Peter Flaschel

Topics in Classical Micro- and Macroeconomics

Elements of a Critique of Neoricardian Theory

Springer

Topics in Classical Micro- and Macroeconomics

Peter Flaschel

Topics in Classical Micro- and Macroeconomics

Elements of a Critique of Neoricardian Theory

Springer

Prof. Dr. Peter Flaschel
Department of Business Administration
and Economics
Bielefeld University
Universitaetsstrasse 25
33615 Bielefeld
Germany
pflaschel@wiwi.uni-bielefeld.de

ISBN 978-3-642-00323-3 e-ISBN 978-3-642-00324-0
DOI 10.1007/978-3-642-00324-0
Springer Heidelberg Dordrecht London New York

Library of Congress Control Number: 2009933993

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)

Preface

This book on Classical micro- and macrodynamics includes revised versions of papers which were written between 1983 and 2000, some jointly with co-authors, and it supplements them with recent work on the issues which are raised and treated in them. It attempts to demonstrate to the reader that themes of Classical economics, in particular in the tradition of Smith, Ricardo and Marx, can be synthesized into a coherent whole, from the perspective of formal model building.

This is accomplished by means of mathematical techniques which, on the one hand, provide a consistent accounting framework (labor values and prices of production) as point of reference for Classical micro- and macro-dynamics and which, on the other hand, attempt to apply these accounting schemes – or suitable extensions of them – by showing their usefulness as tools of analysis of the implications of technological change (labor values) and as potential tools for understanding the dynamics of market prices and of income distribution around their centers of gravity (production prices and the wage-profit curve).

It is, however, one finding of this book that the imposition of a uniform profit rate should give way sooner or later to the consideration of significant (more or less stable) profit rate differentials, to make production price schemes applicable to real world phenomena, as this is done in Flaschel et al. (2008) by way of a critical appraisal of the relevance of Han and Schefold's (2006) recent empirical application of Sraffian capital theory. We here act on suggestions made by Farjoun and Machover (1983) 25 years ago that the imposition of a uniform rate of profit on price formation for all sectors of a given economy is far too restrictive to be of empirical relevance. This should be obvious on the ultra-micro level of actual physical input–output data, but it is also inadequate for highly aggregated input–output data as we shall show in Chap. 8.

The first set of the above two tasks is solved through the application of the so-called Perron–Frobenius theory of eigenvectors and eigenvalues of non-negative matrices and will supply us with a Classical System of National Accounts (SNA), based on labor values, that helps to classify what is going on behind the surface of competition in real terms, comparable to the SNA established by Richard Stone and his research group (see the United Nations' (1968) System of National Accounts). Such an SNA provides measures of real output, labor productivity, real growth of both of them and more, constructed both in the Classical theory and in Stone's system as instruments to describe real tendencies behind the nominal aggregates. On the basis of this understanding, Classical labor values are not competing in the first instance with production prices about being the better theory of market prices, but are indeed providing a framework for National Accounting that should be compared with the current (the UN's) System of National Accounts with respect to their weak points and their strong analytical implications.

Considering the current SNA (not Stone's original version) one may hold the view that its various measures do not construct something "real" behind the "nominal", for example when real value added is calculated on an industry level in terms of prices of a more or less distant past. Likewise, one could claim that the construction of labor values is nothing that can be considered as "real". But what is the meaning of "real" here? In our view this can only be substantiated by showing mathematical propositions that demonstrate important implications of the measures proposed by the employed System of National Accounts, be it Classical or Stone's, for the understanding of the capitalist mode of production and its process of creative destruction at all levels of society.

This is the setup in which the Classical Theory of Value and Competition has to be confronted with the achievements of Stone's SNA. We shall show in the first part this book that there is no conflict between the two approaches to National Accounting, but in fact some complementarity, with labor values originating from the input–output part of Stone's accounting system and this even at the highest levels of generality that is present in Stone's input–output methodology.

Labor values are built on the principle that only labor is productive. Keynes (1936, p. 213/4), not at all a proponent of the labor theory of value, is indeed expressing a somewhat similar view, when he writes:

> It is preferable to regard labor, including, of course, the personal services of the entrepreneur and his assistant, as the sole factor of production, operating in a given environment of technique, natural resources, capital equipment and effective demand. This partly explains why we have been able to take the unit of labor as the sole physical unit which we require in our economic system, apart from the units of money and of time.

Our view on the role of labor values for economic analysis is a pragmatic one. Labor Values should be well defined for general models of production (see Chap. 5 for an example) and they should first of all be applied to generally understandable scientific topics like the implications of technological change in the capitalist mode of production (see Chap. 3). There they can be used at the theoretical level for example to show that capital-using labor-saving technical change systematically lowers such labor values, and at the empirical level to measure whether this actually is the case.

Approaching labor values from this pragmatic perspective indicates that there is not really a "transformation problem" to be solved (as in the example of Marx's (1977) *Capital* Vol. III), since the role of labor values is not primarily one of explaining the movements of prices.[1] Labor values – when based on Richard Stone's SNA, as done in this book – are nothing counterfactual, but can be calculated and used as measures of the total labor costs or of labor productivity characterizing the various commodities produced in the economy. Such a pragmatic, application-oriented approach to the LTV does not exclude the view however, that labor values, viewed as representation of abstract labor, can be used as in Marx's (1977) Capital, Vol. I, also from a philosophical perspective, as a concept with which one can interpret and analyze the socio-economic relationships (of classes) of human beings in a certain society at a certain time.

Prices of production, our second accounting measure (besides labor values), based on the assumption of a uniform rate of profit (and of wages) between industries and a given numéraire commodity, can also be derived from Stone's input–output methodology and thus be determined empirically and compared with the profit rate differentials that actually exist in the economy. At the theoretical level they can be used as long period prices for modeling capitalist competition and induced directions of technological change among other things. They are also defined by an application of the Perron–Frobenius theory, with the uniform rate of profit given through a simple transformation of the dominant eigenvalue that this theory investigates.

While labor values are characteristics of the sphere of production and devoted to an understanding of what is going on there, prices of production apply to the sphere of circulation and the distribution of net national product. Labor values may be useful in understanding the conflict between capital and labor in the transformation of commodity inputs into commodity outputs, while production prices may be of use for the understanding of capital flows between the sectors of the economy, of investment decisions of firms, and for the comparison of the newest with the average and the oldest production techniques and – at the macro-level – for the study of the conflict about income distribution between capital and labor.

With these two instruments, labor values and production prices (appropriately generalized), we thus have concepts at hand that in specific ways allow the analysis of the production and the distribution of commodities in capitalist economies from supply-side and long-period perspectives. Keynesian effective demand problems concerning the short-run evolution of the economy and the business cycle need to be integrated into such a supply-side framework, a task that is not really approached in this book. It is however an implication of the book that prices of production may be considered as an unnecessary intermediate step in this reflection of the relationship between production-based labor values and average market prices, in particular when the latter are measured in wage-units (as the "real" magnitudes underlying Keynes' (1936) theory of the business cycle).

Authors working in the Neoricardian tradition have indeed produced little evidence that prices of production are point attractors of market prices and that uniform

[1] Though it may be an empirical outcome that total labor costs are a fairly significant component in actual price formation.

profitability is a tendency in capitalism in its earlier or later phases. We will see in part II of the book that the latter may be very questionable (if stock-flow relationships are taken properly into account). Moreover, as part III of the book will show, the theoretical stability of the Classical long-period prices is far from being well proven. We consequently conclude here – until the opposite is clearly shown – that prices of production may represent an unnecessary detour in the study of the results of the capitalist circulation process and that the direct link between labor values and actual average market prices may be the better choice for theoretical as well as empirical investigations (see Chap. 3 in particular) than the addition of prices or production to this link, since the latter may be irrelevant for the actual choice of technique under capitalism.

The second set of tasks described at the beginning concerns dynamics, both on the micro- as well as on the macro-level. It may be claimed with respect to the above that the Classical authors would have created the Perron–Frobenius eigenvalue theory if they, like Marx, had attempted to go extensively into the mathematical literature that existed at their time. Similarly, they could have established the Lotka-Volterra mathematics underlying the investigation of population dynamics if they – in particular Marx – would have attempted to formalize the Classical ideas on the dynamics of market prices and capital flows and – on the macro-level – Marx's general law of capitalist accumulation by the mathematical formulation of these laws of motion.

In Classical ruthless competition, financial capitals are moving into the sectors with a rate of profit higher than the average and are leaving the sectors that are characterized by the opposite. But in doing so they increase the supply of commodities in the profitable sectors and reduce it in the unprofitable ones. Prices will therefore tend to fall in the profitable sectors and rise in the latter ones, thereby providing a check to this type and direction of capital flows. From a predator–prey perspective, price reactions counteract profitability levels and are thus the predator in this Classical approach to competition, capital mobility, the law of demand and supply and their consequences.

Price-determined profitability acts positively on supply and supply acts negatively on prices, which is exactly the Lotka-Volterra predator–prey mechanism, applied to a multi-sectoral economy and thus to microeconomic price and quantity adjustment processes. At the macro-level, in the theory of employment and income distribution, we know of course from Goodwin's (1967) formalization of Marx's general law of accumulation that the roots of his modeling of this law are indeed given by the Lotka-Volterra predator–prey dynamics, with employment as the prey, acting positively on the wage share, and with the wage share the predator, acting negatively on investment and thus on future employment possibilities of the workforce. This is again a cross-dual or cross-over type of dynamics with one positive feedback mechanism and one negative feedback channel, when looked at from this general perspective.

We thus have the result that, from a mathematical perspective Classical value and price accounting are intimately related with the theory of non-negative matrices (or more generally, matrix bundles) and the eigenvalue theory that can be based on them, while the Classical theory of competition between industries and between labor and capital shows significant analogies to Lotka-Volterra types of dynamics, and thus not only of the overshooting predator–prey type, but also with respect to other types of interacting population dynamics.

The Classical approach to economics thus not only supplies us with two – from the definitional point of view – clear-cut factual accounting schemes for the investigation of the tendencies that govern the capitalist mode of production and circulation, but also provides us with micro and macro laws of motion around these accounting schemes (when appropriately formulated). The total labor costs accounting schemes, in addition, remind us of the fact that only labor is productive (as the only really indispensable factor of production) and they provide us with an analytical instrument which allows us to detect the tendencies that characterize the capitalist mode of production.

On this background, this book is structured as follows: In its part I, we define labor values for general models of production and show that this type of definition not only mirrors the factual cost-accounting behavior of firms, but is also – which came as a surprise – closely related to the principles that characterize Stone's input–output methodology when applied to measures of total labor costs of produced commodities in general models of production.

This starting point for the investigation of the Classical concept of labor values should make sense to all schools of economic thought and thus not only be of interest to scientists working on the so-called Marxian transformation problem (which is an issue only when labor values are interpreted as some sort of physical magnitude like energy in place of considering them as a mathematical definition, the usefulness of which must be proved by mathematical theorems and their empirical examination). While Chaps. 1 and 5–7 are based on work published in the 1980s, Chaps. 1–4 show that this earlier work is still relevant for the current debate on labor values and measures of total labor costs.

Part II considers the Classical theory of competition in the form of the long-period prices this theory starts from. It provides – in Chap. 8 – an introduction to the results implied by Classical ruthless competition, the perfectly competitive prices of production and the theorems this second Classical accounting scheme gives rise to. Since these pricing procedures and the wage–profit relationship they imply have already been investigated in numerous articles and books, we can be brief here. We therefore concentrate in the remaining chapters of part II on two issues, namely: on the usefulness of Sraffa's concept of basic commodities in general models of production and on the uselessness of his concept of a Standard commodity of a given input–output structure, by which the theory of income distribution is in fact not simplified, but obscured.

Part III is on Classical microdynamics and starts this topic in fact from a Walrasian perspective. Walras (1954) has indeed – as we shall see there – reformulated Classical cross-dual microdynamics between prices, profitability and quantities supplied, at the level of production economies, by way of a tâtonnement process between firms, households and the auctioneer. This dynamic process is reformulated by means of differential equations in Chaps. 13 and 14 and shown to be of fairly stable nature if a further aspect of actual market dynamics is taken into account, namely that derivative forces, showing the influence of the direction of change of the interacting imbalances, also matter in this abstract formulation of the forces of competition in capitalist economies.

Chapter 15 applies these considerations to the Classical von Neumann model and the theory of production prices it implies. We there find a clear indication for the proper working of the Lotka-Volterra predator–prey mechanism, and can also apply its features that concern the extinction, in our context, of economic processes and marketed commodities. Chapter 16 finally adds Keynesian dual dynamics to the Classical cross-dual ones, which with respect to quantities is of dynamic multiplier type and with respect to prices uses iterated markup pricing procedures.

The overall outcome of part III of the book is that Classical cross-dual dynamics can be successfully formalized in mathematical terms (and also be extended by Keynesian short-run forces). However, these dynamical structures in no way depend on the assumption that the restrictive concept of prices of production is to be used as their center of gravity. There may instead exist many reasons that differentiate average profit rates also in the longer run so that average market prices are to be confronted with a long-period price accounting scheme that is more flexible than the conventional formulation of prices of production.

In part IV we reconsider the Classical growth cycle model of Goodwin (1967) from various perspectives, concerning its structural instability, endogenous aspirations in pricing procedures, low-skilled and high-skilled labor solidarity – or cooperations of the latter group with capital in place of labor. We also reformulate the Goodwin growth cycle as a limit cycle that surrounds and tames explosive forces around the steady state caused by the conflict of labor and capital over income distribution and we confront – as in Solow (1990) – this overshooting, but stable dynamics with empirical phase plots of the Goodwin growth cycle type for various OECD economies as well as – in a new paper, see Chap. 21 – with modern econometric investigations (for the US economy) of the long phase cycle that is implied by this cross-dual cycle generator. Finally, its relationships to a general model of Keynes-Wicksell type are explored in Chap. 22.

Summing up the preface, we stress that labor values can be investigated in their role to reflect what is happening in capitalist competition and the technological dynamics it implies by contrasting them directly with average market prices (in terms of the wage-unit as in Keynes General Theory). Prices of production (with their strict assumption of a uniform rate of profit) may be a useful intermediate step, at least when reformulated in an appropriate way, yet this is currently far from being obvious. This holds true in particular when they are formulated as in Sraffa (1960) from a purely academic physical perspective and not as in Bródy (1970) from an applicable Leontief approach at some intermediate level of aggregation.

If prices of production are not close to market prices, their role for analyzing technical change may indeed be very limited. It may therefore well be the case – as Farjoun and Machover (1983) indirectly argue – that Samuelson's (1971) eraser principle does in fact not apply to the usefulness of labor values, as it is repeatedly stated in Steedman (1977), but instead to the alternative accounting concept of prices of production — for which no empirically relevant application may exist.

In closing, I thank Reiner Franke for supplying material for the empirical sections of Chaps. 3 and 8. Roberto Veneziani has read part of the manuscript and contributed many valuable suggestions. Christian Proaño has done a marvelous job in formatting the manuscript according to the style files supplied by Springer Verlag. Finally, the chapters of the book owe much to controversial and non-controversial discussions with colleagues, too numerous to be mentioned in person, working in the areas covered by this book, in particular very recent ones at two conferences on Marx's Capital in Bristol upon Avon and in Kingston-upon-Thames, as well as at the 10th annual conference of the Association of Heterodox Economists in Cambridge in 2008. Of course, the usual caveats apply.

Bielefeld Peter Flaschel
July 1, 2009

References

Bródy, A. (1970). *Proportions, Prices and Planning*. North Holland: Amsterdam.
Farjoun, E., & Machover, M. (1983). *Laws of Chaos*. London: Verso Edition and NLB.
Flaschel, P., Franke, R., & Veneziani, R. (2008). 'Neoricardian Theory and the Measurement of Prices of Production': An Alternative Approach. Bielefeld University: CEM working paper.
Goodwin, R. (1967). A growth cycle. In C. H. Feinstein (Ed.), *Socialism, capitalism and economic growth* (pp. 54–58). Cambridge: Cambridge University Press.
Han, Z., & Schefold, B. (2006). An empirical investigation of paradoxes: reswitching and reverse capital deepening in capital theory. *Cambridge Journal of Economics, 30*, 737–765.
Keynes, J. M. (1936). *The general theory of employment, interest and money*. New York: Macmillan.
Marx, K. (1977). *Capital, Vol. I–III*: London: Lawrence and Wishart.
Solow, R. (1990). Goodwin's growth cycle: Reminiscence and rumination. In K. Velupillai (Ed.), *Nonlinear and multisectoral macrodynamics. Essays in honour of Richard Goodwin* (pp. 31–41). London: Macmillan.
Steedman, I. (1977). *Marx after Sraffa*. London: New Left Books.
Samuelson, P. (1971). Understanding the Marxian notion of exploitation: A summary of the so-called transformation problem between Marxian values and competitive prices. *Journal of Economic Literature, 9*, 399–431.
Sraffa, P. (1960). *Production of commodities by means of commodities*. Cambridge: Cambridge University Press.
United Nations. (1968). *A system of national accounts*. New York: Studies in Methods, Series F, No.2, Rev.3.
Walras, L. (1954). *Elements of pure economics*. London: George Allen and Unwin.

Acknowledgements

The following journal articles have been used as foundations for various chapters of the book. The chapters have to some extent been reworked in the light of new developments in their subject areas and are not necessarily identical in their titles to the ones of the original papers. I thank the editors/publishers of these journals, the Econometric Society (Chap. 5), Elsevier (chaps. 1, 7, 14, 16, 18, 22), JITE (Chap. 12), Routledge (Chap. 21), Wiley-Blackwell (Chaps. 11, 15, 20), for permission to reuse the original articles in this book on Classical micro- and macroeconomics.

- Chap. 1: The so-called "transformation problem" revisited. A comment. *Journal of Economic Theory*, 33, 1984, 349–351.
- Chap. 5: Actual labor values in a general model of production. *Econometrica*, 51, 1983, 435–454.
- Chap. 6: The derivation and comparison of employment multipliers and labour productivity indexes using monetary and physical input-output tables. *Economics of Planning*, 16, 1980, 118–129.
- Chap. 7: Input-output technology assumptions and the energy requirements of commodities. *Resources and Energy*, 4, 1982, 359–389.
- Chap. 9: On two concepts of basic commodities for joint production systems. *Zeitschrift für Nationalökonomie*, 42, 1982, 259–280.
- Chap. 11: The Standard Commodity and the theory of income distribution. *Australian* Economic Papers, 23, 1984, 123–129.
- Chap. 12: Sraffa's Standard Commodity: No fulfillment of Ricardo's dream of an 'invariable measure of value'. *Zeitschrift für die gesamte Staatswissenschaft*, 142, 1986, 588–602. Reprinted in: M. Blaug (Ed.): Pioneers in Economics, 44: Piero Sraffa (1898–1983). Aldershot: Edward Elgar, 1992, 120–134.
- Chap. 13: Dressing the emperor in a new dynamic outfit. *Annals of Operations Research*, 37, 1992, 33–49.
- Chap. 14: Stability - independent of economic structure? A prototype analysis. *Structural Change and Economic Dynamics*, 2, 1991, 9–35. Reprinted in: D.A. Walker (Ed.): The Legacy of Léon Walras. Northhampton, MA: Edward Elgar, 2001.

- Chap. 15: Classical and neoclassical competitive adjustment processes (with W. Semmler). *The Manchester School*, 55, 1987, 13–37. Reprinted in: D.A. Walker (Ed.): Equilibrium. Critical Ideas in Economics, Volume I. Northhampton, MA: Edward Elgar, 2000, 37–61.
- Chap. 16: On the integration of dual and cross-dual adjustment processes in Leontief Systems (with W. Semmler). *Ricerche Economiche*, 42, 1988, 403–432.
- Chap. 17: Some stability properties of Goodwin's growth cycle. A critical elaboration. *Zeitschrift für Nationalökonomie*, 44, 1984, 63–69.
- Chap. 18: A model of cyclical growth using an endogenous aspiration factor (with M. Krüger). *Ricerche Economiche*, 38, 1984, 588–612.
- Chap. 19: Cooperation vs. solidarity in a complete version of the classical growth cycle. In: A. Riedl, G. Winckler and A. Wörgötter (eds.): *Macroeconomic Policy Games,* Physica, 1995, 57–85.
- Chap. 20: The Classical growth cycle: Reformulation, simulation and some facts (with G. Groh). *Economic Notes*, 24, 1995, 293–326.
- Chap. 21: The Classical growth cycle after fifteen years of new observations (with G. Groh, G. Kauermann, T. Teuber). In: P. Flaschel and M. Landesmann (eds.): *Mathematical Economics and the Dynamics of Capitalism. Goodwin's Legacy Continued*. London: Routledge, 2008.
- Chap. 22: Classical dynamics in a general model of the Keynes-Wicksell type (with R. Sethi). *Structural Change and Economic Dynamics*, 7, 1996, 401–428.

Contents

Part I
Labor Values: Theory and Measurement

This part considers the sphere of commodity production and the question which tools can be of use to analyze its evolution in its own right, a very dynamic process of creative destruction as Schumpeter has characterized it. We therefore are here abstracting from the process of commodity circulation and the explanation or theory of the price signals that drive this latter process. On this basis we assert that only Classical labor values can be of use to analyze the dynamic processes of production and technical change in depth.[1]

We share in this field the opinion of Keynes (1936, p. 213/4),[2] who formulated a pragmatic position with respect to production, when he wrote:

> It is preferable to regard labor, including, of course, the personal services of the entrepreneur and his assistant, as the sole factor of production, operating in a given environment of technique, natural resources, capital equipment and effective demand.

In contrast to Marx's Labor Theory of Value (LTV), he however uses prices divided by the wage unit, as the real unit underlying his theory of effective demand. We will see in Chap. 3 that it may indeed be meaningful to consider labor values and prices measured in the wage unit side by side, in particular, since the latter are an upper estimate of labor values in general. However, labor values (total labor costs) are more closely related to the evolution of the technological structure and thus serve to measure its historical phases in a better way than Keynes' prices in terms of the wage-unit, where income distribution is involved to a significant degree. Keynes' measure of real magnitudes may be useful in demand constrained n-sectoral economies that are using marginal cost pricing principles. This topic however concerns the sphere of commodity circulation and thus not production in its own right. We take here the view that the traditional approach to defining labor values (appropriately generalized) is the more fruitful one, regarding changes in the sphere of production, and it is firmly rooted in general input-output routines established by Richard Stone, see United Nations (1968), as part of a complete System

[1] This part also considers in its Chap. 3 a measure of total energy costs, but we believe that such measures are of a partial usefulness only and are not related very much with the core relationship within capitalism, i.e., the conflict between capital and labor about production conditions and income distribution.

[2] The General Theory of Employment, Interest and Money. New York: Macmillan.

of National Accounts, as we shall see. Part I therefore provides a production-based approach to labor values (total labor cost or – in reciprocal form: indexes of labor productivity). This approach is in general distinct from an alternative measure of total employment effects, the so-called employment multipliers, which can be negative in joint production economies in a meaningful way. Our concept of labor values (total labor costs) does not allow for this, but allows instead for Classical propositions of the LTV also in quite general models of production (and for a variety of price-theoretic approaches). It is however not directly oriented towards a solution of the so-called "transformation problem", an issue that we consider to be secondary in nature in the relationship between average actual prices and average total labor cost, the "real" behind the "nominal" as part of Stone's Systems of National Accounts and the categories it uses as real magnitudes.

Labor value accounting therefore primarily provides a scientific framework that may allow to understand the results of capitalist production. This interpretation of the role of the LTV is quite independent of whether and how labor values can be transformed into price of production (or any other price system) such that certain aggregate expressions remain unchanged under such a transformation. This latter view runs into the danger that a formal scientific definition that attempts to characterize produced commodities qualitatively and quantitatively in an applicable way is reinterpreted as "object" inherent in these commodities, a substance that can transferred between the firms which constitute the considered economy.

In Chap. 1 we provide a sketch of one interpretation of the LTV, primarily concerning the understanding of Marx's rate of exploitation, as the fundamental entity behind profit creation. It shows in addition that central ratios based on labor value accounting may provide measures for the systematic component in their corresponding price ratios. Chapter 2 gives a survey on approaches to the LTV that can be classified as single or dual systems. Its main conclusion is that a synthesis between the new interpretation and traditional labor value measurement á la Stone can provide a fruitful approach to an extended LTV. This gives labor values an independent role in National Accounting and separates them methodologically from their potential use as price indicators and their interpretation from a purely price-theoretic perspective.

Chapter 3 shows factual uses to which such labor value measurement can be put, concerning technical change and sectoral productivity growth, in contrast to what the United Nation is nowadays proposing as sectoral measures of labor productivity in its System of National Accounts. The chapter also provides important propositions on the relationships between types of technical change, actual prices measured in wage-units and labor values. It thus in particular shows that Sraffian prices of production are not needed to understand the interrelationships between commodity production and commodity exchange and are therefore secondary for a proper understanding of the LTV.

In Chap. 4 we show by means of examples from Steedman (1977) that neither pure joint production nor fixed capital create problems for labor value accounting from a proper input-output perspective, since labor values are not just prices of production at a zero rate of profit in general. Instead, labor values are related to the full cost accounting principles of firms where it is well-known that physical data are in general insufficient to perform such a task.

General joint production models are considered in Chap. 5 where it is shown that many propositions of the LTV that hold for square single-product systems can be meaningfully generalized to these economies. This task is solved by using a price system that is not based on Sraffian prices of production accounting which again shows that the LTV is not dependent on the very special Sraffian approach to the determination of long-period prices. Chapter 6 relates these issues again to Stone's formulation of a SNA and its consideration of input-output techniques and the measures that can be derived from them.

Chapter 7, finally, applies these general input-output accounting procedures to a commodity called "energy" and shows how total energy consumption and total energy costs can be calculated in joint production systems. It shows that the definition of such magnitudes is not restricted to the case of "labor", but is also meaningful for other primary factors of production. The differences between "labor" and "energy" are, however, that energy is a produced commodity (which labor is not), that only labor is truly indispensable for social reproduction, that the commodity "labor" is traded between interacting social groups and that there is awareness of the conditions of capitalist reproduction only within this particular exchange relationship.

Summing up, this part of the book shows that definitions of labor values not based on and related to input-output methodology and its considerations of labor productivity are of a very questionable nature. This concerns all approaches which attempt to solve the transformation problem by an appropriate static or temporal description of labor values that make them more or less an outcome of the sphere of commodity circulation in place of commodity production. Our finding therefore is that the traditional approach to labor values – appropriately extended to general models of production by means of Stone's input-output methodology – is the only approach allowing to detect the 'real' evolution of capitalism behind the nominal interactions on its surface, the sphere of commodity circulation. In principle, we believe, that this result is compatible with the approaches suggested by Foley, Duménil and Lévy and Shaikh, though these authors consider these issued from their own and to a certain degree different angle.

By contrast, Steedman's claim of the redundancy of labor value calculations (for prices of production calculations) does not at all imply that labor values are completely redundant as this part of the book shows. In the next part we will instead see that prices of production accounting procedures may in fact be the redundant element, as far as the sphere of commodity circulation is concerned.

References

Keynes, J. M. (1936). *The general theory of employment, interest and money*. New York: Macmillan.

Sraffa, P. (1960). *Production of commodities by means of commodities*. Cambridge, UK: Cambridge University Press.

Steedman, I. (1977). *Marx after Sraffa*. London: NLB.

United Nations. (1968). *A system of national accounts*. New York: Studies in Methods, Series F, No.2, Rev.3.

Chapter 1
The So-Called "Transformation Problem" Revisited

> Thus, even if the transformation problem could be solved mathematically, the resulting model would not only rest on the fallacious assumption of the uniformity of the rate of profit, but would actually be inferior to the original unmodified model (of Capital, Vol.I, P.F.) in respect of prices (Farjoun and Machover 1983, p. 134).

1.1 Introduction and Overview

This chapter on the "transformation problem between labor values and prices of production"[1] shows that Lipietz's analysis of the Marxist transformation procedure represents but a simple, though useful reinterpretation of obvious mathematical consequences of a standard Sraffa model – by making appropriate use of its known degrees of freedom. Labor values are not involved in this new interpretation of conventional prices of production. A proposal is therefore made how the role of labor values can be investigated further in such a framework, from the perspective of Marx's "Capital" and on the basis of Lipietz's theorem and its reinterpretation of the "value of labor power". Our additions to Lipietz' definitional procedures suggest that important labor value aggregates such as the average value rate of profit and the value rate of exploitation may be of use in analyzing the systematic consequences of changes in the sphere of capitalist production, while the effects of the actual price dynamics that drive these changes (not yet accounted for by total labor costs) may be unsystematic and may therefore represent distortions of secondary importance. The issues considered here will be further investigated in the next chapters where also Marx's (1954, p. 48) view that labor values are measures of labor productivity, and thus also important in their own right, is explored from the perspective of Richard Stone's System of National Accounts. From this perspective, labor values concern the accounting side of an economy, constructed from the observed dynamics of nominal magnitudes in order to understand in a conventional way or in a Marxian sense what is going on behind the surface of nominal magnitudes.

---

[1] This chapter provides an extended version of Flaschel's (1984) comments on Lipietz (1982), cf. also the comments on his paper by Duménil (1984) in the same Journal and Foley's (1982) contribution to these issues.

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_1,

1.2 Lipietz's Theorem

In the Journal of Economic Theory, Lipietz (1982) has presented a new version of a "Marxist transformation theorem". This chapter argues that Lipietz's theorem is contained in a conventional Sraffa model in a mathematically trivial way. This does not imply that his idea how to reformulate Marx's transformation problem must be regarded as useless. Indeed, I find his idea convincing or at least worthy for elaboration. Yet, Lipietz's mathematical formulation obscures what in fact has been achieved by him. Furthermore, if – as we shall see – the transformation problem becomes a trivial exercise in definitions, one is asked to point to at least one useful application of this exercise. Such an application will be sketched at the end of this chapter.

Let the symbols $A, l, I, x, y = x - Ax$ be defined as is customary in input–output analysis ($x = Y$ in Lipietz (1982)), i.e., we start from a simple Sraffa input–output system with given vectors of gross and net outputs x, y. It is assumed that the input–output–matrix A is productive. If wages w are paid ex ante we get instead of Sraffa's prices of production the price equations

$$p = (1 + r)(pA + wl), \quad py = lx. \tag{1.1}$$

It is well known that eq. (1.1) can be uniquely solved for each given $w \in [0, 1]$ with regard to prices p and the rate of profit r, in an economically meaningful way (cf. also (H2)–(H2") in Lipietz (1982) and note that his symbol p^* in (H2) – and in his following text – should be replaced by p (or $v.v.$) to clear up the formulae employed by him). Solving (1.1) for $w = 1$ $(r = 0)$ defines labor values $v = vA + l, vy = lx$ with regard to which the transformation problem then has to be re-formulated.

In his transformation theorem, Lipietz (1982, p. 78) takes the vector y and wages $w \in (0, 1)$ as given and defines – as I interpret his formulations – a capitalist redistribution of value by a solution p of (1.1) with respect to these data. That such a solution exists and is uniquely determined has already been noted to be a well–known fact. Furthermore, Sraffa's prices (1.1) of course fulfill

$$r(pAx + wlx) = py - wlx = vy - wlx, \tag{1.2}$$

i.e., profits, of course, must equal (or are a redistribution of) surplus values if w is interpreted to represent Marx's "value of labor power". Finally, if the rate of surplus value e is defined by $e = (1 - w)/w$, there immediately follows from (1.2)

$$r = \frac{(1 - w)lx}{pAx + wlx} = \frac{1 - w}{w} \frac{wlx}{pAx + wlx} = e\frac{V}{C + V}, \tag{1.3}$$

i.e., the third assertion of Lipietz's theorem.

1.3 Labor Value Ratios: The Systematic Component in Their Price Expressions?

We conclude that Lipietz' theorem is but a simple reinterpretation of a modified conventional Sraffa model (see Sraffa (1960, Chap. 3)) by making appropriate use of its degree of freedom w. This corresponds to Robinson's (1969, pp. 333/4) proposal that Marx's rate of surplus value e should best be measured by the ratio profits/wages, i.e., by $(1-w)/w$, which also implies the above redistribution property. Yet, what is the use to which such a reinterpretation of Sraffa's prices – besides redefining certain Marxian aggregates – can be put?

With regard to Marx's aims this cannot be demonstrated by Lipietz's final equation on p. 80, since this equation is but a formal reformulation of (1.1) in terms of $e = (1-w)/w$ and $v = l(I-A)^{-1}$, the conventional definition of labor values, the independent use of which we are looking for. This equation consequently does not leave the sphere of Sraffa's price calculations. Lipietz's in our view meaningful reinterpretation of the value of labor power (in particular, if workers are allowed to save) by means of the wage rate (the wage share) of system (1.1) can, however, be supplemented by the value rate of profit ρ, the central link in Marx's own transformation procedure in a meaningful way. This rate is to be defined as follows

$$\rho = \frac{v(I-A)x - wlx}{vAx + wlx} = \frac{(1-w)lx}{vAx + wlx} = \frac{e}{vAx/wlx + 1}, \tag{1.4}$$

$$e = \frac{1-w}{w}, \quad vy = lx \tag{1.5}$$

For the relative deviation between the price rate and the value rate of profit we easily obtain from (1.1), (1.2), and (1.4) the expressions

$$\frac{r-\rho}{\rho} = \frac{(v-p)Ax}{pAx + wlx} = \frac{(v-p)x}{pAx + wlx} \quad [= 0, \text{ if } x = \alpha y, \alpha > 0] \tag{1.6}$$

This in our view represents the fundamental formula on the basis of which Marx's value theory of the price rate of profit r, i.e., its deviation from the value rate ρ, and thus the transformation problem should be evaluated further – by means of suitable theoretical *as well as* empirical examinations of the difference shown by (1.6).[2] Hence, Marx's central aim can be examined further and can in particular be subjected to test by means of the labor values or productivity indexes v as measured by input–output analysts (see Gupta and Steedman (1971) for an example of such a measurement), indexes which play no role in Lipietz's rate of profit formula (1.3). The real issue for a Marxian analysis of profits, therefore, is to test whether the production–based rate ρ can provide a proxy for the uniform (or average) rate of profit or not. Lipietz's redefinitions in this respect only serve to pose the problem anew.

[2] The above result also holds for all average price rates of profit in place of the uniform rate of profit we have considered so far.

We get that the price and the value rate of profit (for any given price vector p with $py = lx$) in fact differ only by unsystematic historically determined price-value deviations from each other which tend to neutralize themselves in the aggregate at least to a certain degree, see Chaps. 3–5 for more details. The systematic forces of capitalism primarily concern the evolution and laws of motion of production, and not so much the many interacting (opposing) forces that determine actual price dynamics. A rising organic composition of capital vAx/wlx will therefore in general not only lower the value rate of profit, but also the price rate of profit if not offset by a rising rate of exploitation e, see Farjoun and Machover (1983, Chap. 7) on how such an argument can be made more precise from a probabilistic point of view. Note here also that their argument that actual prices and their Marxian ratios should be investigated form the viewpoint of Marxian labor value categories is shared by the chapters that follow, since all of the above does not depend on the use of a production price system which may be a very hypothetical and restrictive (micro or meso) construct in the globalized world we are experiencing now in the age of the internet.

Supplement: If workers do not save and their yearly consumption is given by C_w we can define – in correspondence to the rate e – the value rate of exploitation by:

$$\epsilon = \frac{1 - vc_w}{vc_w}, \quad c_w = C_w/lx$$

and compare it with the price rate of exploitation $e = (1-w)/w$ we have used in the above calculations. Since there must hold $pc_w = w$ then, we get for their difference:

$$e - \epsilon = \frac{(v-p)c_w}{vc_w pc_w} \quad [= 0, \text{ if } c_w = \alpha y, \alpha > 0]$$

We thus also get that the price and the value rate of exploitation (for any given price vector p with $py = lx$) differ only by unsystematic, historically determined price-value deviations from each other which may neutralize themselves in the aggregate to a larger degree. One may therefore claim that the systematic forces behind an increase in the price rate of profit are the forces that lower either v or c_w (or both) or that increase the labor time the worker family has to work for their consumption bundle c_w. The consideration of the value rate of exploitation therefore directs our view to central causes of increasing exploitation which are not equally well visible if this ratio is expressed in money terms as the actual profit share divided by the actual wage share, as it was discussed above.

1.4 Conclusions

We have shown in this brief chapter how central aggregates of Marx's theory of capitalistic reproduction can be defined within a system of Sraffian production prices and also for all actual price vectors (fulfilling $py = lx$ for later comparison with labor value analogues). We have moreover shown that the systematic changes in profit and exploitation rates should be represented by labor value expressions rather than by price expressions, due to the chaotic nature of the interacting processes of commodity exchange in space, time and with respect to contingencies. We thus regard the evolution of labor value (or total labor cost) expressions as capturing the essence and the inertial laws of motion of capitalism, while the corresponding price expression are to a larger degree chaotic in their daily worldwide motions, an arbitrariness which may however only be of a secondary degree as far as deviations between the considered price and value aggregates are concerned.

Labor power is the only commodity which (in a systematic way) is not produced by firms and where no profits accrue in the course of its production (in contrast to slavery). Moreover labor power is indispensable for social reproduction, while all other commodities can in one way or another be substituted through each other. Reducing the value of labor power – through a lengthening of the workday (of families), a reduction in workers per hour consumption or most importantly: through technological change – therefore is the central mechanism by which the average rate of profit of an actual economy can be increased.

For further thoughts on such issues the reader is referred to the following chapters and their discussion of the role of labor values for an explanation of the forces that drive technical change in a capitalist economy. We here state already however that it may well be that the so-called "Marxian transformation problem" can be replaced by a System of National Accounts, calculated in Marxian labor time expressions as the underlying "real structure" to be used for the explanation of the ways actual price-quantity interactions are determining the accumulation and innovation dynamics of capitalist economies.

References

Duménil, G. (1984). The so–called 'transformation problem' revisited: A brief comment. *Journal of Economic Theory,* 28, 340–348.

Farjoun, E., & Machover, M. (1983). *Laws of chaos*. London: Verso Edition and NLB.

Flaschel, P. (1984). The so–called 'transformation problem' revisited: A comment. *Journal of Economic Theory,* 28, 349–351.

Foley, D. (1982). The value of money, the value of labor power and the Marxian transformation problem. *Review of Radical Political Economy,* 14, 37–47.

Gupta, S., & Steedman, I. (1971). An input–output study of labour productivity in the British economy. *Oxford Bulletin of Economics and Statistics,* 33, 21–34.

Lipietz, A. (1982). The so–called 'transformation problem' revisited. *Journal of Economic Theory,* 26, 59–88.

Marx, K. (1954). *Capital. A critique of political economy* (Vol. 1). New York: Lawrence + Wishart.

Sraffa, P. (1960). *Production of commodities by means of commodities*. Cambridge, UK: Cambridge University Press.

Chapter 2
Baseline Approaches to the Labor Theory of Value

> A scientific theory cannot confine itself to dealing with what is directly observable, to the exclusion of abstract theoretical concepts. The attempt to expunge theoretical concepts, such as labour-content, from economic theory, leaving only directly observable quantities, such as prices, is a manifestation of instrumentalism, an extreme form of empiricism, which is destructive of all science. Without the concept or labour-content, economic theory would be condemned to scratching the surface of phenomena, and would be unable to consider, let alone explain, certain basic tendencies of the capitalist mode of production (Farjoun and Machover 1983, p. 97).

2.1 Introduction

The dominant price theory from the perspective of models of general equilibrium is in terms of rigor the Arrow-Debreu General Equilibrium Theory (GET) of so-called (neoclassical) perfect competition. The most developed framework for national accounting is the System of National Accounts (SNA) of the United Nations in its current form. Both approaches towards a classification and analysis of microeconomic structures flourished in the 1960s and 1970s, but lost in importance thereafter, in the first case, due to the internal limitations of GET in the fulfillment of Smith's conjecture on the working of market economies and, in the second case, due to a dilution of the current SNA as a rigorous and coherent approach to input–output structures within the System of National Accounts as it was originally formulated by Richard Stone and his research group.

Moreover, the Arrow-Debreu world pays little attention to the need for a System of National Accounts (though there have been some attempts to combine these two approaches in the study of the "real" magnitudes usable to characterize market economies).[1] It is therefore basically a purely "nominal" approach,[2] despite the fact that it is in fact solely a theory of relative prices and thus faces the problem

[1] See Fisher and Shell (1972) for a prominent example.

[2] The expression "nominal" is here used in contradistinction to the concept of "real" ("quantity"-oriented) magnitudes of national accounting systems.

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_2,

of the choice of a numéraire, which however is not supposed to reflect something truly "real". It therefore seems to suggest that there is nothing "real" behind the "nominal", not even as a theoretical construction that can help to understand the movement of "nominal" magnitudes. In addition to its pure "surface" orientation, GET pursues a theory of competition that does not reflect any competition at all, since all individuals and firms are isolated utility or profit maximizing price-takers without any interaction with each other.

The United Nations' System of National Accounts (SNA), now from 1993, scheduled to be revised again in 2008 and based on Stone's SNA, is a rigorously developed classification system for the economic activities of a whole economy. It considers many complexities of real life, as for example joint production, and attempts to construct from detailed economic data, not only stock and flow matrices that can characterize the evolution of economies, but also real magnitudes like real GDP, physical input–output tables, and labor productivity measures. Quite obviously, its constructions of real magnitudes have to be considered as theoretical concepts intended to increase our understanding of what goes on in actual economies behind their nominal categories and not as representing something "real" in the sense that we can find it in the real world. The United Nations' System of National Accounts provides therefore a language (with precise qualitative and quantitative meanings) with which we can discuss the progress or regress in the (world) economy.

In my investigation of the United Nations' Systems of National Accounts I have come to the opinion that this system is more Classical than Neoclassical in nature, where Classical here simply means that its concepts stress more the evolution of average magnitudes than of marginal ones obtained under the assumption of perfect competition. Classical theory, moreover, can be characterized as providing an approach to indeed ruthless competition, where households and more significantly firms interact (sometimes with brute force) such that all differential advantages are swept away. The result are so-called prices of production which are conceived of as the centers of gravity of market prices and which provide some sort of long-period moving averages for the many concrete pricing actions that take place in daily economic life, a process assumed to be working already in this way at the time of the industrial revolution and maybe even with more ruthless sectoral inflows and outflows of capital nowadays. The theory of ruthless Classical competition and its theoretical gravity concept, the prices of production, is one of the building blocks from which this chapter will start its investigations. The other building block will be Marx's labor theory of value which in my interpretation has the basic objective of finding the "real" or the "essence" behind the surface of nominal magnitudes, from a Marxian perspective,[3] by way of the qualitative concept of "abstract labor" and its quantitative expression "labor content", measured by the average amount of labor time that is "embodied" in the various commodities (in the sense of full-cost accounting in terms of labor time spent on average in the production of commodities).

[3] But based on Marxian categories.

We stress here that we take Classical prices of production only as one point of reference (besides actual average prices in terms of the wage unit for example), the properties of which have to be compared with those of the labor values, including the theoretical links that exist between these two types of theoretical accounting systems. Our approach to labor values is however independent from this type of comparison and in fact a purely factual one which needs as inputs the production data (the depreciation of stocks and the flows) of the year that is under consideration and also actual prices in some places (when joint production, heterogeneous labor and the like are taken into account). We thus use only production data (and some price data in addition) for a given economy in a given year in our formulation of the "labor time directly and indirectly embodied" in the various commodities. These data can of course also be supplied from some equilibrium approach like the von Neumann model and the like which then only means that we impute them into this type of framework as an additional tool of analysis.

Marx's labor theory of value has of course many qualitative and quantitative aspects which cannot be treated adequately in a single chapter.[4] The aspects of it that I will stress and investigate is that its methodological status is that of a Classical System of National Accounts, with the basic objective of analyzing and explaining what really goes on in a capitalist market economy. As the UN's SNA it therefore aims at categorizing in real terms what the (dis-)achievements of such an economy actually were in a certain year, not in terms of the very limited concept of Pareto efficiency, but in terms of real growth, productivity progress, exploitation, increasing or decreasing tensions between capital and labor and the like. It is thus not at all of the status of a price theory as Samuelson and others have claimed it to be over and over again, a status that nobody would seriously associate with the SNA of the United Nations as established by Stone.

The aim of the presentations in this part of the book is to demonstrate that Classical price (production prices and labor commanded prices) and value theory are at least as far-reaching in their theoretical and empirical potential as the only loosely connected neoclassical price theory and the accounting principles of the conventional SNA (based on constant price data of a certain base year, which indeed needs to be rebased often in order not to loose contact with the ongoing economic evolution). Classical (labor) value theory is a theoretical concept that can be determined simultaneously with actual prices and prices of production and thus does not need a base year for its proper formulation. The question then however is what rigorous relationships there are between such labor value accounting and the Marxian SNA that is based on it and the prices of production, not in the sense of some sort of transformation theorem, but in the sense of detecting the qualitative and quantitative relationships between the theoretical concept of non-nominal economic reasoning and the centers of gravity of the purely nominal development of actual market prices.

In this respect the chapter will in particular discuss in the next section a list of properties that may help to understand (here primarily) the quantitative relationships

---

[4] See Eatwell et al. (1992) for a summary of Marx's economics.

of theorizing the "real" behind the dynamics of the nominal magnitudes like profit, wages, value added and more. Concerning the so-called Marxian transformation problem, we start from the state of the art in economic accounting, the United Nations' SNA (of the 1960s rather than of the 1990s), where magnitudes measured in terms of current prices and constant prices coexist without raising the issue whether one scheme can be transformed into the other one in order to obtain a meaningful relationship between the two. It is obvious that our perspective will provide a dual approach to Marxian economics, with labor values providing the means to analyze the "real" behind the nominal resulting from the interactions of the human beings that constitute a certain society at a certain time and certain place in the history of mankind. Yet, as we shall see, this Marxian dual is embedded (from the quantitative perspective) in what is provided by the United Nations' SNA (with all its details for deriving physical input–output tables in the presence of many technological complications as they exist in modern market economies). We simply have to take its measure total labor costs and to interpret it from the perspective of Marx's Capital.

This can be done in competition with or in contrast to the categories provided by the conventional SNA and thus provides an ideal scenario by which the explanatory power of the two SNA's, the conventional one and the Marxian one, can be compared and evaluated, potentially also allowing the conclusion that both systems for a "real value accounting" (labor values vs. magnitudes based on constant prices) have their own advantage in certain areas of their application. The United Nations' SNA starts from the nominal to construct its "real" magnitudes on this basis, while Marx started from labor values in order to show their explanatory power for the price-quantity dynamics of capitalist economies. Nevertheless, the two "real" SNA's thereby obtained are both not meant to provide a substitute for a price theory, which is obvious for the United Nations SNA and which was totally confused in its objectives by the discussion on the transformation problem that followed reasonings of Samuelson (1971) and others.

From today's perspective the task simply is to formulate and prove propositions that show the usefulness of the real SNA of the United Nations and of Marx's valuation scheme and also maybe to show that they both can face common application problems. This places them on an equal footing with respect to what they claim to be the "real" behind the nominal, which in my view creates a scientific approach that can proceed with rigor and without any necessity for heated ideological debates and terminology. We shall consider here as possible theoretical outcomes either a result that is of the type of Keynes' (1936) wage units construction, an approach that attempts to have a *single*, basically proportional to prices, reconstruction of values from the sphere of nominal price magnitudes or a *dual* one – which we favor – where an accounting system is created that differs in structure from the one supplied by the nominal price magnitudes.

With respect to such possibilities, we provide in Sect. 2.2 a set of assertions that can be used – if accepted – to test competing theories of labor values against each other. Section 2.3 then briefly presents various contemporary approaches to the labor theory of value from the unifying perspective of a system of national accounts' point of view. Section 2.4 concludes.

In Chap. 4 we will provide the details of our approach to the definition of labor values, there exemplified for the case of Steedman's (1977) joint production and fixed capital examples, in order to show the working of our definition of labor values in a context where conventionally defined labor values would become negative and thus meaningless (as Steedman has shown). This chapter also compares our procedure of defining labor values using accounting principles from the cost accounting methodology of firms with procedures introduced by Richard Stone into generalized input–output compilations and analyses. We find that there is indeed a close correspondence between these two ways of approaching a definition of total labor costs (if the so-called industry technology assumption is used for input–output table compilation and the so-called sales value method from the accounting perspective of single firms).

2.2 Labor Value Accounting: Some Propositions

The aim of this section is to provide lists of properties that may be of use in evaluating the various proposals for a definition of labor values (or total labor costs) that have been put forward in the literature, and their application to theoretical as well as empirical investigations. This list is not intended to exclude any approach that violates one or another of its principles (maybe with quite different objectives in mind) from serious consideration. Instead, they should help the reader to systemize (and form preferences for) the different approaches to Marx's LTV with respect to the features they explicitly or implicitly exhibit. We believe however that these list are by and large in accordance with what is stated in Marx' Capital on the various properties his definition of labor values should give rise to.

1. *Simple quantitative features of the Labor Theory of Value (LTV)*:
   (a) *Aggregation Theorem*. The (labor) value of net production of a given year equals the total labor time expended in this period. A simple matter of the proper definition of labor values.
   (b) *Profit–Rate Theorem*. The average (labor) value- and price-rate-of-profit are of the same magnitude in situations of uniform rates of growth. A very weak side-condition (see also Chap. 1 on this matter).
   (c) *Price / Value Theorem*. Uniform ratios of profits to wages (in terms of whatever prices) in all sectors of production imply proportionality between labor values and these prices. A methodologically important proposition of Marx's labor theory of value.
   (d) *Redistribution Theorem*. Total profits are equal to total surplus values (and the rate of exploitation is given by the ratio of total profits to total wages). A simple matter of choosing an appropriate definition of the value of labor power (and net output y as numéraire commodity, see Chap. 1).
   (e) *"Fundamental" Marxian Theorem*. The rate of exploitation is positive if and only if the uniform price rate of profit is positive. A very weak side-condition.

(f) *Labor–Commanded Theorem*. Labor values are smaller than actual prices when these prices are normalized by the money wage rate (assuming that all sectors earn positive profits). A proposition with important empirical content.

Most of these assertions are known to hold true in single non-joint production systems (no fixed capital), but some of them are not easy to generalize to general production systems, see Chap. 4 for example.

2. *Basic principles, when generalizing Labor Values (LVs)*:

(a) *Commodity Correspondence Principle (Free good rule)*. The sign of the price of a good equals the sign of the labor value of the good. In particular: The labor values of free goods are zero. This is not a trivial property of labor values in the light of the discussion of their proper definition for general joint production systems in the 1970s and 1980s.
(b) *Value–added principle*. Value added (per commodity) equals direct labor (per commodity). This is not a trivial property of labor values in the light of the discussion of their proper definition in the 1970s and 1980s.
(c) *Individual– and Market–value Principle*. Labor values are averages of individual values, which in turn are derived from actual production data of multiple activity systems by means of average labor values. A basic construction principle that has been stressed by Marx already.
(d) *Labor–Value Continuity Principle*. Labor values change continuously with technology. This is not a trivial property of labor values in the light of the discussion of their proper definition in the 1970s and 1980s.
(e) *Labor–Unit Principle*. Labor is to be homogenized by means of wage differentials. One prominent approach towards the solution of the so-called reduction problem which allows for the generalization of the price-value theorem stated above.
(f) *Imputation Principles*. If full–cost accounting (of any type) is not possible by means of actual physical input–output data alone, the existing practices of firms have to be analyzed and to be applied appropriately to close the then existing degrees of freedom in the definition of such total costs (principles like the sales value method, e.g., see later sections of this chapter).

Most proposed concepts for generalized labor values in the 1970s and 1980'[5] for general production systems are hurting one or more of these principles so that either these value definitions or some of the above principles must be discarded from a further discussion on the meaningfulness of the labor theory of value.

3. *Pragmatic uses of the notion of LVs*:

(a) *Leontief Multiplier Theorem*. Monetary input–output calculations of total labor costs per unit of output value determine the value/price ratios of individual commodities also in general production systems – if input–output

[5] By Morishima, Okishio, Steedman, Wolfstetter, Krause, Holländer, and others.

tables are calculated appropriately (by means of the so-called industry technology assumption, see later sections of this chapter).

(b) *Inflation Measurement*. The "monetary equivalent of labor time" (MELT) is to be determined by total nominal net output (NNP) per unit of labor time expended which leads to an index formula of the type

$$\epsilon = py/lx = \sum_i (p_i/v_i)\frac{v_i y_i}{\sum v_i y_i},$$

see the preceding chapter and note that change of this expression in time can be used to determine the rate of inflation of the economy.

(c) *Labor Productivity Measurement*. The reciprocal values of labor values are the appropriate measures of labor productivity of the corresponding sectors of commodity production.

(d) *Technical Change Theorem (one example)*. Capital–using labor–saving technical change which is profitable raises labor productivity (in the sense just defined).

(e) The General Law of Capitalist Accumulation (Marx's Capital I, Chap. 23) implies the need for a macroeconomic presentation in real terms that is independent of base periods as they are needed – and often rapidly updated – in the measurement of real magnitudes in the conventional system of national accounts.

These assertions attempt to link the theoretical concept of labor values to actual data and the measurement of so-called real magnitudes and try to avoid the pessimistic conclusion: "The only real in a capitalist production economy are the nominal (price times quantity) expressions" as judgement on the value of conventional accounting practices in so-called real terms (and all the fallacies they may exhibit).

The purpose of the presentation of the above lists of features of and assertions on Marx's labor theory of value lies in the suggestion that all these points can be considered as systematic outcomes of the reflection of Marx's labor theory of value in the 1970s and 1980s – and this on the level of simple two-sectoral models as well as general n-sectoral models of production – on the basis of which the remaining possibilities for a coherent and applicable LTV can then be investigated and judged in detail.

In the next section we will provide a brief survey of baseline definitions and approaches to the Marxian concept of a value rate of profit and an underlying value rate of exploitation that are still proposed, including a comparison with the status of the United Nations' SNA and its considerations of total labor costs. We will however not go into a detailed discussion here, that confronts the above list of assertions with the approaches to be presented next, but leave this for future research and debate of the issues that are raised in this chapter.

2.3 Four Baseline Approaches to Marx' Labor Theory of Value

The following discussion of various approaches to a value accounting in Marxian or in other terms will be very short, since we solely want to provide a framework where all four approaches that are here discussed can be compared from the unifying perspective of National Accounting and a specific angle, namely by their provision of a "real" accounting system, in addition to the official purely nominal one and its categorization of economic activities, stocks, flows and the growth processes the interaction of stocks and flows gives rise to.

One possibility to evaluate the following approaches (where we consider the UN approach here from the perspective of its nominal categories and its definitions of inflation and growth, but not of so-called real magnitudes) is to briefly apply the criteria of the preceding section to these approaches in order to evaluate their proposals for the determination of labor values or total labor costs. Ultimately the theoretical and empirical application of the proposed definitions and the quantitative expressions derived therefrom will decide which approach is the more fruitful one in constructing something behind the UN's nominal magnitudes that can be of help in the understanding of what is actually observed in nominal terms for capitalist market economies in space and in time. We stress that the statements made in the following subsections are still somewhat preliminary and need further discussion and elaboration, in particular of those contributions that are not considered as appropriate in this book.

2.3.1 The Temporal Single System Interpretation (TSSI)

In this approach, labor values v_{t+1} are derived from the physical and labor input costs of firms, see McGlone and Kliman (1996, p. 46),[6] the former evaluated at current prices p_t and divided through a given scalar ϵ, called the monetary expression of labor time (MELT) in the literature, which renormalizes the price expressions for the input costs towards a measurement in terms of labor units: $v_{t+1} = (p_t/\epsilon)A + l$.[7] Otherwise, the definitional procedure is as in the conventional algebraic approach to labor values, with the important difference however that input costs (in prices) are taken from the beginning of the production period and the labor values of outputs are defined as end of period values (beginning of the next one).[8] Labor values – and prices of production, see below – therefore are here employed in a dynamic fashion, one that leads from exogenously given prices (of production) to an appended updating of labor values (and prices of production). We set the MELT expression ϵ equal

[6] I have to thank Andrew Kliman for detailed comments on this section of the chapter which contributed to improving its presentation. Of course, the usual caveats apply.

[7] A, l are the unit input data of standard input–output analysis, see also Chaps. 1/3, that is augmented by workers average consumption data.

[8] In contrast to the simultaneous equations approach there are however no linear equation systems to be solved here.

to 1 for expositional simplicity (and in order to avoid confusion with the value rate of exploitation we defined in the preceding chapter). On the basis of the notation of this chapter we can then define the average rate of profit of the value system by[9]

$$\rho_t = \frac{p_t(I - A^+)x_t}{p_t A^+ x_t} = \frac{(1 - w_t)lx_t}{p_t A^+ x_t} = \frac{e_t}{p_t A^+/w_t l x_t + 1}, \quad e_t = \frac{1 - w_t}{w_t}, w_t = p_t c_w \tag{2.1}$$

since there holds $p_t y = l x_t$ due to $\epsilon = 1$.

We define next[10] the uniform rate of profit system (the prices of production in this dynamic setup) by:

$$p_{t+1} = (1 + \rho_t) p_t A^+$$

It is easy to show on this basis that there holds (r_t the average price rate of profit):

1. $p_{t+1} x_t = v_{t+1} x_t$
2. $\Pi_t := \rho_t p_t A^+ x_t = S_t := l x_t - w_t l x_t = p_t y_t - p_t c_w l x_t$
3. $r_t = (p_{t+1} x_t - p_t A^+ x_t)/p_t A^+ x_t = \rho_t = (1 - w_t) l x_t / p_t A^+ x_t$

These equations provide the core equations of the TSSI solution to the Marxian transformation problem, an interpretation which preserves the Marxian accounting identities in his transformation example. If iterated in time, they give – on the basis of what was assumed above – in the limit (if it exists) rise to:

$$v = vA + l, \quad p = \frac{px}{pA^+ x} pA^+, \quad i.e.,$$

the conventional equations for labor values and prices of production, see Bródy (1970) for example and for convergence proofs. As temporal values, old prices (and values) determine the average value rate of profit and the amount of surplus value that is produced,[11] while the next periods values and prices of production are just appended to the current situation's characteristics (and may need adjustment with respect to the MELT condition).

The basic question here is (as in any scientific approach that deals with phenomena of real life) which theoretical and empirical propositions can be obtained from these definitions of the value and price schemes v_{t+1}, p_{t+1}, apart from the three identities they give rise to by definition. Following Mohun (2004) we would also stress here that the central point of a quantitative expression or definition is to be able to use it in the form of proposition on v_{t+1}, p_{t+1} relationships and in empirical investigations of the actual behavior (measured in terms of actual prices) of the economy with respect to production and technical change on the one hand and competition and exchange on the other hand.

---

[9] See McGlone and Kliman (1996, p. 46). Note that p_t is here interpreted in terms of a historically given vector v_t.

[10] See McGlone and Kliman (1996, p. 46).

[11] Constant capital, variable capital and surplus value are thus all given magnitudes when the price-value iteration is started.

Due to the dynamic nature of the definition of labor values this is labelled a temporal approach, since it implies an evolving system of labor values even if all technological data are given (where also prices of production are updated by an iteration procedure as proposed by the TSSI). The advantage of this definitional procedure is that it preserves Marx's basic aggregate accounting identities. This approach is discussed and evaluated in detail in Duménil and Levy (2000a,b), Foley (1997, 2000), Freeman and Carchedi (1996), Freeman et al. (2004), Kliman and McGlone (1999), Mavroudeas (1999), Mohun (2003), Mongiovi (2002).

There is also the question how such labor values can be properly generalized[12] to the treatment of pure joint production systems (with a rectangular output matrix B), in particular if the jointly produced commodities are used again in production (in different processes), without giving rise to negative values for some commodities, indeterminacy of value accounting or other quantitative "anomalies".[13] This is a topic where in our view also actually employed methods of dealing with joint production within firms should be taken into account (an empirical orientation of the labor theory of value clearly found in Marx's Capital, Vol. II). The further question is how the definition of labor values in the TSSI can be related to Marx's (1954, p. 48) understanding of the relationship between labor values and the measurement of labor productivity. The latter should change systematic fashion (ignoring "secondary" influences of actual prices on labor values as they are discussed in the Chaps. 4 and 5) when methods of production are changing, for example in the simple input–output system considered by McGlone and Kliman (1996, p. 46), while labor values according to the TSSI can change in proportions when the proportions of prices (of production) are changing in the iteration procedure they propose for labor values and prices of production.

In our view, the most basic problem of this approach to values and prices however is that it makes use of a uniform point-input (t) point–output (t + 1) assumption for all production processes happening in the considered economy. This is extremely implausible from the empirical perspective.[14] Input–output flow data are accumulated data transformed into averages by appropriate normalizations and input–output stock data measure inventories needed for production at certain moments in time, also transferred to averages by appropriate normalization procedures. We thus have average items for capital consumed (including wages) as well as for capital advanced (also including wages). To assume that all flows are consumed uniformly at the beginning of the year and all outputs sold uniformly at its end is introducing an

[12] This seems to be a general problem for the presentations of the TSSI in the literature, since there meanwhile exist numerous examples for its formulation, but by and large no compact, concise definition for general models of production which avoids the various shortcomings of the examples.

[13] A possible solution could be found here by using the distinction between individual and market values in the way proposed in Flaschel (1983a) or alternatively of the kind proposed in Duménil and Levy (1989).

[14] If at all, a continuous-input continuous-output model type would here be the more appropriate starting point for the modelling of a capitalist economy, see Foley (1986) for a formulation of this type of approach in the context of Marxian economics.

abstraction that is not adequate in the context of a Marxian approach to reality, see Marx's detailed factual analysis of the turnover of capital in Capital, Vol. II. We have production processes that use up inputs and produce new outputs each day during the year as well as processes where even one year is not sufficient to produce a finished commodity. Turnover times of inputs therefore can vary in extreme ways and should thus not be forced into a purely theoretical Austrian point-input point-output approach to capital theory.

Instead input–output averages moving continuously in time (based on data that are changing on a daily basis, but normally only measured once per year) should be used to measure labor values and prices of production (both pure accounting concepts in such a framework) which therefore also represent moving averages to be defined at each moment in time and thus necessarily not of the temporal type we considered above. The task then is to state laws of motion for such moving averages and their interactions and to show their theoretical as well as empirical validity. Definitions – whether temporal or simultaneous – therefore must be based on empirically relevant formulations of the production processes of a capitalist economy and be employed to a theoretical and empirical understanding of what we observe in reality through more or less conventional statistical procedures.

Following Kliman (2007) the TSSI is primarily concerned with refuting the myth of inconsistency of Marx's solution to the transformation problem from labor values to prices of production. It provides a specific solution to this problem and is as such concerned about value – price relationships, where production prices are just the first step when going from theory and essence (abstract labor) to the surface of price-quantity adjustment processes (including commercial capital, banking capital, international exchange and so on). Yet, handling the transformation problem in our view leads to a combination of value and price expressions that distorts the distinction between essence (abstract labor) and surface (price and quantity interactions). It runs the risk of not separating Marx's System of Labor Value Accounts (Capital, Vol. I) in a persuasive way from what happens on the surface of capitalist competition.

We close this brief section on the TSSI with the conclusion that its primary contribution is to make the TSSI comparable – from our perspective – to the treatments of the LTV that are now following. There has been an extensive debate in the literature on the merits and the deficiencies of this interpretation of Marx's Capital which we will not discuss here any further, see however – besides the contributions already mentioned – the papers by Veneziani (2004, 2005), Mohun and Veneziani (2007) and also the response by Kliman and Freeman (2006).

2.3.2 The Aggregate Single System Interpretation (ASSI)

To a certain degree this approach is similar to Keynes' (1936) approach who considered the working of the economy from the perspective of prices normalized by the wage unit, i.e. in his case, neoclassical marginal cost prices in terms of labor commanded, representing the amount of labor that is exchanged for one unit of the considered commodity. In the ASSI interpretation of Marxian categories, prices of production or actual prices) are normalized in terms of the labor time expended in the year under consideration, leaving actual prices p as remainder the expression py/lx, the monetary equivalent of labor time (MELT) we have already considered in the preceding subsection. The ASSI approaches to the labor theory of value share, on the one hand, a common core in their understanding of Marxian price ratios, but are also and on the other hand to a certain degree significantly distinguished from each other. Original contributions that are related to what was discussed in Chap. 1 are given by the works of Duménil (1983, 1984) [15] and Foley (1982, 1983, 1986)[16] and – with a different twist – in Germany by work of Krause (1980a,b, 1998) and Picard (1979), where the postulate of a uniform rate of exploitation is discarded in favor of a single value and price interpretation. Mohun (1993, 1994, 2003) has considered the Duménil-Foley (DF) interpretation in detail, while we have done so (indirectly) in Chap. 1.[17]

The DF single system approach rescales actually observed market prices (or prices of production) such that they represent the price of net product py by the amount of labor $L = lx = vy$ expended in its production, see here Chap. 1, in order to define on this basis Marxian categories like the value of labor power, surplus value, the rate of exploitation and more. Assuming that workers do not only consume, but also save, makes it necessary to depart from the subsistence definition of the value of labor power as measured in terms of labor values applied to the assumed subsistence basket. A new interpretation of the value of labor power is then provided by money wages divided by MELT, i.e., the wage share in national income, see again Chap. 1, whereby the sum of wages (divided by MELT), i.e., measured relative to $py = lx$, becomes identical to Marx's concept of variable capital and the sum of profits becomes identical to Marx's notion of surplus value. The accounting identities of this particular framework are therefore given by these three sets of equations. The attractive thing with this approach lies in the fact that it is empirically the least demanding one to be implemented and that it therefore can progress rapidly from a given nominal system of national accounts to the consideration of the tendencies that are implicitly contained in these data sets and their evolution over time (including the determination of the rate of inflation, see the next subsection).

The ASSI therefore interprets the existing data in a new way and is immediately applicable to the analysis of the evolution of capitalist economies, such as in the study of Duménil and Levy (1993) on the economics of the profit rate. In this work however, in the appendix on pp. 48/49, a brief account of the transformation of

[15] Duménil and Levy (2000a,b).

[16] See also Foley (2000).

[17] See also Mohun (2004) for further remarks on the literature and an outline of some recent approaches to an accounting structure which relates observable prices to Marxian labour values.

values into prices of production is provided that stresses the difference between appropriation and realization of surplus value, stating that (p. 49):

> Surplus value is appropriated proportionally to labor inputs, but realized (under ordinary circumstances) proportionally to capital advanced. This separation between appropriation and realization hides the existence of exploitation.

With respect to the use of conventionally defined labor values and their role in defining rates of profit and exploitation, see here Chap. 1, the ASSI is therefore somewhat inconclusive and does in any case not erase this definition as it was proposed by Samuelson (1971). In our view, the statement from Duménil and Levy (1993) can be associated with the approach to the definition of labor values and the value rate of profit we have considered in Chap. 1, which bases the stated difference again on dual concepts of value and price and the proximate relationships they imply for central Marxian aggregates (which re-direct the focus again on capitalist production and the forces that are shaping it). It is in principle also obtained from what is supplied by the United Nations' System of National Accounts and its application to the data of particular economies if one replaces their concepts of (aggregate or sectoral) labor productivity by labor values and their aggregates as indexes of labor productivity, see the following two subsections.

2.3.3 The Conventional Dual System Approach (CDSA)

In theoretical debates on Classical economics and their considerations of value and price in the framework of given input–output data the work of Piero Sraffa (1898–1983) is clearly of outstanding importance, represented in particular by his 1960 book "Production of Commodities by Means of Commodities" which may be considered the Classical equivalent to Debreu's "Theory of Value", both very compact publications with an overwhelming impact on the corresponding scientific communities. Both contributions are heavily concentrated on the sphere of competition and thus on price theory, in one case long-period production prices and in the other case short-run market prices. From a Marxian perspective these theories therefore concern "surface phenomena" that do not penetrate what is going on behind commodity exchange in the sphere of capitalist production.

Be that as it may, conventional economics goes beyond such categories of competition in significant ways in that it constructs accounting concepts on the micro as well as on the macro-level that are intended to provide insights on the dynamics of a capitalist economies by snapshots of its real behavior underlying by definitional construction its nominal magnitudes and their movement in time. These efforts have been started on a larger scale, since the appearance of Keynes' General Theory and have found their culmination point in the work of Nobel Laureate Richard Stone (1913–1991) and his co-authors, in their joint efforts to establish a coherent framework for national accounting, published in compact form as "A System of National Accounts" by the United Nations in 1968. Reading both Sraffa's and Stone's work (who both lived in Cambridge, UK) reveals striking common features (for example between Stone's Commodity Technology Assumption and Sraffa's Standard Commodity in the case of joint production), interrelationships that have been totally ignored in the mainly academic debate on capital controversies, but also in the pragmatically oriented, but theoretically very refined work of Stone and his followers.

With the abbreviation CDSA we here simply mean the current practices in the System of National Accounts of the United Nations as far as the calculation of real magnitudes, besides nominal expressions, based on double deflating procedures are concerned. Such an approach is clearly dual in nature, since it employs besides a full set of nominal categories a constructed set of so-called real magnitudes, calculated at constant prices (where inputs and outputs are deflated differently), or prices of a certain base year, like real GDP, real growth, real value added, labor productivity measures and more. We may also call this approach a temporal one, because it gets into trouble when the base period departs too much from the current period, in which case magnitudes have to be rebased in some way or another. Furthermore, it is questionable what is really measured when one calculates for example real value added at prices of a base year, i.e., at prices that may be quite different in structure from the one of the present period, leading for example to potentially virtual income expressions thereby. This however does not mean that the double deflating methods applied in this accounting approach are generally suspect from a theoretical point of view, for example when they are used as in input–output methodology where different things have to be deflated differently. The important thing here however is that such differently deflated things (the inputs) should then still be treated as different and not deducted from separately deflated output in order to arrive at a difference, then called real value added, with which indeed no economic meaning can be associated.

This has lead some researchers in this area to declare that the only real object of investigation in the SNA is the purely nominal one, or less strictly that only a single deflator should be applied throughout (the so-called single deflating method) when going from nominal magnitudes to real ones. Yet, the example of input–output compilation shows that double deflation can in principle be applied to certain areas of the System of National Accounts, though of course subject to well-known aggregation problems as well as changes in process and product properties. The current system of national accounts – as routinized by the methodology published since the 1950s by the United Nations Statistical Division – provides however a wealth of categories, classifications and definitions which demand for closer inspection from the perspective of advanced economic theory, in particular in the area where quantity expressions for real magnitudes are derived and applied.

In this part of the book, we make the general assumption that there is something "real" behind the dynamics of nominal magnitudes, and that these real magnitudes are given by theoretically sound definitions and not by some substance hidden behind the interaction of nominal expressions as we observe them as individuals and from a scientific perspective. These real magnitudes of an economy with many production and household sectors are to be constructed with great care and precision and they of course are only justified if we can use them to measure, explain and predict what is going on in the economy in greater depth than is possible by means of nominal prices and their aggregates, regardless of whether market prices or prices of production are used for this purpose. We view Stone's SNA as a big step forward into such a direction, in particular what its detailed and very general input–output methodology is concerned. From this perspective, microeconomics of any type is nowadays always characterized by a dual system approach, the accounting system on the firm as well as on the national level (which have to correspond to each other) and the theory of prices, be it a Classical or a Neoclassical one. We will call the combination of Stone's SNA with the Sraffian theory of long-period prices the Conventional Dual System Approach (CDSA) in this section. Their common origin is Cambridge, UK in the 1950s and 1960s and their treatment of input–output data is in many respects interrelated as we have tried to show in Flaschel (1984). In a subsequent section we shall moreover show how value theory fits into such a framework, indeed by correcting for undesirable developments that have taken place in its further evolution, since the seminal contributions of Stone (1968), see United Nations (1993).

From a macroeconomic perspective the most important measures provided by a SNA are the rate of inflation and the rate of growth. With respect to inflation rates π_t one starts from expressions of the type:

$$\begin{aligned} 1+\pi_t &= \frac{\sum_i p_{i,t+1} y_{i,t}}{\sum_i p_{i,t} y_{i,t}} = \sum_i \frac{p_{i,t} y_{i,t}}{\sum_i p_{i,t} y_{i,t}} \frac{p_{i,t+1}}{p_{i,t}} \\ &= \sum_i \alpha_{it}(1+\pi_{i,t}) = 1 + \sum_i \alpha_{it}\pi_{i,t}, \quad \text{with } \alpha_{it} = \frac{p_{i,t} y_{i,t}}{\sum_i p_{i,t} y_{i,t}} \end{aligned}$$

From these expressions there easily follows by iterative extension:

$$1+\pi_{t,o} = (1+\pi_t)(1+\pi_{t-1})\cdots(1+\pi_o) = \frac{\sum_i p_{i,t+1} y_{i,t}}{\sum_i p_{i,o} y_{i,t}}$$

i.e., accumulated inflation factors are just given by the value of current output levels divided by their value measured in prices of the base period $t = 0$. So far, everything is fine. We measure inflation by a specific weighted average of sectoral inflation rates where the weights are given by the relative sectoral output value in the current value of total output. The weights therefore depend on the current price vector, but having taken note of this, we just have an average of sectoral inflation rates at our disposal to measure and apply inflation rates for a whole economy.

In the same way we can measure the average growth rate of an economy by:

$$\begin{aligned} 1+\gamma_t &= \frac{\sum_i p_{i,t} y_{i,t+1}}{\sum_i p_{i,t} y_{i,t}} \\ &= \sum_i \frac{p_{i,t} y_{i,t}}{\sum_i p_{i,t} y_{i,t}} \frac{y_{i,t+1}}{y_{i,t}} \\ &= \sum_i \alpha_{it}(1+\gamma_{i,t}) = 1 + \sum_i \alpha_{it}\gamma_{i,t}, \quad \text{with } \alpha_{it} = \frac{p_{i,t} y_{i,t}}{\sum_i p_{i,t} y_{i,t}} \end{aligned}$$

From these expressions there again easily follows by iterative extension:

$$1 + \gamma_{t,o} = (1 + \gamma_t)(1 + \gamma_{t-1}) \cdots (1 + \gamma_o) = \frac{\sum_i p_{i,t} y_{i,t+1}}{\sum_i p_{i,t} y_{i,o}}$$

which in a specific way provides an expression for accumulated growth factors. It is also easy to show that the growth factor of nominal output fulfills the equations

$$1 + \eta_t = \frac{\sum_i p_{i,t+1} y_{i,t+1}}{\sum_i p_{i,t} y_{i,t}} = \sum_i \alpha_{it}(1 + \pi_{i,t})(1 + \gamma_{i,t}), \quad i.e.$$

there holds approximately $\eta_t = \sum_i \alpha_{it}\pi_{i,t}\gamma_{i,t} = \pi_t + \gamma_t$.

We here concentrate on the determination of inflation rates and now show that they are identical to the fractions formed from the MELT expressions used in the preceding section if the net output vector $y = (y_1, ..., y_n)$ is the vector used in above summations for average inflation rates. This follows easily from

$$MELT_{t+1}/MELT_t = \frac{p_{t+1} y_t / l x_t}{p_t y_t / l x_t}$$

if the data characterizing production are kept constant (since lx_t can be canceled in these expressions).

Growth rate calculations, whether for prices or for output, therefore enrich the consideration of nominal data such as $p_t y_t$ in that they separate price level effects from output level effects in terms of their rates of change, i.e., as dimensionless percentages. This adds information to the consideration of the time series $p_t y_t$ and thus helps to distinguish price level growth from output level growth. A big error however occurs in the United Nations's (1993) SNA when one proceeds from there to an interpretation of the fraction $Y_t = \sum_i p_{i,o} y_{i,t}$ in the denominator of the accumulated inflation rate expressions, by calling it the real NNP of period t and by proceeding from there to the measurement of average labor productivity in terms of $Y_t / L_t = p_o y_t / l x_t$:[18]

$$\frac{Y_t}{L_t} = \frac{\sum_i p_{i,o} y_{i,t}}{\sum_i L_{i,t}} = \sum_i \frac{L_{i,t}}{L_t} \frac{p_{i,o} y_{i,t}}{L_{i,t}}$$

Viewed from its bare definition, Y_t is nothing but the current net output basket valued at price of a base period 0 which remains a price expression, based on a price vector of some arbitrary past. Output at hypothetical past prices cannot be used to measure labor productivity in a technically convincing way. This will be shown in detail in the next Chap. 3, but should be already relatively obvious here from an input–output theoretic perspective. Similarly, since y_i are the net output levels of a whole economy (where intermediate inputs have been deducted) we cannot use

[18] $y = (I - A)^{-1} x$ as usual.

$y_i/L_i = ((I - A)^{-1}x)_i/L_i$ as a sectoral measure of labor productivity, since this is providing an expression that cannot be considered as isolated from the other sectors of the economy. On the other hand, using x_i/L_i is but a partial measure of sector's i performance, since it neglects its capital consumption in the form of intermediate inputs. Finally, the United Nations (1993) measure of sectoral labor productivity $(p_{oi}x_{i,t} - p_o A_i x_{i,t})/L_{i,t}$, i.e., value added of sector i in terms of arbitrary base years prices p_o divided by the total labor input of this sector is again contaminated by arbitrary price-dependent aggregators which prevents that anything characterizing the production side of the economy can be defined meaningfully in this way.

We conclude that the measurement of labor productivity should be left to the consideration of input–output theory and not become a byproduct of the measurement of real GDP or NDP as it is the case in the Systems of National Accounts in their current form (which differs from what was originally proposed by Stone himself). To show this in detail is the task of Chaps. 3 and 4. Here we only conclude that the construction of SNA's behind the evolution of nominal magnitudes is a meaningful activity, independently of whether it is classically oriented or neoclassical in nature. SNA's provide theoretical concepts intended to measure evolution not visible from the consideration of purely nominal magnitudes and aggregates and in this sense they are dual in nature as compared to the sphere of competition, exchange and money prices. As economics is taught and investigated today it is indeed dual in nature. This however does not automatically imply that all of its categories are well-defined and coherently applicable, but they may sometimes be flawed by erroneous definitional attempts. The next subsection will argue on this basis that Marx's Capital I–III forms such a dual system of national accounts and long-period or market prices where one should not immediately proceed to the conclusions that the labor values of the Classical System of National Accounts are but – in the majority of interpretations of the Labor Theory of Value: bad – predictors of prices of production or even market prices. It is not the central task of a System of National Accounts to provide price predictors, but its foremost duty is to provide categories (including their quantification and measurement) that are of use for the understanding of the dynamics of nominal magnitudes in the working of capitalist economies.

The structured macro-data as supplied by the United Nations' System of National Accounts will be the point of departure and also a point of reference for our proposal, in the next section, to formulate a system of indexes of labor productivity by means of labor values from a Marxian perspective. We stress that the United Nations' System of National Accounts (in the original version as formulated by Stone and his research group in 1968) indeed defines labor productivity indices (and thus implicitly labor values, there called total labor costs) in the tradition of the Classical authors, and does so in the presence of joint production and even more general modes of production, see the concluding section of this chapter.

2.3.4 The Marxian Dual System Approach (MDSA)

With respect to the single commodity production system A, l, as considered already above in our representation of McGlone and Kliman's (1996) transformation procedure of the TSSI, the MDSA approach is based on the traditional algebraic and simultaneous type of labor value accounting in line with the work published by Okishio and Morishima among others in the 1960s and 1970s, and also in line with the measures for direct and indirect or total labor costs in the United Nations' System of National Account based on the work of Richard Stone, i.e., its definition of labor values is simply given by the matrix equation $v = vA + l$. This approach is therefore the conventional approach in the literature on Marxian economics and thus seems to offer nothing really new for the interpretation of Marx's Capital, Vols. I–III.[19] Yet, first of all, this conventional approach to the definition of labor values is quite general in nature. It has been generalized to the treatment of multiple activities for the production of a single commodity, pure joint production, fixed capital and heterogeneous labor in Flaschel (1980, 1983a, 1983b, 1995) making use of certain accounting practices actually applied by firms, certain accounting practices of input–output methodology and above all of the averaging approach put forth by Bródy (1970), see also Bródy (1987) and Simonovits and Steenge (1996), in place of Steedman's (1977) generalizations of labor values by means of Sraffian zero-profit approaches to joint production and fixed capital. Moreover, and more importantly, the conventional approach to the definition of labor values is not only providing a very general accounting framework for the determination of total labor costs, but in addition allows for various theoretical as well as empirical applications of this valuation scheme that prove the meaningfulness of this approach. We will consider some of these applications below, after some short comments on the generality of the conventional approach to the definition of labor values.

Multiple activities lead in a natural way to the distinction of market from individual values, the former being certain averages of the latter as in Marx (1954), and as in the aggregation procedures of input–output methodology. Pure joint production is compatible (with respect to a disentangling of joint input costs that is neutral with respect uniform rates of profit) with only one allocation method of firms' actual cost accounting procedures, the so-called sales value method. This method is applied, but barely understood in standard books on cost accounting. It in fact represents the only method that allows to allocate costs in pure joint production activities that does not introduce a distortion in the profitability statements of the whole process as compared to its single disentangled activities. From the perspective of Marx's Capital, Vol. II (where the actual behavior of firms is always paid attention to) it thus recommends itself from the practical and the empirical point of view. Astonishingly enough, this method reappears (unnoticed) in the treatment of secondary products in input–output methodology designed by Richard Stone, by way of the so-called industry technology assumption for the reallocation of such secondary products towards the sector where they are produced as main products. This happens without

[19] An interesting non-standard approach to a definition of labor values – which includes capitalists' consumption basket into the "means of production" in a stationary economy – has been provided recently by Wright (2007).

any reference to the actual accounting practices of firms and may be interpreted as fact driven behavior on the level of firms as well as on the level of national accounting.

Fixed capital is already treated in a detailed way in Bródy (1970), there too by the application of actual accounting techniques that define the concept of turnover times and its relationship to capital advanced as opposed to capital consumed. Such a distinction makes the relatively arbitrary or even hypothetical distinction between circulating and fixed capital superfluous, since nearly every means of production appears in the form of capital advanced and capital consumed, by referring to an accounting period of one year in general (or one quarter), with respect to which turnover times are then measured as being less or larger than one. The sharp distinction between circulating and fixed capital by contrast refers to a hypothetical period of production with no factual content and thus assumes that turnover times are either exactly one or – if larger than one – lead to a vintage approach with close connection to joint production and fairly academic valuation schemes for the various vintage types of fixed capital.

Skill differences with respect to labor inputs finally are here evaluated by way of actual wage differentials, which may be subject to purely arbitrary valuation conventions in different countries and at different times, which thus includes a historical dimension into labor value accounting. Like the TSSI the ASSI needs market prices, now however only in certain accounting procedures, namely when disentangling joint productions activities (where relative sales values are used) and also in the solution of the so-called reduction problem of skilled to simple labor. It is a purely ex post approach and can be directly applied to actual input–output tables when these tables have been constructed by way of the industry technology assumption. It distinguishes between stocks and flows in the same way as firms do it in their accounting procedures and also as in the stock-flow distinction in the United Nations' Systems of National Accounts. In sum this approach in fact allows for all the assertions summarized at the beginning of this chapter, without any need to construct data for labor value calculations that are not already provided by the conventional System of National Accounts, at least in principle. It in addition bears relationships with the work provided by Shaikh and Tonak (1994). These authors also discuss the United Nations' Accounting methodology to a certain extent (as it derives from make or supply matrices and use or absorption matrices), quite independent from the question of whether their use of the data is already a convincing one, see Mohun (2005) in this regard.

Duménil and Levy (1989), see also Duménil and Levy (1987, 1988), have reconsidered the labor value definition of the joint production approach of Flaschel (1983a) from a more general perspective that initially makes use of physical relationships (market shares) solely. Such an approach allows for more than just one definition of labor values, with Flaschel's (1983a) case as a special example. We would however maintain here that firms' actual behavior should be taken into account when searching for a determined labor value definition. Firms indeed reverse the order in cost allocation procedures in the case of pure joint products (by using relative sales values to obtain the costs to be allocated to a single item in the joint output basket) in order to get determinacy. We should therefore also be prepared to use such values in total labor cost allocation, since joint production exhibits unavoidable degrees of freedom that must be closed in reference to factual procedures in firms' behavior.

In a comparable case, Rowthorn (1974) has solved the reduction problem of skilled to simple labor in terms of a physical approach solely. The question here too is to what extent market prices should have an impact on labor value accounting or not. In view of the preceding section and its principles (also with respect to the rule for free goods) we believe that the contact to actual accounting procedures on the level of the firm and the level of the whole economy is a necessary one in order to arrive at a concept of labor values that is factual in nature and applicable to the data generated by the evolution of capitalist economies. Yet, in this respect the ASSI has surely its own merits, in categorizing and measuring facts of this evolutionary process based on nominal magnitudes solely and has in this respect for example received recent reconsideration and application in the work of Mohun (2004) and others. Our dual approach (of this subsection) is more difficult to handle than this approach, and in fact an extension of it, and is directed towards a total cost measure of labor inputs into the production of the various commodities which can be applied to an analysis of the labor productivity implications of price- and profitability-driven capitalist technological change, an important issue at least on the level of macroeconomics (where for example productivity slowdowns have been discussed intensively), but similarly on the level of industries whose productivity changes are to be measured and evaluated.

Turning now to applications of labor value accounting (in the case of the single production system A, l so far considered), we use actual prices p to show the relationship between input–output tables A^n that are measured in nominal terms (and their corresponding labor usage vector l^n), which show the \$-inputs (labor inputs) per \$ of output value and the ones measured in physical terms. Denoting by $\hat{p}$ the diagonal matrix which can be obtained from the price vector p the relationship between the monetary and the physical tables are then given by: $A^n = \hat{p} A \hat{p}^{-1}$ ($l^n = l \hat{p}^{-1}$). There follows that the measurement of total labor costs per \$ of output value, v^n, is given by the matrix equation $v^n = v^n A^n + l^n$, while labor values per unit of output are of course still given by $v = vA + l$. It is straightforward to show that there holds $v^n = v \hat{p}^{-1}$. We thus get that labor values can immediately (in principle) be calculated from monetary input–output data which in fact even deliver the value-price relationship at one and the same time.

Conventional labor values are therefore (and this also holds for joint production when the industry technology assumption of input–output analysis is used, see the next chapters) factual magnitudes that can in principle be measured and studied in their evolution in time. In the following chapters we will consider uses of these accounting magnitudes in detail, which will here only be summarized in their essential features. The principles we have considered in Sect. 2.2 in this chapter can all be applied to the now considered dual to the sphere of prices (of production), but we shall concentrate here our efforts on the fundamental properties our MDSA gives rise to.

A first basic property of labor values is that they are always smaller than prices measured in terms of the wage unit (if profits are positive in all sectors of the economy), i.e., we have $p_w = p/w > v$. The labor time commanded by the various commodities thus provide an upper estimate of the labor time that was embodied (imputed) into them. This provides an important bridge to what Keynes considered as real magnitudes in the General Theory, namely the nominal expressions divided by the wage unit.

A second important property of conventionally defined labor values or the total labor costs of commodities is that they fulfill the following proposition:

> Assume that technical change is profitable (as measured by actual prices) and in a strict sense capital-using and labor saving. Then: the total labor costs of commodities as measured by the above vector v (all) decrease (if the input–output matrix is indecomposable).

This theorem will be formulated and proved in detail in the next chapter. It shows that there are deterministic foundations for the statistical "law of decreasing labor content" that is formulated and proved in Farjoun and Machover (1983, Chap. 7). Such a law is assumed to exists on the macrolevel by nearly every macro-theory (if applicable) and it here receives a fundamental formulation through a comparison of prices in terms of the wage unit and our labor value accounting scheme.

A third important property of labor values (in their own right) is that they can be used to measure labor productivity, as proposed in Marx's Capital I (by means of the reciprocal values $1/v_i$), and in the United Nations (1968) SNA. Using the matrix equation $x_i = Ax_i + e_i$, where e_i is the ith unit vector, i.e., calculating the total input basket in order to produce one unit of commodity i immediately implies the relationship:

$$L_i = lx_i = l(I - A)^{-1}e_i = v_i, \quad i.e.,$$

the labor time needed to produce one extra unit of net output of commodity i is given by the labor content of this commodity. This property of labor values will be investigated in detail in the next chapter.

Final important properties of a system of labor value accounts have already been studied in Chap. 1 where we have identified the average value rate of profit as the systematic (production oriented) component in the average price rate of profit which – following again Farjoun and Machover (1983) may be subject to chaotic influences from the sphere of commodity exchange that are statistically viewed of second order type.

Summarizing this subsection we would claim here that the conventional type of labor value accounting has important roots in firms accounting procedure as well as national input–output accounting procedures that not only imply that such labor content are well-defined in general models of production, but also give rise to meaningful proposition concerning labor productivity, technical change, the price rate of profit and Classical labor commanded prices and that allow to proceed with the labor theory of value as expressed by the principles formulated at the beginning of this chapter. This implies that the line of research which has been put forward in Bródy (1970) can be continued successfully in very general and applicable terms.

2.4 Conclusions

Summing up, we would conclude that the CDSA and the MDSA are closely related with each other and can supplement each other. The CDSA lays more stress on (however sometimes questionable) macroeconomic real accounting, obtained from single or double deflating methods, like real GDP, real values added and the like in order to characterize the performance of capitalist economies. By contrast, the MDSA puts more emphasis on multisectoral flow matrices and, with respect to the them, on the derivation and application of measures of directly and indirectly embodied labor efforts (labor content or total labor costs) and their implications for the measurement of labor productivity, see Stone's productivity considerations in United Nations (1968, p. 69) for a bridge between the two approaches. The two approaches to a system of national accounts should therefore be further integrated with each other in future research, paying also attention to the contributions provided by the ASSI of G. Duménil, D. Foley and others on the level of price aggregates normalized by the labor efforts of the yearly production cycle.

The TSSI, by contrast and on the one hand, is in our view however not of help here because its definition of labor values is too temporarily oriented or too futile to provide an anchor for actual productivity measurements as they are discussed in Marx (1954, p. 48) Capital, Vol. I:

> In general, the greater the productiveness of labour, the less is the labour-time required for the production of an article, the less is the amount of labour crystallised in that article, and the less is its value; and *vice versâ*, the less the productiveness of labour, the greater is the labour-time required for the production of an article, and the greater is its value. The value of a commodity, therefore, varies directly as the quantity, and inversely as the productiveness, of the labour incorporated in it.

This quotation is much more in line with what is proposed in Stone's SNA as labor productivity indices, see United Nations (1968, p. 69, 4.42) which in our notation and slightly simplified reads:

$$\Lambda^{***} = \frac{x'(1)(I - A'(1))(I - A'(0))^{-1}l'}{x'(1)l'(1)} = \frac{l(I - A(0))^{-1}(I - A(1))x(1)}{l(1)x(1)}$$
$$= \frac{v(0)y(1)}{v(1)y(1)}$$

where 0, 1 denote points in time and x, y are feasible vectors of gross and net production. Stone's measure Λ^{***} thus exactly describes (in inverted form) the change in (the conventional) labor value of the net vector of period 1 that occurs through the technical change leading from $A(0), l(0)$ to $A(1), l(1)$. The increase of his measure thus shows increasing labor productivity in the sense of Marx (1954, p. 48).

On the other hand, the TSSI concept of prices of production and their uniform rate of profit (on which level of dis-aggregation?) may not be a well-suited one as far as an analysis of capitalist competition, in particular in the age of globalization – we are currently subject to – is concerned. Therefore, identities between labor value aggregates and production price aggregates are not the most important thing a Marxian theory of value (essence) and prices (surface) has to investigate. We would follow here at least partly the suggestions of Farjoun and Machover (1983) that we should use empirically applicable measures of labor content and actual prices (normalized by the wage-unit) to further study in particular their law of decreasing labor content from the theoretical as well as from the empirical perspective. Actual prices normalized by the MELT condition on the other hand may be used to study the conflict about income distribution in the spirit of Marx (1954, Chap. 23) formulation of a general law of capitalist accumulation.

Another subject for future research may be to reconsider the concept of abstract labor introduced by Marx in Capital, Vol. I, and to provide a sociological framework where Marx's objective to understand the laws of motion of capitalism from the perspective of "equivalent exchange" (as the underlying link between human beings, but covered by the laws that regulate actual commodity exchange) can be substantiated from the quantitative point of view under general production relationships. In this respect, Keynes' (1936, p. 213/4) is indeed expressing a somewhat similar point of view, when he writes:

> It is preferable to regard labour, including, of course, the personal services of the entrepreneur and his assistant, as the sole factor of production, operating in a given environment of technique, natural resources, capital equipment and effective demand.

From such a point of view it may then be a worthwhile attempt to understand the reported stock-flow interaction of capitalist economies from the angle of the single factor of production that allows this interaction to continue in time, by help of the Classical concepts for the analysis of the evolution of capitalist economies were labor commanded prices (Keynes' concept of prices in terms of the wage unit) and total labor costs (Marx's concept of value) are of central importance. The present section has argued in this regard (see the next chapter for details) that there are links between these two measures (expressed in terms of labor) that may be relevant for the understanding of the general laws of capitalist accumulation, the technical changes that drive this accumulation and the forces of competition by which these laws are implemented.

In the next chapters we concentrate on the MDSA approach to labor values and show that this dual approach to social accounting is indeed compatible with the accounting practices on the level of firms as well as on the level of whole economies, as provided by the United Nations' (1968) System of National Accounts, not only as far as pure joint production is concerned, but also in the treatment of fixed capital. Marx's labor theory of value therefore performs quite well when compared in the details of its accounting with the accounting practices of the conventional SNA, its proper counterpart when comparisons have to be made. Prices of production by contrast – which are just another accounting scheme – must prove their relevance as centers of gravity of market prices theoretically as well as empirically and it may happen here, if one follows and extends Farjoun and Machover's (1983) arguments, that Samuelson's (1971) eraser must be applied to them as a suggested "relevant" link between the sphere of production (labor values) and the sphere of competition (actual prices). Theoretical as well as empirical relevance decides what type of accounting concepts are of help in the analysis of capitalist reproduction and here it may happen that the law of decreasing labor content is much more to the point than the law of equalizing profit rates in a globalized world with agricultural production, manufacturing and industrial as well as consumer services production.

Summing up, we view Marx's (1954) Capital, Vol. I as providing through appropriate definition the essential categories and theoretical (internally consistent) structure the underlying the analysis of competition discussed in Vol. III of "Das Kapital". This marxian System of National Accounts need not be transformed to the interaction of price and quantities happening on the surface of the economy. Instead it must show its usefulness by its application to what happens empirically in the monetary dynamics of a capitalist economy, the observed real phenomena, like productivity increases as well as productivity slowdowns. Here it may be that the Marxian definition of the rate of exploitation and its changes provides the essential source for increases in the rate of profit, though there are of course secondary elements – like changes in the turnover time of capital – that may lead to increases in the observed average rate of profit as well. All concerned magnitudes – and the input–output data they are based on – have to be understood as moving averages (which may only be measured once a year). This latter fact should therefore not lead us to the empirically false conclusion that we can use point-input point-output models in the discussion of the baseline approaches to the labor theory of value.

References

Bródy, A. (1970). *Proportions, prices and planning*. North Holland: Amsterdam.

Bródy, A. (1987). Prices and quantities. In J. Eatwell, M. Milgate, & P. Newman (Eds.), *The new Palgrave series: Problems of the planned economy* (pp. 218–226). London: Macmillan.

Duménil, G. (1983). Beyond the Transformation Riddle: A Labor Theory of Value. *Science and Society, 47*, 427–450.

Duménil, G. (1984). The So-Called 'Transformation Problem' Revisited: A Brief Comment. *Journal of Economic Theory, 33*, 340–348.

Duménil, G., & Levy, D. (1987). Values and Natural Prices Trapped in Joint Production Pitfalls. *Zeitschrift für Nationalökonomie, 47*, 15–46.

Duménil, G., & Levy, D. (1988). Linear joint production models. Prelude to a reassessment of the classical legacy (value, equilibrium, and disequilibrium). *Political Economy. Studies in the Surplus Approach, 4*, 185–212.

Duménil, G., & Levy, D. (1989). Labor Values and the Imputation of Labor Contents. *Metroeconomica, 40*, 159–178.

Duménil, G., & Levy, D. (1993). *The Economics of the profit rate.* Cheltenham, UK: Edward Elgar.

Duménil, G., & Levy, D. (2000a). The conservation of value. A rejoinder to Alan Freeman. *Review of Radical Political Economy, 32*, 119–146.

Duménil, G., & Levy, D. (2000b). Paolo Guissani's defense of sequential values. *Review of Political Economy and Sciences, 4*, 179–83.

Eatwell, J., Milgate, M., & Newman,P. (Eds.). (1992). *The new Palgrave series: Marxian economics*. London: Macmillan.

Farjoun, E., & Machover, M. (1983). *Laws of chaos.* London: Verso Editions and NLB.

Fisher, F. M., & Shell, K. (1972). *The economic theory of price indices. Two essays on the effects of taste, quality, and technological change*. New York: Academic.

Flaschel, P. (1980). The derivation and comparison of employment multipliers and labour productivity indexes using monetary and physical input–output tables. *Economics of Planning. 16*, 118–29.

Flaschel, P. (1983a). Actual labor values in a general model of production. *Econometrica, 51*, 435–454.

Flaschel, P. (1983b). *Marx, Sraffa und Leontief. Kritik und Ansätze zu ihrer Synthese.* Europäische Hochschulschriften. Frankfurt/M.: Peter Lang.

Flaschel, P. (1984). The so-called 'transformation problem' revisited. A comment. *Journal of Economic Theory, 33*, 349–351.

Flaschel, P. (1995). *Fixkapital und Profitabilität. Gesamt- und einzelwirtschaftliche Aspekte.* Frankfurt/M.: Peter Lang.

Foley, D. (1982). The value of money, the value of labor power and the Marxian transformation problem. *Review of Radical Political Economy, 14*, 37–47.

Foley, D. (1983). On Marx's theory of money. *Social Change,* 1.

Foley, D. (1986). *Understanding capital*. Cambridge, MA.: Harvard University Press.

Foley, D. (1997). Marx and non-equilibrium economics. *Eastern Economic Journal, 23*, 493–496.

Foley, D. (2000). Recent developments in the labor theory of value. *Review of Radical Political Economics, 32*, 1–39.

Foley, D. (2002). Temporality and value in Marxian economic theory. New School University, mimeo.

Freeman, A., & Carchedi, G. (Eds.). (1996). *Marx and non-equilibrium economics.* Cheltenham, UK: Edward Elgar.

Freeman, A., Kliman, A., & Wells, J. (Eds.). (2004). *The new value controversy and the foundations of economics.* Cheltenham, UK: Edward Elgar.

Keynes, J. M. (1936). *The general theory of employment, interest and money*. New York: Macmillan.

Kliman, A., & McGlone, T. (1999). A temporal single-system interpretation of Marx's value theory. *Review of political economy, 11*, 33–59.

Kliman, A., & Freeman, A. (2006). Replicating Marx: A reply to Mohun. *Capital and Class, 88*, 117–123.

Kliman, A. (2007). *Reclaiming Marx's capital: A refutation of the myth of inconsistency*. London: Lexington Books.

Krause, U. (1980a). Abstract labor in general joint systems. *Metroeconomica*, XXXII, 115–135.

Krause, U. (1980b). *Money and abstract labour: On the analytical foundations of political economy*. London: Verso.

Krause, U. (1998). Abstract labour and money. In H. D. Kurz & N. Salvadori (Eds.), *The Elgar companion to classical economics* (pp. 6–10). Cheltenham UK: Edward Elgar.

Marx, K. (1954). *Capital. A critique of political economy* (Vol. 1). New York: Lawrence + Wishart.

McGlone, T., & Kliman, A. (1996). One system or two? The transformation of values into prices of production vs. the transformation problem. In A. Freeman & G. Carchedi (Eds.), *Marx and non-equilibrium economics*. Cheltenham, UK: Edward Elgar.

Mavroudeas, S. (1999). Marx and non-equilibrium economics. *Capital and Class*, 67.

Mohun, S. (1993). A note on Steedman's 'joint production and the new solution to the transformation problem'. *Indian Economic Review, XXVIII*, 241–246.

Mohun, S. (1994). A re(in)statement of the labour theory of value. *Cambridge Journal of Economics, 18*, 391–412.

Mohun, S. (2003). On the TSSI and the exploitation theory of profit. *Capital and Class, 81*, 85–102.

Mohun, S. (2004). The labour theory of value as foundation for empirical investigations. *Metroeconomica, 55*, 65–95.

Mohun, S. (2005). On measuring the wealth of nations: the U.S. economy, 1964–2001. *Cambridge Journal of Economics, 29*, 799–815.

Mohun, S., & Veneziani, R. (2007). The incoherence of the TSSI: A reply to Kliman and Freeman. *Capital and Class, 92*, 139–146.

Mongiovi, G. (2002). Vulgar economy in Marxian garb: a critique of temporal single system marxism. *Review of Radical Political Economy*, *34*, 393–416.

Picard, R. (1979). Gibt es ein Transformationsproblem? In: H. G. Backhaus et al. (Eds.), *Gesellschaft – Beiträge zur Marxschen Theorie*, 13, 54–68. Frankfurt, a.M.: Edition Suhrkamp.

Rowthorn, R. (1974). Skilled labour in the Marxian system. *Bulletin of the Conference of Socialist Economists*, 1–12.

Samuelson, P. (1971). Understanding the Marxian notion of exploitation: A summary of the so-called transformation problem between Marxian values and competitive prices. *Journal of Economic Literature, 9*, 399–431.

Shaikh, A., & Tonak, A. (1994). *Measuring the Wealth of Nations. The Political Economy of National Accounts.* Cambridge, UK: Cambridge University Press.

Simonovits, A., & Steenge, A. (Eds.). (1996). *Prices, growth and cycles. Essays in honor of Andras Bródy*. Basingstoke: Macmillan.

Steedman, I. (1977). *Marx after Sraffa*. London: New Left Books.

United Nations. (1968). *A system of national accounts*. New York: Studies in Methods, Series F, No.2, Rev.3.

United Nations. (1993). *System of national accounts*. http://unstats.un.org/unsd/ sna1993/handbooks.asp

Veneziani, R. (2004). The Temporal Single-system Interpretation of Marx's Economics: A Critical Evaluation. *Metroeconomica, 55*, 96–114.

Veneziani, R. (2005). Dynamics, disequilibrium, and Marxian economics: A formal analysis of temporal single-system Marxism. *Review of Radical Political Economics, 37*, 517–529.

Wright, I. (2007). Nonstandard labor values. Milton Keynes: Open University Discussion Papers in Economics.

Chapter 3
Using Labor Values: Labor Productivity and Technical Change

> The law of decreasing labour-content is a prime example of a tendency that operates "behind the backs" of the social protagonist, as though it were a law of nature. The fact that it nevertheless does operate must be explained by the existence of some systematic connection between the visible and the invisible - between price and labour-content. Without such a "black law" (to use Rosa Luxemburg's apt phrase) it is quite incomprehensible why individual actions motivated by considerations of price should in the long term result in a systematic effect on labour-content (Farjoun and Machover 1983 p. 84).

3.1 Introduction

In this chapter we provide theoretical and empirical examples for the usefulness of the Marxian Dual System Approach (MDSA) to the LTV, here from a very pragmatic perspective, namely concerning their role to serve (in reciprocal form) as measures of labor productivity which allow to discuss the sectoral implications of technological change as we observe it happening now in more and more rapid terms. Already Marx supplied such an interpretation of his measure of labor values, see for example Marx's (1954, p. 48), where he discusses the reciprocal relationship between labor values (total labor costs) and the measurement of labor productivity in the production of commodities by means of commodities and labor. Moreover, there are theoretical links between cost-reducing (profitable) technical changes and decreases in the labor content of commodities that assume various forms, depending on the type of technical change that is occurring.

We show in the next section of the chapter that conventional measures of labor productivity, see United Nations (1993), can be very misleading in grasping what is going on in the production side of the economy.[1] Compared to these measures the Marxian view to use labor values for this purpose in reciprocal form is shown to be well-defined and superior. This indicates again that the strength of the labor theory of value lies in its use as a system of national accounts and not as a means to understand price formation in a capitalist economy (though total labor

[1] See Flaschel (1983) for the original article (in German) on the arguments considered here in Sect. 3.2.

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_3,

costs are clearly an important – if not the central – component in prices as measured by Keynes (1936) wage-units. In addition we will consider in Sect. 3.3 theorems that reveal the consequences of cost-reducing technical change for labor values as well as their interpretation as indexes of labor productivity. These propositions provide theoretical explanations for the law of decreasing labor content, in addition to what has been shown by Farjoun and Machover (1983) from a probabilistic point of view. In this section we also provide some empirical applications of these propositions and show that this law has much wider applicability then is suggested on the purely theoretical level.

3.2 Labor Productivity. A Marxian Critique of its Value-Added Decomposition

3.2.1 The Measurement of Labor Productivity

Measures of labor productivity play an important role as foundation for our understanding of the evolution of economies with respect to growth, employment, but also inflation (income distribution) and international competitiveness. On the one hand, one may ask how much additional labor is needed for a given change in final demand, which leads to the idea of so-called employment multipliers or, in single product systems, a measure of the total labor costs that can be imputed to the production of single commodities.[2] On the other hand, one may use real GDP per worker or work-hour or – with respect to single branches of the economy – "real value added" per work hour to measure the performance of (the sectors of) the economy.

In this section we show however that any measure of the latter type can be quite misleading if it is based on constant price data (measured in terms of the prices of some base year) and must be considered as inadequate if only a single deflator – for example the GDP deflator – is used to measure the performance of single sectors, besides its use to measure the inflation rate in a certain economy for a certain time period, see the discussion of the CDSA in Chap. 2. The basic conclusion of this section will be that measuring sectoral and total labor productivity should be based on technological data as much as possible (though never totally, due to the unavoidable existence of a certain degree of aggregation) and not on data that by their very formulation explicitly employ a set of price expressions in addition. Measures of labor productivity are thus concepts that are to be based on input–output calculations and analysis and not on the real value added concepts of the System of National Accounts. These latter concepts at best measure purchasing power, but not real output on the sectoral level as will be shown in this section.[3]

[2] These two measures are identical for basic Leontief matrices, but can become very different for more general systems of production, see Chap. 4.

[3] Revisions of the United Nations' System of National Accounts (SNA) have reintroduced the conventional measures of labor productivity (in place of the indices proposed here and – in

Such magnitudes have been criticized in the (German) literature as being even devoid of content if they are obtained by means of the method of double deflating where outputs and inputs are deflated by their respective price deflators, see for example Neubauer (1978), Meyer (1981). Neubauer (1978, p. 123) for example writes:

> Einen Nettoproduktionswert der doppelten Deflationierung zu unterwerfen heißt: Eine Deflationierungsmethode, die nichts anderes leisten kann und soll als die Isolierung der physischen Komponente aus den Veränderungen eines Wertaggregats, auf einen Wertausdruck anzuwenden, der eine isolierbare physische Komponente nicht enthält. Das Resultat muß ein gänzlich fiktives sein, dem es an Validität gebricht.

We fully agree with this negative judgment on the practices of actual statistical measurement and will criticize the resulting measures of labor productivity similar to what has been shown in Meyer (1981) for measures of sectoral net production, here for the SNA concept of total labor productivity and its conventional disaggregation into measures of sectoral labor productivities. To do this we will reflect such productivity measures from the perspective of the underlying input–output table. Thereafter we will introduce our alternative measures of sectoral and economy-wide labor productivity indices which by construction will represent purely technological concepts as long as input–output coefficients can be interpreted in this way. We will compare both concepts in their potential to grasp the consequences of final demand changes and also technological changes. We will do this on the general n-sectoral level as well as for two-sectoral numerical examples. They will in particular show that input–output oriented measures of labor productivity may show (correctly) the opposite direction of change when compared with measures that are based on sectoral "real net production values".

We conclude this section by showing that this misleading disaggregation of real GDP into sectoral components can be replaced by a decomposition that is based on total labor costs (equal to the employment multipliers of input–output calculations if only single product industries are considered) in a sound and intuitively understandable way. The conclusion will therefore be that input–output tables should always be considered as an integral part of the System of National Accounts and should be the point of reference for all concepts that try to provide "quantity measures" on the macro or the meso-level of economic activity. In the case of our ADSA-based measure of labor productivity, see Chap. 2, we will close this chapter with the assertion – based on some examples we have calculated for a 7 sectoral model of the German economy – that labor values (the total labor costs of commodities) will tend to fall in time, see also Farjoun and Machover (1983), implying rising labor productivity, a fact that is not at all easily visible by just looking at the sequence of input–output tables from which this sequence of labor values is constructed.

---

input–output methodology– by Richard Stone) and are therefore now proposing again measures which this section seeks to show are devoid of empirical content.

3.2.2 Input–Output Tables and Measures of Real Value Added

Point of departure for our reconsideration of measures of labor productivity that are proposed in the literature is Table 3.1 and its representation of a standard and single input–output table. It shows in the usual way the interindustry transactions of a given economy for a particular year including final demands, total outputs and values added for and in the n branches of the economy.[4]

Meyer's (1981) article compares the methods of double and single or real value deflating with regard to determining real gross production values and GDP. According to Neubauer (1978), Meyer's method of real value calculations results in the recommendation of a single deflator, e.g. of the Paasche type

$$p_p = \sum_i f_i(t)p_i(t) \Big/ \sum_i f_i(t)p_i(0) \tag{3.1}$$

The therewith deflated gross production values or real quantities of value added $Y^*(t) = F^*(t)$ then fulfill

$$Y^*(t) = Y(t)/p_p = \sum_i Y_i^*(t) = \sum_i f_i(t)p_i(0) = F^*(t). \tag{3.2}$$

In contrast to this inflation adjusting of value added, the method of double deflating attempts to count everything in constant prices, i.e., with regard to Table 3.1 to work with prices $p_1(0), ..., p_n(0)$ of the base period 0 instead of current prices $p_i(t)$. This method is problematic in that, initially, it provides no insight on how to deal with the row of values added in the above Table 3.1.

This problem is then solved by applying the accounting consistency requirement of the nominal input–output Table 3.1 also to its analogue in constant prices $p_i(0)$.

Table 3.1 The standard form of an input–output table

Delivery from ↓ to →	Sector 1 ... Sector n	Final demand	Row sum
Sector 1	$x_{11}(t)p_1(t) \dots x_{1n}(t)p_1(t)$	$f_1(t)p_1(t)$	$x_1(t)p_1(t)$
.	. .	.	.
.	. .	.	.
.	. .	.	.
Sector n	$x_{n1}(t)p_n(t) \dots x_{nn}(t)p_n(t)$	$f_n(t)p_n(t)$	$x_n(t)p_n(t)$
Value added	$Y_1(t) \dots Y_n(t)$	–	$Y(t)$
Column sum	$x_1(t)p_1(t) \dots x_n(t)p_n(t)$	$F(t)$	

[4] We stress that the use of a price vector $p(0)$ of a base year 0 (in place of $p(t)$) is not sufficient to get the physical input structure behind the nominal Table 3.1, but that these prices have to be removed too in order to solve this task. Value added in base year prices thus remains a value magnitude and is thus no volume measure that is independent of relative prices as it is commonly assumed in the SNA.

This means that the value added $Y_j^o(t)$ of the double deflating method is simply the value added that would have resulted if the prices in Table 3.1 had remained constant after their base year determination. It is obvious from (3.2) that in this aggregate case the identity

$$\sum_i Y_i^*(t) = Y^*(t) = Y^0(t) = \sum_i Y_i^0(t)$$

must apply when both methods are compared. It is however not possible to transfer these results to the sectorial quantities of income $Y_i^*(t)$ and $Y_i^o(t)$.

This suggests that fictitious (price deflated) quantities of income, $Y_j^o(t)$, which do not satisfy any argument of real purchasing power, have no economic content. This has also been demonstrated by Neubauer (1978) and Meyer (1981) by a different set of arguments.

As already carried out introductorily, there are meaningful economic structural coefficients, e.g. macroeconomic labor productivity, the content of which is closely related to the intention behind the above mentioned values $Y_j^o(t)$. Following Stobbe (1980, p. 313/335) it is possible to decompose the value of macroeconomic labor productivity $\pi^o(t)$ in the following way:

$$\begin{aligned}\pi^o(t) = Y^o(t)/L(t) &= \sum_j (L_j(t)/L(t)) \cdot (Y_j^o(t)/L_j(t)) \\ &= \sum_j g_j(t)\pi_j^o(t), \ \sum_j g_j(t) = 1 \end{aligned} \tag{3.3}$$

Here, $L(t)$ and $L_j(t)$ indicate the employed work hours for the entire economy and the sector j, respectively.

Following the two authors mentioned above, one might expect that criticism of the gross production value $Y_j^o(t)$, in constant prices, can be extended to the sectoral labor productivity indices $\pi_j^o(t)$. However, Neubauer and Meyer give only short or implicit evidence to support such a view. In contrast, Härtel (1981, S. 190/1) holds the view that value added in constant prices as a "quasi-quantity-aggregate" is, especially in this case, a construct rich in content, answering correctly the question of which sectors can be accredited with the rise of macroeconomic labor productivity. Therefore, it is of interest to further reflect on such aspects so far only briefly deduced from the work of Neubauer and Meyer.

It is possible to approach a critical examination of the above coefficient (3.3) of labor productivity of the system of national accounts and its structure $(\pi_1^o, ..., \pi_n^o)$ due to the following circumstances. On the one hand, distinct from the view of "gross production value in constant prices," labor productivity can supposedly be understood as a well-defined technological concept. On the other hand, given that the volume-oriented (double) deflation of the input–output Table 3.1 - neglecting the row on values added - is no doubt meaningful, we have an instrument to examine the relation of labor productivity to technology (see Sect. 3.2.1). To do so, one has to make use of the quantitative input–output structure which has been formulated above. Furthermore, input–output calculations which are based on such structures, have led to the separate term "sectoral labor productivity" which we have to compare with the concepts introduced above.

Remark. We remark briefly that we will considered later on and in later chapters expressions such as

Double deflation: $p_2(0)C/L_2$ as well as Single deflation: $p_2(t)C/L_2/p_p$

as problematic expressions as far as the measurement of labor productivity in the consumption good sector C (sector 2) is concerned, in an economy where the net output consists of investment goods I and consumption goods C in such a two-sector framework. Since the single deflation measure is clearly not suitable for such a measurement we will refrain in the following from considering this deflation method any further as far as the determination of volume or productivity indices is concerned.

3.2.3 Labor Values as Measures of Labor Productivity

It is common practice in the technological evaluation of input–output tables to choose the units of n produced commodities in a way that one can assume $p(0) = (p_1(0), ..., p_n(0)) = (1, ..., 1)$ (with regard to the base period), i.e., this base price vector is set equal to the summation vector e'. In the following, we denote diagonal matrices, which have been composed with the vector e as the diagonal, as $I = \hat{e}$, the unity matrix. It is then possible, to express Table 3.1, double or row-wise "price deflated" in matrix notation as shown in Table 3.2. where $'$ denotes – as usual – row vectors instead of column vectors.[5] It is also common practice in input–output-analysis to transform the matrix of intermediate inputs X into the so-called matrix of input coefficients $A = X\hat{x}^{-1}$. This is done - as shown above - by dividing all columns (processes) in X by the corresponding output value x_j. Accordingly, we also get $\ell = l\hat{x}^{-1}$ with regard to the vector of the labor inputs $l = (L_1, ..., L_n)$. In the following we will abstain from the dating t — as already done – unless needed for clarification.

Table 3.2 Elementary input–output table in matrix notation

\	$1 \ldots n$		
1 · · · n	X	f	x
	$y^{o'}$	-	Y^o
	x'	F^o	-

[5] We use this symbol if the original vector was a column vector, but stress that prices, etc. are always given as rows (without any $'$-superscript.

The macro-identity $Y^o = p(0)f = F^o$ which is behind Table 3.2 can be made explicit with help of the above symbols as follows

$$Y^o = y^{o'}e = e'(I - A)x = e'f = F^o \tag{3.4}$$

It can be reasonably completed by the following identity which is obvious by definition of the vector $v = \ell(I - A)^{-1}$ but important nevertheless

$$L = le = \ell x = \ell(I - A)^{-1}f = vf \tag{3.5}$$

Vectors of the type v are commonly used in input–output-analysis. They are applied to acquire indirect effects, here with regard to the matrix of intermediate inputs; the vector v then represents in particular labor time spent directly or indirectly in the production of the n types of commodities. Equation (3.5) thus simply shows that labor time vf that is directly and indirectly necessary to manufacture the net product f is equal to the total employment (L). The components v_i of the vector v are also referred to as system measures of labor productivity in sector i, when they are expressed in a reciprocal form $\pi_i^m = 1/v_i$.

We will show that such a denomination indeed makes sense. It will allow us to detect and to correct the frailty of (3.2), i.e., the decomposition of macroeconomic labor productivity in terms of the conventional system of national accounts. In order to avoid problems of interpretation, it is useful and common to understand the structural coefficients A, ℓ added to Table 3.2 as parameters of a linear technology. The equations are then valid for all nonnegative net product vectors f and the corresponding levels of activity x, without the need to consider them as average expressions.

We have now outlined the necessary elements needed to analyze increases of labor productivity within the frame of the given input–output structure.

Definition 3.1. We say that labor productivity has increased with regard to commodity i if an increase of the net product f by one unit of commodity i demands less additional labor with regard to this commodity than was necessary in the base period, i.e. formally:[6]

$$\text{For:}\quad l_i(t) = \ell(t)x_i(t), \quad x_i(t) = (I - A(t))^{-1}e_i \quad \text{there holds:}\quad l_i(t) < l_i(0).$$

Note that the original net product f is irrelevant in this comparison due to the supposed linearity of the technological relationships. Secondly, we say that the labor productivity has increased with regard to the total input–output system if it has not decreased for any $i = 1, ..., n$ and if an increase is observed for at least one $i \in \{1, ..., n\}$.

[6] $e_i = (0, ..., 1, ..., 0)'$ the ith unity base vector.

Proposition 3.2. *The first and the second formulation in Definition 3.1 are equivalent to: $v_i(t) < v_i(0)$ and $v(t) \leq v(0)$, respectively, where $\leq$ means semipositivity, i.e., the case of $v(t) = v(0)$ is excluded there.*

The proof of this proposition is evident, since $l_i(t) = v(t)'e_i = v_i(t)$ applies due to (3.5). It demonstrates that $1/v_i(t)$ can be meaningfully understood as a system indicator of labor productivity with regard to the production of commodity i though with the well-known disadvantage that it is not possible to deduce $v_i(t)$ simply and solely from data that characterize sector i.

It seems to be possible to avoid this "disadvantage" by starting instead from the following measure of the System of National Accounts, namely the measures of labor productivity $\pi_j^o = Y_j^o/L_j$ obtained from (3.2), instead of the input–output-theoretical sectoral measure $\pi_j^m = 1/v_j$. Yet, the advantage of using data from sector j alone to define π_j^o is only apparently an advantage: the dependence on the other sectors is solely not as evidently as it is the case of the definition of vector v. It is indeed not possible to formulate and to interpret value added Y_j – as well as value added in constant prices Y_j^o – without reference to a price system (even if this price system does not appear explicitly because of the base year assumption $p(0) = e$). The employed prices thus only reveal the dependence on the data of the other sectors – in difference to $1/v_j$ – and does so in a way that depends in an unexplained manner on the actual institutional and market specifications of the base year $t = 0$. The distinct technological foundation which characterizes the indicators v_j is here absolutely missing. Therefore, it cannot be agreed upon that the increase of gross production in constant prices per labor hour π_j^o correctly identifies which sectors were responsible for the macroeconomic increase of total labor productivity of the SNA (as Härtel (1981, p. 191) argues and as it is intended by (3.3)).[7]

We show below that attempting to identify such sectors cannot be expected to make sense in general. With regard to a basic comparison of the indicators $\pi^m = 1/v_i$ and π_j^o the following proposition however applies:

Proposition 3.3. *The identity $\pi_j^o = 1/v_j, j = 1, ..., n$ applies iff all sectors have the same (uniform) labor productivity indexes $\pi_1^o = ... = \pi_n^o = \pi^o$. Deviations between these two productivity indicators therefore must be examined and estimated in relation to sectoral productivity differences.*[8]

The proof of Proposition 3.3 also is very simple. Uniform labor productivity π^o immediately signifies that $e' - e'A = \pi^o \ell$ has to be fulfilled. It follows $(1/\pi^o)e' = \ell(I - A)^{-1} = v$.

Proposition 3.4. *Consider as given with regard to the base and the actual time period the same vector of final demands f (which furthermore is assumed for simplicity to be strictly positive).*

[7] Note here again that the indexes v_i are commodity oriented and the measures π_j^o industry oriented.

[8] The net product transformation curve exhibits in this case slope -1.

1. $\pi^o(t) > \pi^o(0)$ *follows from* $v(t) \leq v(0)$, *i.e., the technologically progressive changes as described in the second part of the definition of this subsection imply a corresponding reaction of the conventional measure (3.4) of macroeconomic labor productivity.*
2. $\pi^o(t) > \pi^o(0)$ *implies* $v(t)'f < v(0)'f$, *i.e., the expenditure of human labor for the production of the given vector of final demand* f *has necessarily decreased with this change in technology.*

To prove this proposition it suffices to remember that $\pi^o(t)$ can be defined by $p(0)'f/L = e'f/L$ due to the result in sect. 3.2.2 and that the following applies due to (3.5): $L = vf$. The change in π^o can thus completely be ascribed to changes of the input–output-theoretical measure of labor productivity.

Figures 3.1 and 3.2 illustrate the results of Proposition 3.4 focusing on the case $n = 2$. They furthermore show the technological relations addressed in the

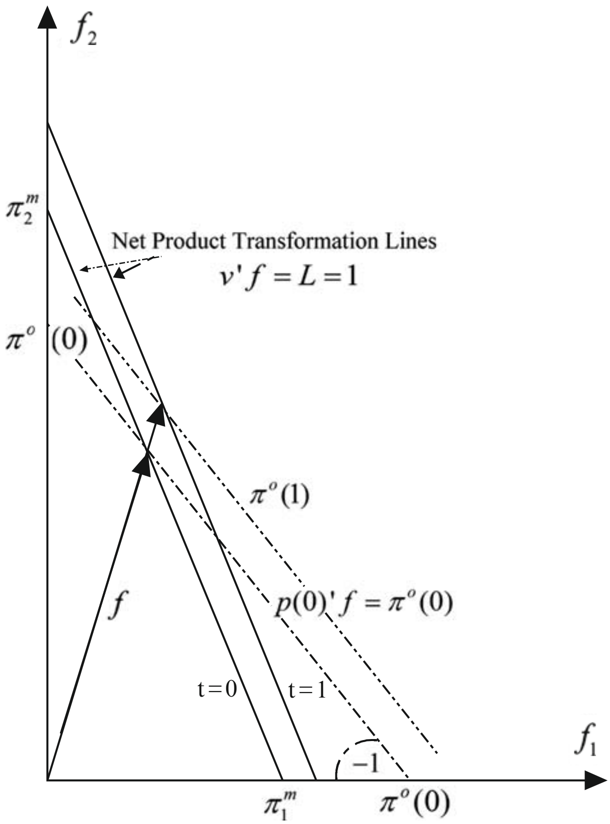

Fig. 3.1 $v(1) < v(o) \Rightarrow \pi^o(1) > \pi^o(0)$:. The case of an unambiguous increase of labor productivity and its impact on π^o based on physical transformation curves (net production lines): $v'f = L = 1$)

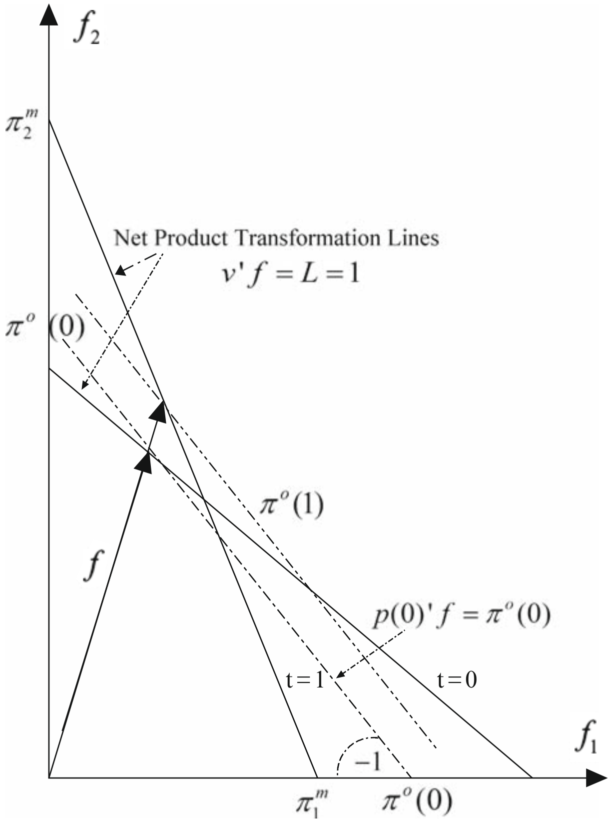

Fig. 3.2 $\pi(1)^o > \pi^o(0) \Rightarrow v(1)f < v(0)f = 1$:. The case of an increase of labor productivity that is depending on final demand

definitions and in Proposition 3.2. Note that the defining equation of labor values or total labor costs $v_1 f_1 + v_2 f_2 = L$ implies as net product transformation curve the linear equation: $f_2 = \frac{L - v_1 f_1}{v_2}$.

It can be stated as a result - at the level of macroeconomic aggregate relationships of the usual SNA measures - that macroeconomic labor productivity and the vector measures of labor productivities, which have been obtained with the help of input–output-accounting, do not act contrariwise under certain conditions.

In the following subsection we will show that this need not be the case on the sectorial level and will identify there which conclusions can be drawn from this result.

3.2.4 Notes on Technological Change

We will now consider the simple example of a two-sector-economy as described in Table 3.3 where process 1 is subject to technological change between time $t = 0$ and $t = 1$.[9]

The following now applies to this technical change:

1. The cost of production of sector 1 decrease from 0.95 to 0.85 (per unit) i.e., the technical change is profitable.
2. The value added in constant prices per unit of labor, i.e., labor productivity in terms of the SNA measure in Sect. 3.1 increases from 1.33... to 1.6 (while the labor productivity of Sect. 3.2, of course, remains constant: 2.66).
3. The indicators of labor productivity of the input–output analysis (π_1^m, π_2^m) — obtained from $vA + \ell = v$ – decrease from approx. (2.02, 2.30) to (1.85, 2.18), i.e., the considered technical change is of regressive type.
4. Therefore, the labor time which is necessary for the production of a certain net product f, has to increase when changing from t=0 to t=1, e.g. in the case of $f = (10, 5)'$ from approximately $L = 7.19$ to $L = 7.71$.

This means graphically that the line of net production, i.e., the set of combinations of both commodities, which can be produced alternatively as net output by the given technology, given the amount of employment $L(= 1)$, has to shift towards the origin, as Fig. 3.3 illustrates. The vectors f^1 and f^2 in Fig. 3.3 furthermore demonstrate with regard to the dashed budget line $pf^1 = pf^2 = const.$, that there can be cases in which $vf^1 = vf^2 = 1$ holds true, i.e., the necessary labor time L remains constant for this change in final demand, and where $e'f^2 = p(0)f^2 > p(0)f^1 = e'f^1$ is the case, where labor productivity in terms of the SNA has increased. It follows that close to these situations vectors of final demand can be found, with regard to which the necessary labor time and the labor productivity in terms of the SNA behave "perversely" to each other. This leads to the question to which extent changes in the structure of final demand should be related at all to changes of macroeconomic labor productivity. We will deal with this question in more detail in the next, concluding subsection.

Table 3.3 A two-sector economy with profitable C(apital)S(aving)-L(abor)U(sing) technological change (based on constant prices data $p(0) = e', w = 1$)

Structure\ Period	$t = 0$		t = 1	
Matrix of intermediate input A	0	0.4	0.2	0.4
	0.8	0.2	0.4	0.2
Inputs of labor ℓ	0.15	0.15	0.25	0.15

[9] Note that the terms Capital-Saving/Using here (and in the following) only apply to the aggregate value of intermediate inputs and are thus much less restrictive concepts than the ones used in the theorems of the next section.

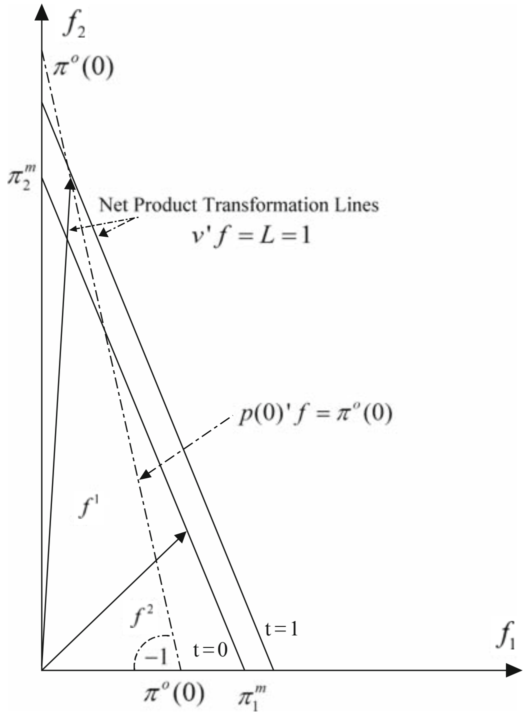

Fig. 3.3 Increase of overall labor productivity π^o in the case of a simultaneous decrease of labor productivity indices π_1^m, π_2^m (L=1 being constant, while final demand f has changed from $f^1 \rightarrow f^2$)

Table 3.4 A two-sector economy with profitable CU-LS technological change (based on constant prices $p(0) = e', w = 1$)

Structure \ Period	$t = 0$	t = 1
Matrix of intermediate inputs A	0.1 0.3	0.44 0.3
	0.4 0.3	0.1 0.3
Labor inputs ℓ	0.4 0.05	0.32 0.05

We have presented in Table 3.3 an example in which capital-saving, labor-using technical change has taken place. It has still to be examined if the opposite direction of substitution in which labor is replaced by capital, also allows the same opposition between the development of labor productivity of the SNA and of the input–output analysis. The example in Table 3.4 shows that this is indeed the case.

This technical change is also profitable, since the unit costs (in the first sector) decrease from 0.9 to 0.86. With regard to sectoral labor productivity π_1^o and π_2^o the following applies in terms of the SNA measure:

$$\pi_1^o(1) \approx 1.44 > \pi_1^o(0) = 1.25, \qquad \pi_2^o(1) = \pi_2^o(0) = 8,$$

i.e., as in the first example, there is an increase with regard to sector 1. On the other hand, we get with regard to the indexes of labor productivity $\pi_1^m = 1/v_1, \pi_2^m = 1/v_2$ of input–output analysis contrary to the just stated increase of productivity:

$$\pi_1^m(1) \approx 1.58 < \pi_1^m(0) \approx 1.70 \qquad \text{and}$$
$$\pi_1^m(1) \approx 3.04 < \pi_1^m(0) \approx 3.09.$$

and thus again an unambiguously regressive type of technical change.

This example is even more drastic compared to the first example (Table 3.3), since it shows, that the conventional indicators of labor productivity in a narrow and a broad sense - the reciprocal values of the necessary labor time per output unit $1/\ell_i$ as well as the value added in constant prices per labor hour - can be monotonically changing in a direction which contradicts the direction of change of the sectoral labor productivities of input–output-analysis (which is monotonically decreasing). It holds, however, with regard to Figs. 3.1–3.3 that the latter constitute the coefficients, which reflect the factual technological changes that have been taking place. Therefore we have to state as a result of this section that the sectoral labor productivities of the SNA are no suitable structural coefficients for estimating to which sectors an increase of the macroeconomic labor productivity has to be assigned. The reader is referred to Sect. 3.3.5 for further observations on this matter, in particular to a two sectoral example of the atheoretical character of the measures π_j^o.

3.2.5 Disaggregating Aggregate Measures of Labor Productivity

We have just seen that the decomposition (3.3) of the macroeconomic coefficient of labor productivity as conventionally defined in the SNA can be very misleading. This holds because this procedure is based on profit-like balances that depend on income distributions instead on technological gains and necessary labor time. These balances are in addition suspect, since they have been acquired by an arbitrarily given price structure judged from the current point of view. It follows that the effects of labor effort versus productivity structure as discussed in Stobbe (1980, p. 314) do not meet the necessary standards from the viewpoint of measuring labor productivity. We have therefore to ask whether it is possible to find a decomposition of the concept of average labor productivity, which is based on, as an alternative to the indices π_j^o, the measures of technological labor productivity of the SNA? This is indeed possible and can be done in the following simple way.

We know – due to the reasoning in Sect. 3.2.2 – that the there discussed methods of deflating always lead to the same result with regard of the aggregate Y(t) = F(t), i.e.

$$Y^o(t) = \sum_j Y_j^o(t) = Y^*(t) = \sum_i p_i(0) f_i(t) \qquad (3.6)$$

This formula shows that the value-added oriented decomposition used in (3.3) may have been the wrong way since we cannot expect a quantitative structure in the background of an income magnitude as was stressed several times. Such a structure is only existent in the last term of (3.6), i.e., referring to the vector of final demand. As a net product of the given input–output system, it is clearly of physical nature, but – on the other hand – it does not give rise to an obvious decomposition which allows an imputation to the different sectors in a way such that production outcomes per labor unit can be determined on a sectoral level. This can only be done with the help of the formulae (3.4), (3.5). Applied to $\pi^o = Y^o/L$, they deliver with regard to the new point of reference, the vector of final demand f:

$$\pi^o = Y^o/L = e'(I - A)x/\ell x = e' f/v f \tag{3.7}$$

Note here the underlying assumption $p(0) = e'$. There follows from (3.7) in comparison to (3.3) by means of the weights $\breve{g}_i = v_i f_i/v f$, which express the labor time directly or indirectly used in the production of the corresponding components of final demand as part of the totally expanded labor hours, the decomposing relationship

$$\pi^o = \sum_i \breve{g}_i(\pi_i^m) \tag{3.8}$$

We regard this structuring of macroeconomic labor productivity π^o, as a weighted arithmetic mean of the sectoral labor productivities in terms of SNA, as superior to its conventional pendant (3.3) due to the arguments in Sect. 3.3.3 and the there indicated content of the latter indicators. Yet, we will not conceal here that our opinion is more based on the weakness of (3.3) as analyzed in sects. 3.3.3 and 3.3.4 than on an evidence of the validity of (3.8), since this evidence has still to be provided with regard to the coefficients v_i in spite of the existing theoretical and empirical explorations. This question is however left for future research here.

Instead, we will briefly deal with the question whether the initial measure π^o itself is problematic and would thus need to be re-examined in view of its structure as reformulated in (3.8). Does it really make sense to aggregate the (meaningful) indicators $\pi_i^m = 1/v_i$ in the form of the one-dimensional mean value as in (3.8)?

An example: If $v_1/v_2 > p_1(0)/p_2(0) = 1$ holds, a movement along the net product line $L = 1$ towards the left is accounted for as an increase of macroeconomic labor productivity by the measure π^o, on the basis of a given linear technology A, ℓ.. This situation is illustrated in Fig. 3.4.

From our point of view, one should not speak of an increase of macroeconomic labor productivity in the case of a pure change of the structure of final demand (of the above kind). A first reason for this is given by the fact that in empirical examinations of the development of aggregates like $L(t) = v(t)f(t)$ in general a splitting-up is done in a term conditional on technology and in a term which is referring to final demand as for example in

$$L(t) - L(0) = (v(t) - v(0))f(0) + v(t)(f(t) - f(0)) \tag{3.9}$$

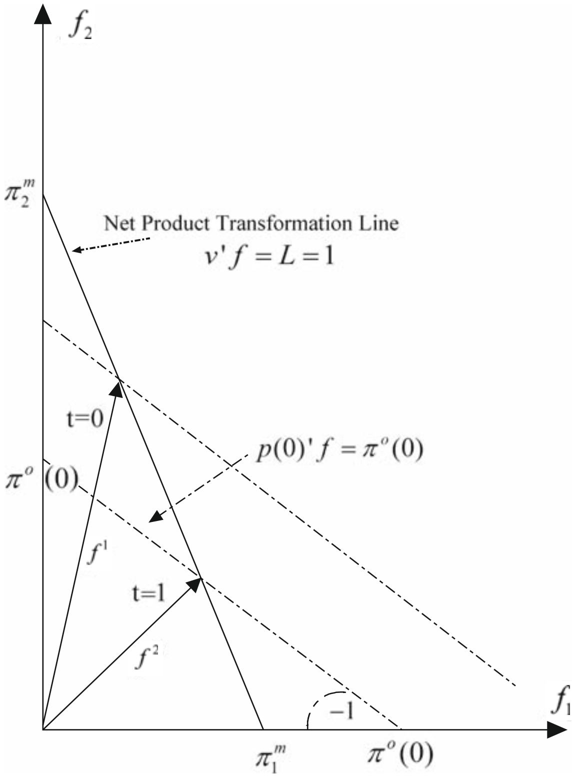

Fig. 3.4 Two vectors of final demand, which demand the same labor input, but for which $\pi^{o2} = e'f^2/L < \pi^{o1} = e'f^1/L$ holds true

These measures should allow to differentiate changes in factor inputs due to changes in consumer habits from changes that are due to technological change. How is it possible to describe this reduction of the change $L(t) - L(0)$ to its technology-dependent part $(v(t) - v(0))f(0)$ (= 0 in the above example) in the case of the measure π^o?

As the above figure shows, the increase of productivity stated there with regard to π^o has to be assigned to the fact, that the price structure and the structure of the indicators of labor productivity of the input–output calculation fall apart. This points to the way how the measure π^o could also be improved in its present form. First, we have to state, that the above presentation (3.8) of the indicator π^o:

$$\pi^o = \sum_i p_i(0) f_i(t) \Big/ \sum_i v_i(t) f_i(t) \tag{3.10}$$

shows a significant arbitrariness, i.e., its price structure p(0) which is not at all understood in its underlying causes and which thus turns out to be completely arbitrary.

Here, the question occurs whether $p(0)$ should be replaced by the indicators $v(0)$ provided at time 0 with a clearly defined content, i.e., succinctly, whether one unconsciously refers to a Paasche-index of the evaluation scheme v when using π^o. A correction could therefore be done in the following way

$$\breve{\pi}^o = \sum_i v_i(0) f_i(t) \Big/ \sum_i v_i(t) f_i(t) \tag{3.11}$$

This index would express by its increase that the labor input to produce today's net product f has decreased in relation to the base period. It would have the advantage to be homogeneous with regard to dimensions and furthermore to be really price deflated. It avoids a deficit in (3.8) which is hidden due to a clever choice of physical units, i.e., that it would have been necessary to present them in the form

$$\pi^o = \sum_i \breve{g}_i [p_i(0)/v_i(t)] \tag{3.12}$$

It again appears that two theoretical non-related, dimensionally different indicators $p_i(0)$ and $v_i(t)$ are related to each other in order to get the mean value π^o.

Therefore, our concluding proposal here is to deliver the task of productivity measurement only to the theoretically well-founded, though at present still afflicted with several severe practical problems, vectorial measure v, or to test it further with regard to this task. Besides the above Paasche-index other indices of labor productivity are noteworthy in addition, such as one of the following Laspeyres-type

$$\tilde{\pi}^o = \sum_i v_i(0) f_i(0) \Big/ \sum_i v_i(t) f_i(0) = v(0) f(0)/v(t) f(0)$$

In such a re-orientation of the macroeconomic measure of labor productivity it need not necessarily follow that measures of type (3.10) have become completely irrelevant. First, it would be necessary to clarify whether such measures may be suitable approximations for their pure theoretical forms (3.11) with good reasons, in spite of the objections raised and in spite of their inhomogeneity with regard to dimensions. This means, however, that one has to give the (often made measurements of the indicators of labor productivity of the input–output calculations within the theory of allocation) a price-theoretical fundament which is still mainly missing.

In spite of such missing allocation-theoretical reflections of the input- output measures v_i it can already be stressed here that their applicability is by no means limited to the assumptions of Table 3.3, see, e.g., Gupta and Steedman (1971) with regard to the treatment of fixed capital and imports as well as Flaschel (1980) with regard to procedures that can be applied when joint production is involved.

The technological aspect of measuring labor productivity by means of the indicators v_j as examined in this section is therefore not subject to principal definitional barriers. This may be regarded as a further argument for leaving the judgement of the development of labor productivity to an input–output calculation as an integral part of the System of National Accounts. The conventional price deflated measures of value added are neither necessary nor meaningful for this purpose. This also means that the second theoretical aspect of our analysis of labor productivity which we have mentioned earlier cannot be associated with it without problems. It should therefore be excluded from this discussion.

3.2.6 A Summing Up

We are now in the position to compare the following three per capita expressions with each other with respect to their economic content on the macroeconomic as well as on the microeconomic level (where they are decomposed into their sectoral components).

$$\frac{Y^o(t)}{L(t)} = \sum_j \frac{L_j(t)}{L(t)} \cdot \frac{Y_j^o(t)}{L_j(t)} = \sum_j \frac{v_j(t) f_j(t)}{v(t) f(t)} \cdot \frac{p_j(0)}{v_j(t)} \tag{3.13}$$

$$\frac{Y_w(t)}{L(t)} = \frac{Y^n(t)}{wL(t)} = \sum_j \frac{v_j(t) f_j(t)}{v(t) f(t)} \cdot \frac{p_{wj}(t)}{v_j(t)} = \sum_j \frac{v_j(t) f_j(t)}{v(t) f(t)} \cdot \frac{p_j(t)}{wv_j(t)} \tag{3.14}$$

$$\frac{L(t)}{L(0)} = \frac{\sum_j v_j(t) f_j(t)}{\sum_j v_j(0) f_j(t)} = \sum_j \frac{v_j(0) f_j(t)}{v(0) f(t)} \cdot \frac{v_j(t)}{v_j(0)} = \frac{v_j(0) f_j(t)}{v(0) f(t)} \sum_j \cdot \frac{\pi_j^m(0)}{\pi_j^m(t)} \tag{3.15}$$

We consider (3.13) as a meaningless decomposition of GDP at constant prices and also stress that this average is more of an income – expenditure type than of a volume index. In place of this virtual income expression we would propose to deflate National Income (as in Keynes' (1936) General Theory) by means of the wage unit as in expression (3.14) which has moreover the advantage that it does not have a trend in the very long run (since it is the inverse of the wage share in national income). It can be decomposed – with relative labor costs as weights – into a sum of ratios representing the relation of labor commanded prices to total labor costs for the n goods of the considered economy, respectively (the amounts of hours bought by a particular commodity divided by labor embodied in this commodity). As this measure is formulated it represents an income distribution oriented representation of what is going on in an actual economy and thus not a measure of labor productivity in the technological sense of this word. Such a concept can better be represented by the ratio considered in (3.15), the labor time needed in t to produce a given vector of net output f as compared to the time the technology at time 0 would have needed for this vector of final demands. The weights needed in this case are the relative value magnitudes at time 0 needed for the given net output vector and they are applied to the relative labor values of the n commodities between period t and period 0.

We thus propose the second measure as measure of per capita real income and the third as measure of technological progress in terms of labor productivity. In macro-models of the wage-price spiral – see for example Flaschel and Krolzig (2006) – one could however employ the growth rate of measure 1 as proxy for the growth in labor productivity (based on Proposition 3.6 and due to its representation in terms of final demand). Taylor type interest rate i policy rules can in this context be reformulated as

$$i = \alpha_{iw}(\dot{w}/w - const) + \alpha_{iy}(Y_w - \bar{Y}_w)$$

They then are concentrated on wage inflation and the wage share which in a closed economy is indeed sufficient to control inflation. There is thus also here no need to use base year prices to deflate output to its real value and to calculate and employ the inflation rate for commodity prices in order to control inflationary pressure in the economy. If needed one can of course add the unemployment rate gap $U - \bar{U}$ as a further measure such a monetary policy is responding to.

3.3 Technical Change and the Law of Decreasing Labor Content

> In input output theories of prices and profit the notion of labour- content can be defined, and the law can certainly be formulated. But it cannot be deduced or explained, because in these theories there is no general systematic connection between labour-content and price (Farjoun and Machover, 1983, p. 141).

It is one purpose of the following pages to show that there is some theoretical and deterministic connection in the relationship between price and labor content which adds insights to the approach chosen by Farjoun and Machover, but does not question the probabilistic framework they have chosen in their book.

3.3.1 Basic Propositions on Price-Value Relationships

We will only briefly consider here the case of joint production, see the following chapters for more details, but will concentrate in the following propositions again on the case of a simple (single) input–output matrix A and thus on the case where the output matrix $B = I$ is the identity matrix. We will now study the relationships between prices p_i, and firm's profitability, and labor values v_i and labor productivity, and different forms of technical change, so called capital–using labor–saving (CU–LS) technical change and capital–saving labor–using (CS–LU) technical change. One typical result, in fact the one that may be characteristic of many phases in the evolution of capitalism will be that CU–LS technical change that is profitable will increase labor productiveness $\pi_i^m = 1/v_i$ at least in certain sectors, i.e., will decrease the total labor costs of producing commodities i, but there still remain CU–LS technical changes that have this latter property, but are not profitable and thus are not implemented in a capitalistic economy.

We will consider such theorems for simplicity for matrices A that are not only productive, see Chap. 1, but also indecomposable and would thus get that the above proposition will extend to all commodities i, i.e., the measures π_i^m will increase for all i. We assume for simplicity again also $l = (l_1, \ldots, l_n) > 0$ for the direct labor used up in each sector and thus know that labor values must all be positive, due to

$$v = vA + l \geq l > 0.$$

In the preceding section we have moreover already provided arguments that $\pi_i^m = 1/v_i$ is indeed the only sensible measure of labor productivity in the considered environment, since $1/l_i$ is too limited as a measure for labor productivity in the production of commodity i and since the conventional measures of labor productivity of the System of National Accounts, where the trend in the price level is eliminated or where constant prices are the basis of their calculation of real value added, do not provide workable alternatives to the measures π_i^m.[10]

Definition 3.5.

1. Technical change $(A, l) \mapsto (A^*, l^*)$ is profitable iff at initially given prices (where each sector is assumed to earn a positive rate of profit) we have for the vector p_w of prices in terms of labor commanded:

$$p_w A + l \geq p_w A^* + l^*$$

2. Technical change $(A, l) \mapsto (A^*, l^*)$ is progressive iff

$$v = vA + l \geq v^* A^* + l^* = v^*.$$

3. Technical change $(A, l) \mapsto (A^*, l^*)$ is of type CU–LS, CS–LU and CS–LS, respectively if

$$A \leq A^*, l \geq l^*; \quad A \geq A^*, l \leq l^*; \quad A \geq A^*, l \geq l^*,$$

respectively.

Proposition 3.6.[11]

1. *All CU–LS technical changes which are profitable are progressive, but there are CU–LS progressive changes which are not profitable.*
2. *All CS–LU technical changes which are progressive are profitable, but there are CS–LU changes which are profitable, but not progressive.*
3. *Technical changes of type CS–LS are always profitable and progressive.*

---

[10] The following is based on Roemer (1977), see also Roemer (1977, 1981).

[11] Note here that the assumptions in this propositions are much more restrictive than what was considered as technical change in some of the examples in the preceding section.

Remark. We thus have that profitable CU–LS changes do not fully exploit the potential of technical change to increase the labor productivity indices π_i^m, while there can be profitable CS-LU changes which may decrease the labor productivity indices π_i^m. Of course CS–LS changes are always welcome to capitalist firms and are also always increasing labor productiveness $\pi_i^m, \quad i = 1, \ldots, n.$

Note that technical change here occurs only in quantitative terms and not in terms of qualitative product changes. Note also that we at present only consider capital consumed and not capital advanced. The validity of the above proposition is thus still fairly limited, though its conclusions become more and more likely the closer total labor costs resemble actual market prices in their structure (in the limit, the above propositions indeed provide conditions that are necessary and sufficient).

We now present proofs of the above three assertions which to some extent mirror economic intuition and thus can be read also from this perspective.

Proofs.

Assertion 1. We know from work on uniform rates of profit (which we do not consider in this chapter, unless it is explicitly stated) that $0 < v < p_w$ holds true, i.e., labor values are always smaller than prices of production measured in terms of the wage–unit. This assertion however also holds for all (profitable) price systems p_w which fulfill

$$p_w A + l < p_w,$$

since the iteration[12]

$$p_w(t+1) = p_w(t)A + l < p_w(t),$$

$t = 0, 1, 2, \ldots, \quad p_w(0) = p_w$ provides us with a sequence of positive vectors which is bounded from below by zero and monotonically decreasing and thus converging to our definition of labor values:

$$p_w(\infty) = p_w(\infty)A + l = v.$$

Therefore $v < p_w$ for all actual price systems p_w in terms of labor commanded that allow for positive profits $p_w - (p_w A + l)$ in all sectors. Labor commanded prices are thus a useful upper estimate for total embodied labor costs under quite general conditions.

This proposition is now applied to the change $A \leq A^*, l \geq l^*$ which fulfills

$$p_w(A^* - A) - (l - l^*) \leq 0$$

with the terms in brackets both being nonnegative due to the above assumptions. Due to $v < p_w$ we therefrom get

$$v(A^* - A) - (l - l^*) \leq 0$$

[12] Note that the consideration of this sequence bears some relationships with the TSSI system considered in Chap. 2.

and thus[13]

$$vA^* + l^* \leq vA + l = v.$$

By the recursive application of this inequality we then again get:

$$v(t+1) = v(t)A^* + l^* \leq v(t)$$

with $v(0) = v$ and for $t = 0, 1, 2, 3, \ldots$. This sequence is again bounded from below and monotonically decreasing and thus converges to the vector

$$v(\infty)A^* + l^* = v(\infty) = v^*$$

the vector of labor values of the technology A^*, l^*. This proves assertion 1., when account is taken of the fact that $\leq$ must lead to $<$ in the case of indecomposable A's and $l > 0$. It also shows that there may be situations with $v^* < v$, $A^* - A, l - l^* \geq 0$ which are not profitable when judged from the perspective of the initial price structure $p_w > p_w A + l$, which is not proportional to v in general.

Assertion 2. We now have

$$A - A^* \geq 0, \quad l^* - l \geq 0, \quad v > v^*$$

and thus

$$v = vA + l > v^* A^* + l^* = v^*.$$

Without loss of generality we assume that technical change only occurs in sector 1. We thus have $vA_j + l_j = vA_j^* + l_j^*$ for all $j > 1$. Assume now that

$$vA_1 + l_1 < vA_1^* + l_1^* \tag{3.16}$$

would hold true (which we want to show not to be possible). Note here that, so far, we only know that

$$vA_1 + l_1 > v^* A_1^* + l_1^*$$

must hold true, i.e., labor costs at initial labor values are not yet known to decrease. If (3.16) holds, we can find however:

$$\breve{A}_1^* < A_1^*, \quad \breve{l}_1^* < l_1^* \quad \text{s.t.} \quad vA_1 + l_1 = v\breve{A}_1^* + \breve{l}_1^*$$

i.e.

$$vA + l = v\breve{A}^* + \breve{l}^*$$

[13] Note that this condition is all that we need to get the main conclusion of the proposition, i.e., the new technology must be labor value saving with respect to the initially given system of labor values. This enlarges considerably the set of technological changes that allow for the asserted result.

must hold true, implying for the labor values of the system $\breve{A}^*, \breve{l}^*$ the result $\breve{v}^* = v$. But A^*, l^* was already progressive and $\breve{A}^*, \breve{l}^*$ is therefore progressive, too, implying $\breve{v}^* < v$, i.e., a contradiction.

Therefore, we must have $vA_1 + l_1 > vA_1^* + l_1^*$ (since 0 would imply a contradiction again). We therefore get:

$$vA + l \geq vA^* + l^*, \text{ i.e.}$$
$$v(A - A^*) - (l^* - l) \geq 0.$$

But since $v < p_w$, $A - A^* \geq 0$ holds true we finally get

$$p_w(A - A^*) - (l^* - l) \geq 0 \text{ or}$$
$$p_w A + l \geq p_w A^* + l^*,$$

with strict inequalities again following from our assumptions on the matrices A, A^*, l, l^*. The change is therefore profitable with respect to initially given labor commanded prices p_w. Again, due to the presented inequalities, there is room for profitable CS–LU changes which are not progressive and which therefore allow for the possibility of technical change undertaken by capitalist firms that does not increase labor productivity unambiguously. Periods, where capital–saving labor–using technical change occurs may therefore be plagued by productivity decreases.

Assertion 3. Obvious □

In an economy without joint production, fixed capital and a single primary input we thus have some systematic relationships between certain forms of technical change, profitability driven substitutions in production and the total labor costs that characterize the production of the considered n commodities. In particular, the conflict over income distribution may introduce a bias into the direction of technological change (towards CU-LS changes if CS-LS changes are not available) that may explain to some extent the significant increases in labor productivity observed for capitalist market economies at least on the aggregate level.

3.3.2 Notes on the Law of Decreasing Labor Content

Farjoun and Machover's (1983) prove in their Chap. 7 in probabilistic terms that the law of decreasing labor content is implied as a cumulative result of a sequence of technological changes of physical inputs that reduce the costs of these inputs (assuming a given labor input per commodity as a side condition). They are therefore considering capital-saving (and labor-preserving) technical change where capital-saving is now only in the form of the aggregate price of the inputs consumed in production. As we understand their argument this however makes a statement of the following type

$$p_w A^* \leq p_w A \Rightarrow vA^* \leq vA$$

and not really one on the new labor content vector v^* to be determined from the matrix equation $v^* = v^*A^* + l$. Yet as our proofs in the preceding subsection shows, one simply has to iterate then the sequence of decreasing vectors v^k

$$v^{k+1} = v^k A^* + l, v^0 = v = vA + l \rightarrow v^* = v^* A^* + l$$

in order to get the result that $v^* \leq v$ must hold true (with strict inequalities in the case of indecomposable matrices and with somewhat weaker formulations in the case of existence of non-basic commodities).

They therefore provide a fairly strong result on the law of decreasing labor content, allowing significant deviations from the purely physical formulation of the definition of capital-using in the preceding subsection. This represent a first important step that takes from physical inequalities to sectoral price aggregates. The assumption needed for the above sequence to work as suggested in fact only is that $p_w(A^* - A) \leq 0$ implies $v(A^* - A) \leq 0$ which always holds when p_w are prices of production and when the composition of capital is uniform across sectors (since values are then proportional to prices measured in terms of the wage unit). Their law is therefore based on assumptions concerning probability distributions of p_w (relative to v) and a sequence of changes of the type described above (and not just the one step change there considered), by which they can deduce the probability of the statement: $p_w(A^* - A) \leq 0$ implies $v(A^* - A) \leq 0$.

However, technical change need not be of their capital-saving labor time preserving type, but may be of capital-saving labor-using type (in the service sector) or even – in particular in the case of fixed capital consumption (depreciation of machines over a certain lifespan) – be of capital-using labor-saving type. In order to approach a brief discussion of these cases let us first consider as alternative to the above labor-saving capital-preserving technical change (where the matrix A is now kept constant. This immediately gives rise to always falling labor content by considering the above sequence under this new assumption, since $l^* - l \leq 0$ implies $vA + l^* \leq v$. The case $CS - LS$ (in terms of prices should therefore strengthen their argument and not create new problems for their formulation of the law of decreasing labor content.

But what happens in the case $CS - LU$ if we go from the detailed physical inequalities of the preceding subsection to the consideration of sectoral price aggregates in the measurement of capital-saving? In this case it is no longer sufficient to consider only the input matrix A. We now need the condition

$$p_w A^* + l^* \leq p_w A + l \Rightarrow vA^* + l^* \leq v = vA + l$$

in order to make use again of the proof strategies of the preceding subsection. Therefore Farjoun and Machover's (1983) probabilistic reasoning must now be applied to the above implication and its probabilistic relevance when sequences of technical change are considered as carriers of the law of decreasing labor content. Such a reconsideration of their argument must be left here for future research however.

Considering finally (in terms of prices p_w) CU-LS technical change, we have to justify how such a type of innovation is occurring under capitalists' choice of technique, since it is not likely that the use of raw material or auxiliary means of production (energy) is systematically increasing per unit of output that is produced. Yet, input matrices A also contain items that stem from fixed capital consumption (depreciation of buildings and machinery) and here it is conceivable that the amount of fixed capital per worker and thereby also the consumption of these items is increasing in the sense we have used it above in terms of prices measured in the wage unit (no longer only in the strict physical sense of the preceding subsection). We stress that we consider here the input–output approach to fixed capital (capital advanced with lifespans that exceed the yearly production cycle of the economy) as point of reference and not the Sraffian one with its hypothetical uniform profit rate calculations over all vintages of the existing machinery, see Bródy (1970) for its detailed discussion in terms of capital advanced, capital consumed and turnover times of capital advancements. The input–output matrix A is therefore now to be augmented by durable investment goods in its dimension and by the addition of a depreciation matrix A^δ for these commodities. On this basis of this extension, the definition of labor values is the same as before, namely: $v = v(A + A^\delta) + l$.

Investment criteria for the choice of technique in the case where items with a longer life span than one year are used in capitalist production are much more difficult to formulate, but we still assume that they will by and large imply cost-reduction with respect to the matrix $(A + A^\delta)$ so that we may still consider the relevance of the condition

$$p_w(A^* + A^{\delta *}) + l^* \leq p_w(A + A^\delta) + l \Rightarrow v(A^* + A^{\delta *}) + l^* \leq v(A + A^\delta) + l = v$$

Again, Farjoun and Machover's (1983) probabilistic reasoning may be applied to this condition in place of $p_w(A^* - A) \leq 0$ implies $v(A^* - A) \leq 0$ which when applicable would then give rise to the same reasoning on decreasing labor content as we have used it before. Again, this further extension of their argument must be left for future research here.

3.3.3 Multiple Activities and Joint Production: Some Observations

We want to indicate here verbally how cases of more than one activity per sector or even of multiple activities with joint production can be treated in order to generalize in an empirically relevant way what has been shown beforehand for the case of a square input–output system with no joint production. The topic of pure joint production (including multiple activities) and its implications for relationship between labor value accounting and profitable substitutions with specific forms of technical change will be treated from the formal point of view in Chaps. 4–7 of the book in detail.

Let us consider the case of multiple activities first. In this case input output methodology derives a standard square input–output table by aggregating the activities of the given sectors by way of the activity levels that characterize the single activities. The $A_{\star j_l},\ l = 1, ..., k(j)$ of a single sector are thus combined to a single column vector by the following operation:

$$\bar{A}_{\star j} = (x_{j_1} A_{\star j_1} + ... + x_{j_{k(j)}} A_{\star j_{k(j)}})/(x_{j_1} + ... + x_{j_{k(j)}})$$

which sums activities on their activity level and then divides by the total output generated in this sector (the same happens with the labor inputs of the various activities). Labor values are then to be defined as solution $v\bar{A} + \bar{l} = v$, since they are understood to represent average total labor costs of producing the various commodities with respect to the multiple activities that are operated in each sector. This procedure is closely related to Marx's distinction between average and individual labor values, see Chap. 5 for details on this.

If all single activities have been profitable in the sense of this section then of course also their above aggregate is profitable with respect to the initially given prices. And if, for example, CU-LS technical change is taking place with respect to some activities and activities levels are assumed to be invariant, then of course this type of technical change will also characterize the average inputs $\bar{A}, \bar{l}$. These observations indicate that the propositions of this section can find obvious generalizations to the case of multiple activities, at least as long as the market of single activities remains the same. Furthermore, these reformulations are in line with standard input output methodology and thus keep the definition of labor values also in line with the definition of labor productivity indexes as they are made in Stone's System of National Accounts and its input–output substructure.

