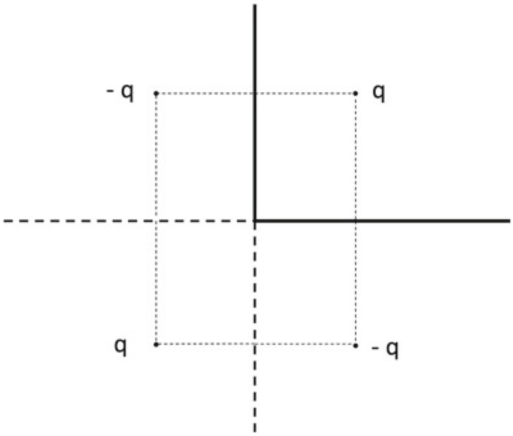


Fig. 3.6 Point charge near a conductive corner

This energy is one half the interaction energy between the charge and its image charge. As pointed out in the text the electrostatic energy for the charge near the conductive plane is one half of that of two opposite charge at the same distance.

3.2 Reminding the case of a point charge near a conductive plane, we can realize that by setting three image charges as in Fig. 3.6, the orthogonal planes are at a null potential. Then the conductive planes which form the corner are at ground. The potential outside of the conductive corner is the sum of the potentials of the four point charges. The field is the vector sum of the fields of the four charges and the force on the charge q is the vector sum of the forces from the three images charges.

3.3 For a point charge near a conductive sphere, adding a image charge $q' = -q(R/d)$, at distance R^2/d from the center, between the charge q and the center of the spherical surface, the potential on the sphere is null. If we add this charge the hemispherical bump that lies on the sphere is at ground. Adding two more image charges $q'' = -q$ and $q''' = -q'$ symmetric to q and q' relative to the plane, both the bump and the plane are at a null potential as demanded in the problem (see Fig. 3.7).

3.4 We have to introduce an image dipole symmetric to the dipole with respect to the plane as in Fig. 3.8. The two dipoles are coplanar and their interaction energy was found in Problem 2.7. Now $\theta' = -\theta$, $p = p_{imag}$ and $r = 2d$ so the interaction energy is:

$$U = \frac{1}{4\pi\epsilon_0}\frac{p^2}{(2d)^3}\left[\cos 2\theta - 3\cos^2\theta\right] = -\frac{1}{4\pi\epsilon_0}\frac{p^2}{(2d)^3}\left[1 + \cos^2\theta\right].$$

The force on the dipole is:

$$F = -\frac{\partial U}{\partial(2d)} = -\frac{3p^2}{4\pi\epsilon_0}\left[\frac{1+\cos^2\theta}{(2d)^4}\right]$$

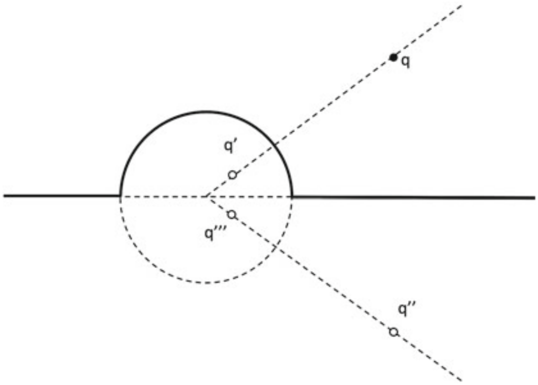


Fig. 3.7 Hemispherical bump on a conductive plane

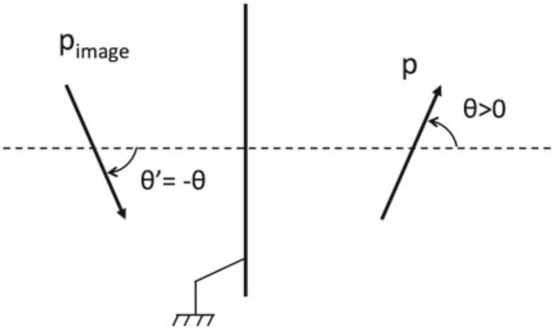


Fig. 3.8 Electric dipole in front of a grounded conductive plane

and the torque:

$$M = -\frac{\partial U}{\partial \theta} = -\frac{1}{4\pi\epsilon_0}\frac{p^2}{(2d)^4}\sin 2\theta$$

$M < 0$ for $0 < \theta < \pi/2$ and $M > 0$ for $\pi/2 < \theta < \pi$.
For the dipole oriented parallel and normal to the plane the attractive forces are:

$$\mathbf{F}_{parallel} = -\frac{3}{4\pi\epsilon_0}\frac{p^2}{(2d)^4}\hat{\mathbf{n}} \qquad \mathbf{F}_{normal} = -\frac{6}{4\pi\epsilon_0}\frac{p^2}{(2d)^4}\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is the normal versor out of the plane.
The force between two dipoles was found also in Problem 2.6. Here the image dipole $\mathbf{p}_{imag}$ replaces $\mathbf{p}_1$, the real dipole $\mathbf{p}$ is $\mathbf{p}_2$ and $\mathbf{r}$, going from $\mathbf{p}_{imag}$ to $\mathbf{p}$, has length $2d$.
So the force on the dipole is:

$$\mathbf{F} = \frac{1}{4\pi\epsilon_0}\left[3\frac{(\mathbf{p}_{imag}\cdot\mathbf{p})\mathbf{r} + \mathbf{p}(\mathbf{p}_{imag}\cdot\mathbf{r}) + (\mathbf{p}_{imag}\cdot\mathbf{r})\mathbf{p}}{r^5} - 15\frac{(\mathbf{p}_{imag}\cdot\mathbf{r})(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^7}\right].$$

From $\theta' = -\theta$, $p = p_{imag}$ and noting that:

$$\mathbf{p}_{imag}\, p\, r\ \cos\theta + p_{imag}\, \mathbf{p}\, r\ \cos\theta' = 2\, p_{imag}\, p\, \mathbf{r}\ \cos^2\theta = 2\, p^2\, \mathbf{r}\ \cos^2\theta$$

the force acting on the dipole becomes:

$$\mathbf{F} = \frac{p^2}{4\pi\epsilon_0}\left[\frac{3\cos 2\theta - 9\cos^2\theta}{(2d)^4}\right]\hat{\mathbf{n}} = -\frac{3p^2}{4\pi\epsilon_0}\left[\frac{1+\cos^2\theta}{(2d)^4}\right]\hat{\mathbf{n}}\,.$$

3.5 This problem is the inverse of that with a point charge q at distance d from the center of a conductive sphere. There, to get a null potential on the spherical surface, we have added an image charge $q' = -qR/d$ inside the sphere at distance $a = R^2/d$ from the center. It is evident that if a charge q is inside the sphere, at distance a from the center, we can get the spherical surface at null potential adding outside of the sphere an image charge $q^* = -qd/R$ at distance $d = R^2/a$ from the center of the sphere.
The inner charge q induces a charge $-q$ on the inner surface of the conductor. For an uncharged conductor, a charge q has to be uniformly distributed on the outer surface of the conductor. Therefore the potential of the conductor is:

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{R}.$$

For the Coulomb's theorem the charge density induced on the inner surface, in a point with the radius at an angle θ with respect to the line from the center of the conductor to the charge q, is:

$$\sigma(\theta) = -\epsilon_0 E_r = -\frac{q}{4\pi}\frac{R^2-a^2}{R\left(R^2+a^2-2Ra\ \cos\theta\right)^{\frac{3}{2}}} \qquad (3.5)$$

where E_r is the radial component of the sum of the fields from the charges q and q^* on the inner surface. The charge densities at $\theta = 0$ and $\theta = \pi$ are:

$$\sigma(\theta = 0) = -\frac{q}{4\pi}\frac{R+a}{R\,(R-a)^2} \qquad \sigma(\theta = \pi) = -\frac{q}{4\pi}\frac{R-a}{R\,(R+a)^2}\,.$$

It is easy to verify that the integral of the charge density (3.5) is equal to total induced charge $-q$ on the inner surface.
Outside of the spherical surface, at a distance $r > R$ from the center, the potential is generated by the charge q uniformly distributed on the outer surface:

$$V = \frac{1}{4\pi\epsilon_0}\frac{q}{r}\,.$$

3.6 The problem is a simple extension of the point charge and the conductive plane. An image wire with charge density $\lambda' = -\lambda$ can be set symmetric to the real wire with respect to the plane. The potential and the field in the half space with the wire are given by the superposition of the potentials and the fields of the two wires.

3.7 For a charged wire with linear density λ at distance $d > R$ from the axis of a conductive cylinder, the surface of the cylinder is equipotential if we place an image wire, with opposite linear charge, inside the cylinder at distance $a = R^2/d$ from the axis. From this result we can infer that for a charged wire with density λ at distance $a < R$ from the axis of the cylindrical surface, we can write the field inside the cylindrical surface by setting an image wire with charge density $-\lambda$ at distance $d = R^2/a > R$ from the axis.
The charge per unit length on the inner surface of the cylindrical surface is $-\lambda$. Its distribution can be found from the component of the electric field normal to the surface. On the outer surface of the cylindrical conductor the charge is null, thus also the electric field is null outside the conductor.

3.8 The problem is similar to that of conductive sphere in a uniforme electric field. If we dispose the axis of the cylinder on the z axis, the field $\mathbf{E}_0 = E_0\hat{x}$ can be produced by two wires on the xz plane, parallel to the axis of the cylinder: one with density λ at $x = -d$ and the other with density $-\lambda$ at $x = d$. It easy to show that in the limit $d \to \infty$ the two wires produce nearby the z axis a uniform electric field:

$$\mathbf{E}_0 = \frac{\lambda}{\pi\epsilon_0 d}\,\hat{\mathbf{x}}\,. \tag{3.6}$$

The potential and the cylindrical components of this field are:

$$V = -E_0 x = -E_0 r\cos\theta \qquad E_{0r} = E_0\cos\theta \qquad E_{0\theta} = -E_0\sin\theta \qquad E_z = 0\,.$$

To have an equipotential cylindrical surface, from the result for a charged wire near a conductive cylinder, we have to add two image wires: one with charge density $-\lambda$ at $x = -R^2/d$ and the other with charge density λ at $x = R^2/d$. In the limit $d \to \infty$ the distance between the image wires become infinitesimal and for $r > R$ they produce a potential:

$$V'(r,\theta) = \frac{-\lambda}{2\pi\epsilon_0}\log\left(1 - \frac{2x}{r}\cos\theta\right) \simeq \frac{\lambda}{\pi\epsilon_0}\frac{x}{r}\cos\theta = \frac{\lambda}{\pi\epsilon_0 d}\frac{R^2}{r}\cos\theta \tag{3.7}$$

to which is associated a field $\mathbf{E}'$ with components:

$$E'_r = \frac{\lambda}{\pi\epsilon_0 d}\frac{R^2}{r^2}\cos\theta \qquad E'_\theta = \frac{\lambda}{\pi\epsilon_0 d}\frac{R^2}{r^2}\sin\theta.$$

By substituting in these two expressions the density λ found from the relation (3.6), for $r > R$ the total field $\mathbf{E}_{tot} = \mathbf{E}_0 + \mathbf{E}'$ has components:

$$E_{tot_r} = E_0 \left(1 + \frac{R^2}{r^2}\right) \cos\,\theta \qquad E_{tot_\theta} = -E_0 \left(1 - \frac{R^2}{r^2}\right) \sin\,\theta\,.$$

For $r = R$ the component E_{tot_θ} is zero, as expected for the cylindrical equipotential surface, and from the value of E_{tot_r} the Coulomb's theorem gives the charge density induced on the conductive cylinder:

$$\sigma = 2\epsilon_0 E_0\,\cos\theta\,.$$

3.9 We apply the solution for a charged wire parallel to the axis of a conductive cylinder. Considering the Fig. 3.9, if the wire f_1, with linear charge density λ, is at distance d from the axis of the cylinder C_2, to get the cylindrical surface of C_2 equipotential we have to introduce an image wire f_2 with charge $-\lambda$, at distance R^2/d from the axis of C_2. The potential of C_2 is:

$$V_2 = -\frac{\lambda}{2\pi\epsilon_0}\log\frac{d}{R}.$$

Symmetrically to have the equipotential surface of C_1 in presence of the wire f_2 with charge density $-\lambda$ at distance d from its axis, we have to add just the image wire f_1 with charge density λ at distance R^2/d from the axis of C_1. The potential of C_1 is $V_1 = -V_2$ and the potential difference between the two cylindrical surfaces is:

$$\Delta V = V_1 - V_2 = \frac{\lambda}{\pi\epsilon_0}\log\frac{d}{R}.$$

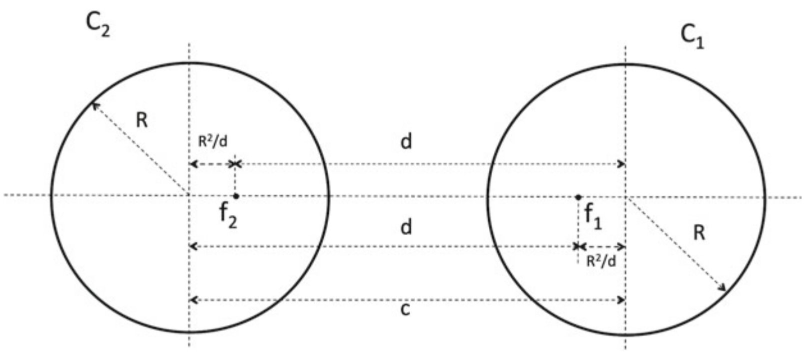


Fig. 3.9 Conductive cylinders of same radius with parallel axis

If the distance of the axis of the cylinders is c, for the distance d we can write the relation:

$$c = d + \frac{R^2}{d} \quad \text{and we get:} \quad d = \frac{c + \sqrt{c^2 - 4R^2}}{2}$$

thus we get:

$$\Delta V = \frac{\lambda}{\pi\epsilon_0} \log \frac{c + \sqrt{c^2 - 4R^2}}{2R}$$

and the capacitance:

$$C = \frac{\lambda}{\Delta V} = \frac{\pi\epsilon_0}{\log \frac{c+\sqrt{c^2-4R^2}}{2R}}$$

We can write ΔV also in the form:

$$\Delta V = V_1 - V_2 = \frac{\lambda}{2\pi\epsilon_0} \log\left(\xi + \sqrt{\xi^2 - 1}\right) \quad \text{where:} \quad \xi = \frac{c^2}{2R^2} - 1.$$

and the capacitance per unit length:

$$C = \frac{\lambda}{\Delta V} = \frac{2\pi\epsilon_0}{\log\left(\xi + \sqrt{\xi^2 - 1}\right)} = \frac{2\pi\epsilon_0}{\operatorname{acosh} \xi}\,.$$

3.10 We have to find the relative position of two parallel wires f_a, with charge density $-\lambda$, and f_b, with charge density λ, which produce two cylindrical equipotential surfaces C_a and C_b with radii a and b and with their axes at distance c. Looking at Fig. 3.10: if the wire f_b is at distance d_a from the axis of C_a, the wire f_a has to be at distance a^2/d_a from the axis of C_a. Symmetrically if the wire f_a is at distance d_b

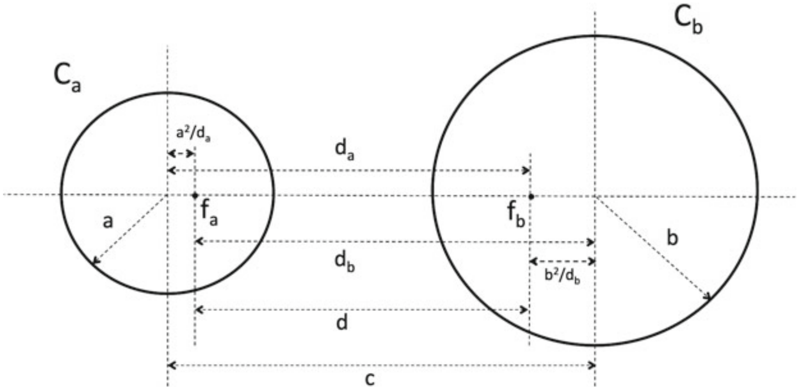


Fig. 3.10 Conductive cylinders of different radii with parallel axis at distance $c > a + b$

from the axis of C_b, the wire f_b has to be at distance b^2/d_b from the axis of C_b. The potentials of the two cylindrical surfaces are:

$$V_a = -\frac{\lambda}{2\pi\epsilon_0}\log\frac{d_a}{a} \qquad V_b = \frac{\lambda}{2\pi\epsilon_0}\log\frac{d_b}{b}.$$

We have to determine d_a and d_b. If d is the distance of the two wire, we can write the system:

$$\begin{cases} d + \frac{a^2}{d_a} + \frac{b^2}{d_b} = c \\ d = d_a - \frac{a^2}{d_a} \\ d = d_b - \frac{b^2}{d_b} \end{cases}$$

and solving we find:

$$d_b = \frac{c^2 + b^2 - a^2 + \sqrt{(c^2 - a^2 - b^2)^2 - 4a^2b^2}}{2c}$$

$$d_a = \frac{c^2 + a^2 - b^2 + \sqrt{(c^2 - a^2 - b^2)^2 - 4a^2b^2}}{2c}.$$

The potential difference becomes:

$$V_b - V_a = \frac{\lambda}{2\pi\epsilon_0}\log\left(\frac{d_b}{b}\cdot\frac{d_a}{a}\right) = \frac{\lambda}{2\pi\epsilon_0}\ \log\frac{c^2 - a^2 - b^2 + \sqrt{(c^2 - a^2 - b^2)^2 - 4a^2b^2}}{2ab}$$

that can be rewritten as:

$$V_b - V_a = \frac{\lambda}{2\pi\epsilon_0}\log\left(\xi + \sqrt{\xi^2 - 1}\right) = \frac{\lambda}{2\pi\epsilon_0}\text{acosh}\,\xi \quad \text{where } \ \xi = \frac{c^2 - a^2 - b^2}{2ab}.$$

Finally the capacitance per unit length is:

$$C = \frac{2\pi\epsilon_0}{\log\left(\xi + \sqrt{\xi^2 - 1}\right)} = \frac{2\pi\epsilon_0}{\text{acosh}\,\xi}\,.$$

If the two radii are equal ($a = b$) this result becomes that of the previous problem.

3.11 We have to find two parallel opposite charged wires that produce the two equipotential cylindrical surfaces C_a and C_b, inside each other, with radii a and b ($a > b$) and their axes at distance $c < a - b$. A charged wire f_1 has to be located inside the inner cylinder and a wire f_2 outside the outer one as in Fig. 3.11. To get C_a equipotential f_1 and f_2 have to be respectively at distance a^2/d_a and d_a from the axis

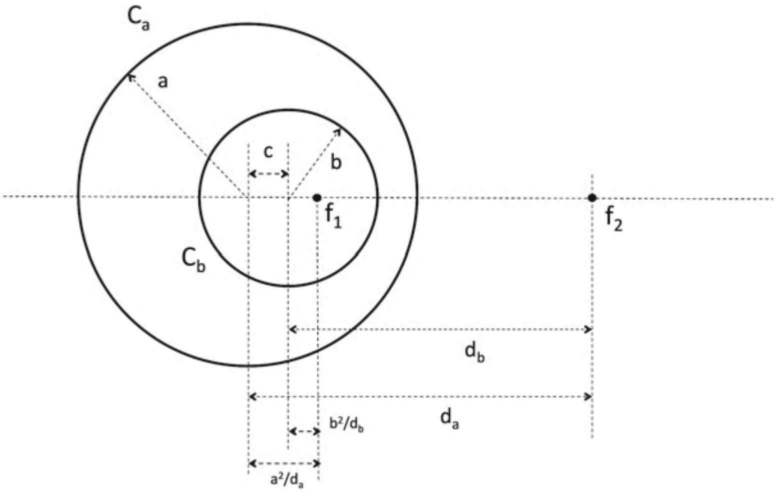


Fig. 3.11 Cylindrical conductors of radii a and b, inside each other, with distance of the axis $c < |a - b|$

of C_a. At the same time to have equipotential C_b the two wires have to be distant b^2/d_b and d_b from the axis of C_b. To find d_a and d_b we can write the system:

$$\begin{cases} c = \frac{a^2}{d_a} - \frac{b^2}{d_b} = d_a - d_b \\ d_a - \frac{a^2}{d_a} = d_b - \frac{b^2}{d_b} \end{cases}$$

with solutions:

$$d_a = \frac{a^2 - b^2 + c^2 + \sqrt{(a^2 + b^2 - c^2)^2 - 4a^2b^2}}{2c}$$

$$d_b = \frac{a^2 - b^2 - c^2 + \sqrt{(a^2 + b^2 - c^2)^2 - 4a^2b^2}}{2c} .$$

The potentials of C_a and C_b are:

$$V_a = -\frac{\lambda}{2\pi\epsilon_0}\log\frac{d_a}{a} \qquad V_b = -\frac{\lambda}{2\pi\epsilon_0}\log\frac{d_b}{b}$$

and the difference is:

$$V_a - V_b = \frac{\lambda}{2\pi\epsilon_0}\log\frac{a}{d_a}\frac{d_b}{b} = \frac{\lambda}{2\pi\epsilon_0}\log\frac{a^2 + b^2 - c^2 + \sqrt{(a^2 + b^2 - c^2)^2 - 4a^2b^2}}{2ab}$$

that can be written in the form:

$$V_a - V_b = \frac{\lambda}{2\pi\epsilon_0}\ \log\left(\xi + \sqrt{\xi^2 - 1}\right) \quad \text{where:} \quad \xi = \frac{a^2 + b^2 - c^2}{2ab}\,.$$

The capacitance per unit length of the two cylindrical conductors is:

$$C = \frac{\lambda}{\Delta V} = \frac{2\pi\epsilon_0}{\log\left(\xi + \sqrt{\xi^2 - 1}\right)} = \frac{2\pi\epsilon_0}{\operatorname{acosh}\xi}\,. \tag{3.8}$$

For two cylindrical conductors of different radius ($a \neq b$) which are coaxial ($c = 0$) the capacitance (3.8) becomes that of a simple cylindrical capacitor $C = 2\pi\epsilon_0/\log(a/b)$.

Chapter 4
Image Charges in Dielectrics

The method of image charges is extended to systems with dielectric media. In some problems it is easy to find the solution for the Poisson equation with the boundary conditions on the fields derived from the constitutive relations. First we consider a point charge near a planar surface separating two dielectric media. Then we study the case of a dielectric sphere embedded in a different dielectric where a uniform electric field is present. More examples are examined in the problems at the end of the Chapter.

4.1 Electrostatics in Dielectric Media

The field in an isotropic and uniform dielectric is provided by the solution of the Poisson equation provided that the permittivity ϵ_0 of the empty space is replaced with the permittivity ϵ of the medium:

$$\Delta V(x, y, z) = -\frac{\rho(x, y, z)}{\epsilon}.$$

The solutions are those for the empty space but with the potential and the fields reduced by the factor $\kappa = \epsilon/\epsilon_0$, the dielectric constant of the medium.

In the presence of different dielectric media the problem becomes more complicated: the fields E and D are not continuous at the interfaces between the media, so that the fields are not differentiable and the Poisson equation cannot be written across these surfaces. The recipe is to solve the Poisson equation inside each of the dielectrics ($i, j, ...$) with the boundary conditions on the fields $E^{(i)}$ and $D^{(i)}$ derived by the Maxwell equations and the relations $D^{(i)} = \epsilon^{(i)} E^{(i)}$. These are that the parallel component of the electric field E and the normal component of the displacement field D are continuous at the (uncharged) interfaces between the media i and j:

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9_4

$$E_{\parallel}^{(i)} = E_{\parallel}^{(j)} \qquad D_{\perp}^{(i)} = D_{\perp}^{(j)} \tag{4.1}$$

In this chapter we apply the method of image charges in presence of two dielectric media[1] in order to satisfy the Poisson equation with the boundary conditions.

4.2 Point Charge Near the Plane Separating Two Dielectric Media

A point charge q is located at distance d apart from a plane separating two dielectric media. The z axis can be fixed normal to the plane placed at $z = 0$ and passing on the charge. The dielectric constants are κ_1 for $z > 0$ and κ_2 for $z < 0$. We have to find a solution for the potential in each of the two media requiring that the boundary conditions (4.1) are satisfied on the plane of separation. The solution seen for a point charge in front of a conductive plane, suggests to write the potential in the semispace $z > 0$ assuming all of the space of constant κ_1 and adding[2] an *image* charge q' on the z axis at $z = -d$ (see Fig. 4.1). For the potential in the semispace $z < 0$ we consider all the space of constant κ_2 and we position only an image charge q'' at $z = d$.

For $z > 0$ we have to find the solution of the Poisson equation: while the potential due to q is the particular solution, the potential from the charge q' is a solution of the homogeneous (Laplace) equation. For $z < 0$ we need the solution of the Laplace equation and this can be given by potential of the charge q'' at $z > 0$.

For a point P at $z > 0$ and at a distance r from the z axis, the potential is:

$$V_1 = \frac{1}{4\pi\epsilon_1}\left(\frac{q}{R_1} + \frac{q'}{R_2}\right)$$

where $R_1 = \left[(z-d)^2 + r^2\right]^{\frac{1}{2}}$ and $R_2 = \left[(z+d)^2 + r^2\right]^{\frac{1}{2}}$.

The cylindrical components z and r of the electric field are:

$$E_z^{(1)} = -\frac{\partial V_1}{\partial z} = \frac{1}{4\pi\epsilon_1}\left[\frac{q(z-d)}{R_1^3} + \frac{q'(z+d)}{R_2^3}\right]$$

$$E_r^{(1)} = -\frac{\partial V_1}{\partial r} = \frac{1}{4\pi\epsilon_1}\left[\frac{qr}{R_1^3} + \frac{q'r}{R_2^3}\right].$$

[1] For the *image charges method* in dielectrics media see also: J.D.Jackson, *Classical Electrodynamics*, cited, Sect. 4.4; L.D.Landau–E.M. Lifšits, *Electrodynamics of continuous media*, Chap. II, Sect. 7, with problems.

[2] This choice includes the possibility of the same dielectric filling all the space. In fact in this case the charges have to be $q' = 0$ and $q'' = q$, as we will find in the general solution for q' and q''.

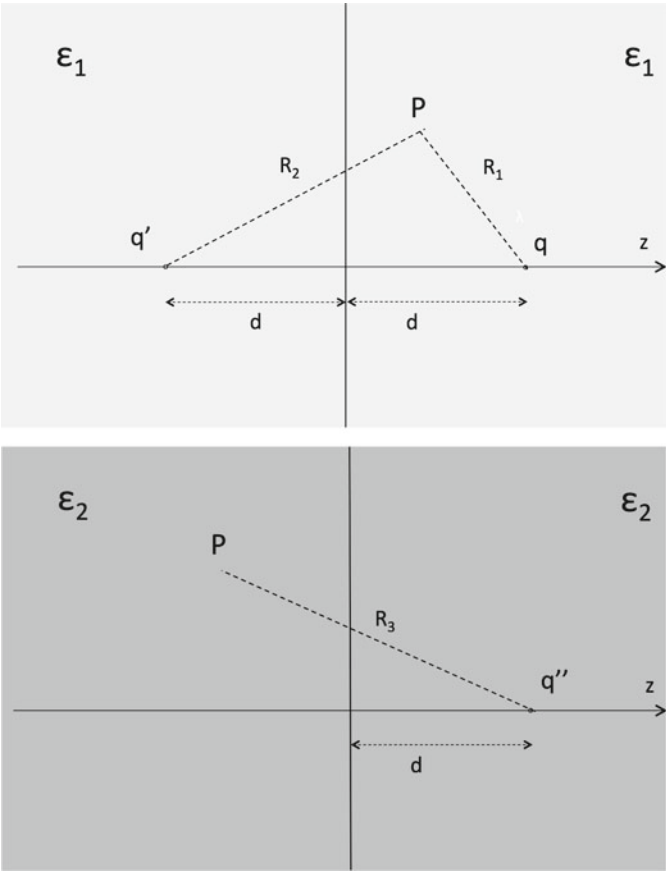


Fig. 4.1 Image charges for a point charge q in front of a plane separating two dielectric media. For the field in the semispace $z > 0$ an image charge q' is located at $z = -d$ in a space of dielectric ϵ_1; for the field in $z < 0$ only an image charge q'' at $z = d$ in a space of dielectric ϵ_2

For a point P at $z < 0$ the potential is:

$$V_2 = \frac{1}{4\pi\epsilon_2}\left(\frac{q''}{R_3}\right)$$

with $R_3 = \left[(z-d)^2 + r^2\right]^{\frac{1}{2}}$ and the components of the field are:

$$E_z^{(2)} = -\frac{\partial V_2}{\partial z} = \frac{1}{4\pi\epsilon_2}\frac{q''(z-d)}{R_3^3}$$

$$E_r^{(2)} = -\frac{\partial V_2}{\partial r} = \frac{1}{4\pi\epsilon_2}\frac{q''r}{R_3^3}\,.$$

In every point of the plane separating the two dielectric media, the fields have to satisfy the boundary relations (4.1) $E_{\parallel}^{(1)} = E_{\parallel}^{(2)}$ and $D_{\perp}^{(1)} = D_{\perp}^{(2)}$ that in this problem become $E_r^{(1)} = E_r^{(2)}$ and $\epsilon_1 E_z^{(1)} = \epsilon_2 E_z^{(2)}$.

For $z = 0$ $R_1 = R_2 = R_3$, and from the previous relations we find:

$$\frac{q+q'}{\epsilon_1} = \frac{q''}{\epsilon_2} \qquad q - q' = q''$$

and finally the values for the *image* charges:

$$q' = -q\,\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1} \qquad q'' = q\,\frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}\,.$$

The field lines for $\epsilon_2 > \epsilon_1$ and $\epsilon_2 < \epsilon_1$ are sketched in Fig. 4.2. It is easy to verify from the found components of the fields E and D, the known law of refraction for the field lines:

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\epsilon_1}{\epsilon_2}$$

where θ_1 and θ_2 are the angles between the fields and the normal to the separating plane.

For $\epsilon_2 = \epsilon_1$ the same dielectric medium is present in the two semispaces. The point charge is embedded in a dielectric that fills all the space and from the previous relations we have: $q' = 0$ and $q'' = q$ as expected (see the footnote in a previous page).

The polarization surface charge density on the two sides of the plane between the two media, is found by the relation $\sigma_P = \mathbf{P}\cdot\hat{\mathbf{n}}$ where $\mathbf{P} = \epsilon_0(\kappa - 1)\mathbf{E}$ is the induced polarization and $\hat{\mathbf{n}}$ is the versor, out of the dielectric, normal to the plane:

$$\sigma_P^{(1)} = -\epsilon_0(\kappa_1 - 1)E_z^{(1)} = \frac{qd}{2\pi R^3}\,\frac{\epsilon_1 - \epsilon_0}{\epsilon_1}\,\frac{\epsilon_2}{\epsilon_2 + \epsilon_1}$$

$$\sigma_P^{(2)} = \epsilon_0(\kappa_2 - 1)E_z^{(2)} = -\frac{qd}{2\pi R^3}\,\frac{\epsilon_2 - \epsilon_0}{\epsilon_2 + \epsilon_1}$$

where $R = R_1 = R_2 = R_3$.

The total surface polarization charge density and the total polarization charge are:

$$\sigma_P = \frac{qd}{2\pi R^3}\,\frac{\epsilon_0}{\epsilon_1}\,\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} \qquad Q_{pol} = q\,\frac{\epsilon_0}{\epsilon_1}\,\frac{\epsilon_1 - \epsilon_2}{\epsilon_2 + \epsilon_1}$$

and are $\gtreqless 0$ for $\epsilon_1 \gtreqless \epsilon_2$.

For $\epsilon_2 \gg \epsilon_1$ the dielectric at $z < 0$ becomes a conductor and from the previous formulas $q' = -q$, the components of the electric field $E^{(2)}$ go to zero and the field lines are normal to the plane of separation ($E_r^{(1)} = 0$ and $\theta_1 = 0$).

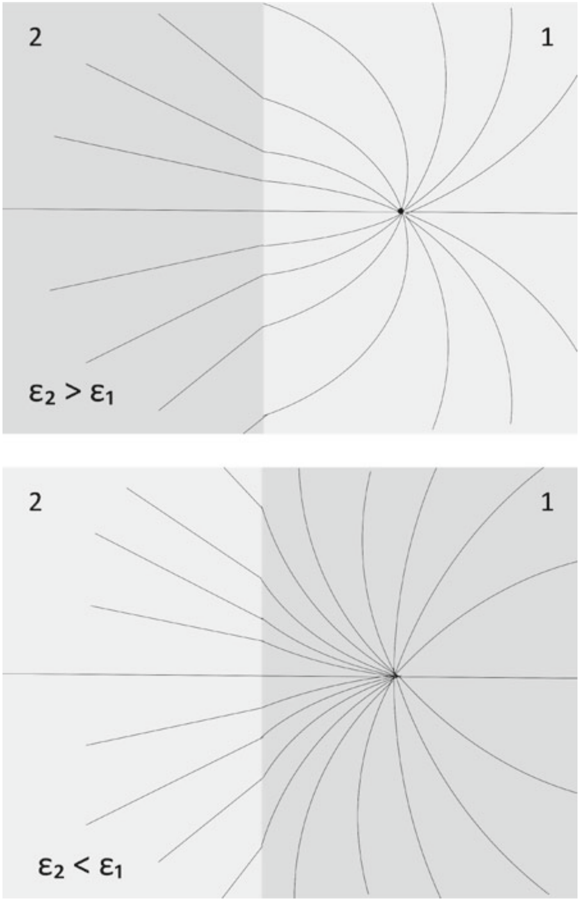


Fig. 4.2 Field lines for a point charge in front of a plane separating two dielectric media: *top* for $\epsilon_2 > \epsilon_1$, *bottom* for $\epsilon_2 < \epsilon_1$

The force on the charge q due to the polarization charge on the plane at $z = 0$ can be written as the force between the charge q and the image charge q' in the dielectric with ϵ_1:

$$F = -\frac{1}{4\pi\epsilon_1}\frac{q^2}{(2d)^2}\frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}.$$

This force is attractive if $\epsilon_2 > \epsilon_1$ and repulsive if $\epsilon_2 < \epsilon_1$.

4.3 Dielectric Sphere in an External Uniform Electric Field

A sphere of dielectric with permittivity ϵ_2, of radius R, is embedded in a dielectric medium with permittivity ϵ_1, where it is already present an external uniform electric field $\mathbf{E} = E\hat{\mathbf{z}}$. We suppose, as proved correct a posteriori by the solution, that inside the sphere the electric field E' is uniform and with the same direction of the external field E, while outside the field is the superposition of the uniform field E and of a field produced by the polarization of the dielectric media. Furthermore we assume this last field is that produced by an image electric dipole $\mathbf{p} = p\hat{\mathbf{z}}$ placed at the center of the sphere.

We position the center of the sphere in the origin of the frame. The potential associated to the external field $\mathbf{E}$ is:

$$V_e = -Ez = -E\,r\,\cos\theta$$

and, in spherical coordinates, the net potential outside the sphere is:

$$V(r,\theta) = -E\,r\,\cos\theta + \frac{1}{4\pi\epsilon_1}\frac{p\,\cos\theta}{r^2}\,. \tag{4.2}$$

with p to be determined. The components of the field are:

$$E_r = -\frac{\partial V}{\partial r} = E\,\cos\theta + \frac{1}{2\pi\epsilon_1}\frac{p\,\cos\theta}{r^3} \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial\theta} = -E\sin\theta + \frac{p\,\sin\theta}{4\pi\epsilon_1 r^3}. \tag{4.3}$$

Inside the sphere the potential is:

$$V' = -E'\,z = -E'\,r\,\cos\theta$$

and the components of the field are:

$$E'_r = -\frac{\partial V'}{\partial r} = E'\,\cos\theta \qquad E'_\theta = -\frac{1}{r}\frac{\partial V'}{\partial\theta} = -E'\,\sin\theta\,.$$

From the boundary conditions (4.1) for the fields $\mathbf{E}$ and $\mathbf{D}$ on the surface of the sphere (at $r = R$), we get two independent equations:

$$E_\parallel = E'_\parallel \quad\Longrightarrow\quad E_\theta = E'_\theta \quad\rightarrow\quad E - \frac{1}{4\pi\epsilon_1}\frac{p}{R^3} = E'$$

$$D_\perp = D'_\perp \quad\Longrightarrow\quad \epsilon_1 E_r = \epsilon_2 E'_r \quad\rightarrow\quad \epsilon_1 E + \frac{1}{2\pi}\frac{p}{R^3} = \epsilon_2 E'$$

and solving the system we find:

$$E' = \frac{3\epsilon_1}{2\epsilon_1+\epsilon_2}\,E \qquad p = \frac{\epsilon_2-\epsilon_1}{2\epsilon_1+\epsilon_2}\,\epsilon_1\,4\pi R^3\,E\,. \tag{4.4}$$

If $\epsilon_2 > \epsilon_1$ (see Fig. 4.3a): $p > 0$, the dipole has the same orientation of E; inside the sphere the field is $E' < E$, that is a lower field due to the field E_P produced by

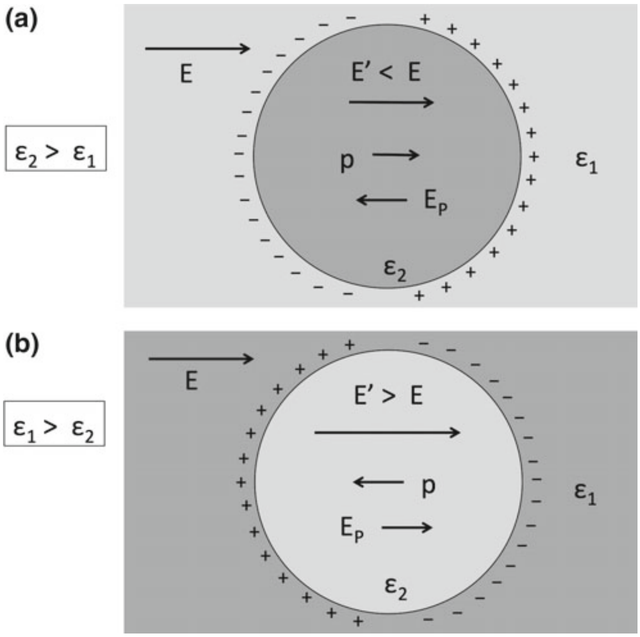


Fig. 4.3 Charge of polarization, image dipole p, E' internal electric field, E_P field from the polarization charges in a dielectric sphere of permittivity ϵ_2 embedded in a dielectric of permittivity ϵ_1 with an external electric field E: **a** $\epsilon_2 > \epsilon_1$, **b** $\epsilon_2 < \epsilon_1$

the polarization charges, opposite to E. Outside the sphere, the field lines tend to converge around the sphere.

For $\epsilon_1 > \epsilon_2$ (see Fig. 4.3b): $p < 0$, the dipole is opposite to E; inside the sphere $E' > E$, the strength of the field is larger due to the field E_P, from the polarization charges, that adds to the field E; outside the sphere the field lines tend to diverge from the sphere.

If $\epsilon_1 = \epsilon_2$, the same dielectric fills the space, and the relations (4.4) clearly give $p = 0$ and $E' = E$.

Replacing the values of p and E' from (4.4) in (4.2) and (4.3) we get the potential and the field components outside the sphere:

$$V(r,\theta) = -\left(1 + \frac{\epsilon_1 - \epsilon_2}{2\epsilon_1 + \epsilon_2} \cdot \frac{R^3}{r^3}\right) E\ r\ \cos\theta$$

$$E_r(r,\theta) = \left(1 + 2\frac{\epsilon_2 - \epsilon_1}{2\epsilon_1 + \epsilon_2} \cdot \frac{R^3}{r^3}\right) E\ \cos\theta$$

$$E_\theta(r,\theta) = -\left(1 + \frac{\epsilon_1 - \epsilon_2}{2\epsilon_1 + \epsilon_2} \cdot \frac{R^3}{r^3}\right) E\ \sin\theta.$$

At the surface of the sphere ($r \to R$) the potential is:

$$V(R,\theta) = -\frac{3\epsilon_1}{2\epsilon_1 + \epsilon_2}\, E\, R\, \cos\theta$$

and the components of the electric field:

$$E_r(R,\theta) = \left(\frac{3\epsilon_2}{2\epsilon_1 + \epsilon_2}\right) E\, \cos\theta \qquad E_\theta(R,\theta) = -\frac{3\epsilon_1}{2\epsilon_1 + \epsilon_2}\, E\, \sin\theta\,.$$

The total charge density of polarization on the surface is:

$$\sigma_p = 3\epsilon_0 \frac{\epsilon_2 - \epsilon_1}{2\epsilon_1 + \epsilon_2} E \cos\theta = \sigma \cos\theta \quad \text{where} \quad \sigma = 3\epsilon_0 \frac{\epsilon_2 - \epsilon_1}{2\epsilon_1 + \epsilon_2} E.$$

For a *dielectric sphere* of permittivity ϵ placed in an external field E_0 in vacuum, we substitute $\epsilon_2 = \epsilon,\ \epsilon_1 = \epsilon_0$ and we get:

$$p = \frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon} \epsilon_0 4\pi R^3 E_0 \ > 0 \qquad E' = \frac{3\epsilon_0}{2\epsilon_0 + \epsilon} E_0 < \ E_0$$

$$\sigma_p = \sigma \cos\theta \qquad \sigma = 3\epsilon_0 \frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon} E_0 > 0\,.$$

For a *spherical cavity* in a dielectric with permittivity ϵ in an external field E, $\epsilon_1 = \epsilon,\ \epsilon_2 = \epsilon_0$ and we find:

$$p = \frac{\epsilon_0 - \epsilon}{2\epsilon + \epsilon_0} \epsilon 4\pi R^3 E \ < 0 \qquad E' = \frac{3\epsilon}{2\epsilon + \epsilon_0} E > \ E$$

$$\sigma_p = \sigma\ \cos\theta \qquad \sigma = 3\epsilon_0 \frac{\epsilon_0 - \epsilon}{2\epsilon + \epsilon_0} E < 0\,.$$

Problems

4.1 Determine the electric field produced by a charged infinite wire parallel to the plane separating two different dielectric media. Write the force acting on the wire.