Let us next consider the case of pure joint production. In this case, input–output methodology disentangles the joint outputs of one sector or activity – if its so-called industry technology assumption is applied – by splitting up all inputs in proportion to the relative value of output in the output basket of the joint production activity. In the full-cost accounting techniques of business administration this is called the sales value method, since the relative proceeds of the items in a joint bundle then determine the amount of joint inputs these single items have to bear. It is obvious that this disentangling of jointly produced products preserves profitability (since it is intended to be neutral with respect to profit generation capabilities). We thus again obtain the situation just considered for multiple activities and can apply the same conclusions, namely that labor values can be defined by the square input matrix (and its corresponding labor input vector) obtained from input–output methodology in the case of the industry technology assumption and the subsequent aggregation of the resulting disentangled multiple activities. The propositions of this section will then also apply to this general case of capitalist commodity production, if again the market shares of activities are assumed as unchanged. We shall come back to the details of such generalizations of labor value accounting (with positive labor values obtained in the conventional way) and their comparison with actual prices or production prices in terms of the wage unit in chaps. 4–6 of this book.

3.3.4 *The Okishio Theorem and the Tendency of the Profit Rate to Fall*

Before closing the theoretical part of the section we add one further proposition on the implications of profitable technical change, the so-called Okishio theorem, which basically states that sectoral profitable technical change – starting from the position of balanced rates of profit – will never lower the equilibrium (uniform) rate of profit when wages remain at their subsistence level s. In fact, if the square input matrix A^s (augmented by subsistence consumption of workers per activity) is assumed to be indecomposable, the equilibrium rate of profit will necessarily rise for all conceivable forms of profitable technical change, i.e., the extra profits generated by the applied technological substitution in one or more sectors of the economy will always – when prices and quantities have adjusted to the new equilibrium – increase the uniform rate of profit of the economy compared to the one before the technical change. We stress that this theorem in general holds only when we start the process of substitution from the position of uniform rates of profit and is thus not necessarily true in the more general profitability consideration that were the basis of our comparison of profitable and progressive technical change in the preceding subsections.

Proposition 3.7. *Assume as given an equilibrium* $(1 + r)pA^s = p, p > 0$ *of the currently prevailing input–output technology. We here only consider input–output systems where the sector of basics (of which at least one is assumed to exist) determines the Frobenius root of the matrix* A^s.

Technical changes A^{s*} *– occurring in one or more columns of the input–output matrix* $A^s = A + S = A + sl$ *with respect to the inputs* A, l *– which are profitable (cost-reducing) for each modified activity* j *with respect to the given prices of production* p : $pA^{s*}_{\star j} < pA^s_{\star j}$, *never lower the balanced rate of profit of this input–output structure and will in fact increase it if the matrix* A^s *is assumed to be indecomposable.*

Proof. By assumption we have as relationship before and after the technical change for the augmented input matrix A^s:

$$pA^s \geq pA^{s*} \quad \text{and thus} \quad (1 + r)pA^{s*} \leq p$$

According to Woods (1978, p. 21) we have the proposition that

$$pA^{s*} \leq \frac{1}{1+r}p, \quad p > 0 \qquad \Rightarrow \qquad \lambda(A^{s*}) \leq \frac{1}{1+r} = \lambda(A^s)$$

for the dominant root $\lambda(A^{s*}) = \frac{1}{1+r^*}$ of the non-negative matrix A^{s*}. This implies

$$\lambda(A^s) \leq \lambda(A^{s*}), \quad \text{and thus} \quad r^* \geq r.$$

In the case of an indecomposable matrix A^s the situation just considered applies to an indecomposable matrix $\breve{A}^{s*}$ where the innovating sectors are assumed to exhibit slightly higher inputs than is actually the case. We therefore know from Perron-Frobenius theory, see part II of the book, that its dominant root must be larger than the one of the actual matrix A^{s*}.

We thus have that there cannot be a falling rate of profit in input–output systems $A, l, s, B = I$ with a given subsistence wage s if innovation occurs in a cost-reducing way, completely independent of the physical type of technical change that is in fact occurring. The question here however is whether wages will indeed stay at their subsistence level in the face of more or less rapid labor-saving technical change, in particular if labor productivity increases significantly under this type of technical change. It is our view that the Marxian assertion of a falling rate of profit is to be based on a given rate of exploitation $\epsilon = (1-w)/w$, as considered in Chap. 2 in the ASSI of Duménil, Foley and others, where therefore there are real wage increases occurring that by and large keep the share of wages in national income constant, see Foley (1986) for a detailed investigation of such an argument. Here we will only state our belief that the rate of profit would have been increased considerably over the past two hundred years if the real wage would had stayed constant over this period. But this is a purely academic assertion that has not much to do with the actual evolution of capitalist economies.

3.3.5 *The Law of Decreasing Labor Content: Empirical Results*

In this subsection we provide some empirical illustrations of the concepts and propositions we have considered in this chapter.[14] For this purpose we are using input–output data from Kalmbach et al. (2005) utilized in their study of the role of the service sector in the German economy, 1991 – 2000. Their subdivision of the economy into 7 sectors is a very interesting one, since it shows the importance of the service sector in an advanced economy and also the characteristics that distinguish its three subsectors from more traditional industries.

The industry sector is in their work split up into agriculture, manufacturing, and construction. Within manufacturing itself, they separate out another subsector which for an export-oriented country like Germany should be of particular importance. It comprises the four single production sectors (among the 71 originally given sectors) with the highest exports: chemical, pharmaceuticals, machinery, and motor vehicles. For short, they call this macro sector the *export core*. On the other hand, also the services sector is made up of very different sorts of output "goods". They distinguish between three main types: business-related services, consumer services, and social services. The term business-related services needs a further clarification. There is

[14] See Gupta and Steedman (1971) for another contribution on the empirical content of labor values.

an extra sector among the 71 original sectors of the German input–output tables carrying this label, which has grown considerably over the 1990s and whose output share in 2000 has risen to almost seven percent (it is thus larger than the construction sector). This single sector may be viewed as business-related services in a narrow sense. For their aggregation, however, they understand this term in a broader sense and in addition count the following sectors within it: wholesale trade, communications, finance, leasing, computer and related services, research and development services. The concept of what they have specified as business-related services (in a broader sense) will become even clearer by briefly indicating what they have not assigned to it and what they rather include in the consumer services, namely: retail trade, repair, transport, insurance, real estate services, and personal services. Table 3.5 summarizes the seven (macro) sectors thus obtained. For a better assessment of their relative importance, it also indicates the sectoral output shares (again for the year 2000).[15]

The technological coefficients of the 7-sectoral aggregation are reported in Table 3.6 and show the input–output matrix A of the German economy for the year 2000 per 10^6 Euro of output value. Before having a look at it, the reader may ask himself or herself in which row he or she expects the largest and smallest coefficients, respectively; and perhaps even in which cells. In addition to the plain presentation of input–output coefficients $\tilde{a}_{ij}$ in Table 3.6, an overall impression of the input–output relationships can be gained from the three-dimensional plot of the matrix A shown in Fig. 3.5. We have used $\tilde{a}_{ij}$ to characterize the entries of the considered input–output table A. The reason for this is that input–output calculations are based on constant prices, but still contain the prices of their base period in the

Table 3.5 The 7-sectoral structure of the economy

1:	Agriculture	1.33
2:	Manufacturing, the export core	12.37
3:	Other manufacturing	22.55
4:	Construction	6.29
5:	Business-related services	21.36
6:	Consumer services	23.35
7:	Social services	12.75

Table 3.6 Technological coefficients of the 7-sectoral aggregation (Germany, year 2000)

	1	2	3	4	5	6	7
1:	0.028	0.000	0.045	0.000	0.000	0.002	0.002
2:	0.090	0.282	0.050	0.022	0.003	0.008	0.011
3:	0.142	0.232	0.324	0.287	0.030	0.055	0.065
4:	0.007	0.003	0.006	0.017	0.006	0.028	0.016
5:	0.142	0.121	0.140	0.107	0.332	0.134	0.096
6:	0.036	0.053	0.051	0.108	0.072	0.152	0.049
7:	0.031	0.006	0.011	0.007	0.007	0.013	0.024

[15] The numbers in the last column are the sectoral output shares (in percent) for Germany in 2000. See text for an explanation of the sectors.

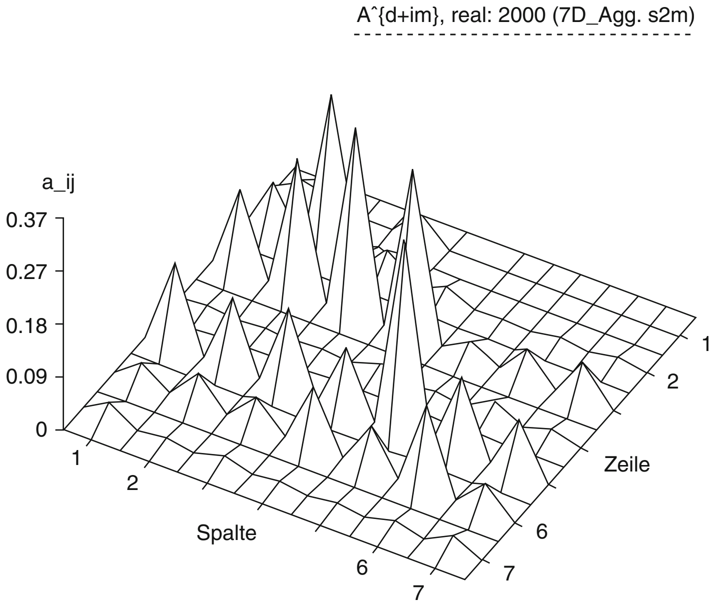

Fig. 3.5 3D plot of the technological coefficients (The German labels "Zeile" and "Spalte" mean row and column, respectively)

definition of their coefficients, i.e., we have as relationship between technological coefficients a_{ij} and $\tilde{a}_{ij}$ the equation $\tilde{a}_{ij} = p(0)_i a_{ij} / p(0)_j$. Input–output methods normally simply assume $p(0) = (1, ..., 1) = e$ and thus conceal thereby that relative prices (of the base year) are involved in the measurement of input–output tables in so-called real terms.[16]

For the purposes of this subsection we are using the column sums of the above input–output matrix.[17] This gives for the ten years and seven sectors under consideration the table of *intermediate consumption coefficients*:

1991	0.475	0.698	0.627	0.548	0.450	0.392	0.263
1992	0.502	0.601	0.605	0.481	0.415	0.375	0.240
1993	0.479	0.611	0.611	0.496	0.410	0.379	0.247
1994	0.466	0.616	0.612	0.509	0.410	0.387	0.249
1995	0.480	0.622	0.614	0.517	0.409	0.382	0.244
1996	0.482	0.630	0.627	0.523	0.412	0.374	0.251
1997	0.471	0.653	0.624	0.537	0.418	0.373	0.252
1998	0.470	0.654	0.628	0.538	0.429	0.373	0.253
1999	0.469	0.671	0.638	0.536	0.429	0.380	0.255
2000	0.475	0.702	0.637	0.550	0.434	0.388	0.263

[16] We similarly have $\tilde{l}_j = l_j / p(0)_j$ and thus get for labor values the same relationship $\tilde{v}_j = v_j / p(0)_j$ by definition.

[17] Note however that we are neglecting the depreciation of the fixed capital stock in each sector here.

Subtracting these items from 1 gives real value added (at prices of the assumed base year 1995) per 10^6 Euros of output value. Similarly the *labor coefficients* (number of employed workers per 10^6 Euros output) are given by:

1991	34.9	8.5	10.2	14.2	11.5	13.6	21.1
1992	29.1	8.0	9.5	13.2	11.3	13.3	20.5
1993	27.1	8.4	9.2	13.2	11.1	13.3	20.6
1994	26.8	7.5	8.7	12.8	11.1	13.1	20.5
1995	25.4	6.6	8.5	13.3	11.0	12.8	20.4
1996	22.0	6.4	8.2	13.2	10.7	12.6	20.5
1997	21.6	6.1	7.9	12.9	10.4	12.6	20.3
1998	21.3	5.7	7.7	12.9	10.2	12.5	20.2
1999	20.4	5.6	7.4	12.6	9.8	12.4	20.1
2000	20.2	5.3	7.0	12.5	9.7	12.5	20.0

For calculating the labor values of the seven sectors and the ten years under consideration we have to use the formula $v = l(I - A)^{-1}$ for each year for their determination. The 10 input–output matrices of the type shown in Table 3.5 have been supplied to me by Reiner Franke (including fixed capital consumption) and gave rise to the following columns of *labor values* for the seven sectors and the 10 years these tables were available for. In contrast to the preceding table, showing the labor coefficients of the seven sectors, we here have the accumulated number of employed workers per 10^6 Euros output, augmented by the indirect effects as they are contained in the application of the Leontief inverse $(I - A)^{-1} = \sum_{k=o}^{\infty} A^k$. Clearly, the labor values must all be larger than the corresponding direct labor input vectors of labor coefficients.

1991	47.64	22.72	25.98	25.82	20.40	22.17	26.82
1992	40.50	21.84	24.47	24.69	19.81	21.72	26.17
1993	37.96	22.19	23.84	24.76	19.46	21.78	26.20
1994	37.72	20.87	22.92	24.20	19.34	21.28	25.91
1995	36.17	19.86	22.79	24.67	19.18	20.69	25.83
1996	32.17	19.79	21.88	24.55	18.86	20.34	25.85
1997	31.57	19.15	21.31	24.05	18.68	20.23	25.58
1998	31.16	19.03	21.26	23.91	18.39	20.19	25.47
1999	30.14	19.45	20.61	23.63	17.87	20.13	25.43
2000	29.76	18.56					

Dividing, on the one hand, each of the 70 real value added items (per 10^6 Euro output value) by the corresponding labor coefficient (per 10^6 Euro output value) in order to get conventional measures of labor productivity for the considered economy and taking, on the other hand the inverse of the above labor values (which are also expressed per 10^6 Euro output value), we get magnitudes that are all in the interval [0, 1] and thus easy to compare with each other with respect to their size and their direction of change as the graphical representations for the seven sectors given in Fig. 3.6 show. These figures show the measures of productivity $\pi_j^o, 1/v_j$ we have discussed extensively in Sect. 3.2 of this chapter. We here repeat again by way of

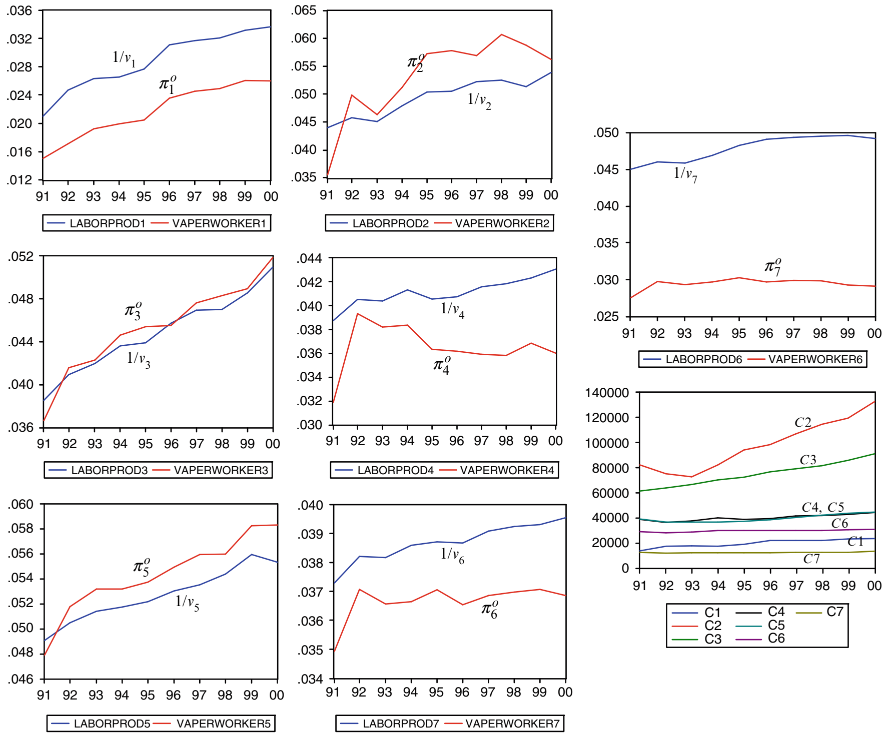

Fig. 3.6 Comparing conventional and Marxian labor productivity indices: $\pi_j^o, 1/v_j$

a simple example why the latter terms are the better measure of labor productivity and in fact the only reasonable one.

For this purpose we use again the price vector $p(0)$ of the base period in explicit form in the presentation of the data of the economy. We consider a simple economy with one capital and one pure consumption good, characterized therefore by the input–output matrix:

$$\tilde{A} = (p_i(0)a_{ij}/p_j(0)), \quad 0 < a_{11} < 1, a_{12} > 0, a_{21} = a_{22} = 0$$

Since the investment good sector is homogeneous with respect to inputs and outputs we there immediately get:

$$\pi_1^o = \frac{1 - p_1(0)a_{11}/p_1(0)}{l_1/p_1(0)} = \frac{1 - a_{11}}{l_1} = 1/v_1, \quad i.e.,$$

no discrepancy between the two measures of labor productivity. But for the consumption good sector we get:

$$\pi_2^o = \frac{1-p_1(0)a_{12}/p_2(0)}{l_2/p_2(0)} = \frac{p_2(0)-p_1(0)a_{12}}{l_2} \neq$$
$$\frac{1}{v_2/p_2(0)} = \frac{1}{(v_1/p_1(0))p_1(0)a_{12}/p_2(0)+l_2/p_2(0)} = \frac{1}{(v_1a_{12}+l_2)/p_2(0)}$$

It is true that also labor values are here measured relative to output value (not output level), but this only means that each time series of labor values has been divided through the constant price of the corresponding commodity which does not distort the internal structure of each time series. Yet, the numerator of the conventional measure of labor productivity $p_2(0) - p_1(0)a_{12}$ depends on relative prices and is thus not independent of their structure (and thus of the base period that is used). The measure π_j^o (for given j) can therefore change erratically without any change in the production conditions of the economy, while the measure $1/v_j$ is only rescaled (for given j) in case of a change in the price vector of the base period (in fact it is totally independent of prices if its rate of growth is considered).

Generalizing this situation (and returning to $p(0) = e$) for notational simplicity) we expect to get for

$$\pi_j^o = \frac{e(I-A)_j}{l_j} \neq \frac{1}{v_j} = \frac{1}{l(I-A)_j^{-1}}$$

that these two series should behave quite differently over time due to the (arbitrarily chosen) relative base year prices on which the first measure depends. The comparison of these time series in Fig. 3.6 therefore came a bit as a surprise, since it shows by and large that the two measures do not develop in different directions as time goes by. Our theoretical comparison of the labor content of commodities (or better its reciprocal value "labor productivity") with real value added per worker in the same sector thus does not give rise to extreme differences in the behavior of the two indices of labor productivity considered from the perspective of highly aggregated input–output tables (which exhibit significant difference in their capital to labor ratios c_j)[18] shown in Fig. 3.6 bottom right.[19] Note however that the tables in Fig. 3.6 all show (with mild exceptions) that Farjoun and Machover (1983) law of falling labor content holds for the German economy over the considered period. Note also that construction is the single sector where productivity in Marxian terms has risen over the considered period while the conventional measure shows only a sharp increase directly after the German reunification and significant decline thereafter. Consumer services are characterized (much weaker however) by a similar occurrence, as are social services (a sector that is however subject to processes that are in general not controlled by private enterprises). Be that as it may, the general conclusion is that indexes π_j^o must be considered as atheoretical and thus have to be treated with care, and should moreover only be relied upon if indices $1/v_j$ provide by and large the same qualitative behavior.

[18] Measured in million Euros per worker.

[19] This may be due to the larger variations in the inputs l as compared to the inputs A.

3.4 Conclusions

We have considered in this chapter the question of a proper measurement of labor productivity and argued that the Marxian measurement is the theoretically sound one, while the conventional measure of the SNA, real value added per worker is questionable both from the theoretical as well as from the empirical point of view. Moreover, labor values also convince by explaining rising labor productivity in conjunction with the competitive choice of techniques (by cost-reducing new methods of production). The law of decreasing labor content has thus been justified here by theoretical propositions and has also been clearly working for the German economy after reunification.

In our view this demonstrates that labor values as defined in this chapter in the conventional way (concerning Marxian economics as well as Leontief type input–output analysis) are the relevant measure for total labor costs and – in reciprocal form – commodity-specific indexes of labor productivity.

References

Bródy, A. (1970). *Proportions, prices and planning*. Amsterdam: North Holland.

Farjoun, E., & Machover, M. (1983). *Laws of chaos.* London: Verso Editions and NLB.

Flaschel, P. (1980). The derivation and comparison of employment multipliers and labor productivity indexes using monetary and physical input–output tables. *Economics of Planning, 16*, 3.

Flaschel, P. (1983). Gesamtwirtschaftliche Arbeitsproduktivität. Eine mengenorientierte Kritik ihrer wertorientierten Strukturierung. *Allgemeines Statistisches Archiv, 67*, 317–333.

Flaschel, P., & Krolzig, H.-M. (2006). Wage-price Phillips curves and macroeconomic stability: Basic structural form, estimation and analysis. In C. Chiarella, P. Flaschel, R. Franke, & W. Semmler (Eds.), *Quantitative and empirical analysis of nonlinear dynamic macromodels.* Contributions to Economic Analysis (Series Editors: B. Baltagi, E. Sadka and D. Wildasin). Amsterdam: Elsevier.

Foley, D. (1986). *Money, accumulation and crisis*. London: Harwood Academic Publishers.

Gupta, S., & Steedman, I. (1971). An input–output study of labor productivity in the British economy. *Bulletin of the Oxford University Institute of Economics and Statistics*, 33.

Härtel, H.-H. (1981). Deflationierung der Bruttowertschöpfung. In: U.-P. Reich & C. Stahmer (Eds.), Input–Output-Rechnung: Energiemodelle und Methoden der Preisbereinigung. Frankfurt/M.: Campus, 179–193.

Kalmbach, P., Franke, R. , Knottenbauer, K., & Krämer, H. (2005). Die Interdependenz von Industrie und Dienstleistungen - Zur Dynamik eines komplexen Beziehungsgeflechts. Berlin: Edition Sigma.

Keynes, J. M. (1936). *The general theory of employment, interest and money.* New York: Macmillan.

Marx, K. (1954). *Capital. A critique of political economy* (Vol. 1). New York: Lawrence + Wishart.

Meyer, U. (1981). Doppel-Deflationierung versus Realwert-Deflationierung. Eine vergleichende Gegenüberstellung. In U.-P. Reich und C. Stahmer (Eds.), Input–Output-Rechnung: Energiemodelle und Methoden der Preisbereinigung. Frankfurt/M.: Campus, 147–178.

Neubauer, W. (1978). Reales Inlandsprodukt: preisbereinigt oder inflationsbereinigt? Zur Deflationierung bei veränderter Preisstruktur. *Allgemeines Statistisches Archiv, 62*, 115–160.

Roemer, J. (1977). Technical change and the ‘tendency of the rate of profit to fall’. *Journal of Economic Theory, 16*, 403–424.

Roemer, J. E. (1980). Innovation, rates of profit, and the uniqueness of von Neumann prices. *Journal of Economic Theory, 22*, 451–464.

Roemer, J. E. (1981). *Analytical foundations of Marxian economics*. Cambridge, UK: Cambridge University Press.

Stobbe, A. (1980). *Volkswirtschaftslehre I: Volkswirtschaftliches Rechnungswesen*. Heidelberg: Springer.

United Nations. (1993). *System of National Accounts*. http://unstats.un.org/unsd/sna1993/handbooks.asp

Woods, J. E. (1978). *Mathematical economics*. Suffolk: Richard Clay (The Chaucer Press).

Chapter 4
Marx After Stone: The Marxian Contribution to the UN's SNA

4.1 Introduction

In this chapter we reconsider the examples of Steedman (1977, Chaps. 10, 11) from the viewpoint of input–output methodology. Steedman's examples wee intended to show that Marxian labor values can exhibit anomalies – which deprive them of economic content – when models of production are considered that include pure joint production or fixed capital. We will show that his claims are due to a very narrow understanding of the definition of labor values in Marx's capital, namely that they are under all circumstances defined in Marx's Capital in a purely additive way, ignoring here in particular that Marx's distinguished between average and individual values whenever multiple activities for producing one and the same commodity were present. Moreover, as in particular obvious from the first part of second volume of Marx's Capital, Marx's was always eager to learn *what firms actually do* when a new aspect of value accounting was to be considered and not so much in what economists think firms should do in such a situation.

Keeping this in mind it is easy to understand that joint production must be disentangled into separated single product activities if firms want to calculate the total costs of single items. Firms here often use the so-called sales value method, by which joint costs are split and allocated to the single outputs on the basis of the relative proceeds generated by these outputs. This method is always applicable and it has the remarkable property that the rate of profit of the considered joint production activity is not distorted when the profitability of the single items is calculated. Production prices on the level of a joint production system thus remain production prices after joint products have been disentangled in the above way. Surprisingly enough the same method is used under the name "industry technology assumption" in the UN's input–output methodology when they construct single input–output matrices from originally give make and use matrices, if the make matrix exhibits joint production, here generally meaning that there are secondary products produced in each sector beside his main product. Firms and input–output theorists have therefore instinctively done the same thing without taking notice of each other and without stressing that their method preserves the uniformity of profit rates and is therefore a neutral type of accounting principle.

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_4,

Using Steedman's examples and the data they contain we show in this chapter that labor values can be generalized by means of such principles in a meaningful way without giving rise to anomalies as in Steedman (1977) anymore. They are treated in the same way as firms and input–output theorists do it in their full cost accounting methods and lead here to the calculation of total labor costs embodied in single commodities in direct analogy to the cost accounting methods on the firm and on the national level. But even if the examples of Steedman (1977), intended to disqualify the quantitative content of Marxian labor values, can be shown to be misguided by the Sraffian methodology to use linear equations without any extra qualification and for all models or production as the single principle (up to the rate of profit which of course enters in a multiplicative way), his basic argument may still hold, namely that labor values are but a detour when going from the data of production to the prices of production scheme of the Classical theory of competition.

Samuelson's (1971) eraser principle is just another way of expressing this opinion. We will see however in the next part of the book that just the opposite is true in the relationship between labor values and prices of production, since labor values are firmly rooted in the total cost accounting techniques of the UN's System of National Accounts (and their application to specific data). Price of production, however, represent a very extreme assumption on the dynamics of capitalist economies which when taken literally in a Sraffian way would suggest that all activities (old and new) exhibit a uniform rate of profit for all vintages of their fixed capital, for all semifinished products and throughout the globalized economy we are now living in. Billions of purely technological equations are therefore to be solved here under the application of a unique principle, namely that there prevails a uniform rate of return on total costs throughout. Empirically seen this is a task that can neither be solved by the market nor by the computer, since it abstracts from to many things that drive capitalist economies in the real world. We conclude that purely technological determined production prices are accounting prices of a very hypothetical nature and of purely academic interest.

Input–output theorists would approach the issue of uniform profitability (with stock matrices representing capital advanced, not with a Sraffian vintage approach), on the level of industries (appropriately classified). An example for this has been given in the preceding chapter with respect to which we will show in the next part of the book that nothing that comes near to a uniform rate of profit can be expected to hold even in such a highly aggregated seven sectoral example. The sectors there considered are just to diverse in their fundamentals and their evolution to expect that a single rate of return can characterize this evolution approximately. The irony in this situation (and its implications) is that we should in fact apply Samuelson's (1971) eraser principle to prices of production and not to labor values, since the latter are proper and usable theoretical accounting concepts for the analysis of labor productivity changes and their implications, while the former are just an inconvenient detour to the theories of competition we have to apply to understand the behavior of actual average market prices and their implications for productivity changes (see the preceding chapter).

Defining labor values in an appropriate way for examples of input–output structures which include joint production proper and fixed capital (machinery) is the contribution of this chapter. It shows in particular that accounting procedures actually employed by capitalist firms for the treatment of joint products and fixed capital must be reflected and applied in such generalized situations. This then gives rise to labor value concepts that – in contrast to the literature – are direct generalizations of the conventional approach to the labor theory of value and its propositions (see Chap. 3 on the MDSA approach). An additional advantage of this way of generalizing the Labor Theory of Value is that it is in line with procedures of the United Nations' (1968) System of National Accounts, based on work of Richard Stone and his research group, in particular its conventions for the treatment of joint products and capital advancements. In this sense, the LTV even contributes to an understanding of the measures of total labor costs as they are formulated in the UN's SNA, and their applications and implications.

4.2 Employment Multipliers and Labor Values in Pure Joint Production Systems

In the following[1] we will compare the notion of "employment multipliers" exemplified in Steedman (1977, Chap. 11) in the frame of a specific two–sector model with the definition and interpretation of labor values we have briefly sketched in the preceding chapter, here also confined to Steedman's example. Our aim is to show the difference between these two definitions with respect to concrete numbers, and to exemplify that the latter concept is based on a joint–product convention,[2] which allows us to generalize Marx's ideas on the additivity of the value creation process to joint production in a meaningful way, without running into the anomalies described in Steedman (1977). In contrast to Steedman's employment multipliers, which reflect the system's labor requirements to produce additional quantities of net output, our concept of labor values tries to incorporate the idea that *average* values of jointly produced goods should vary with prices (and proceeds) in a *continuous* fashion such that the labor value of a good, which becomes free, must become zero. This idea implies that by–products which shrink in price should exhibit this fact also with regard to their labor value, i.e., *labor values* cannot be defined independently of prices (as is the case for Steedman's (1977) definition of *employment multipliers*).

We have already shown for the von Neumann model, see Flaschel (1983b) that both concepts of total labor requirements share the feature that they allow to preserve Marx's methodological device to explain the general nature of profits at first under the assumption that commodities exchange at their labor values, adopted by Marx in the first two volumes of "Capital" by assuming a uniform composition of

[1] The sections of this chapter were originally published in Flaschel (1983b).

[2] Cf. also Sen (1978, p. 178).

capital. Their difference consequently first arises when we consider how additivity of value creation is to be described for joint production systems in case of a nonuniform composition of capital. For such a case Steedman has demonstrated by means of his example that a purely technologically determined notion of "labor costs" does not remain sensible with regard to the aims pursued by Marx.[3] Our alternative extension of Marx's labor values to the case of differing organic compositions of joint activities, is based on principles of full–cost accounting for jointly produced products, see also Flaschel (1983b). It allows to further examine Marx's point of view that the actual production plays a dominant part in the process of profit creation.[4]

4.2.1 Employment Multipliers

In Steedman (1977, pp. 151f.) the following two–sector examples of a von Neumann model (where, however, wages are paid ex post) are explored with regard to the definition of employment multipliers and the validity of the so-called Marxian FMT (Fundamental Marxian Theorem), see Morishima (1973) for the details of this theorem:

$$A = \begin{pmatrix} 5 & 0 \\ 0 & 10 \end{pmatrix}, L = (1, 1) \text{ material and labor input matrices}$$

(processes are represented by columns),

$$B = \begin{pmatrix} 6 & 3 \\ 1 & 12 \end{pmatrix} \text{output matrix}$$

(circulating capital goods and pure joint production only),

$$C = (3/6, 5/6) \text{ or } (6/5, 3/5) \text{ or } (3/7, 6/7)$$

(various wage baskets per unit of labor input[5])

The relevant von Neumann equilibrium values (measured in wage–units) are given in each of these cases by: $p_1 = 1/3, p_2 = 1, w = 1 = (p_1, p_2)C^t$. On the quantity side we get with respect to the three levels of labor supply $L^1 = 6, L^2 = 5$ or $L^3 = 7$ associated in this order with the above wage baskets:[6]

$$x_1^1 = 5, x_2^1 = 1 \text{ or } x_1^2 = 3, x_2^2 = 2 \text{ or } x_1^3 = 6, x_2^3 = 1.$$

[3] This does not mean, however, that they cannot be usefully applied to other purposes.

[4] Cf. also Sen (1978, p. 179) on this matter.

[5] $A + CL$ would then give the usual augmented von Neumann input matrix in each of these cases which, however, is of no use here, because of the assumed ex post payment of wages.

[6] We shall write $x^i = (x_1^i, x_2^i)^t, Y^i$, etc. $[i = 1, 2, 3]$ in order to distinguish the employed concepts with respect to the three given wage baskets C^i and labor supplies L^i.

In each of these three cases the growth rate is equal to the rate of profit equal to 20%. As net output $Y = (B - A)x$ we obtain for these three cases $Y^1 = (8,7)^t, Y^2 = (9,7)^t, Y^3 = (9,8)^t$.

In comparing equilibrium 1 and 2, Steedman finds that a change in employment $\Delta L = L^2 - L^1 = -1$ corresponds to a change in output $\Delta Y = Y^2 - Y^1 = (1,0)^t$ by one unit of commodity 1. And when comparing 2 with 3 he finds:

$$\Delta L = +2 \mathrel{\hat{=}} \Delta Y = (0,1)^t.$$

These two calculations demonstrate the general fact that a meaningful change in output by one unit of commodity 1 (respectively 2) will be accompanied by a change in total employment of "−1" (respectively "+2"), numbers which therefore describe "employment multipliers" of the technique under consideration, cf. Steedman (1977, p. 158) for further details.

It is easy to prove that these well–defined employment multipliers z_1, z_2 fulfill the following equation system

$$25z_1 + 5 = 30z_1 + 5z_2 \text{ or } 5z_1 + 1 = 6z_1 + z_2 \tag{4.1}$$

$$10z_2 + 1 = 3z_1 + 12z_2 \text{ or } 10z_2 + 1 = 3z_1 + 12z_2 \tag{4.2}$$

i.e., they can be calculated from the given technological data in the usual purely additive way.

Should we regard these employment multipliers as the successful extension of the concept of labor value as defined by Marx under simpler conditions of technology with respect to the aim pursued by him? Steedman who restricts the aim of value calculation to the exact quantitative determination of the uniform rate of profit does deny this, though he claims that his employment multipliers give the correct extension of the Marxian definition of labor value. But how can we judge "correctness" if not by means of a *successful* application to the aims pursued by Marx?

And, if we compare Steedman's strictly additive definition of value (4.1), (4.2) with characterizations of labor value found in Marx's *Capital* the following problems can be stated in addition:

1. The view that labor gets "incorporated" in the commodities produced with it (actually or by means of a theoretical imputation) has become meaningless in the present situation if it is understood in terms of "technology" alone. This is due to the fact that there no longer exists a sensible chain of commodity inputs purely determined by facts of technology which can be characterized as having gone into the production of single units of net output. Subsystems $(B - A)^{-1}Y$ of gross output needed to produce a certain basket of net output Y no longer represent a sensible way to determine the amount of labor "embodied" in it. This is demonstrated in the above example by the fact that "$-1''$ hours of labor cannot be characterized as having been "incorporated" into commodity 1.

2. Additivity with respect to labor value determination is restricted in Marx's Capital to *average* conditions of production. Are we sure that such average conditions are immediately expressed by the above example, though one of the two processes is absolutely inferior to the other?
3. The above presented change which results in an extra production $(1,0)^t$ obviously goes hand in hand with an *increase* in labor productivity: $(8,7)^t/6 \rightarrow (9,7)^t/5$. Marx did connect changes in labor productivity with changes in labor value. Steedman's "correct" value calculations, however, will show no change at all.

We conclude that, regarding actual technological data alone, there remains but an unacceptable possibility to define "labor values" in the present situation, namely jointly and by their strict and narrow interpretation as the sum of values of the means of production and the new value added by "living labor". This interpretation of "labor values" destroys their dual characterization as presented in Morishima's (1973, Chap. 1) for single–product activities. It applies Marx's treatment of constant and variable capital to the case of joint inputs without any need for further qualifications. And it thereby lays the ground on which Marx's "Fundamental Theorem" can be proved to be incorrect.

4.2.2 Labor Values

We shall now consider the definition of labor values introduced in Flaschel (1983b) with respect to the example here employed. It will be shown that this definition gives a better approximation to the above mentioned characteristics of labor value than employment multipliers do. Yet, it is not clear which descriptive, predictive or normative power the new concept will have besides the uses already described in models without joint production. Nevertheless it may be useful to know an extension of Marx's concept of labor values which allows for a sensible reformulation of the conventional LTV.

Since our method of labor value calculation is a strict ex post calculation, we have to introduce further data for its execution first. In general these date may be given by all actual quantity and price configurations of the period under consideration. With respect to the above example this means that we must take an equilibrium situation of this model as given. We therefore start from the knowledge of Steedman's actual commodity prices $p_1 = 1/3$, $p_2 = 1$ and (e.g.) from the following absolute input–output configurations (realized in the above case 1):

	Com.1	Com.2	Labor		Com.1	Com.2
Process 1	25	0	5	→	30	5
Process 2	0	10	1	→	3	12

Our method of labor value calculation takes its starting–point in the separation of joint activities in proportion to proceeds relationships within each joint output basket. These ratios are here given by:

$$\beta_{11} = \frac{p_1 b_{11}}{pB^1} = \frac{(1/3)\cdot 6}{(1/3)\cdot 6 + 1\cdot 1} = \frac{2}{3}, \beta_{21} = \frac{1}{3} \qquad \text{(process 1)}$$
$$\beta_{12} = \frac{(1/3)\cdot 3}{(1/3)\cdot 3 + 1\cdot 12} = \frac{1}{13}, \beta_{22} = \frac{12}{13} \qquad \text{(process 2)}$$[7]

for each of the three cases considered. With the help of these coefficients we can derive four single–product “processes”. In a second step we then have to aggregate the processes “producing” the same good according to market shares, thereby arriving at a two–equations system for the two unknown labor values: v_1, v_2:

$$(v_1, v_2) = (v_1, v_2)A\left(\frac{p_i b_{ij} x_j}{(pB^j)(B_i x)}\right)^t + L\left(\frac{p_i b_{ij} x_j}{(pB^j)(B_i x)}\right)^t \tag{4.3}$$

The two steps which have led to this equation system may be exemplified with respect to case 1 as follows:

$$A' = \begin{pmatrix} \left(\frac{2}{3}\right)\cdot 25 & \left(\frac{1}{3}\right)\cdot 25 & 0 & 0 \\ 0 & 0 & \left(\frac{1}{13}\right)\cdot 10 & \left(\frac{12}{13}\right)\cdot 10 \end{pmatrix} \to B' = \begin{pmatrix} 30 & 0 & 3 & 0 \\ 0 & 5 & 0 & 12 \end{pmatrix}$$
$$L' = ((2/3)\cdot 5, (1/3)\cdot 5, 1/13, 12/13)$$
$$\breve{A} = \begin{pmatrix} \left(\frac{2}{3}\right)\cdot 25 & \left(\frac{1}{3}\right)\cdot 25 \\ \left(\frac{1}{13}\right)\cdot 10 & \left(\frac{12}{13}\right)\cdot 10 \end{pmatrix} \qquad \to \breve{B} = \begin{pmatrix} 30+3 & 0 \\ 0 & 5+12 \end{pmatrix}$$
$$\breve{L} = \left(\frac{2}{3}\cdot 5 + \frac{1}{13}, \frac{1}{3}\cdot 5 + \frac{12}{13}\right)$$

Equations (4.3) can then be obtained by normalizing the matrices $\breve{A}, \breve{L}$ to unit–output levels $(\cdot \breve{B}^{-1})$.

The calculation of labor values by $v\breve{A} + \breve{L} = v\breve{B}$ or $v\breve{A}\breve{B}^{-1} + \breve{L}\breve{B}^{-1} = v$ leads to the following (approximate) values with respect to cases 1, 2 and 3: $v = (v_1, v_2) \approx (0.24, 0.59)$ or $(0.24, 0.60)$.

In difference to Steedman’s unchanged employment multipliers we here recognize a rise (fall) in labor productivity, when comparing case 2 with 1 or case 3 with 1 (case 3 with 2), changes which obviously correspond to the changes which take place with regard to the ratios Y^i/L^i. Furthermore, individual values can now be calculated by help of relative dales values as:

$$\begin{aligned}(\widetilde{v}_1, \widetilde{v}_2, \widetilde{v}_3, \widetilde{v}_4) &\approx (0.24, 0.18, 0.72, 0.54) \quad \text{case 1} \\ &\approx (0.22, 0.14, 0.66, 0.42) \quad \text{case 2} \\ &\approx (0.25, 0.18, 0.75, 0.54) \quad \text{case 3}\end{aligned}$$

[Index 1, 2 = Com.1, Index 3, 4 = Com.2].

[7] b_{ij} are the coefficients, B^j the columns and B_i the rows of the employed matrix B.

We can see from these values that both commodities are produced by less labor with respect to the second process which reflects the fact that the first process is absolutely inferior to the second one with regard to a timeless comparison of input–output relationships: one unit of labor is converted into more than twice as much net output by the second process as compared to the first. But as the inputs of each of our two commodity sectors indeed stem from both activities – the distribution of inputs is governed by the ratios in output–value – both commodity sectors employ inputs produced under favorable as well as under unfavorable conditions. Productivity differences therefore equalize to a certain extent, thereby narrowing the gap in individual values actually observed.

Still another way to perceive the productivity differences between the two sectors in question is to calculate

$$L^0 = v(B - A) = v\begin{pmatrix} 1 & 3 \\ 1 & 2 \end{pmatrix} \approx \begin{cases} (0.83, 1.90) & \text{(case 1)} \\ (0.65, 1.50) & \text{(case 2)} \\ (0.84, 1.92) & \text{(case 3)} \end{cases}$$

and to regard these numbers as "socially necessary direct labor inputs". These values show the hypothetical technological changes in labor inputs that have to take place in order that all processes will look "normal"[8] with respect to the average labor values determined above.[9]

To achieve this kind of normality labor productivity measured in the usual narrow sense (output per labor input) has to rise in the first and to fall in the second process, indicating thereby again the productivity differences between the two considered activities.

It is stated in Marx (1977, Vol. II, p. 153) that

> *"capitals of equal magnitude yield equal profits in equal periods, applies only to capitals of the same organic composition, even with the same rate of surplus value. These statements hold good on the assumption which has been the basis of all our analyses so far, namely that the commodities are sold at their values."*

Slightly reformulated this means that Marx's analysis is based on the assumption of a uniform composition of capital in the first two volumes of *"Capital"*. To give an example of this situation for Steedman's and our definition of labor values consider the following modification of the above presented input–output configurations.

	Com.1	Com.2	Labor		Com.1	Com.2
Process 1	25	0	5	→	30	5
Process 2	0	10	6	→	3	17

[8] It is perhaps in this sense that Sen's (1978, p. 178) characterization of "socially necessary labor time" as "involving counterfactuals" has to be understood.

[9] The system A, B, L^0 will not necessarily generate the same v or even $\widetilde{v}$ as before when our method of calculation is applied to this hypothetical technology.

With regard to this table our former prices $p_1 = 1/3$, $p_2 = 1$, $w = 1$ again represent prices of production, now with the additional property that they imply a uniform composition with regard to the two processes shown: $(25/3)/5 = 10/6$. This fact is not changed by the reallocation A', L' and $\breve{A}, \breve{L}$ of the inputs now underlying equation system (4.3) by virtue of the method employed to disentangle joint products. It follows that the reallocated production data $\breve{A}\breve{B}^{-1}, \breve{L}\breve{B}^{-1}$ also exhibits a uniform composition of capital with regard to the above prices. In a well–known fashion this implies the proportionality between our labor values (4.3) and these prices $p = (1/3, 1)$ (with a proportionality factor that can be determined by: $(v_1, v_2)(8, 12)^t = 11$, implying $(v_1, v_2) = 1/4, 3/4)$). By proposition 8, see Flaschel (1983b), we already know that Steedman's employment multipliers must then be identical with our labor values v, i.e., we in this case also have

$$v = (5,6)\left(\begin{pmatrix}30 & 3\\ 5 & 17\end{pmatrix} - \begin{pmatrix}25 & 0\\ 0 & 10\end{pmatrix}\right)^{-1} = (5,6)\begin{pmatrix}7 & -3\\ -5 & 5\end{pmatrix} \Big/ 20$$

Disagreement with Steedman's understanding of labor values thus can arise solely, when joint production and a non–uniform composition of capital are considered in conjunction. For such a situation we would claim that a straightforward application of Marx's additive process of value creation (of "Capital", Vol. I) – with no averaging process involved – is misplaced and cannot discredit the established formulation of the LTV by means of the anomalies which are thereby obtained.

Be that as it may, it is in any case *most important to recognize* that the inadequacy of Steedman's technological value–accounting method $vB = vA + L$ is first given at the same level of abstraction where the "transformation problem" poses itself (in Chap. IX of Marx's Capital, Vol. III). It is our opinion that from this point onward, the methodological status of "labor values" has not yet been discussed sufficiently. Thus, it may well be that our way of extending joint–product labor values to the case of unequal compositions of capital is not so unacceptable in the end as is suggested at first glance by the chosen approach.

It must be admitted, of course, that these results, see also Flaschel (1983b), with regard to our labor value definition cannot yet be regarded as sufficiently strong to imply the "correctness" of this extended approach to the LTV. The main weakness of this extension – as will be argued surely – is the *direct* dependence of the employed labor values on given prices p [Note in this respect that our method of definition will imply for individual values $\widetilde{v}$ that $\widetilde{v}_1/\widetilde{v}_3 = \widetilde{v}_2/\widetilde{v}_4 = p_1/p_2 = 1/3$ must hold true, see above examples and proposition 7 in note II of Flaschel (1983b)].

Three reasons can already be presented by which such an approach to generalize the LTV may be defended:

1. Labor values depend on price already before any introduction of joint production and its peculiarities, since, e.g., the choice of technique depends on it. Thus their basic methodological status is not changed by our extension of their definition.
2. The rule we have employed to allocate joint costs to the different units of costing is important to the extent it is the only general rule of cost accounting

which allows for the possibility that a uniform rate of profit may result from price–setting behavior and competition. Yet, this rule makes *costs* and profits accruing to a particular item of joint outputs dependent on the price of this product. It thus seems to undermine the practical possibility of a price–setting behavior by reversing the order of "causation" to a certain extent. This circular problem involved in the formulation of a tendency towards uniform profit rates on the basis of a factual method of price–setting is logically similar to the above formulated type of price–dependency of labor values.

3. If full–cost accounting for jointly produced commodities by means of the above market value method is accepted – on the level of prices – as *the* general method of allocating joint costs to the various outputs produced, then it is only sensible to act accordingly with regard to Marx's LTV, i.e., to apply this rule to full labor–cost accounting, too. This, in our view, is the only meaningful way to further examine Marx's procedure which uses labor values as a sort of real–cost accounting by which one can relate price behavior to the results achieved in the process of production, such that the coercive nature of the latter may be made visible.

Labor cost accounting in our view should apply the same principles as are customary on the level of ordinary cost accounting. This gives the main reason why we would conjecture that our definition of labor values may provide a successful method for a general formulation of the Marxian LTV.

Finally it may be useful to look for simplifications and approximations to the calculations we have made. Since the three units of commodity 1 produced in process 2 do not contribute very much to output value in that process (and also to total output with respect to commodity 1) and since the proceeds–relationship within process 1: $2/3 : 1/3$ is not too far away from one, it may already give a first impression of the labor value situation if we neglect the three units of commodity 1 in process 2 and divide inputs into halves to separate the joint outputs in process 1. We then arrive at the following "technological" description of our joint production system (in case 1):

$$A' = \begin{pmatrix} 5 \cdot 2.5 & 5 \cdot 2.5 & 0 \\ 0 & 0 & 10 \end{pmatrix} \qquad B' = \begin{pmatrix} 30 & 0 & 0 \\ 0 & 5 & 12 \end{pmatrix}$$
$$L' = (2.5, \quad 2.5, \quad 1)$$

which gives (when aggregated in the same way as before):

$$\breve{A} = \begin{pmatrix} 5 \cdot 2.5 & 5 \cdot 2.5 \\ 0 & 10 \end{pmatrix} \qquad \breve{B} = \begin{pmatrix} 30 & 0 \\ 0 & 17 \end{pmatrix}$$
$$L' = (2.5, 3.5)$$

This (now decomposable) equation system $v\breve{B} = v\breve{A} + \breve{L}$ has the solution $v_1 = 0.14, v_2 = 0.75$ and thus points into the right direction [but overstates, since the output of process 2 and the input imputed to commodity 1 in process 1 have both been reduced].

4.2.3 Summary

Summarizing we can state to have exemplified that the term “incorporated labor” may remain sensible, when price valuations help to decide how much labor to impute to the different commodities created by joint activities.[10] This method of calculation provides us with a valuation scheme of the same qualitative behavior as prices, which – when filled with reasonable magnitudes – (presumably) will be determined primarily through the production side of the system nevertheless. Steedman’s association of $L^2 - L^1 = -1$ with $Y^2 - Y^1 = (1,0)^t$ is now represented by

$$v^2 Y^2 - v^1 Y^1 = v^2(Y^2 - Y^1) + (v^2 - v^1)Y^1 = L^2 - L^1 = -1,$$

i.e., the change in the labor value of net output equal to the change in employment is now of a double nature. This shows that this change cannot be simply associated with a change in net output (1× com.1) solely. The valuation scheme needed to pursue Marx’s aims of analysis in the presence of joint production is not only dependent on relative input–output proportions, but in fact involves scale–dependent average considerations, whenever one commodity is produced by more than one process. Conventional additivity of labor value determination is now present with regard to the original technological data on the level of individual values only. And related average calculations then guarantee that the exemplified change in physical labor productivity: $(8,7)^t/6 \rightarrow (9,7)^t/5$ obtained by switching from case 1 to case 2 is in fact reflected in the labor value calculations made.

Incorporated labor cannot be measured in general without the use of some convention[11] needed to complete the pure facts of technology in such a way that the degrees of freedom, which “technology” in general entails for economic costing–procedures, are again removed. This is also demonstrated by Steedman by means of the concept of “employment multipliers” – the sole concept which makes use of the technological data of actual production only – when he shows that they behave “perverse” in the light of their interpretation as measures of labor costs or embodied labor time.

Finally “…one may be interested in the relationship between values and prices even if this is not a convenient way of calculating prices with given physical data, or a good way of predicting future prices. Value is then treated not as an ‘intermediate product” in some calculational or predictive exercise but as a concept of interest in its own right’ Sen (1978, p. 182). This is the way – we think – one has to conceive the usefulness of labor value derivations. There already exists a variety of operational theorems relating such labor value to questions of allocation and of technical change. Such considerations have to be extended further (instead of an interference into “transformation procedures” which are of purely algebraic kind) if the true content of Marx’s labor values is to be discovered. The existence of joint products in our view is no hindrance to that.

---

[10] For this purpose, the expression $(B - A)^{-1}Y$ of point 1) in the section on “employment multipliers” is now to be replaced by $(\breve{B} - \breve{A})^{-1}Y$, see the notation introduced above.

[11] Cf. again Sen (1978, p. 178).

4.3 Measurements of Total Labor Requirements Using Input–Output Methodology

One important and early motivation for the laborious construction of input–output tables with a considerable degree of sectoral disaggregation was the practical interest to calculate the indirect employment effects associated with the expected change in activity levels for the U.S. economy at the end of world war two. Associated "total employment multipliers" were considered by Leontief already in 1944, cf. his paper: *Output, employment, consumption and investment* (Quarterly Journal of Economics 58) and also the various editions of his book: *The Structure of the American Economy 1919 - 1939.*[12]

Later writers have adopted Leontief's procedure to construct indexes of total labor productivity in an – on the surface – identical manner (see Evans (1953), Stobbe (1980) for example). This new interpretation of the same formal measure strengthened the (today still prevailing) view that employment multipliers will be positive numbers under all economically sensible circumstances. Yet, we have seen in the preceding section that this conclusion is incorrect if joint production is assumed to occur. We conclude that the additional characterization of employment multipliers as indexes of labor productivity must be premature, since the latter, of course, should be positive by their intended purpose. A notion which tries to capture atemporal total changes in employment associated with definite changes in final demand need not at the same time express intertemporal changes in labor productivity in a satisfying way if technological situations of more general type than the simple Leontief model prevail. It is astonishing to see that input–output analysts have since long accepted such problematic situations as their methodological starting–point, yet did not arrive at the conclusion that employment multipliers could be negative in principle. The obvious reason for this fact in my view is that they in the main accepted the procedure that data on input and output should be rearranged in a way which finally allows the simple Leontief model – and the interpretation of total or system labor requirements it offers – to become applicable. A consequence of this attitude was that negative input–output coefficients were declared to be "manifestly absurd" whenever they were observed, cf. United Nations (1968, p. 39) for example. Thus, no attempt was made to interpret the product × product labor–content calculations in the light of the product × sector accounts they were actually derived from.

This section offers, such an interpretation for the two basic mechanical rules which are in use to move from product × sector accounts to ordinary input–output accounts, methods which transfer inputs and outputs to achieve this aim [cf. United Nations (1968, pp. 39f.)]. In addition to this, we shall see that methods which rest on the transfer of outputs alone cannot be sensibly interpreted in general and thus should be ignored in actual applications, even if the numerical difference they give rise to (in comparison with those methods we show to make sense) is neglectable, unless they are stated explicitly as a numerical simplification of a rule which is

[12] Taken from Flaschel (1983b).

theoretically sound. Such an advice may help to avoid surprises if the repeated application of non-reflected approximations suddenly leads to absurd results.

4.3.1 A Physical Input–Output Example

To pursue our aim we shall employ the example of the preceding section, see also Flaschel (1983b), on joint production. Yet, in difference to the presentations and calculations we have provided in this section, we shall now employ input–output tabulation methods throughout to exemplify the various ways by which input–output tables may be derived. Consider as given the following physical data on input and output which are arranged in the way as it is proposed by input–output methodology.

Note that certain standard row and column sums of this basic input–output tabulation are not yet available because of the heterogeneity of products still involved. To fill this gaps the elements of Table 4.1 have to be recalculated in price terms first. Employing the prices $p_1 = 1/3, p_2 = 1$ for products $1, 2$ and $w = 1$ for labor the following completed system of monetary accounts results:

This is the basic schematic arrangement of monetary data on input and output of the United Nations' (1968) System of National Accounts, from which input–output tables are then to be derived (note that the displayed factor incomes represent labor income and residual capital income, cf. the preceding section). Four cases to derive a single input–output or Leontief table from the above data are now considered in succession and examined with regard to their content.

4.3.2 Case 1: Industry Coefficients

In this simplest case no transfer of outputs or inputs is made between different sectoral accounts in order to obtain a Leontief–table Q of ordinary type. Instead, products of each sector are regarded as if they were homogeneous and of the type of the characteristic product (i.e., the one which is principally produced) for each sector. Table 4.2 thereby changes to the following Table 4.3. This rearranged Table 4.2

Table 4.1 Steedman's joint production example

	Products		Sectors		Final demand	Totals
Product 1			25	0	8	33
Product 2			0	10	7	17
Sector 1	30	5				
Sector 2	3	12				
Primary Inputs			5	1		
			–	–		
Totals	33	17				

Table 4.2 Disentangling joint production, step 1

	Products		Sectors		Final demand	Totals
Products			25/3	0	8/3	11
			0	10	7	17
Sectors	10	5				15
	1	12				13
Factor incomes			5	1		6
			5/3	2		11/3
Totals	11	17	15	13	29/3	

Table 4.3 Disentangling joint production, step 2

	Products		Sectors		Final Demand	Totals
Products			25/3	0	20/3	15
			0	10	3	13
Sectors	15	0				15
	0	13				13
Factor Incomes			5	1		6
			5/3	2		11/3
Totals	15	13	15	13	29/3	

Note here that final demand must be adjusted to reflect the hypothetical change in homogeneity that is here assumed.

is now of the classical Leontief type. Input coefficients Q_{ij}, u_j of intermediate products and labor can therefrom be obtained in the known way, by normalizing outputs to "one" which gives

$$Q = \begin{pmatrix} 5/9 & 0 \\ 0 & 10/13 \end{pmatrix}, \quad u = (1/3, 1/13).$$

Note that Q is identical with the matrix S of *industry coefficients* which we have considered in notes IV and V in Flaschel (1983b). The above Leontief structure Q, u thus simply gives the unit–costs (per \$ of output value) structure of the various sectors j irrespective of the type of product that is produced therewith.

Data of kind Q, u are employed in Rettig (1979, cf. p. 93) to calculate sectoral employment multipliers by means of the classical multiplier formula of single–product systems. For our example this formula gives

$$\begin{aligned} X &= u(I - Q)^{-1} \\ &= (1/3, 1/13)\begin{pmatrix} 1 - 5/9 & 0 \\ 0 & 1 - 10/13 \end{pmatrix}^{-1} \\ &= (3/4, 1/3) \end{aligned}$$

The resulting total labor inputs X_i per \$ of commodity of sector i thus seem to look perfectly normal. They imply that index "1" is associated with the larger total employment effect.

One problem with this simple Leontiefian approach, however, is that it does not make unambiguously clear what actually should be done if an increase of employment is to be achieved for the economy as a whole. Should we stimulate the production of commodity 1 (irrespective of where it is in fact produced) or should we stimulate sector 1 (irrespective of what this sector will produce in the end)? If we do the first, then the preceding section implies that aggregate employment may exhibit a decrease instead of an increase, since the physical employment multiplier there calculated is equal to "−1". And if we do the second then the example of the preceding section tells us that we are stimulating the less productive sector and that the resulting increase in employment simply derives from this fact. That such a rearrangement of outputs is completely arbitrary can also be demonstrated by considering an isolated (hypothetical) change of the price p_1 from $p_1 = 1/3$ to $p_1 = 1$. By this change the above employment multipliers are changed from $(3/4, 1/3)$ to $(1/2, 1/5)$. We conclude that Rettig's method of calculating employment multipliers cannot be regarded as theoretically sound. The product × sector difficulties of joint production are thereby in no way overcome.

4.3.3 Case 2: The Output Method

The output method, cf. United Nations (1968, p. 39), differs from case 1 to the extent as it in addition tries to remove the arbitrary homogeneity assumption we have employed to motivate the use of industry coefficients S_{ij} as entries of the Leontief matrix Q_{ij}. In addition to Table 4.3 it is now (hypothetically) assumed that co–products are sold and thus transferred to the production accounts of the industry in which they are characteristic products (or where they are principally produced). This implies that such co–products now appear both as an input and as an output (of equal amount) of their characteristic industry. Thereby, one may hope to overcome the lack of homogeneity involved in Table 4.3, since this method acknowledges that the output matrix of Table 4.3 is based on inhomogeneous entities, which in a second step are then transferred (sold) to that sector which characteristically produces them. An inspection of Table 4.2 shows that by this second step we have to add one (respectively: five) units of value to the input and output accounts of sector 1 (respectively: sector 2), see Table 4.4:

The vector of total employment effects is now given by the following calculations:

$$
\begin{aligned}
X &= (5/16, 1/18)\begin{pmatrix} (1-28/48) & 0 \\ 0 & (1-5/6) \end{pmatrix}^{-1} \\
&= (5/6, 1/3).
\end{aligned}
$$

Table 4.4 The output method

	Products		Sectors		Final Demand	Totals
Products			28/3	0	20/3	16
			0	15	3	18
Sectors	16	0				16
	0	18				18
Factor Incomes			5	1		6
			5/3	2		11/3
Totals	16	18	16	18	29/3	

Again this vector is subject to considerable changes if isolated changes in commodity prices occur. Furthermore, since sector 1 now seems to be related to the production of commodity 1 only (cf., however, United Nations (1968, p. 39) for some interpretational doubts), it seems now even more plausible that the production of commodity 1 should be subsidized, which – as we have already argued – will lead to a reduction and not an increase in total employment. The output method thus only superficially corrects the arbitrariness of method 1 to calculate the employment multipliers associated with the joint production system of Table 4.1.

Both methods do not produce the meaningful magnitudes we have defined in the preceding section (there per unit of physical output!). We conclude that some proportions of the inputs U_{ij} have to be transferred along with outputs if better results are to be achieved. Thus, while transferring outputs is a comparatively simple matter because the outputs of uncharacteristic products appear as the off–diagonal elements in the make matrix V and though the simultaneous transfer of inputs is much more difficult, this task must be approached nevertheless, if the theoretical measures of the preceding section are to be reconstructed from the monetary input and output data of Table 4.2.

Remark. The output method is generally described in the following mechanical way (cf. United Nations (1968, 1973)): Transfer all off–diagonal outputs to the diagonal by horizontal as well as by vertical displacement. Eliminate the double–counting of outputs thereby involved by adding all off–diagonal elements of the make matrix V to the corresponding elements of the absorption matrix U. This method, on the one hand, does not change the "value added" in each sector, but it, on the other hand and in contrast to method 1, now involves the transfer of outputs between different production accounts. Of course,this method depends on the assumption that the principal product of each sector can be unambiguously identified.

4.3.4 *Case 3: The Commodity–Technology Hypothesis*

We have seen that neither the method where jointly produced goods are simply considered as homogeneous nor its "improvement" by hypothetically assuming that co–products are "sold" to their characteristic sector (which then sells them to final demanders) represent suitable hypotheses with which the total employment effects of the given joint production example can be analyzed. Outputs cannot be simply separated from inputs in the way proposed by the output method if an examination of the labor requirements of commodities is intended in the end.

A possibility to overcome this weakness is given by the so–called commodity technology hypothesis. This hypothesis assumes that there exists a uniquely determined unit–cost structure of *commodities* from which Table 4.2 is derived by means of (institutionally determined) product mixes based on constant \$ –economies of scale. Of course, by the very example chosen,this hypothesis cannot be true with regard to the situation considered in Table 4.1.

Let us nevertheless calculate the input/output rearrangements and the employment multipliers derived therefrom to analyze the working of this assumption. Denote in accordance, see also Flaschel (1983b), by α_{il} the uniquely determined costs which result from commodity i as employed in the production of commodity l.

For the example considered in Table 4.2 there then follows:

$$10\alpha_{11} + 5\alpha_{12} = 25/3 \tag{4.4}$$

$$10\alpha_{21} + 5\alpha_{22} = 0 \tag{4.5}$$

$$\alpha_{11} + 12\alpha_{12} = 0 \tag{4.6}$$

$$\alpha_{21} + 12\alpha_{22} = 10 \tag{4.7}$$

In matrix notation (4.4)–(4.7) read:

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} 10 & 1 \\ 5 & 12 \end{pmatrix} = \begin{pmatrix} 25/3 & 0 \\ 0 & 10 \end{pmatrix}$$

The input–output table or cost–structure of commodity × commodity type is therefore given by

$$Q = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = \begin{pmatrix} 20/23 & -5/69 \\ -10/23 & 20/23 \end{pmatrix}$$

Wage costs to produce commodities 1 and 2 must, of course, be calculated in the same way:

$$10\alpha_{01} + 5\alpha_{02} = 5 \tag{4.8}$$

$$\alpha_{01} + 12\alpha_{02} = 1 \tag{4.9}$$

Equations (4.8), (4.9) imply $u = (\alpha_{01}, \alpha_{02}) = (11/23, 1/23)$.

The above input–output table Q represents an example for what would be declared an absurd result according to the understanding of matrix Q in input–output methodology [cf. our introduction, and United Nations (1968, p. 39)]. Nevertheless, the application of the classical Leontief–multiplier formula here gives

$$X = u(I - Q)^{-1} = (11/23, 1/23) \begin{pmatrix} 9 & 5 \\ -30 & 9 \end{pmatrix} = (-3, 2) \tag{4.10}$$

Table 4.5 The Commodity Technology procedure

	Products		Sectors		Final demand	Totals
Products			$\frac{220}{23}$	$-\frac{85}{69}$	$\frac{8}{3}$	11
			$-\frac{110}{23}$	$\frac{340}{23}$	7	17
Products	11	0				11
	0	17				17
Factor incomes			$\frac{121}{23}$	$\frac{17}{23}$		6
			$\frac{22}{23}$	$\frac{187}{69}$		$\frac{11}{3}$
Totals	11	17	11	17	$\frac{29}{3}$	

i.e., the technological employment multipliers (1),(2) of note III in Flaschel (1983b) now reckoned per \$ of output value and not per unit of product ($p = (1/3, 1)$). This results comes about, since (4.10) is but a mathematically twisted form of the original employment multiplier equations we have established and motivated in note III. To see this, re–write (4.10) in the form $X(I - Q) = u$ and postmultiply this equation with the transposed of the make matrix V^t. This leads us to the matrix equations

$$(5,1) = X(I - Q)\begin{pmatrix} 10 & 1 \\ 5 & 12 \end{pmatrix} = X\left[\begin{pmatrix} 10 & 1 \\ 5 & 12 \end{pmatrix} - \begin{pmatrix} 25/3 & 0 \\ 0 & 10 \end{pmatrix}\right]$$

which is the original system of multiplier equations expressed in monetary terms.

We conclude, that the above procedure of deriving input–output data Q, u from which employment multipliers are then calculated in the classical Leontief way represents but a complicated detour in the application of the true multiplier formula which is based on the inverse of $V^t - U$ instead of $I - Q$ in the case of multiple production. This view finds support from the virtual "technology" assumption we had to make to derive matrix Q and from the "absurd" elements this matrix Q now exhibits from the standpoint of input–output computations. Nevertheless no harm is done, if the results obtained are interpreted by means of the employment multipliers we have considered in the preceding section, see also Flaschel (1983b), where there is no need for dubious intermediate steps of doubtful technological content. Thus, Table 4.2 should not be replaced by the following "absurd" Table 4.5 of product × product type in the hopeless attempt to show the inputs of commodities into *commodities* as in the classical Leontief model despite the more general technological data now given. (cf. Flaschel (1980) for a general examination of the misconceptions underlying this treatment of the original data on input and output of Table 4.2).

4.3.5 Case 4: The Industry–Technology Hypothesis

A further basic procedure invented to overcome the weakness of the output method (of paying no attention to inputs while transferring outputs between accounts) is given by the so–called industry–technology hypothesis. Here it is now assumed that

Table 4.6 The Industry Technology procedure

	Products		Sectors		Final demand	Totals
Products			$\frac{50}{9}$	$-\frac{25}{9}$	$\frac{8}{3}$	11
			$-\frac{10}{13}$	$\frac{120}{13}$	7	17
Products	11	0				11
	0	17				17
Factor incomes			$\frac{133}{39}$	$\frac{101}{39}$		6
			$\frac{148}{117}$	$\frac{281}{117}$		$\frac{11}{3}$
Totals	11	17	11	17	$\frac{29}{3}$	

the cost–structure of commodities is determined by their industry of origin, i.e., is given by the industry coefficients we have considered in case 1.

This assumption implies that the cost–structure of commodities is no longer uniquely determined if joint production is involved. Average cost–structures of commodities therefore have to be derived to obtain an input–output table of ordinary type. This task is performed in the usual way by means of market shares as weights. We have already seen, that this procedure will lead us to input–output data Q, u on intermediate and labor inputs which by the conventional Leontief–multiplier formula just measure the indexes of labor productivity we defined in notes II and III in Flaschel (1983b): $X \approx (0.72, 0.59)$.

We shall conclude our present investigation of input–output procedures employed to overcome the problems of observed multiple production (Table 4.2) by the representation of the input–output data of Table 4.2 if the above methodology is applied to transfer co–products to their characteristic industry, see Table 4.6.

This rearranged table exemplifies what is proved in Flaschel (1983a) for the general case, namely that no "absurd" input–output coefficients are possible in this case. The very choice of our example (Table 4.1) again shows, however, that this cannot be due to any property of the given technology as the chosen denomination "industry technology hypothesis" for the assumption employed to disentangle joint inputs wrongly suggests. Furthermore, a comparison of case 3 and 4 shows that it is the commodity–"technology" and not the industry–"technology" hypothesis which should be employed if employment multipliers are to be measured in the presence of multiple production, in contrast to the choice which is made in Dominion Bureau of Statistics (1969, p. 140).

4.3.6 Concluding Remarks

We have shown in this section that the matrix of industry coefficients S because of its arbitrary homogenization of jointly produced goods should not be considered as a final type of input–output table; instead, this table is but an intermediate step in the derivation of the input–output table of case 4 (where $Q = ST$). Manipulations of outputs alone which aimed at their rearrangement such that homogeneity of output items might be claimed, furthermore, was also not a successful strategy to provide an interpretable type of labor–requirements measurement in the presence of joint products. Instead, two basic methods of transfer of outputs *and* inputs (though devoid of the technological content normally ascribed to them) could be shown to imply sensible measures of such requirements. These two measures in addition made clear that employment effects associated with final demand changes under constant technical conditions and productivity effects associated with technical changes depart from each other once joint production is taken into consideration [cf. Flaschel (1980) for a more complete demonstration of this fact].

Hence, measuring employment multipliers as in Dominion Bureau of Statistics (1969, p. 140) or measuring indexes of labor productivity à la Evans (1953)/Stobbe (1980) in principle demands for a particular derivation procedure of the input–output table to be employed for this purpose. It is only after this choice has been made explicit for the reader in the intended investigation that one may pose the question whether, e.g., the output method (though devoid of theoretical content) may serve as a numerical approximation to the type of labor requirements that is to be measured. Finally, it should be noted that the methods 1, 2, 4 here considered are incapable of analyzing supply bottlenecks which stem from joint production, since by these methods all rigidities due to joint production are removed, from the production accounts that are used for the analysis.

4.4 Actual Labor Values vs. Zero–Profit Prices in Sraffian Models of Fixed Capital

4.4.1 Introduction

In his book "Marx after Sraffa" Steedman (1977) has collected a number of problems and calculations which are intended to show that the Marxian definition of value will lead to nonsensical quantitative expressions when applied to general technological situations and that, furthermore, no such value calculations are needed for a correct determination of the rate of profit. The main points of his examination of the properties which a quantitative expression of "embodied labor time" ought to have (but will not have) are developed in his chapters on fixed capital and pure joint production, but attention is also paid to problems arising from the existence of multiple activities and from heterogeneous labor.[13]

For the case of pure joint production we have already shown in note II in Flaschel (1983b) that the raised problems can be overcome when prices are taken into account in the formation of labor values to disentangle joint labor costs. This is not as

[13] This chapter was first presented at the conference "The value of value theory" in Bielefeld, March 1980. I like to thank Heinz Kurz and Ian Steedman for comments on the original version of the chapter.

"unnatural" as it may appear to be at first sight, if it is remembered that labor values are deduced in Marx's *Capital* from the notion of exchange value (things may be useful *and* the product of human labor, yet their labor values are zero because they have no exchange value, cf. Marx (1957, Vol. I, p. 40). Note also that relative wages are sometimes used by Marx to make labor "homogeneous").

The particular difficulties that arise with fixed capital and labor value depreciation have been treated in Flaschel (1979) only briefly and in relation to the procedures used for pure joint production in the main. The present section now elaborates the there noted particularities of value depreciation to derive a definition of labor values for the case of fixed capital that is economically meaningful (i.e., is additive, positive, and unique) and that in our view generalizes Marx's original description by integrating more flexible depreciation rules than are taken into account by him.

As fixed capital models embrace the existence of production alternatives, the introduction of the Marxian notion of "individual" (see note I in Flaschel (1983b)) complementing its average: "labor value" will become unavoidable in the following. This additional concept will be defined first in its natural surroundings: input–output tables which contain production alternatives but neglect fixed capital and its depreciation. To enable direct comparison this will be done at the same level of generality as is applied in Steedman (1977) chapter on fixed capital, i.e., we will use a two–sector presentation throughout. The next sections will then employ these pre–considerations to show how negative or incoherent book (labor) values of old machines can be avoided completely in Steedman's example of falling efficiency.

Furthermore, it will be argued that there is no necessity to view rising efficiency and rising book values as anomalies, no matter whether Steedman's "correct value calculation rule" or our definition is applied to his example of rising efficiency. Our conclusion will be: Nothing unusual can be observed on the side of labor value determination, when the notion of "individual value" is taken properly into account (though surely such definitional considerations do not suffice to give the label "true" or "correct" to the here proposed definition).

There remains the question of the usefulness of labor values. To that end certain results on the connection of labor–value and price magnitudes are derived for multiple–activity as well as fixed–capital models. Yet, despite the shown relationship between prices and labor productivity considerations, the explanatory power of labor values for a theory of profits must still be regarded as an open question.

Consequently, the main purpose of the following considerations is to push the discussion back to the real issue, i.e., the assertion of redundancy with respect to the utility of labor values which is the second type of critique that is raised by Steedman and others. How to generalize labor values and relating known propositions in a way that preserves the standard properties of this notion is not the crucial question of the labor theory of value, though it may make the methodological status of this ex post theorizing more precise. Instead it is the basic Marxian model presented, e.g., in Morishima's (1973) book which needs further elaboration.

However, the still existing crude versions of the "transformation problem", which give labor values a direct role in the *determination* of the rate of profit (a point which is rightly criticized by Steedman), should not directly lead to the conclusion that there is nothing to be gained from such an elaboration - as a view into existing input–output methodology should make clear, see also notes IV-VI in Flaschel (1983b).

4.4.2 Average and Individual Labor Values in Single Product Systems

To demonstrate the possibility of a sensible definition of labor values in the framework of fixed–capital–using economies as considered in Steedman (1977, pp. 14ff.) we have to treat single–product systems first. Contrary to the customary presentation of such systems our interest here lies in the examination of the case where several activities are employed in at least one sector of the economy.

Apart from this alteration our assumptions are the same as those made by Steedman (1977, pp. 140/1). In particular, we shall only consider two–sector models of simple reproduction with a given real wage s (corn) paid in advance. We assume as given two (circulating) capital goods M and C, where C (corn)[14] is used for consumption purposes of both workers and capitalists. Labor will be symbolized by "L". The period of production (a year) will be uniform between and within both sectors of the considered economy.

In direct correspondence to Steedman's (1977, p. 141) Table I we assume the following input–output table as *given* (Table 4.7), representing the quantities realized during the year under consideration.

As simple reproduction prevails, we have $M_2 + M_2' = M_1$ and $Q - C$ as net output of the system.

Let v_m, v_c *labor values* of "material" and "corn", respectively. According to Marx (1977, Vol. I, pp. 316f) we have to consider individual values, too, when more than one technique – here for the production of corn – is applied. The relation between individual values and labor values can be described in the following way:

Table 4.7 Disentangling fixed capital processes

Material	Corn	Labor		Material	Corn	
0	C_1	L_1	$\rightarrow$	M_1	0	M-sector
M_2	C_2	L_2	$\rightarrow$	0	Q_2	C-sector
M_2'	C_2'	L_2'	$\rightarrow$	0	Q_2	
$M_2 + M_2'$	C =	L =	$\rightarrow$	M_1	Q = $Q_2 + Q_2'$	Totals
	$C_1 + C_2 + C_2'$	$L_1 + L_2 + L_2'$				

[14] M = material and later: machines.

Let w_c, w'_c be the two *individual values* of corn with respect to the two given processes (of course no such distinction has to be made for the first sector here). Then we should have:

$$\frac{w_c Q_2 + w'_c Q'_2}{Q} = v_c \tag{4.11}$$

i.e., the labor value of corn is the average of the individual values, the weights being given by the output shares of the processes of the corn–producing sector. And by Marx's verbal definition of individual values, cf. Marx (1977, Vol. I, p. 317), we on the other hand have:

$$v_c C_2 + v_m M_2 + L_2 = w_c Q_2 \tag{4.12}$$

$$v_c C'_2 + v_m M'_2 + L'_2 = w'_c Q'_2 \tag{4.13}$$

i.e., the process–dependent individual value of a commodity – here of one unit of corn – is the sum of the labor values of physical inputs plus the direct labor time individually needed.

Substituting (4.11) into (4.12) and (4.13) and adding the equation for sector 1 – also transformed into w_c, w'_c by help of equation (4.11) – we get a system of 3 equations with 3 unknowns w_c, w'_c, v_m:

$$\frac{w_2 Q_2 + w'_c Q'_2}{Q} C_1 + L_1 \qquad = v_m M_2 \tag{4.14}$$

$$\frac{w_c Q_2 + w'_c Q'_2}{Q} C_2 + v_m M_2 + L_2 = w_c Q_2 \tag{4.15}$$

$$\frac{w_c Q_2 + w'_c Q'_2}{Q} C'_2 + v_m M'_2 + L'_2 = w'_c Q'_2 \tag{4.16}$$

(the fourth unknown: v_c is then immediately given by (4.11)).

Assumption. There exist prices $p_m, p_c > 0$ for "material" and "corn" such that (in matrix notation) :

$$(p_m, p_c)\begin{pmatrix} 0 & M_2 & M'_2 \\ C_1 & C_2 & C'_2 \end{pmatrix} < (p_m, p_c)\begin{pmatrix} M_1 & 0 & 0 \\ 0 & Q_2 & Q'_2 \end{pmatrix}$$

i.e., the proceeds from output exceed the costs of physical inputs for each of the three given processes. No other price systems will be taken into account in the following.

Proposition 4.1. *The system (4.14), (4.15), (4.16) has exactly one solution* (v_m, w_c, w'_c)*. This solution is strictly positive.*

In matrix notation the system (4.14)–(4.16) can be written as

$$(v_m, w_c, w'_c)\begin{pmatrix} 0 & M_2 & M'_2 \\ x_2 C_1 & x_2 C_2 & x_2 C'_2 \\ x'_2 C_1 & x'_2 C_2 & x'_2 C'_2 \end{pmatrix} + (L_1, L_2, L'_2) = (v_m, w_c, w'_c)\begin{pmatrix} M_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q'_2 \end{pmatrix}$$

with $x_2 = Q_2/Q$ and $x'_2 = Q'_2/Q$. Let A symbolize the matrix of inputs, B that of outputs, and let "a'' stand for the vector of direct labor inputs of this system. The last equation can then be abbreviated by

$$(v_m, w_c, w'_c)A + a = (v_m, w_c, w'_c)B \quad \text{or}$$
$$(v_m, w_c, w'_c)AB^{-1} + aB^{-1} = (v_m, w_c, w'_c)$$

And by help of the above assumption we have (because of $x_2 + x'_2 = 1$):

$$(p_m, p_c, p_c)A \quad < (p_m, p_c, p_c)B \quad \text{or}$$
$$(p_m, p_c, p_c)AB^{-1} < (p_m, p_c, p_c)$$

which implies Proposition 4.1 by known theorems on non–negative matrices (see, e.g., Nikaido (1968, Chap. II)).

Proposition 4.2.

$$(v_m, v_c)\begin{pmatrix} 0 & M_2 + M'_2 \\ C_1 & C_2 + C'_2 \end{pmatrix} + (L_1, L_2 + L'_2) = (v_m, v_c)\begin{pmatrix} M_1 & 0 \\ 0 & Q \end{pmatrix}$$

This result follows immediately by summing (4.15) and (4.16). It says that (average) labor values can be computed from specifically aggregated data without any interference from individual values. Note that the above assumption on the positivity of "value added" can be reduced to this aggregated system without loss of implications.

By summing the two remaining equations of result 2 we furthermore get:

Proposition 4.3. *There holds the value added identity:* $L = v_c(Q - C)$ *i.e., total employment is equal to the labor value of the net product* $Q - C$ *of Table I.*

Proposition 4.4.

$$\varepsilon = \frac{1 - v_c s}{v_c s} = \frac{v_c(Q - C) - v_c s L}{v_c s L}$$

i.e., the ratio of surplus value to the total value of labor power is equal to ε*, the rate of exploitation per labor hour bought.*

Proposition 4.5. *The rate of exploitation is positive if and only if there exist positive prices* (p_m, p_c)*, which imply positive profits (in the aggregate).*

Positive (aggregate) profits can be expressed by

$$P_c Q + p_m M_1 - p_c C - p_m(M_2 + M'_2) - p_c s L > 0,$$

where $p_c s$ describes the money wage per unit of labor. Now, because of $M_1 = M_2 + M_2'$, this is equivalent to $p_c(Q - C - sL) > 0$, which in turn is equivalent to $\varepsilon > o$.

The following definitions and proposition 4.8 have been taken from Roemer (1977) by extending his proof, cf. the mathematical appendix, to the case where several activities coexist in at least one sector.

Definition 4.6. Technical change is of capital–using labor–saving type, i.e., of type CU–LS, if and only if material or corn inputs must be increased for labor input to be decreased (technical change of type CS–LU or CS–LS is defined in a similar way).

Definition 4.7. Technical change is progressive (or neutral or retrogressive) if and only if the labor values (v_m^*, v_c^*), computed after the technical change has occurred, fulfil

$$(v_m^*, v_c^*) \leq (v_m, v_c) \qquad (\text{ or: } = (v_m, v_c), \text{ or: } \geq (v_m, v_c)) \tag{4.17}$$

with respect to the labor values (v_m, v_c) originally given.

Proposition 4.8. *(Roemer, 1977, p. 411)):*

(a) *All technical changes of type CU–LS which reduce average cost*[15] *are progressive (but there are progressive CU–LS–changes which do not reduce average cost).*
(b) *All technical changes of type CS–LU which are progressive reduce average cost (but there are CS–LU changes which reduce average cost but are not progressive).*

Since Marx (1977, Vol. I, p. 40) regards labor values also as indicators of labor productivity:

> *"In general, the greater the productiveness of labor, the less is the labor time required for the production of an article, ... the less is its value"*,

we can reformulate the above two assertions in the following way: Technological changes of the type that Marx considered as being of dominant kind in capitalism (average cost reducing CU–LS changes) always raise labor productivity, though not all of the CU–LS changes which raise labor productivity will be of this kind. On the other hand, CS–LU changes which raise labor productivity are always "profitable" for capitalists, but here there may exist average–cost reducing CS–LU changes which will lower labor productivity in at least one sector.

The results 1–6 and especially the definitions of average and individual labor values are not restricted to two–sector models, constant returns to scale, stationary economies and the like. All that in fact is needed is an ex–post Leontief–table with multiple activities together with a price vector p, such that value added is positive for any sector of this table.

[15] At the given initial prices $p_m, p_c, p_c s$.

Summarizing we have shown that the presence of production alternatives leads in a natural way to the distinction of individual vs. average labor values, where the latter are defined as properly weighted averages of the former, while the former are given by the column sums of the given input–output table made homogeneous by help of the latter. And though such individual values are not really necessary to compute the corresponding labor values, they are nevertheless needed to explain the latter, i.e., they help to depict the nature of labor values. This should be particularly obvious for an interpretation of labor values as productivity indexes. Here, individual values can be used not only to calculate their mean; but also to consider the variance that may exist within each sector with respect to such average productivity indexes.

Finally, our presentation has shown that additive calculations indeed form the basis of the determination of individual and average labor values, though – as deviations between them are possible – it cannot normally be expected that average labor values will fulfil the originally given rectangular input–output system in a strictly additive way. To my knowledge the necessity of such individual and average calculations has been completely ignored in the literature published so far.[16] These calculations are crucial for the now following arguments which show that Steedman's "correct value calculations" for fixed capital models are not so compelling as they seem to look at first glance.

4.4.3 Individual Values in the Case of Fixed Capital: Steedman's Example of Falling Efficiency Reconsidered

Consider the following technological description of a process which employs a durable capital good M ("machines" instead of "materials" as in the preceding section) to produce the consumption– and seed–commodity C:

$$\underbrace{\overset{C}{(-49,}\quad \overset{M}{-3,}\quad \overset{L}{-30,}}_{\text{date 1}}\ \underbrace{\overset{C}{88-3,}\quad \overset{M}{0,}\quad \overset{L}{-30,}}_{\text{date 2}}\ \underbrace{\overset{C}{30,}\quad \overset{M}{0,}\quad \overset{L}{0)}}_{\text{date 3}}$$

production period 1 production period 2

In this example, 3 machines together with 49 units of corn and 30 units of labor have to be advanced to produce 88 units of corn at the end of the first period and can be employed again (but with falling efficiency) to produce 30 units of corn at the end of the second period by using up 3 units of corn and 30 units of labor at its beginning. At the end of the second period the employed machine is physically worn out with no use value left.

[16] See however Kurz's (1979) consideration of "rent" for an exception.

Following Sraffa's methodology, Steedman (1977) gives the following alternative description of the above example of fixed capital (cf. his Table III on p. 145):

$$49C + 3M_{new} + 30L \to 88C + 3M_{old} \tag{4.18}$$

$$3C + 3M_{old} + 30L \to 30C \tag{4.19}$$

Here, the two phases of the above process are shown separately by means of the introduction of one further good: the one–year–old machine: M_{old}.

It is the aim of such a presentation to make applicable the price equations of Sraffa's square joint production systems for the case of fixed capital models, too. By this approach, prices of production are calculated in such a way that a uniform rate of profit will emerge on each stage of the machine–using process under consideration. To perform this task in the here needed economy–wide sense it is necessary to complete the above characterization of corn production (4.18), (4.19) by a machine–producing process, which is done by Steedman in the following way:

$$3C + 3L \to 3M_{new} \tag{4.20}$$

Taking corn as numéraire he then obtains with respect to a given real wage of $s = (2/3)C$ as prices for the two types of machines

$$p_n = 2, \; p_0 = 2/3 \tag{4.21}$$

corresponding to a uniform rate of profit of 20%.

With regard to these prices – which are taken as actual ones in the following – two thirds of the value of the new machine is written down in the first period and the remaining (book) value in the second.

To calculate his "correct labor values" Steedman now applies the principle of uniform profitability to a zero rate of profit, too, i.e., to a hypothetical real wage s, and he characterizes Marx's labor value calculations as being purely additive as far as new commodities are concerned and strictly linear with regard to fixed capital and depreciation. As result he then obtains a negative book value for the old machine by his first method and incoherent results, i.e., an overdetermined equation system by his second one.

But is it really plausible that value depreciation – which as any depreciation process is not a matter of technology alone – should be conducted by ideal rules of uniformity – here of zero profits – which are not related to existing facts, i.e., do not pay regard to our actual depreciation rates $2/3, 1/3$? It is our opinion that not only factual data of technology but also *factual economic data* have to be used for the calculation of "embodied labor time" in case the former do not suffice to solve this task. And it is a well–established fact in managerial cost–accounting that one cannot arrive at a determination of full costs on purely technological grounds in general, i.e., without help from more or less plausible economic imputations. Why should things be better with respect to the calculation of "real costs" - here with regard to labor?

We have elsewhere shown, cf. note II in Flaschel (1983b), how labor values can be defined without anomalies in the presence of "pure joint production" (corn and straw). There, the existing joint commodity outputs and their inputs have been disentangled by help of economic imputations which reflected the benefit received from each unit of costing.

The above prices of production (4.21) and the implied depreciation rates $2/3, 1/3$ now obviously obey the same "benefit principle" (which again imputes joint effort in such a way that uniform profitability, here over stages of production, will emerge). By employing this principle now for the case of fixed capital, the methodology of the foregoing section will again suffice to prevent Steedman's anomalies of labor value depreciation as we shall now show.

The above treatment (4.18), (4.19) of the corn–producing sector already indicates that fixed capital systems contain the side by side existence of production alternatives, i.e., that the foregoing section will find application. But compared to the case of pure joint production it is now only the fixed capital good which has to be imputed to the different stages of production [and not the whole input basket (as was necessary in that former case). This method, which replaces the artificial output: M_{old} by the actual deduction that is made from the (value of the) input: $3M_{new}$, reduces the considered situation to the case of multiple activities already considered.

Treating Table 4.8 now in the same way as Table 4.7 in the preceding subsection, we get for the two labor values of "corn" and "machines" (by means of Proposition 4.2):

$$(v_m, v_c)\begin{pmatrix} 3 & 0 \\ 52 & 3 \end{pmatrix} + (3, 60) = (v_m, v_c)\begin{pmatrix} 3 & 0 \\ 0 & 118 \end{pmatrix}, \text{ i.e. }, (v_m, v_c) = (2, 1),$$

which are the same values as those obtained by Steedman (1977, p. 145), as they stem from the same aggregated system. But instead of $v_0 = -1$ we now have $(1/3)v_m = 2/3$ for the book value of the old machine. Furthermore: Instead of extending this additive approach to a three equations system we here have two further (4.12),(4.13) to consider, which determine the two individual values of corn for the two existing stages of production:

$$w_c \approx 0.943, \; w_c^{'} \approx 1.167$$

which are connected to the (average) labor value of corn by means of (4.11):

$$v_c = \frac{88w_c + 30w_c^{'}}{118}.$$

Table 4.8 Rising machinery efficiency

M	C	L		M	C	
0	3	3	→	3	0	M–sector
$(2/3) \cdot 3$	49	30	→	0	88	C–sector
$(1/3) \cdot 3$	3	30	→	0	30	
3	55	63	→	3	118	Totals

Hence, nothing abnormal can be observed with respect to labor values in Steedman's example of falling efficiency, when reference is given to the notion of individual value in order to take into account productivity differences at the two stages of the corn–producing process. As against that Steedman's approach produces its anomalies because it starts from the assumption that: $v_c = w_c = w_c^{'}$ (then having only three equations to consider, instead of 5), an assumption, that will be justified in very special cases only.

But what about Marx's original value calculations?

> "*...Marx worked in terms of linear value depreciation but was aware that it was an over–simplification*", Steedman (1977, p. 140)).

Let us thus adopt $\alpha_1 = \alpha_2 = 1/2$ instead of Sraffa's $2/3, 1/3$ as depreciation rates in our Table II. We then obtain (besides v_m, v_c as before) the individual values: $w_c \approx 0.932, w_c^{'} \approx 1.200$ – now with respect to the book value $(1/2) \cdot 2 = 1 = v_0$ for the one–year–old machine.

Thus Marx's oversimplification' produces little difference to the preceding calculations (and surely no "internal incoherence"), but has the advantage of getting rid of price–determined non–linear depreciation procedures. It may, therefore, be regarded as an approximation, an approximation that works the better, the less fixed capital is employed per unit of output.

The above calculations clearly indicate, that (at least as long as $\alpha_1, \alpha_2 \geq 0$, $\alpha_1 + \alpha_2 = 1$ holds true) no perverse (negative) labor values (book values) can be obtained from the application of our first subsection to such examples with falling efficiency. These values remain well-defined, whatever the process may be that actually determines the depreciation rates α_1, α_2.

This shows that it is not the exact form of the employed actual depreciation rates which is crucial for the above obtained quite normal results, but that it is simply the methodology of our approach that prevents the occurrence of "anomalies". Our understanding of what is factual with respect to depreciation may change, but as long as it is acknowledged that fixed capital must embrace the existence of production alternatives and that this makes the concept of "individual values" unavoidable, nothing unusual will happen as far as the determination of "real labor costs" is concerned. It is the neglect of this latter concept (and the derivation of depreciation rates $\alpha_1 = 1.5, \alpha_2 = -0.5$ outside the above admissible range [0, 1] by a formal procedure [17]) which makes Steedman's "correct value calculations" unsuitable for a proper generalization of Marx's labor values.

There are two notes of warning to be appended to our method of definition. First: The above indicated range of admissible choices of depreciation rates α_1, α_2 is not sufficient to establish proportionality between prices of production and labor values (in fact the above labor values v_m, v_c do not even depend on α_1, α_2!) And second: This independence of v_m, v_c from α_1, α_2 and their equality with zero–profit prices

[17] which ensures $w_c = w_c^{'}$ and therefore makes our average calculations avoidable zero–profit condition.

is due to the assumption that "simple reproduction" prevails, in which case the aggregated system of Proposition 4.2 does not depend on α_1, α_2. Hence, it will not be possible in general to deduce the positivity of our labor values from the positivity of Sraffa's zero profit prices for finished goods.

Summarizing, we can state that uniform profitability implies uniform productivity on all stages of production of the actually chosen technique in very special cases only, i.e., different stages of production will be characterized by different real costs with respect to their homogeneous output in general. Thus, averages have to be formed to arrive at unique labor values per type of commodity.

4.4.4 Rising Efficiency and Rising Book Values of Machinery

We have excluded negative depreciation rates from the above reasoning. Yet, Steedman also provides an example, where a situation with $\alpha_1 < 0, \alpha_2 > 0$ will arise. The technological description of this example of rising efficiency (expressed in integrated form) reads (Table 4.9):

The output of corn in the first period does not even exceed the input necessary for the second period, but there is a net output of $25 + 10 - (15 + 9)$ units of corn with respect to the whole process.

To apply Sraffa's book values to the determination of labor values, we have to determine Sraffa's prices first (which have been omitted by Steedman in this example). Let the real wage s be $0.2C$ and set $p_c = 1$. We then have to solve:

$$(1 + r)(1 + 0.2 \cdot 5) = 5p_m$$
$$(1 + r)(9 + 5p_m + 0.2 \cdot 10) = 10 + 5p_0$$
$$(1 + r)(15 + 5p_0 + 0.2 \cdot 25) = 25$$

As solution we get:

$p_m \approx 0.4204$ the price of the new machine

$p_0 \approx 0.7540$ the book value of the old machine

$r \approx 5.1\%$ the uniform rate of profit

The given phenomenon of rising (physical) efficiency here finds its expression in rising vintage prices and *"might be the result of the 'running–in' of the new machine"*, see Steedman (1977, p. 142). We would suppose, however, that the "running–in" of the new machine resulting in risen efficiency during its second year

Table 4.9 Disentangling fixed capital processes

C	M	L	C	M	L	C	M	L	
−1	0	−5	0	5	0	0	0	0	M
−9	−5	−10	10 − 15	0	−25	25	0	0	C
	date 1			date 2			date 2		

cannot be reasonably thought of if it is not accompanied by running–in costs[18] – not necessarily easily identifiable in each case. Therefore (and for a simple presentation of our arguments), we shall assume that 3 of the 9 units of the corn– input at the beginning of the first period are necessary (e.g.):

(a) To have a sufficient output in period one, or
(b) To ensure the possibility of using the one–year–old machine once again[19] or
(c) To raise the output of corn in each period by two units (from 8 to 10 and from 23 to 25 respectively).[20]

Yet, let us neglect these possible interpretations of the above 3 units of corn–costs for the moment and apply instead the methods of definition considered in the preceding subsection without further qualification.

Steedman (1977, p. 144) – identifying again v_c with w_c and w_c' by his "correct value accounting" method – has to solve the equation system

$$\begin{aligned} v_c + 5 &= 5v_m \\ 9v_c + 5v_m + 10 &= 10v_c + 5v_0 \\ 15v_c + 5v_0 + 25 &= 25v_c \end{aligned}$$

and gets as solution $v_c = 4, v_m = 1.8, v_0 = 3$.

He concludes *"that with machines of rising efficiency, value depreciation can be negative"* (Steedman, 1977, p. 144). Yet there is nothing paradoxical in the fact that additional costs (3C) may lead to rising book values of machinery - both at a rate of profit of 5.1% or of 0%! The only thing to be explained here is the way in which this simple fact expresses itself in the above used system of price equations, where no distinction is made between those corn–costs which will circulate with their product and those which will remain fixed with the machinery during its useful life. This is a topic that cannot however be pursued here any further.

Now, in contrast to Steedman's method of definition we have proposed to distinguish between v_c and w_c, w_c' when labor values are to be defined. By use of Sraffa's vintage prices (see above) we get the depreciation rates:

$$\alpha_1 = \frac{p_m - p_0}{p_m} \approx -0.794, \ \alpha_2 \approx 1.794$$

and thereby – through an economically determined allocation of the 5 machines to the two stages of corn production – in complete analogy to the preceding section (Table 4.10):

Employing once more the definitions initially introduced (especially (4.12), (4.13)) we here get:

[18] A special kind of investment expenditure.

[19] In this case there exists the alternative profile $(-6, -5, -10, 10, 0, 0)$ which can be neglected, because it is inferior at the given equilibrium prices.

[20] This again will lead to a more profitable solution.

Table 4.10 Fixed capital and the measurement of embodied labor time

M	C	L	→	M	C	
0	1	5	→	5	0	M–sector
$-0.794 \cdot 5$	9	10	→	0	10	C–sector
$1.794 \cdot 5$	15	25	→	0	25	
5	25	40	→	5	35	Totals

$$w_c \approx 3.886, w_c^{'} \approx 4.046, \quad v_c = \frac{10w_c + 25w_c^{'}}{35}(= 4)$$
$$v_0 = (1 + 0.794) \cdot v_m \approx 3.229 \quad (v_m = 1.8).$$

While in the example with falling efficiency our distinction between individual and average labor values made it possible to prevent Steedman's negative book–values completely, our definition of v_0 (and $w_c, w_c^{'}$) applied to his example of rising efficiency stays in close contact with the qualitative features found by him. This can be so, since there is no anomaly in rising book values, as we have noticed above; on the contrary, it is quite natural that the labor value of machine vintages should rise if investment expenditures (implying "appreciation") exceed depreciation.

We now come to a consideration of the three kinds of investment expenditures a),b),c) stated at the beginning of this section. As truncations can be excluded here, there will be no difference between them with regard to Sraffa's method of price determination. This method will only discriminate between these types of investment, when different truncations are implied by them, and it performs this task by representing the combined effect of depreciation allowances and "appreciation" efforts at each stage by a single number: the (then possibly positive) change in book–value.

But (at least) when measuring labor value and productivity there might be a difference between the three stated types. With respect to our example of 3 units of corn as "running–in" costs this may be illustrated in the following way. Neglecting these costs we obtain for the two stages of the corn–producing sector:

$$6C + 5M_{new} + 10L \to 10C$$
$$15C + 5M_{old} + 25L \to 25C.$$

Consequently, we get as pure depreciation rates for period 1 and 2

$$\alpha_1 = 2/7; \quad \alpha_2 = 5/7,$$

if we assume proportionality between activity level and depreciation and regard constant efficiency to be in line with the application of linear depreciation.[21] But in addition to that we have distinguished three types of "running–in" costs. We now

[21] This is not the case when Sraffa's book–keeping method is applied.

would propose to interpret case a) as implying *"appreciation rates"* $\beta_1 = 1, \beta_2 = 0$[22] and case c) as implying (approximately) $\beta_1 = 2/7, \beta_2 = 5/7$.

We have shown in the subsection on multiple activities that in the case of simple reproduction (average) labor values do not depend upon such rates. But with regard to individual values and (4.12), (4.13) we would propose to define these values in the presence of "appreciation" by:

$$6v_c + \beta_1 3v_c + \alpha_1 5v_m + 10 = 10w_c$$
$$15v_c + \beta_2 3v_c + \alpha_2 5v_m + 25 = 25w_c^{'}.$$

In case a) we then get: $w_c \approx 4.857, w_c^{'} \approx 3.657$ and $v_0 = [\beta_2(3v_c) + \alpha_2(5v_m)]/5 \approx 1.286$, i.e., we get rising labor productivity within the process considered and a falling book value.

In case b) we get: $w_c \approx 3.657, w_c^{'} = 4 = v_c$ and $v_0 \approx 3.686$, i.e., the opposite result in comparison to a).

And in case c) we get: $w_c = w_c^{'} = 4 = v_c$ and $v_0 = 3$, which is the case that Steedman describes as "correct value accounting" (a special situation, where "appreciation" exceeds depreciation).

But no matter how any given pattern of α_i, β_i will be determined in the end, once they are given it is always possible to associate with these depreciation and "appreciation" coefficients a positive vector of individual values describing productivity changes *within* the fixed–capital–using process(es) in a meaningful way in comparison to their (positive) averages: the Marxian labor values. There is no need to dwell upon the question of how to determine the practically most relevant depreciation procedures to show that labor values and the transfer of labor value from durable means of production to commodity output can be defined without anomalies.

4.4.5 Final Remarks

We have shown that the first part of Steedman's (1977, pp. 148/9) conclusions may be due to a too narrow definition of labor values. Neither does Marx's simplification (to assume linear value depreciation) necessarily lead to incoherent results nor are Steedman's "properly calculated additive values" as conclusive as they seem to be.

We have reached this result by the inclusion of prices into the definition of labor values itself. Will this new feature of labor values deprive them of their formerly possibly existing usefulness? We do not believe this to be the case.

Though other opinions indeed exist, we would insist that labor values 1 have to be considered as ex–post defined magnitudes, i.e., they are derivatives of the physical data Q in existence. As such they will, of course, depend on prices p, since actual physical data will depend on them. Labor values v, therefore are characterized by the following dependency:

$$v = v(Q(p))$$

[22] This may be the only case which is really covered by Sraffa's method of price computation.

But is there really a new quality involved between this understanding of labor values and the now reached stage of their interpretation

$$v = v(Q(p), p),$$

where prices exercise direct influence on their determination to some extent? And is it really such a surprise that prices may have further influence on labor value determination apart from their influence on the choice of technique, e.g., by regulating the amount of labor value depreciation? In fact, we do not believe that the above described change in interpretation – from the first type of dependency to the second – will give further reasons for the claim of redundancy of labor values with regard to the calculation, determination or explanation of profits and prices.

Both kinds of the comprehension of labor values on the one hand as pure derivatives of actual technology, and on the other hand as the attempt to erect a sensible definition of full costs in real terms (which is known to be impossible on purely technological grounds) in my view share the common and more important methodological difficulty of how to explain the uses that can be drawn from such post factum magnitudes. It is the scientific content of such derivatives of price–quantity–interrelationships which presents the most crucial question.

There are two basic and related strategies – besides methodological contemplation – to attack this difficulty. First: to enrich the standard Marxian model as in Morishima (1973) by further theoretical insights into the relationship of price-value calculations, e.g., by exploiting further the labor productivity characterization of labor values (cf. Proposition 4.8) and secondly: by integrating the empirically oriented work done by input–output analysts with respect to similar magnitudes to enrich the above model by hypotheses which can be judged by empirical observation.

It may be that in doing so we shall find out that the question of generalization will be solved simply by practical constraints, conventions or insights which still lack a thorough theoretical consideration.

4.5 Conclusions and Outlook

We have proposed in this chapter an alternative way of measuring labor values in general joint production systems (see the following chapters for its general formulation) which allows us to use the nominal and "real" statistics supplied by the United Nations' System of National Accounts without much change when the input–output tables and their data are derived on the basis of the so-called industry technology hypothesis. We have moreover seen that this hypothesis is in line with what firms are actually doing in cost accounting when they face the situation of pure joint production. Our approach to labor values, their definition and their application, is on the one hand of a very general nature, but nevertheless operational and applicable in the way concepts of input–output analysis are defined and applied. This also holds for the treatment of fixed capital, semi-finished products, skilled labor and more, see Flaschel (1983b) for details. The only big and still unsettled issue for the approach we are propagating is in fact the treatment of open economies and international labor values, a treatment where we also hope to learn from procedures applied in United Nations' System of National Accounts. As in the United Nations' System of National Accounts – though from a different angle – we thus suggest that a dual way of interpreting and analyzing (ex post, without any dependence on a particular equilibrium or disequilibrium approaches) the data supplied by the working of capitalist market economies is the appropriate one, since we interpret Marx' value theory as a System of National Accounts meant to allow an analysis of what is "really" going on behind the reported nominal statistics.

Mathematical Appendix

In this section we provide a proof for the proposition 4.8 formulated in Sect. 4.4.[23] As the case of switch–points (under the assumption of a finite set of substitution possibilities) is an isolated phenomenon, it is not sensible to restrict the following proof to constant returns to scale and switch point situations solely. Hence, the following simple generalization of a switch point situation is assumed to be given:

Assumption. There exist positive prices p_c, p_m and positive numbers $r_1, r_2, r_2^{'}$ (rates of profit) such that the following equations hold for the data of Table 4.7:

$$\begin{aligned} (p_c C_1 + L_1)(1 + r_1) &= p_m M_1 \\ (p_c C_2 + p_m M_2 + L_2)(1 + r_2) &= p_c Q_2 \\ (p_c C_2^{'} + p_m M_2^{'} + L_2^{'})(1 + r_2^{'}) &= p_c Q_2^{'} \end{aligned} \tag{4.22}$$

Let $v\overline{A} + \overline{L} = v$ denote the equation system of Proposition 4.2 (with respect to $M_1 = 1, Q = 1$ by appropriate choice of units).

Lemma.

$$(0, 0) < v = (v_m, v_c) < p = (p_m, p_c)$$

Proof. Equation system (4.22) implies

$$(p_m, p_c)\left[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} - \begin{pmatrix}0 & (1+\overline{r}_2)(M_2+M_2^{'})\\ (1+r_1)C_1 & (1+\overline{r}_2)(C_2+C_c^{'})\end{pmatrix}\right] = \begin{pmatrix}(1+r_1)L_1\\ (1+r_2)(L_2+L_2^{'})\end{pmatrix}^t$$

with respect to $1 + r_2 = \frac{p_c}{p_c(C_2+C_2^{'})} + p_m(M_2 + M_2^{'}) + (L_2 + L_2^{'}) > 1$.

Since the right hand side of this equation system

[23] cf. also Roemer (1977, p. 420) for a proof with respect to the standard Leontief model.

$$p(I - D) = (1 + r_1)L_1, (1 + \overline{r}_2)(L_2 + L_2^{'}))$$

is positive, the matrix $I - D$ on the left hand side must be nonnegatively invertible (cf. Nikaido (1968, p. 95)), i.e., we get

$$(p_m, p_c) = ((1 + r_1)L_1, (1 + r_2)(L_2 + L_2^{'}))(I - D)^{-1} \text{ and}$$
$$(I - D)^{-1} = 1 + D + D^2 + D^3 + \ldots \geq 0.$$

Furthermore, we have:

$$((1 + r_1)L_1, (1 + r_2)(L_2 + L_2^{'})) > (L_1, L_2 + L_2^{'})$$
$$\text{and } (I - D)^{-1} + \overline{A} + \overline{A}^2 + \ldots, \text{ where } \overline{A} = \begin{pmatrix} 0 & M_2 + M_2^{'} \\ C_1 & C_2 + C_2^{'} \end{pmatrix}.$$

Therefore we get

$$(p_m, p_c) > (L_1, L_2 + L_2^{'})(I + \overline{A} + \overline{A}^2 + \ldots) = (L_1, L_2 + L_2^{'})(I - \overline{A})^{-1} = v. \quad \square$$

CU–LS technical change which reduces average cost is progressive:
Let $\check{A}$, $\check{L}$, $\check{Q}$ be the input–output coefficient matrices of Proposition 4.2 after the change has been made. In direct generalization of the case of constant returns to scale, the CU–LS assumption can be understood to hold "on an average":[24]

$$CU : \overline{A} \leq \check{A}\check{Q}^{-1}, \quad LS : \overline{L} \geq \check{L}\check{Q}_{-1} \tag{4.23}$$

The assumption of a reduction in average costs can be expressed by:[25]

$$p\overline{A} + \overline{L} \geq p\check{A}\check{Q}^{-1} + \check{L}\check{Q}^{-1} \tag{4.24}$$

with respect to the price system initially given (see above).
Combining (4.23) and (4.24) gives:

$$p(\check{A}\check{Q}^{-1} - \overline{A}) - (\overline{L} - \check{L}\check{Q}^{-1}) \leq 0, \quad \check{A}\check{Q}^{-1} - \overline{A} \geq 0, \quad \overline{v} - \check{L}\check{Q}^{-1} \geq 0 \tag{4.25}$$

By the above lemma we have $v < p$, which implies

$$v(\check{A}\check{Q}^{-1} - \overline{A}) \leq p(\check{A}\check{Q}^{-1} - \overline{A}),$$

[24] Multiplication by $\check{Q}^{-1}$ gives the average input coefficients per sector and per unit of output. Note that we have assumed $\overline{Q} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

[25] Inequality (4.24) bears no close resemblance with the criteria used to describe the choice of technique when durable equipments are considered. Therefore, the possibility of the occurrence of profitable CU–LS changes which do not raise labor productivity is considerably increased then.

because of the non–negativity condition $\check{A}\check{Q}^{-1} - \overline{A} \geq 0$. Inserting this result into (4.25) and rearranging the obtained terms we get:

$$v\check{A}\check{Q}^{-1} + \check{L}\check{Q}^{-1} \leq v\overline{A} + \overline{L} = v.$$

And by recursive application of this inequality we finally get:

$$\begin{aligned} v' &= v\check{A}\check{Q}^{-1} + \check{L}\check{Q}^{-1} \leq v \\ v'' &= v'\check{A}\check{Q}^{-1} + \check{L}\check{Q}^{-1} \leq v' \\ v''' &= v''\check{A}\check{Q}^{-1} + \check{L}\check{Q}^{-1} \leq v'', \quad \text{etc.} \end{aligned}$$

This monotonic sequence is bounded from below (by 0) and therefore converges to a nonnegative vector v^* fulfilling:

$$v \geq v^*\check{A}\check{Q}^{-1} + \check{L}\check{Q}^{-1} = v^* > 0 \quad \text{(by Proposition 4.1)} \qquad \square$$

Because of the gap between v and p (cf. the lemma) there exists, on the other hand, the possibility for a $CU--LS$ change to be progressive, i.e., to fulfil $v^* \leq v$, without leading to a reduction in average cost, cf. (4.24), i.e., (4.25) is not a necessary condition for the occurrence of $v^* \leq v$.

The proof of Proposition 4.8(b) is obtained by reversing the above inequalities and the order of the argument.

References

Dominion Bureau of Statistics. (1969). *The input–output structure of the Canadian economy 1961*, 1. Ottawa: The Queen's Printer.

Evans, D. W. (1953). Indexes of labor productivity as a partial measure of technological change. In *Input–output relations*. Leiden: The Netherlands Economic Institute.

Flaschel, P. (1979). Wert und Arbeitswert: Wertproduktivität kapitalistischer Produktionsweise auf Basis des Begriffs der Arbeitsproduktivität im Systemzusammenhang? *Gesellschaft. Beiträge zur Marxschen Theorie* 13. Frankfurt/M.: edition Suhrkamp.

Flaschel, P. (1980). The derivation and comparison of employment multipliers and labor productivity indexes using monetary and physical input–output tables. *Economics of Planning, 16*, 118–29.

Flaschel, P. (1983a). Actual labor values in a general model of production. *Econometrica, 51*, 435–454.

Flaschel, P. (1983b). *Marx, Sraffa und Leontief. Kritik und Ansätze zu ihrer Synthese.* Europäische Hochschulschriften. Frankfurt/M.: Peter Lang.

Kurz, H. D. (1979). Wahl der Technik und Arbeitswertlehre in einem einfachen Modell mit zwei originären 'Faktoren'. *Jahrbuch für Sozialwissenschaft, 30*, 27–51.

Marx, K. (1977). *Capital, Volume I – III*: London: Lawrence and Wishart.

Morishima, M. (1973). *Marx's economics. A dual theory of value and growth*. London: Cambridge University Press.

Nikaido, H. (1968). *Convex Structures and Economic Theory*. New York: Academic Press.

Rettig, R. (1979). Sektorale Beschäftigungseffekte. Essen: *Mitteilungen des Rheinisch–Westfälischen Instituts Essen*, 30.

Roemer, J. (1977). Technical change and the "tendency of the rate of profit to fall". *Journal of Economic Theory, 16*, 403–424.

Samuelson, P. (1971). Understanding the Marxian notion of exploitation: A summary of the so-called transformation problem between Marxian values and competitive prices. *Journal of Economic Literature, 9*, 399–431.

Sen, A. (1978). On the labor theory of value: some methodological issues. *Cambridge Journal of Economics, 2*, 175–190.

Steedman, I. (1977). *Marx after Sraffa.* London: NLB.

Stobbe, A. (1980). Volkswirtschaftslehre I: Volkswirtschaftliches Rechnungswesen. Heidelberg: Springer.

United Nations. (1968). *A system of national accounts*. New York: Studies in Methods, Series F, No.2, Rev.3.

United Nations. (1973). *Input–output tables and analysis*. New York, Studies in Methods, series F, no.14, rev.1.

Chapter 5
Actual Labor Values in a General Model of Production

5.1 Introduction

In contrast to counterfactual linear programming definitions of labor values this chapter introduces an at least equivalent definition of such values for the general case of joint production which is exclusively based on actual data. This task is accomplished by extending Marx's concept of "individual values" from multiple activities to joint production by means of certain price ratios at those points where production data are insufficient to ensure the positiveness of "embodied labor time". Our approach generalizes the simple formula that relates labor values to monetary input–output tables, and it reformulates the labor theory of value in such a way that the "theoretical priority" of the case of a uniform composition of capital is reaffirmed.

In the concluding pages of his book, *Marx's Economics*, it is suggested by Morishima that Marxian economists "ought radically to change their attitude towards the labor theory of value" since in general the value system of the methods of production actually adopted in a capitalistic economy "may be determined to be negative, indefinite or even contradictory to the postulate of the uniform rate of exploitation" (Morishima 1973, p. 193). He then continues by proposing that *"optimal labor values",* the shadow prices of certain linear programming problems which minimize the amount of labor needed to produce given bundles of net output, should take the place of *"actual labor values"* when production models of a more general kind than the simple Leontief model are to be considered, an interpretation of labor values which has been worked out in detail in Morishima (1974) and has become a widely accepted approach to labor values and the labor theory of value for all types of multiple production systems.

In contrast to these theoretical refinements in the determination of such – direct and indirect or – *system labor requirements*, input–output analysts have continued to measure them in the standard way, i.e., by means of the Leontief inverse of recorded transactions tables. Yet they, too, became aware of the problems of multiple and in particular joint production in the course of verifying their basic assumptions of "the identity of industry and product" and of "product homogeneity." In the process of aggregating production statistics collected from establishments into those of industries, by means of the method of "principle products," it was usually observed that some subsidiary production remained in existence despite efforts towards the

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_5,

classification of establishments and products into suitable groups. Data on output and input of industries, accordingly, arise in a form for which no obvious procedure is available for reducing them to the single input–output table finally presented. To solve this problem formal methods have been invented which suitably transfer subsidiary production from the production accounts of their industry of origin to those of their principal industry. This methodology is meanwhile well established, and is surveyed in the United Nations' SNA, the System of National Accounts (see United Nations (1968, Chap. III)). Despite the tenacious existence of multiple products of various sorts, therefore, input–output methodology has succeeded in deriving single input–output tables to which the conventional formula of measuring system labor requirements can be applied. Probably because of the minuteness of the reallocations involved in the derivation of such tables (at present), however, no theoretical penetration of their effect on the obtained measure of labor requirements, when assuming one or another definite technological basis, has been undertaken so far.

Taking both approaches into consideration we are thus confronted with two distinct attitudes towards the determination of such system labor requirements: one that is theoretically sound but which, by the counterfactual use it makes of all kinds of potential methods of production out of a given set of blueprints, cannot be checked by observation, and one that performs this task in the conventional way by first removing all multiple production through seemingly arbitrary manipulations of input and output coefficients. Are we consequently forced to conclude that the identity between these two types of measures, shown to exist for simple Leontief models in Okishio (1963, p. 291) through a simple translation of physical into monetary terms, is restricted to this latter type of model, thereby in general invalidating his conclusion that "value" in the Marxian sense is not metaphysical as is often claimed but an observable and operational magnitude'?

It is the purpose of this chapter to show that this need not be the case. A definition of labor values can be given which again is based on the methods of production that are actually adopted, but which nevertheless avoids the anomalies ascribed to it in the introductory quotation from Morishima's book, and by which Okishio's result can be generalized into the von Neumann world of joint production. The conceptional discrepancy pointed out above between theory and measurement, therefore, can be bridged again, and, to be sure, by way of a new interpretation of labor values which applies the so–called "sales value method" of the accounting principles of firms to the case of labor costs, and through an intimate relationship of this method with the allocation rule proposed for the case of joint products in the SNA.

For a detailed demonstration of these assertions we shall employ the general equilibrium model of capitalistic reproduction that was introduced by Roemer (1980). This model, together with Roemer's reformulation of the "optimum labor theory of value," our basis of comparison, will be considered briefly in Sect. 5.2. The actual data supplied by means of its equilibrium concept of "reproducible solutions" will then be utilized in Sect. 5.3 and will suffice to present our new formulation of labor values for the case of joint production. To motivate this formulation we shall first consider the case of multiple, but single product, activities (Sect. 5.3) – and will do this by the very method used in Marx's *Capital* for this case, i.e., by way of a formalization of his important distinction between *average* and *individual values*. Section 5.3 then shows how the case of joint production can be reduced to this preliminary study of multiple activities per commodity, that is, by the sales value method already mentioned. In Sect. 5.3, on the other hand, the employed actual equilibrium solution will be transformed into a single input–output table of conventional type by way of the SNA methods recommended for the case of joint production. As one of our basic findings we shall then demonstrate that the resulting system labor requirements per unit of output value must simply be multiplied by corresponding commodity prices to arrive at the labor values as defined in Sect. 5.3, i.e., Okishio's result for the case of joint production. In Sect. 5.4, finally, basic propositions as, e.g., the Marxian case where "prices are equal to values," (only) known to be valid in the conventional equations presentation of Marx's labor theory of value, are shown to remain true for this general model of production. The narrow limits of the "optimum labor theory of value," discussed in Sect. 5.2, will thereby be considerably overcome. Morishima's (1973, p. 193) claim that the techniques actually adopted will allow for "no satisfactory theory at all" thus cannot be regarded to represent a good working hypothesis when measured against the standards set by his own "optimum labor theory of value."

It may be argued, however, that there is a serious cost involved in our rehabilitation of conventional propositions of the labor theory of value, since by our use of the sales value method (to disentangle joint products in such a way that Marx's notion of "individual value" can again be applied) relative prices are now introduced into the definition of labor values itself. Yet, this does not impose far–reaching restrictions on our version of a labor theory of value in comparison with its elementary form (Morishima, 1973). In fact, our approach will neither sacrifice the fundamentals of established theory nor lose contact with common practices of measurement. Should we, therefore, be bothered by the fact that one further step has been taken to dispense with the curious postulate that labor values have to be "independent of what happens in the market" (see Morishima (1973, p. 181))?

5.2 A General Equilibrium Approach to Marxian Economics

In elementary mathematical formulations of Marxian economics in general, and of the labor theory of value in particular, all basic definitions (of values, prices and quantities) are given in terms of simultaneous input–output equations, as for example in Morishima's (1973) well known presentation of the theory. In the final chapter of that work, however, this equations approach is severely criticized by Morishima for its narrow range of validity and is replaced by an inequality approach: the von Neumann model with regard to prices and quantities plus a linear programming formulation of labor values and the rate of exploitation (see Morishima (1974) and Morishima and Catephores (1978) for extensions and modifications).

The central theme of this way of generalizing labor values to include all kinds of joint products has been the so–called Fundamental Marxian Theorem (FMT): the equivalence of positive profits and a positive rate of exploitation (see Morishima and Catephores (1978, p. 30) for a brief history). This theorem and its current formulation in terms of "optimum values" is also included in the general equilibrium approach to Marxian economics presented in Roemer (1980), which will provide the formal framework, the price and quantity data, and the point of reference for the alternative approach we shall develop in the following.

5.2.1 Reproducible Solutions

We shall employ the following notation in this chapter: lower case letters are used to represent vectors (upper case letters, matrices). A prime (') denotes transposition. Untransposed vectors are column vectors. The circumflex ($\wedge$) over a vector indicates the matrix formed by diagonalizing the vector. $\Re^n_+$ denotes the subspace of nonnegative elements ($\geqq 0$) of $\Re^n$, while semipositivity (strict positivity) is characterized by $\geq 0 (> 0)$. Furthermore, $e = (1, \ldots, 1)'$ is the summation vector (of appropriate dimension in each case) and $I = \widehat{e}$ the identity matrix. A_i, A^j indicate the ith row and the jth column of a matrix A, respectively; the vectors $I^1, \ldots, I^n$, therefore, represent the canonical basis of $\Re^n$.

To introduce briefly the required components of Roemer's general equilibrium model[1] consider as given a commodity space of dimension $2n + 1$, i.e., $\Re^{2n+1}$, that is based on one elementary region, two elementary time intervals (the present and the future), and $n + 1$ physically distinguished commodities, n produced and one unproduced commodity (homogeneous labor). Production takes time; both labor and the means of production have to be advanced. There are m capitalists ($j = 1, \ldots, m$), each facing a production possibilities set $P^j \subset \Re^{2n+1}$ which contains 0 (the possibility of inaction) and is closed and convex. Vectors $\alpha^j \in P^j$ are written as $\alpha^j = (-\alpha^j_0, -\underline{\alpha}^j, \overline{\alpha}^j)$, where $\alpha^j_0 \in \Re_+, \underline{\alpha}^j, \overline{\alpha}^j \in \Re^n_+$ denote the input of direct labor and the inputs and outputs of produced commodities for capitalist j, respectively. Net production $\widetilde{\alpha}^j$ is defined by $\overline{\alpha}^j - \underline{\alpha}^j$. The symbols $\alpha = (-\alpha_0, -\underline{\alpha}, \overline{\alpha}) \in P, \widetilde{\alpha}$ refer to the economy as a whole, i.e., they are obtained from the corresponding individual data of capitalists j by summation over j.

Workers do not save, but simply obtain a consumption bundle $s \in \Re^n_+, s \geq 0$ per hour of labor sold at the wage rate $w = p's$, where $p \in \Re^n_+$ denotes the vector of commodity prices. We normalize prices by taking the wage rate as 1. Capitalist j is assumed to possess the physical capital $\omega^j \in \Re^n_+$ and to maximize profits, i.e., to choose his activities from the set

$$A^j(p) = \{\alpha^j \in P^j / \alpha^j_0 + p'\underline{\alpha}^j \leqq p'\omega^j \wedge p'\widetilde{\alpha}^j - \alpha^j_0 \to \max\} \subset P^j.$$

[1] We stress that we use Roemer's equilibrium concept (of reproducible solutions) as providing the data for the calculation of labor values. These calculation can however equally well performed with any other equilibrium or disequilibrium price-quantity configuration of a given period of time.

We shall employ the following equilibrium concept as representation of our "actual data" in all that follows.

Definition 5.1. $\alpha \in A(p) = \sum_{j=1}^{m} A^j(p)$ is called a *reproducible solution* with respect to the price vector p and the above specified economy if $\underline{\alpha} + \alpha_0 s \leqq \omega = \sum_{j=1}^{m} \omega^j$ and $\widetilde{\alpha} \geqq \alpha_0 s$, i.e., if the profit maximizing solution α is feasible and if it reproduces at least the amount of consumption and production goods needed for its execution (see Roemer (1980, 1981) for a discussion and various generalizations of this approach).

It is not necessary, for the purpose of the following investigation, to enumerate the set of assumptions which would assure the existence of "reproducible solutions" for this general model of production. Our understanding of labor values, as measures of *actually* embodied labor time, only demands that they be defined with respect to the actual data of a *given* equilibrium situation, no matter how its existence in fact is assured. We, therefore, may conclude this section by providing an example of the above type of economy and its equilibrium concept which (with appropriate modifications) will serve to illustrate all basic definitions which will follow.

Example E1. Consider the following economy:

$$
\begin{aligned}
P^1 &= \{\mu_1(-2;0,-2;8,0)' + \mu_2(-0.5;0,-3;8,0)'/\mu_1,\mu_2 \geqq 0\}, \quad \omega^1 = (6,0)',\\
P^2 &= \{\mu_1(-4;0,-1;0,4)' + \mu_2(-0.1;0,-3;0,4)'/\mu_1,\mu_2 \geqq 0\}, \quad \omega^2 = (0,3)',\\
P^3 &= \{\mu_1(-1;0,-4;12,0)'/\mu_1 \geqq 0\}, \quad \omega^3 = 4,2.5)'\\
P^4 &= \{\mu_1(-3;0,-3;0,6)'/\mu_1 \geqq 0\}, \quad \omega^4 = 0,4.5)',
\end{aligned}
$$

i.e., where four capitalists with distinct methods and endowments are engaged in the production of only two commodities, the first a pure consumption good and the second a pure capital good. Let the workers' consumption bundle s be given by $(1,0)'$. It can easily be checked then that the vectors $p = (1,2)'$ and

$$
\begin{aligned}
\alpha^1 &= (-2;0,-2;9,0)' \in P^1, \qquad \alpha^2 = (-4;0,-1;0,4)' \in P^2,\\
\alpha^3 &= (-1;0,-4;12,0)' \in P^3, \qquad \alpha^4 = (-3;0,-3;0,6)' \in P^4,
\end{aligned}
$$

describe a reproducible solution for this economy.

5.2.2 *The Optimum Labor Theory of Value*

The following definitions of labor values and exploitation adopt Morishima's concept of "optimal (or true) values" for the general model of production here under consideration.

Definition 5.2. The *labor value* of a bundle $h \geqq 0$ of commodities is defined by

$$v^*(h) = \min\{\alpha_0/(-\alpha_0, -\underline{\alpha}, \overline{\alpha}) \in P, \tilde{\alpha} \geqq h\}.$$

Definition 5.3. The *rate of exploitation* at $\alpha = (-\alpha_0, -\underline{\alpha}, \overline{\alpha}) \in P$ is

$$e(\alpha) = (\alpha_0 - wv^*(\alpha_0 s)/v^*(\alpha_0 s) \quad \text{if} \quad v^*(\alpha_0 s) > 0$$

(see Roemer (1980, 2B) and Morishima and Catephores (1978, 2.3/2.5) for further explanations).

It has already been noted that it is mainly the FMT at which this construction of the rate of exploitation is aimed. This theorem is declared to be of decisive importance to Marxism and Marxian economics, and claimed to be valid only if it is interpreted in terms of the above type of labor values (see Morishima and Catephores (1978, p. 38)). In simplified form Roemer's version of this theorem is the following (see Roemer (1980, p. 519) and note the restrictive assumption of "independence of production" is required there for its proof).

Theorem 5.4 (FMT1). *There exists a reproducible solution yielding positive total profits π if and only if there exists a reproducible solution yielding a positive rate of exploitation, in which case both characteristics extend to the set of all reproducible solutions.*

Example 2. With regard to the reproducible solution considered in E1 we have $\pi = 10 > 0$ and $\alpha_0 = 10$. The labor value of a bundle $h \geqq 0$ will be determined by the two basic processes of P^1, P^2 that are not activated by the profit–maximizing capitalists 1, 2 in the reproducible solution presented in Example El, which in particular implies:

$v^*((1,0)') = 0.1$, $v^*((0,1)') = 0.1$, and $v^*(\alpha_0 s) = v^*((10,0)') = 1$. The rate of exploitation, therefore, will be $(10 - 1)/1 = 9$ or 900 percent and this in view of an actual price rate of profit of 33.3 percent.

To conclude this section we shall list some properties of optimal values which in our view do not reflect Marx's understanding of labor values in an adequate way :

1. Optimal labor values can be positive quite independent of the question of whether the product concerned is actually exchanged or – still weaker – is an *object of utility* (see, e.g., Marx (1977, Vol. I, p. 48) for opposing views).
2. No close relationship need exist between the labor time actually expended and the (optimal) labor value of actual net production, as is obvious from Example E2, where $\alpha_0 = 10$ and $v^*(\widetilde{\alpha}) = v^*((20,0)') = 2$. Optimal labor values are of a counterfactual nature and may depend solely on purely potential methods of production (compare again Example E2). They – and the rate of exploitation – thus may be subject to change without any change in the conditions of production that actually prevail.

3. As these labor values do not define a *linear functional* on the commodity space $\Re^n$, homogeneous ratios such as the *rate of surplus value* $s(\alpha)$ and the *value rate of profit* can no longer be defined in an unambiguous way, since, e.g., the first rate $s(\alpha)$ may be defined by $v^*(\widetilde{\alpha} - \alpha_0 s)/v^*(\alpha_0 s)$ or by $(v^*(\widetilde{\alpha}) - v^*(\alpha_0 s))/v^*(\alpha_0 s)$, or even by further disaggregated terms. (With regard to E1,2, the above two expressions for $s(\alpha)$ are equal to "1", which is quite different from the "bastard" ratio $e(\alpha) = 9$.)
4. Labor values cease to be proportional to prices of production in the *case of equal composition of capital* for this particular extension of Marx's labor theory of value to quite general conditions of production. Adding to the difficulties in finding proper expressions for important ratios of Marx's theory, the Marxian intention of deriving a set of basic quantitative relationships between value and price expressions, which preserve or only modify the basic insights, developed in *Capital*, Vol. I, II, for the case of uniform composition of capital, thereby loses its basis completely.

5.3 A New and Measurable Definition of Labor Values for Joint Production Systems

We have tried to give a brief indication of the fact that apart from the FMT not much is left by Definition 5.2 of Marx's original perception of value and price relationships in the general case of joint production. It is now the intent of this and the following section to show that this loss of labor values in theoretical as well as in factual content can be avoided completely if a new concept of labor values is adopted which (though fulfilling the FMT) differs sharply from the one given by Definition 5.2. To prove this we shall first consider an important special case of Roemer's general model of production, where a plausible extension of the simple equations approach can be given simply by integrating properly Marx's concept of individual values. Related averages will then replace the optimal values hitherto used and will guide our approach to the general case, to begin in Sect. 5.3.

We did not consider it necessary in Sect. 5.2 to state the assumptions made by Roemer to assure the existence of reproducible solutions. Our suggestion, on the contrary, was to start from such an equilibrium $\alpha = (-\alpha_0, -\underline{\alpha}, \overline{\alpha})$, p – however derived. And then *only one assumption* must be made with regard to this given reproducible solution in order to ensure all following constructions of labor values from such data.

Assumption 1. The given reproducible solution $\alpha = (-\alpha_0, -\underline{\alpha}, \overline{\alpha})$ is assumed to fulfill: $\overline{\alpha}^j \geq 0 \rightarrow \alpha_0^j > 0$ $(j = 1, \ldots, m)$ and $\overline{\alpha} \geq 0$, i.e., we assume that labor is indispensable whenever production occurs (which is supposed to be the case at least once).

This assumption is not implausible in view of the fact that capitalists, i.e., institutions, have been defined to provide the smallest possible unit of production. Furthermore, it only serves to relieve the considerations below from unnecessarily detailed technicalities.

As a simple consequence of the assumed possibility of inaction, implying that profits $\pi^j = p'(\widetilde{\alpha}^j - \alpha_0^j s) = p'\widetilde{\alpha}^j - \alpha_0^j$ have to be nonnegative in each case, there immediately follows this lemma:

Lemma 5.5. $p'\overline{\alpha}^j > 0, p'\widetilde{\alpha}^j > 0$ *for all* j *and* $\overline{\alpha} > 0$*, since capitalists* j *and commodities* i *that do not take part in production* $(\overline{\alpha}^j = 0, \overline{\alpha}_i = 0)$ *can be safely neglected in all that follows (note in this connection that a commodity* i *which is not produced will not be used as an input; see Definition 5.1).*

The above properties together with Assumption 1 represent the information that is necessary and admissible for our view of a proper definition of embodied labor time. We continue to employ m, n to denote the number of activities and commodities now left for consideration.

5.3.1 Marx's Case of Multiple Activities

Consider the simple case where each capitalist j is engaged in the production of one commodity $i(j)$ only, i.e., the $\overline{\alpha}^j$'s will have but one positive component (which implies $n \overset{\leq}{=} m$). Note that, by the very choice of our equilibrium model, we are not constrained to situations of equal profitability.

To introduce Marx's (and our own) approach to labor value determination for this case it is best to first let Marx speak for himself:

> "Now let some one capitalist contrive to double the productiveness of labor, and to produce in the working–day of 12 hours, 24 instead of 12 such articles. The value of the means of production remaining the same, the value of each article will fall .... The individual value of these articles is now below their social value ...." "The real value of a commodity is, however, not its individual value, but its social value; that is to say, the real value is not measured by the labor–time that the article in each individual case costs the producer, but by the labor–time socially required for its production" (Marx, 1977, Vol. I, pp. 300/1).

This states that the *individual value* of a commodity produced by a capitalist j is given by the sum of *labor values* of the commodities used in its production plus the direct labor time *individually* needed, and that labor values are an *average* of individual values, an average which in our view should best be formed by means of *market shares* as weights.

To formalize this idea with regard to the given data let $v_1 = (v_1, \ldots, v_n)'$ denote the vector of labor values and let $v(j)$ denote the individual value of the commodity which is produced in process j, the index of which we have already denoted by $i(j)$. The foregoing twofold connection between individual values and their averages, labor values, can then be represented by

$$v(j)\overline{\alpha}^{j}_{i(j)} = v'\underline{\alpha}^{j} + \alpha^{j}_{0} \qquad (j = 1, \ldots, m), \tag{5.1}$$

$$v_k = \sum_{j=1}^{m} v(j) D_{jk} \qquad (k = 1, \ldots, n), \tag{5.2}$$

$$D_{jk} = \overline{\alpha}^{j}_{k}/\overline{\alpha}_{k}, \text{ the (potential) market share of capitalist } j \text{ in the market for commodity } k. \tag{5.3}$$

Proposition 5.6. *The equation system (5.1), (5.2), (5.3) has exactly one solution* $v = (v_1, \ldots, v_n)'$, $v_I = (v(1), \ldots, v(m))'$ *for average as well as individual labor values. These solution vectors will be positive throughout.*

Proof. See Proposition 5.13. □

Example E3. For the reproducible solution Example E1 we by definition (5.1)–(5.3) obtain $v = (1/2, 7/6)'$ and $v_I \equiv (0.54, 1.29, 0.47, 1.08)'$, where $j = 1, 3(2, 4)$ relate to commodity 1(2), respectively (see (17) for a simplification in computation). This example shows that individual values may provide an expression of individual productivity when compared with average or system labor requirements: $1/2$ and $7/6$ and may be usefully employed in that direction. Furthermore, these actual Marxian values differ considerably from the hypothetical "optimal values" of Example E2 (both equal to 0.1), which shows that "optimal values" cannot be regarded as a proper extension of Marx's ideas on "value."

It is argued in Morishima and Catephores (1978, p. 35) that the value system will easily be liable to fluctuate once alternative methods of production are allowed for (because firms will be indifferent between equally profitable alternative processes then). With regard to this opinion, it is strange to see how an obvious indeterminacy of "equilibrium production" is turned against Marx's concept of labor values, since it is quite natural that these latter values should fluctuate if the same is true for the actually prevailing methods of production. Such fluctuations, if a problem, *are a problem of the equilibrium concept employed*, and not one of labor value determination – the uniqueness of which, moreover, is assured by an appropriate equations approach despite the possibility of rectangular systems of input–output data (of type $n \leq m$).

5.3.2 Joint Production

We deduce from the preceding section that the Marxian notion of "individual value" may be of decisive importance if such hypothetical labor values as considered in Example E2 are to be avoided. Recent models of the labor theory of value, however, have not paid much attention to this intermediate case of multiple activities, but have in general progressed immediately to a consideration of the case of joint production. The question arises whether Marx's ideas on individual values can be made general enough to overcome the difficulties that have been attributed to the equations approach in this general case.

In accordance with Morishima and Catephores (1978, p. 54) we believe "that there was no serious Marxian economic analysis of joint production." Yet it is not our aim here to examine what Marx himself might have done in this case. It is our opinion that Marx's labor theory of value should be understood as a dual manner of cost accounting attempting to connect real (labor) costs with normal prices. And it is completely in line with the purely factual approach to real cost accounting we have employed to this point that in this question, too, we will look for a factual principle, in use to solve the problems of unit–cost determination for joint products in everyday life, but also applicable to the general situation here under consideration. Such a principle is indeed available in the form of the so–called sales value method, a method which enjoys great popularity among cost accountants and which sometimes is viewed as "the only logical way to pro–rate joint costs" to the various items produced Matz and Usry (1976, p. 189).

The definition which follows provides the basis for incorporating this sales value method into the abstract setting which here serves as foundation for the envisaged construction of embodied labor time.

Definition 5.7. The vectors

$$\delta^{j,k} = (-\delta_0^{j,k}, -\underline{\delta}^{j,k}, \overline{\delta}^{j,k}) = (-C_{kj}\alpha_0^j, -C_{kj}\underline{\alpha}^j, \overline{\alpha}_k^j I^k) \neq 0 \tag{5.4}$$

where $C_{kj} = p_k\overline{\alpha}_k^j / p'\overline{\alpha}^j$ denotes the *relative sales value* of good k with respect to activity j, are said to prorate the joint costs of activity j, $j = 1, \ldots, m$, according to the *sales value method*, to its various outputs $\overline{\alpha}_k^j (\neq 0)$.

It is known that the above rule for disentangling joint costs – here traced back to the level of physical magnitudes – is not unanimously accepted by cost accountants as the only solution to the problem of joint–product accounting in all of its possible settings. Yet, to the extent that as cost determination for single commodities – here with respect to labor costs – is considered a meaningful task at all, it appears to us as the only truly general method as far as an abstract approach to joint production is concerned. (Note in this connection that our data source Assumption 1 may comprise questions of partially separable costs and the like and that some aggregation may already be involved, both of which problems which, however, must remain excluded by assumption from this general reconsideration of Marx's labor theory of value.)

In addition to its universality the employed rule is known to have the advantage of representing a costing procedure for which it cannot occur that some co–products in a jointly produced group appear to be consistently unprofitable, while others are profitable. Under its regime each unit of value of jointly produced goods must contain the same amount of profit $1 - (p'\underline{\alpha}' + \alpha_0^j)/p'\overline{\alpha}^j$, which in particular means that we will have $p'\underline{\delta}^{j,k} + \delta_0^{j,k} \leq p'\underline{\alpha}^j = p_k\overline{\alpha}_k^j$ (because of $p'\underline{\alpha}' + \alpha_0^j \leq p'\overline{\alpha}^j$) for any reproducible solution under consideration. The conclusion that value added cannot be negative (because of the assumed possibility of inaction) hence also holds for all components of joint production if the sales value method is used to allocate their joint costs.

From the formulation of Definition 5.7 it should be obvious how the application of the sales value method will lead us back to the labor value determination envisaged by Marx and worked out here in Sect. 5.3.

Definition 5.8. Let $p, \alpha^1, \ldots, \alpha^m$ be a reproducible solution as postulated in Assumption 1. *Labor values* v_k (and *individual values* $v(j)_k$) of commodities k (as produced by capitalist j) are determined by equations (5.1)–(5.3) applied to the set of "activities" $\delta^{j,k}$ of Definition 5.7.

Proposition 5.9. *Labor values* v_k *as defined in Definition 5.8 are uniquely determined and of the same sign as prices* p_k *for any reproducible solution which satisfies Assumption 1.*

Proof. See Proposition 5.13 in Sect. 5.3. □

We herewith have formulated a way in which Marx's ideas on actual labor values can be extended to the case of joint production without running into the anomalies of negativity or non-uniqueness normally attributed to them in this situation. The *"apparent cost"* to obtain this result – the inclusion of certain price ratios into the equations determining individual values – will be examined in Sect. 5.4, where the resulting form of labor theory of value is considered.

Example E4. Consider the following modification of Example E1. There exist two capitalists, capitalist 1, who employs the basic activities that formerly formed p^1, p^2 (on the basis of the endowment $\omega^1 + \omega^2$), yet with the joint activity $\beta^1 = (-6; 0, -3; 8, 4)'$ instead of its formerly separable components $\alpha^1 = (-2; 0, -2; 8, 0)'$ and $\alpha^2 = (-4; 0, -1; 0, 4)'$, and capitalist 2, who in a similar fashion employs $\beta = (-4; 0, -7; 12, 6)'$ instead of $\alpha^3 = (-1; 0, -4; 12, 0)$ and $\alpha^4 = (-3; 0, -3; 0, 6)$ on the basis of the endowment $\omega^3 + \omega^4$. The activities which in Example E1 gave rise to a reproducible solution with regard to prices $p = (1, 2)'$ are thereby transformed into two joint production activities, a fact which, however, does not disturb their feature to provide a reproducible solution with regard to the given prices p. Furthermore, the two potential methods of production included in the definition of P^1 and P^2 in Example E1 will experience no changes here, which implies that optimal labor values will remain the hypothetical ones determined in Example E2. Hence, no change will result for the optimum labor theory of value through the assumed change in the institutional and technological set–up of the employed model.

Things are different when our definition of labor values is utilized instead. Their technological background, which by our definition is given by the activities β^1, β^2 that are actually employed, now has to be disentangled by the sales value method first to allow for the application of Marx's description (5.1), (5.2), (5.3) of labor values v_1, v_2. In the above case, relative sales values are given by 0.5, 0.5 for each of the two joint activities β^1, β^2. Prorating their inputs according to Definition 5.7 then leads to

$$(-3; 0, -1.5; 8, 0) \text{ and } (-3; 0, -1.5; 0, 4) \text{ in the first, and}$$
$$(-2; 0, -3.5; 12, 0) \text{ and } (-2; 0, -3.5; 0, 6) \text{ in the second case,}$$

a situation which clearly differs from the four activities on which the calculations in Example E3 were based (compare $\alpha^1, \ldots, \alpha^4$ in Example E1). Through this difference we now obtain $v = (1/2, 1)'$ instead of $v = (1/2, 7/6)'$, the labor values of the non joint case Example E3.

5.3.3 The Input–Output Approach to Joint Production

We have seen how the theoretical–quantitative side of labor value determination may be treated without anomalies in a very general model of production. It is now the purpose of the following to show the same for the empirical–quantitative side of this problem (as far as its methodology is concerned) in a way that will lead us back to Definition 5.8 of the preceding section. Let us start again from the reproducible solution α, p underlying Assumption 1. These data can be translated into a single (monetary) input–output table by means of standard SNA *methodology* in the following way (see United Nations (1968, 3.67–3.86 and 3.40)):

$$U = (U_{ij}), \quad U_{ij} = p_i \underline{\alpha}_i^j \qquad (i = 1, \ldots, n; j = 1, \ldots, m) \quad (5.5)$$

$$V = (V_{ji}), \quad V_{ji} = p_i \overline{\alpha}_i^j \qquad (j = 1, \ldots, m; i = 1, \ldots, n) \quad (5.6)$$

$$u = (u_j), \quad u_j = \alpha_0^j = w\alpha_0^j \qquad (j = 1, \ldots, m) \quad (5.7)$$

define the so–called *absorption matrix, make matrix*, and the *vector of wage incomes*, respectively. From these basic arrangements the following auxiliary vectors or matrices are then derived:

$$D = V\widehat{q}^{-1}, \ q \text{ given by } \ q_k = (V'e)_k = p_k\overline{\alpha}_k \text{ if } p_k > 0 \text{ (1 otherwise)}, \quad (5.8)$$

i.e., *the matrix of market shares* $D_{jk} = p_k\overline{\alpha}_k^j / q_k$;

$$g = V\widehat{q}^{-1}, \text{ i.e., } g_j = p'\overline{\alpha}^j > 0 \qquad (j = 1, \ldots, m; \text{ see Lemma 1}) \quad (5.9)$$

which gives the *vector of output values of activities* $1, \ldots, m$ and

$$B_p = U\widehat{g}^{-1} \text{ and } b_p' = u'\widehat{g}^{-1}, \quad (5.10)$$

i.e., the *unit–cost structures* of current production activities. It is from these unit–cost structures of activities that *input–output tables* A_p (and corresponding rows a_p' of primary inputs, here of wage incomes) are finally derived and, to be sure, for the case of joint production in the following way:

$$A_p = B_p D \text{ and } a_p' = b_p' D. \quad (5.11)$$

This derivation of an nxn–matrix A_p (of commodity × commodity type) – and a corresponding n-vector a'_p – from the originally given joint production system U, V is motivated as follows:

1. "The construction of an input–output table which is necessarily square (because of the intention of applying standard input–output theory, P.F.) from separate input and output tables involves transfers of inputs and outputs between categories" (United Nations, 1968, 3.15) in order to show the inputs of commodities in the production of the single commodities.
2. In the presence of joint production the most sensible way to accomplish these transfers seems to be given by the so–called *"industry technology assumption,"* whereby it is assumed that each dollar of activity j's output should have the same cost structure B_p^j, irrespective of the particular product behind this dollar of output value (see United Nations (1968, 3.40)) for such a recommendation).
3. This assumption about the cost structures of activities is then used to define the *cost structure of commodity* k, i.e., A_p^k, by their weighted sum, the weights being the *market shares* D_{jk} activities j have with regard to commodity k. It is exactly this summation which is described by $A_p^k = B_p D^k$, i.e., by $A_p = B_p D$ with regard to the set of all commodities $k = 1, \dots, n$, see (5.11). The same explanation applies to a'_p in (5.11).

It is a bit strange to find in the *SNA* that a technological characterization is associated throughout with the procedure just described though quite obviously no technological feature is involved, at least in its application to the joint production system here under review. Be that as it may, having derived the matrices A_p, a'_p by means of the above reallocations, system labor requirements are then measured in the conventional way by the following definition:

Definition 5.10. *System labor requirements* $t = (t_1, \dots, t_n)'$ of commodities k per dollar of output value are defined by $t' = t'A_p + a'_p$.

Before proceeding to a further analysis of the formula just presented, a final definition has to be added for the purpose of later comparison.

$$C = V'\widehat{g}^{-1}, \text{ i.e., } C_{kj} = p_k \overline{\alpha}_k^j / p' \overline{\alpha}^j \tag{5.12}$$

defines the *matrix of commodity* mixes or relative sales values, the coefficients of which we have already employed in our Definition 5.8 of labor values in Sect. 5.3.

We have described how monetary joint production tables U, V can be converted into a form where the ordinary Leontief inverse may become applicable. No attention is paid in this application to the effect the proposed mechanical transfers (see United Nations (1968, 3.16)) may have on the values to be measured. That measurement is possible is all that is of interest, and this can be assured by the following:

Proposition 5.11. *The equation $t' = t'A_p + a'_p$ will allow for exactly one solution vector $t \in \Re^n$, which is nonnegative in addition. System labor requirements t_k will be zero if and only if product k is free with regard to the given reproducible solution $p, \alpha^j, j = 1, \dots, m$.*

Sketch of Proof. By Lemma 2 $e'B_p = e'U\widehat{g}^{-1} = (p'\underline{\alpha}^j/p'\overline{\alpha}^j) < e'$. Furthermore, $e'D = e'V\widehat{q}^{-1} \leq q'\widehat{q}^{-1} = e$, i.e., $e'A_p = e'B_pD < e'$. The matrix A, therefore, is nonnegatively invertible, i.e., $t' = a'_p(I - A_p)$ is well defined and nonnegative. Furthermore, $b'_p > 0$, i.e., $p_k = 0$ if and only if $(a'_p)k = 0$ and $A^k_p = 0$ if and only if $t_k = 0$. □

For the purpose of comparing the above system labor requirements t_k with the labor values defined in Definition 5.8 let us restate this definition in more explicit terms:

Definition 5.12.

(i) The individual value $v(j)_k$ of product k (if) produced by capitalist j is defined by

$$v(j)_k\overline{\alpha}^j_k = C_{kj}(v'\underline{\alpha}^j + \alpha^j_0) \qquad (k = 1, \ldots, n; j = 1, \ldots, m) \tag{5.13}$$

(and, of course, by "0" if $\overline{\alpha}^j_k = 0$, in which case C_{kj} will be "0", too), where $v = (v_1, \ldots, v_n)'$ is the vector of labor values and C_{kj} the relative sales value of product k with respect to process j.

(ii) The *labor value* v_k of product k is defined by

$$v_k = \sum_{j=1}^{m} D_{jk}v(j)_k \qquad (k = 1, \ldots, n) \tag{5.14}$$

where D_{jk} is the market share of capitalist j with regard to the kth product market.

Proposition 5.13. *The equation system (5.13), (5.14) has exactly one set of solution vectors $v, v(1), \ldots, v(m) \in \Re^n$, which are nonnegative throughout and which fulfill*

$$t_k = v_k/p_k > 0 \text{ if } p_k > 0 \textit{ (and } t_k = v_k = 0 \textit{ if } p_k = 0) \tag{5.15}$$

$$v(j)_k = 0 \qquad \textit{if and only if } p_k = 0 \textit{ or } \overline{\alpha}^j_k = 0. \tag{5.16}$$

Proof. Note first that the equation system (5.13), (5.14) is equivalent to the system of equations that is given by (5.13) (plus the qualification made there!) and

$$v_k\overline{\alpha}_k = \sum_{j=1}^{m} C_{kj}(v'\underline{\alpha}^j + \alpha^j_0) \qquad (k = 1, \ldots, n). \tag{5.17}$$

Equations (5.17) are obtained by summing equations (5.13) with respect to j and by inserting the equation $v_k\overline{\alpha}_k = \sum_{j=1}^{m} v(j)_k\overline{\alpha}^j_k$ (which is a direct consequence of (5.14), $\overline{\alpha}_k > 0$ and $v_k, v(j)_k = 0$ if $p_k = 0$) into the left hand side of this sum. Equations (5.14) in turn follow from (5.13) and (5.17) by the insertion of (5.13) into (5.17). Note further, that the vector x defined by v_k/p_k if $p_k > 0$ (and by "0" otherwise) fulfills $x'\widehat{p} = v'$, because we have $v_k = 0$ if $p_k = 0$ by (5.14). With these preliminaries behind us we can now exploit the fact that individual values $v(j)_k$ have been eliminated through the introduction of equations (5.17) in place of (5.14). Transformed to the above defined vector x the equations (5.17) read

$$x_k p_k \overline{\alpha}_k = \sum_{j=1}^{m} C_{kj}(x'\widehat{p}\underline{\alpha}^j + \alpha_0^j), \qquad \text{i.e., in input–output notations}$$

$$x_k q_k = \sum_{j=1}^{m} C_{kj}(x'U^j + u_j) \qquad \text{(compare (5.5), (5.7), (5.8)).}$$

In matrix notation this results in

$$x'\widehat{q} = x'UC' + u'C' = x'U\widehat{g}^{-1}V' = x'B_p V' + b_p' V' \tag{5.18}$$

(compare (5.10), (5.12) and recall that $q > 0$ by definition). Premultiplying equations (5.18) with $\widehat{q}^{-1}$ then gives

$$x' = x'B_p v'\widehat{q}^{-1} + b_p' V'\widehat{q}^{-1} = x'B_p D + b_p' D = x'A_p + a_p'$$

by (5.8), (5.9), (5.11), which by Proposition 5.11 implies that x must equal the vector of system labor requirements t defined in Definition 5.10. It is hereby obvious that the equations (5.17), (5.13), which can be solved in this order, will have exactly one solution set of vectors $v, v(j), j = 1, \ldots, m$, since the above transformations of (5.17) can all be reversed again, and that these solution vectors will fulfill (5.15) and (5.16). □

Remark. Summing (5.13) over k implies

$$v(j)'\overline{\alpha}^j = v'\underline{\alpha}^j + \alpha_0^j, \qquad v(j) = (v(j)_1, \ldots, v(j)_n)' \tag{5.19}$$

and from (5.13) and (5.19) there follows

$$C_{kj} = v(j)_k \overline{\alpha}_k^j / v(j)'\overline{\alpha}^j, \tag{5.20}$$

i.e., the relative sales values employed in Definition 5.12 are identical to those determined by individual values (once the latter have been defined by means of the former). Utilizing the relationships (5.14), equations (5.13) hence may be expressed in terms of individual values exclusively, yet will not represent a determinate system unless sales values C_{kj} in (5.20) have been qualified further, i.e., by prices p as far as our intents are concerned.

There does, however, exist a way which avoids any use of prices in Definition 5.12 without changing the method of definition employed, namely by inserting sales values C_{kj} defined by $v_k \overline{\alpha}_k^j / v'\overline{\alpha}^j$ instead of $p_k \overline{\alpha}_k^j / p'\overline{\alpha}^j$ into equations (5.17) which gives the equation system

$$v'\widehat{\overline{\alpha}} = (v'(\underline{\alpha}^j) + (\alpha_0^j)')\widehat{v'(\overline{\alpha}^j)}^{-1}(\overline{\alpha}^j)'\widehat{v}, \quad \text{with } (\underline{\alpha}^j) = (\underline{\alpha}^1, \ldots, \underline{\alpha}^m), \text{ etc.}$$

No attempt will be made here to examine this technologically self–contained (though "sales value" dependent) definition of labor values, because (a) there is no need for labor values to be price–independent with regard to the labor theory of value (see Sect. 5.4), (b) the labor value of a jointly produced free good should be zero, but should become positive if non–technological changes make this good an economic one, and (c) labor values should be identical to prices (of production) in the case of a uniform internal composition of capital (see again Sect. 5.4), facts which will not be fulfilled if Definition 5.12 is changed in this way.

That *labor value depends on price* is meanwhile also acknowledged by Roemer (see Roemer (1982, VI.)), who arrives at this result from a completely different point of view, which also leads him to conclude that the FMT is not a sufficiently discriminating tool to indicate the superiority of the "optimum labor theory of value"!

We have shown by Proposition 5.13 that Marx's labor values in the extended version of Definition 5.12 remain "observable" despite the general type of production considered, which generalizes Okishio's claim (we have quoted in Sect. 5.1) in a way which preserves his simple relationship between system labor requirements t_k per unit of output value and labor values v_k per unit of product, and, to be sure, without any marked change in its final presentation, as the following corollary shows.

Corollary 5.14. *Assume* $p > 0$ *for simplicity and define matrix* A *by* $(\underline{\alpha}^1, \ldots, \underline{\alpha}^m) C'\widehat{\overline{\alpha}}^{-1}$ *and vector* a' *by* $(\alpha_0^1, \ldots, \alpha_0^m) C'\widehat{\overline{\alpha}}^{-1}$*. Then* $A_p = \widehat{p} A \widehat{p}^{-1}$, $a'_p = a'\widehat{p}^{-1}$*, i.e.* $t' = t'\widehat{p} A \widehat{p}^{-1} + a'\hat{p}^{-1}$ *or* $v' = v'A + a'$.

Proof. Compare (5.18) and its subsequent transformation. □

The employed mechanical allocations of cost structures U, u by means of output structures V thereby become interpretable (however large the degree of jointness of production may be), while theoretically motivated imputations of labor costs to single products remain measurable without any discernible change in the matrix multiplier formula that is applied. This final result in our view should be particularly stressed, as it, quite independent from any interest in Marxian economic analysis, can provide a theoretical underpinning, as far as joint production is concerned, to the measurement of system labor requirements actually performed, as, e.g., by Gupta and Steedman (1971), where it is, e.g., discovered in the empirical part of this chapter that "a falling tendency in the direct labor input vector was combined with a rising tendency in the input–output matrix to produce a fall in total (or system) labor use" Gupta and Steedman (1971, p. 29). Such a finding on the impact of technological change on direct and indirect labor use and also the whole approach that is taken in demonstrate that labor values (as system labor requirements or productivity indexes) may constitute at least an analytical concept of interest in itself.

5.4 Values, Prices and Profits

It has been demonstrated in Sect. 5.3 how the anomalies of actual labor values, extensively discussed in Steedman (1977), can be avoided completely through an appropriate choice of their definition. To provide further justification for this choice of Definition 5.12 we shall now show that it also resolves the deficiencies of "optimal values" pointed out in Sect. 5.2, and that it fulfills the standard "quality specification": the validity of the "Fundamental Marxian Theorem" (FMT).

The following propositions again refer to the reproducible solution α, p of Assumption 1 and the notation introduced in Sect. 5.2.

Proposition 5.15. $v'\widetilde{\alpha} = \alpha_0$ *The influence of relative prices p drawn into the definition of labor values cancels with respect to the bundle* $\widetilde{\alpha}$*, reestablishing the* identity between the labor value of actual net national product and the total labor time actually expended *(compare comment 2 in Sect. 5.2 and note that 1 has already been treated in Proposition 5.13).*

Proof. Summing equations (5.17) over k gives

$$v'\overline{\alpha} = \sum_{k=1}^{n}\sum_{j=1}^{m} C_{kj}(v'\underline{\alpha}^j + \alpha_0^j) = \sum_{j=1}^{m}\left(\sum_{k=1}^{n} C_{kj}\right)(v'\underline{\alpha}^j + \alpha_0^j)$$
$$= \sum_{j=1}^{m}(v'\underline{\alpha}^j + \alpha_0^j) = v'\underline{\alpha} + \alpha_0 \qquad \square$$

Definition 5.16. The *rate of surplus value* ε^j *with regard to capitalist* j is defined by $\varepsilon^j = (v'\widetilde{\alpha}^j - \alpha_0^j v's)/\alpha_0^j v's$ and *for the economy as a whole* by $\varepsilon = (v'\widetilde{\alpha} - \alpha_0 v's)/\alpha_0 v's$, the ratio of surplus value to the total value of labor power.

Corollary 5.17. $\varepsilon = (\alpha_0 - \alpha_0 v's)/(\alpha_0 v's) = (1 - v's)/v's.$

Note that the above rates of surplus value are well defined ($w = 1$) and that the individual rates ε^j will be equal to the overall rate of surplus value ε, whenever $v(j)'\overline{\alpha}^j = v'\overline{\alpha}^j$ holds (compare (5.19)). It follows that Steedman's (1977) straightforward equations approach to labor value determination will coincide with our approach exactly in that case where a uniform rate of surplus value prevails with regard to our Definition 5.12. Furthermore, it should now be apparent that it is illegitimate to speak of a "postulate of the uniform rate of exploitation" Morishima (1973, p. 193) with regard to Marx's conception of value rather than of a number of explicitly stated or implicitly contained conditions by which such a uniformity is implied (see Marx (1977, Vol. I, p. 302) for some such conditions).

By means of Proposition 5.13 the rate of surplus value (as any other labor value expression) can be transformed into a measurable magnitude in the following way

$$\varepsilon = (t'\widehat{p}\widetilde{\alpha} - \alpha_0 t'\widetilde{p}s)/(\alpha_0 t'\widehat{p}s) = (1 - t'\widehat{p}s)/t'\widehat{p}s$$

where t is determined by Proposition 5.11. It is by such an expression that the rate of surplus value is in fact estimated in Wolff (1979). And quite in contrast to the consequences drawn from Definition 5.3 this procedure is not deprived of its theoretical foundations if the degree of jointness of production is assumed to have reached dimensions worthy of notice, as we have shown in Sect. 5.3.

Example E5. By Example E4 we have

$$\varepsilon = (1 - (0.5, 1)(1, 0)')/(0.5, 1)(1, 0)' = 1,$$

which is quite different from the optimal rate $\varepsilon(\alpha) = 9$ of Example E2, 4.

Theorem 5.18 (FMT2). *The rate of surplus value ε is positive if and only if total profits $\pi = p'\widetilde{\alpha} - \alpha_0$ are positive.*

The proof of this theorem is simple and follows from the fact that the economic goods of the two accounting schemes employed, v and p, have been shown to be identical. It, as well as the proofs of the following two propositions, can be obtained on request from the author as part of the original discussion paper from which this chapter has been drawn. Note, that in comparison to Theorem 5.4 (see also Roemer (1981, pp. 64 ff.)), no additional assumptions are necessary here to ensure the validity of this theorem.

Definition 5.19. The *average price and value rates of profit ρ_p, ρ_v* are given by $p'(\widetilde{\alpha}-\alpha_0 s)/p'(\underline{\alpha}+\alpha_0 s)$, $v'(\widetilde{\alpha}-\alpha_0 s)/v'(\underline{\alpha}+\alpha_0 s)$ respectively. Note that the deficiencies of optimal values of point 3 in Sect. 5.2 cannot arise for the ratios introduced here.

Proposition 5.20. *For all $\gamma \in \Re$ and for κ defined by $p'(\underline{\alpha} + \alpha_0 s)/v'(\underline{\alpha} + \alpha_0 s)$ there holds*

$$\rho_p - \rho_v = (1 + \rho_p)(e' - \kappa t')\widehat{p}(\overline{\alpha} - (1 + \gamma)(\underline{\alpha} + \alpha_0 s))/p'\overline{\alpha}. \qquad (5.21)$$

Note that, here again, all expressions involved can be checked by observation. Both rates have been determined separately for the U.S. economy for the years 1947, 1958, 1963 and 1967 in Wolff (1979, p. 335) by means of the simple equations approach to labor values and have been found to be closely related to each other. Equation (5.21) now shows that as in the simple equations approach there exist two independent reasons which may account for this observation: (i) $\overline{\alpha} \approx (1 + \gamma)(\alpha + \alpha_0 s)$, i.e., commodity production approximately grows in proportion to physical inputs including the necessities of the labor force, and (ii) $e \sim \kappa t$, i.e., labor values are virtually proportional to given prices p. With regard to this second case, it is, of course, worthwhile to know the conditions that may lead to such a situation. Here again, our approach allows us to preserve the results of the simple equations approach and to add a condition which directly applies to the derived input–output data A_p, a'_p of Sect. 5.3.