4.2 Consider the problem of an infinite wire with linear charge density λ in a dielectric of permittivity ϵ_1 parallel, at distance d, to the axis of a cylinder of infinite length and radius $R < d$, composed by a medium of permittivity ϵ_2. Find the force acting on the wire.

4.3 Find the field for an infinite wire with linear charge density λ, placed at distance $h < R$ parallel to the axis of a infinite long cylinder, of radius R and permittivity ϵ_2, embedded in an other medium with permittivity ϵ_1. Give the force acting on the wire.

4.4 Study the case of a cylinder of permittivity ϵ_2, embedded in a medium of permittivity ϵ_1, where in its absence an electric field is present that is uniform and with direction normal to the axis of the cylinder. Assume, as seen for the dielectric sphere, that the field inside the cylinder is uniform and oriented as the external field at large distance.

Solutions

4.1 We say $z = 0$ the plane separating the dielectric with ϵ_1 in the semispace $z > 0$ from that with ϵ_2 at $z < 0$. Then we position the infinite wire f with linear charge density λ at distance d on the line $(z = d,\ x = 0)$ as shown in Fig. 4.4. From the solution for a point charge near the plane of separation, we can infer to write the field in the medium with ϵ_1 as the superposition of the field from the wire f and of a field due to an *image wire* f_1 with linear charge density λ_1, located symmetric to f with respect to the plane. Furthermore we write the field in the other dielectric (at $z < 0$) as produced by an *image wire* f_2 with density λ_2 located at $z = d$ in the same position of f.

The potential at a point P in the semispace $z > 0$ depends on its distances r and r_1 from the wires f and f_1[3] and is:

$$V(P) = -\frac{\lambda}{2\pi\epsilon_1}\log\frac{r}{r_0} - \frac{\lambda_1}{2\pi\epsilon_1}\log\frac{r_1}{r_0} = -\frac{\lambda}{4\pi\epsilon_1}\log\frac{(z-d)^2+x^2}{r_0^2} - \frac{\lambda_1}{4\pi\epsilon_1}\log\frac{(z+d)^2+x^2}{r_0^2}.$$

The components of the electric field at $z > 0$ are:

$$E_x^{(1)} = -\frac{\partial V}{\partial x} = \frac{\lambda}{2\pi\epsilon_1}\frac{x}{(z-d)^2+x^2} + \frac{\lambda_1}{2\pi\epsilon_1}\frac{x}{(z+d)^2+x^2} \qquad E_y^{(1)} = -\frac{\partial V}{\partial y} = 0$$

$$E_z^{(1)} = -\frac{\partial V}{\partial z} = \frac{\lambda}{2\pi\epsilon_1}\frac{(z-d)}{(z-d)^2+x^2} + \frac{\lambda_1}{2\pi\epsilon_1}\frac{(z+d)}{(z+d)^2+x^2}.$$

For a point P in the semispace $z < 0$ the potential is:

$$V' = -\frac{\lambda_2}{2\pi\epsilon_2}\log\frac{r_2}{r_0} = -\frac{\lambda_2}{4\pi\epsilon_2}\log\frac{(z-d)^2+x^2}{r_0^2}$$

[3] We can assume a reference potential at a point on the plane so that the distance r_0 is the same for the two wires.

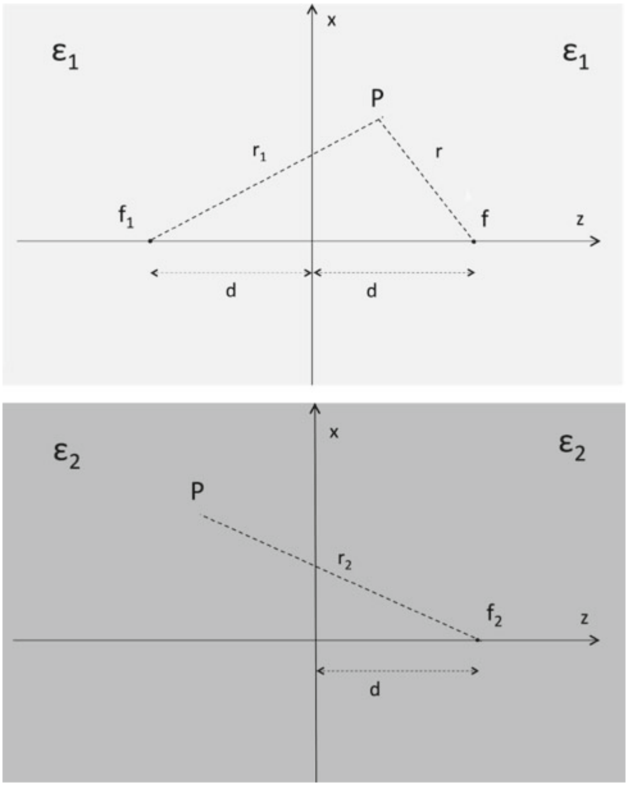


Fig. 4.4 Charged wire near the plane, at $z = 0$, separating two dielectric media. At *top* the wire f with charge density λ and the *image wire* f_1 with charge density λ_1 for the field in the semispace $z > 0$; at *bottom image wire* f_2 with density charge λ_2 for the field in the semispace $z < 0$

with r_2 the distance of P from the wire f_2. The components of the field are:

$$E_x^{(2)} = -\frac{\partial V'}{\partial x} = \frac{\lambda_2}{2\pi\epsilon_2}\frac{r}{(z-d)^2+x^2} \qquad E_y^{(2)} = -\frac{\partial V'}{\partial y} = 0$$

$$E_z^{(2)} = -\frac{\partial V'}{\partial z} = \frac{\lambda_2}{2\pi\epsilon_2}\frac{(z-d)}{(z-d)^2+x^2}\,.$$

The boundary conditions for the fields E and D on the plane give the relations between the charge densities:

$$E_{\parallel}^{(1)} = E_{\parallel}^{(2)} \quad\Longrightarrow\quad E_x^{(1)} = E_x^{(2)} \quad\rightarrow\quad \frac{\lambda+\lambda_1}{\epsilon_1} = \frac{\lambda_2}{\epsilon_2}$$

$$D_{\perp}^{(1)} = D_{\perp}^{(2)} \quad\Longrightarrow\quad \epsilon_1 E_z^{(1)} = \epsilon_2 E_z^{(2)} \quad\rightarrow\quad \lambda - \lambda_1 = \lambda_2$$

and from these, the values for λ_1 and λ_2 are:

$$\lambda_1 = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\,\lambda \quad \lambda_2 = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2}\,\lambda.$$

For the same dielectric filling all the space $\epsilon_1 = \epsilon_2$ thus $\lambda_2 = \lambda$ and $\lambda_1 = 0$.

The force per unit length on the wire due to the polarization is equal to the force from the wire f_1:

$$F = \frac{\lambda^2}{2\pi\epsilon_1}\frac{1}{2d}\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}$$

attractive for $\epsilon_1 < \epsilon_2$.

4.2 We locate the axis of the cylinder on the z axis and the wire f with linear charge density λ at distance d from the axis. The solution for the dielectric sphere in an external uniform field suggests to take into account the effect of the polarization for $r > R$ with two *image wires* f_1 and f_2, parallel to the z axis, respectively with charge densities λ_1 and λ_2. The former is located between the wire f and the axis at a distance $x < R$ from the axis, the latter on the axis of the cylinder. To describe the field inside the cylinder we can position an *image wire* f_3 with charge density λ_3 in the same place of the wire f. See Fig. 4.5.
The potential at a distance $r > R$ from the axis is:

$$V = -\frac{1}{4\pi\epsilon_1}\left[\lambda\ \log\left(\frac{r^2 + d^2 - 2rd\cos\theta}{r_0^2}\right) + \lambda_1\ \log\left(\frac{r^2 + x^2 - 2rx\cos\theta}{r_0^2}\right) + \lambda_2\ \log\frac{r^2}{r_0^2}\right]$$

and inside the cylinder is:

$$V' = -\frac{1}{4\pi\epsilon_2}\left[\lambda_3\ \log\left(\frac{r^2 + d^2 - 2rd\cos\theta}{r_0^2}\right)\right].$$

The electric field components outside are:

$$E_r = \frac{1}{2\pi\epsilon_1}\left[\lambda\,\frac{r - d\cos\theta}{r^2 + d^2 - 2rd\cos\theta} + \lambda_1\,\frac{r - x\cos\theta}{r^2 + x^2 - 2rx\cos\theta} + \lambda_2\,\frac{1}{r}\right]$$

$$E_\theta = \frac{1}{2\pi\epsilon_1}\left[\lambda\,\frac{d\sin\theta}{r^2 + d^2 - 2rd\cos\theta} + \lambda_1\,\frac{x\sin\theta}{r^2 + x^2 - 2rx\cos\theta}\right]$$

and inside:

$$E'_r = \frac{1}{2\pi\epsilon_2}\left[\lambda_3\,\frac{r - d\cos\theta}{r^2 + d^2 - 2rd\cos\theta}\right] \qquad E'_\theta = \frac{1}{2\pi\epsilon_2}\left[\lambda_3\,\frac{d\sin\theta}{r^2 + d^2 - 2rd\cos\theta}\right].$$

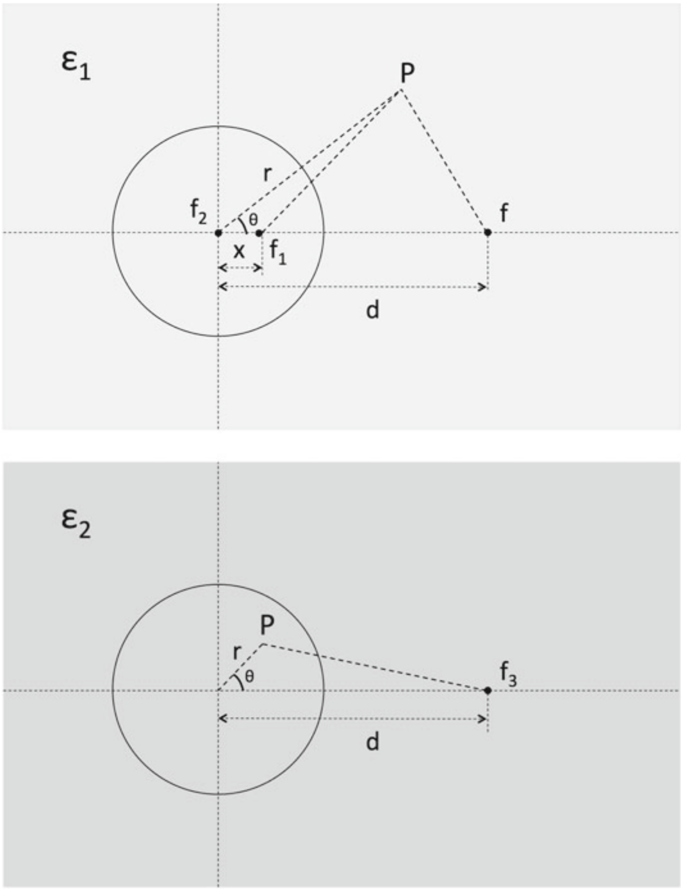


Fig. 4.5 Wire f with linear charge density λ external to a cylinder of dielectric permittivity ϵ_2 embedded in a medium of permittivity ϵ_1: *top* wire f and *image wires* f_1 and f_2 for the field outside the cylinders; *bottom image wire* f_3 for the field inside the cylinder

From the continuity of the tangent component of the electric field: $E_\theta = E'_\theta$ on the surface of the cylinder for any value of θ, we have:

$$x = \frac{R^2}{d} \qquad \text{and} \qquad \frac{\lambda}{\epsilon_1} - \frac{\lambda_3}{\epsilon_2} + \frac{\lambda_2}{\epsilon_1} = 0\,.$$

From the continuity of the normal component of the displacement field: $D_r = D'_r \rightarrow \epsilon_1 E_r = \epsilon_2 E'_r$ for any value of θ, and from the previous relations, finally we get:

$$\lambda_1 = \frac{\epsilon_1 - \epsilon_2}{\epsilon_2 + \epsilon_1}\,\lambda \qquad \lambda_2 = -\lambda_1 \qquad \lambda_3 = \frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}\,\lambda\,.$$

If $\epsilon_1 = \epsilon_2$: $\lambda_3 = \lambda$ and $\lambda_1 = \lambda_2 = 0$.
The force per unit length on the wire is the sum of the forces exerted by the wires f_1 and f_2:

$$F = \frac{\lambda}{2\pi\epsilon_1}\left(\frac{\lambda_1}{d-x} + \frac{\lambda_2}{d}\right) = \frac{\lambda^2}{2\pi\epsilon_1}\,\frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2}\,\frac{R^2}{d(d^2 - R^2)}$$

repulsive for $\epsilon_1 > \epsilon_2$.

4.3 We locate the axis of the cylinder on the z axis. The result of the previous problem suggests to write the field inside the cylinder (at $r < R$) with an *image wire* f_1 with charge density λ_1 at distance $d > R$ from the axis to be determined (see Fig. 4.6). For the field outside we can introduce an *image wire* f_2 with charge density λ_2 in the position of the wire f and a second *image wire* f_3 with charge density λ_3 on the axis of the cylinder.
The potential inside the cylinder (at $r < R$) is:

$$V = -\frac{1}{4\pi\epsilon_2}\left[\lambda\;\log\left(\frac{r^2 + h^2 - 2rh\cos\theta}{r_0^2}\right) + \lambda_1\;\log\left(\frac{r^2 + d^2 - 2rd\cos\theta}{r_0^2}\right)\right]$$

and outside ($r > R$):

$$V' = -\frac{1}{4\pi\epsilon_1}\left[\lambda_2\;\log\left(\frac{r^2 + h^2 - 2hd\cos\theta}{r_0^2}\right) + \lambda_3\;\log\frac{r^2}{r_0^2} + \right].$$

The components of the electric field inside are:

$$E_r = \frac{1}{2\pi\epsilon_2}\left[\lambda\,\frac{r - h\cos\theta}{r^2 + h^2 - 2rh\cos\theta} + \lambda_1\,\frac{r - d\cos\theta}{r^2 + d^2 - 2rd\cos\theta}\right]$$

$$E_\theta = \frac{1}{2\pi\epsilon_2}\left[\lambda\,\frac{h\sin\theta}{r^2 + h^2 - 2rh\cos\theta} + \lambda_1\,\frac{d\sin\theta}{r^2 + d^2 - 2rd\cos\theta}\right]$$

and outside:

$$E'_r = \frac{1}{2\pi\epsilon_1}\left[\lambda_2\,\frac{r - h\cos\theta}{r^2 + h^2 - 2rh\cos\theta} + \lambda_3\,\frac{1}{r}\right] \quad E'_\theta = \frac{1}{2\pi\epsilon_1}\left[\lambda_2\,\frac{d\sin\theta}{r^2 + h^2 - 2rh\cos\theta}\right].$$

From the continuity of the tangent component of the electric field $E_\theta = E'_\theta$ on the lateral surface for any value of θ we find:

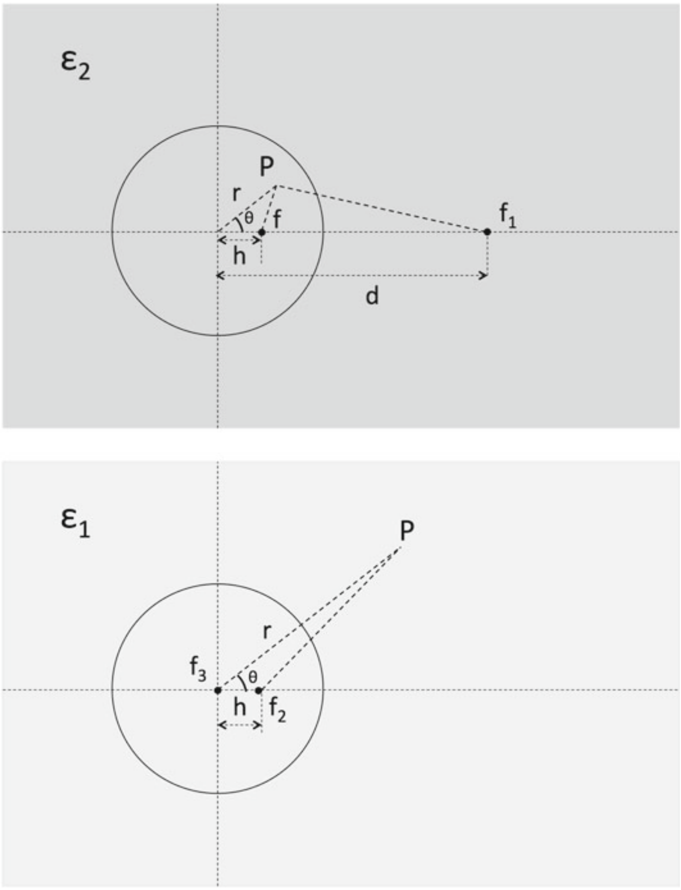


Fig. 4.6 Wire f with linear charge density λ inside a cylinder of permittivity ϵ_2 embedded in a medium of permittivity ϵ_1: *top* wire f and *image wire* f_1 for the field inside the cylinder; *bottom image wires* f_2 and f_3 for the field outside

$$d = \frac{R^2}{h} \qquad \text{and} \qquad \frac{\lambda}{\epsilon_2} - \frac{\lambda_2}{\epsilon_1} + \frac{\lambda_1}{\epsilon_2} = 0.$$

From these relations and from the continuity of the normal component of the displacement field for any value of θ: $D_r = D'_r \;\rightarrow\; \epsilon_2 E_r = \epsilon_1 E'_r$ we find:

$$\lambda_2 = \frac{2\epsilon_1}{\epsilon_1 + \epsilon_2}\lambda \qquad \lambda_1 = \lambda_3 = \frac{\epsilon_2 - \epsilon_1}{\epsilon_1 + \epsilon_2}\lambda\,.$$

If the same medium fills the space $\epsilon_1 = \epsilon_2$: $\lambda_2 = \lambda$, $\lambda_1 = \lambda_3 = 0$.
The force per unit length on the wire is that exerted by the wire f_1:

$$F = \frac{\lambda\lambda_1}{2\pi\epsilon_2}\frac{1}{d-h} = \frac{\lambda^2}{2\pi\epsilon_2}\frac{\epsilon_2-\epsilon_1}{\epsilon_1+\epsilon_2}\frac{h}{R^2-h^2}$$

repulsive for $\epsilon_2 > \epsilon_1$ (wire pushed to the center).

4.4 The problem is similar to Problem 4.2 for the dielectric sphere in an external uniform field. The solution is also suggested by the solutions of the Problem 4.2 and of the Problem 3.7.

We choose as z axis the axis of the cylinder of radius R. We can consider the uniform field due to two wires with opposite charge densities (of absolute value λ), located symmetric with respect the z axis, at distance $d \gg R$: with density λ at $x = -d$ and $-\lambda$ at $x = d$ as shown in Fig. 4.7. To these wires are associated two parallel *image wires*, in the dielectric ϵ_1, with opposite charge densities: λ' at $x = -\delta$ and $-\lambda'$ at $x = \delta$.

The field outside the cylinder is the superposition of the uniform field E, produced by the wires at large distance in the limit $d \to \infty$, and of the field E_{im} from the *image wires*. Inside the cylinder, as proved correct by the result, the field E' has the same direction of the uniform external field.

The potential of the *image wires* at distance r from the axis of the cylinder is:

$$V_{im} = -\frac{\lambda'}{4\pi\epsilon_1}\left(\log\frac{r^2+\delta^2-2r\delta\cos\theta}{r_0^2} - \log\frac{r^2+\delta^2+2r\delta\cos\theta}{r_0^2}\right)$$

$$\text{with: } \lambda' = \frac{\epsilon_1-\epsilon_2}{\epsilon_2+\epsilon_1}\lambda \ \text{ from the result of Problem 4.2,}$$

and in the limit $\delta \ll r$ the potential and the components of the electric field E_{im} become:

$$V_{im} = \frac{\lambda'}{\pi\epsilon_1}\frac{\delta}{r}\cos\theta \qquad E_{imr} = \frac{\lambda'}{\pi\epsilon_1}\frac{\delta}{r^2}\cos\theta \qquad E_{im\theta} = \frac{\lambda'}{\pi\epsilon_1}\frac{\delta}{r^2}\sin\theta\,.$$

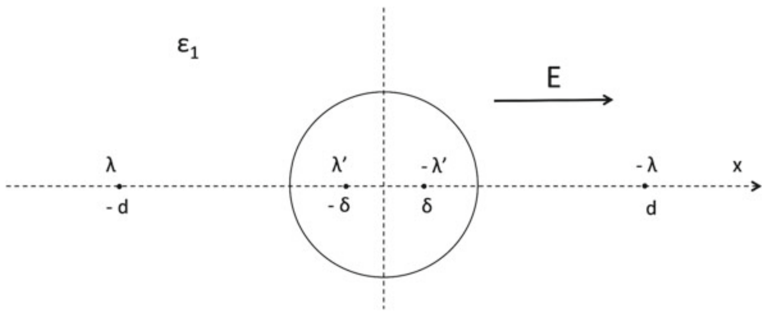


Fig. 4.7 Wires with linear charge density λ to produce the external field E and relative *images wires* with density λ' for a dielectric cylinder embedded in another dielectric medium

The components of the field E are:

$$E_r = E\cos\theta \qquad E_\theta = -E\sin\theta\,.$$

The components for the electric field outside the cylinder are:

$$E_{est,r} = E_r + E_{imr} = E\cos\theta + \frac{\lambda'}{\pi\epsilon_1}\frac{\delta}{r^2}\cos\theta$$

$$E_{est,\theta} = E_\theta + E_{im\theta} = -E\sin\theta + \frac{\lambda'}{\pi\epsilon_1}\frac{\delta}{r^2}\sin\theta$$

and inside:

$$E'_r = E'\cos\theta \qquad E'_\theta = -E'\sin\theta\,.$$

From the continuity of the tangent component of the electric field and of the normal component of the displacement field on the surface of the cylinder, we have the equations:

$$E' = E - \frac{\lambda'}{\pi\epsilon_1}\frac{\delta}{R^2} \qquad \epsilon_2 E' = \epsilon_1 E + \frac{\lambda'}{\pi}\frac{\delta}{R^2}$$

with the solution for E' and $\lambda'\delta$:

$$E' = \frac{2\epsilon_1}{\epsilon_2+\epsilon_1}\,E \qquad \lambda'\delta = \frac{(\epsilon_2-\epsilon_1)}{\epsilon_2+\epsilon_1}\,\epsilon_1\pi R^2 E\,.$$

The components of the electric field outside the cylinder are:

$$E_{est,r} = E\left[1 - \frac{\epsilon_1-\epsilon_2}{\epsilon_2+\epsilon_1}\frac{R^2}{r^2}\right]\cos\theta$$

$$E_{est,\theta} = -E\left[1 + \frac{\epsilon_1-\epsilon_2}{\epsilon_2+\epsilon_1}\frac{R^2}{r^2}\right]\sin\theta\,.$$

Note that the two *image wires* form a sort of *dipole of image wires* of moment[4]:

$$p = 2(-\lambda')\delta = 2\frac{(\epsilon_2-\epsilon_1)}{\epsilon_2+\epsilon_1}\,\epsilon_1\pi R^2 E\,.$$

[4]The negative sign in the formula for p comes from the choice of a positive λ to produce a field E oriented as the z axis.

It is possible to get the same result if we start considering a *dipole* of wires *image* of the wires at distance d in the limit $d \to \infty$. To have a field E from the distant wires the density has to be $\lambda = \pi\epsilon_1 dE$ and then from the result of Problem 4.2, the moment of the *dipole of image wires* is:

$$p = (-\lambda')2\delta = \frac{(\epsilon_2 - \epsilon_1)}{\epsilon_2 + \epsilon_1}\lambda\, 2\frac{R^2}{d} = 2\frac{(\epsilon_2 - \epsilon_1)}{\epsilon_2 + \epsilon_1}(\pi\epsilon_1 E)R^2$$

that is the result already found.

Chapter 5
Functions of Complex Variables and Electrostatics

A useful and elegant indirect approach to solve some two-dimensional problems in Electrostatics, comes from the analytic functions of a complex variable. The real and imaginary parts of these functions are solutions of the Laplace's equation and if they satisfy the boundary-values of the problems, they offer the solutions to these problems. After a brief introduction to the analytic functions, some interesting examples are discussed: the potentials for a quadrupole, a wedge, the edge of a thin plate and a wire. At the end of the chapter the same potentials are derived from the direct solution of the Laplace's equation.

5.1 Analytic Functions of Complex Variable

The functions of complex variables are studied in the courses of mathematics and mathematical methods of physics. Some notions useful to apply these functions to the solution of some electrostatic problems will be briefly discussed.

A complex number can be represented in the Cartesian plane $O(x, y)$ with a correspondence between the complex number $z = x + iy$ and the position of the point P with coordinates (x, y), see Fig. 5.1.

A complex number can also be expressed using polar coordinates (ρ, φ):

$$z = \rho e^{i\varphi} \qquad z = \rho(\cos\varphi + i\sin\varphi)$$

so that $x = \rho\ \cos\varphi$ and $y = \rho\ \sin\varphi$.

A function $f(z)$ of the complex variable z can always be written in the form $f(z) = u + iv$ with $u = u(x, y)$ and $v = v(x, y)$ two real functions of the real variables x and y.

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9_5

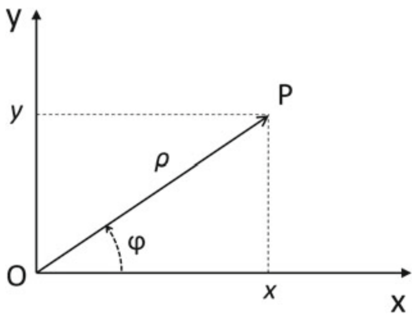


Fig. 5.1 A complex number $z = x + iy$ represented by a point on the complex plane

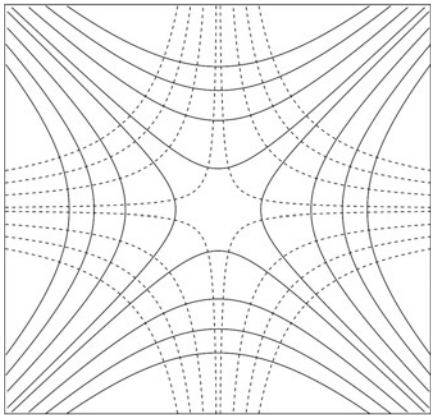


Fig. 5.2 Functions $u(x, y) = x^2 - y^2 = A$ (*solid line*) and $v(x, y) = 2xy = B$ (*dashed line*)

As an example, the function $f(z) = z^2$ can be written as:

$$f(z) = (x + iy)^2 = (x + iy)(x + iy) = x^2 - y^2 + i2xy$$

with $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$ (see Fig. 5.2).

From the theory of the functions of complex variable it is known that a function $f(z) = u(x, y) + iv(x, y)$ is differentiable in a simply connected domain if the Chauchy-Riemann or monogenity equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{5.1}$$

are satisfied with $u(x, y)$ and $v(x, y)$ two continuous functions with continuous partial derivatives.

A function which satisfies these equations is called *analytic* or *holomorphic function* and the existence of its first derivative implies the existence of the derivatives at any order and, as a direct consequence, the function can be expanded in power series. Moreover the integral of the function along a closed path on the complex plane is null and as a consequence its integral along a an arbitrary line depends only on the extremes of the paths ($\gamma_1 \neq \gamma_2$):

$$\oint f(z)\,dz = 0 \qquad\qquad \int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz\,.$$

Taking the derivative of the first relation in (5.1) with respect to $x(y)$ and of the second relation with respect to $y(x)$ and adding (subtracting) the two relations, we get:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad\qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

and then the functions $u(x, y)$ and $v(x, y)$ of any analytic function, satisfy the Laplace's equations:

$$\varDelta u = 0 \qquad\qquad \varDelta v = 0\,.$$

If one of the two functions u or v is given, the other is obtained from (5.1) up to an arbitrary constant. Indeed if u is known, we can write:

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

which can be integrated to find v.

The vectors:

$$\mathbf{u}\left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \quad \text{and} \quad \mathbf{v}\left(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right)$$

respectively normal to $u(x, y) = const$ and $v(x, y) = const'$, as a consequence of (5.1), are normal:

$$\mathbf{u} \cdot \mathbf{v} = 0$$

and thus also the curves $u(x, y) = const$ and $v(x, y) = const'$ are locally normal to each other in their intersection point.

5.2 Electrostatics and Analytic Functions

For a system of N conductors of known geometry and with fixed potentials V_i ($i = 1, \dots, N$), in order to find the potential $V(x, y, z)$ at each point in space, we have to solve the Laplace's equation $\Delta V = 0$ in the region outside the conductors, assuming the potential at the surfaces of the conductors as boundary conditions. This is the Dirichelet's problem.

The functions $u(x, y)$ and $v(x, y)$, associated to an analytic function, are solutions of the Laplace's equation and can be the solution for the potential when the charge distributions depend on two coordinates only and are conformal in all the planes normal to the third axis. These are two-dimensional electrostatic problems.

Indeed, if for the system of the examined conductors it is possible to find a function $u(x, y)$ (or $v(x, y)$) with the value V_i on the surface of the ith conductor, then, for the uniqueness of the solution of the Laplace's equation, $u(x, y)$ (or $v(x, y)$) is the function that describes the potential in the space outside the conductors.

The equipotential curves $u(x, y) = V_i$ (or $v(x, y) = V_i$) describe the equipotential borders of the conductors and the equipotential surfaces outside the conductors are given by the same function by changing the value of the potential. Moreover the field lines are described by the family of curves $v(x, y)$ (or $u(x, y)$) normal in each point to $u(x, y)$ (or $v(x, y)$).

The subject of this chapter is presented in the Feynman Lectures on Physics[1] with an interesting introduction. A very short but effective presentation is given by Pauli in his Electrodynamics.[2]

5.3 The Function $f(z) = z^\mu$

5.3.1 *The Quadrupole:* $f(z) = z^2$

Let us consider the function $f(z) = z^2$ in Electrostatics. This function describes the electric field between the four polar expansions of a quadrupole as sketched in Fig. 5.3.

If the functions $u(x, y) = x^2 - y^2 = \pm A$ are the equipotential surfaces of the poles, the equipotential surfaces inside the quadrupole are similarly described by $x^2 - y^2 = A'$ with $|A'| < A$ and the field lines are represented by the function $v(x, y) = 2xy = B$ with different values for B (positive or negative). This is evident from Figs. 5.2 and 5.3.

---

[1] R.P. Feynman, R.B. Leighton, M. Sands, *The Feynman Lectures on Physics*, Vol. II, Sects. 7.1.2.

[2] W. Pauli, *Electrodynamics*, Sect. 12.2.

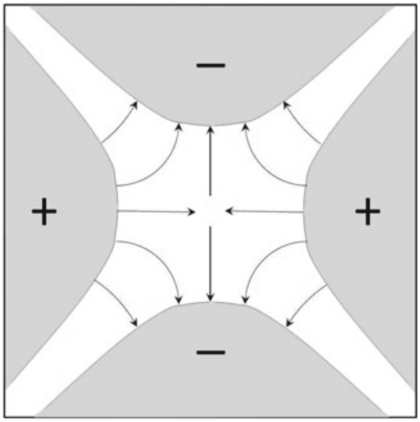


Fig. 5.3 Positive and negative poles in an electric quadrupole and electric field lines

In order to get the potential from the function u(x, y), this has to be multiplied by a constant $C = V/d^2$ with $\pm V$ being the potentials of the polar expansions and d their distance from the center.[3]

The electric field is:

$$\mathbf{E} = -\nabla\left(\frac{V}{d^2}\left(x^2 - y^2\right)\right) \qquad \mathbf{E} : \left(-\frac{2V}{d^2}\,x,\ \frac{2V}{d^2}\,y\right)$$

focusing in the horizontal direction a positive particle crossing the quadrupole while defocusing it in the vertical direction (Fig. 5.3).

The field for a quadrupole rotated by 45° can be obtained simply by exchanging the functions u and v. It is easy to see that he function v describes the field in the corner formed by two conductive planes crossing each other on the z axis at a 90° angle.

5.3.2 The Conductive Wedge at Fixed Potential

Consider the more general function:

$$f(z) = z^\mu = \rho^\mu e^{i\mu\varphi} = \rho^\mu(\cos\mu\varphi + i\sin\mu\varphi)\,,$$

its imaginary part:

$$v = \rho^\mu \sin\mu\varphi \tag{5.2}$$

---

[3]The function $u = x^2 - y^2$ is also the potential for the electric field at the center of a quadrupole composed by four charged wires parallel to the z axis: two with linear charge density λ which cross the plane xy in the points $(a, 0)$ and $(-a, 0)$, and two with linear charge density $-\lambda$ located at $(0, a)$ and $(0, -a)$. The field is that between the poles of the quadrupole if $a = d\sqrt{(\lambda/\pi\epsilon_0 V)}$.

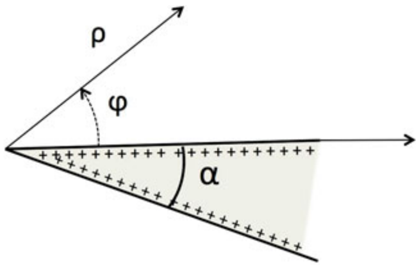


Fig. 5.4 Charged wedge of opening angle α

gives the solution for the potential[4] outside a conductive wedge with an opening angle α (see Fig. 5.4) extended to infinity along the direction orthogonal to the ρ, φ

[4]The formula (5.2) is only a mathematical one, in order to describe a potential it has to be multiplied by a constant C having dimension $[Vl^{-\mu}]$ and then an arbitrary constant can be added:

$$V = C\rho^{\mu} \sin \mu\varphi + V_C \tag{5.3}$$

where V_C is the voltage of the conductive wedge while the constant C depends on the strength of the field near the wedge. We can fix the potential V_1 at distance ρ_1 on the field line $\varphi = \frac{2\pi - \alpha}{2}$, $\mu\varphi = \frac{\pi}{2}$, the bisector of the angle external to the wedge (note that we cannot choose a null potential at an infinite distance because the charge extends to infinity). We get:

$$C = (V_1 - V_C)\rho_1^{-\mu}$$

and the formula for the potential becomes:

$$V = (V_1 - V_C)\left(\frac{\rho}{\rho_1}\right)^{\mu} \sin \mu\varphi + V_C\,.$$

On the bisector line we can write:

$$\frac{V - V_C}{\rho^{\mu}} = \frac{V_1 - V_C}{\rho_1^{\mu}}$$

and if we fix also the potential V_2 at distance ρ_2, from:

$$\frac{V_2 - V_C}{\rho_2^{\mu}} = \frac{V_1 - V_C}{\rho_1^{\mu}}$$

we find the potential to be applied to the wedge:

$$V_C = \frac{\rho_2^{\mu} V_1 - \rho_1^{\mu} V_2}{\rho_2^{\mu} - \rho_1^{\mu}}$$

and the value of the constant C:

$$C = \frac{V_2 - V_1}{\rho_2^{\mu} - \rho_1^{\mu}}\,.$$

The voltage of the wedge is therefore determined by the voltage difference $V_2 - V_1$ between two points at distances ρ_2 and ρ_1 from the edge, that is by the strength of the electric field near the conductive wedge ($C > 0$ for a negative charge and $C < 0$ for a positive charge). For a conductive charged semispace (see in the following) $\mu = 1$ and if positive/negative $C = \mp E_0$.

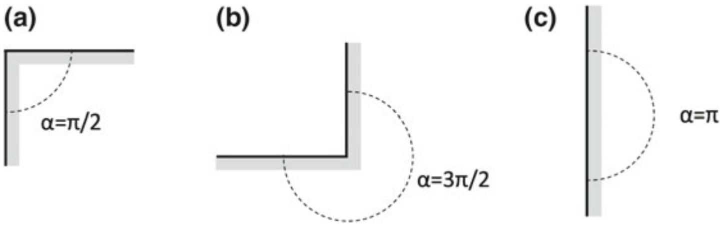


Fig. 5.5 Special configurations for the function z^μ: **a** corner with $\alpha = \pi/2$, **b** corner with $\alpha = 3\pi/2$, **c** plate ($\alpha = \pi$)

plane:

$$V = C\rho^\mu \sin \mu\varphi + V_C \,. \tag{5.4}$$

For the equipotential conductor it has to be:

$$V(\varphi = 0) = V(\varphi = 2\pi - \alpha)$$

and from this:

$$\mu = \frac{\pi}{2\pi - \alpha}. \tag{5.5}$$

Special configurations are:

$\alpha = \frac{\pi}{2}$ $\mu = \frac{2}{3}$ corner with inner right angle (Fig. 5.5a),
$\alpha = \frac{3}{2}\pi$ $\mu = 2$ corner with outer right angle (Fig. 5.5b),
$\alpha = \pi$ $\mu = 1$ conductive plate (semispace) (Fig.5.5c).

The cylindrical components of the electric field are:

$$E_\rho = -\nabla_\rho V = -\frac{\partial V}{\partial \rho} = -\mu C\rho^{\mu-1}\ \sin \mu\varphi$$

$$E_\varphi = -\nabla_\varphi V = -\frac{1}{\rho}\frac{\partial V}{\partial \varphi} = -\mu C\rho^{\mu-1}\ \cos \mu\varphi$$

with $E_\rho = 0$ for $\varphi = 0$ and $\varphi = (2\pi - \alpha)$ as required for the two equipotential conductive planes. For $\varphi = 0$ and $E_\varphi = -\mu C\rho^{\mu-1}$ (opposite to $\hat{\varphi}$), the electric field enters normal into the plane at $\varphi = 0$, while for $\varphi = (2\pi - \alpha)$ and $E_\varphi = \mu C\rho^{\mu-1}$ (oriented as $\hat{\varphi}$), the field enters normal into the plane at $\varphi = 2\pi - \alpha$.

The surface charge density can be obtained using the Coulomb theorem:

$$\sigma \ = \ -\epsilon_0 \mu C\rho^{\mu-1} \,.$$

The charge per unit length at distances $<\rho$ can be calculated from the Gauss theorem. Considering the flux through the cylindrical surface of radius ρ and axis the edge of the wedge, we have:

$$\frac{Q(\rho)}{\epsilon_0 \Delta z} = \int_0^{2\pi-\alpha} \mathbf{E} \cdot \hat{n}\, dS$$

$$\frac{Q(\rho)}{\Delta z} = \epsilon_0 \int_0^{2\pi-\alpha} E_\rho\, \rho d\varphi = \epsilon_0 \int_0^{2\pi-\alpha} -\mu C \rho^{\mu-1} \sin \mu\varphi\, \rho d\varphi = -2\epsilon_0 C \rho^{\mu} \quad .$$

Corner with inner right angle (Fig. 5.5a):

$\alpha = \frac{\pi}{2} \quad \mu = \frac{2}{3} \quad f(z) = z^{\frac{2}{3}} \quad V = C\rho^{\frac{2}{3}} \sin(\frac{2}{3}\varphi) + V_C \quad \sigma = -\frac{2}{3}\epsilon_0 C \rho^{-\frac{1}{3}}$
$E_\varphi = -\frac{2}{3} C \rho^{-\frac{1}{3}}$ for $\varphi = 0$ and $E_\varphi = \frac{2}{3} C \rho^{-\frac{1}{3}}$ for $\varphi = \frac{3}{2}\pi$.

$$\frac{Q(\rho)}{\Delta z} = -2\epsilon_0 C \rho^{\frac{2}{3}}.$$

Corner with outer right angle (Fig. 5.5b):
this is one quarter of the quadrupole (the first quadrant).
$\alpha = \frac{3}{2}\pi \quad \mu = 2 \quad f(z) = z^2 \; V = C\rho^2 \sin(2\varphi) + V_C \quad \sigma = -2\epsilon_0 C \rho$
$E_\varphi = -2C\rho$ for $\varphi = 0$ and $E_\varphi = 2C\rho$ for $\varphi = \frac{3}{2}\pi$.

$$\frac{Q(\rho)}{\Delta z} = -2\, \epsilon_0 C \rho^2.$$

Conductive plate (semispace) (Fig. 5.5c):

$\alpha = \pi \quad \mu = 1 \quad f(z) = z \quad V = C\rho \sin(\varphi) + V_C \quad \sigma = -\epsilon_0 C$
$E_\varphi = -C$ for $\varphi = 0$ and $E_\varphi = C$ for $\varphi = \pi$,

$$\frac{Q(\rho)}{\Delta z} = -\, 2\epsilon_0 C \rho.$$

For a positive charged plate $C = -E_0$ (see footnote) we have $E_\varphi = E_0$ for $\varphi = 0$, $E_\varphi = -E_0$ for $\varphi = \pi$, and the charge density $\sigma = \epsilon_0 E_0$.

5.3.3 *Edge of a Thin Plate*

For $\alpha = 0$ the wedge becomes the edge of a thin conductive plate. In this case $\mu = \frac{1}{2}$ and the function $f(z)$ becomes:

$$f(z) = z^{\frac{1}{2}} = \rho^{\frac{1}{2}}\, e^{i\frac{\varphi}{2}} = \rho^{\frac{1}{2}} \left(\cos\left(\frac{\varphi}{2}\right) + i\ \sin\left(\frac{\varphi}{2}\right)\right) \ . \tag{5.6}$$

The potential $V = C\rho^{\frac{1}{2}} \sin(\frac{\varphi}{2}) + V_C$ is $V = V_C$ for $\varphi = 0$ and $\varphi = 2\pi$, while for $\varphi = \pi$ is:

$$V = C\rho^{\frac{1}{2}} + V_C\ .$$

The components of the electric field are:

$$E_\rho = -\frac{1}{2} C \rho^{-\frac{1}{2}} \sin\left(\frac{\varphi}{2}\right)$$

$$E_\varphi = -\frac{1}{2} C \rho^{-\frac{1}{2}} \cos\left(\frac{\varphi}{2}\right)$$

with $E_\rho = 0$ for $\varphi = 0$ and $\varphi = 2\pi$, and $E_\varphi = -\frac{1}{2} C \rho^{-\frac{1}{2}}$ for $\varphi = 0$ and $E_\varphi = \frac{1}{2} C \rho^{-\frac{1}{2}}$ for $\varphi = 2\pi$.

The function (5.6) can easily be written in terms of Cartesian coordinates:

$$f(z) = \rho^{\frac{1}{2}} \left\{ \sqrt{\frac{1 + \cos\varphi}{2}} + i \sqrt{\frac{1 - \cos\varphi}{2}} \right\}$$

$$= \left\{ \left[\frac{\left(x^2 + y^2\right)^{\frac{1}{2}} + x}{2} \right]^{\frac{1}{2}} \right\} + i \left\{ \left[\frac{\left(x^2 + y^2\right)^{\frac{1}{2}} - x}{2} \right]^{\frac{1}{2}} \right\}$$

with the equipotential surfaces given by:

$$\left[\frac{\left(x^2 + y^2\right)^{\frac{1}{2}} - x}{2} \right]^{\frac{1}{2}} = A$$

and the field lines by:

$$\left[\frac{\left(x^2 + y^2\right)^{\frac{1}{2}} + x}{2} \right]^{\frac{1}{2}} = B$$

with different values for the constants A and B (see Fig. 5.6).

5.4 The Charged Wire: $f(z) = \log z$

The field due to an infinite charged wire aligned with the z axis is given by the function:

$$f(z) = \log z\,.$$

By substituting $z = \rho\, e^{i\varphi}$ we have:

$$f(z) = \log \rho \, + i\varphi\,.$$

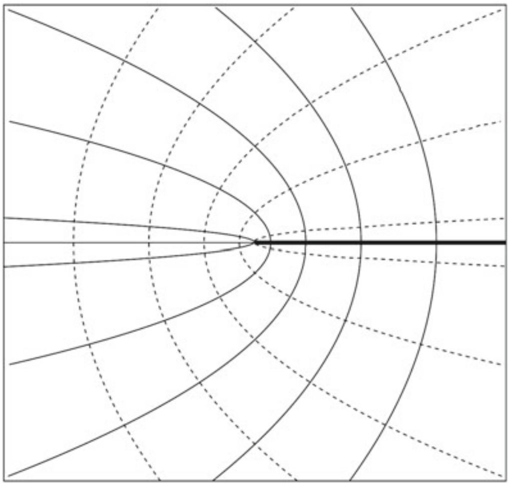


Fig. 5.6 Equipotential curves (*dotted*) and electric field lines (*solid*) near the edge of a thin charged plate (*thick line*)

The real part, multiplied by a factor accounting for charge density and units, gives the well known logarithmic potential of the charged wire:

$$V(\rho) = -\frac{\lambda}{2\pi\epsilon_0}\ \log\rho$$

while the imaginary part $\varphi = const$ represents the equations of the electric field lines.

5.5 Solution of the Laplace's Equation for Two-Dimensional Problems: Wire and Corners

It is interesting to find the potentials for the two-dimensional charge configurations examined in this chapter: the wire and the conductive corners. We have to solve the Laplace's equation with the potentials on the conductors[5] as boundary values.

Using polar (cylindrical) coordinates, as suggested by the symmetry of the problem, the Laplace equation is:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial V}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial\varphi^2} = 0\,.$$

[5] This paragraph is based on the more detailed presentation given in Jackson, *Classical Electrodynamics*, cited, Sect. 2.11.

This can be solved using the separation of variables. By substituting:

$$V(\rho, \varphi) = R(\rho)\Phi(\varphi)$$

and upon multiplication for ρ^2/Φ we have the equations:

$$\rho \frac{\partial}{\partial \rho}\left(\rho \frac{\partial R}{\partial \rho}\right) = A^2 R \qquad \frac{\partial^2 \Phi}{\partial \varphi^2} = -A^2 \Phi$$

For $A^2 < 0$ the solutions are exponential in φ and not acceptable for our problems, therefore we can have only $A^2 \geq 0$. It is easy to show that the solutions are:

$$\begin{array}{lll} \text{for } A^2 = 0 & R(\rho) = a_0 + b_0 \ \log(\rho) \ \text{ and } & \Phi(\varphi) = A_0 + B_0\varphi \\ \text{for } A^2 > 0 & R(\rho) = c\ \rho^A + d\ \rho^{-A} \ \text{ and } & \Phi(\varphi) = a\ \cos A\varphi + b\ \sin A\varphi\ . \end{array}$$

For a full symmetry around the z axis only the solution $A^2 = 0$ is valid and $B_0 = 0$ because the function does not depend on φ. For instance this is the case for the field between the two conductive surfaces of a cylindrical condenser. The potential becomes:

$$V = A_0\,(a_0 + b_0\ \log \rho)$$

with the known coefficient $A_0 b_0 = -\frac{\lambda}{2\pi\epsilon 0}$ and $A_0 a_0$ an additive constant potential.

For a conductive wedge of angle α we have to consider the solutions for $A^2 \geq 0$. To include in the solution the point at $\rho = 0$ we have to set $b_0 = 0$ and $d = 0$. Then the most general solution is:

$$V = a_0 A_0 + a_0 B_0 \varphi + c\ \rho^A\,(a\ \cos A\varphi\ +\ b\ \sin A\varphi)\ .$$

In order to get the same potential V_0 for any value of ρ at $\varphi = 0$ and at $\varphi = 2\pi - \alpha$, we have to put $a = 0$, $a_0 B_0 = 0$, $a_0 A_0 = V_0$ and $A = \frac{n\pi}{2\pi - \alpha} = n\mu$ where μ is the coefficient defined in (5.5).

The solution is reduced to:

$$V = V_0 + \sum_{n=1}^{\infty} V_n \rho^{n\mu}\ \sin\ n\mu\varphi\ .$$

It is evident that the solutions with $n > 1$ are of no interest for a simple conductive wedge. Therefore for $n = 1$ we have $A = \mu = \pi/(2\pi - \alpha)$ and finally the solution is:

$$V = V_0 + V_1 \rho^{\mu}\ \sin\ \mu\varphi$$

that is the formula (5.4) with the coefficient (5.5).

Problems

5.1 Show the sum:

$$f(z) = -\sum_j \frac{\lambda_j}{2\pi\epsilon_0} \log(z - z_j)$$

yields the potential for a system of wires with linear charge density λ_j, parallel to the z axis and crossing the plane xy in the points (x_j, y_j).

5.2 Verify that the function:

$$f(z) = \frac{\lambda\delta}{2\pi\epsilon_0}\frac{1}{z}$$

gives the potential at large distance due to two opposite charge parallel wires. Be λ the linear charge density and δ the distance of the wires. The two wires are a two-dimensional analog of an electric dipole.

Remind the power series:

$$\log\ (1+z)\ = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots \qquad |z| < 1.$$

5.3 Use the formula in problem 1 to show that the field inside a quadrupole is that produced near the axis of a quadrupole by four wires: two with linear charge density λ at $(a, 0)$ and $(-a, 0)$ on the complex plane, and two with $-\lambda$ at $(0, ia)$ and $(0, -ia)$.

Solutions

5.1 From $z = \rho e^{i\varphi}$, the position of the point P, and $z_j = \rho_j e^{i\varphi_j}$ that of the jth wire, we have:

$$z - z_j = \rho e^{i\varphi} - \rho_j e^{i\varphi_j} = r_j e^{i\alpha_j}$$

with:

$$r_j^2 = \rho^2 + \rho_j^2 - 2\rho\rho_j \cos(\varphi - \varphi_j) \qquad tg\ \alpha_j = \frac{\rho\sin\varphi - \rho_j \sin\varphi_j}{\rho\cos\varphi - \rho_j\cos\varphi_j}$$

where r_j is the distance of the point P from the wire jth. The complex potential becomes:

$$f(z - z_j) = -\frac{\lambda_j}{2\pi\epsilon_0} \log r_j e^{i\alpha_j}.$$

Taking the real part of this potential due to wire jth and summing on all the wires we get:

$$V = -\sum_j \frac{\lambda_j}{2\pi\epsilon_0} \log r_j .$$

5.2 Locate the wire with charge density λ at $(0, \delta/2)$ and that with $-\lambda$ at $(0, -\delta/2)$. From the previous problem the complex potential is:

$$f(z) = -\frac{\lambda}{2\pi\epsilon_0}\left[\log\left(z-\frac{\delta}{2}\right) - \log\left(z+\frac{\delta}{2}\right)\right] .$$

We can write:

$$\log\left(z-\frac{\delta}{2}\right) - \log\left(z+\frac{\delta}{2}\right) = \log\ \left(1-\frac{\delta}{2z}\right) - \log\left(1+\frac{\delta}{2z}\right)$$

replacing $z = \rho e^{i\varphi}$ and reminding the power series for $\log(1+z)$ we get:

$$\log\left(1-\frac{\delta e^{-i\varphi}}{2\rho}\right) - \log\left(1+\frac{\delta e^{-i\varphi}}{2\rho}\right) \simeq -\frac{\delta e^{-i\varphi}}{\rho} = -\frac{\delta}{z} .$$

The complex potential is:

$$f(z) = \frac{\lambda}{2\pi\epsilon_0}\frac{\delta}{z} = \frac{\lambda}{2\pi\epsilon_0}\frac{\delta e^{-i\varphi}}{\rho}.$$

Its real part gives the potential for the two wires:

$$V(\rho,\varphi) = \frac{\lambda}{2\pi\epsilon_0}\frac{\delta\cos\varphi}{\rho}$$

already seen in the solution of the Problem 3.8 (formula 3.7).

5.3 The complex potential from the four wires is:

$$V(z) = -\frac{\lambda}{2\pi\epsilon_0}\left[\log(z-a) + \log(z+a) - \log(z-ia) - \log(z+ia)\right]$$

$$= -\frac{\lambda}{2\pi\epsilon_0}\left[\log(a-z) + \log(a+z) - \log(ia-z) - \log(ia+z)\right]$$

that expanded in a power series neglecting terms of third order or higher becomes:

$$V(z) = -\frac{\lambda}{2\pi\epsilon_0}\left[-\frac{2z^2}{a^2} - i\pi\right]$$

and removing the constant term we get:

$$V(z) = \frac{\lambda}{\pi \epsilon_0} \frac{z^2}{a^2} = \frac{\lambda}{\pi \epsilon_0 a^2} \left[(x^2 - y^2) + i2xy \right]$$

where the real part, for $\lambda/\pi\epsilon_0 a^2 = V_0/d^2$ as anticipated in a footnote, gives the potential $V(x, y)$ of the quadrupole, with distance d of the poles from the axis, at voltage V_0.

Chapter 6
Relativistic Transformation of E and B Fields

The invariance of the electric charge and the coordinate transformations in special relativity determine the transformation laws for the electric charge and current densities. Changing densities implies different electric and magnetic fields in reference frames in uniform motion relative to one another. The example of a charged particle moving parallel to a wire carrying a current is presented. From this and other examples it is possible to derive the relativistic transformations for electric and magnetic fields.

6.1 From Charge Invariance to the 4-Current Density

The charge Q is the same in all reference frames: the charge is a relativistic invariant and therefore it is a *scalar*. This is a direct consequence of the invariance[1] of the elementary charge e value in all inertial frames because a charge Q is always a multiple of the elementary charge.

Many experiments[2] have established limits of the order of 1 part over 10^{21} on the relative difference between the charge of the proton and the charge of the electron in atoms and molecules where electrons are moving with velocities greater than $0.01c$ while protons in nuclei are moving with velocities about $0.3c$. Further evidence for charge invariance comes from the validity of the Lorentz force when applied to particle accelerators that confine moving charged particles into near circular orbits by

---

[1]The elementary electric charge e, the velocity of the light c and the Planck constant h are three scalar quantities, therefore these three constants are the same in all reference frames. The reason has probably to be searched in some symmetry beyond the Electrodynamics and the Standard Model of elementary particles.

[2]See for instance J.D. Jackson, *Classical Electrodynamics*, cited, Sect. 11.9.

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9_6

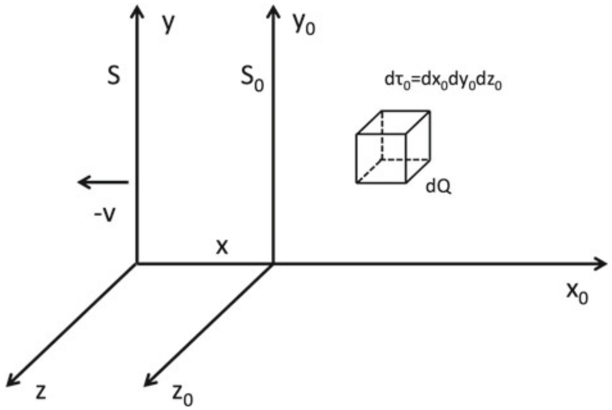


Fig. 6.1 The charge dQ in the small volume $d\tau_0$ at rest in S_0. The frame S moves relative to S_0 with velocity −v along the x axis

using magnetic fields.[3] All these considerations support the invariance of the electric charge under Lorentz transformations.

Consider[4] two inertial reference frames: $S(x, y, z)$, that we assume at rest, and $S_0(x_0, y_0, z_0)$ in motion with uniform velocity v, along the x axis, relative to $S(x, y, z)$ as in Fig. 6.1. The charge dQ inside a small volume $d\tau_0 = dx_0dy_0dz_0$ is at rest in the frame S_0. In this frame the charge density is:

$$\rho_0 = \frac{dQ}{d\tau_0} = \frac{dQ}{dx_0dy_0dz_0}\,.$$

In the frame S, in motion with velocity $-v$ along x relative to S_0, the distances transverse to the motion do not change: $dy = dy_0$, $dz = dz_0$ while along x we have $dx = dx_0\sqrt{1-\frac{v^2}{c^2}}$. Therefore, because of the Lorentz contraction, the charge density in S is:

$$\rho = \frac{dQ}{d\tau} = \frac{dQ}{dxdydz} = \frac{dQ}{dx_0dy_0dz_0}\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{\rho_0}{\sqrt{1-\frac{v^2}{c^2}}} = \rho_0\gamma$$

[3]The formula for the Lorentz force predicts the orbit curvature from non relativistic velocities up to $\gamma = 6500$, namely $(c - v)/c \sim 1.2 \cdot 10^{-8}$, for protons in the LHC collider and $\gamma = 2 \times 10^5$ $((c - v)/c \sim 1.25 \cdot 10^{-13})$ for electrons in the LEP200 collider. Anyway, due to the synchrotron radiation losses and to the corrections to the energy and to the path of the beams, the limit on the relativistic charge invariance is modest $(\delta q/q \lesssim 10^{-9})$.

[4]The approach used in this chapter can be found also in other textbooks: C. Mencuccini and V. Silvestrini, *Fisica 2, Elettromagnetismo-Ottica*, Sect. 5.8, Zanichelli Ed.; R.P. Feynman, R.B. Leighton, M. Sands, *The Feynman Lectures on Physics*, Vol. II, Sect. 13.6; E.M. Purcell, *Electricity and Magnetism, Berkeley Physics Course*, Vol II, Chapter V, McGraw-Hill; F. Lobkowicz and A.C. Melissinos, *Physics for Scientists & Engineers*, Vol. II, Sect. 13.2.

with:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\,.$$

In the frame S the charge moves with a velocity $\mathbf{v}$ and it is seen as a current with current density $\mathbf{J} = \rho\mathbf{v} = \rho_0\mathbf{v}\gamma$. We can realize that $J_0 = \rho c$ and $\mathbf{J}$ are the time and space components of a 4-vector current density[5] $\underline{J} = (\rho c, \mathbf{J}) = (\rho_0 c\gamma,\ \rho_0\mathbf{v}\gamma)$:

$$\underline{J} = \left(\frac{\rho_0 c}{\sqrt{1-\frac{v^2}{c^2}}}, \frac{\rho_0\mathbf{v}}{\sqrt{1-\frac{v^2}{c^2}}}\right) \tag{6.1}$$

with norm $|\underline{J}|^2 = J_0^2 - |\mathbf{J}|^2 = \rho_0^2 c^2$. In the frame S_0 where the charge is at rest, the 4-current density has components: $(\rho_0 c,\ 0)$.

The transformations of the 4-current density from a reference frame S to a frame S' in motion along x with velocity V, can be found easily[6] using the transform matrix L:

$$L = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \beta = \frac{V}{c} \qquad \gamma = \frac{1}{\sqrt{1-\frac{V^2}{c^2}}}\,.$$

Writing $\underline{J} = (\rho c, J_x, J_y, J_z)$ and $\underline{J'} = (\rho' c, J'_x, J'_y, J'_z)$ as column matrices, and taking the product $\underline{J'} = L\ \underline{J}$:

$$\begin{pmatrix} \rho' c \\ J'_x \\ J'_y \\ J'_z \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \rho c \\ J_x \\ J_y \\ J_z \end{pmatrix} \tag{6.2}$$

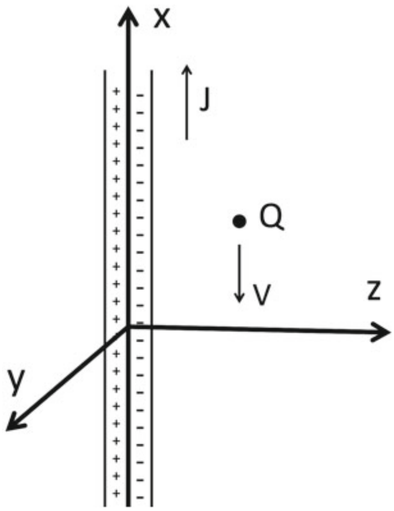
for the transformations of charge and current densities we get the relations:

$$\begin{cases} \rho' = (\rho - \frac{V}{c^2} J_x)\gamma \\ J'_x = (J_x - \rho V)\gamma \\ J'_y = J_y \\ J'_z = J_z \end{cases}\,.$$

[5] Note that (6.1) can be found as $\underline{J} = \rho_0 \underline{u}$ where $\underline{u}$ is the 4-vector velocity (7.13) similarly to the 4-momentum of a particle $\underline{p} = m_0\underline{u}$ given by the relations (7.14).

[6] For the coordinates $\underline{x'} = (ct', x', y', z')$ and $\underline{x} = (ct, x, y, z)$ the product $\underline{x'} = L\ \underline{x}$ gives $ct' = (ct - \beta x)\gamma,\ x' = (x - \beta ct)\gamma,\ y' = y,\ z' = z$. See also next chapter.

Fig. 6.2 The wire carrying a current and the charge Q with velocity V parallel to the wire



Switching from the S_0 frame where the charge is at rest and the 4-current density is $\underline{J} = (\rho_0 c,\ 0)$ to the S frame, the transformation (6.2) with $V = -$v, gives the charge and current density $\underline{J} = (\rho_0 c\gamma,\ \rho_0 \mathbf{v}\gamma)$ already given in (6.1).

6.2 Electric Current in a Wire and a Charged Particle in Motion

An infinite wire placed along the x axis carries a current flowing in the positive direction. A particle with charge Q moves parallel to the wire, at distance r, with velocity $\mathbf{V} = (-V, 0, 0)$ as shown in Fig. 6.2. The magnetic field, due to the current, produces a Lorentz force $\mathbf{F}_L = Q\mathbf{V} \times \mathbf{B}$ on the moving particle which gives to the particle a radial acceleration. In the frame moving along the x axis with the same velocity of the particle, the particle is at rest and so a Lorentz force cannot be present. For the principle of relativity we have to foresee the presence of an electrostatic field that gives the same acceleration to the particle at rest.

Defining n the number of charges per unit volume and $\mathbf{v}$ $(-v, 0, 0)$ the drift velocity of the electrons in the wire, the 4-current positive and negative densities,[7] observed in the laboratory frame S where the neutral wire is at rest, are $\underline{J}_+(nqc, 0, 0, 0)$ and $\underline{J}_-(-nqc, nqv, 0, 0)$. The total current density is:

$$\underline{J} = \underline{J}_+ + \underline{J}_- = (0, nqv, 0, 0)\,. \tag{6.3}$$

[7]The negative charges are the free electrons and the positive charges are the charges left after the electrons are freed from the atoms.

Upon the transformation (6.2) from the laboratory frame S to the frame S' where the charge Q is at rest, with $\beta = -\frac{V}{c}$, we find:

$$\underline{J}' = \underline{J}'_{+} + \underline{J}'_{-} = (nqv\frac{V}{c}\gamma, nqv\gamma, 0, 0)\,. \tag{6.4}$$

Thus in the frame S' in addition to the current density $\mathbf{J}' = nqv\gamma\,\hat{\mathbf{x}}$, that produces a magnetic field, there is a charge density $\rho' = nqv\frac{V}{c^2}\gamma$ that generates an electrostatic field.

It is interesting to compare how the motion of the particle is described in the two frames.

Laboratory frame (wire at rest). If Σ is the area of the cross section of the wire, there is a current $I = J\Sigma = nqv\Sigma$ which at distance r produces a magnetic field:

$$B = \frac{\mu_0}{2\pi}\frac{J\Sigma}{r} = \frac{\mu_0}{2\pi}\frac{nqv\Sigma}{r} \tag{6.5}$$

and, for the Lorentz force, the equation of motion for the charge Q is:

$$\frac{dp_r}{dt} = QVB = \frac{\mu_0}{2\pi}\frac{nqv\Sigma}{r}QV \tag{6.6}$$

with p_r the radial component of the momentum.

Frame with the charge at rest. In this system, as seen before, the wire has a charge density $\rho' = nqv\frac{V}{c^2}\gamma$, that is a linear charge density $\lambda' = \rho'\Sigma$, which produces a radial electric field:

$$E' = \frac{1}{2\pi\epsilon_0}\frac{\lambda'}{r} = \frac{1}{2\pi\epsilon_0}\frac{nqv\Sigma}{r}\frac{V}{c^2}\gamma \tag{6.7}$$

and the equation of motion[8] is:

$$\frac{dp_r}{dt'} = QE' = \frac{1}{2\pi\epsilon_0}\frac{nqv\Sigma}{r}\frac{V}{c^2}Q\gamma = \frac{\mu_0}{2\pi}\frac{nqv\Sigma}{r}QV\gamma \tag{6.8}$$

where we have used the relation $1/c^2 = \mu_0\epsilon_0$.

Observing that the proper time interval of the charge dt' is related to dt by the relation $dt = dt'\gamma$, the Eq. (6.6) becomes:

$$\frac{dp_r}{dt'} = \frac{\mu_0}{2\pi}\frac{nqv\Sigma}{r}QV\gamma$$

that coincides with Eq. (6.8).

[8] The radial component of the momentum is the same in the two frames because it is normal to their relative velocity V.

In the two reference frames S and S' the Eqs. (6.6) and (6.8) are the equivalent of the covariant equation of motion (7.15):

$$\frac{dp^\mu}{d\tau} = f^\mu$$

that we will see in the next chapter.

From this example we get the transformation for a magnetic field $B_\perp$ transverse to the relative motion of the two frames. From (6.4), (6.5) and (6.7) we have:

$$B_\perp \quad \Longrightarrow \quad B'_\perp = \frac{\mu_0}{2\pi}\frac{J'\Sigma}{r} = B_\perp\gamma \qquad E'_\perp = \frac{1}{2\pi\epsilon_0}\frac{\lambda'}{r} = VB_\perp\gamma\,. \tag{6.9}$$

Note that the fields depend on the charge and current densities that are different in the two frames.

6.3 Transformation of the E and B Fields

As seen in the example of the current in a wire and the charge Q, the 4-current density in the frame S is $\underline{J} = \underline{J}_+ + \underline{J}_- = (0, nqv, 0, 0)$ and only an electric current is present. In a frame S', in motion with velocity $V\hat{\mathbf{x}}$ relative to S, the 4-current density is $\underline{J}' = \underline{J}'_+ + \underline{J}'_- = (-nqv\frac{V}{c}\gamma, nqv\gamma, 0, 0)$ and a current density and a non null charge density are both observed in the wire. This implies that the magnetic field $B_\perp$ in S is seen in S' as the superposition of a magnetic field $B'_\perp$ and of an electric field $E'_\perp$ as in (6.9). In a similar way if in S we have a non null charge density $\underline{J} = \underline{J}_+ + \underline{J}_- = (nqc, 0, 0, 0)$, the 4-current density in S' is $\underline{J}'' = \underline{J}''_+ + \underline{J}''_- = (nqc\gamma, -nqV\gamma, 0, 0)$ and thus a charge density and a current density are both present, so that an electric field E in S is seen in S' as an electric field E'' plus a magnetic field B''. This can be seen for instance in the parallel plate condenser in motion considered in Problem 6.1 where we have:

$$E \quad \Longrightarrow \quad E'_\parallel = E_\parallel \qquad B'_\parallel = 0 \qquad E'_\perp = E_\perp\gamma \qquad B'_\perp = \frac{V}{c^2}E_\perp\gamma$$

while in the case of a solenoid in motion along its axis, in Problem 6.2, we find only: $B'_\parallel = B_\parallel$.

From these considerations and from the analysis of the forces applied to the charges it is possible to find the transformation laws for the fields E and B from one frame to another frame in relative motion with velocity V ($\beta = \frac{V}{c}$). These are:

$$\begin{cases} E'_x = E_x \\ E'_y = \gamma(E_y - c\beta B_z) \\ E'_z = \gamma(E_z + c\beta B_y) \end{cases} \qquad \begin{cases} B'_x = B_x \\ B'_y = \gamma(B_y + \beta\frac{E_z}{c}) \\ B'_z = \gamma(B_z - \beta\frac{E_y}{c}) \end{cases} \tag{6.10}$$

or in terms of the components parallel and transverse to the velocity **V**:

$$\begin{cases} \mathbf{E}'_{\parallel} = \mathbf{E}_{\parallel} \\ \mathbf{E}'_{\perp} = \gamma(\mathbf{E}_{\perp} + \mathbf{V} \times \mathbf{B}_{\perp}) \end{cases} \qquad \begin{cases} \mathbf{B}'_{\parallel} = \mathbf{B}_{\parallel} \\ \mathbf{B}'_{\perp} = \gamma(\mathbf{B}_{\perp} - \frac{1}{c^2}\mathbf{V} \times \mathbf{E}_{\perp}) \, . \end{cases}$$

In next chapter these transformations will be derived on the basis of more general principles.

It is evident that the mixture of the electric and magnetic fields changes when considered in different frames. Indeed the two fields are different components of the same field, the electromagnetic field. This will be evident when the tensorial nature of the electromagnetic field will be introduced in next chapter.

In this chapter the transformations of the electric and magnetic field have been derived from the charge invariance and the relativistic transformation of the coordinates. Historically Einstein was led to the special theory of relativity to get a coherent description of phenomena in electrodynamics differently interpreted when observed in different frames.

6.4 The Total Charge in Different Frames

From the expressions of the 4-current density (6.3) and (6.4) the wire seems neutral in the laboratory frame but charged in the frame with the charge Q in motion. It seems $Q_w = 0$ in the laboratory while $Q'_w \neq 0$ in the charge rest frame, in evident contrast with the invariance of the charge.

Consider the charge present in the wire taking into account also the drift motion of the electrons with velocity v. In the laboratory the 4-current densities are:

$$\underline{J}_+(nqc, 0, 0, 0) \qquad \underline{J}_-(-n'qc\gamma', n'qv\gamma', 0, 0)$$

where n' is the density of the free electrons in their rest frame (only the drift velocity is considered) and:

$$\gamma' = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \, .$$

The total 4-current density is:

$$\underline{J} = \underline{J}_+ + \underline{J}_- = (qc(n - n'\gamma'), \; n'qv\gamma', 0, 0) \, . \tag{6.11}$$

The 4-current density in the rest frame of the charge moving parallel to the wire is:

$$\underline{J}' = \left[nqc\gamma - n'qc\left(1 - \frac{vV}{c^2}\right)\gamma'\gamma, \; nqV\gamma + n'q(v - V)\gamma\gamma', 0, 0 \right].$$

The positive charge density in the laboratory frame S is $\rho_+ = nq$ and the charge in an element of length l of the wire is $Q_{w+} = nq\Sigma l$. In the charge Q rest frame S' the positive charge density is $\rho'_+ = nq\gamma$ and the charge in the same element of wire is $Q'_{w+} = nq\gamma\Sigma l'$ with $l' = l/\gamma$ the length seen in S'. So that we find $Q'_{w+} = nq\Sigma l = Q_{w+}$, the positive charge is the same in the two frames.

The negative charge density in the frame of the free electrons at rest is $\rho_- = -n'q$ and the charge in an element of wire of length l_e in this frame is $Q_{w-} = -n'q\Sigma l_e$. In the rest frame of the charge Q the negative density is:

$$\rho'_- = -n'q\left(1 - \frac{vV}{c^2}\right)\gamma\gamma'$$

and the charge in the same wire element, seen of length l'_e by the charge in motion, is:

$$Q'_{w-} = -n'q\left(1 - \frac{vV}{c^2}\right)\gamma\gamma'\Sigma l'_e\,.$$

For the length l'_e we can write: $l'_e = l_e/\gamma''$ with γ'' relative to the transformation from the rest frame of the free electrons to the frame of the charge.

From the addition-velocity formula the velocity of the charge relative to the electrons is:

$$V'' = \frac{V - v}{1 - \frac{vV}{c^2}}$$

thus we have:

$$\gamma'' = \frac{1}{\sqrt{1 - \frac{V''^2}{c^2}}} = \left(1 - \frac{vV}{c^2}\right)\gamma\gamma'$$

and for the negative charge Q'_{w-}:

$$Q'_{w-} = -n'q\left(1 - \frac{vV}{c^2}\right)\gamma\gamma'\Sigma\frac{l_e}{\gamma''} = -n'ql_e\Sigma = Q_{w-}$$

therefore the negative charge in the frame of the charge in motion is equal to the charge in the frame of the free electrons at rest.

Of course if the wire in the laboratory is neutral we have $Q_{w+} = Q_{w-}$ and this relation is valid in any other frame. The neutral wire implies $n = n'\gamma'$.

Due to the Lorentz contraction, the length of an element of a charged wire changes, but the charge density changes inversely and thus in any frame the charge in the same element of wire is the same. The charge is an invariant, but passing from a frame to another the charge density and the current density change and also the electric and magnetic fields which depend on these densities as seen for instance in (6.5) and (6.7).

Problems

6.1 Determine in the laboratory frame the electric field between the rectangular parallel plates of a charged condenser moving with a velocity parallel to a side of the plates or with a velocity normal to the plates. Is there any magnetic field?

6.2 Find the magnetic field inside a long solenoid with a current I in its turns, which moves with velocity V in the direction of its axis.

6.3 Find the force per unit length between two infinite parallel wires carrying the same current, in terms of an electrostatic force as seen for a charge moving parallel to a current in a wire.

6.4 A rectangular loop, with sides a and b and cross section Σ, carries a current with density J. This loop moves in the laboratory with velocity V parallel to the side b. Find the densities of charge and current in the four sides of the loop. Calculate the electric dipole moment seen in the laboratory.

Solutions

6.1 We consider a condenser with parallel rectangular plates with sides a and b at a negligible distance with respect to these dimensions. When a charge Q is present on the plates, the electric field between the plates is:

$$E_0 = \frac{\sigma}{\epsilon_0} = \frac{Q}{ab\epsilon_0} .$$

If the condenser in the laboratory moves with velocity V in the direction of the side b, due to the Lorentz contraction, the area of the plates is $S' = ab'$ with $b' = b\sqrt{1 - V^2/c^2}$ and the electric field is:

$$E_0' = \frac{\sigma'}{\epsilon_0} = \frac{Q}{ab'\epsilon_0} = \frac{Q}{ab\epsilon_0} \frac{1}{\sqrt{1 - V^2/c^2}} = E_0\gamma .$$

In the laboratory to the surface charge density σ' is associated a surface current density $J_s' = \sigma' V$ with opposite direction on the two plates. It is easy to find the magnetic field between the plates:

$$B = \mu_0 J_s' = \mu_0 \sigma' V = \frac{V}{c^2}\gamma E_0$$

normal to the velocity and parallel to the plates. If $\mathbf{V} = V\hat{\mathbf{x}}$ is the velocity of the condenser in the laboratory and $\mathbf{E}_0 = E_{0y}\hat{\mathbf{y}}$ is the electric field in the rest frame of the condenser, the electric and the magnetic fields in the laboratory are:

$$E'_{0y} = E_{0y}\gamma \qquad B'_{0z} = -\beta\left(\frac{E_{0y}}{c}\right)\gamma \qquad \beta = -\frac{V}{c}$$

in agreement with (6.10).

If the velocity is normal to the plates, in the two frames the surface charge density σ is the same and also the electric field $E'_0 = E_0 = \sigma/\epsilon_0$ as in (6.10).

It is easy to see that no magnetic field is observed in the laboratory. We can divide a plate in rings with center on an axis normal to the plates. Then we consider the magnetic field at a point P on this axis produced by the motion of the charge dq present on an element of one of the rings. The field $\mathbf{dB'}$ produced[9] by the moving charge dq at a point on the axis is:

$$\mathbf{dB}' = \frac{\mu_0}{4\pi} dq \frac{\mathbf{V}\times\mathbf{r}}{r^3}$$

with $\mathbf{r}$ the vector fom the charge dq to the point P and $\mathbf{V}$ the velocity of the charge. The element of the ring symmetric relative to the axis gives an opposite $d\mathbf{B}'$, thus the total magnetic field from each ring is null.

6.2 Assume as x axis the axis of the solenoid. In the solenoid rest frame, the current density in the turns is $J = I/\Sigma$ with Σ the area of the cross section of the wires, the current density is $\underline{J}(0, 0, J_y, J_z)$ and the magnetic field inside the solenoid is:

$$B_x = \mu_0 I n \text{ with } n = \frac{N}{l} \quad N \text{ the number of turns, } l \text{ the length of the solenoid.}$$

In the laboratory the 4-current density is $\underline{J}'(0, 0, J_y, J_z)$ because the transverse components are unchanged in the Lorentz transformation, but due to the Lorentz contraction, the cross section is $\Sigma' = \Sigma/\gamma$ and the length is $l' = l/\gamma$. The magnetic field in the laboratory is:

$$B'_x = \mu_0 I' n' = \mu_0 \Sigma' J' \frac{N}{l'} = \mu_0 \frac{\Sigma}{\gamma} J \frac{N}{l}\gamma = B_x$$

and no other field is present. As a general result in the Lorentz transformations we have for the component of the magnetic field parallel to the relative velocity of the frames: $\mathbf{B}_\parallel \implies \mathbf{B}'_\parallel = \mathbf{B}_\parallel$.

6.3 The force acting on a charge Q moving with velocity V parallel to a wire carrying a current, in the frame of the charge is given in Eq. (6.8). To apply this formula to an electron drifting in the second wire we have to replace $Q = -q$ and $V = v$ (currents in the wires with same orientation). The force on a segment of wire of length l in the laboratory is the force on all the electrons in a segment l' in their rest frame, and for the Lorentz contraction $l = l'\sqrt{1-\beta^2}$. The number of drifting

[9] From Biot and Savart law (1.12).

electrons in this segment is $N = n'l'\Sigma = n'l\gamma\Sigma$. From (6.8) the force on a segment of length l of the second wire is:

$$\frac{dP_r^{tot}}{dt'} = -\frac{\mu_0}{2\pi}\frac{nq^2v^2n'\gamma^2\Sigma^2 l}{r}\,.$$

We have seen $dt = dt'\gamma$ and for a neutral wire $n = n'\gamma' = n'\gamma$ because $\gamma' = \gamma$, thus the force per unit length is:

$$\frac{F}{l} = \frac{1}{l}\frac{dP_r^{tot}}{dt} = -\frac{\mu_0}{2\pi}\frac{n^2q^2v^2\Sigma^2}{r} = -\frac{\mu_0}{2\pi}\frac{i^2}{r}$$

where $i = nqv\Sigma$ is the current in the wires. This is the well known result in terms of Biot and Savart law and Lorentz force. For two same direction currents the force is attractive. If the currents are in opposite directions $V = -v$ and the force becomes repulsive.

6.4 The loop is shown in Fig. 6.3. We have to find for each side the 4-current density $\underline{J'}$ seen in the laboratory. We apply the transformation (6.2) with $\beta = -V/c$ to the 4-current density $\underline{J}$ in the rest frame of the loop:

side 1: $\underline{J_1}$ (0, 0, J, 0) $\underline{J'}_1$ (0, 0, J, 0)
side 2: $\underline{J_2}$ (0, J, 0, 0) $\underline{J'}_2$ $(-\beta\gamma J, \gamma J, 0, 0)$
side 3: $\underline{J_3}$ (0, 0, −J, 0) $\underline{J'}_3$ (0, 0, −J, 0)
side 4: $\underline{J_4}$ (0, −J, 0, 0) $\underline{J'}_4$ $(\beta\gamma J, -\gamma J, 0, 0)$.

The charge densities in the laboratory frame are:

$$\rho_1' = \rho_3' = 0 \qquad \rho_2' = -\beta\gamma J/c = V\gamma J/c^2 \qquad \rho_4' = \beta\gamma J/c = -V\gamma J/c^2$$

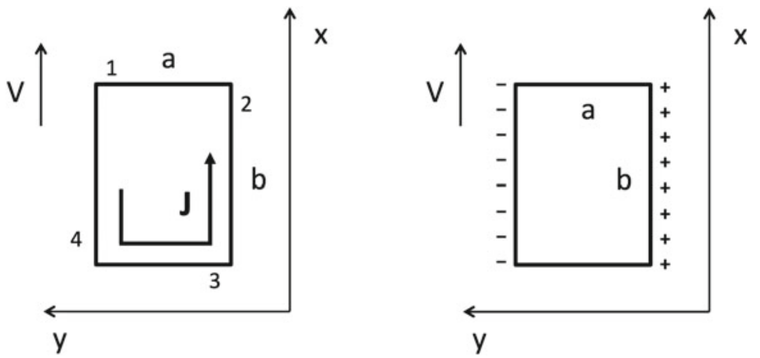


Fig. 6.3 *At left* the rectangular loop with current density J moving with velocity V; *at right* the charge density observed in the laboratory

and the current densities:

$$J_1' = J \qquad J_2' = \gamma J \qquad J_3' = -J \qquad J_4' = -\gamma J \, .$$

The charge $q_2' = \rho_2' b\sqrt{(1-\beta^2)}\Sigma = VJb\Sigma/c^2$ is seen in the side 2 and the charge $q_4' = \rho_4' b\sqrt{(1-\beta^2)}\Sigma = -VJb\Sigma/c^2$ is seen in the side 4.

An electric dipole is seen in the laboratory with moment:

$$\mathbf{p} = -q_2' a \, \hat{\mathbf{y}} = -\frac{V}{c^2} Jab\Sigma \, \hat{\mathbf{y}}$$

transverse to the motion of the loop.

In its rest frame the loop produces only a magnetic field. In the laboratory is observed the superposition of a different magnetic field from the current in the loop and of an electric field from the electric dipole.

Chapter 7
Relativistic Covariance of Electrodynamics

The Maxwell equations accounted for all the electricity and magnetism phenomena and predicted the electromagnetic waves but were in contrast with the Galileian relativity. The inconsistency was solved by Einstein's theory of special relativity. After a short introduction to 4-vectors, 4-tensors and differential operators with their covariant and contravariant components, the equations of Electrodynamics are written in covariant form, that is the same in all the inertial frames. The introduction of the electromagnetic tensor with its Lorentz transformations clarifies the link between the electric and magnetic fields.

7.1 Electrodynamics and Special Theory of Relativity

The Maxwell equations, written in 1859, unified electricity and magnetism, were able to account for all known phenomena of electromagnetism and at the same time predicted the electromagnetic waves later observed by Hertz in 1888. A non negligible difficulty was that, as opposed to the laws of Classical Mechanics, the Maxwell equations and the equation of the electromagnetic waves are not invariant with respect to Galileian transformations.[1]

[1]For two inertial frames S' and S related by a Galileian transformation:

$$x' = x - vt \qquad y' = y \qquad z' = z \qquad t' = t$$

$$x = x' + vt' \qquad y = y' \qquad z = z' \qquad t = t'$$

$$\frac{\partial}{\partial x'} = \frac{\partial}{\partial x} \qquad \frac{\partial}{\partial t'} = \frac{\partial}{\partial t} + v\frac{\partial}{\partial x}$$

$$\frac{\partial^2}{\partial x'^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} = 0 \quad \rightarrow \quad \left(1 - \frac{v^2}{c^2}\right)\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - 2v\frac{\partial}{\partial x}\frac{\partial}{\partial t} = 0\,.$$

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9_7

Since all the experiments proved the validity of the Maxwell equations, one possibility was that the Galileian relativity was wrong[2] and the hypothesis was done of a preferred reference frame where the Maxwell equations and the wave equation were correct. At the same time it was necessary to suppose the existence of medium, called *ether*, at rest in that frame, in which the electromagnetic waves were propagated as the sound in air, independently from the speed of the source. On the contrary in free space the velocity of propagation in the equation of the electromagnetic waves is the same in all the reference frames and is determined by the values of the vacuum permittivity ϵ_0 and of the permeability μ_0 which are the same in all the reference frames. This is in evident disagreement with the Galileian law of composition of the velocities.

All the experiments to observe the presence of the ether failed. In particular the Michelson and Morley experiment of 1887 excluded the possibility of a motion of the Earth relative to the rest frame of the ether. These experiments are usually presented in the textbooks on special relativity.