Proposition 5.21.

(i) $\pi = p'(\alpha - \alpha_0 s) = 0$ *implies* $t = e$, *i.e.,* $\kappa = 1$ *and* $p = v(= v(j), j = 1, \ldots, m$, *where this latter equality should be understood to apply to produced commodities only).*

(ii) $\exists x \in \Re : e'(I - A_p) = x a'_p$ *implies* $x = \kappa = (v's)^{-1}$ *and* $\kappa t = e$, *i.e.,* $\kappa v = p$ *(and* $p > 0$*).*

(iii) The assumption, which implies assertion (ii), is fulfilled if $p'\underline{\alpha}^j/\alpha_0^j$ *is independent of* j *and if* $\rho_p^j = p'(\widetilde{\alpha}^j - \alpha_0^j s)/p'(\underline{\alpha}^j + \alpha_0^j s) = \rho_p$ *for all* j *(and* $p > 0$*).*

This proposition describes (in part (iii) and generalizes (in part (ii)) Marx's classic case of proportionality between labor values and prices of production, whereby the one remaining deficiency (4 in Sect. 5.2) of optimal labor values has now turned out to be avoidable as well, and it provides a simple test for how far the central link in Marx's labor theory of value, the similarity between the rates of profits ρ_v and ρ_p, may or may not be due to a relationship between prices and labor values alone. The degree of price variations that can be explained by variations in labor values thereby remains a question of empirical interest and verification, which, though probably less central than the direct comparison of corresponding rates of profit, seems to lead to noteworthy answers, too (see Nell (1980, p. 189), Wolff (1979, p. 335) and Gupta and Steedman (1971, p. 27) for some interesting statements in that direction).

Example E6. It can easily be calculated that the $\beta's$ (and p) of Example E4 do not fulfill Proposition 5.21 (iii) (the internal composition of capital is not uniform) and that commodity production does not exhibit a uniform rate of growth. Nevertheless, there holds $v \sim p$ (and thus $\rho_v = \rho_p$), because of $a_p' = 0.25(1,1) \sim e'(I - A_p) = 0.5(1,1)$ (compare also Example E4).

A Final Remark. We have already pointed to a possible serious cost involved in our rehabilitation of the conventional equations approach to the labor theory of value in the final paragraph of the introduction to this chapter. For our intents, labor values v were made to depend on prices p by use of sales values C_{kj} (compare (4)). But how much do we lose by this kind of extension of labor values – to take account of joint production – with regard to their known core of applicability? We guess, nothing that is of real importance! There is, of course, no sense in the claim that prices are but derivatives of labor values, i.e., we also do not allege *"logical priority"* to labor values with this in mind. On the contrary, actual labor values quite obviously depend on prices, since the choice of technique, symbolically expressed by $A(p)$, depends on them. What we have added to this evident fact simply is that their basis $A(p)$ may be of a more complex form than hitherto believed (see Corollary 5.14). Hence, little is lost through our extended interpretation of actual labor values with regard to a proper understanding of their relationship to prices p. Just the opposite is the case, since basic aspects of Marx's analysis of values, prices and profits, invalid under the regime of optimal labor values, can then be established under quite general assumptions on production, in a way which does not disqualify the empirical investigations which have been made so far with regard to Marxian (or other) hypotheses based on system labor requirements, i.e., Definition 5.10.

What then is the rationale of claiming "logical priority" for labor values over prices (of production)? In our view the answer to this question is provided by Proposition 5.21 (iii) which insures that labor values can still be employed *theoretically prior* to any consideration of prices, i.e., in the way they in fact are used in *Das Kapital* through the systematics chosen by Marx.

5.5 Conclusions

After having discussed the wide range of technological possibilities covered by the von Neumann model, Morishima (1973, p. 175) concludes: "Once we get rid of the world where $n = m$, the Marxian analysis of value and production in terms of simultaneous equations loses its foundation entirely." It has been the aim of this chapter to show that there is no necessity in this view. In fact, Marx's labor theory of value can be extended to quite general models of production through the inclusion of only one new relationship (represented by the matrix of commodity mixes C) into the Marxian equations approach (5.1), (5.2), (5.3), leaving intact Marx's methodology, the main conclusions he related with his simple transformation example Marx (1977, Vol. III, Chap. IX) to the extent they could be demonstrated within the simple equations approach (Morishima, 1973), and also Okishio's conclusion on the observability of labor values. This implies, that there is no gain involved, but indeed a loss, if optimal labor values are employed in place of our actual ones. Thus, some Marxian economists may have to change their attitude towards the labor theory of value once again.

References

Gupta, S., & Steedman, I. (1971). An Input–Output Study of Labor Productivity in the British Economy. *Oxford Bulletin of Economics and Statistics, 33*, 21–34.

Marx, K. (1977). *Capital, Volume I–III*. London: Lawrence and Wishart.

Matz, A., & Usry, M. F. (1976). *Cost accounting. Planning and control.* Cincinnati, Ohio: South Western Publishing Co.

Morishima, M. (1973). *Marx's economics*. Cambridge: Cambridge University Press.

Morishima, M. (1974). Marx in the light of modern economic theory. *Econometrica, 42*, 611–632.

Morishima, M., & Catephores, G. (1978). *Value, exploitation and growth*. London: McGraw–Hill.

Nell, E. J. (1980). Value and capital in Marxian economics. *Public Interest*, *58*, 174–200.

Okishio, N. (1963): A Mathematical Note on Marxian Theorems. *Weltwirtschaftliches Archiv, 91*, 287–299.

Roemer, J. E. (1980). A General Equilibrium Approach to Marxian Economics. *Econometrica, 48*, 505–530.

Roemer, J. E. (1981). *Analytical foundations of Marxian economic theory*. Cambridge: Cambridge University Press.

Roemer, J. E. (1982). New directions in the Marxian theory of exploitation and class. *Politics and Society, 11*, 253–287.

Steedman, I. (1977). *Marx after Sraffa*. London: New Left Books.

United Nations. (1968). *A system of national accounts.* Studies in Methods, Series F., No.2, Rev.3. New York: United Nations.

Wolff, E. N. (1979). The rate of surplus value, the organic composition, and the general rate of profit in the U.S. economy, 1947–67. *American Economic Review, 69*, 329–341.

Chapter 6
Employment Multipliers and the Measurement of Labor Productivity

6.1 Introduction

There have been numerous attempts to measure total employment effects and productivity indexes since the early work of Leontief (1944). These measurements have been based on the Leontief inverse $(I - A)^{-1}$ of recorded input–output tables. Examples of this are Bezdek's (1973) manpower analysis and the employment multiplier study of Diamond (1975). However, not much attention has been paid in these investigations to the derivation of the coefficients of the input–output matrix A, although at least two fundamentally different procedures exist in this respect. This may be due to the fact that the two procedures lead to different input–output tables only when subsidiary production is present in the classification of industries and commodity groups.

The objective of this chapter is to show that these two input–output accounting schemes, recommended by the United Nations (1968) in the case of insufficient information about subsidiary production, generate two different measures of total factor requirements (Sect. 6.2). In the Sect. 6.3 attention is focused on the case of pure joint production. Sraffa's interpretation of joint production and employment multipliers is summarized and a new concept of average labor content, or in reciprocal form: of labor productivity, is introduced and compared. It is shown in Sect. 6.4 then, that the UN method for calculating total labor requirements per dollar of output value from input–output tables leads either to Sraffa's or to the new interpretation of physical total input requirements, depending on how the A matrix is derived. Subsequent sections examine additional properties of these employment multiplier calculations and productivity concepts.

In sum, this chapter therefore demonstrates that it is in fact unnecessary to reduce to a simpler form the 'technological' input and output data underlying the input–output accounting framework of the UN. Homogeneity of activity outputs (United Nations 1973, p. 20) is in particular not required for a proper calculation of the total labor requirements in the production of commodities by means of commodities.

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_6,

6.2 The Measurement of Total Factor Requirements Using Input–Output Tables

The accounting framework of the United Nations' (1968, 1973) System of National Accounts is represented in the Table 6.1 below. In this table U represents the *absorption matrix*, which shows the input of commodities into industries, and V the *make matrix*, which gives the output of commodities by industries (in transposed form). The identity vector $i = (1, \ldots, 1)$ is used for summation. The column vectors e and q represent final demand and total output for commodities. Row vectors y', g' represent value added and total output for industries. All coefficients in the above table are expressed in value terms.

Three simple reduced-form matrices are normally derived from Table 6.1.[1] These are:

1. The matrix of *industry input coefficients* $G = U\widehat{g}^{-1}$
2. The *matrix of commodity mixes* (of industries)
   $C = V'\widehat{g}^{-1}$
3. The *matrix of market shares* (of industries)
   $D = V\widehat{q}^{-1}$

In order to measure the direct and indirect effects of changes in final demand on primary inputs (factor incomes), it is helpful to derive from the commodity × industry U matrix a commodity × commodity input–output table (there are also procedures to transform such tables to industry × industry version, cf. United Nations 1968, pp. 39–49).

There are two distinct ways to make such a derivation. These depend upon the following assumptions:

1. The *industry technology assumption (ITA)*: the inputs into commodities are in proportion to the value share of these commodities in the total output of their industry.
2. The *commodity technology assumption (CTA)*: there is a unique input structure (per unit of output value) for each of the given commodities $1, \ldots, n$.

These two assumptions can also be combined in a number of ways to allow a more flexible derivation of input–output tables. However, the impact of each of these assumptions on the measurement of direct and indirect requirements should be studied separately.

Table 6.1 The UN's representation of input-output tables

	Commodities	Industries	Final Demand	Totals
Commodities		U	$e = q - Ui$	q
Industries	V			$g = Vi$
Factor incomes		$y = g - iU$		$\zeta = yi$
Totals	$q' = iV$	g'	$\zeta = ie$	

[1] $\hat{g}$ the diagonal matrix formed from a vector g.

In order to understand the implications of the ITA Assumption 1, one should note that each dollar of output value of commodity i is on the average obtained from the different industries $j = 1, \ldots, m$ in proportion to their market shares D_{ji}. The ITA then implies that the vector of average inputs per unit of output of i is given by

$$D_{1i}G^1 + \ldots + D_{mi}G^m, \; i = 1, \ldots, n \tag{6.1}$$

where n is the number of commodities and D_{ji}, G^j the elements and columns of the matrices D and G, respectively. The implied input–output table of commodity × commodity type is therefore given by

$$A_I = GD \tag{6.2}$$

It is customary to restrict attention to cases where the number of industries m equals the number of commodities n. (United Nations 1973, pp. 37f.). In this case the CTA will allow the straightforward determination of the unique input structure $A_C^i \in \Re^n$ of commodities i. By assumption we have

$$G^j = C_{1j}A_C^1 + \ldots + C_{nj}A_C^n, \; j = 1, \ldots, n \text{ or } G = A_C C \tag{6.3}$$

Post-multiplying G by the inverse of C gives

$$A_C = GC^{-1} \tag{6.4}$$

which is the underlying input–output table of commodity × commodity type.

The measure of *total effects of final demand on factor incomes* (in our case the vector of wage incomes (u)) is then obtained by transforming this vector u in the same way as was done with intermediate inputs U (i.e., to industry coefficients $u\widehat{g}^{-1}$ and then to $a_I = u\widehat{g}^{-1}D$ or $a_C = u\widehat{g}^{-1}C^{-1}$ *and* by applying the customary formula for the determination of these effects:

$$\begin{aligned} x_I &= x_I A_I + a_I = a_I(I - A_I)^{-1} \\ x_C &= x_C A_C + a_C = a_C(I - A_C)^{-1} \end{aligned} \tag{6.5}$$

Of course these two formulae will yield different results only when industries are recorded as producing several commodities through the make matrix V. In actual tables the amount of this subsidiary production is often quite small but it increases with the size of the table, see Armstrong (1975, 5.6). Therefore the two vectors x_I and x_C have to be distinguished. The first type of equation has been used by the Dominion Bureau of Statistics (1969) to calculate employment effects. But is this really the proper choice to calculate these effects? Must x_C be regarded as a proxy for these employment multipliers x_I, the quality of which depends on the extent of joint production in the matrix V?

To answer these questions it is necessary to examine the content of the two definitions further. Initially one should ignore the aggregation problem and contrast monetary calculations with physical ones. Aggregation then comes as a second step (not considered here) which may cause additional difficulties, but which cannot correct an interpretation of magnitudes x_I, x_C, which is not justified on the level of physical relationships.

6.3 Joint Production

This section concentrates on the central technological feature that gives rise to subsidiary production in input–output tables, that is, pure joint production.

Following the approach of Sraffa (1960, Chap. VII) – which is in some respects close to that described by Table 6.1 – we assume a positive solution for prices $p_1, \ldots, p_n$ in the following system of price equations:

$$\begin{aligned} (p_1 A_{11} + \ldots + p_n A_{n1})(1 + r) + w l_1 &= p_1 B_{11} + \ldots + p_n B_{n1} \\ &\vdots \\ &\vdots \\ (p_1 A_{1n} + \ldots + p_n A_{nn})(1 + r) + w l_n &= p_1 B_{1n} + \ldots + p_n B_{nn} \end{aligned} \tag{6.6}$$

Here, A_{ij} and B_{ij} denote the physical input and output of commodity i for process j, and l_j denotes the corresponding labor input, each item in terms of absolute production of the year under consideration.[2] The first bracket to the left describes the total value of intermediate inputs for each process, on which the rate of profit r is to be calculated. Wages are considered to be paid *ex post*. The wage rate w and the rate of profit r are assumed to be uniform. Equation (6.6), therefore, describes a system of production prices p.

This system of equations can be put in following matrix form:

$$pA(1 + r) + wl = pB, \ \ p = (p_1, \ldots, p_n) \tag{6.7}$$

(columns A^j, B^j of A, B represent the input and output vectors of activities j, respectively). Combining these symbols with those of the United Nations' framework (Table 6.1) one can obtain:

$$U = \widehat{p}A, \quad V' = \widehat{p}B \quad (\text{ or } B'\widehat{p}) \tag{6.8}$$

[2] Note that 'homogeneity' of activity outputs is no longer assumed and that by Sraffa's methodology (to start from given production conditions) 'proportionality' is not involved unless specific considerations (of change) make such an assumption necessary, cp. Sect. 6.7 for an example and United Nations (1973, p. 20) with regard to the above terminology. Note also in the following that the vectors u, l, p all represent rows (where the 'prime' has been suppressed as no column representation of these vectors will appear in this chapter).

The remaining data then follow from the definitions already included in Table 6.1. Factor incomes y' can now be decomposed into two components:

$$u = wl \text{ (already considered) and } rpA$$

where only u can be considered to be based on a primary factor with regard to the employed model and Sraffa's intentions.

6.4 System Indicators of Employment and Productivity

On the basis of the physical data introduced in the last section, definitions are made of employment multipliers and of labor productivity. These are compared with those definitions established in Sect. 6.2. Employment multipliers can be defined on the basis of Sraffa:

> Now, if we wish to increase by a given amount the quantity in which a commodity enters the net product of the system, while leaving all the other components of the net product unchanged, we normally must increase the total labor employed by society. It is, therefore, natural to conclude that the quantity by which labor has to be increased for this purpose goes in its entirety, whether directly or indirectly, to produce the additional quantity of the commodity in question. The commodity added will, at the price corresponding to a zero rate of profits, obviously be equal in value to the additional quantity of labor (Sraffa 1960, p. 57).

The implied formula for the determination of *employment multipliers* thus reads:

Definition 6.1.

$$zB = zA + l, \; z = (z_1, \ldots, z_n) \tag{6.9}$$

(See Sraffa (1960, Chap. IX) and Steedman (1977, Chap. 11) for further details.)

There is one aspect of this definition which should be stressed here. Due to the assumption of joint production (with fixed coefficients) there is no need for employment multipliers $z = l(B - A)^{-1}$ to be *non-negative* with regard to every component. Technological rigidities[3] may cause the emergence of negative employment effects in connection with final demand stimulation. Although this possibility is only hypothetical, it nevertheless provides a suitable conceptual means to distinguish employment multipliers from the definition of labor contents to be given below – or (in reciprocal form) from labor productivity – which under no circumstances can become negative.

---

[3] which here are only due to the *ex-post* character of the employed input–output table (or alternatively: to Sraffa's consideration of 'frozen' production conditions, cf. the preface in Sraffa 1960), i.e., which do not necessarily exist with regard to time (or reality).

Indeed, Sraffa (1960, p. 56) questions ‘whether it makes any sense to speak of a separate quantity of labor as having gone to produce one of a number of jointly produced commodities’. If one considers such quantities with respect to technology alone these doubts seem to be justified: there is ‘no obvious criterion for apportioning the labor among individual products’ (p. 56).

However, the measurement of labor content (i.e., real total costs) is, in contrast to the definition of purely technologically determined employment multipliers, simultaneously a question of technological interdependence *and* of cost allocation. In order to elaborate this premise, managerial cost accounting should be considered.

It is well-known from the literature on cost-accounting that not all costs can be allocated to final products on the basis of technological causation alone. In the case of joint costs the use of somewhat arbitrary economic conventions is unavoidable: ‘Joint costs can be allocated, but all bases for allocation imply assumptions which cannot be objectively verified.’ NACA (1957, p. 2). However, the market or sales value method

> ‘enjoys great popularity because of the argument that the market value of any product is a manifestation of the costs incurred in its production .... Therefore, the only logical way to prorate joint costs is on the basis of respective market values of the items produced’. Matz and Usry (1976, p. 189). ‘Many, if not most, cost accountants believe that joint costs should be allocated to individual products according to their ability to absorb joint costs’. Dickey (1960, 13.11).

Of course, there are also authors who are doubtful about the usefulness of this procedure of cost allocation. It is however not possible, in this chapter, to discuss the usefulness of the method in the allocation of joint costs referred to above. The quotations cited must suffice to indicate that this rule has some advantage over existing alternatives.[4] This method is now applied to determine *real* total labor costs. In defense of the definition of ’real total labor costs‘ presented below, two arguments may, however, be presented.

First, an empirically minded determination of such real costs should keep contact with the principles which govern the calculation of total costs in general and in practice, i.e., the introduction of a theoretical notion of this kind should to some extent reflect the practical state of affairs on the level of activities and the firm. Second, it will be shown below that the application of the sales value method to our physical data A, B, l or to an apportioning of direct and indirect labor among individual products to determine their labor content will lead us back to the monetary measures x_I of Sect. 6.2, there derived from an input–output table which has been defined by help of market shares of products (instead of the sales value method employed to disentangle joint costs, which lies behind the type of mechanical rearrangement of inputs described in (6.1) to obtain a square monetary table of commodity $\times$ commodity type.

---

[4] Note, however, that the sales value convention is the only allocation method which is compatible with Sraffa’s equilibrium prices (6.7), if such prices are based on full costs, cp. NACA (1957, p. 47) with regard to the practical relevance of this point.

In light of the foregoing remarks consider now the following definition:

Definition 6.2. The vector of *labor content* $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$ of commodities $1, \ldots, n$ is the solution of the following system of equation:

$$\bar{z}\widehat{Bi} = \bar{z}AC' + lC' \tag{6.10}$$

where $C' = \hat{g}^{-1}V$ is the matrix of commodity mixes (in transposed form), cp. Sect. 6.2, and where Bi represents the vector of total outputs of commodities $1, \ldots, n$.

It is customary to speak of *indexes of labor productivity* when $1/\bar{z}$ instead of $\bar{z}$ is considered. By (6.10) the total output of commodity k, i.e., $(Bi)_k = B_k i$ is now associated with an input vector of commodities $1, \ldots, n$ (a column) of kind:

$$(AC')^k = A^1 C_{k1} + \ldots + A^n C_{kn}$$

and an amount of direct labor inputs of kind:

$$(lC')_k = l_l C_{k1} + \ldots + l_n C_{kn}$$

The vector $A^j C_{kj}$ (and similarly $l_j C_{kj}$) represents what firms would allocate to the production of their output B_{kj} with regard to the jth process on the basis of the sales value method, since by definition of the matrix of commodity mixes C we have

$$C_{kj} = p_k B_{kj} \Big/ \sum_k p_k B_{kj}.$$

The above sum of vectors, therefore, represents the total amount of commodity inputs which would have been allocated to the total amount $(Bi)_k = \sum_j B_{kj}$ that is produced of commodity k if the sales value convention is applied throughout. This reallocation of inputs leads to an equation system (6.10) for labor contents of commodities that is of the usual kind, where joint inputs have been disentangled by the described sales value procedure and where joint outputs have all been placed on the diagonal of the matrix B by horizontal summation (which just gives the matrix $\widehat{Bi}$).

Note that relative prices are now involved in the definition of labor contents, because they are used – by means of the coefficients C_{kj} – to determine how much labor is absorbed by the different products of each joint basket. Hence, the above defined labor contents $\bar{z}_k$ can be regarded as 'physical' only up to the applied rule that disentangles formerly joint labor effort of direct and indirect kind.

This is the price that has to be paid if a notion of 'real total factor costs' is desired which remains operational even in the presence of joint production.

It is in any case this vector of labor contents (6.10) which is measured by x_I (on a per \$ base, cp.(6.5)), i.e., by the standard procedure, if an input–output table is used which is based on the industry technology assumption ITA described in Sect. 6.2 as the following proposition will now show in particular. In contrast to this finding (that the vector x_I cannot be interpreted in purely physical terms if joint production is involved) it will be shown furthermore that the alternative definition of labor costs,

x_C (cp. (5) in Sect. 6.2), which employs the CTA instead of the ITA, will be of a purely technological nature throughout since it on the whole is identical with the definition of employment multipliers (6.9) given above.

Proposition 6.3.

1. $x_I = w\bar{z}\widehat{p}^{-1}$. *Vectors* $x_I, \bar{z}$ *will exist, be unique and positive, if* $iG < i$, *i.e., if the unit costs of intermediate goods are less than one with respect to every process.*
2. $x_C = wz\widehat{p}^{-1}$. *Vectors* x_C, z *will exist and be uniquely determined, whenever* $B - A$ *is invertible (which can always be ensured by a slight variation of up to* n *coefficients in the matrices* B *or* A*).*

Proof.

1. Definition (6.10) implies

$$\begin{aligned}(w\bar{z}\widehat{p}^{-1})\widehat{p}\widehat{Bi} &= (w\bar{z}\widehat{p}^{-1})\widehat{p}AC' + wlC', \qquad \text{i.e.} \\ w\bar{z}\widehat{p}^{-1} &= (w\bar{z}\widehat{p}^{-1})\widehat{p}AC'\widehat{q}^{-1} + wlC'\widehat{q}^{-1} \\ &= (w\bar{z}\widehat{p}^{-1})\widehat{p}A\widehat{g}^{-1}D + u\widehat{g}^{-1}D \\ &= (w\bar{z}\widehat{p}^{-1})A_I + a_I.\end{aligned}$$

   And because of $A_I \geq 0$, $iA_I = iGD < iD = i$ the solution $x_I = w\bar{z}\widehat{p}^{-1}$ of this latter equation is uniquely determined and positive as is well-known from theorems on non-negative matrices, see Chap. 7.
2. Definition (6.9) implies

$$\begin{aligned}(wz\widehat{p}^{-1})\widehat{p}B &= (wz\widehat{p}^{-1})\widehat{p}A + wl, \text{i.e.} \\ (wz\widehat{p}^{-1})\widehat{p}B\widehat{pB}^{-1} &= (wz\widehat{p}^{-1}\widehat{p}A\widehat{pB}^{-1} + wl\widehat{pB}^{-1}.\end{aligned}$$

With the notations used in Sects. 6.2,6.3 this gives

$$\begin{aligned}(wz\widehat{p}^{-1})C &= (wz\widehat{p}^{-1})G + u\widehat{g}^{-1}, \text{i.e.} \\ wz\widehat{p}^{-1} &= (wz\widehat{p}^{-1})GC^{-1} + u\widehat{g}^{-1}C^{-1}, \text{i.e.} \\ wz\widehat{p}^{-1} &= x_C\end{aligned}$$

□

The above proposition shows that employment multipliers and labor contents as defined in this section will lead to the formal definitions of total costs (6.5) – which were based on mechanical procedures of input–output methodology – when considered in terms of wage income (i.e., when multiplied by w) and reckoned per unit of value of commodities $1, \ldots, n$ instead of their physical units. These formal definitions, therefore, have now been filled with 'physical content' as far as the occurrence of joint production is concerned.

The kind of physical foundation introduced, however, implies that neither the ITA nor the CTA can be considered as assumptions on 'technology' in this surrounding.

In the case of the CTA post-multiplication of G by C^{-1} only re-establishes the 'technological rigidities' that have been lost in the calculation of the industry coefficient matrix G, i.e., this multiplication only represents a mathematical device to reduce joint production calculations of total labor costs to the kind of formula that is well known from simple Leontief models. The possibility of such a reduction, however, does not imply anything about the input of commodities for the production of *commodities*. This should be obvious from the general type of joint production which is under consideration. And with regard to the ITA we have already stressed the necessity of economic imputations to supplement the usual technological data in the derivation of Definition 6.2, which again shows that there are no facts of technology hidden behind the construction of $A_I = CD$ from the physical data A, B. Here, possibly existing rigidities remain dissolved even after post-multiplication by D which is the reason why no negative values can occur in the corresponding notion of labor requirements.

6.5 Some Results for an Analysis of Technical Change

Let us consider the case of employment multipliers first – with the help of a simple example taken from Steedman (1977, Chap. 11).

Let the matrix of outputs B be defined by $B = \begin{pmatrix} 30 & 3 \\ 5 & 12 \end{pmatrix}$, intermediate inputs A by $\begin{pmatrix} 25 & 0 \\ 0 & 10 \end{pmatrix}$ and labor inputs l by $(5, 1)$. Definition 6.1 then implies

$$z = (z_1, z_2) = (l_1, l_2)(B - A)^{-1} = (l_1, l_2) \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}^{-1}$$
$$= (5, 1) \begin{pmatrix} 5 & 3 \\ 5 & 2 \end{pmatrix}^{-1} = (-1, 2)$$

which in particular means that employment effects with regard to the first commodity may be negative, a not too unexpected result in the light of the given technology.

In general terms the appearing inverse can be calculated as:

$$\frac{1}{x_{11}x_{22} - x_{12}x_{21}} \cdot \begin{pmatrix} x_{22} & -x_{12} \\ -x_{21} & x_{11} \end{pmatrix}$$

and the above vector of 'employment multipliers' by:

$$(z_1, z_2) = \frac{1}{x_{11}x_{22} - x_{12}x_{21}} (l_1 x_{22} - l_2 x_{21}, -l_1 x_{12} + l_2 x_{11})$$

Table 6.2 Comparative Statics: Summary

	ϑx_{11}	ϑx_{21}	ϑx_{12}	ϑx_{22}	ϑl_1	ϑl_2
ϑz_1	−	+	+	−	−	+
ϑz_2	+	−	−	+	+	−

Calculating partial derivatives then implies the following table of 'signs' for the derived matrices of the above example:

This table in particular shows that an isolated increase in $x_{22} = B_{22} - A_{22}$, e.g., an increase in output B_{22}, which should imply an increase in labor productivity (if such an interpretation were justified) in fact causes a fall in z_2^{-1}, i.e., a fall in 'labor productivity' with respect to that commodity. Similarly, a decrease in l_i will not lead to a rise in 'labor productivity' $1/z_i$. Both facts, therefore, clearly show that an interpretation in terms of 'labor productivity' (or in reciprocal for: of 'labor content') cannot be the appropriate one.

On the other hand, the alternative interpretation – in terms of total employment effects – is not invalidated by the occurrence of such 'anomalies' as has already been remarked in connection with (9), cf. also Steedman (1977, Chap. 11).

Better results in the direction of the first interpretation can be obtained for the second definition: $\bar{z}$ and x_I.

To show this let us abbreviate the matrix $AC'\widehat{Bi}^{-1}$ by $\overline{A}$ and the vector $lC'\widehat{Bi}^{-1}$ by $\bar{l}$. From Definition 6.2 and the proof of Proposition 6.3, part 1 we then obtain:

$$\bar{z} = \bar{z}\overline{A} + \bar{l} \quad \text{and} \quad A_I = \hat{p}\overline{A}\hat{p}^{-1} \tag{6.11}$$

where the latter equation describes the usual connection between input/unit–output tables of monetary and physical kind. Since both A_I and $\overline{A}$ are non-negative and since the viability of A_I – cf. Proposition 6.3 – by (6.11) implies the viability of $\overline{A}$, it follows that $I - \overline{A}$ is non-negatively invertible, too. From known facts on the simple open Leontief model it is then clear that changes in the coefficients $\overline{A}_{ik}, \bar{l}_k$ will imply equally directed changes in all components of $\bar{z}$ (up to problems of decomposability).

To calculate the effects of (isolated) changes in A_{ij}, B_{ij}, l_j and p_i on the indexes $\bar{z}_i$ it therefore suffices to look for unidirectional changes in $\overline{A}, \bar{l}$ with respect to these coefficients. From the definition of C and $\widehat{Bi}$ we get

$$(C'\widehat{Bi}^{-1})_{jk} = \frac{p_j B_{jk}}{(\sum_k B_{jk})(\sum_j p_j B_{jk})}$$

Changes in A_{in} and l_j, accordingly, will imply changes in $\overline{A} = AC'\widehat{Bi}^{-1}$ of equal direction solely and will thus be reflected correctly by our measure of labor content $\bar{z}_k$, i.e., of labor productivity.

The productivity effect of changes in B_{ij}, however, cannot be judged so easily. These coefficients will give rise to (desirable) unambiguously opposite (or zero) effects in the ith row and the jth column of $C'\widehat{Bi}^{-1}$ up to their intersection $(C'\widehat{Bi}^{-1})_{ij}$ only. There will be no effect in the remaining entries of $C'\widehat{Bi}^{-1}$. Yet, with regard to the entry $(C'\widehat{Bi}^{-1})_{ij}$ the effect of a change in B_{ij} will be indeterminate and this implies – despite further summations – that the logical possibility for exceptional cases (where a rise in B_{ij} will lead to a rise in some $\bar{z}_i$) may exist. This possibility is due to the adopted rule of imputation and should be examined in this regard in case it happens to occur.

In the light of the foregoing example (which has to be completed by price data then, e.g., 1/3, 1, cf. Steedman 1977, p. 152) the vector of labor contents can be calculated as $(\bar{z}_1, \bar{z}_2) \approx (0.24, 0.59)$, which is very different from the above calculated employment multipliers $(-1, 2)$. Yet, it must be noted here that this very example can give rise to the exceptional cases (with respect to B_{21}, B_{12}) just mentioned and may therefore be used to examine the content of the proposed measure x_I in more detail.

Finally, changes in p_i may cause any effect, that is no general rule will be obtainable here. This is to be expected from a measure which employs economic imputations for its full determination.

In sum, at least the saving of intermediate and primary inputs will give rise to positive effects on such indexes of labor productivity $1/\bar{z}_i$ in each possible case, thereby showing that the first type of 'total factor requirements' of United Nations' input–output calculations, cf. (6.5), may be useful for empirical investigations. The discovered dependency of this notion on the use of economic imputations, on the other hand, has shown that it cannot be regarded as the first candidate for the calculation of employment multipliers – up to certain special cases as will be seen in the next section.

6.6 The Case of a Uniform Composition of Capital

Proposition 6.4. *Let pA be proportional to wl. We then have:*

$$\bar{z} = z = p \tag{6.12}$$

if prices p have been normalized as to fulfill: $p(B - A)i = li$.

Lemma 6.5. *With respect to x_I, x_C of Sect. 6.2 there holds:*

$$(x_I - x_C)(I - A_I) = x_C(I - CD), \text{ i.e.,} \tag{6.13}$$

$$x_C - x_I = a_C(I - A_C)^{-1}(I - CD)(I - A_I)^{-1}. \tag{6.14}$$

Proof. According to Sect. 6.2 we have

$$x_C = a_C + x_C A_C \text{ and } x_I = a_I + x_I A_I \text{ or}$$
$$x_C C = u\widehat{g}^{-1} + x_C G, \; x_I D^{-1} = u\widehat{g}^{-1} + x_I G.$$

From this there follows:

$$x_C C - x_C D^{-1} + x_C D^{-1} - x_I D^{-1} = (x_C - x_I)G \quad \text{or}$$
$$x_C (C - D^{-1}) + (x_C - x_I) D^{-1} = (x_C - x_I)G.$$

Post-multiplication by D then implies:

$$x_C (CD - I) = (x_C - x_I)(GD - I), \quad \text{i.e.,}$$
$$x_C - x_I = x_C (I - CD)(I - GD)^{-1}.$$

Note, that (as always) we have neglected the case where the inverse of C or D (or $B - A$) does not exist. □

By the same method a similar formula may be obtained with respect to the physical Definitions 6.1, 6.2:

$$z - \bar{z} = \bar{z}\widehat{Bi}(C^{-1} - D)'(B - A)^{-1} \tag{6.15}$$

Proof of the proposition: The assumption $pA \sim wl$ implies that the wage rate w and the rate of profit r in system (7) can be changed in such a way as to yield a rate of profit $r = 0$ without any change in the structure of p. It follows that there exists a positive w_0 such that

$$pA + w_0 l = pB$$

is fulfilled. But due to the assumed normalization of prices p the number w_0 must be equal to one, which implies $p = z$ by Definition 6.1. It follows that x_C will be equal to wi, which by help of the above lemma implies:

$$x_C - x_I = wi(I - CD)(I - A_I)^{-1} = 0$$

as we have $iC = i$ and $iD = i$.

Formulae like (6.13), (6.15) may be helpful to estimate the extent of deviations that may occur between employment multipliers and indexes of labor productivity in the case of a non-uniform composition of capital. These estimates could be still improved. However, it is already important to know that there are two cases: the above case of similar compositions of capital and the case of a small proportion of subsidiary production ($C^{-1} \approx D \approx I$, cf. also Armstrong 1975, pp. 88f.) for which a similarity between z and $\bar{z}$ can be claimed.

6.7 Conclusions

We have shown that for a pure joint production system some care is required in the choice of the input–output table when one's objective is to measure interindustry employment effects. This is a point which is completely ignored n the existing UN proposals for such a measurement.

It can be argued of course that the extent of joint production in observed input–output data of the kind as shown in Table 6.1 does not make it necessary to be precise to such an extent. However, knowledge of the correct procedure may be worthwhile in itself, even though resulting quantitative improvements may still be small. Furthermore, technological constraints on supply are in fact recognized in applied work (Dominion Bureau of Statistics 1969, A. 7). It therefore makes little sense to choose tables as A_I (where such constraints have been eliminated by economic imputation) when such questions are approached. Finally, it has been shown that negative entries do occur when the commodity technology assumption is used in the calculation of input–output tables (A_C). As a consequence, methods have been invented to eliminate those negative numbers again in a second step, cf. Armstrong (1975, pp. 78–81), without noticing that such procedures may just endanger the uses that can be drawn from the original table of type A_C.[5]

Instead of such mechanical manipulations the search for further distinctive features of the two basic procedures for the derivation of input–output tables of commodity × commodity type may be worthwhile. In closing we shall, therefore, comment briefly on two such possibilities.

One difference between tables A_C and A_I of Sect. 6.2 is given by the occurrence of rectangular, instead of only square, make and absorption matrices U, V, (United Nations 1973, III.D). With respect to this occurrence the special case of multiple, but single product activities will be of particular interest (this situation is already included in the foregoing derivations insofar as no inverse matrices are employed). In the case of the CTA we will find then that this assumption in fact is now hurt *by assumption*! Furthermore, since it is not clear which activities of each sector will be stimulated by a change in final demand, the concept of an employment multiplier will become rather vague in such a situation. The ITA, on the other hand, will give rise to very simple and perfectly sensible calculations: activities producing the same commodity will now be simply summed by use of the matrices C', $\widehat{Bi}$, thereby leading back to square matrices of the usual kind which again give rise to sensible calculations of average labor contents of commodities.

A second point which allows one to distinguish between the two basic procedures of input–output calculations here employed lies in their usefulness for projection work, i.e., in the question of stability of the input–output matrices A_C, A_I.

The proof of Proposition 6.3 in Sect. 6.4 has shown that the two tables in use can be reduced to 'technological' terms in the following way:

$$A_C = GC^{-1} = \hat{p}AB^{-1}\hat{p}^{-1} \tag{6.16}$$

$$A_I = GD = \hat{p}A\widehat{pB}^{-1}D = \hat{p}AC'\widehat{Bi}^{-1}\hat{p}^{-1} \tag{6.17}$$

Assuming that the technology used for (6.16) and (6.17) is based on fixed coefficients and constant returns, that no technological changes occurs and

[5] For a similar dangerous argument cp. United Nations (1968, p. 39), where the occurrence of negative entries in A_C is characterized as being 'manifestly absurd'.

that distribution and therefore price ratios remain constant (the latter is not really necessary), it can then be seen from (6.16) that changes in final demand and therefore activity levels x will leave A_C invariable (because of $AB^{-1}=A\widehat{x}\widehat{x}^{-1}B^{-1}=(A\widehat{x})(B\widehat{x})^{-1}$, i.e., we then have 'constant technical conditions' in the language used for input–output tables. Under these conditions, the table A_C can, of course, be used for input–output projections with respect to final demand and resulting employment effects.

The matrix A_I, on the other hand, will not be invariable even under these most favorable conditions, since market shares normally will have to change with final demand in the presence of joint production. But this fact will create no discomfort here, since the proper task of A_I – and of resulting labor productivity indexes $\bar{z}$ – in fact lies just in the analysis of technological change and the factors that have influenced the resulting change in labor productivity. Therefore, the often needed assumption of 'constant technical conditions' very nicely is associated to exactly that case where it really is needed.

A final word may be in order about the effect on the application to empirical analysis of issues examined in this chapter. Inspecting reported make matrices V makes it obvious that not much difference between the two measures x_C and x_I can be expected to result in actual computations, since joint production here only appears in the form of (minor) by-products as far as empirical tables are concerned (because of their degree of aggregation), cp. e.g., Armstrong (1975, pp. 81f.) for a comparison of A_C and A_I for tables of type 35×35 and 70×70. This chapter, therefore, should be considered more as a contribution to the methodological and theoretical state of input–output considerations; in particular, it should help to prevent misplaced manipulations with regard to negative entries (for example in A_C) and in general offer an interpretation of the things which are being measured. This latter task has been performed by confronting the absorption matrix U and the make matrix V with physical data of the same qualitative type, which, however, cannot be reduced to 'technical conditions' of form A, I as is customary in input–output analysis, but which nevertheless allow sensible interpretations of the formula for total labor requirements based on such artificial reductions.

References

Armstrong, A. G. (1975). Technological assumptions in the construction of UK input–output tables. In R. I. G. Allen & W. F. Gossling (Eds.), *Estimating and projecting input–output coefficients*. London: Input–Output Publishing Company

Bezdek, R. H. (1973). Interindustry manpower analysis: theoretical potential and empirical problems. *The American Economist, XVII*, 147–153

Diamond, J. (1975). Inter-industry indicators of employment potential. *Applied Economics, 7*, 265–273

Dickey, R. I. (Ed.) (1960). *Accountant's cost handbook.* New York: Ronald

Dominion Bureau of Statistics (1969). *The input–output structure of the canadian economy 1961*, 1. Ottawa: The Queen's Printer

Leontief, W. (1944). Output, employment, consumption and investment. *Quarterly Journal of Economics, 58*, 290–314

Matz, A. & Usry, M. F. (1976). *Cost accounting, planning and control*. Cincinnati, Ohio: South–Western

NACA (1957). *Costing joint products*. Research Series No. 31. New York: National Association

Sraffa, P. (1960). *Production of commodities by means of commodities*. Cambridge: Cambridge University Press

Steedman, I. (1977). *Marx after sraffa*. London: NLB

United Nations (1968). *A system of national accounts, studies in methods, series F, no.2, rev.3*. New York: United Nations

United Nations (1973). *input–output tables and analysis, studies in methods, series F, no.14, rev.1*. New York: United Nations

Chapter 7
Technology Assumptions and the Energy Requirements of Commodities

Since Stone's original work, input output methodology has taken a great step forward in standardizing the procedures employed for the derivation of monetary input output tables which in turn, e.g., are used to measure the energy requirements of commodities. In this methodology there exist two basic (mechanical) ways, or "technology" assumptions, by which such measures under circumstances of joint production can be defined. The physical effects of these assumptions on the measures obtained, however, have not yet been analyzed sufficiently. This chapter presents such an analysis on the basis of technological conditions of a quite general kind, including joint production, and shows that even then these two measures of energy requirements will give rise to economically well-grounded expressions, i.e., do not represent measurement without theory. The expressions obtained will allow us to prove several assertions within their respective ranges of applicability, which complement each other to some extent.

7.1 An Overview on Problems and Results

Input–output methods have been applied to a variety of questions of theoretical as well as of empirical interest. Examples are found in the discussion of the labor requirements of commodities within the theory of capital, or in the empirical study of technological change in its effects on measures of total labor productivity (see Sraffa 1960 and Gupta and Steedman 1971 for a typical example of each). The interaction of both kinds of economic investigation, however, often leaves much to be desired in their further development, as can be briefly indicated with reference to the two works cited. Taking for example their treatment of fixed capital, in theory we find the decidedly idealistic attitude of considering all kinds of old capital goods as merely a special case of joint production, while in Gupta and Steedman's examination of the British economy very crude assumptions are made to derive (and utilize) replacement coefficient matrices from reported matrices of capital stocks. Yet despite such discrepancies between theory and measurement there also exists a range of topics where recent developments can be fruitfully combined to explain factual findings as

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_7,

well as to avoid theoretical constructions – of the kind just referred to – that are too ideal to meet the needs of daily economic life. With the present chapter we hope to contribute to the required interaction.

The measurement of total (direct and indirect) factor requirements of commodities has in general been approached with regard to factors such as labor, capital and imports, then considered primary factors in the underlying input–output methodology. However, the oil embargo and rising energy prices have also directed attention to the measurement of total energy requirements, an input which is frequently situated in the intermediate part of the complete System of National Accounts. Regarding such requirements, questions are then raised as to how far sectoral energy requirements differ from their average, how these requirements have changed over the past two decades, and in particular if and how the energy requirement pattern of the United States – the level of which is roughly twice as that of Western European Countries – can be influenced to approach the standard provided by the latter countries (see Krenz 1977 and Howe 1979 for further details).

As already stated, the purpose of this chapter is to help to bridge the gap between certain theoretically and empirically motivated inquiries of similar kind. To do this we shall take as our topic the determination, interpretation and intertemporal comparison of the energy requirements of commodities in their relation to an important feature (and trouble-maker) of observed technology: the widely prevalent existence of joint production in modern industry. This technological characteristic has become of increased theoretical interest since the publication of Sraffa's (1960) critique of economic theory, and is meanwhile also thoroughly reflected in the empirical input–output methodology of the United Nationss' (1968) SNA, its System of National Accounts. But, despite an existing relationship, our following theoretical considerations will make no explicit use of Sraffa's complex analysis of relative prices and their movements. Instead, the instruments provided by the SNA together with some knowledge of the properties of non-negative matrices (practically comparable to that required for the simple Leontief model) will suffice to prove the assertions we shall make. Furthermore, joint production will not be considered in its myriad details here, but will be treated in the abstract and general manner that is the basis of the mechanical manipulations applied to it in the SNA. Since our interest lies in the first place in the understanding of these mathematical manipulations with respect to their economic content, all noteworthy concrete cases of joint products (of which the energy sector is particularly rich, e.g., crude oil and natural gas, gasoline and heating oil, electricity as an economically motivated by-product of industrial heat production, for example, in chemical and petrol-refining industries) must be left to a separate and more specific treatment then can be provided here (compare, however, James 1980, p. 177, where the output data of five industries engaged in the production of 21 energy commodities are presented in a format which is closely related to the abstract approach which we shall utilize in this chapter).

The mathematical formula conventionally used in input–output analysis to determine the total requirements of commodities j per unit of output value with regard to, say, an energy sector with index 1 is simple. Allowing for slight variations employed by the various writers in this field (see Wright 1975, p. 31 for an example) it is given by the coefficient A^*_{1j} of the Leontief-inverse $A^* = (I - A)^{-1}$ of the central transaction matrix A of the whole input–output system. This coefficient – by its well-known multiplier interpretation (see Lancaster 1968, p. 86ff.) – describes the sum of direct and all indirect requirements of commodity 1 in the production of commodity j.

Yet a number of important additions have to be given when such an expression, calculated from reported monetary transactions tables, is used for the analysis of energy requirements in order to ensure their proper interpretation. These additions can concern (1) the role of fixed capital and depreciation (see Gupta and Steedman 1971; Herendeen 1974, p. 147; and Wright 1975, p. 35 for some details), (2) the treatment of imports (see again Gupta and Steedman 1971 and Wright 1975, p. 32, but also James 1980), (3) the fact of price changes (see here in particular Stobbe 1959, p. 250ff. and United Nations 1968, 1973), and *last but not least* (4) the existence and handling of *joint products*. As already said it is this latter complication and the interpretational problems deriving from it that our following considerations are addressed to, *and this by a strict exclusion of all the other problems* just enumerated (though these are by no means less important than the problem here selected). What we want to show in concentrating on the case of joint products, concerns, among other things, the following assertions:

1. The two notions "total energy consumption" and "total energy costs", which are identical if the simple Leontief model is applied to assign a physical content to their definition by means of A^*_{1j}, and which are used interchangeably in this case *and* in general (see, e.g., Reardon 1973 and Herendeen 1974), are shown to give rise to two different and both economically meaningful measures, once joint products are introduced into the technological situation assumed to underlie the employed transactions table A.
2. Despite their difference in physical description and content both measures can be expressed in the same conventional way depending (in this order) on whether the "commodity technology" assumption or the "industry technology" assumption of the SNA has been used to derive the monetary input–output table A that is being utilized for their determination.
3. The two notions (at least initially) apply to situations of quite different kinds. The notion of "energy consumption" per commodity finds ideal application whenever the technical conditions of input *and* output can be assumed to be constant; it does not depend on changing activity levels nor on changing prices. Its principle characteristic, therefore, is to measure the energy requirements of commodities in a purely technological way. The notion of "energy costs" of commodities, on the other hand, will not remain invariable under the above stated conditions. It in fact depends on the choice of activity levels and also on prices by way of its implicit reliance on factual economic conventions that are used for unit-cost calculations in the presence of joint products, i.e., it no longer is of a purely technological kind. Its principal field of application, after all, lies in the analysis of technological change – just the opposite of constant input and output coefficients, the ideal case for the measurement of energy consumption.

4. Thus, the two concepts to be introduced in Sects. 7.3 and 7.4 should be regarded as complementary rather than as alternatives, the first being best suited for short-run analysis and scenarios of final demand changes, while the second more naturally applies to an analysis of the long-run consequences of technological change, that is, to intertemporal productivity comparisons.
5. In the light of the given characterization of "energy consumption" coefficients it is not astonishing to find that the application of only one input and one output table for their determination may lead to situations where some of these coefficients may become negative, i.e., where an extra net output of commodities is accompanied by a reduction and not a rise in energy use. There is no reason for a purely technologically determined notion to be positive under all circumstances, since it is not automatically ensured in the presence of joint products that an unproductive use of "energy" cannot occur with a profit-maximizing choice of technique. The appearance of negative entries in input–output tables A of "commodity technology" type, as reported in Armstrong (1975, p. 78ff.), hence allows a quite natural explanation which in fact disqualifies the efforts made to suppress such entries in the tables finally presented for use.
6. "Energy costs", on the other hand, cannot become negative under economically viable conditions by their very definition, and that is one of the reasons that we have attached the term "costs" to this type of measurement.

We have already cited Krenz (1977) and Howe (1979) for possible questions and applications with respect to input–output computations of the energy requirements of commodities. Further computations of such energy requirements can be found in Wright (1975), where an intertemporal comparison of these requirements is provided for the British economy of the years 1963 and 1968. But, though there is reported there a tendency of these requirements to fall (with an average ratio of 1:1.29), Wright (1975, p. 35) ascribes the relevant part of this tendency in the last instance to changing prices rather than to "technological change affecting energy intensity". The time behavior of energy-use coefficients is also estimated in Reardon (1973) for the US economy of the years 1947, 1958 and 1963. In this chapter a substantial improvement is found in energy use per 1958 dollar's worth of final demand for the first of the two mentioned time periods, but none for the second. In addition, a resolution of fuel-use changes into final demand changes and changes due to technology is also presented in this chapter, showing that the changes due to increases in final demand more than offset the reported increases in efficiency. The computations mentioned so far in their essence relate more to an analysis of intertemporal changes of energy requirements (i.e., in principle to "energy costs"). An analysis confined to changes in final demand in their effects on energy consumption (i.e., the first of our two notions) is provided by Folk and Hannon (1974) in conjunction with an accompanying analysis of employment effects by means of simulations based on the 1963 input–output table of the United States.

Clearly, there still exists a considerable gap between these generally very laborious practical estimations of the energy requirements of commodities, where the proper choice of the table A to be employed is rarely considered, and their economic foundation presented here in relation to joint production and standard input–output methodology (which moreover is still far from being universally applied, yet see Herendeen (1974, p. 147) for some comments on the strange practices found in use instead). A final remark may therefore be in order to further clarify the value of the analysis which follows.

As a special case, general joint production systems contain all kinds of multiple, but single-product, activities, a fact which tends to increase the number of activities relative to the number of produced commodities (whereas the existence of joint production tends to reverse this relationship). In view of these technologically based tendencies the two economic interpretations we shall associate with the "commodity technology" and the "industry technology" assumption, respectively, should not be restricted to the case of *square* multiple product systems (a case one is inclined to assume in the light of the existing input–output methodology; compare James 1980, for example). Instead, our two concepts of energy consumption and energy costs should in principle be applicable to all kinds of rectangular systems as well – provided that the technological data on which each of these concepts rests have not been aggregated in a way that is inconsistent with their respective definition. Thus our association of two physically based definitions with the two basic methods of transfer used to derive monetary input–output tables A of commodity × commodity type intends to prove in the first place that these methods do not lead to measurement without theory *despite the existence of joint products (and thus "common costs") of any degree of generality*. Both methods of transfer can be shown to give rise to theoretically sound and applicable concepts of energy requirements even in the most general case, and it is "only" the lack of sufficiently detailed technological information (e.g., caused by the aggregation of single-product activities into multiple production not clearly distinguishable from intrinsic joint production in the industrial census) which introduces all kinds of difficulties into their precise measurement. This does not mean that such problems of (dis-)aggregation (e.g., through application of the hybrid technology assumptions proposed in the SNA) are of minor importance as compared to the theoretical solutions here proposed. On the contrary, they by far represent the more difficult subject, which is also the reason why their treatment cannot be simply appended to the considerations made here.

Yet it is by no means established that the mechanical methods of transfer employed in the SNA to solve the problems arising from joint and other kinds of multiple production indeed make sense and can be usefully applied in theory as well. Were this not the case, however, no subsequent aggregation would be able to give economic content to concepts for which no economic foundation in general exists. Therefore, our considerations evidently have to precede any treatment of the possible disturbances caused by the lack of information and by the aggregation that may be involved in the data finally employed for measurement.

To supply the general foundations for such a measurement consequently is the main purpose of the analysis that follows. After some preliminaries which will make standard input–output methodology accessible for the questions to be approached (Sect. 7.2), we shall first consider the concept of "energy consumption" and some of its properties (in Sect. 7.3). Section 7.4 will then make a similar contribution with regard to the concept of "energy costs". In Sect. 7.5, finally, we shall give a brief comparison of these two concepts accompanied by an examination of the applicability of the second one for the study of technological change.

7.2 Analytical Preliminaries

We have already presented the basic expression in use for measuring the (value of) total energy requirements of the different commodities j (per unit of their output value). In a slightly modified form which excludes the amount of energy commodity "1" that is used for final demand, this expression reads

$$A^*_{1j} - \delta_{1j} = (I - A)^{-1}_{1j} - \delta_{1j} = A_{1j} + (A^2)_{1j} + (A^3)_{1j} + \ldots \qquad (*)$$

where the matrix A represents a conventional input–output table in monetary terms and where δ_{1j} is defined by "1" for $j = 1$ and by "0" otherwise. To apply the pure theory of input–output to such coefficients it is common practice, then, to assume a direct correspondence between the coefficients A_{kj} if A and the physical quantities $\overline{A}_{kj}$ of commodities k needed for the production of one physical unit of commodity j (in general supplemented by the assumption of a linear technology with regard to the resulting matrix $\overline{A}$; see United Nations 1973, p. 20 for an example). The implied relationship between the foregoing monetary measure $(*)$ of energy requirements and the corresponding physical one is of a very simple nature in this case, as can be seen from the following set of equations:

$$A_{kj} = p_k \overline{A}_{kj}/p_j, \quad \text{e.g.,} \quad A^*_{kj} = p_k \overline{A}^*_{kj}/p_j, \quad \overline{A}^* = (I - \overline{A})^{-1} \qquad (**)$$

Here, symbols p_k and p_j denote the prices of the commodities concerned, which should not be eliminated from the above equations through an appropriate choice of physical units (as is often done) if an analysis of intertemporal changes in energy requirements is finally intended.

An early example of such an analysis (of primary factors such as labor) which explicitly includes changing prices and, therefore, the choice of appropriate index numbers and which thus starts from the above set of relationships $(*)$, is presented in Stobbe (1959). And there too the problem of how to disentangle complex multiple-product activities into single-product processes is declared to be of decisive importance for both a theoretical-quantitative as well as an empirical–statistical point of view – on the final solution of which the value of the whole analysis presented in Stobbe (1959) is viewed to depend. Shortly thereafter (and independent of this work) the methods employed to transfer secondary products – the typical form of multiple production that is observed in input–output data collection – from the accounts of their industry of origin to those of the industry where they are principally produced came to be studied in a more systematic way, leading to their first rigorous mathematical treatment in the work of Stone et al. (1963). The methodology of input–output computations which resulted has since become a basic and indispensable part of the United Nations' System of National Accounts (see United Nations 1968, 1973). It has also been further analyzed and extended in various papers as, e.g., in Gigantes (1970) and Armstrong (1975). This progress in the treatment of data on multiple production, however, remained confined to the empirical–statistical side of the approached problems due to the unchanged theoretical position which tries to transform all recorded data on the input and output of industries in such a way that the simple technological interpretation of type (∗) can be applied. The methodological development of input–output analysis thus shows an improvement in the systematic treatment of monetary coefficients that is not accompanied by an equally explicit understanding of its results on the side of physical units.

In what follows we shall adopt the SNA by means of the notation used in Armstrong (1975). We shall assume that the reader has some familiarity with the basic schematic matrix arrangements and rearrangements of this system, but shall, of course, provide a short description of the concepts and symbols which will be adopted. The suggested notation in particular employs the following general conventions:

Matrices will be characterized by capital letters $A, B, \ldots$ in general. A prime ($'$) superscript is used to indicate transposition. Vectors (without a prime) represent column vectors. The symbol $\hat{\cdot}$ above a vector is used to indicate the diagonal matrix that can be formed by means of this vector. The summation vector i is defined by $(1, \ldots, 1)'$ – of appropriate dimension in each case. The matrix $I = \widehat{i}$ then gives the identity matrix (already employed). The jth column and the kth row of a matrix $A = (A_{kj})$ are denoted by A^j and A_k, respectively implying that $I^1, \ldots, I^n$ can be used to represent the canonical basis of $\Re^n$ if $i \in \Re^n$. Finally, the symbols "$\geq$", "$>$" are used to denote non-negativity, semi-positivity and strict positivity for the difference of the vectors (and matrices) that are thereby compared.

The data base of the accounting framework employed in the SNA is given by the so-called make- and absorption-matrices M and X, which by their columns M^j and X^j represent the total output and input of commodities of industry j in monetary terms for the year under consideration. The convention for M^j just described – and this is a point that the reader is asked to keep in mind throughout the chapter – represents a *digression*, the *only one* we will allow for, from the notation employed by Armstrong (1975). To ease theoretical presentation it is recommendable to break with the accounting convention which represents M^j by a row in contrast to the inputs X^i, a column. This implies that, e.g., net output can be denoted simply by $M^j - X^j$ instead of $(M^j)' - X^j$, which would be the correct way if Armstrong's (1975) notation were applied without this slight, single change in convention.

Let m denote the number of industries j and n the number of commodities k represented by the two matrices X and M, numbers which are normally set equal as far as practical classifications and computations are concerned – a practice we shall adopt unless the opposite is stated explicitly. We have already presented some arguments in the introduction as to why problems of aggregation should be excluded from the present study. On the basis of this exclusion, the above data X^j and M^j of industries j can be translated again into physical terms in a very simple way. Let us denote the vector of prices of commodities $1, \ldots, n$ by $p = (p_1, \ldots, p_n)' \in \Re^n$. The vectors $\overline{X}^j = \widehat{p}^{-1} X^j$ (or equivalently $\widehat{p}\overline{X}^j = X^j$)

and $\overline{M}^j = \widehat{p}^{-1}M^j$ $(\widehat{p}\overline{M}^j = M^j)$ then describe the physical inputs and outputs of industry j in the year of report and give rise to physical absorption- as well as make-matrices $\overline{X} = \widehat{p}^{-1}X$ and $\overline{M} = \widehat{p}^{-1}M$.[1] This immediate correspondence between monetary and physical data will allow us to use the terms "industry", "activity", "method of production", etc., interchangeably. We shall employ the symbol $\bar{\cdot}$ throughout to denote the physical analog of a monetary expression of Armstrong's (1975) set of symbols, whenever it exists. But, though prices p are shown in all calculations where they are involved, no analysis of changing prices will be presented in this chapter. Prices p will remain exogenous; yet it is not sensible to set them equal to one through an appropriate choice of physical units since their explicit role in the proposed reallocations is an important source of information, and, of course, a necessary prerequisite for any subsequent analysis of intertemporal changes.

The number of energy commodities explicitly considered in the input–output applications we have so far cited ranges from "2" in Krenz (1977) to "21" in James (1980) and may in both cases also be aggregated into one single "energy commodity" by means of the common unit of measurement petajoule (or Btu, etc.) employed in these works (see Krenz 1977, p. 121 for an example). Yet it is well-known that the formal principles for measuring the energy requirements of commodities remain the same regardless of whether two, five, 21 or only one aggregated energy commodity are taken into consideration (since their treatment by matrices with a corresponding number of rows can be dissolved into a separate treatment of each single row). It suffices therefore to restrict our considerations to only one energy commodity, which will be given the index "1" in all that follows.

For purpose of completeness and for later reference let us conclude here with the following list of concepts and assumptions (see Armstrong 1975, p. 71ff. for further details):

$$\begin{aligned} g' = i'M > 0 &= \text{ vector of the output values of industries} \\ \overline{q} = \overline{M}i > 0 &= \text{ vector of total commodity outputs} \\ B = X\widehat{g}^{-1} \geq 0 &= \text{ unit cost structure of industries} \\ C = M\widehat{g}^{-1} \geq 0 &= \text{ matrix of product mixes and} \\ D = \overline{M}'\widehat{\overline{q}}^{-1} \geq 0 &= \text{ matrix of market shares} \end{aligned}$$

7.3 Energy Consumption

Consider now as given the two matrices $\overline{X}, \overline{M} \geq 0$ of order $n \times n$, which we derived in the previous section from the basic matrices X and M of the schematic input–output data arrangements of the SNA and which by our slight deviation from the conventions represent by their jth columns the inputs and outputs of commodities

[1] We thereby dispense with the "homogeneity assumption" customarily made (see United Nations 1973, p. 20).

$k = 1, \ldots, n$ of the given range of industries $j = 1, \ldots, n$. Note that by our very point of departure there is no reason to expect that these data can be reduced to the single matrix $\overline{A}$ of the purely technological kind postulated in connection with $(*)$. On the contrary, the situation given above in particular may include all types of joint production where, of course, no such reduction can be hoped for. Assuming two matrices $\overline{X}$ and $\overline{M}$ instead of only one $(\overline{A})$ for grasping the technological situation that underlies the recorded matrices X and M is but the logically compelling theoretical counterpart of this observed and widely accepted situation, and thereby the correct starting point for pursuing our aim of analyzing that which is measured by the expressions A^*_{1j} in $(*)$ under the assumed circumstances. To perform this task we shall reverse the order of analysis in the sense that we shall start from the presentation of a physically based measure of total energy requirements here and in the following section in order to subsequently examine what kind of monetarily based measurement these coefficients will give rise to.

Definition 7.1. Let us denote by $\Delta i = (\Delta i_1, \ldots, \Delta i_n)'$ a change in the activity levels of industries $j = 1, \ldots, n$. Then the *change in the consumption of energy*, i.e., here of commodity 1, induced by this change in activity levels is defined by

$$\overline{X}_1 \Delta i = \sum_{j=1}^{n} \overline{X}_{1j} \Delta i_j,$$

and is ascribed to the accompanying change in net output,

$$\Delta \overline{f} = (\overline{M} - \overline{X}) \Delta i \in \Re^n,$$

in total.

Note that by this definition only the (intermediate) consumption of energy of the producing sector is taken into account, leaving out the amount $\Delta \overline{f}_1 = (\overline{M} - \overline{X})_1 \Delta i$ that is produced for other purposes (final demand broadly defined). Adding both amounts $\overline{X}_1 \Delta i + \Delta \overline{f}_1 = \overline{X} \Delta i + (\overline{M}_1 - \overline{X}_1) \Delta i$ establishes the measure $\overline{M}_1 \Delta i$, which could have equally well been used in place of the one proposed above in all that follows. Note further that this definition by its very formulation contains the *implicit assumption* of fixed input and output coefficients – up to variations of scale. Note finally that due to our choice of starting point $\overline{X}, \overline{M}$, the activity levels originally prevailing must be described by $i = (1, \ldots, 1)'$, and thus a constraint of kind $\Delta i \geq -i$ is involved in the above definition (the constraint need not be of this particular form if the original activity levels i are recalculated to allow for greater scales by taking smaller output baskets as units).

To simplify our following computations additional assumptions will be made which, however, are implied in the SNA's application of the "commodity technology" hypothesis to be used in the following.

Assumption 1. The inverse matrices of the two matrices $\overline{M}$ and $\overline{M} - \overline{X}$, i.e., $\overline{M}^{-1}$ and $(\overline{M} - \overline{X})^{-1}$, are assumed to exist for all square systems $\overline{M}, \overline{X}$ that are considered in this section.

Consequently $\overline{\Delta i}$ can be uniquely calculated from any given change in net output $\Delta \overline{f}$ by means of $\Delta i = (\overline{M} - \overline{X})^{-1} \Delta \overline{f}$, which reverses the order of assignment used in the Definition 7.1.

Definition 7.2. The vector $\overline{r} \in \Re$ defined by $\overline{r}' = \overline{X}_1 (\overline{M} - \overline{X})^{-1}$ is said to describe – through its corresponding components – the *total energy requirements* necessary for the production of one (extra) unit of commodities $k = 1, \ldots, n$.

This definition finds justification through the following:

Proposition 7.3. $\overline{r}_k = \overline{X}_1 \Delta i^k$, $\Delta i^k = (\overline{M} - \overline{X})^{-1}, k = 1, \ldots, n$. *The kth component of the vector $\overline{r}$ thus exactly describes the extra energy consumption that results from the change in activity levels Δi which is necessary to satisfy the postulated rise in final demand $\Delta \overline{f}$ by one unit of commodity k: $\Delta \overline{f} = l^k$ (independent of the levels of activity and final demand that form the basis of this change).*

Proof. $\overline{r}_k = \overline{r}' I^k = \overline{r}'(\overline{M} - \overline{X}) \Delta i^k = \overline{X}_1 \Delta i^k$.

Proposition 7.4. *The vector $\overline{r}$ will be non-negative if and only if for every Δi that fulfills $\Delta \overline{f} = (\overline{M} - \overline{X}) \Delta i \geqq 0$ we have $\overline{X}_1 \Delta i \geqq \underline{0}$, i.e., if and only if any change in the economy's activities which raises net output $\overline{f}$ cannot lower the energy consumption associated with it.*

Proof. From Definition 7.2 follows $\overline{r}' \Delta \overline{f} = \overline{X}_1 \Delta i$ for all $\Delta i = (\overline{M} - \overline{X})^{-1} \Delta \overline{f}$ which implies the assertion by means of Proposition 7.3. □

The vector $\overline{r}$ thus contains negative components exactly in those cases where there exist vectors of activity levels i^1 and i^2 such that

$$f^2 = (\overline{M} - \overline{X}) i^2 \geqq (\overline{M} - \overline{X}) i^1 = f^1 \text{ and } \overline{X}_1 i^2 < \overline{X}_1 i^1.$$

The reader accustomed to the behavior of single-product systems and Leontief models may wonder whether there can exist any reasonable case where some of the components of $\overline{r}$ may become negative. To demonstrate the plausibility of such an occurrence consider the following:

Example.

$$\overline{X} = \begin{pmatrix} 3 & 0.2 & 0.2 & 8 \\ 0 & 2 & 3 & 16 \\ 0 & 4 & 2 & 4 \\ 15 & 2 & 3 & 0 \end{pmatrix} \qquad \overline{M} = \begin{pmatrix} 12 & 2 & 2 & 0 \\ 12 & 10 & 0 & 0 \\ 2 & 0 & 10 & 0 \\ 0 & 0 & 0 & 30 \end{pmatrix}$$

With regard to our vector of initial intensities $i = (1, 1, 1, 1)'$ – the choice of which, however, is of no particular relevance for the arguments that follow – we have

$$\overline{X} i = (11.4, 21, 10, 20)' \text{ and } \overline{M} i = (16, 22, 12, 30)',$$

i.e., the above system is *physically viable* utilizing in particular 11.4 units of "energy" to produce, among others, 16 units of "energy". Now consider the change in intensities Δi equal to $(-1, 3, 2, 0)'$. We then have

$$(\overline{M} - \overline{X})\Delta i = (0, 6, 2, 3)' \text{ and } \overline{X}_1 \Delta i = -2,$$

which implies a saving of energy despite the unambiguous increase in the bundle of commodities available for final demand. According to Proposition 7.4 at least one of the components of the vector $\overline{r}$ in this example hence has to be negative, as remains to be demonstrated.

The explanation of this result lies in the fact that the methods of production "2" and "3" taken in conjunction, e.g., with the operating levels "three" and "two" are more efficient than the single method "1", e.g., they then need less input and produce more output than is achieved by operating activity "1" on the level "one". It is the occurrence of such situations – which as the above example shows cannot be excluded on *a priori* grounds – that is reflected by the existence of (some) negative coefficients in the vector $\overline{r}$ of the energy consumption of commodities, situations which in general may be far from being obvious if high-dimensional joint production systems are considered.

Proposition 7.5. *Define matrix A by $A = BC^{-1}$ (with $g' = i'M, B = X\widehat{g}^{-1}$, $C = M\widehat{g}^{-1}$),*[2] *i.e., the matrix A is the input–output table derived from monetary data X and M by means of the so-called "commodity technology" assumption of the SNA's input–output methodology. For the monetary clothing r of the vector $\overline{r}$ of Definition 7.2 we then have:*

$$r' = p_1 \overline{r}\,\widehat{p}^{-1} = A_1 A^* = A_1(I - A)^{-1} = ((I - A)^{-1} - I)_1$$

Proof.

$$\begin{aligned} A = BC^{-1} &= (X\widehat{g}^{-1})(M\widehat{g}^{-1})^{-1} = X\widehat{g}^{-1}\widehat{g}M^{-1} \\ &= XM^{-1} = \widehat{p}\,\overline{X}\,\overline{M}^{-1}\widehat{p}^{-1} \end{aligned} \tag{7.1}$$

From (7.1) there then follows:

$$\begin{aligned} A_1(I - A)^{-1} &= p_1 \overline{X}_1 \overline{M}^{-1}\widehat{p}^{-1}(\widehat{p}(I - \overline{X}\,\overline{M}^{-1})\widehat{p}^{-1}) \\ &= p_1 \overline{X}_1 \overline{M}^{-1}\widehat{p}^{-1}\widehat{p}(I - \overline{X}\,\overline{M}^{-1})\widehat{p}^{-1} \\ &= p_1 \overline{X}_1 \overline{M}^{-1}(\overline{M}\,\overline{M}^{-1} - \overline{X}\,\overline{M}^{-1})^{-1}\widehat{p}^{-1} \\ &= p_1 \overline{X}_1 \overline{M}^{-1}\overline{M}(\overline{M} - \overline{X})^{-1}\widehat{p}^{-1} \\ &= p_1 \overline{X}_1 (\overline{M} - \overline{X})^{-1}\widehat{p}^{-1} = p_1 \overline{r}\,\widehat{p}^{-1}, \end{aligned} \tag{7.2}$$

[2] See the final remark in Sect. 7.2 for a description of these symbols.

by Definition 7.1. Moreover:

$$(I - A) + A = I \text{ implies } I + A(I - A)^{-1} = (I - A)^{-1} \tag{7.3}$$

which quite generally gives

$$A(I - A)^{-1} = (I - A)^{-1} - I,$$

independent of the confinement to the first row of these two matrices which is considered in the last equation of $(***)$. Equations (7.2) and (7.3) taken in conjunction then prove the whole assertion since by Assumption 1 all inverse matrices employed in the above proof in fact exist. □

Proposition 7.5 states that our physical measure of energy consumption $\overline{r}$ can be obtained from an ordinary Leontief table in the customary way $(*)$, provide that this table has been constructed by means of the above "technology" assumption, which – briefly characterized – assumes *that a commodity has the same cost structure in whichever industry it in fact is produced*.[3] The part of table $\overline{A}$ in $(*)$ is now played by the matrix $\overline{XM}^{-1}\widehat{p}^{-1}$ [as equations (7.1) in the above proof shows] – a matrix which in general, i.e., up to the special case where the "commodity technology" assumption in fact represents a true description of reality, no longer will be interpretable in physical terms. Yet the applicability of the matrix $A = \widehat{p}\overline{XM}^{-1}\widehat{p}^{-1}$ to the determination of energy requirements $\overline{r}$ as shown by $(***)$ is not confined to cases where the "commodity technology" assumption makes sense, but remains possible as long as Assumption 1 can be established. We simply do not have to bother about the kind of technological relationships that lie behind the given make- and absorption-matrices M and X (at least as long as the implicitly involved assumption of constant returns to scale is not put to question).

Furthermore, the computation of energy requirements $\overline{r}$ by means of the matrix $A = BC^{-1} = XM^{-1}$, i.e., by $X_1M^{-1}(I - XM^{-1})^{-1}$ (see Proposition 7.5) is but a mathematical detour to obtain a situation that pretends to be of the conventional Leontief type $(*)$.[4] In this regard, (7.2) in the above proof clearly demonstrates that the computation of $\overline{X}_1(\overline{M}-\overline{X})^{-1}$ or $X_1(M-X)^{-1}$ is all that is needed to determine the vectors $\overline{r}$ or r, respectively, thereby showing that a direct computation of such energy requirements from the basic data X and M is possible and recommendable.

Finally, in applying the "commodity technology" assumption it has been observed since the early work of Stone et al. (1963) that negative entries may appear in the thereby derived input–output table A. Having the simple Leontief model in mind, the general attitude toward their appearance has been to remove them in a second step by more or less sophisticated mathematical methods to obtain a final matrix A,

[3] Postmultiplying the matrix B of unit-cost structures of industries by C^{-1} (to obtain the table A) just converts these structures into the unit cost structures of commodities.

[4] This, however, is devoid of economic content as no meaningful interpretation of the matrix XM^{-1} will exist in general (see also the following examples).

which at least does not immediately disqualify the applicability of a technological interpretation of kind $(*)$. This illusion with regard to the technological nature of the matrix A continues to exist to this day, as demonstrated by the following quotation from James (1980, p. 178): "...the mechanical methods of deriving input–output matrices from make and absorption matrices described above do not always work smoothly. For example, negative coefficients sometimes appear. Methods of overcoming this problem can, however, be applied", methods – it must be added – which just endanger the use that can be made of this table A as shown by our Propositions 2 and 3! Such methods are surveyed in Armstrong (1975, p. 79ff.) where it is also stated that it is basically Stone's procedure (to replace all negative entries simply by zeros) plus some further consistency considerations that have been used in computing the input–output table A of the UK economy for 1963.

Example. Regarding the example of this section, the matrix A of Proposition 7.5 is given by

$$A = \frac{1}{13{,}800}\begin{pmatrix} 4{,}080 & -540 & -540 & 3{,}680 \\ -4{,}500 & 3{,}660 & 5{,}040 & 7{,}360 \\ -7{,}800 & 7{,}080 & 4{,}320 & 1{,}840 \\ 18{,}000 & -840 & 540 & 0 \end{pmatrix} \approx \begin{pmatrix} 0.296 & -0.039 & -0.039 & 0.267 \\ -0.326 & 0.265 & 0.365 & 0.533 \\ -0.565 & 0.513 & 0.313 & 0.133 \\ 1.304 & -0.061 & 0.039 & 0.000 \end{pmatrix}$$

(where p is assumed to equal $(1, 1, 1, 1')$ for simplicity).

By means of $I_1((I - A)^{-1} - I)$ this matrix leads to

$$r' = \overline{r}' \approx (1.75, -0.44, -0.36, 0.45) \tag{7.4}$$

as the correct vector of energy requirements of commodities $1, \ldots, 4$. Removing all negative entries from the above matrix A (and neglecting Armstrong (1975) additional consistency considerations for simplicity), however, would lead to the, in this case necessarily, non-negative vector

$$r' = \overline{r}' \approx (1.94, 0.05, 0.07, 0.82). \tag{7.5}$$

With respect to this example we already know that an additional net output of $\Delta\overline{f} = (0, t, 2, 3)'$ is associated with a saving of "energy" of amount "2" and a change in operating levels i given by $(-1, 3, 2, 0)'$. From Definitions 1 and 2 it is furthermore obvious that $\overline{r}'\Delta\overline{f}$ must be equal to $\overline{X}_1\Delta i$ (quite generally), an equality that is confirmed by (7.4). With regard to(7.5), however, we obtain $\overline{r}'\Delta\overline{f} = 2.93$, which suggests wrongly that an extra energy requirement of 2.93 units would be necessary to produce the increase $(0, 6, 2, 3)'$ in net output $\overline{f}$.

The above equation $\overline{r}'\Delta\overline{f} = \overline{X}_1\Delta i$ (and also $\overline{r}'\overline{f} = \overline{r}'(\overline{M} - \overline{X})i = \overline{X}i)$ for totals of industrial energy consumption based on a single input–output table $\overline{A}$ of conventional, and thus obscure, type is applied in Folk and Hannon (1974, p. 163) to compute the total energy requirements for different scenarios of final demand – an undertaking which, in the last instance, requires the application of the "commodity technology" assumption, as we have shown.

In sum, negative entries are to be expected not only in the theoretically misleading input–output table A where they have been frequently observed,[5] but also for the measure $r = A_1(I - A)^{-1}$ – though with much less probability then.[6] See Armstrong's (1975) appendix for an example of the first, where one can easily check that

$$(M - X)^{-1} \approx 0.0001 \begin{pmatrix} 14.0 & 3.8 & 0.3 \\ 2.0 & 5.1 & 0.4 \\ 2.3 & 1.0 & 7.8 \end{pmatrix} \tag{7.6}$$

and

$$A^* = (I - A)^{-1} \approx \begin{pmatrix} 1.259 & 0.341 & 0.026 \\ 0.727 & 1.477 & 0.190 \\ 0.480 & 0.290 & 1.483 \end{pmatrix} \tag{7.7}$$

are positive, despite the occurrence of one negative entry in his input–output table $A(\times 1{,}000)$,

$$A = \begin{pmatrix} 87 & 213 & -11 \\ 422 & 207 & 94 \\ 213 & 86 & 311 \end{pmatrix}$$

derived from

$$X = \begin{pmatrix} 10 & 60 & 0 \\ 40 & 60 & 20 \\ 20 & 30 & 60 \end{pmatrix} \text{ and } M = \begin{pmatrix} 90 & 0 & 0 \\ 10 & 280 & 10 \\ 0 & 20 & 190 \end{pmatrix},$$

facts which are obviously due to the small amount of subsidiary production in the make-matrix chosen by Armstrong. Yet, though we have a normal-looking Leontief inverse (7.7′) in this case, this does not imply the existence of a meaningful interpretation for the matrix A in addition to that which exists for matrix (7.6) (see Proposition 7.3). And, adding to this difficulty, there is no obvious theoretical reason which automatically ensures the *technologically efficient* use of the given energy resource (in the sense of Proposition 7.4) under all *economically viable* circumstances, i.e., even situation (7.4) cannot be dismissed on *a priori* grounds. Finally, the possible excuse that the negative entries which occur in practice are of no great importance is of no help in this case, since such an argument only leads to the conclusion that there is no use in supplying a systematic treatment of joint and other kinds of multiple products (as done in the United Nations' SNA).

In spite of all difficulties, however, the recommendation for avoiding faulty interpretations or manipulations is quite simple. Having acknowledged the

[5] See Armstrong (1975, p. 78ff.).

[6] This is because (at least at present) voluminous negative entries of A (and thus negative entries in A^*) are not likely to be reported in the light of the methods currently applied – a fact, which, however, does not deprive the hypothetical situation we have exemplified above of its conceptual importance for a proper development on input–output analysis.

meaningfulness of starting from a data base of the kind X and M, it is only sensible to act accordingly with regard to the underlying technology, i.e., to apply $\overline{X}$ and $\overline{M}$ to interpret the computations performed (and not $\overline{A} = \overline{XM}^{-1}$, in an attempt to rescue the conventional Leontief model despite the more general observations made).

7.4 Energy Costs

To derive an alternative to the concept of energy consumption given by Definition 7.2, let us start again from known physical make- and absorption-matrices $\overline{M}$ and $\overline{X}$ of order $n \times m$, with no restriction placed now on the number n of commodities and the number m of industries.

The definition of energy consumption in the previous section was characterized by technological considerations throughout, since all prices involved in its conventional monetary measure $r = A_1(I - A)^{-1}$ could be made to disappear by simple and economically reasonable algebraic operations. Furthermore, no non-negative property was needed for this technological characterization of energy consumption to be meaningful.

This will change with regard to the now proposed economically motivated concept of "energy costs" where by choice of terminology the non-negativity of the coefficients to be measured is considered to represent one of their basic features. By Definition 7.2 we get for the total energy requirements of activity j the equation

$$\overline{r}_1\overline{M}_{1j} + \ldots + \overline{r}_n\overline{M}_{nj} = \overline{r}_1\overline{X}_{1j} + \ldots + \overline{r}_n\overline{X}_{nj}, \quad j = 1, \ldots, m. \tag{7.8}$$

It is tempting to regard the set of these equations as an equation system which determines the energy *costs* (or energy *contents* of commodities $1, \ldots, n$ if the "primary" input $\overline{X}_{1j}$ on the right-hand side of (7.8) is interpreted as an input from "nature" and $\overline{r}_1 X_{1j}$ to represent the energy costs of the input 1 of industry j measured in terms of such primary "natural" units. Such an interpretation of (7.8) seems possible because of their close resemblance to standard procedures employed in the field of cost-price determination. There is, however, one problem involved in this straightforward economic reinterpretation of the components of the vector $\overline{r}$. Of course, total costs $\overline{r}_1\overline{M}_{1j} + \ldots + \overline{r}_n\overline{M}_{nj}$ of finished goods are computed by accumulating manufacturing costs (here in terms of the assumed input from "nature"). But in the presence of joint products[7] we have the problem of *common costs* and there the majority opinion is – as far as conventional cost accounting is concerned – that "to some extent all methods of allocating joint costs to individual co-products rest upon opinion rather than upon objective measurements" (Dickey 1960, 13.7). It is to be expected, then, that we will be in a better position when approaching the problem of energy costs under such circumstances from an economy-wide perspective, and that

[7] Not used here in the narrow sense of fixed output proportions (see Proposition 7.12).

(7.8) will provide us with a basis for an "objective measurement" in this particular situation? We do not believe this to be the case and propose as evidence for our opinion the results that have been obtained in Sect. 7.3 with regard to this measure (7.8).

In our view the need for practicable economic conventions cannot be avoided in the field of energy cost accounting of single commodities too, if joint products and common costs are involved – which thus bring in an element of arbitrariness to be closed by an appropriate definition. There are several possibilities for evaluating the different techniques of joint cost accounting that are in use, namely: (a) by the degree to which they are actually employed, (b) by their degree of generality in covering existing as well as in principle possible situations, and, last but not least, (c) by the use to which they can be put – for making a selection which suits our purposes best.

Since we do not have the space here to enter into a satisfactory discussion on managerial cost accounting, let it suffice with respect to point (a) to point to the remarks made in Dickey (1960, 13.11ff.) and in Matz and Usry (1976, p. 189) in favor of the so-called *market value method* of allocating joint costs. Regarding point (b) this method can furthermore claim to be applicable to all kinds of joint products, quite independent of the technological or economic peculiarities involved in each single case. Hence there remains point (c) – in any case the most important one – to be considered when arguments for defending the customary and, in particular, our own application of the market value method to the problem of joint costs are needed. Such a defence of the way by which we shall now use this method in the question of energy cost accounting will be postponed here to Sect. 7.5, where the (dis)advantage of "energy costs" in comparison with last section's measure of energy consumption is investigated.

The *market value method* functions by apportioning joint costs, e.g., of the process j considered in (7.8), to the different units of costing "k" in relation to their relative market values $p_k \overline{M}_{kj} / p' \overline{M}^j$, i.e., by means of the entries C_{kj} of the jth column of the matrix C which we have defined in Sect. 7.2. Applied to our problem of total energy cost determination this then leads to the following modification of "cost equations" (7.8):

$$\overline{c}_k^j \overline{M}_{kj} = C_{kj}(\overline{c}' \overline{X}^j + \overline{X}_{1j}), \quad k = 1, \dots, n, \quad j = 1, \dots, m, \tag{7.9}$$

(and $\overline{c}_k^j = 0$ if $\overline{M}_{kj} = 0$, i.e., if k is not produced by j).

In these equations, the coefficients $\overline{c}_k^j$ describe the *individual energy costs* of commodity k induced by process j, while the vector $\overline{c} = (\overline{c}_1, \dots, \overline{c}_n)'$ represents the *average energy costs* of commodities $1, \dots, n$. The sum $\overline{c}' \overline{X}^j + \overline{X}_{1j}$ consequently represents the average energy costs that fall upon process j, which by means of (7.9) are then apportioned to its output components $\overline{M}_{kj}$ by taking relative market values C_{kj} as weights.[8] It is the introduction of the terms C_{kj} (and thereby also

[8] Note that by this procedure it is not necessary to know the industrial origin of the inputs (or the destination of the outputs).

of individual energy costs $\overline{c}_k^j$) which distinguishes (7.9) from (7.8) (and coefficients $\overline{c}_k$ from $\overline{r}_k$), but which also brings about the situation that the equation system (7.9) now contains may more unknowns than equations. This indicates that the description of individual and average energy costs of commodities has not yet been finished. To complete this description it is evident that the way these two types of costs relate to each other remains to be determined.

The simplest way to perform this task, which at the same time makes the best of a considerable lack of information, is to utilize the market shares D_{jk} (compare Sect. 7.2) of industries j in the market for commodity k as weights to form the average of individual energy costs of commodity k, which leads to

$$\overline{c}_k = \sum_{j=1}^{m} D_{jk}\overline{c}_k^j, \quad D_{jk} = \overline{M}_{kj} / \sum_{j=1}^{m} M_{kj}, \quad k = 1, \ldots, n. \tag{7.10}$$

By means of such market shares, D_{jk}, the individual efficiency, $\overline{c}_k^j$, with regard to average energy costs is again reduced appropriately to these latter average numbers, which in fact is all that is needed for a completion of (7.9) to make them determinate.

If u_j is the number of products produced by process j, (7.9) and (7.10) lead in their totality to the consideration of $u_1 + \ldots + u_m$ values for individual energy costs plus n values for average energy costs. The task of determining all these values can, however, considerably be simplified by inserting (7.9) into (7.10), which gives the following equation system for (average) energy costs $\overline{c}$ of commodities $k = 1, \ldots, n$:

$$\overline{c}_k = \left(\sum_{j=1}^{m} C_{kj} (\overline{c}'\overline{X}^j + X_{1j}) \right) / \left(\sum_{j=1}^{m} \overline{M}_{kj} \right). \tag{7.11}$$

Translated into matrix notation this finally motivates the following:

Definition 7.6. The energy costs of commodities $k = 1, \ldots, n$ are determined by

$$\begin{aligned} c' &= c'\overline{X}C'\widehat{\overline{q}}^{-1} + \overline{X}_1 C'\widehat{\overline{q}}^{-1} \qquad \overline{q} = \overline{M}i \\ &= \overline{c}'\overline{A} + \overline{A}_1, \qquad \overline{A} = \overline{X}C'\widehat{\overline{q}}^{-1} \end{aligned} \tag{7.12}$$

where the matrix C is identical with the one introduced in Sect. 7.2.

We have thus succeeded in obtaining an equation system of $n \times n$ type, uniquely solvable in principle, which replaces Definition 7.2 which we used for the determination of the energy requirements of commodities in Sect. 7.3. To demonstrate the solvability of (7.12) we shall employ the following:

Assertion 4. $p'\overline{X} < p'\overline{M}$, i.e., value-added is positive for all industries $j = 1, \ldots, m$.

We shall employ this assumption in place of Assumption 1 used for the purposes of Sect. 7.3.

Furthermore, and in contrast to the in general purely artificial matrix $\overline{A} = \overline{XM}^{-1}$, we have now introduced a matrix $\overline{A} = \overline{X}C'\widehat{\overline{q}}^{-1}$ of commodity $\times$

commodity type which not only is non-negative by definition, but which also allows a sensible interpretation for each constellation of the numbers n and m with regard to all kinds of technological situations (the same, of course, holds true with regard to the vector of "primary" inputs $\overline{X}_1 C'\widehat{\overline{q}}^{-1}$), to be made in the following way: By (7.12) we have for the kth column $\overline{A}^k$ of $\overline{A}$ the equation

$$\overline{A}^k = \sum_{j=1}^{m} C_{kj}\overline{X}^j / \overline{M}_{ki}, \tag{7.13}$$

the numerator of which gives the vector of total inputs that are allocated to the total production of commodity k by means of relative market values $0 \leq C_{kj} \leq 1$ attached as weights to the total input vectors $\overline{X}^j$ of industries j. Subsequent division by $\overline{M}_{ki} = \sum_{j=1}^{m} \overline{M}_{kj}$, the total production of commodity k, then, of course, expresses this input vector per unit of output of commodity k and gives the vector $\overline{A}^k$.

On the basis of Assumption 4, we now have:

Proposition 7.7. *Define the matrix A by $A = BD$, where the $n \times m$ matrix B is defined as in Proposition 7.5 and the $m \times n$ matrix D by the coefficients D_{jk} of (7.10), i.e., the matrix A is the input–output table derived from monetary data X and M by means of the so-called "industry technology" assumption*[9] *of the SNA's input–output methodology. For the monetary clothing c of the vector $\overline{c} \in \Re^n$ of Definition 7.6 we then have*

$$c' = p_1\overline{c}'\widehat{p}^{-1} = c'A + A_1, \quad i.e., \quad c = A_1(I - A)^{-1}. \tag{7.14}$$

Proof. By Definition 7.2 there holds

$$\begin{aligned} c' &= p_1\overline{c}'\widehat{p}^{-1} = p_1\overline{c}'\overline{X}C'\widehat{\overline{c}}^{-1} + p_1\overline{X}_1 C'\widehat{\overline{q}}^{-1}\widehat{p}^{-1} \\ &= c'\widehat{p}\overline{X}C'\widehat{q}^{-1} + p_1\overline{X}_1 C'\widehat{q}^{-1} = c'X\widehat{g}^{-1}\widehat{g}C'\widehat{q}^{-1} + X_1\widehat{g}^{-1}\widehat{g}^{-1}\widehat{g}C'\widehat{q}^{-1}, \end{aligned}$$

where g, q are defined by $i'M$ and $\widehat{p}\overline{q}$, respectively ($\overline{q} = \overline{M}i$). By the definitions made in Proposition 7.5 these equations then give rise to

$$c' = c'BM'\widehat{q}^{-1} + B_1 M'\widehat{q}^{-1} = c'BD + B_1 D = c'A + A_1,$$

because

$$D = \overline{M}'\widehat{\overline{M}i}^{-1} = \overline{M}'\widehat{p}\widehat{p}^{-1}\widehat{\overline{M}i}^{-1} = m'\widehat{q}^{-1},$$

for the matrix $D = (D_{jk})$ defined in (7.10), proving the first part of the proposition. To prove the second part, i.e., the invertibility of the matrix A, it suffices to realize that

[9] This, in contrast to the "commodity technology" assumption, assumes that all commodities produced by an industry are produced with the same input structure (see Armstrong 1975, p. 71, and compare again Sect. 7.2 with regard to the notation employed).

$$p'\overline{X} < p'\overline{M} \text{ implies } i'A < i'.$$

This follows, since $p'\overline{X} < p'M$ (Assumption 4) is equivalent to

$$i'X < i'M \text{ or } i'X\widehat{i'M}^{-1} < i' \text{ or } i'B < i',$$

which implies

$$i'A = i'BD < i'D = i'$$

by the very definition of D [see (7.10] and regard the non-negativity of all the matrices involved). By well-known assertions on such non-negative matrices (see, e.g., 6.3 in Nikaido 1968, p. 95) we therewith know that the matrix $I - A$ is non-negatively invertible, i.e., the Leontief inverse A exists (and is non-negative).

Proposition 7.8.

(1) $A = \widehat{p}\overline{A}\widehat{p}^{-1}$, $\overline{A} = \overline{X}C'\widehat{\overline{q}}^{-1}$,
(2) c and $\overline{c}$ *are uniquely determined and non-negative, and positive if* $\overline{X}_1 > 0$.[10]
(3) Individual energy costs $\overline{c}^j_k$ *can be determined on the basis of the computed vector* $\overline{c}$ *by means of (7.9).*
(4) For any bundle of commodities $\overline{b} \geq 0$*, we have* $p_1\overline{c}'\overline{b} = c'b$ *where b – as usual – is defined by* $\widehat{p}\overline{b}$.
(5) $\overline{c}'(\overline{M} - \overline{X})i = \overline{X}_1 i$*, i.e., the energy costs of net output are equal to the amount of energy that is consumed in its industrial production.*

Proof. Assertion (1)–(4) are immediate consequences of Proposition 7.7 and its proof. To prove (5), note that by (7.11) we have

$$\overline{c}_k(\overline{M}i)_k = \sum_{j-1}^{m} C_{kj}(\overline{c}'\overline{X}^j + \overline{X}_{1j}).$$

Summation over k then implies

$$\begin{aligned}\overline{c}'\overline{M}i &= \sum_{k=1}^{n} \overline{c}_k(\overline{M}i)_k = \sum_{k=1}^{n}\left(\sum_{j=1}^{m} C_{kj}(\overline{c}'\overline{X}^j + \overline{X}_{1j})\right) \\ &= \sum_{j=1}^{m}\left(\sum_{k=1}^{n} C_{kj}\right)(\overline{c}'\overline{X}^j + \overline{X}_{1j}) = \sum_{j=1}^{,}(\overline{c}'\overline{X}^j + \overline{X}_{1j}) \\ &= \overline{c}'\overline{X}i + \overline{X}_1 i.\end{aligned}$$

[10] Neither this assumption nor Assumption 4 are of the weakest possible kind to allow for these conclusions. This topic is, however, not central to our problem of analyzing the two different concepts of energy requirements $\overline{r}$ and $\overline{c}$ defined here. Weaker assumptions than the one considered above (which assumes that energy is a direct input with regard to every industry) can be obtained, e.g., from the analysis of basic commodities that is supplied in Flaschel (1982). For simplicity, however, the assumption $\overline{X}_1 > 0$ is retained for the remainder of this chapter.

□

Example. With regard to the example of the previous section we here obtain as input–output table $A = \overline{A} = BD$ $(p = i)$,

$$A \approx \boxed{\begin{array}{cccc} 0.090 & 0.071 & 0.033 & 0.267 \\ 0.052 & 0.076 & 0.208 & 0.533 \\ 0.063 & 0.152 & 0.139 & 0.133 \\ 0.485 & 0.390 & 0.304 & 0.000 \end{array}},$$

which by means of

$$c' = \overline{c}' = A_1(I - A)^{-1} = I_1((I - A)^{-1} - I),$$

leads to

$$c' \approx (0.56, 0.51, 0.45, 0.74), \tag{7.15}$$

and, e.g., with respect to the energy commodity "1" to its following individual energy costs:

$$c_1^1 \approx 0.61, \quad c_1^2 \approx 0.38, \quad c_1^3 \approx 0.41, \quad c_1^4 = 0. \tag{7.16}$$

Comparing these results with those obtained for A and r' in Sect. 7.3 [see (7.4)] reveals a remarkable difference between them, in particular with regard to the negative entries allowed for in the latter case, but also with respect to the "corrections" (7.5) that have been used to prevent their occurrence.

By apportioning joint energy costs to individual products by means of (7.9) [replacing (7.8) which were used for the analysis of the previous section], we have managed to establish a notion of such costs which (1) is based on non-negative magnitudes throughout (independent of any relation between n and m), (2) confirms again the procedures (∗) and (∗) that are customarily applied to measure the energy costs of commodities[11] – here under the provision that table $\overline{A}$ of definition e, i.e., the market value method (and thus the "industry technology" assumption) are used as foundations, (3) shows by (2) that it is in fact included among the measures that are actually proposed for use – in particular with regard to the occurrence of joint products (see United Nations 1968, 3.40), and (4) allows the consideration of cost differences in the usage of "energy" [as exemplified by (7.16) with regard to the average energy costs $\overline{c}_1 = 0.56$ per unit of energy commodity 1 produced]! Finally prices p, which have been utilized (by means of C_{kj}) to conduct the proposed

[11] In the light of the demonstrated possibility of reducing both of our measures $\overline{r}$ and $\overline{c}$ to the monetary expression $A_1(I - A)^{-1}$ (depending only on what type of "technology" assumption is being used), the formula (3) which is proposed in James (1980, p. 176) does not appear entirely convincing to us. Translated into our notation it would be based on an expression of the kind $\overline{X}_1\widehat{g}^{-1}(I - A)^{-1}$, that is, one where no full transformation of the vector of energy inputs X_1 (into A_1), corresponding to the complete set of rearrangements made with regard to the matrix A, is accomplished.

apportioning of joint costs and which therefore enter the definition of such energy costs to some extent, have been shown to exercise no influence on the determination of the energy costs of total net production $(\overline{M} - \overline{X})i$ (see Proposition 5(5)).

Despite these achievements it must be admitted that our notion of energy costs is still to this point of an essentially definitional kind when compared with its counterpart, the vector of energy consumption $\overline{r}$, although its factual existence is given whenever the "industry technology" assumption is employed. The actual use to which such an instrument of measure can be put, therefore, still remains to be clarified. Steps into this direction will be undertaken in the now following section.

7.5 Comparing Energy Consumption and Energy Costs

In the preceding two sections we have shown that the two defined notions of energy requirements exhibit essentially the following common characteristics: (1) They satisfy the same formula with regard to their monetary transactions tables A. (2) The accompanying "technology" assumptions are not based on facts of technology in both situations at least as far as joint products are concerned). 3) They are identical if joint products and alternative ways of producing the same commodity are excluded from consideration $\overline{M} = I$; a more general condition which implies this result – by summation of (7.9) over k – is: individual energy costs c_k^{-j} are equal to average energy costs $\overline{c}_k$ whenever k is produced by j.

Despite these common features there are, however, also notable differences between the two definitions presented, as for example with regard to the often employed "physical" analog $\overline{A} = \widehat{p}^{-1} A \widehat{p}$ of the above matrix A. In the case of energy costs $\overline{c}$ it permits an economically sensible interpretation throughout, while it merely represents a mathematical detour in the calculation of energy consumption coefficients $\overline{r}$. A second difference in nature can be illustrated by considering the special case of multiple, but single-product, activities. Energy consumption coefficients $\overline{r}_k$ are (quite generally) independent of changes in operating levels, in market shares, and can then even be defined in this unambiguous way if the number of industries m exceeds n, the number of commodities, by means of the following linear programming problem (compare Definition 7.1):

$$\min\{\overline{X}_1 \Delta i / (\overline{M} - \overline{X}) \Delta i = \Delta \overline{f}, i + \Delta i \geq 0\},$$ [12]

independent of the given change in final demand $\Delta \overline{f}$ (see Lancaster 1968, p. 91ff. and note that the proof of this assertion, i.e., the non-substitution theorem,[13] will

[12] A solution to this program describes the extra energy consumption that is absolutely necessary to allow for the assumed change in final demand with regard to the technological alternatives that are now available.

[13] This theorem is often considered as a justification for the assumption $n - m$ we made in Sect. 7.3.

in general also apply to those multiple-product systems which exhibit the order of magnitude of subsidiary production that is actually observed.) The above invariance properties of the coefficients $\overline{r}_k$ and their purely technological character thus show that they best apply to an (optimizing) investigation of the short run where (hypothetical) shifts in expenditures are examined under the provisional assumption of constant technical conditions. Energy costs $\overline{c}_k$, on the other hand, are defined as pure averages in the above assumed special case (without any reference to prices p then) on the basis of simple summation of the activities which manufacture the same commodity k and a subsequent division of the thereby obtained vector of inputs by the total number of unites k produced [which gives the kth column $\overline{A}^k$ of $\overline{A}$ in this case; see (7.13)]. Energy costs $\overline{c}_k$ thus depend on and specifically summarize the properties of the whole set of given activities and not only those of the energy-minimizing subsystems implied by the non-substitution theorem. They therefore do not possess the above invariance properties of coefficients $\overline{r}_k$ even in this special case, which means that they can at best be applied to an analysis of the factual changes over time of the whole system of existing processes, i.e., to the longer run. Both notions are thus already clearly distinguishable in the special case of multiple activities, complementing each other rather than standing in any direct competition for superiority.[14]

The study of the long-run tendencies of technological change hence may represent the primary filed of exploration for our second notion, the energy costs $\overline{c}$ of commodities. One goal of the remaining pages is to present some fundamentals for such an analysis hoping thereby to point out the relevance of this newly formulated notion of energy costs (in part based on conventional imputations) whose factual existence has already been proved in the last section.

Proposition 7.9. *Define the matrix $\overline{A}^0$ by setting the first row of $\overline{A}$ (see Definition 7.6) equal to zero. The vector $\overline{d} \in \Re^n$ defined by $\overline{d} = (1+\overline{c}_1)^{-1}\overline{c}$ then fulfills*

$$\overline{d}' = \overline{d}'\overline{A}^0 + \overline{A}_1 = \overline{A}_1(I - \overline{A}^0)^{-1}, \quad \overline{d}_1 < 1, \quad \overline{c} = (1-\overline{d}_1)^{-1}\overline{d}.$$

Proof. From Definition 7.6 we get

$$\overline{c} = \overline{c}'\overline{A} + \overline{A}_1 = \overline{c}'\overline{A}^0 + \overline{c}_1\overline{A}_1 + \overline{a}_1 = \overline{c}'\overline{A}^0 + (1+\overline{c}_1)\overline{A}_1,$$

i.e.,

$$(1+\overline{c}_1)^{-1}\overline{c}' = (1+\overline{c}_1)^{-1}\overline{c}'\overline{A}^0 + \overline{A}_1.$$

This implies the first part of the assertion, since $I - \overline{A}^0$ must be non-negatively invertible, because $\overline{A}^0 \leq \overline{A}$ (see Proposition 7.7 and Nikaido 1968, Theorem 7.2(iv)).

[14] Related with this distinction between optimal and average energy requirements is the fact that the latter (but not the former) can be supplemented by the important concept of *individual energy costs* [compare (7.9)].

By the very definition of $\overline{d}$ we furthermore get

$$\overline{d}_1 = \overline{c}_1/(1+\overline{c}_1) < 1,$$

since $\overline{c}_1$ is non-negative. There follows

$$(1+\overline{c}_1)/\overline{c}_1 = 1/\overline{d}_1,$$

i.e.,

$$1/\overline{c}_1 = (1-\overline{d}_1)/\overline{d}_1 \text{ or } \overline{c}_1 = \overline{d}_1/(1-\overline{d}_1),$$

which implies

$$1+\overline{c}_1 = (1-\overline{d}_1)^{-1}.$$

□

The vector $\overline{d}$ defined in Proposition 7.9 thus fulfills an equation system of type (7.12) too, where, however, all intermediate energy input has been removed from these equations. Energy inputs – in contrast to (7.12) – now appear only once and in the form of a "primary" input vector $\overline{x}_1$, implying that the vector $\overline{d}$ represents accumulated (and apportioned) costs, or cost-prices, now *in terms of commodity I* and not in terms of the hypothetical "natural" units utilized to explain the content of the vector $\overline{c}$ in the preceding section. It is, of course, only natural then to expect that the energy costs $\overline{d}_1$ of producing commodity 1 should be less than one (under viable conditions), as we have shown above.

Proposition 7.10. $p_1\overline{d} < p$, *i.e., the energy costs just defined of commodities* $1,\ldots,n$ *are always less than the given market prices* p *of these commodities (if multiplied by* p_1*).*

Proof. By Proposition 7.7 and its proof we already know that $p'\overline{A} < p'$, i.e., that $p'\overline{A}^0 + p_1\overline{A}_1 < p'$. By footnote 10 we in addition get $\overline{A}_1 > 0$, implying that there exists a vector $x > i$ such that

$$p'\overline{A}^0 + p_1\overline{A}_1\widehat{x} = p'.$$

Applying Proposition 7.9 then gives

$$p' = p_1\overline{A}_1\widehat{x}(I-\overline{A}^0)^{-1} > p_1\overline{A}_1(I-\overline{A}^0)^{-1} = p_1\overline{d}',$$

because

$$(I-\overline{A}^0)^{-1} \geq 0.$$

□

Note that the foregoing proof cannot be applied to energy consumption $\overline{r}$ as well because of the negative entries which often appear in the matrix $\overline{A}$ to be employed in this case. This deficiency in the definition of energy consumption also extends to the considerations on technological change which now follows.

Definition 7.11. Technical change is said to be of (purely) *energy-saving, capital-utilizing type* if the following inequalities hold: $U_1^+ \leq U_1$ (energy-saving) and $(U^+)^0 \geq U^0$ (capital-using: a matrix inequality), where U is given by $\overline{X}\widehat{g}^{-1}$, the physical input structure per \$ of output value, where "+" represents the situation after the technical change has occurred, and where U^0 and $(U^+)^0$ are used to denote the matrices obtained from U and U^+ by setting their first rows U_1 and U_1^+ (representing the inputs of commodity 1) equal to zero.

Note that in this distinct situation the problematic cases where the saving of energy is accompanied by a partial saving of capital goods, which makes "viable" (with regard to given prices p) greater amounts of inputs of some of the other capital goods $2, \ldots, n$, are excluded from consideration. Instead, reductions in direct energy input – in some sectors – are always accompanied by unambiguous increases with respect to (some of) the capital goods $2, \ldots, n$ employed by theses sectors [compare (7.17) for more general situations]. Note further that our formulation does not rule out changes in scale (and in product) which may be associated with the given technical change, and that it is by no means clear whether a saving in total energy usage will result from the assumed kind of direct energy input savings, even under conditions of an unchanged final demand, since the necessary extra production of intermediated goods $2, \ldots, n$ by its uses of energy commodity 1 may more than compensate for the assumed reduction in direct energy inputs. To describe situations where this latter possibility can be safely excluded we can now usefully employ our concept of energy costs $\overline{c}$ and thus provide further justification for its somewhat arbitrary definition in Sect. 7.4.

Proposition 7.12. *If the above assumed type of technical change is cost-reducing (viable) with regard to prices p it will be cost-reducing with regard to energy costs $\overline{c}$ and $\overline{d}$ as well,*[15] *provided that the market shares of industries D_{jk} have remained stable.*[16]

Proposition 7.13. *Under the assumptions just made the energy requirements for producing a fixed vector of final demand $\overline{f}$ will fall.*

Proof of Proposition 7.13. By Proposition 5(5) we know the equality

$$\overline{c}'\overline{f} = \overline{c}'(\overline{M} - \overline{X})i = \overline{X}_1 i$$

[15] A similar situation and proposition with regard to single-product activities, prices of production and the primary factor labor is examined in Roemer (1977).

[16] Note in this connection that the employed assumption on cost reduction $p'U^+ \leq p'U$ (a vector inequality in terms of initial prices p solely!) means that the new activities, i.e., the columns of matrix U which actually change, will be regarded as superior and will therefore be adopted by entrepreneurs (if static price expectations prevail). As a consequence of the implied technical change, however, prices p in all probability will be subject to change. But, though this may cause further changes with regard to energy costs $\overline{c}$ which may endanger the derived inequality $\overline{c}^+ \leqq \overline{c}$, this cannot invalidate the assertion made on total energy saving (Proposition 7.13) if no further change in $\overline{c}_1^+$ takes place.

to be true before and after the technical change considered.[17] Now, $\overline{f}$ is assumed to be constant and energy costs $\overline{c}$ unambiguously fall to $\overline{c}^1 \le \overline{c}$ (Proposition 7.12), implying the desired inequality $\overline{X}_1^+ i < \overline{X}_1 i$ by means of the foregoing equality. □

Proof of Proposition 7.12. Utilizing the symbols introduced in Definition 7.11 the assumption on cost reduction can be expressed by

$$p'U^+ \le p'U \text{ or } p'(\Delta U) = p'(U^+ - U) \le 0,$$

(the symbol Δ will always be used to denote differences of the just described kind). Postmultiplying U by $D\widehat{p}$ gives

$$UD\widehat{p} = \overline{X}\widehat{g}^{-1}D\widehat{p} = \overline{X}\widehat{g}^{-1}\overline{M}'\widehat{\overline{q}}^{-1}\widehat{p} = \overline{X}(\widehat{p}\overline{M}\widehat{g}^{-1})'\widehat{\overline{q}}^{-1} = \overline{X}C'\widehat{\overline{q}}^{-1} = \overline{A}$$

(see Definition 7.11 and Sect. 7.2, I–IV). By assumption the matrix D can be applied to U^+ as well, thereby giving rise to

$$p'\Delta UD\widehat{p} = p'\Delta A = p'\Delta A^0 + p_1\Delta A_1 \le 0,$$

because of the non-negativity of the matrix $D\widehat{p}$. The cost reduction assumed with regard to industries j thus is also true for the derived tables $\overline{A}$ and $\overline{A}^+$ describing the "production of commodities by means of commodities". And because of the assumed kind of technical change we furthermore have $\Delta U^0 \ge 0$, i.e., $\Delta\overline{A}^0 \ge 0$. By Proposition 7.10 we then get $p'\Delta\overline{A}^0 \ge p_1\overline{d}'\Delta\overline{A}^0$, i.e., we – a fortiori – will find reduced costs with regard to our initial vector of energy costs $\overline{d}$,

$$p_1\overline{d}'\Delta\overline{A}^0 + p_1\Delta\overline{A}_1 \le 0,$$

or

$$\overline{d}'\Delta\overline{A}^0 + \Delta\overline{A}_1 \le 0,$$

or

$$\overline{d}'(\overline{A}^+)^0 + \overline{A}_1^+ \le \overline{d}'\overline{A}^0 + \overline{A}_1 = \overline{d}'.$$

Let us now define x^1 by $\overline{d}'(\overline{A}^+)^0 + \overline{A}_1^+$, x^2 by $x^1(\overline{A}^+)^0 + \overline{A}_1^+$, x^3 by $x^2(\overline{A}^+)^0 + \overline{A}_1^+$ (and so forth). By the last inequality presented we gain the inequality $x^1 \le \overline{d}'$, which implies $x^2 \le x^1$, which in turn implies $x^3 \le x^2$, etc. We consequently obtain a monotonically falling sequence of vectors x^k which is bounded from below by

[17] With an index "+" in the second case, of course. Note that the change in total energy use $\overline{X}_1 i - \overline{X}_1^+ i$ will be equal to the sum of two effects $\bar{c}' - (\bar{c}^{+1})')\bar{f} + (\bar{c}^+)'(\bar{f} - \bar{f}^+)$ if the assumption $\overline{f} = \overline{f}+$ is dropped, which thus may lead to an *increase* in total energy consumption if the demand effect $(\overline{f}^+ \gtreqless \bar{f})$ is sufficiently strong of offset the saving of energy that is implied by the first item in the above sum (see Reardon 1973, p. 41ff. for a practical computation of this kind).

$0 \in \Re^n$ and which, therefore, must converge to a vector $x^* \geq 0$. Simple continuity arguments then imply that this vector x^* must fulfill $x^*(\overline{A}^+)^0 + \overline{A}_1^+ = x^*$, i.e., by Proposition 7.9, will define the vector of energy costs $\overline{d}^+$ based on the new technical conditions given. This vector thus fulfills $\overline{d}^+ \leq \overline{d}$ and hence also $\overline{c}^+ \leq \overline{c}$ (see again Proposition 7.9). □

Remark 1. Technical change will seldom occur in practice in the extreme form we used to prove Proposition 7.12, a fact which, however, does not render this proposition worthless, since its proof provides important insights into the scope for the assertion of a general reduction in energy costs when less clear (mixed) types of viable and energy-saving technical change are considered. The strategic points which allowed the derivation of an unambiguous reduction for the vector $\overline{c}$ are given by the following two inequalities (compare the proof of Proposition 7.12):

$$0 \geq p' \Delta \overline{A}^0 + p_1 \Delta \overline{A}_1 \text{ and } p' \Delta \overline{A}^0 \geq p_1 \overline{d}' \Delta \overline{A}^0, \tag{7.17}$$

i.e., the fact of cost reduction with regard to the employed input–output table $\overline{A}$ and an inequality which transfers this result to energy costs (on the basis of $p > p_1 \overline{d}$ if $\overline{A}^0 \geq 0$). it is the second inequality which represents the most crucial point in the arguments made and which, therefore, should be ensured by more detailed assumptions or facts in the case where $\overline{A}^0 \geq 0$ does not hold in particular since a falling vector $\overline{c}$ indeed is reported in some of the empirical computations mentioned (compare Sect. 7.1).

Remark 2. Inequalities (7.17) and $p > p_1 \overline{d}$ in addition show that there is room for technical change which reduces energy costs but which is not viable, i.e., which will not be adopted in a profit-maximizing economy (see Krenz 1977, p. 128 for related, more concrete considerations of this kind).

Remark 3. By inverting the order of inequalities used to prove Proposition 7.12 it can finally be shown that technical change which *utilizes* energy to *save* capital (goods $2, \ldots, n$) in the ideal way described in Definition 7.11 will reduce the vector of energy costs $\overline{c}$ at most in those cases where it is viable, yet that there exist viable changes of this kind which do not reduce the vector $\overline{c}$ unambiguously.

Example. Consider the following case of technical change:

$$\Delta \overline{X}^4 = (-6, 0, +4, 0)',$$

on the basis of the example employed in Sect. 7.3 (all other columns of $\overline{X}$ and the matrix $\overline{M}$ remain as they were). This change in the production of commodity 4 provides a simple example for Definition 7.11, which is also viable ($p = i$). The vector of post-change energy costs $\overline{c}^+$ is then given by

$$\overline{c}^+ = (0.27, 0.24, 0.20, 0.27)', \tag{7.18}$$

and is considerably lower than the original one [see (7.15)]. The vector $\overline{r}$ of energy consumption, on the other hand, will change from

$$\overline{r} \approx (1.75, -0.44, -0.36, 0.45)'$$

to

$$\overline{r}^{+} \approx (0.56, -0.23, -0.21, -0.k07)',$$

and thus will not behave in a similar way, i.e., the considered viable technical change (which is energy-saving in the sense of Proposition 7.13) will have no clear-cut (normal) effect on energy consumption coefficients $\overline{r}_k$ here. This provides us with a further argument to stress constant technical conditions – and not technological change – as their proper field of application, a situation where energy consumption $\overline{r}$ is known to be uniquely defined independently of any change in operating levels of activities. This property in return is not shared by energy costs $\overline{c}$. Considering, for example, the change in activity levels $\Delta i = (-1, 3, 2, 0)'$ employed in Sect. 7.3 we indeed get a new vector $\overline{c}$ approximately given by $(0.22, 0.21, 0.23, 0.47)'$ which in its own (averaging) way reflects the rise in productivity brought about by deleting the inefficient activity 1 from the set of applied activities [see again (7.15)]. There thus seems to be some theoretical guideline for choosing between energy consumption $\overline{r}$ and energy costs $\overline{c}$, i.e., between the application of the "commodity technology" assumption and the "industry technology hypothesis" when measuring the energy requirements of commodities in practical applications.

Summarizing we can now state the constitutive result that the measure $\overline{c}$ of Sect. 7.4 indeed allows for useful application in the analysis of change over time, applications which in part depend on the non-negativity of its corresponding input–output table $\overline{A}$. This non-negativity of the underlying matrix $\overline{A}$, which is based on the whole set of existing activities, has become possible in the general case considered by an equal treatment of all kinds of jointly produced commodities by means of the market value method, which in our view by its general and abstract applicability very sensibly – though not necessarily uniquely – supplements the average computations involved in the customary case of multiple, but single-product, activities. In our opinion this further justifies our special choice of an economic convention to allow for an applicable notion of energy costs.[18] Yet in order to prove this applicability the restrictive and pregnable assumption of stable market shares D had to be made. To defend this assumption let us state here briefly that it is not necessarily less plausible than the assumption of stable product mixes C,[19] which we have made (together with the assumption of a constant matrix B, i.e., of constant dollar

[18] Note in this regard that our object here has not been to consider (marginal) investment decisions based on given prices (see Helliwell and Cox 1979 for a joint product example of electricity cogeneration of this kind – making no use, of course, of the market value method), but to introduce a sensible generalization of average energy costs which overcomes the difficulties of joint production on the abstract level of input–output methodology.

[19] See Gigantes (1970, p. 282) for such an assertion.

economies of scale) in Sect. 7.3 to ensure the existence of a single vector of energy consumption $\overline{r}$ for valuing different scenarios of final demand. Thus, the provisional assumption made to show the usefulness of the vector $\overline{c}$ can bear comparison with those we employed in the case of energy consumption $\overline{r}$.

7.6 Summary

In this article the two basic procedures of input–output methodology available to derive monetarily based measures of the energy requirements of commodities in the presence of multiple production of various kind have been examined with regard to their technological and economic content. It was established that the measure which is based on the so-called "commodity technology" assumption allowed an interpretation in terms of a concept which applies to the determination of the energy consumption of all admissible changes of final demand under constant technical conditions, while the second measure, based on the "industry technology" assumption, led us to a calculation of average (real) energy costs which relied on the market value method of managerial cost-accounting to the extent that joint products are present. Our analysis showed that input–output measurement of such general kind contains definite theoretical foundations which permit an explanation of "anomalies" observed with respect to the first measure and an illustration of how to apply the second with regard to an analysis of technical change, thereby supplying us with suitable conceptual means to distinguish energy consumption from energy costs.

The analysis presented, it is hoped, provides an abstract but useful counterpart to the various concrete computations of the energy requirements of commodities which are made. It should be extendable by means of already existing practically oriented methodology to those questions we have excluded from consideration, i.e., in particular to the problem of how to treat fixed capital and depreciation when measuring the energy requirements of commodities.

References

Armstrong, A. G. (1975). Technology assumptions in the construction of U.K. input–output tables. In R. I. G. Allen & W. F. Gossling (Eds.), *Estimating and projecting input–output coefficients* (pp. 68–91). London: Input–Output Publishing Company.

Dickey, R. I. (Ed.) (1960). *Accountant's cost handbook.* New York: Ronald

Flaschel, P. (1982). On the two concepts of basic commodities for joint production systems. *Journal of Economics, 42*, 259–280.

Folk, H., & Hannon, B. (1974). An energy, pollution, and employment policy model. In M. S. Maerakis (Ed.), *Energy: Demand, conservation, and institutional problems* (pp. 159–173). Cambridge, MA: MIT Press.

Gigantes, T. (1970). The representations of technology in input–output systems. In A. P. Carter & A. Bródy (Eds.), *Contributions to input–output analysis* (pp. 270–290) Amsterdam: North-Holland.

Gupta, S., & Steedman, I. (1971). An input–output study of labour productivity in the British economy. *Oxford Bulletin of Economics and Statistics, 33*, 21–34.

Helliwell, J. F., & Cox, A. J. (1979). Electricity pricing and electricity supply. *Resources and Energy, 2*, 51–74.

Herendeen, R. A. (1974). Use of input–output analysis to determine the energy cost of goods and services. In M. S. Maerakis (Ed.), *Energy: Demand, conservation, and institutional problems* (pp. 141–158). Cambridge, MA: MIT Press.

Howe, Ch. W. (1979). Natural resource economies: Issues, analysis and policy. New York: Wiley.

James, D. E. (1980). A system of energy accounts for Australia. *The Economic Record, 56*, 171–181.

Krenz, J. H. (1977). Energy and the economy: An interrelated perspective. *Energy, 2*, 115–130.

Lancaster, K. (1968). *Mathematical economics*. New York: Macmillan.

Matz, A. & Usry, M. F. (1976). *Cost accounting, planning and control*. Cincinnati, Ohio: South–Western

Nikaido, H. (1968). *Convex structures and economic theory*. New York: Academic Press.

Reardon, W. A. (1973). Input–output analysis of U.S. energy consumption. In M. F. Searl (Ed.), *Energy modeling: Art, science, practice*. Washington, DC: Resources for the Future.

Roemer, J. E. (1977). Technical change and the 'tendency of the rate of profit to fall'. *Journal of Economic Theory, 16*, 403–424.

Sraffa, P. (1960). Production of commodities by means of commodities. London: Cambridge University Press.

Stobbe, A. (1959). Produktivitätsmessung auf der Grundlage von Input–Output–Tabellen, *Weltwirtschaftliches Archiv, 82*, 237–271.

Stone, R. et al. (1963). A programme for growth 3: Input–output relationships 1954–1966. London: Chapman and Hall.

United Nations (1968). A system of national accounts. *Studies in methods F* (no. 2, rev. 3). New York: United Nations.

United Nations (1973). Input–output tables and analysis. *Studies in methods F* (no. 14, rev. 1). New York: United Nations.

Wright, D. J. (1975). The natural resource requirements of commodities. *Applied Economics, 7,* 31–39.

Part II
Production Prices and the Standard Commodity. A Critical Reassessment

The characterization of this part of the book could be brief, if it basically were intended to provide some additional material (qualifications) for the well-known theory of production prices (of Sraffa or von Neumann type) and also for the underlying mathematical theory. However, at the close of the first chapter of this part (on prices of production), we will argue that these physically based input-output approaches to the treatment of Classical ruthless competition and its production price scheme are going much too far in the choice of their disaggregation level to allow for a uniform rate of profit based on physical data and a physically determined uniform production period and most importantly – also going much too far – in its treatment of fixed capital vintages as just special cases of joint *commodity* production. There simply is no meaningful technologically determined periods of production in developed market economies by which we can distinguish on a physical level so-called fixed capital goods from so-called circulating ones.

Instead we will, here too, favor the pragmatic Leontief approach which is distinguishing instead between flow matrices (capital consumed) and stock matrices (capital advanced) within the yearly accounting framework of firms and the turnover times these two matrices imply for the use of each input commodity in each activity. From this perspective each commodity gives rise to amounts tied up in their usage and to amounts in which it has been consumed during the typical accounting period, the year. This topic is however not treated in the present book in detail, but is only touched here upon to argue that firm based input-output considerations may be the more appropriate choice when factual economic outcomes are to be theorized. If – as production prices assume – a uniform rate of profit across sectors is justified at all, this becomes a (not trivial) matter of choosing the right aggregation level, in place of an application of the principle of uniform profitability on all conceivable sub-stages of the daily input-output processes of billions of firms that are competing with each other in the world economy.

Our basic findings in the chapter 8 therefore is that prices of production (nicely defined by help of Perron-Frobenius theory) – if useful at all – have to be investigated from the applied perspective in order to find out where they can really be used in a meaningful way. Our empirical findings (for the German economy with only 7 sectors) are not too supportive in this regard. It may therefore well be that

Farjoun and Machover (1983) are correct when they reject the above hard version of the uniformity assumption as fallacious (see their p. 19).

We go on, in chapter 9, to considering Sraffa's concept of basic commodities extended to the case of pure joint production systems where – as we show – there are in fact two fundamental possibilities for their definition and investigation. As already suggested through chapter 8 we here too favor the Leontief – Stone (empirically oriented) input-output approach to such issues as against the Sraffa approach to the investigation of the basic / non-basic distinction in general models of production. Our reason for this is that so-called Leontief-basics are defined quite generally, while the concept of Sraffa basics rests on restrictive mathematical assumptions which moreover do not allow for economic interpretation.

In chapter 10 we return to single product systems. We argue there – on the basis of various continuity propositions – that the distinction between basics and non-basics is fairly arbitrary if one applies these concepts on the physical level, since 'energy' and 'pencils' may then belong to the same category, the basics. Instead aggregation is again needed for a sensible application of the basic / non-basic distinction.

Part II is closed by two chapters on Sraffa's Standard Commodity. It is shown there of not providing a simplifying device for the theory of income distribution and of not providing an invariable measure of value (as searched for by Ricardo).

In sum, part II may therefore appear to the reader as providing a fairly negative contribution to the Classical theory of production prices and the income distribution between capital and labor. It is nevertheless meant to support the Classical theory of long-period prices and the inverse relationship between real wages and the rate of profit they imply, but it suggests that empirically oriented input-output concepts as discussed in detail in Bródy (1970), stressing for example the stock-flow distinction and turnover times, and building quite generally on cost and revenue allocation techniques that are actually used by firms, should be employed for the treatment of general production systems in place of the often purely theoretical concepts of Sraffa's (1960) book.

Sraffa's early contribution was surely a very important one and also a precisely formulated step forward to the revitalization of Classical economics in the 1960's and thereafter, but in order to turn it into an applicable framework it must be revised significantly in view of the contributions by Leontief, Stone and Brody and subsequent work on input-output analysis, including the question to what extent and on what level profit rates tend to approach each other to a certain degree.[20]

The outcome of such a research programme may well be that Samuelson's (1971) eraser principle should be in fact applied to the theory of production prices instead to labor values (which are firmly rooted in the analysis of the implications of technical change as we have argued in part I). Such a result would imply that Steedman's (1977) critique is just the opposite of the truth, in addition to what we have already stated in the ch. 4.

[20] See Flaschel, Franke, and Veneziani (2009) for such a revision of production price schemes that takes account of differentiated profit rates.

Sraffian production price accounting may have been a useful starting point for the revitalization of Classical ideas in the 1960's, but they are definitely inferior to their general definition in Bródy (1970), to be extended by the United Nations' (1968) treatment of joint production. Moreover, their may be a fairly stable distribution of profit rate differentials as discussed in Flaschel, Franke, and Veneziani (2009), distribution that may also be governed by statistical laws as Farjoun and Machover (1983) have argued. We have shown in part I that Marx's (1977) Capital, Vol. I analysis does not depend on the specific price theory that may be the outcome of a coherent revision of Marx's (1077) Capital, Vol. III and chances are that this will not be based on the Sraffian approach to production prices as long-period centers of gravity or even point attractors of market prices.

References

Bródy, A. (1970): *Proportions, Prices and Planning*. North Holland: Amsterdam.

Farjoun, E. and M. Machover (1983): *Laws of Chaos*. London: Verso Editions.

Flaschel, P., Franke, R. and R. Veneziani (2009): 'Neoricardian Theory and the Measurement of Prices of Production': An Alternative Approach. Bielefeld University: CEM working paper.

Marx, K. (1977): *Capital, Volume I – III*: London: Lawrence and Wishart.

Samuelson, P. (1971): Understanding the Marxian notion of exploitation: A summary of the so-called transformation problem between Marxian values and competitive prices. *Journal of Economic Literature*, 9, 399 – 431.

Sraffa, P. (1960): *Production of Commodities by Means of Commodities*. Cambridge, UK: Cambridge University Press.

Steedman, I. (1977): *Marx after Sraffa.* London: NLB.

United Nations (1968): *A System of National Accounts*. New York: Studies in Methods, Series F, No.2, Rev.3.

Chapter 8
In Search of Foundations for a Classical Theory of Competition

> In our own probabilistic analysis, labor value does not determine prices. We have no 'prices of production', or any other kind of ideal 'natural' price, but only actual market prices. The latter are connected to labor content, but the connection is a 'probabilistic' one (Farjoun and Machover 1983, p. 84).

8.1 Classical Ruthless Competition

In this chapter we go from the sphere of the production of commodities to the sphere of their circulation, by means of the theoretical as well as the empirical investigation of the Classical concept of prices of production which is based on the principle that there is a single uniform rate of profit for all activities that are organized on a profit-oriented basis. We start the Classical theory of price formation at first in its most basic setup, and then with increasing generality, in order to show how long-period prices of production may be formulated in more and more general models of production. We also provide a brief survey on the mathematical tools that are needed for the proof of basic Classical assertions on the properties of such long-period accounting prices which shows that indeed quite sophisticated mathematical theorems are needed for this purpose, for proving things that the Classical authors (including Marx) simply took for granted.

Our findings will be that the search for the foundations of a Classical theory of competition has by no means been a successful one so far. On the theoretical level, we find that the analysis of the process of the circulation of capital has by and large ignored the many factual accounting principles that are involved in and indeed are governing this process and that are needed for proper production price calculations in general production systems. And on the empirical side, we come to the conclusion that the principle of imposing a single uniform rate of profit on all profit-oriented activities of the (world) economy is simply going too far in the pursuit of finding useful and applicable long-period prices for the factual analysis of existing economies. Sectoral profitability studies are urgently needed for the proper formulation of long-period prices but are rarely done by the proponents of the Classical theory of prices of production.

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_8,

8.2 Two-Sector Economies

8.2.1 The Crude State of the Society

We start this chapter with the consideration of a simple two-sector economy, with only 'corn' as intermediate physical input into the production of both corn and manufacturing commodities, to investigate in detail output values and prices and wage-profit relationships on the basis of the assumption of uniform wages and uniform rates of profit. Before that we in fact even consider an economy with *two produced commodities*, good 1 and 2, which are *produced by means of labor solely*, i.e., an economy exhibiting Adam Smith's crude state of society (deer and beaver hunt). In such an economy, the Marxian theory of labor values and the rate of exploitation and their relationship to the rate of profit are totally transparent, a situation which already changes significantly on the two sector level with produced means of production. For the time being we here however even ignore all produced means of production.

We assume that the result of one production period can be represented ex post (with frozen production conditions) by

$$L_1 \mapsto \bar{x}_1, L_2 \mapsto \bar{x}_2$$

with L_1 the amount of labor spent in the production of the quantity $\bar{x}_1$ of good 1 (beaver hunt in Smith's example) and $L_2, \bar{x}_2$ with respect to good 2 (deer hunt). Per unit of output we thus would get as average labor input structure

$$l = (l_1, l_2) = (L_1/\bar{x}_1, L_2/\bar{x}_2), \quad A \left(= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right) \quad \text{still irrelevant}$$

which is again augmented by the assumption of constant returns to scale, i.e. on all levels of production, we need labor l_1 to produce one extra unit of good 1 (and nothing else) and, similarly, labor l_2 to produce one extra unit of good 2. It is obvious that the reciprocal values $1/l_1, 1/l_2$ can be considered as full measures of labor productivity. Moreover it is not meaningful to consider an aggregate of these two measures, since they are commodity-specific and incommensurable and since this would involve the use of commodity prices in some way or another.

We could also replace nominal wages w by $p_1 c$ where c is the given commodity 1 wage basket of workers per hour worked and thus formally return to a Leontief input–output matrix, but have here to consider nevertheless labor effort explicitly in order to allow for the definition of labor values as is shown below. We will not consider given subsistence wage baskets in this section however, but will use instead arbitrarily chosen numéraire commodities that do not necessarily represent the wage basket of wage earners.

The above is indeed a crude state of society. Our current representation of the technological side of the economy will nevertheless already be very useful for defining the basic concepts of labor embodied, natural prices (or prices of production in Marx's terminology) and labor commanded prices per unit of commodity of the Classical approach to political economy (Smith, Ricardo and Marx in particular). Accordingly, we define the commodity valuations:

$$v_1 = l_1, \quad v_2 = l_2 \qquad \text{(labor embodied)}$$
$$p_1 = (1+r)wl_1, \quad p_2 = (1+r)wl_2 \qquad \text{(nominal natural prices)}$$
$$p_{1w} = (1+r)l_1, \quad p_{2w} = (1+r)l_2 \qquad \text{(labor commanded)}$$

on the basis of a given uniform rate of profit r and a given uniform nominal wage rate w. Labor commanded prices may also be measured by dividing actual nominal prices $p_1(t), p_2(t)$ by the wage level w, in which case we may have differentiated rates of profit (which we assume to be positive). Assertions on prices in terms of the wage-unit will also apply to such factual prices in place of the long-period prices of the Classical authors.

Labor embodied type valuation schemes or briefly *labor values* are defined by the total amount of labor spent on an average on the production of one unit of output of the considered economy. Such total labor costs will later on also include the labor time spent on the means of production, their means of production and so on and is in principle just a definition that attempts to as general as possible and to be applicable to real world situations, for example by providing meaningful measures of labor productivity, here defined by $1/\ell_i = 1/v_i, i = 1,2$ that are of use in the empirical investigation of whole economies.

Natural prices, in the presently considered crude state of the society, where production is nevertheless assumed to be organized by capitalist firms in the pursuit of positive profits, are based on these labor costs, but expressed in terms of wages costs now, and multiplied by $1+r$ in order to include normal profits at the rate r into these prices. As natural prices are defined, it appears that increases in w or r just lead us to increases in both p_1 and p_2 which would then just include increased components of wage payments and profit receipts. This 'component approach' to the formation of natural prices represents however only a partial understanding of the results of Classical ruthless competition, more concerning 'surface phenomena', and will be supplemented by Ricardo's analysis of the wage-profit relationship below.

Labor commanded prices, finally, are natural prices normalized by the wage rate, i.e., these prices represent the amount of labor that can be bought on the market by one unit of good 1 or 2, respectively. In popular statements this is expressed by the phrase that 'a household has to work p_{w_1} time units in order to get one unit of commodity 1 for consumption purposes', which therefore 'commands' this amount of labor.

In the crude state of society here considered, these value and price concepts of course all imply the same relative magnitudes

$$v_1/v_2 = p_1/p_2 = p_{1w}/p_{2w}$$

and thus here all give rise to one and the same *same relative price* on the real side of the economy. Labor time embodied, natural prices and labor commanded prices thus appear to be just three different ways to look at one and the same thing, a view that cannot however be maintained outside the considered crude state of the society as we shall see in this section.

8.2.2 Some Observations

Prices measured in terms of the wage unit (actual, natural or competitive ones) are used by Keynes (1936) when he discusses the choice of units for his macroeconomic theory and in particular the measurement of output as a whole. It is an unsettled question whether Keynes' General Theory is better represented when output is measured in this way as compared to the national output measures that are used in the now standard system of national accounts.

Natural prices as they were formulated above at first sight do not seem to imply anything that looks like a conflict between profit-oriented capitalist households and consumption-oriented worker households. Yet, if one introduces a real consumption basket of workers $c \in \Re_+$, measured per work-hour and here for simplicity only represented through commodity 1, and fulfilling $w = p_1 c$ one can rewrite the equations for natural prices as follows:

$$1 = (1 + r)cl_1, \quad p_2 = (1 + r)p_1 cl_2, \quad i.e., \quad r = \frac{1 - cl_1}{cl_1} = \frac{1}{cl_1} - 1$$

and thus get a strictly negative relationship between the current consumption level c of workers and the rate of profit that is earned by capitalists. Ricardo, who was held responsible for having formulated this relationship with analytical rigor, was accused later on by economists of his time for having formulated by it a theory of class conflict. From today's perspective this relationship is no longer viewed in this way and it in fact only represents a relationship on the distribution of the domestic product and thus describes, here still in very simple terms, the conflict about income distribution between labor and capital.

Marx attempted to dig deeper into the capital-labor relationships and he formulated for this purpose – based on the concept of labor values – the rate of exploitation, the ratio of surplus labor to necessary labor in for example one hour spent in production. By means of the above labor value expressions and the wage basket c per work-hour we can express this rate of exploitation ϵ as follows

$$\epsilon = \frac{1 - v_1 c}{v_1 c} = \frac{1 - l_1 c}{l_1 c}$$

and thus see that it coincides with the rate of profit r in the considered crude state of society. Marx's two sources for an increase of the rate of profit can thus in a strictly one to one fashion represented by the forces that imply increases in the rate of exploitation. These two reasons are: An increase of the (family) hours worked per week (which decreases c, subsistence level of the worker household obtained per hour worked (or a direct reduction of the consumption basket that a household can afford with its weekly wage rate w). Alternatively, there may be an increase in labor productivity through technological change (or an increase in labor intensity). Marx called the first possibility the creation of absolute surplus value and the second one the creation of relative surplus value. The class conflict that was considered by Ricardo on the level of the distribution of the national product thus here becomes a deeper one, since it already starts on the level of the production of the national product with the conflict over the length of the work week (of the whole worker household, including child labor) and with the conflict over the of the permanent revolution of labor division and the intensity of work in the firm.

In the presently considered extremely simple economy Ricardo therefore by and large only considered the conflict about the determination of c on the external labor market while Marx added the conflict about the work conditions of workers.

8.2.3 Production of Commodities by Means of Commodities

Let us in a next step extend the considered production technology to a two-sectoral one, i.e., we now assume as given an average (or linear) input–output structure (here still coupled with exogenous, i.e., frozen or stationary output levels) of the following type

$$l = (l_1, l_2) > 0, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ 0 & 0 \end{bmatrix} \geq 0, \quad x^{'} = (\bar{x}_1, \bar{x}_2) \quad \text{with } y = x - Ax \geq 0.$$

We thus now simply add the following simple intermediate input structures to the production of commodities 1 and 2. The accounting equations for *labor embodied* or *labor values* or *total labor costs* read in the case of a single intermediate input:

$$v_1 = l_1 + v_1 a_{11} \tag{8.1}$$

$$v_2 = l_2 + v_1 a_{12} \tag{8.2}$$

since the amount of labor embodied in intermediate inputs a_{11}, a_{12} has now to be added to the labor directly embodied into the production of goods 1 and 2. In matrix notation this equation system is given by $v = l + vA$ and it fulfills the aggregate relationship $vy = v(I - A)x = lx = L$, where L is the amount of labor used up in the production of the gross output vector x, which is as shown equal to the total labor cost embodied in the production of net output y.

The solution to system (8.1), (8.2) reads in the case of a productive economy ($a_{11} < 1$):

$$(v_1, v_2) = (l_1, l_2)(I - A)^{-1} = \sum_{k=0}^{\infty} (l_1, l_2) A^k, \quad A^0 = I.$$

It is uniquely determined and strictly positive if the matrix A is productive ($a_{11} < 1$) and if $(l_1, l_2) > 0$ holds. In the presently considered case the solution is however also immediately obtained by solving the first equation with respect to v_1 $(= l_1/(1 - a_{11})$ and by inserting this result into the second equation $(v_2 = l_2 + v_1 a_{12})$. The positivity of labor values $v_i, i = 1, 2$ of course is needed if the concept of 'embodied labor' is to be considered an economically meaningful one. We note that labor values do not depend on income distribution, in difference to the price vectors to be considered below. As the above abstract presentation shows, they can be interpreted as the accumulated sum of labor efforts on all (simultaneously considered) stages of production needed to produce one unit of output of good 1 or 2.

Let us next consider the accounting equations for long-period *natural prices* assuming a given nominal wage rate w and characterized by a rate of profit r that is uniform across sectors. The equations for these prices read in the case of the above extended input–output model[1]

$$(1 + r)[wl_1 + p_1 a_{11}] = p_1 \tag{8.3}$$
$$(1 + r)[wl_2 + p_1 a_{12}] = p_2 \tag{8.4}$$

These equations express that both labor and intermediate inputs are paid at the beginning of the production period and thus represent capital advanced, the basis of the calculation of the rates of profit in both sectors:[2]

$$r = \frac{p_1 - [wl_1 + p_1 a_{11}]}{wl_1 + p_1 a_{11}} = \frac{p_2 - [wl_2 + p_1 a_{12}]}{wl_2 + p_1 a_{12}}.$$

In the presently considered case the solution to the system (8.3), (8.4) is easy to obtain if one observes that only two of the potential unknown p_1, p_2, w, r can be determined by this system. In view of this, we take w as being exogenously given and normalize p_1 by setting it equal to '1', i.e., we now consider the real wage $\omega = w/p_1$ in place of the nominal wage w.[3] We then obviously get:

$$1 + r = \tfrac{1}{\omega l_1 + a_{11}} \quad \text{or} \quad r = \tfrac{1 - [\omega l_1 + a_{11}]}{\omega l_1 + a_{11}} = \tfrac{1}{\omega l_1 + a_{11}} - 1 \tag{8.5}$$

$$p_2 = (1 + r(\omega))[\omega l_2 + a_{12}] \tag{8.6}$$

[1] Note that we here assume as in the crude state of the society that wages are paid ex ante and thus represent capital advanced on which profits have to be earned.

[2] The type wage profit curve here considered need not be strictly convex in higher dimensional commodity spaces.

[3] In the present case the real wage ω is still equal to the consumption basket c of worker households, since this basket here consists of the corn commodity solely. We therefore allow now for variable real wages (which leads us away from the pure subsistence level assumption), but not yet for multiple commodities in workers' physical consumption.

as solution of system (8.3), (8.4). Note that commodity 1 is a basic commodity and commodity 2 a non-basic. The rate of profit is therefore only dependent on the sector of basic commodities, since a_{22} holds in the non-basic sector.

Since A is productive ($a_{11} < 1$), we thus know that r (and p_2) are positive for wages rates ω close to zero. There is, however, a maximum ω where r stops to be positive, given by: $\omega^{\max} = (1 - a_{11})/l_1$. The first equation therefore defines what is called a wage-profit curve $r(\omega)$ in the literature. It is easy to show that

$$Dr(\omega) < 0, \quad D^2 r(\omega) > 0$$

holds true, i.e., this curve is always of the following strictly convex type ($r^{\max} = \frac{1-a_{11}}{a_{11}}, \quad \omega^{max} = \frac{1-a_{11}}{l_1}$):[4]

Ricardo's theory of class conflict over income distribution thus holds in this extended input–output structure (and also in general as we shall see), i.e., real wages can only be increased through a reduction in the rate of profit that is earned by capitalists. Since we consider good 1, besides being the physical input into the production of goods 1 and 2, as consumption good (corn) of the workforce (and thus good 2 as luxury good not present in workers consumption basket), our normalization $p_1 = 1$ implies that the wage rate ω in fact represents the real wage or 'corn' wage of workers. In the case $\omega^{\max}$ the surplus $1 - a_{11}$ obtained in the production of corn is fully paid out as wages, while it is paid out as profit at the rate $r^{\max}$ if $\omega = 0$ holds. The above then shows that there is a strictly convex inverse relationship between corn wages and the rate of profit, so that this rate can only go up if there is a decrease in these real wages. In these real terms there is thus no basis for a component theory of prices, adding up nominal wages and nominal profits (and rent) to the price charged for the considered product, but indeed a subdivision of the surplus $1 - a_{11}$ between capital and labor that furthermore does not vary linearly with changes in the wage rate ω.

With respect to the price of the luxury good we then find $p_2(\omega) = (1 + r(\omega))[\omega l_2 + a_{12}]$. This gives (due to $Dr(\omega) = -\frac{l_1}{(\omega l_1 + a_{11})^2}$):

$$\begin{aligned} Dp_2(\omega) &= -\frac{l_1}{(\omega l_1 + a_{11})^2}(\omega l_2 + a_{12}) + \frac{1}{\omega l_1 + a_{11}} l_2 \\ &= \frac{a_{11} l_2 - a_{12} l_1}{(\omega l_1 + a_{11})^2} = \frac{a_{11}/l_1 - a_{12}/l_2}{l_1 l_2 (\omega l_1 + a_{11})^2} = \frac{k_1 - k_2}{l_1 l_2 (\omega l_1 + a_{11})^2} \end{aligned}$$

The terms $k_1 = a_{11}/l_1$ and $k_2 = a_{12}/l_2$ represent the capital intensities of the two processes under consideration ($k_i = K_i/L_i = a_{1i}\bar{x}_i/(l_i \bar{x}_i)$). We thus get the result that the price p_2 will increase with the wage rate ω iff $k_1 > k_2$ and it will decrease with ω iff $k_1 < k_2$, while it will stay constant in the singular case $k_1 = k_2$ and that $p_2(\omega)$ is strictly concave in the first case and strictly convex in the second situation (i.e., we have $D^2 p_2(\omega) < 0$ and $D^2 p_2(\omega) > 0$, respectively).

[4] since real wages $\omega = w/p_1$ are paid ex ante.

Labor values thus cannot be considered as providing a theory of long period prices (up to the cases where $r = 0$ of $k_1 = k_2$ holds) and this is indeed also not the role they play in national accounting and economic theorizing as we will argue below and will continue to argue throughout the book).

Digression

The rate of exploitation and indexes of labor productivity

1. In the still very simple type of economy here considered we have for the uniform rate of profit r the following relationships:

$$\begin{aligned} r &= \frac{1 - [\omega l_1 + a_{11}]}{\omega l_1 + a_{11}} \\ &= \frac{v_1 - [v_1 \omega l_1 + v_1 a_{11}]}{v_1 \omega l_1 + v_1 a_{11}} \\ &= \frac{l_1 - v_1 \omega l_1}{v_1 \omega l_1} \frac{v_1 \omega l_1}{v_1 \omega l_1 + v_1 a_{11}} \\ &= \epsilon \frac{v_1 \omega l_1}{v_1 \omega l_1 + v_1 a_{11}} \\ &= \epsilon \frac{1}{1 + v_1 a_{11}/(v_1 \omega l_1)} = \epsilon \frac{1}{1 + C/V} \end{aligned}$$

   The ratio C/V is called the organic composition (here in corn production) in Marx's Capital and it now drives a wedge between the rate of exploitation in production ϵ and the rate of profit r that is characterizing long-period prices. Up to this new element in the exploitation-profit relationship we have however again that a decrease in the consumption basket of workers c or a decrease in the labor value of commodity 1 are the only means by which the rate of profit r can be increased in such an economy. In particular, the so-called Marxian Fundamental Theorem, see Morishima (1973), which claims that the rate of profit is positive if and only if the rate of exploitation is positive quite obviously holds in our model economy. Class conflict as in Marx's Capital therefore already starts in the sphere of production and is only augmented later on in the sphere of commodity exchange. Moreover, there is no possibility for the full exchange of equivalents in an economy with positive profitability, since this fact demands that workers have to supply more labor in production than they get back in the form of their consumption basket.
2. Sectoral labor productivity in Marx's (1954) Capital, Vol. I is defined by

$$\pi_1 = 1/v_1 = \frac{1 - a_{11}}{l_1}, \quad \pi_2 = 1/v_2 = \frac{1}{l_2 + v_1 a_{12}}$$

In the basic corn sector it is thus given by the net corn product of this sector per unit of hour worked which is quite obviously a meaningful expression. In the 'luxury good sector' it is the net (=gross) output of this sector divided by the direct and indirect labor spent in producing this output level. Productivity in the corn production increases if labor per unit of output can be decreased of if intermediate corn inputs are decreased per unit of output. The same holds for the production of the luxury good, but there we also get productivity increases if the total labor costs of producing one unit of corn have decreased. Conventional national accounting defines sectoral labor productiveness in different ways, either by using single deflation indices or the so-called double deflation procedure.
In the first case on uses *a single price deflator*, for example defined by

$$sd(t) = (p_1(t)\bar{x}_1 + p_2(t)\bar{x}_2)/(p_1(0)\bar{x}_1 + p_2(0)\bar{x}_2)$$

where 0 denotes the base year for this deflation procedure. Sectoral labor productiveness are then defined by sectoral values added deflated with this single deflator and divided by the direct labor input of this sector:

$$\pi_1(t) = \frac{p_1(t)(1-a_{11})\bar{x}_1/sd(t)}{l_1(t)\bar{x}_1} = \frac{p_1(t)(1-a_{11})/sd(t)}{l_1(t)},$$

$$\pi_2(t) = \frac{(p_2(t)\bar{x}_2 - p_1(t)a_{12}\bar{x}_2)/sd(t)}{l_2(t)\bar{x}_2} = \frac{(p_2(t) - p_1(t)a_{12})/sd(t)}{l_2(t)}$$

These expression may measure real value added per laborer (in terms of real purchasing power), but quite obviously have little to do with labor productivity as a concept that relates itself closely to the production technology that is in operation. For example, labor productivity in sector 1 can here changed due to changes that concern sector 2, though the activities in sector 1 (the basic sector) are completely independent of those of sector 2.
In *the case of double deflating procedures* one would deflate each commodity by means of its own price deflator and not by means of a price deflator that concerns GDP. In this case one would get for values added per unit of labor input the expressions:

$$\pi_1(t) = \frac{p_1(0)(1-a_{11})\bar{x}_1}{l_1(t)\bar{x}_1} = \frac{p_1(0)(1-a_{11})}{l_1(t)},$$

$$\pi_2(t) = \frac{(p_2(0)\bar{x}_2 - p_1(0)a_{12}\bar{x}_2}{l_2(t)\bar{x}_2} = \frac{p_2(0) - p_1(0)a_{12}}{l_2(t)}$$

This implies that labor productivity in the first sector is distorted in its measure by the price of its commodity in the base year which is clearly undesirable. Furthermore, and even more disturbing, is that labor productivity in sector 2 here depends in addition on the relative price $p_2(0)/p_1(0)$ of the base year which may be totally meaningless with respect to what one hopes to measure by such a concept of labor productivity. We conclude that only labor values provide a sound basis for measuring labor productivity, since the final alternative where

labor productivity is measured by $1/l_i, i = 1, 2$ is not paying attention to the indirect effects in a technological system and thus only partial in nature.
Let us now come to the other normalization of natural prices in terms of *labor commanded* prices or prices measured in terms of the wage-unit (as Keynes introduced them in his 'General Theory'). These purchasing power prices (where $w = 1$ is achieved) fulfill:

$$(1 + r)[l_1 + p_{1,w}a_{11}] = p_{1,w}$$
$$(1 + r)[l_2 + p_{1,w}a_{12}] = p_{2,w}$$

which in matrix notations gives rise to

$$p_w \left[\frac{1}{1+r} I - A \right] = l = (l_1, l_2), \qquad p_w = (p_{1,w}, p_{2,w}).$$

The solution to this matrix equation is, of course, proportional, but not equal to:

$$\tilde{p}_w = \begin{bmatrix} 1/w \\ p_2(\omega)/w \end{bmatrix}$$

where $p_2(w)$ is the price function we have considered above. It is thus again economically meaningful for all $w \in [0, w^{\max}]$. We now however take r as exogenous (and $w = 1$) as basis for solving the system with respect to natural prices and get on this basis

$$\begin{aligned} p_w = l \left[\frac{1}{1+r} I - A \right]^{-1} &= l \left[\frac{1}{1+r} (I - (1+r)A) \right]^{-1} \\ &= (1+r) l \left[I - (1+r)A \right]^{-1} \\ &= (1+r) l (I + (1+r)A + (1+r)^2 A^2 + (1+r)^3 A^3 ...) \\ &= (1+r) l \sum_{k=0}^{\infty} ((1+r)A)^k = (1+r) l \begin{bmatrix} \frac{1}{1-(1+r)a_{11}} & \frac{(1+r)a_{12}}{1-(1+r)a_{11}} \\ 0 & 0 \end{bmatrix} > 0 \end{aligned}$$

This follows as long as $r < r^{max}$ holds, since $I - (1 + r)A$ is then profitable as well as productive and therefore nonnegatively invertible, i.e., the Leontief-Inverse exists and is representable by means of the above geometric matrix series in the usual way. Note here in addition that the equation systems

$$p_w(I - (1+r)A) = (1+r)l; \quad p_w \left[\frac{I}{1+r} - A \right] = l$$

are equivalent to each other.

Proposition 8.1. *Assume, as employed above, that the matrix A is productive in the sense of Definition 1.1 (i.e., here $a_{11} < 1$). Then*

1. $p_w > v > 0$, *i.e., labor commanded prices are always strictly larger than labor values (if the rate of profit is positive).*
2. $p_{2,w}/p_{1,w}$ *is strictly increasing, decreasing or constant iff* $a_{11}/l_1 > a_{12}/l_2$, $k_1 = a_{11}/l_1 < k_2 = a_{12}/l_2$ *or* $a_{11}/l_1 = a_{12}/l_2$ *holds, respectively.*

Assertion 1 follows from:

$$p_w = (1+r)l\sum_{k=0}^{\infty}((1+r)A)^k > l\sum_{k=0}^{\infty}A^k = l\begin{bmatrix}\frac{1}{1-a_{11}} & \frac{a_{12}}{1-a_{11}}\\ 0 & 0\end{bmatrix} = v > 0,$$

since $l > 0$.

Assertion 2 has already been shown, due to the following equalities

$$p_{2,w}/p_{1,w} = p_2/p_1 = p_2(\omega) \text{ for } p_1 = 1$$

Natural prices in terms of labor commanded thus can indeed command more labor per unit of commodity than is embedded in this commodity.

Furthermore, relative natural prices p_2/p_1 are identical to relative values v_2/v_1 – independent of their normalization – iff the composition $k_1 = a_{11}/l_1$ in the production of commodity 1 is the same as $k_2 = a_{12}/l_2$ in the production of commodity 2. This follows from the facts that labor values are equal to natural prices in the case of zero profits ($r = 0$) and that relative prices do not change in the considered case of a uniform capital intensity in both of the considered processes. Note that the wage profit curve does not become a linear curve in the considered situation.

These assertions apply to the case $r > 0$, while of course we always have

$$(p_{1,w}, p_{2,w}) = (v_1, v_2)$$

in the case $r = 0$ $[\omega = \omega^{\max}]$. With respect to $p_1 = 1$ we had that p_2 could decrease or increase with the rate of profit. Due to the above representation of $p_{1,w}$ and $p_{2,w}$ we however then have that these normalized prices will always strictly increase with the rate of profit r, the faster increase taking place in the sector i with the higher composition $k_i = a_{1i}/l_i$. We note in closing these considerations that the quantity theory of money may be used in addition to determine which absolute levels of prices and wages actually may come about:[5]

$$p_{1,w}\bar{x}_1 + p_{2,w}\bar{x}_2 = \bar{\upsilon}\bar{M}/w$$

This extra equation implies that either $p_{i,w}$, X_i or w must then adjust appropriately in view of the given quantity of money $\bar{M}$, implying that r, X_i or w are then to be determined endogenously.

[5] $\bar{\upsilon}$ the velocity with which the quantity of money M is turned over.

This section has shown that embodied labor values and natural prices are in general not proportional to each other and thus not both candidates around which actual market prices may be assumed to oscillate. However, labor embodied is always smaller than labor commanded for a given commodity (if positive profits are postulated), which at least provides a first useful comparison between such valuations of one and the same commodity. Adam Smith has not been successful in distinguishing clearly natural prices (in terms of wages or in terms of any other numéraire) from labor values or the labor time embodied (or imputed) into the considered commodities. This also holds true with respect to Ricardo's analysis of values and prices. Though they are there clearly distinguished from each other, Ricardo could not solve for them the problem of the choice of a suitable numéraire (labor in the case of labor values), by means of which the causes behind changes in nominal price magnitudes could be detected in a way that did not make use of labor values as an approximation to prices of production.

The next section goes in this regard a considerable step forward in that it provides generalized expressions for labor values, natural prices and their renormalization in terms of labor commanded by way of powerful mathematical theorems on the dominant eigen-value of nonnegative and square input–output matrices of an arbitrary dimension n.

8.3 Sraffian Multisectoral Economies

In this section, we consider a nonnegative, square and productive $n \times n$ intermediate-input–output matrix $A = (a_{ij}) \geq 0$, which may here also be augmented by a consumption basket $c' = (c_1, ..., c_n)$ (of workers) to be used as numéraire commodity in the place of the numéraire 'corn' of the preceding section.[6] We later add a vector of labor inputs $l = (l_1, ..., l_n)$, assumed as strictly positive, which together with the columns of the matrix A characterizes the (average) inputs of the n activities per unit of output. Whenever convenient we will augment the matrix A by the $n \times n$ matrix $cl = (c_i l_j)$ which then adds the reference basket of workers weighted by the direct labor input l_j to the intermediate inputs of each activity j. We denote the resulting $n \times n$ matrix by A^c. If $\omega = w/pc$ denotes the real wage in terms of the basket c we denote the then resulting augmented matrix by $A^c(\omega) = A + \omega cl$ which then varies with the real wage ω in a linear way. All quantity expressions will be considered as given magnitudes in the present section.

[6] We do not consider here as in Sraffa (1960) wages that are paid ex post, but refer the reader to Chap. 9 for their treatment as surplus wages or deficit wages with respect to the here considered given consumption basket of workers. The reader is referred to Kurz and Salvadori (1995), Schefold (1997) and Bidard (2004) for detailed treatments of many aspects of Sraffa's (1960) approach to the production of commodities by means of commodities.

From the mathematical point of view, but motivated by economic analysis long before the mathematical theorems of this section have become available, we can investigate the eigenvalue problems

$$pA = \lambda(A)p, \lambda(A) \geq 0, p \geq 0, Ax = \lambda(A)x, \lambda(A) \geq 0, x \geq 0,$$

or from the economic point of view the balanced situations

$$(1+r)pA = p, (1+r) \geq 0, p \geq 0, (1+g)Ax = x, (1+g) \geq 0, x \geq 0,$$

i.e., we are here looking for non-negative (non-zero) price systems and activity vectors such that the vector of production cost pA and the vector of intermediate input consumption Ax are proportional to the price system p and the activity vector x, respectively. This is again the question of uniform profitability with the rate of profit formula $1+r = 1/\lambda(A)$, and of balanced expansion with rate of expansion formula $1+g = 1/\lambda(A)$ (assuming $\lambda(A) > 0$ here of course). Note that both r and g can still be negative here, but are assumed to be larger than -1. We shall concentrate here on the price equation, but observe that everything obtained for this case can be obtained for the case of the activity equation as well. The latter case will however need the assumption of a linear technology. while the consideration of (average) price systems is independent of this extra assumption. Note that the properties of dominant eigenvalues considered below of course apply (with different dominant eigenvalues) to he matrices $A, A^c, etc.$

8.3.1 Economic Properties

As is well known, such eigenvalues are obtained from an equation of the type $\det(\lambda I - A) = 0$, the so-called characteristic equation, with solutions in the complex plane in general. In the case of a nonnegative matrix economic intuition suggests that solutions of the above problems should exist. Independently from economics however mathematicians have established at the beginning of last century that this is indeed the case and that such an eigenvalue $\lambda(A)$ majorizes any other eigenvalue λ in modulus $\lambda(A) \geq \|\lambda\|$ and is thus uniquely determined (but may be a multiple root of the above characteristic equation). This root is called the dominant root in the following presentation of the mathematical theorems characterizing such dominant roots. The proofs of all the theorems considered in this section can be found in Nikaido (1968, II).

With respect to such dominant roots, see the theorems that follow below, we have as a first result the Proposition 8.2 which relates our statements on the Leontief-inverse in the preceding section to such roots:

Proposition 8.2.

1. *We define the set $\Phi(A)$ by those r for which the matrix $I - (1+r)A$ is nonnegatively invertible, i.e., these r allow for the calculations of the preceding section in*

particular. Then: The set $\Phi(A)$ *is given by* $(-1, r^{\max})$ *with* $r^{\max}$ *a number larger than* -1, *i.e.,* $1 + r^{\max}$ *is always positive (and may be infinite).*

2. *For the above* $r^{\max}$ *we have that there is some* $p \geq 0$ *such that there holds:*

$$(1 + r^{\max})pA = p.$$

Remark. From the viewpoint of mathematics this amounts of course to the consideration of eigenvalues $\lambda = 1/(1 + r^{\max})$ in the place of such maximum profit rates, with the same vector as eigenvector. From the economic point of view we are however asking whether there is a maximum uniform rate of profit for the sectors of the economy if intermediate inputs are the only costs that are considered. This rate of profit indeed exists and all rates of profit below it allow for the conventional Leontief multiplier formula (but not for this maximal value).

We assume as given for the remainder of this section an input–output matrix A that satisfies the Hawkins-Simon conditions (i.e., that it is productive and profitable). We know that this matrix has a nonnegative Leontief-inverse, which implies according to the above that there exists a maximum *positive* rate of profit $r^{\max} = 1/\lambda(A) - 1$, i.e., the matrix $M = I - (1 + r)A$ is nonnegatively invertible for all rates of profit $r \in [0, r^{\max})$. We now add the vector $l = (l_1, ..., l_n) > 0$ of labor inputs into the n sectors to the considered situation.

Proposition 8.3. *In the considered situation, i.e., for profit rates* $r < r^{\max}$ *we have:*

1. *There is a unique, strictly positive vector* $v \in \Re^n$ *which fulfills*

$$v = l + vA, \qquad v = (v_1, \ldots, v_n) > 0 \tag{8.7}$$

called the vector of embodied labor or briefly the vector of *labor values.*

2. *There is a unique, strictly positive vector* $p \in \Re^n$ *which fulfills*

$$p = (1+r)[\omega l + pA] = (1+r)p[\omega cl + pA], \qquad p = (p_1, \ldots, p_n) > 0 \tag{8.8}$$

for given positive $\omega \in \Re$ *measured in terms of the consumption basket* c $(pc = 1, \omega = w/pc)$, *the* *natural prices* *in terms of this commodity basket that belong to a given rate of profit.*

3. *There is a unique, strictly positive vector* $p_w \in \Re^n$ *which fulfills*

$$p_w = (1 + r)[l + p_w A], \qquad p_w = (p_{w1}, \ldots, p_{wn}) > 0, \tag{8.9}$$

the vector of *labor commanded prices,*[7] *i.e., measured in terms of or the wage unit.*

4. *There always holds:* $p_w > v$, *i.e.,* $p_{wi} > v_i$ *for all* $i = 1, ..., n$ *if* $r > 0$ *holds true.*

[7] This price concept is not restricted to the consideration of production prices schemes.

All these assertions follow again immediately from the fact that we have for all considered rates of profit r the well defined (convergent) sequence of matrix multipliers

$$(I-(1+r)A)^{-1} = I + (1+r)A + (1+r)^2A^2 + (1+r)^3A^3 ... \geq (I-A)^{-1}$$

and from the facts that $(1+r)l$ and $v = l(I-A)^{-1}$ are strictly larger than $l = (l_1, ..., l_n) > 0$. We note that labor values are equal to labor commanded prices in the case $r = 0$ and that they will be proportional to labor values for all admissible $r > 0$ if vA is proportional to l, i.e., if the ratio of labor indirectly embodied in the means of production to direct labor are the same for all n sectors. This follows, since αv for positive α is then a solution of the type considered in Proposition 8.3. A similar assertion of course applies to Proposition 8.2.