Furthermore some inconsistencies were present in the explanation of phenomena with sources of electric or magnetic fields in motion as for instance for a conductive loop moving in the field of a magnet.[3] Such inconsistencies and the failure in detecting the ether are both mentioned in the introduction of Einstein's famous paper on the electrodynamics of moving bodies.[4]

Einstein's theory of special relativity is based on two postulates:

(1) The postulate of relativity: *the laws of nature and the results of all the experiments are the same in all the reference frames in uniform motion relative to one another (the inertial frames).*
(2) The postulate of constancy of the speed of the light in free space: *the speed of light c is the same in all the reference frames and it is independent from the motion of the source.*

The special theory of relativity is introduced in the courses of mechanics where the kinematics and the dynamics are presented. After the heuristic introduction given in the previous chapter, in the present chapter we show that in the framework of the special theory of relativity the electrodynamics is a theory coherent and correct with the same equations holding in all the inertial frames.[5] In the four dimensional space

[2] For more details see for instance J.D. Jackson: *Classical Electrodynamics*, cited, Chap. 11.

[3] In the frame of the magnet the electromotive force can be explained with the Lorentz force acting on the charges inside the loop and moving with the loop in the field of the magnet. In the frame of the loop the Lorentz force cannot act because the loop is at rest and the electromotive force is associated to the rate of change of the magnetic flux enclosed by the circuit. In relativity this incoherence is removed because an electric field arises from the transformation of the magnetic field of the magnet to the loop reference frame.

[4] A. Einstein: *On the Electrodynamics of Moving Bodies*.

[5] This subject can be found in many Electrodynamics textbooks as for instance L.D. Landau-E.M. Lifšits, *The classical theory of fields*, Chapters III and IV, or J.D. Jackson, *Classical Electrodynamics*, cited, Chap. 11. In these two books with an introduction to the theory of relativity, the Kinematics and the Dynamics are also considered.

the nature of the electric and magnetic fields, as components of the electromagnetic field, will be evident.

7.2 4-Vectors, Covariant and Contravariant Components

In a frame S in the four-dimensional Minkowski space, a point event is determined by the coordinates of the position $\mathbf{r}(x, y, z)$, where it occurs in the three dimensional space, and by the time t when it occurs. t and $\mathbf{r}$ are the time and space components of the 4-vector position $\underline{x}$ with components $x^\mu = (x_0, x_1, x_2, x_3) = (ct, x, y, z) = (ct, \mathbf{r})$ where $\mu = 0, 1, 2, 3$. Similarly for an arbitrary 4-vector $\underline{a} = a^\mu = (a_0, a_1, a_2, a_3)$.

For the second postulate of the relativity the speed of light c is the same in all the reference frames and it is independent from the motion of the source.

Consider two systems S and S', in relative uniform motion, which have the origin and the axes coincident at the time $t = t' = 0$. If at that time a light spherical wave is emitted at the origin of the axes, at any later time this wave has to be seen in both reference frames as a spherical wave that propagates with speed c. Thus it has to be:

$$s^2 = c^2t^2 - (x^2 + y^2 + z^2) = c^2t'^2 - (x'^2 + y'^2 + z'^2) = 0\,.$$

Similarly the distance ds between two points in the space-time has to be the same in all the inertial frames. For the two frames S and S' we have:

$$(ds)^2 = c^2(dt)^2 - \left[(dx)^2 + (dy)^2 + (dz)^2\right] = c^2(dt')^2 - \left[(dx')^2 + (dy')^2 + (dz')^2\right]. \tag{7.1}$$

In the four dimensional space the inner or scalar product of the two 4-vectors $\underline{A}$ and $\underline{B}$ is defined by the relation:

$$\underline{A} \cdot \underline{B} = g_{\mu\nu} A^\mu B^\nu \tag{7.2}$$

where $g_{\mu\nu}$ is the *metric tensor* that determines the metric of the space-time. As usual in this relation we assume the sum over any index that appears twice.

In particular we can write the *norm* for $\underline{ds}$:

$$(ds)^2 = \underline{ds} \cdot \underline{ds} = g_{\mu\nu} ds^\mu ds^\nu \tag{7.3}$$

and comparing this with the relation (7.1), for the metric tensor $g_{\mu\nu}$ we have: $g_{00} = 1$, $g_{\mu\nu} = -1$ per $\mu = \nu = 1, 2, 3$; $g_{\mu\nu} = 0$ per $\mu \neq \nu$.

The tensor $g_{\mu\nu}$ can be written as a matrix:

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{7.4}$$

From the above definitions we have the scalars:

$$s^2 = \underline{s} \cdot \underline{s} = g_{\mu\nu} x^\mu x^\nu = c^2 t^2 - (x^2 + y^2 + z^2)$$

$$a^2 = \underline{a} \cdot \underline{a} = g_{\mu\nu} a^\mu a^\nu = a_0^2 - (a_x^2 + a_y^2 + a_z^2)$$

$$\underline{a} \cdot \underline{b} = g_{\mu\nu} a^\mu b^\nu = a_0 b_0 - (a_x b_x + a_y b_y + a_z b_z) .$$

A *scalar* is a quantity which is unchanged by the transformation from one frame to another. It is also called a Lorentz invariant.

If we say x^μ and x'^μ the components of the 4-vector $\underline{x}$ in S and S' we can write:

$$dx'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} dx^\nu .$$

For a 4-vector $\underline{A}$ it is defined the *contravariant vector* A^μ with components (A^0, A^x, A^y, A^z) that are transformed from S to S' according to the rule:

$$A'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} A^\nu . \tag{7.5}$$

From the relations:

$$dx^\mu = \frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\rho} dx^\rho$$

it follows:

$$\frac{\partial x^\mu}{\partial x'^\nu} \frac{\partial x'^\nu}{\partial x^\rho} = \frac{\partial x^\mu}{\partial x^\rho} = \delta^\mu_\rho .$$

The factors

$$\frac{\partial x'^\mu}{\partial x^\nu}$$

in the relation (7.5) are the elements of a 4×4 matrix, $L^\mu{}_\nu$, that can be found from the Lorentz transformation of the coordinates. For the transformation of the 4-vector $\underline{x}(ct, \mathbf{x})$ we have:

$$x'^\mu = L^\mu{}_\nu x^\nu \tag{7.6}$$

and, if the x' axis of S' is moving with velocity $\mathbf{v} = \beta c \hat{x}$ over the x axis of S and the three axis of the two frames are coincident at time $t = t' = 0$, the matrix $L^{\mu}{}_{\nu}$ is[6]:

$$L^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} . \tag{7.7}$$

The 4-vector $\underline{x}$ can be written as a column vector and the transformation (7.6) becomes a product of matrices:

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} (ct - \beta x)\gamma \\ (x - vt)\gamma \\ y \\ z \end{pmatrix} .$$

For the 4-vector $\underline{A}$ is also defined the *covariant vector* A_μ with the components that are transformed from S to S' by the relation:

$$A'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} A_\nu . \tag{7.8}$$

The scalar product of the two 4-vectors $\underline{A}$ and $\underline{B}$ can be also written in the form:

$$\underline{A} \cdot \underline{B} = A'_\mu B'^\mu = \frac{\partial x^\rho}{\partial x'^\mu} \frac{\partial x'^\mu}{\partial x^\nu} A_\rho B^\nu = \delta^\rho_\nu A_\rho B^\nu = A_\nu B^\nu . \tag{7.9}$$

In particular for the norm of ds:

$$(ds)^2 = dx_\mu dx^\mu$$

and comparing with the relation (7.3) the result:

$$dx_\mu = g_{\mu\nu} dx^\nu$$

or more generally, from the (7.2) and (7.9) it follows the relation between the covariant and contravariant components of the 4-vector $\underline{A}$:

$$\underline{A} \cdot \underline{B} = A_\mu B^\mu = g_{\mu\nu} A^\nu B^\mu \qquad A_\mu = g_{\mu\nu} A^\nu .$$

[6]In this simple case the Lorentz transformations are:

$$ct' = (ct - \beta x)\gamma \qquad x' = (x - vt)\gamma \qquad y' = y \qquad z' = z \qquad \beta = \frac{v}{c} \qquad \gamma = \frac{1}{\sqrt{1 - \beta^2}} .$$

So the covariant components of the 4-vector $\underline{A}$ are A_μ $(A_0, -A_x, -A_y, -A_z)$.

As for the 4-vector, by the relations (7.5) and (7.8), it is possible to define the *contravariant tensor* of rank 2:

$$F'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta}F^{\alpha\beta},$$

the *covariant tensor* of rank 2:

$$F'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu}F_{\alpha\beta},$$

and the *mixed tensors* of rank 2:

$$F'_\mu{}^\nu = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\beta}F_\alpha{}^\beta \qquad F'^\mu{}_\nu = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x^\beta}{\partial x'^\nu}F^\alpha{}_\beta.$$

Tensors of higher rank can also be defined.

If f is a scalar function, its differential df is also a scalar, and from:

$$df = \frac{\partial f}{\partial x^\mu}dx^\mu$$

being dx^μ a contravariant vector, it follows that $\frac{\partial f}{\partial x^\mu}$ is a covariant 4-vector. Thus it is usual to write $\partial_\mu f = \frac{\partial f}{\partial x^\mu}$ and it is evident that $\partial_\mu = \frac{\partial}{\partial x^\mu}$ is a *covariant differential operator*:

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \ \left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, \boldsymbol{\nabla}\right).$$

Similarly from:

$$df = \frac{\partial f}{\partial x_\mu}dx_\mu$$

it follows that $\frac{\partial f}{\partial x_\mu}$ is a contravariant 4-vector and $\partial^\mu = \frac{\partial}{\partial \mathrm{x}_\mu}$ is a *contravariant differential operator*:

$$\partial^\mu = \frac{\partial}{\partial \mathrm{x}_\mu} \ \left(\frac{1}{c}\frac{\partial}{\partial t}, -\frac{\partial}{\partial x}, -\frac{\partial}{\partial y}, -\frac{\partial}{\partial z}\right) = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\boldsymbol{\nabla}\right).$$

Like the product $A_\mu B^\mu$, the D'Alambertian operator:

$$\partial_\mu\partial^\mu = \frac{1}{c}\frac{\partial}{\partial t}\frac{1}{c}\frac{\partial}{\partial t} - \left(\frac{\partial}{\partial x}\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\frac{\partial}{\partial z}\right) = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box$$

is also a scalar and has the same form in all the Lorentz reference frames.

7.3 Relativistic Covariance of the Electrodynamics

In the three dimensional space, the equation $\mathbf{F} = m\mathbf{a}$ is a vector equation. It is the same in all the inertial frames related by a translation, a rotation or a Galileian transformation. The components of $\mathbf{F}$ and $\mathbf{a}$ are changed after transformations in different frames, but the three scalar equations between the same type components of the two members of the vector equation, are valid in any reference frame because the two components are transformed in the same way.[7] We say that, in the three dimensional space, a *vector equation* is in *covariant* form. From this property of invariance for the vector relations the invariance of the Classical Mechanics with respect to the Galileian transformations is derived.

The first postulate of special theory of relativity demands that the laws of Physics be the same in any inertial frame of reference. The equations of Electrodynamics are not invariant with respect to Galileian transformations but with respect to Lorentz transformations. To prove the Lorentz invariance of the Electrodynamics it is sufficient that, for all its equations, the two members are transformed in the same way. This condition is satisfied if the two members have the same tensor properties: they are both scalar, 4-vectors or tensors of equal rank. This means that the equations have to be written in *covariant* form.

We will show that all the equations of the Electrodynamics can be written in a *covariant* form and this will prove the Lorentz invariance of Electrodynamics.

7.4 4-Vector Potential and the Equations of Electrodynamics

In the Lorentz gauge the equations for the potentials $\mathbf{A}$ and V in free space, seen in Chap. 1, are:

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0 J \tag{1.29}$$

$$\nabla^2 V - \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} = -\frac{\rho}{\epsilon_0}\,. \tag{1.30}$$

Defining the 4-vector potential $\underline{A} = A^\mu(V/c, \mathbf{A})$, considering the 4-current density $\underline{J} = J^\mu(\rho c, \mathbf{J})$, given by the Eq. (6.1), and the relation:

$$\frac{1}{\epsilon_0} = \mu_0\frac{1}{\mu_0\epsilon_0} = \mu_0 c^2$$

[7] The vector equation exists by itself in the space and of course it is independent from the frame. When considering the equation in a frame we take the projections of both members of the vector equation along the three axes of the frame. The projections are different in the different frames but the vector equation is the same, it is therefore written in covariant form.

it is easy to see that the two equations are the four components of the equation:

$$\partial_\mu \partial^\mu A^\nu = \mu_0 J^\nu \qquad \text{or} \qquad \Box \underline{A} = \mu_0 \underline{J}$$

for $\nu = 0, 1, 2, 3$.

This last equation is an equality between two 4-vectors. Changing the reference frame this 4-vector equation is unchanged, it is the same in all inertial frames. The components of same type of the two 4-vectors are transformed in the same way, therefore the equality between the four components is conserved. This equation is written in *covariant form* and since this equation corresponds to the four fundamental Eqs. (1.29) and (1.30), that summarize all the Electrodynamics, this is sufficient to prove the *relativistic covariance* of the Electrodynamics.

7.5 The Continuity Equation

The continuity Eq. (1.10):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \qquad \frac{1}{c}\frac{\partial \rho c}{\partial t} + \frac{\partial J_x}{\partial x} + \frac{\partial J_y}{\partial y} + \frac{\partial J_z}{\partial z} = 0$$

can be written in covariant form as a simple scalar product:

$$\partial_\mu J^\mu = 0\,.$$

7.6 The Electromagnetic Tensor

Consider now the fields $\mathbf{E}$ and $\mathbf{B}$. From $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$, taking the x component, we have:

$$B_x = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}$$

that can be written as the component of a rank 2 tensor:

$$F^{32} = \partial^3 A^2 - \partial^2 A^3 = -\frac{\partial A_y}{\partial z} + \frac{\partial A_z}{\partial y} = \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = B_x\,.$$

Similarly for $F^{13} = B_y$ and $F^{21} = B_z$.

Then it is possible to introduce the *antisymmetric* tensor $F^{\mu\nu}$:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu\,. \tag{7.10}$$

Varying the indexes $\mu, \nu = 1, 2, 3$ we have the components of the field **B** as we have seen, but if one of the indexes is the time index 0, we have the components of the electric field. For instance for F^{01} it is:

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = \frac{1}{c}\frac{\partial A_x}{\partial t} + \frac{1}{c}\frac{\partial V}{\partial x} = -\frac{E_x}{c}\,.$$

Replacing the index 1 with 2 or 3, we get the y and z components of the electric field.

$F^{\mu\nu}$ is the *electromagnetic tensor*. Its elements are the components of the **E** and **B** fields:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E_x}{c} & -\frac{E_y}{c} & -\frac{E_z}{c} \\ \frac{E_x}{c} & 0 & -B_z & B_y \\ \frac{E_y}{c} & B_z & 0 & -B_x \\ \frac{E_z}{c} & -B_y & B_x & 0 \end{pmatrix}.$$

It is evident now why the fields **E** and **B** are closely connected in the frame transformations: they are the components of the same tensor. As shown in the next paragraph, the components of this tensor are mixed when transformed in different reference frames, and as a consequence the components of the fields **E** and **B** are mixed. In a reference frame only the electric field **E** or the magnetic field **B** could be observed, while in other frames a combination of the two fields appears as we have seen in the previous chapter.

7.7 Lorentz Transformation for Electric and Magnetic Fields

To transform the **E** and **B** fields we have to transform the components of the rank 2 electromagnetic tensor:

$$F'^{\mu\nu} = L^{\mu}{}_{\rho} L^{\nu}{}_{\sigma} F^{\rho\sigma}\,.$$

If we consider the simple transformation (7.7), for E'_y there are only two non null terms:

$$\frac{E'_y}{c} = F'^{20} = L^2{}_{\rho} L^0{}_{\sigma} F^{\rho\sigma} = L^2{}_2 L^0{}_0 F^{20} + L^2{}_2 L^0{}_1 F^{21} = \gamma\frac{E_y}{c} - \beta\gamma B_z$$

and the transformation is:

$$E'_y = \gamma(E_y - vB_z)\,.$$

For B'_y:

$$B'_y = F'^{13} = L^1{}_{\rho} L^3{}_{\sigma} F^{\rho\sigma} = L^1{}_0 L^3{}_3 F^{03} + L^1{}_1 L^3{}_3 F^{13} = \beta\gamma\frac{E_z}{c} + \gamma B_y$$

$$B'_y = \gamma\left(B_y + \frac{v}{c^2}E_z\right) .$$

Repeating the exercise for all the components of the **E** and **B** fields we get the transformations (6.10):

$$\begin{cases} E'_x = E_x \\ E'_y = \gamma(E_y - c\beta B_z) \\ E'_z = \gamma(E_z + c\beta B_y) \end{cases} \qquad \begin{cases} B'_x = B_x \\ B'_y = \gamma(B_y + \beta\frac{E_z}{c}) \\ B'_z = \gamma(B_z - \beta\frac{E_y}{c}) \end{cases} \tag{7.11}$$

that, in terms of the components parallel and transverse to the relative velocity **v** of the two frames, can be written as:

$$\begin{cases} \mathbf{E}'_\parallel = \mathbf{E}_\parallel \\ \mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \times \mathbf{B}_\perp) \end{cases} \qquad \begin{cases} \mathbf{B}'_\parallel = \mathbf{B}_\parallel \\ \mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \frac{1}{c^2}\mathbf{v} \times \mathbf{E}_\perp) \end{cases} .$$

7.8 Maxwell Equations

7.8.1 Inhomogeneous Equations

The first and the fourth Maxwell equation in covariant form are:

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu . \tag{7.12}$$

For $\nu = 1$ we have:

$$\partial_\mu F^{\mu 1} = \mu_0 J^1$$

$$\frac{\partial F^{01}}{\partial x^0} + \frac{\partial F^{11}}{\partial x^1} + \frac{\partial F^{21}}{\partial x^2} + \frac{\partial F^{31}}{\partial x^3} + = \mu_0 J^1$$

but $F^{11} = 0$ and it follows:

$$\frac{1}{c}\frac{\partial F^{01}}{\partial t} + \frac{\partial F^{21}}{\partial y} + \frac{\partial F^{31}}{\partial z} = \mu_0 J^1 \qquad -\frac{1}{c^2}\frac{\partial E_x}{\partial t} + \frac{\partial B_z}{\partial y} - \frac{\partial B_y}{\partial z} = \mu_0 J_x$$

that is the x component of the fourth equation:

$$(\nabla \times \mathbf{B})_x = \mu_0 J_x + \mu_0 \epsilon_0 \frac{\partial E_x}{\partial t}.$$

Similarly the y and z components are derived for $\nu = 2$ and 3.

For $\nu = 0$ the Eq. (7.12) becomes:

$$\partial_\mu F^{\mu 0} = \mu_0 J^0$$

$$\frac{\partial F^{00}}{\partial x^0} + \frac{\partial F^{10}}{\partial x^1} + \frac{\partial F^{20}}{\partial x^2} + \frac{\partial F^{30}}{\partial x^3} = \mu_0 J^0 \quad \frac{1}{c}\frac{\partial E_x}{\partial x} + \frac{1}{c}\frac{\partial E_y}{\partial y} + \frac{1}{c}\frac{\partial E_z}{\partial z} = \mu_0 \rho c$$

because $F^{00} = 0$. By substituting $\mu_0 c^2 = 1/\epsilon_0$ this relation can be written:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}$$

that is the first Maxwell equation.

7.8.2 Homogeneous Equations

The second and the third equation are four scalar equations that are written in the form:

$$\partial^\gamma F^{\mu\nu} + \partial^\mu F^{\nu\gamma} + \partial^\nu F^{\gamma\mu} = 0$$

where the three indexes μ, ν, γ are one of the four possible combinations of indexes with no equal indexes. For two equal indexes the equation is null.

For the three space indexes the equation is:

$$\partial^x F^{yz} + \partial^y F^{zx} + \partial^z F^{xy} = 0$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_x}{\partial x} + \frac{\partial B_x}{\partial x} = 0 \qquad \nabla \cdot \mathbf{B} = 0$$

that is the second equation. For the indexes x, y and 0:

$$\partial^x F^{y0} + \partial^y F^{0x} + \partial^0 F^{xy} = 0$$

$$-\frac{\partial}{\partial x}\frac{E_y}{c} + \frac{\partial}{\partial y}\frac{E_x}{c} - \frac{1}{c}\frac{\partial B_z}{\partial t} = 0 \qquad (\nabla \times \mathbf{E})_z = -\frac{\partial B_z}{\partial t}$$

the component z of the third equation is found and similarly the x and y components can be derived.

7.9 Potential Equations

By substituting in the Maxwell equation (7.12) the expression for the electromagnetic tensor $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ given in (7.10), the four coupled equations for the potentials in covariant form are found:

$$\partial_\mu \partial^\mu A^\nu - \partial_\mu (\partial^\nu A^\mu) = \mu_0 J^\nu \qquad \partial_\mu \partial^\mu A^\nu - \partial^\nu (\partial_\mu A^\mu) = \mu_0 J^\nu$$

where:

$$\partial_\mu A^\mu = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} + \frac{1}{c^2}\frac{\partial V}{\partial t} = \mathbf{\nabla} \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}$$

is a scalar or Lorentz invariant, thus if the Lorentz gauge $\partial_\mu A^\mu = 0$ is chosen in a reference frame, the choice is valid in all the reference frames.

In the Lorentz gauge the uncoupled equations for the potentials in covariant form become:

$$\partial_\mu \partial^\mu A^\nu = \mu_0 J^\nu \quad \to \quad \Box A^\nu = \mu_0 J^\nu$$

already seen in Sect. 7.4.

7.10 Gauge Transformations

The gauge transformations for the potentials (1.22) and (1.23):

$$\mathbf{A}' = \mathbf{A} + \mathbf{\nabla}\varphi \qquad V' = V - \frac{\partial \varphi}{\partial t}$$

in covariant form are:

$$A'^\mu = A^\mu - \partial^\mu \varphi$$

and the Eq. (1.28) to find the potentials satisfying the Lorentz gauge is:

$$\Box \varphi = \partial_\mu A^\mu \, .$$

7.11 Phase of the Wave

Introducing the wave 4-vector k^μ $(\omega/c, \mathbf{k})$, the phase of a monochromatic wave is:

$$k_\mu x^\mu = \omega t - \mathbf{k} \cdot \mathbf{r} = -(\mathbf{k} \cdot \mathbf{r} - \omega t)$$

that is a scalar relativistic invariant equal in all the reference frames.

For an electromagnetic wave the norm of the wave vector is null:

$$k_\mu k^\mu = \frac{\omega^2}{c^2} - \mathbf{k} \cdot \mathbf{k} = 0\,.$$

7.12 The Equations of Motion for a Charged Particle in the Electromagnetic Field

We have to remind here some formulas from the introductory courses on special relativity.

The 4-velocity is:

$$u^\mu = \frac{dx^\mu}{d\tau} \quad \left(c\frac{dt}{dt}\gamma, \frac{dx^i}{dt}\gamma\right) = (c\gamma, \mathbf{v}\gamma) \tag{7.13}$$

where $d\tau = dt/\gamma$ is the proper time. The 4-momentum is $p^\mu = m_0 u^\mu$:

$$p^\mu \ (m_0 c\gamma, m_0 \mathbf{v}\gamma) = \left(\frac{\mathcal{E}}{c}, \mathbf{p}\right) \tag{7.14}$$

where $\mathcal{E}$ is the energy of the particle. The 4-vector force is:

$$f^\mu = \left(\frac{\mathbf{F} \cdot \mathbf{v}}{c}\gamma, \mathbf{F}\gamma\right)$$

and the equations of the motion are:

$$\frac{dp^\mu}{d\tau} = f^\mu\,. \tag{7.15}$$

The force acting on a point charge in the electromagnetic field

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

is the space component of the 4-force:

$$f^{\mu} = qF^{\mu\nu}u_{\nu}\,.$$

By substituting this relation in the (7.15) we have:

$$\frac{dp^{\mu}}{d\tau} = qF^{\mu\nu}u_{\nu}\,.$$

The x component of this equation is:

$$\frac{dp^1}{d\tau} = qF^{10}u_0 + qF^{12}u_2 + qF^{13}u_3 \qquad \frac{dp_x}{dt} = qE_x + qB_z v_y - qB_y v_z$$

that is the x projection of the three dimension vector equation:

$$\frac{d\mathbf{p}}{dt} = \mathbf{F} = q\mathbf{E} + \mathbf{v} \times \mathbf{B}$$

while the time component of this relativistic equation:

$$\frac{dp^0}{d\tau} = qF^{01}u_1 + qF^{02}u_2 + qF^{03}u_3 \qquad \frac{d\mathcal{E}}{dt} = qE_x v_x + qE_y v_y + qE_z v_z$$

is equal to the power, the change of the energy $\mathcal{E}$ per unit time, due to the action of the electric field:

$$\frac{d\mathcal{E}}{dt} = q\mathbf{E} \cdot \mathbf{v}\,.$$

Chapter 8
The Resonant Cavity

The displacement current in the fourth Maxwell equation can be neglected in steady or quasi steady conditions, but it is relevant in fast processes and produces new phenomena if connected to the electromagnetic induction. The most relevant of these are the electromagnetic waves. An elegant and intriguing example of the transition from low to high frequency is the capacitor. As shown by Feynman in one of his lectures[1] at very high frequency, the capacitor becomes a resonant cavity. This chapter aims to introduce shortly this example.

8.1 The Capacitor at High Frequency

Consider a capacitor with parallel circular plates of radius R at distance h as in Fig. 8.1. If a sinusoidal voltage with angular frequency ω is applied to the plates, the electric field inside the capacitor is:

$$E = E_0 e^{i\omega t}$$

and its field lines are perpendicular to the plates. A displacement current density is associated to this electric field:

$$\epsilon_0 \frac{\partial E}{\partial t} = \epsilon_0 (i\omega) E_0 e^{i\omega t}$$

with the same direction of the field E.

According to the fourth Maxwell equation, a magnetic field B, with circular field lines around the axis of the capacitor, is present between the plates. The line integral

[1]R.P. Feynman, R.B. Leighton, M. Sand, *The Feynman Lectures on Physics*, Vol. II, Chap. 23. We suggest to read the Chaps. 16 and 17 on the electromagnetic induction.

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9_8

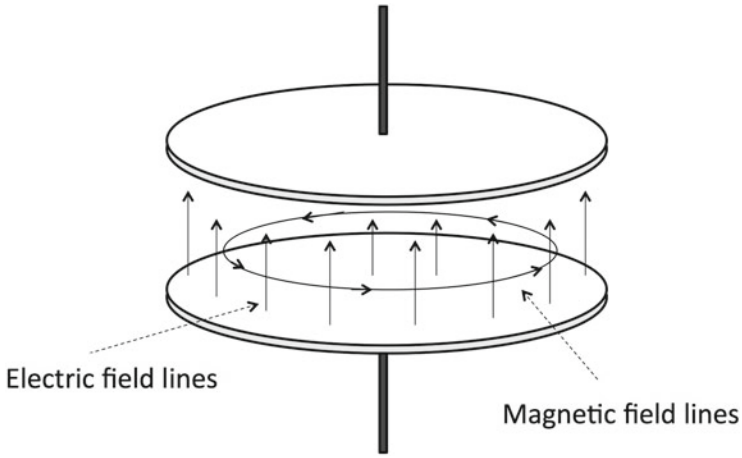


Fig. 8.1 Electric and magnetic field lines in the condenser

of B, along a circular field line of radius r, is equal to μ_0 times the flux of the displacement current through the area inside the circle:

$$2\pi r B = \mu_0 \pi r^2 (\epsilon_0 \frac{\partial E}{\partial t}) = \mu_0 \epsilon_0 \pi r^2 (i\omega) E_0 e^{i\omega t} \tag{8.1}$$

so we find:

$$B(r) = i\omega \frac{r}{2c^2} E_0 e^{i\omega t}$$

where we have used the relation $\mu_0 \epsilon_0 = 1/c^2$.

According to the third Maxwell equation, the magnetic field changing in time induces an electric field E_2 parallel to the axis of the condenser. This field is added to the electric field applied between the plates hereafter called E_1. Thus the total electric field should be:

$$E = E_1 + E_2.$$

From Faraday's law:

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\frac{d\Phi(B)}{dt}$$

the line integral along a path as in Fig. 8.2 gives[2]:

$$-hE_2(r) = -\frac{d}{dt}\int_0^r B(r')h dr' \tag{8.2}$$

[2]The induced electric field is null on the axis of the capacitor and the contour integral of E_1 is clearly null.

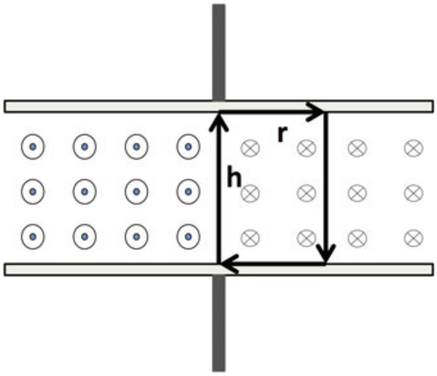


Fig. 8.2 Closed path for the line integral of the field E_2

$$E_2(r) = \frac{d}{dt}\int_0^r \frac{i\omega r'}{2c^2} E_0 e^{i\omega t} dr' = \int_0^r \frac{(i\omega)^2}{2c^2} E_0 e^{i\omega t} r' dr'$$

and after integration:

$$E_2(r) = \frac{1}{2^2}\left(\frac{i\omega r}{c}\right)^2 E_0 e^{i\omega t} = -\frac{1}{2^2}\left(\frac{\omega r}{c}\right)^2 E_0 e^{i\omega t}$$

that is a negative contribution to be added to E_1. The electric field becomes:

$$E = \left[1 - \frac{1}{2^2}\left(\frac{\omega r}{c}\right)^2\right] E_0 e^{i\omega t}$$

with a strength decreasing for increasing distances r from the axis of the capacitor. The correction to the electric field E_1 is negligible if $\omega R/2c \ll 1$ equivalent to periods:

$$T \gg \pi \frac{R}{c}$$

where $\pi R/c$ is about the time of propagation of the electromagnetic signal over a distance comparable with the dimensions of the plates. This condition is fulfilled for slow voltage changes applied to the terminals of the capacitor, because these variations can be considered instantaneously propagated over the whole plates of the capacitor. This is the condition of quasi steady state in which the displacement current can be neglected.

A displacement current is also associated to the field E_2 and to this current is associated a field B_2 that, similarly to (8.1), has a circular integral:

$$2\pi r B_2 = \mu_0 \epsilon_0 \int_0^r \left(\frac{\partial E_2}{\partial t}\right) 2\pi r' dr' .$$

By substituting the expression found for E_2, after integration, we find:

$$B_2 = \frac{1}{2^2}\frac{1}{4}\frac{(i\omega r)^3}{c^4}E_0 e^{i\omega t}\,.$$

From Faraday's law, an electric field E_3 is associated to the field B_2 and, as for the field E_2 seen in (8.2), its closed path integral is:

$$-hE_3(r) = -\frac{d}{dt}\int_0^r B_2(r)hdr = -\frac{d}{dt}\left(-i\frac{\omega^3}{c^4}\frac{1}{16}E_0 e^{i\omega t}\right)\int_0^r r^3 hdr$$

and from this:

$$E_3(r) = \frac{1}{2^2 4^2}\left(\frac{i\omega r}{c}\right)^4 E_0 e^{i\omega t}.$$

To this field E_3 is associated a field B_3 as it was for B_2 associated to E_2:

$$B_3 = \frac{1}{2^2 4^2 6}\frac{(i\omega r)^5}{c^6}E_0 e^{i\omega t}\,.$$

Iterating up to the fifth term, for the electric field we have:

$$E(r,t) = \left[1 - \frac{1}{(1!)^2}\left(\frac{\omega r}{2c}\right)^2 + \frac{1}{(2!)^2}\left(\frac{\omega r}{2c}\right)^4 - \frac{1}{(3!)^2}\left(\frac{\omega r}{2c}\right)^6 + \frac{1}{(4!)^2}\left(\frac{\omega r}{2c}\right)^8\right]E_0 e^{i\omega t}$$

and it is easy to see that this expression is the beginning of a series of corrections with alternate signs that, when extended to infinity, becomes:

$$E(r,t) = \left[\sum_{n=0}^{\infty}(-1)^n \frac{1}{(n!)^2}\frac{1}{2^{2n}}\left(\frac{\omega r}{c}\right)^{2n}\right]E_0 e^{i\omega t}$$

where the expression in brackets is the Bessel function[3] of first kind J_0 shown in Fig. 8.3. So we can write the electric field in the form:

$$E(r,t) = J_0(z)\; E_0 e^{i\omega t} \qquad z = \frac{\omega r}{c}\,. \tag{8.3}$$

Similarly, the first five terms for the magnetic field give:

$$B(r,t) = \left[\frac{1}{2}\frac{i\omega r}{c^2} + \frac{1}{2^2 4}\frac{(i\omega r)^3}{c^4} + \frac{1}{2^2 4^2 6}\frac{(i\omega r)^5}{c^6} + \frac{1}{2^2 4^2 6^2 8}\frac{(i\omega r)^7}{c^8} + \frac{1}{2^2 4^2 6^2 8^2 10}\frac{(i\omega r)^9}{c^{10}}\right]E_0 e^{i\omega t}\,.$$

[3] See for instance: I.M. Gradshteyn and I.M. Rizhik, *Table of Integrals, Series, and Products*, 8.441, 1.

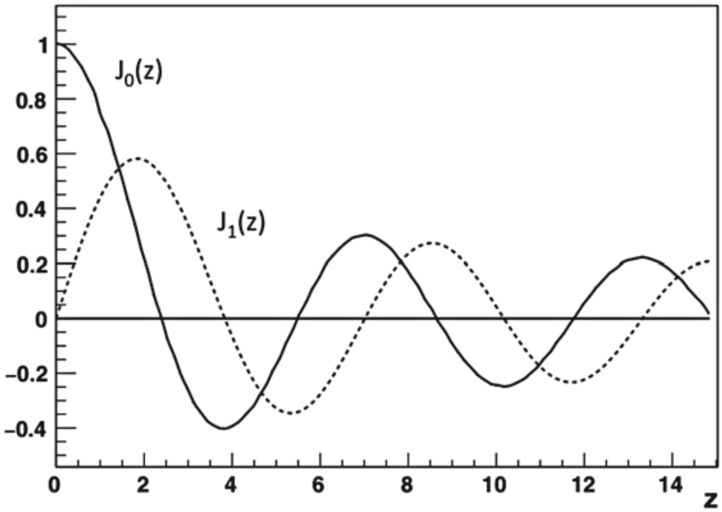


Fig. 8.3 $J_0(z)$ (*solid line*) and $J_1(z)$ (*dotted line*) Bessel functions of first kind

Adding all the terms we get:

$$B(r,t) = \frac{i}{c}\left[\frac{1}{2}\frac{\omega r}{c}\ \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^{2n}}\left(\frac{\omega r}{c}\right)^{2n}\frac{1}{n!}\frac{1}{(n+1)!}\right] E_0 e^{i\omega t}\,.$$

and, since the expression in brackets is the Bessel function[4] of first kind J_1, shown in Fig. 8.3, the field B becomes:

$$B(r,t) = \frac{i}{c}J_1(z)\ E_0 e^{i\omega t} = \frac{1}{c}J_1(z)\ E_0 e^{i(\omega t+\frac{\pi}{2})} \qquad z = \frac{\omega r}{c}\,. \tag{8.4}$$

Taking the real part of expressions (8.3) and (8.4), the fields E and B are:

$$E(r,t) = J_0(z)\ E_0\ \cos(\omega t) \tag{8.5}$$

$$B(r,t) = \frac{1}{c}J_1(z)\ E_0\ \cos(\omega t + \frac{\pi}{2})\,. \tag{8.6}$$

The fields E and B are sinusoidal functions of the time and, for a fixed ω value, the amplitudes of E and B are determined by the functions J_0 and J_1 and depend only on the distance r from the axis of the capacitor as sketched in Fig. 8.4. As shown in Fig. 8.3, these functions can be positive, negative or null, so at the same time but at different distances r, the electric and the magnetic fields can also have opposite

[4] Gradshteyn-Rizhik, 8.441, 2.

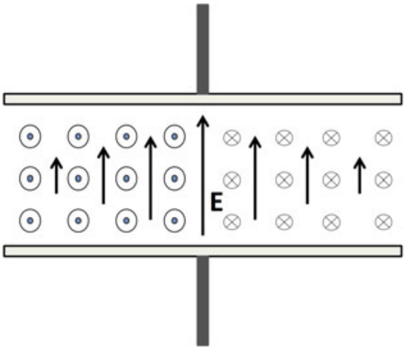


Fig. 8.4 Strength of the electric field between the plates of the condenser at high frequency

directions. Moreover E and B have a difference in phase equal to $\pi/2$ so that when the first has its maximum the second is null and vice versa.

8.2 The Resonant Cavity

For a given angular frequency ω, the field E is null at the distances $r = zc/\omega$ from the axis, where z are the values for which the $J_0(z)$ function is null. The first three of these values are: $z = 2.4048, 5.5201, 8.6537$.

If we connect the plates with a conductive cylindrical surface at the radius corresponding to the first zero:

$$r_1 = 2.4048\ \frac{c}{\omega}$$

the electric field across this surface is null, then the voltage difference is null and there is no current on this surface. If we remove the connections between the plates and the external sinusoidal generator, and we remove the parts of the plates at a radius larger than r_1, inside the metallic cylindrical box that we have built, there can be still present an electromagnetic field with E and B given by (8.5) and (8.6). Indeed we can repeat all the considerations done in the previous paragraph with the only difference that, in order to have the charge on the plates changing its sign periodically, there has to be a current I_C flowing from a plate to the other one, through the cylindrical lateral surface as shown in Fig. 8.5. This current is equal to the total displacement current I_S flowing between the plates inside the box:

$$I_S = \int_0^{r_1} \epsilon_0 \left(\frac{\partial E}{\partial t}\right) 2\pi r' dr' = \epsilon_0 E e^{i(\omega t + \frac{\pi}{2})} \int_0^{r_1} J_0\left(\frac{\omega r}{c}\right) 2\pi r' dr' \,.$$

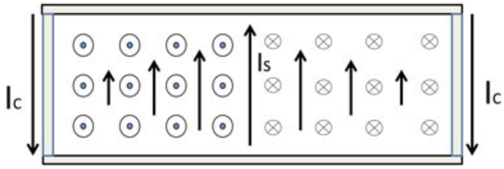


Fig. 8.5 Conduction current I_C and displacement current I_S in the resonant cavity

Fig. 8.6 Resonant cavity of the *collider* e^+e^- ADONE, 3.1 GeV center of mass energy, operating in years 1970 at the INFN Laboratori Nazionali in Frascati, Italy

I_C is a sinusoidal current with angular frequency ω, with the same phase of the magnetic field (8.4), and closes the loop of the current so that for $r > r_1$ the magnetic field is null.

Inside the box the fields E and B are sinusoidal and in quadrature: at a given time only the electric field is present and after a quarter of period only the magnetic field is non null. Thus the electromagnetic energy oscillates between two states: in the first state it is only associated to the electric field and the system behaves as a capacitor while in the second only the magnetic field is present and the system is equivalent to an inductance. The metallic cylinder behaves as a resonant LC circuit: we have built a resonant cavity.

We have fixed the angular frequency ω, while by fixing the radius r_1 of the plates of the resonant cavity, the resonance frequency would be determined by the relation $\omega = 2.4048\, c/r_1$.

The resonance frequencies are very high: for $r_1 = 10$ cm the frequency is about 1 GHz.

In real resonant cavities the plates and the lateral surfaces have a resistance and the equivalent circuit of the cavity has to include with the inductance and the capacitor also a resistor so that the amplitudes of the fields are damped. Anyway it is possible to restore the energy dissipated in the system to keep steady oscillating fields inside the cavity.

For comments and furthers details we suggest to read the Feynman's lecture where it is also shown how a low frequency circuit with a capacitor and an inductance has to transform into a resonant cavity at high frequency.

Resonant cavities as the one in Fig. 8.6 are used for instance in circular particle accelerators: at each turn the particles entering the cavity in phase with the E field, are accelerated and their energy is increased.

Chapter 9
Energy and Momentum of the Electromagnetic Field

When the electromagnetic field accelerates the charged particles and the particles radiate electromagnetic waves, energy and momentum are exchanged between particles and field. In isolated systems with charges and electromagnetic field, energy and momentum are conserved.

Considering the work and the force applied by the field on the charges, the expressions for the densities of energy and momentum of the electromagnetic field can be determined. The Poynting's vector is associated to energy and momentum fluxes and the conservation laws for energy and momentum in the presence of an electromagnetic field are derived.

9.1 Poynting's Theorem

The electromagnetic field and the electric charges can exchange energy: the electric field does work accelerating the charges and the charges can radiate electromagnetic energy. In all these processes the energy is conserved. The Poynting's theorem states the conservation of energy for the electromagnetic field interacting with charges.

Consider a volume τ, surrounded by the closed surface Σ, with the electromagnetic field and the charges inside. If n is the number of q charges per unit of volume and $\mathbf{v}$ their velocity, the force $\mathbf{df}$ applied by the electromagnetic field on the charges in a small volume $d\tau$ is: $d\tau$ is:

$$\mathbf{df} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) n d\tau \,. \tag{9.1}$$

The work dL on the charges inside $d\tau$ in a time interval dt is:

$$dL = \mathbf{df} \cdot \mathbf{ds} = \mathbf{df} \cdot \mathbf{v}\, dt = nq(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \mathbf{v}\, dt\, d\tau = nq\mathbf{v} \cdot \mathbf{E}\, dt\, d\tau = \mathbf{J} \cdot \mathbf{E}\, dt\, d\tau$$

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9_9

where $\mathbf{J} = nq\mathbf{v}$ is the density current and $(\mathbf{v} \times \mathbf{B}) \cdot \mathbf{v} = 0$ (the Lorentz force is normal to the velocity).