We thus first of all have that the labor v_i embodied in commodities $i = 1, ..., n$ or the total labor costs needed for producing these commodities (the summed labor costs contained in the intermediate inputs plus the direct labor added in each production sector), to be defined as the equation system $v = l + vA$, is uniquely determined and always positive. Due to the multiplier formula it is in fact, for commodity j, the direct labor l_j used in the production of j plus the direct labor used in the production of the intermediate inputs $lA_{\star j}$ plus the direct labor used in the production of the intermediate inputs $A(A_{\star j}) = A^2e_j$ and so on (e_j the jth vector in the canonical basis of $\Re^n$). The labor embodied concept or the labor values of commodities j is therefore a not immediately meaningless definition which however is not directly defined with respect to principles that govern price formation and which therefore needs sensible applications via economically meaningful propositions in order to justify the usefulness of such a definition.

A first important result in this direction is the assertion 4.4 which states that – due to the inclusion of profits into the definition of labor commanded by the various commodities – that one can always get more labor in exchange for one such commodity than is embodied into the production of it. Dividing actual prices, not only natural prices, by the wage rate w should therefore always (as long as there are positive profits in each of the considered sectors) lead to a magnitude that is larger than the labor that has been embodied into this commodity or that represents is total cost of production in terms of labor. The first empirically important assertion thus here is that we can easily get an upper estimate for labor values by just dividing actual prices – not only natural prices – by the money wage rate.

8.3.2 Mathematical Foundations

We now provide a formulation of the fundamental Frobenius–Perron theorem on nonnegative square matrices that is based on Nikaido (1968, p. 102) and that leads us on a route with many intuitively plausible properties of the dominant eigenvalue of such matrices. Note that the following theorems are all concerning maximal rates of profit for intermediate input–output relationships augmented or not augmented by the numéraire wage basket c and the real wage ω.

According to the fundamental theorem of algebra, the roots of the polynomial $\det(\lambda I - A)$ of degree n in the variable λ are always n in number when solved in plane of complex numbers. The generally complex-valued eigen-vectors corresponding to these eigenvalues may not reach the same dimensionality if multiple roots occur in the above characteristic polynomial of the matrix A. There are thus a number of difficulties when arbitrary matrices A and all of their eigen-values and eigen-vectors are considered. In the case of square semipositive matrices however we can – in close correspondence to the Classical considerations of natural prices of balanced growth paths – concentrate on so-called dominant roots and can then formulate the following set of propositions which by and large are quite intuitive in nature when the dominant roots are transformed to the uniform rate of profit they imply.

Theorem 8.4. *Let A be nonnegative and square. Then:*

1. *A has a nonnegative eigenvalue. A nonnegative eigenvector is associated with the largest λ, to be denoted by $\lambda(A)$, of all the nonnegative eigenvalues λ.*
2. *$I - (1+r)A$ is nonnegatively invertible if and only if $r < 1/\lambda(A) - 1$.*
3. *If $Ax \geq \lambda x$ for a real number λ and a semipositive vector $x \geq 0$, then $\lambda(A) \geq \lambda$.*
4. *$\lambda(A) \geq \|\lambda\| = \sqrt{a^2 + b^2}$ for any eigenvalue $\lambda = a + bi$ of A.*

There are further remarkable, to some extent intuitively plausible, properties of the dominant root $\lambda(A)$ of Theorem 8.4.

Theorem 8.5. *Let A be nonnegative. Then:*

1. *$\lambda(A) = \lambda(A')$.*
2. *$\lambda(\alpha A) = \alpha\lambda(A)$ for $\alpha \geq 0$.*
3. *$\lambda(A^k) = (\lambda(A))^k$ for any positive integer k.*
4. *$\lambda(A) \geq \lambda(B)$ if $A \geq B \geq 0$.*
5. *$\lambda(A) \geq \lambda(B)$ for any principal minor matrix B of A.*
6. *$\lambda(A) = 0$ if and only if $A^k = 0$ for some positive integer k.*

The reader is again referred to Nikaido for the proofs of the assertions of this theorem. The Proposition 8.3 in particular allows to establish the result on the wage-profit frontier shown in Fig. 8.1 for the case of a multisectoral economy too. To show

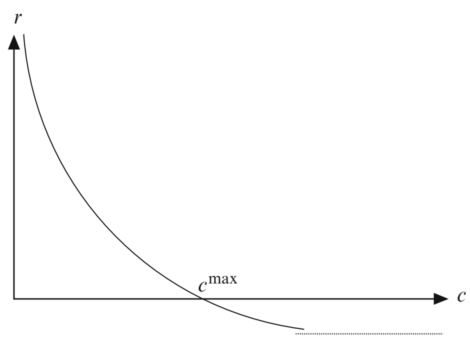

Fig. 8.1 The real wage/profit curve underlying the conflict about income distribution and about work conditions in the crude state of the society

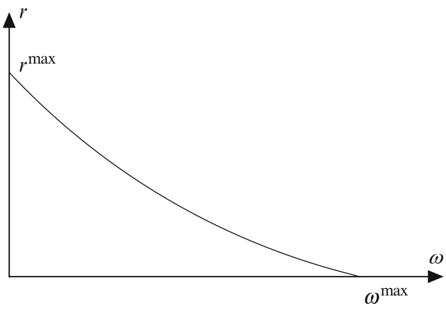

Fig. 8.2 The wage-profit curve and the conflict over income distribution in the production of commodities by means of commodities case

this we consider the matrix $A^c(\omega)$ presented at the beginning of this section. Using again the normalization rule $pc = 1$ for natural prices p in place of $w = 1$ then provides us with the equation system:

$$p = (1+r)pA^c(\omega) = (1+r)p[A+\omega cl] = (1+r)[pA+\omega pcl] = (1+r)[pA+\omega l]$$

where ω is the real wage in terms of the wage basket c and determines the scale of consumption of this basket that workers can realize.

We know that $r = r^{\max}$ if $\omega = 0$ holds true. And we expect that the normalization $pc = 1$ here defines a wage-profit curve $r(\omega)$ or $\omega(r)$ that is decreasing, see Fig. 8.2. This is shown with the help of theorems 2/3 as follows. We first recall that cl is a square matrix of the same type as A, since c is a column vector of dimension n and l a row vector of this dimension. The $n \times n$ matrix $\omega cl = \omega(cl_1, ..., cl_n)$ in fact provides the consumption of workers of the wage basket c for each production sector j here considered (cl_j), now in the same form as the consumption of intermediate inputs $A_{\star j}$. Since the thus augmented matrix $A^c(\omega) = A + \omega cl$ is decreasing with ω, we thus immediately get that also $\lambda(A^c(\omega)$ is decreasing with ω and $r = 1/\lambda(A(\omega)) - 1$ therefore increasing with decreasing ω.

However, in order to obtain strictly decreasing (increasing) in the considered situation we have to restrict our considerations to matrices A^c that are indecomposable (or that represent a non-empty basic sector of the considered economy) as the following theorem makes clear.

Theorem 8.6. *Let A be nonnegative and indecomposable. Then:*

1. *Any nonnegative eigenvector associated with $\lambda(A)$ is positive. Moreover $\lambda(A) > 0$.*
2. *The eigenvector of A associated with $\lambda(A)$ is unique up to multiplication by scalars.*
3. *$\lambda(A)$ is a simple root of its characteristic equation.*
4. *If $A \geq B \geq 0$, and one of A or B is indecomposable, then $\lambda(A) > \lambda(B)$.*

We thus have positive profit factors and positive and unique (up to scale) prices $1 + r$, p associated with such situations. Furthermore, considering the case $A^c(\omega)$ and varying ω now leads to strictly decreasing reaction of the rate of profit r. Note however that neither convexity nor concavity will characterize the resulting strictly decreasing wage profit curve in general.

In the simple case of only two commodities we have as formula for the eigenvalues of the matrix $A = (a_{ij})_{i,j=1,2}$:

$$\lambda_{1,2} = trA/2 \pm \sqrt{(trA)^2/4 - \det A} = \frac{a_{11} + a_{22}}{2} \pm \sqrt{\left(\frac{a_{11} - a_{22}}{2}\right)^2 + a_{12}a_{21}}$$

and in particular only real solutions in this case, since at least must be real. In the case of only one basic commodity ($a_{21} = 0$) this in particular gives

$$\lambda_{1,2} = \frac{a_{11} + a_{22}}{2} \pm \sqrt{\left(\frac{a_{11} + a_{22}}{2}\right)^2 - a_{11}a_{22}} = \frac{a_{11} + a_{22}}{2} \pm \frac{\|a_{11} - a_{22}\|}{2}$$

In the case $a_{11} \geq a_{22}$ this gives as eigen-values $a_{11} > a_{22}$ and in the opposite case ($<$) just the opposite order of these eigen-values (with right and left hand eigenvectors $x' = (1, 0), p = (1, a_{12}/(a_{11} - a_{22})$ and $x' = (1, a_{11}/(a_{22} - a_{11}), p = (0, 1))$. In the even simpler case where also $a_{22} = 0$ holds we finally simply get $\lambda(A) = a_{11}$.

It is indeed not implausible to assume that the matrix $A^c(\omega) = A + \omega cl$ is indecomposable, since this amounts to assuming that each good is used directly or indirectly for the production of intermediate goods or consumption goods of workers (so that not too many goods are excluded from consideration). There is thus no selection of commodities possible that do not feed back into the sector of basics and are thus of the type of luxury goods to be used outside the sector of worker households. For indecomposable $A^c(\omega) = A + \omega cl$ we then indeed get from theorems 2/3 that $\lambda(A^c(\omega))$ is strictly decreasing with ω and thus $r = 1/\lambda(A^c(\omega)) - 1$ strictly increasing with this measure of the real wage. Furthermore, there is a unique value $\omega^{\max}$ where

$$\lambda(A^c(\omega^{\max})) = 1, \quad i.e., \quad r(\omega^{\max}) = 0$$

must hold true, where therefore ω has become so large that the uniform rate of profit has become zero. There is thus in the multisectoral economy a wage-profit curve as shown in Fig. 8.1 (there measured in terms of commodity 1 in place of the basket c), but one that need not be convex (or concave) and that is in sum defined by:

$$p = (1 + r)[\omega l + pA], \qquad pc = 1, \quad \omega \in [0, \omega^{\max}]$$

for some maximum real wage $\omega^{\max} \in \Re$.

We note finally that in the case of decomposable matrices $A, A^c, A^c(\omega), \ldots$ we shall consider only cases where at least one basic commodity exists and we assume with respect to its canonical form in terms of basic and non-basic commodities (represented by the matrices A_{11}, A_{22}):

$$\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}: \qquad \lambda(A) = \lambda(A_{11}) \geq \lambda(A_{22}),$$

with A_{11} a primitive matrix in addition (no cyclical hierarchies in the ordering of industries).[8] Following U.Krause's suggestion we call such a matrix a Sraffa matrix and note once again that the matrix A_{22} can be further structured in the way we have structured the matrix A, and so on. We note also that the assumption of the existence of basics implies that all columns in the matrix A_{12} must be semi-positive, but that A_{22} may consist of zeros throughout (as in the basis example of the preceding section). Matrices where $\lambda(A_{11}) < \lambda(A_{22})$ holds and where thus the non-basic commodities dominate the basic ones, since their maximum profit rate restricts the profit rate of the basic sector, have no real economic meaning, as the Frobenius-Perron theorems suggest (the prices of the basic commodities are then zero in general), and are therefore excluded from all following considerations.

Due to what has just been assumed, the maximum rate of profit supported by the basic sector is at most equal to the one of the sector of non-basics[9] implying that the Leontief-inverse for non-basics is always well-defined if this holds true for the sector of basic commodities. In such a case the Leontief-inverse of the full matrix A can be shown to be of the form:

$$(I-(1+r)A)^{-1} = \begin{bmatrix} (I-(1+r)A_{11})^{-1} & (I-(1+r)A_{11})^{-1}A_{12}(I-(1+r)A_{22})^{-1} \\ 0 & (I-(1+r)A_{22})^{-1} \end{bmatrix}.$$

In the opposite situation the sector of non-basics would represent a limitation for the rate of profit of the sector of basics and would imply economically strange results if its maximum rate of profit is approached.

8.4 A Streetcar Named Desire: The von Neumann Production Price Model

So far we have considered only output matrices B of the following type

$$B = I = \begin{pmatrix} 1 & \dots & 0 \\ \dots & & \dots \\ \dots & \ddots & \dots \\ 0 & \dots & 1 \end{pmatrix}, \text{ i.e.,}$$

each sector produced a single and unique commodity with a single production method. We now go to the other extreme of a general system of joint production, allowing in addition for a multiplicity of production techniques even if there is no joint production, so for example a sector which produces a single and unique commodity

[8] Note also that the matrix A_{11} represents a principal minor of the matrix A.

[9] This is a quite natural assumption, since the matrix A_{22} neglects all inputs of the basic sector into the sector of the non-basic commodities.

j by means of k production techniques, i.e., activity vectors $A_{*j_1}, ..., A_{*j_{k(j)}}$. This special case of multiple production techniques would then be represented by the partial input–output structure

$$(A_{*j_1}, \ldots, A_{*j_{k(j)}}) \mapsto (e_j, \ldots, e_j),$$

with $e'_j = (0, ..., 0, 1, 0, ...0)$ the jth vector in the canonical basis of $\mathbb{R}^n$. In complete generality we would then proceed to assume given input and output matrices A, B with m columns (representing m different activities) and n rows (representing n different commodities) and would only demand that $A \geq 0, B \geq 0$ holds true. In the extreme it could therefore be that every process produces every commodity ($B > 0$) and that the only restriction then is that the number of such processes is finite.

From the empirical point of view, regarding firms and the input–output tables of the System of National Accounts, the natural accounting unit is one year. We thus do not follow here the practice of von Neumann models to assume a fictitious period of production (of which all real periods of production are a multiple or of which it is a common divisor). This modeling technique would imply the introduction of semifinished products after each unit period[10] and their treatment as if they were marketed commodities, which they are not. We consider the period of production as a non-technological, indeed an accounting concept which represents information on costs and proceeds (inputs and outputs) on a calendar basis. Leaving such conventional timing of inputs and outputs aside, would allow for no real alternative, since there is no common unit for the production period of the considered n commodities in real life and, if it would exist, it would imply an enormous increase in the dimension of the employed matrices A, B that would create enormous calculation difficulties and expenses in reality which only purely theoretical oriented von Neumann modelings can neglect.

We therefore restrict ourselves to input matrices A and output matrices B, the latter representing pure joint production only, with the calendar year as implicit reference time unit. On the level of actually constructed input and output tables, these two matrices are called absorption and make matrices, respectively. We here finally observe that, in the treatment of fixed capital, leading to fictitious, i.e., generally non-marketable outputs (used capital goods) as well, we also do not follow the philosophy of the von Neumann model. In this approach used capital goods are also treated as if there were markets where they can be sold, which is rarely the case. Instead, we shall (in Part II of the book) follow again established accounting practices of firms and in the System of National Accounts, and will introduce an appropriate stock matrix C in this Part II of the book when the treatment of fixed capital is introduced into the classical approach of values and prices.

Having redefined the range of applicability of the von Neumann model, we now proceed to its standard formulation and the assumptions that are generally made to show existence and uniqueness of its solution concept.

[10] and old machine vintages if fixed capital is considered in addition.

We assume in the following that the input and output matrices $A = (a_{ij}), B = (b_{ij}),\ i = 1, ..., n,\ j = 1, ..., m$, represent the unit intensities of a linear technology with a_{ij} the quantity of good i required per unit intensity of technique j and b_{ij} the quantity of good i produced per unit intensity of technique j. We denote by ℓ_j the labor inputs of these techniques and by s_i the quantity of good i required as subsistence per worker. The matrix $S = (s_i \ell_j), i = 1, ..., n, j = 1, ..., m$ is of the same dimension as the matrix A of physical inputs. We define the augmented input matrix A^s by $A + S$ and thus have that the consumption of the workforce is included in input matrix A^s in physical terms, as necessities of life consumed by workers in each activity. We assume finally that each techniques uses at least one input ($A_{\star j} \geq 0$) and that each good can be produced by at least one techniques ($B_{i\star} \geq 0$) and finally that labor supply is unlimited and thus does not represent a bottleneck for the following solution of the model.

We denote by $p = (p_1, ..., p_n)$ the price-vector for the given n commodities and by $x' = (x_1, ..., x_m)$ the vector of intensities at which the given techniques are operated . We are interested in vectors x and p that, in the first case, represent a balanced expansion path with a common expansion factor $1 + g$ for those goods that are in fact commodities and that, in the second case, represent a balanced profitability situation with a common gross rate of profit $1 + r$ for those techniques that are operated. This amounts to assuming

$$\sum_{j=1}^{m} b_{ij} x_j \quad \geq \quad (1+g) \sum_{j=1}^{m} a^s_{ij} x_j, \quad i = 1, \ldots, n \tag{8.10}$$

with $p_i = 0$ for all i where $' >'$ applies, and

$$(1+r) \sum_{i=1}^{n} p_i a^s_{ij} \quad \geq \quad \sum_{i=1}^{n} p_i b_{ij}, \quad j = 1, \ldots, n \tag{8.11}$$

with $x_j = 0$ for all j where '>' holds true. Goods in access supply with respect to the factor g thus become free goods in this search for balanced growth and profitability situations, and processes with inferior profitability become extinct.

The above model, the von Neumann (1945) model of economic growth, can be rewritten in matrix notation as follows:

$$(B - (1+g)A^s)x \geq 0, \quad x \geq 0 \tag{8.12}$$

$$p(B - (1+g)A^s)x = 0 \tag{8.13}$$

$$p(B - (1+r)A^s) \leq 0, \quad p \geq 0 \tag{8.14}$$

$$p(B - (1+r)A^s)x = 0 \tag{8.15}$$

If x, p are solutions, then, of course, $\alpha x, \beta p$ are also solutions for any $\alpha, \beta > 0$. The model (8.12)–(8.15) therefore involves the determination n+m unknown, including the common rate of growth g and the common rate of profit r.

Proposition 8.7.

1. *There is a solution* (x, g, p, r) *of the model (8.12)–(8.15) which satisfies* $pBx > 0$ *and:*

$$p_i = 0 \quad \textit{iff} \quad (1+g)\sum_{j=1}^{m} a^s_{ij} x_j < \sum_{j=1}^{m} b_{ij} x_j$$
$$x_j = 0 \quad \textit{iff} \quad (1+r)\sum_{i=1}^{n} p_i a^s_{ij} > \sum_{i=1}^{n} p_i b_{ij}$$

2. *There holds* $g = r > -1$ *for any* (x, g, p, r) *which satisfies (8.13), (8.15) and* $pBx > 0$.
3. *We get that* $g = r > o$ *holds if there is* $x \geq 0$, *s.t.* $(B - A^s)x > 0$, *i.e., if the von Neumann model is productive.*

Proof. See Nikaido (1968, pp. 145–147).

Proposition 8.7 ensures an economically meaningful solution to (8.12)–(8.15) if the productiveness condition $Bx > A^s x$ is made. It, however, does not ensure the uniqueness of the common growth and profitability factor $1 + g = 1 + r$. Nikaido (1968, p. 147) gives in this matter the following simple example for nonuniqueness:

$$A^s = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$$

where one has the solutions

$$x' = (1, 1), \quad p = (1, 0), \quad g = r = 1 \tag{8.16}$$
$$x' = (0, 1), \quad p = (1, 1), \quad g = r = 2 \tag{8.17}$$

This is an economically decomposable system of the simple sort considered in standard input–output analysis and thus requires no sophisticated output matrix B in order to get this result. Von Neumann (1945) himself assumed $A^s + B > 0$ in order to get the uniqueness of $g = r$ and justified this assumption by means of arbitrarily small additions to the originally given matrices, for example, as follows

$$A^s = \begin{pmatrix} 1 & 3\epsilon \\ 2\epsilon & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}, \qquad \epsilon \quad \text{small}$$

□

In our view this does however not solve the problem, since the size of the additions influences the solution to be obtained in a radical way. In the next section, we shall therefore simply assume uniqueness (for square-systems) in order to investigate the stability of the activity vector x and the price system p if the economy is not situated in a von Neumann equilibrium situation. In Woods (1978, p. 282) it is briefly stated that decomposability (appropriately applied to joint production systems[11]) is a necessary, but not a sufficient condition for multiple growth rates to occur.

We continue this section with a brief discussion of the important special case of multiple activities, but no joint production. In this case we have a rectangular output matrix of the following type

$$B = \begin{pmatrix} 1\ldots 1\ 0\ldots 0\ldots\ldots\ldots 0\ 0\ldots 0 \\ 0\ldots 0\ 1\ldots 1\ldots\ldots\ldots 0\ 0\ldots 0 \\ \ldots \quad \ldots\ldots\ldots\ldots\ldots \quad \ldots \\ \ldots \quad \ldots\ldots\ldots\ldots\ldots \quad \ldots \\ \ldots \quad \ldots\ldots\ldots\ldots\ldots \quad \ldots \\ 0\ldots 0\ \ldots\ldots\ldots\ldots\ldots 0\ 1\ldots 1 \end{pmatrix}$$

with k_1 activities for the production of commodity 1, k_2 for commodity 2 and so forth, up to k_n, each case represented by the '1's in one row of the matrix B. The matrix B, and also A^s, are therefore of dimension $n\cdot(k_1+...+k_n)$ in this case. Selecting one process per commodity in order to form square subsystems $A^s_n, B_n = I$ of the given input–output structure allows for the selection of $\prod_{i=1}^n k_i$ possible square subsystems of this type.

Let us now select that matrix A^{s*}_n from the above subsystems which has the smallest Frobenius or dominant root $\lambda(A^{s*}_n) \leq \lambda(A^s_n)$ and thus the fastest expansion path among all these alternatives. We assume here for simplicity that A^{s*}_n is indecomposable and thus know that

$$\lambda(A^{s*}_n) < \lambda(A^s_n)$$

must hold true if $A^s_n \neq A^{s*}_n$, up to flukes (special cases which we here exclude from consideration). We furthermore assume that this matrix fulfills $\lambda(A^{s*}_n) < 1$ and thus have a unique solution

$$A^{s*}_n x^*_n = \lambda(A^{s*}_n)x^*_n, \quad p^* A^{s*}_n = \lambda(A^{s*}_n)p^*$$

with $x^*_n, p^* > 0$ and $r^*_n = 1/\lambda(A^{s*}_n) - 1 > 0$. In this situation we then get the following proposition:

Proposition 8.8.

1. *The equations (8.12)–(8.15), with* $pBx > 0$ *added, have the solution*

[11] The input and output structure A, B may be called decomposable if there is a proper subset of goods that can be produced by using only inputs from this proper subset.

$$x^* = \check{x}_n^*, \quad p^*, \quad g^* = r^* = 1/\lambda(A_n^{s*}) - 1$$

where x^* denotes the activity vector of $\mathbb{R}_+^m$ where all components corresponding to activities in A^s not present in A_n^{s*} are zero and all other given by the vector x_n^*.

2. *If $x, p, g = r$ solve (8.12)–(8.15), with $pBx > 0$, then $x = c_1\check{x}^{s*}, p = c_2 p^*$ for positive constants c_1, c_2 and $g = r = 1/\lambda(A_n^{s*}) - 1 > 0$.*

Proof. See Woods (1978, pp. 274ff.).

Selecting the square subsystem with the smallest Frobenius root therefore provides us (when trivially expanded) with the unique von Neumann solution of a system with multiple activities and no joint production.

Returning to the case of joint production, we close this section with a result from Bidard (1986, p. 412) which reads as follows: □

Proposition 8.9. *From an indecomposable generic (non-exceptional[12]) von Neumann model of production, A^s, B, it is possible to extract one square system A^{s*}, B^* consisting of operated methods and commodities (goods with positive prices) at the rate $g^* = r^*$, such that*

1. *Production methods, resp. goods, inside the truncation are efficient, resp. not overproduced.*
2. *The row and column vectors p^*, x^* of A^{s*}, B^* associated with $\frac{1}{1+r^*}$ are positive and (up to a factor) unique and when completed by zeros represent equilibrium prices and equilibrium activity levels of the whole von Neumann model of production.*
3. *There exists an open interval $(\bar{r}, r^*)$ on which $B^* - (1+r)A^{s*}$ is nonnegatively invertible.*

Proof. See Bidard (1986). In almost all cases, we can therefore assume that the number of efficient activities (with respect to the rate of profit they allow for) equals that of commodities (goods not produced in excess) and, moreover, that the square active part A^{s*}, B^* of A, B we determined above satisfies the side-condition $(B^* - (1+r)A^{s*})^{-1} > 0$ for all rates of profit r sufficiently close, but strictly below the rate r^*. This latter property becomes important if (part of the) labor effort l is not included by means of a subsistence basket into the activities shown which then allows again the direct calculation of prices of production, in terms of labor commanded, by means of an expression of the type $p^* = l^*(B^* - (1+r)A^*)^{-1}$. □

We have explicitly excluded semifinished goods and fixed capital from our interpretation of the equations of the von Neumann model. Our reason for this is that their inclusion in the output matrix B, has little to do with what firms actually do in treating these complications in their production activities. Moreover, the uniform period of production that the von Neumann model is assuming for all of its activities is a purely hypothetical one in this approach and can in fact be of any length (whereby the structure of semifinished goods and vintages of machinery becomes

[12] See Bidard (1986) for details.

purely arbitrary). Yet, if the restriction of long-period price accounting to a uniform rate of profit matters at all, it must be applicable to the rate of return calculations that firms actually perform.

This implies that the proper period of time for such profitability calculations is the year and that moreover the book-keeping methods that the accountants of firms actually apply will matter for the determination of prices of production. The relevant concepts here are: capital consumed (a flow magnitude) and capital advanced (a stock magnitude) and corresponding turnover times which should replace the common usage of the terms circulating and fixed capital, at least in their interpretation in physical terms. For all commodities employed in the production process (raw of auxiliary materials, plant and equipment, semi-finished goods and cash – and also labor) there exist some capital advancements, since their are some funds tied up in their usage.

Fixed capital is normally associated with plant and equipment and their lifespan, but in cost accounting it is not life span that matters, but their turnover time τ_{ij} (of commodity j in process j (which is normally less than the physical and also the economic life span of such commodities, due to a flexible financial management of firms.[13] Moreover, the distinction between funds tied up and consumed also matters for so-called circulating capital which in sum implies that we get from such a procedure a flow matrix A (as before, but with respect to the year as time unit now) and a stock matrix K, which is related to the flow matrix by means of turnover times as follows: $k_{ij} = \tau_{ij} a_{ij}$ for all commodities i that enter the production of commodity j (in the single output case).

The details of this approach to a definition of prices of production are explained in Bródy (1970), who also discusses the importance that Marx gave to it in the second volume of 'Capital'. We will use such an approach in the remainder of this chapter from the empirical point of view (for the German economy as it was already considered in Chap. 3 here). Another issue that should be mentioned in this respect is the level of aggregation to which such calculation of prices of production should be applied. Due to the co-existence of old and new techniques in industries and sub-industries at each moment of time it seems that the industry level is the lowest disaggregation level to which the concept of prices of production can be meaningfully applied. People using the von Neumann approach (or Sraffians) normally assume that their input–output coefficients are physical magnitudes and thus apply their approach on the highest level of disaggregation (and to an unspecified period of production). They even assume that the choice of technique is driven by prices of production on this no-aggregation level and thus assume that every action is driven by this uniformity principle (for every vintage of fixed capital). This is an assumption that is far beyond anything that is happening in reality. One need not go as far as Farjoun and Machover (1983) in ones characterization of what is happening on the level of actual price formation to see the purely hypothetical nature of such a calculation of accounting prices of production.

[13] With straight line depreciation and immediate reinvestment of depreciation turnover-time is one-half of life span.

Moreover, accounting prices of production, if applicable on a certain level of aggregation, have not much to do with the actual choice of technique at least in modern capitalist societies (not mirrored by the two sectoral treatment we gave at the beginning of this chapter). Investment decisions primarily concern the choice of most profitable new activities (in an environment where average and even more old fashioned methods of production may still be applied for a considerable span of time). Such investment criteria are much more complicated than the average rate of return calculation offered by Classical prices of production. These prices of production may show how successful firms were in the past compared to their competitors in the same branch, but they have little to do with switches from one technique to another on the level of aggregation where this concept may be a meaningful one.

In sum we would therefore conclude that the burden of proof of the meaningfulness of prices that are based on the principle of a uniform profitability throughout the economy lies in the hands of those that use such prices for the discussion of the evolution of capitalist economies. This concerns the level of aggregation to be chosen, the use of their calculation as in Bródy (1970) augmented by output matrices that exhibit subsidiary production, the demonstration that these prices are centers of gravity of actual market prices and the empirical investigation of how close actual economies are to the Classical situation of a uniform rate of profit as a meaningful restriction for price formation. This last point will be the focus of interest of the now following sections.

8.5 Differentiated Sectoral Wage and Profit Rates

An economy in the real world cannot be expected to be characterized by a uniform rate of profit as it is formulated by the equations of the von Neumann model in the circulating capital framework.[14] One objection is the conceptual problems of different sectoral turnover times of the intermediate inputs, that is, the time it takes until these capital outlays are recovered. Moreover, the sectoral profitabilities will differ from each other for various systematic reasons. Even if the profit rates were directly comparable, some sectors will persistently maintain a profit rate above average as a premium for a higher risk they incur, or because of certain oligopolistic or monopolistic features, which are mainly connected to the degree of concentration, the extent of entry barriers, and the degree of collusion between firms.[15]

As an example for a systematic discrepancy of profit rates we may refer to a study by Duménil and Lévy (2002) where, on the basis of several definitions of profits accounting for interest, taxes and inventories, the authors find out that industries in

[14] This part of the chapter is based on Flaschel and Franke (2008, Chap. 4) where further details on its arguments can be found. I have to thank Reiner Franke for allowing to reuse the material from there for the following sections. The reader is moreover referred to Bródy (1970) for the general input–output methodology that is underling the following sections.

[15] See, e.g., Semmler (1984, p. 106) for this classification.

Table 8.1 Sectoral price components for Germany 1995

	1	2	3	4	5	6	7	sum or average
X_i:	43,910	347,001	732,445	245,606	571,317	740,861	429,290	3,110,430
II_i:	21,162	218,765	459,154	128,434	235,144	277,395	107,706	1,447,761
W_i:	9,382	99,663	196,801	78,819	180,355	178,708	253,172	996,900
D_i:	7,871	17,267	46,502	5,860	52,430	94,088	42,452	266,470
$\pi_i^{(1)}$:	13,366	28,573	76,490	38,353	155,818	284,758	68,412	665,769
$\pi_i^{(2)}$:	5,495	11,306	29,988	32,493	103,388	190,670	25,960	399,299
$r_i^{(1)}$:	63.16	13.06	16.66	29.86	66.26	102.65	63.52	45.99
$r_i^{(2)}$:	18.93	4.79	5.93	24.20	35.95	51.33	17.29	23.29
L_i :	1,117	2,301	6,216	3,266	6,272	9,449	8,761	37,382
w_i:	8,399	43,313	31,660	24,133	28,756	18,913	28,898	26,668

Note: X_i, II_i, W_i, D_i are sectoral gross outputs, intermediate inputs, wage payments, and depreciation, respectively (in mill. Euro; nominal and real values are identical in 1995). $\pi_i^{(1)}$ and $\pi_i^{(2)}$ are total profits, where $\pi_i^{(1)} = X_i - II_i - W_i$ and $\pi_i^{(2)} = \pi_i^{(1)} - D_i$. $r_i^{(k)}$ are the corresponding profit rates, here defined as $r_i^{(1)} = \pi_i^{(1)}/II_i$ and $r_i^{(2)} = \pi_i^{(2)}/(II_i + D_i)$. L_i is the number of persons (in 1,000) working in the sector, w_i is the sector's (makeshift) average wage rate per year (in Euro), $w_i = W_i / L_i$.

the U.S. with very large capital-labor ratios are totally different from other industries in that they persistently earn an extremely low rate of profit.[16]

Evidently, wage rates are not uniform, either. This will even be true if they are aggregated across larger sectors such as in our 7-sectoral standard aggregation. To get a first impression of the sectoral differentiation of profits and wages, we compile the basic data for Germany in Table 8.1, again for the year 1995 and with respect to our standard aggregation.

We need not bother about real and nominal values in the table since for 1995 the two are identical. The coincidence is especially convenient for depreciation, which does not refer to the single capital goods installed in the sector but is only reported as a monetary aggregate.[17] Including depreciation in the specification of (gross) profits, Table 8.1 makes clear that this item is of significant numerical importance. Its impact is seen by a direct comparison of the two sums of sectoral profits $\pi_i^{(1)}$ and $\pi_i^{(2)}$, where $\pi_i^{(1)}$ are sector i's output minus intermediate inputs minus wages, and in $\pi_i^{(2)}$ costs include depreciation D_i as an additional factor. The corresponding sectoral rates of profit are $r_i^{(1)}$ and $r_i^{(2)}$, which are obtained by relating these profits to the intermediate inputs. The last entry in these two rows of Table 8.1 is the respective average rate of profit.

[16] A discussion of older studies on differential profit rates or profit margins is given in Semmler (1984), especially in Chap. 4.

[17] 1995 is the last year for which presently real data on depreciation are available. This is the main reason why in the empirical tables before we have not presented more recent data.

The sectoral differences in the profit rates are conspicuous, both for $r_i^{(1)}$ and $r_i^{(2)}$. It is hardly imaginable that they could be explained by different turnover times alone. Even if the low rates $r_i^{(2)}$ for the industrial sectors $i = 2, 3$ are corrected for their presumably short turnover times, we would still expect them to be below average. On the other hand, this feature would be well compatible with Duménil and Lévy (2002) result from above that sectors with large capital-labor ratios tend to have low rates of profit. Though we do not know the capital-output ratios of sectors 2 and 3, they should in any case not be too low, while the low labor-output ratios have been observed earlier.

The differences in the sectors' (average) wage rates as well as the order will not come as a great surprise. It should nonetheless not go unnoticed that all wage rates are somewhat downward biased. As already mentioned, the L_i-statistics reproduced in Table 8.1 include independent business men, too, whereas W_i are the wage payments to the employed persons only. Especially for $i = 1$ and, to a lesser degree, $i = 6$, the agricultural and consumer services sectors, the ratios $w_i = W_i/L_i$ will thus involve a certain error. It can, however, be accepted in the wage variations we will study below, since the wage rates relative to each will assumed to remain fixed.

Despite the strong dispersion of wage rates w_i and profit rates $r_i^{(1)}$, there is a remarkable correspondence to the standard wage-profit curve under the supposition that the underlying wage basket has the same composition as the economy's actual consumption vector in 1995. If we assume a real wage rate equivalent to 26,668 Euro, then a uniform rate of profit $r = 45.5\%$ results. This value comes very close to the average profit rate of 45.99% which we obtain in Table 8.1. So, is the assumption of uniform wage and profit rates not too bad after all?

The astonishing numerical match of the theoretically motivated uniform rate of profit and the empirical average profit rate notwithstanding, we take Table 8.1 and the aforementioned paper by Duménil and Lévy (2002) as evidence that a nonnegligible dispersion of wage and profit rates is a relevant empirical phenomenon. In the following we entertain the view that the dispersion is not just temporary but, for whatever reasons in details, exhibits some persistence. Accordingly, we seek to incorporate it into the production price modeling approach, though we still neglect capital depreciation and continue to divide profits by the intermediate inputs.

The formulation of differentiated wage and profit rates is straightforward. Considering both versions with wages paid ex-post and ex-ante, it reads

$$p_i = (1 + r_i)\,(pA)_i \; + \; w_i\,\ell_i \tag{8.18}$$

$$p_i = (1 + r_i)\;[\,(pA)_i \; + \; w_i\,\ell_i\,] \tag{8.19}$$

where, of course, i runs from 1 to n. We study this set of relationships from three different angles. First the profit rates $r_i \geq 0$ and the nominal wages rates $w_i > 0$ are taken as exogenously given and we ask for the conditions on the profit rates that permit an economically meaningful solution for the price vector p. Second, the real sectoral wage rates are treated as given and we characterize the set of profit rates $r_1, \ldots, r_n$ for which a meaningful price vector p exists. Finally, in a third point, we postulate a fixed structure of the profit and wage rates, in the form of constant ratios, and derive a wage-profit frontier in this setting.

To begin with the first question, it should be clear that here the real wage rates are determined as residuals. As before, they should come out rather low in the presence of large profit rates r_i. In the special case of equal profit rates, we know that they must fall short of the maximum rate of profit R. Dropping the equalization assumption, it may be expected that some sectors may raise their rate above this boundary, provided that the other sectors remain below it. But certainly, there should be other upper-bounds (for the sectors producing basic commodities). The following proposition, which we have borrowed from an Italian article by Grillo (1976), gives a precise condition (we correct the quoted paper for a slight imprecision). What thus comes about is also a nice connection to the quantity side.

Proposition 8.10. *Let $r_i \geq 0$ and $w_i > 0$ be given in (8.18) and (8.19). Then each of the two price systems has a meaningful solution $p \geq 0$ if and only if the sectoral rates of profit are sufficiently low in the following sense: there exists a gross output vector $x \geq 0$ together with a corresponding net output vector $y = x - Ax \geq 0$ such that*

$$r_i \leq \frac{y_i}{x_i - y_i} = \frac{y_i}{(Ax)_i} \qquad \textit{for all } i$$

and in at least one component the inequality is strict (in the presence of non-basics, the formulation admits the case $(Ax)_i = 0$, when r_i may become arbitrarily large).

In each sector i, therefore, the profit rate must not exceed the "surplus ratio", which is given by the ratio of the final demand y_i for good i over the good's material inputs that are required by this and the other sectors to produce the economy's entire net product $y \in \mathbb{R}^n_+$. Note, however, that x and y need not be the quantities actually produced and demanded but any skillfully chosen vectors may do. In particular, with $x = (I - A)^{-1} y$ the condition for (8.18) and (8.19) to have a meaningful solution $p \geq 0$ may be rewritten as

$$r_i \leq R_i = R_i(y) := \frac{y_i}{[A\,(I - A)^{-1}\,y]_i} \quad \textit{for all } i\textit{, and '<' for at least one } i \tag{8.20}$$

Proof. To prepare the ground, put $q_i = 1 + r_i$ and observe that $r_i \leq y_i/(x_i - y_i)$ is equivalent to $q_i \leq x_i/(x_i - y_i)$. Furthermore, let Q be the diagonal matrix with entries q_i and abbreviate $\tilde{\ell}_i = w_i \ell_i$. $\tilde{\ell}$ being the corresponding row vector, (8.18) can then in compact form be written as $p = p\,AQ + \tilde{\ell}$. As for (8.19), simply specify $\tilde{\ell}_i = (1 + r_i) w_i \ell_i$. We use the fact that the solution $p = \tilde{\ell}\,(I - AQ)^{-1}$ exists and is semipositive if and only if the dominant eigen-value of AQ is less than unity.

Thus, to demonstrate that the proposition's condition is necessary, suppose $\lambda^\star(AQ) < 1$. Then there exists $d \in \mathbb{R}^n$, $d \geq 0$, such that $(I - AQ)d \geq 0$. Next, put $x = Qd$ and $y = (I - A)x$. We thus get $0 \leq (I - AQ)d =$

$(I-Q)d+(I-A)Qd = (I-Q)d+(I-A)x = d-Qd+y = Q^{-1}x-x+y$ and, by premultiplication with Q, $x - Q(x-y) \geq 0$. Considering this vector inequality component-wise and dividing the components by $x_i - y_i$ if the expression is positive yields the condition ($x_i - y_i$ is nonnegative anyway, so that the inequality sign is maintained; if $x_i - y_i = 0$, q_i and thus r_i may be arbitrarily large).

To show the reverse, suppose the condition is satisfied. Accordingly, let x, y be two semipositive vectors related by $x = Ax + y$ that entail $q_i \leq x_i/(x_i - y_i)$ for all i, where one inequality is strict (and $x_i - y_i = 0$ is admitted). Since $x_i - y_i = (Ax)_i \geq 0$, it is easily seen that the inequalities are equivalent to $x - Q^{-1}x \leq y$. Putting $d = Q^{-1}x$, this in turn is equivalent to $y - (Q-I)d \geq 0$. In this way we obtain $y = x - Ax = Qd - AQd = (Q-I)d + (I-AQ)d$, and furthermore $(I - AQ)d = y - (Q-I)d \geq 0$. Since $d \geq 0$, $\lambda^\star(AQ) < 1$ follows. □

It is intuitively clear that a situation where (8.20) is satisfied with equality for all i corresponds to overall zero wages $w_1 = w_2 = \cdots = w_n = 0$ in (8.18) or (8.19).[18] With respect to an underlying (hypothetical) net output vector y, the sectoral rates of profit $r_i = R_i(y)$ are maximal in the sense that no sector can possibly achieve a higher profit rate, unless (presumably) another sector lowers its rate (this presumption is verified below). In other words, a sector's maximum rate of profit depends on the other sectors' profit rates, and generally also on their distribution; it will make a difference whether two distinct profit rates r^a and r^b are, in that order, prevailing in sector j and k, or in sector k and j.

The *set $\mathcal{R}$ of all sectoral maximum rates of profit* can be described by letting the vector y vary over a suitable (by normalization) subset of the nonnegative orthant in the $\mathbb{R}^n$, a simplex let us say. We then have the characterization

$$\mathcal{R} = \{(r_1, \ldots, r_n) : \exists y \in \mathbb{R}^n_+ \text{ with } \textstyle\sum_j y_j = 1 \text{ and } r_i = R_i(y) \text{ for all } i\} \quad (8.21)$$

A sector i's maximum of all its (conditional) maximum rates of profit is obtained if all other sectors consent to zero profits. It can be immediately computed by substituting the i-th unit vector e_i for y in the expression $R_i(y)$,

$$\max_y R_i(y) = R_i(e_i) \quad (8.22)$$

which entails $r_j = R_j(e_i) = 0$ for all other sectors $j \neq i$. Observe that normally the $R_i(e_i)$ will not be equal across the sectors.

In a two-sector world, for example, the set $\mathcal{R}$ traces out a downward-sloping curve in the (r_1, r_2)-plane from end-point $(0, R_2(e_2))$ to the other end-point $(R_1(e_1), 0)$. The inner points of the curve can be computed by letting y_1 vary from 0 to 1 and putting $y_2 = 1 - y_1$, $r_1 = R_1(y)$, $r_2 = R_2(y)$. Since the $R_i(y)$ are fractions of the y-components, the connection between the two end-points will not be straight line.

[18] Formally, it suffices to replace the inequality signs with an equality sign in the proof of Proposition 8.10.

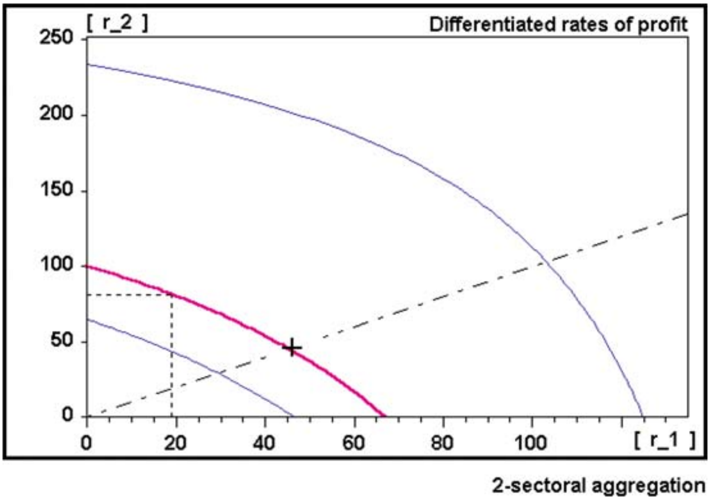

Fig. 8.3 Differentiated sectoral rates of profit under exogenous real wage rates
Note: With respect to the two-sectoral aggregation for $A = A_T$, the outer curve is the set of sectoral maximum rates of profit. The bold line represents the set $\mathcal{R}(\mathbf{!})$ of Proposition 8.11, where $\mathbf{!} = (\omega_1, \omega_2)$ are the empirical real wage rates expressed in the consumption basket c (see Table 8.2), while the lower curve depicts the same set for a 25% increase in real wages. The dotted lines indicate the empirical profit rates in 1995, $r_1 = r_2$ on the dash-dotted line, and $r_1 = r_2 = 45.99\%$ at the cross, the empirical average rate of profit.

Applying this procedure to the empirical two-sectoral matrix A_T, which we here treat as matrix A in the formal expressions above, yields the upper curve in Fig. 8.3. This geometric locus of the sectoral maximum rates of profit has obviously a concave shape. The area below this boundary represents the set of all sectoral profit rates (r_1, r_2) that are *a priori* admissible in (8.18) and (8.19).

After the complete characterization of admissible sectoral rates of profit and their upper boundary, we turn to the second point in the analysis of (8.18) and (8.19), where it is now the real wages that are assumed to be exogenously given. Measuring them again in a wage basket $c \in \mathbb{R}^n_+$, let ω_i be the corresponding real wage rates. Real and nominal wages are related by

$$w_i = \omega_i \, pc \tag{8.23}$$

Equations (8.18) and (8.19) can be more compactly written if we set up a matrix C defined by

$$c_{ij} := \omega_j \, c_i \, \ell_j \,, \qquad i, j = 1, \cdots, n \tag{8.24}$$

Matrix C is helpful in that it allows us to write the row vector $(w_1\ell_1, \ldots, w_n\ell_n)$ conveniently as pC. Entry c_{ij} is the quantity of good i consumed by workers that are occupied in producing one unit of good j, for which they spend a corresponding fraction of their nominal wage $w_i\ell_i$.

Furthermore, let Q be the diagonal matrix with entries $1+r_i$. To emphasize the dependence on the profit rates, we introduce the notation $\mathbf{r} = (r_1, \dots r_n)$ and write $Q = Q(\mathbf{r})$.[19] That is,

$$Q(\mathbf{r}) := \mathrm{diag}\,[1+r_1, 1+r_2, \cdots, 1+r_n] \tag{8.25}$$

Using (8.23), (8.24), (8.25) it is easily checked that (8.18) and (8.19) can be reformulated as

$$p = p\,[\,AQ(\mathbf{r}) + C\,] \tag{8.26}$$
$$p = p\,[\,(A + C)\,Q(\mathbf{r})\,] \tag{8.27}$$

It is clear that for (8.26) and (8.27) to admit a meaningful solution, wages must not be too high. This means a suitable production vector x should exceed the material inputs and workers' consumption associated with it. Expressed in more technical terms, not only matrix A but the matrix augmented by the consumption coefficients, $A+C$, is required to be productive. Beyond the existence of solutions, Proposition 8.11 establishes the trade-off of the sectoral profit rates that with respect to zero wages has already been alluded to above: one rate can only rise at the cost of some other. This relationship is strict if (not A but) $A + C$ is assumed to be indecomposable, which, however, we consider perfectly plausible (at least at positive wages).

Proposition 8.11. *Let the sectoral real wages $\boldsymbol{\omega} = (\omega_1, \dots, \omega_n)$ be incorporated in the matrix C as specified by (8.24) and suppose that $A + C$ is productive as well as indecomposable. Then the set $\mathcal{R}(\boldsymbol{\omega})$ of all sectoral profit rates $\mathbf{r} = (r_1, \dots, r_n)$ fulfilling (8.26) or (8.27), respectively, is a non-empty one-dimensional manifold in $\mathbb{R}^n_+$, where for every $\mathbf{r} \in \mathcal{R}(\boldsymbol{\omega})$ the corresponding price vector p is strictly positive. Choosing any sector k, its rate of profit r_k is represented by a differentiable function f_k of the other profit rates,*

$$\mathbf{r} \in \mathcal{R}(\boldsymbol{\omega}) \quad \textit{if and only if} \quad r_k = f_k(r_1, \cdots, r_{k-1}, r_{k+1}, \cdots, r_k).$$

In addition, all partial derivatives of f_k are negative (given that $r_k > 0$),

$$\partial r_k / \partial r_j = \partial f_k / \partial r_j < 0 \qquad \textit{for } j \neq k$$

We note that the proposition only says that a function f_k with the stated property exists, it does not claim that f_k can be expressed by an explicit closed formula.

[19] We should not, as it is possible with the other sectoral variables, use r to denote the profit rate vector, because this might lead to confusion in other parts of the book where r as a uniform rate is just a scalar. On the other hand, we do not wish to employ another letter for this purpose. Therefore the compromise with the (otherwise unnecessary) vector arrow above the letter r, which we below we equally apply to the vector of the differentiated real wage rates $(\omega_1, \dots \omega_n)$.

In fact, the computation of r_k involves (iteratively) solving an eigen-value equation, so that in practice one has to resort to special numerical methods requiring a computer.

Proof. We formulate the proof with respect to (8.26). The treatment of (8.27) is completely analogous.

The important thing to note is that, by virtue of the indecomposability assumption, an increase (decrease) of any profit r_i causes the dominant eigen-value of $AQ(\mathbf{r}) + C$ to increase (decrease) strictly. Moreover, a solution of (8.26) exists if $\mathbf{r}$ induces $\lambda^\star[AQ(\mathbf{r}) + C] = 1$; p is then the left-hand eigen-vector, which we know is strictly positive. Thus, one may begin with a sufficiently small vector $\mathbf{r}$ such that $\lambda^\star[AQ(\mathbf{r}) + C] < 1$, which is possible since the productivity assumption implies $\lambda^\star[AQ(0) + C] < 1$. Then any sector i may be chosen and its profit r_i increased until the dominant eigen-value equals unity.

Increasing a rate r_i in a situation where $\mathbf{r} \in \mathcal{R}(\boldsymbol{\omega})$ raises the eigen-value above 1. It can be brought back to this level by sufficiently lowering r_k. Existence of a function f_k and their partial derivatives follows from applying the Implicit Function Theorem to the equation in $\mathbf{r}$, $F(\mathbf{r}) := \det[\lambda I - AQ(\mathbf{r}) - C] = 0$, where the eigen-value λ is fixed at $\lambda^\star = 1$ (and since the determinant is differentiable to any desired order). □

If one has studied the proof of Proposition 8.10, one may note that the argument goes equally through if vector $\tilde{\ell}$ is replaced with the zero vector, the inequality signs with the equality sign, and the matrix A with the augmented matrix $A+C$. What is obtained in this way is the situation of (8.27). The profit rates satisfying this equation can thus be readily characterized by the next proposition, where in extension of Proposition 8.10 the concept of 'net output' now also means net of workers' consumption.

Proposition 8.12. *Let the sectoral real wages $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_n)$ be incorporated in the matrix C as specified by (8.24) and suppose that matrix $A + C$ is productive. Then for the sectoral profit rates $\mathbf{r} = (r_1, \ldots, r_n)$ a price vector $p \geq 0$ satisfying (8.27) exists if and only if there exists a gross output vector $x \geq 0$ together with a corresponding 'net output' vector $y = x - Ax - Cx \geq 0$ such that*

$$r_i = \frac{y_i}{x_i - y_i} = \frac{y_i}{(Ax)_i + (Cx)_i} . \qquad \textit{for all } i$$

Curiously enough, it does not seem possible to derive a similar statement for system (8.26) with wages paid ex-post, since the relationships showing up in the proof cannot in a likewise manner be solved for r_i on one side of the equation and no other profit rates on the other side.

In the framework of a two-sectoral world, the frontier of the maximum sectoral rates of profit of Proposition 8.10 has already been illustrated by the outer curve in Fig. 8.3. We now want to do the same for Proposition 8.11, limiting ourselves to wages paid ex-post. To this end we have to determine empirical values for the two-dimensional vectors ℓ, c and $\boldsymbol{\omega}$, which are entering the proposition. On the

Table 8.2 Empirical two-sectoral data for (8.26)

	1	2
c_i:	32.95	67.05
ℓ_i:	9.42	14.06
w_i:	29,819	25,008
r_i:	18.95	82.06

Note: Data are from Germany 1995. The components of the consumption vector are in percent; ℓ_i are persons per 1 mill. Euro output; the w_i are given in 1,000 Euro per job; (corresponding to the $r_i^{(1)}$ in Table 8.1) the profit rates r_i are the thus resulting profit rates, $r_i = r_i^{(1)} = [p_i - (pA)_i - w\ell_i] / (pA)_i$, where $p = (1,1)$ for 1995, and $A = A_T$ (rounding errors apart).

basis of the German data for 1995, this is done in Table 8.2. We also add the two-sectoral rates of profit (which correspond to the $r_i^{(1)}$ for the 7-sectoral aggregation in Table 8.1, and which yield the same average rate of profit).

If the consumption basket c is normalized such that $pc = p_1 c_1 + p_2 c_2 = 1$ ($p_1 = p_2 = 1$ for 1995), then the wage rates w_i reflect directly the real wages, $\omega_i = 0.001 \cdot w_i$ (given that the quantities are measured in 1 mill. Euro in 1995-prices). $r_i = r_i^{(1)}$ are the thus resulting profit rates for wages paid ex-post.

Equipped with these data, we can determine the consumption matrix C and then the set of sectoral profit rates $\mathcal{R}(\boldsymbol{\omega})$ for (8.26). Concretely, we fix r_1 successively at different values and solve (8.26) for the corresponding value of r_2, which is the value that causes the dominant eigen-value of $AQ(\mathbf{r}) + C$ to be unity. In the notation of Proposition 8.11, this procedure yields the function $r_2 = f_2(r_1)$. Since the present example is only two-sectoral, it would be possible to summarize it in an explicit formula for f_2.

It may, however, be noted that in a wider perspective another approach is more useful, which applies an iterative method. To begin with, it-correctly-presupposes that an algorithm is available to compute the dominant eigen-value of semipositive matrices. Given r_1, we exploit the fact that the function $\phi = \phi(r_2) := \lambda^\star[AQ(r_1, r_2) + C] - 1$ is strictly increasing in r_2 and choose two, possibly extreme, values r_2^a and r_2^b that entail $\phi(r_2^a) < 0$ and $\phi(r_2^b) > 0$. Then a straight line is drawn from $(r_2^a, \phi(r_2^a))$ to $(r_2^b, \phi(r_2^b))$ and the value r_2^c is determined where it intersects the zero line. If $\phi(r_2^c) < 0$, r_2^a is replaced with r_2^c, if $\phi(r_2^c) > 0$, it is r_2^b that is replaced with r_2^c. After that, a new round is started. In this way the points of intersection approximate, step by step, the (unique) value r_2 at which the function $\phi(r_2)$ vanishes up to any desired degree of precision.

The procedure just described is the *regula falsi*, which for well-behaved functions such as our ϕ is well-known to converge quite rapidly, and certainly. In practice, it is a good investment to programme *regula falsi* once and then apply it to any function, rather than develop an explicit formula (if this is possible at all) for every new situation.

In any way, using *regula falsi* the bold curve in Fig. 8.3 was obtained, which is the geometric locus $\mathcal{R}(\boldsymbol{\omega})$ of all pairs of profit rates r_1 and r_2 that are compatible in (8.26) with the empirical wages and consumption structure. Of course, the curve is situated considerably below the frontier of the sectoral maximum rates of profit. The dotted lines indicate the two profit rates from Table 8.2 that were actually prevailing in 1995. The dash-dotted line, which depicts equal rates of profit $r_1 = r_2$, elucidates how much the empirical configuration deviates from the stylized uniform rate of profit.

The cross outlines a situation where $r_1 = r_2 = 45.99\%$, which is just the average rate of profit in 1995. This point is very close to the bold curve. It is thus seen that the error made by hypothesizing a uniform rate of profit in (8.26) is very small, if we compare the resulting profit rate with the empirical average rate of profit. By contrast, the associated prices will be quite different from the empirical prices $p_1 = p_2 = 1$.

Finally, the lowest curve in Fig. 8.3, below the bold line, is the set $\mathcal{R}(\boldsymbol{\omega})$ that results from a uniform 25% increase in the empirical real wage rates. It shows that the sectoral profitabilities would be severely by such a (at present purely hypothetical) event, unless it is not compensated by falling labor coefficients.

In the analysis of (8.18), (8.19) it now remains to take up the third point, which is a study of the wage-profit relationship in this framework. If for that purpose profit and wages are each to be represented by a single variable, we need to fix the structure of either side. The most natural assumption in this respect is to postulate constant ratios of the sectoral wage and profit rates.[20]

To formalize this idea, choose an arbitrary sector k which will serve as a reference. We hypothesize constant ratios $d_i^r = r_i/r_k$ for the sectoral rates of profit, and $d_i^w = w_i/w_k$ for the sectoral wage rates. Equation (8.18) can thus be rewritten as

$$p_i = (1 + r_k d_i^r)\,(pA)_i \;+\; w_k d_i^w\,\ell_i \tag{8.28}$$

Here and in the following, (8.19) can be dealt with in much the same manner, so that we omit this case. Next, define the diagonal matrices

$$D_k^r \;=\; \text{diag}[\,d_1^r, d_2^r, \ldots, d_n^r\,]\,, \qquad D_k^w \;=\; \text{diag}[\,d_1^w, d_2^w, \ldots, d_n^w\,] \tag{8.29}$$

for a more compact formulation (in order not be too cumbersome, the coefficients d_i^r and d_i^w themselves neglect the reference to k). In this way (8.28) becomes

$$p = p\,A\,(I + r_k D_k^r) \;+\; w_k\,\ell D_k^w \tag{8.30}$$

Evidently, sector k's profit rate r_k takes the role of the uniform rate of profit r, and its wage rate w_k the role of the uniform wage rate w. The sectoral distortions are captured by the two diagonal matrices.

[20] The basic idea of the following treatment can be found in Giannini (1976), though his wages are still uniform.

Before inquiring into the variations of the profit rate, the maximum value $\tilde{R}_k$ that r_k can possibly attain has to be determined. In the notation we apply a tilde to R_k because the context is distinct from that of R_i in (8.20). $\tilde{R}_k$ will in fact be dependent on the distribution of the other sectors' profit rates, so that $\tilde{R}_k = \tilde{R}_k(D^r_k)$ in general.

Despite the similarity with equations concerning uniform profitability, the formal determination of $\tilde{R}_k$ is slightly different from the determination of the maximum uniform rate of profit. To derive $\tilde{R}_k$, put $w_k = 0$ in (8.30) and isolate $r_k = \tilde{R}_k$ on the right-hand side of the equation, which gives us $p(I - A) = \tilde{R}_k\, pAD^r_k$. Postmultiplying both sides by $(I - A)^{-1}$, $\tilde{R}_k$ is seen to be the reciprocal of the dominant eigen-value of the thus resulting (likewise semipositive) matrix,

$$\tilde{R}_k = 1 / \lambda^\star[AD^r_k(I - A)^{-1}] \tag{8.31}$$

The special case $D^r_k = I$ of uniform profit rates, in which we have $R = \tilde{R}_k = 1 / \lambda^\star[A(I - A)^{-1}]$, should of course be equivalent to $R = [1 - \lambda^\star(A)]/\lambda^\star(A)$. To verify this, abbreviate $\tilde{\lambda} = \lambda^\star[A(I - A)^{-1}]$ and note that the eigen-value equation $\tilde{\lambda}p = pA(I - A)^{-1}$ is equivalent to $\tilde{\lambda}p(I - A) = pA \Leftrightarrow \tilde{\lambda}p = (1 + \tilde{\lambda})\, pA \Leftrightarrow [\tilde{\lambda}/(1 + \tilde{\lambda})]\, p = pA = \lambda^\star(A)p$, from which $\tilde{\lambda} = \lambda^\star(A)/[1 - \lambda^\star(A)] = 1/R$ can be concluded.

Normalizing sector k's nominal wage at $w_k = 1$ and treating r_k as the exogenous distribution variable, (8.30) can be solved for the prices, which are here prices in terms of sector k's labor commanded,

$$p^w \;=\; p^w(r_k) = \ell\, D^w_k\, [\, I \;-\; A\,(I + r_k D^r_k)\,]^{-1}\,, \qquad 0 \le r_k < \tilde{R}_k \tag{8.32}$$

The corresponding real wage rate of sector k, measured in the consumption basket $c \in \mathbb{R}^n_+$, is given by

$$\omega_k \;=\; \omega_k(r_k) = 1 \,/\, p^w(r_k)\, c \tag{8.33}$$

Regarding an increase of r_k, the argument from the uniform profit rate exactly carries over. This change increases all entries of the inverse matrix in (8.32) if A is indecomposable, which entails that the real wage rate ω_k is a strictly decreasing function of r_k. Representative of the whole economy, the antagonistic character of wages and profits is thus succinctly summarized by the inverse relationship between these two sectoral distribution variables.

To illustrate the wage-profit frontier brought about by (8.32), (8.33), we again take matrix $A = A_T$ from Flaschel and Franke (2008, Chap. 3) and similarly ℓ and c, while the diagonal coefficients d^r_i, d^w_i are obtained from Table 8.1 by dividing the $r^{(1)}_i$ through $r^{(1)}_k$ and the w_i through w_k. With respect to the latter magnitudes, we pick 'other manufacturing' as the reference sector, i.e., $k = 3$. For convenience, the resulting coefficients are presented in Table 8.3.

Table 8.3 Coefficients of D_k^r, D_k^w (derived from Table 8.1)

	1	2	3	4	5	6	7
d_i^r:	3.791	0.784	1.000	1.792	3.977	6.161	3.813
d_i^w:	0.265	1.368	1.000	0.762	0.908	0.597	0.913

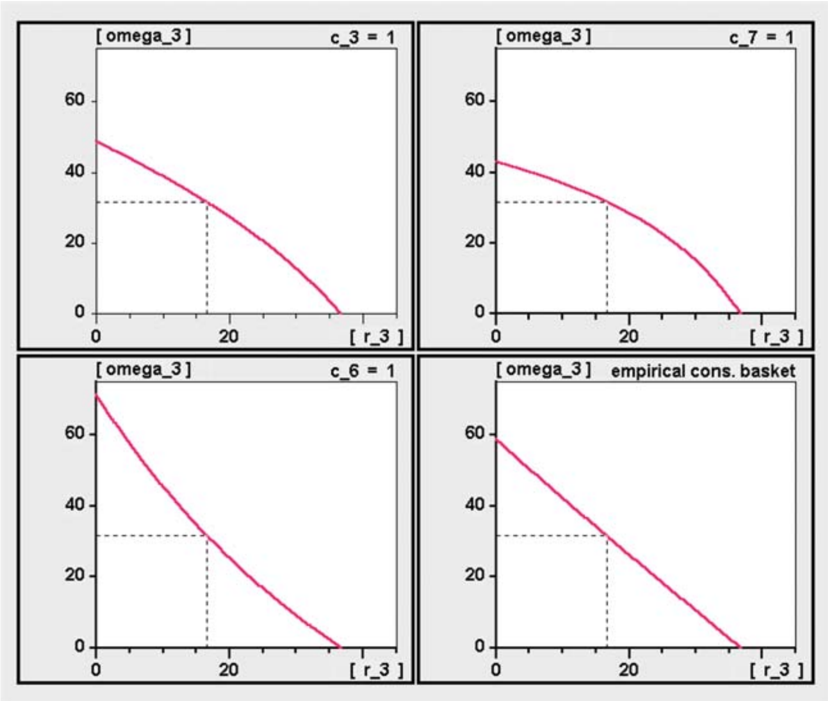

Fig. 8.4 Empirical wage-profit curves for differentiated r_i and w_i

Note: Computation of (8.32), (8.33) for the 7-sectoral aggregation with $k = 3$ as the reference sector. Values for D_k^r, D_k^w from Table 8.3. The dotted lines indicate the empirical 1995-values

Employing the values of Table 8.3 for $D_k^r = D_3^r$ and $D_k^w = D_3^w$, two benchmark positions can be computed. First, the maximum rate of profit (8.31) of sector 3 turns out to be

$$\tilde{R}_k = 36.79\,\%$$

Second, if sector 3's profit rate of 16.66% from Table 8.1 is substituted for r_3 in (8.32), the empirical situation of 1995 is recovered. Here the prices are all equal, $p_1 = p_2 = \cdots = p_7$, and the real wage rate ω_3 buys 31.660 consumption baskets worth 1,000 Euro in those prices of 1995.

Figure 8.4 plots the entire ω_3-r_3 relationship. The four underlying consumption baskets are the same as for the uniform rates of profit. A comparison of the two figures shows that the shape of the curve is different for the baskets containing only good 1 or good 6, respectively, while it has not changed much in the other two cases. Interestingly, if the real wages are measured in the actual consumption basket, the relationship is almost linear.[21]

The dotted lines indicate the empirical situation prevailing in 1995, which has just been described. Because of the uniform prices, the real wage rate is here independent of the composition of the consumption basket. Hence all four curves depicted in Fig. 8.4 go trough the same point $r_3 = 16.66\%$ and $\omega_3 = 31.660$.

8.6 Capital Stock Matrices and Sectoral Profit Rates

8.6.1 Capital Consumed and Capital Advanced

So far we have differentiated capital goods into fixed and circulating ones with respect to a base period of one month (the time unit underlying wage payments due to institutional arrangements in the economy). The time unit is therefore not chosen on the basis of technological relationships and the separation of capital goods into fixed ones ($\delta < 1$) and circulating ones ($\delta = 1$) is from this point of view somewhat arbitrary, but also quite generally a fairly artificial one. Circulating capital can be as differentiated as fixed capital, since it can for example circulate on a daily, a weekly and any other time basis, not well reflected by the choice of a single parameter $\delta = 1$. Furthermore capital goods should not be distinguished from each other on a technological basis, but from an economic point of view concerning capital advanced and capital consumed. Finally, rate of return calculations have a definitely institutional characteristic, namely that they are basically calculated on a yearly basis (which may change in the course of the evolution of capitalism and under specific circumstances). 'Natural' prices thus should be investigated first of all on the background of their 'natural' time period of calculation, i.e., on a yearly basis. If this empirically motivated choice of 'the period of production' is accepted, it follows – also by empirical reasoning – that it is no longer meaningful to assume ex post payment of wages literally. Instead, wages as well as the so-called circulating capital and, of course, the so-called fixed capital must all be advanced to a certain degree, that is to be determined still. From the institutional perspective of a yearly evaluation of the activities of firms, concerning their inputs and outputs and the rate of returns to be calculated on this basis, the proper distinction is indeed

[21] We have checked this by magnifying and also distorting the proportions of this panel. The phenomenon is interesting since the consumption vector still differs (not too much but significantly) from the right-hand eigen-value x^k of the matrix $AD_k^r(I-A)^{-1}$, where the relevance of this observation derives from the fact that the ω_k-r_k relationship is linear if ω_k is expressed in terms of x^k.

between capital advanced (on an average) and capital consumed for any particular year under consideration and this for any item that is used as an input in the yearly evaluations of the returns of firms.

On the basis of these arguments we now reformulate our two sector economy as follows. In the case of machines we still stick to our technological assumption of radioactive decay and assume that all such machines always remain unchanged in quality, but that the portion δ of them disappears without scrapping costs from the sphere of production every year. Capital advanced (per unit of output and continuously reproduced) is in this case now represented by k_{11} and capital consumed by δk_{11} as far as the sector of machine production and his inputs are concerned. In the case of machinery as the capital input into machinery production we thus have as price equation (with $p_2 = 1$ again):

$$rp_1k_{11} + rwk_{01} + \delta p_1k_{11} + \tau_0 wk_{01} = p_1,$$

where the k_{11} are just a new type of notation for input-coefficients a_{11}, we have used so far in their place, while the k_{01} are now different from $a_{01}(= \tau_o k_{01})$, since wages advanced, broadly speaking, are now just $1/24$ of wages consumed in the production of one unit of machinery in the year under consideration, if wage payments continue to be based on a monthly basis. Ex post payment of wages now at best applies to each month, so that wage funds have to be accumulated during each month to allow wage payments at the beginning or end of each such month. Approximately speaking, firms thus keep $1/24$ of yearly wage payments as average stock for these payments, i.e., there is some capital tied up to guarantee the timely payment of wages within each 'production' or better 'accounting' period. Besides δ we thus now also have a parameter τ_o that relates wage funds wk_{01} held on an average to actual yearly wage payments wa_{01} per unit of output. A similar relationship of course applies to the consumption goods sector and gives there rise to

$$rp_1k_{12} + rwk_{02} + \delta p_1k_{12} + \tau_0 wk_{01} = p_2 = 1,$$

with τ_o, δ again assumed as uniform throughout the economy for reasons of simplicity.

With

$$K = \begin{pmatrix} k_{11} & k_{12} \\ 0 & 0 \end{pmatrix}, \quad k_0 = (k_{01}, k_{02}),$$

we thus get for price determination now the matrix equation

$$(r + \delta)pK + (r + \tau_0)wk_0 = p = (p_1, p_2),$$

where we can use the numéraires $p_1 = 1$, $p_2 = 1$ or $w = 1$ depending on the focus of the analysis. In terms of the former A, a_0 matrices this equation would instead read:

$$(r + \delta)pA + (r/\tau_0 + 1)wa_0 = p.$$

This equation shows again that part of wage payments must be considered as advanced capital and to be used in this way in a proper calculation of the (uniform) yearly rate of profit r. The solution to the first of the above equation is

$$(r + \tau_0)wk_0(I - (r + \delta)K)^{-1} = p,$$

which allows a unique solution under standard productiveness assumptions and, for example, given r and $p_2 = 1$.

In the two-commodity economy of this chapter we cannot treat machinery and intermediate inputs side by side, but have to present them as two separate aspects of the circular flow of capital. We therefore now contrast the above consideration of machinery and labor inputs into the production of machinery and consumption goods with the situation where intermediate inputs are combined with labor in the two production activities under consideration.

In the case of intermediate inputs or simply raw material, like 'corn' one has to distinguish again the corn consumed in production in a given year per unit of output from the 'corn' that must be kept as average inventory for an uninterrupted process of production. If corn is ordered for example on a weekly basis in order to allow continuous production, then average inventories of 'corn' are approximately $1/104$ of the corn that is consumed in production during the course of a year. More generally, we thus assume that $k_{1j}, j = 1, 2$ represent the amount of corn tied up on an average in the production of commodity $j = 1$ and $j = 2$, while $\tau_1 k_{1j}$ (again with uniform τ_1 for simplicity) is the amount of corn consumed per unit of output ($\tau_1 = 104$ in our example). As price equations for intermediate inputs $j = 1$ and consumption goods $j = 2$ we now get on the basis of these observations ($p_2 = 1$):

$$\begin{aligned} rp_1k_{11} + \tau_1 p_1 k_{11} + rwk_{01} + \tau_0 wk_{01} &= p_1, \\ rp_1k_{12} + \tau_1 p_1 k_{12} + rwk_{02} + \tau_0 wk_{02} &= p_2 = 1, \end{aligned}$$

or

$$(r + \tau_1)pK + (r + \tau_0)wk_0 = p,$$

with

$$K = \begin{pmatrix} k_{11} & k_{12} \\ 0 & 0 \end{pmatrix}, \quad k_0 = (k_{01}, k_{02}).$$

This matrix equation is of the same formal structure as the one in the case of machinery, with the sole (stylized) distinction that $\delta < 1$ holds in the earlier situation, while we have $\tau_1 \geq 1$ now. The former apparently technological distinction between fixed and circulating capital is thus fairly besides the point and is to be replaced by the distinction between capital advanced and capital consumed where the turnover time $1/\tau$ or $1/\delta$ may be larger or smaller than the accounting period of one year. This proper distinction of capital goods turnover times will be further refined and investigated in Part II of the book, as the alternative to Classical (Neo-Ricardian) or Neo-Classical price and value theory.

We conclude this subsection with the observation that we – due to our above considerations – will spend no effort in discussing Classical or Neo-Classical treatments of fixed capital goods (by means of hypothetical joint production approaches to 'natural' price formation). The same applies to the treatment of semifinished products ('ocean liners' not finished within year of production). Such things must be subjected later on to the approach developed in this section and are not be included in an output matrix B as if they would represent commodities that are to be valued in the same way as the commodities that truly arrive at the market place. The way firms calculate the value of used machinery and semi-finished product cannot be identified without empirical justification with the values that may would be established in a Classical world of production prices where resale markets are assumed to exist for all vintages of machinery and all goods under construction.

8.6.2 Makeshift Construction of Empirical Depreciation and Capital Stock Matrices

The two major shortcomings from which our empirical examples have suffered so far[22] are the neglect of replacement investment in the specification of profits, and that in the profit rate definition profits were related to the flow of material inputs, rather than to the sectors' capital stock that ties up the money invested. The reason for accepting the weaknesses was, of course, the lack of suitable data. We now take a step further and see what we can nevertheless infer from the existing statistics. There are two kinds of data available in the input–output tables that show at least a certain relationship to what we need. On the one hand, the tables offer data on the sectors' total depreciation. We can use them to get an indication of the different levels of capital installed in the single sectors. On the other hand, the reported investment vector as a component of final demand may give us a faint idea of the composition of the sectoral capital stocks. Combining this information we can also construct coefficients that, distinguished by goods and sectors, proxy replacement investment. Clearly, a number of heroic assumptions have to be employed in this endeavor. They are made explicit in the following five steps.

Step 1: Our first heroic assumption postulates that the capital stock has in all sectors the same composition of capital goods. Formally, let the composition be represented by proportions $\kappa_1, \ldots, \kappa_n$ (which sum up to unity, though this is not essential). If k_{ij} designates the capital good i installed in sector j per unit of its output, the assumption says that in all sectors the capital good vector $(k_{1j}, \ldots, k_{nj})$ is

[22] The following sections are based on Flaschel and Franke (2008) and the reader is referred to this work for the flow matrices here referred to. The inclusion of the following discussion, taken from Flaschel and Franke (2008), here serves the sole purpose to show that prices of production are much too simplistic and restrictive in their formulation from the empirical point of view. Moreover they are not needed for the analysis of the implications of factual average price changes as we have shown in the second part of Chap. 3.

proportional to the composition vector $(\kappa_1, \dots, \kappa_n)$. Denoting the proportionality factor in sector j by α_j, the relationship reads

$$k_{ij} = \alpha_j \, \kappa_i \ , \qquad i, j = 1, \cdots, n \tag{8.34}$$

Step 2: The composition vector $(\kappa_1, \dots, \kappa_n)$ is directly obtained from the composition of the economy's investment vector (gross investment, inclusive of imports). Regarding good i, the input–output tables distinguish between its investment as plant, I_i^{plt}, and its investment as equipment, I_i^{eqt}. I being overall investment, $I = \sum_k (I_k^{plt} + I_k^{eqt})$, the proportions κ_i are specified as

$$\kappa_i = (I_i^{plt} + I_i^{eqt}) \, / \, I \qquad i = 1, \cdots, n \tag{8.35}$$

Step 3: We hypothesize that a pure plant capital good deteriorates at a rate δ^{plt} per year, and a pure equipment capital good at rate δ^{eqt}. The depreciation rate δ_i of good i is then supposed to be a weighted average of the two polar rates, where the weight derives from the proportions of the two types of investment I_i^{plt} and I_i^{eqt}. Concretely,

$$\delta_i = \eta_i \, \delta^{plt} \ + \ (1-\eta_i) \, \delta^{eqt} \ , \qquad \text{where} \quad \eta_i := I_i^{plt} / (I_i^{plt} + I_i^{eqt}) \tag{8.36}$$

As we have no definite hints, the rates δ^{plt} and δ^{eqt} themselves are set free-hand. The guideline for our choice will be the level of total capital in the economy to which they eventually give rise.

Step 4: Although the depreciation of a firm's accounting and the (theoretical concept of the) physical depreciation of the capital stock are quite different things, we put them on an equal footing. We determine the level of the capital goods in sector j by the condition that the total depreciation resulting from the δ_i equals sector j's empirical depreciation. Relating the latter to sector j's output and designating this ratio as d_j, the coefficients k_{ij} have therefore to fulfill the equation $\sum_i \delta_i \, k_{ij} = d_j$.[23] The proportionality factor α_j in (8.34) is then easily obtained by substituting $\alpha_j \kappa_i$ for k_{ij} and solving the equation for α_j, which yields

$$\alpha_j = d_j \, / \, \textstyle\sum_i \delta_i \, \kappa_i \ , \qquad j = 1, \cdots, n \tag{8.37}$$

Step 5: Accepting the above assumptions, it is now natural to suppose that replacement investment is identical to the physical deterioration of the capital stock. Hence, denoting replacement investment of good i per unit of output j by $a_{\delta;ij}$, we have

$$a_{\delta;ij} = \delta_i \, k_{ij} \tag{8.38}$$

[23] The coefficients k_{ij} can be added up in a column j if we recall that empirically they all have the unit 'worth 1 mill. Euro in prices of 1995'. Naturally, the same applies to the κ_i.

On the basis of empirical data on d_i as well as I_i^{plt} and I_i^{eqt}, our recipe of constructing the coefficients k_{ij} and $a_{\delta;ij}$ thus goes at follows: obtain the composition vector $(\kappa_1, \ldots, \kappa_n)$ from (8.35); compute the depreciation rates δ_i by means of (8.36); use (8.37) to determine the α_j; get k_{ij} from (8.34); lastly, get $a_{\delta,ij}$ from (8.38). The corresponding matrices are K, the capital stock matrix, and A_δ, the replacement investment or, synonymously, the depreciation matrix.

Regarding the data on d_i, I_i^{plt} and I_i^{eqt}, we again exploit the German input–output tables. For depreciation we take the 1995 data because this is the last year for which real data on depreciation and (which we need below) wage payments are available. It is also convenient that for this year real and nominal data are identical. The investment data are taken from 2000 (the particular year is here rather inessential since only the composition of the investment matters, which does not vary much). With respect to our 7-sectoral aggregation, these data are reported in the first three rows of Table 8.4.

Clearly, the vector of the weights κ_i in the fourth row is the sum of the two preceding rows (one rounding error apart); cf. (8.35). It appears perhaps somewhat peculiar that agricultural products, consumer services and even social services (sectors 1, 6 and 7) can statistically become plant or equipment, but we do not mind since the percentages are fairly low anyway. On the other hand, it certainly accords better with common economic sense that the output of the construction sector 4 is exclusively used as plant and makes up 43.06% of total investment. Similarly with the major part of the investment goods produced by the two industrial subsectors 2 and 3.

The investment data are also employed do deduce the annual depreciation rates δ_i of the capital goods from (8.36). However, they require us first to decide on the two polar rates δ^{plt} and δ^{eqt} for plant and equipment. The δ_i reported in the fifth row of Table 8.4 are based on $\delta^{plt} = 1/20 = 5.0\%$ and $\delta^{eqt} = 1/8 = 12.5\%$ (values which are justified in a moment). The first value is exactly attained by δ_4 and δ_7 for construction and social services as capital goods, while almost all of the capital goods bought from the industrial subsector 2 (the so-called export core) are equipment, so that the aggregate depreciates at not much less than 12.5%.

The main idea of how to arrive at reasonable 'guestimates' of the two depreciation rates δ^{plt} and δ^{eqt} has already been indicated in a remark on (8.36), namely,

Table 8.4 Data underlying the construction of matrices K and A_δ

	1	2	3	4	5	6	7	$\sum$
d_i:	17.93	4.98	6.35	2.39	9.18	12.70	9.89	
I_i^{plt}/I:	0.73	0.52	3.68	43.06	3.46	0.94	0.35	52.72
I_i^{eqt}/I:	0.02	17.14	19.96	0.00	7.11	3.04	0.00	47.28
κ_i:	0.74	17.66	23.64	43.06	10.57	3.98	0.35	
δ_i:	5.16	12.28	11.33	5.00	10.05	10.73	5.00	

Note: All ratios in percent. d_i is depreciation per unit of output (Germany 1995); I_i^{plt} and I_i^{eqt} are investment in plant and equipment, respectively, I is total investment (in 2000); κ_i and δ_i result from (8.35) and (8.36), given δ^{plt} and δ^{eqt} from (8.39) below.

Table 8.5 CVAR resulting from different values of δ^{plt} and δ^{eqt}

δ^{plt}	δ^{eqt}	CVAR
1/25	1/10	2.34
1/25	1/8	2.00
1/25	1/7	1.81
1/25	1/6	1.60
1/20	1/10	2.18
1/20	1/8	1.88

Note: CVAR stands for the economy-wide capital / (gross) value added ratio. See text for more detailed explanation.

to look for the economy's total capital stock that is implied by them. For the order of magnitude that we would like to obtain we have the following information. In 1994, the economy-wide ratio of gross capital to (gross) value added was 2.9, while in 1995 the net capital stock was 63.1% of the gross capital stock. The notion more appropriate for us is the net capital stock, since here over the lifetime of the capital goods depreciation is deducted from their initial value. This consideration gives us a desired capital / value added ratio (CVAR) of $0.631 \cdot 2.9 = 1.83$.

Once the matrix K is determined from δ^{plt}, δ^{eqt} and (8.34)–(8.37), we therefore have to compute the ratio CVAR $= pKx/p(I - A_T)x$, where x and p are the empirical vectors of 1995 (p simply being the summation vector). Table 8.5 reports these ratios for several selected combinations of δ^{plt} and δ^{eqt}.

Our *a priori* idea of these rates is that plant deteriorates at a rate between $1/20$ and $1/30$ per year, while deterioration of equipment is $1/10$ per year or faster. Beginning with a pair $\delta^{plt} = 1/25$, $\delta^{eqt} = 1/10$, the table gives us a ratio CVAR $= 2.34$, which is much too high. Higher values of δ^{plt} and δ^{eqt} would increase this ratio even more (cf. (8.36), (8.37)), so the rest of the table is concerned with lower rates. The small (incomplete) grid of pairs δ^{plt}, δ^{eqt} and the results are self-explanatory. On this basis the two pairs with CVAR $= 1.81$ and CVAR $= 1.88$ are equally good. We have also checked that the differences in the sectoral rates of profit are rather small. Since a depreciation rate of 1/8 appears slightly less arbitrary than $1/7$, and since a depreciation rate of $1/20$ for plant is perhaps too low after all, we decide on

$$\delta^{plt} = 1/20 = 5.0\% , \qquad \delta^{eqt} = 1/8 = 12.5\% \tag{8.39}$$

It has already been mentioned that the depreciation rates δ_i in Table 8.4 are based on these reference rates. Table 8.6 documents the capital stock matrix K that is brought about in (8.34) by the values of κ_i and δ_i in Table 8.4, together with the α_j computed in (8.37).

The last row in Table 8.6 computes the column sums. The number in column 2, for example, indicates that per 1 mill. Euro output in prices of 1995, the sector has capital goods installed that in prices of 1995 are worth 0.5823 mill. Euro. This being understood, the column sums can be said to represent the sectoral capital-output ratios (the ratios of capital to gross output, more precisely). At first glance it is

Table 8.6 Capital stock matrix K from (8.34)–(8.37)

	1	2	3	4	5	6	7
1:	0.0156	0.0043	0.0055	0.0021	0.0080	0.0111	0.0086
2:	0.3,705	0.1,028	0.1,312	0.0493	0.1,897	0.2,625	0.2,044
3:	0.4,959	0.1,377	0.1,756	0.0660	0.2,539	0.3,513	0.2,736
4:	0.9,031	0.2,507	0.3,199	0.1,202	0.4,624	0.6,398	0.4,982
5:	0.2,217	0.0615	0.0785	0.0295	0.1,135	0.1,570	0.1,223
6:	0.0835	0.0232	0.0296	0.0111	0.0427	0.0591	0.0460
7:	0.0074	0.0020	0.0026	0.0010	0.0038	0.0052	0.0041
$\sum$:	2.0975	0.5,823	0.7,429	0.2,792	1.0739	1.4,861	

Table 8.7 Empirical sectoral rates of profit (Germany 1995)

	1	2	3	4	5	6	7	average
II_j:	48.19	63.04	62.69	52.29	41.16	37.44	25.09	46.55
W_j:	21.37	28.72	26.87	32.09	31.57	24.12	58.97	32.05
D_j:	17.93	4.98	6.35	2.39	9.18	12.70	9.89	8.57
P_j:	12.51	3.26	4.09	13.23	18.10	25.74	6.05	12.84
K_j:	209.75	58.23	74.29	27.92	107.39	148.61	115.72	100.21
r_j:	5.97	5.60	5.51	47.39	16.85	17.32	5.23	12.81

Note: $\mathrm{II}_j, \mathrm{W}_j, \mathrm{D}_j, \mathrm{P}_j, \mathrm{K}_j$ are intermediate inputs, wage payments, depreciation, profits, and capital of sector j, all expressed in percent of gross output; r_j is the rate of profit, $r_j = 100 \cdot \mathrm{P}_j/\mathrm{K}_j$. Profits are output minus $(\mathrm{II}_j + \mathrm{W}_j + \mathrm{D}_j)$.

perhaps somewhat surprising that the three services sectors 5, 6 an 7 have significantly higher ratios than the two industrial subsectors 2 and 3. In fact, if a high capital-output ratio were really an indicator of a high ‘degree of industrialization’, then agriculture (sector 1) would be the most industrialized. We also draw attention to the construction sector’s capital-output ratio, which is by far the lowest among our seven macro sectors.

Our main motive for constructing the capital stock matrix K, however, was that it enables us to work with the sectoral rates of profit proper, where in the denominator the material inputs $(pA_T)_j\, x_j$ are replaced with the sectoral capital stocks $(pK)_j\, x_j$. We have now all the information to compute them for the year 1995. The relevant data are collected in Table 8.7.

The sectoral capital stocks per unit of output in the fourth row of Table 8.7, which are temporarily denote by K_j, are given by the expressions $(pK)_j\, x_j \,/\, x_j$. Since empirically in 1995 nominal and real magnitudes coincide, so that $p = (1, \ldots, 1)$, the numbers in the fourth row are nothing else than the column sums of Table 8.6 (multiplied by 100). It should be clear from the above discussion that they are strictly proportional to the sectoral depreciation statistics D_j.

The figures in row 1–4 are the same as in Table 8.1, only that they are here directly measured as percentages of the sectoral outputs. In particular, P_j corresponds to $\pi_j^{(2)}$ in Table 8.1, meaning that now profits have not only the costs of material inputs and labor deducted from the sales of output but, of course, also depreciation. Nevertheless, the $r_j^{(2)}$ in Table 8.1 are for some sectors totally different from the r_j computed in Table 8.7.

Table 8.8 German sectoral rates of profit over the 1990s, computed from (8.34)–(8.37)

	1	2	3	4	5	6	7	average
1991:	3.75	7.79	11.51	51.56	19.49	16.84	4.68	13.82
92:	5.26	4.04	9.06	57.49	17.78	16.27	5.19	12.93
93:	4.70	−0.21	7.14	51.13	17.37	15.62	4.71	11.91
94:	5.62	1.87	7.17	52.52	17.84	16.24	5.67	12.72
95:	5.97	5.60	5.51	47.39	16.85	17.32	5.23	12.81
96:	7.05	4.28	6.56	42.56	16.46	17.97	5.20	13.05
97:	7.44	6.87	7.10	40.54	16.70	18.37	4.99	13.50
98:	6.77	7.86	7.71	39.88	17.76	18.17	5.05	13.84
99:	5.83	5.73	8.19	40.17	18.01	17.74	4.85	13.67

The most striking profit rate is the excessively high value for the construction sector 4. It is mainly because of this phenomenon that we also document the data in the first four rows. We thus see that the sector's share of profit in gross output is not very noticeable. Hence, what makes the sector's profit rate so outstanding is its comparatively low capital in use.

Although the rate might exhibit a certain upward bias, the large gap to the other sectors remains a remarkable, even puzzling phenomenon. On the other hand, the profit rates accruing to the two industrial subsectors 2 and 3 are remarkably low.

Table 8.8 employs the procedure of (8.34)–(8.37) to derive the sectoral rates of profit over the 1990s, until 1999 as the last year for which all the data categories we need are available.[24] It demonstrates that regarding the order of magnitudes as well as the sectoral order, the profit rates in 8.7 for the year 1995 are largely representative. One exception are sector 2 and 3, where 1995 is the only year in which sector 2 scores a higher profit rate.

On the whole, the profit rates are relatively stable over the 1990s. One exception here is the steady decline of the rate of profit in the construction sector, after its extraordinary peak in 1992. The profit rates reflect the general phenomenon, some more and some less, that after 1991 (the short boom following the German unification) the first half of the 1990s was for most sectors a rather poor period, while they showed a substantially better output performance over the rest of the decade.[25] In contrast, for the construction sector it was just the other way round.

Having additionally available the coefficients of the technological matrix A_T of the intermediate inputs, the labor coefficients ℓ_j, as well as the sectoral wage rates w_j and the goods prices p_i, the sectoral rates of profit are given by the following

[24] Since after 1995 wages and depreciation are only available as nominal data, we used nominal data for the whole decade. This, in particular, means that the capital stock coefficients here obtained are nominal magnitudes, which does not matter as long as we are only interested in the profit rates.

[25] The best year was 2000, for which unfortunately we have not sufficient data to continue our computations.

expression, where the profits are now related to the (replacement) value of the sectoral capital stocks:

$$r_j = \frac{1 - \sum_i p_i \, (a_{T;ij} + a_{\delta;ij}) - w_j \ell_j}{\sum_i p_i \, k_{ij}}, \qquad j = 1, \cdots, n \tag{8.40}$$

8.7 Conclusions and Outlook

We have considered in this chapter the Classical theory of ruthless competition and the assumption of prices of production (representing long-period prices) it leads to. In models with only circulating capital goods one may be inclined to consider this hypothesis as reasonably justified, since physical capital is then very mobile. Yet, in the real world, we have plant and equipments on a large scale (at least in manufacturing) and thus generally find multiple activities producing commodities of one and the same class at various levels of sophistication throughout the world. This raises the question whether profit rates can indeed be assumed to equalize in the longer run. Obviously it makes no sense to assume such a tendency on the level of physical input–output data on a worldwide scale, since the markets that are operating on such data cannot be assumed to solve or approximate billions of equations over a certain time span (in particular since these data are also constantly changing in time).

Yet, we have also found no such tendency even on a very high level of aggregation over a time span of ten years in the case of the German economy and may indeed draw the conclusion from these findings that the three fundamental sectors of the economy: ‘agriculture’, ‘manufacturing’ and ‘services’ are not subject to such a tendency towards equalizing profit rates. It may therefore be that accounting prices based on the principle of uniform profitability are a much too restricted concept to provide insights into the actual working of capitalist economies on any of their aggregation levels. However, more empirical work needs to be done in order to further test the empirical relevance of prices of production.

There are two extreme attitudes in the literature with respect to this problem. In their book ‘Laws of Chaos’ Farjoun and Machover (1983) completely reject the concept of prices of production, while Han and Schefold (2006) apply this price concept empirically to existing flow input–output matrices A in order to investigate by means of them the empirical occurrence of reswitching and the like. Our judgment here is – on the basis of the last two sections – that empirical investigation based on uniform profitability that ignore durable means of production cannot be used for a significant study of the implications of all blueprint combinations the considered flow matrices give rise to. The existence of fixed capital prevents most of these combinations and also alters the calculation of prices of production in a decisive way (including the size of the profit rates to be measured). We would tend to accepting the approach of Farjoun and Machover (1983), which stresses the applicability of labor value calculations, see here Part I, and which denies that the behavior of actual average prices and their implications can be understood by production price accounting, which in their view imposes too much regularity on price formation.

The implication of this viewpoint, ironically, is that Samuelson's (1971) 'eraser principle' is to be applied to prices of production and not to labor values. The 'detour' argument of Steedman (1977) thus seems to apply more to prices of production rather than to labor values, in a complete reversal of the arguments advanced in Steedman's book, compare also Gupta and Steedman (1971) in this regard. It may very well be that there is nothing in between labor values, as characteristics of production (Marx, Stone), and actual average market prices in terms of the wage unit, used to characterize real effective demand and income distribution (Keynes, 1936), that can be based on a single and uniform profitability accounting principle.

In the next Part III of the book we shall – despite this negative conclusion – investigate Classical cross-dual dynamics as well as Keynesian dual ones. The findings there however can be applied to Walrasian general equilibrium prices, to von Neumann type production prices and growth rates modelling as well as to other possibilities of determining a point of rest of this type of price-quantity adjustment dynamics. These processes therefore describe how capital moves between sectors and how prices react to such movements without being constrained to Neoclassical or Neoricardian theories of price formation. The search for a theory of long-period prices as foundation for a Classical theory of competition and point of rest of a dynamical system that can applied to actual stock-flow input–output data stays therefore an undecided issue, implying that the Classical theory of competition – quite apart from the level of aggregation and averaging it must necessarily assume[26] – is built on shaky foundations.

References

Bidard, C. (1986). Is von Neumann square? *Journal of Economics*, *46*, 407–419

Bidard, C. (2004). *Prices, reproduction, scarcity.* Cambridge, UK: Cambridge University Press

Bródy, A. (1970). *Proportions, prices and planning*. North Holland: Amsterdam

Duménil, G. & D. Lévy (2002): The profit rate: Where and how much did it fall? Did it recover? *Review of Radical Political Economy*, *34*, 437–461

Farjoun, E., & Machover, M. (1983). *Laws of Chaos* London: Verso Edition and NLB

Flaschel, P., & R. Franke (2008). *Value and competition*. Bielefeld University: Book manuscript

Giannini, C. (1976). Saggi di profitto difformi e merce tipo. *Giornale degli Economisti e Annali di Economia*, *35*, 349–355

Grillo, M. (1976). Introduzione di saggi del profitto differenti in uno schema di interdipendenze settoriali. *Giornale degli Economisti e Annali di Economia*, *35*, 201–208

Gupta, S., & Steedman, I. (1971). An input–output study of labor productivity in the British economy. *Bulletin of the Oxford University Institute of Economics and Statistics*, *33*

Han, Z., & Schefold, B. (2006). An emprical investigation of paradoxes:.reswitching and reverse capital deepening in capital theory. *Cambridge Journal of Economics*, *30*, 737–765

Keynes, J. M. (1936). *The General theory of employment, interest and money.* New York: Macmillan

[26] see Bródy (1970) for the needed changes in treating such issues.

Kurz, H. & Salvadori, N. (1995). *Theory of production. A long-period analysis.* Cambridge, UK: Cambridge University Press

Marx, K. (1954). *Capital* (Vol. I). London: Lawrence and Wishart

Morishima, M. (1973). *Marx's economics. A dual theory of value and growth.* London: Cambridge University Press

Neumann, J. von (1945). A model of general economic equilibrium. *Review of Economic Studies*, *13*, 1–9

Nikaido, H. (1968). *Convex structures and economic theory*. New York: Academic

Samuelson, P. (1971). Understanding the Marxian notion of exploitation: A summary of the so-called transformation problem between Marxian values and competitive prices. *Journal of Economic Literature, 9*, 399–431

Schefold, B. (1997). *Normal prices, technical change and accumulation.* London: Macmillan

Semmler, W. (1984). *Competition, monopoly and differential profit rates*. New York: Columbia University Press

Sraffa, P. (1960). *Production of commodities by means of commodities*. London: Cambridge University Press

Steedman, I. (1977). *Marx after Sraffa.* London: NLB

Woods, J. E. (1978): *Mathematical economics*. Suffolk: Richard Clay (The Chaucer Press)

Chapter 9
Two Concepts of Basic Commodities for Joint Production Systems

9.1 Introduction

The concept of a basic commodity as introduced in Sraffa's (1960) well-known book: 'The Production of Commodities by Means of Commodities' has often been treated and reformulated in the literature under the presupposition of a square single-product system. Its various formulations in terms of direct and indirect – or solely in terms of direct – relationships between the production accounts of the commodities produced can be easily understood from a mathematical as well as from an economic point of view. This situation, however, changes drastically once joint products are taken into account. In this general case the various definitions at hand not only lose their equivalence, but are – following a proposal made by Sraffa (1960, 60) and formalized by Manara (1980) – in fact replaced by a new and much more complicated definition, which, in addition, does not give a complete generalization of the original concept of basic commodities (as we shall see in Sect. 9.4).

The proposed new formulation of basic commodities for joint production systems is based on linear combinations of the *direct* relationships which describe the production of commodities by means of commodities for this case, and Sraffa (1960, 57) explicitly states that 'the criterion previously adopted…(in terms of *direct and indirect* relationships, P.F.) now fails, …'. It is the aim of the present chapter to demonstrate that this view need not be conclusive. In fact, we shall see in Sect. 9.2 that our four equivalent ways of generalizing the single product approach to the case of joint production (among them one which generalizes this approach in a very natural way), give rise to a concept of 'basics' which differs from the Sraffian one. Section 9.3 presents several properties of this alternative notion of basics and it also provides a motivation for their denomination: Leontief-basics, to be suggested in this chapter. In Sect. 9.4 we shall then briefly consider their known alternative, the Sraffa-basics, by utilizing two simplifications of their original definition which have been provided by Steedman (1980) and Pasinetti (1980). This section also completes their up to now incomplete characterization if the set of singular output matrices is excluded from consideration.

To ease the comparison of the two concepts of basics to be examined in the following, we shall restrict joint production to the same situation as it is considered in

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_9,

Sraffa (1960, 51), i. e., we assume as given two nonnegative input matrices $A = (a_{ij})$ and $l = (l_j)$ of physical capital and labor, respectively, and one nonnegative output matrix $B = (b_{ij})$, where $i, j = 1, \ldots, k$. The columns B^j and the rows B_i of matrix B – referring to processes j and to the distribution of commodity outputs of type i, respectively – are supposed to be semipositive (≥ 0) in each case. We use e to denote the summation vector $(1, \ldots, 1)' \in \mathbb{R}^k$ and a prime ($'$) to denote transposition. The foregoing semipositivity can then be expressed equivalently through

Assumption 1. The vectors $e'B$, Be are strictly positive (> 0).

Each process thus produces at least one commodity and each commodity is produced by at least one process. A circumflex ($\hat{}$) will be used to denote the diagonal matrix $\widehat{x}$ which can be formed from a given vector x. Finally, to keep the mathematical part of the chapter within reasonable bounds, we shall presume that the reader is familiar with basic matrix operations and, in addition, with the decomposability properties of the single square matrices we shall employ in the following (see Pasinetti (1977) and here Chap. 8 for a presentation of these mathematical tools).

9.2 Basic Leontief-Commodities

The criterion for a basic commodity is whether this commodity 'enters (no matter whether directly or indirectly) into the production of all commodities', Sraffa (1960, 6). This criterion, which is used by Sraffa in the elementary situation of a square input–output table of physical type, will now be applied to the square and nonnegative matrix L defined by AB', i.e., to a matrix where the columns A^j of the given input matrix A have been aggregated by means of the output coefficients b_{hj}, $j = 1, \ldots, k$ of commodity h to form the h-th column L_h of matrix $L = (l_{ih})$ ($h = 1, \ldots, k$). This latter matrix thus shows a positive entry (i) in its h-th column in exactly those cases where commodity i is an input into the production of commodity h for at least one process j. The usefulness of this matrix L, which provides the foundation for the following definition, will become progressively clearer from the equivalent formulations we shall derive for the definition of Leontief-basics in this section.

Definition 9.1. A commodity i will be called a *Leontief-basic* (L-basic), if for all commodity indexes $h \in \{1, \ldots, k\}$ there exists a sequence of commodity indexes $i_1, \ldots, i_r$ (of finite length r), such that the product:

$$l_{ii_1} \cdot l_{i_1 i_2} \cdot \ldots \cdot l_{i_r h} \text{ is positive} \qquad (A)$$

Commodities which do not fulfill such a condition are called non-basics (NL-basics).

This condition states the existence of a technological link between all commodities h and the commodity i under consideration. Commodity i, consequently, is directly or indirectly involved in the production of any commodity, but in a way which has still to be found out, because of the rearrangements of inputs by means of outputs that have taken place in the formation of the table L.

Assumption 2. We shall assume throughout that the system A, B contains at least one L-basic commodity and that (after suitable relabeling) the set $\{1, \ldots, n\}$ denotes the set of L-basic commodities.

Proposition 9.2. *The subdivision of table L which is implied by the number n (of L-basics)*

n	k-n	commodities
L_1^1	L_1^2	n commodities
L_2^1	L_2^2	k-n commodities

has the following three properties: the submatrix L_2^1 is zero, the submatrix L_1^1 is irreducible (i.e., this matrix cannot be subdivided any longer in the way just described), and *there does not exist a commodity index $m \geq n$ (and a relabelling of NL-basics $n+1, \ldots, k$) such that the subdivision of the table L that is implied by index m will fulfill $L_1^2 = 0$.*[1]

The first two parts *of the proposition are well-known [see, e.g., Pasinetti (1977, pp. 104 f.)], and they illuminate in a different way the fact that L-basics are directly or indirectly necessary for each other, while no NL-basic is used in their production ($L_2^1 = 0$). The* third part *then adds that there is no sector of NL-basics which is completely independent from the rest of the economy, a condition which quite obviously is fulfilled because of the assumed existence of at least one L-basic commodity.*

Remark 9.3. It is easy to see that Proposition 9.2 in fact represents an equivalent way of defining the set of basic commodities, also in the case of arbitrary square matrices S (instead of our $L \geq 0$), if 'positive' is replaced by 'nonzero' throughout. In the following two propositions nonnegativity will, however, be crucial for the results to be obtained.

[1] This third condition – later shown to lead back to the existence of at least one basic – is normally neglected when basics are represented in the form of Proposition 9.2; see, e.g., Pasinetti (1977, pp. 104f.), Abraham-Frois and Berrebi (1979, pp. 39 f.), and also Varri (1979, pp. 57/58), in particular his (2) and the definition following it, where only the case $n = m$ is considered, which, however, is insufficient to allow his following direct/indirect characterization of 'basics' (compare, e.g., the matrix $B'A$ in our Example 9.15).

Proposition 9.4. *The characteristics of table* L *as formulated in Proposition 9.2 imply that the underlying physical input and output tables* A *and* B *fulfill:*

A_1^1	A_1^2	B_1^1	0
0	A_2^1	B_2^1	B_2^2

for some $j \in \{1, \ldots, k\}$ *– to be chosen in a minimal way below and based on a suitable rearrangement of the* k *processes at hand.*

The matrix A_1^1 *will be irreducible, and there will not exist a number of commodities* $m \geq n$ *and a number of processes* j *together with a rearrangement of the* k *given activities and the* NL*-basic commodities, such that* $A_1^2 = 0$, $B_2^1 = 0$ *for the then resulting subdivision of the tables* A, B.

This proposition states in particular that the processes $1, \ldots, j$ *which actually produce the* n L*-basics defined above (by definition: the* basic processes *which form* the basic sector*) have no physical input from the sector of* NL*-basic commodities (which is as it should be), and that there does not exist a collection of processes* $\tilde{j} + 1, \ldots, k$ *which are the only ones that produce a set of* NL*-basic commodities, and which in turn suffice to operate this subsector of processes* $\tilde{j} + 1, \ldots, k$.

Proof of Proposition 9.4: Let the last $k - j$ processes be defined as those which do not produce the first n (basic) commodities, i.e., we have $B_1^2 = 0$ by definition and no vanishing column or row in B_1^1 (note that the case $k = j(k - j = 0)$ is included here!). The product AB' then is of the form (where the blank parts are of no interest in the following):

n	k-n	*commodities*
$A_1^1(B_1^1)'$		n *commodities*
$A_2^1(B_1^1)'$		k-n *commodities*

From Proposition 9.2 it follows that the matrix $A_1^1(B_1^1)$ is irreducible and that the matrix $A_2^1(B_1^1)'$ must be zero. But in defining B_1^1 we have noted that all columns and rows of this matrix have to be semipositive. Therefore A_2^1 must be identically zero (as has been asserted).

To prove the last assertion it suffices to note that a subdivision of the kind shown below:

$A =$

$\widetilde{A}_1^1$	0	m
		k-m

$B =$

$\widetilde{B}_1^1$		m
0		k-m

would imply the following situation:

	m	$k\text{-}m$	
$AB' =$	$\widetilde{A}_1^1(\widetilde{B}_1^1)'$	0	m
			$k\text{-}m$

which contradicts the last statement made in Proposition 9.2. □

Proposition 9.5. *A subdivision of commodities* as considered in Proposition 9.4 *is characterized by the following property: For any* $i \in \{1, \ldots, n\}$ *and any* $h \in \{1, \ldots, k\}$ *there exist finite sequences of commodity indexes* $i_1, \ldots, i_r$ *and of process indexes* $j_1, \ldots, j_{r+1}$, *such that*

$$(a_{ij_1} b_{i_1 j_1}) \cdot (a_{i_1 j_2} b_{i_2 j_2}) \cdot \ldots \cdot (a_{i_r j_{r+1}} b_{hj_{r+1}}) > 0. \tag{B}$$

Note that this seemingly complicated expression simply states that commodity i *is an input for commodity* i_1 *with respect to at least one activity, namely* j_1, *that commodity* i_1 *similarly is an input for commodity* i_1 *with respect to an activity* j_2, *etc. ... up to commodity* h. *This situation represents a straightforward generalization to the case of joint production of the direct/indirect relationship of the type shown in the first table, i.e., of Sraffa's intuitive notion of a commodity which enters (directly or indirectly) into the production of all commodities.*

Proof of Proposition 9.5: By definition, the element $(AB')_{ih}$ of the product AB' is given by:

$$(AB)'_{ih} = \sum_j A_{ij} B_{hj}.$$

An element of the matrix AB', therefore, is positive if and only if there exists an activity j such that $A_{ij} B_{hj}$ is positive. It follows that relationship (B) in fact describes the set of basics with regard to Sraffa's simple definition, see (A), formally applied to the square matrix AB. By assumption we know that the $n \times n$ matrix $A_1^1(B_1^1)'$ considered in Proposition 9.4 is irreducible, hence Proposition 9.5 is already known to be true for all $i, h \in \{1, \ldots, n\}$. Suppose then, that a commodity $h \in \{n + 1, \ldots, k\}$ exists, for which property (B) is not true. Consider the set of these commodities: $\{m + 1, \ldots, k\}$ (after suitably reordering them). Regarding this set, a situation as described by Table AB' must then be true, since all commodities $1, \ldots, m$ can be connected to the basics by way of (B). But the situation in Table 9.6 can only be true if a renumbering of processes $j = 1, \ldots, k$ exists, such that Tables A,B will arise. This, however, contradicts the assumptions, on which Proposition 9.5 was based. The assumed set $\{m + 1, \ldots, k\}$, therefore, must be empty, i.e., formula (B) will hold for all $h \in \{1, \ldots, k\}$. □

The following proposition is now obvious and it implies that our four characterizations of L-basics in table L, Proposition 9.2, Proposition 9.4, and (B) are in fact all equivalent to each other.

Proposition 9.6. *The commodities $i = 1, \ldots, n$ which fulfill condition (B) are exactly the L-basics of Definition 9.1.*

Including the qualifications made in each case, L-basics are consequently characterized equivalently by direct and indirect relationships of kind (A) or (B) or by a structure of direct relationships of the kind shown in Proposition 9.2, Proposition 9.4 with regard to tables $L = AB'$ or A, B. They represent a uniquely *determined set of commodities defined in purely* qualitative *terms (where all quantitative rigidities of joint production are excluded from consideration). And finally: the definition of L-basics reduces to Sraffa's definition in the case of $B = \widehat{e}$, i.e., in the case where no joint production exists, as is obvious from Definition 9.1.*

9.3 L-Basics: Further Discussion

Below we shall briefly describe some simple properties of L-basics and provide some examples to illustrate this definition:

Proposition 9.7. *(a) A change in the conditions of production of the L-basics $1, \ldots, n$, i.e., in the basic sector $\{1, \ldots, j\}$ (see Proposition 9.4), will have no technologically determined demand effect on the production of NL-basics. (b) If no process exists which produces L-basics and NL-basics jointly, then Sraffa's rate of profit r can be obtained from a consideration of the production conditions of the basic sector only.*

Proof. (a) See Proposition 9.4. (b) By assumption, we now in addition have: $B_2^1 = 0$, i.e., the matrix B is completely decomposable with respect to the employed classification of commodities. From Sraffa's well-known price equation

$$pB = (1 + r)pA + wl \qquad (C)$$

where r is the uniform rate of profit, w the uniform rate of wages, and $p = (p_1, \ldots, p_k)$ the corresponding prices of production [see, e.g., Abraham-Frois and Berrebi (1979, pp. 65 f.) for further details], we then get

$$(1 + r)p^1 A_1^1 + wl^1 = p^1 B_1^1, \; p^1 = (p_1, \ldots, p_n), \quad l^1 = (l_1, \ldots, l_n),$$

i.e., the above assertion (note that we have suppressed primes (p', etc.) in the last two equations). The following example is designed to show that the above type of price independence may even arise in the case where no NL-basics exist (all activities may even be reported to consume all commodities directly with regard to the table

AB'), i.e., the above decomposition of price interdependence is not bounded by the number of L-basics from below. □

Example 9.8.

$$A = \left[\begin{array}{cc:cc} 0.1 & 0.1 & 0.1 & 0.1 \\ 0.1 & 0.1 & 0.1 & 0.1 \\ \hdashline 0 & 0 & 0.1 & 0.1 \\ 0 & 0 & 0.1 & 0.1 \end{array}\right], \quad B = \left[\begin{array}{cc:cc} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ \hdashline 0 & 0 & 2 & 1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

We then have

$$AB' = \left[\begin{array}{cc|cc} 0.3 & 0.3 & 0.3 & 0.3 \\ 0.3 & 0.3 & 0.3 & 0.3 \\ \hline 0.2 & 0.2 & 0.3 & 0.3 \\ 0.2 & 0.2 & 0.3 & 0.3 \end{array}\right]$$

despite the fact that the prices p_1, p_2 obviously do not depend on the production conditions of the last two processes.

Proposition 9.9. *The L-basics of Schefold's (1977) model of fixed capital are exactly the ordinary basics of type (A) of his integrated system $\widetilde{A}(r)$ (or $\widetilde{A}(0)$) of finished goods [see his p. 419] plus the machines which are used in the production of these basics (if each newly begun process employs at least one basic finished good).*

According to Sraffa (1960,73) 'Fixed Capital is the leading species' of joint products. The above proposition thus indicates that L-basics may find useful application in important parts of Sraffa's analysis outside the narrow range of square single-product systems.

We do not prove this proposition here, since this would lead us too far into a formal presentation of Schefold's fixed capital model (the proof, however, is simple once this model has been established), but shall only illustrate this proposition by means of a simple example:

Example 9.10. Let M stand for 'machine' and let C denote the consumption good, the only further finished good that is assumed to exist besides M. Let M_0 denote the one-year-old machine in the production of machines and M_0' that of C-production (once installed machines are no longer transferable and will have a useful life of two years in each case).

A typical example of a Sraffian fixed capital model can then be represented as follows

$$\begin{array}{c} \\ A = \end{array}\begin{array}{c} \textit{processes} \\ \boxed{\begin{array}{cc:cc} M & 0 & M & 0 \\ 0 & M & 0 & 0 \\ \hdashline 0 & C & 0 & 0 \\ 0 & 0 & 0 & M_0' \end{array}} \end{array} = \begin{array}{c} \textit{processes} \\ \boxed{\begin{array}{cc:cc} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ \hdashline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}} \end{array} \textit{commodities}$$

$$B = \boxed{\begin{array}{cc:cc} M & M & 0 & 0 \\ M_0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & C & C \\ 0 & 0 & M_0' & 0 \end{array}} = \boxed{\begin{array}{cc:cc} 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{array}} \text{commodities}$$

i.e., ‘machines’ are used in the production of ‘machines’ as well as for ‘consumption goods’, while ‘consumption goods’ only become necessary in the second stage of the machine-producing process (which furthermore uses the one-year-old machine turned out by its first stage).

In the light of Proposition 9.9 we should now expect all four goods to be L-basics: The integrated system $\widetilde{A}(0)$ is of type $\begin{pmatrix} M & M \\ C & 0 \end{pmatrix}$, i.e., it is irreducible (both finished goods are basics) and each newly begun process employs at least one basic in this ordinary sense (the machine M). And indeed, the calculation of AB' gives

$$AB' = \boxed{\begin{array}{cccc} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}}$$

which allows the ’commodity-chain‘: $1 \rightarrow 2 \rightarrow 1 \rightarrow 4 \rightarrow 3 \rightarrow 1$, which, despite the many zeros contained in AB', shows that this matrix is irreducible. Note that in the case of $C = 0$ (with respect to matrix A) there exist only two L-basics: M, M_0, to which Proposition 9.7b may then be applied.

Definition 9.11. Let Q be a $k \times k$ matrix. Entries $1, \ldots, n$ $(n \leq k)$ are called weakly basic, if the implied subdivision of Q

$$Q = \begin{array}{c} \\ \end{array}\overset{\begin{array}{cc} n & k-n \end{array}}{\boxed{\begin{array}{c|c} Q_1^1 & Q_1^2 \\ \hline Q_2^1 & Q_2^2 \end{array}}}\begin{array}{l} n \\ k-n \end{array}$$

fulfills: Q_1^1 is irreducible, $Q_2^1 = 0$ and $Q_1^2 \neq 0$.

Remark 9.12. The above definition can be applied to any subset of entries of the set $\{1, \ldots, k\}$ after a suitable (simultaneous) reordering of the rows and columns of the matrix Q. Note that – contrary to the set of basic entries – the set of weakly basic entries need not be uniquely determined, as the following matrix Q immediately exemplifies:

$$Q = \boxed{\begin{array}{ccc} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{array}}$$

This again stresses the importance of the third part of Proposition 9.2 for a proper definition of basic commodities, which, nevertheless, is normally neglected in the standard formulations of this proposition (compare footnote 1).

Proposition 9.13. *Assume with regard to Proposition 9.4: $e'A_1^1 > 0$ ($e \in \mathbb{R}^n$ and $A_1^2 \neq 0$, i.e., the basic processes and the sector of NL-basics each employ at least one basic commodity. It follows that the sets of L-basic commodities and processes as described by n and j in the tabular representation of Proposition 9.4 are given by the sets of weakly basic entries of AB' and $B'A$, respectively, which both are uniquely determined in this case.*

Proof. With regard to the set of basic commodities this has already been proved in Sect. 9.2 (note in this connection that the there assumed existence of at least one basic commodity immediately implies that the matrix AB' will have only one set of weakly basic entries, namely $\{1, \ldots, n\}$). Hence, it remains to be shown that the processes $1, \ldots, j$ as determined in Proposition 9.4 implies a subdivision of the kind

$$B'A = \overset{\begin{array}{cc} j & k\text{-}j \end{array}}{\boxed{\begin{array}{c|c} (B'A)_1^1 & (B'A)_1^2 \\ \hline 0 & \end{array}}}\begin{array}{l} j \\ k-j \end{array}$$

where $(B'A)_1^1$ is irreducible and where $(B'A)_1^2$ is nonzero, and, furthermore, that $\{1, \ldots, j\}$ forms the only set of weakly basic entries with regard to the matrix $B'A(= Q)$. Starting from the tables in Proposition 9.4, it in fact follows:

1. $(B'A)_2^1 = 0$,
2. $(B'A)_1^1 = (B_1^1)'A_1^1$ is irreducible, since $A_1^1(B_1^1)'$ is irreducible by Proposition 9.4,[2] and

[2] This can be seen by assuming the converse and by applying to this situation the argument of Proposition 9.4 which we have used to show that A_2^1 equals zero, a result which contradicts the

3. $(B'A)^2_1 = (B^1_1)'A^2_1 + (B^1_2)'A^2_2 \geq 0$, since we have $A^2_1 \geq 0$ and $e'(B^1_1)' > 0$, $e = (1, \ldots, 1)' \in \mathbb{R}^n$.

To prove the last part of the assertion, assume finally that a further subset $\{1, \ldots, j\}$ of $\{1, \ldots, k\}$ of weakly basic entries exists – again brought in this canonical order by a suitable renumbering of commodities. We thus have

$$Q = B'A = \begin{array}{c|c|c|l} & \tilde{j} & k & \\ \hline & Q^1_1 & Q^2_1 & \tilde{j} \\ \cline{2-3} & 0 & & k \end{array}$$

where Q^1_1 is irreducible and $Q^2_1 \geq 0$ (see Definition 9.11).

In direct analogy to the proof of Proposition 9.4, we now choose indexes $1, \ldots, m$ to represent those commodities which are not produced by processes $j + 1, \ldots, k$ (it is not yet excluded that this set of commodities can be empty). This choice again implies a corresponding subdivision of matrices A, B such that we get the situation shown below):

$$Q = B'A = \overset{m}{\begin{array}{|c|c|} \hline (B^1_1)' & (B^1_2)' \\ 0 & (B^2_2)' \\ \hline \end{array}} \; \tilde{j} \; \overset{\tilde{j}}{\begin{array}{|c|c|} \hline A^1_1 & A^2_1 \\ A^1_2 & A^2_2 \\ \hline \end{array}} \; m$$

and $(B^2_2)'A^1_2 = Q^1_2 = 0$, i.e., $A^1_2 = 0$, because there is no vanishing column in $(B^2_2)'$ by choice of the set $\{1, \ldots, m\}$. For the product AB' we then get

$$AB' = \begin{array}{c|c|c|} & m & k \\ \hline m & & \\ \hline k & 0 & \\ \hline \end{array}$$

and $m \geq 1$, because of $(B^1_1)'A^1_1 = Q^1_1 \neq 0$, which by Sect. 9.2 implies $\{1, \ldots, m\} \supset \{1, \ldots, n\}$. From the definitions of j, $\tilde{j}$ we consequently get $\{j + 1, \ldots, k\} \supset \{\tilde{j} + 1, \ldots, k\}$, which in turn implies that $Q^1_1 = (B^1_1)'A^1_1$ cannot be irreducible unless $j = \tilde{j}$ (compare the above table and the assertions 1., 2. preceding it). □

Example 9.14.

$$A = \begin{bmatrix} 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

given irreducibility of $A^1_1(B^1_1)'$! Note in this connection that the two matrices $(B'A)^1_1$ and $(AB')^1_1$ can differ in dimension.

i.e.,

$$AB' = \begin{array}{|ccc|} \hline 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline \end{array}, \qquad B'A = \begin{array}{|ccc|} \hline 0.5 & 0.5 & 0.5 \\ 0 & 0 & 0 \\ 0.5 & 0.5 & 0.5 \\ \hline \end{array}$$

which implies that the number of L-basics n is one (commodity 1) and that the number of basic processes j is two (processes 1 and 3).

We therefore obtain presentation Proposition 9.4 simply by interchanging processes 2 and 3, which gives:

$$A = \begin{array}{|c|cc|} \hline 0.5 & 0.5 & 0.5 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline \end{array}, \quad B = \begin{array}{|c|cc|} \hline 1 & 1 & 0 \\ \hline 1 & 0 & 1 \\ 0 & 1 & 1 \\ \hline \end{array}$$

To find the arrangement $\{1, \ldots, n\} \times \{1, \ldots, j\}$ of basic commodities and processes as described in Proposition 9.4 it is thus not necessary to treat the matrices A, B in the simultaneous fashion presented there [which amounts to a simultaneous application of the two notions of technological and economic decomposability introduced and applied in Abraham-Frois and Berrebi (1979, pp. 118 f.)]. This task can now be decomposed into two separate steps based on only one matrix in each case, as we have shown by the Proposition 9.13.

Example 9.15.

$$A = \begin{array}{|c|cc|} \hline 1 & 1 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 1 & 0 \\ \hline \end{array}, \quad B = \begin{array}{|c|cc|} \hline 2 & 0 & 0 \\ \hline 0 & 2 & 2 \\ 2 & 0 & 2 \\ \hline \end{array}$$

$$AB' = \begin{array}{|c|cc|} \hline 2 & 2 & 2 \\ \hline 0 & 2 & 0 \\ 0 & 2 & 0 \\ \hline \end{array}$$

i.e., $n = 1$ is the only basic commodity, and

$$AB' = \begin{array}{|c|cc|} \hline 2 & 4 & 0 \\ \hline 0 & 2 & 0 \\ 0 & 4 & 0 \\ \hline \end{array}$$

i.e., $j = 1$ is the only basic process.

Note that the entry $j = 1$ of matrix $B'A$ is *not basic in the sense of Definition 9.1* but only in the weaker sense supplied by Definition 9.11, a fact which exemplifies the necessity of the additional efforts made to prove the uniqueness of weakly basic entries $\{1, \ldots, j\}$ of matrix $B'A$.

Note finally, that by means of Definition 9.11 (and Proposition 9.6) our Definition 9.1 can be reformulated in the following equivalent way: A set of entries $\{1, \ldots, n\}$ is L-basic if it is weakly basic and if there does not exist a subset of entries of the set $\{n+1, \ldots, k\}$ which is weakly basic, too (always with regard to the matrix $L = AB'$).

Remark 9.16. On the notion of L-basic commodities.

We have seen in this and the preceding section how L-basics can be defined and treated if a square physical joint production structure A, B is assumed to be given. Such a structure – then, however, based on monetary aggregates and given positive 'prices' of commodity groups instead of prices (C) – also represents the basic starting-point of today's input–output *methodology* [see United Nations (1968, pp. 48 ff.)]. But to make conventional input–output *analysis* applicable, it is in general the unquestioned aim of this methodology[3] to reduce the given situation of multiple production U, V (instead of A, B!) to a single matrix of either commodity $\times$ commodity or industry $\times$ industry type (though the original data U, V are clearly recognized to be of institutionally or technologically determined commodity $\times$ industry type). For the case of joint production, the input–output table of commodity $\times$ commodity type recommended for use [see United Nations (1968, 3.40)] is derived in the following way:

1. Through appropriate choice of units the 'prices' p_i of the given commodity groups are all set equal to one, i.e., $p = e \in \mathbb{R}^k$.
   The matrix $X = U\widehat{e'V}^{-1}$ thus represents the structure of average unit-costs of the k given industries.
2. It is assumed that all commodities produced by an industry j have been produced with its structure of unit-costs X^j on the basis of constant returns to scale.
3. To obtain the structure of *average unit-costs of commodities* – which may have been produced by several industries in the assumed situation – averages which relate to the market shares of the industries in question are introduced. This can be done very simply by forming the matrix $Y = V'\widehat{Ve}^{-1}$ – the columns of which describe the market shares of the given industries – and by premultiplying it with the above matrix X. The resulting matrix $A_I = XY$ is called an input–output table of *commodity* $\times$ *commodity type, which is based on the 'industry technology assumption'* (see point 2).
4. This table represents the final form of input–output table, to which conventional input–output analysis can then be applied. We do not intend to judge this methodology here, by which input–output analysis of conventional type is made applicable;[4] but rather our interest lies in establishing and evaluating the following simple assertion:

Proposition 9.17. *The basics of the matrix A_I are exactly the L-basics of the originally given input–output data U, V, which therefore are given by the strictly positive*

[3] See Rosenbluth (1968) for an early, yet widely unknown critique of this attitude.

[4] See again Rosenbluth (1968) for several critical remarks from a statistical as well as an analytical point of view.

rows of the Leontief inverse $(I - A_I)^{-1}$ *of the matrix* A_I *(which in general is also published if* A_I *has been calculated).*

Proof. $A_I = XY = U\widehat{e'V}^{-1}V'\widehat{Ve}^{-1}$ is equivalent to UV' as far as its distribution of zeros is concerned (since the two diagonal matrices employed are, of course, well-defined in any input–output application).

The above proposition shows that the results of Sect. 9.2 and 9.3 can be applied to input–output tables of type A_I as well [see, e.g., United Nations (1968, p. 43) for a numerical example of this type], though they, of course, then relate to matrices U, V based on monetary aggregates instead of facts of technology. This change in framework, however, does not prevent their useful application in this field of investigation, and it furthermore shows that (monetary) L-basics are in fact in use, whenever input–output tables of type A_I are examined with regard to their set of basic sectors.

This analogy to empirically motivated methodology recommended for the case of joint products also explains why the denomination 'Leontief-basics' has been chosen to distinguish the basics of Sect. 9.2 and 9.3 from the following consideration of Sraffa-basics. □

9.4 Basic Sraffa-Commodities

The criterion of direct and indirect technological relationships of commodities, which formed the basis of our investigation in Sect. 9.2 and 9.3, is characterized as leading to 'uncertainty' in Sraffa (1960, 57), if joint products are present. Yet we have seen that at least four very clear-cut descriptions of such relationships exist, if the term 'direct' is used in the following weaker sense: commodity i is a direct input with regard to *commodity* h if a process j exists, which both uses i and produces h. Sraffa's neglect of this possibility of generalizing the concept of basic commodities finds explanation in our view, however, if the purpose of his alternative construction of generalized basic commodities is taken into account, namely, to find a generalized version for his Standard System. For this purpose, a different extension of the notion of 'basics' is in fact more appropriate. This notion will be reformulated below simply by utilizing the matrix AB^{-1} instead of AB, whereby the complete characterization of basics as formulated in Proposition 9.2 again becomes exploitable (but not that of Propositions 9.4 and 9.5), and it will be compared briefly with our previous concept of L-basics.

To avoid more complicated mathematical investigations the following assumption will be employed in this final section:

Assumption 3. The matrix B of Sraffa's joint production system A, B is a regular matrix ($\det B \neq 0$).

This assumption may be justified as follows. Assume that $\det B = 0$, but that there exists a reordering of processes $j = 1, \ldots, k$ such that the diagonal of the output matrix B becomes strictly positive, which means that some similarity with single-product systems is still retained. From 'Satz 12' in Zurmühl (1964, p. 163) we can then conclude that $\det(B + \varepsilon\widehat{e})$ will be nonzero for all sufficiently small $\varepsilon > 0$, i.e., an arbitrarily small perturbation of, at most, *n positive* elements of the matrix B will make this matrix regular. Assuming the possibility for a positive diagonal of matrix B, we thus may conclude that Assumption 3 can be assured without any change as far as the qualitative features of this output matrix B are concerned.

Remark 9.18. Yet, familiar examples of joint products exist, to which the above procedure cannot be applied, e.g., in the case of wool and mutton [if produced by only one process; see also Sraffa (1960, 59) for a similar example]. In such a case small and perhaps inadmissible qualitative changes (of non-outputs into outputs) have to be allowed for, in addition, to ensure the regularity of matrix B.

To motivate the following definition of Sraffa's basics let us (despite better knowledge) presume that there exists a unique single-product technology S (à la Sraffa) behind our joint production system A, B from which this system is derived by (e.g., institutionally determined) process mixes in the customary linear way. Hence, matrix S times the output program of process j: B^j gives the inputs A^j necessary to support this output program. The presumed conventional technology underlying system A, B can then be determined in a very simple way: The assumed equations $SB^j = A^j, j = 1, \ldots, k$, immediately imply the equation $SB = A$, which by means of Assumption 3 in turn implies $S = AB^{-1}$. With regard to this hypothetical single-product technology $S = (s_{ih})$ underlying system A, B we can now define in complete analogy to Definition 9.1 of Sect. 9.2:

Definition 9.19. A commodity i will be called a *Sraffa-basic* (S-basic), if for all commodity indexes $h \in \{1, \ldots, k\}$ there exists a sequence of commodity indexes $i_1, \ldots, i_r$ (of finite length r), such that the product

$$s_{ii_1}s_{i_1i_2}, \ldots, s_{i_rh} \text{ is nonzero}$$

Commodities which do not fulfill such a condition are called non-basics (NS-basics).

Note that the product shown in Definition 9.19 in contrast to what we considered in (A) may now also become negative, since the employed 'technology' S generally will not fulfill $S \geq 0$.

Assumption 4. The joint production system A, B contains at least one S-basic. Commodities $1, \ldots, k$ are assumed to be relabelled in such a way that the set of S-basics is given by $\{1, \ldots, n\}$.

Remark 9.20.

1. By Remark 9.3 of Sect. 9.2 we know that Proposition 9.2 applies to Definition 9.19 as well, thus leading to an equivalent description of S-basics solely in terms of rearrangements of 'direct' relationships s_{ih}. This proposition completes by its third condition the incomplete description of S-basics given in Pasinetti (1977, p. 53).

2. Assume, in addition to Assumption 3, that the matrix AB^{-1} does not have an eigenvalue equal to one. In this case, the matrix $AB^{-1}(I - AB^{-1})^{-1}$ can be formed and is equal to Pasinetti's matrix $H = A(B - A)^{-1}$ of 'vertically integrated units of productive capacity', which thus can be used instead of matrix S to characterize S-basics.
3. A proposal for making the original Sraffa–Manara approach to S-basics complete (with regard to the third condition stated in Proposition 9.2) may be given in the following way: (a) A set of (reordered) commodities $i = m + 1, \ldots, k$ is NS-basic if a regular matrix X exists, such that we get for the matrices $\widetilde{A} = AX, \widetilde{B} = BX$ the condition $\widetilde{A}_2^1 = \widetilde{B}_2^1 = 0$ with regard to the subdivision implied by the number m [see Manara (1980)]. (b) A set of NS-basics is maximal if a set of NS-basics (as defined in (a)) which contains it as a proper subset does not exist. (c) If there is only one such maximal set $\{m + 1, \ldots, k\}$, then commodities $1, \ldots, m$ are called the S-basics of system A, B (no S-basic commodity will exist in the opposite case). That points (a)–(c) in fact generalize our Definition 9.19 (as reformulated by means of Proposition 9.2) can easily be seen as follows: By Assumption 3, the matrix $\widetilde{B}$ must be regular, and from Pasinetti (1977, p. 243) we know that $\widetilde{B}$ and $\widetilde{B}^{-1}$ are always block-triangular matrices of the same type, a fact which then must also hold for the matrix $\widetilde{A}\widetilde{B}^{-1} = AB^{-1} = S$ as well.
4. Points 1–3 can be considered to present a similar set, yet no longer a set of incomplete descriptions of S-basics as supplied by Pasinetti (1980) and Steedman (1980), read in conjunction. With regard to the third description 3) it must, however, be noted that it has the disadvantage of making a complete examination of all possible sets of NS-basics necessary first (as in this case no careful reformulation of the third condition of Proposition 9.2 is available as far as the Sraffa–Manara procedure is concerned). The direct approach to S-basics by means of matrices S or H thus may be preferable (in case the assumed regularities can be assured), in particular since it is of customary type so that no new definition of S-basics has to be learned.

Examples (in continuation of the Examples 9.8, 9.14, 9.15 of Sect. 9.3):

Example 9.21.

$$B^{-1} = \left[\begin{array}{cc|cc} 1 & 0 & -1/3 & -1/3 \\ 0 & 1 & -1/3 & -1/3 \\ \hline 0 & 0 & 2/3 & -1/3 \\ 0 & 0 & -1/3 & 2/3 \end{array}\right], \quad AB^{-1} = \frac{1}{10}\left[\begin{array}{cc|cc} 1 & 1 & -1/3 & -1/3 \\ 1 & 1 & -1/3 & -1/3 \\ \hline 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 1/3 & 1/3 \end{array}\right]$$

In contrast to the four L-basics found out in the last section, only two S-basics (commodities 1 and 2) exist in this case.

Example 9.22.

$$B^{-1} = \begin{array}{|ccc|} \hline 0.5 & 0.5 & -0.5 \\ -0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 \\ \hline \end{array}\,, \quad AB^{-1} = \frac{1}{4}\begin{array}{|c|cc|} \hline 1 & 1 & 1 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 0 \\ \hline \end{array}$$

The set of L-basics here coincides with the set of S-basics.

Example 9.23.

$$B^{-1} = \begin{array}{|ccc|} \hline 0.5 & 0 & 0 \\ 0.5 & 0.5 & -0.5 \\ -0.5 & 0 & 0.5 \\ \hline \end{array}\,, \quad AB^{-1} = \begin{array}{|ccc|} \hline 1 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 \\ \hline \end{array}$$

i.e., three S-basics (but only one L-basic, see Sect. 9.3) exist. It follows that no general relationship between the number of L- and S-basics is to be expected.

These examples make absolutely clear that our definition of S-basics has been based on direct/indirect relationships of purely imaginary kind, generally with no meaningful (technological) interpretation (see Pasinetti 1980, p. 54 for a similar conclusion).

By their very definitions, S-basic and NS-basic commodities thus seem to be only of a very artificial nature. However, Sraffa (1960, 65) claims that the 'chief economic implication' of this distinction (with regard to prices and the rate of profit) will still be found to be true under this new definition. This implication now reads:

Proposition 9.24. *(compare Steedman 1980, pp. 47/48):*

(a) A proportional change in the input or output of a non-basic commodity with regard to all sectors will not influence the prices of basics and the rate of profit.
(b) Given the conditions of production, the relation between the prices of basics and the rate of profit can be presented independently of the relation between the prices of non-basics and the rate of profit, while the converse is not true (aside from very exceptional cases).

Proof. The proof of these two assertions is a simple consequence of (C) rewritten in the form

$$(1 + r)pAB^{-1} + waB^{-1} = p$$

where commodities $1, \ldots, k$ are imagined to have been reordered such that Proposition 9.2 [see Remark 9.3] can be applied. It will not be considered here in its details. Note, however, that an equiproportional change in the i-th row of both matrix A and B is equivalent to the imposition of an *ad valorem* tax on commodity i. □

Remark 9.25. In the presence of joint production, Proposition 9.24b is no longer applicable to *changes* in the conditions of production of NS-basics as well [aside from the exceptional cases considered in 9a; see Steedman (1980) for further remarks on this deviation from single-product systems]. Furthermore, it is not yet clear whether Proposition 9.9 of Sect. 9.3 will apply to S-basics as well, a question which may be of importance for a proper construction of Sraffa's Standard Commodity in that case (see Schefold 1977, pp. 432 f. for some investigations into this problem, and note that in our examples 5/6 the set of S-basics is in fact equal to the set of L-basics in the two cases considered). Finally, it should be noted that arbitrarily small changes in the quantities of outputs *actually produced* ($b_{ij} > 0$) may cause the number of S-basics to jump, e.g., from zero to k. A simple example of this type of behavior is provided by

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 12+2\varepsilon & 17+2\varepsilon & 24+\varepsilon \\ 5 & 7 & 10 \\ 8+\varepsilon & 8+2\varepsilon & 4+2\varepsilon \end{bmatrix}$$

if ε changes from zero to positive values, since AB^{-1} then changes from

$$AB^{-1} = \left[\begin{array}{cc|c} 3 & -7 & 0 \\ -2 & 5 & 0 \\ \hline 0 & 0 & 1/4 \end{array}\right]$$

to a matrix with no zeros at all. This is due to the lack of non-negativity of the matrix S and therefore cannot occur with regard to the matrix L.

Remark 9.26. (compare Remark 9.16): We have seen that L- and S-basics can be defined in the same formal way, but that the first concept relates more to facts of technology, while the second is meant to ease certain price-theoretic considerations. Despite their differing contents there is, however, one further formal analogy which should be mentioned here briefly.

Returning to the level of monetary aggregates, corresponding input and output matrices U, V, and their input–output methodology as considered in Remark 9.16, we should observe that a fundamental alternative to the 'technology assumption' employed there exists, i.e., the so-called 'commodity technology assumption' (see United Nations 1968, pp. 48 f. for details). By means of this assumption the table A_I of Remark 9.16 is replaced by another fundamental type of final input–output table, namely:

$$A_C = X(B\widehat{eB}^{-1})^{-1} = A\widehat{eB}^{-1}\widehat{eB}B^{-1} = AB^{-1}$$

where X is defined as in Remark 9.16. Hence, on the level of reported monetary aggregates, the S-basics of observed make matrices and absorption matrices V, U are exactly the ordinary basics of the input–output table A_C derived from U, V (see United Nations 1968, p. 42 for an example).

9.5 Conclusion

Summing up, we can state that Sraffa's notion of basic commodities for single product systems gives rise to two different, both well-defined, notions of such commodities, if general joint production systems are immediately used in place of square single-product technologies. This disparity arises because technologically based decompositions and price-theoretically motivated ones lose their equivalence in this general case despite the fact that both notions can indeed be defined in the same formal way (depending only on whether 'input–output table' AB' or AB^{-1} is chosen as basis of their definition). However, the latter approach is confined to square joint production, while the first one is also meaningful in general and firmly rooted in the input–output methodology of the United Nations' (1968) SNA.

References

Abraham-Frois, G., & Berrebi, E. (1979). *Theory of value, prices and accumulation*. Cambridge, UK: Cambridge University Press

Manara, C. F. (1980). Sraffa's model for the joint production of commodities by means of commodities. In L. Pasinetti (Ed.), *Essays on the theory of joint production* (pp. 1–15). London: McMillan

Pasinetti, L. (1977). *Lectures on the theory of production*. London: Macmillan

Pasinetti, L. (1980). A note on basics, non-basics and joint production. In L. Pasinetti (Ed.), *Essays on the theory of joint production* (pp. 51–54). London: Macmillan

Rosenbluth, G. (1968). Input–output analysis: A critique. *Statistische Hefte, 9*, 255–268

Schefold, B. (1977). Fixed capital as a joint product. *Jahrbücher für Nationalökonomie und Statistik, 192*, 415–439

Sraffa, P. (1960). *Production of commodities by means of commodities*. Cambridge, UK: cambridge University Press

Steedman, I. (1980). Basics, non-basics and joint production. In L. Pasinetti (Ed.), *Essays on the theory of joint production* (pp. 44–50). London: Macmillan

United Nations (1968). *A system of national accounts*. Studies in Methods, Series F, No. 2, Rev. 3

Varri, P. (1979). Basic and non-basic commodities in Mr. Sraffa's price system. *Metroeconomica, 30*, 55–72

Zurmühl, B. (1964). Matrizen. Berlin: Springer.

Chapter 10
Some Continuity Properties of a Reformulated Sraffa Model

10.1 Introduction

In this chapter we provide some continuity results for Sraffa price system which include subsistence consumption in the matrix of intermediate inputs, but which allow for wage level fluctuations around the subsistence wage level, assuming thereby ex post wage payments only with respect to this deviation from the subsistence wage level. These continuity results all concern a neighborhood of the maximum rate of profit R derived from the subsistence wage situation.

An economic consequence of theses mathematical results is that the concept of 'basic commodities' needs reformulation from the empirical point of view, since it may (on the physical level) include basics of very minor importance ('pencils'). These types of commodities must in some way or another be classified as non-basics. They should also not be considered as giving rise to switches of techniques when ball-pens are replacing pencils, in case this latter commodity is no longer available.

From a broader perspective, these observation again suggest that the basic/non-basic distinction cannot be meaningfully applied to the highest level of disaggregation (the physical level), but must be reconsidered in its usefulness after some aggregation for appropriately chosen aggregated sets of commodities and aggregated methods of production. This again suggests that an input–output oriented approach as in Bródy (1970), concerning fixed capital, semi-finished products, and now also the notion of basic commodities will be the better choice compared to the physical one that was chosen by Sraffa and his followers.

10.2 Limit Cases of Sraffian Models of Production Prices

In the following we shall consider the elementary Sraffian price equations

$$(1+r)pA + wl = p = (p_1, \ldots, p_n) \in \Re^n, \quad py = p(x - Ax) = 1 \qquad (10.1)$$

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_10,

in their dependence on the parameters A, r.[1] Assuming A to be productive (cf. footnote 1) it can be shown[2] that $I - (1+r)A$ is non-negative invertible for any r in the interval $[0, R)$,[3] where $\lambda = \frac{1}{1+R} < 1$[4] is the dominant characteristic root of A. This fact can be used to solve the above price equations in a unique way:

$$p = wl(I - (1+r)A)^{-1} = w \cdot \sum_{v=0}^{\infty}(1+r)^v l A^v > 0 \tag{10.2}$$

$$w = \frac{1}{l(I - (1+r)A)^{-1}(I - A)x} > 0 \tag{10.3}$$

These two expressions define continuous functions

$$p : [0, R) \longrightarrow \Re^n, \quad w : [0, R) \longrightarrow \Re.$$

What we intend to show in the following is, that plausible assumption, which ensure that the equation system

$$(1 + R)pA = p, \quad p(x - Ax) = 1 \tag{10.4}$$

has a unique and positive solution $p(R)$, will also be sufficient to prove that the definitions

$$p := p(R), \quad w(R) := 0$$

lead to a *continuous extension* of the above two functions to the interval $[0, R]$ (now including R). Furthermore, if A_n, l_n is a sequence of input coefficients, which converges to our given system A, l (with respect to each coefficient), then the corresponding sequence of functions p_n, w_n will *converge uniformly* to the functions p, w.

Such continuity properties are necessary ingredients for a sensible interpretations of the prices $p(R)$ as a 'pure capital theory of value' (cf. Pasinetti 1977, pp. 78–80) and the reinterpretation of the wage w as 'surplus wage'.

Corresponding continuity properties can be established for the quantity side (the right hand side) of the given input–output system A, l, I (i.e., with respect to Sraffa's Standard Commodity especially), but there they also reveal that Sraffa's distinction: basics vs. non-basic commodities may declare commodities as basic which are of minor interest only.

---

[1] $A \geq 0$ is the quadratic matrix of physical inputs of a simple input–output system and is assumed to be productive, i.e., $y = x - Ax > 0$ for a given $x = (x_1, \ldots, x_n)^t$ (a column), the gross industry output vector. $l > 0$ (a row) is the vector of direct labor inputs and $p, w, r \geq 0$ denote the usual system of production prices (a row) and its wage and profit rate; for details see Pasinetti (1977, Chap. 5) or Weizsäcker (1971, Part II).

[2] I the identity matrix

[3] $\lambda(A)$, $R(A)$, if explicit reference to the matrix A is necessary.

[4] This notation means that the point R is to be excluded.

The two assumptions we shall employ in the following are already well-known (cf. e.g. Zaghini 1967; Pasinetti 1977, p. 109).[5] They can be expressed as follows: (1) There exists at least one basic commodity, (2) the physical own rate of reproduction of the non-basic commodities is greater than the corresponding rate in the sector of basics.

In mathematical terms these two conditions are equivalent to:

There exists a suitable ordering of the given commodities: $1, \ldots, n$, such that the input matrix A takes the form (see also Chap. 8):

$$A = \begin{pmatrix} A_1 & X \\ 0 & A_2 \end{pmatrix}$$

where A_1 is quadratic (of dimension s, $1 \le s \le n$) and irreducible and where $X \neq 0$ (condition 1). Furthermore $\lambda(A) = \lambda(A_1) > \lambda(A_2)$, i.e., $R(A) = R(A_1) < R(A_2)$ (condition 2).[6] While condition 1 can be defended easily with respect to developed economies[7], there may exist peculiar cases of (groups of) non-basic commodities, whose own rate of reproduction will be less than that of the basic sector.[8] But as this simple model already neglects the consideration of fixed capital, pure joint production, inhomogeneous labor and more, it can be regarded as reasonable or adequate to neglect such peculiar situations too.

Lemma 10.1. *Under the above two assumptions we have: The equations*

$$(1+R)pA = p \quad [p(x - Ax) = 1] \tag{10.5}$$

$$(1+R)Aq = q \quad [lq = 1] \tag{10.6}$$

(where $R = R(A) = R(A_1)$) have uniquely determined solutions p, q with $p > 0$ and $(q_1, \ldots, q_s)^t > 0$, $(q_{s+1}, \ldots, q_n)^t = 0$ ($\{1, \ldots, s\}$ the subsector of basic commodities; cf. condition 1).

Proof. The (10.5) imply $(1+r)(p_1, \ldots, p_s)A_1 = (p_1, \ldots, p_s)$ with respect to the irreducible submatrix A_1 of A. It is well-known that $R = R(A_1)$ is a simple characteristic value of A_1, i.e., the vector $(p_1, \ldots, p_s)$ is uniquely determined – and strictly positive.[9] And because of $R < R(A_2)$ we have

$$(p_{s+1}, \ldots, p_n) = [(1+R)(p_1, \ldots, p_s)X](I - (1+R)A_2)^{-1} > 0,$$ [10]

which proves (10.5).

[5] cf. also the exchange of views between Sraffa and Newman in Bharadwaj (1970).

[6] The first s commodities then describe the basic sector of the given input–output system.

[7] Example: Electric energy. This condition implies that R is finite and that the economy cannot be decomposed into two unconnected parts.

[8] cf. Sraffa (1976, Appendix B).

[9] For details see Nikaido (1968, Chap. 2).

[10] Cf. Appendix 6 in Sraffa (1976).

The (10.6) on the other hand imply:

$$(1+R)A_2 \begin{pmatrix} q_{s+1} \\ \vdots \\ q_n \end{pmatrix} = \begin{pmatrix} q_{s+1} \\ \vdots \\ q_n \end{pmatrix}, \quad \text{i.e.,}$$

$\lambda = (1+R)^{-1}$ must be a characteristic value of A_2, unless all $q_i = 0, s+1 \leq i \leq n$. This latter characterization indeed must be the case, since otherwise we would have $\lambda(A) \leq \lambda(A_2)$ in contradiction to our Assumption 2.

Now from $(q_{s+1}, \ldots, q_n)^t = 0$ there follows

$$(1+R)A_1 \begin{pmatrix} q_1 \\ \vdots \\ q_s \end{pmatrix} = \begin{pmatrix} q_{s+1} \\ \vdots \\ q_s \end{pmatrix}$$

which implies (10.6) – again by the irreducibility of the non-negative matrix A_1. □

10.3 Some Propositions

On the basis of these preliminaries we are now able to formulate the main proposition of this chapter:

Proposition 10.2. *For the considered Sraffa price system there holds:*

(a) The functions $p : [0, R] \longrightarrow \Re^n$ and $w : [0, R] \longrightarrow \Re$ we defined above are continuous. There exists a unique continuous extension of p, w to an interval $[0, K], K > R$ with respect to a fulfillment of (10.7), (10.8) (see below).

(b) For any sequence of non-negative matrices A_n sufficiently close to A and converging to A the induced sequences $p_n(r), w_n(r)$ converge to $p(r), w(r)$ with respect to a suitable chosen interval $[0, K], K > R$.

Proof. Part a: It suffices to consider a neighborhood of $r = R$ (for $r < R$ these continuity properties are already well-known, cf. (10.2) and (10.3)).

The following system of equations

$$p - (1+r)pA - wl = 0 \tag{10.7}$$
$$p(I - A)x - l = 0 \tag{10.8}$$

which has been used to define $p(r)$ and $w(r)$, can be written in the following functional form:

$$H(r; p, w) = 0 \qquad \text{keeping } A, l \text{ constant}$$
$$G(A, r; p, w) = 0 \qquad \text{keeping } l \text{ constant only,}$$

thereby defining continuously differentiable functions[11]

$$H : \Re^{n+2} \longrightarrow \Re^{n+1} \text{ and } G : M(n \times n; \Re) \times \Re^{n+2} \longrightarrow \Re^{n+1}.$$

Let us first employ the function H to prove part (a) of the proposition: Consider the price vector $p(R)$ determined by (10.4), which therefore fulfills: $H(R; p(R), 0) = 0$. Partial differentiation of H with respect to p, w gives:

$$\frac{\partial H}{\partial (p, w)}(R, p(R), 0)^t = \begin{pmatrix} I - (1+R)A & \vdots & (I - A)x \\ \cdots\cdots\cdots & \vdots & \cdots\cdots \\ -l & \vdots & 0 \end{pmatrix} =: D_2 H^t.$$

It is impossible that there exists a vector $\binom{x}{0} \in \Re^{n+1}$ such that $D_2 H^t \binom{x}{0} = 0$: The product $(I - (1+R)A)x$ would be zero then and by lemma (10.6) the vector x would have to be equal to Sraffa's Standard Commodity $q^t \geq 0$, which would imply $-lx < 0$.

The first n columns of $D_2 H^t$ therefore are linearly independent. But then all $n + 1$ columns must be linearly independent, because otherwise we would have

$$0 \leq (I - A)\tilde{x}) = [I - (1 + R)A]\tilde{x} \text{ with respect to a vector } \tilde{x} \neq 0.$$

If such a vector $\tilde{x} \in \Re^n$ would exist, the equation system $p(I - (1 + R)A) = 0$ would have no positive solution (cf. Stiemke's Theorem in Nikaido 1968, p. 36) contrary to lemma (10.5). Therefore, the matrix $D_2 H^t$ must be regular.

By the implicit function theorem (see Dieudonné 1960, p. 265) it then follows, that there exists an open interval U around R and a unique continuous mapping $(p, w) : U \longrightarrow \Re^{n+1}$ such that $(p, w)(R) = (p(R), 0)$ and $H(r, (p, w)(r)) = H(r, p(r), w(r)) = 0$ for any $r \in U$. But for $r \in U$, $r < R$ only $p(r), w(r)$ as defined by (10.2), (10.3) will fulfill this equation. This completes the proof of part (a) of our proposition.

Part b: The function G (defined above) represents a continuously differentiable extension of the function H with the same partial derivative with respect to (p, w) (at $(A, R; p(R), 0)$ now). Therefore the necessary assumptions for the application of the implicit function theorem are fulfilled with respect to G too. The existence of the above continuous mapping (p, w) then extends to a connected neighborhood of $(A, R(A))$, which implies Proposition 10.2(b). Note that all non-negative matrices A_n sufficiently close to A must fulfill the same conditions as are assumed for our given system A, whereby lemma (10.5, 10.6) and Proposition 10.2(a) can be applied to each of these matrices too. □

Corollary 10.3. *The price vector sequence $p_n(R(A_n))$ converges to the price vector $p(R(A))$.*

---

[11] $M(n \times n; \Re)$ the vector-space of all $n \times n$-matrices with real coefficients.

Proof. The function $A_n \longrightarrow R(A_n)$ is continuous, see e.g. Blackley and Gossling (1967, p. 430). Therefore $(A_n, R(A_n))$ will lie in the domain of the above function (p, w) for any non-negative matrix A_n sufficiently close to the given A. The desired result then follows by a suitable combination of continuous functions (including (p, w)). □

Corollary 10.4. *The same continuity property holds with respect to q_n – the right hand characteristic vector of $A_n, R(A_n)$ – normalized by $lq_n = 1$, i.e., with respect to Sraffa's Standard Commodity.*

Proof. In this case we have to consider the equation system

$$q - (1+g)Aq - c \cdot s = 0 \tag{10.9}$$
$$1 - lq = 0, \tag{10.10}$$

where $g, c \in \Re$ and $s \in \Re^n, s > 0$ (an exogenously given 'consumption vector'). As in the above proposition this set of equations defines a function G (regarding l as constant again) with the following partial derivative, now with respect to (q, c):

$$\frac{\partial G}{\partial(q,c)}(A, R(A); q(R), 0) = \begin{pmatrix} I-(1+R)A & s \\ -l & 0 \end{pmatrix}.$$

Now

$$\det \begin{pmatrix} I-(1+R)A & -s \\ -l & 0 \end{pmatrix} \quad = \quad -\det \begin{pmatrix} I-(1+R)A & s \\ -l & 0 \end{pmatrix}.$$

The proof of Proposition 10.2(a) then shows that both determinants must be non-zero (both matrices are regular), i.e., the implicit functions theorem can again be applied. As in Corollary 10.3 the continuous function $A_n \longrightarrow (A_n, R(A_n))$ can then be used to complete the proof (since q_n is uniquely determined by $A_n, R(A_n)$). □

Remark. From the continuity of the functions (10.2), (10.3), from the continuity of the dominant characteristic root $R(A)$ with respect to the non-negative matrix A and from the continuity property established in our proof of Proposition 10.2(b) there further follows that the sequences considered in Proposition 10.2(b) will be uniformly convergent, i.e., we even have:

$$\lim_{n\to\infty} \sup_{r\in[0,K]} ||p_n(r) - p(r)|| = 0 \text{ and } \lim_{n\to\infty} \sup_{r\in[0,K]} |w_n(r) - w(r)| = 0.$$

Supplement. There exists a further sensible assumption on the coefficient matrix A_{11}[12] of the basic sector by which a certain stability result (cf. Nikaido 1968, p. 99) can be established:

[12] for simplicity written as 'A' in the following.

Assumption 3. A is a *primitive* matrix.

There are several equivalent characterizations of this property (cf. Seneta 1973; Varga 1962 or Nikaido 1968):

(a) The (irreducible) matrix A is not cyclic, i.e., there does not exist a renumbering of the basic commodities such that A takes the form:

$$\begin{bmatrix} 0 & C_{12} & 0 & \cdots\cdots & 0 \\ 0 & \vdots & C_{23} \; 0 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \vdots & \vdots & \vdots & & 0 \\ 0 & 0 & 0 & \cdots\cdots & C_{k-1,k} \\ C_{k1} & 0 & 0 & \cdots\cdots & 0 \end{bmatrix}$$

with square null sub*matrices* in the diagonal of A.

(b) $B := \lim_{k\to\infty}(\frac{A}{\lambda(A)})^k = \lim_{k\to\infty}(1 + R(A))^k A^k$ exists and is equal to qp, where the vectors p, q – defined in the above lemma, but restricted to the basic vector – have been chosen such as to fulfill: $pq = 1$. It follows that the rows (columns) of the matrix B all are proportional to $p(q)$.

(c) There exists $k \in \aleph$ such that $A^k > 0$.

We do not think it probable that the basic sector of any realistic input–output system will be of the simple completely cyclical kind as described in (a), i.e., will be of the Austrian linear type (Weizsäcker 1971, p. 32), but with its 'final' goods (and only these) employed again on the first stage of production.

Furthermore there does exist an even simpler condition which will ensure the primitivity of the irreducible matrix A:

Assumption 4. There exists a basic commodity that is used directly in its own production, i.e., there exists a positive element a_{ii} in A (cf. Varga 1962, p. 43).

If Assumptions 3 or 4 are accepted, the following proposition can readily be established:

Proposition 10.5. *For the considered Sraffa input–output system there holds:*

(a) $\sqrt[k]{(A^k)_{ij}} \longrightarrow \lambda(A) = \frac{1}{1+R(A)}$

(b) $(\frac{A}{\lambda(A)})^k y_0 \longrightarrow \alpha \cdot q \, [y_0^t(\frac{A}{\lambda(A)})^k \longrightarrow \beta \cdot p]$ *for any* $y_0 \geq 0$ *and some* $\alpha > 0 \, [\beta > 0]$.

(c) The mapping ψ*:*

$$A \xrightarrow{\psi} \lim_{k\to\infty}(\frac{A}{\lambda(A)})^k =: B > 0$$

is continuous on the set of all primitive matrices (which is open and dense in the set of all non-negative matrices).

Proof.

(a) $\frac{(A^k)_{ij}}{\lambda(A)^k} \longrightarrow B_{ij} > 0$ implies that $\frac{\sqrt[k]{(A^k)_{ij}}}{\lambda(A)} \to \lim_{k\to\infty} \sqrt[k]{B_{ij}} = 1$.

(b) $\lim_{k\to\infty} \frac{A^k}{\lambda(A)} y_0 = By_0 = (py_0)q = \alpha q$.

(c) It suffices to examine the continuity of ψ under the additional assumption: $\lambda(A) < 1$, since the mapping ψ is homogeneous of degree zero. But then the claimed continuity follows from Corollaries 10.3, 10.4 in connection with part (b) of the present discussion. □

10.4 Conclusions

We have already stressed the importance of the above continuity results for a 'pure theory of capital' (or a 'surplus wage' theory[13]) in our introductory remarks. In addition we can now state that the qualitative behavior of Sraffa's objects of investigations: the functions $p(r), w(r)$ (and – if necessary – also: $q(R)$) can be approximated as close as we like by using strictly positive input matrices only, i.e., by using Perron–Frobenius theory in its strongest form. This fact can be combined with Schefold's (1976) conclusion 'that the normal case is one where all roots of the system are simple', whereby a simple expression for the vector of production prices (in terms of the wage rate) can be obtained (cf. there p. 30).

But to restrict our attention to strictly positive matrices means that the difference between basic and nonbasic commodities has been abolished – or has been transformed to a situation with major and minor basics (where the latter could be described as the (largest) set of types of commodities whose technical conditions do not exhibit a significant influence on the relative prices of the former). But such a 'definition' – replacing 'no influence' by 'no significant influence' – is not easily rendered precise. And furthermore it may also contain a certain discontinuity: the set of major basics may be a good deal smaller than the set of basics (defined with respect to Sraffa's original intention there may exist basics with neglectable influence on the relative prices of the other basics).[14] The question arises whether Sraffa's original dividing line within the set of all commodities must be regarded as being too 'ideal', if quantitative dimensions are taken into account. Is it really important to have a classification which regards 'electric energy' and 'pencils' as being 'equal' but which normally excludes wage goods from the list of such basics?

To give the notion of a basic commodity operational significance as well as economic content, it may be useful to look for criteria which help to eliminate 'minor basics' from the list of basics, i.e., which serve to eliminate corresponding weak price dependencies as far as possible. Such a procedure tries to go the opposite

[13] We do not question here the analytical usefulness of such a theory.

[14] One may think of a commodity which normally will be used as a consumption good only, but is also used as an input in the production of a 'true basic' in a very small amount.

way as compared to that followed in our text or by Schefold (1976). But it is not without economic interest to look for such peculiarities of given input–output systems A, l, I.

This fact can be illustrated with respect to a further classification, the classification of the different sectors of our system A, l according to the similarities in the value composition $\frac{pA_j}{wl_j}$ of capital. These similarities can be used to eliminate certain price independencies (with respect to movements of the rate of profit r), thus leading to non-distorting aggregation procedures (cf. e.g. Miyao 1977).

But here we have been interested mainly in the question whether the far reaching properties of positive matrices can be expected for our given input–output system A, l, I too, which especially meant, that we had to look for assumptions on A, l by which discontinuities in the derived notions could be prevented. Fortunately the necessary assumptions are well-known and of sensible nature too.

References

Bharadwaj, K. (1970). On the maximum number of switches between two production systems. *Schweizerische Zeitung für Volkswirtschaft und Statistik*, *106*, 409–429

Blackley, G. R., & Gossling, W. F. (1967). The existence, uniqueness and stability of the standard system. *Review of Economic Studies*, *34*, 427–431

Bródy, A. (1970). *Proportions, prices and planning*. North Holland: Amsterdam

Diedonné, J. (1960). *Foundations of modern analysis*. New York: Academic

Miyao, T. (1977). A generalization of Sraffa's standard commodity and its complete characterization. *International Economic Review*, *18*, 151–162

Nikaido, H. (1968). *Convex structures and economic theory*. New York: Academic

Pasinetti, L. (1977). *Lectures on the theory of production*. London: Macmillan

Schefold, B. (1976). Relative prices as a function of the rate of profit: A mathematical note. *Zeitschrift für Nationalökonomie*, *36*, 21–48

Seneta, E. (1973). *Non-negative matrices*. London: George Allen & Unwin

Sraffa, P. (1976). *Warenproduktion mittels Waren*. Frankfurt a.M.: Edition Suhrkamp, Bd. 780

Varga, R. (1962). *Matrix iterative analysis*. Englewood Cliffs, New Jersey: Prentice Hall

Weizsäcker, C. C. V. (1971): *Steady State Capital Theory.* Heidelberg: Springer

Zaghini, E. (1967). On non-basic commodities. *Schweizerische Zeitschrift für Volkswirtschaft und Statistik*, *103*, 257–266

Chapter 11
The Standard Commodity and the Theory of Income Distribution

11.1 Introduction

Since the publication of Sraffa's (1960) book: *Production of Commodities by Means of Commodities* numerous comments have been published with regard to its Chaps. 4 and 5, the construction and the proof of uniqueness of his Standard Commodity. While most of the commentators have stressed the importance of Sraffa's Standard Commodity as an 'invariable measure of value', there are also some, cf., e.g., Pasinetti (1977, pp. 119–120), Woods (1978, p. 92), who find that its most remarkable theoretical implication lies in the demonstration that it is possible to treat the distribution of income logically prior to and independently of prices.