The power per unit volume transferred from the field to the medium enclosing the charges, is:

$$\frac{dL}{d\tau dt} = w = \mathbf{J} \cdot \mathbf{E}\,.$$

From the fourth Maxwell equation we get:

$$\mathbf{J} = \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}$$

and we can write:

$$\mathbf{J} \cdot \mathbf{E} = \mathbf{E} \cdot \nabla \times \mathbf{H} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}\,.$$

Reminding the vector relation:

$$\nabla \cdot (\mathbf{E} \times \mathbf{H}) = \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \nabla \times \mathbf{H}$$

we have:

$$\mathbf{J} \cdot \mathbf{E} = -\nabla \cdot (\mathbf{E} \times \mathbf{H}) + \mathbf{H} \cdot \nabla \times \mathbf{E} - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t}$$

and using the third Maxwell equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}$$

$$\mathbf{J} \cdot \mathbf{E} = -\nabla \cdot (\mathbf{E} \times \mathbf{H}) - \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} - \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t}\,.$$

If we assume linear relations between the fields for the considered medium: $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$, we find:

$$\mathbf{J} \cdot \mathbf{E} = -\nabla \cdot (\mathbf{E} \times \mathbf{H}) - \frac{\partial}{\partial t}\left(\frac{1}{2}\mathbf{E} \cdot \mathbf{D} + \frac{1}{2}\mathbf{H} \cdot \mathbf{B}\right)\,. \tag{9.2}$$

In this equation we have the density of electromagnetic energy:

$$u = \frac{1}{2}\mathbf{E} \cdot \mathbf{D} + \frac{1}{2}\mathbf{H} \cdot \mathbf{B}$$

thus the Eq. (9.2) becomes:

$$-\frac{\partial u}{\partial t} = \mathbf{J} \cdot \mathbf{E} + \nabla \cdot (\mathbf{E} \times \mathbf{H})\,. \tag{9.3}$$

The electromagnetic energy in a finite volume τ is:

$$U = \int_\tau u\, d\tau = \int_\tau \left(\frac{1}{2}\mathbf{E}\cdot\mathbf{D} + \frac{1}{2}\mathbf{H}\cdot\mathbf{B}\right) d\tau$$

and the integral of (9.3) over the volume τ is:

$$-\frac{dU}{dt} = \int_\tau \left[\nabla\cdot(\mathbf{E}\times\mathbf{H}) + \mathbf{J}\cdot\mathbf{E}\right] d\tau\,. \tag{9.4}$$

If we introduce the *Poynting's vector*:

$$\mathbf{S} = \mathbf{E}\times\mathbf{H}$$

the relation (9.4) becomes:

$$-\frac{dU}{dt} = \int_\tau \nabla\cdot\mathbf{S}\, d\tau + \int_\tau \mathbf{J}\cdot\mathbf{E}\, d\tau \tag{9.5}$$

and applying the Gauss-Green theorem we have:

$$-\frac{dU}{dt} = \int_\Sigma \mathbf{S}\cdot\hat{n}\, d\Sigma + \int_\tau \mathbf{J}\cdot\mathbf{E}\, d\tau\,. \tag{9.6}$$

The Poynting's vector **S** is the flux of energy through a unitary normal surface.

The last formula represents the *Poynting's theorem* which states the conservation of energy for the electromagnetic field interacting with charges: the decrease, in the time unit, of the electromagnetic energy inside a volume is equal to the flux of electromagnetic energy, associated to the Poynting vector, through the closed surface around the volume plus the power transferred to the charges inside the volume.

Equation (9.3) can be written as:

$$-\frac{\partial u}{\partial t} = \nabla\cdot\mathbf{S} + \mathbf{J}\cdot\mathbf{E} \tag{9.7}$$

that is the local form of the Poynting's theorem.

For an electromagnetic wave propagating in direction $\hat{\mathbf{v}}$, the energy that flows through a unit normal surface in a unit time is that in a cylinder with unitary base and height v, the velocity of the electromagnetic field, thus we have:

$$\mathbf{S} = uv\hat{\mathbf{v}} \tag{9.8}$$

where u is the energy density in the wave.

9.2 Examples

9.2.1 *Resistor*

Consider a cylindrical resistor or a piece of resistive wire, of radius r and length l, of uniform resistivity, with its axis along the z axis as in Fig. 9.1a.

A uniform current I flows inside the resistor in the $-\hat{\mathbf{z}}$ direction. From the Ampère theorem on the lateral surface of the cylinder there is a magnetic field[1]:

$$2\pi r H = -I \qquad \mathbf{H} = -\frac{I}{2\pi r}\hat{\boldsymbol{\varphi}}\,.$$

Nearby the lateral surface the electric field is equal to the electric field inside the resistor. If we assume a uniform voltage drop along the resistor, the electric field $\mathbf{E}$ is:

$$\mathbf{E} = -\frac{V}{l}\hat{\mathbf{z}}\,.$$

So at the surface of the cylinder there is a Poynting's vector:

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = -\frac{V}{l}\frac{I}{2\pi r}\hat{\mathbf{r}}$$

that integrated on the whole lateral surface Σ gives the flux of energy entering the resistor:

$$\int_{\Sigma} \mathbf{S}\cdot\hat{\mathbf{r}}\,d\Sigma = S\,2\pi r l = -VI\,.$$

The power dissipated in the resistor[2] is just the flux of energy of the electromagnetic field through its lateral surface (the minus sign implies that the flux of $\mathbf{S}$ is entering the cylinder).

9.2.2 *Solenoid*

Consider the case of a solenoid of radius r and length $l \gg r$, with n_S turns per unit length. If a current $I(t)$ flows in the solenoid, the inner magnetic field is $\mathbf{B}(t) = \mu_0 n_S I(t)\hat{\mathbf{z}}$. For the Faraday-Neumann law, on the lateral surface of the solenoid there is an electric field:

[1] We use cylindrical coordinates.

[2] This is correct for any component with a current I flowing inside and a voltage drop V.

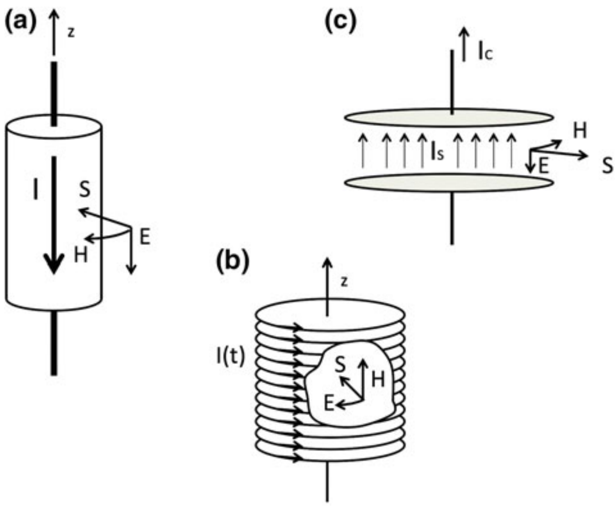


Fig. 9.1 Poynting's vector on the lateral surfaces: of a resistor with a flowing current (**a**), of a solenoid with a variable current (**b**), of a condenser during the discharge (**c**)

$$\oint \mathbf{E}\cdot\mathbf{dl} = -\frac{d\Phi(\mathbf{B})}{dt} \qquad 2\pi r E = -\pi r^2 \frac{\partial B(t)}{\partial t} \qquad \mathbf{E} = -\frac{r}{2}\frac{\partial B(t)}{\partial t}\hat{\boldsymbol{\varphi}}$$

and a Poynting's vector:

$$\mathbf{S} = \mathbf{E}\times\mathbf{H} = \frac{E_\varphi B_z}{\mu_0}\hat{\mathbf{r}} = -n_S I(t)\frac{r}{2}\frac{\partial B(t)}{\partial t}\hat{\mathbf{r}} = -\frac{r}{2}H(t)\frac{\partial B(t)}{\partial t}\hat{\mathbf{r}}$$

as in Fig. 9.1b.

The flux of the Poynting's vector through the lateral surface Σ of the solenoid, when the current ($I = 0$ at $t = 0$) and the field change, is:

$$\int_0^t \int_\Sigma \mathbf{S}\cdot\hat{\mathbf{n}}\, d\Sigma\, dt = -(2\pi r l)\frac{r}{2}\int_0^t H\frac{\partial B}{\partial t}\, dt = -\pi r^2 l\frac{HB}{2} = -\pi r^2 l\frac{\mu_0 {n_s}^2 I^2}{2} = -\frac{1}{2}LI^2$$

where we have used the formula $L = \mu_0 {n_S}^2 l\pi r^2$ for the solenoid inductance in the infinite length approximation.

The magnetic energy inside the solenoid enters through the lateral surface as a flux of electromagnetic energy associated to the Poynting's vector (the minus sign accounts for the entering flux).

9.2.3 Condenser

Suppose we have a condenser with parallel circular plates of radius r at distance δ. Initially the condenser is charged. While it is discharged, the electric field $\mathbf{E} = E_z \hat{\mathbf{z}}$ between the plates, changes and there is a displacement current:

$$\mathbf{J}_S = \epsilon_0 \frac{\partial E_z}{\partial t} \hat{\mathbf{z}} .$$

From the line integral of the magnetic field $\mathbf{B}$ around a circular closed curve of radius r, coaxial with the condenser, for $\mathbf{B}$ we find:

$$\oint \mathbf{B} \cdot \mathbf{dl} = 2\pi r B = \mu_0 \pi r^2 J_S \qquad \mathbf{B} = \frac{\mu_0 \epsilon_0}{2} r \frac{\partial E_z}{\partial t} \hat{\boldsymbol{\varphi}} .$$

As shown in Fig. 9.1c, on the lateral surface of the condenser we have the Poynting's vector:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = -E_z \frac{B_\varphi}{\mu_0} \hat{\mathbf{r}} = -\epsilon_0 \frac{r}{2} E_z \frac{\partial E_z}{\partial t} \hat{\mathbf{r}}$$

with a total flux:

$$\Phi(S) = -\epsilon_0 \frac{r}{2} E_z \frac{\partial E_z}{\partial t} \; 2\pi r \delta = -\pi r^2 \delta \epsilon_0 \frac{1}{2} \frac{\partial E_z^2}{\partial t} .$$

The integral of the flux over the time interval in which the electric field changes from the initial value $E_i = V/\delta$ to zero, is the total energy going through the lateral surface:

$$\int_0^\infty \Phi(S)\, dt = \pi r^2 \delta \frac{\epsilon_0 E_{zi}^2}{2} = \frac{1}{2} (\epsilon_0 \pi r^2 \delta) \left(\frac{V}{\delta} \right)^2 = \frac{1}{2} \left(\frac{\epsilon_0 \pi r^2}{\delta} \right) V^2 = \frac{1}{2} C V^2 \tag{9.9}$$

and it is equal to the electrostatic energy initially stored in the condenser.[3]

---

[3]We get the same result if we consider the discharge of the condenser with a resistor R. The charge Q as a function of the time is:

$$Q = Q_0 e^{-\frac{t}{\tau}}$$

where $\tau = RC$, and the electric field can be written:

$$E = \frac{\sigma}{\epsilon_0} = \frac{1}{\epsilon_0} \frac{Q_0}{\pi r^2} e^{-\frac{t}{\tau}} .$$

The Poynting's vector is:

$$S = \frac{1}{\tau} \left(\frac{Q_0}{\pi r^2} \right)^2 \frac{r}{2\epsilon_0} e^{-\frac{2t}{\tau}}$$

and its integral over the lateral surface during the discharge gives the electrostatic energy initially in the condenser as in (9.9).

9.3 Energy Transfer in Electrical Circuits

The Poynting's vector describes the flux of energy of the electromagnetic field and it is usually considered for the electromagnetic waves while the energy transfer in the electrical circuits is associated to the electrical currents. The previous examples instead clearly show that the flux of the Poynting's vector accounts also for the energy transfer in the components of the electrical circuits and even in quasi steady or steady conditions. The power dissipated in a resistor and the energy stored in a capacitor or in an inductor, flow inward (or outward) through the sides of the components. This surprising and apparently strange description is correct and consistent with the Maxwell equations.

After all, the transfer of energy in electrical circuits, associated to the flux of the electric and magnetic fields, is to be expected because also in electrostatics and magnetostatics the energy is located in the space around the charges and the electric currents. While moving, the charges drag the fields around them and also the energy associated to their fields.

A. Sommerfeld[4] has solved the problem of an infinitely long straight resistive wire carrying a steady current with a return path through a hollow coaxial conductive cylinder surrounding the wire[5] and has shown that the electromagnetic energy flows in the space between the two conductors and is dissipated in the wire. A qualitative but clear and effective description of the energy transfer from a battery to a resistive load has been given by Galili and Goihbarg.[6] It is useful to shortly report here their analysis.

Consider the circuit as sketched in Fig. 9.2a. The terminals of the battery are electrically connected to the terminals of the resistor R by two cylindrical conductors. These two conductors, at constant voltages, are charged and a surface charge is present on their surface. The inner electric field E is clearly null while outside there is a radial field E_n due to the surface charge. Inside the resistor, the applied voltage generates a uniform non null electric field E_τ parallel to the axis while a radial field is present on the lateral surface due to the surface charge as in the two conductors. This radial field reverses by 180° along the outer part of the resistor.

In the battery is present an electrostatic field from the anode to the cathode and all the chemical processes can be accounted by an electromotive field,[7] opposite to the electrostatic field and slightly larger, that, pushing the positive charges from the cathode to the anode and the negative charges in the opposite direction, generates the current in the circuit. When an electrical current flows in the circuit, around the battery, the conductors and the resistor, there is also a magnetic field with circular field lines coaxial with these components.

---

[4]A. Sommerfeld, *Electrodynamics*, (1952), Academic Press, New York, pp. 125–130.

[5]See Problem 9.2 of this chapter.

[6]I. Galili and E. Goihbarg, *Energy transfer in electrical circuits: a qualitative account*, American Journal of Physics **73**, pp. 141–144, (2005).

[7]Other generators are based on thermal, luminous or mechanical phenomena but the electromotive field description is still valid.

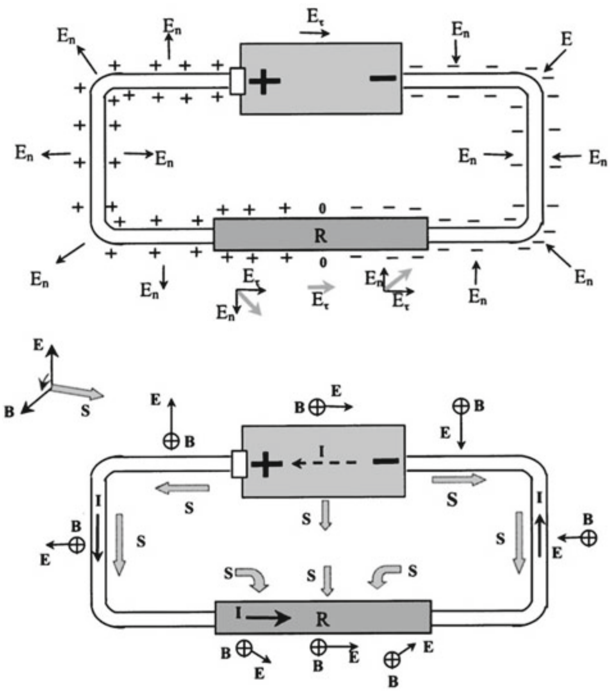


Fig. 9.2 The circuit with the battery, the conductors and the resistor: *up* the surface charge and the external electric field are shown; *down* the electric field, the magnetic field and the Poynting's vector S along the lateral surfaces of the circuit (Figures reproduced with permission from I. Galili and E. Goihbarg, cited. Copyright 2005, American Association of Physics Teachers.)

Consider the energy flux as shown in Fig. 9.2b. Inside the two conductors the electric field is null: the Poynting's vector is null and there is no energy flux. Instead a non null Poynting's vector is present around the lateral surface of the conductors as a consequence of the magnetic field and of the radial electric field. This vector is parallel to the conductors and accounts for the energy transfer from the terminals of the battery to the terminals of the resistor. At the surface of the resistor the electric field is the sum of the parallel field E_τ and of the normal field E_n and with the magnetic field give a Poynting's vector transferring the flux of energy from the outside of the conductors to the surface of the resistor. Inside the resistor the electric field E_τ, parallel to the resistor axis, and the magnetic field give a radial Poynting's vector that transfers the energy in the resistor. In conclusion, the energy flows from the battery outside of the conductors without any loss, while the current is carried inside the conductors with no energy transfer. Then the energy enters through the sides of the resistor and it is absorbed and converted into heat inside it. While the energy is carried in non conductors (air), the current is carried inside the conductors.

9.4 The Maxwell Stress Tensor

Consider a volume τ, enclosed by a surface Σ, and inside it an electromagnetic field interacting with electric charges q in motion. We have seen in (9.1) that the force applied by the electromagnetic field to a small volume $d\tau$ is:

$$\mathbf{df} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})\, n\, d\tau$$

where n is the number of charges per unit volume and $\mathbf{v}$ is their velocity.

Substituting $\rho = nq$ and $\mathbf{J} = nq\mathbf{v}$ the force $d\mathbf{f}$ becomes:

$$\mathbf{df} = (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})\, d\tau\,.$$

The force on the matter in the finite volume τ is:

$$\mathbf{F} = \int_\tau \mathbf{df} = \int_\tau (\rho\mathbf{E} + \mathbf{J} \times \mathbf{B})\; d\tau$$

and since from the first and the fourth Maxwell equations we have:

$$\rho = \nabla \cdot \mathbf{D} \qquad \mathbf{J} = \nabla \times \mathbf{H} - \frac{\partial \mathbf{D}}{\partial t}$$

the force can be written:

$$\mathbf{F} = \int_\tau \left[\mathbf{E}\,(\nabla \cdot \mathbf{D}) + (\nabla \times \mathbf{H}) \times \mathbf{B} - \frac{\partial \mathbf{D}}{\partial t} \times \mathbf{B}\right] d\tau\,.$$

From the relation:

$$\frac{\partial(\mathbf{D} \times \mathbf{B})}{\partial t} = \frac{\partial \mathbf{D}}{\partial t} \times \mathbf{B} + \mathbf{D} \times \frac{\partial \mathbf{B}}{\partial t}$$

we have:

$$\frac{\partial \mathbf{D}}{\partial t} \times \mathbf{B} = \frac{\partial(\mathbf{D} \times \mathbf{B})}{\partial t} - \mathbf{D} \times \frac{\partial \mathbf{B}}{\partial t}$$

and the force $\mathbf{F}$ becomes:

$$\mathbf{F} = \int_\tau \left[\mathbf{E}\,(\nabla \cdot \mathbf{D}) + (\nabla \times \mathbf{H}) \times \mathbf{B} - \frac{\partial(\mathbf{D} \times \mathbf{B})}{\partial t} + \mathbf{D} \times \frac{\partial \mathbf{B}}{\partial t}\right] d\tau\,.$$

Using the third Maxwell equation and adding the null term $\mathbf{H}(\nabla \cdot \mathbf{B})$ finally we can write:

$$\mathbf{F} + \int_\tau \frac{\partial(\mathbf{D} \times \mathbf{B})}{\partial t}\, d\tau = \int_\tau [\mathbf{E}\,(\nabla \cdot \mathbf{D}) - \mathbf{D} \times (\nabla \times \mathbf{E}) + \mathbf{H}(\nabla \cdot \mathbf{B}) - \mathbf{B} \times (\nabla \times \mathbf{H})]\; d\tau$$

where in the right-hand side of the equation the fields **E** and **D** and the fields **H** and **B** enter into the equation in the same way.

Consider first the x component:

$$F_x + \int_\tau \frac{\partial(\mathbf{D} \times \mathbf{B})_x}{\partial t}\, d\tau \tag{9.10}$$

$$= \int_\tau \{E_x\,(\nabla \cdot \mathbf{D}) - [\mathbf{D} \times (\nabla \times \mathbf{E})]_x + H_x(\nabla \cdot \mathbf{B}) - [\mathbf{B} \times (\nabla \times \mathbf{H})]_x\}\; d\tau\ .$$

An explicit calculation of the two terms for the electric field:

$$E_x\,(\nabla \cdot \mathbf{D}) - [\mathbf{D} \times (\nabla \times \mathbf{E})]_x$$

gives:

$$= E_x\left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) - D_y\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) + D_z\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)$$

$$= \epsilon\frac{\partial}{\partial x}\left(\frac{E_x^2 - E_y^2 - E_z^2}{2}\right) + \epsilon\frac{\partial}{\partial y}\left(E_xE_y\right) + \epsilon\frac{\partial}{\partial z}\left(E_xE_z\right)$$

and a similar expression can be written for the two terms of the magnetic fields:

$$\mu\frac{\partial}{\partial x}\left(\frac{H_x^2 - H_y^2 - H_z^2}{2}\right) + \mu\frac{\partial}{\partial y}\left(H_xH_y\right) + \mu\frac{\partial}{\partial z}\left(H_xH_z\right)\ .$$

Defining the variables:

$$T_{xx} = \frac{1}{2}\epsilon(E_x^2 - E_y^2 - E_z^2) + \frac{1}{2}\mu(H_x^2 - H_y^2 - H_z^2)$$

$$T_{xy} = \epsilon E_xE_y + \mu H_xH_y$$

$$T_{xz} = \epsilon E_xE_z + \mu H_xH_z$$

the Eq. (9.10) becomes:

$$F_x + \frac{\partial}{\partial t}\int_\tau (\mathbf{D} \times \mathbf{B})_x\; d\tau = \int_\tau \left(\frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z}\right)\, d\tau \tag{9.11}$$

$$= \int_\Sigma \left(T_{xx}n_x + T_{xy}n_y + T_{xz}n_z\right)\; d\Sigma$$

where we have used the Gauss-Green theorem and we have defined $\hat{\mathbf{n}}$ (n_x, n_y, n_z) the ouward-pointing unit vector normal to the element of surface $d\Sigma$.

Similarly for the y and z components of the force $\mathbf{F}$:

$$F_y + \frac{\partial}{\partial t}\int_\tau (\mathbf{D}\times\mathbf{B})_y \; d\tau = \int_\Sigma \left(T_{yx}n_x + T_{yy}n_y + T_{yz}n_z\right)\; d\Sigma \tag{9.12}$$

with:

$$T_{yy} = \frac{1}{2}\epsilon(E_y^2 - E_z^2 - E_x^2) + \frac{1}{2}\mu(H_y^2 - H_z^2 - H_x^2)$$

$$T_{yx} = \epsilon E_y E_x + \mu H_y H_x$$

$$T_{yz} = \epsilon E_y E_z + \mu H_y H_z$$

and:

$$F_z + \frac{\partial}{\partial t}\int_\tau (\mathbf{D}\times\mathbf{B})_z \; d\tau = \int_\Sigma \left(T_{zx}n_x + T_{zy}n_y + T_{zz}n_z\right)\; d\Sigma \tag{9.13}$$

with:

$$T_{zz} = \frac{1}{2}\epsilon(E_z^2 - E_x^2 - E_y^2) + \frac{1}{2}\mu(H_z^2 - H_x^2 - H_y^2)$$

$$T_{zx} = \epsilon E_z E_x + \mu H_z H_x$$

$$T_{zy} = \epsilon E_z E_y + \mu H_z H_y \,.$$

The terms T_{ij} are the components of the tensor:

$$T_{ij} = E_i D_j + H_i B_j - \frac{1}{2}\delta_{ij}(\epsilon E^2 + \mu H^2) \tag{9.14}$$

$$= \epsilon[E_i E_j + v^2 B_i B_j - \frac{1}{2}\delta_{ij}(E^2 + v^2 B^2)]$$

which is clearly symmetric.

We can introduce the vector $\mathbf{g} = \mathbf{D}\times\mathbf{B}$ and its integral $\mathbf{G}$:

$$\mathbf{G} = \int_\tau \mathbf{g}\, d\tau \,.$$

The equation of the Mechanics for the force $\mathbf{F}$ applied to the matter is:

$$\mathbf{F} = \frac{\mathbf{dP}^{\mathrm{Mech}}}{dt}$$

where $\mathbf{P}^{\mathrm{Mech}}$ is the total momentum of the matter inside the volume τ.

The Eq. (9.11) becomes:

$$\frac{dP_x^{\mathrm{Mech}}}{dt} + \frac{dG_x}{dt} = \int_\Sigma \sum_{i=1}^{N} T_{xi} n_i \, d\Sigma$$

and similar equations can be written for the components y and z.

If we consider a volume τ expanded up to regions where the fields E and B are null, the right-hand side is null and we get:

$$\frac{d}{dt}(P_x^{\mathrm{Mech}} + G_x) = 0$$

and thus:

$$\mathbf{P}^{\mathrm{Mech}} + \mathbf{G} = const\,.$$

Since for an isolated system with charges and electromagnetic fields the sum of these two vectors is conserved, if $\mathbf{P}^{Mech}$ is the total momentum of the charged particles in motion, we have to conclude that the vector $\mathbf{G}$ is the total momentum of the electromagnetic field inside the volume τ. As a consequence the vector $\mathbf{g}$ is the momentum density of the electromagnetic field (momentum per unit volume).

The three Eqs. (9.11), (9.12) and (9.13) can be written as:

$$\frac{d}{dt}(P_x^{\mathrm{Mech}} + G_x) = \int_\Sigma \left(T_{xx}n_x + T_{xy}n_y + T_{xz}n_z\right)\, d\Sigma$$

$$\frac{d}{dt}(P_y^{\mathrm{Mech}} + G_y) = \int_\Sigma \left(T_{yx}n_x + T_{yy}n_y + T_{yz}n_z\right)\, d\Sigma$$

$$\frac{d}{dt}(P_z^{\mathrm{Mech}} + G_z) = \int_\Sigma \left(T_{zx}n_x + T_{zy}n_y + T_{zz}n_z\right)\, d\Sigma\,.$$

The terms $T_{ij}n_j d\Sigma$ in the integrals are the contributions to the total force applied to the charged system in the volume τ, along the directions x, y and z, across any small element $d\Sigma$, oriented with normal $\hat{\mathbf{n}}(n_x, n_y, n_z)$, of the surface Σ surrounding the volume τ. This is the reason why, in analogy with the stress tensor in Mechanics of the continuous systems, the tensor T_{ij} is called *Maxwell stress tensor*.

From $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{B} = \mu\mathbf{H}$ we can find the relation between the momentum density $\mathbf{g}$ and the Poynting's vector $\mathbf{S}$:

$$\mathbf{g} = \mathbf{D} \times \mathbf{B} = \mu\epsilon\mathbf{E} \times \mathbf{H} = \frac{\mathbf{E} \times \mathbf{H}}{v^2} = \frac{\mathbf{S}}{v^2} = \frac{u}{v}\hat{\mathbf{v}} \tag{9.15}$$

where we have used relation (9.8).

The momentum of the electromagnetic field crossing a normal unit surface in a time unit is:

$$\mathbf{p}_{rad} = \mathbf{g}v = \frac{\mathbf{S}}{v}\,. \tag{9.16}$$

We have seen that $\mathbf{S}$ is the energy through the same surface in the time unit, so we have that the (9.16) corresponds to the relation between the momentum and the energy $h\nu$ of the photon in Quantum Mechanics:

$$p_\gamma = \frac{h\,\nu}{c}\,.$$

9.5 Radiation Pressure on a Surface

Consider an electromagnetic wave incident normally on a fully absorbing surface of area Σ. The average pressure applied by the radiation to the surface is:

$$\overline{p_\Sigma} = \frac{\overline{F}}{\Sigma} = \frac{1}{\Sigma}\frac{\overline{dP^{Mech}}}{dt} = -\frac{1}{\Sigma}\frac{\overline{dP^{rad}}}{dt} = \overline{|\mathbf{g}v|} = \frac{\overline{|\mathbf{E}\times\mathbf{H}|}}{v} = \frac{\overline{|\mathbf{S}|}}{v} \tag{9.17}$$

where we have considered that the momentum change for the absorbing plane is opposite to the momentum change of the radiation, and that $\mathbf{g}v$ is the momentum carried by the radiation through a normal unit surface per unit time.

For a perfectly reflective surface, the final momentum of the radiation is exactly opposite to the initial one. Thus the change in momentum is twice that for the fully absorbing surface and the pressure is:

$$\overline{p'_\Sigma} = \frac{\overline{F'}}{\Sigma} = \frac{1}{\Sigma}\frac{\overline{dP^{Mech}}}{dt} = -\frac{1}{\Sigma}\frac{\overline{dP^{rad}}}{dt} = 2\overline{|\mathbf{g}v|} = 2\frac{\overline{|\mathbf{E}\times\mathbf{H}|}}{v} = 2\frac{\overline{|\mathbf{S}|}}{v}\,. \tag{9.18}$$

The pressure of the electromagnetic wave on the absorbing wall can be also calculated from the Maxwell stress tensor. We assume a plane linearly polarized monochromatic wave, moving in the $\hat{z}$ direction, with the field components:

$$E_x = E_0\,\cos(kz-\omega t) \qquad B_y = B_0\,\cos(kz-\omega t) \qquad E_0 = B_0 v$$

incident on the fully absorbing half-space at $z \geq 0$. Then we consider a cylindrical volume, of negligible height, crossed by the plane $z = 0$ that is parallel to the two basis as in Fig. 9.3. The force applied by the wave can be calculated from the relations (9.11), (9.12) and (9.13).

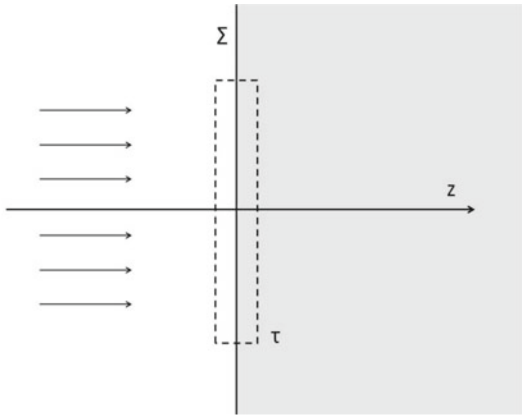


Fig. 9.3 Cylinder for the calculation of the pressure applied by a plane wave on a fully absorbing wall

Only the z component of the vector $\mathbf{D} \times \mathbf{B}$ is non null:

$$\frac{\partial}{\partial t} (\mathbf{D} \times \mathbf{B})_z = -\epsilon E_0 B_0 \, 2\omega \sin(kz - \omega t) \cos(kz - \omega t)$$

but its time average is zero. The components of the stress tensor at $z > 0$ are null because the fields E and B are null. For $z < 0$ $\;T_{xx}, T_{yy}, T_{xy}, T_{yz}$ and T_{zx} are zero and the only non null component is T_{zz}:

$$T_{zz} = -\frac{1}{2}\epsilon E_x^2 - \frac{1}{2}\mu H_y^2 = -\epsilon E_0^2 \, \cos^2(kz - \omega t)$$

that averaged over the time gives:

$$\overline{T}_{zz} = -\frac{1}{2}\epsilon E_0^2 \, .$$

From these components for T_{ij} and from the relations (9.11) and (9.12) we find $F_x = F_y = 0$ while inserting $\overline{T}_{zz}$ in the (9.13) and integrating over the external surface of the cylinder we find a non null contribution only for $z < 0$ (where $\hat{n}_z = -1$) and for the force applied to the cylinder that, in the limit of an infinitesimal height, coincides with the absorbing surface, we get:

$$\overline{F}_z = -\overline{T}_{zz}\Sigma = \frac{\epsilon}{2}E_0^2\Sigma = \overline{u}\Sigma$$

and the average pressure: $\;\overline{p}_\Sigma = \dfrac{\overline{F}_z}{\Sigma} = \overline{u} = \dfrac{\overline{|\mathbf{S}|}}{v}$

equal to (9.17).

9.6 Angular Momentum

An angular momentum $\mathbf{dL}$ can be associated to the momentum $\mathbf{g}d\tau$ of the electromagnetic field in the small volume $d\tau$. If $\mathbf{r}$ is the vector from the pole Ω to the volume $d\tau$, the angular momentum is:

$$\mathbf{dL} = \mathbf{r} \times \mathbf{g}\, d\tau \,. \tag{9.19}$$

The angular momentum of the electromagnetic field in a volume τ is the integral of $\mathbf{dL}$:

$$\mathbf{L} = \int_\tau \mathbf{r} \times \mathbf{g}\, d\tau \,. \tag{9.20}$$

This is the angular momentum associated to the motion of the field. For the electromagnetic radiation there is also a spin, the intrinsic angular momentum. For a circular polarised monochromatic plane wave of frequency $\nu = \omega/2\pi$ this is:

$$\mathbf{L} = \pm\frac{\mathbf{S}}{\omega}$$

with the sign $+$ or $-$ for a clockwise or counterclockwise polarization.[8]

We remind that in Quantum Mechanics the spin of the photon, relative to a direction, is quantized and is $\pm\hbar$ with $\hbar = h/2\pi$, where h is the Planck constant, while for the electron the spin is $\pm 1/2\ \hbar$.

In the next chapter we will see examples of the angular momentum of the electromagnetic field.

9.7 The Covariant Maxwell Stress Tensor

The tensor T_{ij} defined in (9.14) is the space part of the 4-dimension symmetric tensor[9] $T^{\mu\nu}$:

$$T^{\mu\nu} = \epsilon c^2 \left[F^{\mu\rho} F^{\nu}{}_{\rho} - \frac{1}{4} g^{\mu\nu} F^{\rho\sigma} F_{\rho\sigma} \right]$$

$$= T^{\mu\nu} = \begin{pmatrix} -u & -\frac{S_x}{c} & -\frac{S_y}{c} & -\frac{S_z}{c} \\ \frac{-S_x}{c} & T_{xx} & T_{xy} & T_{xz} \\ \frac{-S_y}{c} & T_{yx} & T_{yy} & T_{zz} \\ -\frac{S_z}{c} & T_{zx} & T_{zy} & T_{zz} \end{pmatrix}$$

---

[8] See for instance: F.S. Crawford Jr, *Waves, Berkeley Physics Course, Vol. 3*, (1965), McGraw-Hill, Sect. 7.4.

[9] For more details see for instance: J.D. Jackson, *Classical Electrodynamics*, cited, Sect. 12.10; L.D. Landau-E. M. Lifšits, *The Classical Theory of Fields*, Chapter IV, Sect. 33.

also called *energy-momentum tensor* of the electromagnetic field. The space components correspond to the stress tensor T_{ij}, T^{00} is equal to $-u$ where u is the energy density of the electromagnetic field, $T^{0i} = T^{i0} = -S_i/c$ for $i = 1, 2, 3$ are proportional to the components of the Poynting's vector.

The force per unit volume:

$$f^{\mu} = F^{\mu\nu}J_{\nu} \quad \left(\frac{\mathbf{E}\cdot\mathbf{J}}{c},\ \rho\mathbf{E} + \mathbf{J}\times\mathbf{B}\right)$$

is related to the stress tensor by the covariant expression:

$$f^{\mu} = \partial_{\nu}T^{\mu\nu}\,.$$

For $\mu = 0$ this relation gives Eq. (9.7):

$$\mathbf{E}\cdot\mathbf{J} = -\frac{\partial u}{\partial t} - \nabla\cdot\mathbf{S}$$

and for $\mu = i = 1, 2, 3$ the equations:

$$f^{i} + \frac{\partial g^{i}}{\partial t} = \frac{\partial T^{ix}}{\partial x} + \frac{\partial T^{iy}}{\partial y} + \frac{\partial T^{iz}}{\partial z}$$

which are the local expressions of the integral Eqs. (9.11), (9.12), (9.13).

Problems

9.1 A battery of voltage V is connected to a resistor R with a coaxial cable. This cable is composed by two coaxial hollow cylinders of radii a and b ($b > a$) as in figure. Find the Poynting's vector in the coaxial cable. Show that the total flux of this vector in the cable is equal to the power dissipated in the resistor (Fig. 9.4).

9.2 Sommerfeld has given the solution for a wire of radius a and conductivity σ, surrounded by a cylindrical hollow conductor with inner radius b and an infinite outer radius. An electrical current, with uniform density J, flows in the central conductor

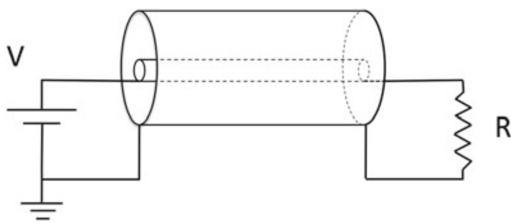


Fig. 9.4 The coaxial cable connecting the battery to the resistor

from $z = -\infty$ to $z = +\infty$ and returns in the outer conductor. The wire lies on the z axis.

Find the solution for the field and consider the Poynting's vector or proceed through the following steps:

(a) define the boundaries conditions for the electric field in the space between the two conductors;
(b) verify that the solution of the Laplace equation in the space between the conductors is:

$$V(r, z) = -\frac{J}{\sigma}\frac{\log\frac{r}{b}}{\log\frac{a}{b}}z$$

(c) find the charge densities over the surfaces of the conductors;
(d) write the Poynting's vector between the two conductors;
(e) consider the Poynting's vector inside the central conductor.

9.3 In the solar system a space vehicle might be propelled by the pressure of the solar radiation on a large fully reflective sail. Determine the minimal area of the sail to push a satellite of mass $m_v = 300$ kg when it is distant from the Sun as the Earth where the solar radiation power is 1.35 kW/m^2. (M$_{Sun}$ = $1.99 \cdot 10^{30}$ kg, distance Sun-Earth d_{SE} = $1.5 \cdot 10^{11}$ m, G$_{Gravit.}$ = $6.67 \cdot 10^{-11}$ Nm^2/kg^2)

9.4 Find the force on the plates of a charged parallel plate condenser by using the stress tensor.

9.5 Calculate the pressure on the turns of a long solenoid with a constant current.

Solutions

9.1 The inner conductor is at voltage V while the outer is at ground. Say λ the linear charge density on the inner conductor, and $-\lambda$ that on the outer one. Consider a cylindrical surface of length l and of radius r ($a < r < b$) coaxial with the cable. The axis of the cylinders is the z axis. From the Gauss theorem we get the electric field inside the cable:

$$E \cdot 2\pi rl = \frac{\lambda l}{\epsilon_0} \quad \rightarrow \quad \mathbf{E} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\hat{\mathbf{r}}\,.$$

From the voltage between the two conductors we can get λ:

$$V = \int_a^b E\,dr = \frac{\lambda}{2\pi\epsilon_0}\int_a^b \frac{1}{r}dr = \frac{\lambda}{2\pi\epsilon_0}\log\frac{b}{a} \qquad \lambda = \frac{2\pi\epsilon_0 V}{\log\frac{b}{a}}$$

and the electric field becomes:

$$\mathbf{E} = \frac{V}{\log \frac{b}{a}} \frac{1}{r} \hat{\mathbf{r}} .$$

A current $I = V/R$ is flowing in the conductor from the battery to the resistor. The field has circular field lines around the axis. From the Ampère law we get the field B at a radius r:

$$B \cdot 2\pi r = \mu_0 I \quad \mathbf{B} = \frac{\mu_0}{2\pi} \frac{V}{R} \frac{1}{r} \hat{\boldsymbol{\varphi}} .$$

The field $\mathbf{E}$ and $\mathbf{B}$ are perpendicular in any point between the coaxial surfaces and the Poynting's vector is:

$$\mathbf{S} = \frac{|\mathbf{E} \times \mathbf{B}|}{\mu_0} \hat{\mathbf{z}} = \frac{1}{2\pi} \frac{1}{\log \frac{b}{a}} \frac{V^2}{R} \frac{1}{r^2} \hat{\mathbf{z}}$$

oriented in the direction of the axis from the battery to the resistor. The total flux of this vector through a cross section normal to the axis, is:

$$\Phi(S) = \int_a^b 2\pi r S dr = \int_a^b 2\pi r \frac{1}{2\pi} \frac{1}{\log \frac{b}{a}} \frac{V^2}{R} \frac{1}{r^2} dr = \frac{1}{\log \frac{b}{a}} \frac{V^2}{R} \int_a^b \frac{1}{r} dr$$

$$= \frac{1}{\log \frac{b}{a}} \frac{V^2}{R} \log\ r|_a^b = \frac{V^2}{R} .$$

The flux of energy between the cylindrical surfaces is just the power dissipated in the resistor.

9.2 **(a)** Nearby the surface of the wire, the outside z component of the electric field has to be equal to the inside z component that is $E_z = \sigma J$. Near the surface of the external conductor, at $r = b$, $E_z = 0$ because inside this conductor the field is null. The radial component of the electric field can be non null on the surface of the wire and on the inner surface of the external conductor.
(b) It is easy to verify that the function $V(r, z)$ is a solution of the Laplace equation $\Delta V = 0$. The component E_z is:

$$E_z(r, z) = -\frac{\partial V}{\partial z} = \frac{J}{\sigma} \frac{\log \frac{r}{b}}{\log \frac{a}{b}} .$$

Thus for $r = a$ $E_z = J/\sigma$ while for $r = b$ $E_z = 0$ as requested by the boundary conditions. The radial component of the electric field is:

$$E_r(r, z) = -\frac{\partial V}{\partial r} = \frac{J}{\sigma} \frac{z}{\log \frac{a}{b}} \frac{1}{r}$$

that for $r = a$ gives:

$$E_r(a, z) = \frac{J}{\sigma} \frac{z}{\log \frac{a}{b}} \frac{1}{a}$$

and for $r = b$:

$$E_r(b, z) = \frac{J}{\sigma} \frac{z}{\log \frac{a}{b}} \frac{1}{b}.$$

(c) From the Coulomb's theorem the surface charge densities on the conductors are:

$$\text{at } r = a \quad \sigma = \epsilon E_r(a, z) = \epsilon \frac{J}{\sigma} \frac{z}{\log \frac{a}{b}} \frac{1}{a}$$

$$\text{at } r = b \quad \sigma = -\epsilon E_r(b, z) = -\epsilon \frac{J}{\sigma} \frac{z}{\log \frac{a}{b}} \frac{1}{b}.$$

The electric field lines and the equipotential surfaces inside the conductors are shown in the Fig. 9.5.
(d) The magnetic field due to the current in the wire is:

$$\mathbf{H} = \frac{r}{2} J \hat{\boldsymbol{\varphi}} \qquad r < a$$

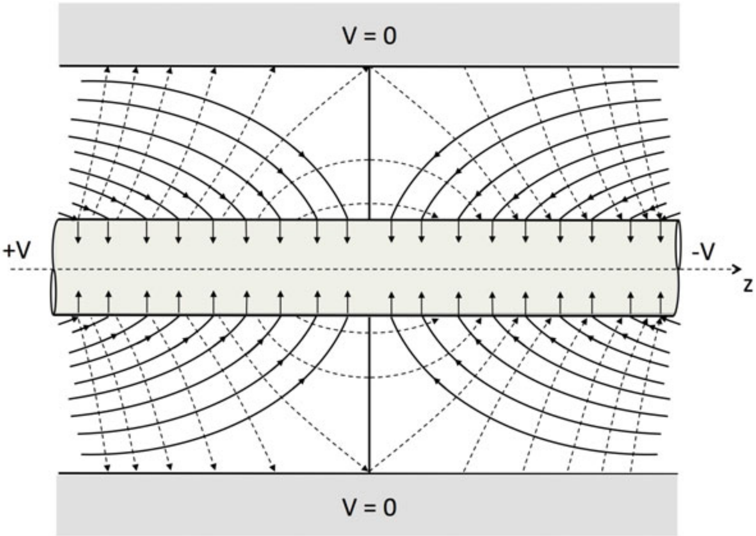


Fig. 9.5 The resistive wire with a coaxial hollow conductor. The *dotted lines* are electric field lines; the *full lines* represent the equipotential surfaces (only *outside* the wire) and also the Poynting's vector field lines

$$\mathbf{H} = \frac{a^2}{2r} J \hat{\boldsymbol{\varphi}} \qquad a < r < b$$

$$\mathbf{H} = \frac{a^2}{2r} J \hat{\boldsymbol{\varphi}} \qquad b < r < \infty\,.$$

The Poynting's vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ is normal to the field $\mathbf{H}$ and to the field $\mathbf{E}$ thus its field lines are the curved lines determined by the crossing of the rz planes with the equipotential surfaces, as shown in figure. These curves show the energy flux from the generator to the wire.
(e) Inside the wire the electric field has only the component E_z. Thus the Poynting's vector $\mathbf{S} = \mathbf{E} \times \mathbf{H}$ has radial direction oriented to the center of the wire. It gives the flux of energy dissipated in the core of the resistive wire.

9.3 The solar power per unit area is just the time averaged Poynting's vector $\overline{S} = 1.35$ kW/m^2. From (9.18) we have the average force per unit surface:

$$\overline{F} = 2\overline{\frac{\Delta p_{rad}}{\Delta t}} = 2\,\overline{g}c = \frac{2\overline{S}}{c}\,.$$

The force on the sail of area Σ has to be larger than the gravitational force from the Sun:

$$\frac{2\overline{S}}{v}\Sigma > G\frac{M_{Sun}m_v}{d_{SE}^2}$$

and so we get: $\Sigma > 19.66 \cdot 10^4$ m^2.

9.4 Assume the z axis normal to the plates of area Σ and a cylinder of very small height across the inner side of the upper plate as in Fig. 9.6. Inside the plate the electric field is null while between the plates it is $\mathbf{E}_0\ (0, 0, E_{0z})$. Inside the plate the T_{ij} are null and between the plates the only non null components of T_{ij} are:

$$T_{xx} = -\frac{1}{2}\epsilon_0 E_{0z}^2 \quad T_{yy} = -\frac{1}{2}\epsilon_0 E_{0z}^2 \quad T_{zz} = \frac{1}{2}\epsilon_0 E_{0z}^2\,.$$

The normal to the upper plate is $\hat{\mathbf{n}}\ (0, 0, -1)$. From relations (9.11), (9.12), (9.13), we have:

$$F_x = 0 \quad F_y = 0 \quad F_z = -\int_\Sigma T_{zz}\, d\Sigma = -\int_\Sigma \frac{1}{2}\epsilon_0 E_{0z}^2\, d\Sigma = -\frac{1}{2}\epsilon_0 E_{0z}^2 \Sigma = -u_e \Sigma$$

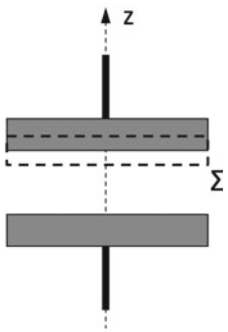


Fig. 9.6 The cylinder across the *inner side* of the plate of the condenser to calculate the force on the plate

where u_e is the density of electrostatic energy. The same result can be found[10] by writing the force over the charge $\sigma\Sigma$ of one plate in the field $E_{0z} = -\sigma/2\epsilon_0$ of the other plate (Fig. 9.6):

$$F = -\frac{\sigma}{2\epsilon_0}\sigma\Sigma = -\frac{1}{2}\epsilon_0 E_{0z}^2 \Sigma\,.$$

9.5 We choose as z axis the axis of the solenoid and we consider a small cylinder of base $d\Sigma$ and infinitesimal height across the surface of the solenoid. $\hat{\mathbf{n}}\,(-\cos\theta, -\sin\theta, 0)$ is the normal of the inner side. Inside the solenoid the magnetic field is $\mathbf{H_0}\,(0, 0, H_{0z})$ and the only non null components of T_{ij} are:

$$T_{xx} = -\frac{1}{2}\mu_0 H_{0z}^2 \qquad T_{yy} = -\frac{1}{2}\mu_0 H_{0z}^2 \qquad T_{zz} = \frac{1}{2}\mu_0 H_{0z}^2$$

Outside the solenoid the field and the T_{ij} are null.

From the relations (9.11), (9.12), (9.13) applied to the volume of the small cylinder for the components of the force on inner $d\Sigma$ we get:

$$dF_x = \frac{1}{2}\mu_0 H_{0z}^2 \cos\theta\; d\Sigma \qquad dF_y = \frac{1}{2}\mu_0 H_{0z}^2 \sin\theta\; d\Sigma \qquad dF_{zz} = 0$$

so the normal force and the pressure on $d\Sigma$ are:

$$dF = \mu_0 H_{0z}^2\, d\Sigma \qquad p = \frac{1}{2}\mu_0 H_{0z}^2 = u_m$$

where u_m is the density of magnetic energy.

[10] The force can be also calculated from the principle of virtual work.

For a constant current the magnetic force can be derived from the magnetic energy[11]:

$$F_{magn} = \left[\frac{dU_m}{ds}\right]_{i=const}.$$

For the solenoid the magnetic energy is:

$$U_m = \frac{1}{2}Li^2 \quad \text{with } L \text{ the inductance: } L = n^2\mu_0\Sigma l$$

where n is the number of turns per unit length, l the length and $\Sigma = \pi r^2$ the area of the cross section of the solenoid. For a virtual expansion of the the radius of the solenoid we have:

$$F_{magn}\,dr = d\left(\frac{1}{2}\mu_0 n^2 i^2 \pi r^2 l\right) = \mu_0\, n^2 i^2 \pi r l\, dr$$

$$p = \frac{F_{magn}}{2\pi r l} = \frac{1}{2}\mu_0 n^2 i^2 = \frac{1}{2}\mu_0 H_z^2 = u_m\,.$$

[11]This formula can be found from the principle of virtual work. In a electrical network with a generator, a solenoid and a resistor, the sum of the work δL_{ext} from an external force and of the work $\delta L_{gen} = fidt$ of the generator of electromotive force f, which keeps constant the current i, has to be equal to the sum of the changes of the magnetic energy δU_m and of the energy $\delta W_R = i^2Rdt$ dissipated in the resistor. We can write:

$$\delta L_{ext} + \delta L_{gen} = \delta U_m + \delta W_R \qquad \text{where:} \qquad U_m = \frac{1}{2}Li^2.$$

At the same time the equation of the circuit is:

$$f - \frac{d\Phi(B)}{dt} = Ri \qquad \Phi(B) = Li.$$

We find:

$$\delta L_{ext} + id(Li) + i^2Rdt = \delta\left(\frac{1}{2}Li^2\right) + i^2Rdt$$

and from $\mathbf{F}_{magn} = -\mathbf{F}_{ext}$

$$\delta L_{ext} = \mathbf{F}_{ext}\cdot\mathbf{ds} = -\mathbf{F}_{magn}\cdot\mathbf{ds} = -\delta U_m \quad\rightarrow\quad F_{magn} = \left[\frac{dU_m}{ds}\right]_{i=const}.$$

Chapter 10
The Feynman Paradox

The Feynman paradox or paradox of the angular momentum is very intriguing. An insulating disk, with charged spheres at its edges, is at rest in the steady magnetic field produced by a solenoid at its center. The system is isolated, but if the magnetic field vanishes, the disk begins to rotate in apparent contrast with angular momentum conservation. This chapter presents the analysis of the process. The conservation of the angular momentum is verified by taking into account the angular momentum of the electromagnetic field. Then, an original example of two cylindrical shells with opposite charges immersed in a damped magnetic field is examined.

10.1 The Paradox

In his lectures[1] on the induction laws, Feynman presents the following paradox. Consider a thin plastic disk (see Fig. 10.1) supported by a concentric shaft with excellent bearings so that it is quite free to rotate in the horizontal plane. Some metal spheres are uniformly distributed near the edge of the disk. Each sphere has a charge q and is insulated by the plastic disk. Fixed and coaxial with the disk there is a solenoid that carries a steady current provided by a small battery. The system is isolated and at rest. If with no intervention from the outside[2] the current is interrupted, the magnetic

[1] R. P. Feynman, R. B. Leighton, M. Sand, *The Feynman Lectures on Physics*, Vol. II, Sect. 17.4. We suggest to read the two Chaps. 16 and 17 on the electromagnetic induction.

[2] This can be determined by a suitable device mounted on the disk or the solenoid could be done with a superconducting wire initially at very low temperature but later, the exchange of heat with the ambient, increases the temperature over the transition temperature, so that the wire becomes resistive and the current is brought to zero.

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9_10

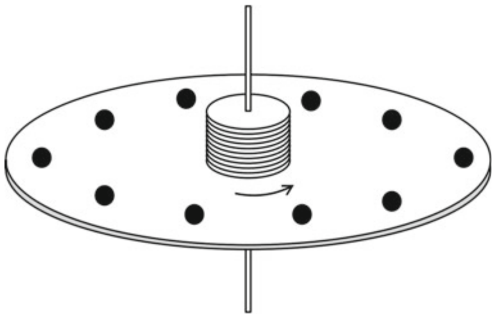


Fig. 10.1 The disk with the charged spheres and the solenoid at the center

field inside the solenoid vanishes and for the induction law an electric field E is generated. This field has a field line tangent to the circle centered on the axis of the system and passing on the small spheres. Thus on each sphere is applied a force tangent to the circle and going in the same direction and so a net torque is applied to the disk that begins to rotate.

If R_{Sol} is the radius of the solenoid, R the distance of the spheres from the center, and B the initial magnetic field inside the solenoid, the field E is given[3] by the Faraday-Neumann law:

$$\oint \mathbf{E} \cdot \mathbf{dl} = -\frac{d\Phi}{dt} \qquad 2\pi RE = -\pi R_{Sol}^2 \frac{dB}{dt} \qquad E = -\frac{R_{Sol}^2}{2R} \frac{dB}{dt}$$

and the net torque relative to the axis of the system is $\mathbf{M} = \mathbf{R} \times Nq\mathbf{E}$.

From the equation of the rotation the final angular momentum L is:

$$L = Nq \frac{R_{Sol}^2}{2} B_{in} \tag{10.1}$$

where B_{in} is the initial magnetic field inside the solenoid.

But we could also say that the system is isolated and for the conservation of the angular momentum that initially is null, the disk should not rotate.

Feynman asks which of the two arguments is correct: the disk begins to rotate or not? He also warns that: *the solution is not easy, nor is it a trick*, and adds: *when you figure it out, you will have discovered an important principle of electromagnetism*.

[3] The return flux of B in the region with a distance from the axis between R_{Sol} and R can be neglected in the approximation of a solenoid of length $l \gg R_{Sol}$.

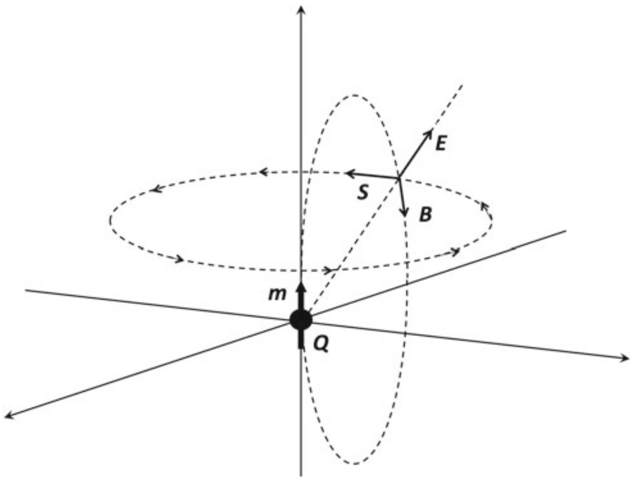


Fig. 10.2 Point charge Q at the center of a small bar magnet of moment **m**. Electric and magnetic fields with the associated Poynting's vector in a point and their *field lines*

10.2 A Charge and a Small Magnet

In the lecture[4] on the Poynting's vector, Feynman proposes the case of a point charge near the center of a small bar magnet both at rest. The relative electric E and magnetic B fields are static and clearly the electric and magnetic energy densities do not change with time. But the Poynting's vector $\mathbf{E} \times \mathbf{B}/\mu_0$ is not null and has circular field lines centred on the line of the bar magnet as shown in Fig. 10.2.

This can appear strange because we have to conclude that while the electromagnetic energy is conserved, there is a steady circular flow of this energy around the line of the magnet. Moreover, as seen in the previous chapter in relation (9.15), a flow of momentum is associated to the flow of the Poynting's vector, then there is also an angular momentum of the electromagnetic field relative to the axis. This is not strange at all because the Ampère currents in the magnet (the spinning electrons inside) or the current in the solenoid seen at the beginning of this chapter, circulate in one of the two possible directions and the charges, carried in the currents, drag with them their electric fields.

In the Feynman paradox the initial presence of the magnetic field of the solenoid and of the electric field of the charged spheres, determines a flow of the Poynting's vector and then an associated angular momentum of the electromagnetic field. When the magnetic field vanishes, the initial angular momentum of the electromagnetic

[4] *The Feynman Lectures on Physics*, cited, Vol. II, Sect. 27.5.

field is transferred (in part) to the disk that begins to rotate and the energy of the magnetic field is (in part) transformed into kinetic energy.[5]

10.3 Analysis of the Angular Momentum Present in the System

An interesting analysis of the Feynman paradox is given by Bettini[6] in the approximation of the solenoid with a vanishing small magnetic dipole at the center of the disk. The total flux of the field B at distances from the axis smaller than R, the distance of the spheres, can be easily calculated by observing that it is equal in modulus but opposite to the flux at larger distances when this is evaluated over the plane normal to the dipole and passing through its center. The field B on this plane at a distance r from the axis is:

$$B = -\frac{\mu_0}{4\pi}\frac{m}{r^3}$$

and the flux for $r > R$ is:

$$\Phi_{(r>R)}(B) = \int_R^\infty B(r) 2\pi\, r\, dr = -\frac{\mu_0}{2R} m \, .$$

Then the flux for $r < R$ is:

$$\Phi_{(r<R)}(B) = -\Phi_{(r>R)}(B) = \frac{\mu_0}{2R} m$$

and from this relation, the electromotive field E_e associated to the change of dipole moment m, from the Faraday-Neumann law, is:

$$2\pi R E_e = -\frac{d\Phi_{(r<R)}}{dt} = -\frac{\mu_0}{2R}\frac{dm}{dt} \, .$$

The equation for the angular momentum $\mathbf{L}$ of the system with the N charges at distance R is:

$$\frac{\mathbf{dL}}{dt} = \mathbf{R} \times Nq\mathbf{E}_e = \frac{\mu_0}{4\pi}\frac{1}{R} Nq \frac{\mathbf{dm}}{dt}$$

[5] The balance of energy and angular momentum between the field and the rotating system will be clear in the example examined in next section.

[6] A. Bettini, *A Course in Classical Physics 3—Electromagnetism*, Springer 2016, Sect. 10.5.

that, when integrated over the time interval needed for the dipole moment of the magnet to become null, gives the final angular momentum of the system:

$$\mathbf{L} = \frac{\mu_0}{4\pi} \frac{1}{R} Nq \, \mathbf{m} \,. \tag{10.2}$$

It is not easy to find directly the angular momentum of the electromagnetic field but it is possible to get it from a calculation of the angular momentum transferred to the system while the system with the charges is assembled. Initially only the magnetic dipole is present and the spheres on the disk are uncharged. Then the charges are carried on the spheres from an infinite distance. To carry a charge dq along a radial direction in the plane of the disk, an external force $\mathbf{dF}_{ext}$ has to contrast the force $\mathbf{dF}$ on the charge from the fields present in the system:

$$\mathbf{dF}_{ext} = -\mathbf{dF} = -dq(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

where $\mathbf{E}$ is the force due to the charges already carried on the small spheres and $\mathbf{B}$ is the field from the magnetic dipole. To the force $\mathbf{dF}_{ext}$ is associated an applied external torque[7]:

$$\mathbf{dM}_{ext} = \mathbf{r} \times \mathbf{dF}_{ext} = -dq \, \mathbf{r} \times (\mathbf{v} \times \mathbf{B}) = dq \, (\mathbf{r} \cdot \mathbf{v}) \, \mathbf{B} \,.$$

The angular momentum transferred to the system to carry dq in a radial direction from infinite to the distance R, being $vdt = dr$, is:

$$\mathbf{dL} = \int \mathbf{dM}_{ext} \, dt = \int_\infty^R dq \, \mathbf{B} \, rdr = -\frac{\mu_0 \mathbf{m}}{4\pi} \int_\infty^R dq \, \frac{1}{r^3} \, rdr = \frac{\mu_0 \mathbf{m}}{4\pi} \frac{dq}{R}$$

and summing over all the charges Nq deposited on the spheres we get the total angular momentum transferred to the system during the formation:

$$\mathbf{L} = \frac{\mu_0 \mathbf{m}}{4\pi} \frac{Nq}{R}$$

that is equal to the angular momentum (10.2). Since the disk is at rest, this angular momentum has to be associated to the electromagnetic field of the system composed by the charges and the magnetic dipole. This angular momentum remains stored in the space around the system.

[7]The contribution of the electrostatic force to the torque $\mathbf{dM}_{ext}$ is null.

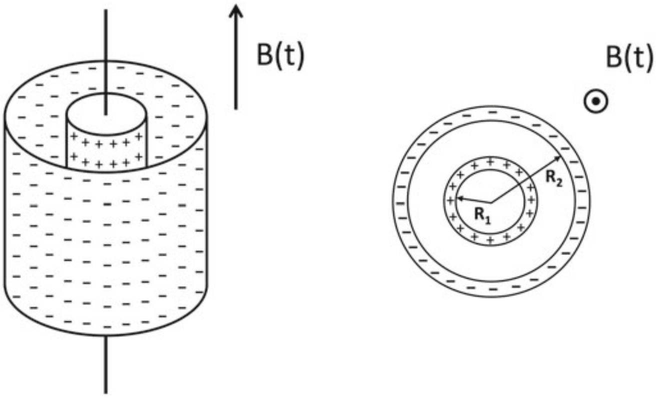


Fig. 10.3 The two insulating cylindrical shells with opposite sign surface charge densities in the magnetic field $B(t)$

10.4 Two Cylindrical Shells with Opposite Charge in a Vanishing Magnetic Field

An easy example of the Feynman paradox is the charged insulating cylindrical shell coaxial with an opposite charged wire in a vanishing magnetic field that is suggested as a problem at the end of this chapter. Here we want to examine the case of a rigid system of two charged insulating coaxial cylindrical shells, of radii R_1 and R_2, with $R_2 > R_1$, and length $l \gg R_2$, as shown in Fig. 10.3. The electric charges are deposited on the shells, with surface density $\sigma_1 = \lambda/2\pi R_1$ and $\sigma_2 = -\lambda/2\pi R_2$ where $\lambda > 0$ is the charge per unit length. The two shells are inside a solenoid[8] of radius $R \gg R_2$, coaxial with the shells and of length $l' \gg R$. Initially the current flowing in the solenoid produces a uniform magnetic field B_{in} parallel to the axis. Then the current decreases slowly and vanishes. As a consequence also the magnetic field B inside the solenoid decreases and, for the Faraday-Neumann law, at distance r from the axis there is an induced electric field E_{in}, tangent to the circle of radius r and centered on the axis, given by:

$$2\pi r E_{in} = -\pi r^2 \frac{dB}{dt} \qquad E_{in} = -\frac{r}{2}\frac{dB}{dt}$$

so that the fields on the shells at distances R_1 and R_2 are:

$$E_1 = -\frac{R_1}{2}\frac{dB}{dt} \qquad E_2 = -\frac{R_2}{2}\frac{dB}{dt}\,.$$

[8]The solenoid and the two shells can be considered isolated. An internal device can decrease to zero the current in the solenoid.

Thus on each element $d\Sigma$ of the cylindrical shells is applied a force $dF = \sigma E_{in} d\Sigma$ tangent to the surface and there is a torque $\mathbf{dM} = \mathbf{R} \times \mathbf{dF}$ relative to the axis. For a unit length of the shells the net torques are:

$$M_1 = R_1 \lambda E_1 \qquad M_2 = -R_2 \lambda E_2$$

and after substitution with the fields expressions:

$$M_2 + M_1 = \frac{\lambda}{2}\left(R_2^2 - R_1^2\right)\frac{dB}{dt}\,.$$

When the system begins to rotate with angular velocity ω (negative for our system) the charges deposited on the two surfaces, are equivalent to two surface currents directed as $\hat{\boldsymbol{\varphi}}$:

$$i_1 = \frac{\lambda l}{2\pi}\omega \qquad i_2 = -\frac{\lambda l}{2\pi}\omega$$

and between the two cylindrical charged surfaces a magnetic field[9] appears:

$$B_a = -\mu_0 \frac{\lambda\omega}{2\pi}\,.$$

While ω changes a self-induced electric field E_a is present on the outer shell:

$$E_a\, 2\pi R_2 = -\frac{d}{dt}\left[\pi\left(R_2^2 - R_1^2\right) B_a\right]$$

$$E_a = \left(R_2^2 - R_1^2\right)\frac{1}{2R_2}\mu_0 \frac{\lambda}{2\pi}\frac{d\omega}{dt}$$

and on the element $d\Sigma_2$ of the outer shell is applied a force:

$$dF_2' = \sigma_2 d\Sigma_2 E_a$$

[9] As shown at the end of the chapter, the field B_a is a consequence of the motion of the electrostatic field between the two rotating shells when seen in the laboratory frame.

From the fourth Maxwell equation, the equation for B_a is:

$$\nabla \times \mathbf{B}_a = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial E_i}{\partial t}\hat{\boldsymbol{\varphi}} \quad \nabla \times \mathbf{B}_a = \mu_0 \mathbf{J} - \mu_0 \epsilon_0 \frac{r}{2}\frac{\partial^2 B}{\partial t^2}\hat{\boldsymbol{\varphi}}$$

that after integration gives:

$$\mathbf{B}_a(r) = \left[-\mu_0 \frac{\lambda\omega}{2\pi} + \frac{\mu_0\epsilon_0}{4}\left[r^2 - R_1^2\right]\frac{\partial^2 B}{\partial t^2}\right]\hat{\mathbf{z}}\,.$$

The last term is null if $dB/dt = const$ but can be neglected if, for usual charges and dimensions, the frequency of rotation is $\ll 10^{11}/B(Tesla)$.

and the net axial torque, per unit length of the outer surface, is:

$$M_2' = -\frac{\lambda}{2}\left(R_2^2 - R_1^2\right)\mu_0 \frac{\lambda}{2\pi}\frac{d\omega}{dt}\,.$$

Thus the equation of the angular momentum $L = I\omega$, per unit length of the rigid system, is:

$$\frac{dL}{dt} = M_2 + M_1 + M_2'$$

$$I\frac{d\omega}{dt} = \frac{\lambda}{2}\left(R_2^2 - R_1^2\right)\frac{dB}{dt} - \frac{\lambda}{2}\left(R_2^2 - R_1^2\right)\mu_0\frac{\lambda}{2\pi}\frac{d\omega}{dt}$$

that can be written as:

$$I\frac{d\omega}{dt} + \frac{\lambda}{2}\left(R_2^2 - R_1^2\right)\mu_0\frac{\lambda}{2\pi}\frac{d\omega}{dt} = \frac{\lambda}{2}\left(R_2^2 - R_1^2\right)\frac{dB}{dt}$$

and integrating over the time that the external magnetic field B_{in} takes to become null, we get the relation:

$$I\omega_{fin} + \frac{\lambda}{2}\left(R_2^2 - R_1^2\right)\mu_0\frac{\lambda\,\omega_{fin}}{2\pi} = -\frac{\lambda}{2}\left[R_2^2 - R_1^2\right]B_{in} \tag{10.3}$$

$$I\omega_{fin} - \frac{\lambda}{2}\left(R_2^2 - R_1^2\right)B_a^{fin} = -\frac{\lambda}{2}\left[R_2^2 - R_1^2\right]B_{in} \tag{10.4}$$

where B_a^{fin} is the final self-induced magnetic field.

The relation (10.4) is the expression of the conservation of the angular momentum when the initial and final angular momenta of the electromagnetic field are taken into account.

From the relation (10.3) the final angular velocity ω_{fin} is:

$$\omega_{fin} = -\frac{\frac{\lambda}{2}\left[R_2^2 - R_1^2\right]B_{in}}{I + \frac{\lambda}{2}\left(R_2^2 - R_1^2\right)\mu_0\frac{\lambda}{2\pi}}$$

and, as expected, it is negative and depends on the moment of inertia of the rigid system.

To understand the meaning of the Eq. (10.4) we have to consider the electric and magnetic fields present while the system begins to rotate.

Inside the outer cylindrical shell (at $r < R_2$) there are:

- the magnetic field of the external solenoid $\mathbf{B} = B\hat{\mathbf{z}}$,
- the electric field induced[10] by the change of B:

[10]This field can be found from the third Maxwell equation. In cylindrical coordinates the components are:

$$\mathbf{E}_{in} = -\frac{r}{2}\frac{dB}{dt}\,\hat{\boldsymbol{\varphi}}\,.$$

Between the two cylindrical shells ($R_1 < r < R_2$) there are:

- the electrostatic field from the two surface charges:

$$\mathbf{E}_s = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\,\hat{\mathbf{r}}\,,$$

- the self induced magnetic field:

$$\mathbf{B}_a = B_a\hat{z} = -\mu_0\frac{\lambda\omega}{2\pi}\,\hat{\mathbf{z}}\,,$$

- the self induced electric field[11]:

$$\mathbf{E}_a = \frac{1}{2}\left(r - \frac{R_1^2}{r}\right)\mu_0\frac{\lambda}{2\pi}\frac{d\omega}{dt}\,\hat{\boldsymbol{\varphi}}\,.$$

Thus the Poynting's vector is:

$$\mathbf{S} = \mathbf{E}\times\frac{\mathbf{B}}{\mu_0} = (\mathbf{E}_s + \mathbf{E}_{in} + \mathbf{E}_a)\times\frac{(\mathbf{B}+\mathbf{B}_a)}{\mu_0}$$

with six components:

(Footnote 10 continued)

$$\frac{1}{r}\frac{\partial E_z}{\partial\varphi} - \frac{\partial E_\varphi}{\partial z} = 0 \qquad \frac{\partial E_r}{\partial z} - \frac{\partial E_z}{\partial r} = 0 \qquad \frac{1}{r}\left[\frac{\partial}{\partial r}(rE_\varphi) - \frac{\partial E_r}{\partial\varphi}\right] = -\frac{\partial B_z}{\partial t}\,.$$

Since E_r, E_φ e E_z cannot depend on z and on φ, from the first two equations we get $E_z = const = 0$, and from the third equation we find:

$$\frac{1}{r}\frac{\partial}{\partial r}(rE_\varphi) = -\frac{\partial B_z}{\partial t} \qquad \text{that integrated gives:} \qquad E_\varphi = -\frac{r}{2}\frac{dB_z}{dt} + \frac{C}{r}$$

where the constant C has to be null since $E_\varphi(r=0) = 0$, then:

$$E_\varphi = -\frac{r}{2}\frac{dB_z}{dt}\,.$$

[11] To find this self induced electric field E_a we have to solve the same equation seen in the previous note replacing B with the field B_a and since this is limited into the space between the two shells we have to determine the constant C imposing E_a is null on the shell at $r = R_1$. So it follows:

$$E_a = \frac{1}{2}\left(r - \frac{R_1^2}{r}\right)\mu_0\frac{\lambda}{2\pi}\frac{d\omega}{dt}\,.$$

$$\mathbf{S} = \mathbf{E}_s \times \frac{\mathbf{B}}{\mu_0} + \mathbf{E}_s \times \frac{\mathbf{B}_a}{\mu_0} + \mathbf{E}_{in} \times \frac{\mathbf{B}}{\mu_0} + \mathbf{E}_a \times \frac{\mathbf{B}_a}{\mu_0} + \mathbf{E}_{in} \times \frac{\mathbf{B}_a}{\mu_0} + \mathbf{E}_a \times \frac{\mathbf{B}}{\mu_0}$$

$$\mathbf{S} = S_{\varphi 1}\hat{\boldsymbol{\varphi}} + S_{\varphi 2}\hat{\boldsymbol{\varphi}} + S_{r1}\hat{\mathbf{r}} + S_{r2}\hat{\mathbf{r}} + S_{r3}\hat{\mathbf{r}} + S_{r4}\hat{\mathbf{r}}$$

the first two with direction $\hat{\boldsymbol{\varphi}}$ and the other four with radial direction $\hat{\mathbf{r}}$.

We have to examine now the variation of the energy of the electromagnetic field of the system by analysing the energy flux through the cylindrical shells at R_1 and R_2.

The first radial component $S_{1r}\hat{\mathbf{r}}$:

$$S_{1r}\hat{\mathbf{r}} = \mathbf{E}_{in} \times \frac{\mathbf{B}}{\mu_0} = -\frac{r}{2}\frac{dB}{dt}\frac{B}{\mu_0}\,\hat{\mathbf{r}}$$

corresponds to the energy flux, with radial direction, that escapes from the shell of radius R_2 while B vanishes. The outgoing energy per unit time over a length l of a cylindrical surface of radius r is:

$$-\frac{dU_1}{dt} = S_{1r}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\, 2\pi rl = \left(-\frac{r}{2}\frac{dB}{dt}\frac{B}{\mu_0}\right) 2\pi rl = -\frac{1}{\mu_0}\frac{r}{4}\frac{dB^2}{dt}\, 2\pi rl$$

and the total energy out from the surface of radius R_2 when B is null, is:

$$-\Delta U_1 = \frac{1}{4}\frac{B_{in}^2}{\mu_0}\, 2\pi R_2^2 l = (\pi R_2^2 l)\,\frac{1}{2}\frac{B_{in}^2}{\mu_0} \tag{10.5}$$

that is just the magnetic energy initially inside the cylinder of length l and radius R_2.

The second radial component $S_{2r}\hat{\mathbf{r}}$:

$$S_{2r}\hat{\mathbf{r}} = \mathbf{E}_a \times \frac{\mathbf{B}_a}{\mu_0} = -\frac{1}{2}\left(r - \frac{R_1^2}{r}\right)\mu_0 \frac{\lambda\omega}{2\pi}\frac{\lambda}{2\pi}\frac{d\omega}{dt}\,\hat{\mathbf{r}}$$

corresponds to a flow of energy with radial direction entering the solenoid. The energy entering per unit time over a length l of the cylindrical shell of radius R_2 is:

$$-\frac{dU_2}{dt} = S_{2r}\hat{\mathbf{r}} \cdot \hat{\mathbf{r}}\, 2\pi R_2 l = -\frac{1}{2}\left(\frac{R_2^2 - R_1^2}{R_2}\right)\mu_0 \frac{\lambda\omega}{2\pi}\frac{\lambda}{2\pi}\frac{d\omega}{dt}\, 2\pi R_2 l$$

$$\frac{dU_2}{dt} = \frac{1}{2}\left(R_2^2 - R_1^2\right)\frac{1}{2}\mu_0 \frac{\lambda^2}{2\pi}\frac{d\omega^2}{dt}\, l$$

and the total energy flowed through the surface of radius R_2 is:

$$\Delta U_2 = \pi\left(R_2^2 - R_1^2\right) l\,\frac{1}{2}\mu_0\left(\frac{\lambda\omega}{2\pi}\right)^2 = \pi\left(R_2^2 - R_1^2\right) l\,\frac{1}{2}\frac{B_a^2}{\mu_0} \tag{10.6}$$

that is the magnetic energy located at the end between the cylindrical shells for a length l. This energy is due to the magnetic field produced by the currents associated to the rotation of the two charged shells.

Then there is the flow of the component $S_{r3}\hat{\mathbf{r}}$:

$$S_{r3}\hat{\mathbf{r}} = \frac{r}{2}\frac{1}{\mu_0}\frac{dB}{dt}\left(\frac{\lambda\omega}{2\pi}\right)\hat{\mathbf{r}}$$

that has to be integrated over the cylindrical surfaces of radii R_1 and R_2:

$$-\frac{dU_3}{dt} = \frac{2\pi l}{\mu_0}\left(\frac{R_2^2 - R_1^2}{2}\right)\frac{dB}{dt}\left(\frac{\lambda\omega}{2\pi}\right)$$

and the flow of the component $S_{r4}\hat{\mathbf{r}}$:

$$S_{r4}\hat{\mathbf{r}} = \frac{1}{2}\left(r - \frac{R_1^2}{r}\right)B\frac{d}{dt}\left(\frac{\lambda\omega}{2\pi}\right)\hat{\mathbf{r}}$$

to be integrated only over the surface of radius R_2 (S_{r4} is null at $r = R_1$):

$$-\frac{dU_4}{dt} = 2\pi l\left(\frac{R_2^2 - R_1^2}{2}\right)\frac{d}{dt}\left(\frac{\lambda\omega}{2\pi}\right)\frac{B}{\mu_0}.$$

The sum of these two terms is:

$$\frac{dU_3}{dt} + \frac{dU_4}{dt} = -\frac{2\pi l}{\mu_0}\left(\frac{R_2^2 - R_1^2}{2}\right)\frac{d}{dt}\left(\frac{\lambda\omega}{2\pi}B\right) = \pi l\left(R_2^2 - R_1^2\right)\cdot\frac{1}{2}\frac{d}{dt}(2B_a B)\frac{1}{\mu_0}$$

and is given by the superposition of the fields B and B_a in the magnetic energy density while the fields are changing. This term integrated over the time from the initial state, when $\omega = 0$ and $B_a = 0$, up to the final state, when $B = 0$, gives a net null contribution to the magnetic energy inside the two cylindrical shells.

Finally we have to consider the components $S_{\varphi 1}\hat{\boldsymbol{\varphi}}$ and $S_{\varphi 2}\hat{\boldsymbol{\varphi}}$ of the Poynting's vector:

$$S_{\varphi 1}\hat{\boldsymbol{\varphi}} = \mathbf{E}_s \times \frac{\mathbf{B}}{\mu_0} = -\frac{\lambda}{2\pi}\frac{1}{\epsilon_0\mu_0}\frac{1}{r}B\,\hat{\boldsymbol{\varphi}}$$

$$S_{\varphi 2}\hat{\boldsymbol{\varphi}} = \mathbf{E}_s \times \frac{\mathbf{B}_a}{\mu_0} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\frac{\lambda\omega}{2\pi}\,\hat{\boldsymbol{\varphi}}$$

both with circular field lines around the axis of the system. From (9.16) a flow of momentum:

$$\mathbf{p} = \frac{S_\varphi}{c}\hat{\boldsymbol{\varphi}}$$

is associated to each of these components, with the same field lines of $S_\varphi \hat{\boldsymbol{\varphi}}$ and momentum density:

$$\mathbf{g} = \frac{S_\varphi}{c^2} \hat{\boldsymbol{\varphi}} \,.$$

The momentum of the electromagnetic field in a volume $d\tau = r\,dr\,d\varphi\,dz$ is:

$$\mathbf{dp} = \frac{S_\varphi}{c^2}\, \hat{\boldsymbol{\varphi}}\, r\,dr\,d\varphi\,dz \,.$$

and the associated angular momentum is:

$$\mathbf{dL} = \mathbf{r} \times \mathbf{g}\, d\tau \,.$$

In our case, due to the symmetry, there is a non null axial angular momentum that can be found by integration over the volume of height l, from R_1 to R_2 and for $0 < \varphi < 2\pi$.

For the component $S_{\varphi 1} \hat{\boldsymbol{\varphi}}$ we get an angular momentum per unit length of the cylindrical system:

$$L_1 = \frac{1}{l} \int_{R_1}^{R_2} \int_0^{2\pi} \int_0^{l} \frac{S_{\varphi 1}}{c^2} r dr d\varphi dz = \frac{1}{l} \int_{R_1}^{R_2} \int_0^{2\pi} \int_0^{l} \frac{1}{c^2} \left(-\frac{\lambda}{2\pi \epsilon_0} \frac{B_{in}}{\mu_0} \frac{1}{r} \right) r dr d\varphi dz$$

that, reminding $c^2 = 1/\epsilon_0 \mu_0$, results exactly:

$$L_1 = -\frac{\lambda}{2} \left(R_2^2 - R_1^2 \right) B_i \,. \tag{10.7}$$

That is the initial angular momentum of the electromagnetic field between the cylindrical shells per unit length, and is at the second member of the Eq. (10.3). Similarly for the component $S_{\varphi 2} \hat{\boldsymbol{\varphi}}$ we get:

$$L_2 = \frac{\lambda}{2} \left(R_2^2 - R_1^2 \right) \mu_0 \frac{\lambda\, \omega_{fin}}{2\pi} = -\frac{\lambda}{2} \left(R_2^2 - R_1^2 \right) B_a^{fin} \tag{10.8}$$

that is the final angular momentum of the electromagnetic field between the cylindrical shells per unit length, in the first member of the Eq. (10.4).

Comments

- From the equation for the motion of the rotating system and the Faraday-Neumann law we have found the relation (10.4). This is the expression of the conservation of the total angular momentum of the system: when the external magnetic field B_{in}

decreases and becomes null, the initial angular momentum (10.7) of the electromagnetic field (magnetic field B_{in} and electrostatic field E_s), is transferred to the rigid system of moment of inertia I and to the final angular momentum (10.8) of the electromagnetic field (magnetic field B_a^{fin} and electrostatic field E_s) that is present between the two shells as a consequence of the rotation. At the same time the initial magnetic energy that was inside the cylindrical shell of radius R_2 escapes from the outer shell (10.5) and between the two charged shells enters the magnetic energy (10.6) associated to the self induced magnetic field B_a^{fin} that appears between the two shells.

- The final angular momentum (10.8) of the electromagnetic field is due to the rotation of the electrostatic field between the shells. The initial angular momentum (10.7) cannot be associated to the motion of the fields because the shells are at rest. As for the angular momentum in the system with a magnetic dipole and a point charge, the initial angular momentum of the present system is given to the electromagnetic field when assembling the system.
- Every point between the two shells can be considered as the origin of a frame S at rest in the laboratory, with the z axis parallel to the axis of the system, the y axis in the radial direction and the x axis to form with the other two a right handed frame. Then we consider the frame S' that at time $t = t' = 0$ has the origin and the axis coincident with those of S, but moves with the rotating system. At time $t = t' = 0$ the axis x' is in motion relative to x axis with velocity $v_x = -\omega r$. In the system S' the only non null component of the electric field is E'_y:

$$E'_y = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\,.$$

At the initial time the system S is in motion relative to S' with velocity $v'_x = \omega r$ and transforming the field E'_y from S' to S, from the relations (7.11) we find:

$$E_x = 0 \quad E_y = \gamma E'_y \quad E_z = 0$$

$$B_x = 0 \quad B_y = 0 \quad B_z = \gamma\left(\frac{-v'_x}{c^2}\,E'_y\right)$$

with

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta = \frac{\omega r}{c}\,.$$

By substituting the expressions of v'_x and of E'_y in B_z we find:

$$B_z = \gamma\left(-\frac{\mu_0}{2\pi}\omega\lambda\right)\,.$$

In the limit $\omega r \ll c$ this field becomes B_a and so it is evident that this field originates from the motion of rotation of the electrostatic field seen in the laboratory frame.

The Poynting's vector:

$$S_{\varphi 2}\hat{\boldsymbol{\varphi}} = \frac{\mathbf{E}_s \times \mathbf{B}_a}{\mu_0}$$

is also a consequence of the rotation of the electrostatic field seen in the laboratory. We can associate to this vector a momentum density in the direction $\hat{\boldsymbol{\varphi}}$:

$$g = \frac{S}{c^2} = \frac{1}{c^2}\frac{E_y B_z}{\mu_0} = -\frac{1}{\mu_0}\gamma^2 \frac{E_y'^2}{c^3}\beta = -\epsilon_0 \gamma^2 \frac{E_y'^2}{c}\beta$$

and an energy density[12]:

$$u = \frac{1}{2}\epsilon_0 E_y^2 + \frac{1}{2}\frac{1}{\mu_0}B_z^2 = \frac{1}{2}\epsilon_0 E_y'^2 \gamma^2 + \frac{1}{2}\frac{1}{\mu_0}\gamma^2 \frac{E_y'^2}{c^2}\beta^2 = \frac{1}{2}\epsilon_0 E_y'^2 \gamma^2 \left(1+\beta^2\right).$$

Problem

10.1 An insulating cylindrical shell, of radius r_c and length l, has its axis on the z axis where lies also an infinite long wire with linear charge density $-\lambda$ (see Fig. 10.4). The cylindrical shell has moment of inertia I and a uniform surface charge density $\sigma = \lambda/2\pi r_c$. The system, initially al rest, is embedded in an external magnetic field $B_{ext}\hat{\mathbf{z}}$ that at the time $t = 0$ begins to decrease slowly to zero.