Regarding these views we shall show in the following that this assertion is a completely misleading one. The Standard Commodity and the therewith derived linear wage-profit curve $\widetilde{w}(r)$ do not simplify the theory of income distribution, but in fact make it more obscure, i.e., either incomplete ore more complex, as compared to its presentation by means of a conventional nonlinear $w(r)$-curve, e.g., based on a *numéraire* which keeps national income fixed when changes in the distributional variables w and r are considered. Furthermore, our way of demonstrating this claim will to some extent indicate that Sraffa's composite commodity may be of dubious economic significance quite generally, i.e., will not serve as an appropriate 'measure of value', also cf. Burmeister (1968, p. 86–87) for an early remark on this and Flaschel (1980). The present chapter thus should be conceived as a first argument towards the conclusion that an analysis based on Sraffa's *Production of Commodities by Means of Commodities* should dispense with his hypothetical composite commodity, the 'Standard Commodity'.

11.2 The Sraffian Approach to Income Distribution

To substantiate this claim as far as its application to the theory of income distribution is concerned, we shall make use of simple examples which are similar to one used by Sraffa (1960, Chap. 6) to criticize Böhm–Bawerk's concept of an average period

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_11,

of production. In reference to the Austrian school which thought of production as a linear process with a finite number of stages, Sraffa considers two such processes to show the possibility of a reversal in the direction of change of the relative price of the assumed two final goods, a fact which 'cannot be reconciled with any notion of capital as a measurable quantity independent of distribution and prices' (Sraffa 1960, p. 38).

This method of critique will now be applied to the above claims of Pasinetti and Woods by employing the following examples of data on production. Consider a standard single-product system with n processes where labor inputs are described by the vector $l = (l_1, \ldots, l_n) \in \Re^n$ and where the structure of physical inputs per unit of commodity output is given by a matrix of type

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ & & & \\ & & 0 & \\ 0 & & 1 & \\ x & 0 & & 0 \end{pmatrix} \tag{11.1}$$

We thus assume as given an almost linear production structure, where commodities pass through n stages, but a structure which exhibits one important circularity (represented by the number x) by which the final stage n is related with the first stage of production. We must, of course, assume that $0 < x < 1$ holds. The matrix A is then productive as well as irreducible; all 'intermediate' goods $1, \ldots, n-1$ and the 'final' good n are basic commodities, cf. Pasinetti (1977) with regard to the definition of these concepts.

It is customary to normalize l by setting $L = l_1 + \ldots + l_n$ equal to 'one'. We may also assume that inputs A, l describe the production of commodities in absolute terms. Net production is thus simply given by $1 - x$ units of commodity n and is positive by assumption. On the basis of these data Sraffa's prices $p = (p_1, \ldots, p_n)$ are then given by:

$$p = (1 + r)pA + wl, \quad p_n(1 - x) = L = 1 \tag{11.2}$$

where r denotes the uniform rate of profit and w the uniform rate of wages.

Example 11.1. For A as in (11.1) and $l = (1, 0, \ldots, 0)$ we get for prices (11.2) the expressions ($i = 1, \ldots, n$):

$$p_i = \frac{(1 + r)^{i-1} w}{1 - (1 + r)^n x}, \quad p_n(1 - x) \equiv 1,$$

an equation system which can be uniquely solved if the rate of profit r is assumed to be given.

Note that by the very definition of the above prices p_i the value of national income is kept constant (equal to one), a fact which will be of importance, when Sraffa's change of *numéraire* by means of his Standard Commodity is considered.

11.3 The Standard Commodity

This Standard Commodity is defined by the corn-like property that it must be proportional to the inputs needed for its production (Sraffa 1960, p. 20), i.e., it is given by the right-hand eigenvector of the matrix A which fulfills

$$(1 + R)Aq = q, \quad lq = 1, q \geq 0 \tag{11.3}$$

It is known that (11.3) has a unique and positive solution vector $q \in \Re^n$ if – as here is the case – the employed matrix A is irreducible. Furthermore, the scalar R gives the maximum rate of profit, which can be obtained from (11.2) by setting $w = 0$.

Example 11.2. The Standard Commodity of Example 11.1 is given by

$$q = (1, x^{1/n}, \ldots, x^{(n-1/n)})^t, \quad R = x^{-1/n} - 1.$$

This can immediately be checked by inserting q and R into the above eigenvector equation (11.3). Referring again to Burmeister (1968, p. 87) it is in fact difficult to see why such particular weights $q_i = x^{(i-1)/n}$ – which have not much in common with actual net production $y = (0, \ldots, 0, 1 - x)^t$, a column vector as is indicated by the superscript 't' – should be singled out as a proper 'measure of value', i.e., here with respect to a better understanding of income distribution by means of the linear wage-profit curve implied by them (see (11.5)).

11.4 Hiding Nonlinearities: The Role of the Standard Commodity

It is mathematically easy to separate the two distributional variables w, r of (11.2) from prices p. For $0 \leq r < R$ the first equation in (11.2) can be rewritten as $p = wl(I - (1 + r)A)^{-1}$, where I denotes the identity matrix. Post-multiplying this equation with the column vector $y = (0, \ldots, 0, 1 - x)^t \in \Re^n$ we thus obtain from the second equation in (11.2):

$$w = w(r) = [l(I - (1 + r)A)^{-1}y]^{-1} \tag{11.4}$$

which gives the well-known falling wage-profit curve of system A, l. The situation described by (11.4) can be summarized as far as its qualitative characteristics are concerned by means of the Fig. 11.1.

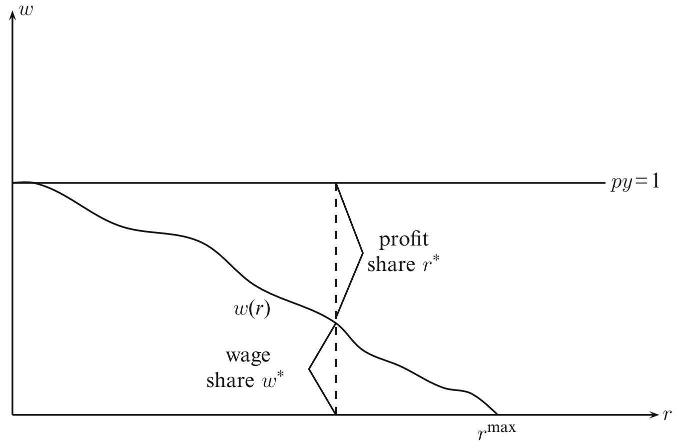

Fig. 11.1 Wages w measured in terms of net national product $y : py = Y \equiv 1$

With regard to a given rate of profit r^* this diagram immediately informs us about the actual share of wages and of profits: w^* and $1 - w^*$ in income $Y = py \equiv 1$, and this is true for all possible situations of income distribution within the range (0, R).

Example 11.3. The wage-profit curve (11.4) of Example 11.1 is given by

$$w = (1 - (1 + r)^n x)/((1 - x)(1 + r)^{n-1}),$$

which quite obviously is a strictly decreasing function of $r \leq R = x^{-1}/n - 1$.

In what way will the application of the Standard Commodity (11.3) now improve the situation? To show this, the conceded procedure is to replace the *numéraire* commodity y by the Standard Net National Product $y(q) = q - Aq$, i.e., to normalize relative prices $p = wl(I - (1+r)A)^{-1}$ by $\tilde{p}y(q) \equiv 1$ and no longer by $py \equiv 1$. By this change in the choice of the employed *numéraire* commodity – leading to $\tilde{p}$, $\widetilde{Y}$, etc. – we immediately get (see (11.2) and (11.3):

$$\begin{aligned}(\widetilde{p} - \widetilde{p}A)q &= \widetilde{p}(q - Aq) = 1 = r\widetilde{p}Aq + \widetilde{w} \quad \text{and} \\ R\widetilde{p}Aq &= 1,\ i.e.,\ 1 = r/R + \widetilde{w} \text{ or} \\ \widetilde{w} &= \widetilde{w}(r) = 1 - r/R \end{aligned} \tag{11.5}$$

This is the linear relation between wages $\widetilde{w}$ (expressed in units of $y(q)$) and the rate of profit r, which according to Pasinetti (1977, pp. 119–120) should enable us to see the alternative possible distributions of net national income py between wages and profits now under ideal conditions, i.e., independently of prices p (see footnote 1).

With regard to this latter point it should, however, be clear that – mathematically speaking – this had already been the case with respect to (11.4), since prices p are not explicitly involved in the representation of this relationship as shown there. Yet, prices p are, of course, implicitly contained in this functional relationship as can be seen from its above derivation. It is from this fact that the alleged disadvantage of (11.4) as compared to (11.5) seems to originate, since the latter curve no longer contains the expression $l(I - (1+r)A)^{-1}$ – which is proportional to prices p – as a constituent part. Despite this true observation, however, there is no gain, but indeed a loss of insight involved if we replace (11.4) by (11.5) in the study of questions of income distribution. Instead of supplying 'ideal conditions' for such an investigation this latter equation is characterized by a loss of information in this respect. The following related points are intended to make this assertion obvious step by step.

It has been stated in commenting on Fig. 11.1 that for any given rate of profit r^* the related data w^*, $1 - w^*$ will represent the actual share of wages and of profits in national income py if y is chosen for *numéraire*. This conclusion, however, is not true if the new standard $y(q)$ is adopted as *numéraire* commodity. In this case the curve $w(r)$ of Fig. 11.1 must be replaced by the ('ideal') straight line which connects the points '1' and 'R' and its value $\widetilde{w}(r^*) = \widetilde{w}^*$ must now be used in place of the former wage rate $w^*(r^*)$. This value $\widetilde{w}^*$ describes that proportion of the Standard Net Product $y(q)$ which can be bought by wage-earners from their wage-income at the rate of profit r^*. However, this (hypothetical) proportion of Standard Net Product $y(q)$ now used to measure wage-earners income is no longer related to the actual wage share (w^*) or profit share $1 - w^*$ in any simple manner. This follows from the fact that because of the change in *numéraire* commodity ($y \to y(q)$) actual income $\widetilde{Y} = \widetilde{p}y$ will no longer be invariable with respect to changes in the rate of profit r (since $py \equiv 1$ has been replaced by $\widetilde{p}y(q) \equiv 1$). The distribution of income in the new situation must thus be represented now by

$$\widetilde{w}^*/\widetilde{p}y, (\widetilde{p}y - \widetilde{w}^*)/\widetilde{p}y, \widetilde{p}y = \widetilde{Y}$$

with an unknown value $\widetilde{Y}$ of actual national income in terms of the new standard adopted. Hence, the description of income distribution must be regarded as rather incomplete if the 'ideal' (11.5) is used in place of its ordinary counterpart (11.4).

It may, however, be claimed that we did not prove that the identity $py \equiv 1$ will get lost if the *numéraire* $y(q)$ is adopted in place of y. To show that this is indeed the case, i.e., that (1) the above is to the point, the calculations of Examples 11.1 and 11.2 can now be usefully employed. It is not necessary to recalculate the prices p_i of Example 11.1 by means of the vector q of Example 11.2 according to the new normalization rule deriving from it, since we already know that its wage-profit curve in contrast to that of Example 11.3 must be given by $\widetilde{w} = 1 - r/R, R = x^{-1/n} - 1$ (see (11.5)). Inserting this relationship into Example 11.1 gives for the price of commodity n in terms of Sraffa's Standard Net Product

$$\widetilde{p}_n = (1+r)^{n-1}(1 - r/R)/(1 - (1+r)^n x)$$

which shows that $\widetilde{p}_n(1 - x) = \widetilde{p}y$ may be equal to one only in the case where r is equal to zero. Choosing, e.g., $n = 5$ and $x = 1/32$ it can easily be calculated in

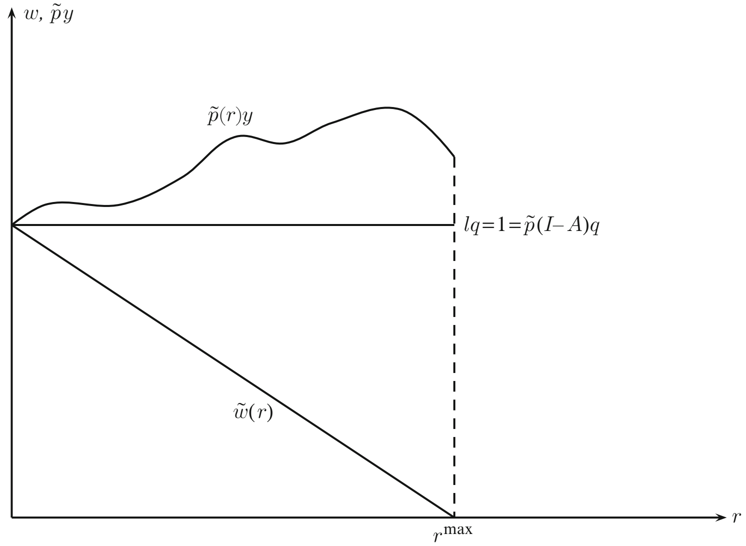

Fig. 11.2 The illusion of a linear subdivision of national income when $\widetilde{w}$ and $\widetilde{Y}$ are measured in terms of the Standard Product $y(q)$

addition, that $\widetilde{p}y$ will be strictly increasing then for all rates of profit $0 \le r < 1$ (we have $R = 1$ in this case). The 'ideal' distributional relationship (11.5) must consequently be completed as shown in Fig. 11.2.

This again demonstrates the incompleteness of a consideration that is based on relation (11.5), i.e., the $\widetilde{w}(r)$-curve solely, as is usually the case. Furthermore, it should be obvious by now that the dependency of income distribution on prices which with regard to the old *numéraire* found its expression in the nonlinearity of the wage-profit curve itself (see Fig. 11.1) has only been transferred here to the shape of the national income curve $\widetilde{Y}(r)$ of Fig. 11.2, which, however, remains unconsidered if the 'ideal'character of (11.5) is stressed. Alternative distributions of national income between wages and profits – in contrast to what is claimed by Pasinetti (1977, p. 120) – cannot be studied free from any interference due to price changes specific to the commodity used as the unit of measurement even in the case of the Standard Commodity. The opposite claim simply is an illusion generated by concentrating on its wage-profit curve $\widetilde{w} = 1 - r/R$ and neglecting the (formerly trivial) corresponding curve of income determination with regard to this unit of measurement.

Let us consider finally the situation underlying Fig. 11.2, i.e., $n = 5$ and $x = 1/32$ $(R = 1)$, at the particular value $r^* = 0.25$ and let us compare it with the same numerical situation where, however, the vector of labor inputs now become necessary only on the last instead of the first stage of production. Considering (11.3) there follows that the scale of q has to be adjusted to this change in the distribution of labor inputs. Yet, it is not necessary to make this change in scale explicit, since it – as in point (2) – suffices to insert the relationship $\widetilde{w} = 1 - r$ into the first equation of

(11.2) to get Sraffa's prices in terms of his Standard Net Product, whatever its actual form may be. In case of the vector $l = (0, \ldots, 0, 1)$ Sraffa's prices are now given by [see Example 11.1 for a comparison with those of the vector $l = (1, 0, \ldots, 0)$]:

$$\widetilde{p}_i = (1+r)^i \widetilde{w} x/(1-(1+r)^n x), \quad i = 1, \ldots, n-1$$
$$\widetilde{p}_n = \tilde{w}/(1-(1+r)^n x)$$

At the particular values $n = 5$, $x = 1/32$, $r^* = 0.25$, i.e., $\widetilde{w}^* = 0.75$ $(R = 1)$ we therewith get in comparison to the situation defined by the labor input vector $l = (1, 0, \ldots, 0)$:

$l = (1, 0, \ldots, 0)$	$1 = (0, 0, \ldots, 1)$
$\widetilde{p}_5 \approx 2.02$	$\widetilde{p}_5 \approx 0.83$
$\widetilde{Y} \approx 1.96$	$\widetilde{Y} \approx 0.80$
$\widetilde{w}^* \approx 0.75$	$\widetilde{w}^* \approx 0.75$
$\widetilde{Y} - \widetilde{w}^* \approx 1.21$	$\widetilde{Y} - \widetilde{w}^* \approx 0.05$
$\widetilde{w}^*/\widetilde{Y} \approx 0.38$	$\widetilde{w}^*/\widetilde{Y} \approx 0.94$
p $(\widetilde{Y} - \widetilde{w}^*)/\widetilde{Y} \approx 0.62$	$(\widetilde{Y} - \widetilde{w}^*)/\widetilde{Y} \approx 0.06$

We thus find quite different situations of income distribution (characterized by the last two rows of this table) behind our harmless-looking distributional equation (11.5):

$$\widetilde{w} = 1 - r, \text{ here } : 0.75 = 1 - 0.25$$

There again follows that the information which can be obtained from (11.5) must be considered as far too meager for a proper analysis of income distribution.

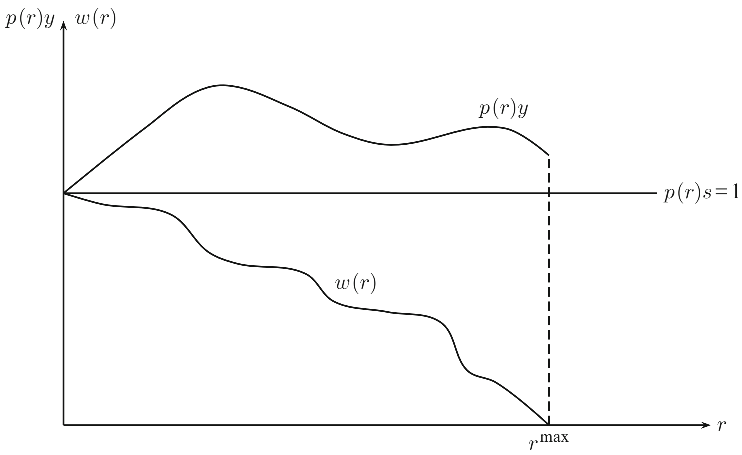

Fig. 11.3 Variable National Income and the implied wage-profit curve (based on the normalization conditions $l(I - A)^{-1}s = 1 = p(r)s$)

11.5 Conclusions

Summing up, we can thus state that the analysis of income distribution is not improved, but in fact obscured if the Standard Commodity is used for *numéraire*. This obscurity may, of course, be avoided if the functional relationship $\widetilde{Y}(r)$ is added to its 'ideal' and universal counterpart $\widetilde{w} = 1 - r/R$, a relationship, for which no qualitative characteristics have been established so far. A closer examination of this function in our view is, however, avoidable and of no great interest since it is very questionable whether the then given tools (of the type considered in Fig. 11.2) will inform us better on income distribution than the originally given ones, i.e., (11.2) and Fig. 11.1, where there is not need to rack one's brains over Sraffa's Standard Commodity. The Fig. 11.3 summarizes the problem consisting for Sraffa's Standard Commodity: We consider here an arbitrary basket of commodities s, for example the subsistence basket of workers (if an interpretation of it is needed):

We see that in such a case both distributable income py and the wage rate w will depend on the rate of profit r in a nonlinear fashion. A change in the numéraire commodity can remove one of these nonlinearities, but never both of them simultaneously. We thus searched in vain for a worthwhile result to be obtained by use of the Standard Commodity, and a similar conclusion seems to be drawn in Broome (1977, p. 236). Yet, no 'brute force' as he claims is required to recognize that such results cannot exist.

References

Broome, J. (1977). Sraffa's Standard Commodity. *Australian Economic Papers*, *16*

Burmeister, E. (1968). On a theorem of Sraffa. *Economica*, *35*

Flaschel, P. (1980). Notes on Sraffa's Standard Commodity and related questions. *Discussion paper* 1. Berlin: Freie Universität

Pasinetti, L. (1977). *Lectures on the theory of production*. London: Macmillan

Sraffa, P. (1960). *Production of commodities by means of commodities*. Cambridge, UK: Cambridge University Press

Woods, J. E. (1978). *Mathematical economics*. London: Longman

Chapter 12
Sraffa's Standard Commodity: No Fulfillment of Ricardo's Dream of an 'Invariable Measure of Value'

The construction of Sraffa's Standard Commodity is often motivated by the claim that it allows to isolate the price-movements of any other commodity, so that they may be observed as in a vacuum. This chapter shows that

(a) A specific and complete solution to the problem of finding the conditions for such a Standard is indeed available, yet that
(b) Sraffa's Standard does not fulfill this list of conditions.

The source of Sraffa's error is then isolated and the true set of 'invariable measures' is determined and shown to be devoid of economic content.

12.1 Introduction

> The search for what has been called 'the chimera of an invariable standard of value' preoccupied Ricardo to the end of his life. However, the problem which mainly interested him was not that of finding an actual commodity which would accurately measure the value of corn or silver at different times and places; but rather that of finding the conditions which a commodity would have to satisfy in order to be invariable in value ... (Sraffa 1970, p. xli)

With regard to this quotation from Sraffa's introduction into Ricardo's Principles, we shall demonstrate in the following:[1] (a) that Sraffa (1960), Chap. 3 indeed offers a specific solution to the above problem of 'finding the conditions ..., and (b) that the construction of an actual commodity of this type in Sraffa (1960), Chap. 4 does not satisfy this very list of conditions. It therefore is to be expected – and it will indeed be the case – that Sraffa's actual commodity will not have the properties derived from the list of conditions preceding its construction. This logical discrepancy escaped the notice of Sraffa and his followers and, as a consequence,

[1] I am very grateful to E. Nell and B. Schefold for critical and helpful comments. The following Sect. 12.2 – as should be stressed – owes much to the presentation given in Schefold (1976, pp. 221–5), though my original motivation to reexamine Sraffa's Chaps. 3 and 4 and the conclusions reached differ very much from those of B. Schefold, see also the preceding chapter as well as Flaschel (1986a,b).

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_12,

the properties (of the conditions) for an 'invariable measure of value' were ascribed to a 'commodity' which in fact is not 'more invariable' in value with respect to changes in income distribution than any other commodity. Hence, Ricardo's dream of an 'invariable measure' of value (cf. Pasinetti (1977, p. 120)) is *not* fulfilled by Sraffa's Standard Commodity![2]

Below we shall first question Sraffa's verbal descriptions which accompany the construction of his standard of value. This will be done on the basis of a simple two-sector example in particular. We then continue by laying bare the reason for his dubious descriptions in the sense pointed out above, i.e., we shall show that one of his two conditions for a commodity to be 'invariable' in value is in fact neglected in his construction of the Standard Commodity. The true set of 'invariable measures of value' will be determined in the final paragraph of this chapter. We will show that this set is normally devoid of economic content. Sraffa's search for an 'invariable measure of value' therefore fails to lead to a sensible solution of Ricardo's great problem. In contrast to recent reconsiderations of Sraffa's Standard Commodity (see, e.g., Burmeister (1984), Samuelson (1983)), this chapter consequently does not consider problems of generalizing this commodity to input–output models of the more complicated type. Instead, it already questions the meaningfulness of the whole approach within the basic circulating-capital model of Sraffa's analysis.

12.2 Flaws in the Interpretation of the Standard Commodity

The starting point of Sraffa's (1960), Chap. 3 price theoretic considerations is the assumption of a given input–output system A, l, I. A denotes the $n \cdot n$ matrix of physical inputs of n given types of commodities (with columns as processes), l denotes direct labor inputs (a row with $l_1 + \ldots + l_n = 1$) by suitable choice of unit) and I – the identity matrix – represents the matrix of outputs (e the vector given by its diagonal). On the basis of these data Sraffa then considers the following equations for prices $p = (p_1, \ldots, p_n)$ measured in terms of net national product $e - Ae \in \Re^n$:[3]

$$p = (1 + r)pA + wl, \quad p(I - A)e = 1. \tag{12.1}$$

Here r and w denote the uniform rate of profit and of wages, respectively.

Let $\lambda(A)$ denote the dominant characteristic value of the matrix A and define $1+R$ by $\lambda(A)-1$. Under well-known assumptions it can be shown for any $r \in [0, R]$ that the above system of $n + 1$ equations in the $n + 1$ unknowns $(p_1, \ldots, p_n)$, w will have a unique and economically meaningful solution. The thereby introduced mapping $w : [0, R] \to [0, l]$ is differentiable: $w'(r) < 0$ and $w(0) = 1, w(R) = 0$.[4]

[2] See however Schefold's comments and my reply in the *Zeitschrift für die gesamte Staatswissenschaft*, 142, 1986, 603ff. for a further discussion of the issues treated in this chapter.

[3] All gross output levels are set equal to one here and in the following.

[4] For further details, cf. Pasinetti (1977), Chap. 5.

It is the purpose of Sraffa's construction of the Standard Commodity to help to penetrate the complexities in the movement of relative prices p generated by exogenous changes in the rate of profit r.

From Sects. 22 and 26 in Sraffa (1960) it can be seen that Sraffa's ultimate definition of the Standard Commodity is that of a (normalized) right hand characteristic vector q of the input matrix A, corresponding to its dominant characteristic value $\lambda(A) = (1 + R)^{-1}$:

$$(1 + R)Aq = q, \qquad lq = \sum_{i=1}^{n} l_i = 1. \tag{12.2}$$

Consider for an example:

$$A = \begin{pmatrix} 1/6 & 1/2 \\ 1/9 & 0 \end{pmatrix}, \; l = (1/2, 1/2).$$

As Standard Commodity we here get

$$q = \begin{pmatrix} 1.5 \\ 0.5 \end{pmatrix}, \; R = 2 \quad (lq = l_1 + l_2 = 1)$$

And as prices in terms of the chosen numéraire $e - Ae = \begin{pmatrix} 1/3 \\ 8/9 \end{pmatrix}$ we get:

$$p = w\frac{1}{14 - 5r - r^2}(10 + r, 12 + 3r), \quad w = \frac{14 - 5r - r^2}{14 + 3r} \tag{12.3}$$

In what respect does the introduction of Sraffa's Standard Commodity now represent an aid in the study of these prices?

For purposes of illustration, consider the price of commodity 1 of this example in terms of the Standard Commodity q:

$$\frac{p_1}{pq} = \frac{\frac{10 + r}{14 + 3r}}{\frac{2}{3}\frac{10 + r}{14 + 3r} + \frac{1}{2}\frac{12 + r}{14 + 3r}} = \frac{\frac{10 + r}{14 + 3r}}{\frac{76 + 13r}{6(14 + 3r)}} = \frac{60 + 6r}{76 + 13r}. \tag{12.4}$$

It is with regard to such a price relationship that the following quotation from Sraffa (1960, p. 18) has to be analyzed:

> 'It is true that, as wages fell, such a commodity (q, P. F.) would be no less suspectable than any other to rise or fall in price relative to other individual commodities; but we should know for certain that any such fluctuation would originate exclusively in the peculiarities of production of the commodity which was being compared with it (here: commodity 1, P. F.), and not in its own'.

The properties of the Standard Commodity therefore have to be of a kind that makes it possible to ascribe the above dependency (12.4) of p_1 as well as that of pq on the rate of profit r wholly to the conditions of production of commodity 1. The change in p_1/pq corresponding to a change in r is thus claimed to be due to the conditions of production behind the numerator only and viewed to be independent from the conditions which govern the denominator!

Yet, how is it possible to reach a causal conclusion from a pure system of general interdependence?

Similarly, a r-dependent change in $p(e - Ae)/pq$ should originate in the peculiarities of production of $e - Ae$ only. But, the vectors $e - Ae$ and q often are both positive, i.e., they depend on the same processes, though of course with different scales of operation. The above discriminating conclusion consequently rests on a difference in scales of operations only. Furthermore, Sraffa's (1960), Chap. 3 analysis of 'invariability' is based on the *assumption* that $p(e-Ae)$ equals '1' throughout. But his final construct, the Standard Commodity q, then reverses the order of 'invariability': the change in pq with respect to $p(e-Ae) = 1$ is now claimed to be due to the peculiarities of actual net product $e-Ae$ and not to the Standard product q. This strange *verbal* discrimination between q and the commodities not proportional to q seems to be the only possible 'result' with respect to the original price system 1.

This questionable result suggests that the simplifying use of Sraffa's Standard Commodity must be understood to derive from a change in numéraire: from actual net product $e - Ae$ to $d = q - Aq$, the Standard net product. But such a change has nothing in common with the question of 'invariability'; it is a change by definition. Within the model in use the advantage of this new numéraire therefore has to be explained in terms of something else.

This is normally done with respect to its property to imply a linear wage profit relationship:

$$w = 1 - \frac{r}{R}, \qquad r \in [0, R]$$

the only – possibly – relevant *implication* that can be drawn from the use of this new numéraire d.

We have already shown elsewhere[5] by means of the curve $p(e - Ae)$ that no additional information – superior to that already contained in the original curve $w(r)$ – can be gained from such an expression.[6] The choice of the Standard Commodity for numéraire thus only conceals that no useful *implication* can be reached by means of this commodity.

---

[5] cf. Flaschel (1984) and here Chap. 11.

[6] Furthermore, it is simply not true that 'the particular proportions of the Standard Commodity ... aid in comprehension of the connection that exists between changes in the distribution of income and the system of relative prices' as is argued in Roncaglia (1978, p. 75). With respect to our example this would mean that the expression

$$p = \left(\frac{10 + r}{14 + 2r}, \frac{12 + 3r}{14 + 2r}\right)$$

should give an aid in comprehending the originally given prices:

$$p = \left(\frac{10 + r}{14 + 3r}, \frac{12 + 3r}{14 + 3r}\right)$$

($pd = 1$ in the first and $p(e - Ae) = 1$ in the second case).

In addition to this result, one may wonder, however, what the exact reasons are that no explicit formulation – in terms of the given model – of the usefulness of the Standard Commodity can be given. We shall argue in the next paragraph that it is in fact the unnoticed elimination of one of Sraffa's sufficient conditions for a commodity to be really 'invariable in value' from the construction of his Standard Commodity that is responsible for its poor performance. The resulting failure of the 'famous Standard Commodity – *not to be* – a commodity with a price completely independent of changes in income distribution' (see Bacha et al. (1977, p. 45) for the opposite view) prevents it from becoming an analytical useful device.

12.3 Flaws in the Construction of Sraffa's Standard of Value

Sraffa (1960, Chap. 3) analyzes the causes for relative prices to change (or not to change) *under the assumption of a constant net national income:* $p(e - Ae) = 1$! By giving the wage rate w successive values between '1' and '0' Sraffa there especially determines certain watershed properties or critical ratios for 'invariability'. These conditions are subsequently used (in modified form!) in the construction of his Standard Commodity (Chap. 4). Yet, we have already exemplified that this 'commodity' is not invariant in value with respect to changes in income distribution in the surrounding where the conditions for 'invariability' have been established. Sraffa's conditions for an invariable measure of value and their relationship to the properties of the Standard Commodity therefore have to be regarded anew to find the source of this divergence.

> 'We now revert to the 'critical' proportion which has been mentioned before (Sect. 17) as constituting the borderline between 'deficit' industries and 'surplus' industries. Suppose that there was an industry which employed labor and means of production in that precise proportion, so that with a wage-reduction, and on the basis of the initial prices, it would show an exact balance of wages and profits'. (Sraffa 1960, Sect. 21)

Utilizing (12.1) the output q of such an industry must fulfill (cf. also Schefold (1976, pp. 221, ff.)):

$$\Delta r p A q + \Delta w l q = 0 \quad (\text{ on the basis of } \Delta p = 0), \text{ i.e.}$$

$$w'(r) = -\frac{pAq}{lq} \text{ for all } w \in [0, 1] \text{ or } r \in [0, R]. \tag{12.5}$$

Note that this condition only makes sense if a numéraire commodity, i.e., $p(I - A)e \equiv 1$ in the context of Sraffa's Chap. 3, has already been adopted!

Equation (12.5) describes the critical proportion just mentioned, a proportion which normally depends on the rate of profit r. We note that this condition has been established by partial equilibrium analysis. It is therefore not immediately clear, whether it already constitutes a sensible condition for the price measure looked for. In addition, condition (12.5) is also not yet sufficient for the desired invariability. Our quotation from Sraffa (1960, Sect. 21) therefore should continue: 'suppose further ...'. The then following condition is well known. It suffices therefore to describe it shortly by:

$$\frac{pAq}{lq} = \frac{pA^2q}{lAq} = \frac{pA^3q}{lA^2q} = \ldots = c(r) \tag{12.6}$$

Proposition 12.1. *Conditions (12.5) and (12.6) are sufficient to imply* pq *= constant with regard to the prices (12.1).*

Proof. By differentiating (12.1) we get for $r \in [0, R]$:

$$\begin{aligned}
p' &= pA + (1+r)pA + w'(r)l, \quad \text{i.e.} \\
p'(I-(1+r)A) &= w'(r)l + pA, \quad \text{i.e.} \\
p' &= (w'(r)l + pA)(I-(1+r)A)^{-1} = (w'(r)l + pA)\sum_{v=0}^{\infty}(1+r)^v A^v \\
&= \sum_{v=0}^{\infty}(1+r)^v [w'(r) + c(r)] l A^v q.
\end{aligned}$$

□

From (12.6) we then get:

$$p'q = \sum_{v=0}^{\infty}(1+r)^v [w'(r) + c(r)] l A^v q.$$

Because of $c(r) = \frac{pAq}{lq} = -w'(r)$ (cf. (12.5)), we finally obtain

$$p'(r)q = 0 \text{ for all } r \in [0, R].$$

> 'The commodity produced by such an industry (which fulfills conditions (12.5), (12.6), P. F.) would be under no necessity, arising from the conditions of production of the industry itself, either to rise or to fall in value relative to any other commodity when wages rose or fell (in fact we have proved that its output value is fixed, P. F.).... A commodity of this type would in any case (a retreat?, P. F.) be incapable of changing in value relative to the aggregate of its own means of production since the recurrence of the same 'proportion' would apply equally to them'. (Sraffa 1960, Sect. 21)

This last assertion

$$\frac{pq}{pAq} = \text{ const } (=1+R) \quad \text{or} \quad \frac{p(I-A)q}{pAq} = R$$

is a simple consequence of the above proposition (or of its two assumptions (12.5), (12.6)). We already know that

$$0 = p'q = rp'Aq + pAq + w(r)lq.$$

Together with condition (12.5) this implies:

$$rp'Aq \equiv 0 \text{ or } p'Aq \equiv 0.$$

Therefore pq and pAq (and $pA^2q\ldots$) are all invariant with respect to changes in income distribution, which means that $\frac{pq}{pAq}$ has to be invariant, too,[7] and equal to $1+R$.
After stating these propositions, Sraffa (1960, Sect. 21) continues:

> 'Two separate conditions have been assumed to obtain this result, namely, (1) that the 'balancing' proportion is used, and (2) that one and the same proportion recurs in all successive layers of the industry's aggregate means of production without limit. We shall, however, find that the first condition is necessarily implied in the second for, . . . within any one system complete 'recurrence' is only possible with the balancing proportion'.

We already know that this conclusion *must be false* in the context of the chapter from which this quotation has been drawn: the Standard Commodity does fulfill condition (12.6) but not (12.5) (cf. our Sect. 2 for an example). How can Sraffa nevertheless succeed in 'proving' this last assertion? *He can do so simply by redefining his former 'critical proportions' now as 'balancing proportions' by means of a 'convenient' non-hybrid ratio*, i.e., by employing

$$\frac{p(q-Aq)}{pAq} \equiv R \quad \text{instead of} \quad \frac{pAq}{lq} = -w'(r)$$

(see his Sect. 22; R the maximum rate of profit). Sraffa thereby implicitly reformulates the conditions (12.5), (12.6) in the following way

$$\frac{pq}{pAq} = \frac{pAq}{pA^2q} = \frac{pA^2q}{pA^3q} = \ldots \equiv 1+R. \tag{12.7}$$

But since neither this nor the original condition (12.6) implies condition (12.5), the replacement of the 'critical proportion' (12.5) by the non-hybrid ratio $pq/pAq \equiv 1+R$ means that the correct 'balancing proportions' have got lost.

Therefore Sraffa gives a solution to the problem of *'finding the conditions'* (cf. the introduction), but when he proceeds to the problem of *'finding the commodity'*, he indeed neglects one of these conditions, thereby singling out commodities which are only partially connected to the initially raised question. By dismissing one condition Sraffa consequently gains the necessary degree of freedom in construction (but also in content) for his 'invariable measure of value'.

Condition (12.6) – now in non-hybrid form (12.7) – implies

$$pA^{\nu}[q-(1+R)Aq] = 0 \text{ for all } \nu \in N.$$

[7] For further (equivalent) properties, cf. Miyao (1977).

This suggests that the equality $q - (1 + R)Aq = 0$ will then hold true, wherefrom Sraffa's Standard Commodity is finally obtained.[8] But such a Standard is *not* 'capable of isolating the price movements of any other product so that they could be observed as in a vacuum' (Sraffa 1960, p. 18), as it does not obey the assumption (12.5) of Sraffa's Chap. 3 investigation.

Proposition 12.2. *The Standard Commodity q will fulfill condition (12.5) if and only if the wage-profit curve (12.1) is of the form $w = 1 - \frac{r}{R}, w \in [0, R]$.*

Proof. With respect to $\frac{pAq}{lq} = pAq = \frac{1}{1+R}pq$ one can easily calculate

$$\begin{aligned}\frac{pAq}{lq} &= \frac{w(r)}{1+R} l(I-(1+r)A)^{-1}q \\ &= \frac{w(r)}{1+R}\sum_{v=0}^{\infty}(1+r)^v lA^v q = \frac{w(r)}{1+R}\sum_{v=0}^{\infty}\left(\frac{1+r}{1+R}\right)^v \\ &= \frac{w(r)}{1+R}\frac{1}{1-\frac{1+r}{1+R}} = \frac{w(r)}{R-r}.\end{aligned}$$

The curve $w = 1 - \frac{r}{R}$ consequently fulfills

$$w'(r) = -\frac{1}{R} = -\frac{1-\frac{r}{R}}{R-r} = -\frac{pAq}{lq}.$$

It remains therefore to prove that the equation

$$w'(r) = -\frac{pAq}{lq} = \frac{w(r)}{R-r}, \quad \text{i.e.,} \quad \frac{w'(r)}{w(r)} = \frac{1}{r-R}$$

in turn implies that $w(r)$ must be equal to $1 - \frac{r}{R}$.

By integrating the last equation we get

$$\ln w = \int_0^r \frac{1}{t-R}dt \quad (\text{ for } r < R, \text{ i.e., } w > 0).$$

Now, the integral on the right hand side is equal to $\ln\left(1 - \frac{r}{R}\right)$ as can be proved by taking derivatives and by paying attention to the condition $\ln w(0) = \ln 1 = 0$. This completes the proof of the above proposition. □

From the above proposition it follows that the two types of 'balancing ratios' considered in Sraffa (1960, Chap. 3) have to be kept apart very carefully: while the 'balancing-ratio'-condition

[8] We neglect here minor problems of non-uniqueness (cf. Miyao (1977) in this regard).

$$\frac{p(q - Aq)}{pAq} = \frac{p((1 + R)Aq - Aq}{pAq} = R \qquad (12.8)$$

is a trivial consequence of the definition (12.2) of q, the 'balancing ratio'-condition (12.5)

$$w'(r) = -\frac{pAq}{lq} \quad (q \text{ the Standard Commodity })$$

represents a further and severe restriction on the given input–output system:[9] the wage-profit curve then has to be linear in its original setting (12.1) already.[10] This implication will not be fulfilled in most of (especially the relevant) input–output systems. Therefore (12.5) had to be dismissed by Sraffa to avoid that the search for an actual commodity of the desired kind would become a hopeless task.

The switch from hybrid to non-hybrid conditions is, however, damaging to Sraffa's intended interpretation of commodity q. What originally seemed to emerge *as an implication* now has to be regulated by *assumption*:

$$p(q - Aq) \equiv 1.$$

> 'Particular proportions, such as the standard ones, may give transparency to a system and render visible what was hidden, but they cannot alter its mathematical properties'. (Sraffa 1960, p. 23)

This purpose of simplification is not fulfilled by this change in numéraire.

The theory of income distribution as is now well known, cannot be simplified by use of 'surrogate capital' – but by use of 'surrogate corn' neither (cf. also Flaschel (1984)). Contrary to Bacha et al. (1977, p. 40), who declare the Standard System as Sraffa's main analytical contribution, we accept Steedman's (1979, p. 72) position that the Standard Commodity is not a central part of Sraffa's work. Yet, central or not, our final conclusion is that the Standard Commodity is of no use in the analysis of prices and distribution, i.e., it must be characterized as being redundant.

12.4 On the Non-Existence of an 'Invariable Measure of Value'

We have shown in the preceding paragraph why Sraffa's Standard Commodity does not give an answer to the problem of 'invariability' considered in Sraffa (1960, Chap. 3): 'Ricardo's dream' does not find its fulfillment in Sraffa's pseudocorn commodity (12.2). There exists one important reason which contributes to an understanding of the delayed occurrence of this negative result: the problem which is

---

[9] This fact is ignored most explicitly in Bacha et al. (1977, p. 45), there leading to the absurd conclusion – not supported by Sraffa – that the price of the Standard Commodity is 'completely independent of changes in income distribution.'

[10] For consequences of this property, cf. Miyao (1977).

to be solved by such a commodity simply has not been stated clearly in the literature so far. The first thing which should be noticed in this regard is, that the problem of invariability *cannot be described unless a measure of value has already been assumed*. This fact is implicitly taken into account by Sraffa (1960, Chap. 3) by his assumption $p(e - Ae) \equiv 1$. The question of 'invariability' consequently is of a relative nature only: the search for (conditions for) a 'measure of value' relative to an already given measure of value! But what can be expected from the solution of such a problem? Commentators on Sraffa's Standard Commodity have been astonishingly silent in this regard.

Let $q \in R^n, q \geqq 0$ be any bundle of commodities. It is well-known that the value pq of this bundle as a function of the rate of profit (see (12.1)) can be quite irregular in its behavior, depending on the choice of both q and the numéraire ($e - Ae$ in our case).

Following Sraffa we have, however, found out two conditions (see (12.5), (12.6)) which imply that commodity bundles q which fulfill these conditions will be invariant in value (pq) relative to $e - Ae$. Such commodities q thus would provide the ideal yardstick whereby *changes in income distribution* can be analyzed without running into the problem of a simultaneous change in the *amount of income* to be distributed.

Below we shall give a description of the complete set of such 'measures' $q \in R^n$. We shall contrast this set with the set of all Standard Commodities as determined by Miyao (1977). The set of all vectors $q \geqq 0$ which fulfill Sraffa's original conditions for invariability (12.5), (12.6) is then determined as the intersection of these two sets, i.e., Sraffa's original conditions for 'invariability' are more restrictive than necessary. Yet, the complete set of 'invariable measures of value' also is of no interest, since it will normally consist of the vector $e - Ae$ only. Furthermore, it is not admissible to consider Miyao's Standard Commodities as exponents of such 'measures'. The search for an 'invariable measure of value' therefore generally fails to lead to a sensible solution.

Proposition 12.3. *A vector $q \geqq 0$ will be an 'invariable measure of value', i.e., $pq \equiv 1$ with regard to equations (12.1), if and only if q is proportional to*

$$(e - Ae) + z \geqq 0 \tag{12.9}$$

where z is any vector orthogonal to the set of vectors:

$$\{l, lA, lA^2, lA^3, \ldots\}.$$

Proof. [11] From Miyao (1977, Lemma 6) we know that the set of vectors z, which fulfill $lA^v z = 0$ for all $v = 0, 1, 2, \ldots$ is identical to the set of vectors z, which fulfill

[11] Following Miyao (1977) we shall assume for simplicity that A is irreducible.

$pz = 0$ for all $r \in [0, R]$. It suffices to consider those $q \geqq 0$ where $p(0)q = 1$ holds true. But then $pq - p(e - Ae) \equiv 0$ is equivalent to $q - (e - Ae) \equiv z$, which proves the proposition. □

Proposition 12.4. *The set of vectors $q \geqq 0$ which fulfill (12.6) coincides with the set of (generalized) Standard Commodities as defined in Miyao (1977):*

$$q = q^* + z \geqq 0, \tag{12.10}$$

where z is determined as in Proposition 12.3 and $q^* > 0$ is determined by $(1 + R)Aq^* = q^*$.

Proof. It suffices to consider

$$\frac{p(R)Aq}{lq} = \frac{p(R)A^2q}{lAq} = \frac{p(r)A^3q}{lA^2q} = \ldots = c(R). \tag{12.11}$$

From $p(R) = (1 + R)p(R)A$ we get:

$$\frac{pA^{\nu+1}q}{lA^\nu q} = \frac{p(1+R)^{-(\nu+1)}q}{lA^\nu q} = \frac{pq}{l(1+R)^{\nu+1}A^\nu q} = \text{const},$$

which in turn implies

$$\frac{l(1+R)^{\nu+1}A^\nu q}{l(1+R)^{\nu+2}A^{\nu+1}q} = 1 \quad \text{or} \quad \frac{lA^\nu q}{lA^{\nu+1}q} = 1 + R \quad \text{for} \quad \nu = 0, 1, 2, \ldots.$$

This latter situation is identical to condition (12.7) in Miyao (1977, p. 154), which implies that q must be of the form $q^* + z$. Conversely, such a vector q evidently fulfills condition (12.6), since it fulfills

$$lA^\nu q = lA^\nu q^* = lq^*(1+R)^{-\nu}$$
$$p(r)A^\nu q = p(r)A^\nu q^* = p(r)q^*(1+R)^{-\nu}$$

because of $p(r) = w(r)l(I - (1+r)A)^{-1}$, $w(r) = [l(I - (1+r)A)^{-1}(e - Ae)]^{-1}$. □

Proposition 12.5. *The vectors q which fulfill Sraffa's conditions (12.5) and (12.6) (and $p(0)q = 1$) are of the form*

$$q = \begin{cases} (e - Ae) + z \geqq 0 \\ q^* + \widetilde{z} \geqq 0 \end{cases} \tag{12.12}$$

where $z, \widetilde{z}$ are given as in Proposition 12.3.

Proof. From (12.6) there follows that q must be of the form $q = q^* + \widetilde{z}$ (cf. Proposition 12.4). And from (12.5) to (12.6) there further follows that $pq =$ const. must be fulfilled, which implies $q = (e - Ae) + z$ by Proposition 12.3.

Conversely, let q be given by (12.12). Then q must fulfill condition (12.6), cf. again Proposition 12.4. And from Proposition 12.3 we know furthermore $p'q \equiv 0$. The vector Aq consequently is of type $q^* + \widetilde{z}$, since Aq^* is proportional to q^* and $(lA^v)(Az) = lA^{v+1}z = 0$ for all $v \in N$. We therefore have $p'Aq \equiv 0$.

Differentiation of the (12.1) (with respect to r) then especially gives:

$$(1 + r)p'(r)A + pA + w'(r)l = p', \text{ i.e.}$$

$$(1 + r)p'(r)Aq + pAq + w'(r)lq = p'(r)q \text{ or}$$

$$pAq + w'(r)lq = 0, \text{ i.e. condition (12.5) must hold true.}$$

□

Proposition 12.6. *There exist vectors q which fulfill conditions (12.5) and (12.6) iff*

(a) $e - Ae$ is a generalized Standard Commodity, i.e.: $e - Ae = q^* + z \geqq 0$ (cf. Proposition 12.4) or:
(b) the wage-profit curve $w(r)$ of (12.1) is of the form $w = 1 - r/R$.

Proof.

(a) cf. Proposition 12.5.
(b) cf. Miyao (1977, p. 154).

We conclude from the above assertions that the 'invariable measures of value' (12.9) will in general not fulfill conditions (12.5) and (12.6) and that the (generalized) Standard Commodities (10) will not satisfy condition (12.5). While all commodities $A^v q$ and $(I - A)^{-1}q$ will be Standard Commodities, if q is of this type, such a property cannot be established for 'invariable measures of value' (12.9), since $A^v(e - Ae)$ will not be proportional to $e - Ae$ in general.

A number of conclusions can be drawn from the above propositions:

(1) Sraffa's original conditions (12.5), (12.6) are unnecessarily restrictive with respect to the search for an 'invariable measure of value' (12.9): net national product has to be of Standard Commodity type (12.10) in this case, a condition which is nearly as special as

$$(1 + R)lA = l,$$

the case where labor values coincide with the above price theory (in both cases the wage-profit curve $w(r)$ of (12.1) will be a linear curve).
(2) Though the imposition of (12.5) and (12.6) is more restrictive than necessary with respect to the determination of a 'measure of value' q, the reduction to only one of these two conditions will not give an admissible generalization. Therefore Sraffa's (1960, Sect. 21) procedure – to consider the commodities which

obey condition (12.6) or (12.7) – as the candidates for a 'measure of value' – is misleading. The thereby determined (generalized) Standard Commodities (cf. Proposition 12.4) in general have nothing in common with the originally raised question of 'invariability', but do generalize input–output relations of corn-type only.

(3) The complete set of 'invariable measures of value' (12.9) has up to now been neglected in the discussion of Sraffa's work. This discussion has been concentrated on Standard Commodities $(1, 0)$ which satisfy (12.6), cf. Miyao (1977).

(4) The final situation now presents itself as follows: The set of 'measures of value' (12.9) normally will be of a very uninteresting or even trivial type, i.e., it will be identical with $e - Ae$, its definitional element, in essence. On the other hand, the set of Standard Commodities always contains a non-trivial element: Sraffa's Standard Commodity q^*, but such a commodity is devoid of economic content. □

12.5 Conclusions

We have seen that – given a numéraire commodity, e.g., net national product $e - Ae$ – Sraffa's Standard Commodity q will be an 'invariable measure of value' if and only if the wage-profit curve $w(r)$ for the above numéraire is a linear curve. Sraffa's proposal *to choose* his Standard Commodity $q - Aq$ for numéraire may thus be characterized as follows: Assuming this numéraire, i.e., $p(q - Aq) \equiv 1$, establishes conditions (12.5), (12.6) which indeed imply that this commodity $q - Aq$ will be invariant in value, which, however, is already true by assumption! This type of circular reasoning identifies the problem of choosing a numéraire with the question of invariability, which thereby becomes an impenetrable whole. This explains why there are no useful implications for Sraffa's Standard Commodity (see Sect. 12.2).

The original question of Sraffa's Chap. 3 was, however, whether there exists a non-trivial commodity which does not change in value when the distribution of a *given* value of net national product is changed. This question has been answered in the negative. We have shown that Sraffa's implicit change in the conditions which characterize 'invariability' represents the basic explanation for the result and for the meager performance of his Standard Commodity. There is consequently no need to consider additional complications as, e.g., multiple activities, fixed capital, further primary factors, etc. (as is done in Burmeister (1984), Samuelson (1983), cf. also the discussion in the *Zeitschrift für Nationalökonomie* between Samuelson and Baldone in 1985) to prove that Sraffa's Standard has no real analytical meaning.

The failure of the Standard Commodity of not fulfilling the true conditions for 'invariability' should, however, not be confused with the problem of properly generalizing to general models of production the classical concepts of prices of production and labor values (indexes of labor productivity), such that in particular certain classical insights into their relationship remain true. This latter task can be attacked in an economically meaningful way, if attention is paid to factual methods of cost-accounting as well as rate-of-return calculations. It is not meaningful in this regard, however, to concentrate solely on the 'reference case' of a uniform composition of capital and this with respect to the perfect von Neumann world of capital depreciation and the like (see e.g., Samuelson (1983)), if a proper study of the classical theories of value and price is intended. To show the inadequacy of this latter approach and the superiority of the first more factually-oriented one would, however, demand more than just another chapter.

References

Bacha, E., Carneiro, D. & Taylor, L. (1977). Sraffa and classical economics: Fundamental equilibrium relationships. *Metroeconomica*, *29*, 39–53

Burmeister, E. (1984). Sraffa, labor theories of value and the economics of real wage determination. *Journal of Political Economy*, *92*, 508–526

Flaschel, P. (1984). The standard commodity and the theory of income distribution. A critical note. *Australian Economic Papers*, *23*, 123–129

Flaschel, P. (1986a). Sraffa's standard commodity: No fulfillment of Ricardo's dream of an 'invariable measure of value'. *Zeitschrift füer die Gesamte Staatswissenschaft*, *142*, 588–602

Flaschel, P. (1986b). The standard commodity as a tool of economic analysis: A reply to Schefold. *Zeitschrift füer die Gesamte Staatswissenschaft*, *142*, 623–625

Miyao, T. (1977). A generalization of Sraffa's standard commodity and its complete characterization. *International Economic Review*, *18*, 151–162

Pasinetti, L. (1977). *Lectures on the theory of production*. London: Macmillan

Roncaglia, A. (1978). *Sraffa and the theory of prices*. Chichester: Wiley

Samuelson, P. (1983). Durable capital inputs: Conditions for price ratios to be invariant to profit-rate changes. *Zeitschrift für Nationalökonomie*, *43*, 1–20

Sraffa, P. (1960). *Production of commodities by means of commodities*. London: Cambridge University Press

Sraffa, P. (1970). *The works and correspondence of David Ricardo*, I., London: Cambridge University Press

Schefold, B. (1976). *P. Sraffa: Warenproduktion mittels Waren*. Frankfurt a.M.: Suhrkamp

Schefold, B. (1986). The standard commodity as a tool of economic analysis: A comment on Flaschel. *Zeitschrift füer die Gesamte Staatswissenschaft*, *142*, 603–622

Steedman, I. (1979). On an alleged inconsistency in Sraffa's economics. *Capital and class*, *9*, 71–74

Part III
Gravitation or Convergence in Classical Micro-Dynamics

In this part of the book we demonstrate that the Classical process of ruthless competition, formulated so far in terms of algebraic equations, representing prices of production, can also be investigated from the perspective of dynamics, in addition to the simultaneous equations determination of these long-period equilibrium positions considered in the preceding part of the book. There is a description available in the Classical theory of capital flows and market price dynamics that can be rigorously modeled and also analyzed in detail and that may – or may not – imply convergence to the Classical long-period prices of Neoricardian theory (with their uniform wage and profit rates).

It may however also be the case that only gravitation around such long-period positions is implied, which would then loose their character as attracting equilibrium positions. Of course, there may in addition exist fairly stable profit rate differentials in actual economies (as was shown in ch. 8), a situation which would then demand for a significant revision of the concept of prices of production. In sum, this means that the cross-dual dynamics which we shall investigate in this part of the book remain still somewhat ambiguous with respect to the steady state position they are to be anchored to. And from a Schumpeterian perspective it may even be possible that Classical capital movements and the gravitation processes suggested by them may not have a point of rest at all, but are subject to perpetual shifts into new technological environments, via more or less basic innovations, their subsequent bunching and their diffusion, until new basic innovations change the world again.

The Classical adjustment process towards or around prices of production is in its baseline formulation of cross-dual type, like the predator-prey interaction of population dynamics. Capital is suggested to flow into highly profitable sectors and the resulting increases in the supply of the commodities of these sectors are supposed to act negatively on prices and profitability and therefore provides a check for the further inflow of capital or even a reversal of these flows when other sectors of the economy have become more profitable thereby. This process can exhibit overshooting characteristics and may thus initiate persistent capital flows between industries with no tendency towards the establishment of a uniform profit rate across these industries.

In addition, within the industries of actual economies, there always exists side by side newly established, modern, but also outdated, more or less rapidly dying methods of production which of course prevents the establishment of uniform profit rates inside these industries. Finally, the fundamental sectors of the economy (agriculture, manufacturing, services) may be subject to conditions that establish definite profit rate differentials between them. The Classical process of gravitation may therefore only represent a starting point for a further analysis of the laws of motion of capitalist competition within the setup of a variety of market structures.

Profitability driven capital flows between industries, the conflict over market shares and the implied pricing decisions of firms, also constitute a dynamic interaction between quantities and prices in such market economies which differ significantly from the market dynamics that is – if at all – generally considered in mainstream economics.[1]

We know from Lotka – Volterra predator-prey dynamics (the pure crossover type of dynamics where one group of state variables is acting positively on another group of state variables, while this latter group is acting negatively on the former one) that these interactions will produce cyclical stability, but not asymptotic stability. We will however show that – when the direction of change of profitability differentials and market excess demands is taken into account in these dynamics in addition (where increases in excesses are assumed to have stronger dynamic effects than decreases) – we get convergence as a fairly robust result. This will be considered in Walrasian production economies in chs. 13 and 14 and for the Classical von Neumann model in ch. 15.

In a final chapter we add Keynesian dual forces to the Classical cross-over process of ruthless competition, in fact basically a dynamic multiplier story in the case of quantities and a dynamic markup pricing procedure in the case of prices. These additions may be considered the short-run or fast forces within the dynamics of capital flows, market shares and market prices, while capital flows and their implications for profitability may be considered as working slower and thus only in the medium run if not even only in the longer run.

Summing up we thus get in this part of the book somewhat mixed results where, on the one hand, convergence to so-called long-period prices can be shown to exist in certain idealized economic structures. On the other hand, capitalist economies are subject to rapid and radical changes in technology, in particular in a globalized world, so that the idea of getting in the average or on the margin significant convergence to uniform rates of profit and to the production prices they imply may be completely illusionary from the empirical perspective. While labor values represent an accounting scheme which allows to discuss technological change in a rapidly changing world, prices of production rely on an accounting framework that may be purely academic from the applied perspective (as the empirical analysis conducted in ch. 8 has exemplified).

[1] There are exceptions, most notably Walras himself, who indeed reformulated a cross-over tâtonnement process in the case of production economies as we shall see in chs. 13 and 14.

Sraffa's (1960) production price accounting schemes therefore, in the minimum, need revision from the perspective of input-output theory as suggested in Bródy (1970), but may even need to be further modified as in Flaschel, Franke, and Veneziani (2009) in order to allow for the discussion of a stable distribution of profit rate differentials as moving centers of market prices. The dynamical price-quantity adjustment processes we have discussed in this part of the book can be applied to all of these approaches to long-period or average price determinations and are thus not proving anything special for the Neo-Ricardian theory of long-period production prices.

References

Bródy, A. (1970): *Proportions, Prices and Planning*. North Holland: Amsterdam.

Flaschel, P., Franke, R. and R. Veneziani (2009): 'Neoricardian Theory and the Measurement of Prices of Production': An Alternative Approach. Bielefeld University: CEM working paper.

Marx, K. (1977): *Capital, Volume I – III*: London: Lawrence and Wishart.

Samuelson, P. (1971): Understanding the Marxian notion of exploitation: A summary of the so-called transformation problem between Marxian values and competitive prices. *Journal of Economic Literature*, 9, 399 – 431.

Sraffa, P. (1960): *Production of Commodities by Means of Commodities*. Cambridge, UK: Cambridge University Press.

Steedman, I. (1977): *Marx after Sraffa.* London: NLB.

Chapter 13
Dressing the Emperor in a New Dynamic Outfit

13.1 Introduction

This chapter shows that there is a natural extension of the conventional price tâtonnement procedure of pure exchange economies which significantly increases the stability properties of such adjustment processes from a local as well as a global point of view. This extension is motivated by the observation that market price adjustments should not only depend on the levels of excess demand, but also on their direction (and magnitude) of change. Taking these additional forces appropriately into account implies an adjustment process which is formally similar to the Generalized Newton Methods which have been construed in the search for price mechanisms that are 'universally stable'. Furthermore, this adjustment process also generalizes the stability proof for the ordinary tâtonnement procedure in the case of gross substitutes (by means of a suitable Liapunov function) in a straightforward way.

Since the late sixties the stability properties of Walrasian general equilibrium models have been regarded with more and more skepticism or even strong pessimism with regard to their relevance and generality. Dierker (1974) and Hahn (1984, Chap. 4) provide important examples of judgments of a fairly skeptical type.[1] Furthermore, many publications on Walrasian Economics which have appeared since Arrow and Hahn's (1971) extensive presentation of the stability of general equilibrium have treated this topic as one of only very secondary importance. Kirman (1988, 1989) has recently reconsidered the state of art in this field of investigation and finds that 'instability' is a far better description of this state than 'stability'. He discusses some new ways out of this instability problem which, however, are not convincing enough to allow for a revision in the prevailing skepticism. Yet, this state of stability analysis is considered by him to be a severe problem for modern economic theory: 'Introducing more sophisticated adjustment processes does not, unfortunately, help.': The emperor has no clothes.

[1] See also Ingrao and Israel (1990) for a recent survey on the dynamic properties of the 'invisible hand'.

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_13,

There are others, Hildenbrand (1983) in particular, who have shown that the equilibria of distribution – as well as exchange-economies[2] can be uniquely determined and globally stable with respect to the standard tâtonnement procedure under appropriate assumptions on the distribution of incomes or endowments.[3] Furthermore, such an approach also allows for useful comparative static results, since it gives rise to the so-called 'Law of Demand'.[4] It is stressed in Hildenbrand (1989) in this regard that one should strive for such definite results because it is the ordinary job of the economist to make use of comparative statics – and not to contemplate very general relationships solely.

Yet, from a purely theoretical perspective the following ranking of analytical necessities and objectives seems adequate and should be adopted:

1. Supplying proofs of existence and characterizations of the set of equilibria,
2. Establishing necessary and sufficient conditions for the stability of such equilibria,
3. Finding conditions for the uniqueness of general equilibrium,
4. Formulating conditions which allow for meaningful comparative static results.

In considering these four points it should be obvious that the first two are indispensable steps for any kind of equilibrium analysis in order to make it meaningful (nevertheless stability is generally only assumed instead of being proved), while the remaining two must be considered as a result of special circumstances which may or may not represent a good model of 'reality'.

It is a well known fact in micro- as well as in macro-dynamics that conceivable types of disequilibrium reactions have been investigated much less than the concepts of equilibrium that are in use. In the case where one observes instability for a chosen adjustment mechanism there is therefore much room left for changing the dynamic structure in order to increase its stability properties. In our view it is sensible and often also necessary to start from a conventional Walrasian (1954) tâtonnement – as it is formalized in Arrow and Hahn (1971, 11.2) – as the (appropriately purified) basic type of price adjustment procedure and add further disequilibrium price reactions to it. In addition, since Walrasian demand and supply functions neglect the existence of quantity constraints in disequilibrium it is even methodologically adequate to make use of a tâtonnement in order to formulate and test basic dynamic features of conceivable market price reactions (the 'pure price effects' according to Arrow and Hahn (1971, p. 265)) as simply as possible.

In addition to the necessity of looking for further and more reliable economic forces in case of a weak performance of the disequilibrium adjustment processes under consideration, it is, of course, also possible to assume or prove further

[2] With finitely many commodities and a continuum of individuals.

[3] See Kirman (1988, 1989) for an evaluation of these results in view of a related theorem of the Sonnenschein-Debreu variety.

[4] cf. Hildenbrand (1989) for a detailed description of this 'Law'.

restrictions for the excess demand function that are used[5] in order to obtain better stability results. As already stated it is, however, not an absolute necessity here that theoretically or empirically motivated refinements of the assumed givens of general equilibrium analysis *must* also lead to global uniqueness[6] or clear-cut reactions of equilibria to exogenous shocks or endogenous evolution.

We shall concentrate here on the issue of stability and propose an extension of the conventional Walrasian tâtonnement process which significantly increases the stability properties of this process. This extension can be briefly described as follows: Consider two identical states of excess demand which are changing in opposite directions, one towards a higher level of excess demand, the other towards a lower level. Assuming that this fact is observed by the 'auctioneer' it seems plausible to integrate this into his reaction and to assume that his revision of prices p will be different in these two situations. A natural proposal here is that price changes should be more pronounced in the case of increasing market disequilibria than in the opposite case (for a given state of excess demand). This idea gives rise to a simple extension of the auctioneer's rules used for tâtonnement as they are formulated in Arrow and Hahn (1971) with interesting local as well as 'global' stability results.[7]

This extension of the standard Walrasian tâtonnement process will in addition show that the conditions which guarantee the stability of such price adjustment processes are of a quite different nature than those needed for establishing the existence or uniqueness of equilibrium (more disequilibrium information for the 'auctioneer' instead of more structure for the given economy in the case of uniqueness for example). And – since this revised tâtonnement indirectly incorporates information on the Jacobian of the excess demand function by its use of the time rate of change of excess demands – it will in addition provide a new route of escape from the instability problem of Walrasian Economics in line with the conditions that have been stressed by Saari and Simon (1978) and others as being absolutely necessary for a tâtonnement to work 'universally'. Our approach will thus improve the Walrasian tâtonnement procedure as an economic adjustment method in a way that provides an example for a so-called Generalized Newton Method as defined in Jordan (1983). Introducing a more sophisticated adjustment process in the above way may thus indeed help to provide the emperor with clothes.

The following Sect. 13.2 will give a brief presentation and motivation of the intended extension of the standard tâtonnement mechanism in the context of exchange economies. Section 13.3 will then show that this more sophisticated adjustment mechanism exhibits strong local as well as 'global' stability properties. These properties are illustrated in the final Sect. 13.4 by means of simple examples and numerical simulations.

---

[5] See Keenan (1990) for a recent attempt which starts immediately on the level of excess demand functions and Hildenbrand (1983, 1989) for a micro-justifications of such approaches.

[6] See Kaldor (1940) for an interesting dynamic exploitation of the non-uniqueness of equilibria.

[7] See Flaschel (1991) with respect to possible modifications for production economies and an alternative analysis of the local stability properties of such an extension.

13.2 An Extension of the Walrasian Tâtonnement Process

We shall consider in this chapter a standard pure exchange economy and assume as given in this context the following kind of a tâtonnement procedure:

$$\dot{p}_i = X^i(p), \quad i = 1, ..., n, \quad p \in \Re^n_{++} \tag{13.1}$$

where X denotes the excess demand function of the given exchange economy,[8] and where – in line with the approach chosen in Jordan (1983) described below – the numéraire p_{n+1} has already been excluded from consideration: $p = (p_1, ..., p_n)'$[9] We briefly remark that – up to certain boundary conditions – the excess demand function X may be an arbitrary vector field according to theorems of Sonnenschein, Debreu and others and may therefore be assumed as non-linear as desired and to exhibit a given finite set of points of $\Re^n_{++}$ as its equilibrium set and thus as points of rest of the dynamics (13.1).

Due to this last fact there is, of course, no hope, that the above price adjustment mechanism will allow for general propositions on its stability. As Scarf (1960) has shown this process can fail to be even locally stable at a unique equilibrium (cf. Sect. 13.4). Such observations have lead to various reactions of how to deal with this 'instability problem'. One particularly interesting approach – which developed from an important paper by Saari and Simon (1978)[10] – is to look for modifications of the above dynamics (for example by using the Jacobian of X in some way in addition) such that it will become locally 'universally stable', i.e., asymptotically stable at p^* for all excess demand functions X which have p^* as a regular equilibrium. An important mathematical example for such a mechanism is given by the following Generalized Newton Method:

$$\dot{p} = \begin{cases} -X'(p)^{-1}X(p) & : \quad \det X'(p) \neq 0 \\ X(p) & : \quad \det X'(p) = 0 \end{cases}$$

Since, however, this Newton Method has no economic meaning, it is natural to ask whether there exist equally potent and economically interesting price mechanisms. This question is also posed at the beginning of Jordan's (1983) article on (universally) locally stable price mechanisms. The object of his paper then is to provide 'some strong, necessary conditions characterizing price mechanisms which achieve local stability'. To this end Jordan (1983) defines the concept of a price mechanism in the following way:

Let $\mathcal{E}$ denote the space of C^2 excess demand functions of the above kind (with the topology of C^1 uniform convergence on compact subsets of $\Re^n_{++}$ and let $\mathcal{E}_p$

[8] See the next section for a brief explanation of the employed notation.

[9] See Arrow and Hahn (1971) and Hahn (1982) for details and various special stability results that exist for such *'numéraire processes'* and note here that variable adjustment speeds – as they are assumed by these authors – make the use of the conventional numéraire $p \cdot p = 1$ much less plausible.

[10] See Jordan (1983) in particular.

be defined by $\{X \in \mathcal{E} : X(p) = 0 \quad \text{and} \quad \det X'(p) \neq 0\}$. A *price mechanism* is a function $M : \Re^n_{++} \times \mathcal{E} \to \Re^n$ such that for each (p,X) we have:

(a) $M(p, X) = 0$ if and only if $X(p) = 0$.
(b) There is an open set $U \subset \Re^n_{++} \times \mathcal{E}$ with $p \times \mathcal{E}_p \subset U$ such that (1) M is continuous on U, and (2) $M(\cdot, X)$ is C^1 on the open set $\{q : (q, X) \in U\}$.

Such a price mechanism is then viewed to give rise to the following ordinary differential equation

$$\dot{p} = M(p, X), \quad p(0) = p_0$$

for any given excess demand function $X \in \mathcal{E}$ and the initial condition $p_0 \in \Re^n_{++}$.

Finally, a price mechanism will be called a *Generalized Newton Method* if there is a C^1 function $A : \Re^n \times \mathcal{L} \to \Re^n$, where $\mathcal{L}$ is the space of non-singular $n \times n$ matrices, such that

$$M(p, X) = A(X(p), X'(p)) \quad \text{for all} \quad (p, X) \in \Re^n_{++} \times \mathcal{E}$$

with $\det X'(p) \neq 0$.

Such Generalized Newton Methods will give rise to *(hyperbolically) locally stable price mechanisms* if $A_1(0, \ell)\ell$ is non-singular for every $\ell \in \mathcal{L}$ and if all eigenvalues of $A_1(0, \ell)$ have negative real parts.

The above Generalized Newton Method provides an example for such a price mechanism, but – as noted – not one of economic interest. This method, however, illustrates why the regularity conditions in (b) are not imposed on all of $\Re^n_{++} \times \mathcal{E}$. This fact will also apply to the dynamics introduced and studied in this chapter, which indeed provides an – economically motivated – example for the various concepts of price mechanisms formulated above.

Our version of a price mechanism is obtained from the simple economic observation that the differences in the time rates of change of excess demands which accompany the above type of tâtonnement dynamics should in fact play a role in its formulation and should thus be incorporated into the rules followed by the auctioneer as they are described in Arrow and Hahn (1971, 11.2/3). In this way we provide just another example[11] of what these two authors call a *tâtonnement* in order to analyze the working of 'pure price effects' in the same way as they propose it for their basic kind of such a tâtonnement[12] – in the hope that 'to understand these price effects in a full story, it is reasonable to suppose that a study of them in isolation is a hopeful procedure' (cf. their page 265). Our aim here is to provide thereby an economically motivated example for a locally 'universally stable' price mechanism as in Jordan (1983) which has even certain global stability properties.

[11] We shall neglect here their adjustment rule for boundary values of $\Re^n_{++}$ for simplicity.

[12] Note here, that they, too, employ a *numéraire* tâtonnement approach – due to the fact that they use adjustment functions G_i instead of fixed parameters for the various components of the excess demand function.

It is a common belief that process (13.1) is something like the skeleton of the dynamics of a market economy – even when formulated in a tâtonnement environment. Such a view, however, neglects that there is further basic 'information' that has to be taken into account in disequilibrium (here by the 'auctioneer') along the price adjustment path generated by the dynamics (13.1). In a continuous framework such basic information is contained in the time-derivative of the excess demand function X along the price trajectories: $\dot{X}(p)$, which in our view should be integrated into the dynamics (13.1). This proposal can be motivated as follows:

Compare two hypothetical situations where $X^i(p)$, the excess demand on the ith market, is a given magnitude, but where its time rate of change $\dot{X}^i(p)$ differs in sign and in magnitude. If tâtonnement dynamics is not just purely mathematical in nature, but if it is believed that it can, however remotely, mimic what goes on in actual markets [so that this method allows to test disequilibrium adjustment processes of the real world and their complicated set of feedback mechanisms as in a vacuum], then it should appear as a natural suggestion that the price reaction $\dot{p}_i$ is to be formulated differently for these two hypothetical situations. Already in the setup of Walrasian tâtonnement we should thus have that the same level of excess demand will lead to more pronounced reactions when it is still increasing in comparison to the situation where it is instead decreasing. This simple idea (and critique) of the standard formulation of tâtonnement processes (13.1) – where only the levels of excess demand are taken into account as forces which act on prices p – leads to their following reformulation:

$$\dot{p}_i = X^i(p) + g_i \dot{X}^i(p), \quad g_i > 0, \quad i = 1, ..., n, \qquad \text{or} \tag{13.2}$$

$$\dot{p} = X(p) \quad + \quad < g > \dot{X}(p) \tag{13.3}$$

Here, $< g >$ denotes the diagonal matrix obtained from the set of adjustment coefficients $g_i > 0$. Note, that we have suppressed all adjustment parameters d_i in front of the level magnitudes by an appropriate choice of units.[13] The parameters g_i are consequently to be interpreted as being determined relative to the given d_i of a standard tâtonnement procedure and should – as the d_i – be made variable in the end (cf. Sect. 13.4, Fig. 13.4). Note also, that our motivation for changing the standard tâtonnement procedure has been guided by economic reasons. We did not modify this process from a mathematical point of view with the aim of obtaining a more reliable calculation method for equilibrium prices as, for example, in a recent paper by Kamiya (1989) where a weighted average of the conventional tâtonnement and Smale's global Newton method is used for this purpose. It came therefore as a surprise when eigenvalue calculations showed that this process is closely related to the abstract discussion of effective price mechanisms and (hyperbolically locally stable) Generalized Newton Methods as in Saari and Simon (1978) and Jordan (1983) and that it shared their power to serve as universal adjustment mechanisms.[14]

[13] or by a suitable redefinition of the function X.

[14] See Flaschel (1991) for details in this matter.

We shall, however, not make use of such eigenvalue calculations in our following analysis of this price dynamics. Instead, we are here interested in formulating certain 'global' properties of this dynamics (with multiple equilibria) by means of a suitably chosen Liapunov function.

The dynamics (13.3) can be reformulated locally as follows:

$$\dot{p} = X(p) + <g> X'(p)\dot{p}, \qquad \text{i.e.,} \tag{13.4}$$

$$\dot{p} = (I - <g> X'(p))^{-1} X(p), \tag{13.5}$$

under the assumption that the matrix $I- <g> X'(p)$ is regular at the various equilibrium points $p = p^*$.

We shall show in the following that this system of differential equations exhibits strong stability properties despite the described arbitrary nature of excess demand functions. This demonstrates in particular that the negative definiteness of the Jacobian matrix $X'(p^*)$ (or any other restriction of this kind) is of no importance as far as questions of stability are concerned – if the above basic extension of the conventional tâtonnement dynamics is accepted as an indispensable component of pure price adjustment processes, i.e., if one agrees that disequilibrium adjustment processes should be based on more information for price revisions than just the levels of prevailing disequilibria – even in the remote perspective of a tâtonnement analysis.

13.3 Global Stability by Derivative Control

Let $\| \cdot \|$ denote the Euclidean norm on $\Re^n$ (obtained from the Euclidean product $\ll .,. \gg$), and let $\Re^n_{++} \subset \Re^n$ be the positive orthant of $\Re^n$. We denote by $\dot{p}, X'$ (time) derivatives and by $<g>$ the diagonal matrix corresponding to a given vector g. Let $X : \Re^n_{++} \to \Re^n$ be the excess demand function of a (n+1)-goods exchange economy. We assume for the following that X is C^2 and that it exhibits a finite number of strictly positive equilibria p^*, each with a regular Jacobian $X'(p^*)$.

Continuing the approach of the last section we have now to investigate the stability properties of the following autonomous system of differential equations

$$\dot{p} = (I - <g> X'(p))^{-1} X(p), \quad g = (g_1, ..., g_n) \in \Re^n_{++}. \tag{13.6}$$

We start with the local stability analysis of an arbitrary equilibrium p^* of the given excess function X: $X(p^*) = 0$. Assuming $\det(I- <g> X'(p^*)) \neq 0$,[15]

[15] Note that this regularity condition must hold true for all sufficiently large parameter values g_i. This can be shown by means of continuity arguments applied to the expression $\det(<g>)\det(<g>^{-1} I - X'(p^*))$ – due to the assumed regularity of the matrix X' at each of the equilibrium points of X.

where I denotes the identity matrix, implies[16] that the dynamical system (13.6) is well defined and C^1 in a neighborhood of p^*. In order to investigate the local stability of (13.6) at each p^* we will make use of the following auxiliary function L:

$$L(p) = \|p - p^* - <g> X(p)\|^2 \tag{13.7}$$

which is well-defined and C^2 on $\Re^n_{++}$. This function provides a Liapunov function at p^* for the above dynamical system and is a direct generalization of a Liapunov function used in the special case $<g>= 0$:

Proposition 13.1. *The given equilibrium vector p^* is an asymptotically Liapunov-stable point of rest of the dynamical system (13.6) if all parameters g_i are chosen sufficiently large (such that* $\det(I- <g> X'(p^*)) \neq 0$ *is again assured in particular).*

Proof. This assertion is proved as follows: By the definition of L we have $L(p^*) = 0$. Consider now the following function $Z(p) = p - p^* - <g> X(p)$. Since the Jacobian of this function Z is by assumption regular at p^* we know that this mapping is a homeomorphism for an open neighborhood U of p^* such that the inverse mapping of Z is C^2 on the open set Z(U). And since $Z(p^*) = 0$ we know that Z(p) must be different from zero in a neighborhood of p^*, i.e., $L(p) > 0$ must hold true in $U - p^*$ for such a neighborhood U of the equilibrium p^*.

According to Liapunov's stability theorem[17] there remains to be shown that the condition $\dot{L} < 0$ is true for a neighborhood $V - p^*$ of the given equilibrium.

Since p^* is a regular point of X there exists a positive constant $c \in \Re$ such that

$$\|X^{-1}(q) - X^{-1}(0)\| \le c \cdot \|q - 0\| \qquad \text{or} \qquad \|p - p^*\| \le c \cdot \|X(p)\|$$

is true for suitably chosen neighborhoods $U' \subset U$, V' of $0 = X(p^*)$ – due to the mean value theorem applied to the mapping X^{-1} on the open set V'.[18] Let $g_{min} := \min\{g_i / i = 1, ..., n\}$. By assumption we have $g_{min} > 0$. Differentiating L along the trajectories of (13.6) then gives (cf. Dieudonné (1960, p. 144)):

$$\begin{aligned}
\dot{L} &= 2 \ll (I- <g> X'(p))\, \dot{p} \quad , \quad p - p^* - <g> X(p) \gg \\
&= 2 \ll X(p), p - p^* - <g> X(p) \gg \\
&= 2[\ll X(p), p - p^* \gg - \ll X(p), <g> X(p) \gg] \\
&\le 2[\|X(p)\| \|p - p^*\| - g_{min}\|X(p)\| \|X(p)\|] \\
&\le 2(c - g_{min})\|X(p)\|^2 < 0
\end{aligned}$$

[16] cf. for example Dieudonné (1960, pp. 268/9).

[17] cf. Hirsch and Smale (1974, p. 193).

[18] cf. Dieudonné (1960, p. 155).

for all $p \in U' - p^*$ if $g_{min} > c$. We have thus shown that L *is a strict Liapunov function* at p^*, i.e., p^* is a sink for suitably chosen parameters g_i. □

Corollary 13.2. *All the equilibria p^* of the given economy can be made simultaneously asymptotically Liapunov-stable points of rest of the dynamical system(s) (13.6) if the parameters g_i are chosen sufficiently large.*

Note here, that this means the manipulation of a different Liapunov function for each of the given equilibria p^* just as in the proof of Proposition 13.1. The corollary reformulates and generalizes a theorem in Flaschel (1991) where it has been shown by means of eigenvalue calculations for the special case $g := g_1 = ... = g_n > 0$ that the real parts of all eigenvalues of all the equilibria of the given economy can be made negative simultaneously by the choice of a sufficiently large value for the parameter g. It follows from the index theorem, cf. e.g., Mas-Colell (1985), that this cannot be done by a smooth modification of the original dynamics (13.1). This is easily exemplified in the case of our process (13.6), see Fig. 13.1 in the next section.

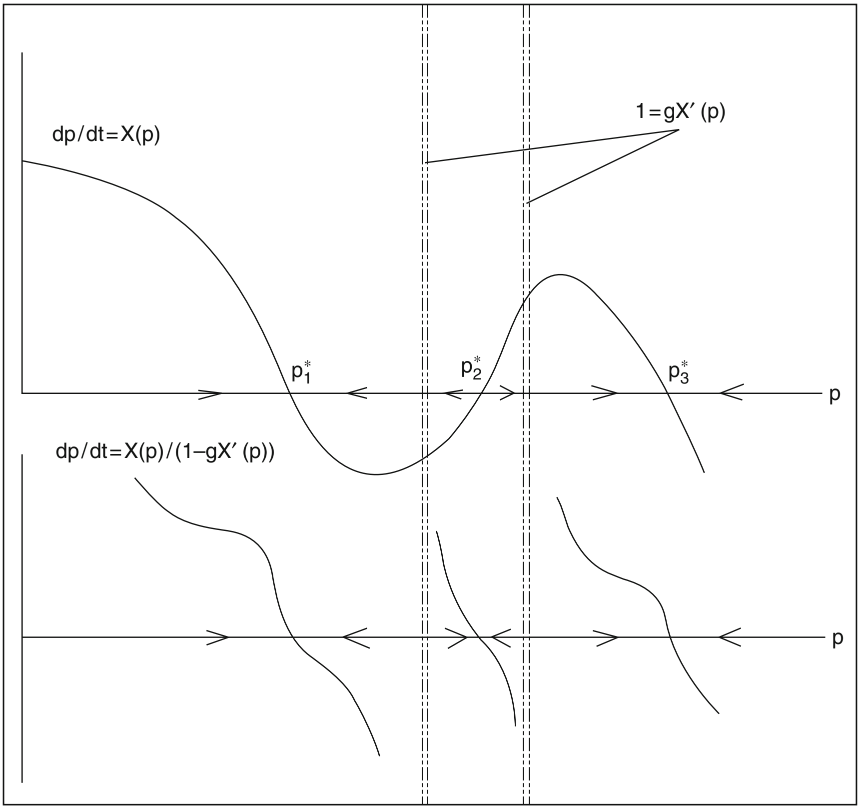

Fig. 13.1 An illustration of the two-goods case

We have considered in Sect. 13.2 a mathematical example of a price mechanism, namely the following Generalized Newton Method (GNM):

$$\dot{p} = \begin{cases} -X'(p)^{-1}X(p) & : \quad \det X'(p) \neq 0 \\ X(p) & : \quad \det X'(p) = 0 \end{cases}$$

Our proposal for an economic example of such a price mechanism obviously is fairly similar to this basic form of a GNM:

$$\dot{p} = \begin{cases} (I- < g > X'(p))^{-1}X(p) & : \quad \det(I- < g > X'(p)) \neq 0 \\ X(p) & : \quad \det(I- < g > X'(p)) = 0 \end{cases}$$

This is indeed a price mechanism and a GNM as they were defined in Sect. 13.2 and it is furthermore (hyperbolically) locally stable and thus also locally stable as in Jordan (1983).[19] In contrast to further mathematical alternatives to the above basic example of a GNM (such as the orthogonal Newton method, cf. Jordan (1983, p. 247)), the above provides for the first time an economically motivated example for such a universally stable price mechanism. This shows that the search for interesting price mechanism is not without hope – in particular since the various concepts of ‘universal stability’ that exist may still be too demanding for a proper economic characterization of such processes.[20]

It is not our aim to pursue these local approaches to universal stability any further in this chapter. We are here instead interested in somewhat global results for our example of a price mechanism – now again for a given excess demand function X – by exploiting further the properties of the above Liapunov function at the various isolated equilibria of the function X. The strongest result into this direction is given by the following:

Proposition 13.3. *Let us denote by P – in place of p – the situation where all n+1 goods are considered ($P_{n+1} = 1$). Assume that all goods are gross substitutes at all price vectors P, i.e., according to Arrow and Hahn (1971, p. 230) we in particular have* $- \ll X(P), P^* \gg \; < \; 0$ *for all P which are not equilibria. Assume furthermore that the set* $\Re^n_{++}$ *is positively invariant with regard to the dynamics (13.6) – whenever it is defined.*[21] *The given economy then has exactly one equilibrium which is furthermore globally stable with regard to the dynamics (13.6) for any choice of positive parameters* g_i.

[19] for suitable choices of the parameters g_i for each excess demand function X, see Flaschel (1991) for details.

[20] An obvious example for a too demanding definition is Jordan's (1983, p. 253) concept of a *market mechanism* which keeps markets in equilibrium once they have reached it – independent of what happens in the other markets. If this definition were economically sensible it would exclude our proposal for a price mechanism – and indeed any derivative feedback mechanism – from the set of economically meaningful adjustment processes.

[21] We shall see in the proof of this proposition that (13.6) is well-defined on $\Re^n_{++}$. The invariance of $\Re^n_{++}$ can then be obtained from the condition that $\|X(p^n)\| \to \infty$ if the sequence p^n approaches the boundary of $\Re^n_{++}$.

Proof. The uniqueness of equilibrium is a well known fact in the case of gross substitutes, see Arrow and Hahn (1971, p. 222). Note next, that all matrices $I- < g > X'(p)$ must be regular in this particular situation – for any choice of the parameters g_i. This can be seen as follows: Let us denote by S the excess supply function corresponding to the excess demand function X: $S(p) = -X(p)$. It is a known fact that $S'(p)$ will possess a dominant diagonal in the gross substitute case, cf. e.g., Hahn (1982, pp. 758/9). A straightforward calculation then shows that the matrix $I+ < g > S(p)$ will possess a dominant diagonal as well. According to Kemp and Kimura (1978, p. 7) this matrix must therefore be nonsingular.

From our proof of Proposition 13.1 we furthermore get $\dot{L} < 0$ for all admissible g and all $p \neq p^*$, since we have for such p

$$\ll X(p), p - p^* \gg = \ll X(P), P - P^* \gg = - \ll X(P), P^* \gg \quad < 0$$

due to Walras' Law.[22]

And this inequality also implies that $L(p) > 0$ must be true for $p \neq p^*$. Otherwise we would get $p - p^* = < g > X(p)$, i.e., a contradiction with respect to the preceding paragraph.

By Theorem 2 in Hirsch and Smale (1974, p. 196) we thus know that all trajectories of (13.6) must converge to the unique equilibrium of this economy, since the inequality $\dot{L} < 0, p \neq p^*$ does not allow for a entire orbit in the set $\Re^n_{++} - p^*$ on which L is constant. □

The above two propositions and their proofs show that our dynamics strengthens the qualitative properties of the conventional Walrasian process (13.1) in a fairly direct way. It, of course, must confirm these properties for $< g > = 0$ and it otherwise adds to the well-known situation of gross substitutability a further stabilizing influence for any choice of the excess demand function X.

Proposition 13.4. *Let p^* be an equilibrium of the function X and let C be a compact domain in $\Re^n_{++}$ on which the dynamics (13.6) is well defined and which contains p^* as the only equilibrium of X. Let $0 \leq k$ denote an upper bound for $\ll X(p), p - p^* \gg$ on C,[23] let $c, U' \subset C$ be determined as in the proof of proposition 13.1, i.e., we have $\|p - p^*\| \leq c \cdot \|X(p)\|$ for all p in U' and let ϵ be defined by $\inf\{\|X(p)\|/p \in C - U'\}$.[24] Assume furthermore that $g_{min} = \min\{g_i/i = 1, ..., n\} > \max\{c, k/\epsilon^2\}$ holds true and that $b > 0$ has been chosen such that the set $L_b = L^{-1}([0, b])$ is contained in C, where L is the Liapunov-function (13.7).*

[22] Note that these expression are in addition equal to $\ll X(p) - X(p^*), p - p^* \gg = \ll X(P) - X(P^*), P - P^* \gg$ and are thus related to the notion of monotonicity.

[23] See Fig. 13.1 for an example where such an upper bound will exist for the whole positive orthant $\Re^n_{++}$ and not only for sets C of the above kind and note that we had $k = 0$ in the preceding proposition.

[24] $\epsilon > 0$, since C is a compact set which contains no other equilibrium of X.

With regard to the basin of attraction $B(p^*) \subset \Re^n_{++}$ *of the equilibrium* p^*[25] *we then have:*

$$L_b \subset B(p^*)$$

Proof. From the proof of Proposition 13.1 we know that

$$\dot{L} = 2[\ll X(p), p - p^* \gg - \ll X(p), < g > X(p) \gg]$$

must be negative for all $p \in U' - p^*$ if $g_{min} > c$. And for the set $L_b - U'$ we get by assumption

$$\dot{L} = \leq 2[k - g_{min}\|X(p)\|\|X(p)\|] < k - (k/\epsilon^2)\epsilon^2 = 0.$$

There follows that $\dot{L} < 0$ must hold true in $L_b - p^*$ for the above choice of g_{min}. The set L_b is thus positively invariant and must be contained in $B(p^*)$ – again due to Theorem 2 in Hirsch and Smale (1974, p. 198). □

13.4 Examples

The simplest way to illustrate the results of the foregoing section is given by a two-goods exchange economy, i.e., by assuming n = 1 for the model of Sect. 13.2. A typical excess demand function is shown in the upper part of Fig. 13.1 which also shows two asymptotically stable price equilibria of the conventional tâtonnement process (and one unstable equilibrium).

The lower part of this figure depicts our revision of this tâtonnement procedure on the assumption that there are exactly two points (here one to the left and one two the right of p_2^*) where $gX'(p) = 1$ holds true. The new dynamics is therefore no longer well-defined and smooth on the set of all positive prices p, but gives rise here to three vector fields, one for each equilibrium (which must be global sinks in this simple case with respect to the vector field they correspond to). Note here that the two border lines which separate the domains of definition of these vector fields need not enclose the formerly unstable equilibrium point p_2^* and that they need not exist at all (if $g < 1/X'(p)$ for all prices p where $X'(p) \geq 0$ holds true). The area enclosed by the two lines may therefore exhibit no equilibrium at all (and may thus give rise to falling or rising prices throughout). In such a case the equilibrium in the middle remains an unstable one. It is obvious, however, that g can always be chosen large enough ($> 1/X'(p_2^*)$) so that p_2^* will become a sink for the vector field to which it corresponds.

[25] cf. Hirsch and Smale (1974, p. 190) for a description of this intuitive concept.

The choice of the parameter g may consequently be used for various purposes. We know that it can always be chosen sufficiently large such that all equilibria become asymptotically stable simultaneously. This implies the necessity of cutting the original vector field (13.1) by its extension (13.2), (13.5) in at least as many pieces as there are equilibria. The range of definition of each vector field, of course, depends on the exact choice of g which gives this choice further importance, since the basins of attraction of the various equilibria will depend on it. And finally, the parameter g may also be used[26] for increasing the speed of adjustment toward the various stable equilibria of (13.1) solely without any change in the range of definition of the original vector field. There is therefore no need to accept only *large* parameter values g_i. Instead, the whole range of these g_i may be worth while for further consideration.

Such possibilities may also be of help in overcoming some objections which – quite plausibly – can be raised against the above rigid type of a derivative control mechanism. Though this mechanism removes an implausible feature from the ordinary tâtonnement procedure, [27] namely that the kind of change of disequilibrium does not count in the adjustment of market prices, it has its own and new problems. It may thus happen, e.g., that prices will fall in our revised adjustment procedure though there is excess demand in their markets. Also, points near the separating lines in the above Fig. 13.1 may give rise to extraordinary strong price reactions. These simple examples suggest that our reformulation of Walrasian tâtonnement is not yet really satisfactory from a global point of view. This is, however, not astonishing since the role of adjustment speeds is not considered very thoroughly in this first attempt towards an analysis of derivative forces.[28]

A famous example of global instability of a unique equilibrium of the Walrasian tâtonnement process has been provided by Scarf (1960). For an economy with three consumers and three commodities he in particular derived the following form for the adjustment process (13.1) – by choosing utility functions and endowments in a certain circular way (and by putting $p_3 \equiv 1$):

$$\dot{p}_1 = \frac{-p_2}{p_1 + p_2} + \frac{1}{1 + p_1}$$
$$\dot{p}_2 = \frac{p_1}{p_1 + p_2} + \frac{-1}{1 + p_2}$$

This dynamical system gives rise to the closed orbit structure shown in Fig. 13.2 (which is characterized by $p_1 p_2 e^{-0.5(p_1^2+p_2^2)} = const.$):

[26] not necessarily by increasing it further!

[27] When it is believed that it can, however remotely, mimic what goes on in actual markets, see Arrow and Hahn (1971, p. 265)!

[28] See, however, Fig. 13.4 for a simple, more elaborate choice of adjustment speeds.

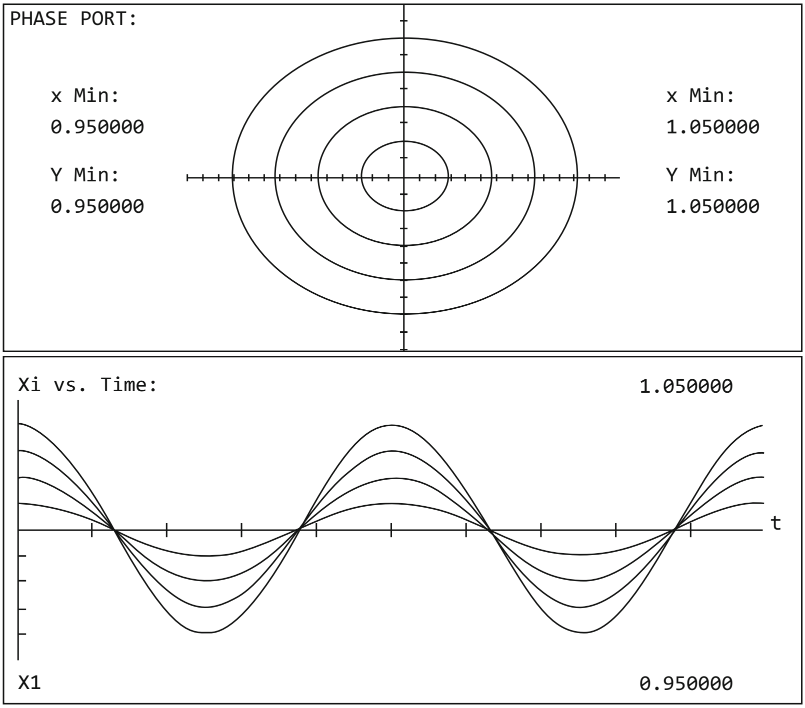

Fig. 13.2 Scarf's Example (x $= p_1, y = p_2$)

This figure exemplifies Scarf's claim of global instability in the particular case where commodity 3 has been taken for numéraire (which – following Jordan (1983) – is the choice of this article, but not typical for Scarf's paper).

Dierker (1974, pp. 54/5) remarks: 'One might object that the phenomenon of Scarf's (or Gale's) example perhaps would not happen for other or more realistic adjustment processes. ... If we find out, however, that each differentiable vector field on S can be approximated by the excess demand of some economy, then we can no longer put the blame on the specification of the price adjustment rule and we have to expect the worst in dimensions high enough to provide space for various 'pathological' features.' In contrast to this statement we have been able to show in this chapter – and from another perspective also in Flaschel (1991) – that this pessimistic conclusion need not be accepted. There are adjustment rules which work well 'universally' – without the need of putting more structure into the economic model at this level of their formalization. This is now exemplified by means of Scarf's example.

Calculating the eigenvalues of this dynamics at its unique equilibrium (1,1)' gives ± 0.25 which confirms the situation of a center dynamics as it is depicted in Fig. 13.2. On the basis of propositions provided in Flaschel (1991) we should expect that any $g = g_1 = g_2 > 0$ will turn the above equilibrium into a sink. This expectation is indeed confirmed by simulations of the dynamics (13.1) for this example, see Fig. 13.3. Furthermore, increasing the parameter g over a certain range

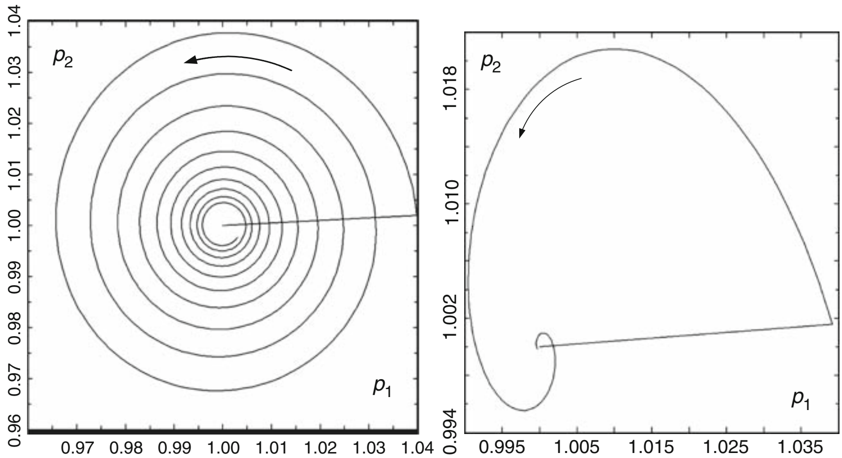

Fig. 13.3 Scarf's Example with Derivative Control ($g = 0.2$, $g = 2$, respectively)

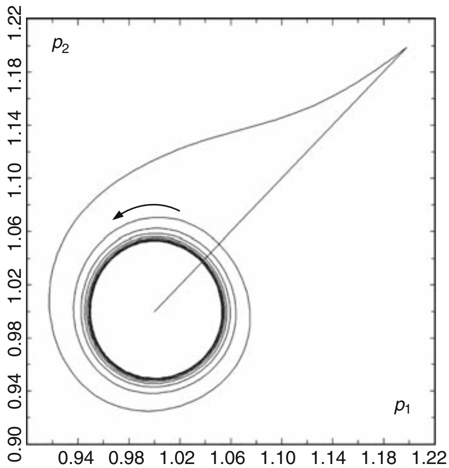

Fig. 13.4 A Variable Derivative Feedback Mechanism

leads systematically to increased stability away from cyclical convergence towards a monotonic kind of behavior. This fact was also observed for various other examples and may thus represent an important feature of this new price adjustment procedure.

Finally, a plausible modification of the dynamics (13.5) is implied by the observation that the derivative influence should increase in strength the larger the deviations from equilibrium become. A simple example of this idea is given by the following now endogenous determination of the value of $g = g_1 = g_2$:

$$g = (10p_1 - 10)^4 + (10p_2 - 10)^4.$$

For the above illustration of the dynamics (13.5) one would here expect that there is a nearly monotonic movement towards the equilibrium for larger discrepancies in demand and supply which changes into the original cyclical movement as it approaches the equilibrium. This expectation is indeed confirmed by the simulation shown in Fig. 13.4:

Such a result in our view has important methodological consequences. To exemplify this assume for simplicity that the derivative mechanism is absent for very small deviations from equilibrium (which is very plausible in the light of its motivation given in Sect. 13.2), and that it is then changed in a smooth way as in the preceding example. Local stability analysis can in such a case be totally misleading by telling, for example, that the given equilibrium is not at all supported by stability and should thus not be used for comparative static purposes. Nevertheless, depending on how quick derivative forces come into being as we move away from the equilibrium, this equilibrium may nevertheless be a 'center of gravity' of market prices and may consequently be used for comparative statics – though it is not asymptotically stable! It is obvious that this only an extreme example in a larger variety of cases questioning the value of local stability analysis in a nonlinear environment.

13.5 Conclusions

We have argued in this chapter that the standard tâtonnement price dynamics should be augmented by derivative effects in order to make it more convincing from an economic point of view. Such an extension may give rise to quite potent stability properties if the derivative feedbacks are sufficiently pronounced. The magnitudes of the parameter values g_i needed for the proof of the propositions in Sect. 13.3 or in numeric simulations (where smaller values already account for a good performance) are, however, still be much too high from an economic point of view in many situations. Weaker results may be obtainable from a more detailed analysis of the stabilizing potential of the parameters g_i for an appropriately chosen subset of the whole set of equilibria. Under certain circumstances – in other economic environments – it may also be sensible to assume $g_i = 0$ for certain components i. The question of how much derivative control is necessary or plausible must, however, be left as an open question here. Yet, setting all $g_i = 0$ as in the conventional tâtonnement analysis, should no longer be regarded as a sensible procedure: derivative feedbacks are part of any general disequilibrium analysis, which therefore differs substantially in its approach from general equilibrium analysis!

Revising what might be conceived of as the skeleton of market adjustment processes has led here to the conclusion that it is not absolutely necessary that the structure of general equilibrium models must be much more specific in order to allow equilibrium analysis to be backed up by stability assertions. Highly complex economic systems and their probably fairly nonlinear adjustment dynamics need not give rise to very complicated dynamics (chaos) as it is the fashion today. The emperor may thus be dressed in a dynamic outfit which suits his interest.

References

Arrow, K. J. & Hahn, F. (1971). *General competitive analysis*. Amsterdam: North Holland

Dierker, E. (1974). *Topological methods in walrasian economics.* Heidelberg: Springer

Dieudonné, J. (1960). *Foundations of modern analysis.* New York: Academic

Flaschel, P. (1991). Stability – independent of economic structure? A prototype analysis. *Structural Change and Economic Dynamics, 2*, 9–35

Hahn, F. (1982). Stability. In K. J. Arrow & M. D. Intrilligator (Eds.), *Handbook of mathematical economics, II* (pp. 745–793). Amsterdam: North Holland

Hahn, F. (1984). *Equilibrium and macroeconomics*. Oxford: Basil Blackwell

Hildenbrand, W. (1983). On the 'Law of Demand'. *Econometrica, 51*, 997–1019

Hildenbrand, W. (1989). Facts and ideas in microeconomic theory. *European Economic Review, 33*, 251–276

Hirsch, M. W. & Smale, S. (1974). *Differential equations, dynamical systems, and linear algebra*. New York: Academic

Ingrao, B. & Israel, G. (1990). *The invisible hand. Economic equilibrium in the history of science*. Cambridge, MA: MIT

Jordan, J. S. (1983). Locally stable price mechanisms. *Journal of Mathematical Economics, 11*, 235–259

Kaldor, N. (1940). A model of the trade cycle. *Economic Journal, 50*, 78–92

Kamiya, K. (1989). *A globally stable price adjustment process*. Osaka University

Keenan, D. (1990). Morishima systems and global stability. *International Economic Review, 31*, 11–16

Kemp, M. & Kimura, Y. (1978). *Introduction to mathematical economics.* Heidelberg: Springer

Kirman, A. (1988). On the instability of Walrasian Economics. In M. Galeotti et al. (Eds.), *Nonlinear dynamics in economics and social sciences*. Bologna: Pitagora Editrice

Kirman, A. (1989). The intrinsic limits of modern economic theory: The emperor has no clothes. *Economic Journal, 99*, 126–139

Mas-Colell, A. (1985). *The theory of general equilibrium. A differentiable approach*. Cambridge, UK: Cambridge University Press

Mas-Colell, A. (1986). Notes on price and quantity tâtonnement dynamics. In H. Sonnenschein (Ed.), *Models of economic dynamics*. Heidelberg: Springer

Saari, D. G. & Simon, C. P. (1978). Effective price mechanisms. *Econometrica, 46*, 1097–1125

Scarf, H. (1960). Some examples of global instability of the competitive equilibrium. *International Economic Review, 1*, 157–172

Walras, L. (1954). *Elements of pure economics*. London: George Allen and Unwin

Chapter 14
Stability: Independent of Economic Structure? A Prototype Analysis

14.1 Introduction

Since the early investigations of Arrow et al. in the late fifties and early sixties with their optimistic views on the stability properties of general equilibrium systems, there have been numerous contributions which have favored the opposite view – due to the counterexamples found shortly afterwards and due to the theorems of 'Debreu-Sonnenschein' type which have been formulated in the sequel. A typical statement in this regard is that of Frank Hahn (1970, p. 2): 'What has been achieved is a collection of sufficient conditions, one might almost say, anecdotes, and a demonstration by Scarf and later by Gale, that not much more could be hoped for.'[1] [2]

Similar statements can be found in a variety of publications on the stability issue, cf. for example Dierker (1974, p. 115) and in particular the recent survey article of Kirman (1989)[3]. This latter article also discusses one important possible route of escape from (non-uniqueness and) instability in the context of general equilibrium models, i.e., the approach by Hildenbrand and others who introduce further restrictions on the distribution of the characteristics of economic agents in order to avoid the above problems.

There are, however, also two further routes on which some progress has been achieved for stability analysis of exchange and production economies. These attempts concern:

- The characterization of the properties of adjustment mechanisms which are universally stable for appropriately specified (sub-)spaces of all conceivable exchange economies

[1] See, however, Keenan (1990) for a quite new attempt to obtain global stability for pure price mechanisms on the basis of a Morishima-type sign pattern of the Jacobian of the excess demand function (for all prices).

[2] I thank Dierker and Saari for helpful comments during the period when this chapter took shape. Of course, usual caveats apply.

[3] cf. also Kirman (1988).

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_14,

- The recognition of the fact that Walras' (1954) tâtonnement procedure must be formulated in a different way in the case of production economies than it is done for pure exchange economies.

Investigations of the *first type* date back to the discussion of the so-called Newton Method and its generalization as in Saari and Simon (1978). And advocates of the *second type* of approach have since long been Morishima and Goodwin. Their detailed work on such adjustment mechanisms has now also found entrance into the writings of general equilibrium theorists, cf. in particular Mas-Colell (1986).[4]

In the following an attempt is made to unify and extend the above two types of investigations of the stability question in the context of general equilibrium models

- By showing that the introduction of production does in fact improve the adjustment mechanism in its range of applicability
- By demonstrating how this mechanism can be further enriched – by making use of the recent and classically oriented discussion of this price/quantity reaction pattern[5] – such that its stability properties become fairly universal
- By showing that this originally classical adjustment method provides an interesting *economic* example for the *formal* discussion of universally effective mechanisms as in Saari and Simon (1978).

These three topics are treated in Sects. 14.2, 14.3, and 14.4, respectively. In Sect. 14.5 we shall finally consider weaker forms of our adjustment process in order to indicate that its ultimate design is presently by no means clear, but will demand further reflection and analysis. A concluding section then attempts to summarize the achievements of the chapter and enumerates a variety of topics for future research which should further enhance our understanding of this 'new' type of adjustment process for (capitalistic) market economies.

In order to avoid possible misinterpretations of what follows let us finally stress the following points:

- All assertions (and proofs) of this chapter – with the exception of Sect. 14.5 (Ignorable components) – immediately generalize to the case of n commodities, i.e., to the case treated in Mas-Colell (1986). The one-input one-output case of the present chapter is therefore in general only used to simplify its presentation.

[4] It has also been noticed recently, that there exists a close relationship between Walras' price-quantity tâtonnement process for production economies and the stability analysis for so-called classical long-term positions, cf. in particular Duménil and Lévy (1989), Goodwin (1989), e.g., Essay 1, and Flaschel and Semmler (1987) for such observations. This (formal) similarity in the type of price-quantity adjustment considered by Walras' and the Classics allows that results which have been obtained with respect to one approach may be applicable to the other approach if the differences in their concepts of 'equilibrium' are taken into account in an appropriate way. In the present article we shall study this cross-dual price-quantity adjustment process within the framework of Walrasian equilibrium analysis. [For the alternative approach the reader is referred to Flaschel and Semmler (1987, 1988)].

[5] cf. e.g., Flaschel and Semmler (1987).

- The adjustment process considered here is not confined to Walrasian models of general equilibrium. It can also be easily applied, for example, to the classical von Neumann model (if there are only pure joint products[6]) – due to its assumption of fixed proportions (see, however, Mas-Colell (1986) for problems in the case of joint production systems with smooth substitution possibilities).
- Our use of a tâtonnement procedure in order to test for universal stability should be understood as a preliminary step in the investigation of a new micro dynamic adjustment mechanism. If interesting and strong properties come about in a tâtonnement-like environment, then it might be hoped that these properties will survive when the characteristics of this new process are applied to more realistic types of disequilibrium analysis. This task, however, cannot be solved in the present chapter.
- The stability results of the present chapter are not confined to local considerations and the assumption of a uniform adjustment parameter γ. In a paper the author has shown meanwhile by means of a simple Liapunov function that global stability of the considered (non-linear) adjustment process can also be assured – even if this process is locally unstable.

14.2 Cross-Dual Dynamics in Walrasian Production Economies

In Mas-Colell (1986) the one-input/one-output case of general equilibrium models is used as an example to illustrate some global stability properties of a cross-dual type of Walrasian tâtonnement procedure whose formulation can be directly derived from Walras' writings on disequilibrium *in production economies*. Nevertheless, this extended tâtonnement procedure has been fairly neglected in the literature on general equilibrium systems so far. This process is formulated and analyzed in Mas-Colell's article in great generality and detail. In the following we shall reconsider it, however, for the above simplest general equilibrium model with production – in order to design and explore a quite natural and important extension of it in a way as instructive as possible (we have already noted that our analysis is not confined to this simple case which, however, has the advantage that it allows for diagrammatic representation). In contrast to Mas-Colell's findings on the stability of Walras' cross-dual dynamics our extended adjustment process will exhibit astonishingly *universal* stability properties (as we shall show in Sects. 14.3, 14.4).

Let us start by briefly summarizing the one-input/one-output case of general equilibrium analysis and its Walrasian price/quantity adjustment procedure (a more detailed, but also still partial version of it can be found in Beckmann and Ryder (1969), cf. also Mas-Colell (1986, pp. 64–67)).

We assume as given an economy where commodities are produced solely by means of labor subject to a smooth production function $f(l^d) = y^s$ which may

[6] i.e., fixed capital and the like is here excluded from consideration.

exhibit decreasing, constant or increasing returns to scale. Furthermore, we assume as given a smooth demand function $d(p, \pi)$ for the one produced commodity, where profits π are defined by $\pi = pf(l^d) - wl^d$ (p the commodity price and w the nominal wage rate, by choice of numéraire we assume $w = 1$)[7]. Households' initial endowments consist of labor only and labor supply can be derived from the above demand function by means of Walras' Law

$$pd(p,\pi) = l^s(p,\pi) + \pi = l^s(p,\pi) - l^d(p) + py^s(p).$$

Due to this law (and our choice of numéraire w = 1) we shall neglect the labor market in the following and will investigate the question of stability by reference to the market for goods.

The above demand function d can obviously be rewritten as a function of the two variables p and l^d and will be denoted by $d(p, l^d)$ for simplicity. According to Mas-Colell (1986, p. 65) we have for the partial derivative of this function $d_p \leq 0$ if and only if the weak axiom of revealed preferences holds true, a situation normally not assumed as given in the following. We denote by $l^d = l(y^s)$ the inverse of the production function (i.e., planned employment as a function of planned output) and will abbreviate from now on l^d and y^s by l and y for simplicity. The function $l = l(y)$ thus represents the (minimum) cost function in the present simple model.

Assume now as given an interior equilibrium of the above model, i.e., a situation of the following type

$$d(p^*, l(y^*)) = y^* > 0 \tag{14.1}$$

$$l'(y^*) = p^* > 0 \tag{14.2}$$

such that $0 \leq l''(y)$ holds true for the second derivative of l(y) in a neighborhood of y^* (i.e., locally decreasing or constant returns to scale). Due to the assumption of (dynamic) profit maximization we shall generally not consider those points of rest where increasing returns to scale prevail, though of course segments with such returns to scale may exist for the assumed production function.[8]

Out of equilibrium y^*, p^* the following type of *tâtonnement adjustment process* has been suggested by Mas-Colell (1986) as a formalization of Walras' views on market dynamics in a production economy:

$$\dot{p} = \alpha \cdot [d(p, l(y)) - y] = \alpha \cdot F^1(p, y), \quad \alpha = const > 0 \tag{14.3}$$

$$\dot{y} = \beta \cdot [p - l'(y)] \qquad = \beta \cdot F^2(p, y), \quad \beta = const > 0 \tag{14.4}$$

[7] Profits π are neglected in Mas-Colell's disequilibrium investigation of this basic situation.

[8] Note here, that our stability results can also be applied to such points of rest (which are no equilibria, since profits are at minimum here). These points of rest are of importance in the literature on public utilities and marginal cost pricing.

Verbally stated the dynamics (14.3), (14.4) claims that *prices are adjusted according to the excess demand on the market for goods* and that *supply is adjusted following the discrepancy between the current price for that good and the marginal wage costs of producing this current supply*. Such a process has since long been related with the writings of Walras by a few authors, most notably by Morishima (1959, 1977) and Goodwin (1953, 1989), cf. also the exegetical reconsideration of Walras' disequilibrium-production model in Walker (1987). However, most of the literature on tâtonnement processes has neglected this cross-dual type (Morishima) or cross-field type (Goodwin) of adjustment process for production economies despite its long tradition in classical ass well as neoclassical non-mathematical analysis. Neoclassical analysis has used instead the pure price dynamics of exchange economies also in the context of production economies (assuming thereby that all produced quantities will always be adjusted with infinite speed to the profit maximum throughout the adjustment procedure).

For a single market model, see Fig. 14.1, the above dynamics has been explored in a variety of ways in Beckmann and Ryder (1969) and also in Mas-Colell (1986), there on the assumption $\pi \equiv 0$, which however is not true in disequilibrium. We shall therefore briefly reconsider the above dynamics for $\pi = pf(l) - l \neq 0$ and compare it to the conventional type of Walrasian tâtonnement analysis where profits are always maximal and where only prices adjust in order to remove any goods market disequilibrium.

For the *Jacobian* J of the process (14.3), (14.4) we get *at the equilibrium point* p^*, y^*

$$J = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \begin{pmatrix} d_p & -1 \\ 1 & -l'' \end{pmatrix} \tag{14.5}$$

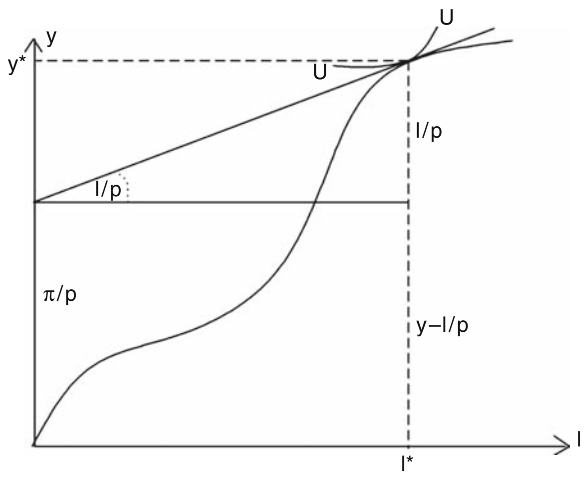

Fig. 14.1 The one-input/one-output case

since $d_y = d_\pi \pi'(l)l'(y) = 0$ at $l^*(y^*)$.[9] Note here, that we will consider only equilibria with $\det J \neq 0$ in the following[10]. In the case of the weak axiom we therefore have trace $J \leq 0$ (since $l''(y^*) \geq 0$) and $\det J > 0$ and thus get *local asymptotic stability* if either $d_p < 0$ or $l''(y) > 0$ holds true,[11] i.e., in case of a negatively sloped $d(\cdot, y)$-curve or for strictly increasing marginal costs.[12]

Because of (14.5) the (in-)stability analysis is exactly the same as in the partial model of Beckmann and Ryder (1969) and thus need not be repeated here in its further details. However, the above model can be considered as being entirely Walrasian (see again Walker (1987)) and need not be characterized as a combination of 'Walrasian' and 'Marshallian' features as these two authors have suggested.

It is illuminating to compare the cross-dual adjustment (14.3), (14.4) with the *conventional one-sided tâtonnement process* of general equilibrium theory which in the present situation is given by

$$\dot{p} = \alpha \cdot [d(p, l(y)) - y] = \alpha \cdot X(p), \qquad y = l'^{-1}(p) \tag{14.6}$$

Here, output y is determined as the profit-maximizing output ($p = l'(y)$) with regard to given prices p. Because of this, output y is now an upward sloping function of prices p in the case of decreasing returns to scale: $y'(p) = \frac{1}{l''(y(p))}$. For $l'' > 0$, the local stability condition for (14.6) is consequently given by

$$d_p - y' = d_p - (\frac{1}{l''}) = \frac{(d_p l'' - 1)}{l''} < 0 \tag{14.7}$$

since the second partial derivative of the function d with respect to p is zero [again due to the profit maximum condition.] Condition (14.7) therefore simply represents - in the case of decreasing returns – one of the two necessary and sufficient conditions for the asymptotic stability of the above cross-dual type of dynamics, namely $\det J > 0$, which in the case of (14.6) is already sufficient for local asymptotic stability. And: For any given demand function d this stability condition can always

[9] The second of the above two matrices will in general be of the form:

$$\begin{pmatrix} A & -B \\ B' & C \end{pmatrix},$$

cf. Mas-Colell (1986, p. 55) for further details. Note also, that we will make use of subscripts in order to denote partial derivatives in the following.

[10] which can be shown to be finite in number under certain simple additional conditions, cf. Dierker (1974, Chaps. 1,10) and Kirman (1989) for details.

[11] which in this case is independent of adjustment speeds (D-stability) because of the negative quasi-definiteness of the matrix J. In the case $d_p > 0$ we will have a unique bifurcation point with regard to the parameter β for any α – instead of D-stability – which separates stable from unstable spirals.

[12] In the case of $d_p > 0$ the second adjustment coefficient β must be chosen sufficiently large in order to obtain local asymptotic stability (if $l''(y) > 0$ holds true).

be ensured by assuming decreasing returns (at the equilibrium point) sufficiently close to the case of constant returns, i.e., the one-sided neoclassical tâtonnement procedure (14.6) can always be stabilized through appropriate assumptions on the production side of our economy. This is due to the – empirically implausible – fact that output reactions with respect to price changes approach infinity when the technology approaches the constant returns case (whereby the destabilizing influence of income effects on the side of demand can always be overcome).

Yet, though process (14.6) may at first appear to be the more convincing one, because the more stable type of dynamics it guarantees in comparison to process (14.3), (14.4) in the case of a production economy, the limit properties of (14.6) in our view nevertheless support the original approach of Walras to disequilibrium production models, i.e., the dynamics (14.3), (14.4), as the more promising route toward the determination of the skeleton of the dynamics of market economies. This latter dynamics – in contrast to (14.6) – is not ill-defined in the limit case of a constant returns economy, but has in this case as Jacobian the matrix

$$J = \begin{pmatrix} \alpha d_p & -\alpha \\ \beta & 0 \end{pmatrix} \tag{14.8}$$

i.e., it is stable whenever $d_p < 0$ holds true. It thus does not lead to an undefined situation for this often employed basic case of economic model building.

Figure 14.2 provides a simple illustration of the arbitrary dynamic behavior that can be expected even in the above simple one-input/one-output economy with regard

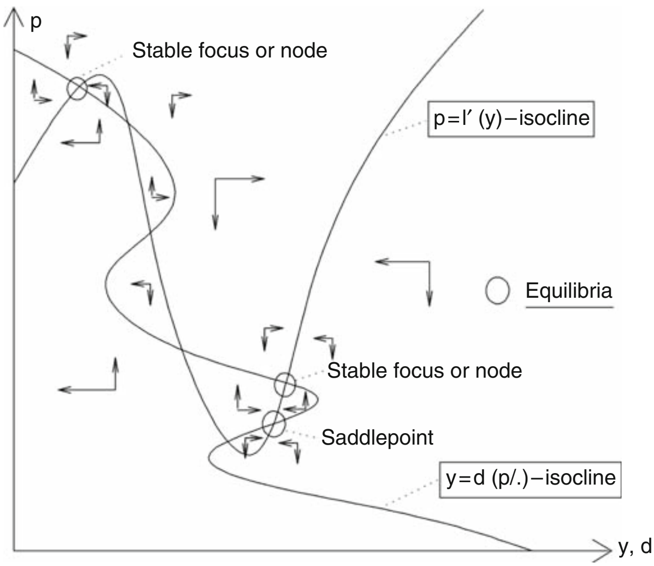

Fig. 14.2 Local cross-dual dynamics at Walrasian equilibria

to its various equilibrium situations (if only standard restrictions are imposed on the properties of such a production economy):[13]

This picture seems to suggest that not too much can be gained from using a cross-dual stability analysis à la Walras instead of the standard one-sided pure price dynamics – even though this type of dynamics appears as more convincing from an economic point of view than the conventional tâtonnement procedure (14.6) of neoclassical general equilibrium theory. However, important similarities in the formulation of Walrasian cross-dual dynamics with the cross-field processes as formulated by the Classics and Marx (their 'tendency of profit-rates to equalize', cf. Flaschel and Semmler (1987) for details) suggests, on the one hand, that this dynamics has a much wider *economic* background and plausibility than its neoclassical one-sided counterpiece. And, on the other hand, the long tradition that this latter process now has in economic theorizing opens up the possibility that further modifications of it can be found – by a careful reflection of the features of, e.g., a classical type of dynamics – which will increase its stability properties. A suggestion of this kind is the natural idea that rising profit-rate differentials will exercise a stronger influence on the conditions of supply than falling ones, cf. again Flaschel and Semmler (1987) in this regard. This idea will now be applied in an extended form to the Walrasian dynamics here under consideration.

14.3 Universal Stability

In this section we shall attempt to improve the stability properties of the above cross-dual dynamics by introducing further plausible reaction patterns into it. These additions concern effects that are caused by the direction of change of excess-profits and -demands and will give rise to a radical improvement in the stability behavior of the considered dynamics. This new dynamics will be investigated further in the following sections.

It has been shown by Kose (1956, Sect. 14.2) for the special situation of purely imaginary eigenvalues (the case of neutral stability), that it is possible to obtain asymptotic stability in such a case by a fairly natural modification of the given center dynamics. To make such a center case converge he extended this type of dynamics as follows ($\gamma = const > 0$ a given parameter):

$$\dot{p} = \alpha \cdot [(d(p, l(y)) - y) + \gamma \cdot \dot{\overbrace{(d(p, l(y)) - y)}}\,] \tag{14.9}$$

$$\dot{y} = \beta \cdot [(p - l'(y)) + \gamma \cdot \dot{\overbrace{(p - l'(y))}}\,] \tag{14.10}$$

---

[13] Note that we have neglected – as in Mas-Colell (1986, pp. 64/4) – the influence of π on d in our Fig. 14.2. Due to this fact we could make use of 'Debreu-Sonnenschein' theorems on the arbitrary nature of demand functions as they are formulated in the context of pure exchange economies, cf. Dierker (1974, pp. 56 ff.) for example and also Kirman (1989). Note further, that there can be no equilibrium in the presence of increasing returns due to the neoclassical assumption of a pure price-taker behavior.

Note, that the points of rest of this new dynamics are the same as in our original dynamics (14.3), (14.4).

In this revised dynamics, prices and quantities do not only react with respect to the level of excess demands and the excess of prices over costs (i.e., a proportional type of automatic control), but also with regard to their time rates of change (i.e., a derivative kind of automatic control), which in economic terms says that *rising discrepancies or disequilibria exercise a different influence than falling ones on the dynamics of prices and quantities*.

Partial derivative control, i.e., on the one hand, the influence of the rate of change of excess profitability on supply conditions, and on the other hand, the impact of the rate of change of excess supply on prices, has already been used in Flaschel and Semmler (1987) in the context of a linear Sraffa/von Neumann model there giving rise to global stability results with process extinction or product extinction. The following study differs from this earlier approach insofar as

- It also allows for decreasing returns to scale
- It neglects 'normal profits'
- It considers the joint effect of above two derivative forces
- It allows for stability even if the original cross-dual dynamics itself is unstable
- It stresses the importance of adjustment speeds for obtaining stability for one or all equilibria of a given economy or even for all conceivable economies.

The following analysis attempts to demonstrate to the reader the strong stabilizing influence supplied by economic forces which relate to the impact of differing rates of change of the two situations of excess here considered and which are thereby characterized as important components of real economic adjustment mechanisms.

Making use of the approach of Kose (1956) the above extension of the dynamics (14.3), (14.4) gives rise to the Proposition 14.1 (which is a special case of the Proposition 14.3 and which already supplies part of the proof for the latter):

Proposition 14.1. *Assume* $l''(y^*) = d_p(p^*, y^*) = 0$. *The dynamics (14.9), (14.10) then is* locally asymptotically stable *at any* p^*, y^* *and for any* $\gamma > 0$.

Proof. We will make use of the following notation:

$$D = \begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}, \quad Q = \begin{bmatrix} d_p(p^*, l(y^*)) & -1 \\ 1 & -l''(y^*) \end{bmatrix}, \quad z = \begin{bmatrix} p \\ y \end{bmatrix}.$$

Equations (14.9), (14.10) can then be rewritten as

$$\dot{z} = D[Q(z - z^*) + \gamma Q \dot{z}] = (I - \gamma DQ)^{-1} DQ(z - z^*) \tag{14.11}$$

as far as their linear part is concerned.

In order to demonstrate this we consider (14.9), (14.10) in the following abbreviated form (cf. (14.3),(14.4),$z = (p, y)'$):

$$\dot{z} = D[F(z) + \gamma \cdot \dot{\overbrace{(F(z))}}] = D[F(z) + \gamma \cdot F'(z) \dot{z}].$$

From this we obtain

$$\dot{z} = (I - \gamma \cdot DF'(z))^{-1} DF(z) \tag{14.12}$$

for those $\gamma > 0$ for which the employed inverse matrix is well-defined.[14] According to (Dieudonné, 1960, p. 146) we get for the derivative of the product operator $\circ$ of matrices $v_0 \circ u_0$ an expression of the type:

$$(s, t) \longmapsto v_0 \circ s + t \circ u_0.$$

Yet, in our case, we have $u_0 = DF(z^*) = 0$, so that the Jacobian J of (14.12) is simply given by

$$J = (I - \gamma DF'(z^*))^{-1} DF'(z^*)$$

This indeed shows that (14.11) gives the linear part of (14.9), (14.10). [Note, that the matrix J is well-defined in our specific situation for all $\gamma > 0$, because $\det(I - \gamma DF'(z^*))$ equals $1 + \alpha\beta\gamma^2$ for the particular type of neutral equilibrium here considered].

Consider next an arbitrary eigenvalue μ of the matrix $J = (I - \gamma DQ)^{-1} DQ$ and an associated eigenvector x, i.e., the following equation:

$$\begin{aligned}(I - \gamma DQ)^{-1} DQx = \mu x, \quad x \neq 0. \qquad & \textit{Then}: \\ DQx = (I - \gamma DQ)\mu x \quad & \text{or} \\ (1 + \gamma\mu) DQx = \mu x \ \text{ or } \ DQx = \frac{\mu}{1 + \gamma\mu} x &\end{aligned}$$

if $\mu \neq \frac{-1}{\gamma}$ (which is impossible, cf. the following calculations). The expression $\frac{\mu}{1+\gamma\mu}$ consequently is an eigenvalue of DQ. By assumption it is purely imaginary, i.e., we have $\frac{\mu}{1+\gamma\mu} = bi$. This gives

$$\mu(1 - \gamma bi) = bi \quad \text{or} \quad \mu = \frac{bi}{1 - \gamma bi} = \frac{bi(1 + \gamma bi)}{1 + \gamma^2 b^2} = \frac{-\gamma b^2 + bi}{1 + \gamma^2 b^2}.$$

Since the value $b = 0$ is impossible in the present situation, we get

$$Re\ \mu = \frac{-\gamma b^2}{1 + \gamma^2 b^2} < 0,$$

which proves the proposition. □

---

[14] The region where the expression (14.12) is not defined may be a complex domain for general excess functions F and in higher dimensions (cf. Fig. 14.3 for a particularly simple illustration of this remark).

Remark 14.2. We have formulated the above proof in such a way that its generalization to any number of commodities should be obvious. Furthermore, its extension to arbitrary equilibria is now a fairly easy matter:

Proposition 14.3. *Consider an interior equilibrium of system (14.9), (14.10). There exists* $\gamma_0 > 0$ *such that* (p^*, y^*) *will be* locally asymptotically stable for all $\gamma > \gamma_0$ *with regard to the adjustment process (14.9), (14.10).*

Proof. We apply the same arguments as in the preceding proof which now gives the relationships

$$\frac{\mu}{1+\gamma\mu} = \lambda \quad \text{or} \quad \mu = \frac{\lambda}{1-\gamma\lambda} \tag{14.13}$$

between the eigenvalues μ of the matrix $(I - \gamma DQ)^{-1} DQ$ and λ of the matrix DQ (of course, γ has to be chosen in such a way that $\gamma\mu \neq -1$ and $\gamma\lambda \neq 1$ will hold true). Note, that the mappings (14.13) are inverse to each other and that they define one-to-one linear transformations of the extended complex plane onto itself (see Ahlfors (1953) for details). Using the expression on the right hand side of (14.13) we get for $\lambda = a + ib$:

$$\mu = \frac{\lambda(1-\gamma\bar{\lambda})}{|1-\gamma\lambda|^2} = \frac{\lambda - \gamma\lambda\bar{\lambda}}{|1-\gamma\lambda|^2} = \frac{a - \gamma(a^2+b^2) + ib}{|1-\gamma\lambda|^2}$$

where $|\lambda| = \sqrt{\lambda\bar{\lambda}}$ denotes the absolute value of a complex number.

Since (p^*, y^*) has been assumed as regular [see Sect. 14.2], we have $a_j^2 + b_j^2 > 0$ for all eigenvalues λ_j of the matrix DQ (j = 1, 2 here). Hence, for each λ_j, we can choose $\gamma_j > 0$ such that the corresponding μ_j fulfills $Re\ \mu_j < 0$ for all $\gamma > \gamma_j : \gamma_j = \frac{a_j}{(a_j^2+b_j^2)}$. Defining γ_0 by

$$\gamma_0 = \max_j \gamma_j$$

then implies the assertion of the proposition, since the number of eigenvalues of DQ is finite (= 2 in the present case). Note here, that we have $det(I - \gamma DQ) \neq 0$ for

$$\gamma > \gamma_0 = \max_j \left[\frac{a_j}{(a_j^2 + b_j^2)} \right],$$

either because of $b_j \neq 0$ or because of $\gamma > \frac{1}{a_j}$ in the case of a real root ($b_j = 0$) □

The above proposition shows how market pressures – in combination with price/cost-comparisons – can be reformulated in such a way that the stability of all economic equilibria comes about. Yet, this proposition also shows that the information which 'markets' need in order to allow for generally stable adjustment processes exceeds the information that they have to provide for the derivation of the existence of equilibria.

Remark 14.4. (1) We have shown for $\lambda_j = a_j + b_j \cdot i$ that the equation

$$Re\ \mu_j = \frac{a_j - \gamma(a_j^2 + b_j^2)}{(1 - \gamma a_j)^2 + (\gamma b_j)^2}$$

must hold true. This expression however shows – in addition to the result that these real parts must all become negative for $\gamma > \gamma_0$ - that this property does not behave too nicely for larger and larger γ. In fact, $Re\ \mu_j$ shrinks to 0 (from below) for $\gamma \to +\infty$, which means that the proven asymptotic stability becomes weaker and weaker for a rising value of the parameter γ.[15]

(2) The γ_0 we have chosen in the proof of the last proposition still depends on the given equilibrium. It may, however, also be chosen large enough to be independent of the particular equilibrium of the given regular economy, since the number of these equilibria is finite. Yet, as noted under 1), this will further weaken the exponential degree of stability of the equilibria of the given economic model. Note also, that γ cannot be chosen independently of the functions d and l', i.e., independently of the particular economy considered.

(3) In (14.9), (14.10) we have added terms which represent a kind of *derivative control* for the system (14.3), (14.4). Similarly, one might be tempted to add an integral control (based on the integral of past deviations from equilibrium) or a derivative control of second order to the initial dynamics (14.3), (14.4). Both approaches, however, do not seem to improve the situation of the simple derivative control analyzed above.

The process which we have introduced with (14.9), (14.10) has been obtained by the inclusion of further *economic forces* into the Walrasian process (14.3), (14.4). We have seen that the added feedbacks greatly improve the stability of the original process (14.3), (14.4). Yet, there also exist features of this dynamics which at the present stage of its investigation appear as somewhat perplexing. Examples of this kind are given by: (a) Larger positive real parts of the dynamics (14.3), (14.4), i.e., greater instability in the original system, demand less of the new stabilizing force (a smaller γ) in order to be turned into stable roots, and, (b) not only the equilibria, but *also the other points of rest* (see Fig. 14.2 for examples) are stabilized by such an integration of derivative dynamic forces.[16]

Figure 14.3 provides an intuitive illustration of the change in the dynamics that is implied by our addition of a derivative control mechanism to the Walrasian cross-dual tâtonnement procedure. For simplicity, we consider in this figure the case of a one-sided price adjustment mechanism (14.6) solely, in which case the comparison between the original type of adjustment and its enrichment by means of derivative

[15] Such an observation quite naturally gives rise to the question of global stability (treated in Flaschel (1990) for stable as well as unstable types of equilibria), since it appears as plausible that adjustment speeds γ should not be chosen to large in order to make sense economically.

[16] The proposed dynamics is therefore also of interest for studies of marginal cost pricing in the presence of increasing returns to scale.

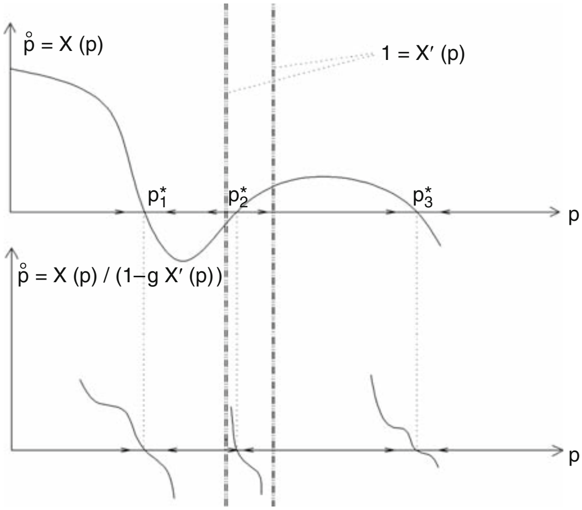

Fig. 14.3 Universal stability for pure price adjustment

feedbacks is particularly illuminating and easy to depict: [17] This figure shows that universal asymptotic stability is obtained through the partitioning of the domain of definition of the original dynamics into isolated basins of attraction for each point of rest.[18] And: there may also exist isolated regions which do not contain a point of rest. These facts are, however, presently of no great importance, because of the local viewpoint we have adopted in this chapter.

The type of universal stability introduced above thus is to some extent still of a questionable type. Despite its problematic features, it is nevertheless of great interest, since

- It provides a first truly *economic* example for a so-called Generalized Newton Method, cf. Jordan (1983).
- It shows how economic agents can economize the information that is claimed to be necessary for universally convergent adjustment processes (they only need to reflect certain rates of change to this end).
- It thus demonstrates the fact that mathematical conditions which have been proved to be absolutely necessary for universal convergence (i.e., the inclusion of the whole Jacobian of the excess-function into the formulation of such mechanisms) cannot be viewed as implying that agents must have this huge amount of knowledge at their disposal in order to be able to implement such quite potent market mechanisms, cf. Saari and Simon (1978) and the following Sect. 14.4.

[17] cf. Dierker (1974) and Kirman (1989) for the details of the conventional upper part of this diagram and its dynamics.

[18] Note here, that the Jacobian at points of rest in the lower part of Fig. 14.3 is given by $\frac{X'(p^*)}{1-\gamma X'(p^*)}$ which must be negative for each γ larger than $\frac{1}{X'(p^*)}$.

Derivatives are important ingredients in the formulation of workable economic adjustment processes, but – as we have seen – they may be involved only in a fairly indirect way as far as their economic representation is concerned.

14.4 Newton Methods: Old and New

In Saari and Simon (1978) the concepts of *locally effective price mechanisms* (LEPM) and of *effective price mechanisms* (EPM) are introduced and analyzed in the context of pure exchange economies. Due to the already mentioned results of Debreu-Sonnenschein and others on the arbitrary nature of (excess) demand functions in such economies, they see the necessity of designing more reliable and powerful price mechanisms than the conventional neoclassical price tâtonnement [see (14.6) for an example of this process] which in their view too often yields unsatisfactory results.

The starting point of their investigations are methods which had been invented to compute the fixed points of a map from any point near the boundary of its region of definition. One such method is discussed in Smale's (1976) paper on convergent price adjustment through '*Global Newton Methods*' (GlN) applied to the excess demand functions of general equilibrium systems. Making use of certain boundary conditions, Smale's approach does not only offer a new method of computing price equilibria, but it also attempts to bridge the gap between the early optimistic views of Arrow, Hurwicz et al. on the stability of general equilibrium systems and the subsequently developing pessimistic characterizations of this problem based on the examples by Scarf and Gale and the theorems of 'Debreu-Sonnenschein' type.

For regular points p of the excess demand function X Smale's GlM is given by the following system of differential equations

$$\dot{p} = -\lambda(p) X'(p)^{-1} X(p) \tag{14.14}$$

where $\lambda(p)$ is often assumed to be determined by det $X'(p)$ (see Keenan (1981, p. 160) for an extended definition of this process which also includes the singular points of X). The boundary restrictions of Smale's approach have been improved with regard to their economic content in Varian (1977), yet as shown in Keenan (1981) this process is not really of interest as an adjustment mechanism toward price equilibria, as it does not behave very well in the interior of relevant regions of the price-domain (in particular it is not an LEPM in the sense of Saari and Simon, see below). Though the GlM may have some advantage over the following so-called '*Generalized Newton Method*' (GeN)

$$\dot{p} = -X'(p)^{-1} X(p) \tag{14.15}$$

– since it (through sign reversal) can pass certain singular points without coming to a halt – it is only this GeN which is of potential interest for designing more potent price mechanisms, since only this latter procedure allows for local stability in general [see Saari and Simon (1978, p. 1103) for a proof].

In the main part of their paper Saari and Simon (1978) investigate, however, whether the informational content for so called (locally) effective price mechanisms can be substantially lower than that of the GeN (which is an LEPM, i.e., all equilibria are turned into sinks by this modification of the conventional excess demand price dynamics). The GeN-Method (14.15) requires knowledge of $X(p)$ and of all partial derivatives $X_i^j(p)$ of $X(p)$ and it furthermore has the unpleasant property that it reduces all excess demands monotonically and in a very strict way (leaving in particular all markets which are already in equilibrium in this position). This implies that it must be considered as a too powerful method from an economic perspective. It is therefore of great interest to ask whether there exist other types of locally effective price mechanisms, which are more plausible from an economic point of view and need less information on the Jacobian of excess demand functions than the Generalized Newton Method.

As shown above, the GlN and the GeN have stimulated renewed interest in the search for alternative effective price mechanisms which behave well universally, i.e., independently from any a priori knowledge concerning excess demand situations. Yet, as Saari and Simon prove, the informational content of the GeN cannot be substantially relaxed in the construction of such alternative effective price mechanisms. Furthermore, it is also generally regretted that all known alternatives to the GeN (see Jordan (1986), e.g.) lack an economic interpretation and are thus no good candidates from an economic point of view in the search for more reliable adjustment processes.[19]

In contrast to these findings on effective adjustment processes, our extended Walrasian tâtonnement dynamics (14.9), (14.10), which also pays attention to adjustments on the side of production and to derivative effects, has been obtained from purely economic considerations, firstly, by taking serious the original approach of Walras for a production economy and, secondly, by observing that the direction of change of the two excess situations that can exist in a production economy should both count in the determination of the price and quantity reactions of such an economy. Furthermore, our process is surprisingly similar from a formal point of view to the GeN (compare (14.12) and (14.15)), so that in the light of Saari and Simon's investigations and results the question as to its degree of effectiveness arises.

We have seen that the dynamics (14.9), (14.10) can be reformulated as follows [20]

$$\dot{z} = \begin{bmatrix} \dot{p} \\ \dot{y} \end{bmatrix} = (I - \gamma \cdot F'(z))^{-1} F(z), \quad z = \begin{bmatrix} p \\ y \end{bmatrix} \tag{14.16}$$

where F denotes our extended excess-function [see (14.3), (14.4)]. Note, that we now make use of the original function F and not of its linear approximation as in

[19] See Kamiya (1989) for a recent example of this kind.

[20] The matrix of adjustment coefficients [see (14.12)] is now suppressed in the above presentation by an appropriate choice of the function F.

the proof of Proposition 14.1. Note furthermore, that (14.16) is also applicable to ordinary excess demand functions $X(p)$ of pure exchange economies. Note finally, that in order to have a well-defined mechanism we should make the parameter γ dependent on the excess function F, i.e., on the particular economy, in such way that $(I - \gamma \cdot F'(z))^{-1}$ is always well defined for regular economies (cf. Proposition 14.3 in the last section and Proposition 14.5 in this one).

We have shown in the proof of Proposition 14.3 that an appropriate choice of γ will turn all equilibria $z^*[F(z^*) = 0]$ of a given regular economy into sinks of the modified system (14.9), (14.10), but have also seen that the adjustment parameter cannot be chosen independently of the particular economy and its excess demand and excess price function F. Despite these difficulties, we can nevertheless prove the local effectiveness of our mechanism (14.9), (14.10) by making use of certain definitions and results from Jordan (1983).

According to Jordan a *Generalized Newton method* (GNM) is an adjustment mechanism (in his case: price mechanism) of the following kind

$$\dot{z} = A(F(z), F'(z)), \quad \det F'(z) \neq 0 \tag{14.17}$$

where the function $A : \Re^n \times \Omega \longrightarrow \Re^n$ is C^1.[21]

Our process (14.15), i.e.,

$$\dot{z} = (I - \gamma \cdot F'(z))^{-1} F(z),$$

obviously is of this type.

However, this process must now be defined for all regular matrices $l = F'(z)$ which demands that γ must be made dependent on l in a smooth way (see the following proposition). Furthermore, the above definition of a GNM also demands that it should have the general continuity properties of a price mechanism as defined in Jordan (1983, p. 241). These properties may be briefly summarized as follows:

Let us denote by Γ the space of C^1 excess-functions F (with the topology of C^1 uniform convergence on compact subsets) and by Γ_p the subset of those functions which have p as a regular equilibrium. *Price mechanisms* à la Jordan

$$\dot{p} = M(p, F)$$

then have the following two properties

(1) They have the same equilibrium set as the original function F.
(2) They are continuous on open neighborhoods U of $\{p\} \times \Gamma_p$ and C^1 on the open set $U_f = \{p/(p, F) \in U\}$.

Furthermore, these mechanisms are conceived as being defined (but not necessarily in a continuous fashion) on a domain as large as possible. The reason behind this fact can be exemplified by means of the GeN as follows

$$\dot{z} = -F'(z)^{-1} F(z) \;\; for \;\; \det(F'(z)) \neq 0, \qquad \dot{z} = F(z) \;\; \text{otherwise}.$$

[21] Ω denotes the set of regular $n \times n$ matrices $l : \det l \neq 0$.

The GeN – applied to the excess function $F(z)$ – has all the properties of a price mechanism à la Jordan and it is, of course, also a GNM as defined above. And with regard to our own approach (14.15) we are now able to show:

Proposition 14.5. *Define* $\gamma(l)$ *by* $\|l^{-1}\| \cdot (1+\epsilon)$ *for all regular* l *by making use of a* C^1 *matrix norm* $\|..\|$ *and an arbitrary* $\epsilon > 0$.[22] *The following adjustment mechanism*

$$\dot{z} = (I - \gamma(F'(z)))^{-1} F(z) \;\; for \det(F'(z)) \neq 0, \qquad \dot{z} = F(z) \;\; otherwise \quad (14.18)$$

is then well-defined and a GNM.

Proof. We abbreviate the matrix $F'(z)$ by l in the following. We need to consider regular matrices l only, i.e., 0 is not an eigenvalue of l. To the real eigenvalues of l there correspond the real eigenvalues $\frac{1}{a}$ of its inverse l^{-1}. It is well-known [see Zurmühl (1964, p. 204) for example] that all matrix norms provide upper bounds for the absolute value of the eigenvalues of matrices l. From this fact we immediately get the inequality $\frac{1}{a} < \gamma(l)$, which means that we have $1 = \frac{1}{a} \cdot a < \gamma(l) \cdot a$ for all real eigenvalues $\gamma(l) \cdot a$ of $\gamma(l)l$. This implies that no eigenvalue of $I - \gamma(l)l = I - \gamma(F'(z))F'(z)$ can be zero, i.e., the inverse of this matrix exists. The above GNM-mapping is therefore well-defined and it is C^1 on the space of regular matrices l. □

Remark 14.6.

(1) Note, that the above does not imply that the real parts of all complex eigenvalues of $I - \gamma(l)l$ must be negative, too.
(2) Note also, that the above dynamics reduces to the Walrasian one if we set $\gamma \equiv 0$ and that it gives some sort of GeN if γ approaches infinity [as is obvious from its reformulation in the form $\frac{1}{\gamma} \cdot (\frac{1}{\gamma} I - l)^{-1} l$].

A GNM $\dot{z} = A(F(z), F'(z))$ is called *hyperbolically locally stable* in Jordan (1983, p. 247) if we furthermore demand that its first partial derivative $A_1(0,l)l$ is non- singular for all regular matrices $F'(z) = l$ and if all eigenvalues of $A_1(0,l)l$ have negative real parts. It is then proved in Jordan that such a mechanism is also *locally stable*, i.e., it is Liapunov-stable and asymptotically stable in the small for each admissible state z and a given neighborhood U^* of z with regard to an open neighborhood V of $z \times \Gamma_z$ of those excess dynamics which have z as a regular equilibrium (see Jordan (1983, p. 242) for the details of this and the preceding definition).

Proposition 14.7. *The adjustment process (14.17) is* locally hyperbolically stable *and thus also locally stable in the sense of Jordan (1983).*

Proof. In the case of process (14.17) we get for the above partial derivative ($F'(z) = l$):

$$A_1(0,l) = (I - \gamma(l)l)^{-1}$$

[22] See, e.g., Zurmühl (1964) for a list of such norms.

which by our choice of $\gamma(l)$ must be nonsingular. Furthermore, $A_1(0,l)l = (I - \gamma(l)l)^{-1}l$ equals the matrix that we have employed in the dynamics (14.11). And we know (by definition) that $\gamma(l) > 0$ must be larger than

$$\gamma_0 = \max\left\{\frac{a}{a^2+b^2}/a+bi \text{ eigenvalue } of \ \ l\right\}$$

since $\frac{a}{a^2+b^2}$ gives the real part of the eigenvalues of $l^{-1}\left[\frac{1}{\lambda} = \frac{\bar{\lambda}}{(\lambda\bar{\lambda})}\right]$. By Proposition 14.3 we therefore get in addition to Proposition 14.5: all eigenvalues of $(I - \gamma l)^{-1}l$ must have negative real parts. Our adjustment process (14.9), (14.10) (as modified above) is thus a hyperbolically locally stable process, and therefore also locally stable in the sense of Jordan. □

Since neither the Generalized Newton Method nor any other known approach, as for example the *Orthogonal Newton Method* (Jordan 1983), represent processes which mirror actual market behavior, it is of great interest that a dynamic process has been found which shares many formal properties with these quite potent mechanisms, but which at the same time has been obtained from economic considerations. Furthermore, this mechanism clearly shows that the criterion which Jordan (1983, p. 239) himself uses to characterize *true market adjustment processes*, i.e.,

$$\text{‘Excess demand for commodity } j = 0' \Rightarrow' \dot{p}_j = 0'$$ [23]

does not represent a good choice This criterion is, on the one hand, very partial in nature (markets in equilibrium cannot be disturbed by the adjustments taking place on all other markets). On the other hand, it provides a rather static view of economic adjustment processes by its assumption that price changes do not depend on the time rate of change of excess demand $X(p)$. In the light of the preceding discussion it is therefore not astonishing that Jordan's definition of a proper market process leads to the seemingly strong result, that there does not exist such a mechanism which is locally stable (if the number of commodities of the given exchange economy exceeds 2, see his p. 254). Yet, in contrast to this statement, our type of adjustment process shows that there is scope for a universal market mechanism if a sufficiently rich reaction pattern is used for the modeling of adjustment behavior.

As defined in Saari and Simon (1978, 1.1) our dynamics does not seem to provide a price mechanism in their use of this word. This, however, is not a weakness of our approach, but one of their concept of such a mechanism, for which indeed too much smoothness has been assumed (cf. also Proposition 6.2 in Jordan, which shows that care must be taken with regard to the singular derivatives of the excess functions). If Definition 1.1 in Saari and Simon is appropriately reformulated with regard to this problem, then it should be true that our mechanism (14.17) will also provide a locally effective adjustment mechanism (LEPM) in the sense of Saari

[23] See also Woods (1978, Sect. 8.5) for remarks on the limitations of this approach.

and Simon (1978). As a *mechanism* or *adjustment method* our type of dynamics is indeed generally effective, though – as we have seen – its parameters, i.e., here: γ, cannot be chosen independently of the particular economy in question. [24]

The above questionable feature of Saari and Simon's early definition of effectiveness is also considered in Saari and Williams (1986), where it is shown that local information should and can be used to select the proper adjustments for particular types of 'economic environments'. Note in this respect, that our procedure (14.17) as an economic adjustment method works for all such environments, so that the dynamical system we employ needs only a minor parameter adjustment with regard to each local situation. This generality of our approach may, however, also represent a disadvantage of our adjustment method. As discussed in Saari and Williams (1986) it may therefore be of interest to tailor the proper adjustment method still closer to the specific types of the economies under consideration. It is indeed not very plausible that an economic adjustment process can be found that works properly in all conceivable economic environments, i.e., in all regular exchange or production economies with only slight modifications in its adjustment parameters. Furthermore, as seen in Sect. 14.3, the use of a universal factor γ for the formulation of the derivative feedback mechanism may be too strong (as an assumption and in its implications). More general mechanisms with weaker stability properties may therefore still be desirable and can indeed be found, cf. the introduction.

14.5 Ignorable Components?

From an economic viewpoint it is very natural to consider the dynamics (14.9), (14.10) with differing adjustment speeds $\gamma_1 \neq \gamma_2$ for the influences of the rate of change of excess demand and excess price. Furthermore, it is tempting to investigate whether the situations $\gamma_1 = 0 < \gamma_2$ or $\gamma_1 > \gamma_2 = 0$, where only one of these additional forces is at work, will suffice for designing a particular type of universal mechanism. For example, from a classical perspective, one could argue that the adjustment of supply due to excess profitability and its time rate of change may be the essential element which together with the conventional law of demand ($\gamma_1 = 0$) will suffice to guarantee something like universal stability (in the small), fairly independent of whether demand is also a stabilizing factor or not. In this final section we will consider this claim as well as other aspects which are introduced by the use of arbitrary and differing adjustment parameters for the two equations (14.9), (14.10).[25]

[24] A formulation of the general kind: $\gamma(F, F')$ – together with appropriate assumptions on such a function γ – may also be of use in developing this matter further.

[25] Note here, that part of the results *of this section* will depend on the particularly simple economy chosen to discuss these matters (cf. Flaschel and Semmler (1987) for more general investigations).

Case 14.8. $\gamma_1 = 0 < \gamma_2$

In this case (14.9), (14.10) can be transformed into an explicit system of differential equations in the following simple way. Equation (14.4), extended by means of $\gamma_2 > 0$, gives rise to [see (14.10)]:

$$[1+\gamma_2\beta l''(y)]\cdot \dot{y} = \beta(p-l'(y))+\beta\gamma_2\,\dot{p} = \beta(p-l'(y))+\gamma_2\beta\alpha(d(p,l(y))-y)$$

The dynamic system thus reads in this case:

$$\dot{p} = \alpha((d(p,l(y)) - y) \tag{14.19}$$

$$\dot{y} = \frac{\beta(p - l'(y)) + \gamma_2\alpha\beta(d(p,l(y)) - y)}{1 + \gamma_2\beta l''(y)} \tag{14.20}$$

which is well-defined around equilibrium values where decreasing or constant returns to scale prevail. Calculating the Jacobian at equilibria (p^*, y^*) gives

$$J = \begin{pmatrix} \alpha d_p & -\alpha \\ \frac{\beta+\gamma_2\alpha\beta\cdot d_p}{1+\gamma_2\beta\cdot l''} & \frac{-\beta l''-\gamma_2\alpha\beta}{1+\gamma_2\beta\cdot l''} \end{pmatrix} \tag{14.21}$$

We consequently get in this case:

$$\text{trace } J = \alpha d_p - \frac{\beta l'' + \gamma_2\alpha\beta}{1 + \gamma_2\beta l''}, \quad \det J = \frac{\alpha\beta(1 - d_p l'')}{1 + \gamma_2\beta l''}$$

We can see from these expressions that not all equilibria can be stabilized by the choice of a sufficiently large parameter $\gamma_2 > 0$, e.g., if $1 < d_p l''$ holds true at the equilibrium. Stability is, however, possible if constant returns are assumed to prevail, in which case the above two expressions reduce to

$$\text{trace } J = \alpha d_p - \gamma\alpha\beta, \quad \det J = \alpha\beta.$$

For constant returns economies we thus simply need

$$\gamma_2 > \frac{d_p}{\beta}$$

to obtain (with appropriate qualifications) all the results of the preceding section.[26] A sufficient sensitivity to changes in price/cost-differentials is thus capable of stabilizing the possibly destabilizing influences of the demand component of a given economy.

[26] In the present case, only one of the two eigenvalues of J will tend to zero as γ_2 approaches infinity.

Case 14.9. $\gamma_1 > \gamma_2 = 0$

In this opposite case [of a derivative influence of excess demand on price changes] we get instead of (14.9)

$$\begin{aligned}(1-\gamma_1\alpha d_p)\,\dot{p} &= \alpha(d(p,l(y))-y)-\gamma_1\alpha\,\dot{y}\\ &= \alpha(d(p,l(y))-y)-\gamma_1\alpha\beta(p-l'(y))\end{aligned}$$

and thus

$$\dot{p} = \frac{\alpha(d(p,l(y))-y)-\gamma_1\alpha\beta(p-l'(y))}{1-\gamma_1\alpha d_p} \tag{14.22}$$

$$\dot{y} = \beta(p-l'(y)) \tag{14.23}$$

which is a well-defined system of differential equations in the neighborhood of equilibria if γ_1 is chosen sufficiently large. For the Jacobian at such equilibria we get this case:

$$J = \begin{pmatrix} \frac{\alpha d_p-\gamma_1\alpha\beta}{1-\gamma_1\alpha\cdot d_p} & \frac{-\alpha+\gamma_1\alpha\beta l''}{1-\gamma_1\alpha\cdot d_p} \\ \beta & -\beta\cdot l'' \end{pmatrix} \tag{14.24}$$

i.e.,

$$\text{trace } J = -\beta l'' + \frac{\alpha d_p-\gamma_1\alpha\beta}{1-\gamma_1\alpha d_p}, \quad \det J = \frac{\alpha\beta-\alpha d_p\beta l''}{1-\gamma_1\alpha d_p}$$

which again allows for instability even if γ_1 is chosen sufficiently large. In this case, however, assuming constant returns is of no help:

$$\text{trace } J = \frac{\alpha(d_p-\gamma_1\beta)}{1-\gamma_1\alpha d_p}, \quad \det J = \frac{\alpha\beta}{1-\gamma_1\alpha d_p}$$

We see from these two expressions that an instability which originates in the demand component of the system ($d_p > 0$) cannot generally be removed by the inclusion of a sufficiently strong derivative influence in the law of demand, since the set

$$\{\gamma_1 > 0/\gamma_1\alpha d_p < 1,\ d_p < \gamma_1\beta\}$$

may be an empty set. Derivative feedbacks from the production side of the economy therefore appear as being more effective (successful) than the more direct derivative feedback of excess demand on prices p in the stabilization of an unstable demand component. This represents an important asymmetry in the analysis of partial derivative controls.

Case 14.10. $\gamma_1, \gamma_2 > 0$

Analogously to (14.15) one gets in this case the following explicit form of a system of differential equations

$$\dot{z} = (I - \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} F'(z))^{-1} F(z) \tag{14.25}$$

The Jacobian of this system at an equilibrium $z^* = (p^*, y^*)$ is given by

$$J = (I - \begin{bmatrix} \gamma_1 & 0 \\ 0 & \gamma_2 \end{bmatrix} F'(z^*))^{-1} F'(z^*) \tag{14.26}$$

As can be seen from this expression the method of the proof of Proposition 14.3 is no longer applicable in this more general situation in order to show that J will be a stable matrix for all γ_1, γ_2 sufficiently large.

Yet, in our simple two-dimensional model we can immediately derive an explicit expression for J, e.g., in the case of constant returns:

$$J = \frac{1}{1 - \gamma_1 \alpha d_p + \gamma_1 \gamma_2 \alpha\beta} \begin{bmatrix} \alpha d_p - \gamma_1 \alpha\beta & -\alpha \\ \beta(1 - \alpha d_p(\gamma_1 - \gamma_2)) & -\alpha\beta\gamma_2 \end{bmatrix}$$

which gives

$$\text{trace} J = \frac{\alpha d_p - \alpha\beta(\gamma_1 + \gamma_2)}{1 - \gamma_1 \alpha d_p + \gamma_1 \gamma_2 \alpha\beta}$$
$$\det J = \alpha\beta$$

These calculations show that γ_2 must be chosen larger than $\frac{d_p}{\beta}$ to ensure that all positive γ_1 allow for asymptotic stability in this extended model of a proportional plus a derivative control.

Returning to our Case 14.8 we can use (14.26) to rewrite this dynamics as follows $(F'(z) = l)$:

$$\dot{z} = (I - \begin{bmatrix} 0 & 0 \\ 0 & \gamma_2 \end{bmatrix} F'(z))^{-1} F(z) \tag{14.27}$$

In Jordan (1983, p. 256), cf. also Saari and Simon (1978, p. 1105) in this regard, an entry ij of such a generalized Newton process (where $\gamma_2(\cdot)$ has to be determined appropriately, see below) is called an *ignorable entry* of this Newton method

$$(z, l) \overset{A(z,l)}{\longrightarrow} (I - \begin{bmatrix} 0 & 0 \\ 0 & \gamma_2 \end{bmatrix} l)^{-1} z = \frac{1}{1 - \gamma_2 l_{22}} \begin{bmatrix} 1 - \gamma_2 l_{22} & 0 \\ \gamma_2 l_{21} & 1 \end{bmatrix} z$$

if the l_{ij} -component of the matrix l can be varied without influencing the values $A(z, l)$ obtained from this method.[27]

[27] $det(l) \neq 0$ and also $\det(I - \begin{bmatrix} 0 & 0 \\ 0 & \gamma_2 \end{bmatrix} l) \neq 0$.

It is then stated in Jordan (1983, p. 256) that a locally stable generalized Newton method admits no ignorable entries. In the light of this assertion and the fact that our Case 14.8 has ignorable coordinates one may ask in what respect the stability properties of this case are not of the locally stable type that is considered by Jordan.

To answer this question consider again the Jacobian (14.20) in the case of constant returns:

$$J = \begin{bmatrix} \alpha d_p & -\alpha \\ \beta + \gamma_2 \alpha \beta d_p & -\gamma_2 \alpha \beta \end{bmatrix} = A_1(0,l)l,$$
$$\text{trace} = \alpha d_p - \gamma_2 \alpha \beta, \quad \det J = \alpha \beta$$

which is a stable matrix for all $\gamma_2 > \frac{d_p}{\beta}$.