Determine the final angular velocity of the cylindrical shell:

(a) from the equation for the motion of the shell,
(b) from the conservation law of the angular momentum of the system.

---

[12] In the frame S' the electrostatic energy density is:

$$u' = \frac{1}{2}\epsilon_0 E_y'^2$$

and from the previous relations it is easy to verify the relation:

$$u'^2 = u^2 - g^2c^2$$

similarly to the case of a particle of mass m at rest in S':

$$\mathcal{E}'^2 = (mc^2)^2 = \mathcal{E}^2 - p^2c^2\,.$$

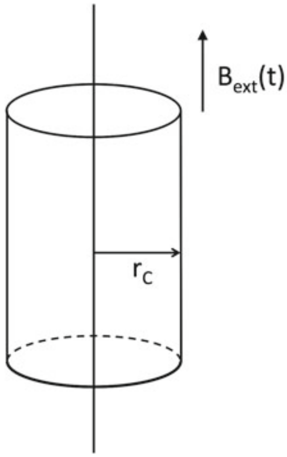


Fig. 10.4 The charged wire with the opposite charged coaxial cylindrical shell in the vanishing magnetic field

This problem has been extensively considered by many authors. See the article by J. Belcher and Kirk T. MacDonald: *Feynman Cylinder Paradox*,
http://www.physics.princeton.edu/~mcdonald/examples/feynman_cylinder.pdf
where relativistic corrections are considered. Many references on the Feynman paradox can be found in this paper.

Solution

10.1 **(a)** *From the equation of motion and the Faraday's law.*

While the magnetic field decreases, there is an induced electric field $\mathbf{E}_i$ tangent to a circle of radius r_c concentric with the shell, that can be found by the Faraday's law:

$$f = -\frac{d\Phi(B)}{dt} \qquad 2\pi r_c E_i = -\pi r_c^2 \frac{dB}{dt} \qquad \mathbf{E}_i = -\frac{r_c}{2}\frac{dB}{dt}\hat{\boldsymbol{\varphi}}\,.$$

The force applied to an element $d\Sigma$ of the lateral surface of the cylinder is:

$$\mathbf{dF} = \sigma \mathbf{E}_i \, d\Sigma$$

and its torque relative to the z axis is:

$$\mathbf{dM} = \mathbf{r}_c \times \mathbf{dF} = \mathbf{r}_c \times \sigma \mathbf{E}_i \, d\Sigma = -\mathbf{r}_c \times \ \sigma \frac{r_c}{2}\frac{dB}{dt}\hat{\boldsymbol{\varphi}}\, d\Sigma\,.$$

The total torque is:

$$\mathbf{M} = -\frac{r_c^2}{2}\frac{dB}{dt}\,(2\pi r_c l \sigma)\ \hat{\mathbf{z}}$$

and the equation of motion is:

$$I\frac{d\omega}{dt} = -\frac{r_c^2}{2}\frac{dB}{dt}\,(2\pi r_c l\sigma) = -\frac{r_c^2}{2}\frac{dB}{dt}\,(\lambda l)$$

that integrated gives:

$$I'\omega = -\frac{\lambda}{2}r^2\left(B_{fin} - B_{ext}\right) \qquad B_{fin} = \mu_0\frac{\lambda\omega}{2\pi} \qquad I' = \frac{I}{l}$$

where B_{fin} is the magnetic field generated by the rotation of the charged shell (see point b).

The final angular velocity is:

$$\omega = \frac{\lambda r_c^2 B_{ext}}{2}\frac{1}{I'\left[1 + \frac{\mu_0\lambda^2 r_c^2}{4\pi I'}\right]}\,.$$

(b) *From the conservation of the angular momentum.*

With the system at rest, the electric field and the Poynting's vector in the space at $r < r_c$ are:

$$\mathbf{E} = -\frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\hat{\mathbf{r}} \qquad \mathbf{S}_{in} = \frac{\lambda}{2\pi\epsilon_0}\frac{1}{r}\frac{B_{ext}}{\mu_0}\hat{\boldsymbol{\varphi}}\,.$$

The momentum density associated to $\mathbf{S}_{in}$ from (9.15) is:

$$\mathbf{g}_{in} = \frac{\mathbf{S}_{in}}{c^2} = \frac{\lambda}{2\pi}\frac{1}{r}B_{ext}\hat{\boldsymbol{\varphi}}$$

and the initial angular momentum of the electromagnetic field per unit length is:

$$\frac{\mathbf{L}_{in}}{l} = \int_0^{r_c}\mathbf{r}\times\mathbf{g}_{in}\,2\pi r dr = \frac{\lambda B_{ext}}{2}r_c^2\,\hat{\mathbf{z}}\,.$$

At the end the shell is rotating with an angular velocity ω and the surface charge becomes a surface current:

$$i = \frac{\lambda l}{T} = \frac{\lambda l}{2\pi}\omega$$

and for $r < r_c$ there is a magnetic field:

$$\mathbf{B}_{fin} = \mu_0\frac{\lambda\omega}{2\pi}\hat{\mathbf{z}}\,.$$

The final momentum density of the electromagnetic field is:

$$\mathbf{g}_{fin} = \frac{\mathbf{S}_{fin}}{c^2} = \frac{\lambda}{2\pi}\frac{1}{r}B_{fin}\hat{\boldsymbol{\varphi}}$$

and the angular momentum per unit length is:

$$\frac{\mathbf{L}_{fin}}{l} = \int_0^{r_c} \mathbf{r} \times \mathbf{g}_{fin}\, 2\pi\, r dr = \mu_0 \frac{\lambda^2 r_c^2}{4\pi} \omega\, \hat{\mathbf{z}}\,.$$

The angular momentum conservation gives:

$$I'\omega + \mu_0 \frac{\lambda^2 r_c^2}{4\pi} \omega = \frac{\lambda B_{ext}}{2} r_c^2$$

and thus the final angular velocity is:

$$\omega = \frac{\lambda r_c^2 B_{ext}}{2} \frac{1}{I' \left(1 + \frac{\mu_0 \lambda^2 r_c^2}{4\pi I'}\right)}\,.$$

Chapter 11
Test of the Coulomb's Law and Limits on the Mass of the Photon

In 1785 Charles Augustin de Coulomb measured the forces between electric charges and deduced[1] the law that governs them. For two point charges the force is proportional to the product of the charges and inversely proportional to the square of their distance apart. The dependence on the inverse square of the distance seems almost obvious to the reader of an introductory textbook on Electricity because usually he has already encountered this dependence in the gravitational force. This was not the case for the physicists who contributed to the formulation of the Electrostatics. Indeed they performed many tests on its validity and gave limits on a possible deviation ϵ from the second power of the distance. Similar tests were performed until the half of '900. More recently physicists started to interpret the tests of the Coulomb's law in terms of limits on the mass of the photon.

A short review of the subjects of this chapter can be found in the Jackson's book.[2] For some topics we refer to the original papers. A.S. Goldhaber and M.M. Nieto have written a paper[3] on the limits for the masses of the photon and the graviton. This is an update of a previous paper of these authors[4] that is useful to

---

[1]The dependence on the inverse square of the distance was already found by John Robinson in 1769. In 1767 Joseph Priestley wrote in a book that he had repeated an experiment, already done by Benjamin Franklin, showing that there is no charge inside a charged closed metallic box. From this result he deduced that the repulsion force between same sign charges has to be proportional to the inverse square of the distance. The results by Robinson and Priestley remained unknown to the scientific community. See note at page 283 of the paper by A.S. Goldhaber and M.M. Nieto, (1971), and Segrè's book mentioned in the following of this chapter. See also D. Halliday and R. Resnick, *Physics for students of Science and Engineering*, Part II, 28.4-5, 2nd Ed., J. Wiley & Sons, 1960.

[2]J.D. Jackson, *Classical Electrodynamics*, cited, Sects. 1.2, 12.8.

[3]A.S. Goldhaber and M.M. Nieto, *Photon and Graviton Mass Limits*, Reviews of Modern Physics, **82**, 939–979, (2010).

[4]A.S. Goldhaber and M.M. Nieto, *Terrestrial and Extraterrestrial Limits on The Photon Mass*, Reviews of Modern Physics, **43**, 277–296, (1971).

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9_11

consider. For the historical part on the birth of Electrostatics we suggest the E. Segrè's book.[5]

11.1 Gauss's Law

To introduce the first tests of the Coulomb's law we recall and comment the Gauss's law.

In order to demonstrate the Gauss's law one has to start considering a point charge inside a closed surface S. The flux of the field $\mathbf{E} = E\,\hat{\mathbf{r}}$ from the charge through an element dS of the surface is $d\Phi = \mathbf{E}\cdot\hat{\mathbf{n}}\,dS$ with $\hat{\mathbf{n}}$ the external versor normal to dS. Since $dS_n = \hat{\mathbf{r}}\cdot\hat{\mathbf{n}}\,dS = r^2 d\Omega$, we can write:

$$d\Phi = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}dS_n = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}r^2 d\Omega\,. \tag{11.1}$$

After the term r^2 at the numerator cancels out with the same term at denominator and integrating over the solid angle, the total flux through the whole closed surface, is:

$$\Phi(\mathbf{E}) = \frac{Q}{\epsilon_0}\,.$$

But the simplification done in (11.1) is not trivial at all because while the term at the numerator comes from a correct mathematical relation, the term at the denominator is a consequence of the Coulomb's law. If this would depend not on r^{-2} but on $r^{-(2+\epsilon)}$ in the result for Φ we should find a dependence on r and the Gauss's theorem would be wrong. Also the geometrical principle of conservation of the number of field lines coming out from a charge would not be valid. Moreover the first Maxwell equation would lose its validity and at the same time all the consequences of the Gauss's law would be incorrect as for instance the well known property that in a conductor the charges are distributed on the external surface.

11.2 First Tests of the Coulomb's Law

The law for the force between two charges was found by Coulomb using a torsion balance. However, in his treatise[6] Maxwell observes that the measurement errors with this apparatus might be not negligible. A more accurate test of the Coulomb's

[5] E.Segrè, *From falling bodies to radio waves. Classical physicists and their discoveries*. W.H. Freeman & Company, New York, 1984.

[6] J.C. Maxwell, *A Treatise on Electricity and Magnetism*, 1873, Cap. II, Sect. 74a, b.

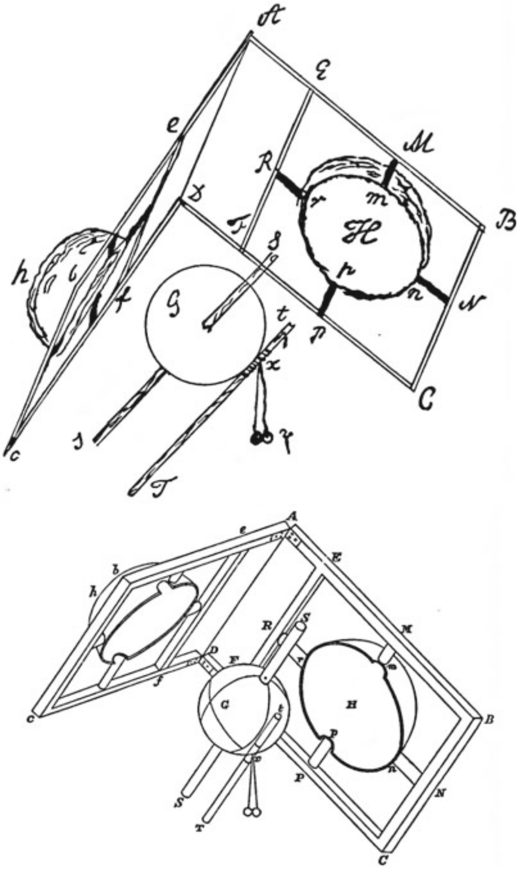


Fig. 11.1 Cavendish's experiment to test the inverse square of the distance dependence for the force between point charges. *Top* the drawing done by Cavendish, *bottom* the drawing redone by Maxwell (Figures reproduced from H. Cavendish, edited by J.C. Maxwell, cited, by permission of the Cambridge University Press.)

law can be performed with the following experiment used by Cavendish to prove the law of inverse square of the distance. A metallic globe was surrounded by two hollow metallic hemispheres that when closed formed a spherical surface concentric with the globe but isolated from that (see Fig. 11.1). Initially the globe was electrically connected to the hemispheres by a conductive wire. After having deposited an electric charge on the hemispheres, the wire was removed and the hemispheres were opened and taken away. At that point the residual charge on the globe was measured with an electrometer but was found smaller than the minimal charge detectable by that instrument. The minimal measurable charge was then measured by depositing on the globe known fractions of electrical charge. From the results of this experiment, Cavendish was able to prove that the dependence of the electrostatic force on the inverse square of the distance was different from a power 2 by a quantity $|\epsilon| \leq 0.02$. This result of course is also expected from the Gauss's law, not stated at that time, for the positioning of an excess charge placed on an isolated conductor.

The experiment was performed by Cavendish in 1773, that is well before Coulomb presented his law, however Cavendish was[7] "an extraordinary experimenter and a major physicist but his eccentricities were equally extraordinary, ... His relations with other scientists were reduced to a minimum, ...", so he did not publish his result that became known only about a hundred years later when Maxwell, at that time Cavendish Professor at Cambridge, undertook the publication[8] of the Cavendish's electrical work. Maxwell repeated the Cavendish's experiment at the Cavendish Laboratory in Cambridge with some differences and more accurate instruments and found for the deviation from the square power $|\epsilon| \leq 5 \times 10^{-5}$.

11.3 Proca Equations

In 1936 Alexandre Proca, under the influence of de Broglie, added a mass term to the Lagrangian density[9] of the electromagnetic field:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \mu_0 J_\mu A^\mu + \frac{1}{2}\, \mu_\gamma^2 A_\mu A^\mu \tag{11.2}$$

where

$$\mu_\gamma = \frac{m_\gamma c}{\hbar}$$

is the inverse of the Compton length of the photon with a non null mass m_γ, while

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \tag{11.3}$$

are the fields **E** and **B** in terms of the potentials:

$$\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V \qquad \mathbf{B} = \nabla \times \mathbf{A} \tag{11.4}$$

and are equivalent to the second and the third Maxwell equations.

The first term in the Lagrangian density (11.2) is the energy density of the electromagnetic field, the second is the term of interaction of the field with the charges and the currents while the last is the mass term of the photon.

From the Euler-Lagrange equations for the fields with the density given in (11.2) the first and the fourth Proca equations can be derived as shown in the Appendix of this Chapter. Thus the Maxwell equation modified by Proca after the introduction of the mass term for the photon are:

---

[7] See E. Segrè, cited, p. 115.

[8] H. Cavendish, edited by J.C. Maxwell, *The Electrical Researches of the Honourable Henry Cavendish, 1771–1781*, Cambridge University Press, 1879.

[9] The Lagrangian density is written in SI units.

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} - \mu_\gamma^2\, V \tag{11.5}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{11.6}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{11.7}$$

$$\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}) - \mu_\gamma^2 \mathbf{A}\,. \tag{11.8}$$

These equations are different from the Maxwell equations because of the presence of the pseudo-charge $-\mu_\gamma^2 V \epsilon_0$ in the first equation and the pseudo-current $-\mu_\gamma^2 \mathbf{A}/\mu_0$ in the fourth one. These terms vanish if $\mu_\gamma = 0$, a null mass for the photon, and the Proca equations reduce to Maxwell equations.

The fields given by (11.3) and (11.4) are invariant under a gauge transformation:

$$A^\mu \rightarrow A'^\mu = A^\mu + \partial^\mu \Lambda \qquad \mathbf{A} \rightarrow \mathbf{A}' + \nabla \Lambda \qquad V \rightarrow V' = V - \frac{\partial \Lambda}{\partial t}$$

that leaves unchanged the Maxwell equations but not the Proca equations (11.5) and (11.8).

The charge conservation is expressed in local form by the continuity equation:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = 0$$

that is included in the non homogeneous Maxwell equations. For a massive photon from (11.5) and (11.8), we get the relation:

$$\nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t} = \frac{\mu_\gamma^2}{\mu_0}\left(\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t}\right) \tag{11.9}$$

that is the conservation equation for the sum of the charge and the pseudo-charge:

$$\nabla \cdot \left(\mathbf{J} - \mu_\gamma^2 \frac{\mathbf{A}}{\mu_0}\right) + \frac{\partial}{\partial t}\left(\rho - \mu_\gamma^2 V \epsilon_0\right) = 0\,.$$

In order for the charge conservation to be separately valid, from (11.9) it follows that also the pseudo-charge has to be conserved and for the potentials the following relation has to be valid:

$$\nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial V}{\partial t} = 0\,.$$

This relation is just the condition for the Lorentz gauge and it implies that in the gauge transformations the scalar function Λ must satisfy the equation:

$$\nabla^2 \Lambda - \frac{1}{c^2}\frac{\partial^2 \Lambda}{\partial t^2} = 0\,.$$

Finally, for a non null mass of the photon, the equations for the potentials are:

$$\Box A^\mu - \mu_\gamma^2 A^\mu = -J^\mu$$

$$\rightarrow \ \Box V - \mu_\gamma^2 V = -\frac{\rho}{\epsilon_0} \qquad \Box \mathbf{A} - \mu_\gamma^2 \mathbf{A} = -\mu_0 \mathbf{J}\,.$$

For a point charge q, in the stationary case, from the last equation for the potential V or from the first Proca equation (11.5) we get the solution:

$$V(r) = \frac{q}{4\pi\epsilon_0}\frac{e^{-\mu_\gamma r}}{r}\,.$$

This 'Yukava' potential represents for the electrostatic force a new possible deviation from the inverse square of the distance. This point was clear only later, when Yukawa wrote the equation for a scalar particle, the meson π discovered afterwards, that with its mass justifies the short range of the strong nuclear interaction.

11.4 The Williams, Faller and Hill Experiment

In 1936, in a test similar to the Cavendish's experiment, Plinton and Lawton[10] found a limit $|\epsilon| \le 1 \times 10^{-9}$ for the deviation from the inverse square of the distance. Afterwards the last experiment of this type was performed by Williams, Faller and Hill published[11] in 1971. Their experimental apparatus, sketched in Fig. 11.2, consisted of five concentric icosahedrons with an external diameter of about 1.5 m. A 4 MHz radio frequency, 10 kV peak-to-peak voltage was applied to the resonant circuit formed by the outer shells (4 and 5 in figure) and a high-Q water cooled coil. After the intermediate shell 3, the voltage difference across the inductor between the inner shells 1 and 2, was measured by a battery-powered lock-in amplifier located inside the innermost shell and then isolated from the outer of the apparatus.

Considering a spherical surface of radius r between conductors 1 and 2 and an approximated[12] potential $V_0 e^{i\omega t}$ in the volume inside shell 4, from Eq. (11.5) and Gauss's law, the electric field is:

[10] S.J. Plinton and W.E. Lawton used two conductive concentric hollow spheres of radii 2.5 and 2.0 ft, both initially at ground. After the outer sphere was connected to a 3000 V generator with a 130 cycles/minute frequency, and the difference of voltage between the two spheres was measured by detecting the current between them with a galvanometer.

[11] E.R. Williams, J.E. Faller and H.A. Hill, Physical Review Letters, **26**, 721 (1971).

[12] A detailed solution is given in the paper.

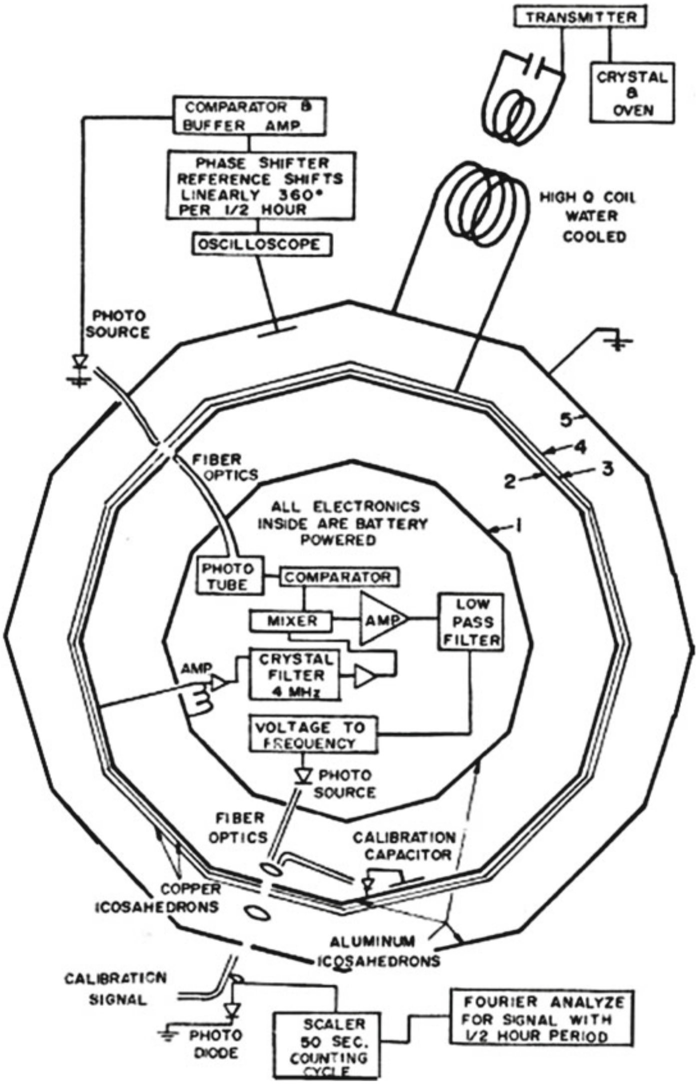


Fig. 11.2 The Williams, Faller and Hill experiment setup (Reprinted figure with permission from E.R. Williams, J.E. Faller and H.A. Hill, cited. Copyright 1971 by American Physical Society. http://dx.doi.org/10.1103/PhysRevLett.26.721)

$$\mathbf{E}(r) = \frac{1}{4\pi\epsilon_0}(\frac{q}{r^2} - \frac{1}{3}\mu_\gamma^2 V_0 e^{i\omega t} r)\hat{\mathbf{r}}$$

where q is the total charge on the inner shell 1. Then if C is the capacitance between conductors 1 and 2, the voltage across the inductor L connecting these two shells is:

$$\Delta V = \frac{q}{C} - \frac{1}{6}\mu_\gamma^2 V_0 e^{i\omega t}(R_2^2 - R_1^2)$$

that is applied to the LC circuit formed by the two shells and the inductor. The proportionality to V_0 for the term with the photon mass suggests to apply an high V_0 voltage.

The measurements put a limit $|\epsilon| \leq (2.7 \pm 3.1) \times 10^{-16}$ for the deviation from the inverse square of the distance, while if interpreted in terms of a non null photon mass the limit was $m_\gamma \leq 1.6 \cdot 10^{-50}$ kg.

11.5 Limits from Measurements of the Magnetic Field of the Earth and of Jupiter

E. Schrödinger proposed a method to get a limit on the photon mass from measurements of the Earth's static magnetic field.

The magnetic field $\mathbf{B}(\mathbf{r})$ generated by a magnetic dipole of moment $\mathbf{D} = D\hat{\mathbf{z}}$, at the position $\mathbf{r}$, from the fourth Proca equation (11.8) is:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{De^{-\mu_\gamma r}}{r^3}\left[\left(1 + \mu_\gamma r + \frac{1}{3}\mu_\gamma^2 r^2\right)\left(3\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}} - \hat{\mathbf{z}}\right) - \frac{2}{3}\mu_\gamma^2 r^2\hat{\mathbf{z}}\right] \tag{11.10}$$

while the field due to the same magnetic dipole from the fourth Maxwell equation ($\mu_\gamma = 0$ in (11.8)) is:

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\frac{D}{r^3}\left(3\hat{\mathbf{z}}\cdot\hat{\mathbf{r}}\,\hat{\mathbf{r}} - \hat{\mathbf{z}}\right) .$$

In the magnetic field in (11.10) there is a component proportional to $-\frac{2}{3}\mu_\gamma^2 r^2\hat{\mathbf{z}}$ antiparallel to the dipole $\mathbf{D}$. As evident from the ratio of this term and the coefficient of the main dipole component:

$$\frac{\frac{2}{3}\mu_\gamma^2 r^2}{\left(1 + \mu_\gamma r + \frac{1}{3}\mu_\gamma^2 r^2\right)}$$

its relative contribution to the field increases with the radius r. The measurements performed on the terrestrial surface (at $r = R_T$) allowed for a long time to set the best limit on the photon mass. This limit was then improved in 1968 by Goldhaber and Nieto,[13] who considered the contributions from different origins.

Later the same measurement at a larger radius was done by the Pioneer10 probe in the Jupiter magnetic field and the new limit was $m_\gamma \leq 4 \times 10^{-52}$ kg. A more sensitive measure could be performed in the solar magnetic field with r the distance from the Sun.

[13] See A.S. Goldhaber and M.M. Nieto, cited, 1971.

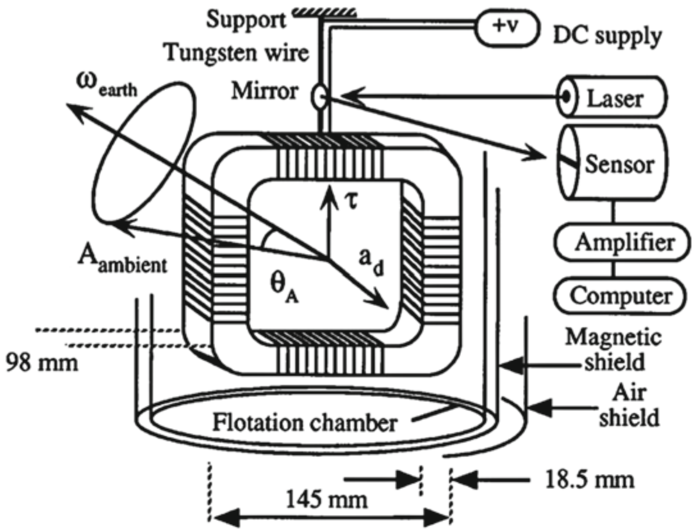


Fig. 11.3 The Lakes's experiment setup (Reprinted figure with permission from R.Lakes, cited. Copyright 1998 by American Physical Society. http://dx.doi.org/10.1103/PhysRevLett.80.1826)

11.6 The Lakes Experiment

In 1998 Lakes[14] proposed a very creative method to measure in a laboratory experiment the mass of the photon in the cosmic magnetic potential.

Before introducing this method, remind that, for the equation $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$, a coil carrying a current I is the source of a field $\mathbf{B}$, and has a dipole moment $m_d = \pi r^2 I$ with $I = J\Sigma$, where Σ is the cross section of the wire and r is the radius of the coil. If the coil is embedded in an external magnetic field $\mathbf{B}_{ext}$, a torque $\boldsymbol{\tau}_d = \mathbf{m}_d \times \mathbf{B}_{ext}$ acts on the coil as a consequence of the energy density of the magnetic field that is proportional to B^2. Then in analogy with the coil, since the equations are formally the same, if we consider an iron toroid, wound with turns carrying a current, since $\nabla \times \mathbf{A} = \mathbf{B}$, the flux of $\mathbf{B}$ inside the toroid, $\Phi = B\Sigma'$ (Σ' the cross section of the toroid), becomes the dipolar source of a field $\mathbf{A}$. This source has a dipole moment $m_{dA} = \pi r^2 \Phi$ parallel to the axis of the toroid, and if the toroid is embedded in the cosmic potential A_{amb}, there is a torque acting on the toroid of momentum $\boldsymbol{\tau}_{dA} = \mathbf{m}_{dA} \times \mu_\gamma^2 \mathbf{A}_{amb}$. The factor μ_γ^2 in the torque is a consequence of the fact that the contribution of the potential A to the energy density is $\mu_\gamma^2 A^2/2$ while that of the field B is $B^2/2$, as evident from (11.2).

In the experiment (see Fig. 11.3) a toroid with turns carrying a current, was suspended by the wire of a Cavendish balance to measure the torque generated by the cosmic magnetic potential $\mathbf{A}_{amb}$. To a first approximation $|\mathbf{A}_{amb}| \approx |\mathbf{B}|R$ where R is

[14]R. Lakes, Physical Review Letters, **80**, 1826 (1998). A more precise experiment of this type was later performed by J. Luo et al., Physical Review Letters, **90**, 081801–1, (2003).

the radius of a cylinder with axis in the direction of **B**, thus even though the cosmic magnetic field is very small, this potential might be very large since R is of the order of galactic or extragalatic dimensions.

From the galactic magnetic field ($\approx 1\,\mu$G) and its reversal position toward the center of the Milky Way (≈ 600 pc, 1 pc $= 3.08 \times 10^{16}$ m) the potential would be $A_{amb} \approx 2 \times 10^9$ Tm. This has to be compared with the potential ≈ 200 Tm from the terrestrial magnetic field and ≈ 10 Tm from the Sun. If the ambient cosmic vector potential were 10^{12} Tm, corresponding to the Coma cluster of galaxies (0.2 μG over 1300 kpc diameter), the limit would be[15] $m_\gamma \leq 10^{-55}$ kg.

11.7 Other Measurements

For a non null mass of the photon, the electromagnetic waves should have a frequency dependent speed and limits on the photon mass could be found by measurements of the dispersion of the electromagnetic waves. The best limit from this method was deduced by N.M. Kroll with measurements on the Schumann[16] resonances. These are low-frequency standing electromagnetic waves traveling through the atmosphere between the Earth's surface and the ionosphere, which are two conductive layers of a waveguide. If these surfaces were planar and parallel, for a special mode the speed could be independent from the frequency and equal to c also for a non null mass of the photon. But these two surfaces are two concentric spherical surfaces and there is a dispersion relation of the special mode which depends on an effective mass $m^2_{\gamma eff} = g\ m^2_\gamma$ where the dilution factor g is equal to the ratio ($\lesssim 1\,\%$) of the distance Earth surface—ionosphere (about 60 km) to the sum of the radii of the Earth (6400 km) and of the ionosphere. Considering the low frequencies of the Schumann resonances (the lowest is equal to 8 Hz), Kroll[17] deduced the limit $m_\gamma \lesssim 4 \times 10^{-49}$ kg.

At present the strongest and most controlled limit on the photon mass has been found by D.D. Ryutov in measurements on the solar wind. In the model of solar wind supported by experiments, the plasma moving radially from the Sun, carries with itself the field lines of the magnetic field and, due to the Sun rotation, these lines wind up like an Archimedes spiral and at large distance have almost an azimuthal direction. If the photon has a non null mass, to maintain this configuration, a real current of plasma **J** should balance the pseudo-current $-\mu^2_\gamma \mathbf{A}/\mu_0$ in the Proca equation (11.8).

---

[15]See A.S. Goldhaber and M.M. Nieto, cited, 2010, and J. Luo et al., cited, 2003.

[16]J.D. Jackson, *Classical Electrodynamics*, cited, Sects. 8.9 and 12.9.

[17]Since the mass energy has to be smaller than the total energy of the wave, from the $m_\gamma c^2 < h\nu$, for the lowest Schumann frequency equal to 8 Hz, considering the dilution factor, the mass limit is $m_\gamma < 6 \times 10^{-49}$ kg.

Table 11.1 Experimental limits for the photon mass m_γ and its Compton wavelength $\lambdabar_C = \hbar/m_\gamma c[\lambdabar_C(\mathrm{m}) = 3.52 \times 10^{-45}/\mathrm{m(kg)}]$

Experimental method	$m_\gamma(\mathrm{kg}) \lesssim$	$\lambdabar_C(\mathrm{m}) \gtrsim$
Dispersion relation for Shumann resonances (Kroll)	4×10^{-49}	8×10^5
Williams, Faller and Hill experiment	2×10^{-50}	2×10^7
Jupiter magnetic field	7×10^{-52}	5×10^8
Measurements in the plasma of the solar wind	2×10^{-54}	2×10^{11}

As a consequence in the local magnetic field an acceleration of the plasma $\mathbf{J} \times \mathbf{B}$ should be observed. The measurements of density, speed and pressure of the plasma beyond the Pluto orbit allow to exclude this current and provide an upper limit $m_\gamma \lesssim 2 \times 10^{-54}$ kg that is the best limit on the photon mass.

11.8 Comments

The daily test of the classical Electromagnetism in many applications and the achievements of the Quantum Electrodynamics in the prediction of phenomena with high accuracy (more than six orders of magnitude) support a null mass of the photon as usually assumed. This tacit assumption implies the hypothesis that an effective mass may be determined only by the uncertainty principle:

$$m_\gamma \approx \frac{\hbar}{\Delta t_{Univ.} \cdot c^2} = 2 \times 10^{-59} \text{ kg}$$

where $\Delta t_{Univ.}$ is the age of the Universe. The fully experimental measurements mentioned in this chapter are the most sensitive to photon mass and are reported in Table 11.1. Lower limits can be derived from the measurements with theoretical hypothesis. A.S. Goldhaber and M.M. Nieto (2010) provide an exhaustive review on the theory and the experiments. A full and updated report on the photon mass limits can be found in the Particle Data Group tables.[18]

[18]K.A. Olive et al. (Particle Data Group), Chin. Phys. C, 38, 090001 (2014), and update in http://pdg.lbl.gov/index.html.

Appendix: Proca Equations from the Euler-Lagrange Equations

The Proca equations can be derived from the Euler-Lagrange equations for the fields:

$$\partial^{\mu}\left(\frac{\partial\mathcal{L}}{\partial\left(\partial^{\mu}A^{\nu}\right)}\right)-\frac{\partial\mathcal{L}}{\partial A^{\nu}}=0$$

using the Lagrangian density of the electromagnetic field with a term to account for a non null mass of the photon:

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-\mu_0 J_{\mu}A^{\mu}+\frac{1}{2}\,{\mu_{\gamma}}^2 A_{\mu}A^{\mu}\,.$$

For the first term of the equation we find:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\left(\partial^{\mu}A^{\nu}\right)}&=-\frac{1}{4}\frac{\partial}{\partial\left(\partial^{\mu}A^{\nu}\right)}\left[F_{\alpha\beta}F^{\alpha\beta}\right]=-\frac{1}{4}\frac{\partial}{\partial\left(\partial^{\mu}A^{\nu}\right)}\left[g_{\alpha\rho}g_{\beta\sigma}F^{\rho\sigma}F^{\alpha\beta}\right]\\&=-\frac{1}{4}\frac{\partial}{\partial\left(\partial^{\mu}A^{\nu}\right)}\left[g_{\alpha\rho}g_{\beta\sigma}(\partial^{\rho}A^{\sigma}-\partial^{\sigma}A^{\rho})(\partial^{\alpha}A^{\beta}-\partial^{\beta}A^{\alpha})\right]\\=-\frac{1}{4}g_{\alpha\rho}g_{\beta\sigma}&\left[(\delta^{\rho}_{\mu}\delta^{\sigma}_{\nu}-\delta^{\sigma}_{\mu}\delta^{\rho}_{\nu})(\partial^{\alpha}A^{\beta}-\partial^{\beta}A^{\alpha})+(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}-\delta^{\beta}_{\mu}\delta^{\alpha}_{\nu})(\partial^{\rho}A^{\sigma}-\partial^{\sigma}A^{\rho})\right]\\&=-\frac{1}{4}\left[g_{\alpha\mu}g_{\beta\nu}F^{\alpha\beta}-g_{\alpha\nu}g_{\beta\mu}F^{\alpha\beta}+g_{\mu\rho}g_{\nu\sigma}F^{\rho\sigma}-g_{\nu\rho}g_{\mu\sigma}F^{\rho\sigma}\right]\\&=\qquad-\frac{1}{4}\left[F_{\mu\nu}-F_{\nu\mu}+F_{\mu\nu}-F_{\nu\mu}\right]=-F_{\mu\nu}\end{aligned}$$

and for the second:

$$\frac{\partial\mathcal{L}}{\partial A^{\nu}}=-\mu_0 J_{\nu}+\frac{1}{2}{\mu_{\gamma}}^2\frac{\partial}{\partial A^{\nu}}[g_{\mu\alpha}A^{\alpha}A^{\mu}]=-\mu_0 J_{\nu}+\frac{1}{2}{\mu_{\gamma}}^2[g_{\mu\nu}A^{\mu}+g_{\nu\alpha}A^{\alpha}]=-\mu_0 J_{\nu}+{\mu_{\gamma}}^2 A_{\nu}$$

thus the Lagrange equation becomes:

$$\partial^{\mu}F_{\mu\nu}-\mu_0 J_{\nu}+{\mu_{\gamma}}^2 A_{\nu}=0$$

or in the usual form:

$$\partial_{\mu}F^{\mu\nu}=\mu_0 J^{\nu}-{\mu_{\gamma}}^2 A^{\nu}\,.$$

This equation corresponds to the two non homogeneous Proca equations (11.5) and (11.8) and for $\mu_{\gamma}=0$ gives the covariant expression (7.12) of the non homogeneous Maxwell equations.

Chapter 12
Magnetic Monopoles

In the absence of electric charges and currents the Maxwell equations are clearly symmetric in the electric and magnetic fields. This symmetry would be conserved in the presence of field sources if, in addition to the electric charges and currents, magnetic charges (*monopoles*) and currents would exist.

In 1931 Dirac showed that the existence in nature of one magnetic monopole could account for the electric charge quantization. Probably the high mass of the monopoles did not allow their production and observation in experiments performed at the particle accelerators so far.

Modern Grand Unification Theories (GUT) of the fundamental interactions predict the existence of massive magnetic monopoles. These magnetic monopoles are too massive to be produced at the present or future accelerators, but if produced in the early Universe, should be observed as relics because the magnetic charge conservation should prevent their decay in other particles.

12.1 Generalized Maxwell Equations

Under the hypothesis of the existence of magnetic charges and currents with densities ρ_m and j_m, similar to the electric charges and currents with densities ρ_e and j_e, Maxwell equations[1] would be:

$$\nabla \cdot \mathbf{D} = 4\pi \rho_e \tag{12.1}$$

$$\nabla \cdot \mathbf{B} = 4\pi \rho_m \tag{12.2}$$

[1] Gaussian units are used in this chapter.

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9_12

$$-\nabla \times \mathbf{E} = \frac{4\pi}{c}\mathbf{j}_m + \frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{12.3}$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\mathbf{j}_e + \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t}\,. \tag{12.4}$$

In addition to these equations, a continuity equation[2] must be formulated to include the conservation of the magnetic charge:

$$\nabla \cdot \mathbf{j}_m + \frac{\partial \rho_m}{\partial t} = 0$$

similar to the continuity equation for the electric charge:

$$\nabla \cdot \mathbf{j}_e + \frac{\partial \rho_e}{\partial t} = 0$$

and a law of interaction of the magnetic charge g with the fields[3]:

$$\mathbf{F} = g\left[\mathbf{B} - \frac{\mathbf{v}}{c} \times \mathbf{E}\right] \tag{12.5}$$

similar to the Lorentz force:

$$\mathbf{F} = q\left[\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right]. \tag{12.6}$$

12.2 Generalized Duality Transformation

The generalized duality transformation:

$$\mathbf{E} = \mathbf{E}' \cos\varphi + \mathbf{H}' \sin\varphi \qquad \mathbf{D} = \mathbf{D}' \cos\varphi + \mathbf{B}' \sin\varphi$$

$$\mathbf{H} = -\mathbf{E}' \sin\varphi + \mathbf{H}' \cos\varphi \qquad \mathbf{B} = -\mathbf{D}' \sin\varphi + \mathbf{B}' \cos\varphi$$

---

[2]This equation determines the sign of $\mathbf{j}_m$ in Eq. (12.3).

[3]This relation, as for the Lorentz force, can be derived directly from the relativistic transformation of the electric and magnetic fields from a frame where the magnetic charge is at rest to a frame where it moves with velocity $\mathbf{v}$.

leaves invariant the quadratic forms of the fields: the Poynting's vector, the energy density and the Maxwell stress tensor. If a similar transformation is applied to the sources:

$$\rho_e = \rho'_e \cos\varphi + \rho'_m \sin\varphi \qquad \mathbf{j}_e = \mathbf{j}'_e \cos\varphi + \mathbf{j}'_m \sin\varphi$$

$$\rho_m = -\rho'_e \sin\varphi + \rho'_m \cos\varphi \qquad \mathbf{j}_m = -\mathbf{j}'_e \sin\varphi + \mathbf{j}'_m \cos\varphi$$

also the Maxwell equations (12.1–12.4) and the Lorentz forces (12.5) and (12.6) are invariant: the fields $\mathbf{E}'$, $\mathbf{D}'$, $\mathbf{H}'$, $\mathbf{B}'$ satisfy the same equations of the fields $\mathbf{E}$, $\mathbf{D}$, $\mathbf{H}$, $\mathbf{B}$. This transformation represents a rotation of an angle φ in a two dimensional frame with the electric charge and the magnetic charge as axes. The electric and magnetic charges of a particle are determined by the value of the angle φ. If the ratio of the magnetic charge to the electric charge is the same for all particles, it is possible to fix the angle φ to have always $\rho_m = 0$ and $j_m = 0$. With this choice the Eqs. (12.1–12.4) take the usual form of the Maxwell equations in the absence of magnetic charges.