Due to the special structure of the matrix J we see that this matrix can easily be turned into a stable matrix through an appropriate choice of γ_2. Yet, the properties of being hyperbolically locally stable (and the implied local stability) demand that any regular matrix l should lead to a stable matrix of the type

$$(I - \begin{bmatrix} 0 & 0 \\ 0 & \gamma_2 \end{bmatrix} l)^{-1} l$$

(see Jordan (1983, p. 255)). This, however, is obviously not true, since we have

$$\begin{aligned} (I - \begin{bmatrix} 0 & 0 \\ 0 & \gamma_2 \end{bmatrix} l)^{-1} l &= \begin{bmatrix} 1 & 0 \\ -\gamma_2 l_{21} & 1 - \gamma_2 l_{22} \end{bmatrix}^{-1} \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} \\ &= \begin{bmatrix} 1 - \gamma_2 l_{22} & 0 \\ \gamma_2 l_{21} & 1 \end{bmatrix} \begin{bmatrix} l_{11} & l_{12} \\ l_{21} & l_{22} \end{bmatrix} / (1 - \gamma_2 l_{22}) \\ &= \begin{bmatrix} (1 - \gamma_2 l_{22}) l_{11} & (1 - \gamma_2 l_{22}) l_{12} \\ l_{21} + \gamma_2 l_{21} l_{11} & l_{22} + \gamma_2 l_{21} l_{21} \end{bmatrix} / (1 - \gamma_2 l_{22}) \end{aligned}$$

For example, choosing a regular matrix l of the form $\begin{bmatrix} + & + \\ + & 0 \end{bmatrix}$ implies for this matrix the sign structure $\begin{bmatrix} + & + \\ + & + \end{bmatrix}$ which clearly describes an unstable matrix.[28] Hence, though, e.g., the choice

$$\gamma_2 = (1 + l_{22}^2)^{-\frac{1}{2}}$$

defines a generalized Newton method

$$A(z,l) = (I - \begin{bmatrix} 0 & 0 \\ 0 & \gamma_2 \end{bmatrix} l)^{-1} z$$

[28] Note, that the above choice of l is not possible for the system (14.3), (14.4) from which the dynamic analysis of this chapter has started.

in the sense of Jordan (1983, p. 255), the domain of definition of this method is left too unrestricted in Jordan in order to allow for his stability concepts in the presence of only a partial derivative adjustment by means of the component γ_2.

This situation is not changed if one applies the less restrictive concept of *local stability at unique equilibria* which stabilizes only those equilibria for which a global demand function exists which near the equilibrium is identical to the given one and which has this equilibrium as a unique equilibrium. These equilibria are equivalently characterized by $\det(-l) > 0$ instead of only $\det l \neq 0$ [see Jordan (1983, pp. 251 ff.) for details]. As is obvious from the above this restriction on the Jacobians of our basic dynamics (14.3), (14.4) is still insufficient to imply asymptotic stability for all matrices $(I - \begin{bmatrix} 0 & 0 \\ 0 & \gamma_2 \end{bmatrix} l)^{-1} l$ as it could be proved for the extended dynamics of Sect. 14.4.

To obtain stability in the above situation the reference matrices l must be additionally restricted by the 'cross-dual' sign conditions $\begin{bmatrix} & - \\ + & \end{bmatrix}$. These side-conditions define an open subset of the matrices l with $\det l \neq 0 (> 0)$. Using this additional information on the Jacobian J of our basic dynamics (14.3), (14.4) [which is not available in the equivalent situation of the exchange economies considered by Jordan] one can show then:

Proposition 14.11. *The matrix*

$$J = (I - \begin{bmatrix} 0 & 0 \\ 0 & \gamma_2 \end{bmatrix} l)^{-1} l$$

is a stable matrix ($Re\ \mu_i < 0$) *for all* $\gamma_2 > 0$ *chosen sufficiently large, if* l *is the matrix of a unique equilibrium and if either* $l_{22} = 0$ *(constant returns) or* $l_{11} < \frac{l_{21} l_{12}}{l_{22}}$ *(in the case of decreasing returns to scale) holds true in addition.*

Proof. The matrix J is known to equal

$$\frac{1}{1 - \gamma_2 l_{22}} \begin{bmatrix} (1 - \gamma_2 l_{22}) l_{11} & (1 - \gamma_2 l_{22}) l_{12} \\ 1_{21} + \gamma_2 l_{21} l_{11} & 1_{22} + \gamma_2 l_{21} l_{12} \end{bmatrix}$$

which gives

$$\text{trace } J = l_{11} + \frac{l_{22} + \overbrace{\gamma_2 l_{21} l_{12}}^{-}}{\underbrace{1 - \gamma_2 l_{22}}_{+}}$$

$$\det J = \frac{\det(l)}{\underbrace{(1 - \gamma_2 l_{22})}_{+}} > 0 \quad \text{if } \det(l) > 0.$$

This implies

$$\text{trace}J = l_{11} - \frac{l_{21}l_{12} + \frac{l_{22}}{\gamma_2}}{l_{22} - \frac{1}{\gamma_2}}$$

and thus the assertion, if γ_2 is chosen sufficiently large.

Remark 14.12. To obtain local stability at unique equilibria on the basis of hyperbolic local stability at unique equilibria as in Jordan (1983) it is consequently necessary to restrict the set of vector fields allowed for in the definition of local stability in the way indicated by the above proposition. This possibility shows that locally stable universal price mechanism may be found despite the existence of ignorable components in the Newton method that is used. This is true, if these ignorable entries can be supplemented by suitable restrictions on the set of vector fields at the various admissible equilibria (in correspondence to the dynamics deriving from conditions of excess which drive the system). In the case of our cross-dual price/quantity-dynamics (14.3), (14.4) these restrictions can be taken from the Jacobian of this dynamics at an equilibrium which is given by

$$J = \begin{bmatrix} \alpha d_p & -\alpha \\ \beta & -\beta l'' \end{bmatrix}$$

and which obviously has a fairly definite structure up to the demand component d_p (cf. also footnote 2).

14.6 Conclusions

In this chapter we have analyzed the local properties of Walras' cross-dual adjustment procedure for production economies. We have seen how this simultaneous price/quantity adjustment enlarges the range of applicability for a tâtonnement stability analysis and how it very typically modifies the structure of the Jacobian matrix at equilibrium points. By adding a particular type of dual derivative control to this Walrasian adjustment procedure we found universal stability for this adjustment process which made this extended cross-dual dynamics nearly independent of the particular kind of economic structure on which it is supposed to work.

In our view this is, however, too strong a result from two related points of view:

- The stabilizing contribution from the side of production which this – in the end classical – cross-dual approach to stability is supposed to lay bare is again obscured by the fact that our special choice of derivative forces will work equally well for all conventional kinds of tâtonnement processes which therefore at present must be viewed as the main carriers of the stability results obtained.
- The choice of adjustment parameters we have allowed for in Sects. 14.3, 14.4 is still too ideal (thus supporting a too universal character of the proposed dynamic process).

We have indicated in the last section how less capable but more realistic processes may be designed, but have also remarked that the obtained results have no general validity in their present formulation. They only serve to illustrate directions for a future analysis of the proper design of the adjustments that are basically taking place in a capitalistic market economy.

In addition, a treatment of the following list of topics may then be added and should be of help in the further refinement of the adjustment mechanisms we have analyzed in this chapter:

- The consideration of multiple activities and of process extinction, cf. Flaschel and Semmler (1987) for an example,
- The formulation of output reactions in joint production systems, cf. Mas-Colell (1986) for a discussion of the problems to be solved in this case,
- The integration of Keynesian price/quantity reaction patterns, cf. Mas-Colell (1986) and Flaschel and Semmler (1988) for their formulation and problems,
- The design of systems with effective functions for demand and supply, cf. Duménil and Lévy (1989),
- The investigation of global stability and its ranges, and the implications of such a topic for an evaluation of short-run vs. long-run theories of value, cf. Mas-Colell (1986) and Flaschel and Semmler (1987) with regard to some results,
- The treatment of discrete time models, cf. Saari (1985) for the implied increase in complexity,
- The question of the entry and exit of firms in the long run, cf. Novshek and Sonnenschein (1987) for simple examples of this additional dynamics.

This list indicates that much remains to be done, in particular in the search for and the testing of more sophisticated tâtonnement procedures than those that have been studied here.

References

Ahlfors, L. V. (1953). *Complex Analysis*. London: McGraw Hill

Beckmann, M. J. & Ryder, H. E. (1969). Simultaneous price and quantity adjustment in a single market. *Econometrica, 37*, 470–48

Dierker, E. (1974). *Topological Methods in Walrasian Economics*. Heidelberg: Springer

Dieudonné, J. (1960). *Foundations of Modern Analysis*. New York: Academic

Duménil, G. & Lévy, D. (1989). The classical legacy and beyond. CEPREMAP: Discussion Paper

Flaschel, P. (1990). Cross-dual dynamics, derivative control, and global stability. A neoclassical presentation of a classical theme. *Political Economy*

Flaschel, P. & Semmler, W. (1987). Classical and neoclassical competitive adjustment processes. *The Manchester School, 55*, 13–37

Flaschel, P. & Semmler, W. (1988). On the integration of dual and cross- dual adjustment processes in Leontief systems. *Ricerche Economiche, 42*, 403–432

Goodwin, R. M. (1953). Static and dynamic general equilibrium models. In R. M. Goodwin (Ed.), *Essays in Linear Economic Structures*. London: Macmillan

Goodwin, R. M. (1989). Essays in Nonlinear Economic Dynamics. In P. Flaschel & M. Krüger (Eds.), *Dynamische Wirtschaftstheorie, 3*. Bern: Peter Lang

Hahn, F.H. (1970): Some adjustment problems. *Econometrica*, *38*, 1–17

Jordan, J. S. (1986). Instability in the implementation of Walrasian allocations. *Journal of Economic Theory, 39*, 301–328

Jordan, J.S. (1983). Locally stable price mechanisms. *Journal of Mathematical Economics, 10*, 235–259

Kamiya, K. (1989). *A globally stable price adjustment process*. Osaka: Mimeo

Keenan, D. (1981). Further remarks on the global Newton method. *Journal of Mathematical Economics*, 159–165

Keenan, D. (1990). Morishima systems and global stability. *International Economic Review, 31*, 11–16

Kirman, A. (1988). On the instability of Walrasian economics. In M. Galeotti et al. (Eds.), *Nonlinear Dynamics in Economics and Social Sciences*. Bologna: Pitagora Editrice

Kirman, A. (1989). The intrinsic limits of modern economic theory: The emperor has no clothes. *Economic Journal, 99*, 126–139

Kose, T. (1956). Solutions of saddle value problems by differential equations. *Econometrica, 24*, 59–70

Mas-Colell, A. (1986). Notes on price and quantity tâtonnement dynamics. In H. Sonnenschein (Ed.), *Models of Economic Dynamics*. Heidelberg: Springer

Morishima, M. (1959). A reconsideration of the Walras-Cassel-Leontief models of general equilibrium. In K. Arrow et al. (Eds.), *Mathematical Social Sciences*. Stanford: Stanford University Press

Morishima, M. (1977). *Walras' Economics*. Cambridge: Cambridge University Press

Novshek, W. & Sonnenschein, H. (1987). General equilibrium with free entry: A synthetic approach to the theory of perfect competition. *Journal of Economic Literature, 25*, 1281–1306

Saari, D. G. (1985). Iterative price mechanisms. *Econometrica, 53*, 1117–1131

Saari, D. G. & Simon, C. P. (1978). Effective price mechanisms. *Econometrica, 46*, 1097–1125

Saari, D. G. & Williams, S. R. (1986). On the local convergence of economic mechanisms. *Journal of Economic Theory, 40*, 152–167

Smale, S. (1976). Exchange processes with price adjustment. *Journal of Mathematical Economics, 3*, 211–226

Varian, H. R. (1977). A remark on boundary restrictions in the Global Newton Method. *Journal of Mathematical Economics, 4*, 127–130

Walker, D. A. (1987). Walras's theories of tâtonnement. *Journal of Political Economy, 95*, 758–774

Walras, L. (1954). *Elements of Pure Economics*. London: George Allen and Unwin

Woods, J. (1978). *Mathematical Economics*. London: Longman

Zurmühl, R. (1964). *Matrizen und ihre technische Anwendungen*. Heidelberg: Springer

Chapter 15
Classical and Neoclassical Competitive Adjustment Processes

15.1 Introduction

In recent neoclassically and classically oriented literature on competitive processes several models have been presented which work not only with the ‘law of demand’ but also with the ‘law of profitability’. In such dynamical models a cross-dual process is stylized in which price changes are initiated by imbalances in supply and demand and changes of outputs are caused by profitability differentials. Such cross-dual dynamics can now be found in neoclassical tradition (Morishima 1976, 1977; Mas–Colell 1974, 1986) and in classical tradition. In classical tradition the analysis of such a cross-dual adjustment process was initiated particularly by some recent publications of Nikaido (1978, 1983, 1985) who questioned the stability of classical competition. In comparison to his results it is the purpose of this article to show that the classical approach to the dynamics of competition may be able to produce stability results which are of at least comparable interest to those of neoclassical stability theory.[1]

To this end, both traditions will be briefly surveyed in Sects. 15.2 and 15.3. The formal treatment of the classical competitive process is currently in an early phase of development (see, e.g., Steedman 1984, in this regard). Definite results are rare and of a somewhat heterogeneous nature regarding the basic components of the process that is to be modeled (see Sect. 15.2). It therefore cannot be expected that our own approach will provide already a fully acceptable formalization of these forces of economic dynamics. Its advantages in comparison to alternative formal investigations of this dynamical process however are, first, that it formulates the classical process of cross-dual dynamics in a more plausible way than e.g., Nikaido (1978, 1983, 1985). Second, it formulates more than other models the dynamics from the point of view of production and the behavior of the firm. Third, it is able to provide a treatment of joint production, multiple activities, process extinction, as well as a proof of global stability for Sraffa as well as von Neumann joint production technologies. We are able to show that a dynamic version of the ‘law of demand’ in combination with

[1] We thank V. Caspari, K. Dietrich, G. Duménil, D. Foley, R. Franke, M. Glick, D. Lévy and A. Shaikh for helpful comments on an earlier version of this chapter.

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_15,

supply changes caused by profit rate differentials will produce a stable movement around specific balanced growth paths of input–output models and, furthermore, that this cross-dual adjustment process will be asymptotically stable if the direction of the change of profit rate differentials is also taken into account. These results stand in stark contrast to problematic stability propositions of standard neoclassical economics, whether of short-run Walrasian nature or of the long-run type of efficient capital accumulation. Our formalizations are intended to provide a simple answer to an important question raised, for example, by Nikaido (1978, 1983) and Steedman (1984) concerning the stability properties of classical dynamics.

A final word of warning should be added, however. The following remarks about the classics and Marx are not concerned with a detailed textual support for our suggested cross-dual dynamics; in particular, we shall not investigate thoroughly whether there exist great differences between the views of the classics, Marx and other writers such as Walras and Marshall. The interested reader is referred to Steedman (1984); Duménil and Lévy (1983); Flaschel and Semmler (1985, 1986a); Semmler (1985); and, for the Walrasian tradition, to Yaffé (1967).

15.2 Neoclassical Stability Analysis in the Short and in the Long Run

Neoclassical short-run equilibrium analysis, operating with excess demand functions, considers also cases where planned supply deviates from planned demand. Walras already noted in his *Elements* (1954):

> 'Generally, however, the total demand will not equal the total offer of each and every commodity, ... What will happen on the market then? If the demand for any one commodity is greater than the offer, the price of that commodity in terms of the numéraire will rise; if the offer is greater than the demand, the price will fall. What must we do in order to prove that the theoretical solution is identically the solution worked out by the market? Our how that the upward and downward movements of prices solve the system of equations of offer and demand by a process of groping ('par tâtonnement')'. (pp. 169–170).

This process of groping has become the most popular form of adjustment process within the neoclassical analysis of (temporary) general equilibrium.[2] The working of this Walrasian price-tâtonnement or so-called 'law of demand and supply' is modeled by constructing Walrasian (or notional) excess demand functions $Z_i(p)$ and by stipulating, for example,

$$\dot{p}_i = d_i Z_i(p), \ i = 2, \dots, n \tag{15.1}$$

[2] We should stress however that there also exists an established body of literature which dispenses with this type of anonymous market price adjustment by allowing individuals to set prices (see Hahn 1982; Fisher 1983 for details). Yet from a classical perspective, the interaction between natural and market prices was the important one, which is the reason why we neglect this part of the literature on the stability of market economics.

where the $d_i > 0$ denote certain adjustment coefficients and where p denotes the vector of prices of the n given commodities ($p_1 = 1$; see Hahn 1982, pp. 759 f. for further details).

However, it is well known that the dynamic process (15.1) generally does not behave well even in the simple exchange economy which underlies the above quotation from Walras's *Elements of Pure Economics*. Hahn (1970, p. 2), e.g., admits that the study of the Walrasian groping has not been very fruitful. Even for the drastically simplified situation of a Walrasian tâtonnement in the context of an exchange economy, the exclusion of trading at false prices was not sufficient *'to lay bare the essentials of the law of 'supply and demand"* and to reveal thereby the validity of the working of the 'invisible hand'.

> 'What has been achieved is a collection of sufficient conditions, one might almost say, anecdotes, and a demonstration by Scarf and later by Gale, that not much more could be hoped for.' (Hahn 1970, p. 2).

In the light of such results it has become fairly common for neoclassical equilibrium theorists to avoid such problems completely by restricting their attention exclusively to equilibrium situations. That is, to support their equilibrium notions they simply assume that there exists some 'law of demand and supply' in operation which works with infinite speed in an asymptotically stable manner. Viewed from the perspective of growth theory, this also means that 'time' becomes dichotomized into *hypothetical time*, where adjustment to short-run equilibrium position takes place, and *real time*, where the evolutionary forces of the system supposedly work to adjust the economy to a long-run steady–state position. In view of this fact we therefore have to examine briefly whether neoclassical economics has demonstrated successfully the stability of this latter type of adjustment.

It has been shown, e.g., in Hahn (1966), that the trajectories of momentary equilibria with myopic price-expectations of the perfect foresight variety will not, in general, approach the steady state. In contrast to such results, it is demonstrated in Burmeister et al. (1968) for a multigood version of Solow's growth model that such a model can be globally stable if particularly simple saving functions are assumed. Subsequent investigations within the context of neoclassical growth models with heterogeneous capital goods have, however, generally revealed the possibility of a saddlepoint behavior of their dynamics. An example of this is provided in Hahn (1970, pp. 12 f.) by means of temporary equilibrium prices of a two-capital-goods model with static price expectations. He concludes from his investigations

> 'that there is no theoretical evidence to suggest that the invisible hand performs better 'asymptotically' than it does 'momentarily' at least in the role in which it has been cast by the recent literature.' (p. 9).

Such unpleasant implications also flow from the works of Burmeister et al. (1973) as well as Caton and Shell (1971), which differ from Hahn's approach in that quantities *and* prices simultaneously evolve or adjust in *real* time in these models.

Burmeister et al. (1973) consider the following extension of a multigood Solovian model

$$\dot{k}_i = y_i - (n + \delta_i)k_i$$

by means of a price-dynamics of type

$$\dot{p}_i = -w_i + (r_0 + \delta_i) = p_i$$

where r_0 is the rate of interest determined by flow equilibrium conditions of the capital market. The dynamics of this model is fairly complex and fully interdependent.

Assuming that per capita outputs y_i and gross rental rates w_i are determined by the usual competitive conditions of neoclassical economics (for given capital intensities k_i and given prices p_i at each moment of time), it is shown by Burmeister et al. (1973) that the above dynamics need not imply asymptotic convergence to its equilibrium point. A 'saddlepoint property' may typically be involved. Efficient capital accumulation (of the perfect foresight variety) consequently does not guarantee a stable economic evolution.

From the point of view of input–output analysis a related situation is described by the so-called dual (in) stability theorem of (closed) Leontief models with fixed coefficients. In such models one assumes as quantity- and price-determining equations, for example

$$x = Ax + C\dot{x} \tag{15.2}$$

$$p = pA + rpC - \dot{p}C \tag{15.3}$$

where the matrix A represents capital consumed and matrix C capital advanced (see Jorgenson 1960; Woods 1978, pp. 189 ff. for further details). As noted by Morishima (1977), such adjustment processes are dual processes only. The dynamic adjustments of outputs and prices are uncoupled, i.e., each is determined by an independent differential equation system. It can be shown for (15.2), (15.3) that the price and output system of such a dynamic Leontief model cannot be simultaneously relatively stable. This behavior of the model (15.2), (15.3) is, however, understandable if two of its underlying assumptions are made explicit: that 'capital' is always fully utilized and that price-expectations are of the perfect foresight type. By removing these two assumptions, Fukuda (1975) shows that price and quantity adjustments can be simultaneously stable (see Aoki 1977 for a similar investigation).

We conclude from all this that the perfect world of neoclassical economics is far from perfects with regard to its stability analysis. Yet neoclassical equilibrium analysis lacks any scientific foundation if the belief in a strong tendency to the assumed type of equilibrium is unfounded.

> 'The study of equilibria alone is of no help in positive economic analysis.... The most intellectually exciting question of our subject remains: is it true that the pursuit of private interest produces not chaos but coherence, and if so, how is it done?' (Hahn 1970, pp. 11–12).

The section which follows will approach this question from the classical rather than the neoclassical point of view. The results obtained in the subsequent parts of the chapter, though still crude with regard to real world complexities, will nevertheless bear comparison with the stability properties of neoclassical models that we have sketched above.

15.3 Classical Competition: Notes on the Literature

Astonishingly, some sort of classical approach to competition[3] may even be 'read' in the work of Walras. When he considers the process of groping in a production economy he, in fact, states (Walras 1954, p. 242) that

> '...entrepreneurs use tickets ['bons'] to represent the successive quantities of products which are first determined at random and then increased or decreased according as there is an excess of selling price over cost of over cost of production or *vice versa*, until selling price and cost are equal.'

If one adds that prices themselves are determined by the 'law of demand', as stated in Sect. 15.2, and if one interprets the notion 'cost of production' appropriately than at least some similarity can be claimed to exist between Walras's and the classical view of the adjustment of prices and quantities. Walras realized that the response of firms to profitability differences had to be included in the process of tâtonnement. He writes in the fourth edition of the *'Elements'* published in 1900 (which is identical to the edition of 1954 used here) that 'the process of groping (tâtonnement) in production entails a complication which was not present in the case of exchange...' (Walras 1954, p. 242). In this edition, however, Walras does not explain any longer why this should constitute a 'complication'. In the second edition of 1889, he still had described these difficulties:

> 'In production (on the other hand) productive services are transformed into products. After certain prices for services have been cried and certain quantities have been manufactured, if these prices and quantities are not equilibrium prices and q quantities, it will be necessary not only to cry new prices but also to manufacture other quantities of products. Taking this necessity into account we must suppose that at each renewal ('reprise') of tâtonnement our entrepreneurs will find in the home countries landowners, workers and capitalists possessing unchanged ('les mêmes') quantities of services and having unchanged ('les mêmes') needs for services and products.' (cited in Yaffé 1967, p. 10).

Surprisingly, in the fourth edition Walras avoids describing those complications by writing in the Preface:

> 'In the theory of production, I no longer represented the preliminary groping toward equilibrium as it takes place effectively, but I assumed, instead, that it was done by means of tickets ('sur bons') and I then carried this fiction through the remainder of the book.' (cited in Yaffé 1967, p. 12).

This cross-dual dynamical process and the problems associated with it. were already well known in Marx. Some similarities of Walras's and Marx's view on the functioning of capitalist competition can, e.g., be seen from the following:

> 'This movement of capitals is primarily caused by the level of market prices, which lift profits above the general average in one place, and depress them below it in another.' (p. 208). And on the other hand: 'Supply and demand determine the mark et price...' (p. 191).

[3] This, for reasons of simplicity, is used to denote the corresponding theories of Smith, Ricardo and Marx as well as their followers.

Of course, there are also considerable differences in the approaches to dynamics between Walras and Marx, most notably between Walras's groping procedure and Marx's considerations of the real difficulties of the transfer of capital from one sphere to another through entry and exit of firms (Marx 1959, p. 208). Yet the important common element is that both authors formulate some sort of cross-dual dynamics: price/cost- or profit-rate-differentials determine the conditions of supply, while demand and supply discrepancies determine price changes, and these in turn again determine changes in profit-rate differentials.

Without claiming that the following investigation of cross-dual dynamics is a consistent interpretation of Marx's or the classics' (or Walras's) views on dynamics, we derive from the above considerations the confidence that it may be worthwhile to examine, instead of the popular one-sided process (15.1), an adjustment process which leads from profit-rate discrepancies to changing supply conditions and, via the 'law of demand', back to changing market prices and changing profit-rate differentials (and so on). The question is whether such a dynamic process will be stable or even asymptotically stable under economically plausible conditions. Marx (1959, p. 208) himself was realistic enough not to claim that his consideration of the equalization of profit rates through competition would necessarily imply the latter:

> 'Experience allows, moreover, that if a branch of industry, such as, say, the cotton industry, yields unusually high profits at one period, it makes very little profit, or even suffers losses, at another, so that in a certain cycle of years the average profit is much the same as in other branches. And capital soon learns to take this experience into account.'

For the Walrasian as well as for the classical tradition there exist meanwhile several studies that have attempted to formalize this cross-dual dynamics. An attempt to formalize the dynamics of competition in the spirit of Walras can be found in Morishima (1976, 1977) who had already in his earlier work surpassed the one-sided view of Walrasian dynamics as represented by the 'law of demand' in neoclassical general equilibrium theory (see Sect. 15.2). For example, in Morishima (1977, pp. 60 ff.) the model of Walras's rules of price and quantity adjustments is described as follows: goods prices p_i and factor prices v_k adjust according to excess demand ξ_i and ζ_k, respectively:

$$\dot{p}_i = u\xi_i \ (i = 1, \ldots, n), \tag{15.4}$$

$$\dot{v}_i = u\zeta_k \ (k = 1, \ldots, \overline{n} - 1; v_{\overline{n}} \equiv 1) \tag{15.5}$$

unless these prices are zero (in which case excess demand is replaced by 'zero' if it is positive). This law of (excess) demand is (and has to be) supplemented by a rule which describes how quantities are adjusted, particularly in the case of a constant returns production economy:

$$\dot{x}_j = wy_j \ (j = 1, \ldots, m) \tag{15.6}$$

Here, x_j denotes the activity level of process j and y_j-according to Morishima – the excess profit from process j (or 'zero' if certain boundary values are reached).

Morishima then assumes that consumption demand and factor supplies are continuous functions of goods and factor prices as well as of aggregate income.

In equilibrium, no positive excess demand exists in any commodity or factor market, and no process can earn positive excess profits with regard to the single product, multiple activity technology assumed. Finally, Morishima also assumes that the equilibrium is uniquely determined. By means of an appropriate distance function as Liapunov function he then shows that the equilibrium is asymptotically stable, as long as substitution effects dominate over income effects (in the household's demand-supply Jacobean).

Hence, Morishima has formulated a cross-dual process which seems to be more classical than neoclassical in nature, particularly if it is restricted to the goods market and long-run equilibrium positions, i.e., if it is freed from the need to divide stability investigations into short-run and long-run adjustment processes as is done in the theory of temporary equilibria.

For the classical tradition, Steedman (1984) provides a brief survey concerning modern formalizations of the cross-dual dynamics. Some comments with regard to the literature considered by him are necessary, however, in order to correct the relatively pessimistic conclusions he draws. Nikaido (1978), for example, does not really consider the above type of 'classical' cross-dual dynamics. Instead he assumes a quantity system of type (15.2) which grows at full capacity in complete independence of the development of prices and profit-rate differentials. He then shows that quantity growth in itself is unstable (a known result, see Jorgenson 1960) and that this fact carries over to the dynamics of prices (whether determined through equilibrium and disequilibrium) and resulting profit rates. The lack of any feedback of differentials between the latter with regard to the conditions of supply implies, however, that such a construction *cannot* refute the conjectured tendency towards equalizing profit rates of the classics and Marx. This type of dynamic instability is modified and refined in Nikaido (1983). The independent movement of capital stock accumulation now appears only as a subcase (the third case on his p. 349). Two further cases exist where potential growth is not fully realized and where price reactions are induced by the price-dependent reallocation of a given money capital (see his equations (21) and (25)). These price-reactions, however, are not really governed by the 'law of demand' (see his equations (50), (53) and (54)) and thus represent another unmotivated departure from the basic features of classical competition. Nikaido's (1983) dynamic model undoubtedly represents an important contribution to dynamic analysis in general, but it does not seem to model properly the classical competitive process. An additional weakness of his approach is that he studies a system which is able to grow on the basis of a fixed vector of final demand and a fixed fund of money capital.

In Nikaido (1985) the classical dynamics of competition is finally modeled for a growing economy and capitalist consumption expenditure is endogenized. Yet, here again, Nikaido does not really model a classical cross-dual adjustment process, since the quantity-side of his dynamics remains basically of the type (15.2) of the dual instability situation (see his equation (19)), only modified insofar as there is a weak feedback from the price side if the savings rate s of capitalists is less than one. In the extreme case $s = 1$ (which is often referred to in Nikaido's paper) the quantity system simply grows because all net outputs are ploughed back into production such that they are fully utilized (see (15.2)). This has, of course, little to do with the classical view on competition, yet lies at the heart of the instability results which Nikaido obtains. Hence, even his latest contribution cannot be considered as a full investigation of the workability of classical cross-dual dynamics.

We shall approach Nikaido's theme in the following by Sraffa/von Neumann growth models with no capitalist consumption, but joint instead of only single product industries. The question will be examined whether pure movements of capital (with no inventory and financial constraints) into sectors with profit rates above the norm and the (sooner or later) induced falling prices of these sectors (and rising prices of other sectors) will produce convergence to the equilibrium ray of quantities and prices. Such a starting point has the advantage that it concentrates on the process of surplus distribution and activity redistribution and abstracts from all additional complexities such as flexible consumption, quantity constraints, and the like.

The situation of maximum growth and its stability is also investigated in Duménil and Lévy (1983, pp. 21 f.), now with a true process of cross-dual dynamics. There it is found by local analysis and with regard to single production that their type of crossover dynamics will not be *asymptotically* stable. (In a more recent paper, Duménil and Lévy 1983, obtain an asymptotically stable growth path, but in contrast to their earlier version this result is based mainly on a simulation study.) Yet we have already indicated that asymptotic stability is not necessarily what we should expect to find in the first instance. Stability, i.e., a self-restricted movement of quantities, prices and profit-rate differentials, may also represent an important analytical result, e.g., as a reference situation for further dynamic investigations (see Sect. 15.4).

Finally, though standard neoclassical theory mainly rests on pure price dynamics (the 'law of demand') as the proper formalization of Walras's process of tâtonnement, recent contributions even in neoclassical theory have also attempted to shift the attention to the 'law of profitability' and Walras's process of groping for a production economy in which the excess of prices over costs initiates supply responses of firms (Sonnenschein 1981, 1982; Mas–Colell 1986). Alternatively, Marshall's notion of the difference of demand and supply price is utilized to explain quantity responses on the supply side (Svensson 1984). In those studies the 'law of profitability' is taken into account in the formalization of the competitive process and Walras's price dynamics is referred to as a fast dynamics either equating supply and demand in each instant of time (Sonnenschein 1981, 1982) or occurring simultaneously with the dual dynamics, i.e., the quantity dynamics (Mas–Colell 1986). Therefore, the analysis of a cross-dual dynamics seemingly becomes more interesting even among neoclassical theorists:

> 'Much of the research... has concentrated on the limit pure price dynamics (with production, if at all there, automatically adjusted to equilibrium) or, to a lesser extent, on the limit pure quantity dynamics... The general case..., where prices and quantities stand on the same footing, is much less familiar and this in spite of the fact, as we shall see, it displays interesting and, relative to the two limit cases, novel and illuminating dynamic features.' (Mas–Colell 1986, pp. 49).

A revival of such a cross-dual dynamics is also intended with the proposed models of dynamic adjustments which we shall investigate in the following section, here however from a classical long-run equilibrium perspective.

15.4 A New Approach to the Stability of Market Economies

15.4.1 Square Joint Production Systems

Let A, B be the $n \times n$ (augmented) input and output matrices of a square joint commodity production system where workers' consumption is included in the inputs a_{ij} of goods i into sectors j. This input–output system may be regarded as a particular type of Sraffa (or von Neumann) model, where constant returns have been added as far as Sraffa's analysis is concerned. We assume for this linear model of production that there exists an equilibrium R^*, x^*, p^*:

$$(B - R^*A)x^* = Cx^* = 0, \quad p^*(B - R^*A) = p^*C = 0 \tag{15.7}$$

such that both the vector of activity levels x^* (a column vector) and the vector of prices p^* (a row) are strictly positive (>0) and are uniquely determined thereby and that $R^* > 1$ holds true for the gross rate of return R^*.

We shall use a prime ($'$) to denote the transposition of vectors and matrices. The symbol $< \cdot >$ is used to denote the diagonal matrix $< x >$ which is formed by means of given vectors x. We use a dot $(\cdot)$ to denote time derivatives of curves in $\mathbb{R}^k$ and shall abbreviate vectors of growth rates $(\dot{x}_i/x_j)_{j=1,\dots,n} =< x >^{-1} \dot{x}$ by $\widehat{x} = (\widehat{x}_j)_{j=1,\dots,n}$. Rows of matrices X are denoted by X_i and columns by X_j. Finally, Ω^k_+ is used to denote the positive orthant $\{x \in \mathbb{R}^k / x > 0\}$ of $\mathbb{R}^k$ and Ω^k the non-negative orthant $(x \geq 0)$; $x \geq 0$ is used for semi-positivity.

Our first problem is to study the stability of the above type of equilibrium R^*, x^*, p^*. (A famous and extreme kind of example for such an equilibrium is provided in Steedman's 1977, Chap. 11, discussion of positive profits with negative surplus value.) We stipulate the following *price and supply dynamics* for the above input–output system and given activities x and prices p in Ω^n_+:

$$\dot{x} = + < d^1 >< x > (B - R^*A)'p' = + < d^1 >< x > C'p' \tag{15.8}$$

$$\dot{p}' = + < d^2 >< p > (B - R^*A)x = - < d^2 >< p > Cx \tag{15.9}$$

where $d^1, d^2 \in \Omega^n_+$ are given vectors of adjustment coefficients.

The first equation states that the time rate of change of activity levels x_j is of the same sign as the term $p(B - R^*A)^j x_j = pC^j x_j$, i.e., proportional to the sum of extra profits of sector j measured by reference to the equilibrium rate R^*.[4] The

[4] Though in our chapter the proofs of the stability properties of the proposed dynamical systems are provided by referring to the equilibrium or natural profit and growth rate R^*, computer simulations have shown that the results are not invalidated if the average profit and growth rate $R(x, p) = pBx/pAx$ instead of R^* is used as benchmark in our dynamical systems (see Flaschel and Semmler 1986a, 1986b). This actual rate, of course, constitutes the more important benchmark and should be used for this purpose in the end (see Steedman 1984, p. 135, for a related observation).

second equation says that market prices react positively (negatively) if the supply $B_i x$ of commodity i falls short of (exceeds) the demand $R^* A_i x$ for commodity i, which is here *defined* by the current input requirements $A_i x$ multiplied by the steady growth factor R^*. Note that the above price changes are percentage changes.

The system (15.8), (15.9) is defined for all $x, p' \in \Re^n$. Since, however, situations where x_j and p_i are equal to zero imply $x_j, p_i = 0$, respectively, the hyperplanes which are tangent to the positive orthant Ω_+^{2n} of $\Re^{2n}$ are all invariant sets of the vector field (15.8), (15.9), i.e., none of its solution curves which start in such a hyperplane can leave this set. This implies that the positive orthant is an invariant set, too. The discussion of the stability of the equilibrium $x^*, p^* > 0$ can therefore safely be restricted to the positive orthant Ω_+^{2n} where (15.8), (15.9) can be reformulated in the following way

$$\widehat{x} = + < d^1 > (B - R^* A)' p' = + < d^1 > C' p' \tag{15.10}$$

$$\widehat{p}' = + < d^2 > (B - R^* A) x = - < d^2 > C x \tag{15.11}$$

Our cross-dual adjustment differs from Morishima's approach (15.4), (15.5), (15.6) in three regards:

(1) We use growth rates instead of time derivatives for x and p, which makes the use of non-negativity constraints redundant.

(2) We represent normal growth and profit rates explicitly in the adjustment equations (whereby we also indicate the difficulty of integrating better proxies for extra profits and excess supplies for a growing system).

(3) We do not use Morishima's numéraire $v_{\overline{n}} = 1$, since we have included workers' consumption in the input matrix A. In addition, Morishima's problem of upper constraints on activity levels is ignored here, since an integrated treatment of shortages in factor supplies should be made from the perspective of real-time dynamics by making use of a growth cycle dynamics à la Goodwin.[5] Finally,

[5] Since our classical-oriented model in prices and quantities still (1) abstracts from the role of non-reproducible inputs, inventory and financial constraints, and (2) formulates demand functions (for a growing system) in a preliminary way, our cross-dual competitive process does not yet present a dynamical process in 'real time'. As discussed above, similar complications also have led Walras to describe his process of groping in a production economy not as a process 'as it takes place effectively'. Our proposed cross-dual process however, lends itself to two possible interpretations. First, the suggested cross-dual dynamics allows an application to the Taylor–Lange iteration to an equilibrium in a planned economy, see also the remarks of Mas–Colell (1986, pp. 60) on the iterative determination of an equilibrium in a planned economy by means of the 'indirect' Walrasian instead of the 'direct', more Keynesian, method. Second our proposed dynamics provides a framework for comparing corporate quantity, price and financial planning with classical competitive adjustment processes (see, in this regard also the remarks of Clifton 1983, pp. 30/31 on 'dynamic tâtonnement' and corporate planning, and also Semmler 1985, Chap. 6). In fact, the actual pricing procedures of large multi-plant and multi-product firms can be viewed as two stage procedures where first, 'base prices' including a normal rate of return on investments are computed to be modified flexibly in a second stage by responding to changing market conditions (capturing the imbalances in the markets where the firms operate).

Morishima's consumption demand functions are also ignored in this basic treatment of adjustment for a production economy with positive rates of profit and growth.

Definition 15.1. An equilibrium z^* of a differential equation system $\dot{z} = f(z)$ is called *stable* if for every neighborhood U of z^* there is a neighborhood U_1 of z^* in U such that every solution $z(t)$ which starts in U_1 is in U for all $t > 0$.

A sufficient condition for this type of stability is the existence of a so-called Liapunov function for z^*, i.e., a continuous function on U (differentiable on $U - \{z^*\}$) such that $V(z^*) = 0$, $V(z) > 0$ if $z \neq z^*$ and $\dot{V} = \text{grad}\, V(z) \cdot \dot{z}(t) \leq 0$ in $U - \{z^*\}$ (see Hirsch and Smale 1974, pp. 185 ff., for some further technical details and explanations). Such a function will now be introduced to prove the following:

Proposition 15.2. *Any equilibrium (15.7) of system (15.10), (15.11) is stable.*

Proof. Making use of the new notation $z = (x, p')' \in \Omega_+^{2n}$ we can represent the differential equations (15.10), (15.11) in the compact form

$$\dot{z} =< d >< z > Qz \text{ or } \widehat{z} =< d > qz \tag{15.12}$$

where d is given by $(d^1, d^2)'$ and where Q stands for the matrix

$$\begin{pmatrix} 0 & C' \\ -C & 0 \end{pmatrix} = \begin{pmatrix} 0 & (B - R^* A)' \\ -(B - R^* A) & 0 \end{pmatrix}.$$

We note that matrix Q is skew-symmetric and that it does not depend on the vectors x, p (in contrast to the case where the average rate of profit is used instead of the equilibrium rate R^*).

We propose as Liapunov function around the given equilibrium z^* the function $V : \Omega_+^{2n} \to \Re$ which is defined by

$$v(z) = q'[(z- < z^* > \ln z) - (z^* - < z^* > \ln z^*)] \tag{15.13}$$

where $q \in \Omega_+^{2n}$ is defined by $q_k = d_k^1$ and $(\ln z)_i$ by lnz_i. By calculating the partial derivatives of first and second order of this differentiable function it can easily be shown that the equilibrium z^* of (15.12) is a strict local minimum of the function V and that $\text{grad}\, V(z)' = q'(E- < z > -1 < z^* >)$, E the identity matrix.

For the derivative $\dot{V}$ of V along the trajectories of (15.12) we thereby get

$$\begin{aligned} \dot{V} &= q'(E- < z >^{-1} < z^* >)\dot{z} \\ &= q'(E- < z >^{-1} < z^* >) < z >< d > Qz \\ &= q' < d > (< z > - < z^* >)Qz \\ &= (z - z^*)'Qz \\ &= z'Qz = 0 \end{aligned}$$

since $z^{*'}Q = 0$ (see (15.4)) and since Q is skew-symmetric. The function V is therefore constant along all trajectories of (15.12) in Ω_+^{2n}, which in particular means that V is a Liapunov function for z^*. □

Cross-duality in its simplest form hence gives rise to stability, but not to asymptotic stability for the equilibrium assumed. This differs from Morishima's result which, in our view, is to be attributed to the differences in the formulation of the excess supply function. To get also asymptotic stability we, however, will not introduce below a modification of the law of demand, but shall apply instead a reformulation of the law profitability, i.e., of the other part of this cross-dual adjustment process, to obtain convergence to the equilibrium z^*. To this end we assume that capitalists also take account of the sign of change of extra profits (or losses) when moving their capitals between sectors. We suggest that rising extra profits will speed up the growth rate of the respective sector, while falling extra profits will tend to reduce the growth effect of supernormal profits. Generally speaking, the growth rates of activity levels consequently should also be influenced by the distribution of signs within the vector

$$s = \frac{d}{dt}(B - R^* A)' p' = (B - R^* A)' \dot{p}' \in \Re^n \tag{15.14}$$

This vector shows the direction into which extra profits (or losses) will change at a point in time t. Integrating (15.14) into (15.8), (15.9) we get as *new dynamical system*

$$\dot{x} = + < d^1 >< x > [C' p' + \gamma s] \tag{15.15}$$
$$\dot{p}' = - < d^2 >< p > C x \tag{15.16}$$

where $\gamma > 0$ is an adjustment parameter.

To investigate the asymptotic stability of this modified cross-dual adjustment process we shall make use of the following stability concept (see Hahn 1982, pp. 749 f., and Fisher 1983, pp. 220 f., for related formulations):

Definition 15.3. The equilibrium z^* of the system (15.15), (15.16) is *globally asymptotically stable* if, and only if, for any $z(o) \in \Omega_+^{2n}$ and the trajectory $z(t, z(0))$ of (15.15), (15.16) which starts at $z(0)$ there exist scalars $\alpha^1, \alpha^2 > 0$ such that

$$\lim_{t \to 0} z(t, z(0)) = (\alpha^1 x^*, \alpha^2 p^{*'}).$$

We should note here that this stability definition is a special case of what Hahn and Fisher call quasi-global stability, appropriately applied to our present case of an equilibrium z^* which is uniquely determined up to scale factors by the rate R^*.

Proposition 15.4. *The dynamical processes (15.15), (15.16) are globally asymptotically stable with respect to their interior steady state position.*

Proof. Equation (15.14) inserted into (15.15) gives

$$s = -C' < p >< d^2 > Cx = Sx = -T'Tx \tag{15.17}$$

where $C = B - R^*A$ and $T = \sqrt{< p >}\sqrt{< d^2 >}C$. The matrix S is therefore negative semi-definite, i.e., $x'Sx = -(Tx)'(Tx) \leq 0$ for all $x \in \Re^n$. Furthermore, $x'Sx = 0$ if, and only if, $Cx = 0$, i.e., $x = \alpha^1 x^*$ for some $\alpha^1 > 0$, as long as $p > 0$ and $x > 0$ can be ensured. With the above notation, the dynamics (15.15), (15.16) can therefore be represented in compact form by

$$\widehat{z} = < d > \begin{pmatrix} \gamma S & C \\ -C & 0 \end{pmatrix} z = < d > Q(\gamma) z \tag{15.18}$$

where as before $z = (x, p')', d = (d^1, d^2)'$ and where $Q = Q(0)$ represents the case we have treated in Proposition 15.2. For the more general system (15.18) we now have the property

$$\overline{Q}(\gamma) := \frac{1}{2}(Q(\gamma) + Q(\gamma)' = \begin{pmatrix} \gamma S & 0 \\ 0 & 0 \end{pmatrix} \tag{15.19}$$

where $\overline{Q}(\gamma)$ is negative semi-definite.

Utilizing the Liapunov function (15.13) we get, with regard to (15.18),

$$\begin{aligned} \dot{V} &= q'(E- < z >^{-1} < z^* >)\dot{z} \\ &= (z - z^*)' Q(\gamma) z = z' Q(\gamma) z - z^{*'} Q(\gamma) z \\ &= z Q(\gamma) z = z' \overline{Q}(\gamma) z \\ &= y x' S x \leq 0. \end{aligned}$$

since $z^{*'}Q(\gamma) = O$ because of $S = -C' < p >< d^2 > C$. This inequality implies that the sets $V^{-1}([0, c]), c > 0$ are positively invariant with regard to the dynamics (15.18), i.e., no trajectory which enters such a set can leave it later on. We note that each set $V^{-1}([0, c])$ is a compact subset of Ω^{2n}_+. This is easily deducted from (15.13), since (15.13) is an additive combination of the strictly convex functions.

$$V^i(z_i) = q_i(z_i - z_i^* \cdot \ln z_i - (z_i^* - z_i^* \cdot \ln z_i^*)) \tag{15.20}$$

Following Hahn (1982, p. 750) it is now easy to demonstrate that process (15.18) is *quasi-globally stable*, i.e., that all limit points of its trajectories are points of rest (equilibria). To this end we have to show that V_{oz} is convergent for the trajectories z of (15.18) and constant if, and only if, such a trajectory describes a point of rest. The first of these conditions has already been shown (since V is monotonically decreasing along all solution curves z of (15.18) and bounded from below. To show the second condition, let us assume that V_{oz} is constant for an entire orbit $z(t) = (x(t), p(t)')', t \geq 0$ in Ω^{2n}_+. The condition $\dot{V} = 0$ implies $x'Sx$ for this orbit, which because of the properties of the sets $V^{-1}([0,c])$ (i.e., because of $x, p > 0$ throughout) implies $0Cx = 0$ and thus $x = \alpha^1 x^*$ for a positive scalar α^1. Inserting this result into (15.16) gives $\dot{p} \equiv 0$, i.e.,

$$\widehat{z} =< d^1 > (B - R^* A)' p' = \text{ const.}$$

Multiplying this equation from the left by $\omega^1 = x^{*'} < d^1 >^{-1}$ gives $\omega^1 \widehat{x} = 0$. Since, however, a constant growth rate $\widehat{x}_j > 0$ is incompatible with our result that the compact sets $V^{-1}([0,c])$ are invariant, we get $\widehat{x}_j = 0$ for all j. Hence: $\dot{x} \equiv 0$, since $\omega_j > 0$ for all j. By Theorem T.1.4 in Hahn (1982, p. 751) the process (15.18) is therefore quasi-globally stable.

By our uniqueness assumption we furthermore know for these limit points of orbits and points of rest $z = (x, p')' > 0$:

$$z = (\alpha^1 x^*, a^2 p^{*'})' \text{ for some } \alpha^1, \alpha^2 > 0.$$

The scalars α^1, α^2 are uniquely determined by the expressions

$$\omega^1 \ln x(0)/\omega^1 \ln x^* \text{ and } \omega^2 \ln p(0)'/\omega^2 \ln p^{*'}$$

where $\omega^1 = x^{*'} < d^1 >^{-1}$ and $\omega^2 = p^* < d^2 >^{-1}$, cf. also the following remark. Process (15.18) is thus globally asymptotically stable. □

If capitalists also pay regard to the time rate of change of extra profits where moving their capital from) one sector to another, we consequently know that the stipulated price reaction of the 'market' will lead to uniform profitability and growth in the limit and to relative prices and activity levels which are those of the given equilibrium $x^*, p^* > 0$. Our cross-dual adjustment process is convergent if $\gamma > 0$ holds true (it will be divergent if $\gamma < 0$ is true.

Remark 15.5. The dynamics (15.15), (15.16) determines only relative activity levels and relative prices which is as it should be in an approach which neither includes a disequilibrium theory of growth nor of inflation. Furthermore, the above defined two vectors ω^1, ω^2 are easily shown to fulfill

$$\omega^1 \widehat{x} = 0, \omega^2 \widehat{p}' = 0,, \text{ i.e., } \omega^1 \ln x = c^1, \omega^2 \ln p' = c^2$$

where $c^1, c^2 > 0$ are given constants which depend only on the initial values $x(0), p(0)$ of each trajectory (cf. the definition of the two scalars α^1, α^2 in the preceding proof). The above conditions of invariance state that certain weighted averages of the percentage changes of activity levels or prices are zero throughout or that activity levels and prices stay within the sets $\{\exp \overline{x}/\omega^1 \widetilde{x} = c^1$, $\{\exp \widetilde{p}'/\omega^2 \overline{p}' = c^2$, which are the images of the hyperplanes perpendicular to ω^1, ω^2 under the exponential mapping. Such normalization rules differ from the usual ones which restrict prices or activity levels to certain simplices. They are, however, not inferior to these alternative ways of normalizing activities and prices as there is no natural condition of invariance (and since 'boundedness' of prices and activities is ensured because of the invariance of $V^{-1}([0, c])$).

Because of this last fact it might even be claimed that the explicit or implicit existence of a numéraire is unnecessary for an investigation of the stability properties of adjustment processes with regard to relative prices and activity levels. Following Fisher (1983, cf. in particular p. 25), a proof of quasi-global stability, a compactness argument, and a demonstration of (local) uniqueness of rest points can be regarded as sufficient to provide an analysis of the asymptotic properties of *adjustment processes*. The stability properties of adjustment processes may change if an *a priori* condition on invariance is added and if the adjustment process is modified accordingly to suffice this invariance condition. Yet, such a procedure is arbitrary as long as the true conditions which restrict prices and activities in such an economy are not revealed thereby. The adoption of a numéraire commodity or of another condition of invariance (e.g., of the above type) therefore only contributes to an analysis of the robustness of the analyzed stability with regard to such additional (and arbitrary) restrictions. Such an analysis will be bypassed in this chapter, which concentrates on the above three steps of stability analysis in the same way as they are applied by Fisher (1983).

15.4.2 Process Extinction

The results of the preceding section can be applied to both Sraffa's and von Neumann's types of analysis (if wages are paid *ex ante* in both cases). In principle, this remains true also for the following more general equilibrium of a rectangular $n \times m$ linear input–output model A, B (with $A^j, B^j \geq 0$ for all sectors $j = 1, \ldots, n$).

Assume, that the linear model of production A, B has an equilibrium $R^* > 1$, $x^*, p^* \geq 0$ which fulfills $p^* > 0$, i.e., which is characterized by

$$o^*C \leq 0, Cx^* = 0, p^*Bx^* > 0 \tag{15.21}$$

The number of commodities (rows) n may now differ from the number of activities m.

An important special case of this situation is the case of multiple, but single product activities where the non-substitution theorem holds true. In this case, as well as in the above more general case, an important question is whether the adjustment processes considered in the last section will not only tell something about the process of equalizing profit rates but also about the extinction of inferior processes $p^*C^j < 0$ (where $x_j^* = 0$ holds true). Note that we do not allow for product extinction (free goods) in the present section and that a uniqueness assumption has no longer been made.

One analytical difficulty in treating this case along the lines of the preceding paragraph is that the domain of definition of the Liapunov function (15.13) must now be extended in a relatively complex fashion. To examine asymptotic stability it should at least include our reference equilibrium (15.21), i.e., boundary values of Ω^{n+m} for those components where process extinction may occur. It is easily seen, however, that the function (15.13) – which depends on the choice of z^* – has a well-defined continuous extension with regard to this new situation, since $z_j^* \cdot \ln z_j$ is identically zero for these components (and $z_j > 0$). The functions (15.20) must therefore be supplemented by the following functions to provide a full picture of the component-by-component forms of the function V:

$$v^i(z_i) = q_i z_i \text{ for } z_i^* = 0$$

On the other hand, function (15.13) cannot be extended to situations $z_i = 0$ with $z_i^* \neq 0$. As domain of definition for (15.13) we consequently now have

$$\Omega_{+,0}^{n+m} = \{z \in \Re^{n+m} / z_j > 0 \text{ if } z_j^* > 0, z_j \geq 0 \text{ if } z_j^* \geq 0\}$$

(cf. Rouche et al. 1977, pp. 263 f., for a related approach).

Proposition 15.6. *The equilibrium (15.21) is stable with regard to the adjustment process (15.8), (15.9) and the domain of definition $\Omega_{+,0}^{n+m}$. The stability is asymptotic for all components j where $p^* C^j < 0$ holds true (which may be a proper subset of $\{j / x_j^* = 0\}$).*

Proposition 15.7. *The adjustment process (15.15), (15.16) is quasi-globally stable, i.e., all trajectories which start in $\Omega_{+,0}^{n+m}$ have only equilibria as limit points.*[6]

The limit set of each trajectory (i.e., the set of points where the trajectory converges to) thus consists of rest points only, but it may now contain different economic *equilibria.*

Proof. Recall first that there is a well-defined continuous extension of the Liapunov function (15.13) to the domain $\Omega_{+,0}^{n+m}$ which contains the given equilibrium $z^* = (x^*, p^{*'})'$. This function allows the same calculations as in the proofs of Propositions 15.2, 15.4 except that we now get:

$$z^{*'} Q(\gamma) = z^{*'} \begin{pmatrix} \gamma S & C' \\ -C & 0 \end{pmatrix} = (-p^* C, 0) \geq 0 \tag{15.22}$$

This implies $z^{*'} Q(\gamma) z = -p^* C x$ which gives

$$\dot{V} = (z - z^*)' Q(\gamma) z \leq z' Q(\gamma) z = \gamma x' S x = 0 \tag{15.23}$$

i.e., an additional inequality term in the estimation of the behavior of V along z. The proof of Proposition 15.6 is then completed by observing that the case $\gamma = 0$ now gives rise to $\dot{V} < 0$ as long as $p^* C x < 0$, i.e., $x_j > 0$ for $p^* C^j < 0$.

[6] See Fisher (1983, Appendix) for further details on quasi-global stability.

In the case $\gamma > 0$ (Proposition 15.7), we first note that $x'Sx = 0$ is again equivalent to $Cx = 0$. This remains true since $V^{-1}([0,c])$ is still compact and invariant and since it has a positive distance from the boundary of Ω_+^{n+m} for all those components z_j where $z_j^* > 0$ is true. The limit points of trajectories which start in $V^{-1}([0,c])$ therefore must have positive components for all j where $z_j^* > 0$, i.e., in particular all prices must stay positive even in the limit. Hence: $x'Sx = -x'C' < p >< d^2 > Cx = O$ iff $Cx = 0$. Note, however, that $Cx = 0$ no longer implies that x is proportional to x^*, as is obvious from the possibility of switches of techniques.

To show the quasi-global stability of process (15.15), (15.16) in this generalized situation there remains again to be shown that V is constant if, and only if, z is a point of rest of (15.15), (15.16), (cf. Hahn 1982, p. 751). Note that Hahn's proof of quasi-global stability also applies to the special type of 'orthant' $\Omega_{+,0}^{n+m}$ on which the above Liapunov function had to be defined, since all limit points of solution curves which start in $V^{-1}([0,c])$ must be contained in this set.

Assume now that $\dot{V} = 0$ for an entire orbit $z(t) = (x(t), p(t)')', t \geq 0$: By (15.23) we then get

$$z^{*'}Q(\gamma)z = 0 \text{ and } x'Sx = 0.$$

This implies $p^*Cx = 0$, i.e., $x_j = 0$ for all j with $p^*C_* < 0$. Because of $Cx = 0$ we get from (15.16) the result $p \equiv 0$ or $p \equiv$ const and $s = 0$, cf. (15.15). There remains the dynamics

$$\dot{x} =< x >< d^1 > C'p' \gtreqless 0 \text{ for } pC \gtreqless 0$$

(if $x_j > 0$; the cases $x_j = 0$ imply already a situation of no change with regard to these components). The case $pC^j > 0$ can, however, be excluded by observing that a constant vector of prices p would then imply a constant and positive rate of growth for x_j in contradiction of the fact that $z(t), t \geq 0$ cannot leave $V^{-1}([0,c])$. And the remaining possibility $pC^j < 0, x_j > 0$ is then also not compatible with the assumed circumstances, since it would imply

$$\dot{V} = \sum_{x_j > 0} q_j(x_j - x_j^*) \cdot \widehat{x}_j \neq 0$$

because of $\widehat{x}_j = \text{const} < 0$ for these components j, i.e., $\ddot{V} = \sum q_j \cdot \dot{x}_j \cdot \widehat{x}_j < 0$.

In sum, we therefore get $\dot{z} = 0$ for the above orbit $z(t)$, i.e., this orbit describes a point of rest z^{**}, which – as the proof has shown – fulfills $p^{**}C \leq 0$, $Cx^{**} = 0$, $p^{**} > 0$, $x^{**} \geq 0$ and $p^{**}Bx^{**} > 0$. □

Corollary 15.8.

(1) The equilibria x^{**}, p^{**} of the system A, B which are limit points of the solution curves of process (15.15), (15.16) are of the same type as the given equilibrium x^*, p^* (see (15.21)).

(2) *The combinations* (x^*, p^{**}) *and* (x^{**}, p^*) *are also equilibria with regard to the rate* R^**. They form a convex subset* Z^* *of the set of all equilibria of the natural rate* R^**.*[7]
(3) *An activity which is inferior with regard to an equilibrium* $z \in / Z^*$ *will not be operated by any equilibrium* $z \in / Z^*$*.*
(4) *Denote by* T^* *the (uniquely determined) maximum set of activities which are operated by the set of equilibria* Z^**. Then:* $j \in T^* \to x_j^{**} > 0$ *for the equilibria which are limit points of (15.15), (15.16). This adjustment process therefore does not (and cannot) discriminate between the optimum activities of the different equilibria in the set* Z^**, but leads always to an equilibrium where these activities are all jointly operated.*

Proof. The above assertions follow from the facts (1) that the functions (15.20) are strictly convex with the absolute minimum z_i^*, (2) that they approach infinity if z approaches zero (for $z_i^* > 0$) and that (3) the invariant sets $V^{-1}[0, c]$) around the equilibrium (15.21) are all compact. This implies that positive components z_i^* must lead to positive components z_i^{**} by our adjustment process (15.15), (15.16) which is *independent* of the particular choice of the equilibrium z^*. Note here that the function V depends on this choice, yet in a manner which is irrelevant for the truth of Proposition 15.7. □

15.4.3 Product Extinction

We have shown stability and even global asymptotic stability for general joint production where free goods have been eliminated from consideration (if the considered equilibrium is uniquely determined). It will be shown below that these assertions cannot be extended to the case of free goods ($p_j^* = 0$) where indeed a modification of the law of excess demand is needed (instead of the law of profitability) to imply their extinction by such a disequilibrium dynamics.

Because of $p^* > 0$, i.e., $Cx^* = 0$ we had the following simple equation in Sect. 15.4. to prove the monotonicity of V along the orbits of (refad10), (15.16), cf. (15.22):

$$z^{*'} Q(\gamma) z = -p^* C x \geq 0$$

In the general case $x^* \geq 0$, $p^* \geq 0$ the expression for $z^{*'} Q(\gamma) z$ however reads

$$-p^* C x + p C x^* - \gamma x^* C' < p >< d^2 > C x \qquad (15.24)$$

Here, only the first two vectors will be unambiguously non-negative, while for the last vector it can, in fact, be shown that it must be non-positive in a small neighborhood of $(x^*, p^{*'})'$, since $C_i x$ and $C_i x^*$ will be of the same sign for all i where $C_i x^* < 0$ in such a neighborhood.

[7] According to Fujimoto (1975), the natural rate R^* is uniquely determined, because of our assumption $p^* > 0$.

The expression (15.24) shows, however, *that Proposition 15.6 can be generalized* to this most general type of equilibrium situation $Cx^* \geq 0$, $p^*C \leq 0$, $p^*Bx^* > 0$ of the joint product system A, B, i.e., the original kind of adjustment process $y = 0$ is stable for any equilibrium R^*, x^*, p^*. To obtain asymptotic stability in the presence of free goods the following modification of this original process is appropriate (this modification has been proposed to us by R. Franke) :

$$\dot{x} = + < d^1 >< x > C'p' \qquad (15.25)$$

$$\dot{p}' = - < d^2 >< p > [Cx + \delta v] \qquad (15.26)$$

where $v = \frac{d}{dt}Cx = C\dot{x}$ and $\delta > 0$. The time rate of change of excess supplies is here assumed to exercise an extra influence on our original dynamics (15.8), (15.9) – now with regard to price instead of quantity adjustments. Instead of (15.18) we here get

$$\dot{z} =< z >< d > \begin{pmatrix} 0 & C' \\ -C & \delta U \end{pmatrix} z$$

with a matrix U which is given by $U = -C < x >< d^1 > C'$. *In the case of no inferior activities* ($x^* > 0$) *this dynamics may then be treated in the same way we have treated the opposite case* $p^* > 0$, i.e., this adjustment will be quasi-globally stable and will now exhibit product- instead of process-extinction. Modifications of our simple version of the law of demand may therefore be exploitable for the treatment of free goods. Such modifications are, however, not of central importance in this chapter which focuses on capital movements and their stabilizing properties. Furthermore, the analysis of both, (15.15) and (15.26). i.e., the investigation of a simultaneous operation of the two additional influences is not straightforward and must be left aside in this chapter.

15.5 Conclusions

We have derived in Sect. 15.4 some stability results for processes of 'classical competition' around von Neumann equilibria of the multiple activity type. These processes are also related to some ideas of Walras (but not of Walrasian economics) on the adjustment to equilibria in a production economy, now, however, from a long-run perspective. In contrast to other dynamical processes as, for example, formulated by Morishima (1976, 1977); Mas–Colell (1974, 1986), our cross-dual process is stabilized through the 'law of profitability' and the sensitivity of the producers to profit changes. Such an approach may be helpful to overcome the problematic separation into short-run and long-run equilibrium dynamics of the standard neoclassical type as discussed in Sect. 15.2. To prove the stability of such unified adjustment processes to prices of production (and to steady growth) may be of great importance, since *'the theory of value is not satisfactory without a description of the adjustment processes that are applicable to the economy and the way in which individual agents adjust* *to disequilibrium. In this sense, stability analysis is of more than merely technical interest. It is the first step in a reformulation of the theory of value.'* (see Fisher 1983, p. 16) – which need not be the theory of value which Fisher has in mind.

On the other hand, our results can bear comparison with stable dual adjustment rules of Keynesian origin. A modern version of such dual adjustment rules from the perspective of input–output models has, e.g., been provided by Fukuda. Fukuda (1975) avoids the saddlepoint property of dynamic Leontief models by claiming that two of its implicit (and ideal) assumptions, those of a fully utilized capital stock and of expectations of the myopic perfect foresight variety should be modified. As an alternative to (15.2), (15.3) he therefore considers, e.g., the following stable adjustment process

$$\dot{x} = < d^1 > [Ax + \overline{g}Bx + \overline{f}x]$$
$$\dot{p}' = < d^2 > [A'p' + \overline{r}B'p' + \overline{w} - p']$$

where $< d^1 >< d^2 >$ are again diagonal matrices of adjustment coefficients and where the rates $\overline{g}, \overline{r}$, final demand $\overline{f} \in \Re^n$ and primary costs $\overline{w} \in \Re^n$ are given exogenously. This attempt that avoids the instability results of efficient capital accumulation described in Sect. 15.2 (for sufficiently small $\overline{r}, \overline{g}$) is, however, immediately addressed to quite modern conditions: output adjusts solely according to excess capacity and pricing is based on full-cost calculations (including a target rate of profit). There is no crossover dynamics involved.

Dynamical processes of the above dual type can be considered to follow a fast dynamics of Keynesian type where quantity imbalances are mapped into quantity changes and differences of price and cost are mapped into price changes. These processes appear, if written in an appropriate form (Flaschel and Semmler 1986a), in the diagonal of the system's matrix (when the system is written in the compact form $\widehat{z} = Qz$) whereas our proposed dynamics appears in the side diagonal, since it is a cross-dual dynamics. In comparison to the dual (Keynesian) dynamics we propose therefore to interpret our dynamics as slow and long-run dynamics whereas the dual dynamics can be called a fast and short-run dynamics. Both types of dynamics do not seem to be exclusive but rather complementary (though the stability properties of a system including both types of dynamics has not been explored so far).

Finally, this chapter also provides a basis for considering the stability of von Neumann equilibria which to our knowledge has been rarely investigated in the literature (see Burley 1974, for an isolated attempt to test the stability properties of a purely formal feed-back mechanism for the von Neumann model by means of computer simulations). And since von Neumann solutions are generalized eigensystems, our procedure therefore also suggests a mathematical method of finding the right- and left-hand eigen-solutions of general input–output systems by means of a single iterative procedure if the maximum eigenvalue is already known or if this value is also iterated by means of the reciprocal value of the average rate of profit (see Flaschel and Semmler 1986b, for details and computer simulations and note that no proof of convergence of the latter process has been supplied so far).

References

Aoki, M. (1977). Dual stability in a Cambridge-type model. *Review of Economic Studies, 44*, 143–151

Burley, S. P. (1974). Dynamic Generalizations of the von Neumann Model. In J. Los & M. W. Los (Eds.), *Mathematical models in economics*. Amsterdam: North Holland

Burmeister, E. et al. (1968). A Note on the global stability of a simple growth model with many capital goods. *Quarterly Journal of Economics, 82*, 657–665

Burmeister, E. et al. (1973). 'The saddle-point property' and the structure of dynamic heterogeneous capital good models. *Econometrica, 41*, 79–95

Caton, C. & Shell, K. (1971). An exercise in the theory of heterogeneous capital accumulation. *Review of Economic Studies, 38*, 14–22

Clifton, J. (1983). Administered Prices in the Context of Capitalist Development. *Contributions to Political Economy, 2*, 23–38

Duménil, G. & Lévy, D. (1983). *'La concurrence capitaliste: un processus dynamique*. Paris, Mimeo: OFCE and CEPREMAP

Duménil, G. & Lévy, D. (1984). *The dynamics of competition. A restoration of classical analysis*. Paris, Mimeo: CEPREMAP

Fisher, F. M. (1983). *Disequilibrium foundations of equilibrium economics*. Cambridge, UK: Cambridge University Press

Flaschel, P. & Semmler, W. (1985). The dynamic equalization of profit rates on multiproduct industry levels. Berlin: Freie Universität. Discussion Paper

Flaschel, P. & Semmler, W. (1986a). The dynamic equalization of profit-rates for input–output models with fixed capital. In W. Semmler (Ed.), *Competition, instability, and nonlinear cycles*. Heidelberg: Springer

Flaschel, P. & Semmler, W. (1986b). Cross-dual dynamics for Sraffa/von Neumann models with a nonsubstitution property. Mimeo: Universität Bielefeld

Fujimoto, T. (1975). Duality and uniqueness of growth equilibrium. *International Economic Review, 16*, 781–791

Fukuda, W. (1975). The output adjustment mechanism in a multisectoral economy. *Kobe University Economic Review, 21*, 53–62

Hahn, F. (1966). Equilibrium dynamics with heterogeneous capital goods. *Quarterly Journal of Economics, 80*, 633–646

Hahn, F. (1970). Some Adjustment Problems. *Econometrica, 38*, 1–17

Hahn, F. (1982). Stability. In K. J. Arrow & M. D. Intrilligator (Eds.), *Handbook of mathematical economics*. Amsterdam: North Holland

Hirsch, M. W. & Smale, S. (1974). *Differential equations, dynamical systems, and linear algebra*. New York: Academic

Jorgenson, D.W. (1960). A dual stability theorem. *Econometrica, 28*, 892–899

Marx, K. (1959). *Capital, Vol. III*. London: Lawrence and Wishart

Mas–Colell, A. (1974). Algunas observaciones solere la theoria del tâtonnement de Walras en economias productivas. *Anales de Economia*, 21–22, 191–224

Mas–Colell, A. (1986). Notes on price and quantity tâtonnement dynamics. In H. Sonnenschein (Ed.), *Models of economic dynamics* (pp. 49–68). Heidelberg: Springer

Morishima, M. (1976). *The economic theory of modern society*. Cambridge, UK: Cambridge University Press

Morishima, M. (1977). *Walras' economics*. Cambridge, UK: Cambridge University Press

Nikaido, H. (1978). *Refutation of the dynamic equalization of profit rates in Marx's scheme of reproduction*. Mimeo: University of Southern California

Nikaido, H. (1983). Marx on competition. *Zeitschrift für Nationalökonomie, 43*, 337–362

Nikaido, H. (1985). Dynamics of growth and capital mobility in Marx's scheme of reproduction. *Zeitschrift für Nationalökonomie, 45*, 197–218

Rouche, N. et al. (1977). *Stability theory by liapunov's direct method*. Heidelberg: Springer

Semmler, W. (1985). *Competition, monopoly and differential profit rates*. New York: Columbia University Press
Sonnenschein, H. (1981). Price dynamics and the disappearance of short-run profits. *Journal of Mathematical Economics, 8*, 201–204
Sonnenschein, H. (1982). Price dynamics based on the adjustment of firms. *American Economic Review, 7*, 1088–1096
Steedman, I. (1977). *Marx after sraffa*. London: NLB
Steedman, I. (1984). Natural prices, differential profit rates and the Classical competitive process. *The Manchester School, 52*, 123–140
Svensson, L. E. O. (1984). Walrasian and Marshallian stability'. *Journal of Economic Theory, 34*, 371–389
Walras, L. (1954). *Elements of pure economics*. London: George Allen and Unwin
Woods, J. E. (1978). *Mathematical economics*. London: Longman
Yaffé, W. (1967). Walras' theory of tâtonnement: A critique of recent interpretation. *Journal of Political Economy, 75*, 1–19

Chapter 16
Composite Classical and Keynesian Adjustment Processes

16.1 Introduction

This chapter attempts an integration of Keynesian dual and Classical cross-dual micro-dynamic adjustment processes in the framework of a standard Leontief model. It investigates why strategies which are capable of proving stability for each separate case cannot in general successfully be applied to the composite system, where both prices and quantities are each revised on the basis of two instead of only one principle, namely supply/demand – as well as price/cost – discrepancies. It will be shown that significant limits to the adjustment speeds in the Classical domain have to be postulated in order to prove stability for the composite dynamics by means of the standard tools of the Walrasian tâtonnement literature. In view of these results an alternative approach to the stability of such composite systems is then introduced and applied to this system. This approach takes explicitly into account the type of composition of our dynamic system, i.e., its set of negative feedback mechanisms and the various interactions that may in addition exist between such substructures, which makes this approach of great methodological interest.

Our central findings are that there exist three different ways which allow to prove stability for our composite Keynesian/Classical structure (diagonal dominance, quasi-negative definiteness and the above new approach with a two-level type of stability analysis). In each of these approaches, however, we have to assume relatively narrow limits for the strength of the Classical component to obtain a stable composite dynamics. In contrast, no such narrow restrictions can be detected when the eigenvalues of numerical examples are calculated for a wide range of adjustment coefficients, even though counterexamples to stability do indeed exist then as well in other cases (see the mathematical appendix, subsection 1).

The exact limits for the stability of our composite system therefore remain an open question in the present chapter. Their determination may, however, be subordinate to another problem, which is the need for a more developed analysis of Classical dynamics itself before the stability properties of its integration with Keynesian types of adjustments processes are discussed in more depth.

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_16,

16.2 Notes on the Literature

This chapter deals with the stability properties of different types of micro-dynamic adjustments which arose from different theoretical traditions in the history of the economic analysis of market systems. In particular it will treat Classical and Keynesian micro-dynamic adjustment processes.

Already in Classical Political Economy an important type of micro- dynamical adjustment for competitive market systems was considered. This competitive process was later stylized as an adjustment process called 'cross-dual' by Morishima (1976, 1977) and 'cross-field' dynamics by Goodwin (1970).

According to the Classicals the dynamics of market systems can be formulated as follows: (a) the output of a commodity is expanded or reduced (through entry or exit of firms) whenever the excess of price over cost (including normal profits) is positive or negative ('law of excess profitability'); and, (b) the price of a commodity is raised or lowered whenever there is an excess demand or supply on the market ('law of excess demand'). Extensive verbal formulations of this dynamics can be found not only in Smith (1974, Chap. 7), Ricardo (1951, Chap. 4), Marx (1967, Chap. 10) , but also in Walras (1977, Chaps. 12, 18) and in Marshall (1947, Chaps. 3 and 5). Mathematical formulations of such dynamics have been provided, for different variations of this Classical dynamics, in more recent times, for example by Goodwin (1953, 1970, 1988), Goodwin and Punzo (1986), Morishima (1960, 1976, 1977), Duménil and Lévy (1987a,b), Franke (1987), and Flaschel and Semmler (1986, 1987) (cf. also Mas–Colell (1986) who discusses the stability of the two components of our composite system from the perspective of Walrasian general equilibrium theory).

In modern neoclassical theory, however, since J. Hicks's and P. Samuelson's writings in the 1930s and 1940s this two-sided dynamical process is generally reduced to a one-sided process of price adjustment, i.e., the 'law of (excess) demand', only. This price dynamics has been shown to be asymptotically stable, e.g., under the assumption of gross substitutes. However, due to the possibility of very general excess demand functions, it proved to be unstable for a wide class of economies.

This type of price dynamics is motivated by means of the celebrated price-tâtonnement dynamics. Quantities, i.e., supply and demand – in contrast to prices – are assumed in this mechanism to adjust infinitely fast to each new price vector. There is in particular no dynamics formulated for the adjustment of supply, i.e., the 'law of excess profitability' is neglected, and out of equilibrium behavior of economic agents – facing possibly quantity or income constraints – is not allowed for. Economic systems with this dynamic adjustment rule for the short run have been called by J. Hicks 'flexprice' systems, cf. Hicks (1965, Chap. 6). In temporary equilibrium theory it is then simply assumed that this short-run price dynamics is sufficiently stable, so that supply and demand can be thought to be in equilibrium in every single period, while in the long-run the capital stock will be adjusted to its steady state value. Time is thereby dichotomized into hypothetical time where fast adjustments to temporary equilibria are taking place and real time where the dynamical forces of evolution are supposed to work adjusting the economy toward a long-run growth path. Such formulations of short-run dynamic adjustments – and more so its long-run version – are considered unsatisfactory, however, particularly in the Keynesian and post-Keynesian tradition.

In Keynesian economics another dynamic process has been favored. This type of dynamic process has been called 'dual dynamics' by Morishima (1976, 1977). The dynamic process can basically be stylized as follows: (a) quantities change due to excess demand (the output reaction of already established firms) and (b) prices change due to the difference of prices and (marked-up) costs. This latter price adjustment procedure is based on the assumption of a discretionary price setting behavior of large firms (oligopolized industries). Such a system has been called by Hicks a 'fixprice' system where 'we no longer assume that the system is in equilibrium in every single period', Hicks (1965, p. 82). In the fixprice system imbalances of supply and demand cause quantities to change and prices are determined from 'outside the model. All that is said about prices is that they must cover cost; more strictly, that a thing will not be produced unless it is profitable to produce it', Hicks (1965, p. 78). This type of price dynamics has been formalized later on by means of the aforementioned mark-up or target rate of return pricing according to which prices respond solely to the difference of marked-up costs and current prices [for details of such a pricing procedure, cf. Kaldor (1985) and Semmler (1984)].

Though there is considerable doubt of whether this dual dynamics can already be found in Keynes' 'General Theory', the 'Keynesian Revolution', however, is usually associated with it and as Hicks already mentions 'there is no question that ... Keynes was moving in the direction of the new method', Hicks (1965, p. 77). Leijonhuvud goes a step further than Hicks and writes more distinctively that 'The main innovation – and virtually the only major innovation – attempted in the 'General Theory' was the effort to provide a systematic analysis of the behavior of a system that reacts to disturbances through 'quantity adjustments' rather than through price-level or wage-rate adjustments', Leijonhuvud (1968, p. 24). This quantity adjustment process has become an essential element in non-Walrasian models on quantity rationing and disequilibrium analysis, yet, as Drazen states 'dynamics in non-Walrasian models is an open area deserving extensive further study', Drazen (1980, p. 303).

Early dynamic formalizations of Keynesian quantity adjustments – decoupled from (but contrasted with) a corresponding type of price dynamics – were given for Leontief-systems by Jorgenson (1960) and others in the form of the so-called dual instability theorem. However, it has been recognized meanwhile that two very restrictive assumptions: (a) full utilization of capacity and (b) perfect foresight, are the basis for the dual instability assertion put forward in this literature.[1] Reconsiderations of this formalization of a dual dynamics of Keynesian type are found in the work of Morishima (1976, 1977), Goodwin (1970, 1988), Aoki (1977), Fukuda (1975) [cf. again also Mas–Colell (1986) for a different interpretation of this dynamics]. Here, for the most part, the two aforementioned assumptions are

[1] This is an early example of a saddle-point instability, the now favored type of dynamics in approaches which make use of 'rational' expectations.

dropped. Our formalization of the dual dynamics in this chapter is based on the work of these authors, in particular on Morishima's and Fukuda's analysis of this dynamics.

In the remainder of the chapter, in Sect. 16.2, the dual dynamics will be introduced formally and its relation to the cross-dual dynamics is considered. Section 16.3 of the chapter elaborates composite or aggregate versions of the cross-dual and dual dynamics generally separately discussed in the history of economic analysis. As will be shown, traditional methods of stability analysis can demonstrate stability for the dynamic behavior of our composite system only under restrictive assumptions on the reaction coefficients and also on the structure of the model. Section 16.4, a new stability method in the tradition of Liapunov's direct approach is therefore introduced for studying the dynamics of such composite systems. This method works with vector Liapunov functions and shows how conclusions may be drawn with respect to the aggregate system by means of its subsystems and its interconnections. In this way, in Sect. 16.5, an alternative stability formulation for our composite Keynesian-Classical system can be provided for certain combinations of adjustment speeds. Computer studies in Sect. 16.6 explore some further conjectures and suggest (in combination with the experience from many eigenvalue calculations) that stability regions with regard to adjustment speeds may be much larger than we are able to prove in the main part of the chapter. Nevertheless – as will become clear from the mathematical appendix, subsection 1, to this chapter – counterexamples to stability do exist, even if asymmetries due to differences in profit- and growth rates and different speeds of adjustment are assumed away.

The proposed integration of Keynesian and Classical views on micro-economic dynamics is therefore not without problems. In this regard the various difficulties which we encounter in the following sections in our view suggest in the end that the next important step should not be the determination of the exact limits of stability for our present form of composite system but should rather attempt an improvement of the stability properties of the Classical cross-dual component first. In the course of writing this chapter it became more and more apparent that the Classical dynamics is still so unfinished in its basic formulation that the addition of the stable Keynesian feedback mechanism – which is not so intimately related to the Classical one as it appeared at first – is generally insufficient to imply overall stability in cases where the Classical component is sufficiently pronounced.[2]

16.3 Dual Dynamics

Following Morishima's work on dual adjustment processes, Fukuda (1975) (cf. also Aoki 1977) investigated further the stability properties of such output and price adjustment mechanisms of the Keynesian type for a multi-sectoral economy. In the

[2] See Flaschel and Semmler (1987) for a more detailed analysis of the Classical dynamics.

second part of his chapter he, in particular, explores the stability of the following dual process (his stock matrix $K = A$ for simplicity):

$$\dot{x} = d_{11}(Ax + gAx + c - x) = d_{11}(C(g)x + c) \tag{16.1}$$
$$\dot{p}' = d_{22}(A'p' + rA'p' + w' - p') = d_{22}(C(r)'p' + w'), \tag{16.2}$$

Here, d_{11}, d_{22} are diagonal matrices with positive diagonals, representing adjustment speeds. The $n \star n$-matrix A is the usual intermediate input matrix, g the rate of growth, c (a column) the n-vector of final consumption, and r and w denote the rate of profit and the n-vector of wage payments per unit of output (a row). The n-vectors x, p (corresponding to c, w) as usual stand for activity levels and prices, respectively, and $\dot{x}, \dot{p}$ denote their time derivatives. Note finally that we shall make use of the abbreviations $C(i) = (1 + i)A - I, i = r, g$ in the following.

For the output dynamics (16.1) thus holds that a positive (negative) excess demand – demand (given by $(1 + g)Ax + c$) minus supply x – will increase (decrease) output and for the price equation (16.2) a markup (or target rate of return) pricing is assumed where an excess of computed prices $(1+r)A'p' + w'$ over actual prices p' will lead to a price increase (and vice versa).

We assume that $c \geq 0$, $w > 0$ and $0 < r, g < R^* - 1 = r^* > 0$ are given exogenously. The scalar $1/R^* = \lambda(A) = \lambda(A') = 1/(1 + r^*)$ is the maximum eigenvalue of the matrix A and A is assumed indecomposable for simplicity, i.e., $\lambda(A)$ is a simple characteristic root. The same holds for A'. In a simulation study below we will partly make r variable such that it will be determined endogenously in each period of time.

Under the above assumptions the stability of both the quantity reaction to demand and supply imbalances as well as the price reaction to the discrepancy between full cost prices (with a target rate r) and actual prices is a very simple matter. Utilizing the spectral displacement property of the eigenvalues Ω of arbitrary matrices A, B (Zurmühl 1964, p. 209), i.e.,

$$\Omega_B = \Omega_A - z \quad \text{if } B = A - zI,$$

we get for $C = C(g) = (1 + g)A - I$ and all eigenvalues Ω_C of $C(g)$:

$$\begin{aligned} \Omega_C &= (1 + g)\Omega_A - 1, \qquad\qquad \text{and thus} \\ Re \;\; \Omega_C &= (1 + g)(Re \;\; \Omega_A) - 1 \geq (1 + g)\lambda(A) - 1 \\ &= (1 + g)/R^* - 1 < 0, \text{ since } 1 + g < R^*. \end{aligned}$$

The same considerations hold for the dynamics (16.2).

The matrices $C(g), C(r)'$ are thus stable Metzler-matrices and consequently also diagonal-stable. The first property means that all off-diagonal elements of C, C' are nonnegative (and all eigenvalues have negative real parts) and the second property says that C, C' will always give rise to stable matrices when multiplied by diagonal matrices with a positive diagonal, cf. Kemp and Kimura (1978, pp. 134 ff.).

For the unique and strictly positive equilibrium of (16.1), (16.2) we have

$$x^* = (I - (1+g)A)^{-1}c = -C(g)^{-1}c \tag{16.3}$$

$$p^{*'} = (I - (1+r)A')^{-1}w' = -C(r)'^{-1}w'. \tag{16.4}$$

Fukuda contrasts the foregoing results on stability with the dual instability theorem of Leontief models mentioned above. He notes that stability is obtained, because the full utilization assumption for the capital stock is dispensed with and the perfect foresight assumption removed from the price dynamics. Less ideal and stringent assumptions therefore increase the stability of the dual adjustment of quantities and prices (given the 'factor prices' w, r). The cost of his approach is, however, that certain imbalances are introduced into the dynamics of the model, the full consequences of which are not thoroughly analyzed. Examples for such imbalances and unanalyzed feedbacks are in particular given by:

1. The creation of inventories
2. Supply-constraints
3. Effects of activities levels on employment, money wages and consumption
4. The existence of profit-rate differentials among activities (measured by $-C(r)'p' - w'$, cf. (16.2))
5. Price reactions due to demand and supply imbalances (measured by $C(g)x + c$, cf. (16.1)).

Of particular interest, in our opinion, are points 4. and 5., since they represent a fundamental incompleteness of the approach given by (16.1), (16.2) and since these aspects can also be integrated into (16.1), (16.2) in a fairly obvious way.

Dual adjustment processes of type (16.1), (16.2) are indeed very implausible because they in particular suggest that quantities change only due to quantity imbalances and prices change only due to differences between actual and marked-up prices. The two systems are considered in complete independence of each other and a mark-up which is 'too high' (or low) with regard to the actual does not at all modify the quantity mechanism. Yet, at least weak or slow influences of profit-rate differentials on the conditions of supply (through entry and exit of firms) and of supply and demand discrepancies on prices should be allowed for. Even if one assumes a target rate of return pricing as formalized in (16.2) one has to admit that the imbalances in (16.1) will have influence on the formation of prices (16.2). Modern corporations, for example, do not only pay attention to certain price targets, but, of course, also consider the state of demand and supply when revising their prices. The above two modifications 4., 5. should therefore be incorporated into the model (16.1), (16.2) of price-quantity adjustments. This will be done in Sect. 16.3 in a way as simple as possible. Nevertheless, considerable complexity will be introduced through our extension of this separable dual adjustment à la Keynes-Leontief (Morishima 1976, 1977) which introduces cross-dual elements and reaction patterns into it.

In closing this section, let us briefly consider an important limit case of the above model (16.1), (16.2).

Assume: $c, w = 0, r = g = R^* - 1, R^* = \lambda(A)^{-1}$. Then:

$$\dot{x} = d_{11}(R^*Ax - x) = d_{11}C^*x \qquad (*)$$
$$\dot{p} = d_{22}(R^*A'p' - p') = d_{22}C^{*'}p' \qquad (**)$$

can, e.g., be interpreted as adjustment rules within the context of a very simple closed von Neumann model [labor inputs included in the A-matrix]. In contrast to (16.1), (16.2) the equilibrium x^*, p^* is now uniquely determined only up to scalars α, β applied to x^*, p^*.

Though our following considerations will not immediately be applicable to this limit case as well, the results that we will obtain for system (16.1), (16.2) and its extensions will – suitably reformulated – also carry over to $(*)$, $(**)$ and its analogous extensions since instead of only negative real parts of eigenvalues we will here have zero as an additional eigenvalue (of multiplicity 1 or 2) as the sole exception.

16.4 The Composite System

Integrating the two cross-dual adjustment components 4., 5. of Sect. 16.2 into system (16.1), (16.2) gives the following new and more complete type of dynamics

$$\dot{x} = d_{11}C(g)x - d_{12}C(r)'p' + q^1 \qquad (16.5)$$
$$\dot{p}' = d_{21}C(g)x + d_{22}C(r)'p' + q^2 \qquad (16.6)$$

with $q^1 = d_{11}c - d_{12}w', q^2 = d_{21}c + d_{22}w'$ (d_{12}, d_{21} of the same type as d_{11}, d_{22}). This composite dynamics can be interpreted as follows. Concerning the output reaction, formalized in (16.5), one can realistically hypothesize that firms do not respond solely to imbalances of supply and demand when revising their production (and investment) decisions but that output is also scaled up (or down) according to whether the actual rates of return are above (or below) the norm or target rate r. Compared with the (Keynesian) quantity reaction due to quantity imbalances the additional (Classical) quantity reaction due to profitability differences may, however, be considered a slow dynamics – mainly initiated through entry and exit of firms ($d_{12} << d_{11}$). Furthermore, it also adds realism to our dynamics if we assume that the price setting behavior of firms formalized in (16.6) follows two decision criteria: first, prices are provisionally set on the basis of a mark-up (or target rate of return) calculation and secondly, are further revised according to the imbalances of supply and demand in the various markets.

Before discussing the dynamics of this composite system, however, existence and uniqueness of the equilibrium must be briefly considered.

Existence of Equilibrium: The equilibrium of (16.5) and (16.6) is the same as the one for (16.1) and (16.2), cf. (16.3) and (16.4).

Uniqueness: For the homogeneous part of (16.5), (16.6) we get

$$0 = d_{12}^{-1}\dot{x} + d_{22}^{-1}\dot{p}' = d_{12}^{-1}d_{11}Cx + d_{22}^{-1}d_{21}Cx = (d_{12}^{-1}d_{11} + d_{22}^{-1}d_{21})Cx.$$

This implies $Cx = 0$, i.e., $x = 0$, which by (16.6) also implies $p = 0$.

System (16.5), (16.6) can be written in compact form as:

$$\dot{z} = \begin{bmatrix} \dot{x} \\ \dot{p}' \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + \begin{bmatrix} d_{21}^{-1}d_{11} & -I \\ I & d_{22}d_{12}^{-1} \end{bmatrix} \begin{bmatrix} d_{21}C(g) & 0 \\ 0 & d_{12}C(r)' \end{bmatrix} \begin{bmatrix} x \\ p' \end{bmatrix} = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix} + SKz$$

with the following properties of the above two matrices S, K:
$S + S'$: a diagonal matrix; $S - S'$: a skew-symmetric matrix
S : a positive definite matrix

$$S^{-1} = \begin{bmatrix} d_{22}d_{12}^{-1} & I \\ -I & d_{11}d_{21}^{-1} \end{bmatrix} \begin{bmatrix} (I + d_{22}d_{12}^{-1}d_{11}d_{21}^{-1})^{-1} & 0 \\ 0 & (I + d_{22}d_{12}^{-1}d_{11}d_{21}^{-1})^{-1} \end{bmatrix}$$

K: a block-diagonal, stable Metzler matrix.

Yet, despite these (and probably further) basic properties of the two matrices S, K a direct proof of asymptotic stability for (16.5), (16.6) does not seem to be easy, despite the fact that the diagonal blocks have already been shown to imply an asymptotically stable dynamics and that the skew-symmetric off-diagonal terms – taken by themselves – imply a stable dynamics in the sense of Liapunov (as we shall show below).

To further analyze the type of stability of the two subsystems just considered and to indicate various difficulties for proving the stability of the composite system is the purpose of the remainder of this section. We shall first reformulate the stability properties of the dual system (16.1), (16.2) by means of suitable Liapunov functions (which will be of use later on). Thereafter we will consider a Liapunov-function for the cross-dual subsystem of (16.5), (16.6). Both Liapunov-functions will, however, fail to provide a stability proof for the complete system (16.5), (16.6).

16.5 Some Preliminaries

Let the $2n$-vector z^* denote the equilibrium of (16.1), (16.2), and (16.5), (16.6). Decompose S into S^1, S^2 [$S = S^1 + S^2$].

$$S^1 \quad \text{(the diagonal terms)} : \begin{bmatrix} d_{11}d_{21}^{-1} & 0 \\ 0 & d_{22}d_{12}^{-1} \end{bmatrix}$$

$$S^2 \quad \text{(the off-diagonal terms)} : \begin{bmatrix} 0 & -I \\ I & 0 \end{bmatrix}.$$

Equations (16.5), (16.6) thereby become:

$$\dot{z} = q + SKz = q + (S^1 + S^2)Kz \quad \text{with } q = \begin{bmatrix} q_1 \\ q_2 \end{bmatrix}.$$

The corresponding homogeneous system is:

$$\dot{y} = SKy = (S^1 + S^2)Ky, \quad y = z - z^*.$$

The solution of this homogeneous system is (with $SK = Q$):

$$y(t) = e^{Qt} y_0, \quad y_0 = y(0)$$

whereas for the nonhomogeneous system with the constant term q we get:

$$z(t) = e^{Qt}(z_0 - z^*) + z^* \qquad (z_0 = z(0))$$

Because of these relationships we need to study only the stability properties of the homogeneous system in the following. This remark also holds true for the subsystems S^1K, S^2K of SK, i.e., the Keynesian system (16.1), (16.2) as well as the Classical cross-dual dynamics. There, q is replaced by

$$\begin{bmatrix} c \\ w' \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -w' \\ c \end{bmatrix} \quad \text{respectively .}$$

16.5.1 Stability of the Keynesian Case and the Composite System

For the homogeneous system of the Keynesian variety

$$\begin{bmatrix} \dot{x} \\ \dot{p}' \end{bmatrix} = \dot{z} = S^1Kz = \begin{bmatrix} d_{11}C(gg) & 0 \\ 0 & d_{22}C(r)' \end{bmatrix} \begin{bmatrix} x \\ p' \end{bmatrix} \tag{16.7}$$

the following theorem holds:

Theorem 16.1. *Let H_1, H_2 be the Liapunov matrices of the stable systems $d_{11}C(g), d_{22}C(r)'$, i.e., these two matrices are symmetric, positive definite and they fulfill*

$$C(g)'d_{11}H_1 + H_1d_{11}C(g) = -I, \text{ and } C(r)d_{22}H_2 + H_2d_{22}C'(r) = -I,$$

respectively (see Lancaster (1969, pp. 267 ff.) and Kemp and Kimura (1978, pp. 124 ff.) for their derivation). The composite function

$$V(x, p') = x'H_1x + pH_2p'$$

is a Liapunov-function for the system (16.7).

Proof. The method of proof for each component immediately applies to the whole system, too, there giving rise to:

$$\begin{aligned}\dot{V} &= \dot{x}'H_1x + x'H_1\dot{x} + \dot{p}H_2p' + pH_2\dot{p}' \\ &= x'C(g)'d_{11}H_1x + x'H_1d_{11}C(g)x + pC(r)d_{22}H_2p' + pH_2d_{22}C(r)'p' \\ &= x'[C(g)'d_{11}H_1 + H_1d_{11}C(g)]x + p[C(r)d_{22}H_2 + H_2d_{22}C(r)']p' \\ &= -x'x - pp' \leq 0[= 0 \text{ iff } x = p' = 0].\end{aligned}$$

□

Remark 16.2. The Liapunov-matrix for (16.7) is thus simply given by

$$H = \begin{bmatrix} H_1 & 0 \\ 0 & H_2 \end{bmatrix}$$

Lemma 16.3.

1) $(S^1K)'(S^1K) = \begin{bmatrix} C(g)'d_{11}^2C(g) & 0 \\ 0 & c(r)d_{22}^2C(r)' \end{bmatrix}$

$(S^1K)S^1K)' = \begin{bmatrix} d_{11}C(g)C(g)'d_{11} & 0 \\ 0 & d_{22}C(r)'C(r)d_{22} \end{bmatrix}$

2) $(SK)'(SK) =$

$$\begin{bmatrix} +C(g)'d_{11}d_{11}C(g) + C(g)'d_{21}d_{21}C(g) & -C(g)'d_{11}d_{12}C(r)' + C(g)'d_{21}d_{22}C(r)' \\ -C(r)'d_{11}d_{21}C(g) + C(r)d_{22}d_{21}C(g) & +C(r)d_{12}d_{12}C(r)' + C(r)d_{22}d_{22}C(r)' \end{bmatrix}$$

$(SK)(SK)' =$

$$\begin{bmatrix} +d_{11}C(g)C(g)'d_{11} + d_{12}C(r)'C(r)d_{12} & +d_{11}C(g)C(g)'d_{21} - d_{12}C(r)'C(r)d_{22} \\ +d_{21}C(g)C(g)'d_{11} - d_{22}C(r)'C(r)d_{12} & +d_{21}C(g)C(g)'d_{21} + d_{22}C(r)'C(g)d_{22} \end{bmatrix}$$

It follows that the following assumptions will not be satisfied in general.

Theorem 16.4. *The conditions* $(S^1K)'(S^1K) = (S^1K)(S^1K)'$ *and* $(SK)'(SK) = (SK)(SK)'$ *as well as* $d_{21} = d_{12}$ *and* $r = g$ *imply that* V *is also a Liapunov-function for* SK*, i.e., for the composite dynamics. (Fig. 16.1)*

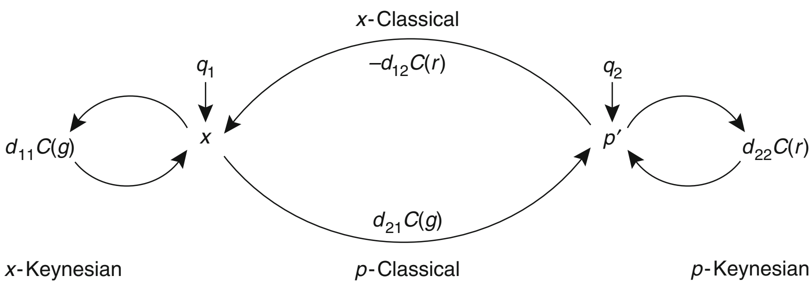

Fig. 16.1 The composite dynamics

Proof. It is well-known that the matrix H can be expressed as

$$H = \int_0^\infty e^{(S^1K)'t} e^{(S^1K)t}\, dt,$$

cf. Lancaster (1969, p. 263) or Kemp and Kimura (1978, p. 124). Furthermore, if $(S^1K)'$ and S^1K commute, we get (cf. Kemp and Kimura 1978, p. 116):

$$H = \int_0^\infty e^{[(S^1K)'+S^1K]t}\, dt = \int_0^\infty e^{[(SK)'+SK]t}\, dt,$$

if $d_{21} = d_{12}, \quad r = g$). And finally:

$$H = \int_0^\infty e^{(SK)'t} e^{SKt}\, dt$$

if $(SK)'(SK) = (SK)(SK)'$, which implies $(SK)'H + H(SK) = -I$.

The above Liapunov stability proof therefore carries over to the system SK. □

Remark 16.5. The above Lemma and Theorem suggest that a general proof of the assertion that V is also a Liapunov function for the composite system SK may be difficult to establish. The proof of stability for the composite system therefore demands more than only an application of the Liapunov technique which characterizes the stability of S^1K. The two subsystems S^1K, S^2K are in this respect not compatible with each other.

A weak positive result is provided by part of the following

Proposition 16.6. *Let r, g be smaller than r^**

(a) The Keynesian system S^1K as well as the composite system SK are Hicksian (but only the former is also Metzlerian).
(b) All real eigenvalues of SK are negative.
(c) $D(SK)$ is stable for at least one positive diagonal matrix D of dimension $2n \star 2n$.
(d) Since SK is only Hicksian (but not Metzlerian), it need not be a stable matrix.
(e) S^1K has a negative dominant diagonal. This property holds also for the composite system if the off-diagonal part S^2K is properly limited.

Proof.

(a) Theorem 14 in Kemp and Kimura (1978, p. 143) states that the stable Metzlerian matrix S^1K is Hicksian. And with regard to SK it suffices to show that the negative of this matrix is a P-matrix, cf. the lemmata on pp. 88 ff. in Kemp and Kimura (1978). According to Sect. 16.3 the matrix $-SK$ can be rewritten as $-DQ = D(-Q)$, so that it remains to be shown that $-Q$ is a P-matrix, where $-Q$ is equal to

$$\begin{bmatrix} -d^1 C(g) & +C(r)' \\ -C(g) & -d^2 C(r)' \end{bmatrix},$$

d^1, d^2 arbitrary positive diagonal matrices. For the determinant of $-Q$ we get up to positive scalars the following expressions

$$\begin{vmatrix} -(d^2 d^1 + I)C(g) & 0 \\ -C(g) & -d^2 C(r)' \end{vmatrix} \triangleq \begin{vmatrix} -(d^2 d^1 + I)C(g) & 0 \\ -(d^2 d^1 + I)C(g) & -(d^2 d^1 + I)d^2 C(r)' \end{vmatrix} \triangleq$$

$$\begin{vmatrix} -(d^2 d^1 + I)C(g) & 0 \\ 0 & -(d^2 d^1 + I)d^2 C(r)' \end{vmatrix} \triangleq \begin{vmatrix} -C(g) & 0 \\ 0 & -C(r)' \end{vmatrix}.$$

This shows how the composite system can be reduced to a Hicksian matrix and its determinant. Since the above calculation is also applicable to any principal minor of $-Q$ it follows that $-Q$ must be a P-matrix, since $S^1 K$ is of this type.

(b) Woods (1978, p. 36).

(c) Woods (1978, p. 307).

(d) See the counterexamples in the mathematical appendix, Sect. 16.1.

(e) Theorem 14 in Kemp and Kimura (1978, p. 143) again implies the assertion for $A = S^1 K$, i.e., we have for this matrix the well-known inequalities

$$d_i |a_{ii}| > \sum_{j \neq 1} d_j |a_{ij}| \qquad (i = 1, \ldots, 2n).$$

These inequalities are modified through the inclusion of $S^2 K$ into the composite system as follows

$$d_i |a_{ii}| > \sum_{j \neq i} (d_j + e_j) |a_{ij}|$$

where e_i, e_j depend on the magnitude of the adjustment coefficients d_{21}, d_{12}. It follows that ranges for these coefficients can be determined such that the above second set of inequalities will be fulfilled. □

16.5.2 Stability of the Classical Case and the Composite System

The cross-dual system reads in the case of $r = g$:

$$\dot{z} = \begin{bmatrix} 0 & -d_{12} C(r)' \\ d_{21} C(r) & 0 \end{bmatrix} z \tag{16.8}$$

Take as Liapunov-function the function $V(z) = z'Q'DQz$ where

$$Q = \begin{bmatrix} 0 & -C(r)' \\ C(r) & 0 \end{bmatrix}, \quad D = \begin{bmatrix} d_{13} & 0 \\ 0 & d_{21} \end{bmatrix}$$

and note that $\dot{z} = DQz$.

V is based on a symmetric matrix $Q'DQ$ which by construction is also positive definite. And finally:

$$\begin{aligned} \dot{V} &= \dot{z}'Q'DQz + z'Q'DQ\dot{z} = z'Q'DQ'DQz + z'Q'DQDQz \\ &= z'Q'D[Q' + Q]DQz = 0 \end{aligned}$$

The origin is therefore stable (but not asymptotically stable) with regard to the Classical system S^2K.

Remark 16.7. Asymptotic stability can be obtained again if a term of the form

$$\delta C(r)'\dot{p} = \delta C(r)'d_{21}C(r)x(r = g, \delta \text{ an adjustment parameter }),$$

indicating the direction of change of profit-rate differentials, is integrated into the x-component of this dynamics, since this additional component will imply a sufficient degree of negative definiteness for the new system matrix

$$Q = \begin{bmatrix} -d_{12}\delta C'd_{21}C & -d_{12}C' \\ d_{21}C & 0 \end{bmatrix}$$

For a demonstration of this assertion, cf. Flaschel and Semmler (1987). The purely cyclical behavior of the simple Classical process (16.8) is thereby turned into damped oscillations, which indicates that (16.8) can only be considered as a starting point for further considerations of this Classical dynamics.

Theorem 16.8. *The Liapunov function $V(z) = z'Q'DQz$ of the system S^2K above does not supply a Liapunov function for the composite dynamics $\dot{z} = SKz$.*

Proof. In the case of the composite system we have:

$$\begin{aligned} SK &= \begin{bmatrix} d_{11}d_{21}^{-1} & -I \\ I & d_{22}d_{12}^{-1} \end{bmatrix} \begin{bmatrix} d_{21}C(g) & 0 \\ 0 & d_{12}C(r)' \end{bmatrix} \\ &= \begin{bmatrix} d_{12} & d_{11} \\ -di22 & d_{21} \end{bmatrix} \begin{bmatrix} 0 & -C(r)' \\ C(g) & 0 \end{bmatrix} \\ &= \begin{bmatrix} di12 & 0 \\ 0 & d_{21} \end{bmatrix} \begin{bmatrix} d_{11}d_{12}^{-1}C(g) & -C(r)' \\ C(g) & d_{22}d_{21}^{-1}C(r)' \end{bmatrix} = DQ. \end{aligned}$$

The most natural type of decomposition of the matrix SK to be used to define V seems to be the third from the above ($SK = DQ$), which as in the preceding proof and with regard to the new decomposition DQ gives for $r = g$ (with $d_{11}d_{21}^{-1} = d_1$ and $d_{22}d_{21}^{-1} = d_2$).

Simplifying further ($d_1 = d_2 = I$) gives for the matrix in the middle of this expression

$$2\begin{bmatrix} (1+r)(A'+A)/2 - I & 0 \\ 0 & (1+r)(A+A')/2 - I \end{bmatrix}.$$

The maximum eigenvalue of $(A' + A)/2 = (A + A')/2$ is, however, in general greater than $(\lambda(A') + \lambda(A))/2 = \lambda(A)$, i.e., $\lambda((1 + r)(A' + A)/2)$ can be larger than 1 (see the examples in the mathematical appendix, Sect. 16.1). The spectral displacement $(1 + r)(A' + A)/2 - I = F$ does therefore in general not imply that the eigenvalues of this matrix all have negative real parts. This implies that F is generally not negative definite (C is not quasi-negative definite) and that $V \leq O$ need not hold true.

The above considerations have shown that the stability of S^2K is not obviously turned into asymptotic stability by the addition of the two asymptotically stable block-diagonal terms in the matrix SK. The complete structure (16.5), (16.6) cannot be treated simply by means of the Liapunov techniques that are available for the diagonal or the off-diagonal terms of the block-matrix

$$\begin{bmatrix} d_{11}C(g) & -d_{12}C(r)' \\ d_{21}C(g) & d_{22}C(r)' \end{bmatrix}$$

Alternative approaches are thus needed to further study the stability of this integrated dynamical system. To introduce and apply such an alternative is the task of the following sections.

However, before closing this section let us briefly state as a proposition the positive contribution it contains for the stability analysis of our composite system: □

Proposition 16.9. *Assume that $r = g < r^*$.*

The composite system SK will be stable if the maximum eigenvalues λ_i of

$$(1+r)(\sqrt{d_i}A\sqrt{d_i}^{-1} + \sqrt{d_i}^{-1}A'\sqrt{d_i})/2, \quad d_i = d_{ii}d_{ij}^{-1}, i = 1, 2, j \neq i$$

are both less than one.

Proof. The matrix $Q + Q'$ in the preceding proof is composed of matrices of the type

$$\begin{aligned} C'd_1 + d_1C &= 2[(1+r)(d_1A + A'd_1) - d_1)] \\ &= 2\sqrt{d_1}[(1+r)(\sqrt{d_1}A\sqrt{d_1}^{-1} + \sqrt{d_1}^{-1}A'\sqrt{d_1}) - I]\sqrt{d_1} \end{aligned}$$

The assumption of the proposition then implies that the eigenvalues of the term in square brackets have all negative real parts, which in turn implies that $\dot{V} < 0$ must hold true for $z \neq z^*$ and the function V of the preceding theorem. □

Example. Assume as matrix A

$$\begin{bmatrix} 0.34 & 0.44 \\ 0.35 & 0.30 \end{bmatrix}$$

Assume furthermore $d_{11} =< (0.6, 0.8) >, d_{22} =< (0.7, 0.6) >,$
$d_{12} =< (0.3, 0.6) >, d_{21} =< (0.6, 0.5) >$, and $R = G \approx 1.20 < R^* \approx 1.40$.
We then have: $\lambda_1 \approx 0.88, \lambda_2 \approx 0.86.$, and

Eigenvalues of $S^1K : -0.77, -0.70, -0.092, -0.096,$
Eigenvalues of $S^2K : 5.56i, 0.65i$, and finally:
Eigenvalues of $SK : -0.745.55i, -0.090.65i$.

Remark 16.10. Note that the real parts of the eigenvalues of the matrix $C(r)$ change monotonically with r, but that this need not be true for the composite system.

16.6 A New Approach to the Stability of Composite Systems

The subsequent part of the chapter will utilize vector differential inequalities and vector Liapunov functions as, e.g., put forward in Siljak (1978)[3] by which it can be shown in an alternative way that a composite system such as (16.5), (16.6) (and its admissible substructures) may be stable if the magnitude of the interactions of its subsystems is again limited in a specific way. The type of stability analysis which is here involved is termed connective stability. Before this concept is introduced in more detail it requires, however, that some definitions which are basic for the notion of connective stability should be briefly explained. Thereafter this new technique of stability analysis is introduced and applied. (Fig. 16.2).

(1) Directed graphs and interaction matrices: The interaction of subsystems can be described by means of directed graphs and interconnection matrices. In our case the directed graph, which represents the complete interconnection of our subsystems, exhibits the following structure where q_1 and q_2 indicate how the external world affects this production oriented interaction (x- and p-Classical give the direction of Classical output and price dynamics, and x- and p-Keynesian the Keynesian output and price dynamics). For our case with two interacting subsystems, for example, the range of interconnection matrices considered so far is as follows

[3] Further useful references on this topic are Berussou and Titli (1982), Medio (1987), Michel and Miller (1977), and part I in Singh and Titli (1979).

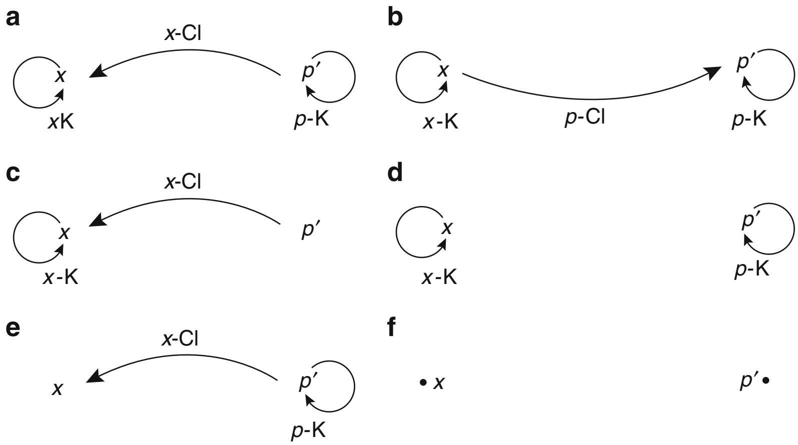

Fig. 16.2 Substructures

$$\text{Full interaction} \qquad \overline{E} : \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$\text{Keynesian subsystem } E_1 : \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} : \text{ Classical subsystem } E_2 : \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

In general, the interaction of two subsystems S_1 and S_2 can be represented as follows:

$$\begin{aligned} S_1 : \dot{z}_1 z_1 &= \overline{A}_{11} z_1 + e_{11} A_{11} z_1 + e_{12} A_{12} z_2 \\ S_2 : \dot{z}_2 z_1 &= \overline{A}_{22} z_2 + e_{21} A_{21} z_1 + e_{22} A_{22} z_2 \end{aligned} \tag{16.9}$$

with $e_{ij} \begin{cases} 1, & S_j \text{ can act on } S_i \\ 0, & S_j \text{ cannot act on } S_i \end{cases}$

where e_{ij} are the elements in the interaction matrix E, A_{ij} are the interacting matrices with regard to the dynamic relationship between the variables of S_i and S_j and A_{11}, A_{22} represent the self-interacting part of the system when all secondary feedbacks between variables have been removed from (16.9). In general there could be a changing on–off participation and interaction between the variables of the various subsystems of a composite system (to be represented then by a time-dependent matrix E).

(2) Structural perturbations: The aforementioned changes in the system's structure by means of changing interaction matrices E are called structural perturbations. They in general can destroy the system's stability properties. Desirable stability properties, however, are such that they remain invariant under structural perturbations. Here it will suffice to introduce possible perturbations by referring to our Classical and Keynesian subsystems. Disconnection and interconnection

resulting in structural perturbations can be illustrated by the following directed graphs (leaving aside the 'environment' q_1, q_2): Here, the directed graph (a) represents the interconnection of the Classical and the Keynesian output dynamics (price dynamics remaining solely Keynesian), (b) is the interaction of Classical and Keynesian price dynamics (quantity dynamics solely Keynesian), (c) is the Classical and Keynesian output dynamics (price dynamics set zero) and (d) the decoupled Keynesian output and price dynamics. We have used here $x - K$, $p - K$ to denote Keynesian output and price dynamics and $x - Cl$, $p - Cl$ for the Classical output and price dynamics.

(3) Connective stability: One-shot stability analysis attempts to prove stability for a dynamic system without going through an analysis of its basic component parts first. In a composite system with interconnected basic subsystems, however, a composite type of stability analysis may be more appropriate than the well-known single step approaches to stability by means of single Liapunov functions. Here, one attempts to show that a dynamic interconnection, properly limited, will remain stable when stable, isolated subsystems are aggregated in various ways. This type of stability analysis is termed connective stability in Siljak (1978). Possible interconnections have already been visualized above for the dynamic model of this chapter by means of the directed graphs (a)–(d). Inspecting these graphs again, one notes, however, that the case c does not represent a meaningful dynamic system in the context of our model. Roughly speaking, connective stability therefore here means that the fully connected system (described by the full interconnection matrix E) is stable as well as all of its structural perturbations which do not remove one of the self-contained subsystems from this structure, i.e., which contain at least the initially given decoupled substructure. Note, that this means that we now consider the Keynesian subsystem as the more 'basic' one and the Classical system as introducing the interactions. Switching on (or off) Keynesian types of adjustment in the Classical context demands an analysis of partially stable composite systems which – due to space limitations – will not be attempted in this chapter.

Interesting methods for studying connective stability for composed systems by a decomposition-aggregation procedure are provided by the concepts of vector differential inequalities and vector Liapunov functions as elaborated in Siljak (1978, Chap. 2). In what follows we want to briefly outline this decomposition-aggregation method of connective stability analysis following Siljak and then apply this method to our composite Keynesian-Classical system.

The connective stability of the equilibrium $z^* = 0$ of a system composed by connecting stable, initially isolated systems can be investigated in three steps.

Step a:

One formulates an interconnected dynamical system from the knowledge of its basic components, their internal dynamic structure and various conceivable interactions between these basic components. In general this may result in a system such as

(16.9). In our case (16.5), (16.6) this general approach can, however, be reduced to the following system:

$$\begin{aligned} S_1 : \dot{z}_1 &= \overline{A}_{11} z_1 + e_{12} A_{12} z_2 \\ S_2 : \dot{z}_2 &= \overline{A}_{22} z_2 + e_{21} A_{21} z_1 \end{aligned} \tag{16.10}$$

In this system the matrices A_{11}, A_{22} represent the independent Keynesian subsystems. The first part of (16.10) thus represents the decoupled system which is not modified by the structural perturbations allowed for [note here, that Siljak (1978, p. 33) uses a different notation to represent this case and that he in general allows for further feedbacks of z_i on z_i which may be switched on and off through e_{ii} and structural perturbations].

Step b:

The asymptotic stability of each decoupled system in (16.10): $\dot{z}_1 = \overline{A}_{11} z_1, \dot{z}_2 = \overline{A}_{22} z_2$ is assumed as given (or proved). As Liapunov functions for the isolated subsystems $\overline{A}_{11}, \overline{A}_{22}$ we can then take (cf. also Sect. 16.3)

$$v_1(z_1) = (z_1' H_1 z_1)^{1/2}, \quad v_2(z_2) = (z_2' H_2 z_2)^{1/2} \tag{16.11}$$

where the positive definite and symmetric matrices H_1, H_2 are determined by

$$\overline{A}_{11}' H_1 + H_1 \overline{A}_{11} = -I, \quad \overline{A}_{22}' H_2 + H_2 \overline{A}_{22} = -I \tag{16.12}$$

The total time derivatives of (16.11) are [see (16.12), $(i = 1, 2)$]:

$$\begin{aligned} \dot{v}_i &= (\text{ grad } v_i) \dot{z}_i = (\text{ grad } v_i)' \overline{A}_{ii} z_i \\ &= (v_i^1 H_i z_i)' \overline{A}_{ii} z_i = -(1/2) v_i^{-1} (z_i' z_i). \end{aligned} \tag{16.13}$$

From (16.11) and (16.13) estimates for these Liapunov functions are then produced as follows ($i = 1, 2$; note the minus sign in (16.13)):

$$\Theta_{11} ||z_i|| \le v_i \le \Theta_{i2} ||z_i||, \dot{v}_i \le -\Theta_{i3} ||z_i||, || \text{ grad } v_i || \le \Theta_{i4}, \tag{16.14}$$

with the following positive scalars Θ_{ij}

$$\Theta_{i1} = \Omega_m^{1/2}(H_i), \qquad \Theta_{i2} = \Omega_M^{1/2}(H_i),$$
$$\Theta_{i3} = \frac{1}{\Omega_M^{1/2}(H_i)}, \quad \Theta_{i4} = \frac{\Omega_M(H_i)}{\Omega_m^{1/2}(H_i)}$$

Here Ω_m and Ω_M denote the minimum and maximum eigenvalues of the symmetric and positive definite matrices H_1, H_2.

Step c:

The functions v_1, v_2 are representatives of the stability of each subsystem $\overline{A}_{11}, \overline{A}_{22}$ and we can now study the stability of the aggregate system S composed of S^1, S^2 by considering appropriate compositions of these two stability indicators, no longer considering the dynamic interaction within each subsystem in its details. The total time derivative along the solutions curves of each interconnected subsystem S_i of (16.10) is $(i, j = 1, 2)$:

$$\dot{v}_i = (\text{grad } v_i)'[\overline{A}_{ii} z_i + e_{ij} A_{ij} z_j] = \dot{v}_i(13) + (\text{grad } v_i)' e_{ij} A_{ij} z_j \tag{16.15}$$

where $\dot{v}_i(13)$ is given by (16.13). This gives rise to

$$\dot{v}_i \leq -\Theta_{i3}||z_i|| + e_{ij}||\text{ grad } v_i|| \quad ||A_{ij} z_j|| \tag{16.16}$$

which together with the constraint on the nonsymmetric interaction matrix A_{ij}

$$||A_{ij} z_j|| \leq \varepsilon_{ij}||z_j||, \quad \varepsilon_{ij} = \Omega_M^{1/2}(A'_{ij} A_{ij}) \tag{16.17}$$

finally gives (because of the minus sign in (16.16)):

$$\begin{aligned} \dot{v}_1 &\leq -\Theta_{12}^{-1}\Theta_{13} v_1 + e_{12}\varepsilon_{12}\Theta_{14}\Theta_{21}^{-1} v_2, \\ \dot{v}_2 &\leq e_{21}\varepsilon_{21}\Theta_{24}\Theta_{11}^{-1} v_1 - \Theta_{22}^{-1}\Theta_{23} v_2 \end{aligned} \tag{16.18}$$

This system can be rewritten by means of the vector Liapunov function $v = (v_1, v_2)'$ as one vector inequality

$$\dot{v} \leq W v \tag{16.19}$$

where the aggregation matrix W is defined by $(i, j = 1, 2)$:

$$w_{ij} = \begin{cases} -\Theta_{i2}^{-1}\Theta_{i3}, & i = j, \\ e_{ij}\varepsilon_{ij}\Theta_{j1}^{-1}\Theta_{i4}, & i \neq j \end{cases} \tag{16.20}$$

The connective stability of the overall system then follows from a stability proof for the aggregated system (16.19).

In order to prove this result Siljak (1978, Chap. 2) introduces the comparison principle for vector differential inequalities by majorizing the function v appropriately. This principle uses for comparison the differential equation

$$\dot{r} = \overline{W} r$$

with the initial condition $v_0 = r_0$, and W the aggregation matrix corresponding to the fundamental interaction matrix $\overline{E}$

$$\overline{w}_{ij} = \begin{cases} -\Theta_{i2}^{-1}\Theta_{i3} & i = j \\ \varepsilon_{ij}\Theta_{j1}^{-1}\Theta_{i4}, & i \neq j \end{cases} \tag{16.21}$$

If the matrix $\overline{W}$ is stable and if we know (for all our interconnections E)

$$v(t) \leq r(t), \quad t \geq t_0,$$

then one can conclude that $\lim_{t\to\infty} v(t) = 0$ holds true for all such E.

We thus obtain connective stability, as defined above, for the whole system S [see Siljak (1978, pp. 37ff.) for further details].

The above type of decomposition-aggregation analysis by means of vector Liapunov functions consequently gives rise to the following.

Theorem 16.11. *Given (1) that asymptotic stability of each decoupled subsystem is established and described by the estimates (16.14) obtained for the Liapunov functions v_1 and v_2, (2) that the constraints (16.17) on the interactions $A_{12}z_2$ and $A_{21}z_1$ between the subsystems S_1 and S_2 hold, and (3) that stability of the aggregate matrix $\overline{W}$ corresponding to the fundamental interconnection matrix $\overline{E}$ has been proved, then the system S is stable for all interconnection matrices E, that is, it is connectively stable.*

A variety of related stability concepts and theorems are in addition to the above investigated in Siljak (1978) – and also in the book of Michel and Miller (1977). We cannot go into the details of all these variants here, yet want to supplement the above theorem by an alternative version of it from in Michel and Miller (1977), p. 49). These authors employ in this version the same constraints (16.14) as in the above theorem (see their property $\widetilde{A}$ for the isolated subsystems and note that $\sigma_i = -1, L_i = \Theta_{i4}$ holds with regard to the notation used by them). The authors then formulate the following variant of the above theorem:

Theorem 16.12. *If in addition to property $\widetilde{A}$ there exist constants $q_{ij} \geq 0 (i, j = 1, 2)$ such that*

$$|z_i| \leq \sum_{j=1}^{2} q_{ij}\Theta_{j3}|z_j|, \quad i = 1, 2$$

and if the leading principal minors of the $2 \star 2$ test matrix $\overline{W}$ are all positive, where

$$\overline{w}_{ij} = \begin{cases} -(\sigma_i + L_i q_{ii}) & i = j \\ -L_i q_{ij} & i \neq j, \end{cases}$$

then the composite system is (uniformly) asymptotically stable (in the large).

Two important observations with regard to applications of this theorem should be added: (1) The above parameters σ_i, q_{ij}, L_i must not be determined by the definitions following (16.14), but need only have the properties assumed in the above theorem, and (2) the real parts of the eigenvalues of $\overline{W}$ can be used to estimate the margin of stability of the composite system, i.e., the degree to which the isolated subsystems may be modified and still leave the composite system asymptotically stable (see Michel and Miller (1977, p. 53) in this regard).

The above theorems decompose a stability analysis of dimension $2n$ *into two stability problems of dimension* n *and one of dimension* 2 *(the choice of dimensions is of course only due to the special type of problem here considered). They will be applied to our composite system in the next section.*

Two final remarks may be added in concluding this section.

Remark 16.13. For a dynamically reliable large-scale system one would expect that the system is allowed to disintegrate itself and then to reintegrate itself during its functioning. The above discussed class of structural perturbations can be generalized into this direction by means of time-dependent interconnection matrices $E(t)$ to allow for on–off participations of subsystems in the course of time (see Siljak 1978).

Remark 16.14. In even more general terms, it is also not necessary that all connected subsystems are stable when isolated. Unstable subsystems may be permitted to be parts of a large composite system, provided, of course, that sufficiently strong stabilizing cross-feedbacks are present at all times. When interconnection matrices are carefully chosen, unstable subsystems[4] can be allowed for and the system may nevertheless exhibit connective stability, cf. Siljak (1978, Chap. 2.6).

16.7 An Alternative Investigation of the Stability of Composite Systems

From our earlier discussion it is known that the problem we are facing concerning the stability of system (16.5), (16.6) is less severe than the one indicated in the second remark above. Our composite system (16.5), (16.6) is of the form (16.10) with two stable decoupled subsystems $\overline{A}_{11} = d_{11}C(g)$, $\overline{A}_{22} = d_{22}C(r)'$ for which therefore the above two Liapunov functions exist.

In order to investigate the asymptotic stability of the totally interconnected system and the interconnection matrices E here allowed for there consequently remains to be considered according to the above theorem:

$$(\overline{w}_{ij}) = \begin{bmatrix} -(\Omega_M^{1/2}(H_1))^{-1}1/2\frac{1}{\Omega_M^{1/2}(H_1)} & \varepsilon_{12}(\Omega_m^{1/2}(H_2))^{-1}1/2\frac{\Omega_M(H_1)}{\Omega_M^{1/2}(H_1)} \\ \varepsilon_{21}(\Omega_m^{1/2}(H_1))^{-1}1/2\frac{\Omega_M(H_2)}{\Omega_m^{1/2}(H_2)} & -\Omega_M^{1/2}(H_2)^{-1}1/2\frac{1}{\Omega_M^{1/2}(H_2)} \end{bmatrix}$$
$$= \begin{bmatrix} -1/2(\Omega_M(H_1))^{-1} & \varepsilon_{12}\frac{\Omega_M(H_1)}{\Omega_m^{1/2}(H_1)\Omega_m^{1/2}(H_2)} \\ \varepsilon_{21}\frac{\Omega_M(H_2)}{\Omega_m^{1/2}(H_1)\Omega_m^{1/2}(H_2)} & -1/2(\Omega_M(H_2))^{-1} \end{bmatrix} \qquad (16.22)$$

with $\varepsilon_{12} = \Omega_M^{1/2}(C(r)d_{12}^2C'(r))$ and $\varepsilon_{21} = \Omega_M^{1/2}(C(g)d_{21}^2C(g)) > 0$.

Sufficient for the stability of our composite system is that the Metzlerian matrix (16.22) is Hicksian, cf. Kemp and Kimura (1978, pp. 141 ff.).

[4] e.g., the price reaction $\dot{p}$ due to excess demand in its dependence on prices p.

Since $\overline{w}_{11}, \overline{w}_{22}$ are negative we therefore have to explore only whether Det $(\overline{W})$ is positive. We have

$$\begin{aligned}\text{Det }(\overline{W}) &= 1/4\frac{1}{\Omega_M(H_1)\Omega_M(H_2)} - \varepsilon_{12}\varepsilon_{21}\frac{\Omega_M(H_1)\Omega_M(H_2)}{\Omega_m(H_1)\Omega_m(H_2)} \\ &= 1/4\frac{1}{\Omega_M(H_1)\Omega_M(H_2}\left[1 - \frac{(2\Omega_M(H_1)\Omega_M(H_2))^2}{\Omega_m(H_1)\Omega_m(H_2)}\varepsilon_{12}\varepsilon_{21}\right] \end{aligned} \quad (16.23)$$

In order to explore situations where Det $(\overline{W} > 0$ holds true we consider the following scalar variations of our reaction coefficients $d_{11}, d_{22}, d_{12}, d_{21}$:

$$\alpha_{11}d_{11}, \alpha_{22}d_{22}, \alpha_{12}d_{12}, \alpha_{21}d_{21} \text{ with } \alpha_{ij} > 0.$$

In this case H_1, H_2 are to be substituted by $H_1/\alpha_{11}, H_2/\alpha_{22}$ as can be seen immediately from the following explication of (16.12)

$$C(g)'d_{11}H_1 + H_1d_{11}C(g) = -I, \quad C(r)d_{22}H_2 + H_2d_{22}C(r)' = -I.$$

The elements of matrix W are thus determined in nearly the same way as before – with the provision that the scalars α_{ij} appear as multipliers at the appropriated places. There remains to be shown therefore that the expression in square brackets [·] in the determinant (16.23) becomes positive for appropriate variations of adjustment speeds, i.e., that we can have

$$1 - \frac{\alpha_{11}^{-2}\alpha_{22}^{-2}}{\alpha_{11}^{-1}\alpha_{22}^{-1}}\frac{(2\Omega_M(H_1)\Omega_M(H_2))^2}{\Omega_m(H_1)\Omega_m(H_2)}\alpha_{12}\alpha_{21}(\varepsilon_{12}\varepsilon_{21}) > 0 \quad (16.24)$$

for a suitably chosen range of α_{ij}.

Now, it is obvious from (16.24) that for given α_{12}, α_{21}, for example, scalars α_{11}, α_{21} chosen sufficiently large will render the system asymptotically stable (or for given α_{11}, α_{22} there exist always sufficiently small α_{12}, α_{21} which will generate connective stability for system (16.5), (16.6)). This shows again the already known fact that the composite system will be asymptotically stable if the Classical dynamics is made sufficiently weak or long run in nature. Furthermore the expression (16.24) can be equivalently rewritten as follows

$$(\Omega_m(H_1)\Omega_m(H_2))/(2\Omega_M(H_1)\Omega_M(H_2)) > (\varepsilon_{21}\alpha_{12}\varepsilon_{21}\alpha_{21})/(\alpha_{11}\alpha_{22}) \quad (16.25)$$

In this form it very clearly shows how the stability characteristics of the Keynesian subsystem (as they are expressed by the two Liapunov matrices H_1, H_2) must dominate the off-diagonal interaction coefficients to obtain overall stability: The larger the interaction delimiters ε_{ij} are, the smaller we have to choose the α_{ij} to make the approach of Sect. 16.4 applicable (the α_{ii} can be set equal to one without loss of generality).

However, the above calculations also show that estimates for $\Omega_m(H_i)$, $\Omega_M(H_i)$, ε_{ij} have to be produced first to obtain more than the above very general statements (cf. Siljak (1978, pp. 110/111) for a simple numerical example in this regard). To this end Kronecker products and their application to matrix equations of type (16.12) can be of use (see Lancaster and Tismenetsky (1985, Chap. 12) for details). Yet, an investigation of the numerics of the above approach is beyond the scope of the present chapter. It should however be noted that neither Siljak's nor Michel and Miller's book does offer much help in this regard. Furthermore, a second problem with this approach is that it – by its very formulation – is insensitive to the typical sign structure of our cross-dual interconnection. The advantage of this method over one-shot approaches must therefore be considered as somewhat limited in the present case. It may, however, be improved either (a) by decomposing the system further into substructures (basic vs. non-basic commodities e.g., cf. Siljak (1978, Chap. 4) for related considerations), or (b) by adding more features to the interconnective (or Classical) part of the dynamics so that its influence on the stability of the system becomes more apparent.

Making use of the extension of Classical dynamics sketched in III.3 we shall indicate here briefly how this latter alternative can be pursued: In the second remark there we have proposed to consider the refined adjustment

$$\dot{x} = -d_{12}[C'p' + \overline{\sigma}C'\dot{p}'] \text{ instead of only } \dot{x} = -d_{12}C'p'$$

where the new term takes account of the fact that equal profit-rate differentials will have different effects on the conditions of supply when these differentials are rising than when they are falling. With regard to such an extension we get for the composite system in terms of (16.10) – due to its more complex price dynamics:

$$\begin{aligned} S_1 &: \dot{z}_1 = \overline{A}_{11}z_1 + e_{12}A_{12}z_2 + e'_{12}\overline{\sigma}A_{12}\dot{z}_2 \\ S_2 &: \dot{z}_2 = \overline{A}_{22}z_2 + e_{21}A_{21}z_1 \end{aligned}$$

which gives rise to:

$$\begin{aligned} S_1 &: \dot{z}_1 = \overline{A}_{11}z_1 + e'_{12}e_{21}\overline{\sigma}A_{12}A_{21}z_1 + e_{12}A_{12}z_2 + e'_{12}\overline{\sigma}A_{12}\overline{A}_{22}z_2 \\ S_2 &: \dot{z}_2 = \overline{A}_{22}z_{22} + e_{21}A_{21}z_1. \end{aligned} \tag{16.26}$$

Making use of the concrete form (16.5), (16.6) of this system (for $r = g$) gives for $A_{12}A_{21}$ the expression $-d_{12}C'd_{21}C = d_{12}Q_{11}$, where Q_{11} is symmetric and negative definite. With $e'_{12} = e_{21} = 1$ we therefore have introduced an extra stabilizing term into the diagonal of S_1, S_2. This new stabilizing term is to be contrasted in its influence with the new off-diagonal term $A_{12}\overline{A}_{22} = -d_{12}C'd_{22}C'$ and the new limit this term imposes on the cross-dual interaction.

Several conclusions may be drawn from this brief extension of system (16.5), (16.6). First, the above argument for increased stability is only valid if the perturbations $e'_{12}, e_{21} = 0$ are no longer allowed for in the notion of connective stability.

Secondly, excess demand functions (as they appear in Walrasian tâtonnement analysis) are here represented through the terms $\overline{A}_{11}, \overline{A}_{21}$. More general excess demand functions – known to create problems for the one-sided conventional tâtonnement process – are thereby made less central in (16.26) for two reasons: (a) Their influence on price dynamics (given by $\dot{z}_2 = e_{21} A_{21} z_1$) is now off-diagonal and thus need only be properly limited (but no longer be stable itself), and (b) their – possibly destabilizing – influence on quantity dynamics is now partly compensated for through the new stabilizing term in S_1.

16.8 Some Simulations Studies

It has been our aim in the preceding section to provide a meaningful application for the new type of stability analysis considered in Sect. 16.4 to indicate its usefulness for economic analysis. We have seen that this method gives rise to definite stability criteria which limit the adjustment speeds d_{ij} in a specific way.

In what follows some simulation results will be presented that exhibit the proven stability properties also for less stringent restrictions on adjustment speeds in the limit case $r = g = r^*$ and in the case of a time dependent rate of profit for the Classical cross-dual, the Keynesian dual dynamics and the composite dynamics.

The Classical cross-dual dynamics was simulated in the following time discrete form

$$\begin{bmatrix} x_{t+h} \\ p_{t+h} \end{bmatrix} = \begin{bmatrix} x_t \\ p_t \end{bmatrix} + h \begin{bmatrix} 0 & -d_{12}C' \\ d_{21}C & 0 \end{bmatrix} \begin{bmatrix} x_t \\ p_t \end{bmatrix}$$

with $h = 0.1$ the step size. As input–output matrix A and as matrices of reaction coefficients we used

$$A = \begin{bmatrix} 0.35 \; 0.55 \\ 0.25 \; 0.45 \end{bmatrix}, d_{12} = \begin{bmatrix} 1 \; 0 \\ 0 \; 2 \end{bmatrix}, d_{21} = \begin{bmatrix} 1 \; 0 \\ 0 \; 2 \end{bmatrix}.$$

Whereas Fig. 16.3 represents the graphs for the time path of relative prices and relative outputs of Classical dynamics for $1 + r = 1 + g = R^* = 1.29 = 1/\lambda(A)$, Fig. 16.4 depicts the relative price and output dynamics for the average and time-dependent rate of profit $R_t = p_t x_t / p_t A x_t$.

Both relative prices and relative outputs exhibit stability for the cases R^* as well as R_t. (The slightly increasing amplitude results from the fact that our time continuous-dynamics is approximated by the above type of a time-discrete system).[5]

[5] See also the observations on Euler's method in Ortega and Poole (1981, pp. 38 ff.), there with regard to models of predator-prey type.

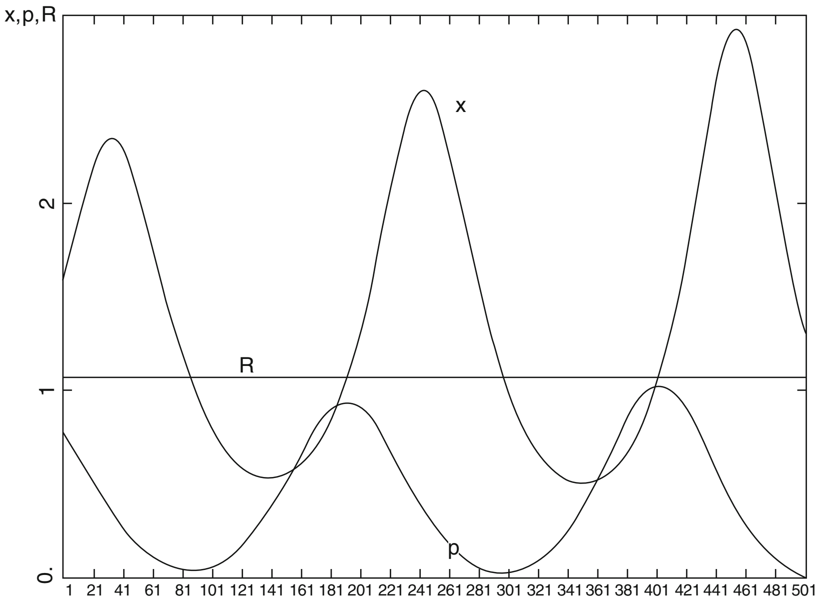

Fig. 16.3 Price-quantity business fluctuations

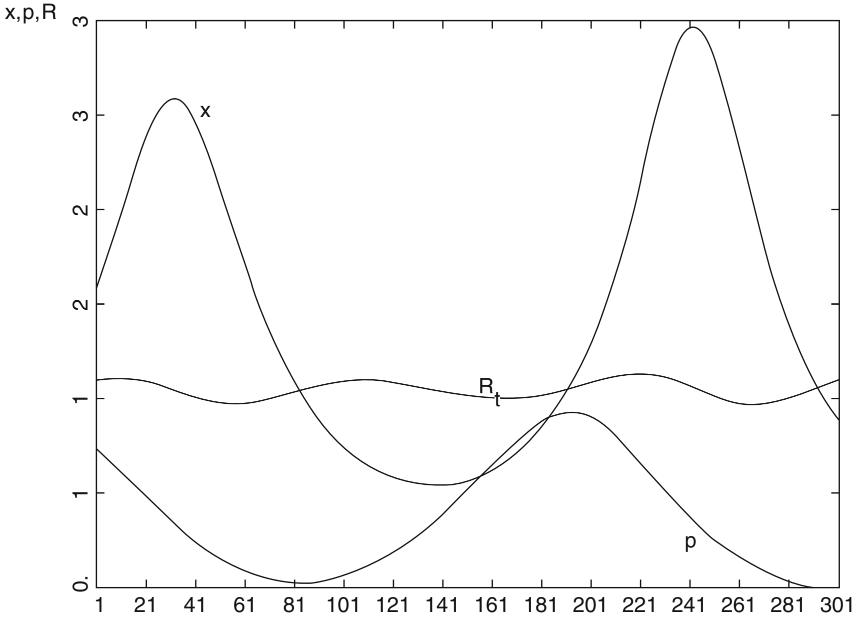

Fig. 16.4 Price-quantity business fluctuations

The Keynesian dual dynamics is captured in the following time discrete version:

$$\begin{bmatrix} x_{t+h} \\ p_{t+h} \end{bmatrix} = \begin{bmatrix} x_t \\ p_t \end{bmatrix} + h \begin{bmatrix} d_{11}C & 0 \\ 0 & d_{22}C' \end{bmatrix} \begin{bmatrix} x_t \\ p_t \end{bmatrix}$$

with the same matrix A in C, C' and reaction coefficients as follows

$$d_{11} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, d_{22} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.4 \end{bmatrix}.$$

As Figs. 16.5 and 16.6 demonstrate asymptotic stability prevails for both the price dynamics and the quantity dynamics.

Asymptotic stability of the composite system of Classical and Keynesian dynamics is demonstrated in the simulation results depicted in Figs. 16.6 and 16.7. In this simulation run the reaction coefficients for the cross-dual dynamics were chosen much larger than those for the dual dynamics. Yet, this (and many similar) simulation results show that stability can be expected even in such cases. This (and further simulation studies) support, therefore, the conjecture that the region of stability is much larger than indicated in the proofs of Sects. III, V. Similar results were obtained for composite systems of dimension 6 (with three prices and three outputs, cf. Fig. 16.8 for the Classical dynamics, and Fig. 16.9 for the composite system).

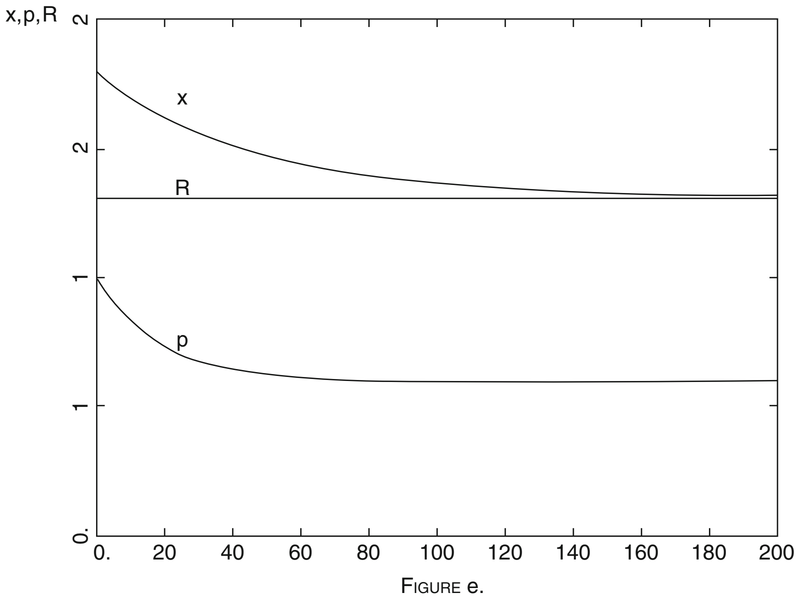

Fig. 16.5 Price-quantity dynamics: Monotonic convergence

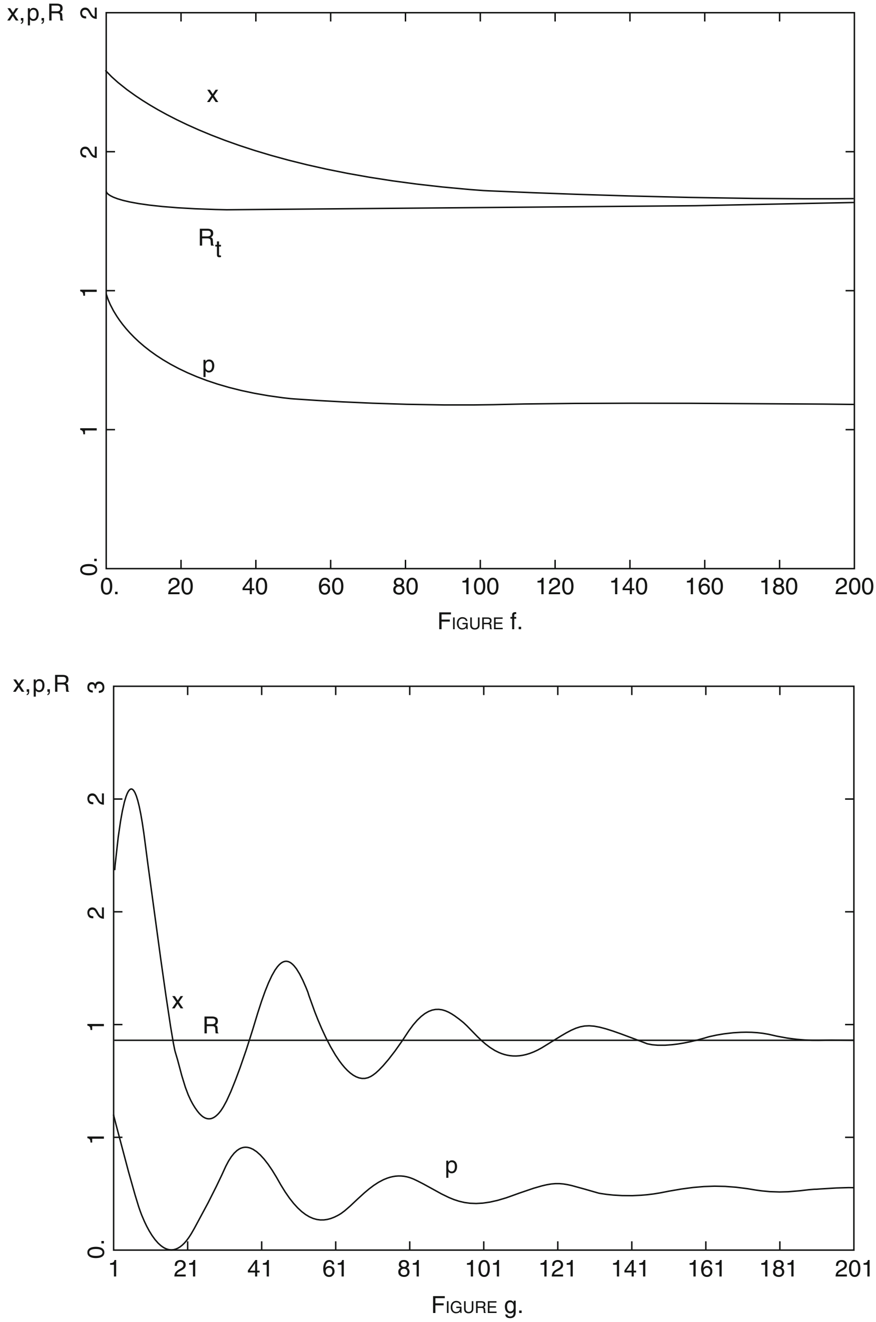

Fig. 16.6 Price-quantity dynamics: Monotonic or cyclical Convergence

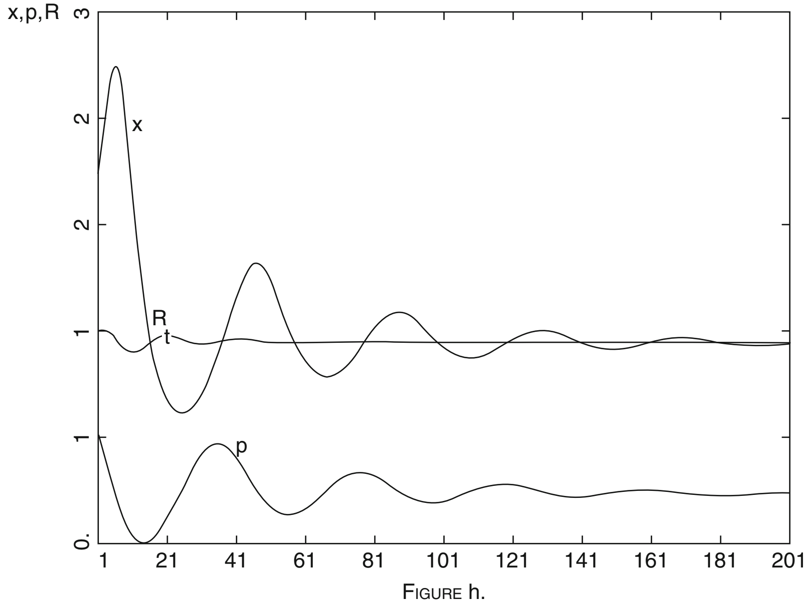

Fig. 16.7 Price-quantity dynamics: Damped oscillations

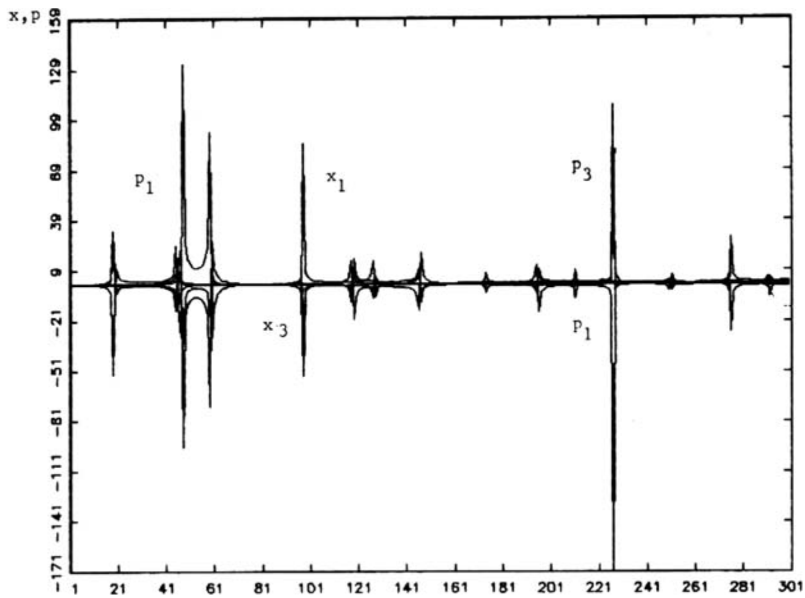

Fig. 16.8 Price-quantity dynamics: Composite systems

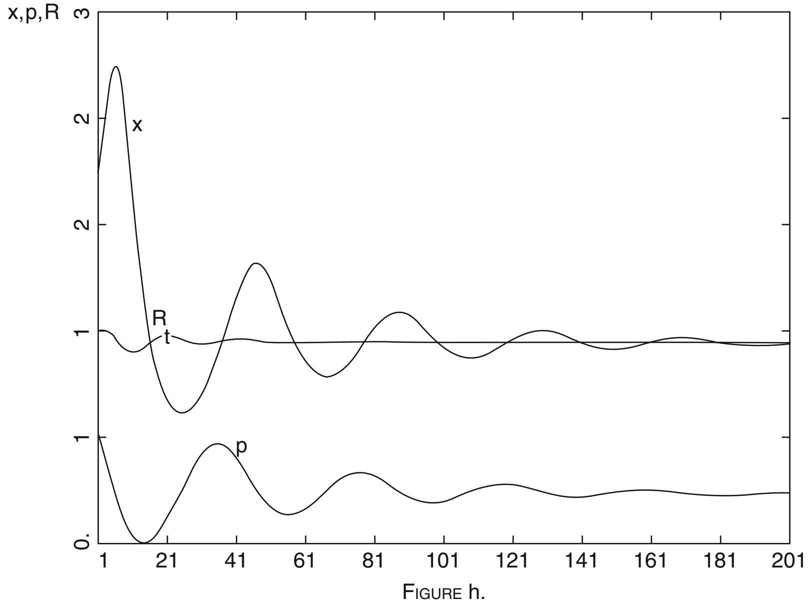

Fig. 16.9 Price-quantity dynamics: Composite systems

16.9 Conclusions

In this chapter we have reconsidered and combined two formulations of micro-dynamic adjustment processes. We have discussed the stability properties of Classical cross-dual and Keynesian dual adjustment processes first isolated for each approach and then in aggregate form. In our view it adds realism to micro-dynamic studies if both approaches are synthesized and their dynamics studied in a composite model. In this way we may move closer to the study of empirically relevant adjustment processes as observed by many econometric studies (cf. Gordon 1983 and Semmler 1984, Chap. 3 for a summary of such studies). We have found stability of the Hicksian kind and furthermore also full stability when the Classical part of the dynamics is limited in such a way that the dominant diagonal of the Keynesian substructure is preserved. We have also seen that quasi-negative definiteness may be utilized in certain other situations to prove composite stability. And, we have applied a fairly new approach to the stability of large scale systems which seemed to be particularly well-suited for the type of composite system we had to analyze (in another respect it was however fairly insensitive to the details of the given composite structure). The obtained results have been extended by means of computer simulations which suggest that stability will also prevail even if a much more general parameter variety than considered, for example, in the above application of the decomposition-aggregation method is allowed for. Yet, as the mathematical appendix will show, there are also definite limits for such a generalizing conjecture – at least at the present stage of the formulation of the Classical part of this composite dynamics.

With regard to the realism of such composite adjustment processes one should also express some caution since our modeled dynamics builds on a fairly simplified version of effective demand. Moreover, we have not analyzed the feedbacks resulting from supply constraints (labor, other inputs, or finance) to the output and price dynamics and the like. However, the proposed synthesized dynamics of Classical and Keynesian tradition should allow for such extensions and generalizations in future research. In that regard particularly the stabilizing aspects of Classical competition for the overall structure should be investigated in more detail than was possible in this chapter, especially in those cases where our composite dynamics might give rise to local stability. In such situations the introduction of (classically motivated) nonlinearities may provide an appropriate method for keeping the system within certain bounds. This final remark also shows that the analysis of the present chapter is in so far limited as it is still more of a local than of a global character.

Mathematical Appendix

Counterexamples

In the case where $r \neq g$ $(R \neq G)$ we get for the classical substructure Q_C of our system Q (here with $d_{ij} = I, i, j = 1, 2$):

$$Q_C + Q_c' = \begin{bmatrix} 0 & (G-R)A' \\ (G-R)A & 0 \end{bmatrix}$$

that is, this matrix is no longer skew-symmetric. This loss of anti-symmetry suggests that examples may be found where such a Q_C will exhibit eigenvalues with positive real parts. This implies that also matrix Q has such eigenvalues if the adjustment coefficients of the classical substructure $d_{ij}, i \neq j$ are made sufficiently large. The following example shows that this is indeed the case.

Example 1. $C = RA - I, C' = GA - I, R = 5.71, G = 1.31 < R^*(= 5.81)$.

$$A = \begin{bmatrix} 0.00 & 0.56 & 0.00 & 0.00 \\ 0.00 & 0.00 & 0.12. & 0.00 \\ 0.00 & 0.00 & 0.00 & 0.10 \\ 0.14 & 0.00 & 0.00 & 0.00 \end{bmatrix}$$

For this example we get the following list of eigenvalues:

Q_K		Q_C		Q	
Real Part	Im.Part	Real Part	Im.Part	Real Part	Im.Part
−1.23	0.00	0.00	12.29	−1.56	4.67
−1.00	0.23	0.00	−12.29	−1.56	−4.67
−1.00	−0.23	0.00	0.58	→ 0.04	3.81
−0.77	0.00	0.00	−0.58	→ 0.04	−3.81
−1.98	0.00	−1.22	6.76	−0.50	0.00
−1.00	0.98	−1.22	−6.76	−0.36	0.00
−1.00	−0.98	→ 1.22	6.76	−2.05	2.98
−0.02	0.00	→ 1.22	−6.76	−2.05	−2.98

Note that Q_C has eigenvalues with positive real parts as does Q. However, we had to choose $6^* Q_C$ in the off-diagonal of Q, i.e., a factor '6' as adjustment coefficient to ensure that the positive real part of the eigenvalue of Q_C was in fact transferred to an eigenvalue of the matrix Q. Our general impression here was that counterexamples are only found under relatively exceptional conditions. For example, we had to choose $r = R - 1 = 471\%$ and $g = G - 1 = 31\%$ to produce the above counterexample to the stability of Q.

Example 2. $G = 1.11 < R = 3.71 < R^* = 3.81$.

$$A = \begin{bmatrix} 0.000 & 0.083 & 1.500 & 2.520 \\ 0.083 & 0.000 & 1.200 & 3.860 \\ 0.000 & 0.000 & 0.080 & 3.310 \\ 0.000 & 0.000 & 0.000 & 0.000 \end{bmatrix}.$$

This matrix leads to the following list of eigenvalues (without any support from increased adjustment coefficients, $d_{ij} \neq I$):

Q_K		Q_C		Q	
Real Part	Im.Part	Real Part	Im.Part	Real Part	Im.Part
−1.00	0.00	0.00	11.85	→ 0.86	12.01
−0.70	0.00	0.00	−11.85	→ 0.86	−12.01
−0.70	0.00	0.00	3.09	−2.74	3.60
−1.29	0.00	0.00	−3.09	−2.74	−3.60
−0.03	0.00	0.00	2.10	−1.92	2.08
−1.97	0.00	0.00	−2.10	−1.92	−2.08
−0.91	0.00	0.00	0.01	−0.01	0.01
−1.00	0.00	0.00	−0.01	−0.01	−0.01

This example is in so far more interesting than the preceding one as it shows that Q can be unstable even if all eigenvalues of the two subsystems behave quite normal, i.e., as if the case where Q_C is skew-symmetric were given. And again: Counterexamples could only be produced by making the difference between R and G implausibly large. Instability only came about when extreme situations were assumed, at least for the low dimensions here investigated.

Remark 16.15. That stability of Q can get lost even though Q_C is of pure center type and thus in a way 'neutrally stable' (as our last example shows) suggests that stability may also get lost in cases where we have $r = g$, but where the adjustment speeds of the off-diagonal terms d_{12}, d_{21} differ significantly from each other (and are sufficiently large in comparison to d_{11}, d_{22}). Differences between d_{12} and d_{21} also destroy the skew-symmetry in the off-diagonal of the matrix Q, but do this in a way which does not modify the purely imaginary character of the eigenvalues of Q_C [see Sect. 16.3 in this regard]. Nevertheless – as the following example shows – some roots of Q may (not typically, but) in certain cases exhibit eigenvalues with positive real parts.

Example 3. The Matrix A and the adjustment coefficients d_{ij} are:

$$A = \begin{bmatrix} 0.0 & 0.2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.8 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.9 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 6.4 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 8.7 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 \\ 0.4 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix};$$

$$\begin{aligned} d_{11} &= < (0.1, 92, 0.3, 0.1, 0.1, 0.5, 0.4, 0.3, 0.7, 0.3, 0.1, 0.6) > \\ d_{22} &= < (13, 0.4, 15, 0.4, 0.2, 0.4, 14, 0.2, 24, 0.6, 0.1, 13) > \\ d_{12} &= < (91, 5, 0.1, 0.5, 0.17, 0.5, 0.1, 33, 0.5, 0.5, 0.3, 25) > \\ d_{22} &= < (0.2, 0.7, 0.3, 0.9, 0.3, 0.5, 0.3, 0.1, 4, 0.4, 0.2, 0.9) > . \end{aligned}$$

The eigenvalues of this example are:

Q_K		Q_C		Q	
Real Part	Im.Part	Real Part	Im.Part	Real Part	Im.Part
0.70	0.00	0.00	11.08	−91.96	0.00
−0.62	0.00	0.00	−11.08	−11.54	5.22
−0.07	0.00	0.00	7.23	−11.54	−5.22
−0.09	0.04	0.00	−7.23	−14.99	0.00
−0.09	−0.04	0.00	3.66	−13.20	1.78
−0.50	0.00	0.00	−3.66	−13.20	−1.78
−0.16	0.00	0.00	2.89	−13.89	0.00
−0.41	0.00	0.00	−2.89	→ 0.30	4.32
−0.23	0.00	0.00	1.86	→ 0.30	−4.32
−0.32	0.05	0.00	−1.86	−1.82	0.78
−0.32	−0.05	0.00	0.23	−1.82	−0.78
−92.00	0.00	0.00	−0.23	−0.16	1.89
−24.00	0.00	0.00	2.15	−0.16	−1.89
−15.00	0.00	0.00	−2.15	−0.13	0.20
−14.00	0.00	0.00	0.77	−0.13	−0.20
−0.60	0.00	0.00	−0.77	−0.25	0.65
−0.46	0.00	0.00	0.58	−0.25	−0.65
−0.38	0.06	0.00	−0.58	−0.00	0.01
−0.38	−0.06	0.00	0.15	−0.00	−0.01
−0.19	0.04	0.00	−0.15	−0.32	0.35
−0.19	−0.04	0.00	0.13	−0.32	−0.35
−13.00	0.00	0.00	−0.13	−0.44	0.00
−13.00	−0.00	0.00	0.01	−0.98	0.00
−0.10	0.00	0.00	−0.01	−0.30	0.00

In this example we have made use of a profit rate $R(= G) = 1 + r \approx 1.21$ in view of a maximal profit rate $R^* \approx 1.94$ (whereas the maximum profit rate of $(A + A')/2$ is ≈ 0.19, i.e., this symmetric part of A has a maximal eigenvalue which is much larger than one, see the theorem in Sect. 16.3 for the meaning of this fact).

Remark 16.16. Since we have by now obtained counterexamples for the stability of our composite system Q even in the case where $r = g$, but $d_{12} \neq d_{21}$, it seems high time to test whether these types of counterexamples can be modified in such a way that even an unstable situation for our basic case

$$Q = \begin{bmatrix} C & -C' \\ C & C' \end{bmatrix}$$

can be found ($n > 2$, see Appendix 2). The difficulties we have experienced in the main part of the chapter and the above last counterexample suggest that difficulties for our composite Keynesian/Classical system may also arise in this most basic situation – due to the incompatible and weak stability properties of the Classical subsystem Q_C. And indeed, our next and final example shows that even then eigenvalues with positive real parts can be generated by the integration of an asymptotically stable Keynesian and a purely ‘cyclical’ Classical type of adjustment process.

Example 4. $R = G \approx 1.54 < R^* \approx 1.64$ $(R^* = 0.12$ with regard to $(A + A')/2)$:

$$A = \begin{bmatrix} 0.0 & 10.2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.2 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.8 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.1 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.9 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \\ \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 & 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.2 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.4 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.7 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.3 \\ 0.4 & 0.0 & 0.0 & 10.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

The composite system is given by:

$$Q = \begin{bmatrix} C & -C' \\ C & C' \end{bmatrix} \text{ with } C = RA - I.$$

The eigenvalues of this example are:

Q_K		Q_C		Q	
Real Part	Im.Part	Real Part	Im.Part	Real Part	Im.Part
−1.92	0.00	0.00	22.31	−7.10	20.65
−1.81	0.46	0.00	−22.31	−7.10	−20.65
−1.81	−0.46	0.00	15.75	→ 3.79	19.49
−1.47	0.81	0.00	−15.75	→ 3.79	−19.49
−1.47	−0.81	0.00	1.94	−1.62	2.02
−0.99	0.93	0.00	−1.94	−1.62	−2.02
−0.99	−0.93	0.00	1.74	−1.58	1.82
−0.54	0.80	0.00	−1.74	−1.58	−1.82
−0.54	−0.80	0.00	1.44	−1.25	1.69
−0.06	0.00	0.00	−1.44	−1.25	−1.69
−0.20	0.47	0.00	1.26	−1.15	1.35
−0.20	−0.47	0.00	−1.26	−1.15	−1.35
−1.91	0.00	0.00	1.12	−1.00	1.23
−1.81	0.46	0.00	−1.12	−1.00	−1.23
−1.81	−0.46	0.00	1.00	−0.85	1.14
−1.47	0.81	0.00	−1.00	−0.85	−1.14
−1.47	−0.81	0.00	0.62	−0.43	0.79
−0.99	0.93	0.00	−0.62	−0.43	−0.79
−0.99	−0.93	0.00	0.57	−0.44	0.77
−0.54	0.80	0.00	0.57	−0.44	−0.77
−0.54	−0.80	0.00	0.42	−0.37	0.51
−0.06	0.00	0.00	−0.42	−0.37	−0.51
−0.20	0.47	0.00	0.01	−0.01	0.01
−0.20	−0.47	0.00	−0.01	−0.01	−0.01

Thus Q is not stable even in this simplest case of a composite Keynesian/ Classical system.

This final example as well as the former ones in our view imply that a reformulation of the Classical process is the most urgent task with regard to an improved stability analysis of such composite systems. The negative feedback of the Keynesian system is in general insufficient to turn the center-type stability of the Classical substructure into asymptotic stability. An improvement of Classical dynamics therefore has to be undertaken first [see Flaschel and Semmler (1987) for an attempt into this direction]. On the basis of this a further attempt of integrating these two different approaches to a stable composite dynamics may be more successful with regard to the formulation of general stability assertions.

Remark 16.17. Goodwin considers in Goodwin and Punzo (1986, pp. 78 ff.) various control mechanisms of market systems, which are closely related to the structures here investigated [which he, however, relates more to the Walrasian approach than to a mixture of Keynesian and Classical views]. On p. 81 he introduces a composite system such as ours which there is of the simple form

$$\begin{bmatrix} \dot{x} \\ \dot{p} \end{bmatrix} = \begin{bmatrix} \Omega - I & -(\Omega - I) \\ \Omega - I & \Omega - I \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}$$

where Ω denotes the diagonal matrix of eigenvalues of a given linear production technology $C = RA - I$, R the gross rate of profit. The counterexamples of this appendix show, however, that the stability analysis of such a system is problematic, if the original production data are used in place of such an eigenvalue composition [which is not mathematically 'similar' to the structure Q]. Therefore, though a composite control structure such as the above appears as a very interesting and natural one, our examples have shown that the eigenvalues Ω of its submatrices Q_C, Q_K (or C, C') do not give us immediate information on the stability of the composite structure Q.

An Application of the Routh–Hurwitz Theorem

We shall here consider and prove the stability of the $2n$-matrix

$$Q = \begin{bmatrix} C & -C' \\ C & C' \end{bmatrix} = Q_C + Q_K, \quad C = RA - I, \quad R = 1 + r < R^*$$

by means of the Routh–Hurwitz conditions for the case $n = 2$.

By well-known theorems on such block-matrices Q, cf. e.g., Gantmacher (1970, II. par.5), we get for the characteristic polynomials of the above matrices Q_K, Q_C, Q the following simplified expressions ($|\ldots|$ the determinant of these matrices):

Lemma 16.18.

(a) $|Q_K \Omega I| = |(C - \Omega I)(C' - \Omega I)| = |C - \Omega I|^2$
(b) $|Q_C - \Omega I| = |CC' + \Omega^2 I|$
(c) $|Q - \Omega I| = |(2C - \Omega I)(2C' - \Omega I) + \Omega^2 I| \cdot 2^{-n}$
$= |(C - \Omega I)(C' - \Omega I) + CC'|$
$= |2CC' + \Omega^2 I - \Omega(C + C')|$

Proof. Case (c) (the two other cases are shown in the same way):

$$|Q - \Omega I| = \begin{vmatrix} 2C - \Omega I & -\Omega I \\ C & C' - \Omega I \end{vmatrix} = \begin{vmatrix} 2C - \Omega I & -\Omega I \\ 2c & 2(C' - \Omega I) \end{vmatrix} \cdot 2^{-n}$$

$$= \begin{vmatrix} 2C - \Omega I & -\Omega I \\ +\Omega I & 2C' - \Omega I \end{vmatrix} \cdot 2^{-n} = 2^{-n} |(2C - \Omega I)(2C' - \Omega I) + \Omega^2 I|,$$

since $2C - \Omega I$ and ΩI or ΩI and $2C' - \Omega I$ are commutative. We thus get:

$$\begin{aligned} |Q - \Omega I| &= 2^{-n} \cdot 2^n |2CC' - \Omega(C + C') + \Omega^2 I| \\ &= |(C - \Omega I)(C' - \Omega I) + CC'| \\ &= |2CC' + \Omega^2 I - \Omega(C + C')|. \end{aligned}$$

□

Remark 16.19.

(1) Assertion (c) again implies that all eigenvalues of Q_C are purely imaginary, since CC' is symmetric and positive definite which means that $-\mu = \Omega^2$ gives rise to positive numbers μ as solutions of $|CC' - \mu I| = 0$.
(2) The matrix C (and therefore also Q_K) by assumption has only eigenvalues with negative real parts. To prove the same for the structure Q means that the perturbation terms CC' (or $\Omega^2 I$) have to be taken into consideration appropriately. In the following we shall follow this approach in the special case where $n = 2$.

Lemma 16.20. *Let n be equal to 2 and let $tr\,C$ denote the trace of matrix C.*

(a) The characteristic polynomial of Q_K is given by

$$\Omega^4 - 2tr\,C\,\Omega^3 + (tr(CC') + |C + C'|)\Omega^2 - 2tr\,C|C|\Omega + |C|^2$$

(b) The characteristic polynomial of Q_C is given by

$$\Omega^4 + tr\,(CC')\Omega^2 + |CC'|$$

(c) The characteristic polynomial of Q is given by

$$\Omega^4 - 2tr\,C\,\Omega^3 + (2tr\,(CC') + |C + C'|)\Omega^2 - 4tr\,C|C|\Omega + 4|C|^2$$

Proof.

(a) The assertion follows from

$$|C - \Omega I|^2 = (\Omega^2 - \text{tr}\, C\Omega + |C|)^2$$
$$= \Omega^4 - 2\text{tr}\, C\Omega^3 + (\text{tr}\,(C)^2 + 2|C|)\Omega^2 - \text{tr}\,(C)|C|\Omega + |C|^2,$$

since $\text{tr}\,(C)^2 + 2|C| = \text{tr}\,(CC') + |C + C|'$ (as can be checked immediately for $n = 2$).

(b) straightforward

(c) From Lemma 2 we know

$$0 = |Q - \Omega I| = |CC' - \Omega(C + C') + \Omega^2 I + CC'|$$
$$= |Q_K(\Omega) + CC'|$$
$$= |Q_K(\Omega)| + \begin{vmatrix} (CC')_1 \\ (Q_K(\Omega))_2 \end{vmatrix} + \begin{vmatrix} (Q_K(\Omega))_1 \\ (CC')_2 \end{vmatrix} + |CC'|$$
$$= |Q_K(\Omega)| + 3|CC'| > -\Omega \begin{vmatrix} (CC')_1 \\ (C + C')_2 \end{vmatrix} - \Omega \begin{vmatrix} (C + C')_1 \\ (CC')_2 \end{vmatrix} + \begin{vmatrix} (\Omega^2 I)_1 \\ (CC')_2 \end{vmatrix}$$
$$+ \begin{vmatrix} (CC')_1 \\ (\Omega^2 I)_2 \end{vmatrix},$$

where subindices denote the rows of the corresponding matrices. Further calculations then give

$$= |Q_K(\Omega)| + 3|C|^2 - \Omega \left[\begin{vmatrix} (CC')_1 \\ (C + C')_2 \end{vmatrix} + \begin{vmatrix} (C + C')_1 \\ (CC')_2 \end{vmatrix} \right] + \Omega^2\, \text{tr}(CC').$$

Since $|Q_K(\Omega)|$ is the characteristic polynomial of Q_K there remains to be shown that the bracket following $-\Omega$ is equal to $2\text{tr}\, C|C|$ to obtain (c) from (a). With regard to this term we get:

$$(C_{11}^2 + C_{12}^2) \cdot 2C_{22} - (C_{11}C_{21} + C_{12}C_{22}) \cdot (C_{13} + C_{21})$$
$$+ 2C_{11}(C_{21}^2 + C_{22}^2) - (C_{11}C_{21} + C_{12}C_{22}) \cdot (C_{12} + C_{21})$$
$$= 2C_{22}C_{11}^2 + 2C_{22}C_{12}^2 + 2C_{11}C_{21}^2 + 2C_{11}C_{22}^2$$
$$- 2C_{11}C_{21}^2 - 2C_{22}C_{12}^2 - 2C_{11}C_{21}C_{12} - 2C_{12}C_{22}C_{21}$$
$$= 2C_{11}(C_{11}C_{22} - C_{21}C_{12}) + 2C_{22}(C_{11}C_{22} - C_{12}C_{21})$$
$$= 2\text{tr}\, C|C|.$$

□

Theorem 16.21. *The characteristic polynomial $\Omega^4 + a_1\Omega^3 + a_2\Omega^2 + a_3\Omega + a_4$ of Q is a Hurwitz polynomial, i.e., it fulfills:*

$$a_1 > 0, \quad a_1a_2 - a_3 > 0, \quad a_3(a_1a_2 - a_3) - a_4a_1^2 > 0, \quad a_4 > 0$$

which implies that all of its roots have negative real part, so that matrix Q is a stable matrix [see, e.g., Gantmacher (1971, p. 172)].

Proof. By Lemma 2 (a) and the properties of the matrix $|C|$ we already know that $|Q_K - \Omega I|$ is a Hurwitz polynomial, i.e., it fulfills the above three inequalities with regard to its coefficients $b_1,\ldots, b_4$.

Case 16.22. $|C + C'| \leq 0$. In this case we get from Lemma 3 for the coefficients $a_1,\ldots, a_4$ of $|Q - \Omega I|$ the (in)equalities:

$$a_1 = b_1, \quad a_2 \geq 2b_2, \quad a_3 = 2b_3, \quad a_4 = 4b_4.$$

This immediately implies that $a_1, \ldots, a_4$ must satisfy the same set of inequalities as $b_1, \ldots, b_4$ which proves the assertion.

Case 16.23. $|C + C'| > 0$. In this case we get from $\text{tr}\,(C + C') = 2\,\text{tr}\,C < 0$, that both eigenvalues of $C + C'$ must be negative (and, of course, real, since $C + C'$ is symmetric). The matrix $C + C'$ – and also $(C + C')/2$ – is in this case negative definite, i.e., the theorem in Sect. 16.3 then leads to a positive conclusion with regard to the stability of Q, cf. also Hahn (1982, p. 752). All roots of $|Q - \Omega I| = 0$ must therefore have negative real parts. □

Remark 16.24. In the case $r = r^*$ of a maximal profit rate we have in particular $|C| = 0$, i.e., $a_3 = a_4 = 0(b_3 = b_4 = 0)$. In this special case the roots of Q are simply given by the summation of the real roots of Q_K and the imaginary roots of Q_C (as can be easily shown). In general, however, no such summation rule is true and the roots of Q bear no simple relationship to the real or complex roots ($n \geq 3$) of Q_K and the purely imaginary roots of Q_C. In particular, they are not necessarily 'in between' these extremes. Yet, numerical calculations have shown that the above summation property may be approximately true if $\lambda(A)$ and $\lambda((A + A')/2)$ are not very different from each other.

References

Aoki, M. (1977). Dual stability in a Cambridge-type model. *Review of Economic Studies, 44*, 143–151

Arrow, K., Karlin, S. & Suppes, P. , (Eds.). (1960). *Mathematical social sciences*. Stanford: Stanford University Press

Berussou, J. & Titli, A. (1982). *Interconnected dynamical systems: stability, decomposition and decentralisation*. Amsterdam: North Holland

Drazen, A. (1980). Recent developments in macroeconomic disequilibrium theory. *Econometrica, 48*, 283–306
Duménil, G. & Lévy, D. (1987a). The dynamics of competition: A restoration of the Classical analysis. *Cambridge Journal of Economics, 11*, 133–164
Duménil, G. & Lévy, D. (1987b). *The stability of long-term equilibrium in a general disequilibrium model*. Paris, Mimeo: CEPREMAP
Flaschel, P. & Semmler, W. (1986). The dynamic equalization of profit rates for input–output models with fixed capital. In W. Semmler (Ed.), *Competition, Instability, and Nonlinear Cycles*. Berlin: Springer, 1–34
Flaschel, P. & Semmler, W. (1987). Classical and neoclassical competitive adjustment processes. *The Manchester School, 55*, 13–37
Franke, R. (1987). *Production prices and dynamical processes of the gravitation of market prices*, Bern: Peter Lang
Fukuda, W. (1975). The output adjustment mechanism in a multisectoral economy. *Kobe University Economic Review, 21*, 53–62
Gantmacher, F. R. (1970/1). *Matrizenrechnung I, II*. Berlin: VEB Deutscher Verlag der Wissenschaften
Goodwin, R. M. (1953). Static and dynamic general equilibrium models, In R. M. Goodwin (Ed.), *Essays in linear economic structures*. London: Macmilan, 75–120
Goodwin, R. M. (1970). *Elementary economics from the higher standpoint*. Cambridge: Cambridge University Press
Goodwin, R. M. (1983). *Essays in linear economic structures*. London: Macmillan
Goodwin, R. M. (1988). *Essays in nonlinear economic dynamics*. Bern: Peter Lang
Goodwin, R. M. & Punzo, L. F. (1986). *The Dynamics of a Capitalist Economy*. Cambridge: Polity
Gordon, R. J. (1983). A century of evidence on wage and price stickiness in the United States, the United Kingdom and Japan. In J. Tobin (Ed.), *Macroeconomics, prices and quantities*. Washington D.C.: Brookings Institutions, 125–147
Hahn, F. (1982). Stability. In K. J. Arrow & M. D. Intrilligator (Eds.), *Handbook of mathematical economics*. Amsterdam: North Holland, 45–67
Hicks, J. (1965). *Capital and growth*. Oxford: Oxford University Press
Jorgenson, D. W. (1960). A dual stability theorem. *Econometrica, 28*, 892–892
Kaldor, N. (1985). *Economics without equilibrium*. New York: M.E. Sharpe
Kemp, M. C & Kimura, Y. (1978). *Introduction to mathematical economics*. Heidelberg: Springer
Lancaster, P. (1969). The Theory of Matrices. New York: Academic Press
Lancaster P. & Tismenetsky, M. (1985). *The Theory of matrices*. New York: Academic
Leijonhuvud, A. (1968). *On keynesian economics and the economics of keynes*. Oxford: Oxford University Press
Marshall, A. (1947). *Principles of economics*. London: Macmillan
Marx, K. (1967). *Capital, Vol.III*. New York: International Publishers
Mas–Colell, A. (1986). Notes on Price and Quantity Dynamics. In H. Sonnenschein (Ed.), *Models of economic dynamics*, Heidelberg: Springer, 86–112
Medio, A. (1987). A multisector model of the trade cycle. In D. Batten, J. Casti & B. Johansson (Eds.), *Economic evolution and structural adjustment* (pp. 291–313). Heidelberg: Springer
Michel, A. N. & Miller, R. K. (1977). *Qualitative analysis of large scale dynamical systems*. New York: Academic
Morishima, M. (1960). A reconsideration of the Walras-Cassel-Leontief model of general equilibrium. In K. Arrow, S. Karlin, & Suppes, P. (Eds.), *Mathematical social sciences*. Stanford: Stanford University Press, 156–177
Morishima, M. (1976). *The economic theory of modern society*. Cambridge, UK: Cambridge University Press
Morishima, M. (1977). *Walras' economics*. Cambridge, UK: Cambridge University Press
Ortega, Y. M. & Poole, W. G. Jr. (1981). *An introduction to numerical methods for differential equations*. Marshfield, Mass: Pitman

Ricardo, D. (1951). *Principles of political economy and taxation*. Cambridge: Cambridge University Press

Semmler, W. (1984). *Competition, monopoly and differential profit rates*. New York: Columbia University Press

Semmler, W., (Ed.). (1986). *Competition, instability and nonlinear cycles*. Lecture Notes in Economics and Mathematical Systems. Heidelberg: Springer

Siljak, D. D. (1978). *Large-Scale dynamic systems. Stability and structure*. New York: North Holland

Singh, M. G. & Titli, A. (Eds.). (1979). *Handbook of large scale systems. Engineering applications*. Amsterdam: North Holland

Smith, A. (1974). *The wealth of nations*. Middlesex: Pinguin

Sonnenschein, H., (Ed.). (1986). *Models of economic dynamics*, Heidelberg: Springer

Tobin, J., (Ed.). (1983). *Macroeconomics, prices and quantities*. Washington D.C.: Brookings Institutions

Walras, L. (1977). *Elements of pure economics*. Fairfield, New Jersey: Kelley

Woods, J. E. (1978). *Mathematical economics*. London: Longman

Zurmühl, R. (1964). *Matrizen und ihre technischen anwendungen* Heidelberg: Springer

Part IV
Gravitation or Convergence in Classical Macro-Dynamics

What we have established in the preceding part for prices of production (and basically a given average rate of profit) in principle also characterizes the macroeconomic level, i.e., the conflict over income distribution between capital and labor and therefore the dynamics around the wage–profit curve of the Classical system of production prices. These dynamics are described by Marx in his formulation of the general law of capitalist accumulation and has been formalized for the first time in a seminal paper by Richard Goodwin (1967) with explicit reference to the Lotka–Volterra predator–prey type of population dynamics. This type of dynamics thus not only applies to the micro-adjustment processes of quantities and prices of a capitalist economy, but also to the macrodynamics of the distribution of income between capital and labor.

Goodwin's growth cycle model is indeed exactly of the type that was investigated by Lotka and Volterra for predator–prey systems and it therefore implies the structurally unstable center type dynamics of the predator–prey model for Goodwin's growth cycle model as well. This means that nearly any modification of these dynamics will destroy their closed orbit structure. A typical example for such an occurrence is presented in Chap. 17, where the wage level and the price level dynamics are formulated separately and not just as real wage dynamics as in Goodwin's (1967) original approach. Nevertheless, the basic message of the Goodwin growth cycle model remains intact there in general (under realistic conditions), namely the overshooting mechanism in the conflict between capital and labor, which is characterized by excessive distribution towards labor income in prosperity phases and excessive redistribution to capital in phases of economic depression (and where both processes continue to work for some time even when prosperity is already faltering or when there is already economic recovery under way). From the macro-perspective, order is therefore generated in this framework by persistent (overshooting) fluctuations in the conflict over income distribution and not by convergence to some sort long-period equilibrium as it was in part suggested by the stability investigations of the preceding part.

We generalize the Goodwin approach in various respect, most notably by assuming – besides unskilled workers – a skilled labor force which can either cooperate with unskilled workers or with capital. The question then is of course what this implies for the Goodwin overshooting mechanism in both cases. We also confront the Goodwin model with the facts, first in the same way as Solow (1990) by just showing the phase plots of the wage share and the employment rate for a number of actual economies and then, in a new contribution, by employing modern econometric techniques which allow to separate the long phase Goodwin cycle from the Keynesian business cycle (applied to the case of the US economy). The general result in this part of the book will be that the Goodwin growth cycle dynamics represents a significant contribution to the analysis of the working of capitalist economies on the macro-level.

We close this final part of the book by positioning the Goodwin (1967) approach to cyclical growth within a larger framework, the so-called Keynes–Wicksell approach to macrodynamics which is basically however still of a supply side type, apart from a Wicksellian approach to inflation dynamics. The integration of Keynesian quantity adjustment processes must here be left to other contributions, see in particular Chiarella et al. (2005) in this regard.

References

Chiarella, C., Flaschel, P., & Franke, R. (2005). *Foundations for a disequilibrium theory of the business cycle. Qualitative analysis and quantitative assessment*. Cambridge, UK: Cambridge University Press.

Goodwin, R. (1967). A growth cycle. In: C. H. Feinstein (Ed.), *Socialism, capitalism and economic growth*. Cambridge, UK: Cambridge University Press.

Solow, R. (1990). Goodwin's growth cycle: Reminiscence and rumination. In: K. Velupillai (Ed.), *Nonlinear and multisectoral macrodynamics. Essays in honour of Richard Goodwin* (pp. 31–41). London: Macmillan.

Chapter 17
Some Stability Properties of Goodwin's Growth Cycle Model

17.1 Introduction

This chapter generalizes Goodwin's (1972) growth cycle model as reconsidered in Velupillai (1979) and it also extends the proofs of some assertions made by the latter author. This extended version which we shall introduce in Sect. 17.2 depends on a money-illusion parameter η in such a way that the Goodwin case becomes a bifurcation point between those parameter values ($\eta > 0$) where the extended model is globally asymptotically stable and those where it is totally unstable ($\eta < 0$). A by-product of this result is that Goodwin's dynamical system obviously cannot be structurally stable. This method of demonstration replaces Velupillai's formal proof by economic reasoning. Section 17.3 then shows why Velupillai's demonstration of the closed-orbit structure of Goodwin's model is not yet complete and it briefly indicates how to fill the existing gaps. Since this chapter is supplementary to Velupillai's (1979) article, the reader should consult his paper for further explanations of the model and the symbols used.

Apart from providing some improvements of the original contribution by Velupillai (1979), the chapter also considers a generalization of Olech's theorem to the case of growth dynamics. Economic models are normally based on growth laws of motion and thus only well-defined in the positive orthant of the mathematical phase space $\Re^2$. Yet such models can be expanded from the positive orthant $\Re^2_+$ to the whole of $\Re^2$ by the use of a variable transformation by means of logarithms. The chapter provides a proposition for the situation after this variable transformation that imply that the original growth dynamics is globally asymptotically stable in its phase space $\Re^2_+$.

17.2 An Extended Goodwin Cycle

The core variables of both Goodwin's 1972 and the following growth cycle model are: u (labor's share) and v (the employment rate). By definition these two variables must fulfill the following two identities: $\widehat{u} = \widehat{w} - \alpha, \widehat{v} = \widehat{Y} - (\alpha + \beta)$, where w

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_17,

denotes the real wage, Y output, α, β the given growth rates of labor productivity Y/L and the labor force, and where $\widehat{w}$ is used to symbolize the growth rate, here for example of real wages w.

Consider now the following generalization of Wolfstetter (1977, pp. 147ff.) reformulation of Goodwin's model:[1]

$$\widehat{w} + \eta\pi = f(v), \quad f' > 0 \tag{17.1}$$

$$\pi = \dot{p}/p = g((1+r)wL/Y - 1), \quad g' > 0, g(0) = 0, \quad r \text{ constant} \tag{17.2}$$

$$\sigma = K/Y, \quad \sigma \text{ constant} \tag{17.3}$$

$$\widehat{K} = s(u)Y/K = s(u)/\sigma, \quad s' < 0. \tag{17.4}$$

Equation (17.1) is Goodwin's Phillips-curve, now in general nonlinear and augmented by a term which accounts for money-illusion ($\eta\pi$), i.e., workers receive a real wage which is lower for $\eta > 0$ (higher for $\eta < 0$) than the target $f(v)$ they did actually bargain for. Equation (17.2) – a mark-up equation – describes how the rate of inflation π is formed. Both equations are also applied and further explored in Wolfstetter (1977), there in linearized form and for $\eta \geqq 0$ solely.[2] Assumptions (17.3) are standard for the Goodwin model. Equation (17.4), finally, simply generalizes Goodwin's accumulation formula $\hat{K} = (1-u)/\sigma$ by assuming a more flexible savings behavior.

Combining (17.1)–(17.4) on the basis of the two foregoing identities gives the following autonomous system of ordinary differential equations

$$\widehat{u} = f(v) - \alpha - \eta g((1+r)u - 1) = \widetilde{f}(u, v, \eta), \tag{17.5}$$

$$\widehat{v} = s(u)/\sigma - (\alpha + \beta) = \widetilde{s}(u). \tag{17.6}$$

We assume that this system exhibits a sufficient degree of differentiability on $\Re^2_+$ and that it has an economically meaningful steady-state solution $0 < u^*, v^* < 1$ (at which $\widehat{v}^*, \widehat{u}^* = 0$ and which by (17.1)–(17.4) is uniquely determined).

To explore the dynamics of this parametrized family of vector fields (17.5), (17.6) a slightly modified version of Olech's Theorem which directly applies to a system *formulated in rates of change*[3] and not in terms of time derivatives is now very comfortable:

---

[1] r is a constant mark-up on labor unit-costs $wL/Y' = u$ per \$.

[2] Note that we here have used the term "money-illusion" for the case $\eta < 0$ as well. This is justified in our view since the case $\eta = 0$ can be interpreted to represent "perfect foresight", i.e., there is no need to restrict the outcome of the wage-bargain to cases where $\eta \geqq 0$ holds true. Furthermore, the parametrized family of deterministic differential equations considered in this chapter can be reinterpreted by stochastic methods employing a random coefficient $\eta \in (-\overline{\eta}, +\overline{\eta}), \overline{\eta} > 0$.

[3] This fact and the known direction of motion of the variables u, v (cf. the following figure) imply the u, v-axes of $\Re^2_+$ are trajectories of (17.5), (17.6) which cannot be approached by those trajectories which start in $\Re^2_+$. Our restricted consideration of the invariant set $\Re^2_+$, which excludes the singular point (0, 0), consequently is legitimate.

Proposition 1. *Assume that the Jacobian* $J = (J_{ik})$ *of system (17.5), (17.6) fulfills: trace* $J < 0$, $\det J > 0$ *and* $J_{12}, J_{21} \neq 0$ *everywhere in* $\Re^2_+$. *Then, the equilibrium* u^*, v^* *of system (17.5), (17.6) is asymptotically stable in the large, i.e., each trajectory which starts in* $\Re^2_+$ *will approach the equilibrium point* (u^*, v^*) *without hitting the boundary of* R^2_+.

Proof. By means of the diffeomorphism $D : \Re^2 \to \Re^2_+$ defined by $D(x, y) = (e^x, e^y)$ we get the following equivalent system of differential equations on $\Re^2$:

$$\dot{x} = \widetilde{f}(e^x, e^y, \eta), \quad \dot{y} = \widetilde{s}(e^x). \tag{17.7}$$

The linear part or the Jacobian of this system reads

$$J(x, y) = \begin{pmatrix} J_{11}e^x & J_{12}e^y \\ J_{21}e^x & 0 \end{pmatrix}$$

and it fulfills the same conditions as were postulated with regard to J. The above transformed system (17.7) consequently allows the application of Olech's original theorem, cf. Ito (1978, p. 312), i.e., it – and therefore also system (17.5), (17.6) – is asymptotically stable in the large.

Corollary. *The systems* $\widetilde{f}, \widetilde{s}$ *and* $-\widetilde{f}, -\widetilde{s}$ *fulfill the assumptions of the above proposition for* $\eta > 0$ *and* $\eta < 0$, *respectively.*

The interaction between the share of wages u and the employment rate v will thus always lead to or away from[4] the steady-state equilibrium depending on what type of money-illusion prevails in the economy. Figure 17.1 roughly summarizes these two results together with an obvious conjecture on what will happen in the limit case $\eta = 0$, i.e., the case which we shall briefly reconsider in the next section (despite the presence of nonlinear terms this type of Hopf-bifurcation is qualitatively of a degenerate type).

This diagram shows that the Goodwin-case $\eta = 0$ cannot be structurally stable, since the topological properties of its dynamics are not preserved if this system is slightly disturbed by the money-illusion term $\eta\pi$. This proves Corollary 1.5 in Velupillai (1979) which is not fully proved by his mathematical Theorem A.1.4 since his dynamical system (17.4), (17.8) does not lie in the restricted set of vector fields considered in this theorem ($f : W \to \Re^2$ does not point inward on D^2). The above situation provides a simple example of a Hopf bifurcation which allows to deduct Velupillai's (1979) stability properties (i)–(iii)[5] and also further results for the case $\eta = 0$ by means of the Hopf theorem (cf., e.g., Marsden and MacCracken 1976, p. 96).

[4] $\eta < 0$ implies a totally unstable system, since the trajectories of this system – when running them backwards: $(u(-t), v(-t)), t \to \infty$ – are globally asymptotically stable.

[5] Note in this connection that Velupillai's presentation still allows for cases where $f' \equiv 0, s' \equiv 0$ (constant proportional savings) holds true around the equilibrium point u^*, v^*, i.e., where the features of Goodwin's phase diagram will not be preserved.

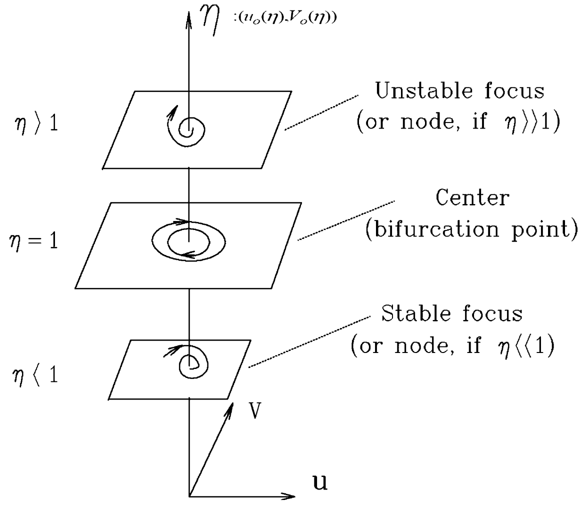

Fig. 17.1 The growth cycle as a degenerate Hopf bifurcation

Remark. By definition of the variables u, v it is, of course, desirable to point to some basic additional mechanisms which ensure that any trajectory which starts in the interval $(0, 1) \times (0, 1]$ cannot leave this subset of $\Re^2_+$. A possibility to obtain such a behavior is given by the following modification of system (17.5), (17.6):

$$\widehat{u} = f(v, u) - \alpha - \eta g((1 + r)u - 1), \quad \eta \geqq -\overline{\eta}, \tag{17.8}$$

$$\widehat{v} = \begin{cases} \min\{0, s(u)/\sigma - (\alpha + \beta)\} & \text{for } v = 1 \\ \sigma(u)/\sigma - (\alpha + \beta) & \text{for } v < 1 \end{cases}, \tag{17.9}$$

where the modified function f is assumed to depend of u if and only if u is close to unity, in which case it may be assumed that the conflict over income distribution will imply that $f(v, u) < 0$ must be fulfilled then. The vector field (17.8), (17.9) thereby is perturbated in such a way that it points inward at the right hand side of $(0, 1) \times (0, 1]$ for $\overline{\eta} > 0$ sufficiently small. And (17.9) implies that $v = 1$ now acts as an upper ceiling, i.e., the implied cycles may hit the upper boundary of $(0, 1) \times (0, 1]$, but they cannot cross it (a similar – Hicksian – idea was first communicated to me by Christian Groth). Hence, (17.8), (17.9) indicate how economically meaningless values $u \geqq 1, v > 1$ can be avoided for cycles which start in $(0, 1) \times (0, 1]$. Yet, the assumed new shape of the function f still represents a very crude extension of Goodwin's approach to accumulation theory, since it is not sensible to consider cycles which approach $u = 1$ without questioning the assumed product market situation of this model. This, however, is a problem which cannot be solved in a brief chapter like the present one.[6]

[6] Cf., however, Cugno et al. (1979) for various modifications of the original Goodwin growth cycle which can be related to the above made proposals (17.8), (17.9).

17.3 The Goodwin Case Reconsidered

Though it is claimed in Velupillai (1979, p. 249) that every non-stationary trajectory of the Goodwin cycle ($\eta = 0$) is closed, his proof (pp. 249ff.) cannot be regarded as being complete. A minor weakness in the demonstration of his claim is that the exclusion of limit cycles (p. 250 below) is not well formulated: closed sets may contain open sets, i.e., it must be shown there, that function H will not be constant on more than one trajectory. Furthermore, the initial argument on the top of p. 251 applies to a neighborhood of his singular point $P_2 = (v_2, u_2)$ solely and is thus at best adequate for sufficiently small RMG-cycles, cf. Hirsch and Smale (1974, pp. 259–261) who show that the doubly infinite sequence they exploit on p. 262 in fact exists – a part of their proof which is not reproduced with regard to his general case by Velupillai (1979). We show in the following that Velupillai's claim, that *every* trajectory of his extended RMG-cycle is closed, is correct. We do this partly in following his (and Hirsch and Smale's) arguments, but shall also introduce some new aspects to indicate the formal difficulties involved in providing a proper global proof for this theorem.

Proposition 2. *The system (17.5), (17.6) taken at $\eta = 0$, i.e.*

$$\widehat{u} = f(v) - \alpha, \quad \widehat{v} = s(u)/\sigma - (\alpha + \beta) \tag{17.10}$$

exhibits a closed-orbit structure as in Goodwin (1972).

Proof. Let G, H be the primitives of $x(u) = -(s(u)/\sigma - (\alpha + \beta))/u$ and $y(v) = (f(v) - \alpha)/v$, respectively, which fulfill $G(u^*) = 0$ and $H(v^*) = 0$, i.e., $G'(u) = x(u)$ and $H'(v) = y(v)$. Interpret the functions G, H as functions which are defined on R^2_+ and define the function K by $G + H$. It is easy to show then that $K(u^*, v^*) = 0$ is a global minimum of the function K and that $K(\Re^2_+) = [0, \infty)$. Furthermore, $K^{-1}[0, c]$ is compact and invariant for each $c > 0$ (cf. Hirsch and Smale 1974, p. 198, with regard to the definition of this latter property). These two properties are also true for the subset $K^{-1}(c)$ of $K^{-1}[0, c]$. Finally, it should be obvious that the function K cannot be constant on any open set in $\Re^2_+$, since this would imply that G and H would be locally constant. This entitles us to apply Velupillai's (1979, p. 250) conclusion that $\Re^2_+$ cannot contain a limit cycle. Consider now an arbitrary point x of the compact, invariant set $K^{-1}(c), c > 0$. According to Hirsch and Smale (1974, p. 198) the ω-limit set $L_\omega(x)$ of x is compact – and thus nonempty (since it is contained in $K^{-1}(c)$). By the Poincaré–Bendixson Theorem, cf. Hirsch and Smale (1974, p. 248), there then follows that $L_\omega(x)$ is a closed orbit, a fact which by the very definition of a limit cycle – the nonexistence of which is already known – implies $x \in L_\omega(x)$, i.e., each such x lies on a closed trajectory (which must enclose the equilibrium point u^*, v^*, see Hirsch and Smale 1974, p. 251). This completes the proof of the above proposition, here without the construction of a doubly infinite sequence as in Hirsch and Smale (1974, p. 259f.).

Remark. Though of exceptional kind (structurally unstable), the above case $\eta = 0$ may nevertheless be very useful as a reference case, e.g., if system (17.8), (17.9) is considered in more detail than was possible in this brief chapter.

References

Cugno, F. et al. (1979). Structural stability in a Goodwin's model. *Serie di Pubblicazioni dell' Istituto Matematico del Politecno di Torino*, Serie I, 16.

Goodwin, R. M. (1972). A growth cycle. In: E. K. Hunt & J. G. Schwartz (Eds.), *A critique of economic theory* (pp.'442–449). London: Pengiun.

Hirsch, M. W., & Smale, S. (1974). *Differential equations, dynamical systems and linear algebra.* London: Academic Press.

Ito, T. (1978). A note on the positivity constraint in Olech's theorem. *Journal of Economic Theory, 17*, 312–318.

Marsden, J. E., & MacCracken, M. (1976). *The Hopf bifurcation and its applications*. Heidelberg: Springer.

Velupillai, K. (1979). Some stability properties of Goodwin's growth cycle. *Zeitschrift für Nationalökonomie, 39*, 245–257.

Wolfstetter, E. (1977). *Wert, Profitrate und Beschäftigung.* Frankfurt: Campus.

Chapter 18
Endogenous Aspirations in a Model of Cyclical Growth

18.1 Introduction

This chapter provides an analysis of the effects on Goodwin's (1967) process of the cyclical accumulation of capital which follow from employing a money wage Phillips-Curve that depends both on the rate of employment and on the rate of growth of the considered economy. This latter influence enters the scenario when the objective of the wage bargaining process is formulated in more detail. Our simple approach here is to assume that this objective is determined by the rate of inflation plus the rate of labor productivity growth and to assume additionally that more will be demanded in periods of very rapid growth, while the opposite occurs in times of subnormal growth. The target of workers' wage claim will therefore be treated as endogenous, i.e., depending on the particular phase of the growth cycle to be analyzed below.

Our above assumption represents one main modification to be incorporated into Goodwin's well-known growth cycle model, whose real wage Phillips-curve is thereby reformulated in a twofold way (besides assuming that the rate of change of money wages depends on employment and the benchmark of a constant share of wages in an endogenous manner). We now have to formulate a hypothesis concerning the price-level, too. For the sake of simplicity we shall adopt a post-Keynesian mark-up equation here, but shall neglect – as in the original Goodwin model – all influences of effective demand. This is not to say that we regard these latter influences as purely secondary, but in the present chapter we prefer to concentrate on the interaction of wage-bargaining, mark-up induced inflation, accumulation, and employment without offering an explanation of the limits for the chosen markup. Focusing on the conflict over income-distribution and its effects on capital accumulation is – at least in our opinion – of some theoretical interest, since this conflict is, e.g., largely neglected in the monetarist approach to the phenomenon of stagflation, where a monotonic relationship between inflation and (un)employment is generally postulated. As we shall see in the following sections this monetarist viewpoint seems to be overly simplified.

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_18,

To demonstrate this assertion is not, however, our main point of interest in this chapter. What we shall mainly strive to show is that:

1. Our proposed modification of Goodwin's model will establish an asymptotically stable equilibrium solution in the large without depriving the model of its cyclical characteristics with respect to its medium term implications.
2. Goodwin's original growth cycles are analytically important to the extent as they will serve as *reference cycles* in the proof of assertion (1).

This result questions the validity of the claim that Goodwin's approach is useless because of the structural instability of the model. Endogenizing the target of wage-bargaining in the above mentioned sense therefore implies the existence of certain additional stabilizers which "point inward" with regard to the cyclicity of the original model, implying that it still remains useful, at least as a point of departure.

We shall discuss our model and its new assumptions in detail in the next section. Section 18.3.2 then investigates the steady state of the system and it provides a proof of local as well as global asymptotic stability (on the basis of an additional steady state assumption in the latter case). We conclude that the proposed modification of the wage-price-sector of Goodwin's model by means of two post-Keynesian hypotheses implies a structurally stable model of cyclical growth à la Goodwin. Our model offers a more enriched story of the inflation-unemployment relationship than the mainstream expectations-augmented models of the interaction of this pair of variables.

18.2 A Model of Cyclical Growth

Our model consists of five definitional equations, six behavioral equations, and one "equilibrium condition" for the product market. Let us begin with definitions which are all well-known.

Average labor productivity (y) is defined as the ratio of national product (Y) to employment (L):

$$Y = Y/L. \tag{18.1}$$

The wage share (u) is given by the product of money wage (w) times employment, divided by the nominal value of national product (pY):

$$u = wL/(pY), \qquad u \in (0, 1). \tag{18.2}$$

The ratio of employment to total labor supply (L^s) is called the rate of employment (v):

$$v = L/L^s, \qquad v \in (0, 1 + \varepsilon), \quad \varepsilon > 0. \tag{18.3}$$

The positive number ε indicates that the state of "full employment" ($v = 1$) should not be interpreted as a state of a completely exhausted labor supply, since some frictional unemployment will always remain.

Our two final definitions concern savings (S) and the capital stock (K):

$$s = S/Y, \tag{18.4}$$
$$g = \dot{K}/K = \widehat{K}. \tag{18.5}$$

Equations (18.4) and (18.5) define the average savings ratio (s) and the accumulation rate (g) respectively.

We turn now to the assumptions on the technology and on economic behavior.

$$\widehat{y} = m, \qquad 0 < m = \text{ const.} \tag{18.6}$$

Labor productivity (y) grows with a constant, positive rate.

$$k = K/Y, \qquad 0 < k = \text{ const.} \tag{18.7}$$

We assume a constant capital–output ratio. Equations (18.6) and (18.7) represent standard assumptions of growth theory, i.e., neutral technical progress in the sense of Harrod.

These assumptions are supplemented in the usual way by an exogenously determined constant rate of growth of labor supply (n):

$$\widehat{L}^s = n, \qquad 0 < n = \text{ const.} \tag{18.8}$$

Note that our set of assumptions is made for convenience, since effects of technical progress and growing labor supply are not essential for the ideas presented here. Next, we stipulate that the savings ratio s is a strictly falling function of the wage share:

$$s = h(u), \qquad h'(u) < 0 \text{ for all } u \in (0, 1). \tag{18.9}$$

With respect to money wage determination we postulate

$$\widehat{w} = f(v) + \eta(g)(\widehat{p} + m), \qquad f'(v) > 0 \text{ for all } v \in (0, 1 + \varepsilon). \tag{18.10}$$

The factor $\eta(g)$, which depends positively on the rate of accumulation $g(\eta'(g) > 0)$, will be called the aspiration factor. Its multiplicative basis is the benchmark: inflation ($\widehat{p}$) + productivity growth ($m = \widehat{y}$). We assume $\eta(g_0) = 1$ for a rate g_0 which may be called "normal", in other words g_0 is near or even equal to $n + m$, the steady state growth rate of the system. Our final behavioral assumption concerns mark-up pricing

$$\widehat{p} = \phi[(1 + a)u - 1], \qquad 0 < \phi'(u), \phi(0) = 0. \tag{18.11}$$

It has been pointed out that we neglect all kinds of problems of effective demand. Thus we apply the standard assumption

$$\dot{K} = S. \tag{18.12}$$

Equation (18.12) may be interpreted as an equilibrium condition for the commodity market. Since we abstract from problems concerning the rate of interest and the money market, it is obvious that the mark-up factor (a) of (18.11) must be given exogenously.

Equations (18.1)–(18.12) constitute the model we are going to study in the remainder of this chapter. Its less common assumptions are given by (18.9)–(18.12) which may therefore deserve some further comments.

The term $(1 + a)u - 1$ in (18.11) can be rewritten in the form $[(1 + a)wL/Y - p]/p$. Thus it represents the deviation of the price target $(1 + a)wL/Y$ from the actual price p as a percentage of this latter price. It is assumed by (18.11) that the rate of inflation depends positively on this percentage deviation.

In addition to earlier comments on (18.10) we may interpret $\eta(g)(\widehat{p} + m)$, i.e., the sum of the rate of inflation and productivity growth, weighted by the aspiration factor, as the target of the wage earners which depends positively on the prevailing rate of accumulation and on the "core target" $\widehat{p} + m$ of maintaining their share in national income. The term $f(v)$, on the other hand, expresses the strength with which the assumed target can be pursued. Finally the (18.9) is but a generalization of the Kaldorian savings function

$$s = s_w + (s_p - s_w)(1 - u), \qquad 0 \leq s_w < s_p \leq 1.$$

Equations (18.9)–(18.11) are therefore nonlinear generalizations of the usual assumptions with respect to the savings function, the money wage determination, and mark-up pricing.

The relationship (18.11) can be supported by econometric studies (see, for example, Rahmann 1977, pp. 374–376). Note, too, that this positive correlation between the wage share u and the inflation rate $\widehat{p}$ does not imply anything definite for the customarily considered inflation-unemployment trade-off $(\widehat{p}, (1-v))$, which will be fairly complex even if the problem of expectations about inflation is short-circuited by adopting a Phillips curve of type (18.10). A relation which looks rather similar to (18.10), namely $\widehat{w} = f(v) + \eta\widehat{p}$, has been interpreted in the recent literature as expressing money illusion by means of the parameter η, see the preceding chapter. Depending on whether $\eta > 1$ or $\eta < 1$, the Goodwin cycle which can be derived from such a relationship has been found to be totally unstable or asymptotically stable (see the preceding chapter). Such a bifurcation in behavior cannot arise in the present model by the very method of our approach. Instead of using the target rate $\widehat{p}$ we assume that wage earners also aim at receiving their share in productivity growth which, under normal conditions of accumulation (g_0), is given by m. Our approach to their respective target $\eta(g)(\widehat{p} + m)$ – which is not due to money illusion – then asserts that periods of subnormal growth imply an endogenous reduction in this target (due to a variety of causes), while the opposite occurs in periods of rapid growth. Neutrality with respect to distribution and growth-dependent deviations from this type of behavior consequently represent the basis of our Phillips curve approach (18.10). We shall show in the next section the resulting direction of redistributional effects will generate "centripetal" forces within Goodwin's growth cycle.

18.3 Discussion of the Model

In a first step we shall reduce the system (18.1)–(18.12) to two nonlinear differential equations in the variables u and v. In Sect. 18.3.2 we analyze the steady state of the system, while Sect. 18.3.3 offers a first impression of the interaction of the wage share, the rate of employment, and the rate of inflation by means of a phase diagram. Subsequently, we shall first consider local stability of the steady state with respect to the dynamic variables u and v (Sect. 18.3.4) and then its stability properties in the large, i.e., for a sufficiently large reference cycle of the original Goodwin model (Sect. 18.3.5).

18.3.1 The Implied Dynamics

Let us derive the differential equation for v first. Logarithmic differentiation of (18.3) gives

$$\widehat{v} = \widehat{L} - \widehat{L}^s = \widehat{L} - n, \qquad \text{see (18.8)}\,.$$

Furthermore, logarithmic differentiation of (18.1) in connection with (18.7) yields

$$\widehat{L} = \widehat{Y} - m = \widehat{K} - m, \qquad \text{i.e.}\,,$$
$$\widehat{v} = \widehat{K} - (n + m).$$

Finally the (18.4), (18.9), and (18.12) imply the following chain of equations:

$$\widehat{K} = \dot{K}/K = S/K = s/k = h(u)/k, \qquad \text{i.e.}\,,$$
$$\widehat{v} = h(u)/k - (n + m). \tag{18.13}$$

This is our first dynamic equation.

In order to derive the second differential equation, for u, we start with expressing (18.2) in terms of growth rates:

$$\widehat{u} = \widehat{w} - \widehat{p} - m.$$

Inserting (18.10) and (18.11) yields

$$\widehat{u} = f(v) + \eta(g)(\widehat{p} + m) - (\widehat{p} + m)$$
$$\widehat{u} = f(f) - (1 - \eta(g))(\widetilde{\phi}(u) + m).$$

The term $\phi(u)$ is an abbreviation for $\phi((1 + a)u - 1)$. We wish to rewrite the term $\eta(g)$ in a slightly different way: $g = h(u)/k$ and we abbreviate $\eta(h(u)/k$ by $\eta(\widetilde{u})$. Our second differential equation can then be written in the form

$$\widehat{u} = f(v) - (1 - \widetilde{\eta}(u))(\widetilde{\phi}(u) + m). \tag{18.14}$$

Note that $\widetilde{\eta}'(u) < 0$ and $\widetilde{\phi}'(u) > 0$ for all $u \in (0, 1)$.

18.3.2 Properties of the Steady State

It is known from Goodwin's model that the parameters involved in the equations have to fulfill certain restrictions in order to establish economically meaningful solutions for the steady state (and the disequilibrium solutions as well). This fact, of course, extends to our system (18.13) and (18.14). We therefore assume that the functions h, $f, \widetilde{\eta}, \widetilde{\phi}$, and the parameters k, n and m are given such that

$$\widehat{v} = 0, \qquad \text{i.e.}, \quad u_0 = h^{-1}(k(n+m)) \tag{18.15}$$

$$\widehat{u} = 0, \qquad \text{i.e.}, \quad v_0 = f^{-1}((1-\widetilde{\eta}(u_0))(\widetilde{\phi}(u_0)+m)) \tag{18.16}$$

do have solutions belonging to the open intervals $(0, 1)$ and $(0, 1+\varepsilon)$, respectively. In this case the uniqueness of the solution $(u_0, v_0) \in \Re^2_+$ follows immediately, because $v_0 = f^{-1}((1-\widetilde{\eta}(u))(\widetilde{\phi}(u)+m))$ is a well-defined function. The solution of (18.15) and (18.16) reduces to $u_0, v_0 = f^{-1}(0)$ if $g_0 = n+m$, since $\widetilde{\eta}(u_0) = \eta(g_0) = 1$.

The steady state (18.15), (18.16) exhibits some well-known facts of growth theory: $\widehat{Y}_0 = \widehat{K}_0 = n + m, \widehat{w/p} = \widehat{w} - \widehat{p} = m, \widehat{(K/L)}_0 = m$, and: $r_0 = (1-u_0)/k_0 =$ const., i.e., in particular: real wages and the average capital intensity grow in line with the rate of growth of labor productivity. Note that the steady state is consistent with a positive rate of unemployment $1 - v_0$ and a positive rate of inflation.

Since the steady state characterizes to some extent the average behavior over the growth cycles which we shall study in the next section, it might be interesting to report some results of comparative statics for u_0 and v_0. As in Goodwin's case, the wage share u_0 reacts negatively with respect to a rising rate of growth of labor productivity. A rising capital–output ratio will – in contrast to Goodwin's case – induce a change of the equilibrium rate of profit, but it is not clear, whether r_0 will tend to rise or to decline. Another difference stems from the fact that the equilibrium rate of employment v_0 is invariant with respect to changes of productivity growth if $\eta(g_0) = 1$, whereas in Goodwin's case v_0 must necessarily rise if m is rising. This latter property will be preserved in our model if and only if $\eta(g_0) < 1$, i.e., if workers demand less than $\widehat{p} + m$ at the steady state of accumulation. Finally we observe a rising rate of employment in response to a rising mark-up whenever $\eta(g_0) < 1$. Thus a higher equilibrium rate of inflation may induce a higher level of employment.

18.3.3 The Phase Portrait of the Model

In order to gain a first qualitative insight into the dynamics of the model (18.1)–(18.12) we shall represent the motion of the rate of employment and the wage share plus the rate of inflation, by constructing the partial equilibrium curves $\dot{u} = 0$, $\dot{v} = 0$, and $\dot{p} = 0$. Inspecting (18.11), (18.13) and (18.14) – compare also (18.15) and (18.16) – it is obvious that $\dot{v} = 0$ and $\dot{p} = 0$ must be vertical in the phase-plane spanned by v and u. Moreover, it is clear that $\dot{p} = 0$ must lie to the left (right) of $\dot{v} = 0$ for sufficiently high (low) mark-up levels. Furthermore, the curve $\dot{u} = 0$ must be strictly increasing in a neighborhood of the "equilibrium wage share" u_0 if, e.g., the mark-up (a) is sufficiently high and $\eta(g_0) = \widetilde{\eta}(u_0) \leq 1$ (compare Sects. 18.3.4 and 18.3.5 for further details). We are now prepared for a graphical representation of the wage share/employment rate interaction and its implications for inflation, at least in a neighborhood of the steady state (u_0, v_0).

The phase plane is divided into four parts by the curves $\dot{u} = 0$, $\dot{v} = 0$, each with a typical direction of the moving variables (u, v). This can be determined from the partial equilibrium curves in the usual way. The result is depicted in Fig. 18.1. Figure 18.1 shows also the domain of inflation (to the right of $\dot{p} = 0$). Now the phase diagram suggests that the dynamics of the variables u, v exhibit cyclical fluctuations, predominantly accompanied by inflation if the mark-up (a) per unit wage costs is sufficiently high. The depicted process is of the same qualitative type as that of Goodwin's original model, i.e., it exhibits the same kind of overshooting of the wage share (u) and the employment rate (v) over their equilibrium values – consult the next section for a formal proof of this proposition. Figure 18.1 shows that – under the set of assumptions described above – only low shares of wages in national income will give rise to situations of a temporarily falling price level, i.e., deflation.

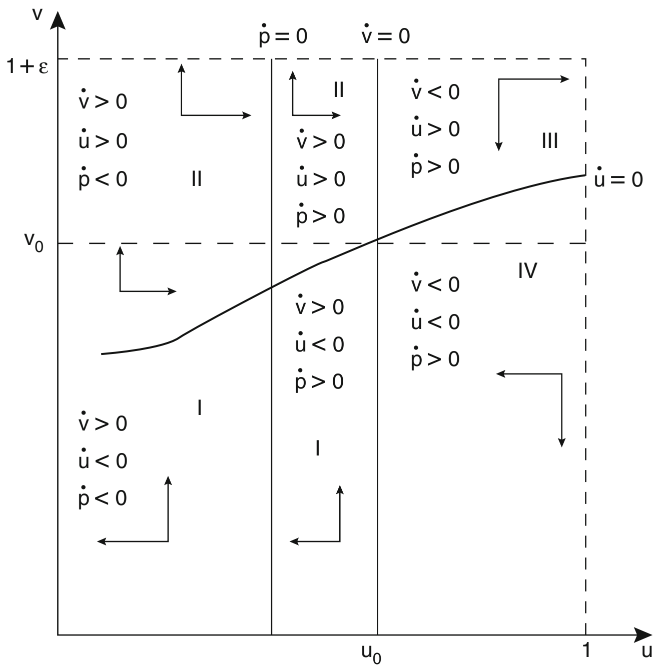

Fig. 18.1 The phase portrait regions of the dynamics

Our simple mark-up hypothesis (and the size of the mark-up chosen) imply that falling employment is always accompanied by inflation (see areas III and IV). This fact (stagnation implies stagflation) is, however, due to the assumed rigidity of the mark-up and thus need not hold true in more general versions of the model.

18.3.4 Local Stability

Let us now turn to the details of stability analysis of our model (18.1)–(18.12), i.e., of the (18.13), (18.14):

$$\widehat{u} = f(v) - (1 - \widetilde{\eta}(u))(\widetilde{\phi}(u) + m), (*)$$
$$\widehat{v} = h(u)/k - (n + m),$$

The functions are assumed to be continuously differentiable, $h'(u) < 0$, $f'(v) > 0$, $\widetilde{\eta}'(u) < 0$, $\widetilde{\phi}'(u) > 0$. Recall that the function f is the first component of the assumed Phillips curve, that $\widetilde{\eta}$ expresses the aspiration factor in its relation to the wage share, and that $\widetilde{\phi}, h$ summarize mark-up and savings behavior respectively.

It should be pointed out explicitly that the system (*) does not fulfill Olech's theorem since the linear part J of the right hand side does not fulfill the conditions: trace $J < 0$, det $J > 0$, $J_{12}J_{21} \neq 0$ everywhere in $\Re^2_+$ (see Flaschel 1984, p. 65, for the formulation of Olech's theorem for dynamical systems in terms of growth rates instead of time derivatives). The system (*) therefore cannot be proved to be asymptotically stable in the large by means of this standard approach to global stability. The reason can briefly be stated by observing that the function $(1 - \widetilde{\eta}(u))(\widetilde{\phi}(u) + m)$ is not monotonic on $\Re_+$.

For *local asymptotic stability* it suffices to have a negative trace of the Jacobian and a positive Jacobian determinant, evaluated at the equilibrium point (u_0, v_0), or, to put it into formal terms:

$$\widetilde{\eta}'(u_0)(\widetilde{\phi}(u_0) + m) - (1 - \widetilde{\eta}(u_0))\widetilde{\phi}'(u_0) < 0 \quad \text{and} \quad - f'(v_0)h'(u_0)/k > 0.$$

While the latter conditions is always fulfilled, the former represents an additional assumption for the functions $\widetilde{\eta}$ and $\widetilde{\phi}$. For the purposes of our next section we like to stress the following important particular case of this assumption:

$$\widetilde{\eta}(u_0) = \eta(n + m) = 1, \qquad \widetilde{\phi}(u_0) + m > 0. \qquad (A)$$

The conditions (A) can be motivated as follows: Assuming $\eta(n + m) = 1$ in particular means that the aspiration of workers will be neutral with respect to distribution if the steady state applies. Thus the condition is a type of consistency condition (as it, e.g., is often applied in the monetarist approach to the Phillips curve and the stagflation phenomenon with respect to the factor which describes the effect of expectations about inflation on the rate of inflation; see however our

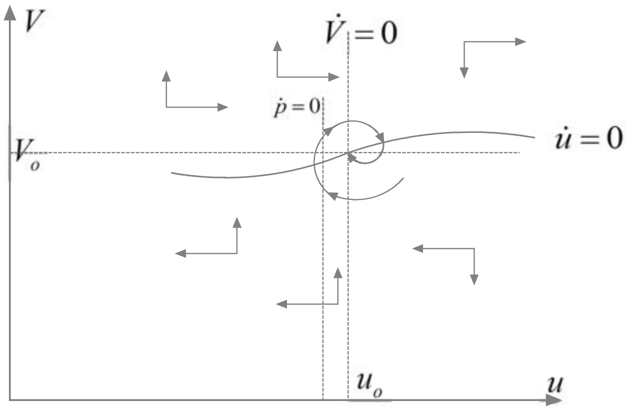

Fig. 18.2 Convergent dynamics

concluding remarks for a critique of this particular approach). Assuming in addition $\widetilde{\phi}(u_0) + m = \phi((1+a)u_0 - 1 > 0$ means that the equilibrium wage share (u_0) and the mark-up (a) do not allow for a rate of deflation, $-\widetilde{\eta}(u_0)$, which exceeds the rate of productivity growth. This is an assumption about the intensity of the conflict over income distribution at the equilibrium value (u_0). It has to be taken into account if the steady state is accompanied by deflation.

Assumption (A) implies that the steady state (u_0, v_0) is a *stable focus*, and not a stable node) if $-\widetilde{\eta}'$ is sufficiently flat relative to the slope of the curves f' and $-h'$ so that the condition $(\text{ trace } J)^2/4 < \det J$ will be fulfilled. Such an assumption seems plausible in the light of our interpretation of $\widetilde{\eta}$ (or η) as an aspiration factor and regarding the fact that η has often been treated as constant in the literature (but see Schlieper and McMahon 1981 for a somewhat similar treatment of a non-constant target). Figure 18.2 shows a typical trajectory of the system (*) under the assumptions of our model:

It might be interesting to have a look at the domain of asymptotic stability in the interval $(0, 1) \times (0, 1 + \varepsilon)$. This question will be examined in the next subsection.

18.3.5 Asymptotic Stability in the Large

We already know that Olech's criteria for global asymptotic stability cannot be applied to the model presented in this chapter. Therefore a different method has to be applied to show that the basin of attraction of the equilibrium point (u_0, v_0) can be extended to a domain which is of real economic interest. To know that there is a neighborhood of (u_0, v_0) where asymptotic stability holds true is only of economic interest if there are no limit cycles "near" to (u_0, v_0) – which then would contain our unique equilibrium point in their "interior" – since the properties of such limit cycles would then govern the economics of the wage share/employment rate interaction. That such limit cycles cannot exist if an assumption of type (A) is true will be shown now by using the original Goodwin model for reference.

Assertion 1: The assumption $\eta(n+m) = 1$ implies that the equilibrium point (u_0, v_0) of system $(*)$ is identical to that of the following system

$$\widehat{u} = f(v), \qquad \widehat{v} = h(u)/k \qquad\qquad (**)$$

This latter system is of the same type as Goodwin's model (Goodwin 1967), (see Flaschel 1984, Sect. 18.2, for the details of the proof), i.e., its trajectories which start in $\Re^2_+$ are all closed orbits around the steady state (u_0, v_0). Note that – due to our assumption on workers' "average target" $\widehat{w} = \widehat{p} + m$ – the term (m) does not appear in the first differential equation, in contrast to Goodwin's result. Each of the orbits of system (**) determines a simply connected region which contains (u_0, v_0) for which assumption (A) holds true. Let us denote by T, the interior of the largest of these regions in which $\widetilde{\eta}(u) + m > 0$ remains true, i.e., in which deflation cannot neutralize or even reverse the inequality implied by the rate of growth of labor productivity $(m > 0)$.

Assertion 2: The region T is an invariant set of the system (*), i.e., all trajectories of system (*) which start in this set cannot leave it, and the point (u_0, v_0) is a global attractor with respect to region T.

The idea of proving assertion 2 is based on the fact that the dynamics of our system (*) essentially point inward with respect to the closed orbits of system (**) which surround the point (u_0, v_0). This implies that the trajectories of (*) which start in T must converge to this point since they must, so to speak, cross every orbit of system (*). The details of a formal proof are given in the mathematical appendix to this chapter.

We have thereby shown that system (*) is asymptotically stable for a simply connected region T around the equilibrium state (u_0, v_0) which should be reasonably large whenever the conflict over income distribution (based on the conflicting claims (18.10) and (18.11) is sufficiently intense so that $\widehat{p} + m$ remains positive on T. In other words we do not observe a falling nominal value of net national income per head within this region T. Furthermore, closed orbits à la Goodwin are useful for the observation that the new elements of our model (18.1)–(18.12) will generate extra forces on the development of the share of wages and the rate of employment which drive the system inward with respect to its "core dynamics" (**) of closed cyclicity. The geometry of assertion 2 is summarized in Fig. 18.3.

Recall that $T \subset (0,1) \times (0, 1+\varepsilon) \subset \Re^2_+$. Now it should be observed that under the assumptions of our model the dynamics of the system point inward on the boundary ∂T *except* at the points X_1 and X_2. It is because of the dynamical behavior of the system at X_1 and $Xi2$ that we have to introduce the concept of quasi-global stability (see the appendix).

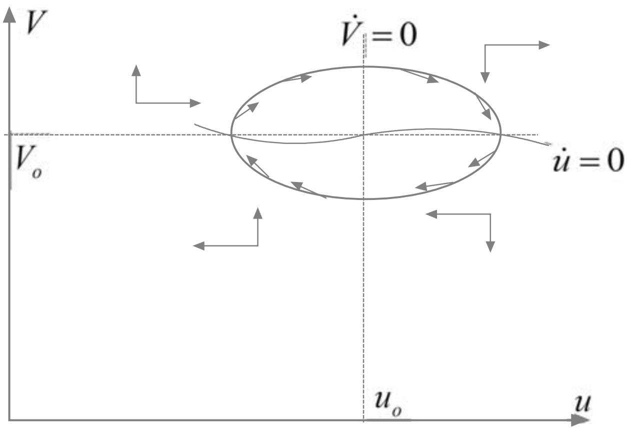

Fig. 18.3 Global stability

18.4 Conclusions

Though neglecting problems arising from effective demand, our model is useful in raising doubts on the robustness of the Phillips curve in the standard expectations augmented version. In conformity with empirical observations we have used a monotonically rising relationship between the rate of inflation ($\widehat{p}$) and the wage share (u) as it may result from mark-up pricing behavior. Furthermore, our money wage Phillips curve (18.10) implicitly assumes that workers have perfect knowledge of the rate of inflation ($\widehat{p}$), though they may be willing to strive for less (or more) in the wage bargaining process. These two components of our model do not, however, imply a positive relationship between the rate of employment (v) and the inflation rate ($\widehat{p}$), since the relationship between the wage share and the employment rate is intrinsically cyclical. This is confirmed by observations (see Rahmann 1977, pp. 376 ff.) and is comprehensible even from the simple analysis of our model of capital accumulation. Now, as the $v - u$ relationship is cyclical and since the $\widehat{p} - u$ relationship can be shown to be monotonic, the $p - v$ relationship must be cyclical, too. This is obvious from our figure 18.1 and is also confirmed by empirical estimations. A monotonic relationship of the standard monetarist version must therefore be rejected as overly simplified if the effects of capital accumulation are really taken into account.

Mathematical Appendix

Proof of assertion 2 of Sect. 18.3.5. Denote by G, H the first integrals of $g(u) = -(h(u)/k - (n+m))/u$ and of $f(v)/v$ on $\Re_+$ which fulfill the conditions $G(u_0) = 0, H(v_0) = 0$. By definition we have $G'(u) = g(u)$ and $H'(v) = f(v)/v$ on $\Re_+$.

Define – by means of a suitable reinterpretation of the functions G, H – the function $K = G + H : \Re^2_+ \to \Re$. Flaschel (1984, Sect. 18.2), see the preceding chapter, has shown that K is a Liapunov function of the dynamical system (**). Furthermore, $K^{-1}[0, c]$ is compact and invariant for each $c > 0$. These two properties hold true for $K^{-1}(c)$, too. The trajectories of system (**) are given by $K^{-1}(c)$ and are shown to be closed orbits.

Consider now a trajectory (u, v) of our system (*) which starts in the set T. Using the partial derivatives K_u, K_v of K then gives

$$\begin{aligned}\dot{\widehat{K(u,v)}} &= k_u\dot{u} + K_v\dot{v} = (K_u u)\widehat{u} + (K_v v)\widehat{v} \\ &= -(h(u)/k - (n+m))\widehat{u} + f(v)\widehat{v} \\ &= -(h(u)/K - (n+m))(\widehat{u} - f(v)) + f(v)(\widehat{v} - (h(u)/k - (n+m))) \\ &= [-h(u)/k - (n+m)][-(1 - \eta(h(u)/k)) \times (\phi((1+a)u - 1) + m].\end{aligned}$$

This product is of type $[-] \times [+]$ if $u < u_0$ and of type $(+) \times (-)$ if $u > u_0$ and it is zero for $u = u_0$. Note that $\phi((1 + a)u - 1) + m = \widetilde{\phi}(u) + m$ is positive within the region T. The function K therefore is a Liapunov function in the sense of Definition D.1.7. in Hahn (1982, p. 751), since it is bounded from below. Furthermore, the assumptions of Hahn's theorem T.1.4. (on p. 751) are quite obviously fulfilled by the function K and the region T. the system (*) therefore is quasi-globally stable with respect to T (for a definition of quasi-global stability see Hahn (1982, p. 750, D.1.6). Since there is only one equilibrium point in T, the system must be globally stable in addition. For all initial conditions $(u(0), v(0)) \in T$ the trajectory (u, v) determined by one initial value $(u(0), v(0)) \in T$ thus converges to the equilibrium point (u_0, v_0) (see again Hahn 1982, pp. 751 f.). □

References

Flaschel, P. (1984). Some stability properties of Goodwin's growth cycle. A critical elaboration. *Zeitschrift für Nationalökonomie, 44*, 63–69.

Goodwin, R. M. (1967). A growth cycle. In: C. H. Feinstein (Eds.), *Socialism, capitalism, and economic growth* (pp. 54–58). Cambridge, UK: Cambridge University Press.

Hahn, F. H. (1982). Stability. In: K. J. Arrow & M. D. Intrilligator (Eds.), *Handbook of mathematical economics* (pp. 745–793). Amsterdam: North-Holland.

Rahmann, B. (1977). Zu den Beziehungen zwischen Inflation, Arbeitslosenquote und Einkommensverteilung in der Bundesrepublik Deutschland 1960–1976. *Kredit und Kapital*, 362–380.

Schlieper, U., & McMahon, P. C. (1981). Behaviour of the real wage rate in the trade cycle in West Germany and the U.K. *Kyoto University Economic Review*, 52–58.

Chapter 19
Partial Cooperation with Capital vs. Solidarity in a Model of Classical Growth

19.1 Introduction

Goodwin's (1967) model of a growth cycle has since long been regarded as a model of class struggle and the conflict over income distribution which mirrors basic aspects of Marx's "General Law of Capitalist Accumulation" in Volume I of "Das Kapital". When rereading this chapter (Marx 1954, Chap. 25) with Goodwin's model and its various extensions in mind, one indeed finds many observations of Marx – in particular in its Sect. 19.1 – which are strikingly similar to the assumptions and conclusions which this growth cycle model exhibits. However, Marx also very often stresses aspects of the behavior of "capital" which are not covered by this approach to cyclical growth (where profits are more or less mechanically invested by "capitalists"). These aspects typically concern the strategic possibilities of capitalists when faced with the profit squeeze mechanism due to a low number of unemployed workers in the reserve army.[1]

Such strategic considerations have, by and large, not found inclusion in the formal discussion of the Goodwin growth cycle. There exist attempts of Balducci et al. (1984), Ricci (1985) and in particular Mehrling (1986) where the theory of differential games is applied to this type of growth cycle model, but this seems to represent all efforts made to incorporate game-theoretic aspects into this conflict over income distribution. In this respect K. Lancaster's (1973) related model on the dynamic inefficiency of capitalism has received much more attention in recent years, cf. Haurie and Pohjola (1987) for a typical article on this subject.

Mehrling's (1986) game-theoretic approach to Goodwin's classical model of the class struggle starts from the following simple generalization of the basic features of this model:

$$\widehat{w} = \dot{w}/w \leq -a + bV \text{ or } \widehat{u} \leq bV - a - m, \tag{19.1}$$

$$0 \leq \widehat{K} \leq \sigma(1-u) \text{ or } -(n+m) \leq \widehat{V} \leq \sigma(1-u) - (n+m), \tag{19.2}$$

$$u \leq 1, \quad V \leq 1, \quad uY \geq cL^s. \tag{19.3}$$

[1] I am grateful to R. Neck, J. Rosenmüller and E. Wolfstetter for helpful comments as well as suggestions for (future) extensions of this chapter. Usual caveats apply.

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_19,

The meaning of the notation used in these equations is the following

$\hat{w}$ the growth rate of real wages,
$u = w/y = (wL)/(yL) = wL/Y$ the share of wages,
$V = L/L^s$ the rate of employment,
$m = \widehat{y}$ the growth rate of labor productivity y,
$\sigma = Y/K$ the output–capital ratio,
$n = \widehat{L}^s$ the growth rate of the labor force,
a, b, c parameters (> 0).

Equation (19.1) says that the growth rate of wages is limited by a labor market reaction curve $-a + bV$, i.e., a real-wage Phillips curve (note that there is no difference between $\widehat{u}$ and $\widehat{w}$ in Mehrling's paper because he excludes technical progress: $m = 0$). Next, (19.2) states that the growth rate g of the capital stock K is limited from above by the amount of profits per unit of capital and from below by zero (because there is no depreciation of existing capital). Finally, the first two inequalities in (19.3) are obvious, while the third is an assumption which is needed in Mehrling's modifications of this growth cycle model to allow the analysis of workers' and capitalists' control problems in this model. This assumption states that the sum of wages must cover the subsistence requirements cL^s of the total workforce and it will play no role in the present chapter (note in this regard, that the inequalities $w > 0$, $u > 0$, $V > 0$ will always be fulfilled due to the growth rate formulations used in this dynamic model).

Mehrling assumes that the right hand side inequalities in (19.1), (19.2) are turned into equalities when agents act as isolated atoms and thus neglect their impact on the economy-wide variables in their "optimization" problem. In this case the known cyclical solutions of the Goodwin model are obtained if none of the other restrictions come into effect (see our Sect. 19.6 for an analysis of how the conditions $u \leq 1$, $V \leq 1$ can be avoided). This standard version of Goodwin's growth cycle will also be the starting point of our own modifications of this model.

If workers or capitalists – or both – act as a group then solutions different from the above are derived in Mehrling's paper with respect to the given constraints of the model. He discusses as alternative solutions a workers' control problem, a capitalists' control problem and a codetermination equilibrium. All these solutions look appealing in their ideality which is based on the assumption that both workers and capitalists maximize their discounted income stream from now to infinity when choosing the variable under their control, i.e., the change in the share of wages $\dot{u}$ on the one hand and the change in employment $\dot{V}$ on the other hand. In the context of a growth cycle model such as Goodwin's – which already through minor extensions generates solutions that cannot be determined explicitly – the attempt to behave in such a way must be based on a huge amount of knowledge, which indirectly also involves costly learning processes, corrective actions, etc. We therefore shall make use of the opposite approach and will start from a very myopic type of behavior of the agents (which only later on should be extended toward more elaborate types of behavior).

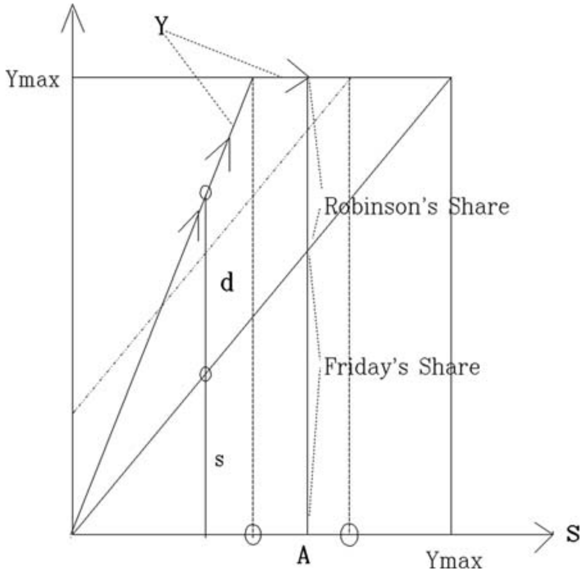

Fig. 19.1 Maarek's model of the conflict over income distribution

A simple further game-theoretic approach to the Marxian analysis of class struggle that we have not mentioned so far is provided in Maarek's (1979) book on Marx's "Capital". In Sect. 8.4 of this book he makes use of a model of bilateral monopoly to describe a *Robinson/Friday* example where Robinson owns the means of production and subsistence and exploits the labor power of Friday in such a way that the initial conditions for exploitation are always set anew.

A simple presentation of Maarek's model is given by the graphical representation shown in Fig. 19.1:

Y : Output corresponding to Friday's remuneration
s : Friday's salary
d : Minimum amount for Robinson which induces him to pay and feed Friday for a workweek instead of working himself
A : Range where "power" decides on the distribution of Friday's product

The great disadvantage of this model is that it only sets limits to the conflict over income distribution, but does not say anything more definite on this matter. It is thus not well-suited for an application to the conventional modeling of the growth cycle related with this conflict.

A way out of this difficulty may be found in the approach of Güth and Selten (1982) who derive a wage-bargaining equation from certain axioms on the bargaining process. This bargaining equation is then used in an extended *linear* multiplier-accelerator model to investigate the implications of this modification for this standard model of a business cycle. Unfortunately, it does not appear to be an easy task to apply their considerations to a nonlinear model such as Goodwin's.

Our approach to introduce aspects of the theory of games into this growth cycle will therefore be much simpler, in that we shall make use of the Nash bargaining solution for the labor–capital relationship in order to close the degree of freedom of the above model of Maarek. We shall make use of a simple extension of Friedman's (1986, pp. 179–180) example of labor/management negotiations in order to apply it to a class-struggle model as in Goodwin (1967). Our approach of modifying this model will consequently be a cooperative one and it will be confronted with an alternative view where cooperation does not take place between capitalists and (part of) the labor force, but where the labor force acts as a single unit. The aim of this chapter thus simply is to explore some aspects of cooperation (or of conflict) between capital and labor in the otherwise harmonious setup of Goodwin's model where capital always fulfills its social role and invests all profits (see also our concluding remarks in this respect). This last observation also indicates that, of course, much work remains to be done if one truly attempts to exploit the (non-) cooperative aspects that can be associated with the Marxian background of Goodwin's approach to cyclical growth.

In contrast to the pessimistic views that Marx was forced to adopt in view of the situation he faced at his times, let us add here, that also his own model of cyclical accumulation exhibits at least two features which he did not analyze very thoroughly in their potential of implying less negative conclusions on the future of capitalism:

1. Labor itself is one of the driving forces in his analysis of the cyclical nature of accumulation.
2. Without the assumption of an ever increasing organic composition of capital his model implies that real wages must grow (on average) in line with labor productivity.

In particular this latter aspect implies that there is considerable scope for cooperation among capitalists and workers which, however, has not yet been analyzed formally in the context of Marx's growth cycle analysis.

To start our own investigations of this approach to cyclical growth we shall show in Sect. 19.2 that the Goodwin growth cycle will not be modified by the assumption of two groups of workers which exhibit a productivity differential (instead of the usually assumed homogeneous type of labor) – as long as these two groups can enforce conditions of equity which exactly mirror their productivity differential. In Sect. 19.3 we then assume that there is cooperation between capital and the more productive segment of the labor force by making use of Friedman's (1986) example of labor – management negotiations in this dynamic setup. A check of the sufficiency conditions for the Nash-solution used will reveal that this solution must be situated on the boundary and not in the interior of the admissible domain. Incorporating this contract between capital and part of the workforce into the Goodwin model will then imply an increase of stability for this model accompanied by a redistribution of income from low skilled workers to high skilled ones. Section 19.4 then attempts to remove an important weakness from the above original Goodwin model – which is comparable to the weakness of the border case of neutral stability of the linear multiplier – accelerator model. In our view, a Marxian completion of

Goodwin's cycle must – even in simple situations – give rise to a kind of limit cycle behavior instead of the three types of dynamics usually associated with this model (i.e., center-type, purely implosive or purely explosive dynamics). We shall establish conditions which will imply such a behavior. On the basis of such a completion we will then also be in the position to show that the partial cooperation considered in this chapter will even be capable of removing such a structurally stable cyclical pattern from this model and again lead to asymptotically stable cyclical growth. This pronounced improvement in stability will again be accompanied by a shrinking average income share of the second group of workers in comparison to the case where there is solidarity between the two groups in the sense of Sect. 19.2. In an appendix we finally will attempt to incorporate into the model a less myopic type of the behavior of firms than has been assumed so far.

19.2 Solidarity Among Workers

As Mehrling (1986) we start from the well-known Goodwin model of cyclical growth

$$\widehat{u} = f(V) - m, \tag{19.4}$$

$$\widehat{V} = \sigma(1-u) - (n+m), \tag{19.5}$$

where the Phillips curve f will for simplicity be assumed to be a linear function of the rate of employment $V: \; f(V) = -a + bV$. We extend this model by assuming in addition that there exist two types of workers, one with *productivity index* y_1 and one with *productivity index* y_2. We denote the *productivity differential* by

$$\gamma_y = y_1/y_2 \tag{19.6}$$

and assume $\gamma_y > 1$ (all other assumptions of the model remain as before, i.e., in particular $\widehat{L}_1^s = \widehat{L}_2^s = n$ for the now two natural rates of growth of the model; $\gamma_y = 1$ will reduce the model to Goodwin's original case). Finally, we shall make use of the following *abbreviations*:

$$\Lambda_i = \sigma K/y_i, \; k_i = \Lambda_i/L_i^s = \sigma K/y_i L_i^s \; (i = 1, 2) \; \text{ and } \; \gamma_L = L_1^s/L_2^s \; (\widehat{\gamma}_L = 0).$$

Having assumed two types of labor, demands that individual and average shares of wages, of productivity indexes, and of rates of employment have to be distinguished. We denote the individual magnitudes by

$$u_1, u_2, V_1, V_2 \; (\text{and} \; y_1, y_2),$$

where $u_i = w_i/y_i$ is defined as the share in individual (not average) productivity.

For the corresponding averages we then get

$$w = \frac{w_1 L_1 + w_2 L_2}{L_1 + L_2} = w_2 \cdot \frac{\gamma_y L_1 + L_2}{L_1 + L_2}$$
$$y = \frac{y_1 L_1 + y_2 L_2}{L_1 + L_2} = y_2 \cdot \frac{\gamma_y L_1 + L_2}{L_1 + L_2},$$

i.e., $u = w/y = w_2/y_2 = u_2 = w_1/y_1 = u_1$ if we assume $w_1 = \gamma_y w_2$, i.e., if relative remuneration corresponds to relative productivity for the two given types of workers. Furthermore

$$V = \frac{L_1 + L_2}{L_1^s + L_2^s} = \frac{L_1}{L_1^s} \cdot \frac{L_1^s}{L_1^s + L_2^s} + \frac{L_2}{L_2^s} \cdot \frac{L_2^s}{L_1^s + L_2^s} = V_1 q_1 + V_2(1 - q_1) = V_1$$

if we assume $V_1 = V_2$. Under the same assumption, output Y can be rewritten as follows

$$Y = y_1 L_1 + y_2 L_2 = y_1 V_1 L_1^s + y_2 V_2 L_2^s$$
$$= V_2 L_2^s y_2(\gamma_y \gamma_L + 1) = V L_2^s y_2(\gamma_y \gamma_L + 1) \tag{19.7}$$

which finally gives

$$\widehat{Y} = \widehat{V} + n + m \tag{19.8}$$

since $\gamma_y \gamma_L + 1$ is a constant.

If now the two groups of workers are *conjointly responsible in that they only accept employment under the conditions*:

1. $w_1(0) = \gamma_y w_2(0)$ (fair relative remuneration at $t = 0$)
2. $\widehat{w}_1 = \widehat{w}_2$ (equality in the results of each wage bargain)
3. $V_1 = V_2$ (equality of employment opportunities)

then it is easily shown that not only the assumptions (up to $\gamma_y > 1$), but also the implications of the present model are the same as that of the original Goodwin cycle:[2]

1. $\widehat{u} = \widehat{w/y} = \widehat{u}_1 = \widehat{u}_2 = f(V_2) - m = f(V) - m$
2. $\widehat{V} = \widehat{Y} - (n + m) = \widehat{K} - (n + m) = \sigma(1 - u) - (n + m)$, see (19.7)

The above assumptions on the type of solidarity among workers consequently produce the same cycle model as before and thus again imply the known kind of neutral cyclical stability which characterizes the interaction between income distribution and accumulation in the Goodwin growth cycle.

[2] Savings $S = (1 - u)Y = \dot{K}$.

19.3 Partial Cooperation Between Labor and Capital

In contrast to the preceding situation let us now assume that there is an inclination to cooperate between capital and the more productive type of labor (while the conditions for the second type of labor remain the same as before). We also assume that the first type of labor is indispensable for production, i.e., $Y = 0$ for L_1 sufficiently small (depending on the amount of capital that is employed). In formal terms we in fact make the assumption that

$$L_1^{\min} = k_1^{\min} \cdot \sigma K / y_1, \quad k_1^{\min} = \text{ const. } > 0$$

describes the minimum amount of labor necessary to operate the economy if the capital stock is presently at the level K. The assumed Leontief technology therefore exhibits a discontinuity for shrinking $L_1 > 0$. Finally – due to "legal restrictions" – binding contracts are considered as admissible only for the above type of restricted cooperation.

To model the basic case of such a cooperation we shall make use of the example of wage negotiations in Friedman (1986, pp. 179–180) and assume for our present case:

$$\text{(a)} \qquad U(w_1, L_1) = w_1^{\alpha} L_1^{1-\alpha}, \alpha \in (0, 1) \tag{19.9}$$

as utility function of (the union of) the first type of workers and

$$\text{(b)} \qquad \Pi(w_1, L_1) = Y - w_1 L_1 - w_2 L_2, \tag{19.10}$$

i.e., profits as the firm's "utility function".

We assume that capital controls the wage rate w_1 and that workers (of type 1) control employment L_1, so that the point $(\bar{w}_1, \bar{L}_1) = (0, 0)$ can be considered to describe the case where no contract on (w_1, L_1) comes about (see our above assumption) implying $(U, \pi) = (0, 0)$ in this case.

Note with respect to the above, that the wage w_2 is given in each moment of time (its motion is governed by $\widehat{w}_2 = f(V_2)$, $V_2 = L_2/L_2^s$). Note furthermore, that output Y is determined by σK in the context of Goodwin's model. Note finally, that the volume of employment L_2 depends on the decision that is made with regard to L_1 in the following way

$$L_2 = \frac{\sigma K}{y_2} - \frac{y_1}{y_2} L_1 = \Lambda_2 - \gamma_y L_1. \tag{19.11}$$

We consider this game as a Nash bargaining game. The Pareto optimal curve may be found by maximizing

$$\delta\, w_1^{\alpha} L_1^{1-\alpha} + (1 - \delta)[\sigma K - w_1 L_1 - w_2(\Lambda_2 - \gamma_y L_1)] \tag{19.12}$$

for $\delta \in [0,1]$ with respect to the wage w_1 and employment L_1. Solving the first order conditions and equating the resulting two equations then gives the expression

$$w_1^* = \frac{\alpha}{2\alpha - 1}\gamma_y w_2 \qquad (L_1^* \text{ indeterminate}) \tag{19.13}$$

which only gives a meaningful expression if $\alpha > 1/2$ holds true.

The Nash bargaining solution may now be found by maximizing

$$w_1^* L_1^{1-\alpha} \cdot [\sigma K - w_1^* L_1 - w_2(\Lambda_2 - \gamma_y L_1)]$$

with respect to L_1 which gives

$$L_1^* = \frac{2\alpha - 1}{2 - \alpha}\left[\frac{\sigma K - \Lambda_2 w_2}{\gamma_y w_2}\right] = \frac{2\alpha - 1}{2 - \alpha}\Lambda_1\left(\frac{1 - u_2}{u_2}\right), \quad \Lambda_1 = \frac{\sigma K}{y_1}. \tag{19.14}$$

This "bargaining solution" has the nice property that w_1^* exceeds the wage which group 1 received in the preceding section, since $\frac{\alpha}{2\alpha-1}$ is always larger than one (for $\alpha > 1/2$ it is monotonically falling, $+\infty$ at $\alpha = 1/2$, and 1 at $\alpha = 1$). Furthermore, w_1 and w_2 exhibit the same rate of growth which makes it easy to incorporate this solution into the Goodwin model in order to analyze the effects of the above type of cooperation.

However, in our two step procedure of analyzing the Nash bargaining solution we did not pay attention to second order conditions and thus did not yet test for true maxima.

With regard to the first step $H = \delta U + (1-\delta)\Pi \longrightarrow \max$ [see (19.9), (19.10)] we get as second order conditions (w_1 the first, L_1 the second variable):

$$\begin{aligned}
H_{11} &= -\delta\frac{\alpha}{w_1^2}u(1-\alpha),\\
H_{12} = H_{21} &= \delta\frac{\alpha(1-\alpha)}{w_1 L_1} - (1-\delta), \qquad \text{and}\\
H_{22} &= -\delta\frac{1-\alpha}{L_1^2}\alpha.
\end{aligned}$$

The symmetric matrix (H_{ij}) has a negative trace, so a positive determinant would imply negative real eigenvalues and thus negative definiteness. However,

$$\det(H_{ij}) = (1-\delta)[\delta(2w_1^{\alpha-1}L_1^{-\alpha}\alpha(1-\alpha) + 1) - 1]$$

which gives

$$\begin{aligned}
&\det = 0 \text{ for } \delta = 1\\
&\det > 0 \text{ for } \delta < 1, \text{ sufficiently close to "1"}\\
&\det < 0 \text{ for } \delta < 1, \text{ sufficiently close to "0"}
\end{aligned}$$

The technique used to determine the Nash contract is therefore problematical. And indeed, when one checks the second order conditions for the above "Nash solution" (19.13), (19.14) of $\widetilde{H} = U \cdot \Pi$ one finds:

$$\widetilde{H_{11}} = -U^*(1-\alpha)\frac{L_1^*}{w_1^*},$$
$$\widetilde{H_{22}} = -U^*\frac{w_1^*}{L_1^*}\left[\frac{1-\alpha}{\alpha}(2-\alpha)\right],$$
$$\widetilde{H_{12}} = \widetilde{H_{21}} = -U^*(2-\alpha), \text{ i.e.,}$$
$$\det(\widetilde{H_{ij}}) = (U^*)^2\frac{(2-\alpha)(1-2\alpha)}{\alpha} < 0 \text{ for } \alpha > 1/2$$

(and again trace < 0), i.e., indefiniteness. The problem of finding a Nash bargaining solution hence must be considered anew.

To obtain the proper Nash solution the characterization of the admissible (w_1, L_1)-domain is of help (as shown in Fig. 19.2).

In this figure $L_1^{\min}$ denotes the minimum amount of labor of type 1 necessary to operate the technology in view of the level of capital K presently in existence. Furthermore, the maximum amount of employment of type 1 is $L_1^{\max} = \sigma K / y_1$ which may be larger or smaller than the supply of labor L_1^s, giving rise to full or less than full employment of type 1 under such conditions. Finally, the curve $\pi = 0$ is given by $w_1 = \sigma K(1 - u_2)/L_1 + u_2 y_1$.

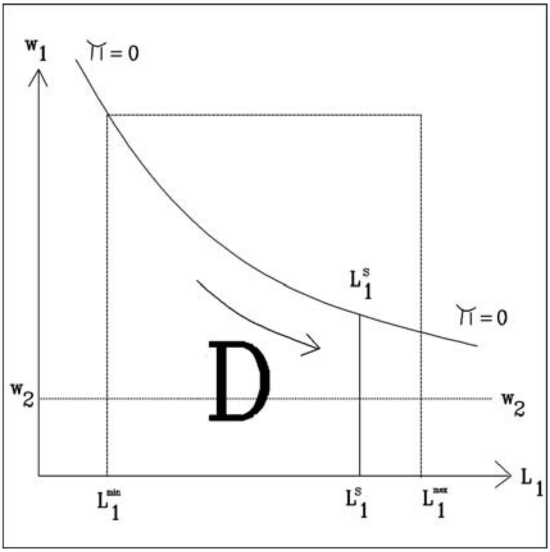

Fig. 19.2 A graphical representation of the admissible domain in the (w_1, L_1) space

Since we already know that there is no interior Nash solution w_1^*, L_1^* for

$$\max_{(w_1,L_1)\in D} U \cdot \Pi \tag{19.15}$$

the solution to this problem can only lie on the right hand or on the left hand boundary of the domain D.

Case A:

Assume that the right hand boundary of D is the relevant one for the above maximum:

1. If we have $L_1^* = L_1^s(k_1 = \sigma K/y_1 \geq L_1^s$, i.e., $V_1^* = 1)$, we get by inserting this into (19.15) and by maximization:

$$u_1^* = w_1^*/y_1 = \frac{\alpha}{1+\alpha}[k_1(1-u_2)+u_2] > \frac{\alpha}{1+\alpha}(\text{ if } u_2 < 1). \tag{19.16}$$

2. In the opposite case $L_1^s > L_1^{\max}$ (unemployed workers of type 1 and $V_2 = 0$) we get for the wage rate w_1^* on the right hand side of D:

$$u_1^* = w_1^*/y_1 = \frac{\alpha}{1+\alpha} \qquad \left(\text{and} \quad V_1^* = \frac{\sigma K}{y_1 L_1^s} = k_1 < 1\right). \tag{19.17}$$

Considering the dynamic implications of the second situation first, we get (because of $V_2 \equiv 0$):

$$\widehat{V}_1^* = \widehat{K} - (m+n) = \sigma(1-u_1^*) - (m+n) = \sigma/(1+\alpha) - (m+n)$$

which with respect to values of σ, m, n corresponding roughly to empirical magnitudes can be assumed to be a positive scalar. This situation can consequently be regarded as being of a temporary nature only, so that sooner or later the situation (1) we have depicted in Fig. 19.2 will come about.

In this latter case we have

$$u_1^* = \frac{\alpha}{1+\alpha}[k_1(1-u_2)+u_2], \quad V_1^* = 1 \quad (\text{see 3.8}),$$
$$V_2 = k_2 - \gamma_y\gamma_L = \gamma_y\gamma_L(k_1-1) = k_2(1-1/k_1),$$
$$u = u_1^*/k_1 + u_2V_2/k_2 = \frac{\alpha}{1+\alpha}(1-u_2+u_2/k_1) + u_2(1-\frac{1}{k_1}) = h(u_2,k_1),$$

(since $V_1 = 1, k_1 > 1, h_{1,2} > 0$) and because of the following relationships:

$$u = \frac{w_1L_1 + w_2L_2}{y_1L_1 + y_2L_2} = u_1V_1\frac{y_1L_1^s}{\sigma K} + u_2V_2\frac{y_2L_2^s}{\sigma K} = u_1V_1/k_1 + u_2V_2/k_2,$$
$$V_2 = k_2 - \gamma_y\gamma_L V_1 \quad \text{and} \quad k_2 = k_1\gamma_L\gamma_y].$$

The above equations now imply

$$\widehat{u}_2 = f(V_2) - m = f(\gamma_L \gamma_y (k_1 - 1)) - m, \tag{19.18}$$

$$\begin{aligned}\widehat{k}_1 &= \widehat{K} - (m+n) = \sigma(1-u) - (m+n)\\ &= \sigma(1 - h(u_2, k_1)) - (m+n)\end{aligned} \tag{19.19}$$

as the final dynamics for *Case A* [where $L_1^s < \sigma K$ (or $k_1 > 1$) holds true].

For the Jacobian of this system we consequently get by our above calculations

$$J = \begin{pmatrix} 0 & + \\ + & - \end{pmatrix}$$

which implies the *asymptotic stability* of the steady state of our model (19.18), (19.19). The *steady state values* of (19.18), (19.19) are:

$$u_2^0 = \frac{[(1-(m+n)/\sigma)(1+\alpha) - \alpha]k_1^0}{k_1^0 - 1} = \frac{[u^0(1+\alpha) - \alpha]k_1^0}{k_1^0 - 1}, \tag{19.20}$$

$$k_1^0 = 1 + \frac{a+m}{b\gamma_L\gamma_y} \quad [f(V_2) = -a + bV_2], \tag{19.21}$$

$$V_2^0 = \frac{a+m}{b} \quad [V_1^0 = 1], \tag{19.22}$$

$$u^0 = 1 - (m+n)/\sigma, \tag{19.23}$$

$$u_1^0 = \frac{\alpha}{1+\alpha}[k_1^0(1-u_2^0) + u_2^0] = \alpha k_1^0(1-u^0) = \alpha k_1^0 \cdot \frac{m+m}{\sigma}. \tag{19.24}$$

Note here, that $u_2^0 > 0$ if $u^0 = 1 - \frac{n+m}{\sigma} > 1/2 \quad (> \alpha/(1+\alpha), \alpha > 1/2)$.

Case A thus exhibits an asymptotically stable steady state, where the first group of workers is fully employed at the wage contract $w_1^* = \frac{\alpha}{1+\alpha}[k_1^0(1-u_2^0) + u_2^0]y_1$.

For the parameter values $\alpha = 2/3$, $\sigma = 1/5$, $m = 0.06$, $n = 0.04$, $b = 1$, $a = 0.9$, $\gamma_y\gamma_L = 1/0.96$, we for example have:

$$k_1^0 = 2, \quad V_2^0 = 0.96, \quad u^0 = 0.5, \quad u_2^0 = 1/3, \quad u_1^0 = 2/3,$$

i.e., a significant difference to the situation we have considered in Sect. 19.2.

And in the second case, i.e., *the temporary situation* $u_1^* = \frac{\alpha}{1+\alpha}$ (and $V_1^* = k_1 \leq 1$, see (19.17)) we get as share of wages $u_1^* = 0.4$ ($\ll 2/3$) and as dynamics the equation

$$\hat{V}_1^* = \sigma/(1+\alpha) - (m+n) = \frac{1}{5} \times \frac{3}{5} - \frac{1}{10} = 0.02$$

which again demonstrates that this situation will lead us to full employment for the first group of workers.

When this state is reached a *regime switching* takes place which, however, will not be analyzed in this chapter in greater depth due to the preliminary type of the above model. Instead, we simply note that the model (19.18), (19.19) will also be globally stable (see the generalization of Olech's theorem in Ito 1978), but we have to stress also that care must be taken with regard to initial, intermediate and boundary conditions to obtain an economically meaningful path leading from

$$u_1^* = \frac{\alpha}{1+\alpha} = 2/5, \quad V_1^* \leq 1 \text{ to the final values } u_1^o = 2/3, \quad V_1^* = 1.$$

For the purpose of comparison with Sect. 19.4 let us add here a simulation example which is based on the same numerical values as those in Sect. 19.4. This example shows – in Fig. 19.3 – the regime switching process from $k_1 = V_1 \leq 1$ to $k_1 > 1$ and $V_1 = 1$. This figure also exemplifies that despite a very rapid and in the end cyclical process toward high employment of the second segment of the labor force, the ratio $u_2 = w_2/y_2$ of their share in their individual productivity is not much higher than the minimum level $u_{2,\min} = 0.2$ which we have assumed here as starting value for an initial employment of this type of labor (Such a minimum level is necessary to allow for a proper regime switching from the employment rate $V_2 = 0$ to rates where $V_2 > 0$).

Note with regard to Fig. 19.3 that the V_2-curve must be steeper than the k_1-curve, since the later measures the employment effects with regard to the first type of labor (which has a higher productivity $y_1 > y_2$). Note furthermore, that this model is asymptotically stable, but that there is a long phase of absorption with regard to the second group of laborers before their effect on income distribution becomes a

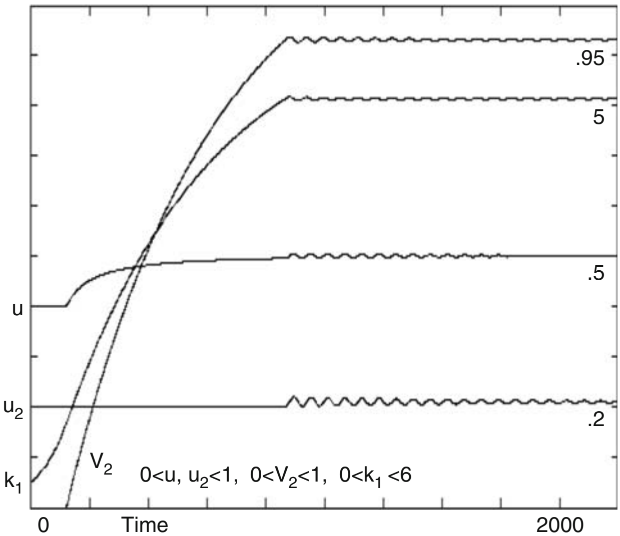

Fig. 19.3 Various adjustment paths

matter. Note finally, that the implied effect on the average share of wages is mainly due to the strong and positive effect on the individual share u_1^* of the first group of workers.

Case B:

We have assumed until now that the solution to (19.15) will always lie on the right hand boundary of the domain D. Let us briefly consider also the other possibility that it is situated on the left hand side of D. In this situation we have instead of the equations which characterize case A.1 (A.2 does no longer exist):

$$\widehat{u}_2 = f(V_2) - m = f(\gamma_L \gamma_y (k_1 - 1)) - m = g(k_1), \; g' > 0,$$
$$\widehat{k}_1 = \widehat{K} - (n + m) = \sigma(1 - u_2(1 - 1/k_1^{\min})/(1 + \alpha) - (n + m) = h(u_2), \; h' > 0,$$

because of $k_1^{\min} = \sigma K/(y_1 L_1^{\min}) \quad (> 1)$ and because of the following equations:

$$k_1/V_1 = k_1^{\min},$$
$$k_2/V_2 = 1 - k_1^{\min},$$
$$u = u_1^* V_1/k_1 + u_2 V_2/k_2,$$
$$u_1^* = \frac{\alpha}{1 + \alpha}[(1 - u_2)k_1^{\min} + u_2].$$

The above two dynamic equations immediately show that this case will lead us back to the original type of Goodwin's growth cycle. In this case the new behavior of the first group of workers does not add anything new to this growth cycle model.

19.4 A Simple Completion of Goodwin's Growth Cycle and the Implications of Cooperation

Goodwin's growth cycle has often been criticized because of its structural instability, i.e., its property of having only closed orbits as solution curves of its dynamics. Furthermore, in view of Marx's (1954, Chap. 25) descriptions of such a cycle, it appears much more adequate to model such a conflict over income distribution in a way such that a limit cycle will come about, i.e., by means of an unstable steady state solution which at the same time does not lead to a purely explosive behavior. Instead, as has already been suggested by Marx, reproductive forces will come into being when employment and income distribution depart too much from their steady state values:

> The rise of wages therefore is confined within limits that not only leave intact the foundations of the capitalistic system, but also secure its reproduction on a progressive scale. (Marx 1954, p. 582)

In the context of a model which centers on the labor market only it is, however, difficult to obtain such a limit cycle behavior purely from the conflict over income distribution. Lags in production would suffice to destabilize the steady state, but would give rise to a complicated delayed differential system. Following Flaschel (1988) we will therefore choose a simpler way to obtain such an instability and we shall only make use of very crude mechanisms to keep the interior explosive behavior within economically meaningful bounds. The purpose of this exercise will be to test whether the results of the preceding section on the partial cooperation between capital and labor can be extended to this more complete type of dynamics and whether this cooperation will give rise to significant changes of its limit cycle behavior, in particular whether it is capable of generating asymptotic stability, i.e., of removing the permanently cyclical nature of the conflict over income distribution governing the case of atomistic competition.

To provide a basis for such a test the following modifications will be assumed with respect to Goodwin's original cycle (see Sects. 19.1 and a19.2):

$$\widehat{w}_2^m = f(V_2) + \widetilde{\eta}(g(u))\widehat{p}, \tag{19.25}$$

$$\dot{p} = \lambda(u)[A(w_1^m L_1 + w_2^m L_2)/(\sigma K) - p], \quad A > 0, \tag{19.26}$$

$$\widehat{K} = i(V_2)\sigma(1-u), \tag{19.27}$$

where:

1. $\widetilde{\eta}(g(u)) = \eta(u)$ fulfills $\eta(u^0) > 1, \eta'(u) < 0$ and $\eta(u) < 1$ for $u > 0$ sufficiently large, u^0 the steady state value of the wage share u
2. $\lambda(u)$ fulfills $\lambda'(u) \geq 0$, $\lambda(u) \to \infty$ for $u \to 1$ [and $\lambda(u) \equiv 0$ for $u \leq u^0 - \delta$, some $\delta > 0$]

and where

3. $i(V_2) = 1$ for $0 \leq V_2 \leq 1-\epsilon, \epsilon > 0$ small and $= 0$ for $V_2 = 1$

(all functions are supposed to be sufficiently smooth).

Since we only want to treat the dynamics (19.18), (19.19) – where V_1 equals 1 – in this extended model we have formulated the above modifications only with respect to V_2, i.e., with respect to that group, which is excluded from cooperation. Equation (19.25) then says that the rate of change of money wages w_2^m depends on the rate of employment V_2 of this group and by means of a factor $\widetilde{\eta}(.)$ on actual inflation $\widehat{p} = \dot{p}/p$. This factor $\widetilde{\eta}(.)$ in turn is assumed to depend on the growth rate g of output Y (and capital K) and is larger than "1" when g equals the "average" conditions $g^0 = g(u^0)$, i.e., the steady state value. Note that this rate g is always equal to $\sigma(1-u)$ in this growth cycle model, so that a positive dependence of $\widetilde{\eta}$ on the rate of growth g is turned into a negative dependence η on the share of wages u. This is the central component used here to obtain an unstable dynamics around the steady state. Note in addition, that the assumption $\eta \equiv 1$ will lead us back to the original growth cycle model.

Equation (19.26) claims that prices $\dot{p}$ change according to the discrepancy between marked-up average wage costs and actual prices p. In addition, the speed of adjustment with regard to these target prices becomes larger and larger if the wage share approaches "1", i.e., inflation will accelerate then. Furthermore, the assumptions on $\lambda(u)$ exclude "significant" deflation (i.e., $\lambda \equiv 0$ for target prices "significantly" less than actual ones) to avoid certain complications which are associated with such an occurrence.

Finally, (19.27) simply states that capital accumulation will stop if there is no further labor available. This, of course, is a very crude, but necessary assumption in a model in which the only alternative to direct investment is consumption.

Equations (19.25) and (19.26) imply

$$\widehat{w}_2 = f(V_2) - (1 - \eta(u))\lambda(u)(Au - 1) \tag{19.28}$$

as new form of the real-wage Phillips curve ($\widehat{u}_2 = \widehat{w}_2 - m$). And from (19.27) we get as our second differential equation [see (19.19) in Sect. 19.3]

$$\widehat{k}_1 = i(V_2)\sigma(1 - u) - (m + n). \tag{19.29}$$

This dynamical system can be transformed into an autonomous one in exactly the same way as the one in Sect. 19.3 [see the equations preceding (19.18), (19.19)], i.e., by making use of the relationships

$$u = \frac{1}{1+\alpha}[(1 - 1/k_1)u_2 + \alpha] \quad \text{and} \quad V_2 = (k_1 - 1)\gamma_y\gamma_L.$$

Let us assume finally that the steady state wage share $u^o = 1 - (m + n)/\sigma$ can be obtained without inflationary pressures, i.e., the parameter A is set equal to $1/u^0$ in the following. This implies that our new dynamical system

$$\widehat{u}_2 = f(\gamma_L\gamma_y(k_1 - 1)) - (1 - \eta(u(u_2, k_1))\lambda(u(.,.))(Au(.,.) - 1) - m \tag{19.30}$$

$$\widehat{k}_1 = i(\gamma_L\gamma_y(k_1 - 1)) \cdot \sigma(1 - u(u_2, k_1)) - (m + n) \tag{19.31}$$

has the same steady state values as the one of Sect. 19.3.

It suffices therefore to check the Jacobian of this system to see whether the cooperative element introduced in Sect. 19.3 will make any difference in comparison to the "solidarity version" of this model. This latter version is obtained from the above model by setting $u = u_2 = u_1$, $V = V_2 = V_1$ and by noting that $\widehat{k}_1$ equals $\widehat{V} = \widehat{V}_1$ in this case ($k_1 = V_1$) (note that V_2 is then no longer determined as a residual with regard to V_1, so that $\gamma_L\gamma_y(k_1 - 1)$ must be replaced by $V_1 = V_2$ in this case). We then get

$$\widehat{u} = f(V) - (1 - \eta(u))\lambda(u)[Au - 1], \tag{19.32}$$

$$\widehat{V} = i(V)\sigma(1 - u) - (m + n). \tag{19.33}$$

This is the more complete version of a Goodwin growth cycle that we would associate with our above quotation from Marx's Capital and the comments we related with it. Note that this more refined dynamics also exhibits the same steady state values as the model of Sect. 19.2 (because of $A = 1/u^0$).

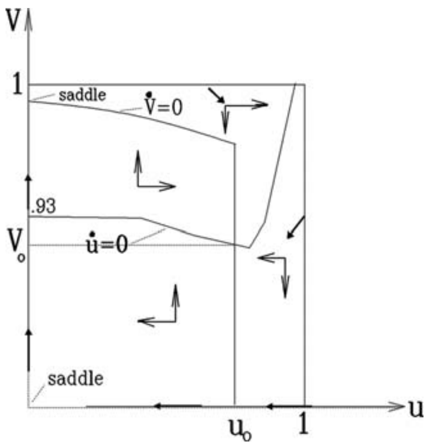

Fig. 19.4 The phase diagram of the generalized Goodwin growth cycle model

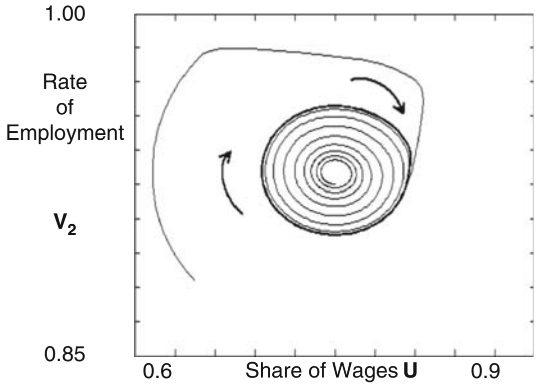

Fig. 19.5 A simulation example of the limit cycle of the model

The dynamics (19.32), (19.33) gives rise to the following phase portrait (see Fig. 19.4) which already indicates that the model (19.32), (19.33) will exhibit a limit cycle (see Fig. 19.5).[3]

The Jacobian of this model at the steady state is given by

$$J = \begin{pmatrix} -(1-\eta(u^0))\widehat{p}'(u^0)u^0 & f'(V^0)u^0 \\ \sigma V^0 & 0 \end{pmatrix}$$

which shows that this steady state is an unstable node or focus (trace > 0, det > 0), as it was intended by the assumption made.

[3] The parameter values are: $m = 0.03$, $n = 0.02$, $\sigma = 0.2$, $A = 4/3$, $f(V) = -a + bV$ with $a = 0.9$, $b = 1$, $i(V) = -50V + 50$ on [0.93,1], $1 - \eta(u) = h(u)$ piecewise linear with $h(0) = -0.5$, $h(0.72) = -0.05$, $h(0.78) = -0.05$, $h(1) = 0.5$ and $\lambda(u) \equiv 0$ on $[0.0\bar{6}]$, $\lambda' \equiv 4$ on $(0.\bar{6}, 0.75]$, $\lambda' \equiv 400$ on (0.75,1].

Figure 19.5 provides a simulation of this modified Goodwin model for the above parameter values. It (as well as Fig. 19.4) clearly exemplify the forces which keep this unstable dynamics within bounds: Inflation is the means by which the rate of profit is defended against a wage share which is rising too high and a marked fall in the rate of growth of capital accumulation prevents that the full employment barrier can be crossed.

The model therefore provides a simple example for Marx's view on the cyclical nature of capitalistic accumulation as well as on the reproduction of its basis of existence, i.e., a rate of profit that is positive throughout.

Let us now turn to the model (19.30), (19.31) and the possible role the cooperative aspect may play in the above type of accumulation dynamics. Calculating the Jacobian at the steady state gives

$$J = \begin{pmatrix} -(1-\eta(u^0))\widehat{p}'(u^0)u^0 u_{u_2} & f'(\cdot)\gamma_L\gamma_y - (1-\eta(u^0))\widehat{p}'(u^0)u^0 U_{k_1} \\ -\sigma u_{u_2}\cdot k_1^0 & -\sigma u_{k_1}\cdot k_1^0 \end{pmatrix}$$
$$= \begin{pmatrix} + & + \\ - & - \end{pmatrix}.$$

The question now is whether there exist plausible situations in which the destabilizing effect of the first term in the diagonal of J is overcome by its new second term, so that the trace of J will become negative (which together with det > 0 then implies local asymptotic stability). Obvious assumptions – which will lead to such a result – are given by

(a) $1-\eta(u^0)$ small
(b) $\lambda'(u^0)$ small $\quad [\widehat{p}'(u^0) = \lambda'(u^0)(Au^0 - 1) + \lambda(u^0)A \approx A\lambda(u^0)]$
(c) k_i^0 close to 1 $\quad [\approx \alpha/(1+\alpha)]$

On the basis of these observations the factual disappearance of the limit cycle of Fig. 19.5, left side, can, e.g., be obtained if we add the following parameter values to this example: $\gamma_L = 0.2$, $\gamma_y = 1.2$, and $\alpha = 2/3$. Note that Fig. 19.6 depicts this result for the average wage share u on the horizontal and the employment rate V_2 on the vertical axes:

Figure 19.6, right side, finally shows the time paths for the variables: u (average share of wages), V_2 (employment rate of the second group of workers), k_1 (capacity to employ workers of type 1) and u_2 (the ratio w_2/y_2). It in particular demonstrates that k_1 remains near to its initial value $k_1(0) = 5$, so that the full employment of the first group of workers is always ensured ($k_1 \in [4.7, 5.1]$). And because of $u_1^* = [u_2(1-k_1)+k_1]\frac{\alpha}{1+\alpha} \approx -1.6\,u_2+2$ we know that u_1^* must follow a time path exactly opposite to that of u_2 (with a higher amplitude and an upward displacement term). Nevertheless, u_2 and u run parallel to each other because of the relative low number of workers of type 1 ($\gamma_L = 0.2$).

In addition to the steady state results of Sect. 19.3 we thus get that even cycles of a persistent nature will be turned into an asymptotically stable dynamics if the assumed type of cooperation between capital and the first type of workers is established.

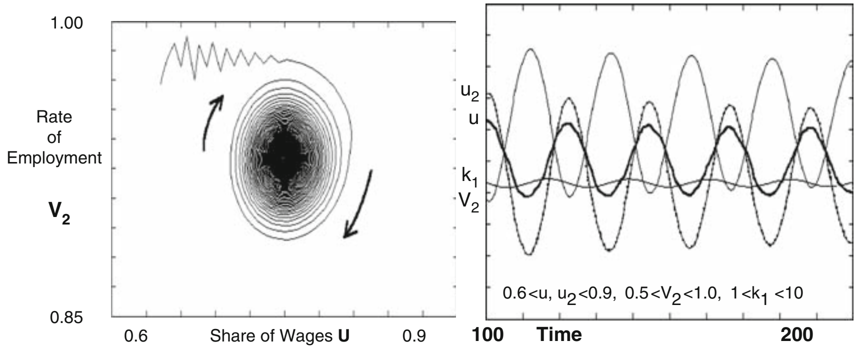

Fig. 19.6 Convergence: phase plot and time series representation

19.5 Conclusions

In Marx's (1954) Chap. 25 on the "General Law of Capitalistic Accumulation" a variety of aspects are considered which influence and modify the basic growth cycle mechanism that he formulates in the first section of this chapter as a critique of the Classical Theory of Accumulation and in particular of the Malthusian Population Law. Among these qualifications of the basic form of his analysis of cyclical accumulation we, for example, find:

(a) The influence of financial asset holdings
(b) The problem of capital export
(c) The influence of "capital" on labor demand (through technical change) *as well* as on labor supply (migration)
(d) Problems of labor market segmentation

We have attempted in this chapter to integrate this last aspect into the well-known growth cycle of R. M. Goodwin, but have done this in a way which at the present stage does not resemble any of the problems which Marx discusses in his analysis of the typical labor market segments of his time. Instead, we have simply assumed two types of labor characterized by different productiveness per "workweek" and have attempted to analyze what new aspects may come about if the more productive group in fact cooperates with "capital" in contrast to the case where it exercises solidarity with the other group of workers.

Our main findings were that the first group will gain from such a cooperation – and the second will loose – and that this cooperation will contribute to the stabilizing features of Goodwin's growth cycle model. Such findings should not, however, come as a surprise to those who use Goodwin's context for a modeling of the class struggle and the conflict over the distribution of income. As recently stated by Wörgötter (1986, p. 225):

> Capitalists ... join the class struggle over income and employment possibilities between different parts of the labor force only indirectly. One could even say that capitalists act as agents for the unemployed part of the labor force. Insofar capitalists resist higher wage demands for the already employed, the rate of accumulation can increase and new employment possibilities arise.

This quotation concerns the time structure of employment possibilities as it is influenced by the conflict over income distribution. What we have done in this chapter is that we have added a further – vertical – component to this structure which – on closer inspection – may reveal a similar type of conflict as the one quoted above. We consider it too early, however, to draw definite conclusions on this matter from the investigations we have made so far.

Nevertheless, it is clear that Goodwin's growth cycle is still fairly incomplete (see our Sect. 19.4), in particular as a model of Marx's views on cyclical accumulation. Much work remains to be done to obtain a more convincing picture of the conflict over income distribution (be it Marxian or otherwise) from this prototype of a growth cycle model. Yet, despite this critique of its preliminary nature, Goodwin's model rightly deserves the attention it has received in the past and hopefully also will receive in the future, since it is explicitly or implicitly involved in a variety of growth models of very different economic schools of thought.

Malinvaud (1980) assumes that the negative effect of rising wages on investment is less severe than their positive effect on consumption in order to obtain a model of a Keynesian depression (instead of a variant of a Goodwin cycle when appropriate modifications of the model are assumed, see Flaschel 1993, Chap. 5, for details).

Marglin (1984) discusses the Neo-Marxian model of the interaction of growth and income distribution verbally, which – when modeled – would have led him to some variant of the Goodwin model, too.

From all this – and also from many other models of cyclical growth – it can be concluded that the problem of integrating questions of income distribution into the macroeconomic analysis of economic evolution must at present be characterized as a very underdeveloped topic, where often aspects of the Goodwin model are involved, yet are not systematically explored.

Appendix: An Extended Objective Function of Firms

Regarding the assumed "utility function" of firms (19.10) an obvious objection against its form is the following: Due to the existence and employment of a second type of labor which gives rise to changing wages according to the labor market reaction curve $\hat{w}_2 = f(V_2)$ it is not plausible that firms will only pay attention to the present levels of wages w_2 and w_1. Instead, they of course will also try to take their future development appropriately into account. In the context of Goodwin's growth cycle model this is, however, in general a very difficult task – due to the fact that this model does not allow for an explicit solution if slight complications are introduced into it.

Because of this fact and because of the in principle myopic perspective which we shall continue to assume with regard to the behavior of the agents of our model, we propose the following simple generalization of the objective functional (19.10) of firms:

$$\int_0^h (\sigma K(t_0+\tau) - w_1(t_0+\tau)L_1(t_0+\tau) - w_2(t_0+\tau)L_2(t_0+\tau))d\tau. \quad (19.34)$$

This new function now allows for the fact that firms may try to choose employment L_1 and thus also L_2 in such a way that they can benefit from the future development of wages with regard to the aggregated profits (19.34). To keep the model tractable we shall furthermore assume that firms use current rates of change (as constants) to calculate the future development of profits. This gives rise to

$$\sigma K(t_0)\int_0^h (1+g\tau)d\tau - w_1(t_0)L_1(t_0)\int_0^h (1+x\tau)(1+g\tau)d\tau$$
$$-w_2(t_0)L_2(t_0)\int_0^h (1+f(V_2(t_0))(1+g\tau)d\tau$$

if we use $(1+..\tau)$ instead of $e^{..\tau}$ as an approximation (g the rate of growth of K and x the rate of growth of wages w_1 assumed by firms). Solving the above integrals and neglecting all terms of type xg, $f(\cdot)g$, etc., then gives rise to

$$\left[\sigma K(t_0)(1+gh/2) - w_1(t_0)L_1(t_0)(\frac{x+g}{2}h) - w_2(t_0)L_2(t_0)(1+\frac{f(V_2)+g}{2}h)\right]\cdot h.$$

An average expected rate of profit r may therefore be obtained from this expression by dividing it by $K(t_o)(1+gh/2)h$ (neglecting again all terms where products of growth rates are involved). This finally gives

$$r = \sigma\{1-[u_1(t_o)V_1(t_o)/k_1(t_o)](1+xh/2) - [u_2(t_o)V_2(t_o)/k_2(t_o)](1+f(V_2(t_o))h/2)\},$$

where as before we use the abbreviation $k_i = (\sigma K)/(y_i L_i^s)$, $i = 1,2$. Note, that $u_i V_i/k_i$ is equal to $(w_i/y_i)(L_i/L_i^s)(y_i L_i^s)/(\sigma K)$ by definition.

This is the final form of function which we shall take as the objective functional of firms in this section. Suppressing the time index and setting for simplicity $h=1$ it reads

$$r = \sigma[1-(u_1V_1/k_1)(1+x/2) - (u_2V_2/k_2)(1+f(V_2)/2)]. \quad (19.35)$$

It should be stressed here that we did not say anything on the growth factor $1+x/2$ of wages w_1 so far. This factor will be regarded as exogenously given in this chapter. The main difference between (19.10) and (19.35) thus is the integration of the wage effect $\hat{w}_2$ which results from the choice of V_1 and its influence on $V_2 = k_2 - \gamma_y\gamma_L V_1$.

The question now is whether firms can gain from taking into account the future development of capital growth and of market wages w_2 in comparison to the situation we have analyzed in Sect. 19.3.

Let us first investigate here the analog to the situation we have considered on the pages following Fig. 19.2, i.e., the case where the Phillips curve $f(V_2) = -a + bV_2$ is sufficiently flat so that the assertion of no interior equilibrium will hold true in the present context, too. Calculating as before the upper boundary of the region $D(r \equiv 0)$ now gives rise to

$$u_1 = \frac{1 - u_2 V_2/k_2(1 - a/2 + bV_2/2)}{(V_1/k_1)(1 + x/2)} \tag{19.36}$$

with $V_2 = k_2 - \gamma_y \gamma_L V_1$, i.e., $V_2/k_2 = 1 - V_1/k_1$. For $1 - u_2 > 0$ this is again a strictly decreasing function of $V_1(k_1, k_2$ given).

In principle we therefore get the same figure as in Sect. 19.3 (where we had $a = b = x = 0$), but now with respect to the variables u_1, V_1 (and the like).

Let us assume again that $V_1^* = 1$ is the typical situation to be investigated for a solution of

$$\max_{u_1, V_1} H = \max_{u_1, V_1} u_1^\alpha V_1^{1-\alpha} \cdot r(u_1, V_1) \tag{19.37}$$

[in the case $V_1^* = k_1 = V_1^{\max} < 1$ we again immediately get $u_1^* = \alpha/(1+\alpha)$ – because of $V_2 = 0$ – and the same result on the temporary nature of such a situation].

When also part of the second type of work force is employed, however, we get (besides $V_1^* = 1$):

$$u_1^* = \frac{1 - u_2(V_2/k_2)(1 - a/2 + (b/2)V_2)}{(1/k_1)(1 + x/2)} \cdot \frac{\alpha}{1+\alpha} \tag{19.38}$$

with $V_2 = k_2 - \gamma_L \gamma_y = \gamma_L \gamma_y (k_1 - 1)$, i.e.

$$u_1^* = \frac{\alpha}{1+\alpha} \cdot \frac{k_1 - u_2(k_1 - 1)(1 - a/2 + (b/2)\gamma_L \gamma_y (k_1 - 1))}{1 + x/2}.$$

For the average wage share $u = u_1^*/k_1 + u_2 V_2/k_2$ we thereby get in analogy to the preceding section the equation:

$$u = \frac{\alpha + u_2(1 - 1/k_1)[-\alpha(1 - a/2 + (b/2)\gamma_L \gamma_y (k_1 - 1)) + (1 + x/2)(1 + \alpha)]}{(1+\alpha)(1 + x/2)}. \tag{19.39}$$

Near the steady state the expression in square brackets is approximately equal to $(1 + x/2)(1 + \alpha) - \alpha(1 - m/2)$ and thus positive. We therefore again get the following functional relationship ($k_1 > 1$):

$$u = u(u_2, k_1), \quad u_1, u_2 < 0$$

and consequently again a dynamical system of the type:

$$\hat{u}_2 = f(\gamma_L \gamma_y (k_1 - 1)),$$
$$\hat{k}_1 = \sigma(1 - u(u_2, k_1)) - (m + n)$$

which is locally asymptotically stable.

We note that the steady state values V_2^0, k_1^0, u^0 are the same as in the preceding section. This implies by (19.39) that u_2^0 will be somewhat lower and therefore that u_1^0 will be somewhat larger than the corresponding values of the case $a = b = x = 0$.

The extension (19.34) thus does not seem to give rise to substantial modifications of the conclusions that we have obtained in Sect. 19.3. Nevertheless, there is a new aspect involved when moving from (19.10) to (19.34) – in the case where the labor market reaction curve $\hat{w}_2 = f(V_2)$ becomes sufficiently steep. Yet, we are only able to treat this aspect in a very preliminary way in this chapter (as follows):

When one investigates the first-order conditions of the Nash solution (19.37) for interior points of the domain D one finds (for a linear curve $f(V_2) = -a + bV_2$):

$$r = \frac{u_1 V_1 (1 + x/2)\sigma}{\alpha k_1} \tag{19.40}$$

and

$$r = \frac{\sigma V_1 (u_1(1 + x/2) - u_2(1 - a/2 + bV_2))}{(1 - \alpha)k_1} \tag{19.41}$$

(note, that these conditions have already been solved for the variable r and recall, that $V_2 = k_2 - \gamma_y \gamma_L V_1$). Equating these two expressions gives

$$u_1 = \frac{\alpha}{2\alpha - 1} \cdot u_2 \cdot \frac{1 - a/2 + bV_2}{1 + x/2}$$

as a necessary relationship between the optimal values u_1^*, $V_2^*(V_1^*)$, $\alpha > 1/2$. And for the second order conditions we obtain at the optimal values u_1^*, V_1^* by means of (19.40), (19.41) the expressions ($H = U \cdot r$):

$$\begin{aligned} H_{11} &= -\frac{\alpha}{u_1} U[(1 - \alpha)r/u_1 + 2(V_1/k_1)(1 + x/2)\sigma] \\ &= -U(1 + \alpha)(V_1/u_1)(\sigma/k_1)(1 + x/2), \\ H_{22} &= -U[(2 - \alpha)\frac{1 - \alpha}{\alpha}(u_1/V_1)(\sigma/k_1) + u_2 \gamma_L \gamma_y \sigma b/k_1], \\ H_{12} &= H_{21} = -U\sigma(2 - \alpha)/k_1. \end{aligned}$$

This again implies trace $(H_{ij}) < 0$. And for the determinant of this matrix we get (up to a positive scalar)

$$\frac{(2 - \alpha)(1 - 2\alpha)}{\alpha} + \frac{V_1}{u_1}(1 + \alpha)u_2 \gamma_L \gamma_y b. \tag{19.42}$$

As is obvious from (19.42) this expression can be made positive if, e.g., the parameter b is chosen large enough [Note again, that the first term is negative for $\alpha \in (0.5, 1)$]. The matrix (H_{ij}) thus can be made negative definite, so that there is now scope for an interior solution of (19.37). Such an interior "Nash solution" is, however, difficult to calculate because of the various quadratic terms involved in its determination. In addition, it is not clear whether a global maximum is given by this new type of solution. And a final problem in the treatment of such a "Nash solution" is that its derivation by means of the Pareto frontier

$$\delta U + (1-\delta)r \to \max_{u_1, V_1}, \delta \in [o, 1]$$

still faces the problem we noted in Sect. 19.3. The first order conditions (solved for δ) are given by

$$\delta = \frac{(V_1/k_1)\sigma}{(\alpha/u_1)U + \sigma V_1/k_1},$$
$$\delta = \frac{(u_1/k_1)\sigma - (u_2/k_2)\sigma\gamma_L\gamma_y[1 - a/2 + bV_2]}{U(1-\alpha)/V_1 + (u_1/k_1)\sigma - \sigma(u_2/k_2)\gamma_L\gamma_y(1 - a/2 + bV_2)}.$$

Instead of (19.13) these two equations now imply a negatively sloped straight line $A_1 - A_2 V_1 = u_1$ with

$$A_1 = \frac{\alpha}{2\alpha - 1}[u_2\gamma_y](1 - a/2 + bk_2) \quad \text{and} \quad A_2 = \frac{\alpha}{2\alpha - 1}[u_2\gamma_y]^2 b.$$

Yet, in contrast to the results on the second order conditions of $U \cdot r$ derived above, one here finds in analogy to the results in Sect. 19.3 that these conditions will allow for negative definiteness only if the parameter δ is again chosen sufficiently large in the interval $(0, 1)$.

References

Balducci, R. et al. (1984). A generalization of R.Goodwin's model with rational behavior of economic agents. In: R. M. Goodwin et al. (Eds.), *Nonlinear models of fluctuating growth* (pp. 47–66). Heidelberg: Springer.

Flaschel, P. (1988). Fiscal policy in an accelerator-augmented classical growth cycle. In: P. Flaschel & M. Krüger (Eds.), *Recent approaches to economic dynamics*. Bern: Peter Lang.

Flaschel, P. (1993). *Macrodynamics. Income distribution, effective demand, and cyclical growth*. Bern: Peter Lang.

Friedman, J. W. (1986). *Game theory with applications to economics*. New York: Oxford University Press.

Goodwin, R. M. (1967). A growth cycle. In: C. H. Feinstein (Ed.), *Socialism, capitalism and economic growth*. Cambridge, UK: Cambridge University Press.

Güth, W., & Selten, R. (1982). Game theoretical analysis of wage bargaining in a simple business cycle model. *Journal of Mathematical Economics, 10*, 177–195.

Haurie, A., & Pohjola, M. (1987). Efficient equilibria in a differential game of capitalism. *Journal of Economic Dynamics and Control, 81*, 1092–1109.

Ito, T. (1978). A note on the positivity constraint in Olech's theorem. *Journal of Economic Theory, 17*, 312–318.

Lancaster, K. (1973). The dynamic inefficiency of capitalism. *Journal of Political Economy, 81*, 1092–1109.

Maarek, G. (1979). *An introduction to Karl Marx's 'Das Kapital'*. Oxford: Martin Robertson.

Malinvaud, E. (1980). *Profitability and unemployment*. Cambridge, UK: Cambridge University Press.

Marglin, S. (1984). *Growth, distribution, and prices*. Cambridge, MA: Harvard University Press.

Marx, K. (1954). *Capital* (Vol. I). London: Lawrence and Wishart.

Mehrling, P. G. (1986). A classical model of the class struggle: A game-theoretic approach. *Journal of Political Economy, 94*, 1280–1303.

Ricci, G. (1985). A differential game of capitalism: A simulations approach. In: G. Feichtinger (Ed.), *Optimal control theory and economic analysis* (pp. 633–643). Amsterdam: North Holland.

Wörgötter, A. (1986). Who's who in Goodwin's growth cycle. *Jahrbuch für Nationalökonomie und Statistik, 201*, 222–228.

Chapter 20
The Classical Growth Cycle: Reformulation, Simulation and Some Facts

20.1 Introduction

In this chapter, we confront a simple extension of the Goodwin growth cycle model and its numerical investigation with corresponding phase plots of data on the wage share, the employment rate, the inflation rate, the profit rate and the output–capital ratio for eight OECD countries. Our modification of this growth cycle model gives rise to a locally unstable growth pattern which is turned into overall stability by an appropriate boundary behavior of investment and the inflation rate. The main finding of the chapter is that for some countries there are some similarities between the plots generated by the model and the data. This suggests that its long swing implications should be investigated further.

Whenever it comes to a consideration of the work of Richard Goodwin there is one particular paper which is most often given special attention: his 1967 growth cycle treatment of the "inherent conflict and complementarity of workers and capitalists".[1] A recent example for this observation is Solow's (1990) contribution to a collection of essays in honor of R. Goodwin where he not only provides a characterization of the merits and weaknesses of this prototype model, but also briefly discusses whether this employment-cycle model of the conflict over income distribution "fits the facts" (and what such a question may mean here).

One of Solow's findings in this article (on p. 39) is that there is a suggestion in the data of a predominantly clockwise motion, but in three separate episodes as far as the phase plot of annual US-data, 1947–1986 of the share of wages and the rate of employment is concerned. On p. 40 he adds: "It is also worth noting that the phase diagram contains the bare hint of a single large long-period clockwise sweep. It is only a hint, at best a hint. One cycle is not a periodic motion".

There are many authors who have questioned the empirical relevance of the period length that is implied by Goodwin's growth cycle model, when its parameters are given empirical content, see in particular Atkinson (1969). Solow himself calculates a period length of 8–10 years even for (linear) Phillips curves that are

[1] See Goodwin (1967, p. 58).

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_20,

very steep. He then concludes (p. 38) that "Goodwin cycles are something else. What could that something be? Here I presume that we must take seriously the story that the model tells: …".

We can add here to Solow's subsequent arguments the fact that Goodwin's model has the Marxian theory of the reproduction of capitalism as its background, see Goodwin (1967, p. 58), i.e., in this case: Marx's (1954) analysis of Chap. XXV, Sect. 1. There it is in particular stated by Marx (p. 582):

> The rise of wages therefore is confined within limits that not only leave intact the foundations of the capitalistic system, but also secure its reproduction on a progressive scale

This is definitely a statement on the stability, but not on asymptotic stability of a capitalistic economy.

Though it is true that Marx relates this observation to the industrial cycle, as he calls it, it is a fairly compelling conclusion here that a growth cycle which recreates endogenously the conditions for rapid capital accumulation may exhibit a phase length (and an amplitude) that not only varies considerably with the social environment in which it operates, but which may also become very large. It is in particular to be expected under modern postwar conditions that the large changes in income distribution among capital and labor which this reproduction process calls for will use up an amount of time which can be drastically higher than the 8–10 years that Solow calculated by means of a steep real-wage Phillips curve.

From our point of view, the attempts in the literature which modified the Goodwin growth cycle in such a way that it comes close to observed business cycles take the wrong track, since they relate this cycle to observations which have not much to do with a far-reaching removal or replacement of economic and social conditions which "seriously imperil the continual reproduction on an ever-enlarging scale, of the capitalistic relation" (Marx 1954, p. 582). Goodwin cycles are something else!

The Goodwin model has often been criticized because of its structural instability, see Farkas and Kotsis (1992, p. 514) for an example. These two authors then proceed to revise this model by introducing aspects of logistic saturation, delays and various types of memory into it which give it a three- or four-dimensional dynamic structure. These reformulations are very interesting, also from a mathematical point of view. They are used by Farkas and Kotsis to analyze in particular the cycle length that is generated by them when supplemented by Hungarian data of the period 1950–1980. In this way they generate a 12 years cycle-length that can be related with an observed "equipment cycle" and a 4 years cycle-length which is close to "inventory cycle time".

It is not the aim of the present chapter to devaluate these extensions of the Goodwin model, but to contrast them at the present stage of only very preliminary investigations of its empirical content with an alternative approach which attempts to argue for a large phase length and amplitude of this growth cycle (at least as far as the development of industrialized countries after World War II is concerned).

As far as the case of no active fiscal policy is considered, we shall favor here the view, that an appropriately modified Goodwin model can only explain the explosive cyclical pattern around its steady-state (through the cross-dual Lotka–Volterra mechanism it contains), and that appropriate mechanisms have to be added to it far off the steady-state to establish its viability in the large. Since capital is always fully utilized in the Goodwin as well as in our following model, the destabilizing role of conflicting income claims is modeled with respect to wage formation in this chapter solely. Missing here as in Goodwin's original formulation is a theory of effective demand which explains the rate of capacity utilization in each moment of time on the basis of which a second Phillips-curve mechanism (relating the rate of price-inflation with the rate of capacity utilization and expected wage inflation) could be added to it. By neglecting this, the instability of the steady-state of our model is therefore achieved in a very preliminary way.

Similarly, the forces which will guarantee the viability of the considered dynamics will also be introduced here in a very preliminary way. The rate of growth of the capital stock falls to (or below) the "natural rate" of growth (including productivity growth) near the absolute full employment ceiling, giving rise to "steady" growth for some time – in an essentially unsteady situation (due to the increase in the share of wages that accompanies this temporarily "steady" growth). Furthermore, if this increase in the wage share goes very far, price inflation in excess of wage inflation will be generated which prevents further increases in the wage share (in particular if this share approaches "one" which is possible in the original Goodwin model). This second barrier to the explosive dynamics of our modified Goodwin growth cycle will remain secondary in our following simulations of such a model, where we mainly rely on the full employment ceiling in order to get viable trajectories for suitably set initial conditions. The present approach is therefore similar in method to the Hicksian multiplier-accelerator theory of the trade cycle, see Hicks (1950), i.e., it does not explain cycles and viability by a single mechanism.

To put it in a different way: We basically retain the "linear" growth structure of the Goodwin model. Beyond locally bifurcating limit cycles, we use forces that only come into being far-off the steady-state growth path (and which in general can only be treated analytically for dynamical systems of dimension 2 by means of the Poincaré–Bendixson theorem).

Such an approach is mathematically seen much cruder and simpler than the model extensions chosen by Farkas and Kotsis. It aims at the construction of a very basic growth cycle model and its simulation which is capable of generating the bare hint of a single large long-period clockwise sweep that Solow observed for the postwar development of the US-economy.

In the next section we shall introduce our reformulation of the Goodwin growth cycle. We shall do this by also integrating into it a government sector allowing for a discussion of the effects of Classical or Keynesian fiscal policy rules as in Wolfstetter (1982). Such policy rules must, however, be interpreted with care in the present context, since we do not question the use of Say's Law of the original Goodwin model.[2] They only mean here that government can influence capital

[2] See Flaschel (1992, 1993) for the introduction of IS- and also LM-components into such a framework and Solow (1990, 3) on the importance of such extensions.

accumulation by means of certain bond market operations while effective demand problems, an "independent" formation of the rate of interest, as well as money and liquidity considerations are completely ignored. We therefore investigate from a very limited perspective the role of government in its role of helping or hindering a Marxian "Bereinigungskrise" to run its course. Nevertheless, the model provides interesting and provocative implications of the use of active fiscal policy in such a growth cycle context which deserve further discussion as to their survival in Keynesian reformulations of the Goodwin model.[3]

Section 20.3 then considers some simulation runs of this model without and with fiscal policy (and also with certain supply side shocks). The aim here is to show that the model can be used to describe (transient) regimes or longer periods of full employment (with a rising share of wages) as well as regimes or longer periods of stagflation with a length that is similar to Solow's observation of a single long-period clockwise sweep. These simulations also serve to illustrate the dynamics of the model and to provide a background with which the data of the following section can then be compared.

The next section performs Solow's "crudest sort of comparison with data" for eight countries of the OECD in order to see whether there is something typical in the pattern observed by Solow for the USA. The conclusions will be the same as in Solow's article (we quoted above) and which he continuous as follows (on p. 40):

> There may be work for Goodwinians here too, theoretical as well as applied. It would make more sense to me if the Goodwinian mechanism were to apply on a time scale considerably longer than the ordinary business cycle. Since the model determines its own period, there is room for some interplay between facts and the theoretical structure.

In this respect, the present chapter provides some preliminary material of theoretical as well as of statistical kind, from which, however, at the present stage no definite conclusions can and should be drawn. It follows Solow's (1990) proposals by extending the model as well as the data set in the hope that this may initiate further work from a theoretical as well as an empirical point of view which – even if the Goodwin growth cycle approach is thereby found to be only partly correct – would be of utmost importance in judging the evolutionary power of capitalistic societies.

The last section of the chapter, finally, lists important exogenous elements of our model which have been kept constant in its present form, but which are definitely not constant in reality, in particular in light of the long period view adopted in this chapter. This list in particular shows that the present model is still much too simple in order to allow for a close resemblance between its simulations and the factual data. We close this chapter by a brief consideration of the wage-share/employment-rate phase plot for the United Kingdom, 1855–1965 which again gives a bare hint that the discussed post-war development may have been uniquely related to this period and that the interaction of the wage share and the rate of employment may have been quite different before World War I.

[3] See Flaschel (1993, 4.7) for an example of this.

20.2 A Growth Cycle Model with a Government Sector

The model we shall make use of employs the following macroeconomic variables. As is customary, we here make use of $\dot{x}$ to denote the time derivative of a variable x and $\hat{x}$ for its rate of growth:

Y	Gross output
K	Capital stock
L	Employment
L^s	Labor supply
$V = L/L^s$	Rate of employment
w	Nominal wages
p	Price level
$\omega = w/p$	Real wages
σ	Output–capital ratio (a constant)
$y = Y/L$	Labor productivity
$u = \omega/y$	Wage share
$r = \sigma(1-u)$	Profit rate
$m = \hat{y}$	Productivity growth (a constant)
$n = \hat{L}^s$	Labor force growth (a constant)
δ	Rate of depreciation (a constant)
$\hat{p}$	Rate of price inflation
$\hat{w}$	Rate of wage inflation
t	Tax rate (a constant, $T = t(Y + rB)$ total taxes)
G	Government expenditures
B	Outstanding government debt

The model is of the same one-sector type as Goodwin's original approach, but it integrates a government sector and assumes more complicated reaction patterns in its wage-price sector as well as for investment behavior:

$$\hat{w} = f(V) + \eta(u)\hat{p}, \quad f' > 0, \eta' \leq 0, \tag{20.1}$$

$$\hat{p} = \lambda(u)[Au - 1], \quad \lambda' \geq 0, A > 1, \tag{20.2}$$

$$\dot{K} = i(V)[(1-t)(Y + rB - \omega L) - \dot{B}] - \delta K, \quad i' \leq 0, \tag{20.3}$$

$$G/Y = t + \mu(\bar{V} - V), \quad \mu \gtreqless 0, \bar{V} \in (0,1), \tag{20.4}$$

$$G = T - rB + \dot{B}, \quad T = t(Y + rB). \tag{20.5}$$

Equation (20.1) describes a money-wage Phillips curve. The strictly increasing function f of the rate of employment V is assumed to vanish at a (unique) $V_0 \in (0,1)$, where the rise in money-wages is equal to the rate of price-inflation $\hat{p}$ times an aspiration factor η which depends negatively on the share of wages u. Lower (higher) rates of employment lead to less (more) money-wage inflation.

Equation (20.2) describes price inflation as being of a delayed markup type $\dot{p} = \lambda(u)[AwL/Y - p]$ with a given markup factor A on average wage costs per unit of output and a speed of adjustment λ that increases with the share of wages u. Equation (20.3) describes capital formation. It assumes that all wages (after taxes) are consumed and that profit and interest income (after taxes) is invested in the new bond supply $\dot{B}$ of the government and in capital goods. The type of bond assumed here is of the fix price variety ($p_B = 1$) and it is furthermore assumed that government makes bonds perfect substitutes for real capital formation by always equating the rate of interest to the actual rate of profit. The value of the function $i(V)$ is normally equal to 1, but it begins to fall near "full employment" $V = 1$ reaching zero for some level of overemployment $V > 1$. This means that firms reduce gross investment significantly (step by step) near and above the full employment barrier. This behavior may be motivated by an increased investment of capital goods in R&D, i.e., in the search for labor-saving techniques which may increase the growth rate m of labor productivity when the economy comes near to the full employment level. For reasons of simplicity we assume m as constant nevertheless. Alternatively, one could have also assumed a positive relationship between the rates n, m and the rate of employment V.

Equation (20.4) describes the fiscal policy rule adopted by the government. For $\mu = 0$ it gives the case of a neutral government. A given parameter $\mu > 0$ describes a Keynesian policy rule (in a non-Keynesian setup), since it implies that the government will increase (decrease) its expenditures when the rate of employment V is below (above) its target level $\bar{V}$. By contrast, $\mu < 0$ represents a (neo)classical policy rule where government reduces expenditures in the depression and increases them in the boom. Equation (20.5), finally, is the government budget equation. It allows to remove the expressions $B, \dot{B}$ from the final presentation of the model (see below) and also implies (together with the assumed consumption and investment behavior) the validity of Say's Law.

A special case of the above model is treated extensively in Wolfstetter (1982) and a related one in Flaschel (1993, Chap. 4, Sect. 5). The following extends this section (where stability was investigated by means of a suitably chosen Liapunov function) to a study of local and global limit cycles via the Hopf-bifurcation and the Poincaré–Bendixson theorem.

Equations (20.1)–(20.4) can be easily reduced to the following two-dimensional dynamic system in the variables u, V :

$$\hat{u} = f(V) + (\eta(u) - 1)\lambda(u)[Au - 1] - m \tag{20.6}$$

$$\hat{V} = i(V)\sigma[(1-t)(1-u) + \mu(V - \bar{V})] - (n + m + \delta) \tag{20.7}$$

since we have $\hat{u} = \hat{\omega} - \hat{y}, \hat{\omega} = \hat{w} - \hat{p}$ and $\hat{V} = \hat{K} + \hat{\sigma} - \hat{y} - \hat{L}^s$ by the definition of $V(= K \cdot \sigma / y / L^s)$ and since (20.4), (20.5) give rise to

$$\dot{B} - (1-t)rB = \mu(\bar{V} - V)Y.$$

The steady state u^*, V^* of this dynamics is given by

$$f(V^*) = m + (1 - \eta(u^*))\lambda(u^*)[Au^* - 1]$$

$$u^* = 1 - [(n + m + \delta)/\sigma) + \mu(\bar{V} - V^*)]/(1 - t),$$

if $i(V^*) = 1$ is assumed by an appropriate choice of the function $i(.)$.

The isoclines $\dot{u} = 0$, $\dot{V} = 0$ of system (20.6), (20.7) read

$$\dot{u} = 0: \qquad V = f^{-1}((1 - \eta(u))\lambda(u)[Au - 1] + m),$$
$$\dot{V} = 0: u = 1 - [(n + m + \delta)/(i(V)\sigma) + \mu(\bar{V} - V)]/(1 - t).$$

For the simulations in the next section we assume for simplicity that f is linear: $f(V) = \rho V - \gamma, \rho, \gamma > 0$. Furthermore, the monotonic functions i, λ, η are there given by simple step functions which are chosen such that the following characteristics of the phase diagram of the above dynamics are implied (for $\mu = 0$) over the economically relevant range of the wage share u:

This diagram assumes $i(V_3) < (n + m + \delta)/(\sigma(1 - t))$ and $\eta(u) > 1$ for $u < u_2$ and $\eta(u) < 1$ for $u > u_2$, which implies that the $\dot{u} = 0-$isocline must be declining to the left of u_2 and rising to its right. A positive μ gives each of the segments of $\dot{V} = 0$ a positive slope and a negative μ a negative one.

Figure 20.1 exemplifies in a simple way the two bounds that we want to impose on our variant of the Goodwin model, namely:

1. That the rate of growth of the capital stock falls below the rate $n + m$ if V is close to its maximum value $V_{max} > 1$.

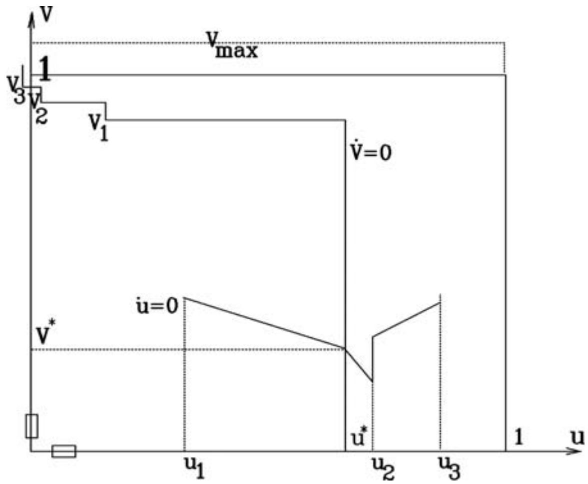

Fig. 20.1 The construction of the isoclines of the phase plot

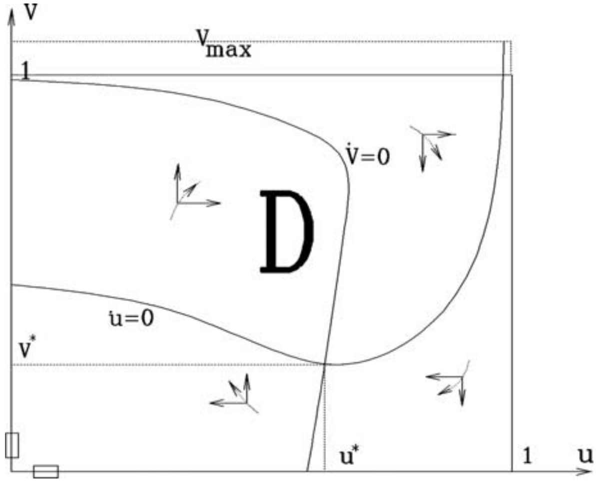

Fig. 20.2 A smooth version of the phase plot

2. That the adjustment speed λ of prices towards their target value becomes sufficiently high for very low rates or shares of profit, i.e.,

$$\lambda(1-\eta)(Au-1) > \rho V_{\max} - \gamma - m$$

holds for all u sufficiently close to 1.

The above situation of step functions i, η, λ will underlie the simulations of the following section. It is there used as a crude approximation to the following smooth situation where we have also added the directions of motion for phase points off the isoclines ($\mu > 0$ here) (Fig. 20.2). Smoothness assumptions are needed when one wants to apply the Poincaré–Bendixson theorem to situations as the above to show the existence of limit cycles.

In order to simplify further computations, we now assume $A = 1/u^*$ and $\bar{V} = V^*$. The latter assumption states that government takes the long-run average when formulating its objective concerning the employment rate. We then get for the Jacobian J of system (20.6), (20.7) at the steady state:

$$J^* = \begin{pmatrix} -(1-\eta(u^*))\lambda(u^*) & f'(V^*)\cdot u^* \\ -\sigma(1-t)V^* & \sigma\mu V^* \end{pmatrix}.$$

Keeping in mind our assumption $\eta(u^*) > 1$ we come to the following results:

Proposition 1. *For all $\mu > \frac{(1-\eta(u^*))\lambda(u^*)}{\sigma V^*}$ the trace of J^* is positive, so that the steady state is locally asymptotically unstable. For $\mu < \frac{(1-t)f'(V^*)u^*}{(1-\eta(u^*))\lambda(u^*)}$ the determinant of J^* becomes negative and thus the instability is of the saddlepoint type.*

Proposition 2. *If* $\frac{(1-\eta(u^*))\lambda(u^*)}{\sigma V^*} > \frac{(1-t)f'(V^*)u^*}{(1-\eta(u^*))\lambda(u^*)}$ *holds, at* $\mu^H = \frac{(1-\eta(u^*))\lambda(u^*)}{\sigma V^*}$ *a Hopf-bifurcation takes place, so that we get a limit cycle for a small range of u's either lower or larger than* μ^H *or a center-type dynamics at* $\mu = \mu^H$. *This will be the case, if, given all the other parameter values, the adjustment speed of the wages in the steady state is sufficiently high, i.e., if the slope of the function* f *is sufficiently steep in* V^*.

This can easily be verified by inserting the special linear form for $f(V)$ used for the simulations: $f(V) = \rho V - \gamma, \quad \rho, \gamma > 0$. Then the condition in Proposition 2 can be rewritten in the following way:

$$\gamma + m > \frac{[(1-\eta(u^*))\lambda(u^*)]^2}{(1-t)\sigma u^*}.$$

So γ has to be large enough to fulfil this inequality. On the other hand, $V^* = \frac{\gamma+m}{\rho} < 1$, so that we get a lower boundary for ρ :

$$\rho > \gamma + m > \frac{[(1-\eta(u^*))\lambda(u^*)]^2}{(1-t)\sigma u^*}.$$

The existence of limit cycles, however, is not restricted to this case. Note that the boundary of the positive orthant is an invariant subset of the phase space of (20.6), (20.7) [which therefore cannot be crossed by the trajectories of (20.6), (20.7)] and that the assumed full employment ceiling guarantees that the dynamics must point into the rectangle D for (nearly all) values of V close to $V_{\max}$. Furthermore, since inflationary processes will diminish the share of wages u for values of it close to 1 (and $V < V_{\max}$), the dynamics is also directed into D for all $V \leq V_{\max}$ close to $u = 1$. The rectangle D therefore determines an invariant subset of the whole phase space of (20.6), (20.7) which gives the standard situation for the application of the Poincaré–Bendixson theorem. We therefore get

Proposition 3. *All trajectories of (20.6), (20.7) which start in* D *must be closed orbits or tend to one (which is then called a limit cycle), if the steady state of the model is locally unstable.*

We consequently have that our modification of the Goodwin growth cycle creates a dynamics which is viable (i.e., stays within economically meaningful bounds when started within these limits) and which comes close to persistent economic fluctuations after some suitable chosen time-interval. Goodwin-like forces here create – due to the steady state assumption $\eta(u^*) > 1$ – explosive motions (cycles near or at the bifurcation value μ^H), which latest are turned into cyclical motions when the two ceilings $V_{\max}$ or $u = 1$ are approached.

Keynesian fiscal policy cannot remove this cyclical pattern, however large it is exercised. Classical fiscal policy, on the other hand, may make the steady state asymptotically stable if it is used with sufficient strength. This asymmetry in the stabilizing potential of the two considered policies also exists for small variations in the parameter μ as simulation runs of the model can show, see the next section. It can also be shown to exist for models with Keynesian demand restrictions, see Flaschel (1993, Chap. 4) for example, and thus provides a further argument – from a quite different angle than more usual ones – why Keynesian full employment policies may be problematic in the long run.

20.3 Some Simulation Results

In order to study the model numerically we shall make use of the following discrete time version of it (which at one and the same time can be interpreted from an economic and a numerical point of view, h being the period length as well as the step length of the iteration procedure):

$$u_{t+h} = u_t + u_t h[f(V_t) + (\eta(u_t) - 1)\lambda(u_t)(Au_t - 1) - m],$$
$$V_{t+h} = V_t + V_t h[i(V_t)\sigma((1-t)(1-u_t) + \mu(V_t - V^*) - (n + m + \delta)].$$

This discrete-time dynamics will be used in the situation we depicted in Fig. 20.1 in order to investigate the limit cycle result we have sketched for the smooth case in the preceding section. Our aim here is to show that this simple model is capable of generating trajectories which at least faintly mirror the full employment regime and the stagflation regime of industrialized post-war economies.[4]

Our choice of the parameters σ and δ is such that a time period of approximately 1 year is involved. The model is simulated (by means of the above Euler method) by choosing for the step length the value $h = 0.1$. Choosing $h = 0.01$, instead, for example, removes some of the kinks from the following simulations, but does not change their overall outlook. However, only each tenth point in time is plotted actually, so that the plots can be interpreted as showing yearly data throughout. The simulations start with the steady-state values which are shocked after some time ($u_t \to u_t \cdot 0.9$) in order to create the starting value for the phase plot.

Figure 20.3 shows the behavior of our simulated economy "T-land" over 75 years. The artificial shock occurs at $t = 15 \quad (\hat{=}1960)$ which, however, is of no real significance for the interpretation of the results.

In the present simulation the stable limit cycle is created by the full employment ceiling only (since $\eta > 1$ over the shown range of the wage share). The orbit remains near full employment for 10–12 years. Thereafter declining employment rates are coupled with further rising inflation rates, i.e., a stagflationary episode is then generated because of the high employment that preceded it. The following limit cycle

[4] The numerical specification of the above model is as follows ($A = 1/u^*$) : $n = 0.01$, $m = 0.03$, $\delta = 0.16$, $\sigma \approx 0.6$, $t = 0.25$. $u^* \approx 0.555$, $V^* = 0.96$, $f(V) = 0.2V - 0.162$, $\eta(u) = 1.05,\ 0.8,\ 0.5$ on $(0.4, 0.6), (0.6, 0.65), (0.65, 0.9)$, respectively. $\lambda(u) = 0.1,\ 1.8,\ 4$ on $(0.4, 0.55), (0.55, 0.65), (0.65, 0.9)$, respectively. $i(V) = 1,\ 0.99,\ 0.95,\ 0.5$ on $(0, 0.99), (0.99, 1), (1, 1.01), (1.01, \infty)$, respectively.

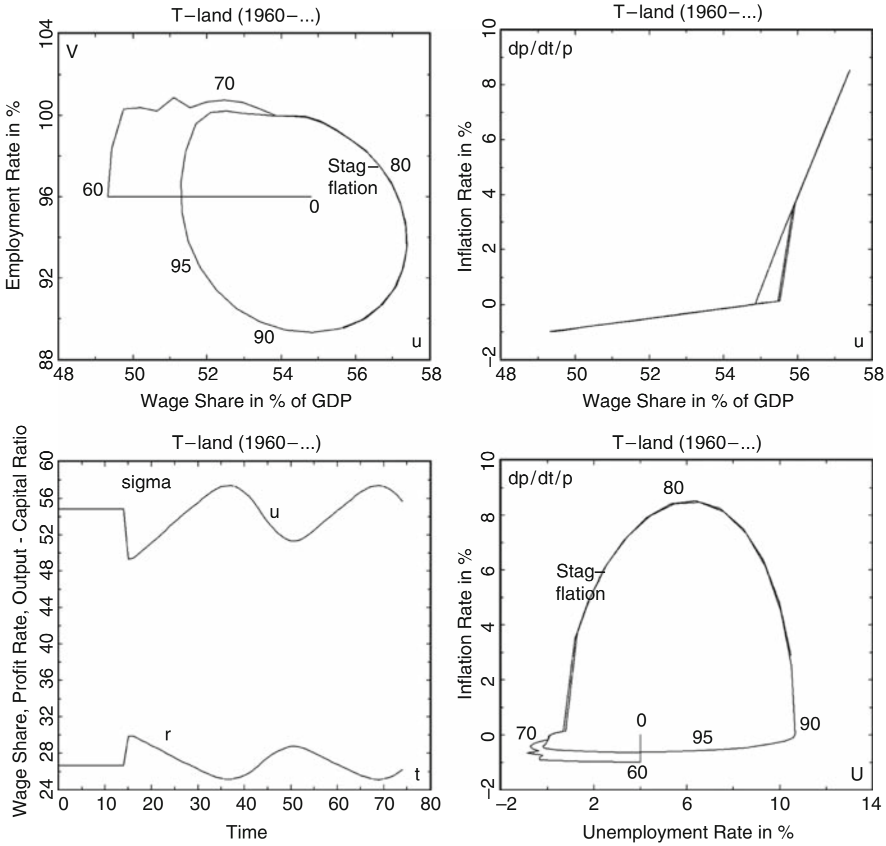

Fig. 20.3 The model without fiscal policy ($\mu = 0$)

behavior allows for wage share variations crudely speaking between 51% and 58% and rate of employment variations between 90% and 100% (= full, but not absolute full employment). The period length of the cycle is approximately 30 years.

The other pictures in the figure show our simple kind of markup theory of inflation (top, right), the behavior of the share of wages and the rate of profit over time (based on our assumption of a constant output–capital ratio: bottom, left) and the phase plot of the unemployment rate against the rate of inflation which is the typical diagram for the standard presentation of stagflation (bottom, right). In sum, these figures show some details of an explosive growth cycle model of Goodwin type in which one further nonlinearity is operating: the full employment ceiling, which keeps the model within economically meaningful bounds. Figure 20.4 provides an example of the working of the Keynesian fiscal policy rule in this context. We here once again stress that this type of behavior can also be shown for models with an IS- or IS-LM-part.

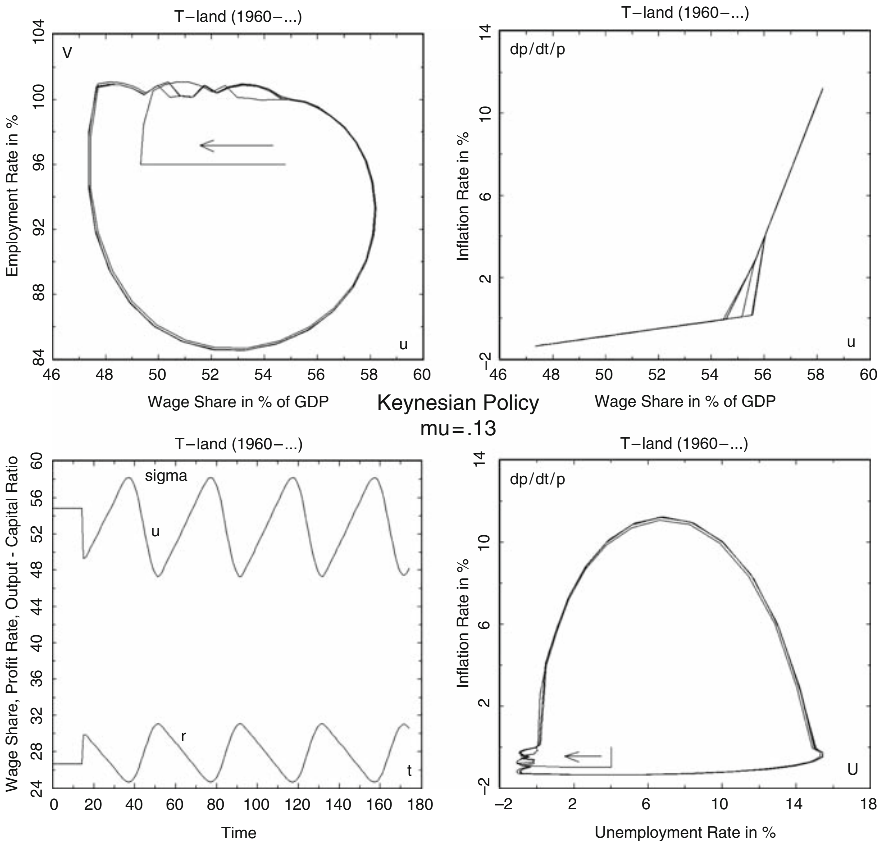

Fig. 20.4 Keynesian fiscal policy ($\mu = 0.13$)

Such a policy increases the amplitude of the cycle and decreases its phase length significantly.

By contrast, a classical fiscal policy can even make the steady-state an asymptotically stable one as the Fig. 20.5 shows.

The phase length as well as the amplitude of the cycle are both decreased considerably in comparison to the case of a neutral policy and the cycle now disappears in the long run.

Oil price-shocks can be interpreted as sudden changes in the markup factor $A > 1$ if there is a fixed ratio x (and y) between the oil price and wages (and oil usage and employment) and if the original markup A is applied to both unit-costs. The factor A is then simply replaced by $A(1 + x)(1 + y)$, if it is furthermore assumed that all income from this source is used for investment, just as all other profit income. Adding three such shocks (two positive and a negative one) to the model without fiscal policy then gives rise to the following dynamics which is our final illustration of the working of the model of Sect. 20.2.