Under the assumption $\varphi_e = 0$ for the electron (that is $q_m^e = 0$), any other particle could have a different angle φ_p given by:

$$\tan\varphi_p = -\frac{q_m^p}{q_e^p}.$$

A body with N_p protons and N_n neutrons would have a magnetic charge:

$$Q_m = N_p q_m^p + N_n q_m^n$$

and considering the negligible contribution (<1 gauss) to the magnetic field of the Earth, where $N_p \simeq N_n \simeq 10^{51}$, the very low upper bound on the magnetic charges of the proton and the neutron is:

$$q_m^p,\ q_m^n < 2 \times 10^{-24}\, e \qquad \text{(in Gaussian units)}.$$

Therefore it is possible to conclude that the particles of the ordinary matter have either no magnetic charges or the same angle φ and the usual convention of taking $q_m = 0$ for all the particles is justified.

For $\varphi = \pi/2$ the transformation gives:

$$\mathbf{E} \to \mathbf{H} \qquad \mathbf{H} \to -\mathbf{E} \qquad \mathbf{D} \to \mathbf{B} \qquad \mathbf{B} \to -\mathbf{D}$$

$$\rho_e \to \rho_m \qquad \rho_m \to -\rho_e \qquad \mathbf{j}_e \to \mathbf{j}_m \qquad \mathbf{j}_m \to -\mathbf{j}_e$$

$$\epsilon \to \mu \qquad \mu \to \epsilon$$

and then it yields the Maxwell equations but with the electric and magnetic field and the relative sources swapped. This transformation can be used to get the properties of the magnetic poles from those of the electric charges.

12.3 Symmetry Properties for Electromagnetic Quantities

The electric charge is scalar with respect to Lorentz transformations and rotations, then it is natural to assume that it is invariant (even) under space and time inversions. The same properties need to be valid for the charge density ρ_e, while the current density is a vector (polar) and is odd under time inversion.

The energy density is invariant under space and time reflections, so the fields E and D and the fields B and H have the same properties with respect to the space and time inversions. This means that ϵ and μ have to be scalar and even under time reversal.

Since the field E is the ratio of a force to a charge, it follows that the field E is a polar vector and is even under time reversal. This is coherent with the first Maxwell equation. From the third Maxwell equation it follows that the field B is a pseudo vector (an axial vector) and is odd under time reversal. The second equation gives the properties of the magnetic charge density ρ_m: it is pseudo scalar and is odd under time reversal. The third equation implies that j_m is also a pseudo vector and it is even under time reversal. These properties are collected in Table 12.1.

Since ρ_m and ρ_e have opposite symmetry properties, an evident symmetry violation with respect to space inversion and time reflection would be present in particles (*dioni*) with both electric and magnetic charges. It is known that an extremely small violation of these symmetries has been observed in the weak interaction but not in the electromagnetic interaction.

Table 12.1 Properties of symmetry for electromagnetic quantities under space inversion (S. I.) and time inversion (T. I.)

	S. I.	T. I.
E, D	Polar vector	Even
H, B	Axial vector	Odd
ρ_e	Scalar	Even
j_e	Polar vector	Odd
ρ_m	Pseudoscalar	Odd
j_m	Axial vector	Even

12.4 The Dirac Monopole

In 1931 Dirac in a brilliant theoretical paper[4] introduced the concept of magnetic monopole to explain the quantization of the electric charge. We shortly summarize his theory.

In quantum mechanics the motion of a particle is described by a wave function $\psi = Ae^{i\gamma}$ with A and γ functions of x, y, z, t. It is possible to add to γ a phase β, function of $x,\ y,\ z,\ t$, that in general is not an exact integral so that its derivatives do not satisfy the conditions of integrability. Since it is demonstrated that the change of phase β along a closed curve has to be the same for all functions (particles), this change has to depend on the field of force in which the particle moves. Indeed, the derivatives of the phase β can be related to the components of the electromagnetic field and its change for a closed curve is equal to the flux of the electromagnetic field through any surface that has the closed curve as a boundary. But the condition on the phase can be relaxed, and for some functions vanishing in some points, the change in phase can be different by multiples of 2π from that of the other functions. The points where these functions vanish have to lie along lines called *nodal lines*. In three dimensions, the flux connected to a close curve is that of the magnetic field and the flux integrated over a closed surface, for all functions (particles), is equal to $2\pi\hbar c/e$ times an integer determined by the *nodal lines* entering the closed surface. The end point P of a *nodal line* is a point of singularity in the magnetic field or a *magnetic monopole* g, while the other end point is at infinity. The magnetic flux coming out in P arrives from infinity along the *nodal line* and returns from the pole to infinity as a radial magnetic field. The line can be regarded as a solenoid of infinitesimal radius with the magnetic flux inside or as a string composed of small magnetic dipoles from infinity to the pole.

The magnetic flux from a pole g in a closed surface is $4\pi g$ and, from the previous considerations, has to be equal to $n \cdot 2\pi\hbar c/e$, thus the relation between the magnetic charge (monopole) and the electric charge is:

$$ge = n\,\frac{1}{2}\,\hbar c \tag{12.7}$$

[4]P.A.M. Dirac, *Quantised singularities in the electromagnetic field*, Proc. Roy. Soc., A 133, 60 (1931). The introduction is particularly interesting because Dirac presents some considerations on the evolution of the mathematics applied to theoretical physics and then he shortly reviews the studies on the negative energy states, inspired to his previous paper *Electrons and protons*, Proc. Roy. Soc., A 126, 360 (1930). Following the suggestion by J.R. Oppenheimer, one year before the discovery of the positive electron by C.D. Anderson (Science, **76**, 238 (1932)), he introduces the idea of anti-electrons, of production and annihilation of electron-antielectron pair, and he also proposes the existence of the antiproton as the antiparticle of the proton. For a comment on the Dirac's paper see the article by E. Amaldi and N. Cabibbo cited in a following note.

with $n = 0, \pm 1, \pm 2, \ldots$ and the relation between the elementary magnetic charge g_D (Dirac monopole) and the electric charge e is:

$$g_D = \frac{1}{2}\frac{\hbar c}{e} = \frac{1}{2}\left(\frac{\hbar c}{e^2}\right)e = \frac{137}{2}e\,. \tag{12.8}$$

In the Gaussian unit system the electric and the magnetic charges have the same dimensions. To the elementary electric charge $e = 4.8 \times 10^{-10}$ e.s.u., in the unrationalized e.m.u. corresponds an elementary magnetic charge $g_D = 3.29 \times 10^{-8}$ oersted·cm^2. In SI units $g_D = 4.12 \times 10^{-15}$ Weber.

The Dirac theory renewed the interest for the search of free magnetic charges and many theoretical and experimental studies[5] followed the 1931 paper.

12.5 Magnetic Field and Potential of a Monopole

The radial field from a magnetic monopole, located at the origin of the frame, is:

$$\mathbf{B} = \frac{g}{r^2}\frac{\mathbf{r}}{r}\,. \tag{12.9}$$

The monopole can be described[6] as the end point of a *nodal line* from infinity to the origin along the negative z axis, and its vector potential can be derived as the integral of the potentials[7] of all the elementary dipoles $\mathbf{dm} = g\,\mathbf{dl}$, from infinity to the origin:

$$\mathbf{A}(\mathbf{r}) = \int_{\infty}^{0} \frac{g\,\mathbf{dl} \times \mathbf{r}}{r^3}\,. \tag{12.11}$$

From this integral, the spherical components of the vector potential $\mathbf{A}$ are:

$$A_r = 0 \quad A_\theta = 0 \qquad A_\varphi = \frac{g}{r}\frac{(1-\cos\theta)}{\sin\theta} = \frac{g}{r}\tan\frac{\theta}{2}.$$

[5] After the first paper, in 1948 Dirac published a detailed analysis of the theory of the magnetic monopole (*Physical Review,* **74**, *817 (1948)*). An extensive review article on the theory and the searches of the magnetic monopole was done by E. Amaldi, *On the Dirac Magnetic Poles*, published in the volume *Old and new problems in Elementary Particles*, in honour of G. Bernardini, edited by G. Puppi, Academic Press, New York, (1968). This review was updated in 1972 by E. Amaldi and N. Cabibbo, *On the Dirac Magnetic Poles*, in *Aspects of Quantum Theory*, a volume in honour of P.M. Dirac, edited by Abdus Salam and E.P. Wigner, Cambridge, University Press, 1972. An introduction to the argument can be found in J.D. Jackson, *Classical Electrodynamics*, cited, Chap. 6.

[6] See for instance E. Amaldi, cited, p. 45.

[7] The potential of a magnetic dipole $\mathbf{m}$ located in the origin is:

$$\mathbf{A}(\mathbf{r}) = \frac{\mathbf{m} \times \mathbf{r}}{r^3}. \tag{12.10}$$

.

This potential is singular along the integration line ($\theta = \pi$). Its curl is:

$$\nabla \times \mathbf{A} = \mathbf{B} + \mathbf{B}^f$$

where $\mathbf{B}$ is the radial field (12.9) with a flux $4\pi g$ outgoing uniformily through any spherical surface centered at the origin, and $\mathbf{B}^f$ is a fictitious field associated to the magnetic flux $-4\pi g$ entering the same sphere along the negative z axis. This can easily be seen by computing the circular integral of $\mathbf{A}$ along a circle of radius $\rho = r \sin\theta$ around the z axis:

$$\oint \mathbf{A} \cdot \mathbf{dl} = \int (\nabla \times \mathbf{A}) \cdot \hat{\mathbf{n}}\, dS = \int \mathbf{B} \cdot \hat{\mathbf{n}}\, dS = \oint A_\varphi \rho d\varphi = 2\pi g (1 - \cos\theta)\,. \tag{12.12}$$

For $\theta \to 0$ the circle of radius $\rho \to 0$ is located around the positive z axis and the flux inside is null; for $\theta \to \pi$ the circle of radius $\rho \to 0$ is around the negative z axis and the entering[8] flux is $4\pi g$ corresponding to a field B^f with components:

$$B_x^f = 0 \qquad B_y^f = 0 \qquad B_z^f = 4\pi g\, \delta(\pi - \theta)\,.$$

Thus the magnetic field $\mathbf{B}$ is given by the relation:

$$\mathbf{B} = \nabla \times \mathbf{A} - \mathbf{B}^f \tag{12.13}$$

where the field $-\mathbf{B}^f$ cancels the singularity on the *nodal line*.

12.6 Quantization Relation

We have seen that the magnetic monopole located at a point P can be considered as the end point of a *nodal line* L (see Fig. 12.1). In the Coulomb gauge its vector potential $\mathbf{A}$ is defined by the equations:

$$\nabla \times \mathbf{A} = \mathbf{B} \qquad \nabla \cdot \mathbf{A} = 0 \tag{12.14}$$

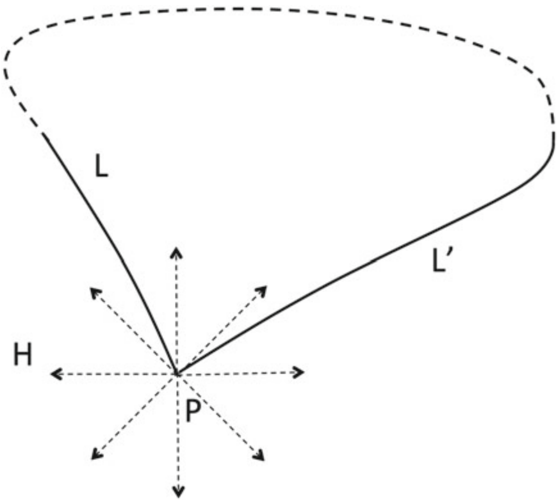
where $\mathbf{B}$ is the radial field given by the relation (12.9) and $\mathbf{A}(P, L)$ can be calculated from (12.11).

Formally the equations (12.14) are identical to the usual equations for a magnetostatic field $\mathbf{b}$:

$$\nabla \times \mathbf{b} = \frac{4\pi}{c}\mathbf{j} \qquad \nabla \cdot \mathbf{b} = 0$$

[8] Here the sign is positive because associated to the direction of the circular path of the integral that, at $\theta = \pi$, is opposite to the direction of the outgoing flux.

Fig. 12.1 Two *nodal lines* L and L' from infinity to the monopole to calculate the vector potential



and thus the integral (12.11) has the same functional form of the Biot and Savart law:

$$\mathbf{b} = \frac{1}{c} \int \frac{I\, \mathbf{dl} \times \mathbf{r}}{r^3} . \tag{12.15}$$

The analogy between the two sets of equations can be used[9] to derive the Dirac relation (12.7). By substituting $4\pi \mathbf{j}/c$ with the radial field $\mathbf{B}$ given by (12.9), we have a current flux $4\pi j r^2 = gc = \Phi(j) = I$ and the *nodal line* is equivalent to a wire carrying a current I equal to gc that from infinity reaches the monopole. To this *nodal line* is associated a vector potential $\mathbf{A}(P, L)$. But it is possible to assume a second *nodal line* L', as in Fig. 12.1, which from infinity goes to the point P and to this line is associated a vector potential $\mathbf{A}(P, L')$. If we consider the circuit composed by the line L' and the line $-L$, we have a loop with a current $I = gc$ and, from the similarity between the relations (12.15) and (12.11), the field $\mathbf{b}$ associated to this current is:

$$\mathbf{b} = \mathbf{A}(P, L') - \mathbf{A}(P, L).$$

From magnetostatics we know that $\mathbf{b}$ can be the gradient of a scalar potential U:

$$\mathbf{b} = -\nabla U$$

where $U = U_0 + 4\pi m I/c = U_0 + 4\pi m g$ is a multiple-valued function with $m = 0, \pm 1, \pm 2, \ldots$ the number of times the path, chosen for the calculation of U from

[9]The quantization proof reported here, was given by E. Fermi (*Acc. Naz. Lincei, Fondazione Donegani Conferenze*, 1950, p. 117) and is reported in the paper by E. Amaldi, cited, p. 47.

infinity to the point where we want **b**, circles around the loop composed by the two *nodal lines*. Thus from the previous relations the result:

$$\mathbf{A}(P, L') = \mathbf{A}(P, L) - \nabla U$$

that is a gauge transformation corresponding to the arbitrary choice of the *nodal line* and to its non observability.

In a semiclassical approximation the quantum wave function of a charged particle in a potential **A** is:

$$\Psi = \Psi_0 e^{\frac{ie}{\hbar c}\int \mathbf{A}\cdot d\mathbf{s}}$$

where Ψ_0 is the the wave function of the free particle. After the gauge transformation the new wave function is:

$$\Psi \to \Psi' = \Psi e^{-\frac{ie}{\hbar c}\int \nabla U\cdot d\mathbf{s}} = \Psi e^{-\frac{ie}{\hbar c}U} = \Psi e^{-\frac{ie}{\hbar c}(U_0 + 4\pi g m)}$$

and this function has the same value only if:

$$\frac{e}{\hbar c}\cdot 4\pi g m = 2\pi n \quad \text{with } \ n = 0, \pm 1, \pm 2, ...$$

that is just the Dirac relation (12.7).

12.7 Quantization from Electric Charge-Magnetic Dipole Scattering

Consider the scattering of a particle,[10] with charge e and velocity **v**, in the radial field **B** (12.9) of a magnetic monopole. The Lorentz force (12.6), normal to **v** and **B**, acts on the particle and for a large impact parameter b, the deflection of the charge is negligible, and the momentum transferred to the particle is:

$$\Delta p_{\perp} = \frac{2eg}{cb}.$$

The change of its angular momentum is:

$$\Delta L_e = \frac{2eg}{c}$$

[10]This argument was proposed by A.S. Goldhaber, *Physical Review*, **140 B**, 1407 (1965). For the calculation of ΔL_e see also J.D. Jackson, *Classical Electrodynamics*, cited, Section 6.13.

in the direction of the velocity and is independent of the impact parameter and of the velocity. Since the angular momentum is quantized, the change has to be equal to an integer multiple of $\hbar$ and this gives the Dirac condition (12.7).

From the conservation of the angular momentum in the isolated system, the change of the angular momentum L_{em} of the electromagnetic field has to be opposite to that of L_e.

The angular momentum L_{em} of the field $\mathbf{E}$ of the electric charge and the field $\mathbf{H}$ of the magnetic monopole, can be found by the integral of the momentum $\mathbf{r} \times \mathbf{g}$ where $\mathbf{g}$ is the electromagnetic momentum density (9.20). L_{em} is independent of the origin because the total momentum of the fields in the isolated system is null. The result[11] is:

$$\mathbf{L}_{em} = \frac{eg}{c}\,\hat{\mathbf{n}}$$

where $\hat{\mathbf{n}}$ is a versor with direction from the charge to the monopole. Of course also this angular momentum has to be quantized but in order to get the Dirac condition it has to be equal to an half-integer multiple of $\hbar$ in some disagreement with the unitary spin of the photon.

12.8 Properties of the Magnetic Monopoles

12.8.1 Magnetic Charge and Coupling Constant

The condition (12.7) with $n = 1$ gives the elementary magnetic charge g_D (Dirac monopole) (12.8):

$$g_D = \frac{\hbar c}{2e} = \frac{1}{2}\frac{e}{\alpha_e} = \frac{137}{2}e = 68.5\,e$$

where α_e is the fine structure constant. From this the adimensional magnetic coupling constant to the electromagnetic field

$$\alpha_g = \frac{g_D^2}{\hbar c} = \frac{1}{4}\left(\frac{\hbar c}{e}\right)^2 \frac{1}{\hbar c} = \frac{\alpha_e}{4} = 34.25$$

that is $\gg 1$ and so a perturbative approximation cannot be used in computations of processes with magnetic monopoles.

[11] The result was first given by J.J. Thompson, *Elements of the Mathematical Theory of Electricity and Magnetism*, Cambridge, University Press, 1900–1904, and can be found in E. Amaldi, cited, p. 16, and in J.D. Jackson, *Classical Electrodynamics*, cited, Section 6.13.

12.8.2 Monopole in a Magnetic Field

A magnetic monopole g_D in a magnetic field B is accelerated by the force (12.5) and after a path of length l its energy increase is $W_m = g_D Bl$ or in numbers $W_m(\text{keV}) = 20.50 \cdot B(\text{gauss}) \cdot l(\text{cm})$. For a cosmic magnetic field $B \simeq 3\,\mu G$ and a path of cosmic length $l \simeq 1$ kpc the energy is $W_m \simeq 1.8 \times 10^{11}$ GeV.

12.8.3 Ionization Energy Loss for Monopoles in Matter

The energy loss due to ionization $-dE/dx$ for a particle of charge ze in the matter is given by the Bethe and Bloch formula[12]:

$$-\frac{dE}{dx} = k\, z^2\, \frac{Z}{A}\, \frac{1}{\beta^2} \left[\frac{1}{2}\, \ln \left(\frac{2\, m_e c^2\, \beta^2\, \gamma^2}{I^2} \right) - \beta^2 \right] \tag{12.16}$$

where $k = 4\pi N_A r_e^2 m_e c^2 = 0.307\ \text{MeVcm}^2/g$ and $I \simeq Z \times 10$ eV is the average excitation energy of the atoms in the crossed medium.

This expression is the weighted sum of all the energies transferred, per unit path, by the electric field E of the charged particle to the electrons of the crossed medium. For a monopole[13] of magnetic charge ng_D the force acting on the electrons is the Lorentz force from the B field of the monopole. This field is proportional to ng_D and in the (12.6) is multiplied by a factor β thus the force acting on the atomic electrons is that from a charge but replacing ze with $ng_D\beta$. By this substitution in (12.16), the energy loss due to ionization for a monopole is:

$$-\frac{dE}{dx} = k\, \left(\frac{ng_D}{e} \right)^2\, \frac{Z}{A} \left[\frac{1}{2}\, \ln \left(\frac{2\, m_e c^2\, \beta^2\, \gamma^2}{I^2} \right) - \beta^2 \right]. \tag{12.17}$$

The differences between this relation and the (12.16) are evident. For $\gamma \gg 1$ a monopole crossing the matter leaves a huge ionization as that of a heavy nucleus of charge $Ze = ng_D = n\, 68.5e$, thus the range of a monopole is much shorter than that for a same momentum charged particle. The factor $1/\beta^2$, important for the energy loss of the charged particles at small β, is missing. Of course these differences in ionization can be exploited in detectors for the direct search of magnetic monopoles. In Fig. 12.2 the ionization energy losses in air for a proton and a Dirac monopole (n=1) are compared.

---

[12] Terms for the density effect and the shell correction have to be added to this simple formula. See the Section *Passage of Particles Through Matter* in K.A. Olive et al. (Particle Data Group), Chin. Phys. C, 38, 090001 (2014).

[13] For this subject and its application to the present experiments, see: C. Bauer et al., *Nucl. Instr. and Methods in Physics Research A* 545 (2005) 503–515. For more details see: L. Patrizii and M. Spurio, *Status of searches for magnetic monopoles*, Annual Review of Nuclear Physics, 2015, 65:279–302.

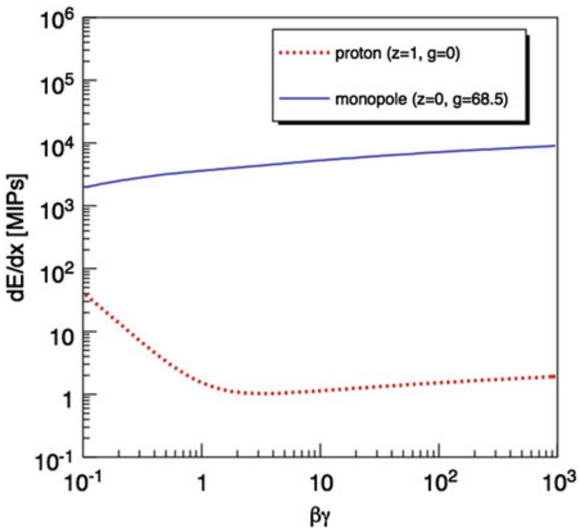


Fig. 12.2 Ionization energy losses in air for a proton and a Dirac monopole, normalized to the value of a minimum ionizing charged particle (Reprinted figure from C. Bauer et al., cited. Copyright 2005, with permission from Elsevier.)

12.9 Searches for Magnetic Monopoles

In a magnetic field a monopole is accelerated and its energy increases while the trajectory is deflected but it is not helicoidal as for charged particles. Its huge ionization can be detected in scintillation detectors, in gaseous detectors or in nuclear track detectors (NTD). Moreover a monopole crossing a superconducting coil induces an electromotive force and a current that can be detected by a SQUID. Similarly, passing samples of materials through the coil, monopoles trapped inside could also be observed.

In spite of its unique properties very different from those of the usual particles, no signal of magnetic monopole has been observed so far. In the following we shortly report the main results. An extensive review of the properties, the detectors and the results of the searches on magnetic monopoles is given in the paper by L. Patrizii and M. Spurio.[14] A periodically updated review of the searches can be found in *The Review of Particle Physics* by the Particle Data Group.[15]

[14]L. Patrizii and M. Spurio, cited; see also G. Giacomelli and L. Patrizii, *Magnetic Monopoles Searches*, arXiv:hep-ex/0506014v1, 7 June 2005. For the first searches see E. Amaldi, cited, 1965, updated by E. Amaldi and N. Cabibbo, cited, 1972.

[15]K.A. Olive et al. (Particle Data Group), cited, updated in url: http://pdg.lbl.gov.

12.9.1 Dirac Monopoles

The mass of the monopole is not predicted by the theory. The Dirac relation (12.8) and the naive assumption that the classical radius of the monopole $r_g = g^2/m_g c^2$ is equal to classical radius of the electron $r_e = e^2/m_e c^2$, give:

$$m_g = \left(\frac{g}{e}\right)^2 m_e = \left(\frac{1}{2}\frac{\hbar c}{e^2}\right)^2 m_e \simeq 4700\, m_e = 2.4\ \mathrm{GeV/c^2}$$

a large mass but much smaller than the present experimental limits.

Direct searches of monopoles produced in experiments at the accelerators (*ee*, $p\bar{p}$, *pp* and *ep* colliders) have set upper limits on the production cross section of monopoles for masses below one TeV. These limits, based on the assumption of monopole-antimonopole pairs production, are model dependent. A recent paper by the ATLAS Experiment[16] at the LHC collider has given the upper limits on the monopole production cross section in proton-proton collisions at 7 TeV in the center of mass: from $\sigma_{MM} < 145$ fb for a 200 GeV monopole mass to $\sigma_{MM} < 16$ fb for a 1200 GeV mass.

Indirect searches of monopole production can be performed in materials deployed near to the interaction points at the colliders or in proton beams interacting in ferromagnetic targets. The monopoles can be extracted by a magnetic field and observed in a superconducting magnetic coil. The MoDAL experiment based on this technique and on the detection of monopoles in tiles of nuclear track detector is at present running nearby an intersection point at LHC.

The present limits on the monopole production cross section as a function of the mass of the monopole, measured by experiments at accelerators are reported in Fig. 12.3.

12.9.2 GUT Monopoles

Grand Unification Theories (GUT) of the electroweak and strong interactions predict the quantization of the electric charge and the production of magnetic monopoles of large mass ($>10^{16} \div 10^{17}$ GeV) in the phase transition corresponding to the spontaneous symmetry breaking that originated the known interactions at temperature of the order of 10^{15} GeV and at an age of the Universe about 10^{-35} s. The present-day abundance of these monopoles would exceed by many orders of magnitude the critical energy density of the Universe. The subsequent cosmological inflation would have reduced their abundance to values that would make difficult the detection. Magnetic monopoles of intermediate mass ($\sim 10^{10}$ GeV) could have been produced in a phase transition at temperature of the order of 10^9 GeV at a time 10^{-23} s.

[16] Aad et al., *Physical Review Letters*, **109**, 261803 (2012).

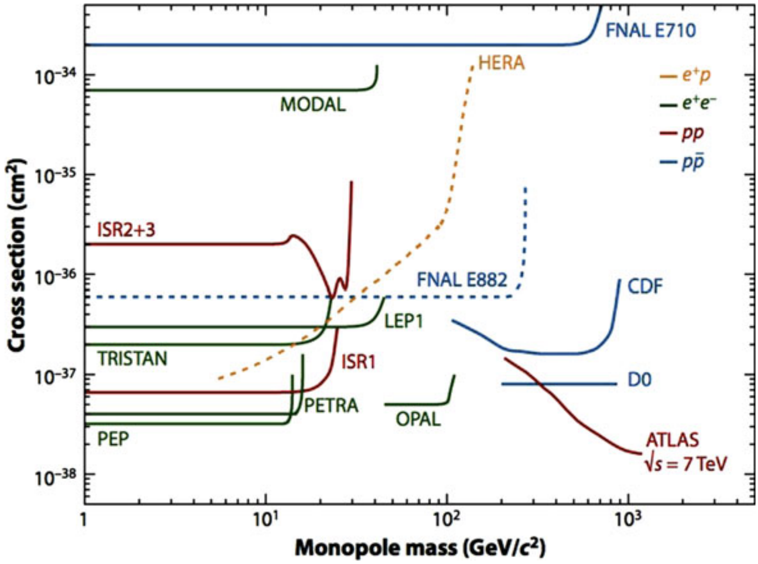


Fig. 12.3 Cross section upper limits (95 % C.L., except 90 % for FNAL E882) versus the mass from searches at colliders. *Dashed lines* are for indirect searches of monopoles trapped in beam pipe or detector materials (Reprinted figure with permission from L. Patrizii and M. Spurio, cited, Copyright 2015 by Annual Reviews.)

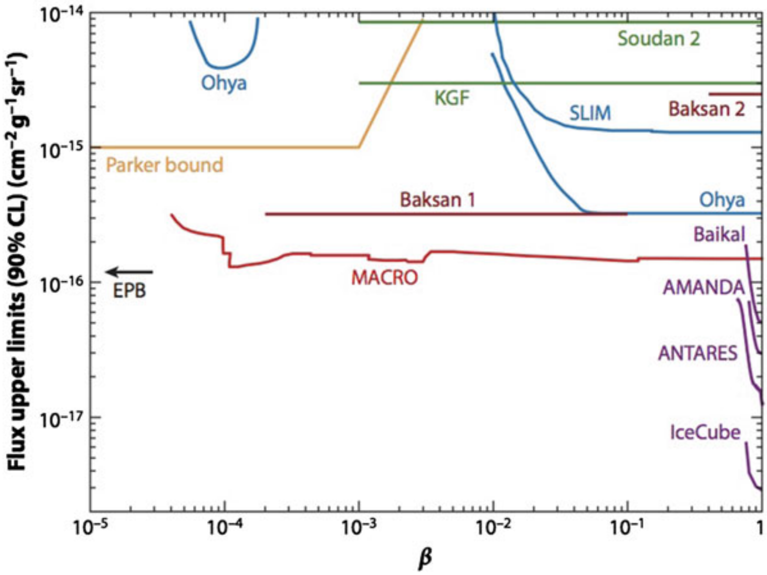


Fig. 12.4 Upper limits (90 % C.L.) versus β for flux of GUT monopoles with magnetic charge g_D (Reprinted figure with permission from L. Patrizii and M. Spurio, cited, Copyright 2015 by Annual Reviews.)

The very massive GUT monopoles are beyond the possibility of production at existing or foreseen accelerators, but, due to the magnetic charge conservation, the lightest magnetic monopoles, expected to be stable and produced in the early Universe, should be present as cosmic relics in the present Universe. Their kinetic energy would be affected by the Universe expansion and by the interaction with galactic and extragalactic magnetic fields.

Considering that the acceleration of the monopoles in the galactic magnetic field would subtract energy to this magnetic field, an upper limit (Parker limit) on the flux of monopoles can be derived from the condition that this energy has to be at most equal to that generated by the dynamo effect connected to the rotation of the Galaxy: $\Phi \sim 10^{-15}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{sr}^{-1}$.

An upper limit on the monopole flux can be found also from the condition that the contribution Ω_{mon} of the magnetic monopoles to the mass of the Universe has to be smaller than the critical density. For masses $\sim 10^{17}$ GeV it follows $\Phi < 1.3 \times 10^{-13}\ \Omega_{mon}\ \beta\ \mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{sr}^{-1}$ where Ω_{mon} can be assumed <0.1.

A direct search of magnetic monopoles was performed in the '90s by the MACRO experiment at the Laboratorio Nazionale del Gran Sasso. The experiment was composed by many tipes of detectors and had an acceptance $\sim 10^4\ \mathrm{m}^2$ sr for an isotropic flux. The non observation of any signal from a monopole set an upper limit on the flux equal to $\Phi = 1.4 \times 10^{-16}\,\mathrm{cm}^{-2}\,\mathrm{s}^{-1}\,\mathrm{sr}^{-1}$ for monopoles of $\beta > 4 \times 10^{-5}$. The results from searches of GUT monopoles are given in Fig. 12.4.

Appendix A
Orthogonal Curvilinear Coordinates

Given a symmetry for a system under study, the calculations can be simplified by choosing, instead of a Cartesian coordinate system, another set of coordinates which takes advantage of that symmetry. For example calculations in spherical coordinates result easier for systems with spherical symmetry.

In this chapter we will write the general form of the differential operators used in electrodynamics and then give their expressions in spherical and cylindrical coordinates.[1]

A.1 Orthogonal curvilinear coordinates

A system of coordinates u_1, u_2, u_3, can be defined so that the Cartesian coordinates x, y and z are known functions of the new coordinates:

$$x = x(u_1, u_2, u_3) \quad y = y(u_1, u_2, u_3) \quad z = z(u_1, u_2, u_3) \tag{A.1}$$

Systems of orthogonal curvilinear coordinates are defined as systems for which locally, nearby each point $P(u_1, u_2, u_3)$, the surfaces $u_1 = const$, $u_2 = const$, $u_3 = const$ are mutually orthogonal.

An elementary cube, bounded by the surfaces $u_1 = const$, $u_2 = const$, $u_3 = const$, as shown in Fig. A.1, will have its edges with lengths $h_1 du_1$, $h_2 du_2$, $h_3 du_3$ where h_1, h_2, h_3 are in general functions of u_1, u_2, u_3. The length ds of the line-element $\overline{OG}$, one diagonal of the cube, in Cartesian coordinates is:

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$$

[1] The orthogonal coordinates are presented with more details in J.A. Stratton, *Electromagnetic Theory*, McGraw-Hill, 1941, where many other useful coordinate systems (elliptic, parabolic, bipolar, spheroidal, paraboloidal, ellipsoidal) are given.

F. Lacava, *Classical Electrodynamics*, Undergraduate Lecture Notes in Physics,
DOI 10.1007/978-3-319-39474-9

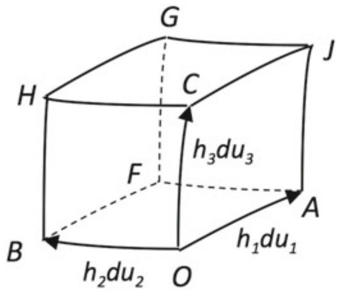


Fig. A.1 The elementary cube in coordinates u_1, u_2, u_3

and in the new coordinates becomes:

$$ds = \sqrt{(h_1du_1)^2 + (h_2du_2)^2 + (h_3du_3)^2}\,.$$

The coefficient h_1, h_2, h_3 can be determined from these two expressions and the relations (A.1).

The volume of the cube is $d\tau = h_1h_2h_3du_1du_2du_3$.

We consider a scalar function $f = f(u_1, u_2, u_3)$ and a vector $\mathbf{A}$ with the components A_1, A_2, A_3, relative to the directions $\hat{\mathbf{u}}_1$, $\hat{\mathbf{u}}_2$ and $\hat{\mathbf{u}}_3$, which are functions of u_1, u_2, u_3.

A.2 Gradient

The differential operator *gradient* of a scalar function f is the vector $gradf = \nabla f$ defined by the relation:

$$df = grad\ f \cdot \mathbf{dl} = \nabla f \cdot \mathbf{dl}$$

where df is the differential of f along an elementary displacement $\mathbf{dl}$ having components $(h_1du_1, h_2du_2, h_3du_3)$ and has is maximum value when $\mathbf{dl}$ is in the same direction of $grad\ f$. Moreover we can write:

$$df = \frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2 + \frac{\partial f}{\partial u_3}du_3$$

and comparing the two formulas:

$$(grad\ f)_1h_1du_1 + (grad\ f)_2h_2du_2 + (grad\ f)_3h_3du_3 = \frac{\partial f}{\partial u_1}du_1 + \frac{\partial f}{\partial u_2}du_2 + \frac{\partial f}{\partial u_3}du_3$$

the gradient components in the new coordinates result:

$$(grad\ f)_i = \frac{1}{h_i}\frac{\partial f}{\partial u_i}\,.$$

A.3 Divergence

To find the general formula for the operator *divergence* of a vector ($div\,\mathbf{A} = \nabla \cdot \mathbf{A}$), we apply the Gauss' theorem to the elementary cube in Fig. A.1. The flux $d\phi$ of the vector $\mathbf{A}$ out of the cube surface is equal to the divergence of the vector multiplied by the volume of the cube:

$$d\phi(\mathbf{A}) = div\,\mathbf{A}\,d\tau\,. \tag{A.2}$$

The flux out of the cube face $OBHC$ on the surface $u_1 = const$ is[2]:

$$-A_1(u_1)\ h_2(u_1)du_2\ h_3(u_1)du_3 = -A_1h_2h_3du_2du_3$$

and the flux out of the face $AFGJ$ on the surface $u_1 + du_1 = const'$ is:

$$A_1(u_1+du_1)\ h_2(u_1+du_1)du_2\ h_3(u_1+du_1)du_3\,.$$

Series expansion at the first order of the three factors of the last expression gives:

$$\left(A_1 + \frac{\partial A_1}{\partial u_1}du_1\right)\left(h_2 + \frac{\partial h_2}{\partial u_1}du_1\right)\left(h_3 + \frac{\partial h_3}{\partial u_1}du_1\right)$$

and neglecting second order terms this flux becomes:

$$A_1h_2h_3du_2du_3 + \frac{\partial}{\partial u_1}(A_1h_2h_3)\ du_1du_2du_3\,.$$

The total flux on the two opposite faces is:

$$\frac{\partial}{\partial u_1}(A_1h_2h_3)\ du_1du_2du_3$$

[2]The flux can be approximated by the value of the component A_1 at the face center multiplied by the face area.

and adding the similar fluxes out of the other faces, for the flux (A.2) out of the cube we find:

$$d\phi(\mathbf{A}) = \left(\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_1 h_3) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right) du_1 du_2 du_3$$
$$= div\, \mathbf{A}\; h_1 h_2 h_3 du_1 du_2 du_3$$

from which we obtain the formula for the divergence:

$$div\, \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(A_2 h_1 h_3) + \frac{\partial}{\partial u_3}(A_3 h_1 h_2)\right).$$

A.4 Curl

To write the general formula for the operator *curl* ($curl\ \mathbf{A} = \nabla \times \mathbf{A}$), we use the Stoke's theorem:

$$\oint \mathbf{A} \cdot \mathbf{dl} = \int curl\, \mathbf{A} \cdot \hat{\mathbf{n}}\, dS\,.$$

We consider the circulation of the vector $\mathbf{A}$ over the curve defined by the loop $OBHCO$ of the elementary cube. The contribution from paths OB and HC, neglecting second order terms, is:

$$A_2(u_3)\; h_2(u_3) du_2 - A_2(u_3 + du_3)\; h_2(u_3 + du_3) du_2 = -\frac{\partial}{\partial u_3}(A_2 h_2) du_2 du_3\,.$$

Similarly from paths BH and CO we have:

$$A_3(u_2 + du_2)\; h_3(u_2 + du_2) du_3 - A_3(u_2)\; h_3(u_2) du_3 = \frac{\partial}{\partial u_2}(A_3 h_3) du_2 du_3$$

and adding the two contributions, accounting for the Stoke's theorem, we have:

$$(curl\, \mathbf{A})_1 h_2 h_3 du_2 du_3 = \left[\frac{\partial}{\partial u_2}(A_3 h_3) - \frac{\partial}{\partial u_3}(A_2 h_2)\right] du_2 du_3$$

from which:

$$(curl\, \mathbf{A})_1 = \frac{1}{h_2 h_3}\left[\frac{\partial}{\partial u_2}(A_3 h_3) - \frac{\partial}{\partial u_3}(A_2 h_2)\right].$$

Similar expressions can be found for the other two components of the curl, therefore we can write:

$$(curl\ \mathbf{A})_i = \frac{1}{h_j h_k}\left[\frac{\partial}{\partial u_j}(A_k h_k) - \frac{\partial}{\partial u_k}(A_j h_j)\right].$$

A.5 Laplacian

The *Laplacian* operator can be written as $\Delta = div\ grad = \nabla \cdot \nabla = \nabla^2$, and using the formulas found for the divergence and the gradient, we get:

$$\Delta f = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial f}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial f}{\partial u_2}\right) + \frac{\partial}{\partial u_3}\left(\frac{h_1 h_2}{h_3}\frac{\partial f}{\partial u_3}\right)\right].$$

A.6 Spherical Coordinates

$$\begin{cases} x = r\ \sin\ \theta\ \cos\ \varphi \\ y = r\ \sin\ \theta\ \sin\ \varphi \\ z = r\ \cos\ \theta \end{cases}$$

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$$

$$\begin{cases} u_1 = r \\ u_2 = \theta \\ u_3 = \varphi \end{cases} \qquad \begin{cases} h_1 = 1 \\ h_2 = r \\ h_3 = r\ \sin\ \theta \end{cases}$$

$grad\ f$:

$$(grad\ f)_r = \frac{\partial f}{\partial r} \quad (grad\ f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta} \quad (grad\ f)_\varphi = \frac{1}{r\ \sin\ \theta}\frac{\partial f}{\partial \varphi}$$

$div\ \mathbf{A}$:

$$div\ \mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\ \sin\theta}\frac{\partial}{\partial \theta}(\sin\theta A_\theta) + \frac{1}{r\ \sin\theta}\frac{\partial A_\varphi}{\partial \varphi}$$

curl **A**:

$$(curl\ \mathbf{A})_r = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\varphi) - \frac{\partial A_\theta}{\partial\varphi}\right]$$

$$(curl\ \mathbf{A})_\theta = \frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\varphi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\varphi)$$

$$(curl\ \mathbf{A})_\varphi = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]$$

Δf:

$$\Delta f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2}$$

Sometime it is useful to replace the first term in the Laplacian with the relation[3]:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf)$$

A.7 Cylindrical Coordinates

$$\begin{cases} x = r\ \cos\ \varphi \\ y = r\ \sin\ \varphi \\ z = z \end{cases}$$

$$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$$

$$\begin{cases} u_1 = r \\ u_2 = \varphi \\ u_3 = z \end{cases} \qquad \begin{cases} h_1 = 1 \\ h_2 = r \\ h_3 = 1 \end{cases}$$

[3]This relation can be easily derived:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) = \frac{1}{r^2}\left(2r\frac{\partial f}{\partial r} + r^2\frac{\partial^2 f}{\partial r^2}\right) = \frac{1}{r}\left(\frac{\partial f}{\partial r} + \frac{\partial f}{\partial r} + r\frac{\partial^2 f}{\partial r^2}\right)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(f + r\frac{\partial f}{\partial r}\right) = \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial}{\partial r}(rf) = \frac{1}{r}\frac{\partial^2}{\partial r^2}(rf)$$

grad f:

$$(grad\ f)_r = \frac{\partial f}{\partial r} \quad (grad\ f)_\varphi = \frac{1}{r}\frac{\partial f}{\partial \varphi} \quad (grad\ f)_z = \frac{\partial f}{\partial z}$$

div **A**:

$$div\,\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\varphi}{\partial \varphi} + \frac{\partial A_z}{\partial z}$$

curl **A**:

$$(curl\ \mathbf{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial \varphi} - \frac{\partial A_\varphi}{\partial z}$$

$$(curl\ \mathbf{A})_\varphi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}$$

$$(curl\ \mathbf{A})_z = \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\varphi) - \frac{\partial A_r}{\partial \varphi}\right]$$

Δf:

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \varphi^2} + \frac{\partial^2 f}{\partial z^2}$$