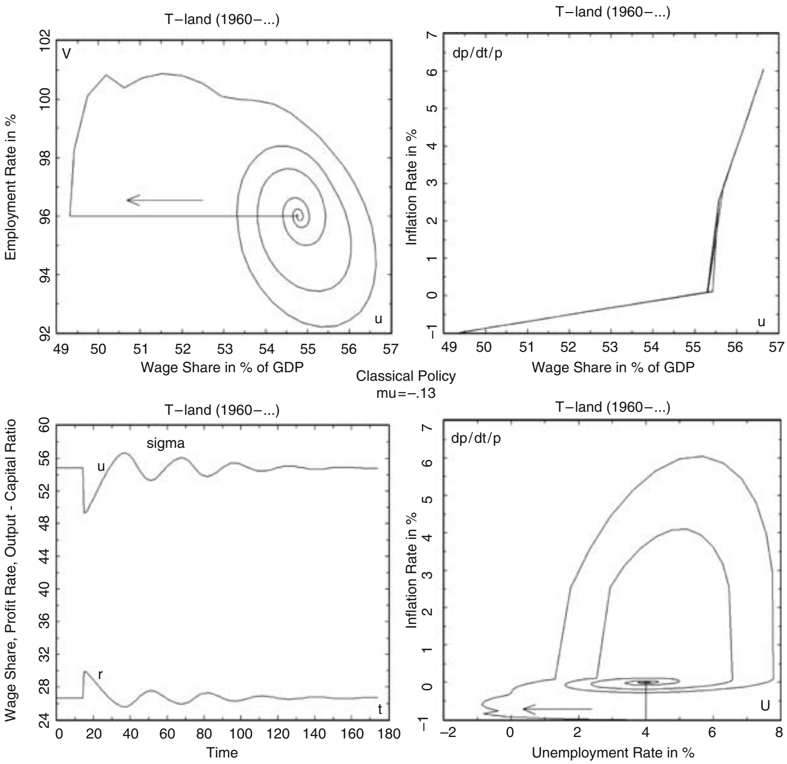

Fig. 20.5 The classical regime ($\mu = -0.13$)

Figure 20.6 show that its first and larger "Goodwin" cycle owes its shape to the three markup shocks and will only give way to the final limit cycle after reaching it again from the full employment ceiling. This indicates that the limit cycle itself may be fairly misleading regarding the long run outcome, if such shocks continue to occur from time to time.

We have shown in this section a few possible outcomes for the model of Sect. 20.2 under highly stylized assumptions on reaction patterns. These few simulations nevertheless clearly indicate that complex patterns may be generated from this simple model if its behavioral underpinning is subject to considerable shifts from time to time.

20.4 A Look at the Data

In this section we confront the behavior of the economic variables considered in the last section with corresponding data for the eight OECD-countries FRG, France, Italy, Belgium, United Kingdom, Ireland, USA and Japan (Figs. 20.7–20.14). It will

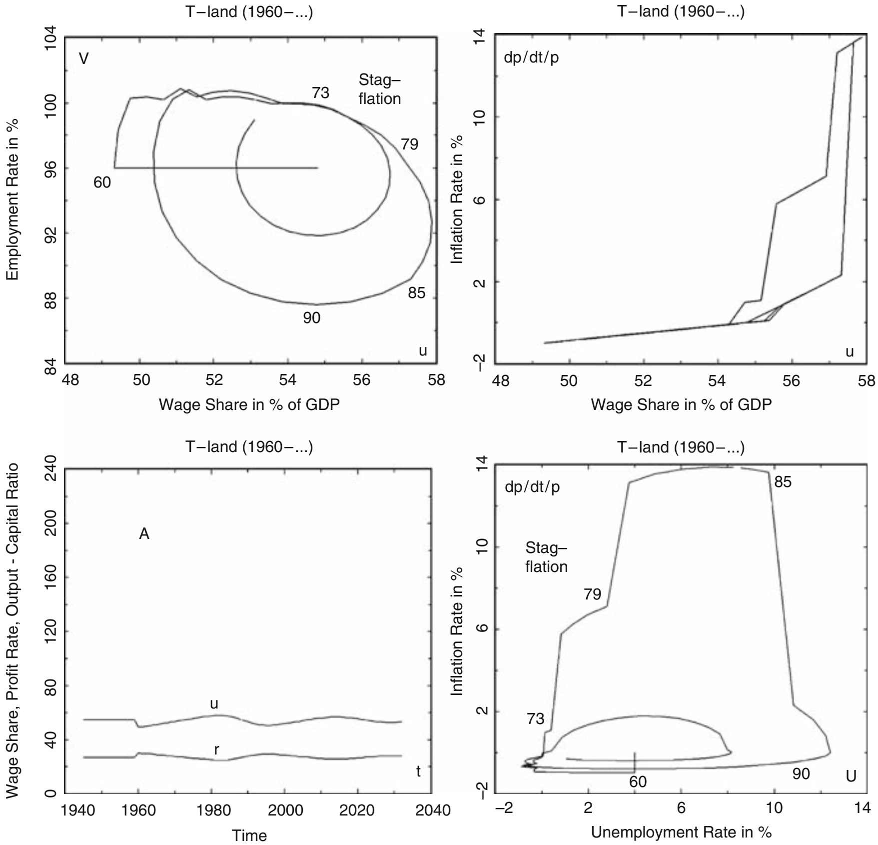

Fig. 20.6 Three exogenous price shocks ($\mu = 0$)

be, however, a very crude comparison, because in reality much more variables and interdependencies have been relevant than are taken into consideration by the above model. So it is already the raw material of the data, which, strictly speaking, should be corrected with regard to the influence of such effects like, e.g., oscillations in the capacity-utilization-rate or the shortening of working-hours per man in the period considered; by the first effect the output–capital ratio and the rate of return of capital are biased in comparison to the underlying model, and by the second one the rate of employment. Only the elimination of these influences from the raw-data would therefore provide an appropriate basis for the application of more advanced statistical and econometric methods, by which, perhaps, some conclusions about the significance of the model's set of variables could be drawn, see M. Desai (1984) for an econometric analysis in this direction. There it is also shown, that the inclusion of additional variables is necessary not only for the sake of a higher degree of completeness but also for technical reasons, i.e., to make the model an econometrically measurable one.

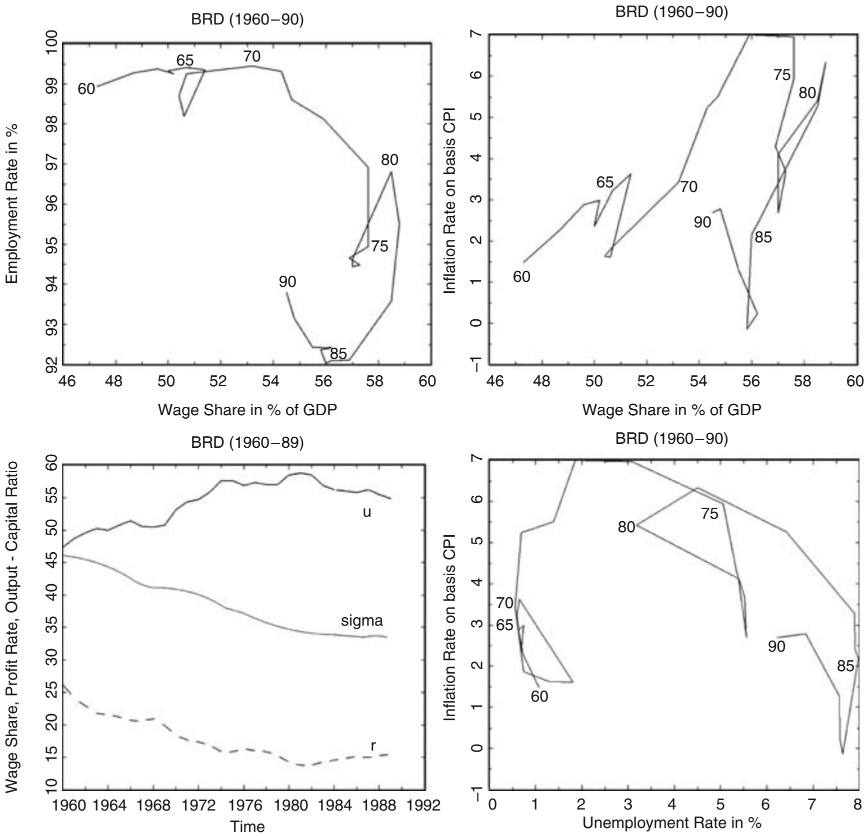

Fig. 20.7 Germany (FRG)

Nevertheless, the empirical results for some of the countries considered seem to confirm the outcome of the model, at least with regard to the interaction between the wage share and the rate of employment. This, of course, does not mean, that reality can be reduced to Goodwin's predator–prey-mechanism and that the story behind the data is necessarily of the same "nice conservative property" that Solow (1990, p. 38) attributes to the Goodwin model. On the other hand, Goodwin-like forces might be one of the factors influencing the long-run outcome of a capitalistic economy.

As the subsequent presentation of the empirical results should offer the possibility for a comparison with the simulations of Sect. 20.3 and the model discussed, the data have been chosen in analogy to the definitions of the variables in Sect. 20.2. Thus, the employment rate is represented by total employment in percent of total working population and the wage share by the compensation of employees in percent of GDP. The inflation rate is computed on the basis of the CPI in the expectation, that it will not differ too much from the more adequate GDP-deflator.

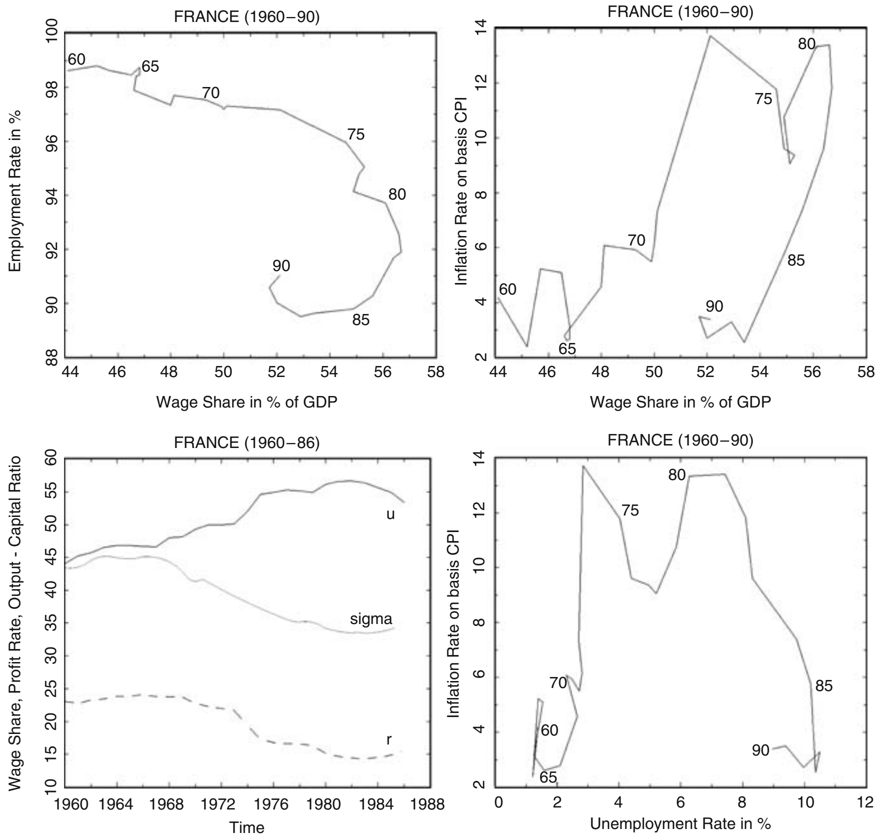

Fig. 20.8 France

Profit-rate and output–capital ratio finally have been ascertained with the help of the stock of fixed capital, as far as corresponding data could be found. In these cases the profit rate was derived as

$$(1 - \text{wage-share}) \times \text{output–capital ratio} = (1 - \text{wage-share}) \times \text{GDP/fixed capital}.$$

In order to plot wage-share, profit-rate and output–capital ratio in one diagram, the latter was multiplied by 100, too.

In the other cases, i.e., those, where capital-stock-data were either missing or of insufficient quality, a "gross-rate of return of the business-sector" was chosen as a proxy for the profit-rate. The corresponding output–capital ratio could then be obtained by dividing the rate of return of the business-sector by the (overall) profit-share ($= 1 -$ wage-share) in the hope, that the wage-share will not differ too much between the business-sector and the economy as a whole.

A great part of the data was taken from the database "International Statistical yearbook 1992" on CD-ROM, which contains statistics from several organizations

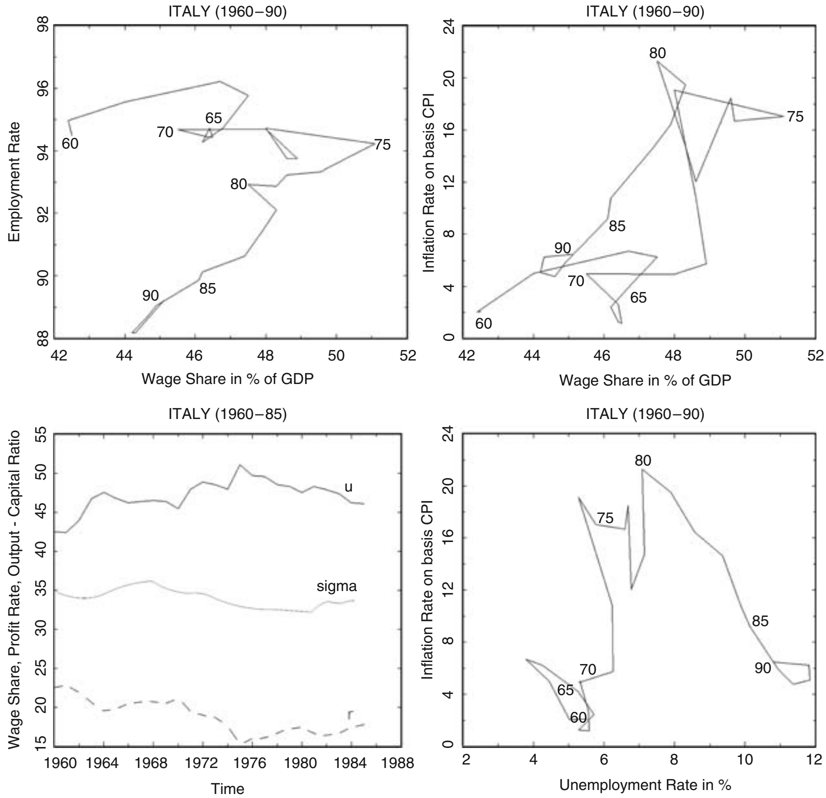

Fig. 20.9 Italy

like the "eurostat/cronos"-statistics from the Statistical Office of the EC, the "International Financial Statistics (IFS)" from the IMF, OECD-statistics and others. So the time-series for the GDP (in current and – as far as needed – in constant prices) and the wage-share, which originate from the "National Accounts – Aggregates (CRONOS/SEC1/Coll. 1–3)", are part of the eurostat-database as well as the components for the employment-rate, which can be found in the "social statistics (CRONOS/SOCI)", whereas the CPI-data for the inflation rate were taken – with the exception of Japan – from the IFS. For Japan, the IFS-data of the annual growth rate of the CPI contained obviously unrealistic values, so that they had to be substituted by corresponding numbers based on the CPI-data in the "ICG-part (General Economic Information)" of the "CRONOS"-statistics.

In order to get – as far as possible – a uniform time-horizon for all countries considered here, which additionally, should not be too short, data were taken from 1960 to 1990; only variables involving the capital-stock have not been available for this range in all cases. For Belgium, the USA and Japan, however, the employment-data from the "CRONOS"-statistics had to be supplemented by another data-source,

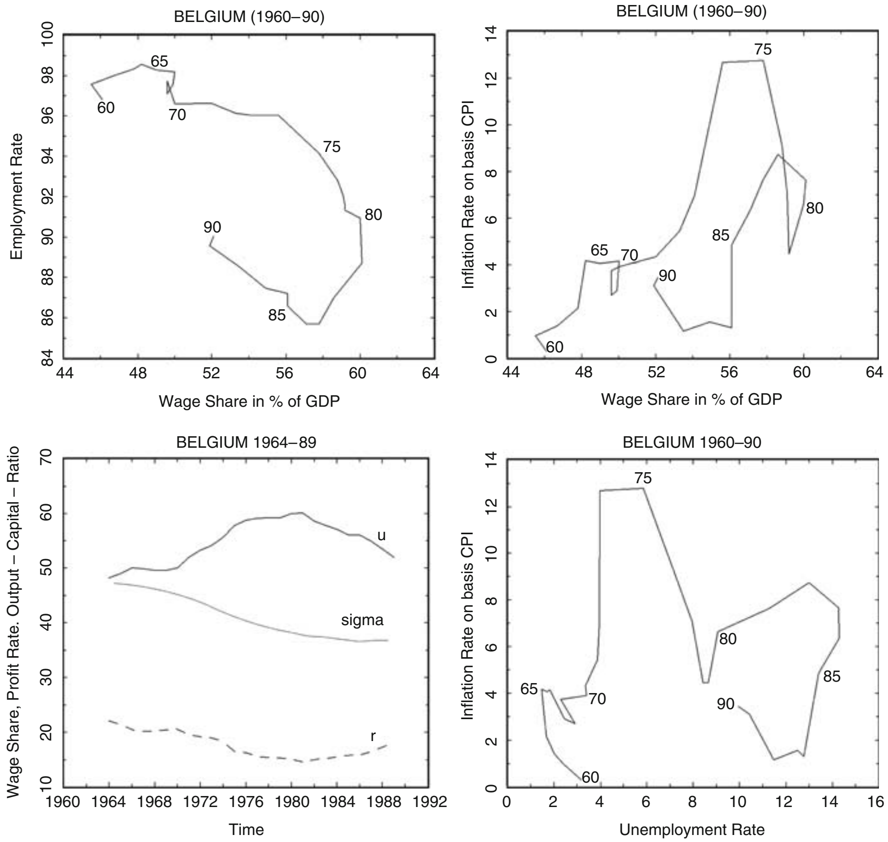

Fig. 20.10 Belgium

because in the case of Belgium the value for 1990 was simply nonsensical, and for the latter two countries it was missing at all. Thus, the time-series for the employment-rate of these three countries were completed in the following way:

At first, the values for the "total working population" as well as for the "total employment" were compared with the corresponding series of "civilian labor force total" and "civilian employment" from the journal "Eurostatistik – Daten zur Konjunkturanalyse" (12/92 p. 38 and p. 41) published by the Statistical office of the EC. While for Japan the values from both sources agreed completely, there was a difference in level in the other two cases, possibly caused by the involvement of the armed forces in the "CRONOS"-statistics. The annual absolute changes in numbers, however, did not differ very strongly, so that the "CRONOS"-data could be supplemented for 1990 with the help of these differences. On this basis the employment-rate for 1990 was calculated. The main problem, however, consisted in the search for data of the stock of fixed capital, because such statistics are not

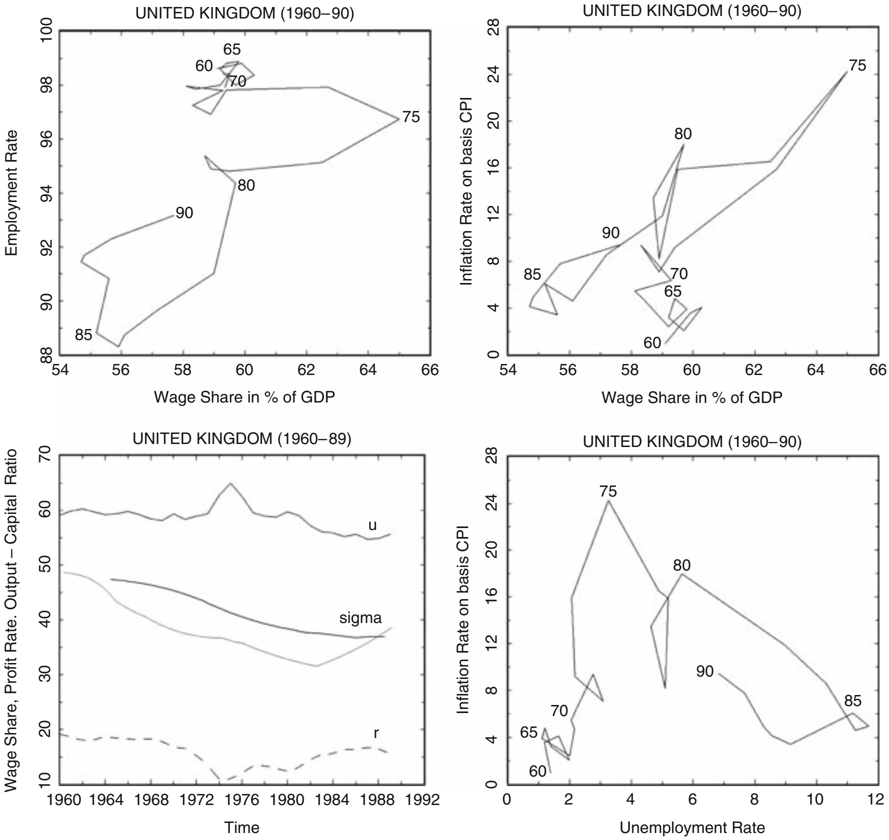

Fig. 20.11 United Kingdom

published in the Annual National Accounts and are generally difficult to obtain. So finally two different data-sources had to be required: For Germany, Belgium, the UK and the USA time-series of the net-capital-stock in current prices (or, in the case of Belgium, in constant prices of 1985) from 1960–1985 and from 1964–1989 were available in the respective journals “Flows and Stocks of Fixed Capital”, published by the department of Economics and Statistics of the OECD.

For France and Italy the “Fixed Capital Stock”-data of the “CRONOS/Sec.2 (National Accounts-Goods & Services)”-statistics were chosen, because in the case of France the OECD-statistics turned out to have the disadvantage of beginning not before 1970, and for Italy these values were missing at all. Unfortunately the “CRONOS”-data refer to the gross-capital-stock, which in contrast to the net-stock does not take into account the depreciation with regard to the declining income-generating-capacity of fixed capital assets through age and additional wear, depending on the extent of its use (see M. Ward 1976, p. 22f. and, especially, p. 31 for further discussion of this item). Therefore, the application of the net stock

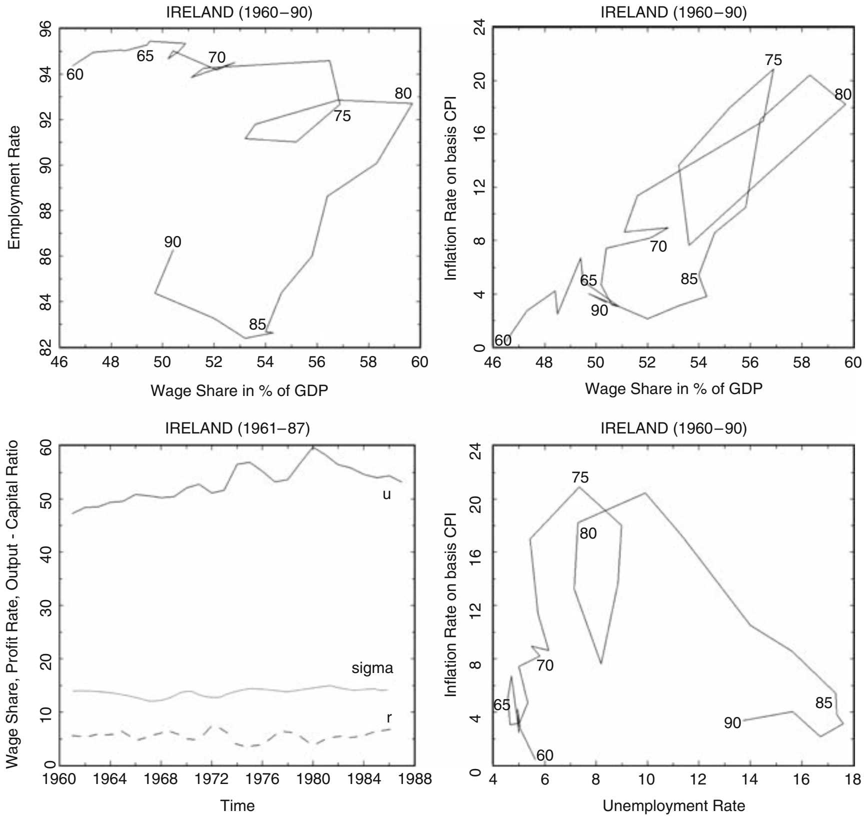

Fig. 20.12 Ireland

concept usually makes more sense, because the definition of depreciated capital is more inclusive and therefore often more appropriate than the gross concept (see "Europäische Wirtschaft", No. 50, Dec. 1991, p. 135 and note that depreciation is explicitly considered in the model discussed in this chapter). So the capital-stock is systematically overestimated in the cases, where the gross-stock concept is used. However, in the fixed-capital-data for the above two countries the sector of non-market-services is missing, in contrast to GDP measurement and the profit-income concept derived from it, which both refer to the whole economy.

With regard to Japan, only gross-capital-stock-data with some sectors missing were available in the OECD-source and no data at all in the "CRONOS"-statistics. For Ireland, in none of these sources a time-series of fixed capital could be found. In order to obtain such data for these countries, the "gross rate of return of the business-sector", published in the already quoted journal "Europäische Wirtschaft" (Table 3, p. 130) has been taken as the profit-rate. This proportion is defined as the gross operating surplus divided by the gross-capital-stock, both with regard to the business sector; the difficulty, which arose from this restriction for the calculation

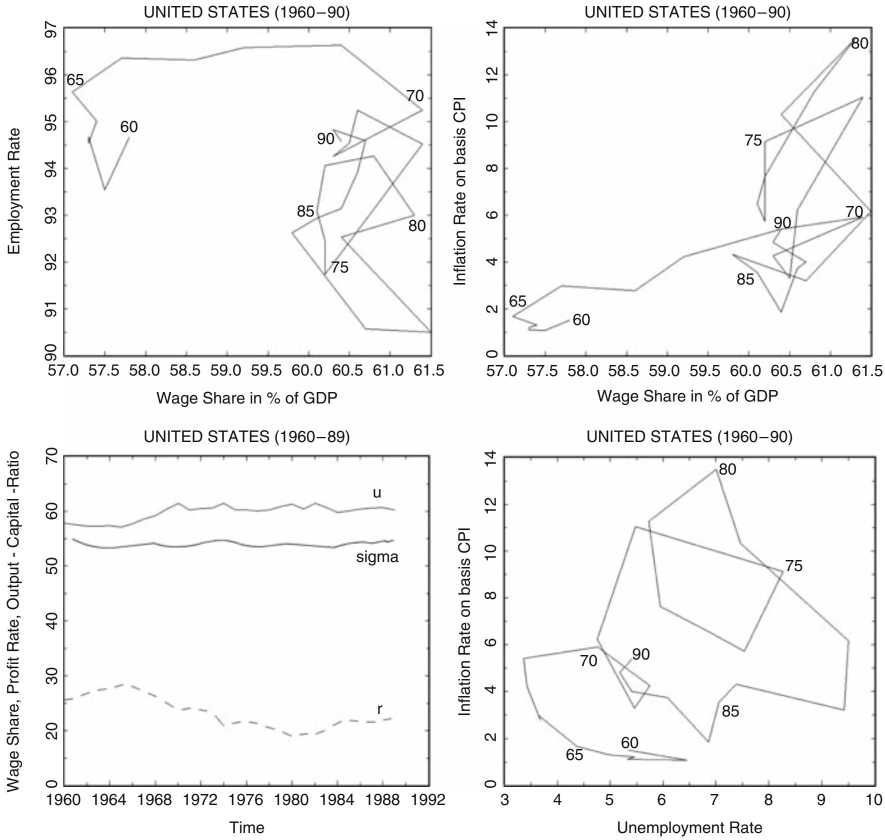

Fig. 20.13 USA

of the output–capital ratio, was already mentioned above. All this shows, that for a given year these data and coefficients should not be taken for a comparison between countries; only the patterns of the time-paths may give some kind of foundation for considerations in this direction (comp. also: "Europ. Wirtschaft", p. 136).

Finally, we should point to the fact, that all capital-stock-data are not really "sound" data, but only estimations of the publishing organizations. The method applied in this context is that of the "perpetual inventory" approach, which tries, crudely speaking, to derive the gross-capital-stock-data from the formation of fixed capital in the past with the aid of a so-called "survival function" employed to generate the weights for the investments of the various years under consideration. These weights "represent the probability, as a function of age, of a capital good still being in operation at a given point in time" (see Keese et al. 1991, p. 13). The net stock of a given year, which takes into account not only the probability of obsolescence, but the whole range of depreciation as described above, can be obtained by

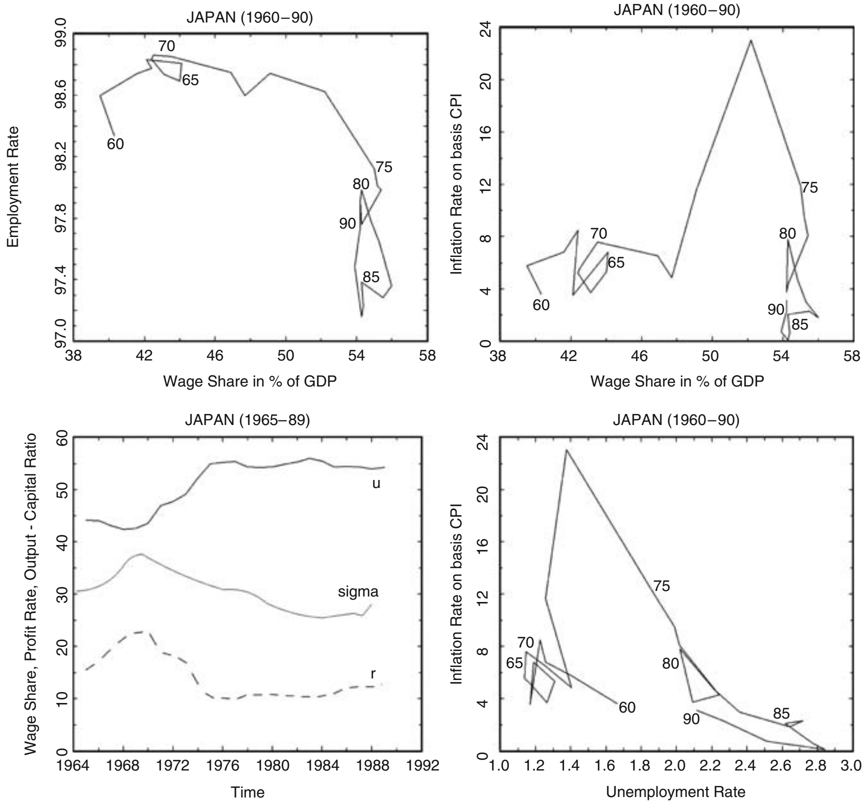

Fig. 20.14 Japan

multiplying the gross-investment-data not only by their "survival-rate" but also by an appropriate capital consumption coefficient (see Ward 1976, p. 31 and p. 56ff.)

After these preliminary remarks we can now focus our attention on the empirical results for each country. In order to make them directly comparable to the simulations of Sect. 20.3, the plots of the time-series are designed and grouped in the same way as in this earlier section. We stress once again that all following considerations are very crude and preliminary, attempting mainly to exemplify that there may be a common mechanism behind the presented observations.

At first, with regard to the depicted "three-quarter" employment-rate and the wage-share "cycle" of Fig. 20.7, one can crudely see an analogy to the corresponding scenario of Fig. 20.5,[5] where the Classical policy regime prevails and where it is shown that each following cycle will be further and further away from the full employment ceiling that dominated the evolution of the economy in the first 15 years.

[5] Note for all our comparisons of data and simulations that there is a significant difference in the length of the periods that are plotted in the two cases.

Yet, since the output–capital ratio σ has declined most of the time during the considered period, part of the movement in the wage share may also be attributable to this fact. Therefore, both Figs. 20.3 and 20.6 can also be related to this "bare hint of a single large long-period clockwise sweep" as it was already observed in Solow (1990, p. 40). It appears as if the fall in the share of wages and the revival in the rate of profit is too weak – in particular in the light of the movement of the output–capital ratio – to allow for an upswing as in Goodwin's growth cycle approach.

The phase plots of the rate of employment against the share of wages are of basically the same quality as for The FRG in the case France, Belgium and Ireland, see Figs. 20.8, 20.10, 20.12. A look at the plots on the right-hand side of Fig. 20.7 furthermore crudely reveals some similarity to the corresponding plots of the "oil-shock-simulation" of Fig. 20.6. The tendency that the relationship between the wage share and inflation rate shifts to the right in the 1980s is again similar in the case of France, Belgium and Ireland.

In the lower left hand plot of Fig. 20.7, the time series of u, r, σ, we have something like a "tendency to fall" for the rate of profit r, which comes to an end after 1981. The situations in Figs. 20.3 and 20.6 come closest to this observation.[6] Note however, that these figures are based on a given output–capital ratio, while we have a rising wage-share in combination with a falling "productivity of capital" σ in the real economy. This demonstrates the need to separate influences of the conflict over income distribution from medium run trends in the output–capital ratio in a more detailed investigation of the observed facts. The above tendentious fall in the rate of profit is by and large confirmed by the time series of the other countries, though there are also considerable differences between them.

The first plot in this figure seems again to have some similarity – perhaps here the closest one – to the case of a "classical fiscal policy rule", since this long period swing possibly will not come up to same high employment rates as prevailed in the 1960s. The fall in the share of wages is here much too weak to allow for a recovery of the Goodwin type. The third plot again shows a striking similarity between the time-paths of profit-rate and output–capital ratio, which in comparison to the FRG, however begin to fall about 10 years later. The last plot, finally, is not so far away from the corresponding simulation of Fig. 20.6, with the exception of the period between 1975 and 1980. It shows in comparison to the other countries in the clearest way that high unemployment may have played a significant role in the cure of the inflation problem, yet so far at the cost of making such high unemployment rates persistent.

While the first two countries exhibited some characteristics comparable to our simulations, this is much harder to assert in the case of Italy. There is a markedly positive correlation between wage-share and employment movements after 1975 and the (still weak) Goodwinian upturn after 1985 is here completely missing. Furthermore, the shift in the $u, \hat{p}$ plot is here more of the opposite type than in Fig. 20.9.

---

[6] It is important to note that the data (1960–1990) correspond to the time-interval 15–45 in the Figs. 20.3, 20.6 of Sect. 20.3.

Remarkably is again the parallel movement of the profit-rate and the capital–output ratio and the persistence of high unemployment rates in the 1980s.

Belgium may be considered as the case which comes closest to the situation depicted in Fig. 20.3, i.e., to the pure form of the cycle. There is nevertheless a significant similarity to the case of France, Ireland and the FRG, as we have already stressed. Again, a strict markup-pricing is not underpinned by the data, although a positive correlation between wage-share and inflation seems to hold in two different phases again. As in most of the other countries, the development of the profit-rate does not stand in contradiction to the simulation, but seems to be strongly influenced by the output–capital ratio, which again behaves in the same way as the profit rate.

Like in the case of Italy and similar to it, no convincing similarity to the simulations can be observed. Again, we have a strong positive correlation between the wage-share and the employment-rate. A unique markup-relation between wage-share and inflation-rate is given a little bit more support here than in the case of the other countries. Again the close relationship between the movement of σ and r is remarkable.

Remarkable in the case of Ireland is the depth of downturn of the employment cycle and its shape. It is tempting to relate this with a Keynesian regime as shown in Fig. 20.4 of the simulations. Yet this may be very premature, in particular, since we do not investigate here the policy regimes that actually prevailed in Ireland during this period.[7] Recall also that our model does yet not allow for a proper treatment of Keynesian demand management. A fall of the profit-rate within the considered time-period, as suggested by the simulations cannot be observed in the case of Ireland.

With regard to the model as well as to the other countries the empirical results for the USA appear as fairly atypically. Although the "full-employment-ceiling" during the second part of the 1960s can be clearly identified in the first plot, these plots have otherwise not much in common with the simulations (they are closest to the case of a Classical policy rule). The most striking result here is, that the wage-share only oscillates only between 60% and 61.5% after 1969, i.e., has remained nearly constant, whereas in all other countries considered (as well as in the simulations) the corresponding difference is over three times as much. Thus, the variation in employment seems here to be mainly caused by factors outside the model. Recall, however, that Solow found a bare hint for a long-period clockwise sweep in his data for the USA and that this is indeed much more visible from his wage share – employment plot than from ours (the source of his data and their exact definition – for US non-farm business – is not given in his article).

There thus remains scope for a Goodwin-type cycle also in the case of the US-economy. Here, the plot of the inflation rate against the unemployment rate is remarkable, since it very neatly appears as a clockwise loop for the 1970s and the 1980s. Note here also that the large swings in the output–capital ratio are here not closely mirrored in the movement of the profit rate.

[7] One may however conjecture that "Keynesian" governments have been rarely in power after 1975.

As in some of the previous cases, a breakdown of employment in Japan follows after a – nearly – "full-employment-ceiling". So we can also in this case interpret the development up to 1990 as two to three quarters of a long-run cycle, although a significant fall in the wage-share did not take place yet. In this respect the situation is very similar to the USA, though employment rates are much larger than in the US-case. In this last data set we have again the close correspondence between the movements in r and σ. Like in the case of Ireland, it should be remembered here, that in the time series plot the profit-rate and the denominator of the output–capital ratio refer to the business-sector, the wage-share and the numerator of σ, on the other hand, to the whole economy. With respect to the movements of the rate of inflation there is no close correspondence to the other countries we have considered in this section.

One of the authors has considered the above graphical presentations for the eight chosen OECD-countries already 10 years ago and has, of course, found a still much weaker evidence than is now available for the Goodwin growth cycle as a long swing (under modern conditions of capitalist production). The most basic implication of the investigations of this section thus surely is that one should continue this attempt of looking at the data from a Goodwinian perspective latest at the turn of this century in order to see how many of the conjectures it gives rise to (exemplified by our simulations of Sect. 20.4) have materialized by then or have not. Of course, some intelligently controlled econometrics – as Solow (1990, p. 40) calls it – may be of great help then or even right now.[8]

20.5 Concluding Remarks and Outlook

Following Solow (1990, Sect. 6) we have confronted our version of the Goodwin growth-cycle of Sects. 20.2, 20.3 with the facts by making use of the same 4 phase plots or time series that were used in Sect. 20.3 to illustrate the working of the model. As Solow we have made this comparison with data on the crudest level to see whether there is some correspondence between the simulated and the factual dynamics. Our findings have been that there is indeed a hint of a single large long-period cycle which, however, cannot be expected to become really a closed orbit before the start of the next century. We have plotted the data for eight OECD countries and have found similarities as well as significant differences. Taking all this into account in our view suggests at the present stage of the investigation that there is something important to be gained in trying the Goodwin model out for further countries, for the 1990s, with more refined data techniques, etc.

In addition, the model of Sects. 20.2, 20.3 can be criticized in many respects and thus may be subject to significant revisions before it can be really taken as a good starting point, since:

[8] See Desai (1984) for a first attempt of this kind.

- It assumes too many things as constant which have not been constant at all in the considered time span: labor supply growth n, labor productivity growth m, the capital–output ratio σ, the depreciation rate δ, the mark-up factor A, and the fiscal policy parameter μ among others.
- It allows for only two factors of production: capital and labor (no substitution between them), with capital being always fully employed.
- It completely neglects effective demand problems ("independent" investment behavior, IS-problems) as well as money and other financial assets ("independent" interest rate behavior, LM-problems).
- It makes use of very primitive and limited adjustment functions $f(V)$, $i(V)$, $\eta(u)$, $\lambda(u)$ in particular in the attempt to limit the explosive behavior of our version of the Goodwin growth cycle model.
- It is a model of a closed economy.

It is, e.g., apparent from the plots of Sects. 20.3, 20.4 that our simple representation of markup pricing does not show too much similarity with the actual phase plot of the share of wages against the rate of inflation. But once again, since the model seems to mirror something in the data, it is worthwhile to consider the Goodwin growth-cycle model further as a possible explanation of the very long run aspects of post-war development.

In addition to this, one should, of course take earlier time periods into account than only from the 1960s to the present if data – in particular on the wage

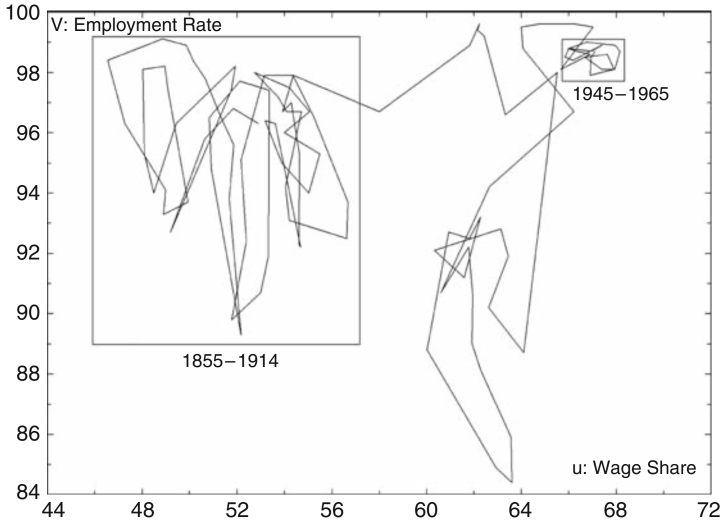

Fig. 20.15 UK-development 1855–1965, phase plot

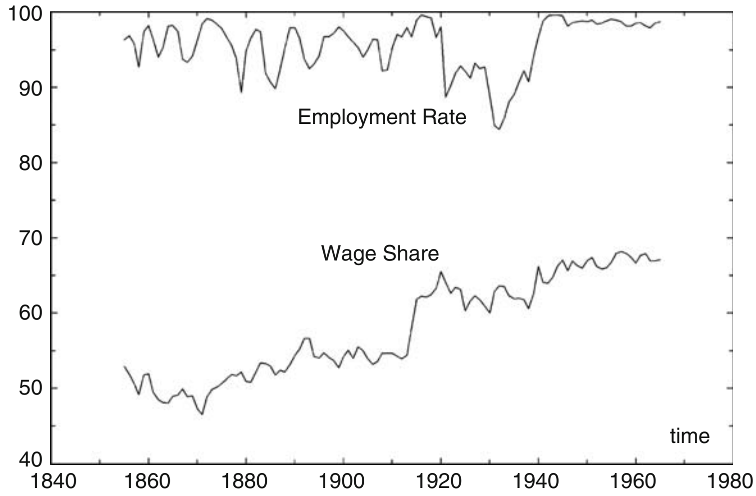

Fig. 20.16 UK-development 1855–1965, time series

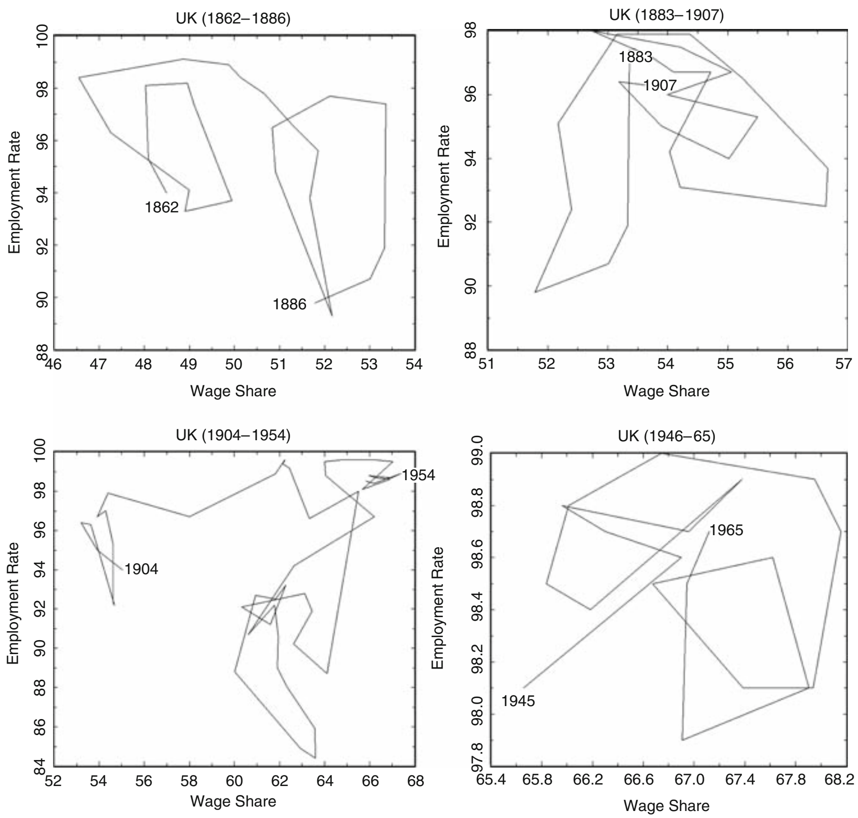

Fig. 20.17 UK-development 1855–1965, sub periods

share – are available. For the United Kingdom there exist such long time-series [9] from 1855 up to 1965 which are summarized by the phase plot and time series diagram shown in Figs. 20.15 and 20.16.

These data[10] seem to indicate that there have been two different types of dynamics with respect to the interaction of the rate of employment and the share of wages:

- The period before 1914 where the employment rate exhibits significant fluctuations of less than 10 years in phase length
- The period after world war II (1945–1965–…) where no such fluctuations can be observed any more

Of course, fluctuations in the employment rate need not be accompanied by Goodwin-type fluctuations in the share of wages. This is to some extent visible when the data of Figs. 20.15 and 20.16 are divided into appropriate subsets as in the Fig. 20.17. The important thing that can be obtained from these diagrams for Great Britain (1855–1965) is that the Goodwin cycle – if it exists – must have been significantly shorter before 1914 (with larger fluctuations in employment during each cycle), and that there has been a major change in it after 1945. This may be explained by significant differences and changes in the adjustment processes of market economies for these two periods: primarily price adjustment before 1914 and primarily quantity adjustments after 1945. Again, this very tentative judgment must be left for future investigations here.

References

Atkinson, A. B. (1969). The timescale of economic models: How long is the long run? *Review of Economic Studies, 36*, 137–152.

Data Source and Information. (1992). *International statistical yearbook 1992* (on CD-ROM).

Desai, M. (1984). An econometric model of the share of wages in national income: UK 1855–1965. In: R. M. Goodwin et al. (Eds.), *Nonlinear models of fluctuating growth* (pp. 253–277). Heidelberg: Springer.

Europäische Wirtschaft. (1992). No. 50.

Farkas, M., & Kotsis, M. (1992). Modeling predator–prey and wage-employment dynamics. In: G. Feichtinger (Ed.), *Dynamic economic models and optimal control. Proceedings of the fourth Viennese workshop on Dynamic Economic Models and Optimal Control* (pp. 513–526). Amsterdam: North Holland.

Flaschel, P. (1992). A Keynes–Wicksell model of the 'growth cycle'. The myopic perfect foresight case. In: G. Feichtinger (Ed.), *Dynamic economic models and optimal control. Proceedings of the fourth Viennese workshop on Dynamic Economic Models and Optimal Control* (pp. 467–489). Amsterdam: North Holland.

[9] See Desai (1984) for the sources of these data and for an econometric approach on the basis of these data with respect to the Goodwin growth cycle model, i.e., here another reformulation of it.

[10] Which do not – at least on the surface – support one of Kaldor's famous stylized fact of growth theory (the constancy of the wage share in the long-run).

Flaschel, P. (1993). *Macrodynamics. Income distribution, effective demand and cyclical growth*. Bern: Peter Lang.

Goodwin, R. (1967). A growth cycle. In: C. H. Feinstein (Ed.), *Socialism, capitalism and economic growth* (pp. 54–58). Cambridge: Cambridge University Press.

Hicks, J. (1950). *A contribution to the theory of the trade cycle.* Oxford: Claredon Press.

Keese, M., Salou, G., & Richadson, P. (1991). *The measurement of output and factors of production for the business sector in OECD countries* (Working Paper 99). Paris: OECD.

Marx, K. (1954). *Capital* (Vol. I). London: Lawrence and Wishart.

OECD, Department of Economics and Statistics. (1987). *Flows and stocks of fixed capital, 1960–1985*. Paris: OECD.

OECD, Department of economics and statistics. (1991). *Flows and stocks of fixed capital, 1964–1989.* Paris: OECD.

Solow, R. (1990). Goodwin's growth cycle: Reminiscence and rumination. In: K. Velupillai (Ed.), *Nonlinear and multisectoral macrodynamics. Essays in honour of Richard Goodwin* (pp. 31–41). London: Macmillan.

Statistical Office of the European Communities. (1992). Eurostatistik – Daten zur Konjunkturanalyse, 12.

Ward, M. (1976). *The measurement of capital*. Paris: OECD.

Wolfstetter, E. (1982). Fiscal policy and the classical growth cycle. *Journal of Economics, 42*, 375–393.

Chapter 21
The Goodwin Distributive Cycle After Fifteen Years of New Observations

21.1 Introduction

In this chapter, we reconsider the simple empirical evidence for the existence of a long-phased cycle in the state variables employment rate e and wage share v that we have collected in Flaschel and Groh (1995) for a number of industrialized market economies, see the preceding chapter. We do this on the basis of 15 years of further observations and now also with quite modern econometric techniques. Our findings will be that – in the case of the US economy – the only three-fourths closed long phase loops we observed in this earlier paper[1] can now be confirmed as by and large closed, giving in sum an approximately 50 years long cycle in employment rates and the wage share in its interaction with the six to seven business cycles observed in the US economy between 1960 and 2006.

As basic theoretical explanation for the occurrence of such long-phased cycles the reference to the seminal papers by Richard Goodwin (1967), augmented by Rose (1967) type limit cycle considerations, and the many papers that are based on this work suggests itself. By and large this means that we consider a growth cycle model where the steady state of the model is repelling and where there are forces far off this steady state which imply constant or falling employment rates at the full employment ceiling $e = 1$ and falling wage shares in situations where income distribution squeezes profitability to a significant degree. We have supplied in Flaschel and Groh (1995) a model example for such a situation and will recapitulate in this chapter only the absolutely necessary ingredients that make a Goodwin–Rose growth cycle dynamics repelling at their interior steady state and bounded within the phase space $[0, 1]^2$. The implication of such a scenario will be a clockwise motion of the state variables (v, e) in the phase space $[0, 1]^2$ as it has been observed for a number of countries in Flaschel and Groh (1995).

Another example for such an observation is Solow's (1990) contribution to a collection of essays in honor of R. Goodwin where he not only provides a characterization of the merits and weaknesses of this prototype model, but also

[1] See the preceding chapter for a revised version.

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_21,

briefly discusses whether this employment-cycle model of the conflict over income distribution "fits the facts" (and what such a question may mean). One of Solow's findings in this article (on p. 39) is that there is a suggestion in the data of a predominantly clockwise motion, in three separate episodes as far as the phase plot of annual US-data, 1947–1986 of the share of wages and the rate of employment is concerned. On p. 40 he adds:

> It is also worth noting that the phase diagram contains the bare hint of a single large long-period clockwise sweep. It is only a hint, at best a hint. One cycle is not a periodic motion.

There are many authors who have stressed the empirical relevance of the period length that is implied by Goodwin's growth cycle model, when its parameters are given empirical content, see in particular Atkinson (1969). Solow himself calculates a period length of 8–10 years, however for an assumed linear real-wage Phillips curve that is very steep. He then concludes (p. 38) that:

> Goodwin cycles are something else. What could that something be? Here I presume that we must take seriously the story that the model tells: ....

The conclusions of this chapter will be the same as in Solow's article (we quoted above) and which he continuous as follows (on p. 40):

> There may be work for Goodwinians here too, theoretical as well as applied. It would make more sense to me if the Goodwinian mechanism were to apply on a time scale considerably longer than the ordinary business cycle. Since the model determines its own period, there is room for some interplay between facts and the theoretical structure.

Last but not least, in addition to the theoretical considerations, the model's contribution to economic policy issues should be recognized as well. Up to the present day, the question regularly comes up in political discussions whether rising wages and an increasing wage share are good or bad for employment. On the one hand, there is the traditional view which claims a negative relationship between both magnitudes due to higher production costs or a lower investment demand which additionally reduces the growth of the capital stock in the future. This is also the view adopted by the Goodwin model and its successors (see below). The opposite view, normally put forward by trade unions, is that in a demand-constrained economy an increasing wage share leads to more purchasing power for those people who have a higher propensity to consume. This effect is perceived to be stronger than the adverse effect on investment demand, provided that the latter is taken into account at all.

The interesting point is now, that the "long-period clockwise sweep" described above contains four different phases two of which supporting the first and the other two supporting the second view, at least at the first glance:

- Phase 1: declining v and rising e
- Phase 2: rising v and rising e
- Phase 3: rising v and declining e
- Phase 4: declining v declining e

Obviously, the first and the third phase – if considered in isolation – are in line with the first view, while the second and the fourth seem to support the second standpoint. Given such a cycle without the concrete "background story" provided by the Goodwin model, one might be tempted to conclude that there are – or have been – times in which the "purchasing-power-effect" dominates and other periods in which the "cost-effect" (or "Rose-effect") overcompensates the former. In the light of the approach considered here, however, the Rose-effect is dominant all of the time and the apparently deviating phases have quite a different explanation. During the second phase, characterized by $v \uparrow$ and $e \uparrow$, the growth rate of effective demand – triggered, among others, by high multiplier effects caused by supernormal investment demand (due to the low wage rate) – is still higher than the growth rate of labor supply, despite the fact, that the increasing wage share already has a dampening effect on the demand side. In the fourth phase, in which $v \downarrow$ and $e \downarrow$ can be observed, aggregate demand is already stimulated by the increasing investment, but its growth rate is still too weak to cope with that of labor supply (caused, as in the previous case, by a constant rate of Harrod-neutral technical progress or population growth).

Against this background, the question about the empirical relevance of the model considered here is of utmost importance. If it turns out not to contradict the data, a straightforward policy implication is that there exists no cogent reason to abandon a restrictive wage policy even in times where a falling wage share goes hand in hand with a further increasing rate of unemployment. This follows because the true reason – according to the model – for the ongoing deterioration on the labor market is not the decline of the wage share, but the fact that the latter does not decline sharply enough. Consequently, policy makers would be well-advised to resist the political pressure resulting from an apparent failure of moderate wage policy, at least as far as the link between wage share and unemployment is concerned.[2] This view is further underpinned by the fact, that due to the long-run nature of the model also prolonged phases with falling v and falling e are not at odds with the model's predictions.

On this background, we shall introduce in the next section a general formulation of the Goodwin growth cycle model. Section 21.3 then summarizes some time series evidence for the Goodwin cycle from the work of Flaschel and Groh (1995). In Sect. 21.4 we provide basic econometric evidence for the existence of this cycle. Section 21.5 makes use of more refined techniques that establish evidence for the Goodwin growth cycle and its mirror image, the unemployment–inflation cycle. Section 21.6 concludes.

[2] This does not deny that restrictive wage policy (if not accompanied by additional measures) might have side-effects which are not desirable with regard to its social or distributional consequences.

21.2 The Growth Cycle Model: Basic Ingredients for a Limit Cycle Result

In this section we sketch the bare essentials that allow for the derivation of a stable limit cycles in the state space $(0, 1)^2$. We start from a general 2D dynamical system[3] of the form:[4]

$$\hat{v} = \dot{v}/v = f(v, e), \tag{21.1}$$

$$\hat{e} = \dot{e}/e = g(v, e), \tag{21.2}$$

where the state variables v, e denote the wage share and the rate of employment, respectively, and where $\hat{x}$ is used to denote the growth rate of a variable x. In this general form the model can be interpreted in the spirit of the original Goodwin model as well as in terms of an approach where the demand constraint determines output and employment. Thus, in the case of $f_v = 0$ and $g_e = 0$ one might think of the classical variant with Say's law applying and a fixed proportions technology which makes employment dependent on the current capital stock. Combined with classical saving behavior (i.e., with savings only stemming from capital income while all wage earnings are devoted to consumption) this leads to the well-known profit-squeeze which occurs if a rising wage share does no longer allow for enough savings and thus investment in order to let the capital stock increase by a sufficient amount to absorb labor supply. The resulting higher unemployment will then exert a dampening effect on further increases of the wage share. Alternatively, as already mentioned, one can interpret the above model in terms of a demand-constrained economy with the wage share affecting consumption and investment demand; this is also the variant pursued here.

Equation (21.1) can be derived from the general two wage/price Phillips curve approach of Chiarella et al. (2005) and we may assume that $f_v < 0$ holds true if a Blanchard and Katz (1999) error correction approach is chosen as foundation of the employed wage and price Phillips curves. The other partial derivative, f_e, is due to the role which employment plays in conjunction with the wage Phillips curve. Since, however, increases of the nominal wage rate have a corresponding impact on the development of prices (via the price Phillips curve), the net effect on the development of the wage share is not unique. Thus, one might think of situations where the wage-effect is dominant and others where the price-effect is stronger. In the following, we will call the case $f_e > 0$ a labor market led situation (since real wage movement is then dominated by nominal wage movements) and the opposite case (where price movements dominate) a goods market led situation.

[3] See Barbosa Filho and Taylor (2006) and Taylor (2004) for detailed analyzes of systems of this type.

[4] The model underlying these two differential equations is typically formulated such that it includes natural growth as well as Harrod-neutral technical change, assumptions that are not visible in the reduced form we use here as a starting point.

Equation (21.2) can be interpreted as a dynamic IS-relationship where income distribution matters in aggregate demand and where the dynamic multiplier process is characterized by the derivative g_e (which can be positive or negative). The case $g_v > 0$ mirrors a situation in which the "purchasing-power-effect" of higher wages (i.e., higher consumption demand due to a higher income share of people with a higher propensity to consume) dominates the "cost-" or "Rose-effect", according to which higher real wages lead to a decline in investment demand. We call the case where $g_v > 0$ holds wage-led and the opposite case profit-led, in line with what is used in the literature.

In sum we thus get the scenario of basically four different interactions of the rate of employment with income distribution as shown in Table 21.1.[5]

From a partial perspective we thus get two cases where real wage or wage share adjustment is stable and two cases where it is not, see Chen and Flaschel (2006) for details on this.

We assume for the phase plot of the considered dynamical system that $e = 1$ is the full employment rate ceiling and that the motion along this ceiling is horizontal for low wage shares and pointing inwards to lower rates of employment for all v sufficiently large, as shown in Fig. 21.1. Similarly, we assume that the rate of

Table 21.1 Four types of real wage or wage share feedback mechanisms

	Wage-led goods demand	Profit-led goods-demand
Labor-market-led real wage adjustment	Adverse = divergent	Normal = convergent
Goods-market-led real wage adjustment	Normal = convergent	Adverse = divergent

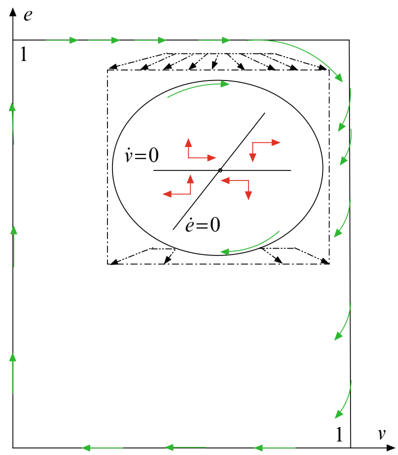

Fig. 21.1 Basic ingredients for persistent and attracting growth cycle dynamics

[5] A labor market led situation is basically of a Marx–Goodwin–Kalecki profit squeeze type and is the generally observed situation in the literature, see also the discussion on pro-cyclical real wages (here real wage growth).

change of the wage share is negative (latest) for wage shares equal to one, see again Fig. 21.1. The basic implication of these two assumptions is that $[0, 1]^2$ is an invariant set of the considered dynamics (cannot be left by it), since the vertical as well as the horizontal axis cannot be crossed by it (are invariant sets themselves).

Next we assume that there is an interior steady state – representing a balanced situation for the economy – around which the right hand sides of the above dynamics can be considered as (locally) linear. We assume for this steady state position that the dynamic multiplier is unstable there ($g_e > 0$) and that the economy is as in Goodwin's original model a labor market led and profit-led one ($f_e > 0, g_v < 0$). Moreover, the Blanchard and Katz error correction terms, see Flaschel and Krolzig (2006) for their discussion, are still absent around this steady state, i.e., $f_v = 0$ holds.[6] For the Jacobian matrix at the steady state we get in this situation:

$$J = \begin{pmatrix} 0 & + \\ - & + \end{pmatrix}.$$

The considered steady state is therefore unstable and of the type shown in Fig. 21.1. It then follows from the Poincaré–Bendixson theorem, see Hirsch and Smale (1974), that each trajectory which does not have a steady state as limit point must converge to a closed orbit. In particular, if there is no further steady state than the one considered in Fig. 21.1, this statement holds true for every trajectory that starts in the invariant domain $(0, 1)^2$ of positive wage shares and positive rates of employment. An example of such a situation was provided in Flaschel and Groh (1995).[7]

21.3 Exploring Growth Cycles for the US Economy: A Brief Reconsideration

In Flaschel and Groh (1995) we have confronted the behavior of the state variables v, e, the wage share and the rate of employment, in their bounded phase space with corresponding yearly data for the eight OECD-countries FRG, France, Italy, Belgium, United Kingdom, Ireland, USA and Japan. We repeat their results here for the case of the US economy in order to show in subsequent sections how these still tentative results have indeed been further confirmed by the evolution in the USA over the past 15 years.

[6] Blanchard and Katz (1999) error correction is only one possibility to justify a negative influence of the wage share on its rate of growth, see for example Barbosa Filho and Taylor (2006) and Taylor (2004) in this regard.

[7] Note however that the model in the general form presented here may allow for a variety of further dynamic outcomes.

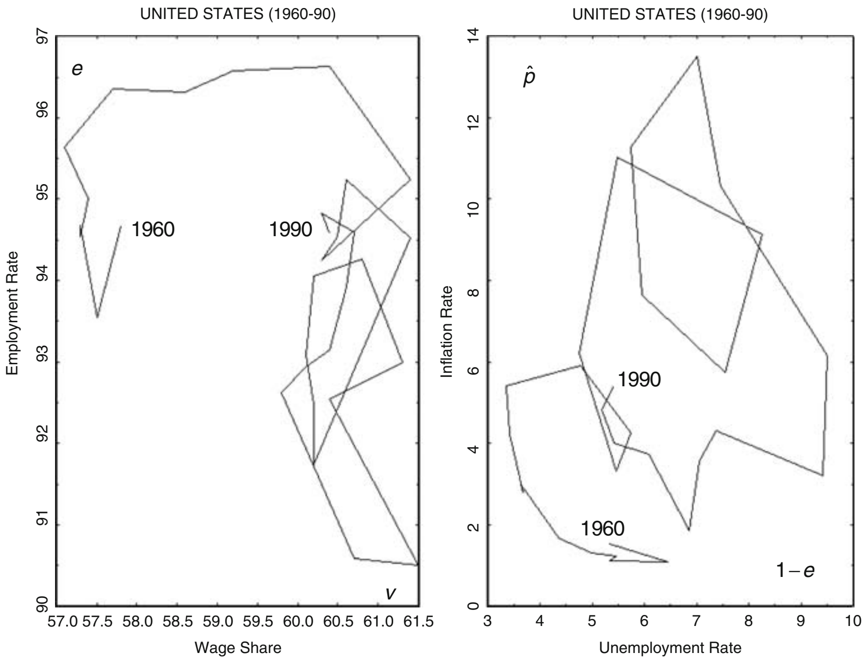

Fig. 21.2 Time series phase plots: the Goodwin growth cycle and the inflation–unemployment nexus

The plot on the left-hand side of Fig. 21.2[8] shows the US time series phase plot that corresponds to Fig. 21.1. There is the indication of a long clockwise loop with one-fourth of the loop however still missing. Nevertheless the figure suggests that the Goodwin growth cycle model and its reformulation in the present chapter is not contradicting the data. In Flaschel and Groh (1995) we have in addition also considered the unemployment/inflation loop as it is generally investigated even in basic textbooks, see Mankiw (1992) for an example. The idea was that price inflation is by and large generated by a constant markup over wage inflation. If this is correct, the Goodwin cycle should be accompanied by a similar nominal cycle in (un)employment and inflation, since the wage share can be reinterpreted as unit real wage costs. We show this loop – following textbook presentations – in the $(1-e,\hat{p})$ phase space in the right-hand plot in Fig. 21.2 and see indeed even on this simple phase plot an also clockwise orientation that follows the left hand cycle to a certain degree.

We conclude from these plots as in Flaschel and Groh (1995) that both phenomena are worthwhile to be considered further and will indeed pursue exactly this in the now following sections.

[8] From Data Source & Information GmbH (1992) *International Statistical Yearbook 1992* (on CD-ROM).

21.4 The US Economy: Extended Data Set and Basic Econometric Issues

In this section we will only consider the Goodwin growth cycle model. We will again consider its mirror image, the unemployment–inflation nexus, in the next section where more powerful econometric techniques will be applied to study both cycle mechanisms. Let us here however first consider a simple 2SlS estimate of a linear version of the Goodwin model (21.1)–(21.2). In this way we estimate the model linearly over the whole phase space and expect that its global features will dominate the outcome, and not its local instability presentation in Fig. 21.1. The estimation result is shown in Fig. 21.3 and it is based on the data set (US Department

Sample: 1958:2 2004:3
Included observations: 186
Total system (balanced) observations 372

	Coefficient	Std. Error	t-Statistic	Prob.
C(11)	0.229038	0.142975	1.601940	0.1100
C(12)	0.562955	0.136181	4.133876	0.0000
C(13)	–0.371814	0.177445	–2.095377	0.0368
C(21)	0.510863	0.071240	7.171027	0.0000
C(22)	0.223516	0.067854	3.294048	0.0011
C(23)	0.563353	0.088415	6.371704	0.0000

Determinant residual covariance 1.10E-07

Equation: 4*@DLOG(V1)=–C(11)*V1(–1)+C(12)*E1(–1)+C(13)
Instruments: E1(–1) V1(–1) C
Observations: 186

R-squared	0.113698	Mean dependent var	–0.000480
Adjusted R-squared	0.104011	S.D. dependent var	0.027606
S.E. of regression	0.026131	Sum squared resid	0.124959
Durbin-Watson stat	1.785956		

Equation: 4*@DLOG(E1)=–C(21)*V1(–1)–C(22)*E1(–1)+C(23)
Instruments: E1(–1) V1(–1) C
Observations: 186

R-squared	0.230329	Mean dependent var	0.000198
Adjusted R-squared	0.221918	S.D. dependent var	0.014761
S.E. of regression	0.013020	Sum squared resid	0.031024
Durbin-Watson stat	1.013713		

Fig. 21.3 The Goodwin growth cycle: two-stage least-squares estimation

of Labor: Bureau of Labor Statistics) employed in the empirical study of Kauermann et al. (2008), see also the next section. The estimation shown in Fig. 21.3 exhibits an acceptable t-statistics, but may be considered as insufficient as far as R-squared and the Durbin Watson coefficient (in the second equation) is concerned. Yet, with this estimate we only intend to provide a first insight into parameter signs and sizes in the case where the typical nonlinearity in our reformulation of the Goodwin growth cycle model is ignored. The estimated coefficients give rise to the following Jacobian matrix for the 2D dynamics in the state variables v, e.[9]

$$J = \begin{pmatrix} -0.23 & 0.56 \\ -0.51 & -0.22 \end{pmatrix} = \begin{pmatrix} - & + \\ - & - \end{pmatrix}.$$

Since the linear model represents the average behavior of the original nonlinear model (21.1)–(21.2), we see that stability is characterizing the matrix J (det $J > 0$, trace $J < 0$. The Blanchard and Katz (1999) error correction mechanism is working with significant strength ($J_{11} < 0$) and the dynamic multiplier process behind the reduced form presentation of the law of motion (21.2) is stable ($J_{22} < 0$). The eigen feedback effects thus indeed work in this overall fashion in a stabilizing way. In the off-diagonal of the matrix J we have the typical sign structure of the Goodwin growth cycle model which is working with considerable strength. With regard to J_{21} the negative sign indeed confirms the dominance of the "Rose-effect" over the "purchasing-power-effect" and thus gives a first answer to one of the key questions raised in the introduction.

On the basis of this matrix J and the estimated constants we can moreover calculate the interior steady state position of this numerical example of the Goodwin model and get approximately $(v_o, e_o) = (0.70, 0.93)$. This is a sensible steady state position, though one whose confidence interval may be large. Nevertheless our system estimate provides us with an idea of how the numerical magnitudes behind a linearized version of model (21.1)–(21.2) may look like.

We next consider the Hodrick–Prescott decomposition into trend and cycle in order to remove the business cycle component from the data. This should allow us to see how the empirical Goodwin cycle indicated in Fig. 21.2 looks like in the case of our extended data set. However determining cyclical trends by means of the Hodrick–Prescott filter leads to the loss of data and this the more the larger the parameter λ in this procedure is set. In Fig. 21.4 we show on its left the usual choice $\lambda = 1{,}600$ and on its right the choice $\lambda = 4{,}800$. By and large we see in both figures a Goodwin growth cycle that is significantly more closed than the one shown in Fig. 21.2.

We conclude from these basic econometric treatments that the vision of the Goodwin (1967) growth cycle model – when augmented as in Rose (1967) to lead to attracting limit cycle results as described in Sect. 21.2 – is further confirmed when now available data for the US economy after World War II are employed.

[9] See Barbosa Filho and Taylor (2006) for estimates of the here considered growth dynamics that can be usefully contrasted with the results of this chapter.

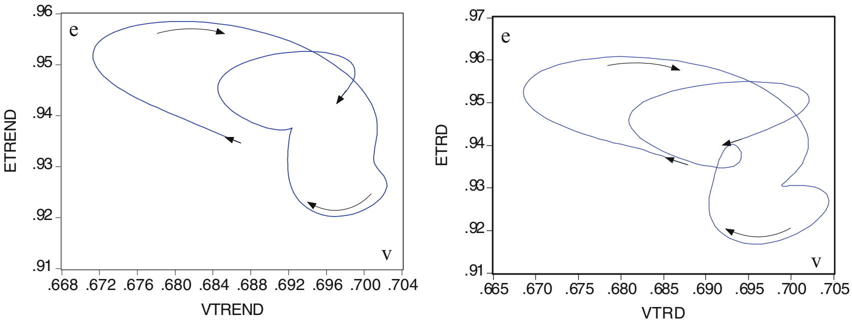

Fig. 21.4 The Goodwin growth cycle: HP-trend visualization

21.5 Business Cycles and Long Phase Cycles in the US Economy: Bivariate P-SPLINE Testing

Econometric studies often investigate on the methodological level as well as in empirical research the problem of how to separate the business cycle from the trend in important macroeconomic time series. Yet, economic growth theory in its advanced form provides us with insights on which economic ratios may exhibit a secular trend (like capital intensity when not measured in efficiency units) and which ones will not (like the output capital ratio or the rate of employment as two measures of macroeconomic factor utilization). In contrast to a variety of econometric studies macrodynamic theory therefore uses appropriate ratios or growth rates in its analytical investigations and there in particular the ones that allow for the determination of steady state positions and which therefore should not exhibit a trend in the very long run when the macrodynamic theory is formulated in a sufficiently general way.

In applying a methodology developed in Kauermann et al. (2008) we will in line with the formulation of the Goodwin growth cycle model (21.1)–(21.2) use secularly trendless magnitudes, namely the employment rate on the external labor market, the wage share in national income and the inflation rate (here of producers' prices). There are a variety of smaller as well as larger macrodynamic models in the tradition of Goodwin (1967) and Friedman (1968) which show the existence of persistent cycles in the interaction between the employment rate and the wage share on the one hand and the employment rate and the inflation rate on the other hand which tend to be long phased when simple constant parameter estimates are used for their numerical investigation. In these models the usual business cycle fluctuations must therefore be explained by systematic variations in the parameters of the model which then add cycles of period lengths of about 8 years to the 50 years cycles these models generate when used with average constant parameter values.

We have considered in Flaschel et al. (2005) models of the employment/income distribution cycle and the employment/inflation cycle in isolation as well as in their direct interaction or more indirect ones in a five dimensional model of goods market, labor market and interest rate dynamics and have calculated numerically the long phase cycles these models generate under plausible parameter values. We take from these models and from the earlier sections of this chapter the working hypothesis that there should be long phase cycles interacting with business cycles in the data as far as employment, income distribution and inflation is concerned. The method developed in Kauermann et al. (2008) now in fact allows us to test this hypothesis in a way much more refined than just by using Hodrick–Prescott filters with an arbitrarily given λ parameter. Moreover it can do this by using an econometric approach that is close in spirit to the two-dimensional phase plots of the employment–income distribution and the employment–inflation cycle of the literature on the Goodwin growth cycle and the Friedman inflation cycle, see Flaschel et al. (2005) for details. The technique developed in Kauermann et al. (2008) can be sketched in the following way. We decompose wage share v and employment rate e in a long phase cycle and business cycle model, that is $v = c_v(t) + b_v(t)$ and $e = c_e(t) + b_e(t)$, with $c(t)$ as long phase cycle and $b(t)$ as business cycle. The cycles themselves are understood as function in time, which ought to be estimated. We pursue a so called non-parametric approach by assuming smoothness for $c(t)$ and $b(t)$, but no specified parametric structure. The functions are then estimated using the technique of penalized splines (p-splines) as generally described in Ruppert et al. (2003). This means, a high dimensional basis is used to capture the shape of the functions, but for fitting a penalty is imposed which leads to smooth fits as those shown in Figs. 21.5 and 21.6. The business cycle itself is fitted in radial basis coordinates, which in fact allows to decompose long phase and business cycle accordingly. Both cycles are shown in Figs. 21.5 and 21.6.

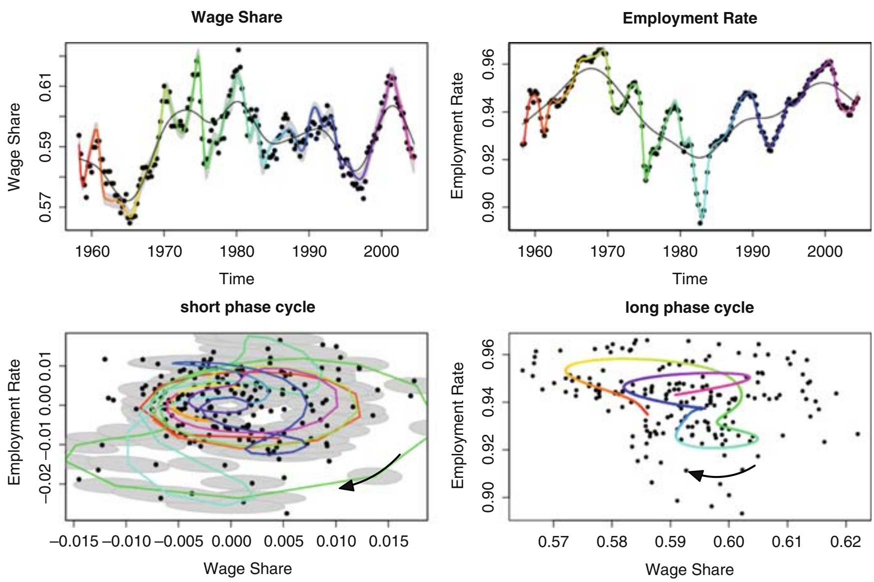

Fig. 21.5 Exploring US income distribution cycles with bivariate loops using penalized spline regression

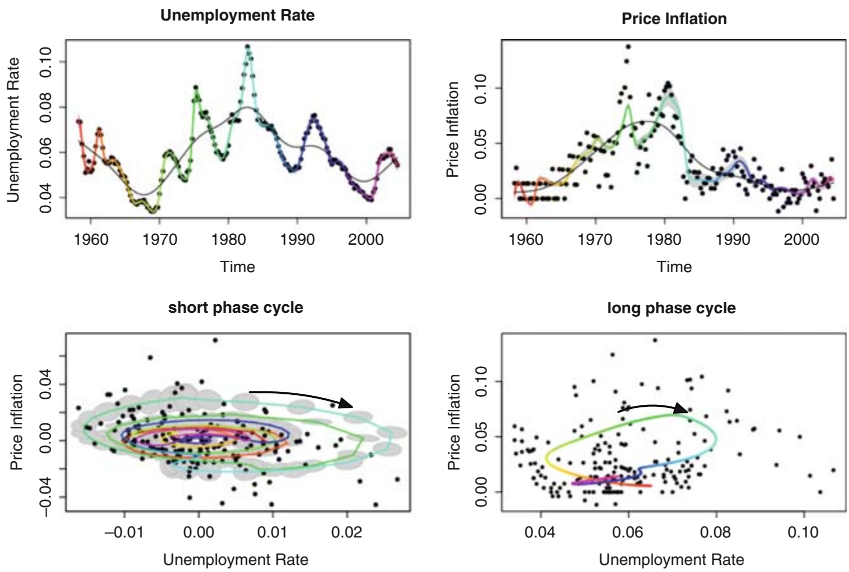

Fig. 21.6 Exploring US inflation cycles with bivariate loops using penalized spline regression

With respect to the long phase real cycle model, the Goodwin (1967) growth cycle model, we have in Fig. 21.5 the following situation. As far as the evolution of the wage share, shown top-left, is concerned we have more volatility as is the case with the inflation rate. This may be due to the involvement of labor productivity as constituent part of the definition of the wage share. Nevertheless one can see a single long phase cycle in the solid line shown in the time series presentation of the wage share. The employment rate is leading with respect to this long phase cycle in the wage share. We know from Goodwin (1967) and the numerous articles that followed his approach that the interaction of the employment rate with the wage share is generating a clockwise motion in the v, e phase space, see again Flaschel et al. (2005) for details. In Fig. 21.5 we can in this regard see that the cycles of business cycle frequency are also moving in a clockwise fashion (bottom left plot). We see (if minor cycles are neglected) by and large seven business cycles overlaid over the long phase cycles as they are also shown in the figure top-right. Figure 21.5 bottom right shows the long phase cycle in isolation.

We stress that our extraction of the business cycle component (the "short" cycle) as shown in Fig. 21.5 is an integral part of our treatment of the long phase evolution of the economy. This differs significantly from the Hodrick–Prescott trend vs. cycle separation shown in Fig. 21.4 which is enforced by an exogenous choice of the parameter λ used in this cycle extraction procedure.

With respect to the mirror image of the real growth cycle, the inflation–unemployment dynamics, we see that the unemployment rate is also leading compared to the inflation rate in the long phase cycle (the solid lines in the two time series plots top-left in Fig. 21.6). Looking closer to the business cycle plot (bottom left), as done in Kauermann et al. (2008), we see moreover that there are now approximately six business cycles surrounding these long phase cycles, in line with what is shown to hold for the US economy in Chiarella et al. (2005). We also get a clockwise rotation of the long phase cycle that is by and large also characterizing the business cycles surrounding it, though there are exceptions to this rule, see also the figure top-right in this regard. Note that we use, as is customary, the unemployment rate in place of the employment rate on the horizontal axis. Using the latter would give rise to an anti-clockwise orientation of the business and the long phase cycles shown in these figures. Figure 21.6 in the present section also shows for better visibility the long-phase cycle in isolation and it indicates that indeed 50 years of data are needed in order to at least indicate the existence of such a cycle.

As in the preceding sections we therefore see again a cycle that is nearly closed (and thus approximately of 50 years length) and that is moving clockwise as suggested by the simple Goodwin (1967) growth cycle model when extended by markup inflation procedures and the like, see Desai (1973) for an early example. The same cyclical pattern therefore holds true for the accompanying unemployment–inflations cycle, the Friedmanian nominal side of the real cycle so to speak. We conclude that the method developed in Kauermann et al. (2008) provides an important approach to the separation of long-phased cycles that describe the evolution from high to low inflation regimes and from high to low wage share regimes from cycles of business cycle frequency. This method therefore allows in a distinct way the discussion of long waves in inflation and income distribution in modern market economies after World War II.

21.6 Conclusions

In this chapter we have provided new empirical evidence for the existence of a long-phased cycle concerning the wage share and the employment rate after World War II along the lines of the original Goodwin (1967) model and later modifications of it. Taking an approach of Flaschel and Groh (1995) as a starting point, the time horizon for the investigation was now extended to 50 years, based on data for the US-economy. For the empirical analysis three different techniques have been applied. First, the two dynamic equations constituting the model have been estimated in their linearized form around the steady state by means of a two-stage LS-estimation. It turned out that at least the signs of the model's "core", the two off-diagonal elements of the Jacobian, have in fact the signs suggested by the model. This means that the assumptions of a "labor-market-led" real wage adjustment as well as of "profit-led" goods demand are not rejected by the data.

In a second step, a Hodrick–Prescott decomposition into trend and cycle was applied in order to remove the business cycle component from the data. As a result, a cycle emerged that was already closed to a higher degree than the one presented in Flaschel and Groh (1995). Thus, the use of new observations led to a further confirmation of the underlying growth cycle model. The issue of separating the business cycle from the trend was then further investigated in a third step by means of a technique developed in Kauermann et al. (2008). It turned out that a single long-phased cycle in the wage share and the employment rate could be detected which was overlaid by approximately seven business cycles. Furthermore, the long-period cycle was indeed characterized by a clockwise motion of the two variables just mentioned in the phase plane as predicted by the theoretical model.

A similar picture emerged with regard to the interaction between unemployment and inflation. Once again, a closed long-phased cycle, moving in a clockwise way, could be established with about six business cycles surrounding it. Thus, earlier theoretical approaches in the tradition of Friedman (1968) could be confirmed as well as recent empirical findings by Chiarella et al. (2005). Furthermore it turned out that a time horizon of 50 years is indeed necessary to derive the results just mentioned.

In addition to what has been shown in this chapter, one should, of course, also try to take earlier time periods into account than just the 1960s to the present if data – in particular on the wage share – are available. For the United Kingdom there exist such long time-series[10] from 1855 up to 1965 which are summarized by the phase plot diagram shown in Fig. 21.7.

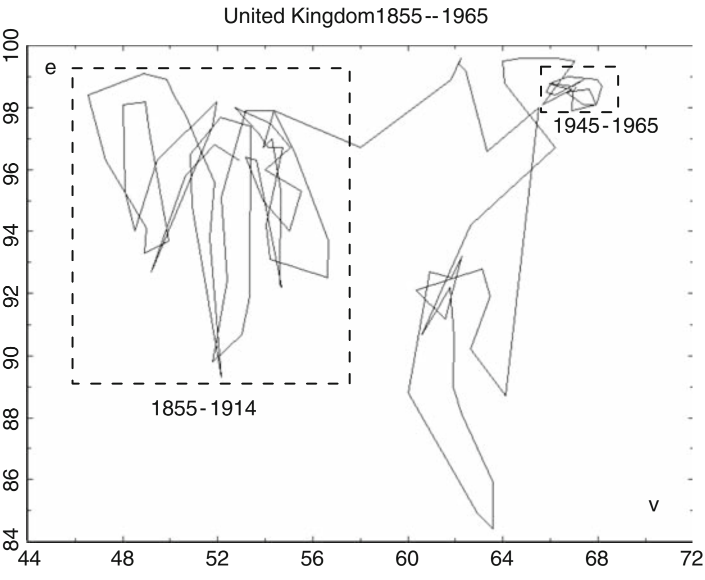

Fig. 21.7 UK income distribution cycles 1855–1965

[10] See Desai (1973) for the sources of these data and for an econometric approach on the basis of these data with respect to the Goodwin growth cycle model.

These data[11] seem to indicate that there have been two different types of dynamics with respect to the interaction of the rate of employment and the share of wages:

- The period before 1914 where the employment rate exhibits significant fluctuations of less than 10 years in phase length.
- The period after world war II (1945–1965) where no such fluctuations can be observed any more.

The important insight that can be obtained from these diagrams for Great Britain (1855–1965) is that the Goodwin cycle – if it exists – must have been significantly shorter before 1914 (with larger fluctuations in employment during each cycle), and that there has been a major change in it after 1945. This may be explained by significant differences and changes in the adjustment processes of market economies for these two periods: primarily price adjustment before 1914 and primarily quantity adjustments after 1945. This very tentative judgment must be left for future research here however.

References

Atkinson, A. B. (1969). The timescale of economic models: How long is the long run? *Review of Economic Studies, 36*, 137–152.

Barbosa Filho, N. H., & Taylor, L. (2006). Distributive and demand cycles in the US economy – A structuralist Goodwin model. *Metroeconomica, 57*, 389–411.

Blanchard, O. J., & Katz, L. (1999). Wage dynamics: Reconciling theory and evidence. *American Economic Review, 89*(2), 69–74.

Chen, P., & Flaschel, P. (2006). Measuring the interaction of wage and price Phillips curves for the U.S. economy. *Studies in Nonlinear Dynamics and Econometrics, 10*, 1–35.

Chiarella, C., Flaschel, P., & Franke, R. (2005). *Foundations for a disequilibrium theory of the business cycle. Qualitative analysis and quantitative assessment.* Cambridge, UK: Cambridge University Press.

Desai, M. (1973). Growth cycles and inflation in a model of the class struggle. *Journal of Economic Theory, 6*, 572–545.

Flaschel, P., & Groh, G. (1995). The classical growth cycle: Reformulation, simulation and some facts. *Economic Notes, 24*, 293–326.

Flaschel, P., Kauermann, G., & Teuber, T. (2005). Long cycles in employment, inflation and real unit wage costs. Qualitative analysis and quantitative assessment. *American Journal of Applied Sciences*, (special issue), 69–77.

Flaschel, P., & Krolzig, H.-M. (2006). Wage-price Phillips curves and macroeconomic stability: Basic structural form, estimation and analysis. In: B. Baltagi, E. Sadka, & D. Wildasin (Series Eds.), *Contributions to economic analysis*. C. Chiarella, P. Flaschel, R. Franke, & W. Semmler (Eds.), *Quantitative and empirical analysis of nonlinear dynamic macromodels*. Amsterdam: Elsevier.

Friedman, M. (1968). The role of monetary policy. *American Economic Review, 58*, 1–17.

[11] Note that they do not – at least on the surface – support one of Kaldor's famous stylized facts of growth theory (the constancy of the wage share in the long-run).

Goodwin, R. (1967). A growth cycle. In: C. H. Feinstein (Ed.), *Socialism, capitalism and economic growth* (54–58). Cambridge: Cambridge University Press.

Hirsch, M., & Smale, S. (1974). *Differential equations, dynamical systems and linear algebra*. London: Academic Press.

Kauermann, G., Teuber, T., & Flaschel, P. (2008). *Exploring US business cycles with bivariate loops using penalized spline regression* (working paper). Bielefeld: CEM.

Mankiw, G. (1992). *Macroeconomics*. New York: Worth.

Rose, H. (1967). On the non-linear theory of the employment cycle. *Review of Economic Studies, 34*, 153–173.

Ruppert, D., Wand, M., & Carroll, D. (2003). *Semiparametric regression.* Cambridge: Cambridge University Press.

Solow, R. (1990). Goodwin's growth cycle: Reminiscence and rumination. In: K. Velupillai (Ed.), *Nonlinear and multisectoral macrodynamics. Essays in honour of Richard Goodwin* (pp. 31–41). London: Macmillan.

Taylor, L. (2004). *Reconstructing macroeconomics: Structuralists proposals and critiques of the mainstream.* Cambridge, MA: Harvard University Press.

Chapter 22
Classical Dynamics in a General Keynes–Wicksell Model

22.1 Introduction

In this chapter, we introduce a general model of monetary growth which contains several existing models as special cases. The model is complete in the sense that it allows for a full interaction among the three major markets (goods, labor and assets), and consistent in that the budget constraints for households, firms and the government are all respected. This gives rise to a four-dimensional differential equation system. The stability properties of some special lower dimensional cases of the model are characterized analytically, and the unrestricted model is then briefly examined numerically.

Stein (1982, p. 191) defines a "Keynes–Wicksell model of money and capacity growth" as "one where there are independent savings and investment functions, but output is always at capacity". The dynamic properties of general models of the Keynes–Wicksell type, consisting of fully interacting money, goods and labor markets, have not been systematically explored in the literature to date. Such models are of a very neoclassical nature with respect to their building blocks. Yet, despite their supply side orientation, our analysis reveals that a monotonic adjustment of real wages to the long-run full employment level cannot be expected to occur once the complexity of the various feedbacks and interactions between the major markets in a capitalist economy are adequately taken into account. This is the main finding of the chapter.

The general model is introduced in Sect. 22.2. We allow for goods market disequilibrium, which means that if goods demand is to be satisfied, then firms must have either unintended inventory changes or unintended changes in the capital stock (or both). In order that their budget constraints are satisfied, these unintended changes in the stocks of firms must be financed in some way. We assume that this is done solely by issuing new equities in balance with the new equity demand of the household sector. Section 22.3 provides a brief presentation of the comparative static properties of the temporary equilibrium positions of the model. In addition, the dynamic laws are derived and their point of rest characterized. In Sect. 22.4 the dynamics of the model are investigated from a medium run perspective where the consequences of growth in factor supplies are neglected. The local

P. Flaschel, *Topics in Classical Micro- and Macroeconomics: Elements of a Critique of Neoricardian Theory*, DOI 10.1007/978-3-642-00324-0_22,

stability of the resulting three-dimensional system is studied. It is shown that the strength of the Keynes effect (the negative dependence of the aggregate demand schedule on the price level) relative to the Mundell effect (the positive dependence of the aggregate demand schedule on the expected rate of inflation) is an important determinant of stability, as is the adjustment speed of wages relative to that of prices. The emergence of cycles in the model is explored by means of the Hopf bifurcation theorem. Here the role of the adjustment speed of inflationary expectations is of particular importance. The stability preserving role of regressively formed expectations is also briefly considered.

In Sect. 22.5 we suppress the stabilizing role of the Keynes effect and the destabilizing role of the Mundell effect by simple assumptions on interest rate flexibility and inflationary expectations. This allows us to discuss the global features of the wage-price adjustment mechanism in isolation, though now in the context of a growing economy. We find that wage *flexibility* and price *inflexibility* support local asymptotic stability. The latter effect is important if there is a negative influence of real wage increases on excess demand in the market for goods. This negative influence, which arises whenever investment is more sensitive than savings to changes in the real wage, may be called the orthodox view of real wage effects on the aggregate demand for goods. If price flexibility is strong enough near the steady state to destabilize it locally, this model can nevertheless be made an economically viable one by assuming that money wages become sufficiently flexible for large deviations of the economy from its steady state path. The degree of wage flexibility may even be tailored in such a way that the viability domain exhibits given boundaries for the fluctuation of the rate of employment. Section 22.6, finally, briefly presents some numerical simulations of the complete model to illustrate the type of cyclical dynamics it generally gives rise to.

22.2 The General Model

Models of the Keynes–Wicksell type were introduced into the macrodynamics literature by Rose, Stein and others in the late 1960s (see Orphanides and Solow 1990 for a brief survey of this literature). In this section, we develop a generalization of such models which resembles that presented in Rose (1990), but with a complete set of budget restrictions for households, firms and the government. Furthermore we shall make use of very specific nonlinearities, namely a nonlinear money wage Phillips curve and a neoclassical production function. These nonlinearities appear also in Rose (1967), and we therefore obtain his employment cycle result as a special case of the present model. All other relationships are kept linear (up to certain growth rate formulations) in order to present the role that these two nonlinearities can play in its purest possible form.

Since the model contains a large number of variables and parameters, we begin with a list of the symbols used:[1]

List of endogenous variables

Statically endogenous		Dynamically endogenous	
$Y > 0$	Output	$K > 0$	Capital stock
$L > 0$	Employment	$w > 0$	Nominal wages
$C > 0$	Consumption	$p > 0$	Price level
I	Net investment	π	Expected inflation
$\rho > 0$	Rate of profit	$M > 0$	Money supply
$r > 0$	Rate of interest	$L^s > 0$	Labor supply
$p_E > 0$	Price of equities	$B \geq 0$	Bonds
$S > 0$	Total savings	$E > 0$	Equities
S_p	Private savings	$W > 0$	Real wealth
S_g	Government savings	$\omega > 0$	Real wage
$T > 0$	Real taxes	$Q > 0$	Inventories
$G > 0$	Government expenditure		

List of exogenous variables and parameters

$\delta = \delta_1 + \delta_2 > 0$	Depreciation plus inventory accumulation rate
$c_i > 0, i = 1, 2, 3$	Consumption function parameters
$\beta_I > 0$	Investment parameter
n, g	Natural growth rates
μ	Growth rate of money supply
$\beta_w \geq 0$	Wage adjustment parameter
$\sigma_w \in [0, 1],\ \sigma_w \cdot \sigma_p \neq 1$	Wage adjustment parameter
$\beta_p \geq 0$	Price adjustment parameter
$\sigma_p \in [0, 1],\ \sigma_w \cdot \sigma_p \neq 1$	Price adjustment parameter
$\beta_\pi^1 \geq 0$	Adaptive expectations parameter
$\beta_\pi^2 \geq 0$	Regressive expectations parameter

Given the above notation, the equations of the model are as follows:

$$Y = F(K, L), \quad F_K, F_L > 0,\ F_{KK}, F_{LL} < 0 \tag{22.1}$$

$$F_L = \omega = w/p \tag{22.2}$$

$$C = c_1(Y - \delta K - T) - c_2(r - \pi) + c_3 W + c_4,\ \ c_i \geq 0,\ i = 1, 2, 3, 4 \tag{22.3}$$

$$I = \beta_I(\rho - (r - \pi))K + gK, \quad \beta_I > 0,\ \rho = F_K - \delta \tag{22.4}$$

$$\begin{aligned} S &= S_p + S_g = (Y - \delta K - T - C) + (T - G) \\ &= Y - \delta K - C - G \end{aligned} \tag{22.5}$$

$$T/K = \tau = \text{const}, \quad G/K = \gamma = \text{const} \tag{22.6}$$

$$M = h_1 pY + h_2(r_0 - r)pK, \ \ r_0 = \rho_0 + \pi_0 \tag{22.7}$$

[1] In addition, the following notation is adopted: for any variable x, we use $\dot{x}$ to denote its time derivative, $\hat{x}$ to denote its growth rate, x_0 to denote its steady state value, and x' and x_y to denote total and partial derivatives respectively.

$$W = (M + B + p_E E)/p \tag{22.8}$$
$$p_E E = (F_K - \delta)pK/(r - \pi) \tag{22.9}$$
$$\hat{M} = \mu = \text{ const} \tag{22.10}$$
$$\hat{L}^s = n \tag{22.11}$$
$$\hat{K} = I/K \tag{22.12}$$
$$\hat{w} = \beta_w(L/L^s - 1) + \sigma_w \hat{p} + (1 - \sigma_w)\pi \tag{22.13}$$
$$\hat{p} = \beta_p(I/K - S/K) + \sigma_p \hat{w} + (1 - \sigma_p)\pi \tag{22.14}$$
$$\dot{\pi} = \beta_\pi^1(\hat{p} - \pi) + \beta_\pi^2(\mu - n - \pi) \tag{22.15}$$
$$\dot{Q} = \delta_2 K + S - I, \quad 0 < \delta_2 < \delta \tag{22.16}$$
$$S = S_p + S_g = (\dot{M} + \dot{B} + p_E \dot{E})/p - (\dot{M} + \dot{B})/p = p_E \dot{E}/p \tag{22.17}$$
$$I = p_E \dot{E}/p - (S - I) \tag{22.18}$$

Equation (22.1) is the conventional neoclassical production function and (22.2) the marginal productivity postulate based on (22.1). We have thus assumed that firms are profit-maximizing price-takers operating on their supply curves (referred to as the classical regime in neo-Keynesian fix-price approaches). Equation (22.3) is a standard consumption function with disposable income $Y_D = Y - \delta K - T$, the real interest rate $r - \pi$, and real wealth W as arguments. Equation (22.4) refers to (net) investment behavior, which is a linear function of the difference between the real rate of return to capital $F_K - \delta$ and the real rate of interest $r - \pi$. In addition, it is assumed that there is a trend growth rate g of the capital stock, which is here set equal to the natural rate of growth n for simplicity.[2] Saving S, in (22.5), is defined as the sum of private and government saving (both obtained from the flow budget constraints of the two sectors) and is equal to the demand gap $Y - \delta K - C - G$ by definition.[3] We assume, following Sargent (1987, Chap. V), that taxes and government expenditures are a constant proportion (τ and γ respectively) of the capital stock, and that the money supply grows at an exogenously given rate μ. These assumptions are reflected in (22.6) and (22.10). The behavior of the government is modeled in this very rudimentary manner since the focus of the present investigation is elsewhere. The government budget constraint is fulfilled through an appropriate adjustment of the rate of change of government debt $\dot{B}$:

$$\dot{B} = p[G - T] - \dot{M} = p[\gamma K - \tau K] - \mu M.$$

[2] This device saves one further adjustment equation; for an attempt at justification, see Sargent (1973, p. 429).

[3] Real taxes T are calculated net of government interest payments. Taxes are lump-sum and thus do not modify rate of return differentials.

If we assume that $c_3 = 0$ in (22.3), there will be no feedback of the evolution of government debt on the rest of the system since the stock of government debt is not a determinant of money demand and since we have assumed particularly simple rules for tax collection and government spending.

Equation (22.8) defines real household wealth, which is composed of money holdings, bonds, and equities. In view of this, it might be considered appropriate to specify the demand for money as $M^d = pWh(r)$, which would imply that for any given value of the interest rate, households wish to hold a fixed proportion of their real wealth in the form of real balances. However, in order to stay close to the specification of Sargent (1987), where $pYh(r)$ is used for representing money demand, we adopt the specification of money demand shown in (22.7). Here real money demand is a linear function of real income and the rate of interest. We assume that equities E and bonds B are perfect substitutes and that all expected profits are distributed as dividends. In this case the rate of return on shares must equal the real rate of interest on bonds. Setting the price of bonds $p_B = 1$, we have the following relation:

$$\frac{p(Y - \delta K - \omega L)}{p_E E} = \frac{p(F_K - \delta)K}{p_E E} = r - \pi$$

which yields (22.9). Money market equilibrium implies that all other asset markets are in equilibrium and can therefore be neglected in the explicit development of the model. Labor supply L^s grows at an exogenously given rate n in (22.11), while (22.12) states that intended net investment is realized and hence leads to the intended growth rate in the capital stock.[4]

Equations (22.13)–(22.15) describe the wage-price sector and have been adopted from Rose's (1990, Chap. 3) general framework. The growth of money wages is driven by excess demand on the labor market (the demand-pull component), augmented by a weighted average of expected medium-run price inflation π and current price inflation $\hat{p}$ (the cost-push components). Analogously, price inflation is driven by excess demand on the goods market, augmented by a weighted average of expected medium-run price inflation π and current wage inflation $\hat{w}$. Medium-run expectations π in turn are revised partly in an adaptive way (β_π^1) and partly in a regressive fashion (β_π^2) as shown in (22.15). This expectation revision rule has a forward looking and a backward looking aspect. Note that $\mu - n$ represents the steady state inflation rate (this is established formally below). When expectations are above (below) current inflation as well as the steady state inflation rate, they are revised downwards (upwards). However, when expectations are above one of these

[4] An alternative specification, requiring that the growth rate of the capital stock be consistent with the savings plans of households (rather than the investment plans of firms) is also explored below. Since we allow for goods market disequilibrium through the adjustment of inventories, these two approaches yield different results.

magnitudes but below the other, the direction of expectation revision is ambiguous. Note that despite this forward looking component of expectation revision, inflationary expectations are self-fulfilling only in the steady state.[5]

We have labor-market disequilibrium as well as goods-market disequilibrium in this model. Labor market disequilibrium is represented here through the use of actual (rather than notional) income $Y_D = Y - \delta K - T$ in the consumption function, and the possibility that actual employment L may deviate from normal employment L^s. Goods market disequilibrium is reflected in two ways in this model of price-taking firms, through changes $\dot{Q}$ in inventories Q and through changes in the financing of investments caused by such unintended inventory changes.

Firms operate in a growing economy here, and are assumed to add $\delta_2 K$ to their inventories in the steady-state in order to keep their inventories growing in line with production.[6] In addition, inventories change by an (unintended) amount equal to the excess demand on the goods market, which yields (22.16). The steady state inventory–capital ratio $q_0 = (Q/K)$ is easily determined by recalling (22.16) and solving:

$$0 = \dot{q} = \frac{\dot{Q}}{K} - \hat{K}q = \frac{S - I}{K} + \delta_2 - \hat{K}q$$

for q, which yields $q_0 = \delta_2/n$ since $S = I$ in the steady state. In this steady state firms retain $\dot{Q} = \delta_2 K$ from current production in order to protect themselves against unforeseen demand when the economy departs from the steady state. It is assumed in the following that the ratio Q/K stays in the neighborhood of q_0 even when the economy is outside the steady state, i.e., goods market disequilibria stay within such limits that inventories are not exhausted or exceed reasonable bounds. Rationing on the goods market is avoided by assuming appropriate inventory changes in a corridor $q \in (\delta_2/n - \alpha, \delta_2/n + \alpha)$, with α sufficiently small. Note that such a condition is normally not made explicit in models which consider the above type of

[5] This method of modeling forward-looking expectations appears, for example, in the influential paper by Dornbusch (1976) where, under certain conditions, it yields self-fulfilling forecasts. Gray and Turnovsky (1979) refer to this as "regressive expectations" and we adopt their terminology here. The same rule is referred to by Stein (1982) as "asymptotically rational expectations" and by Groth (1988) as "monetarist expectations". In the literature on speculative markets, such expectation formation rules are identified as "fundamentalist" (Frankel and Froot 1986; Chiarella 1992). It is important to emphasize that expectations can be *forward-looking* without being *self-fulfilling*. There are a number of reasons, such as the costs associated with the expectation calculation technology (Evans and Ramey 1992) or optimization costs in general (Conlisk 1988; Sethi and Franke 1995) why simple forward (or even backward) looking rules might yield higher returns to forecasting than expensive means of obtaining unbiased forecasts. Note that unlike adaptive expectations, a regressive expectations rule *does* induce instantaneous responses to announcements and news of future events.

[6] Assuming $\delta_2 Y$ as inventory accumulation rule instead of $\delta_2 K$ would only modify the model slightly; our choice allows us to treat planned inventory accumulation as part of depreciation.

goods-market disequilibrium.[7] Inventory changes $\delta_2 K$ are a deduction from gross income just as depreciation $\delta_1 K$ is; hence they do not enter disposable income. We use the composite parameter $\delta = \delta_1 + \delta_2$ to represent the "depreciation and inventory accumulation" rate.

Private savings S_p are assumed in (22.17) to absorb the new supply of money and government bonds: $(\dot{M} + \dot{B})/p$, with the remainder being directed to the purchase of perfectly substitutable equities $p_E \dot{E}/p$. Since firms have to finance net investment I as well as unintended inventory accumulation, their supply of equities is determined by

$$pI + p(S - I) = pS = p_E \dot{E}$$

which gives rise to (22.18). New equity issues by firms are thus equal to (consistent with) the demand for new equities on the part of households. This completes the model, which is basically classical in nature (supply side driven), yet with Keynesian components in the market for goods and the market for money. It is a complete and consistent macroeconomic model of a closed economy.

The literature on Keynes–Wicksell models has often used, instead of (22.12), the equation

$$\hat{K} = \dot{K}/K = S/K$$

to describe capital accumulation. Such a modification makes the outlook of the above model still more classical, since it says that accumulation is supply side driven so that inventory changes (22.16) may be neglected. This comes about by interpreting the case $S > I$ as involuntary additional investment which is financed by voluntary equity demand ($S = p_E \dot{E}/p$). In the opposite case $I > S$ we have that firms have to cut down their investment plans by $I - S$ to be in line with equity demand. Since we can neglect inventory changes, this case reduces the number of dynamic equations by one. A modification of our model which encompasses both cases is the following:

$$\hat{K} = \beta_k I/K + (1 - \beta_k)S/K, \quad \beta_k \in [0, 1]$$

in which case a discrepancy between intended savings and planned investment shows up both as an involuntary change in the capital stock as well as an unintended change in the stock of inventories. We shall not consider this general form in the following, but shall make use of the parameter β_k only to denote which of the two extreme cases we are considering: when $\beta_k = 1$ we have $\dot{K} = I$ as in (22.12), while $\beta_k = 0$ indicates the use of $\dot{K} = S$.

[7] We owe this observation to Reiner Franke, though he is not responsible for this particular solution to the $I \neq S$ problem. We use $\alpha = 0.2$ and $q_0 = 0.2$ in the numerical simulations in Sect. 22.6.

The structure of various well-known models of the Keynes–Wicksell type as well as a variety of other models of cyclical growth can be obtained from the above general approach by specializing it appropriately. The following table provides a brief summary of this claim.[8]

Model type	β_p	β_π^1	β_π^2	σ_w	σ_p	β_w	β_k	$CES(\epsilon)$	r
Goodwin (1967)	0	0	0	1	0	+	1	0	r_0
Fischer (1972)	+	+	0	0	0	0	+	ϵ	r_0
Rose (1967)	+	0	0	0	0	+	0	ϵ	r_0
Sargent (1987)	∞	+	0	0	0	+	I = S	ϵ	r
Sargent (1987)	∞	∞	0	0	0	+	I = S	1	r
Stein (1971)	+	+	0	0	0	0	+	ϵ	r_0
Stein (1971)	+	0	+	0	0	0	+	ϵ	r_0
Stein (1982)	+	0	+	0	1	+	1	ϵ	r

The models by Goodwin (1967), Rose (1967), and Stein (1971) are two-dimensional dynamical systems and thus fairly standard, while Fischer (1972) is three-dimensional and solved (locally) by examining the Routh–Hurwitz conditions. Sargent's adaptive expectations case is treated in its details in Franke (1992), while an extensive discussion of his perfect foresight case ($\beta_\pi^1 = \infty$) is provided in Flaschel (1993). The general model of Stein (1982) is treated comprehensively in Flaschel et al. (1994). We stress once again that the present general model of the Keynes–Wicksell type owes part of its structure (the wage-price sector) to Rose's (1990) general framework, although, unlike Rose, we do not go into a treatment of windfall profits.

22.3 Comparative Statics, Dynamics and the Steady State

The model can be easily reduced to intensive form by making the usual homogeneity assumptions and by employing lower-case letters to denote the corresponding ratios. Hence $y = Y/K, l = L/K$, etc., (although $m = M/(pK)$). For simplicity, assume further that $c_2 = c_3 = 0$ in the consumption function (22.3), so that we may write $C = c(Y - \delta K - T),\ c \in (0,1)$ as in Sargent (1987, Chap. V). Using

[8] We assume for this categorization of various monetary growth models that the production function is of the CES type so that we can use the constant elasticity of substitution expression ϵ to denote special cases: $\epsilon = 1$ corresponds to the Cobb–Douglas case, while $\epsilon = 0$ represents fixed proportions. Blank entries in the table mean that this parameter does not matter in the considered model, while a + indicates that the parameter is positive and finite. Note finally that $\beta_\pi^1 = 0, \beta_\pi^2 = 0$ should be interpreted as $\pi \equiv 0$, and $r = r_0$ as the prevalence of an infinite interest rate elasticity of money demand ($h_2 = \infty$). In the case of the Goodwin model, only its structure is obtained in this way, not its concrete form, since this model relies on Say's Law in the form $I \equiv S$.

$y = f(l) = F(1, L/K)$ to denote the production function in intensive form, the marginal productivity condition (22.2) leads to:

$$\omega = f'(l) \text{ or } l = (f')^{-1}(\omega) = l(\omega).$$

Given the real wage, the output–capital ratio $y(\omega) = f(l(\omega))$ is also determined, as is the profit rate:

$$\rho(\omega) = F_K - \delta = f(l) - lf'(l) - \delta = y(\omega) - l(\omega)y'(\omega) - \delta.$$

The money market equilibrium condition (22.7) in intensive form reads:

$$r = r_0 + \frac{h_1 y(\omega) - m}{h_2} = r(\omega, m).$$

Under the standard assumptions placed on the production function (22.1), the following hold:

$$l'(\omega) < 0, \quad y'(\omega) < 0, \quad r_\omega < 0, \quad r_m < 0.$$

Note also that $\rho'(\omega) = -f''(l) l_\omega l < 0$ since $f''(l) < 0, \quad [l_\omega = l'(\omega)]$.

We shall describe the dynamics of the model (22.1)–(22.18) as an autonomous differential equation system in the variables ω, l^s, m, and π. For notational simplicity, define

$$s(\omega) = S/K = (1 - c)y(\omega) - \delta - \gamma + c(\delta + \tau)$$

as the savings–capital ratio from (22.5), and

$$i(\omega, m, \pi) = I/K = \beta_I \cdot (\rho(\omega) - (r(\omega, m) - \pi) + n$$

as the investment–capital ratio from (22.4). It is a straightforward matter to verify the sign of the following partial derivatives:

$$s_\omega < 0, \quad i_m > 0, \quad i_\pi > 0$$

while $i_\omega = \rho'(\omega) - r_\omega$ is indeterminate in sign. In addition, define the following excess demand functions:

$$X^w(\omega, l^s) = \beta_w(l(\omega)/l^s - 1),$$
$$X^p(\omega, m, \pi) = \beta_p(i(\omega, m, \pi) - s(\omega)).$$

The partial derivatives of these two functions satisfy:[9]

$$X^w(\underset{-}{\omega}, \underset{-}{l^s}), \qquad X^p(\underset{?}{\omega}, \underset{+}{m}, \underset{+}{\pi})$$

[9] $X^w_\omega = \beta_w l_\omega / l^s$ is obviously negative while the sign of $X^p_\omega = \beta_p(i_\omega - s_\omega)$ is ambiguous, since i_ω is of ambiguous sign. The partial derivative X^p_ω will be positive if the production function is approximately characterized by fixed proportions, that is, if $f''(l)$ is sufficiently close to zero, since we would then have $i_\omega \approx -r_\omega > 0$ which is of the same sign as s_ω.

The first of these excess demand functions is typical for growth (cycle) models of the Solow–Goodwin type (smooth factor substitution combined with a real wage Phillips curve), while the second excess demand function summarizes the indeterminate Rose real wage effect on the excess demand for goods, the contractionary Keynes or nominal rate of interest effect and the expansionary Mundell or real rate of interest effect.

Given the above notation, the four dynamic equations in compact form are as follows (see appendix for derivation):

$$\hat{\omega} = \sigma((1-\sigma_p)X^w + (\sigma_w - 1)X^p), \tag{22.19}$$

$$\hat{l}^s = \begin{cases} n - i(\omega, m, \pi) & \text{if} \quad \beta_k = 1 \\ \\ n - s(\omega) & \text{if} \quad \beta_k = 0 \end{cases}, \tag{22.20}$$

$$\hat{m} = \mu - n - \pi - \sigma(X^p + \sigma_p X^w) + \hat{l}^s, \tag{22.21}$$

$$\dot{\pi} = \beta_\pi^1 \sigma(X^p + \sigma_p X^w) + \beta_\pi^2(\mu - n - \pi), \tag{22.22}$$

where $\sigma = (1 - \sigma_w \sigma_p)^{-1}$ in the above.

For the case $\beta_k = 1$ ($\dot{K} = I$), to be considered later, the following is appended as an additional dynamic equation:

$$\begin{aligned} \dot{q} &= \frac{I - S}{K} + \delta_2 - \hat{K} \cdot q \\ &= (1-q)i(\omega, m, \pi) - s(\omega) + \delta_2. \end{aligned}$$

The *steady state* of the model is determined by setting $\hat{\omega} = \hat{l}^s = \hat{m} = \dot{\pi} = 0$ in (22.19)–(22.22). It is uniquely determined and given by (the solution to) the following set of equations:

$$\begin{aligned} y_0 &= f(l_0) = \frac{n - c\tau + \gamma}{1 - c} + \delta, \\ r_0 &= f(l_0) - f'(l_0)l_0 + \mu - n, \\ \omega_0 &= f'(l_0), \\ m_0 &= h_1 y_0, \\ l_0^s &= l_0, \\ \pi_0 &= \mu - n. \end{aligned}$$

In addition, observe that (22.21)–(22.22) and the steady state conditions $\hat{l}^s = \hat{m} = \dot{\pi} = 0$ imply that $X^p + \sigma_p X^w = 0$. Imposing the condition $\hat{\omega} = 0$ on (22.19) yields $(1 - \sigma_p)X^w + (\sigma_w - 1)X^p = 0$. Together, these conditions imply $X^w = X^p = 0$ for $\sigma_p \cdot \sigma_w \neq 1$, so excess demand in both product and labor markets is zero in the steady state. From (22.20), this implies that the steady state growth rate of the capital stock equals n, the growth rate of labor supply. Since the output–capital and labor–capital ratios are constant in the steady state, this implies the following:

$$\hat{K}_0 = \hat{L}_0^s = \hat{L}_0 = \hat{Y}_0 = n.$$

Furthermore, the rates of wage and price inflation are equal to each other and to the expected rate of price inflation, which is given by the difference between the rate of monetary expansion and the rate of growth of the economy:

$$\hat{p}_0 = \hat{w}_0 = \mu - n.$$

The stability of the steady state of this four-dimensional dynamical system will be investigated in the following sections for special medium-run and long-run cases. The properties of the complete system will be investigated numerically in a subsequent section.

22.4 Medium Run Dynamics

In this section we abstract from growth in the labor supply and in the capital stock, and assume that $l^s = l_0^s$ holds at all moments in time. This allows us to consider the real wage dynamics in conjunction with the dynamics of the nominal variables m and π, first for the case of adaptive and then for the case of regressive expectations. The choice between $\beta_k = 0, 1$ (whether it is desired savings or planned investment that is equated with the change in the capital stock) is thus irrelevant in the present section. The resulting system is now three-dimensional, with the dynamic variables being ω, m, and π.

22.4.1 Adaptive Expectations

Setting $\beta_\pi^2 = 0$ in (22.22) yields an expectation formation rule that is purely adaptive. The resulting Jacobian of the three-dimensional dynamics (22.19), (22.21), (22.22), evaluated at the steady state, is the following:

$$J = \begin{pmatrix} \sigma\omega[(1-\sigma_p)X_\omega^w + (\sigma_w - 1)X_\omega^p] & \sigma\omega(\sigma_w - 1)X_m^p & \sigma\omega(\sigma_w - 1)X_\pi^p \\ -\sigma m[X_\omega^p + \sigma_p X_\omega^w] & -\sigma m X_m^p & -\sigma m X_\pi^p - m \\ \beta_\pi^1 \sigma[X_\omega^p + \sigma_p X_\omega^w] & \beta_\pi^1 \sigma X_m^p & \beta_\pi^1 \sigma X_\pi^p \end{pmatrix}.$$

Furthermore, from Sect. 22.3 we obtain

$$X_\omega^w = \beta_w l_\omega / l^s < 0 \qquad \text{(Marginal Productivity Effect)},$$
$$X_\omega^p = \beta_p[\beta_I(\rho_\omega - r_\omega) + c y_\omega - y_\omega] \gtreqless 0 \qquad \text{(Rose Effect)},$$
$$X_m^p = -\beta_p \beta_I r_m > 0 \qquad \text{(Keynes Effect)},$$
$$X_\pi^p = \beta_p \beta_I > 0 \qquad \text{(Mundell Effect)},$$

where $\rho(\omega)$, the net rate of return to physical capital, is defined by $\rho(\omega) = y(\omega) - y'(\omega)l(\omega) - \delta$ as in Sect. 22.3 above.

The negative effect of the real wage on the excess demand for labor mirrors the conventional marginal productivity relationship and has been called the first fundamental postulate of Classical theory by Keynes (1936, p. 5). The sign of X_m^p is positive, since we have the usual positive Keynes effect of a falling price level p (a rising value of $m = M/(pK)$) on the rate of interest r via the money market equilibrium condition; this translates into an increase in excess demand for goods. The sign of X_π^p is also positive, since investment demand depends positively on the expected rate of inflation. This effect has been called the Mundell effect or the real rate of interest effect in the literature. The sign of X_ω^p, however, is ambiguous, since an increase in real wages depresses both investment and savings. It will be negative if the impact is stronger on investment than on savings, which then creates a potentially unstable situation investigated in its details by Rose (1967). We call this effect the Rose effect (of real wage changes on the excess demand for goods). Note that all these effects are formulated in the context of disequilibrium, while they are generally used in equilibrium situations in the literature.

The stability properties of the system may be characterized as follows:

Proposition 1. *Consider the dynamical system (22.19), (22.21), (22.22) under the assumption that* $\beta_\pi^2 = 0$. *Then:*

(a) There exists exactly one parameter value $\beta_\pi^H \geq 0$ *such that the unique steady state of the system is locally stable for all* $\beta_\pi^1 < \beta_\pi^H$ *and unstable for all* $\beta_\pi^1 > \beta_\pi^H$.

(b) This value β_π^H *is positive if* $x = [(1-\sigma_p)X_\omega^w + (\sigma_w - 1)X_\omega^p]\omega - X_m^p m < 0$ *holds at the steady state (that is, if the marginal productivity effect and the Keynes effect outweigh in the sense of this inequality the Rose effect, assuming the latter to be negative. If the Rose effect is positive, the inequality always holds).*

(c) The steady state loses stability in a cyclical fashion at β_π^H *by way of a Hopf-bifurcation.*

(d) The unstable cases $\beta_\pi^1 > \beta_\pi^H$ *are either characterized by one negative characteristic value and two complex conjugates with positive real parts, or by one negative characteristic value and two positive ones.*

Remarks. 1. The condition $x < 0$ which allows for the stability of the model for a bounded set of adjustment coefficients $\beta_\pi^1 > 0$ can always be established by choosing $\beta_w X_\omega^w$ sufficiently large and $\beta_p X_\omega^p$ sufficiently small, since $X_m^p > 0$ holds. Wage flexibility and price inflexibility (if $X_\omega^p < 0$) thus prove favorable for the stability of the steady state.

2. Nevertheless, this stability can always be destroyed by choosing the speed of adjustment of adaptive expectations sufficiently large (for given values of β_p, β_w), since $J_{33} = \beta_\pi^1 \sigma X_\pi^p$ is positive and since J_{11}, J_{22} do not depend on β_π^1.

3. The previous remark indicates that the case $\beta_\pi^1 = \infty$ (myopic perfect foresight) may be problematic. Indeed, we then have the following equations for the wage price sector:

$$\begin{aligned}
\pi &= \hat{p}, \\
\hat{w} &= \hat{p} + \beta_w(l/l^s - 1) \ \text{ or } \ \hat{\omega} = \beta_w(l/l^s - 1), \\
\hat{p} &= \hat{p} + \beta_p(i(\omega, m, \pi) - s(\omega)) + \sigma_p(\hat{w} - \hat{p}).
\end{aligned}$$

There is an obvious contradiction between the last two equations if $\sigma_p > 0$ holds, since the last equation then gives rise to a second, seemingly independent law for the dynamics of the real wage:

$$\hat{\omega} = -\frac{1}{\sigma_p}\beta_p(i(\omega, m, \pi) - s(\omega))$$

unless it is assumed that goods market disequilibrium always stays in a fixed (inverse) relationship to labor market disequilibrium. If this proportionality is added as a restriction to the model, we get, on the one hand, that the real wage must always adjust monotonically to its steady state value, since there is then an independent real wage dynamics described by the Phillips curve of the model, while on the other hand the dynamic law for the rate of inflation must obey a rule that guarantees the above fixed relation at all moments in time. The model therefore has a very strange limit case $\beta_\pi^1 = \infty$ if $\sigma_p > 0$ and $\beta_w, \beta_p \in (0, \infty)$ holds. Note that this case no longer draws a distinction between a medium-run expected rate of inflation and the actual short-run rate.[10]

4. This implicit disequilibrium determination of the rate of inflation is avoided or made an equilibrium determination of the rate of inflation if either $\sigma_p = 0$ is assumed or $\beta_p = \infty$, which both imply that $I = S$. This limit case is the myopic perfect foresight case which Sargent (1987, Chap. V) analyzes to prove some Friedmanite propositions. However, this case is also of a very exceptional nature, since *real* wage movements then only depend on the state of the labor market and not, as in our general model, on the state of the goods market as well [see (22.19)]. With respect to our medium-run type of analysis we have, in this case:

$$\hat{\omega} = \beta_w(l(\omega)/l_0^s - 1), \qquad l' < 0$$

that is, a stable single equation for the real wage dynamics which is independent of the rest of the system. Note again that the distinction between a medium-run expected rate of inflation and the actual short-run rate is no longer present under myopic perfect foresight.

5. The case of adaptive expectations behaves as might be expected in view of earlier results on Keynes–Wicksell or Tobin type monetary growth models (see

[10] The case of myopic perfect foresight is reconsidered from the viewpoint of nonlinear dynamics in the simple Cagan model of money market dynamics in Flaschel and Sethi (1998). We here extend Chiarella's (1990, Chap. 7) investigation of this case by including an accelerator term into the price dynamics that is employed by him.

Orphanides and Solow 1990 in this regard). The expected instability for large adjustment speeds β_π^1 of inflationary expectations is in the present context given in a particularly simple way, since we have $J_{33} = \beta_\pi^1 \sigma X_\pi^p > 0$; a positive feedback of the expected rate of inflation onto itself via the excess demand function in the market for goods and the Mundell effect.

6. We leave open the question of whether the local explosiveness of the model for $\beta_\pi^1 \to \infty$ can be tamed by global considerations by way of the method of so-called relaxation oscillations.
7. We have excluded the case $\sigma_p = \sigma_w = 1$ from of all of our considerations, since it again leads to two different rules for real wage changes. Here too, this "contradiction" can be suppressed by assuming $\beta_p = \infty$, which again gives rise to Sargent's (1987, Chap. V) perfect foresight case.

22.4.2 Regressive Expectations

Setting $\beta_\pi^1 = 0$ in (22.22) yields an expectation formation rule that is purely regressive: regardless of current and historical experience, inflation is expected to move at all times in the direction of its steady state value. This specification captures forward looking aspects of the expectation formation process without adopting the extreme requirement that expectations be self-fulfilling even outside the steady state.

Intuitively, it might be expected that such a specification would lend stability to the dynamics. To explore this question, we require the Jacobian of the three-dimensional dynamics (22.19), (22.21), (22.22), evaluated at the steady state:

$$\tilde{J} = \begin{pmatrix} \sigma\omega((1-\sigma_p)X_\omega^w + (\sigma_w - 1)X_\omega^p) & \sigma\omega(\sigma_w - 1)X_m^p & \sigma\omega(\sigma_w - 1)X_\pi^p \\ -\sigma m(X_\omega^p + \sigma_p X_\omega^p) & -\sigma m X_m^p & -\sigma m X_\pi^p - m \\ 0 & 0 & -\beta_\pi^2 \end{pmatrix}.$$

Note that the positive (destabilizing) Mundell effect disappears from the entry J_{33} of the Jacobian J.

Proposition 2. *Consider the dynamical system (22.19), (22.21), (22.22) under the assumption that $\beta_\pi^1 = 0$ and let x be given as in Proposition 1(b). Then the steady state is locally asymptotically stable if and only if $x < 0$.*

Remarks.

1. A corollary of the above proposition is that the case $\sigma_w = 1$ will always be locally asymptotically stable.
2. Consider the case of a Cobb–Douglas production function $y = l^{1-a}, a \in (0,1)$ which gives rise to a net rate of return to capital $\rho = ay - \delta$. Given our linear money demand function, the Rose stability condition $X_\omega^p > 0$ reads

$$a < h_1/h_2 + (1-c)/\beta_I.$$

A sufficiently small parameter a or a small β_I (or a small h_2) will thus establish stability in this regressive expectations case.
3. The considered dynamics will give rise to growth *cycles* if and only if $(\sigma x)^2 < 4J_1$ holds true. Also, the Hopf theorem is again applicable here, now with respect to the parameter β_p.

Summing up, we have in the case of regressive expectations that $\beta_w \to \infty$ will be good for stability ($X_\omega^w < 0$), while $\beta_p \to \infty$ may be bad for it, if $\sigma_w < 1$ and $X_\omega^p < 0$ hold. Increasing price flexibility may thus lead to stability problems, at least near the steady state of this economy.[11] Note that the size of the parameter β_π^2 is not a problem in the regressive expectations case. Note finally that the results obtained separately for $\beta_\pi^2 = 0$ and $\beta_\pi^1 = 0$ can be easily generalized to the situation where both β_π^1 and β_π^2 are positive.

22.5 Long Run Dynamics and Global Stability

In this section we shall explore the unstable cases considered in the preceding section from a global perspective which includes factor growth. The variable $l^s = L^s/K$ thus is no longer fixed at its steady state value, but evolves according to (22.20) of Sect. 22.3. Of this equation, we shall only consider the case $\dot{K} = S$ ($\beta_k = 0$) in the following and only note here that the alternative case $\dot{K} = I$ can be treated in the same way as a consequence of assumption 1 to be introduced below.[12]

In order to treat the long-run dynamics of the model (22.1)–(22.18) from a global perspective we have to simplify considerably the four-dimensional dynamics it gives rise to. The following two assumptions are sufficient to reduce the dimension of the dynamics to two. This two-dimensional system can then be studied by means of the Poincaré–Bendixson Theorem when the money–wage Phillips curve of the model is made nonlinear in a manner proposed by Rose (1967). The development of the monetary magnitude $m = M/(pK)$ can then be derived from the real dynamics, but will not be investigated here in its details.

Assumption 1. The interest elasticity of money demand is (approximately) infinite at the steady-state value r_0 of the rate of interest r ($h_2 = \infty$)

This assumption implies that the nominal interest rate is (approximately) constant ($r \approx r_0$) over the cycle to be considered below. In addition, we impose:

---

[11] As in earlier situations, stability problems arise if excess demand in the market for goods responds negatively to real wage increases, giving rise to price "decreases" which provide a further stimulus to real wage "increases".

[12] Of course, the case $\dot{K} = I$ ($\beta_k = 1$) requires us to keep track of movements in the inventory–capital ratio q; this is done in the next section.

Assumption 2. $\beta_\pi^1 = 0, \quad \beta_\pi^2 = \infty$

This implies that the expected rate of inflation is at all times equal to its steady state value ($\pi = \mu - n$). While this would necessarily be the case in the steady state, we assume that it holds even in disequilibrium.[13]

It is obvious that these assumptions describe an unrealistic limit case of the general model. Nevertheless computer simulations presented in the section to follow indicate that this limit case gives rise to results that are typical also for dynamics of the general model. Furthermore, the results we obtain below for the special case are of independent interest since they question the neoclassical view of the workings of the labor market in a dynamic setup which consists of fairly standard neoclassical building blocks.

We follow Rose (1967, p. 159) in assuming that the money–wage Phillips-curve (22.13) has a nonlinear shape as far as its first term $\beta_w(L/L^s - 1)$ is concerned. Specifically, let $V = L/L^s = l/l^s$ denote the employment ratio, and write the excess demand in the labor market, X^w, as the function $\beta_w(V)$. Note that the employment ratio can be expressed as a function of ω and l^s since $V = l(\omega)/l^s$; hence we may write $V(\omega, l^s)$. We assume that there exist a, b such that $0 < a < 1 < b$ and $\beta_w(1) = 0, \beta_w(a) = -\infty, \beta_w(b) = +\infty$. As before $\beta_w' > 0 \quad [0 < a < 1 < b]$. Some examples of such functions are shown in Fig. 22.1.[14]

Assumptions 1 and 2 allow us to work with the following two-dimensional system:

$$\hat{\omega} = \sigma((1 - \sigma_p)X^w + (\sigma_w - 1)X^p), \tag{22.23}$$
$$\hat{l}^s = n - s(\omega), \tag{22.24}$$

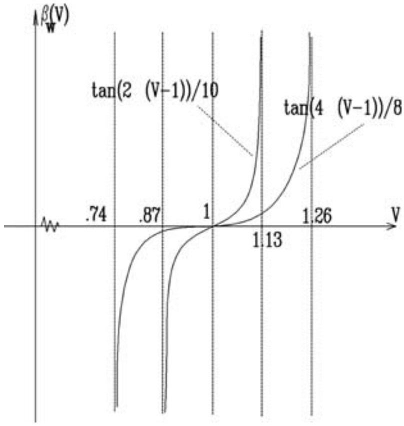

Fig. 22.1 Rose type Phillips curves

[13] We have already seen that the less extreme assumption $\beta_\pi^2 < \infty$ does not make much difference to the results.

[14] Although the figure shows functions that are symmetric, this is not required for the results to follow.

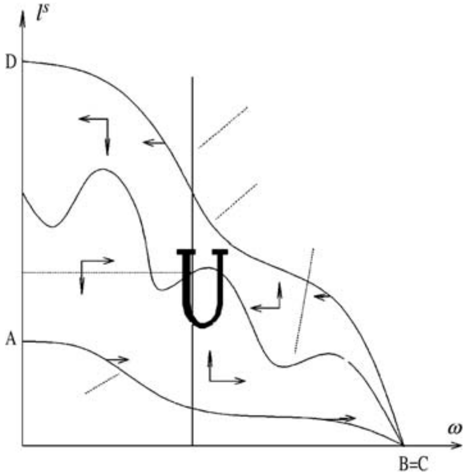

Fig. 22.2 Phase diagram for the two-dimensional dynamics ($x = \omega, y = l^s$)

where $X^w = \beta_w(V(\omega, l^s))$, and $X^p = \beta_p(\beta_I(\rho(\omega) - r_0 + \mu + n) - n - s(\omega))$ and $\rho(\omega) = y(\omega) - y'(\omega)l(\omega) - \delta$ as in Sect. 22.3 above. Note that excess demand in the product market now depends only on the real wage, ω.

This dynamics gives rise the phase portrait shown in Fig. 22.2 (for the interpretation of $l(\omega)/a, l(\omega)/b$ see Fig. 22.1 and the comments preceding it).

Assume, for simplicity, that the production function $y = f(l)$ fulfills

$$f^{'}(0) < \infty, \quad f^{'}(\bar{l}) = 0 \quad \text{for some} \quad \bar{l} < \infty.$$

In this case, the function $l(\omega) = (f^{'})^{-1}(\omega)$ cuts both axes of the positive orthant of $\Re^2$. The two functions $l(\omega)/a$ and $l(\omega)/b$ of ω, of course, have the same properties as the strictly decreasing function $l(\omega)$ and they determine (part of) the boundary of the domain of definition of the dynamics (22.23)–(22.24). To see this, observe that the nonlinearity of the Phillips curve implies that the domain of definition of the dynamics (22.23)–(22.24) requires the condition $l(\omega)/b \leq l^s \leq l(\omega)/a$ to be met. Note in addition that the two segments AD and BC cannot be crossed by the trajectories of (22.23)–(22.24), which implies that the open set U determined by $ABCD$ is an invariant subset of the full domain of definition of the dynamics (22.23)–(22.24): no trajectory which starts in this set can leave it.

Figure 22.2 also depicts the two isoclines of the dynamics (22.23)–(22.24). The calculation of the $\dot{l}^s = 0$ locus is straightforward, since the right-hand side of (22.24) only depends on ω. For $\dot{\omega} = 0$, in the case $\sigma_p < 1$, we get:[15]

$$l(\omega)/b < l^s = \frac{l(\omega)}{\beta_w^{-1}(\frac{1-\sigma_w}{1-\sigma_p} X^p(\omega))} < l(\omega)/a$$

[15] In the case $\sigma_p = 1$, the system (22.23)–(22.24) dichotomizes and thus becomes uninteresting from an economic point of view. the case $\sigma_w = 1$ leads us back to the Goodwin growth cycle model.

since $0 < a < \beta_w^{-1}(\cdot) < b$. This gives a well-defined function $l^s(\omega)$ over $\Re_+$, the relevant portion of which is shown in Fig. 22.2. Note that this function need not be a decreasing one. We shall assume $\sigma_p < 1$ for the remainder of this section.

We state here without proof the following lemma (see Benassy 1986 for a proof in a closely related situation):

Lemma 1. *Any trajectory which starts in the set U (see Fig. 22.2) stays in a compact set K of $\Re^2$ contained in U.*

An immediate consequence of this lemma is that all trajectories of (22.23)–(22.24) in U can be continued up to $t = \infty$ (see the theorem on p. 171 in Hirsch and Smale 1974). Furthermore, the set of all limit points of any trajectory for $t \to \infty$, its ω-limit set L_ω, must then be nonempty and compact. This implies (see Hirsch and Smale 1974, pp. 248 ff. for proof)

Theorem 1 (Poincaré–Bendixson). *Every set L_ω of the above dynamics (in U), which contains no equilibrium point of (22.23)–(22.24), is a closed orbit.*

Corollary 1. *Assume that at the steady state ω_0, l_0^s of the dynamics (22.23)–(22.24) the following holds:*

$$x = (1 - \sigma_p)X_\omega^w + (\sigma_w - 1)X_\omega^p > 0$$

Then any nonstationary trajectory in the set U is a closed orbit or converges to one.

Remarks.

1. It is not excluded here that the stable case $x < 0$ can give rise to limit cycles too. Should this occur we have corridor stability in this model type; see Groth (1992) for such an occurrence in a model of Keynes–Friedman type.
2. The main message of the theorem and its corollary is that wages may be inflexible near the steady state and may thus allow for the condition $x > 0$ and local instability. If wages, however, become very flexible far away from the steady-state, as in Fig. 22.1, the system can be rescued from global instability, thus making the dynamics viable from an economic point of view. This shows that certain types of wage flexibilities may create endogenous cycles rather than convergence to the long-run position in supply-side oriented models.
3. As Fig. 22.2 shows, the viability domain within which all trajectories are attracted by closed curves (or are closed curves) can be made as small as desired by reducing the size of the interval (a, b). The amplitude of the resulting growth cycles can therefore be easily tailored should the need for calibration arise (for example in empirical investigations of the model).
4. The model has been introduced as a general type of Keynes–Wicksell monetary growth model. The results obtained, as well as the model's structure, show that the Keynesian part of the model is underdeveloped, since:

   (a) Wage flexibility is required to rescue the dynamics at least from global instability.

(b) Sufficient flexibility will establish local asymptotic stability of the full employment steady growth path of the model for any given degree of price-flexibility.
(c) Production is supply-determined in this approach.
(d) The IS-LM block of the model mainly serves to determine the rate of price inflation.

By contrast, the model has a definite Wicksellian flavor due to the choice of investment function and goods-market behavior, as well as the property that sufficient price flexibility will destroy at least the local stability of the steady state.[16] We have thus obtained results that conflict with Keynes' (1936, p. 269) claim:[17]

> To suppose that a flexible wage policy is a right and proper adjunct of a system which on the whole is one of *laissez-faire,* is the opposite of the truth.

Here it is *price* rather than *wage* flexibility which may endanger the workings of the economic mechanism. This extends analytical results obtained by Tobin (1975) and confirms the intuition expressed in Tobin (1993) that, contrary to the popular view that rigid prices are the defining characteristic of Keynesian economics, increased price flexibility can destabilize movements in output.

22.6 Some Numerical Results

We have seen in the preceding section that the long-run dynamics (22.23)–(22.24) can be made globally stable within a predetermined range $[a, b]$ for the rate of employment V by means of assumptions on money demand, inflationary expectations, and the production function. Unlimited money wage flexibility for V approaching a or b was crucial to making the dynamics viable. It is also obvious from Fig. 22.2 of the preceding section that perfect downward flexibility (at $V = a$) is more important than perfect upward flexibility (at $V = b$), since $a > 0$ (though not $b < \infty$) is crucial for the application of the Poincaré–Bendixson theorem.

The following numerical investigations of the Keynes–Wicksell model are based on the CES production function $y = (a + (1-a)l^{-\varrho})^{-1/\varrho}$ where $a \in (0, 1)$ and $\varrho \in (-1, \infty)$.[18] Also, we shall only consider the case $\dot{K} = I$ in this section, instead of $\dot{K} = S$ as in the preceding section. The simulations start at the steady state of the model which receives a supply-side shock at time $t = 1$. This procedure allows a simple illustration of whether the dynamics returns to the steady state or cycles around it.

---

[16] For any given degree of wage flexibility and $\sigma_w < 1$ if $X^p_\omega < 0$ holds.

[17] In our model short-run causality still runs from marginal productivity to real wage determination and not the other way round as was intended by Keynes in his theory of effective demand. Goods demand here only helps to determine the rate of inflation in a Wicksellian fashion.

[18] The constant elasticity of substitution of the production function is given by the term $\sigma = 1/(1 + \varrho)$.

Let us first state that numerical investigations of the dynamics (22.23)–(22.24) generally show rapid cyclical adjustment towards the steady state in the special case where $\kappa_w = 1$ holds, i.e., where we have the two-dimensional Goodwin growth cycle structure embedded in a model with neoclassical factor substitution and thus a combination of Goodwin's and Solow's views on capital accumulation (even for elasticities of substitution as low as $1/4$). If we add the Rose goods-market mechanism to the real wage dynamics of the preceding situation and choose a specification where investment is more responsive to real wage changes than savings we get for a linear money–wage Phillips curve the global non-viability of resulting cycles if the adjustment speed of money wages is sufficiently low in relation to the adjustment speed of the price level. The resulting dynamics can be interpreted as an enhanced Goodwin growth cycle where the overshooting mechanism regains (more than) its original power – despite smooth factor substitution – through destabilizing price flexibility as in Rose (1967), reinforced by a sufficiently sluggish adjustment of nominal wages. The result obtained here reveals that the neoclassical argument of the stabilizing role of money and real wage adjustments implicitly relies on the assumption that goods prices are sluggish during the adjustment process. Note that the flexibility of prices is generally assumed as very high (or infinite) in the literature.

Next, we replace the linear Phillips-curve with the nonlinear type considered in the preceding section. This allows us to tame the explosive nature of the preceding growth cycle situation leading to the limit cycle behavior shown in Fig. 22.3. The basic idea behind this employment cycle is that wage changes dominate price

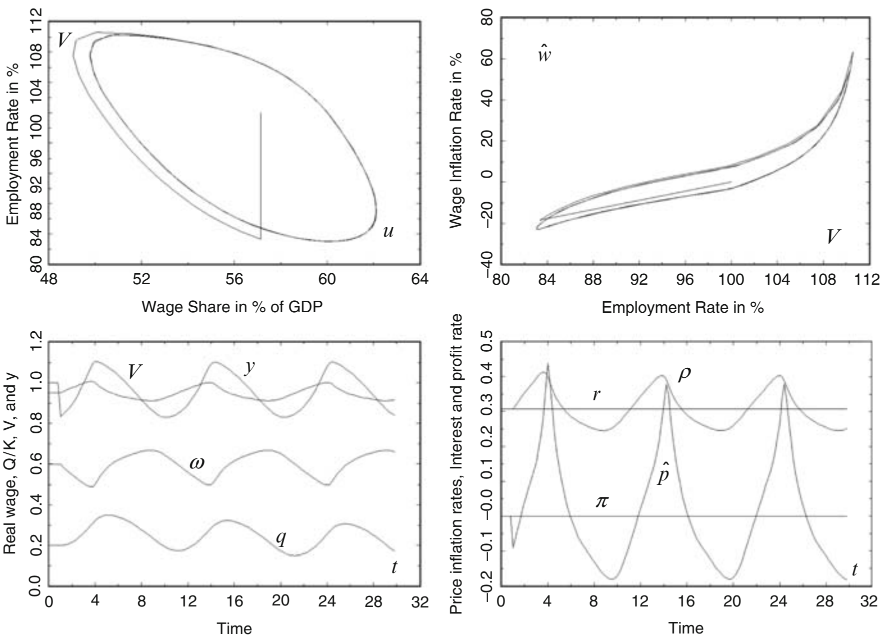

Fig. 22.3 The Rose nonlinear limit cycle situation

changes far from the steady state and can thus exercise their stabilizing potential in the manner assumed by the marginal productivity theory of real wages. Note here that the amplitude of the fluctuations of the rate of inflation is considerable in the present simulation, which is not implausible since this model type relies mainly on price and wage adjustments. The shape of the Phillips curve that is in use here is revealed to some extent by the upper right hand panel where the rate of change of nominal wages is plotted against the rate of employment.

Adding sufficient interest rate flexibility to this situation implies that its limit cycle will disappear and that convergence to the steady state is again obtained. Since such a choice strengthens the static Keynes-effect it is not surprising to see that lower values of h_2 will increase the stability of the dynamics, although this is far from being obvious from a mathematical point of view in this now three-dimensional growth context. Sufficiently low values for h_2 thus make the steady state asymptotically stable, no longer allowing for the persistent business cycle of Fig. 22.3. Wage flexibility and, even more so, interest rate flexibility are thus good for economic stability in the present context.

Simulations of the four-dimensional case with backward looking adaptive expectations, however, show that these two stabilizers are insufficient for ensuring even the boundedness of the dynamics (not to speak of convergence to the steady state). This confirms the results for the medium run dynamics considered in Sect. 22.4. Adaptive expectations, if adjusted sufficiently fast, always destroy the viability of the model. The simulation depicted in Fig. 22.4, where an adjustment

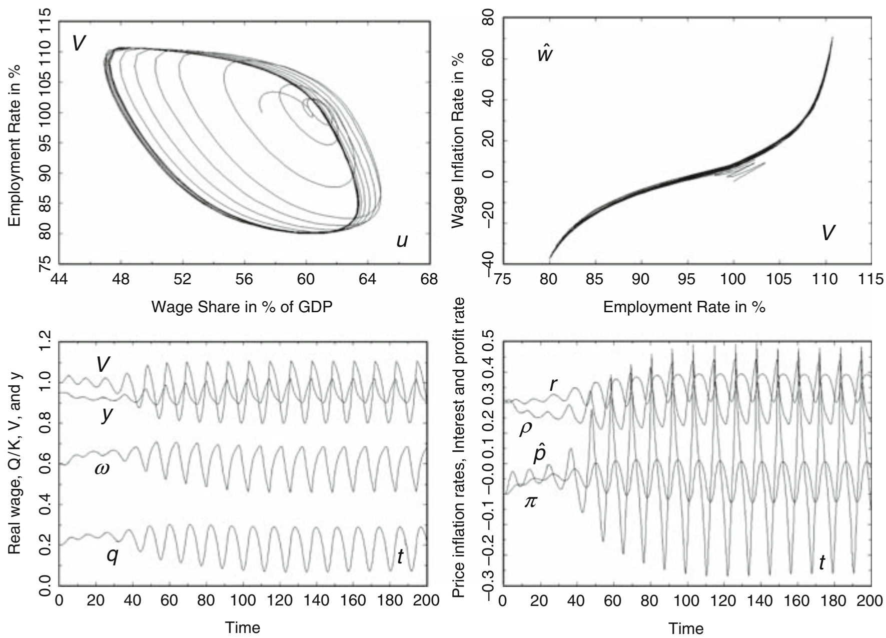

Fig. 22.4 The destabilizing role of the Mundell effect

speed β_π^1 of only 0.15 is used, leads us back to a limit cycle result. Note that we have made use here of a monetary shock (which doubled the growth rate of the money supply), which leads to a long transition phase until the limit cycle is approached. The assumed nonlinearity in wage-adjustment is no longer (close to being) sufficient for making the dynamics (22.23)–(22.24) viable even for smaller sizes of the parameter β_π^1. It appears to be necessary to add further appropriate nonlinearities to the model in order to ensure its viability in general. This, however, is not an easy task and is not pursued here.[19]

Taken together, the analysis of Sects. 22.4 and 22.5, and the shown and further numerical results (not shown) suggest that the common belief in wage-flexibility as the appropriate means of ensuring convergence to full employment steady state growth in a monetary economy cannot be considered as being well-founded even in a "Keynesian" model with as many neoclassical features as possible, i.e., a model which provides a favorable environment for the proof of such a proposition. Of course, most favorable for neoclassical views on the relevance of the full-employment growth path are the assumptions $\beta_p = \beta_w = \infty = \beta_\pi^1 \quad (\beta_\pi^2 = 0)$ which establish the Classical model of Sargent (1987, Chap. I), since these assumptions imply for the model (22.1)–(22.18) that $I = S$, $V = 1$ and $\hat{p} = \pi$. Yet, the relevance of the full employment position is then given by assumption and not by the proof that similar situations with high, but still finite adjustment speeds will give rise to the same qualitative results.

22.7 Conclusions

This chapter has developed and analyzed a monetary growth model of the Keynes–Wicksell type that is more general that any that has appeared in the literature to date. Analytical results obtained for the medium-run dynamics reveal that convergence to the steady state may be endangered if wages are sufficiently *inflexible* and prices are sufficiently *flexible*. Moreover, if the adjustment of inflationary expectations is adaptive and sufficiently rapid, local instability of the steady state is inevitable. Forward-looking expectations make stability more likely, but do not guarantee it. Specifically, if the stabilizing Keynes and marginal productivity effects are outweighed by the potentially destabilizing Rose effect, the steady state even under regressive expectation is unstable. For the case of long-run dynamics with capital accumulation, it is again found that wage inflexibility and price flexibility are destabilizing. The results obtained analytically for special cases of the model are confirmed by simulation results obtained for the complete four-dimensional system, some of which have been considered in the preceding section. Local instability need not imply economic nonviability of trajectories, provided that the model has a

[19] Among the various nonlinearities that might serve to constrain the dynamics at a distance from the steady state are the liquidity effects modeled in Foley (1987) or the balance of payments effects considered in Sethi (1992).

nonlinear structure: the specific type of nonlinearity explored here takes the form of rising wage flexibility as the economy moves away from the steady state employment level.

The destabilizing effects of price flexibility accord with the conjectures of Keynes and "unreconstructed" Keynesians such as Tobin, although the stabilizing effects of wage flexibility do not. This latter feature should not necessarily be surprising due the supply-side nature of Keynes–Wicksell models in which output is always at full capacity and the real wage continuously equals the marginal product of labor. Allowing for excess capacity and the possibility that firms are off their labor demand curve in the short run may well give rise to different results, but this extension is beyond the scope of the present work.

Mathematical Appendix

Derivation of the Dynamic Equations (22.19)–(22.22)

To obtain the dynamic equations of the model, begin by solving (22.13)–(22.14) for the variables $\hat{w} - \pi$, $\hat{p} - \pi$ to get

$$\begin{aligned} \hat{w} - \pi &= \beta_w(l/l^s - 1) + \sigma_w(\hat{p} - \pi) \\ \hat{p} - \pi &= \beta_p(i(\omega, m, \pi) - s(\omega)) + \sigma_p(\hat{w} - \pi) \end{aligned}$$

which for $\sigma_w \cdot \sigma_p \neq 1$ imply

$$\begin{aligned} \hat{w} - \pi &= \frac{1}{1 - \sigma_w\sigma_p}(\beta_w(l/l^s - 1) + \sigma_w\beta_p(i(\omega, m, \pi) - s(\omega))), \\ \hat{p} - \pi &= \frac{1}{1 - \sigma_w\sigma_p}(\beta_p(i(\omega, m, \pi) - s(\omega)) + \sigma_p\beta_w(l/l^s - 1)). \end{aligned}$$

These two equations in turn give rise to our first differential equation:

$$\hat{\omega} = \frac{1}{1 - \sigma_w\sigma_p}((1 - \sigma_p)\beta_w(l(\omega)/l^s - 1) + (\sigma_w - 1)\beta_p(i(\omega, m, \pi) - s(\omega))).$$

The next differential equation (for the dynamics of $l^s = L^s/K$) is given by

$$\hat{l}^s = n - \hat{K} = \begin{cases} n - i(\omega, m, \pi) & \text{if} \quad \beta_k = 1, \\ n - s(\omega) & \text{if} \quad \beta_k = 0. \end{cases}$$

For $\hat{m} = \hat{M}^s - (\hat{p} - \pi) - \pi - \hat{K}$ we get the following, when $\beta_k = 0$:

$$\hat{m} = \mu - \frac{1}{1 - \sigma_w\sigma_p}(\beta_p(i(\omega, m, \pi) - s(\omega)) + \sigma_p\beta_w(l(\omega)/l^s - 1)) - \pi - s(\omega).$$

The corresponding equation when $\beta_k = 1$ is obtained simply by replacing the last term by $i(\omega, m, \pi)$. Finally, for $\dot{\pi}$, (22.15) yields

$$\dot{\pi} = \beta_\pi^1 \left(\frac{1}{1 - \sigma_w \sigma_p} (\beta_p (i(\omega, m, \pi) - s(\omega)) + \sigma_p \beta_w (l(\omega)/l^s - 1)) \right) \\ + \beta_\pi^2 (\mu - n - \pi).$$

Given the definition of the excess demand functions $X^w(\omega, l^s)$ and $X^p(\omega, m, \pi)$ in the text, the four dynamic equations above are equivalent to (22.19)–(22.22).

Proof of Proposition 1.
Parts a/b: By means of the many proportionalities that exist between parts of the rows of the Jacobian matrix, the following expression for the determinant of J is easily obtained

$$\det J = \sigma^2 \omega m \beta_\pi^1 ((1 - \sigma_p) + \sigma_p (1 - \sigma_w)) X_\omega^w X_m^p < 0,$$

since we have $\sigma_p, \sigma_w \in [0, 1], \sigma_p \sigma_w \neq 1$. Note that the sign of the determinant depends only on the sign of the marginal productivity effect and the Keynes effect. Next, for the three principal minors of J, we get

$$J_3 = \begin{vmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{vmatrix} = -\sigma^2 \omega m [((1 - \sigma_p) + \sigma_p (1 - \sigma_w)) X_\omega^w X_m^p] > 0,$$

$$J_2 = \begin{vmatrix} J_{11} & J_{13} \\ J_{31} & J_{33} \end{vmatrix} = \sigma^2 \omega \beta_\pi^1 [((1 - \sigma_p) + \sigma_p (1 - \sigma_w)) X_\omega^w X_\pi^p] < 0,$$

$$J_1 = \begin{vmatrix} J_{22} & J_{23} \\ J_{32} & J_{33} \end{vmatrix} = \sigma^2 m \beta_\pi^1 X_m^p > 0.$$

Note that the signs of these minors are again dominated by the signs of the marginal productivity effect and the Keynes effect, except for J_2 where the Mundell effect is also present. Provided that $\beta_\pi^1 \neq 0$, however, the sign of this minor remains negative as a result of the marginal productivity effect.

For the trace of J we have the expression

$$\text{trace } J = \sigma [(1 - \sigma_p) X_\omega^w \omega + (\sigma_w - 1) X_\omega^p \omega - X_m^p m + X_\pi^p \beta_\pi^1],$$

which shows that the stability condition trace $J < 0$ is supported by the marginal productivity effect and the Keynes effect and endangered by the Rose and Mundell effects. Note also that the marginal productivity and the Rose effect can be suppressed by choosing σ_p or σ_w equal to 1, while the presence and strength of the Mundell effect depends on the size of the parameter β_π^1. The expression for the trace of the matrix J thus shows that there are two real and two monetary forces present which work in opposite directions with respect to the stability of the steady state.

Let x denote the term $(1-\sigma_p)X_\omega^w\omega+(\sigma_w-1)X_\omega^p\omega-X_m^pm$ in the trace of J. The Routh–Hurwitz conditions, which are necessary and sufficient for the local asymptotic stability of the steady state, are

$$a_1 = -\,\text{trace}\;\; J > 0,\; a_2 = J_1 + J_2 + J_3 > 0,$$
$$a_3 = -\det J > 0,\; b = a_1a_2 - a_3 > 0.$$

Case 1 $x \geq 0$ at the steady-state $\Longrightarrow \beta_\pi^H = 0$, since we then have trace $J > 0$ for all parameter values $\beta_\pi^1 > 0$.

Case 2 $x < 0$ at the steady state. We then have at $\beta_\pi^1 = 0$ the situation $a_1 > 0,\; a_2 > 0,\; a_3 = 0,\; b > 0$, so the steady state is stable. By continuity of these expressions with respect to changes in the parameter β_π^1, all Routh–Hurwitz conditions are fulfilled for β_π^1 positive and sufficiently small. Furthermore, there exists exactly one $\beta_\pi^{tr} > 0$ such that trace $J = 0$ at this β_π^1 value (trace $J < 0$ below it, of course). Note that b must be negative when the trace vanishes because $a_3 = -detJ$ is always positive. Also, since b is a quadratic function of β_π^1, there thus exist at least one and at most two values $0 < \beta_\pi^{H1} \leq \beta_\pi^{H2}$ such that $b(\beta_\pi^1) = 0$ holds (since $b(0) > 0$ and $b(\beta_\pi^{tr}) < 0$). It follows that b is negative between β_π^{H1} and β_π^{H2} (or above β_π^{H1} if there is only one positive root of $b(\beta_\pi^1) = 0$). Since $b(\beta_\pi^{tr}) < 0$, the value of β_π^H we are looking for must be smaller than β_π^{tr}. On the other hand, should there exist a second zero β_π^{H2} of $b(\beta_\pi^1)$ it must be greater in value than β_π^{tr}. The choice $\beta_\pi^H = \beta_\pi^{H1}$ thus proves parts (a) and (b) of the proposition, since $a_2 = 0$ implies $b < 0$ which can only happen when $\beta_\pi^1 > \beta_\pi^H$.

Part c: Since $a_3 = -\det J > 0$ holds throughout, there can be only two purely imaginary eigenvalues ($\neq 0$) at β_π^H together with one real and negative one, due to Orlando's formula for the eigenvalues λ_i of the matrix J (see Gantmacher 1954, pp. 173–174):

$$0 = b(\beta_\pi^H) = -(\lambda_1 + \lambda_2)(\lambda_1 + \lambda_3)(\lambda_2 + \lambda_3)$$

Below β_π^H, but sufficiently close to β_π^H we thus have that all eigenvalues have negative real part with two of them being complex (non-real) numbers, while beyond β_π^H, and sufficiently close, we have one negative real eigenvalue and two complex ones with positive real part. This proves part (c) of the proposition.[20]

Part d: Due to Orlando's formula and the fact that det J is negative throughout, the negative eigenvalue of J must remain negative for all $\beta_\pi^1 > 0$, since the other two eigenvalues cannot cross the imaginary axis a second time.

[20] We do not present an elaborate account of the conditions which establish a Hopf bifurcation here since, as is usually the case, one cannot decide whether the bifurcation occurring in the present model is subcritical, supercritical or of a linear (vertical) type. See Benhabib and Miyao (1981) for further details.

Proof of Proposition 2. With respect to the notation employed in the proof of the preceding proposition, we have:

(a) $\det \tilde{J} = J_3 \cdot (-\beta_\pi^2) < 0 \quad (since J_3 > 0)$
(b) $\tilde{J}_2 = -\beta_\pi^2 \sigma\omega[(1-\sigma_p)X_\omega^w + (\sigma_w - 1)X_\omega^p]$
(c) $\tilde{J}_1 = \beta_\pi^2 \sigma m X_m^p > 0$
(d) $\tilde{J}_3 = J_3 > 0$
(e) $\text{trace } \tilde{J} = \sigma[(1-\sigma_p)X_\omega^w \omega + (\sigma_w - 1)X_\omega^p \omega - m X_m^p] = \sigma x$

We thus get $\tilde{J}_1 + \tilde{J}_2 = -\beta_\pi^2 \sigma x$ and both $a_2, a_1 > 0$ if and only if $x < 0$ holds. Furthermore, $b = a_1 a_2 - a_3$ is always positive if both a_1 and a_2 are positive. This proves the proposition.

Proof of Corollary 1. The Jacobian J of (22.23)–(22.24) at the steady state is given by

$$\begin{pmatrix} \sigma x \omega & \sigma(1-\sigma_p)X_{l^s}^w \omega \\ -(1-c)y_\omega & 0 \end{pmatrix} = \begin{pmatrix} ? & - \\ + & 0 \end{pmatrix}$$

which shows that the unique equilibrium ω_0, l_0^s is unstable if $x > 0$ and asymptotically stable if $x < 0$. The set L_ω thus cannot contain the point (ω_0, l_0^s) if $x > 0$ holds, if it is derived from points different from the point of rest itself.

References

Benassy, J.-P. (1986). A non-Walrasian model of the business cycle. In: R. Day & G. Eliasson (Eds.), *The dynamics of market economies*. Amsterdam: North Holland.

Benhabib, J., & Miyao, T. (1981). Some new results on the dynamics of the generalized Tobin model. *International Economic Review, 22*, 589–596.

Chiarella, C. (1990). *The elements of a nonlinear theory of economic dynamics*. Heidelberg: Springer.

Chiarella, C. (1992). The dynamics of speculative behavior. *Annals of Operations Research, 37*, 101–123.

Conlisk, J. (1988). Optimization cost. *Journal of Economic Behavior and Organization, 9*, 213–228.

Dornbusch, R. (1976). Expectations and exchange rate dynamics. *Journal of Political Economy, 84*, 1161–1176.

Evans, G. W., & Ramey, G. (1992). Expectation calculation and macroeconomic dynamics. *American Economic Review, 82*, 207–224.

Fischer, S. (1972). Keynes–Wicksell and neoclassical models of money and growth. *American Economic Review, 62*, 880–890.

Flaschel, P. (1993). *Macrodynamics. Income distribution, effective demand, and cyclical growth*. Bern: Peter Lang.

Flaschel, P., Franke, R., & Semmler, W. (1994). *Nonlinear macrodynamics. Stability, fluctuations and growth in monetary economies*. Bielefeld: Unpublished manuscript.

Flaschel, P., & Sethi, R. (1998). The Stability of models of money and perfect foresight: Implications of nonlinearity. *Economic Modelling, 16*, 221–233.

Foley, D. (1987). Liquidity-profit rate cycles in a capitalist economy. *Journal of Economic Behavior and Organization, 8*, 363–376.
Franke, R. (1992). Stable, unstable, and cyclical behavior in a Keynes–Wicksell monetary growth model. *Oxford Economic Papers, 44*, 242–256.
Frankel, J. A., & Froot, K. A. (1986). Understanding the US dollar in the eighties: The expectations of chartists and fundamentalists. *Economic Record, 62*, 24–38.
Gantmacher, F. R. (1954). *Theory of matrices*. New York: Interscience.
Goodwin, R. M. (1967). A growth cycle. In: C. H. Feinstein (Ed.), *Socialism, capitalism and economic growth*. Cambridge, UK: Cambridge University Press.
Gray, M. R., & Turnovsky, S. J. (1979). The stability of exchange rate dynamics under perfect foresight. *International Economic Review, 20*, 643–660.
Groth, C. (1988). IS-LM dynamics and the hypothesis of combined adaptive-forward-looking expectations. In: P. Flaschel & M. Krüger (Eds.), *Recent approaches to economic dynamics*. Bern: Peter Lang.
Groth, C. (1992). Keynesian-Monetarist dynamics and the corridor: A note. University of Copenhagen: mimeo.
Hirsch, M., & Smale, S. (1974). *Differential equations, dynamical systems, and linear algebra*. New York: Academic Press.
Keynes, J. M. (1936). *The general theory of employment, interest and money*. London: Macmillan.
Orphanides, A., & Solow, R. (1990). Money, inflation and growth. In: F. Hahn (Ed.), *Handbook of monetary economics*. Amsterdam: North Holland.
Rose, H. (1967). On the non-linear theory of the employment cycle. *Review of Economic Studies, 34*, 153–173.
Rose, H. (1990). *Macrodynamics. A Marshallian synthesis*. London: Basil Blackwell.
Sargent, T. (1973). Interest rates and prices in the long run: A study of the Gibson paradox. *Journal of Money, Credit and Banking, 5*(1), 385–449.
Sargent, T. (1987). *Macroeconomic theory*. New York: Academic Press.
Sethi, R. (1992). Endogenous growth cycles in an open economy with fixed exchange rates. *Journal of Economic Behavior and Organization, 19*, 327–342.
Sethi, R., & Franke, R. (1995). Behavioral heterogeneity under evolutionary pressure: Macroeconomic implications of costly optimisation. *Economic Journal, 105*, 583–600.
Stein, J. (1971). *Money and capacity growth*. New York: Columbia University Press.
Stein, J. (1982). *Monetarist, Keynesian and new classical economics*. London: Basil Blackwell.
Tobin, J. (1975). Keynesian models of recession and depression. *American Economic Review, Papers and Proceedings, 55*, 195–202.
Tobin, J. (1993). Price flexibility and output stability: An old Keynesian view. *Journal of Economic Perspectives, 7*, 45–65.